On Generalized Convergence and Related Concepts for Sequences of Fuzzy Numbers

Thesis

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in

MATHEMATICS

by

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Certificate

This is to certify that the thesis entitled **"On Generalized Convergence and related Concepts for Sequences of Fuzzy Numbers"**, submitted by **Pankaj Kumar (Reg. No: 950911011)**, in fulfillment of the requirement for the award of the degree of *Doctor of Philosophy* in School of Mathematics, Thapar University, Patiala is a record of candidate's own independent and original research work carried out by him under our supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any other University or Institute for the award of any degree.

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Declaration

It is certified that the thesis is entirely my own and that the ideas and references cited herein have been duly acknowledged.

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Dedicated

To

My Parents

To my supervisors ...

Gur bin ghor andhar guru bin samajh na avai
Gur bin surat na sidh guru bin mukat na pavai.

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Thapar University, Patiala.
March, 2016

(Pankaj Kumar)

Preface

The present thesis entitled **"On Generalized Convergence and related Concepts for Sequences of Fuzzy Numbers"** comprises certain investigations carried out by me at the School of Mathematics (SOM), Thapar University, Patiala, under the supervision of Dr. S. S. Bhatia, Professor, SOM and Dean of Academic Affairs, Thapar University, Patiala and Dr. Vijay Kaushik, Associate Professor, HCTM Technical Campus, Kaithal, Haryana.

Modern analysis is mainly concerned in finding the limiting value of a sequence. H. Fast [45] provided the major breakthrough in the direction by generalizing the concept of ordinary convergence and called it statistical convergence. Fridy [49] accelerated developed of statistical convergence and presented statistical analogue of many concepts known in the theory of usual convergence.

In recent years much attention has been paid to generalize the basic concepts of classical analysis in fuzzy setting and thus a modern theory of fuzzy analysis is developed. Theory of fuzzy numbers plays an important role in the development of fuzzy analysis. Consequently, in present thesis, we have made efforts to develop or extend certain generalized summablity methods to the sequences in fuzzy environment.

The whole work in the thesis has been divided into six chapters. **Chapter 1** is an introductory one in which we begin with notion of natural density and show how Fast used it to define statistical convergence. After making some remarks on the early development of statistical convergence, we give some interesting extensions of statistical convergence. Finally, in this chapter, we show that how the ideas of statistical convergence are extended to the sequences of fuzzy numbers. Further, a brief plan of the results presented in the subsequent chapters is given.

In **Chapter 2**, the concept of statistical convergence has been extended to the sequences of fuzzy numbers having multiplicity greater than two and develop some of its basic structural properties. In this chapter the concepts of Cauchy sequences and the Cauchy convergence criteria has been presented. Some generalized summability methods as Cesaro summability and strong p-Cesaro summability of multiple sequences of fuzzy numbers have also been introduced. Some connections between the statistical convergence and the new generalized summability methods are discussed. Finally, a stronger generalization of statistical convergence for multiple sequences has been introduced.

The objective of **Chapter 3** is to define the notions of lacunary statistical limit point and lacunary statistical cluster points for sequences of fuzzy numbers with help of a lacunary sequence as these notions help in predicting the behavior of such a sequence which is not lacunary statistical convergent. The sets of lacunary statistical limit points and lacunary statistical cluster points have also been defined and some inclusion relation between these sets and the sets of ordinary statistical limit points and cluster points are obtained in this chapter. Also, a condition for which these sets coincide with the sets of ordinary statistical limit points and cluster points has been obtained. Further, the lacunary statistical limit and cluster points for sequences of fuzzy numbers with the help of difference operator has been studied in this chapter.

The goal of **Chapter 4** is to introduce the notions: lacunary statistical limit inferior, lacunary statistical limit superior, lacunary statistical boundedness and lacunary statistical core for sequences of fuzzy numbers. Some inclusion relations between these notions have been obtained. It has also been proved that if a sequence of fuzzy numbers is lacunary statistical convergent, then lacunary statistical limit inferior and lacunary statistical limit superior coincide with the lacunary statistical limit of the sequence. Subsequently, certain connections between lacunary statistical core and statistical core of a sequence of fuzzy numbers have been obtained. Finally, some new extensions of the above notions have been obtained by using the difference operator.

In **Chapter 5**, concepts of λ-statistical limit inferior, λ-statistical limit superior, λ-statistical bounded sequence and λ-statistical core for sequences of fuzzy numbers have been defined where $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. In addition, for $\beta \in (0, 1]$, concepts of λ-statistical limit points, λ-statistical cluster points, λ-statistical limit inferior and λ-statistical limit superior of order β are also introduced for sequences of fuzzy numbers.

Chapter 6 deals with the concepts of α-statistical convergence and statistical convergence for sequences in fuzzy neighborhood spaces which turn out to be a very useful subcategory of fuzzy topological spaces. Some properties of α-statistical convergence and statistical convergence in these spaces have been developed. Subsequently, certain generalizations of statistical convergence in fuzzy neighborhood spaces have been extended and some relevant connections with α-statistical convergence and statistical convergence have been discussed.

Research Publications

Papers Published/Accepted in Referred Journals

1. Pankaj Kumar, V. Kumar and S.S. Bhatia "Multiple sequences of fuzzy numbers and their statistical convergence", **Mathematical Sciences 2012, 6: 2**. (Springer)

2. Pankaj Kumar, S.S. Bhatia and V. Kumar "On Lacunary Statistical Limit and Cluster Points of Sequences of Fuzzy Numbers", **Iranian Journal of Fuzzy Systems, Vol. 10, No. 6, (2013) pp. 53-61**. (SCI)

3. Pankaj Kumar, V. Kumar and S.S. Bhatia "Lacunary Statistical Limit and Cluster Points of Generalized Difference Sequences of Fuzzy Numbers", **Advances in Fuzzy Systems, Vol. 2012, Article ID 459370, 6 pages, doi.org/10.1155/2012/459370**.

4. Pankaj Kumar, S.S. Bhatia and V. Kumar "On Generalized Core of Sequences of Fuzzy Numbers" is accepted in **British Journal of Mathematics and Computer Science**.

5. Pankaj Kumar, S.S. Bhatia and V. Kumar "Generalized sequential convergence in fuzzy neighborhood spaces", **National Academy Science Letters, Vol. 38, No. 2, (2015) pp. 161-164**. (SCI)

6. Pankaj Kumar, V. Kumar and S.S. Bhatia " $\mathcal{I}$ and $\mathcal{I}^*$-convergence of multiple sequences of fuzzy numbers", **Boletim da Sociedade Paranaense de Matemtica, Vol 33, No. 2, (2015) pp. 69-78**.

7. Pankaj Kumar, S.S. Bhatia and V. Kumar "On λ-statistical Limit Inferior and Limit Superior of order β for Sequences of Fuzzy Numbers", **General Mathematics Notes, Vol. 25, No. 2, (2014) pp. 6-18**.

8. Pankaj Kumar and S.S. Bhatia "A note on λ-statistical Limit Inferior and Limit Superior for Sequences of Fuzzy Numbers", **Pure and Applied Mathematics Letters, Vol. 1, (2013) pp. 34-37**.

Communicated Papers

1. Pankaj Kumar, S. S. Bhatia and V. Kumar, $S_\theta^{\Delta^m}$ -Core of Sequences of Fuzzy Numbers" **Bulletin of Calcutta Mathematical Society.**

List of Symbols

1.	$\mathbb{R}$	the set of real numbers.
2.	$\mathbb{N}$	the set of positive integers.
3.	χ_A	the characteristic function of the set A.
4.	$\delta(A)$	natural density of the A.
5.	$\overline{\delta}(A)$	upper natural density of the A.
6.	$\underline{\delta}(A)$	lower natural density of the A.
7.	s	the set of all statistically convergent sequences of numbers.
8.	ℓ_∞	the normed linear space of all bounded sequences of real numbers.
9.	L_x	the set of all ordinary limit points of the sequence x.
10.	Λ_x	the set of all statistical limit points of the sequence x.
11.	Γ_x	the set of all statistical cluster points of the sequence x.
12.	K-core	the Knopp's core of a real number sequence.
13.	s-core	the statistical core of a real number sequence.
14.	$\delta_\alpha(A)$	α-density of the A.
15.	s^α	the set of all statistically convergent number sequences of order α.
16.	s_θ	the set of all lacunary statistical convergent sequences of real numbers.

17. Λ_x^θ the set of all lacunary statistical limit points of the sequence x.

18. Γ_x^θ the set of all lacunary statistical cluster points of the sequence x.

19. s_λ the set of all λ-statistical convergent sequences of real numbers.

20. s_λ^α the set of all λ-statistical convergent real number sequences of order α.

21. $\Lambda_x(\mathcal{I})$ the set of all $\mathcal{I}$-limit points of the sequence x.

22. $\Gamma_x(\mathcal{I})$ the set of all $\mathcal{I}$-cluster points of the sequence x.

23. $L(\mathbb{R})$ the set of all upper semi-continuous, normal and convex fuzzy numbers.

24. ℓ_∞^F the set of all bounded sequences of fuzzy numbers.

25. S the set of all statistically convergent sequences of fuzzy numbers.

26. $\Lambda_S^F(X)$ the set of all statistical limit points of the sequence $X = (X_k)$ of fuzzy numbers.

27. $\Gamma_S^F(X)$ the set of all statistical cluster points of the sequence $X = (X_k)$ of fuzzy numbers.

28. S_θ the set of all lacunary statistically convergent sequences of fuzzy numbers.

29. S_λ the set of all λ-statistically convergent sequences of fuzzy numbers.

30. $\Lambda_{S_\lambda}^F(X)$ the set of all λ-statistical limit points of the sequence $X = (X_k)$ of fuzzy numbers.

31. $\Gamma_{S_\lambda}^F(X)$ the set of all λ-statistical cluster points of the sequence $X = (X_k)$ of fuzzy numbers.

32. Δ the difference operator.

33. w the set of all sequences of fuzzy numbers.

34. $\delta_3(K)$ the triple natural density of the set K.

35. S_3 the set of all statistical convergent triple sequences of fuzzy numbers.

36. $\ell_\infty^{F_3}$ the set of all bounded triple sequences of fuzzy numbers.

37. $w_p^{F_3}$ the space of all p-Cesàro summable triple sequence of fuzzy numbers.

38. $\mathcal{I}^3$ the set of all $\mathcal{I}_3$-convergent triple sequences of fuzzy numbers.

39. $\Lambda_{S_\theta}^F(X)$ the set of all S_θ-limit points of fuzzy number sequence $X = (X_k)$.

40. $\Gamma_{S_\theta}^F(X)$ the set of all S_θ-cluster points of fuzzy number sequence $X = (X_k)$.

41. $\Lambda_{S_\theta}(\Delta^m X)$ the set of all S_θ-limit points of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy number.

42. $\Gamma_{S_\theta}(\Delta^m X)$ the set of all S_θ-cluster points of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy number.

43. S-core the statistical core of sequences of fuzzy numbers.

44. S_θ-core the lacunary statistical core of sequences of fuzzy numbers.

45. $S_{\lambda_\beta}(X)$ the set of all λ-statistical convergent sequences $X = (X_k)$ of fuzzy numbers of order β.

46. $\Lambda_{S_\lambda^\beta}^F$ the set of all S_{λ^β}-limit points of fuzzy number sequence $X = (X_k)$.

47. $\Gamma_{S_\lambda^\beta}^F$ the set of all S_{λ^β}-cluster points of fuzzy number sequence $X = (X_k)$.

Table of Contents

Chapter 1

Introduction

1.1 General Introduction

Mathematical analysis is primarily concerned with the notion of limit of a sequence of real or complex numbers which forms the basis for the study of infinite series. One important branch of the field of infinite series is the study of summability of divergent sequences (series). This study is an attempt to attach (in some sense) a generalized limit to those sequences which do not converge in the usual sense, realizing at the same time that when the generalized limit is applied to a convergent sequence then it must agree with the limit in the ordinary sense. This procedure of assigning a new limit in generalized sense to divergent sequences is called a summability method. A familiar example of a generalized limit is well known and is given by Cèsaro as follows:

For any sequence $x = (x_k)$, if a new sequence $y = (y_n)$ is defined by

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{k=1}^{n} x_k$$

then, the sequence $x = (x_k)$ is said to be Cèsaro summable of order one to a number

L if the new sequence $y = (y_n)$ converges to L. In this case, we write $x_k \to L$ $(C, 1)$ or $\lim_{k\to\infty} x_k = L$ $(C, 1)$.
For example, if $x_k = (-1)^k (k \in \mathbb{N})$, then

$$y_n = \begin{cases} 0, & \text{for } n = 2, 4, 6, \cdots, \\ -\frac{1}{n}, & \text{for } n = 1, 3, 5, \cdots. \end{cases}$$

Obviously, $\lim_{n\to\infty} y_n = 0$, which shows that $x = (x_k)$ is $(C, 1) - $ *summable* to 0. That is,

$$\lim_{k\to\infty} x_k = 0 \ (C, 1).$$

This shows that a divergent sequence may be $(C, 1) - $ *summable*. Moreover, it is well known that all convergent sequences are $(C, 1) - $ *summable* to their limits. Not all divergent sequences are $(C, 1) - $ *summable* but repeating the process of arithmetic means may lead to a convergent sequence. This idea leads to $(C, 2) - $ *summable* sequences, and further extension leads to $(C, k) - $ *summable* sequences. These types of summability can also be presented by use of infinite matrix transformation. So, we now turn to the fact that how infinite matrix transformation can be used to define generalized limits.

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. For any sequence $x = (x_k)$, the new sequence $y = (y_n)$ defined by

$$y_n = Ax = \left(\sum_{k=1}^{n} a_{nk} x_k \right)_n$$

is called the transform sequence provided that each of the series in the above expression converges. A sequence $x = (x_k)$ is said to A-convergent (or A-summable) to L if Ax exists and is convergent with

$$\lim Ax = \lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} x_k = L.$$

The number L is called the A-limit of the sequence x. In this way, infinite matrix can be used to define a method of sequential convergence which we call a matrix method or a summability matrix method. Thus, the Cèsaro summability (generalized limit) as defined above can be considered as the matrix generalized limit generated by the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} \frac{1}{n}, & \text{for } k = 1, 2, \cdots n, \\ 0, & \text{for } k > n. \end{cases}$$

A matrix summability method is said to be regular or permanent if it preserves the convergence i.e., if $x = (x_k)$ is a convergent sequence with $\lim_{k\to\infty} x_k = L$, then the transform $y = Ax$ satisfies $\lim_{k\to\infty} y_k = L$. In other words $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = L$. Many famous mathematicians including Cèsaro, Hardy, Schur, Teoplitz etc. have made systematic investigations on matrix generalized limits and have contributed fundamentally in this area by using classical analysis.

Although, much contributions have been made by mathematicians to develop matrix summability methods but in 1951, Henry Fast [45] introduced a regular summability method that is not equivalent to any regular matrix method called as the statistical convergence. This well known method has been discussed in literature during past decades and forms the basis for the work presented in this doctoral thesis.

An outlook on historical development of statistical convergence and its generalizations is given in the subsequent sections of this chapter. Many interesting theorems and notions related to statistical convergence could not be mentioned owing to the limitations of the space. However, some of the basic definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

1.2 Statistical Convergence

In this section, we give a brief introduction and historical development on statistical convergence which was introduced over nearly 60 years ago and has become an active area of research nowadays. Although, the idea of statistical convergence goes back to the first edition of the monograph of Zygmund [113] (published in Warsaw in 1935), but formally, it was introduced by Steinhaus [103] and Fast [45] and later reintroduced by Schoenberg [102]. The idea of statistical convergence is based on the notion of natural or asymptotic density of subsets of $\mathbb{N}$, the set of positive integers. Before proceeding further the concept of natural density of subsets of $\mathbb{N}$ is presented below:

For any set $A \subset \mathbb{N}$, the natural density of A, denoted as $\delta(A)$, is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}|$$

provided it exists. Here, $|\{k \leq n : k \in A\}|$ denotes the number of elements of A not exceeding n; χ_A denotes the characteristic function of the set A and the sum appearing in the above expression runs over the cardinality of the set $\{k \leq n : k \in A\}$. If $\delta(A) = 0$, then A is said to be a thin subset, otherwise A is called a non-thin subset of $\mathbb{N}$. Since $\delta(A)$ may not exist for all subsets of $\mathbb{N}$, so it is sometimes convenient to use the upper natural density denoted by $\overline{\delta}(A)$ and is defined by

$$\overline{\delta}(A) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k) = \limsup_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}|.$$

Similarly, the lower natural density $\underline{\delta}(A)$ is defined by

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k) = \liminf_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}|.$$

We now recall some properties of δ, $\overline{\delta}$ and $\underline{\delta}$ which will be used many times throughout the present thesis.

For arbitrary subsets A and B of $\mathbb{N}$,

(i) If $\delta(A)$ exists then $\delta(A) = \underline{\delta}(A) = \overline{\delta}(A)$;

(ii) $\delta(A) \neq 0$ if and only if $\overline{\delta}(A) > 0$;

(iii) If $A \subset B$, then $\overline{\delta}(A) \leq \overline{\delta}(B)$;

(iv) If $\delta(A)$ exists, then $\delta(A^c)$ also exists and that $\delta(A) + \delta(A^c) = 1$;

(v) If A is finite set, then $\delta(A) = 0$;

(vi) If $A = a_1 < a_2 < \cdots < a_n < \cdots$ is an infinite set, then

$\delta(A) = \lim_{n\to\infty} \frac{n}{a_n}$, provided the limit exists;

(vii) $\delta(\mathbb{N}) = 1$;

(viii) $\delta(\{2n : n \in \mathbb{N}\}) = \delta(\{2n-1 : n \in \mathbb{N}\}) = \frac{1}{2}$;

(ix) $\delta(\{n^2 : n \in \mathbb{N}\}) = 0$;

(x) $\delta(\{n^2 : n \quad \text{is a prime}\}) = 0$.

We now turn towards first formal definition of statistical convergence given by Henry Fast [45] with the help of natural density.

Definition 1.2.1. [45] A number sequence $x = (x_k)$ is said to be statistical convergent to a number L provided that for each $\epsilon > 0$, we have

$$\delta(\{k \leq n : |x_k - L| \geq \epsilon\}) = 0.$$

In this case L is called the statistical limit of (x_k) and is abbreviated it as $s-\lim x_k = L$.

Let "s" denotes the set of all statistically convergent sequences of real or complex numbers.

As compared to usual convergence, statistical convergence replaces the set of all terms of the sequence which lies outside the ϵ-neighborhood of L with the set $\{k \leq n : |x_k - L| \geq \epsilon\}$ having natural density zero. Moreover, every convergent sequence is always statistical convergent and converges to the same limit, however converse need not be true in general as can be seen from the following example. Define a sequence

$x = (x_k)$ as follows.

$$x_k = \begin{cases} 1, & \text{if } k \text{ is a cube,} \\ 0, & \text{otherwise.} \end{cases}$$

Since for each $\epsilon > 0$,

$$\{k \leq n : |x_k| \geq \epsilon\} = \{k \leq n : k \quad \text{is a cube}\} \subset \{1^3, 2^3, 3^3, ...\}$$

and the latter set has density zero, it follows that $s - \lim x_k = 0$. But, clearly (x_k) is a divergent sequence.

This simple example shows the power and beauty of statistical convergence. Further, in literature many properties can also be seen which hold for ordinary convergence but do not hold true for statistical convergence. For instance: *"every subsequence of a convergent sequence is convergent to the same limit but this property is not always true for statistical convergence"*. Also, *"usual convergence always implies boundedness but statistical convergent sequences may not be bounded"*, which can be seen from the following example. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} \sqrt{n}, & \text{if } n = m^2, m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, clearly $s - \lim x_k = 0$ whereas (x_k) is an unbounded sequence.

Schoenberg [102] developed some basic properties of statistical convergence and also studied the concept as a summability method. Subsequently, in 1980, *Š*al*á*t [96] obtained the following important characterizations of statistical convergence.

Theorem 1.2.1. *[96] A sequence $x = (x_k)$ is statistically convergent to a number L, if and only if, there exists a set $K = \{k_1, k_2, k_3, ...\} \subset \mathbb{N}$, such that $\delta(K) = 1$ and* $\lim x_{K_k} = L$.

Theorem 1.2.2. *[96] The set of all bounded statistically convergent sequences of real numbers is a nowhere dense subset of ℓ_∞, the normed linear space of all bounded sequences of real numbers.*

The concept "*statistical convergence*" was applied by various authors in different areas of mathematics such as measure theory [79], approximation theory [37], finitely additive set function [32], in the study of subsets of the Stone-Chech compactification of the set of natural numbers [33], trigonometric series [113], locally convex spaces [76], summability theory [48], Banach spaces [31], number theory [41], the fuzzy set theory ([3], [9], [12]) and so on.

Although, statistical convergence was introduced in 1951, but a rapid development in this area started after the paper of John Fridy [49] who obtained an equivalent criterion for statistically convergent number sequences, similar to the Cauchy criterion of convergence as follows:

Definition 1.2.2. [49] A number sequence $x = (x_k)$ is said to be statistically Cauchy if for every $\epsilon > 0$, there exists a positive integer $m = m(\epsilon)$ such that $\delta(\{k \leq n : |x_k - x_m| \geq \epsilon\}) = 0$.

Theorem 1.2.3. *[49] A sequence is statistically convergent if and only if it is statistically Cauchy.*

In 1993, Fridy [50] developed statistical convergence as a sequential limit and following it, he thought about the introduction of the statistical analogue of limit and cluster points for a number sequence. For this purpose, he used the terminology of thin and non-thin subsequences. Later, in 1997, the author along with Orhan used the concepts of statistical limit and cluster points to define the statistical limit inferior and superior for number sequence. Before proceeding further it would to convenient to recall the following definitions from the papers Fridy [50] and Fridy and Orhan [53].

Let $x = (x_k)$ be a sequence and $(x)_K$, where $K = \{k_j : j \in \mathbb{N}\}$ be a subsequence of x. If the natural density $\delta(K) = 0$, then $(x)_K$ is said to be a thin subsequence. On the other hand, $(x)_K$ is said to be a non-thin subsequence of x if K does not have natural density zero. In this case, either $\delta(K) > 0$ or K fails to have natural density.

Definition 1.2.3. [50] Let $x = (x_k)$ be a number sequence. A number ξ is said to be a statistical limit point of x provided that there is non-thin subsequence of x that converges to ξ.

Let Λ_x denotes the set of all statistical limit points of the sequence $x = (x_k)$.

Definition 1.2.4. [50] Let $x = (x_k)$ be a number sequence. A number ξ is said to be a statistical cluster point of x provided that for each $\epsilon > 0$,

$$\limsup_n \frac{1}{n}(\{k \in \mathbb{N} : |x_k - \xi| < \epsilon\}) > 0.$$

Let Γ_x denotes the set of all statistical cluster points of the sequence x.

For a sequence $x = (x_k)$, define the sets

$$A_x = \{a \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k < a\}) \neq 0\};$$
$$B_x = \{b \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k > b\}) \neq 0\}.$$

The statistical limit inferior and superior of the sequence $x = (x_k)$ are defined as follows:

Definition 1.2.5. [53] Let $x = (x_k)$ be a number sequence, then the statistical limit inferior and statistical limit superior of x are respectively defined as follow:

$$s - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \phi, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$s - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi, \\ -\infty, & \text{otherwise.} \end{cases}$$

Note that for any number sequence $x = (x_k)$,

$$\liminf x \leq s - \liminf x \leq s - \limsup x \leq \limsup x.$$

Definition 1.2.6. [53] The real number sequence $x = (x_k)$ is said to be statistically bounded if there is a number B such that $\delta(\{k : |x_k| > B\}) = 0$.

In 2008, Çolak [25] introduced another interesting variation of statistical convergence called as statistical convergence of order α, where $\alpha \in (0, 1]$ and is defined as:

For $K \subset \mathbb{N}$, the α-density of K is defined by

$$\delta_\alpha(K) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : k \in K\}|$$

provided the limit exists.

Any finite subset of $\mathbb{N}$ has zero α-density and $\delta_\alpha(K^c) = 1 - \delta_\alpha(K)$ does not hold for $0 < \alpha < 1$ (in general) however, the equality holds for $\alpha = 1$. Moreover, for $\alpha = 1$ the α-density reduces to the natural density; and for any set $K \subset \mathbb{N}$, $\delta_\beta(K) \leq \delta_\alpha(K)$ where $0 < \alpha \leq \beta \leq 1$.

Definition 1.2.7. [25] A sequence $x = (x_k)$ of scalars is said to be statistically convergent of order α provided that for every $\epsilon > 0$ there is a number L such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $s^\alpha - \lim x_k = L$.

Let s^α denotes the collection of all statistically convergent sequences of order α.

For more work on statistical convergence, we refer- Cacan et al. [20], Cakan and Altay [19], Cacan et al. [21], Çolak et al. [28], Connor [30], Connor et al. [31], Conner and Kline [32], Edely and Mursaleen [39], etc.

1.2.1 Lacunary Statistical Convergence

In 1993, Fridy and Orhan [51] introduced another kind of generalization of statistical convergence called as lacunary statistical convergence with the help of a lacunary sequence. In a different paper, Fridy et al. [52] defined the Cauchy criterion for lacunary statistical convergence and have shown that it is equivalent to lacunary statistical convergence. In this section, we give some definition and results of the papers of Fridy et al. ([51], [52]) along with further developments of lacunary statistical convergence by different authors.

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is denoted by q_r.

Fridy and Orhan [51] defined lacunary statistical convergence by replacing the set $\{k : k \leq n\}$ in Definition 1.2.1. by the set $\{k : k_{r-1} \leq k \leq k_r\}$ as follows:

Definition 1.2.8. [51] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_n)$ of numbers is said to be lacunary statistically convergent or s_θ-convergent to a number L, provided that for every $\epsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$$

In this case, we write $s_\theta - \lim x = L$.

Let s_θ denotes the set of all lacunary statistical convergent sequences.

Definition 1.2.9. [52] A sequence $x = (x_k)$ is said to be lacunary statistically Cauchy or $s_\theta-$Cauchy if there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $k^{'}(r) \in I_r$, for each r, $\lim_{r\to\infty} x_{k'(r)} = L$ and for each $\epsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : |x_k - x_{k'(r)}| \geq \epsilon \right\} \right| = 0.$$

We now demonstrate some important definitions, examples and results on s_θ-convergence which will be frequently used and form the basis for our subsequent chapters.

Theorem 1.2.4. *[51] For any lacunary sequence θ, $s-\lim x = L$ implies $s_\theta - \lim x = L$ if and only if $\liminf_r q_r > 1$. If $\liminf_r q_r = 1$, then there exists a bounded S_θ-summable sequence that is not S-summable (to any limit).*

Theorem 1.2.5. *[51] For any lacunary sequence θ, $s_\theta - \lim x = L$ implies $s - \lim x = L$ if and only if $\limsup_r q_r < \infty$. If $\limsup_r q_r = \infty$, then there exists a bounded s-summable sequence that is not s_θ-summable (to any limit).*

Theorem 1.2.6. *[51] For any lacunary sequence θ, $s_\theta = s$ if and only if*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty;$$

Theorem 1.2.7. *[52] A sequence $x = (x_k)$ is $s_\theta-$convergent, if and only if it is a $s_\theta-$Cauchy sequence.*

Later, Demirci [35] defined $s_\theta-$limit and cluster points which are natural generalization of statistical limit and cluster points of a number sequence as follows:

Definition 1.2.10. [35] A number ξ is a lacunary statistical limit point of the sequence $x = (x_k)$, if there is a set $\{k_1 < k_2 < \cdots < k_r < \cdots\} \subseteq \mathbb{N}$, $\theta-$density of which is not zero, such that $\lim_{r\to\infty} x_{k_r} = \xi$.

Let Λ_x^θ denote the set of all lacunary statistical limit points of $x = (x_k)$.

Definition 1.2.11. [35] A point γ is a lacunary statistical cluster point of the sequence $x = (x_k)$, if for every $\epsilon > 0$,

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : |x_k - \gamma| < \epsilon\}| > 0$$

Let Γ_x^θ denote the set of all lacunary statistical cluster points of $x = (x_k)$. Later, in 2006, Pehliven *et al.* [92] introduced the $s_\theta-$analogue of ordinary cluster point in finite dimensional spaces following Demirci's [35] work and also discussed the closeness and compactness of the set of lacunary statistical cluster points. Apart from this, many aspects of $s_\theta-$convergence have been studied subsequently by different authors. For instance, Çakalli [22] introduced lacunary statistical convergence in metrizable topological groups and proved some inclusion relations between the sets s and s_θ. Çolak [24] studied lacunary strong convergence of difference sequences with the help of modulus function, whereas, Dutta [38] extended this notion for n-norms. For more interesting works on $s_\theta-$convergence, refer: Mursaleen and Mohiuddine [85], Li [73].

1.2.2 $\lambda-$Statistical Convergence

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \le \lambda_n + 1, \quad \lambda_1 = 1.$$

The generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

An interesting generalized summability by using de la Valée-Pousin mean is given by Leindler [72] as follows.

Definition 1.2.12. [72] A sequence $x = (x_k)$ is said to be V_λ-summable to a number L if

$$t_n(x) \to L \quad \text{as} \quad n \to \infty.$$

Let

$$V_\lambda = \left\{ x = (x_k) : \exists\, L \in \mathbb{R}, \quad \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |x_k - L| = 0 \right\}.$$

If $\lambda_n = n$, then V_λ-summablility reduces to $(C,1)$-summability, where

$$(C,1) = \left\{ x = (x_k) : \exists\, L \in \mathbb{R}, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \right\}.$$

Mursaleen [83] used V_λ-summability to generalize the concept of statistical convergence and called it λ-statistical convergence. To define this, Mursaleen used λ-density, a generalization of natural density.

For $K \subseteq \mathbb{N}$, the set of positive integers, the $\lambda-$density of K is defined by

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} \left|\{n - \lambda_n + 1 \le k \le n : k \in K\}\right|.$$

Definition 1.2.13. [83] A number sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent or $s_\lambda-$convergent to a number L if for each $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \left|\{k \in I_n : |x_k - L| \ge \epsilon\}\right| = 0.$$

In this case, we write $s_\lambda - \lim x_k = L$ or $x_k \to L(s_\lambda)$.

Let s_λ be the set of all $\lambda-$ statistically convergent sequences of numbers.

Note that for $\lambda_n = n$, $s_\lambda-$convergence reduces to statistical convergence and that $s_\lambda \subseteq s$ for all λ, since $\frac{\lambda_n}{n}$ is bounded by 1. However, converse of this i.e. $s \subseteq s_\lambda$ holds true if and only if

$$\liminf_n \frac{\lambda_n}{n} > 0.$$

Further, Çolak and Bektas [27] presented a stronger form of $\lambda-$statistical convergence called as "$\lambda-$statistical convergence of order α" where $\alpha \in (0,1]$ as follows:

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. For $\alpha \in (0, 1]$, the $\lambda_\alpha-$density of a subset $K \subseteq \mathbb{N}$ is defined as

$$\delta_\lambda^\alpha(K) = \lim_n \frac{1}{\lambda_n^\alpha}|\{k \in I_n : k \in K\}|,$$

provided the limit exists.

Definition 1.2.14. [27] The sequence $x = (x_k)$ of numbers is said to be $\lambda-$statistically convergent of order α if there is a number L such that

$$\lim_n \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $s_\lambda^\alpha - \lim x_k = L$ or $x_k \to L(s_\lambda^\alpha)$.

The set of all $\lambda-$statistically convergent sequences of order α is denoted by s_λ^α.

(i) For $\lambda_n = n$, $\lambda-$statistical convergence of order α reduces to statistical convergence of order α

(ii) For $\alpha = 1$, $\lambda-$statistical convergence of order α reduces to $\lambda-$statistical convergence.

(iii) For $\lambda_n = n$ and $\alpha = 1$, $\lambda-$statistical convergence of order α reduces to statistical convergence.

After the introduction of $\lambda-$statistical convergence and it's generalizations, many concepts known for statistical and lacunary statistical convergence have been introduced and studied analogously for $\lambda-$statistical convergence. For an extensive view refer: Alghamdi and Mursaleen [1], Çinar, Karakuş and Et [23], Et, Tripathy and Dutta [44], Ghosal [56], Savaş [100], Tuncer and Benli [106] etc.

1.2.3 $\mathcal{I}-$Convergence

In earlier sections, we have presented several new types of convergence of sequences out of which many are related to the statistical convergence as its generalizations.

Kostyrko, Macaj and Salàt [64] introduced $\mathcal{I}-$convergence by use of ideals of subsets of $\mathbb{N}$ which gives a unifying approach to these types of generalized convergence.

Let X be a non-empty set and $\mathcal{P}(X)$ denotes the power set of X, then we have the following definitions:

Definition 1.2.15. [64] A family of sets $\mathcal{I} \subset \mathcal{P}(X)$ is called an ideal in X if and only if

(i) $\emptyset \in \mathcal{I}$;

(ii) For each $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$;

(iii) For each $A \in \mathcal{I}$ and $B \subseteq A$, we have $B \in \mathcal{I}$.

Definition 1.2.16. [64] A nonempty family of sets $\mathcal{F} \subset \mathcal{P}(X)$ is called a filter on X if and only if

(i) $\emptyset \notin \mathcal{F}$;

(ii) For each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$;

(iii) For $A \in \mathcal{F}$ and $B \supseteq A$ we have, $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$.

Proposition 1.2.8. *If $\mathcal{I} \subset \mathcal{P}(X)$ is a non-trivial ideal in X, then the class $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \backslash A : A \in \mathcal{I}\}$ is a filter on X.*

The filter $\mathcal{F} = \mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

Definition 1.2.17. [64] A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(X)$ is called an admissible ideal in X if and only if it contains all singletons *i.e.*, if it contains $\{\{x\} : x \in X\}$.

Definition 1.2.18. [64] An admissible ideal $\mathcal{I} \subset \mathcal{P}(X)$ is said to be satisfy the condition (AP) if for every countable family of mutually disjoint sets $A_1, A_2, \cdots$ belonging to $\mathcal{I}$ there exists a countable family $B_1, B_2, \cdots$ belonging to $\mathcal{I}$ such that $A_i \triangle B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in \mathcal{I}$.

To unify many types of statistical convergence Kostyrko et al. [64] introduced $\mathcal{I}-$convergence as follows.

Definition 1.2.19. [64] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$convergent to L if for every $\epsilon > 0$, the set

$$A(\epsilon) = \{k : |x_k - L| \geq \epsilon\} \in \mathcal{I}.$$

The element L is called $\mathcal{I}-$limit of (x_k) and we write $\mathcal{I} - \lim x_k = L$.

Example 1.2.1. (i) If we take $\mathcal{I}_0 = \{\phi\}$, then clearly $\mathcal{I}_0$ is a non-trivial ideal. Moreover, a sequence is $\mathcal{I}_0$-convergent if and only if it is a constant sequence.

(ii) Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence in $\mathbb{R}$.

(iii) Put $\mathcal{I}_\delta = \{K \subseteq \mathbb{N} : \delta(K) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_\delta$-convergence coincides with the statistical convergence.

Kostyrko et al. [64] used Theorem 1.2.1. to introduce a new type of convergence that is closely related to $\mathcal{I}-$convergence and called it $\mathcal{I}^*-$convergence defined as follows:

Definition 1.2.20. [64] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}^*-$convergent to $L \in X$ if there exists a set $M = \{m_1, m_2, \cdots\} \subset \mathbb{N}$ such that $M \in \mathcal{F}(\mathcal{I})$ and $\lim x_{m_k} = L$.

Following results give the relationship between $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence for a number sequence.

Theorem 1.2.9. *[64] Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^* - \lim x_k = L$, then $\mathcal{I} - \lim x_k = L$.*

Theorem 1.2.10. *[64] $\mathcal{I} - \lim x_k = L$ implies $\mathcal{I}^* - \lim x_k = L$ if and only if $\mathcal{I}$ satisfies the condition (AP).*

Dems [36] used $\mathcal{I}$-convergence to define $\mathcal{I}$-Cauchy sequences and proved that Cauchy type condition is necessary and sufficient for $\mathcal{I}-$convergence for a number sequence. However, Kostyrko et al. [65] introduced the concepts of $\mathcal{I}-$limit and cluster points for a number sequence and obtained some interesting relations.

Definition 1.2.21. [36] For an admissible ideal $\mathcal{I}$, a sequence, $x = (x_k)$ of numbers is said to be $\mathcal{I}-$Cauchy sequence if for each $\epsilon > 0$, there exists a positive integer m such that the set $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in \mathcal{I}$.

Definition 1.2.22. [65] An element $\xi \in X$ is said to be an $\mathcal{I}-$limit point of the sequence $x = (x_k) \in X$ provided that there is a set $M = \{m_1, m_2, \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim x_{m_k} = \xi$.

Let, $\Lambda_x(\mathcal{I})$ denotes the set of $\mathcal{I}-$limit points of the sequence $x = (x_k)$

Definition 1.2.23. [65] An element $\gamma \in X$ is said to be an $\mathcal{I}-$cluster point of the sequence $x = (x_k) \in X$ if and only if $\{n \in \mathbb{N} : |x_k - \gamma| < \epsilon\} \notin \mathcal{I}$, for each $\epsilon > 0$.

Let, $\Gamma_x(\mathcal{I})$ denotes the set of $\mathcal{I}-$cluster points of the sequence $x = (x_k)$.

Following the concepts of $\mathcal{I}-$limit and cluster points Demirci [34] introduced the concepts of $\mathcal{I}-$limit inferior and $\mathcal{I}-$limit superior and obtained their certain properties. Lahiri and Das [71] continued with this study and obtained some other properties of $\mathcal{I}-$limit inferior and $\mathcal{I}-$limit superior which were not discussed by Demirci [34]. In what follows we give several definitions and properties from the papers Demirci

[34] and Lahiri and Das [71].

For a sequence $x = (x_k)$, define the sets

$$A_x = \{a \in \mathbb{R} : \{k \in \mathbb{N} : x_k < a\} \in \mathcal{I}\};$$
$$B_x = \{b \in \mathbb{R} : \{k \in \mathbb{N} : x_k > b\} \in \mathcal{I}\}.$$

$\mathcal{I}$-limit inferior and superior of x is defined as follows:

Definition 1.2.24. [34] Let $\mathcal{I}$ be an admissible ideal and x a real number sequence. Then the $\mathcal{I}$-limit inferior and $\mathcal{I}$-limit superior of x are respectively defined as follow:

$$\mathcal{I} - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \phi, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\mathcal{I} - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi, \\ -\infty, & \text{otherwise.} \end{cases}$$

Theorem 1.2.11. *[34] For any real number sequence x,*

$$\mathcal{I} - \liminf x \leq \mathcal{I} - \limsup x.$$

Definition 1.2.25. [34] The real number sequence $x = (x_k)$ is said to be $\mathcal{I}$-bounded if there is a number B such that $\{k : |x_k| > B\} \in I$.

Theorem 1.2.12. *[34] The $\mathcal{I}$-bounded sequence x is $\mathcal{I}$-convergent if and only if*

$$\mathcal{I} - \liminf x = \mathcal{I} - \limsup x.$$

Theorem 1.2.13. *[71] If $\mathcal{I} - \limsup x$ or $\mathcal{I} - \liminf x$ exits and equal to l, then there exists a subsequence of x that is $\mathcal{I}$-convergent to l.*

In 2008, Sahiner and Tripathy [95] introduced the notions of $\mathcal{I}$-convergence, $\mathcal{I}$-boundedness, $\mathcal{I}$-cluster point, $\mathcal{I}$-limit inferior and $\mathcal{I}$-limit superior and also investigated some further properties of $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of triple sequences.

Although, we have mentioned the introductory works on $\mathcal{I}$ and $\mathcal{I}^*$-convergence but these ideas have attracted many mathematician to work towards this direction. For an extensive view refer: Gezer and Karkus [55], Kostyrko et al. [66], Kumar [67], Mursaleen and Alotaibi [84], Mursaleen and Mohiuddine [86], Mursaleen et al. [87], Nabiey et al. [93], Savaş [99], Savaş and Das [101], Tripathy and Tripathy [105] etc.

1.2.4 Knopp's core and it's generalization

The notion of the core of a sequence was introduced by Knopp [63], who proved a fundamental theorem known as Knopp's core theorem. Knopp defined the core of a complex sequence as follows:

Definition 1.2.26. [63] For any complex number sequence $x = (x_k)$, let $R_k(x)$ be the least convex closed region of the complex plane which contains the points $x_k, x_{k+1}, x_{k+2}, \cdots$ i.e., the convex cover of these points, then obviously $R_1 \supset R_2 \supset R_3 \supset \cdots$. The closed and convex region $R = \cap_{k=1}^{\infty} R_k(x)$ is called the Knopp's core of the sequence $x = (x_k)$.

If the sequence $x = (x_k)$ is real then Knopp's core is define as the closed interval $[\liminf x, \limsup x]$.

For any sequence $x = (x_k)$, Knopp's core is denoted by K-core$\{x\}$.

The fundamental theorem proved by Knopp is as follows:

Theorem 1.2.14. *If A is a non-negative regular matrix, then*

$$K - core\{Ax\} \subseteq K - core\{x\}.$$

Fridy and Orhan [53] replaced limit points with statistical cluster points to define a natural analogue of Knopp's core and presented statistical analogue of Knopp's core theorem.

Definition 1.2.27. [53] If $x = (x_k)$ is a statistically bounded sequence, then statistical core of x is the closed interval $[s - \liminf x, s - \limsup x]$.

If (x_k) is not statistically bounded, then statistical core of x is defined as either $[s - \liminf x, +\infty]$, $[-\infty, +\infty]$ or $[-\infty, s - \limsup x]$ accordingly as $x = (x_k)$ is not statistically bounded above, statistically bounded or statistically bounded below respectively.

Denote the statistical core of x by s-core. It can be easily seen that s-core$\{x\}$ $\subset K$-core$\{x\}$.

Theorem 1.2.15. *If a matrix A satisfies $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$, then*

$$K - core\{Ax\} \subseteq s - core\{x\}$$

for every bounded sequence x.

The idea of Knopp's core is further extended by Demirci [34] by the use of $\mathcal{I}$-convergence and analogously obtained $\mathcal{I}$-analogue of Knopp's core theorem.

Definition 1.2.28. If $x = (x_k)$ is a $\mathcal{I}$-bounded sequence, then $\mathcal{I}$-core of x is the closed interval $[\mathcal{I} - \liminf x, \mathcal{I} - \limsup x]$.

Theorem 1.2.16. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. If a matrix A satisfies $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and the following conditions*

(i) A is regular and $\lim_n \sum_{k\in E} |a_{nk}| = 0$, whenever $E \in \mathcal{I}$,

(ii) $\lim_n \sum_{k=1}^{\infty} |a_{nk}| < 1$, *then*

$$K - core\{Ax\} \subseteq \mathcal{I} - core\{x\}$$

for every bounded sequence x*.*

1.3 Fuzzy Numbers Sequences

In this section, we provide sufficient background on fuzzy number sequences and their developments. We begin with the definition of fuzzy sets which were initially introduced by Zadeh [112] in 1965. These sets have wide applications in almost all areas of science and engineering. For instance; these sets are used not only to resolve problems in non linear dynamical systems [58], control of chaos [47], computer programming [57] , population dynamics [16], quantum physics [77] etc, but are also is used in various branches of mathematics, such as, in metric and topological spaces [2], theory of functions ([18], [61], [110]), the study of matrix and linear systems [91], approximation theory [10], etc.

Definition 1.3.1. [112] A fuzzy set (class) A in X is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point in X, a real number in the interval $[0, 1]$, with the value of $f_A(x)$ at x representing the "grade of membership" of x in A.

In past decades, much attention has been paid to generalize several concepts of classical analysis in fuzzy setting and thus a modern theory of fuzzy analysis is developed in which theory of fuzzy numbers plays a vital role. Infact, the theory of fuzzy numbers is the foundation of fuzzy analysis as the status of theory of real numbers in classical analysis.

In the following, we recall some basic definitions, results and terminology used in the theory of fuzzy numbers.

Given any interval A, we shall denote its end points by $\underline{A}$, $\overline{A}$ and by D the set of all closed bounded intervals on real line $\mathbb{R}$, i.e., $D = \{A \subset \mathbb{R}\colon A = [\underline{A}, \overline{A}]\}$. For $A, B \in D$, we define $A \leq B$ if and only if

$\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$

and

$d(A, B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}$. It is easy to see that d defines a Housdorff metric on D and (D, d) is a complete metric space. Also $\leq$ is a partial order on D. We define the fuzzy number as follows:

Definition 1.3.2. [112] A fuzzy number is a fuzzy set with membership function X from $\mathbb{R}$ to [0,1] which satisfies the following conditions:

(i) X is normal, i.e., there exists and $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$;

(ii) X is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$X(\lambda x + (1 - \lambda)y) \geq \min\{X(x), X(y)\}$;

(iii) X is upper semi-continuous, i.e., if for each $\epsilon > 0$, $X^{-1}([0, a + \epsilon))$ for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R}$.

(iv) The closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$ denoted by X^0 is compact.

The properties (i)-(iv) imply that for each $\alpha \in (0, 1]$, the α-level set,

$[X]_\alpha = \{x \in \mathbb{R} : X(x) \geq \alpha\} = [\underline{X}^\alpha, \overline{X}^\alpha]$

is a non-empty compact convex subset of $\mathbb{R}$. Let $L(\mathbb{R})$ denotes the set of all fuzzy numbers. Define a map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$\overline{d}(X, Y) = \sup_{\alpha \in (0,1]} d(X^\alpha, Y^\alpha)$.

Further, every real number r can be expressed as a fuzzy number $\overline{r}$ as follows:

$$\overline{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{otherwise.} \end{cases}$$

The arithmetic operations $L(\mathbb{R})$ are defined as follows:

$$[X \oplus Y]_\alpha = [\underline{X}^\alpha + \underline{Y}^\alpha, \overline{X}^\alpha + \overline{Y}^\alpha],\ [X \ominus Y]_\alpha = [\underline{X}^\alpha - \overline{Y}^\alpha, \overline{X}^\alpha - \underline{Y}^\alpha].$$

Puri and Ralescu [94] proved that $(L(\mathbb{R}), \overline{d})$ is a complete metric space. The ordered structure on $L(\mathbb{R})$ is defined as: For $X, Y \in L(\mathbb{R})$, we define $X \leq Y$ if and only if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in [0,1]$. We say that $X < Y$ if $X \leq Y$ and there exist $\alpha_0 \in [0,1]$ such that $\underline{X}^{\alpha_0} < \underline{X}^{\alpha_0}$ or $\overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}$. The fuzzy number X and Y are said to be incomparable if neither $X \leq Y$ nor $Y \leq X$.

A subset S of $L(\mathbb{R})$ is said to be bounded above if there exists a fuzzy number β such that $X \leq \beta$ for every $X \in S$. The fuzzy number β is called the upper bound of the set S and β is called the least upper bound (sup) of the set S if β is an upper bound of S and $\beta \leq \beta'$ for all upper bounds $\beta' \in S$. A lower bound and the greatest lower bound (inf) are defined on similar lines. The set S is said to be bounded if it is both bounded above and bounded below. It is noted that if the set $S \subset L(\mathbb{R})$ is bounded, then its supremum and infimum exist (can be seen in Wu and Wu [109]).

By a sequence $X = (X_n)$ of fuzzy numbers, we mean a function from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$. The fuzzy number X_k denotes the value of the function at $k \in \mathbb{N}$ and is called the kth term of the sequence. Let w denotes the set of all sequences of fuzzy numbers.

Matloka [78] introduced bounded and convergent sequences of fuzzy numbers as follows:

Definition 1.3.3. [78] A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for each $\epsilon > 0$, there exists a positive integer m such that $\overline{d}(X_k, X_0) \leq \epsilon$ for every $k \geq m$.

The fuzzy number X_0 is called the ordinary limit of the sequence (X_k) and we write $\lim_{k\to\infty} X_k = X_0$.

Definition 1.3.4. [78] A sequence $X = (X_k)$ of fuzzy numbers is said to bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded.

Let ℓ_∞^F denotes the set of all bounded sequences of fuzzy numbers.

Nanda [88] continued to deal these kind of sequences and introduced Cauchy sequences of fuzzy numbers where he proved that the spaces of bounded and convergent sequences of fuzzy numbers form complete metric space.

Definition 1.3.5. [88] A sequence $X = (X_k)$ of fuzzy numbers is said to be Cauchy sequence if for each $\epsilon > 0$, there exists a positive integer N_0 such that $\overline{d}(X_k, X_m) \leq \epsilon$ for every $k, m \geq N_0$.

Recently, statistical convergence has also been adopted to the sequences of fuzzy numbers. The credit goes to F. Nuray and E. Savaş [90], who first introduced statistical convergent and statistically Cauchy Sequences of fuzzy numbers.

Definition 1.3.6. [90] A sequence $X = (X_k)$ of fuzzy numbers is said to be statistical convergent to a fuzzy number X_0, written as $S - \lim_{k\to\infty} X_k = X_0$, if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Definition 1.3.7. [90] A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer N_0 such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : \overline{d}(X_n, X_{N_0}) \geq \epsilon\}| = 0.$$

Savaş [98] studied some equivalent alternative conditions for a sequence of fuzzy numbers to be statistically Cauchy and obtained the following characterization of statistical convergence for this type of sequences.

Theorem 1.3.1. *[98] A sequence $X = (X_k)$ of fuzzy numbers is statistically convergent to (X_0) if and only if there exists a subset $K = \{k_1, k_2, k_3, \cdots\} \subset \mathbb{N}$ with $\delta(K) = 1$ such that $\overline{d}(X_{k_n}, X_0) \to 0$ as $n \to \infty$.*

Using statistical convergence, S. Aytar [11] introduced statistical limit point and cluster point for sequences of fuzzy numbers. He also demonstrated some inclusion relations among the sets of: ordinary limit points, statistical limit points and statistical cluster points for this kind of sequences. In the following, we will recall some definitions and results of Aytar [11], Aytar et al. [13], Aytar and Pehlivan [14] and Aytar and Pehlivan [15].

Definition 1.3.8. [11] Let $X = (X_k)$ be a sequence of fuzzy numbers. A fuzzy number X_0 is said to be a statistical limit point (s.l.p) of a sequence (X_k) of fuzzy numbers provided that there is a non-thin subsequence of X that is convergent to X_0.

Let $\Lambda_S^F(X)$ denotes the set of all statistical limit points of a sequence $X = (X_k)$ of fuzzy number.

Definition 1.3.9. [11] Let $X = (X_k)$ be a sequence of fuzzy numbers. A fuzzy number Y_0 is said to be a statistical cluster point (s.c.p) of a sequence (X_k) of fuzzy numbers provided that for each $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : \overline{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let $\Gamma_S^F(X)$ denotes the set of all statistical cluster points of a sequence $X = (X_k)$ of fuzzy number.

Theorem 1.3.2. *[11] For a fuzzy number sequence $X = (X_k)$, $\Lambda_S^F(X) \subset \Gamma_S^F(X)$.*

Theorem 1.3.3. *[11] If $X = (X_k)$ and $Y = (Y_k)$ are sequences of fuzzy numbers such that $\delta(\{k \in \mathbb{N} : X_k \neq Y_k\}) = 0$, then $\Lambda_S^F(X) = \Lambda_S^F(Y)$ and $\Gamma_S^F(X) = \Gamma_S^F(Y)$.*

Theorem 1.3.4. *[11] Let $X = (X_k)$ be a sequence of fuzzy numbers. If $S-\lim_{k\to\infty} X_k = X_0$, then $\Lambda_S^F(X) = \Gamma_S^F(X) = \{X_0\}$.*

Definition 1.3.10. [14] A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically bounded from above if there exists a fuzzy number μ (called the statistical upper bound) such that

$$\delta(\{k \in \mathbb{N} : X_k > \mu\} \cup \{k \in \mathbb{N} : X_k \nsim \mu\}) = 0.$$

Similarly, $X = (X_k)$ is said to be statistically bounded from below provided that there exists a fuzzy number ν (called the statistical lower bound) such that

$$\delta(\{k \in \mathbb{N} : X_k < \nu\} \cup \{k \in \mathbb{N} : X_k \nsim \nu\}) = 0.$$

If the sequence $X = (X_k)$ is both statistically bounded from above and statistically bounded from below, then it is called statistically bounded.

Define the following sets:

$$\begin{aligned}
A_X &= \{\mu \in L(R) : \delta(\{k \in \mathbb{N} : X_k < \mu\}) \neq 0\};\\
\overline{A}_X &= \{\mu \in L(R) : \delta(\{k \in \mathbb{N} : X_k > \mu\}) = 1\};\\
B_X &= \{\mu \in L(R) : \delta(\{k \in \mathbb{N} : X_k > \mu\}) \neq 0\};\\
\overline{B}_X &= \{\mu \in L(R) : \delta(\{k \in \mathbb{N} : X_k < \mu\}) = 1\}.
\end{aligned}$$

Definition 1.3.11. [13] If $X = (X_k)$ be a statistically bounded sequence of fuzzy numbers, then the statistical limit inferior and statistical limit superior of $X = (X_k)$ are defined as:

$$S - \liminf X = \inf A_X \quad \text{and} \quad S - \limsup X = \sup B_X.$$

Theorem 1.3.5. *[13] If the sequence $X = (X_k)$ is statistically bounded, then*

$$\inf A_X = \sup \overline{A}_X \text{ and } \sup B_X = \inf \overline{B}_X.$$

Theorem 1.3.6. *[13] For a statistically bounded sequence of fuzzy numbers $X = (X_k)$,*

$$S - \liminf X_k \leq S - \limsup X_k.$$

Theorem 1.3.7. *[13] Assume that the statistically bounded sequence of fuzzy numbers $X = (X_k)$ is statistically convergent to X_0. Then*

$$S - \liminf X_k = S - \limsup X_k = X_0.$$

Theorem 1.3.8. *[15] Let $X = (X_k)$ be a statistically bounded sequence of fuzzy numbers. Let μ is the statistical limit superior of X and for each $\epsilon > 0$,*

$$\delta(\{k \in I_n : X_k \nsim \mu \oplus \bar{\epsilon}\}) = 0 \text{ and } \delta(\{k \in I_n : X_k \nsim \mu \ominus \bar{\epsilon}\}) = 0. \tag{1.3.1}$$

Then μ is the greatest statistical cluster point of X.

Theorem 1.3.9. *[15] Let $X = (X_k)$ be a statistically bounded sequence of fuzzy numbers. Let ν is the statistical limit inferior of X and for each $\epsilon > 0$,*

$$\delta(\{k \in I_n : X_k \nsim \nu \oplus \bar{\epsilon}\}) = 0 \text{ and } \delta(\{k \in I_n : X_k \nsim \nu \ominus \bar{\epsilon}\}) = 0. \tag{1.3.2}$$

Then ν is the least statistical cluster point of X.

Nuray [89] extended the ideas of lacunary statistical convergence and lacunary statistical Cauchy criterion for sequences of fuzzy numbers and proved their equivalence. We also list some results of Kwon and Shim [70] who proved Nurays result in reverse way.

Definition 1.3.12. [89] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary statistically convergent to a fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

In this case, we write $S_\theta - \lim_k X_k = X_0$.

Definition 1.3.13. [89] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary statistically Cauchy sequence if there is a subsequence $(X_{k'(r)})$ of X such that $k'(r) \in I_r$ for each r, $\lim_r X_{k'(r)} = X_0$ and for each $\epsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, X_{k'(r)}) \geq \epsilon\}| = 0.$$

Theorem 1.3.10. *[89] For any lacunary sequence θ, $S_\theta \subset S$ if* $\limsup_r q_r < \infty$.

Theorem 1.3.11. *[70] For any lacunary sequence θ, $S \subset S_\theta$ if* $\liminf_r q_r > 1$.

We next quote some definition and results related to λ-statistical convergence, λ-statistical limit and cluster points, which were respectively extended by Savaş [97] and Tuncer and Benli [107] for sequences of fuzzy numbers.

Recall that $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ and satisfying $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$.

Definition 1.3.14. [97] A sequence $X = (X_k)$ of fuzzy numbers is said to be λ-statistical convergent to some fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

In this case, we write $S_\lambda - \lim X_k = X_0$.

Definition 1.3.15. [107] For any sequence $X = (X_k)$ of fuzzy numbers, let $(X_{k(j)})$ be a subsequence of $X = (X_k)$ and $K = \{k(j) : j \in \mathbb{N}\}$. Denote $(X_{k(j)})$ by $(X)_K$. If

$$\lim_n \frac{1}{\lambda_n} |\{k(j) \in I_n : j \in \mathbb{N}\}| = 0,$$

then $(X)_K$ is called a $\lambda-$thin subsequence. On the other hand, $(X)_K$ is a $\lambda-$nonthin subsequence of X, provided that

$$\limsup_n \frac{1}{\lambda_n} |\{k(j) \in I_n : j \in \mathbb{N}\}| > 0.$$

Definition 1.3.16. [107] A fuzzy number X_0 is said to be a $\lambda-$statistical limit point or $S_\lambda-$limit point of a sequence $X = (X_k)$ of fuzzy numbers provided that there is a $\lambda-$nonthin subsequence of X that converges to X_0.

Let $\Lambda^F_{S_\lambda}(X)$ denotes the set of all $S_\lambda-$limit points of the sequence $X = (X_k)$.

Definition 1.3.17. [107] A fuzzy number Y_0 is said to be a $\lambda-$statistical cluster point or $S_\lambda-$cluster point of a sequence $X = (X_k)$ of fuzzy numbers provided that for each $\epsilon > 0$,

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let $\Gamma^F_{S_\lambda}(X)$ denotes the set of all $S_\lambda-$cluster points of the sequence $X = (X_k)$.

Kumar and Kumar [68] extended the ideas of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}$-Cauchy for sequences of fuzzy numbers and proved their equivalence. We list some definitions and results of Kumar and Kumar [68].

Definition 1.3.18. [68] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A sequence $X = (X_k)$ of fuzzy numbers is said to be $\mathcal{I}-$convergent to X_0 if for every $\epsilon > 0$, the set

$$A(\epsilon) = \{k : |x_k - L| \geq \epsilon\} \in \mathcal{I}.$$

The element X_0 is called $\mathcal{I}-$limit of (X_k) and we write $\mathcal{I} - \lim X_k = X_0$.

Definition 1.3.19. [68] Let $\mathcal{I}$ be a non-trivial ideal in $\mathbb{N}$. A sequence $X = (X_k)$ of numbers is said to be $\mathcal{I}^*-$convergent to a fuzzy number X_0 if and only if there exists a set $M = \{m_1, m_2, \cdots\} \subset \mathbb{N}$ such that $M \in \mathcal{F}(\mathcal{I})$ and $\overline{d}(X_{m_k}, X_0) \to 0$ as $n \to \infty$.

Following results give the relationship between $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence for a fuzzy number sequence.

Theorem 1.3.12. *[68] Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^* - \lim X_k = X_0$, then $\mathcal{I} - \lim X_k = X_0$.*

Theorem 1.3.13. *[68] $\mathcal{I} - \lim X_k = X_0$ implies $\mathcal{I}^* - \lim X_k = X_0$ if and only if $\mathcal{I}$ satisfies the condition (AP).*

Definition 1.3.20. [68] For an admissible ideal $\mathcal{I}$, a sequence, $X = (X_k)$ of fuzzy numbers is said to be $\mathcal{I}-$Cauchy sequence if for each $\epsilon > 0$, there exists a positive integer m such that the set $\{k \in \mathbb{N} : |X_k - X_m| \geq \epsilon\} \in \mathcal{I}$.

Theorem 1.3.14. *[68] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $X = (X_k)$ of fuzzy numbers is $\mathcal{I}$-convergent if and only if it is $\mathcal{I}$-Cauchy.*

The concept of statistical convergence, lacunary statistical convergence and λ-statistical convergence for sequences of fuzzy numbers have been further studied by Biglin [17], Isik [60], Esi [42] by use of difference operator Δ.

For w, the set of all sequences of fuzzy numbers, the operator $\Delta^m : w \to w$ is defined by

$$\begin{aligned}
\Delta^0 X_k &= X_k, \\
\Delta^1 X_k &= X_k - X_{k+1}, \\
\vdots &= \vdots \\
\Delta^m X_k &= \Delta^1(\Delta^{m-1} X_k).
\end{aligned}$$

Definition 1.3.21. [17] A sequence $X = (X_k)$ of fuzzy numbers is said to be Δ-statistically convergent to fuzzy numbers X_0 if for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n}|\{i \leq n : \overline{d}(\Delta X_k, X_0) \geq \epsilon\}| = 0,$$

i.e., $\overline{d}(\Delta X_k, X_0) < \epsilon$ for almost all k. In this case, we write $S^{\Delta}\text{-}\lim_k X_k = X_0$.

Definition 1.3.22. [60] A sequence $X = (X_k)$ of fuzzy numbers is said to be Δ^m-statistically convergent to a fuzzy number X_0 if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \in \mathbb{N} : \overline{d}(\Delta^m X_k, X_0) \geq \epsilon\}| = 0.$$

In this case, we write $S^{\Delta^m}\text{-}\lim_k X_k = X_0$.

Definition 1.3.23. [42] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary Δ^m-statistically convergent to a fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r}|\{k \in I_r : \overline{d}(\Delta^m X_k, X_0) \geq \epsilon\}| = 0.$$

In this case, we write $S_\theta^{\Delta^m}\text{-}\lim X_k = X_0$.

For some more works in this direction refer: Altin et al. [3], Altin et al. [5], Altinok and Mursaleen [9], Çolak [24], Çolak et al. [26], Altinok et al. [8], Et et al. [43], Çolak et al. [28], Subramanian and Esi [104], Altin and Çolak [4], etc.

1.4 Fuzzy Neighborhood Spaces

In this section, we provide a brief introduction on fuzzy neighborhood spaces which are special classes of fuzzy topological spaces and have been discussed in a number of papers: Change [29], Foster [46], Hutton [59], Wong [108], etc. So, before proceeding further, we give the definition of fuzzy topological spaces.

Definition 1.4.1. [29] A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions

(i) $\phi, X \in T$,

(ii) If $A, B \in T$, then $A \wedge B \in T$,

(iii) If $A_i \in T$ for each $i \in I$, then $\vee A_i \in T$.

T is called a fuzzy topology for X, and the pair (X, T) is a fuzzy topological space, or FTS in short.

Lowen [75] introduced fuzzy neighborhood spaces, a subcategory of fuzzy topological spaces, by the use of fuzzy neighborhoods which enables to introduce a local theory of fuzzy topologies.

Let $I = [0, 1]$, $I_0 = (0, 1]$, $I_1 = [0, 1)$ and $I_{0,1} = (0, 1)$.

For any set X, let I^X denote the set of all fuzzy subsets of X. A member ν of I^X is contained in a member μ of I^X denoted $\nu \leq \mu$ if and only if $\nu(x) \leq \mu(x)$, for every $x \in X$. Let ν, $\mu \in I^X$.

Define the following fuzzy sets:

(1) $\nu \wedge \mu \in I^X$ by $(\nu \wedge \mu)(x) = \min\{\nu(x), \mu(x)\}$ for every $x \in X$ (intersection).

(2) $\nu \vee \mu \in I^X$ by $(\nu \vee \mu)(x) = \max\{\nu(x), \mu(x)\}$ for every $x \in X$ (union).

(3) $\nu^C \in I^X$ by $\nu^C(x) = 1 - \nu(x)$ for every $x \in X$.

If X is a set, a non-empty collection $\Sigma(x) \subseteq I^X$ is called a prefilter (on X) iff:

(1) $0 \notin \Sigma(x)$,

(2) $\nu \in \Sigma(x), \mu \in \Sigma(x) \Rightarrow \nu \wedge \mu \in \Sigma(x)$,

(3) $\nu \in \Sigma(x), \nu \leq \mu \Rightarrow \mu \in \Sigma(x)$.

If X is a set, then a non-empty collection $\Delta(x) \subseteq I^X$ is called a prefilter base (on X) iff:

(1) $0 \notin \Delta(x)$,

(2) $\nu \in \Delta(x), \mu \in \Delta(x) \Rightarrow \exists \lambda \in \Delta(x) : \lambda \leq \nu \wedge \mu$.

Definition 1.4.2. [75] A family of prefilters $\Sigma = (\Sigma(x))_{x \in X}$ in a set X is called a fuzzy neighborhood system on X if and only the following are true:

(i) $\forall\, x \in X,\ \forall\, \nu \in \Sigma(x) : \nu(x) = 1;$

(ii) $\forall\, x \in X :\ \Sigma(x)^{\sim} = \Sigma(x);$

(iii) $\forall\, x \in X,\, \nu \in \Sigma(x),\, \forall\, \epsilon \in I_0,\, \exists(\nu_z^{\epsilon})_{z \in X} \in \prod_{z \in X} \Sigma(z) : \forall\, y, z \in X : \nu_x(z) \wedge \nu_z(y) \leq \nu(y) + \epsilon,$

where $\sim$ denotes the saturation operation as given by

$\Sigma^{\sim} = \{\nu : \forall\, \epsilon > 0, \exists \nu_{\epsilon} \in \Sigma : \nu_{\epsilon} - \epsilon \leq \nu\}.$

Given a fuzzy neighborhood system Σ on X; one can obtain a fuzzy topology $t(\Sigma)$, and the pair $(X, t(\Sigma))$ is known as fuzzy neighborhood space.

In the recent years, much work have been published on the sequential convergence in fuzzy neighborhood spaces by various authors Hakeim [40], Lowen [75], Mohannadi et al. [80], Morsi [81], Mashhour and Morsi [82], Wuyts and Lowen [111] etc. We, now quote some of the definitions and results of the papers of Lowen [75] and Wuyts and Lowen [111] as under:

Theorem 1.4.1. *[75] If $\Sigma = (\Sigma(x))_{x \in X}$ is a fuzzy neighborhood system on X, then $\Delta = (\Delta(x))_{x \in X}$ is a basis for Σ if and only if for all $x \in X$, $\Delta(x)$ is a prefilter basis and for all $\nu \in \Sigma(x)$ and for all $\epsilon \in I_0$ there exists $\theta_{\epsilon} \in \Delta(x)$ such that $\theta_{\epsilon} - \epsilon \leq \nu$.*

Theorem 1.4.2. *[75] If $(X, t(\Sigma))$ is a fuzzy neighborhood space and $\alpha \in I_1$, then the level topological space $(X, l_{\alpha}(t(\Sigma)))$ has a neighborhood system the family $(l_{\alpha}(\Sigma(x)))_{x \in X}$ where for all $x \in X$,*

$l_{\alpha}(\Sigma(x)) = \{\nu^{-1}(\beta, 1] : \nu \in \Sigma(x), \beta \in [0, 1 - \alpha)\}.$

Theorem 1.4.3. *[75] If (X, τ) is a topological space, then the associated fuzzy topological space $(X, \omega(\tau))$ has as fuzzy neighborhood system the family $\omega(\Sigma(x)_{x \in X})$, Where*

$\Sigma(x)$ *is the neighborhood system of* τ *and where for all* $x \in X$

$$\omega(\Sigma(x)) = \{\nu \in I^X, for\,each \quad \epsilon \in I_1, \nu^{-1}(0,1] \in \Sigma(x)\}.$$

Definition 1.4.3. [111] A fuzzy neighborhood space $(X, t(\Sigma))$ is said to be NT_2 if and only if for all $x \neq y \in X$ and for all $\epsilon \in I_0$ there exist $\nu_x \in \Sigma(x)$ and $\nu_y \in \Sigma(y)$ such that $\nu_x \wedge \nu_y < \epsilon$.

Khan et al. [62] introduced an interesting notion of convergence which they called α-convergence in fuzzy neighborhood spaces and studied some of its properties.

Definition 1.4.4. [62] Let $(X, t(\Sigma))$ be a fuzzy neighborhood space and (x_k) a sequence in X. For given $\alpha \in I_0$, (x_k) is said to be α-convergent to $x \in X$ if and only if for all $\nu \in \Sigma(x)$, there exists $m \in \mathbb{N}$ such that for all $k \geq m$

$$\nu(x_k) > 1 - \alpha.$$

x is called an α-limit of x_k and indicated as $x_k \rightsquigarrow^{\alpha} x$.

Definition 1.4.5. [62] In a fuzzy neighborhood space $(X, t(\Sigma))$, a sequence (x_k) in X is called convergent to $x \in X$ if and only if for all $\alpha \in I_0$, $x_k \rightsquigarrow^{\alpha} x$. We abbreviate it as $x_k \rightsquigarrow x$.

1.5 Objectives of the study

The research objectives of the present study are defined as:

1. To extend the idea of statistical convergence to sequences of fuzzy numbers of multiplicity greater than two.

2. To define the concepts of lacunary statistical limit and cluster points for sequences of fuzzy numbers and try to obtain some relations among the sets of

ordinary limit points, lacunary statistical limit points and lacunary statistical cluster points.

3. To define the concepts of lacunary statistical limit inferior, lacunary statistical limit superior, and lacunary statistical core for sequences of fuzzy numbers and try to obtain some properties related to these notions.

4. To extend the work of Tuncer and Benlin [2007, 46] by studying the concepts of statistical limit inferior, statistical limit superior and statistical core for sequences of fuzzy numbers.

5. To define generalized statistical convergence of nets in fuzzy topological spaces.

1.6 Thesis organization

The present thesis consists of six chapters. Each chapter is divided into various sections.

In **Chapter 2**, the concept of statistical convergence has been extended to the sequences of fuzzy numbers having multiplicity greater than two and develop some of its basic structural properties. In this chapter the concepts of Cauchy sequences and the Cauchy convergence criteria has been presented. Some generalized summability methods as Cesaro summability and strong p-Cesaro summability of multiple sequences of fuzzy numbers have also been introduced. Some connections between the statistical convergence and the new generalized summability methods are discussed. Finally, a stronger generalization of statistical convergence for multiple sequences has been introduced.

The objective of **Chapter 3** is to define the notions of lacunary statistical limit and cluster points for sequences of fuzzy numbers with help of a lacunary sequence as these notions help in predicting the behavior of such a sequence which is not lacunary statistical convergent. The sets of lacunary statistical limit points and lacunary statistical cluster points have also been defined and some inclusion relation between these sets and the sets of ordinary statistical limit points and cluster points are obtained. Also, a condition for which these sets coincide with the sets of ordinary statistical limit points and cluster points has been obtained. Further, the lacunary statistical limit and cluster points for sequences of fuzzy numbers with the help of difference operator has been studied in this chapter.

The goal of **Chapter 4** is to introduce the notions: S_θ-limit inferior, S_θ-limit superior, S_θ-boundedness and S_θ-core for sequences of fuzzy numbers. Some inclusion relations between these notions have been obtained. Some properties of S_θ-limit inferior and S_θ-limit superior have been derived. Subsequently, certain connections between S_θ-coreX and S_θ-coreX, where denotes a sequence of fuzzy numbers have been obtained. Finally, some new extensions of the above notions have been obtained by using the difference operator.

In **Chapter 5**, concepts of λ-statistical limit inferior, λ-statistical limit superior, λ-statistical bounded sequence and λ-statistical core for sequences of fuzzy numbers have been defined where $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. In addition, for $\beta \in (0, 1]$, concepts of S_λ^β-limit points, S_λ^β-cluster points, S_λ^β-limit inferior and S_λ^β-limit superior for sequences of fuzzy numbers are also introduced.

Chapter 6 deals with the concepts of α-statistical convergence and statistical convergence for sequences in fuzzy neighborhood spaces which turn out to be a very useful subcategory of fuzzy topological spaces. Some properties of α-statistical convergence and statistical convergence in these spaces have been developed.

In the end, a bibliography is given which lists only those books and papers which have been referred to in this thesis so this list is not an exhaustive one.

Chapter 2

Generalized Summability of Multiple Sequences

2.1 Introduction

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has wide range of applications in various fields like population dynamics, chaos control, computer programming, nonlinear dynamical systems, etc. Fuzzy topology is one of the most important and useful tools to deal with such situations where the use of classical theories breaks down. While studying fuzzy topological spaces, we face many situations where the theory of convergence of fuzzy number sequences is needed. The concept of usual convergence of fuzzy numbers sequences was introduced by Matloka [78], where he proved some basic theorems concerning fuzzy number sequences. Nanda [88], later showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. In the recent years, statistical convergence, formally

Some contents of this chapter have been published in **Mathematical Sciences 2012, 6:2 (Springer)** and the rest have been published in **Boletim da Sociedade Paranaense de Matemtica, Vol 33, No. 2, (2015) pp. 69-78** (BRAZIL) introduced by Steinhaus [103] and Fast [45] for real number sequences, has also been adapted to the sequences of fuzzy numbers. The credit goes to Nuray and Savaş [90], who first defined the concepts of statistical convergence and gave the statistically Cauchy criterion for sequences of fuzzy numbers. They proved that a sequence of fuzzy numbers is statistically convergent, if and only if it is a statistically Cauchy. Nuray [89] introduced lacunary statistical convergence of fuzzy numbers sequences, whereas Kwon [69] obtained relationship between statistical convergence and strong p-Cesaro summability of fuzzy numbers sequences. Another interesting generalization of statistical convergence for sequences of fuzzy numbers has been introduced by Kumar and Kumar [68] with the help of an admissible ideal $\mathcal{I}$ of subsets of $\mathbb{N}$. They defined $\mathcal{I}$-Cauchy sequences of fuzzy numbers and studied many interesting properties of $\mathcal{I}$-convergence for sequences of fuzzy numbers. After their pioneer work, many authors have shown keen interest towards the study of sequences of fuzzy numbers. For an extensive view, we refer Aytar [11], Aytar and Mammadov [13], Aytar and Pehlivan [14], Colak [25], Fridy [50], Fridy and Orhan ([51], [53]), Kumar [67], Mursaleen [83], Mursaleen and Alotaibi [84], Savaş [97], Savaş [98], Savaş [99] *etc.*

The purpose of this chapter is to extend some generalized limits namely; statistical convergence and ideal convergence for sequences of fuzzy numbers of multiplicity greater than two. In section 2.2, some basic preliminaries on sequences of fuzzy numbers have been given. The aim of section 2.3 is to define statistical convergence and study some of its interesting properties for sequences of fuzzy numbers of multiplicity > 2. Further in this section, we have also defined statistically Cauchy criterion for these kind of sequences. Subsequently, Cesàro and strongly p-Cesàro summability for multiple sequences of fuzzy numbers have also been defined and their relationships with statistical convergence have been obtained in this section. In the last section, a variety of generalized convergence for multiple sequences of fuzzy numbers with help of an admissible ideal $\mathcal{I} \subset P(\mathbb{N})$ have been studied.

2.2 Preliminaries

By a fuzzy number we mean a fuzzy set with membership function X from $\mathbb{R}$ to [0,1], which satisfy the following conditions

(i) X is normal, i.e., there exists and $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$;

(ii) X is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and

$$\lambda \in [0,1], X(\lambda x + (1-\lambda)y) \geq \min\{X(x), X(y)\};$$

(iii) X is upper semi-continuous;

(iv) the closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by X^0, is compact.

The properties (i)-(iv) imply that for each $\alpha \in (0,1]$, the α-level set,

$$X^\alpha = \{x \in \mathbb{R} : X(x) \geq \alpha\} = [\underline{X}^\alpha, \overline{X}^\alpha]$$

is a non-empty compact convex subset of $\mathbb{R}$. Let $L(\mathbb{R})$ denotes the set of all fuzzy numbers.

In the sequel,we quote some definitions and results of the papers Kumar and Kumar [68], Nuray and Savaş [90] and Savaş [98].

Definition 2.2.1. [90] A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Here, vertical bars denotes the cardinality of the enclosed set. Symbolically, it is written as $S - \lim_k X_k = X_0$.

Definition 2.2.2. [98] A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically Cauchy, if for each $\epsilon > 0$ there is a number N_0 such that

$$\lim_n \frac{1}{n} |\{k \leq n : \overline{d}(X_k, X_{N_0}) \geq \epsilon\}| = 0.$$

For a non empty set X, let $P(X)$ denotes the power set of X, then a family of sets $\mathcal{I} \subset P(X)$ is called an ideal on X if it satisfies

(i) $\phi \in \mathcal{I}$;

(ii) for each $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

(iii) for each $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$.

For a non empty set X, a non-empty family of sets $F \subset P(X)$ is called a filter on X if and only if

(i) $\phi \in F$;

(ii) for each $A, B \in F \Rightarrow A \cap B \in F$;

(iii) for each $A \in F$ and $B \supset A \Rightarrow B \in F$.

A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if it contains all singleton sets i.e., if it contains $\{\{x\} : x \in X\}$.

Definition 2.2.3. [68] Let $\mathcal{I} \subset P(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$. Then, a sequence $X = (X_k)$ of fuzzy numbers is said to be $\mathcal{I}$-convergent to a number X_0 if for each $\epsilon > 0$, the set $\{k \leq n : \overline{d}(X_k, X_0) \geq \epsilon\}$ belongs to $\mathcal{I}$. The fuzzy number X_0 is called the $\mathcal{I}$-limit of the sequence (X_k) and we write $\mathcal{I} - \lim_k X_k = X_0$.

Definition 2.2.4. [68] A sequence $X = (X_k)$ of fuzzy numbers is said to be $\mathcal{I}$-Cauchy, if for each $\epsilon > 0$ there is a number N_0 such that the set $\{k \leq n : \overline{d}(X_k, X_{N_0}) \geq \epsilon\}$ belongs to $\mathcal{I}$.

Definition 2.2.5. [68] A sequence $X = (X_k)$ of fuzzy numbers is said to be $\mathcal{I}^*$-convergent to a fuzzy number X_0 if and only if there exist a set $K = \{m_1 < m_2, \cdots\} \subset \mathbb{N}$ such that $K \in F(\mathcal{I})$ and $\overline{d}(X_{m_k}, X_0) \to 0$ as $n \to \infty$.

2.3 Statistical convergence

For brevity, we state and prove our results only for triple sequences and it can easily be seen that these definitions and results can also be applied to double sequences of fuzzy numbers and to sequences of fuzzy numbers of any multiplicity $d > 3$.

We start this section with the following notion of natural density.

For $K \subset \mathbb{N}^3$, the natural density of K is defined by

$$\delta_3(K) = \lim_{l,m,n \longrightarrow \infty} \frac{1}{lmn} |\{i \leq l, j \leq m, k \leq n : (i,j,k) \in K\}|$$

provided the limit exists. Here the vertical bars denote the cardinality of the enclosed set.

Definition 2.3.1. A triple sequence $X = (X_{ijk})$ of fuzzy numbers is said to be statistical convergent to some fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_{l,m,n \to \infty} \frac{1}{lmn} |\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_0) \geq \epsilon\}| = 0.$$

Symbolically, it is written as $S - \lim X_{ijk} = X_0$.

Here l, m and n tend to infinity independently of each other. Let S_3 denotes the set of all statistically convergent triple sequences of fuzzy numbers. Every convergent triple sequence of fuzzy number is statistical convergent as natural density of any finite subsets of $\mathbb{N}^3$ is zero. But, the following example shows that the converse is not necessarily true.

Example 2.3.1. For every $x \in \mathbb{R}$, define a sequence $X = (X_{ijk})$ of fuzzy numbers as under:

If the numbers i, j and k are all squares, then define

$$X_{ijk}(x) = \begin{cases} 0, & \text{if } x \in (-\infty, ijk-1) \cup (ijk+1, \infty), \\ x - (ijk-1), & \text{if } x \in [ijk-1, ijk], \\ -x + (ijk+1), & \text{if } x \in [ijk, ijk+1]. \end{cases}$$

otherwise $X_{ijk} = X_0$, where X_0 is given by

$$X_0(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \cup (2, \infty), \\ x & \text{if } x \in [0, 1], \\ -x + 2 & \text{if } x \in (1, 2]. \end{cases}$$

Now, for $0 < \epsilon < 1$, we have

$$K(\epsilon) = \{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\} \subset \{(i, j, k) \in \mathbb{N}^3 : i, j \textit{ and } k \textit{ are squares}\}.$$

Since, the later set has triple density zero, it follows that $\delta_3(K(\epsilon)) = 0$, and consequently $X = (X_{ijk})$ is statistically convergent to X_0. But the sequence $X = (X_{ijk})$ is not ordinary convergent to X_0. $\square$

In the following results, we guarantee the uniqueness of statistical limit if exists and present some algebraic characterization of this generalized limit for triple sequences of fuzzy numbers.

Theorem 2.3.1. *If a triple sequence $X = (X_{ijk})$ of fuzzy numbers is statistical convergent to some limit, then it must be unique.*

Proof. Let $X = (X_{ijk})$ be a triple sequence of fuzzy number such that

$$S - \lim X_{ijk} = X_0 \text{ and } S - \lim X_{ijk} = Y_0,$$

where $Y_0 \neq X_0$. Let us suppose $X_0 > Y_0$ and $\overline{\epsilon} = \frac{X_0 - Y_0}{3}$, then the neighborhoods $(X_0 \ominus \overline{\epsilon}, X_0 \oplus \overline{\epsilon})$ and $(Y_0 \ominus \overline{\epsilon}, Y_0 \oplus \overline{\epsilon})$ of X_0 and Y_0 are disjoint.

Since $S - \lim X_{ijk} = X_0$ and $S - \lim X_{ijk} = Y_0$, therefore

$$\delta_3 \left(\{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\}\right) = 0$$

and

$$\delta_3 \left(\{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, Y_0) \geq \epsilon\}\right) = 0.$$

This implies that

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) < \epsilon\} \text{ and } \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, Y_0) < \epsilon\}$$

have triple natural density 1. Therefore

$$\delta_3\left(\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) < \epsilon\} \cap \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, Y_0) < \epsilon\}\right) \neq 0.$$

which is a contradiction to the fact that then the neighborhoods $(X_0 \ominus \overline{\epsilon}, X_0 \oplus \overline{\epsilon})$ and $(Y_0 \ominus \overline{\epsilon}, Y_0 \oplus \overline{\epsilon})$ are disjoint. Hence, we have $X_0 = Y_0$. □

Theorem 2.3.2. *Let $X = (X_{ijk})$ and $Y = (Y_{ijk})$ be two triple sequences of fuzzy numbers, then*

(i) If $X = (X_{ijk})$ is statistically convergent to X_0 and $c \in \mathbb{R}$, then (cX_{ijk}) is statistically convergent to cX_0.

(ii) If $X = (X_{ijk})$ and $Y = (Y_{ijk})$ are statistically convergent to fuzzy numbers X_0 and Y_0 respectively, then $(X_{ijk} \oplus Y_{ijk})$ is statistically convergent to $X_0 \oplus Y_0$.

Proof. (i) Let $\alpha \in (0,1]$, $c \in R$ and $\epsilon > 0$. Let $X^\alpha_{ijk}, Y^\alpha_{ijk}, X^\alpha_0$ and Y^α_0 be α-level sets of X_{ijk}, Y_{ijk}, X_0 and Y_0 respectively. Then, since $d(cX^\alpha_{ijk}, cX^\alpha_0) = |c| d(X^\alpha_{ijk}, X^\alpha_0)$, we have $\overline{d}(cX^\alpha_{ijk}, cX^\alpha_0) = |c|\overline{d}(X^\alpha_{ijk}, X^\alpha_0)$ and this gives

$$\begin{aligned}
&\delta_3(\{i \le l, j \le m, k \le n : \overline{d}(cX_{ijk}, cX_0) \ge \epsilon\}) \\
&\qquad\qquad \le \delta_3\left(\left\{i \le l, j \le m, k \le n : \overline{d}(X_{ijk}, X_0) \ge \frac{\epsilon}{|c|}\right\}\right).
\end{aligned}$$

Hence $S - \lim cX_{ijk} = cX_0$.

(ii) Since

$$\begin{aligned}
d(X^\alpha_{ijk} \oplus Y^\alpha_{ijk}, X^\alpha_0 \oplus Y^\alpha_0) &\le d(X^\alpha_{ijk} \oplus Y^\alpha_{ijk}, Y^\alpha_{ijk} \oplus X^\alpha_0) + d(Y^\alpha_{ijk} \oplus X^\alpha_0, X^\alpha_0 \oplus Y^\alpha_0) \\
&= d(X^\alpha_{ijk}, X^\alpha_0) + \delta_\infty(Y^\alpha_{ijk}, Y^\alpha_0)
\end{aligned}$$

By using Minkowski's equality, we have

$$\overline{d}(X_{ijk} \oplus Y_{ijk}, X_0 \oplus Y_0) = \overline{d}(X_{ijk}, X_0) + \overline{d}(Y_{ijk}, Y_0).$$

Therefore,

$$\frac{1}{lmn}|\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk} \oplus Y_{ijk}, X_0 \oplus Y_0) \geq \epsilon\}|$$
$$\leq \frac{1}{lmn}\left|\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_0) + \overline{d}(Y_{ijk}, Y_0) \geq \epsilon\}\right|$$
$$\leq \frac{1}{lmn}\left|\left\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_0) \geq \frac{\epsilon}{2}\right\}\right|$$
$$+\frac{1}{lmn}\left|\left\{i \leq l, j \leq m, k \leq n : \overline{d}(Y_{ijk}, Y_0) \geq \frac{\epsilon}{2}\right\}\right|,$$

which proves the result. □

Now, we present some important characterizations of statistical convergence for sequences of fuzzy numbers having multiplicity three.

Theorem 2.3.3. *A triple sequence $X = (X_{ijk})$ of fuzzy numbers is statistically convergent to a fuzzy number X_0, if and only if, there exists a subset $K = \{(i_n, j_n, k_n)\} \subset \mathbb{N}^3, n = 1, 2, \ldots$ such that $\delta_3(K) = 1$ and $\lim_{n\to\infty} X_{i_nj_nk_n} = X_0$.*

Proof. Let $X = (X_{ijk})$ be statistically convergent to X_0. For each $\epsilon > 0$, if we denote

$$M = \{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\}$$

and

$$K = \{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) < \epsilon\},$$

then $\delta_3(M) = 0$ and consequently $\delta_3(K) = 1$. Furthermore, $K \cap M = \emptyset$. Since $\delta_3(K) = 1$, it follows that K is an infinite set as otherwise $\delta_3(K) = 0$. Let $K = \{(i_n, j_n, k_n)\} \subset \mathbb{N}^3, n = 1, 2, \ldots$. In order to prove the result, it is sufficient to prove that $(X_{i_nj_nk_n})$ is convergent to X_0. Suppose that $(X_{i_nj_nk_n})$ is not convergent to X_0. By definition, there exists $\epsilon_1 > 0$ such that $\overline{d}(X_{i_nj_nk_n}, X_0) \geq \epsilon_1$ for infinitely many terms. Let

$$K_1 = \{(i_n, j_n, k_n) \in K : \overline{d}(X_{i_nj_nk_n}, X_0) \geq \epsilon_1\}.$$

Clearly $\emptyset \neq K_1 \subseteq K$. Also for all i, j, k and ϵ_1, we have

$$M_1 = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon_1\} \supseteq \{(i_n, j_n, k_n) : \overline{d}(X_{i_n j_n k_n}, X_0) \geq \epsilon_1\}.$$

Thus, $\delta_3(K_1) = 0$ i.e. $K_1 \subseteq M_1$. Further, for $\epsilon < \epsilon_1, M_1 \subseteq M$, which is impossible as $K \cap M = \emptyset$. Hence, $(X_{i_n j_n k_n})$ is convergent to X_0.

Conversely, Suppose that there exists a subset $K = \{(i_n, j_n, k_n)\} \subset \mathbb{N}^3, n = 1, 2, \ldots$ such that $\delta_3(K) = 1$ and $\lim_{n\to\infty} X_{i_n j_n k_n} = X_0$. By definition, there exist a positive integer p such that

$$\overline{d}(X_{i_n j_n k_n}, X_0) < \epsilon, \qquad \forall\, n \geq p.$$

Since

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\} \subseteq \mathbb{N}^3 - \{(i_{p+1}, j_{p+1}, k_{p+1}), (i_{p+2}, j_{p+2}, k_{p+2}), \ldots\},$$

it follows that

$$\delta_3\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\} \leq 1 - 1 = 0.$$

Hence, X is statistically convergent to X_0. $\square$

Theorem 2.3.4. *The set $S_3 \cap l_\infty^{F_3}$ is closed linear subspace of the normed linear space $l_\infty^{F_3}$.*

Proof. Let $X^{(lmn)} = \left(X_{ijk}^{(lmn)}\right) \in S_3 \cap l_\infty^{F_3}$ and $X^{(lmn)} \longrightarrow X \in l_\infty^{F_3}$. Since $X^{(lmn)} \in S_3 \cap l_\infty^{F_3}$, therefore there exist a fuzzy number Y_{lmn}, such that

$$S - \lim_{i,j,k} X_{ijk}^{(lmn)} = Y_{lmn} \qquad (l, m, n = 1, 2, \ldots).$$

Furthermore, $X^{(lmn)} \longrightarrow X$, implies that there exists a positive integer M, such that for every $p \geq l \geq M, q \geq m \geq M$ and $r \geq n \geq M$

$$\overline{d}\left(X^{(pqr)}, X^{(lmn)}\right) < \frac{\epsilon}{3} \tag{2.3.1}$$

Also, by Theorem 2.3.3, there exist subsets $K_{pqr}, K_{lmn} \subset \mathbb{N}^3$ such that $\delta_3(K_{pqr}) = \delta_3(K_{lmn}) = 1$ and

$$\lim_{(i,j,k)\in K_{pqr}; i,j,k\to\infty} X_{ijk}^{(pqr)} = Y_{pqr}. \tag{2.3.2}$$

$$\lim_{(i,j,k)\in K_{lmn}; i,j,k\to\infty} X_{ijk}^{(lmn)} = Y_{lmn}. \tag{2.3.3}$$

Now, the set $K_{pqr}\cap K_{lmn}$ is an infinite set as $\delta_3(K_{pqr}\cap K_{lmn}) = 1$. Choose $(k_1, k_2, k_3) \in K_{pqr} \cap K_{lmn}$, then we have from (2.3.2) and (2.3.3)

$$\overline{d}\left(X_{k_1k_2k_3}^{(pqr)}, Y_{pqr}\right) < \frac{\epsilon}{3} \qquad \text{and} \qquad \overline{d}\left(X_{k_1k_2k_3}^{(lmn)}, Y_{lmn}\right) < \frac{\epsilon}{3} \tag{2.3.4}$$

Hence, for every $p \geq l \geq M, q \geq m \geq M$ and $r \geq n \geq M$, we have from (2.3.1) to (2.3.4)

$$\begin{aligned}\overline{d}\left(Y_{pqr}, Y_{lmn}\right) &\leq \overline{d}\left(Y_{pqr}, X_{k_1k_2k_3}^{(pqr)}\right) + \overline{d}\left(X_{k_1k_2k_3}^{(pqr)}, X_{k_1k_2k_3}^{(lmn)}\right) + \overline{d}\left(X_{k_1k_2k_3}^{(lmn)}, Y_{lmn}\right)\\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

This shows that (Y_{lmn}) is a Cauchy sequence and hence convergent. Let,

$$\lim_{l,m,n\longrightarrow\infty} Y_{lmn} = Y. \tag{2.3.5}$$

Next, we show that X is statistically convergent to Y. Since $X^{(lmn)} \longrightarrow X$, so for each $\epsilon > 0$ there exists l, m, n and $N_0 \in \mathbb{N}$ such that

$$\overline{d}\left(X_{ijk}^{(lmn)}, X_{ijk}\right) < \frac{\epsilon}{3}; \qquad i, j, k \geq N_0.$$

Also from (2.3.5), we have for every $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that

$$\overline{d}(Y_{ijk}, Y) < \frac{\epsilon}{3}; \qquad i, j, k \geq N_1.$$

Furthermore, by virtue of the fact that $\left(X^{(lmn)}\right)$ is statistically convergent to Y_{lmn}, there exists a set $K_{ijk} = \{(i,j,k)\} \subseteq \mathbb{N}^3$ such that $\delta_3(K_{ijk}) = 1$ and for each $\epsilon > 0$, there exists $N_3 \in \mathbb{N}$ such that for $(i,j,k) \in K_{ijk}$, we have

$$\overline{d}\left(X_{ijk}^{(lmn)}, Y_{lmn}\right) < \frac{\epsilon}{3}; \qquad i,j,k \geq N_2.$$

Let $N_3 = \max\{N_0, N_1, N_2\}$. Now, for $\epsilon > 0$ and $(i,j,k) \in K_{ijk}$

$$\begin{aligned}\overline{d}(X_{ijk}, Y) &\leq \overline{d}\left(X_{ijk}, X_{ijk}^{lmn}\right) + \overline{d}\left(X_{ijk}^{lmn}, Y_{ijk}\right) + \overline{d}(Y_{ijk}, Y) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

This shows that X is statistically convergent to Y i.e.$X \in S_3 \cap l_\infty^{F_3}$. This proves that $S_3 \cap l_\infty^{F_3}$ is closed linear subspace of $l_\infty^{F_3}$. □

Now, we introduce the statistically Cauchy sequences for triple sequences of fuzzy numbers and obtain the Cauchy convergence criterion.

Definition 2.3.2. A triple sequence $X = (X_{ijk})$ of fuzzy numbers is said to be statistically Cauchy if for each $\epsilon > 0$ there exists integers $L = L(\epsilon), M = M(\epsilon)$, and $N = N(\epsilon)$ such that

$$\lim_{l,m,n\to\infty} \frac{1}{lmn} \left|\left\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_{LMN}) \geq \epsilon\right\}\right| = 0.$$

Theorem 2.3.5. *A triple sequence $X = (X_{ijk})$ of fuzzy numbers is statistical convergent, if and only if, it is statistically Cauchy.*

Proof. Let $X = (X_{ijk})$ be statistically convergent to X_0. By definition, for each $\epsilon \geq 0$ we have

$$\delta_3(\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\}) = 0.$$

Choose numbers L, M and N such that $\overline{d}(X_{LMN}, X_0) \geq \epsilon$. If we denote

$$A = \{(i,j,k) \in \mathbb{N}^3, i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_{LMN}) \geq \epsilon\};$$

$$B = \{(i,j,k) \in \mathbb{N}^3, i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_0) \geq \epsilon\};$$

$$C = \{(L,M,N) : \overline{d}(X_{LMN}, X_0) \geq \epsilon\},$$

then it is clear that$A \subseteq B \cup C$ and consequently $\delta_3(A) \leq \delta_3(B) + \delta_3(C)$. Hence $X = (X_{ijk})$ is statistically Cauchy.

Conversely, Suppose that $X = (X_{ijk})$ is a statistically Cauchy. We shall prove that (X_{ijk}) is statistically convergent. To this effect, let $(\epsilon_p : p = 1, 2, \ldots)$ be strictly decreasing sequence of numbers converging to zero. Since $X = (X_{ijk})$ is statistically Cauchy, therefore there exist three strictly increasing sequences $(L_p), (M_p)$ and (N_p) of positive integers such that

$$\lim_{l,m,n\to\infty} \frac{1}{lmn} |\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_{L_pM_pN_p}) > \epsilon_p\}| = 0. \tag{2.3.6}$$

Clearly, for each pair p and q $(p \neq q)$ of positive integers, we can select $(i_{pq}, j_{pq}, k_{pq}) \in \mathbb{N}^3$ such that

$$\overline{d}\left(X_{i_{pq}j_{pq}k_{pq}}, X_{L_pM_pN_p}\right) \leq \epsilon_p \quad \text{and} \quad \overline{d}\left(X_{i_{pq}j_{pq}k_{pq}}, X_{L_qM_qN_q}\right) \leq \epsilon_q.$$

It follows that

$$\begin{aligned} \overline{d}\left(X_{L_pM_pN_p}, X_{L_qM_qN_q}\right) &\leq \overline{d}\left(X_{i_{pq}j_{pq}k_{pq}}, X_{L_pM_pN_p}\right) + \overline{d}\left(X_{i_{pq}j_{pq}k_{pq}}, X_{L_qM_qN_q}\right) \\ &\leq \epsilon_p + \epsilon_q \to 0 \quad \text{as} \qquad p, q \to \infty. \end{aligned}$$

Thus, $\left(X_{L_pM_pN_p} : p = 1, 2, ...\right)$ is Cauchy sequence and satisfies the Cauchy convergence criterion. Let, $\left(X_{L_pM_pN_p}\right)$ converges to X_0. Since $(\epsilon_p : p = 1, 2, \ldots) \longrightarrow 0$, so for $\epsilon > 0$, there exists $p_0 \in \mathbb{N}^3$ such that

$$\epsilon_{p_0} < \frac{\epsilon}{2} \qquad \text{and} \qquad \overline{d}\left(X_{L_pM_pN_p}, X_0\right) < \frac{\epsilon}{2}, \qquad p \geq p_0. \tag{2.3.7}$$

Now, consider an arbitrary $(i,j,k) \in \mathbb{N}^3$. By (2.3.7), we have

$$\begin{aligned}\overline{d}\left(X_{ijk}, X_0\right) &\leq \overline{d}\left(X_{ijk}, X_{L_{p_0}M_{p_0}N_{p_0}}\right) + \overline{d}\left(X_{L_{p_0}M_{p_0}N_{p_0}}, X_0\right)\\ &\leq \overline{d}\left(X_{ijk}, X_{L_{p_0}M_{p_0}N_{p_0}}\right) + \frac{\epsilon}{2}\end{aligned}$$

whence, by (2.3.6),

$$\begin{aligned}&\frac{1}{lmn}\left|\left\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_0) > \epsilon\right\}\right|\\ &\qquad\leq \frac{1}{lmn}\left|\left\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_{L_{p_0}M_{p_0}N_{p_0}}) > \frac{\epsilon}{2}\right\}\right|\\ &\qquad\leq \frac{1}{lmn}\left|\left\{i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_{L_{p_0}M_{p_0}N_{p_0}}) > \epsilon_{p_0}\right\}\right|\\ &\qquad\to 0 \,\text{as}\; l,m,n \to \infty.\end{aligned}$$

This shows that $X = (X_{ijk})$ is statistically convergent to X_0 and hence the proof of the Theorem is complete. □

Next, we define Cesàro and strongly p-Cesàro summability for sequences of fuzzy numbers having multiplicity $d = 3$. The Cesàro sum for triple sequences of fuzzy numbers is defined below:

Definition 2.3.3. A triple sequence $X = (X_{ijk})$ is said to be C_{111}-summable or Cesàro summable to X_0 provided that

$$\lim_{l,m,n} \frac{1}{lmn} \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} X_{ijk} = X_0.$$

We, now extend the concept of Cesàro summability to sequences of fuzzy numbers and define strongly p-Cesàro summable sequences of fuzzy numbers.

Definition 2.3.4. Let p be a positive real number. A triple sequence $X = (X_{ijk})$ of fuzzy numbers is said to be strongly p-Cesàro summable if there is a fuzzy number

X_0 such that

$$\lim_{l,m,n} \frac{1}{lmn} \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} \overline{d}(X_{ijk}, X_0)^p = 0.$$

We denote the space of all strongly p-Cesàro summable triple sequences of fuzzy numbers by $w_p^{F_3}$.

Remarks 2.3.1. (i) If $0 < p \leq q < \infty$, then $w_q^{F_3} \subseteq w_p^{F_3}$ (Holder inequality) and

$$w_p^{F_3} \cap l_\infty^{F_3} = w_1^{F_3} \cap l_\infty^{F_3} \subseteq C_{111} \cap l_\infty^{F_3}.$$

(ii) If $X = (X_{ijk})$ convergent but unbounded, then $X = (X_{ijk})$ is statistically convergent, however $X = (X_{ijk})$ need not to be Cesàro nor strongly p-Cesàro summable repectively.

(iii) If $X = (X_{ijk})$ is a bounded convergent triple sequence of fuzzy numbers, then it is C_{111}-summable, strongly p-Cesàro summable and statistical convergent as well.

Theorem 2.3.6. *Let $p \in (0, \infty)$.*

(i) If a triple sequence $X = (X_{ijk})$ of fuzzy numbers is strongly p-Cesàro summable to a fuzzy number X_0, then it is also statistically convergent to X_0.

(ii) If a triple bounded sequence $X = (X_{ijk})$ of fuzzy numbers is statistically convergent to a fuzzy number X_0. Then it is strongly p-Cesàro summable to X_0.

Proof. (i) Let $K_{lmn}(\epsilon) = \big\{(i,j,k), i \leq l, j \leq m, k \leq n : \overline{d}(X_{ijk}, X_0)^p \geq \epsilon\big\}$. Now, we have

$$
\begin{aligned}
&\frac{1}{lmn}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}\overline{d}(X_{ijk},X_0)^p\\
&\qquad=\frac{1}{lmn}\left\{\sum_{(i,j,k)\in K_{lmn}(\epsilon)}\overline{d}(X_{ijk},X_0)^p+\sum_{(i,j,k)\notin K_{lmn}(\epsilon)}\overline{d}(X_{ijk},X_0)^p\right\}\\
&\qquad\geq\frac{1}{lmn}\left\{\sum_{(i,j,k)\in K_{lmn}(\epsilon)}\overline{d}(X_{ijk},X_0)^p\right\}\\
&\qquad\geq\frac{1}{lmn}\left|\left\{(i,j,k),i\leq l,j\leq m,k\leq n:\overline{d}(X_{ijk},X_0)^p\geq\epsilon\right\}\right|.
\end{aligned}
$$

Since, $X=(X_{ijk})$ is strongly p-Cesàro summable to X_0, therefore we have

$$0\geq\lim_{l,m,n}\frac{1}{lmn}|\{(i,j,k),i\leq l,j\leq m,k\leq n:\overline{d}(X_{ijk},X_0)^p\geq\epsilon\}|.$$

Hence,

$$\lim_{l,m,n}\frac{1}{lmn}|\{(i,j,k),i\leq l,j\leq m,k\leq n:\overline{d}(X_{ijk},X_0)^p\geq\epsilon\}|=0$$

as it can not be negative. This shows that $X=(X_{ijk})$ is statistically convergent to X_0.

(ii) Let $K_{lmn}(\epsilon)=\left\{(i,j,k),i\leq l,j\leq m,k\leq n:\overline{d}(X_{ijk},X_0)^p\geq(\frac{\epsilon}{2})^p\right\}$ and $M=\|X\|_{(\infty,3)}+\overline{d}(X_0,\widetilde{0})$, where $\|X\|_{(\infty,3)}$ is the sup-norm for bounded triple sequences $X=(X_{ijk})$.

Since, $X=(X_{ijk})$ is bounded statistically convergent so we can choose a positive integer $r=r(\epsilon)\in\mathbb{N}$ such that for all $i,j,k\geq r$, we have

$$\frac{1}{lmn}\left|\left\{(i,j,k),i\leq l,j\leq m,k\leq n:\overline{d}(X_{ijk},X_0)^p\geq\left(\frac{\epsilon}{2}\right)^p\right\}\right|<\frac{\epsilon}{2M^p}.$$

Now, for all $l, m, n \geq r$,

$$\begin{aligned}
&\frac{1}{lmn}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}\overline{d}(X_{ijk}, X_0)^p \\
&\qquad\qquad = \frac{1}{lmn}\left\{\sum_{(i,j,k)\in K_{lmn}}\overline{d}(X_{ijk}, X_0)^p + \sum_{(i,j,k)\notin K_{lmn}}\overline{d}(X_{ijk}, X_0)^p\right\} \\
&\qquad\qquad \geq \frac{1}{lmn}.lmn.\frac{\epsilon}{2M^p}.M^p + \frac{1}{lmn}.lmn.\frac{\epsilon}{2} \\
&\qquad\qquad = \epsilon.
\end{aligned}$$

This shows that $X = (X_{ijk})$ is strongly p-Cesàro summable to X_0. $\square$

2.4 $\mathcal{I}_3$-convergence

In this section, we introduce and summarize a more generalized convergence with the help of an ideal $\mathcal{I}_3 \subset P(\mathbb{N}^3)$, where $\mathcal{I}_3$ denote the ideal of subsets of $\mathbb{N}^3$. For defining $\mathcal{I}_3$ take $X = \mathbb{N}^3$ in the definition of $\mathcal{I}$. We start by giving the following definition:

Definition 2.4.1. Let $\mathcal{I}_3 \subset P(\mathbb{N}^3)$ be a non-trivial ideal in $\mathbb{N}^3$. A triple sequence $X = (X_{ijk})$ of fuzzy numbers is said to be $\mathcal{I}_3$-convergent to some fuzzy number X_0, if for each $\epsilon > 0$,

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\} \in \mathcal{I}_3.$$

Symbolically, it is denoted as: $\mathcal{I}_3 - \lim X_{ijk} = X_0$.

We shall denote the set of all $\mathcal{I}_3$-convergent triple sequences of fuzzy numbers by $\mathcal{I}^3$.

Theorem 2.4.1. *If a triple sequence $X = (X_{ijk})$ of fuzzy numbers is $\mathcal{I}_3$-convergent to some limit, then it must be unique.*

Proof. Let $X = (X_{ijk})$ be a triple sequence of fuzzy number such that

$$\mathcal{I}_3 - \lim X_{ijk} = X_0 \text{ and } \mathcal{I}_3 - \lim X_{ijk} = Y_0,$$

where $Y_0 \neq X_0$. Let us suppose $X_0 > Y_0$ and $\overline{\epsilon} = \frac{X_0 - Y_0}{3}$, then the neighborhoods

$$(X_0 \ominus \overline{\epsilon}, X_0 \oplus \overline{\epsilon}) \text{ and } (Y_0 \ominus \overline{\epsilon}, Y_0 \oplus \overline{\epsilon})$$

are disjoint. Since $\mathcal{I}_3 - \lim X_{ijk} = X_0$ and $\mathcal{I}_3 - \lim X_{ijk} = Y_0$, therefore

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\} \in \mathcal{I}_3 \text{ and } \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, Y_0) \geq \epsilon\} \in \mathcal{I}_3.$$

This implies that

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) < \epsilon\} \text{ and } \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, Y_0) < \epsilon\}$$

belong to $\mathcal{F}(\mathcal{I}_3)$. Since $\mathcal{F}(\mathcal{I}_3)$ is a filter, therefore

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) < \epsilon\} \cap \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, Y_0) < \epsilon\} \in \mathcal{F}(\mathcal{I}_3).$$

which is a contradiction to the fact that then the neighborhoods $(X_0 \ominus \overline{\epsilon}, X_0 \oplus \overline{\epsilon})$ and $(Y_0 \ominus \overline{\epsilon}, Y_0 \oplus \overline{\epsilon})$ are disjoint. Hence, we have $X_0 = Y_0$. $\square$

We, now prove that a ordinary convergent sequence is also $\mathcal{I}_3$-convergent to the same limit. Furthermore, some algebraic properties of $\mathcal{I}_3$-convergence for sequence of fuzzy numbers are also presented in this section.

Theorem 2.4.2. *Let $X = (X_{ijk})$ and $Y = (Y_{ijk})$ be two triple sequences of fuzzy numbers, then*

(i) If $X = (X_{ijk})$ is convergent to X_0, then (X_{ijk}) is $\mathcal{I}_3$-convergent to X_0.

(ii) If $X = (X_{ijk})$ is $\mathcal{I}_3$-convergent to X_0 and $c \in \mathbb{R}$, then (cX_{ijk}) is $\mathcal{I}_3$-convergent to cX_0.

(iii) If $X = (X_{ijk})$ and $Y = (Y_{ijk})$ are $\mathcal{I}_3$-convergent to fuzzy numbers X_0 and Y_0 respectively, then $(X_{ijk} \oplus Y_{ijk})$ is $\mathcal{I}_3$-convergent to $X_0 \oplus Y_0$.

Proof. (i) Let the triple sequence $X = (X_{ijk})$ of fuzzy numbers is ordinary convergent to a fuzzy number X_0, then for each $\epsilon > 0$ there exist a positive integer m such that

$\overline{d}(X_{ijk}, X_0) < \epsilon$ for every $i, j, k \geq m$. It follows that the set

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\}$$

$$\subset \mathbb{N} \times \mathbb{N} \times \{1,2,3...m-1\} \cup \mathbb{N} \times \{1,2,3...m-1\} \times \mathbb{N} \cup \{1,2,3...m-1\} \times \mathbb{N} \times \mathbb{N} \in \mathcal{I}_3.$$

This shows that $\mathcal{I}_3 - \lim X_{ijk} = X_0$.

(ii) From Theorem 2.3.2, we have $\overline{d}\left(cX^\alpha_{ijk}, cX^\alpha_0\right) = |c|\overline{d}\left(X^\alpha_{ijk}, X^\alpha_0\right)$

this gives that,

$$\left\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(cX_{ijk}, cX_0) \geq \epsilon\right\} \subset \left\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \tfrac{\epsilon}{|c|}\right\} \in \mathcal{I}_3.$$

Hence $\mathcal{I}_3 - \lim cX_{ijk} = cX_0$

(iii) Since $\mathcal{I}_3 - \lim X_{ijk} = X_0$ and $\mathcal{I}_3 - \lim Y_{ijk} = Y_0$, then for $\epsilon > 0$,

$$A = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\} \in \mathcal{I}_3$$

$$B = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(Y_{ijk}, Y_0) \geq \epsilon\} \in \mathcal{I}_3.$$

Let $\quad C = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk} \oplus Y_{ijk}, X_0 \oplus Y_0) \geq \epsilon\}$.

In order to prove the result, we need to prove the $C \in A \cup B$. For $(i,j,k) \in C$, using Minkowski's equality, we have

$$\overline{d}(X_{ijk} \oplus Y_{ijk}, X_0 \oplus Y_0) = \overline{d}(X_{ijk}, X_0) + \overline{d}(Y_{ijk}, Y_0).$$

As both $\overline{d}(X_{ijk}, X_0), \overline{d}(Y_{ijk}, Y_0)$ cannot be $< \frac{\epsilon}{2}$, therefore atleast one of them either $\overline{d}(X_{ijk}, X_0) \geq \frac{\epsilon}{2}$ or $\overline{d}(Y_{ijk}, Y_0) \geq \frac{\epsilon}{2}$. This shows that $(i,j,k) \in A$ or B. This implies $(i,j,k) \in A \cup B$. Hence, $C \subset A \cup B$, which completes the result. □

Theorem 2.4.3. *Let $X = (X_{ijk})$ and $Y = (Y_{ijk})$ be two triple sequences of fuzzy numbers, such that*

(i) $X_{ijk} \leq Y_{ijk}$ for every $(i,j,k) \in K \subset \mathbb{N}^3$ with $K \in F(\mathcal{I}_3)$

(ii) $\mathcal{I}_3 - \lim X_{ijk} = X_0$ *and* $\mathcal{I}_3 - \lim Y_{ijk} = Y_0$.

Then $X_0 \leq Y_0$.

Proof. Let $X_0 > Y_0$ and $\overline{\epsilon} = \frac{X_0 \ominus Y_0}{3}$, so that the neighborhoods

$$(X_0 \ominus \overline{\epsilon}, X_0 \oplus \overline{\epsilon}) \text{ and } (Y_0 \ominus \epsilon, Y_0 \oplus \epsilon)$$

are disjoint. Since

$$\mathcal{I}_3 - \lim X_{ijk} = X_0 \text{ and } \mathcal{I}_3 - \lim Y_{ijk} = Y_0,$$

therefore

$$A = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\} \in \mathcal{I}_3 \text{ and}$$

$$B = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(Y_{ijk}, Y_0) \geq \epsilon\} \in \mathcal{I}_3.$$

It follows that the sets A^c and B^c belongs to $F(\mathcal{I}_3)$. This implies that

$$\phi \neq A^c \cap B^c \cap K \in F(\mathcal{I}_3)$$

and therefore

$$\overline{d}(X_{i_0j_0k_0}, X_0) < \epsilon \text{ and } \overline{d}(Y_{i_0j_0k_0}, Y_0) < \epsilon$$

for some $(i_0, j_0, k_0) \in K$.

This implies that $Y_{i_0j_0k_0} < X_{i_0j_0k_0}$, which is a contradiction to the fact that $X_{ijk} \leq Y_{ijk}$ for every $(i,j,k) \in K \subset \mathbb{N}^3$. Hence $X_0 \leq Y_0$. □

Theorem 2.4.4. *Let* $X = (X_{ijk})$, $Y = (Y_{ijk})$ *and* $Z = (Z_{ijk})$ *be three triple sequences of fuzzy numbers such that*

(i) $X_{ijk} \leq Y_{ijk} \leq Z_{ijk}$ *for every* $(i,j,k) \in K$ *with* $K \in F(\mathcal{I}_3)$

(ii) $\mathcal{I}_3 - \lim X_{ijk} = \mathcal{I}_3 - \lim Z_{ijk} = X_0$.

Then $\mathcal{I}_3 - \lim Y_{ijk} = X_0$.

Proof. Since $\mathcal{I}_3 - \lim X_{ijk} = X_0$ and $\mathcal{I}_3 - \lim Z_{ijk} = X_0$, therefore, the sets

$$A = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) < \epsilon\} \text{ and } C = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(Z_{ijk}, X_0) < \epsilon\}$$

belong to $F(\mathcal{I}_3)$. Consider the set

$B = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(Y_{ijk}, X_0) < \epsilon\}$.

It follows that $A \cap C \cap K \in B$. Since $A \cap C \cap K \in F(\mathcal{I}_3)$, a filter on $\mathbb{N}^3$, therefore $B \in F(\mathcal{I}_3)$. Hence the set

$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(Y_{ijk}, X_0) \geq \epsilon\} \in \mathcal{I}_3$.

This completes the proof. □

Theorem 2.4.5. *Let $\mathcal{I}_3 \subset P(\mathbb{N}^3)$ be an admissible ideal in $\mathbb{N}^3$. Then, the set $\mathcal{I}^3 \cap l_\infty^{F_3}$ is closed linear subspace of the normed linear space $l_\infty^{F_3}$.*

Proof. Let $X^{(lmn)} = \left(X_{ijk}^{(lmn)}\right) \in \mathcal{I}^3 \cap l_\infty^{F_3}$ and $X^{(lmn)} \longrightarrow X \in l_\infty^{F_3}$. Since $X^{(lmn)} \in \mathcal{I}^3 \cap l_\infty^{F_3}$, therefore there exist a fuzzy number Y_{lmn} such that

$$\mathcal{I}_3 - \lim_{i,j,k} X_{ijk}^{(lmn)} = Y_{lmn} \qquad (l,m,n = 1,2,\ldots).$$

Furthermore, $X^{(lmn)} \longrightarrow X$, implies that there exists a positive integer M such that for every $p \geq l \geq M$ and $q \geq m \geq M$ and $r \geq n \geq M$

$$\overline{d}\left(X^{(pqr)}, X^{(lmn)}\right) < \frac{\epsilon}{3} \tag{2.4.1}$$

Also, there exist subsets $K_{pqr}, K_{lmn} \subset \mathbb{N}^3$ such that $K_{pqr} \in F(\mathcal{I}_3)$ and $K_{lmn} \in F(\mathcal{I}_3)$ such that

$$\lim_{(i,j,k)\in K_{pqr}} X_{ijk}^{(pqr)} = Y_{pqr}. \tag{2.4.2}$$

$$\lim_{(i,j,k)\in K_{lmn}} X_{ijk}^{(lmn)} = Y_{lmn}. \tag{2.4.3}$$

Now, the set $K_{pqr} \cap K_{lmn}$ is non-empty in $F(\mathcal{I}_3)$. Choose $(k_1,k_2,k_3) \in K_{pqr} \cap K_{lmn}$, then we have from (2.4.2) and (2.4.3)

$$\overline{d}\left(X_{k_1k_2k_3}^{(pqr)}, Y_{pqr}\right) < \frac{\epsilon}{3} \qquad \textit{and} \qquad \overline{d}\left(X_{k_1k_2k_3}^{(lmn)}, Y_{lmn}\right) < \frac{\epsilon}{3} \tag{2.4.4}$$

Hence, for every $p \geq l \geq M$, $q \geq m \geq M$ and $r \geq n \geq M$, we have from (2.4.1) to (2.4.4)

$$\begin{aligned}\overline{d}(Y_{pqr}, Y_{lmn}) &\leq \overline{d}\left(Y_{pqr}, X^{(pqr)}_{k_1k_2k_3}\right) + \overline{d}\left(X^{(pqr)}_{k_1k_2k_3}, X^{(lmn)}_{k_1k_2k_3}\right) + \overline{d}\left(X^{(lmn)}_{k_1k_2k_3}, Y_{lmn}\right)\\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

This shows that (Y_{lmn}) is a Cauchy sequence and hence convergent. Let,

$$\lim_{l,m,n \longrightarrow \infty} Y_{lmn} = Y. \tag{2.4.5}$$

Next, we show that X is $\mathcal{I}_3$-convergent to Y. Since $X^{(lmn)} \longrightarrow X$, so by the structure of $l_\infty^{F_3}$, it is also coordinate wise convergent. Therefore, for each $\epsilon > 0$, there exist a positive integer $\eta_1(\epsilon)$ such that

$$\overline{d}\left(X^{(lmn)}_{ijk}, X_{ijk}\right) < \frac{\epsilon}{3}; \qquad i, j, k \geq \eta_1(\epsilon). \tag{2.4.6}$$

Also from (2.4.5), for every $\epsilon > 0$, there exists $\eta_2(\epsilon)$ such that

$$\overline{d}(Y_{ijk}, Y) < \frac{\epsilon}{3}; \qquad i, j, k \geq \eta_2(\epsilon). \tag{2.4.7}$$

Let $\eta_3(\epsilon) = \max\{\eta_1(\epsilon), \eta_2(\epsilon)\}$ and choose $l_0, m_0, n_0 > \eta_3(\epsilon)$. Then for any $(i, j, k) \in \mathbb{N}^3$

$$\begin{aligned}\overline{d}(X_{ijk}, Y) &\leq \overline{d}\left(X_{ijk}, X^{l_0m_0n_0}_{ijk}\right) + \overline{d}\left(X^{l_0m_0n_0}_{ijk}, Y_{l_0m_0n_0}\right) + \overline{d}\left(Y_{l_0m_0n_0}, Y\right)\\ &< \frac{\epsilon}{3} + \overline{d}\left(X^{l_0m_0n_0}_{ijk}, Y_{l_0m_0n_0}\right) + \frac{\epsilon}{3} \qquad \text{(by using (2.4.6) and (2.4.7))}\end{aligned}$$

$$\begin{aligned}\text{Let} \quad A_{l_0m_0n_0}\left(\frac{\epsilon}{3}\right) &= \left\{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X^{l_0m_0n_0}_{ijk}, Y_{l_0m_0n_0}) \geq \frac{\epsilon}{3}\right\},\\ A(\epsilon) &= \{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, Y) \geq \epsilon\}\\ A^C_{l_0m_0n_0}\left(\frac{\epsilon}{2}\right) &= \left\{(i, j) \in \mathbb{N}^3 : \overline{d}(X^{l_0m_0n_0}_{ijk}, Y_{l_0m_0n_0}) < \frac{\epsilon}{3}\right\} \quad \text{and}\\ A^C(\epsilon) &= \{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, Y) < \epsilon\}.\end{aligned}$$

So, for any $(i,j,k) \in A^C_{l_0m_0n_0}\left(\frac{\epsilon}{3}\right)$, we have $\overline{d}(X_{ijk}, Y) < \epsilon$ and therefore $A^C_{l_0m_0n_0}\left(\frac{\epsilon}{3}\right) \subset A^C(\epsilon)$. This implies that $A(\epsilon) \subset A_{l_0m_0n_0}\left(\frac{\epsilon}{3}\right)$. Since $A_{l_0m_0n_0}\left(\frac{\epsilon}{3}\right) \in \mathcal{I}_3$, therefore we have $A(\epsilon) \in \mathcal{I}_3$. Hence, X is $\mathcal{I}_3$-convergent to Y and $X \in \mathcal{I}_3$. This proves that $\mathcal{I}_3 \cap l^{F_3}_\infty$ is closed linear subspace of $l^{F_3}_\infty$. This completes the proof of the Theorem. □

We, now define $\mathcal{I}^*_3$-convergence, a notion closely related to $\mathcal{I}_3$-convergence. We use Theorem 2.3.3 to define such a new generalized convergence as follows:

Definition 2.4.2. A triple sequence $X = (X_{ijk})$ of fuzzy numbers is $\mathcal{I}^*_3$-convergent to a fuzzy number X_0, if and only if, there exists a subset $K = \{(i_n, j_n, k_n)\} \subset \mathbb{N}^3, n = 1, 2, \ldots$ such that $K \in F(\mathcal{I}_3)$ and $\lim_{n\to\infty} X_{i_nj_nk_n} = X_0$.

Theorem 2.4.6. *Let $\mathcal{I}_3$ be an admissible ideal. If $\mathcal{I}^*_3 - \lim X_{ijk} = X_0$, then $\mathcal{I}_3 - \lim X_{ijk} = X_0$.*

Proof. Let $X = X_{ijk}$ be a triple sequence of fuzzy numbers such that $\mathcal{I}^*_3 - \lim X_{ijk} = X_0$. Then, by definition, there exist a set $K = \{(i_n, j_n, k_n)\} \subset \mathbb{N}^3$ such that $K \in F(\mathcal{I}_3)(i.e. \mathbb{N}^3 - K = H \in \mathcal{I}_3)$ and $\overline{d}(X_{i_nj_nk_n}, X_0) \to 0 \quad \text{as}\, n \to \infty$.
Then for $\epsilon > 0$, there exists a positive integer n_1 such that $\overline{d}(X_{i_nj_nk_n}, X_0) < \epsilon$ for all $n > n_1$.
Since the set

$$A(\epsilon) = \{(i_n, j_n, k_n) \in K : \overline{d}(X_{i_nj_nk_n}, X_0) \geq \epsilon\} \subset A \cup H,$$

where $A = \{i_1 < i_2 < ... < i_{n_1}, j_1 < j_2 < ... < j_{n_1}, k_1 < k_2 < ... < k_{n_1}\}$ and $\mathcal{I}_3$ is admissible, therefore $A \cup H \in \mathcal{I}_3$. Hence, we conclude that $\mathcal{I}_3 - \lim X_{ijk} = X_0$. □

Definition 2.4.3. An admissible ideal $\mathcal{I}_3 \subset P(\mathbb{N}^3)$ is said to be satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_3$, there exists a countable family $\{B_1, B_2, ...\}$ in $\mathcal{I}_3$ such that $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in \mathcal{I}_3$.

Theorem 2.4.7. *If the ideal $\mathcal{I}_3$ satisfy the property (AP), then $\mathcal{I}_3$-convergence implies $\mathcal{I}_3^*$-convergence for sequence of fuzzy numbers.*

Proof. Suppose that the ideal $\mathcal{I}_3$ satisfies the condition (AP). Let $X = (X_{ijk})$ be a triple sequence of fuzzy numbers such that $\mathcal{I}_3 - \lim X_{ijk} = X_0$. Then for each $\epsilon > 0$, the set $A(\epsilon) = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\}$ belongs to $\mathcal{I}_3$. For $n \in \mathbb{N}$, we define the set A_n as follows: Put

$$A_1 = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq 1\} \in \mathcal{I}_3 \quad \text{and}$$
$$A_n = \left\{(i,j,k) \in \mathbb{N}^3 : \frac{1}{n} \leq \overline{d}(X_{ijk}, X_0) < \frac{1}{n-1}\right\} \in \mathcal{I}_3$$

Now, it is clear that $\{A_1, A_2, ...\}$ is a countable family of mutually disjoint sets belonging to $\mathcal{I}_3$ and therefore by the condition (AP) there is a countable family of sets $\{B_1, B_2, ...\}$ in $\mathcal{I}_3$ such that $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}_3$. Since $B \in \mathcal{I}_3$ so there is set K in $F(\mathcal{I}_3)$ such that $K = \mathbb{N} - B$. Now to prove the result it is sufficient to prove that $\lim_{(i,j,k)\in K} X_{ijk} = X_0$. Let $\eta > 0$ be given. Choose a positive integer q such that $\eta > \frac{1}{q+1}$. Then, we have

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \eta\} \subset \left\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \frac{1}{q+1}\right\} \in \bigcup_{i=1}^{q+1} A_i$$

Since $A_i \Delta B_i, i = 1, 2, ...q+1$ are finite, there exists $(i_0, j_0, k_0) \in \mathbb{N}^3$ such that

$$\left\{\left\{\bigcup_{i=1}^{q+1} B_i\right\} \bigcap \{(i,j,k) : i \geq i_0, j \geq j_0, k \geq k_0\}\right\}$$
$$= \left\{\left\{\bigcup_{i=1}^{q+1} A_i\right\} \bigcap \{(i,j,k) : i \geq i_0, j \geq j_0, k \geq k_0\}\right\}$$

If $(i \geq i_0, j \geq j_0, k \geq k_0)$ and $(i,j,k) \in K$ then $(i,j,k) \notin \bigcup_{i=1}^{q+1} B_i$. Therefore, we have $(i,j,k) \notin \bigcup_{i=1}^{q+1} A_i$. Hence, for every $(i \geq i_0, j \geq j_0, k \geq k_0)$ and $(i,j,k) \in K$, we have $\overline{d}(X_{ijk}, X_0) < \eta$. This completes the proof. $\square$

After the introduction of $\mathcal{I}_3$-convergence and $\mathcal{I}_3^*$-convergence, it is natural to talk about notions of $\mathcal{I}_3$-Cauchy and $\mathcal{I}_3^*$-Cauchy sequences of fuzzy numbers, which we aim in present section.

Definition 2.4.4. A triple sequence $X = (X_{ijk})$ of fuzzy numbers is said to be $\mathcal{I}_3$-Cauchy if for each $\epsilon > 0$ there exists integers $L = L(\epsilon)$, $M = M(\epsilon)$ and $N = N(\epsilon)$, such that $i, p \geq L$, $j, q \geq M$ and $k, r \geq N$

$$\{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_{pqr}) \geq \epsilon\} \in \mathcal{I}_3.$$

Definition 2.4.5. A triple sequence $X = (X_{ijk})$ of fuzzy numbers is said to be $\mathcal{I}_3^*$-Cauchy if there exists a subset $K = \{(i_n, j_n, k_n)\} \subset \mathbb{N}^3, n = 1, 2, \ldots$ such that $K \in F(\mathcal{I}_3)$ and the subsequence $X_{i_n j n k_n}$ is an ordinary Cauchy sequence.

Theorem 2.4.8. *Let $\mathcal{I}_3$ be an admissible ideal. If a triple sequence (X_{ijk}) is $\mathcal{I}_3^*$-Cauchy, then it is $\mathcal{I}_3$-Cauchy.*

Theorem 2.4.9. *If the ideal $\mathcal{I}_3$ satisfy the property (AP) and (X_{ijk}) is a $\mathcal{I}_3$-Cauchy sequences then it is also $\mathcal{I}_3^*$-Cauchy sequence of fuzzy numbers.*

The proofs of above theorems goes on similar lines as for the theorems 2.4.6 and 2.4.7, so are omitted here.

Theorem 2.4.10. *A triple sequence $X = (X_{ijk})$ of fuzzy numbers is $\mathcal{I}_3$-convergent, if and only if, it is $\mathcal{I}_3$-Cauchy.*

Proof. Let $X = (X_{ijk})$ be $\mathcal{I}_3$-convergent to X_0. By definition, for each $\epsilon \geq 0$ we have

$$A = \left\{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \frac{\epsilon}{2}\right\} \in \mathcal{I}_3.$$

This implies that the set $A^C = \left\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) < \frac{\epsilon}{2}\right\} \in F(\mathcal{I}_3)$ is non-empty. So, we can choose $(p,q,r) \in A^C$ such that $\overline{d}(X_{pqr}, X_0) < \epsilon$. If we denote

$$B = \{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_{pqr}) \geq \epsilon\},$$

we need to prove that $B \subset A$. Let $(l,m,n) \in B$, then we have

$$\epsilon < \overline{d}(X_{lmn}, X_{pqr}) \leq \overline{d}(X_{lmn}, X_0) + \overline{d}(X_{pqr}, X_0) \leq \overline{d}(X_{lmn}, X_0) + \tfrac{\epsilon}{2}.$$

This implies that $\frac{\epsilon}{2} < \overline{d}(X_{lmn}, X_0)$ and therefore $(l,m,n) \in A$ and hence $B \subset A$. Since $A \in \mathcal{I}_3$, therefore $B \in \mathcal{I}_3$. This completes the proof.

Conversely, Suppose that $X = (X_{ijk})$ is a $\mathcal{I}_3$-Cauchy. We shall prove that (X_{ijk}) is $\mathcal{I}_3$-convergent. To this effect, let $(\epsilon_p : p = 1, 2, \ldots)$ be strictly decreasing sequence of numbers converging to zero. Since $X = (X_{ijk})$ is $\mathcal{I}_3$-Cauchy, therefore there exist three strictly increasing sequences (L_p), (M_p) and (N_p) of positive integers such that

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_{L_pM_pN_p}) \geq \epsilon_p\} \in \mathcal{I}_3. \tag{2.4.8}$$

This implies that

$$\{(i,j,k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_{L_pM_pN_p}) < \epsilon_p\} \in F(\mathcal{I}_3). \tag{2.4.9}$$

Clearly, for each pair p and q $(p \neq q)$ of positive integers, we can select $(i_{pq}, j_{pq}, k_{pq}) \in \mathbb{N}^3$ such that

$$\overline{d}(X_{i_{pq}j_{pq}k_{pq}}, X_{L_pM_pN_p}) \leq \epsilon_p \quad and \quad \overline{d}(X_{i_{pq}j_{pq}k_{pq}}, X_{L_qM_qN_q}) \leq \epsilon_q.$$

It follows that

$$\begin{aligned}\overline{d}(X_{L_pM_pN_p}, X_{L_qM_qN_q}) &\leq \overline{d}(X_{i_{pq}j_{pq}k_{pq}}, X_{L_pM_pN_p}) + \overline{d}(X_{i_{pq}j_{pq}k_{pq}}, X_{L_qM_qN_q}) \\ &\leq \epsilon_p + \epsilon_q \to 0 \quad as \qquad p,q \to \infty.\end{aligned}$$

Thus, $(X_{L_pM_pN_p} : p = 1, 2, ...)$ is Cauchy sequence and satisfies the Cauchy convergence criterion.

Let $(X_{L_pM_pN_p})$ converges to X_0. Since $(\epsilon_p : p = 1, 2, \ldots) \longrightarrow 0$ so for $\epsilon > 0$, then there exists $p_0 \in \mathbb{N}$, such that

$$\epsilon_{p_0} < \frac{\epsilon}{2} \qquad and \qquad \overline{d}(X_{L_pM_pN_p}, X_0) < \frac{\epsilon}{2}, \qquad p \geq p_0. \tag{2.4.10}$$

Now, we prove that the set $\{(i, j, k) \in \mathbb{N}^3 : \overline{d}(X_{ijk}, X_0) \geq \epsilon\} \subset A_{p_0}$. Consider arbitrary $(i, j, k) \in \mathbb{N}^3$. By (2.4.10)

$$\begin{aligned} \overline{d}(X_{ijk}, X_0) &\leq \overline{d}(X_{ijk}, X_{L_{p_0}M_{p_0}N_{p_0}}) + \overline{d}(X_{L_{p_0}M_{p_0}N_{p_0}}, X_0) \\ &\leq \overline{d}(X_{ijk}, X_{L_{p_0}M_{p_0}N_{p_0}}) + \frac{\epsilon}{2} \end{aligned}$$

and by first half of (2.4.10), $\epsilon_{p_0} < \overline{d}(X_{ijk}, X_{L_{p_0}M_{p_0}N_{p_0}})$. This implies $(i, j, k) \in A_{p_0}$ and therefore $A \subset A_{p_0}$. Since $A_{p_0} \in \mathcal{I}_3$, so that $\mathcal{I}_3$. Hence, (X_{ijk}) is $\mathcal{I}_3$-convergent. $\square$

The concepts and results presented in the previous sections can be extended to d-multiple sequences of fuzzy numbers, where d is a fixed positive integer. Let $\mathbb{N}^d = \{(k_1, k_2, \ldots k_d) : k_j \in \mathbb{N}, \forall\ j\}$. The d-tuple $\mathbf{k} \neq \mathbf{n}$, where $\mathbf{k} = (k_1, k_2, \ldots k_d)$ and $\mathbf{n} = (n_1, n_2, \ldots n_d)$, if and only if, $n_j \neq k_j$ for at least one j. Moreover, the partial order on $\mathbb{N}^d$ is defined as follows:

For $\mathbf{k}, \mathbf{n} \in \mathbb{N}^d$, we say that $\mathbf{k} \leq \mathbf{n}$, if and only if, $k_j \leq n_j$ for each j.
The natural density of a set $\mathbf{S} \subseteq \mathbb{N}^d$ can be defined as

$$\delta_d(\mathbf{S}) = \lim_{\min n_j \longrightarrow \infty} \frac{1}{|\mathbf{n}|} |\{\mathbf{k} \leq \mathbf{n} : \mathbf{k} \in \mathbf{S}\}|,$$

provided that this limit exists. With the help of δ_d-density, the notions of statistical convergence and statistical Cauchy for multiple sequences of fuzzy numbers can be define as follows:

Definition 2.4.6. A d-tuple sequence $(X = X_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is said to be statistically convergent to some fuzzy number X_0 if for each $\epsilon > 0$

$$\lim_{\min n_j \longrightarrow \infty} \frac{1}{|\mathbf{n}|} |\{\mathbf{k} \leq \mathbf{n} : \overline{d}(X_{\mathbf{k}}, X_0) \geq \epsilon\}| = 0,$$

where

$$|\mathbf{n}| = \prod_{j=1}^{d} (n_j).$$

Definition 2.4.7. A d-tuple sequence $(X = X_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is said to be statistically Cauchy if for each $\epsilon > 0$ and $l \geq 0$ there exist $\mathbf{m} = (m_1, m_2, \ldots m_d)$ $\in \mathbb{N}^d$ such that $\min m_j > l$ and

$$\lim_{\min n_j \longrightarrow \infty} \frac{1}{|\mathbf{n}|} |\{\mathbf{k} \leq \mathbf{n} : \overline{d}(X_{\mathbf{k}}, X_{\mathbf{m}}) \geq \epsilon\}| = 0.$$

With the help of a non trivial ideal $\mathcal{I}^d \in P(\mathbb{N}^d)$, the notions of $\mathcal{I}_d$-convergence, $\mathcal{I}_d^*$-convergence, $\mathcal{I}_d$-Cauchy and $\mathcal{I}_d^*$-Cauchy for multiple sequences of fuzzy numbers can be define as follows.

Definition 2.4.8. A d-tuple sequence $(\mathbf{X} = \mathbf{X}_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is said to be $\mathcal{I}_d$-convergent to some fuzzy number $\mathbf{X}_0$ if for each $\epsilon > 0$

$$\{\mathbf{k} \in \mathbb{N}^d : \overline{d}(\mathbf{X}_k, \mathbf{X}_0) \geq \epsilon\} \in \mathcal{I}^d.$$

Definition 2.4.9. A d-tuple sequence $(\mathbf{X} = \mathbf{X}_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is $\mathcal{I}_d^*$-convergent to a fuzzy number $\mathbf{X}_0$, if and only if, there exists a subset $\mathbf{K} = \{\mathbf{k}_n, n = 1, 2, \ldots\} \subset \mathbb{N}^d$ such that $\mathbf{K} \in F(\mathcal{I}_d)$ and $\lim_{n\to\infty} X_{\mathbf{k}_n} = \mathbf{X}_0$.

Definition 2.4.10. A d-tuple sequence $(\mathbf{X} = \mathbf{X}_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is said to be $\mathcal{I}_d$-Cauchy if for each $\epsilon > 0$, there exist $\mathbf{m} = (m_1, m_2, \ldots m_d) \in \mathbb{N}^d$ such that and

$$\{\mathbf{k} \leq \mathbb{N} : \overline{d}(\mathbf{X}_{\mathbf{k}}, \mathbf{X}_{\mathbf{m}}) \geq \epsilon\} \in \mathcal{I}^d.$$

Definition 2.4.11. A d-tuple sequence $(\mathbf{X} = \mathbf{X_k} : \mathbf{k} \in \mathbb{N}^d)$ of fuzzy numbers is said to be $\mathcal{I}_d^*$-Cauchy if there exists a subset $\mathbf{K} = \{(k_1, k_2, \ldots k_d) : k_i \in \mathbb{N}, \forall\ i\} \subset \mathbb{N}^d$ such that $\mathbf{K} \in F(\mathcal{I}_d)$ and the subsequence $\mathbf{X_K}$ is an ordinary Cauchy sequence.

All the results presented in previous sections remain true for d-multiple sequences as well.

Chapter 3

Generalized Limit and Cluster Points of Sequences of Fuzzy Numbers

3.1 Introduction

In recent years, the idea of statistical convergence has grown little faster and many concepts parallel to the classical concepts of ordinary convergence in mathematical analysis have been introduced for statistical convergence. Fridy [50] in 1993, introduced the concept of statistical convergence as a sequential limit and used it to define a statistical limit point of a number sequence $x = (x_k)$ as a number λ that is the limit of a subsequence $(x_{k(j)})$ of x such that the set $\{k(j) : j \in \mathbb{N}\}$ does not have density zero. Similarly, a statistical cluster point of x is a number γ such that for every $\epsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \gamma| < \epsilon\}$ does not have the natural density zero. These concepts, which are not equivalent, are then compared to the usual concept of

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limit point of a sequence x. Fridy [50] obtained many interesting properties of the sets Λ_x: of statistical limit points; Γ_x: of statistical cluster points and L_x: of ordinary limit points of the sequence x. Later in 1995, these ideas were further explored for sequences of fuzzy numbers by Aytar [11], in which the author has given some interesting properties via fuzzy-analogues.

Statistical convergence has also been generalized by Fridy and Orhan [51] by using lacunary sequences. By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ whereas the ratio $\frac{k_r}{k_{r-1}}$ is abbreviated by q_r. Fridy and Orhan [51] introduced a concept of convergence closely related to statistical convergence with the help of a lacunary sequence $\theta = (k_r)$ and called it lacunary statistical convergence or S_θ-convergence. While defining this new generalized concept of convergence, the authors replaced the set $\{k : k \leq n\}$ in the definition of statistical convergence by the set $\{k : k_{r-1} < k \leq k_r\}$.

Following the work of Fridy [50] and that of Fridy and Orhan [51], Demirci [35] introduced the concepts of lacunary statistical limit and cluster points and proved some of their properties on finite dimensional Banach spaces. He also made comparisons between the sets Λ_x and Λ_x^θ, where Λ_x^θ denotes the set of lacunary statistical limit points of the sequence x. Further, Nuray [89] in 2009, extended S_θ-convergence on sequences of fuzzy numbers and gave some inclusion relations (one way) between the sets of statistically convergent and lacunary statistically convergent sequences of fuzzy numbers. These relations in reverse way have been proved by Kwon [70], who observed that Nuray's condition is sufficient as well as necessary.

The aim of the present chapter is to introduce the concepts of lacunary statistical limit point and lacunary statistical cluster point, denoted by $S_\theta-$limit point and $S_\theta-$cluster point respectively for a fuzzy number sequence $X = (X_k)$. The contents of this chapter have been divided into four sections. Section 3.2 deals with the preliminaries related to lacunary statistical convergence of sequences of fuzzy numbers. In Section 3.3, $S_\theta-$limit point and $S_\theta-$cluster point of a sequence of fuzzy numbers have been defined and new sets $\Lambda^F_{S_\theta}(X)$ and $\Gamma^F_{S_\theta}(X)$ have been introduced. Further, some inclusion relations between the sets $\Lambda^F_{S_\theta}(X)$, $\Gamma^F_{S_\theta}(X)$ and the sets $\Lambda^F_S(X)$, $\Gamma^F_S(X)$ (introduced by Aytar [11]) have been studied in this section. Later, we find restriction on $\theta = (k_r)$ for which the sets $\Lambda^F_{S_\theta}(X)$ and $\Gamma^F_{S_\theta}(X)$ respectively coincide with the sets $\Lambda^F_S(X)$ and $\Gamma^F_S(X)$. In the last section, the concepts of $S_\theta-$limit point and $S_\theta-$cluster point for a fuzzy number sequence $X = (X_k)$ have been studied with the help of the generalized difference operator Δ^m.

3.2 Preliminaries

In this section, we present some definitions of the papers due to Altin et al. [6], Aytar [11] and Nuray [89], which will be frequently used in this chapter.

Definition 3.2.1. [11] Let $(X_{k(j)})$ is a subsequence of $X = (X_k)$ and $K = \{k(j) : j \in \mathbb{N}\}$, we abbreviate $(X_{k(j)})$ by $(X)_K$. In case the density of the set K is zero then $(X)_K$ is called thin subsequence. On the other hand, $(X)_K$ is nonthin subsequence of X if K does not has density zero.

Definition 3.2.2. [11] Let $X = (X_k)$ be a sequence of fuzzy numbers. A fuzzy number X_0 is said to be a statistical limit point (s.l.p) of a sequence (X_k) of fuzzy numbers if there exists a non-thin subsequence of X that is convergent to X_0.

Let, Λ^F_S denote the set of statistical limit points of the sequence X.

Definition 3.2.3. [11] Let $X = (X_k)$ be a sequence of fuzzy numbers. A fuzzy number Y_0 is said to be a statistical cluster point (s.c.p) of a sequence (X_k) of fuzzy numbers provided that for each $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n}|\{k \in \mathbb{N} : \overline{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let, Γ_S^F denote the set of statistical cluster points of the sequence X.

Definition 3.2.4. [89] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary statistically convergent to a fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_r \frac{1}{h_r}|\{k \in I_r : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Symbolically, it is written as $S_\theta - \lim_k X_k = X_0$.

Following example illustrates the concept of lacunary statistical convergence:

Example 3.2.1. Let $\theta = (k_r)$ be a lacunary sequence. Define a sequence $X = (X_k)$ of fuzzy numbers as follows:
For $x \in \mathbb{R}$ and $k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r$, $r \in \mathbb{N}$, define

$$X_k(x) = \begin{cases} 0, & \text{if } x \in (-\infty, k) \cup (k+2, \infty), \\ x - k, & \text{if } x \in [k, k+1], \\ -x + k + 2, & \text{if } x \in (k+1, k+2]. \end{cases}$$

otherwise, if k does not satisfy $k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r$, then define $X_k(x) = X_0$, where X_0 is given by

$$X_0(x) = \begin{cases} 0, & \text{if } x \in (-\infty, -1) \cup (1, \infty), \\ 1 + x, & \text{if } x \in [-1, 0], \\ 1 - x, & \text{if } x \in (0, 1]. \end{cases}$$

For $0 < \epsilon < 1$, if we denote the set $K_r(\epsilon) = \{k \in I_r : \overline{d}(X_k, X_0) \geq \epsilon\}$, then

$$\begin{aligned} K_r(\epsilon) &= \{k \in I_r : X_k \text{ with } k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathbb{N}\} \\ &= \{k \in I_r : k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathbb{N}\} \end{aligned}$$

and therefore

$$\frac{1}{h_r}|K_r(\epsilon)| = \frac{1}{h_r}|\{k \in I_r : k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathbb{N}\}| \leq \frac{\sqrt{h_r}}{h_r}.$$

Taking $r \to \infty$, we get

$$\lim_r \frac{1}{h_r}|K_r(\epsilon)| \leq \lim_r \frac{\sqrt{h_r}}{h_r} = 0;$$

which shows that $X = (X_k)$ is $S_\theta-$convergent to X_0. However, $X = (X_k)$ is not convergent to X_0 in the usual sense. □

Altin et al.[6] introduced the concept of lacunary statistical convergence for generalized difference sequences of fuzzy numbers. For the set w of all sequences of fuzzy numbers, the difference operator $\Delta^m : w \to w$ is defined by

$$\begin{aligned} \Delta^0 X_k &= X_k, \\ \Delta^1 X_k &= X_k - X_{k+1}, \\ \vdots &= \vdots \\ \Delta^m X_k &= \Delta^1(\Delta^{m-1} X_k). \end{aligned}$$

Definition 3.2.5. [6] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary Δ^m-statistically convergent to a fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_{r \longrightarrow \infty} \frac{1}{h_r}|\{k \in I_r : \overline{d}(\Delta^m X_k, X_0) \geq \epsilon\}| = 0.$$

Symbolically, it is written as $S_\theta(\Delta^m) - \lim_k X_k = X_0$.
Let $S_\theta(\Delta^m(X))$ denotes the set of all lacunary Δ^m-statistically convergent sequences of fuzzy numbers.

We now present our main results by introducing the concepts of S_θ-limit and S_θ-cluster points for sequences of fuzzy numbers.

3.3 S_θ-limit and S_θ-cluster points

Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Let $(X_{k(j)})$ be a subsequence of $X = (X_k)$ and $K = \{k(j) : j \in \mathbb{N}\}$. We denote $(X_{k(j)})$ by $(X)_K$. If

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| = 0,$$

then (X_K) is called a $\theta-$thin subsequence. On the other hand, (X_K) is a $\theta-$nonthin subsequence of X if

$$\limsup_{r\to\infty} \frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| > 0.$$

Definition 3.3.1. For any two sequences (X_k) and (Y_k) of fuzzy numbers, if

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : X_k \neq Y_k\}| = 0,$$

then we say that $X_k = Y_k$, for almost all k with respect to θ.

Definition 3.3.2. Let $\theta = (k_r)$ be a lacunary sequence. A fuzzy number X_0 is said to be a lacunary statistical limit point or $S_\theta-$limit point of a sequence of fuzzy numbers $X = (X_k)$ if there exists a $\theta-$nonthin subsequence of X that converges to X_0.

Let $\Lambda^F_{S_\theta}(X)$ denotes the set of all $S_\theta-$limit points of the sequence $X = (X_k)$.

Definition 3.3.3. Let $\theta = (k_r)$ be a lacunary sequence. A fuzzy number Y_0 is said to be a lacunary statistical cluster point or $S_\theta-$cluster point of a sequence of fuzzy numbers $X = (X_k)$ provided that for each $\epsilon > 0$,

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let $\Gamma^F_{S_\theta}(X)$ denotes the set of all $S_\theta-$cluster points of the sequence $X = (X_k)$. We, now prove the main results of this section.

Theorem 3.3.1. *Let $\theta = (k_r)$ be any lacunary sequence. Then, for any sequence $X = (X_k)$ of fuzzy numbers, the inclusion $\Lambda^F_{S_\theta}(X) \subseteq \Gamma^F_{S_\theta}(X)$ holds.*

Proof. Suppose $X_0 \in \Lambda^F_{S_\theta}(X)$. Therefore, there exists a subsequence $(X_{k(j)})$ with $\lim_j X_{k(j)} = X_0$ and

$$\limsup_{r} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = d > 0. \tag{3.3.1}$$

For each $\epsilon > 0$, $\{k \in I_r : \overline{d}(X_k, X_0) < \epsilon\} \supset \{k(j) \in I_r : \overline{d}(X_{k(j)}, X_0) < \epsilon\}$ gives $\{k \in I_r : \overline{d}(X_k, X_0) < \epsilon\} \supset \{k(j) \in I_r : j \in \mathbb{N}\} - \{k(j) \in I_r : \overline{d}(X_{k(j)}, X_0) \geq \epsilon\}$; and therefore we have

$$\begin{aligned}\limsup_{r} \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, X_0) < \epsilon\}| &\geq \limsup_{r} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| \\ &\quad - \limsup_{r} \frac{1}{h_r} |\{k(j) \in I_r : \overline{d}(X_{k(j)}, X_0) \geq \epsilon\}|.\end{aligned}$$

Since, $\{k(j) \in I_r : \overline{d}(X_{k(j)}, X_0) \geq \epsilon\}$ is a finite set so $\limsup_r \frac{1}{h_r} |\{k(j) \in I_r : \overline{d}(X_{k(j)}, X_0) \geq \epsilon\}| = 0$. Hence the result follows by (3.3.1). □

Theorem 3.3.2. *Let $\theta = (k_r)$ be any lacunary sequence. Then, for any sequence $X = (X_k)$ of fuzzy numbers,the inclusion $\Gamma^F_{S_\theta}(X) \subseteq L(X)$ holds, where $L(X)$ denotes the set of all ordinary limit points of $X = (X_k)$.*

Proof. Suppose $Y_0 \in \Gamma^F_{S_\theta}(X)$, then for each $\epsilon > 0$

$$\limsup_r \frac{1}{h_r}|\{k \in I_r : \overline{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let $(X)_K$ be a θ-nonthin subsequence of X such that

$$K = \{k(j) \in I_r : \overline{d}(X_{k(j)}, Y_0) < \epsilon\}.$$

Since $\limsup_r \frac{1}{h_r}|\{K\}| > 0$ and there are infinitely many elements in K, so we have a θ-nonthin subsequence $(X)_K$ that converges to Y_0. This shows that $Y_0 \in L(X)$. $\square$

The converse of the Theorem 3.3.2 does not hold in general as can be seen in the following example:

Example 3.3.1. Let $\theta = (k_r)$ be a lacunary sequence. Define a sequence $X = (X_k)$ of fuzzy numbers as follows:

For $x \in \mathbb{R}$, define

$$X_k(x) = \begin{cases} X_0, & \text{if } k_r - [\sqrt{h_r}] + 1 \le k \le k_r; r \in \mathbb{N}, \\ Y_0, & \text{otherwise.} \end{cases}$$

where the fuzzy numbers X_0 and Y_0 are given by

$$X_0 = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \cup (2, \infty), \\ x, & \text{if } x \in [0, 1], \\ 2 - x, & \text{if } x \in (1, 2]. \end{cases}$$

and

$$Y_0 = \begin{cases} 0, & \text{if } x \in (-\infty, 3) \cup (5, \infty), \\ x - 3, & \text{if } x \in [3, 4], \\ 5 - x, & \text{if } x \in (4, 5]. \end{cases}$$

Clearly the sequence $X = (X_k)$ has two different subsequences $(X_0, X_0, \ldots)$ and $(Y_0, Y_0, \ldots)$ which converges to X_0 and Y_0 respectively. Hence $L(X) = \{X_0, Y_0\}$, however, $\Gamma^F_{S_\theta}(X) = \{Y_0\}$.

Theorem 3.3.3. *Let $\theta = (k_r)$ be a lacunary sequence. If $X = (X_k)$ and $Y = (Y_k)$ are two sequences of fuzzy numbers such that $X_k = Y_k$ for all most all k with respect to θ, then $\Lambda^F_{S_\theta}(X) = \Lambda^F_{S_\theta}(Y)$ and $\Gamma^F_{S_\theta}(X) = \Gamma^F_{S_\theta}(Y)$.*

Proof. Assume

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : X_k \neq Y_k\}| = 0$$

and $X_0 \in \Lambda^F_{S_\theta}(Y)$. Then, there exists a $\theta-$nonthin subsequence $(Y)_K$ of (Y_k) that converge to X_0. Since

$$\lim_r \frac{1}{h_r}|\{k \in I_r : k \in K \text{ } and \text{ } X_k \neq Y_k\}| = 0,$$

it follows that

$$\limsup_r \frac{1}{h_r}|\{k \in I_r : k \in K \text{ and } X_k = Y_k\}| > 0.$$

Therefore, the later set yields a θ-nonthin subsequence $(X)_{K'}$ of (X_k) that converges to X_0. Hence,

$$X_0 \in \Lambda^F_{S_\theta}(X) \text{ and } \Lambda^F_{S_\theta}(Y) \subset \Lambda^F_{S_\theta}(X).$$

By symmetry, we see that

$$\Lambda^F_{S_\theta}(X) \subset \Lambda^F_{S_\theta}(Y),$$

whence

$$\Lambda^F_{S_\theta}(X) = \Lambda^F_{S_\theta}(Y) \qquad .$$

The assertion that $\Gamma^F_{S_\theta}(X) = \Gamma^F_{S_\theta}(Y)$ can be proved by a similar argument. □

Theorem 3.3.4. *Let $\theta = (k_r)$ be a lacunary sequence. If $X = (X_k)$ be a sequence of fuzzy numbers such that $S_\theta - \lim_k X_k = X_0$, then $\Lambda^F_{S_\theta}(X) = \Gamma^F_{S_\theta}(X) = \{X_0\}$.*

Proof. We first prove that $\Lambda^F_{S_\theta}(X) = \{X_0\}$. Suppose $\Lambda^F_{S_\theta}(X) = \{X_0, Y_0\}$, where $X_0 \neq Y_0$. Let $0 < \epsilon < \frac{\overline{d}(X_0,Y_0)}{2}$, there are two $\theta-$nonthin subsequences $(X_{k(j)})$ and $(X_{l(i)})$ of (X_k) which respectively converges to X_0 and Y_0. Since, $(X_{l(i)})$ converges to Y_0, then for ϵ,

$$\{l(i) \in I_n : \overline{d}(X_{l(i)}, Y_0) \geq \epsilon\}$$

is a finite set and therefore, we have

$$\limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) \geq \epsilon\}| = 0.$$

Further, for any $\epsilon > 0$

$$\{l(i) \in I_r : i \in \mathbb{N}\} = \{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) < \epsilon\} \cup \{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) \geq \epsilon\},$$

implies that

$$\begin{aligned} 0 < \limsup_r \frac{1}{h_r} |\{l(i) \in I_r : i \in \mathbb{N}\}| &\leq \limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) < \epsilon\}| \\ &\quad + \limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) \geq \epsilon\}| \\ &= \limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) < \epsilon\}|. \end{aligned}$$

That means

$$\limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) < \epsilon\}| > 0 \tag{3.3.2}$$

Since, $S_\theta - \lim_k X_k = X_0$, so for $\epsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0,$$

which implies

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, X_0) < \epsilon\}| > 0.$$

Also for $0 < 2\epsilon < \overline{d}(X_0, Y_0)$,

$$\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) < \epsilon\} \cap \{k \in I_r : \overline{d}(X_k, X_0) < \epsilon\} = \emptyset.$$

But,

$$\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) < \epsilon\} \subset \{k \in I_r : \overline{d}(X_k, X_0) \geq \epsilon\},$$

which immediately implies that

$$\limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) < \epsilon\}| \leq \limsup_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0. \tag{3.3.3}$$

Since left side of (3.3.3) can not be negative, so we must have

$$\limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(X_{l(i)}, Y_0) < \epsilon\}| = 0,$$

which contradicts (3.3.2). Hence, $\Lambda^F_{S_\theta}(X) = \{X_0\}$.

Now, let Z_0 be a $S_\theta-$cluster point of $X = (X_k)$ different from X_0 i.e., $\Gamma^F_{S_\theta}(X) = \{X_0, Z_0\}$, where $X_0 \neq Z_0$. Choose $0 < \epsilon < \frac{\overline{d}(X_0, Z_0)}{2}$, then

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, Z_0) < \epsilon\}| > 0. \tag{3.3.4}$$

Since, for every $0 < \epsilon < \frac{d(X_0, Z_0)}{2}$

$$\{k \in I_r : \overline{d}(X_k, X_0) < \epsilon\} \cap \{k \in I_r : \overline{d}(X_k, Z_0) < \epsilon\} = \emptyset.$$

So,

$$\{k \in I_r : \overline{d}(X_k, X_0) \geq \epsilon\} \supseteq \{k \in I_r : \overline{d}(X_k, Z_0) < \epsilon\}$$

implies that

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, X_0) \geq \epsilon\}| \geq \limsup_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, Z_0) < \epsilon\}|. \tag{3.3.5}$$

Since $S_\theta - \lim_k X_k = X_0$, so left side of (3.3.5) is zero, however, by (3.3.4), the right side of (3.3.5) is strictly greater than zero. This contradiction shows that $\Gamma^F_{S_\theta}(X) = \{X_0\}$. $\square$

We now give some inclusion theorems on the sets $\Lambda^F_{S_\theta}(X)$, $\Gamma^F_{S_\theta}(X)$ and $\Lambda^F_S(X)$, $\Gamma^F_S(X)$.

Theorem 3.3.5. *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers; then*

(i) $\liminf_r q_r > 1$ *implies* $\Lambda^F_{S_\theta}(X) \subseteq \Lambda^F_S(X)$*;*

(ii) $\limsup_r q_r < \infty$ *implies* $\Lambda^F_S(X) \subseteq \Lambda^F_{S_\theta}(X)$*;*

(iii) $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ *implies* $\Lambda^F_S(X) = \Lambda^F_{S_\theta}(X)$*.*

Proof. (i) Suppose $\liminf_r q_r > 1$; there exists a $\delta > 0$ such that $q_r > 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

If $X_0 \in \Lambda^F_{S_\theta}(X)$, there is a subsequence $(X_{k(j)})$ of (X_k) with $\lim_j X_{k(j)} = X_0$ and

$$\limsup_r \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = d > 0 \tag{3.3.6}$$

Now

$$\begin{aligned} \frac{1}{k_r}|\{k(j) \leq k_r : j \in \mathbb{N}\}| &\geq \frac{1}{k_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| \\ &= \frac{h_r}{k_r} . \frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| \\ &\geq \left(\frac{\delta}{1+\delta}\right) . \frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}|; \end{aligned}$$

together with (3.3.6), it implies that

$$\limsup_r \frac{1}{k_r}|\{k(j) \leq k_r : j \in \mathbb{N}\}| > 0.$$

Since $(X_{k(j)})$ converges to X_0 by definition so, it follows that $X_0 \in \Lambda_S^F(X)$. Hence, $\Lambda_{S_\theta}^F(X) \subseteq \Lambda_S^F(X)$.

(ii) If $\limsup_r q_r < \infty$, then there is an H such that $q_r < H$ for all r. We may assume $H > 1$ as $\theta = (k_r)$ is a lacunary sequence. Also for each $r \in \mathbb{N}$, we have

$$\frac{h_r}{k_{r-1}} = \frac{k_r - k_{r-1}}{k_{r-1}} = q_r - 1 < H - 1.$$

If $X_0 \in \Lambda_S^F(X)$, there is a subsequence $(X_{k(j)})$ of (X_k) with $\lim_j X_{k(j)} = X_0$ and

$$\limsup_n \frac{1}{n} |\{k(j) \leq n : j \in \mathbb{N}\}| = d > 0 \tag{3.3.7}$$

Define $N_r = |\{k(j) \in I_r : j \in \mathbb{N}\}|$ and $t_r = \frac{N_r}{h_r}$, then for any integer n satisfying $k_{r-1} < n \leq k_r$; we can write

$$\begin{aligned}
\frac{1}{n}|\{k(j) \leq n : j \in \mathbb{N}\}| &\leq \frac{1}{k_{r-1}}|\{k(j) \leq k_r : j \in \mathbb{N}\}| \\
&= \frac{1}{k_{r-1}}\{N_1 + N_2 + \ldots + N_r\} \\
&= \frac{1}{k_{r-1}}\{t_1 h_1 + t_2 h_2 + \ldots t_r h_r\} \\
&= \frac{1}{\sum_{i=1}^{r-1} h_i} \sum_{i=1}^{r-1} h_i t_i + \frac{h_r}{k_{r-1}} t_r \\
&< \frac{1}{\sum_{i=1}^{r-1} h_i} \sum_{i=1}^{r-1} h_i t_i + (H-1) t_r.
\end{aligned}$$

Suppose $t_r \to 0$ as $r \to \infty$. Since θ is lacunary and the first part on the right side of the above expression is a regular weighted mean transform of the sequence $t = (t_r)$, and therefore it also tends to zero as $r \to \infty$.

Consequently, we get

$$\limsup_n \frac{1}{n}|\{k(j) \leq n : j \in \mathbb{N}\}| = 0$$

which contradicts (3.3.7).

Thus $\lim_r t_r \neq 0$ and therefore $X_0 \in \Lambda^F_{S_\theta}(X)$ as $X_{k(j)} \to X_0$. Hence $\Lambda^F_S(X) \subseteq \Lambda^F_{S_\theta}(X)$.

(iii) This is an immediate consequence of (i) and (ii). □

Theorem 3.3.6. *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers; then*

(i) $\liminf_r q_r > 1$ *implies* $\Gamma^F_{S_\theta}(X) \subset \Gamma^F_S(X)$;

(ii) $\limsup_r q_r < \infty$ *implies* $\Gamma^F_S(X) \subset \Gamma^F_{S_\theta}(X)$;

(iii) $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ *implies* $\Gamma^F_S(X) = \Gamma^F_{S_\theta}(X)$.

Proof. The proof of this theorem can be obtained in a similar way as that of Theorem 3.3.5 and is therefore omitted. □

Finally, we consider the inclusions $\Gamma^F_{S_\theta}(X) \subseteq \Gamma^F_{S_{\theta'}}(X)$ and $\Lambda^F_{S_\theta}(X) \subseteq \Lambda^F_{S_{\theta'}}(X)$ where the lacunary sequence $\theta' = (k'_r)$ is a lacunary refinement of $\theta = (k_r)$ such that $(k_r) \subseteq (k'_r)$.

Theorem 3.3.7. *For any lacunary refinement θ' of a lacunary sequence $\theta = (k_r)$, $\Gamma^F_{S_\theta}(X) \subseteq \Gamma^F_{S_{\theta'}}(X)$ and $\Lambda^F_{S_\theta}(X) \subseteq \Lambda^F_{S_{\theta'}}(X)$.*

Proof. Suppose each I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{v(r)}$ of θ' so that $k_{r-1} < k'_{r,1} < k'_{r,2} < \ldots < k'_{r,v(r)} = k_r$, where $I'_{r,i} = (k'_{r,i-1}, k'_{r,i}]$. Note that for all r, $v(r) \geq 1$. Let$(I^*_j)_{j=1}^\infty$ be the sequence of abutting intervals $I'_{r,i}$ ordered by increasing right end points. Suppose $Y_0 \in \Gamma^F_{S_\theta}(X)$, then for each $\epsilon > 0$,

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \overline{d}(X_k, Y_0) < \epsilon\}| > 0. \tag{3.3.8}$$

As before, write $h'_{r,i} = k'_{r,i} - k'_{r,i-1}$ and $h'_{r,1} = k'_{r,1} - k_{r-1}$. For each $\epsilon > 0$, we have

$$\begin{aligned}\frac{1}{h_r}|\{k \in I_r : \overline{d}(X_k, Y_0) < \epsilon\}| &= \frac{1}{h_r}\sum_{I_j^* \subseteq I_r} h_j^* \frac{1}{h_j^*}|\{k \in I_j^* : \overline{d}(X_k, Y_0) < \epsilon\}| \\ &= \frac{1}{h_r}\sum_{I_j^* \subseteq I_r} h_j^*(C_{\theta'}\chi_K)_j\end{aligned}$$

where χ_K is the characteristics function of the set $K = \{k \in \mathbb{N} : \overline{d}(X_k, Y_0) < \epsilon\}$ and $(C_{\theta'}\chi_K)_j = \frac{|K \cap I_j^*|}{h_j^*}$. Suppose $\lim_j(C_{\theta'}\chi_K)_j = 0$, then the right side of above expression is a regular weighted mean transform of $(C_{\theta'}\chi_K)_j$ and therefore tends to zero as $j \to \infty$.But then we obtain a contradiction to (3.3.8) and hence $\lim_j(C_{\theta'}\chi_K)_j \neq 0$. This shows that $Y_0 \in \Gamma^F_{S_{\theta'}}(X)$. The assertion that $\Lambda^F_{S_\theta}(X) \subseteq \Lambda^F_{S_{\theta'}}(X)$ can be proved by a similar argument. $\square$

3.4 S_θ-limit and S_θ-cluster points for generalized difference sequence $(\Delta^m X_k)$

In this section, we define more generalized notions of statistical limit and cluster points introduced by Aytar [11], by using the generalized difference operator Δ^m and study some analogous results.

Definition 3.4.1. Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. A fuzzy number X_0 is said to be a lacunary statistical limit point (l.s.l.p) or S_θ-limit point of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers if there exists a θ-nonthin subsequence of $(\Delta^m X_k)$ that is convergent to X_0.

Let $\Lambda_{S_\theta}(\Delta^m X)$ denotes the set of all l.s.l.p. of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers.

Definition 3.4.2. Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. A fuzzy number Y_0 is said to be a lacunary statistical cluster point (l.s.c.p) of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers if for each $\epsilon > 0$,

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : \overline{d}(\Delta^m X_k, Y_0) < \epsilon\}| > 0.$$

Let $\Gamma_{S_\theta}(\Delta^m X)$ denotes the set of all l.s.c.p of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers.

These definitions can easily be demonstrated with the help of following example:

Example 3.4.1. Let $\theta = (k_r)$ be a lacunary sequence. Define a sequence $X = (X_k)$ of fuzzy numbers as follows.

For $x \in \mathbb{R}$, define

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} x-k+1, & \text{if } k-1 \le x \le k \\ -x+k+1, & \text{if } k < x \le k+1 \\ 0, & \text{otherwise} \end{array}\right\}, & \text{if} \quad k_r - [\sqrt{h_r}] + 1 \le k \le k_r; r \in \mathbb{N} \\ \left.\begin{array}{ll} x-5, & \text{if } 5 \le x \le 6 \\ 7-x, & \text{if } 6 < x \le 7 \\ 0, & \text{otherwise} \end{array}\right\}, & \text{otherwise.} \end{cases}$$

Then, we obtain

$$[X_k]^\alpha = \begin{cases} [k-1+\alpha, k+1-\alpha], & \text{if } k \in [k_r - [\sqrt{h_r}] + 1, k_r] \\ [5+\alpha, 7-\alpha], & \text{otherwise.} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [-3+\alpha, 1-2\alpha], & \text{if both } k, k+1 \in [k_r - [\sqrt{h_r}] + 1, k_r] \\ [k-8+2\alpha, k-4-2\alpha], & \text{if } k \in [k_r - [\sqrt{h_r}] + 1, k_r] \text{ but not } k+1, \\ [-k+3+2\alpha, -k+7-2\alpha], & \text{if } k+1 \in [k_r - [\sqrt{h_r}] + 1, k_r] \text{ but not } k, \\ [-2+2\alpha, 2-2\alpha], & \text{otherwise.} \end{cases}$$

Thus, for $m = 1$, it is clear that the sequence ΔX_k has two different subsequences which converge to μ_1 and μ_2 respectively, where

$[\mu_1]^\alpha = [-3 + \alpha, 1 - 2\alpha]$ and $[\mu_2]^\alpha = [-2 + 2\alpha, 2 - 2\alpha]$.

Hence, if $L(\Delta X)$ denotes the set of ordinary limit points of (ΔX_k), then $L(\Delta X) = \{\mu_1, \mu_2\}$, however $\Gamma_{S_\theta}(\Delta X) = \{\mu_2\}$. $\square$

Theorem 3.4.1. *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Then, $\Lambda_{S_\theta}(\Delta^m X) \subseteq \Gamma_{S_\theta}(\Delta^m X)$.*

Proof. Suppose $X_0 \in \Lambda_{S_\theta}(\Delta^m X)$. By definition, there is a θ-nonthin subsequence $(\Delta^m X_{k(j)})$ of $(\Delta^m X)$ which is convergent to X_0 and therefore we have

$$\limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = d > 0 \tag{3.4.1}$$

Since, for every $\epsilon > 0$,

$\{k \in I_r : \overline{d}(\Delta^m X_k, X_0) < \epsilon\} \supseteq \{k(j) \in I_r : \overline{d}(\Delta^m X_{k(j)}, X_0) < \epsilon\}$,

so, we have the containment

$$\{k \in I_r : \overline{d}(\Delta^m X_k, X_0) < \epsilon\} \supset \{k(j) \in I_r : j \in \mathbb{N}\} - \{k(j) \in I_r : \overline{d}(\Delta^m X_{k(j)}, X_0) \geq \epsilon\}. \tag{3.4.2}$$

Now, $(X_{k(j)})$ is Δ^m-convergent to X_0, implies that for every $\epsilon > 0$, the set $\{k(j) \in I_r : \overline{d}(\Delta^m X_{k(j)}, X_0) \geq \epsilon\}$ is a finite set for which we have

$$\limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{k(j) \in I_r : \overline{d}(\Delta^m X_{k(j)}, X_0) \geq \epsilon\}| = 0 \tag{3.4.3}$$

Thus from (3.4.2), we obtain

$$\begin{aligned}\limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{k \in I_r : \overline{d}(\Delta^m X_k, X_0) < \epsilon\}| &\geq \limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| \\ &\quad - \limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{k(j) \in I_r : \overline{d}(\Delta^m X_{k(j)}, X_0) \geq \epsilon\}| \\ &\geq \limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = d > 0,\end{aligned}$$

using (3.4.1) and (3.4.3). This shows that $X_0 \in \Gamma_{S_\theta}(\Delta^m X)$ and hence the result is proved. $\square$

Theorem 3.4.2. *Let $\theta = (k_r)$ be a lacunary sequence. Then, for any sequence $X = (X_k)$ of fuzzy numbers, the inclusion $\Gamma_{S_\theta}(\Delta^m X) \subseteq L(\Delta^m(X))$ holds.*

Proof. Assume $Y_0 \in \Gamma_{S_\theta}(\Delta^m X)$. By definition, for each $\epsilon > 0$, we have

$$\limsup_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : \overline{d}(\Delta^m X_k, Y_0) < \epsilon\}| > 0$$

Let $(X)_K$ a θ-nonthin subsequence of $X = (X_k)$ such that

$K = \{k(j) \in I_r : \overline{d}(\Delta^m X_k, Y_0) < \epsilon\}$ for $\epsilon > 0$.

Since $\limsup_{r\longrightarrow\infty} \frac{1}{h_r}|\{K\}| > 0$, it follows that K is an infinite set.

Thus, we have a subsequence $(X)_K$ of X which is Δ^m-convergent to Y_0. This shows that $Y_0 \in L(\Delta^m(X))$. Hence, $\Gamma_{S_\theta}(\Delta^m X) \subseteq L(\Delta^m(X))$. $\square$

Theorem 3.4.3. *Let $\theta = (k_r)$ be a lacunary sequence. If $X = (X_k)$ and $Y = (Y_k)$ are two sequences of fuzzy numbers such that*

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : X_k \neq Y_k\}| = 0,$$

then the inclusions $\Lambda_{S_\theta}(\Delta^m X) = \Lambda_{S_\theta}(\Delta^m Y)$ and $\Gamma_{S_\theta}(\Delta^m X) = \Gamma_{S_\theta}(\Delta^m Y)$ hold.

Proof. We prove this theorem into two parts. In first part, we prove that $\Lambda_{S_\theta}(\Delta^m X) = \Lambda_{S_\theta}(\Delta^m Y)$, whereas in second part, we shall prove $\Gamma_{S_\theta}(\Delta^m X) = \Gamma_{S_\theta}(\Delta^m Y)$.

Part (i): Let $X_0 \in \Lambda_{S_\theta}(\Delta^m Y)$. Therefore by definition, there exists a θ-nonthin subsequence $(Y)_K$ of $Y = (Y_k)$ that is Δ^m-convergent to X_0.

Since

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r}|\{k \in I_r : k \in K \text{ and } X_k \neq Y_k\}| = 0;$$

it follows that

$$\limsup_{r\longrightarrow\infty}\frac{1}{h_r}|\{k\in I_r : k\in K \text{ and } X_k=Y_k\}|>0.$$

Therefore, there exists a θ-nonthin subsequence $(X)_{K'}$ of $X=(X_k)$ that is Δ^m-convergent to X_0. Hence, $X_0\in\Lambda_{S_\theta}(\Delta^m X)$ and therefore we have $\Lambda_{S_\theta}(\Delta^m Y)\subset\Lambda_{S_\theta}(\Delta^m X)$. Also by symmetry one gets $\Lambda_{S_\theta}(\Delta^m X)\subset\Lambda_{S_\theta}(\Delta^m Y)$.
On combining, we get $\Lambda_{S_\theta}(\Delta^m X)=\Lambda_{S_\theta}(\Delta^m Y)$.

Part(ii): Let $Y_0\in\Gamma_{S_\theta}(\Delta^m X)$. By definition, for each $\epsilon>0$,

$$\limsup_{r\longrightarrow\infty}\frac{1}{h_r}|\{k\in I_r : \overline{d}(\Delta^m X_k,Y_0)<\epsilon\}|>0$$

Since $X_k=Y_k$ for all most all k, it follows that for each $\epsilon>0$,

$$\limsup_{r\longrightarrow\infty}\frac{1}{h_r}|\{k\in I_r : \overline{d}(\Delta^m Y_k,Y_0)<\epsilon\}|>0$$

This shows that $Y_0\in\Gamma_{S_\theta}(\Delta^m Y)$ and therefore $\Gamma_{S_\theta}(\Delta^m X)\subset\Gamma_{S_\theta}(\Delta^m Y)$. By symmetry, we see that $\Gamma_{S_\theta}(\Delta^m Y)\subset\Gamma_{S_\theta}(\Delta^m X)$, whence $\Gamma_{S_\theta}(\Delta^m X)=\Gamma_{S_\theta}(\Delta^m Y)$. □

Theorem 3.4.4. *Let $\theta=(k_r)$ be a lacunary sequence. If $X=(X_k)$ be a sequence of fuzzy numbers such that $S_\theta-\lim_k\Delta^m X_k=X_0$, then $\Lambda_{S_\theta}(\Delta^m X)=\Gamma_{S_\theta}(\Delta^m X)=\{X_0\}$.*

Proof. We prove this theorem into two parts. In first part, we prove that $\Lambda_{S_\theta}(\Delta^m X)=\{X_0\}$, whereas in second part, we prove $\Gamma_{S_\theta}(\Delta^m X)=\{X_0\}$.

Part(i): Suppose that $\Lambda_{S_\theta}(\Delta^m X)=\{X_0,Y_0\}$ where $X_0\neq Y_0$, that is Y_0 be a l.s.l.p. of the generalized difference sequence $(\Delta^m X_k)$ different from X_0.
Choose $\epsilon>0$ such that $0<\epsilon<\frac{\overline{d}(X_0,Y_0)}{2}$. By definition, there exist two θ-nonthin subsequences $(X_{k(j)})$ and $(X_{l(i)})$ of the sequence $X=(X_k)$ which are Δ^m-convergent

to X_0 and Y_0 respectively. Since $(X_{l(i)})$ is Δ^m-convergent to Y_0 so for each $\epsilon > 0$, $\{l(i) \in I_n : \overline{d}(\Delta^m X_{l(i)}, Y_0) \geq \epsilon\}$ is a finite set for which

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(\Delta^m X_{l(i)}, Y_0) \geq \epsilon\}| = 0 \tag{3.4.4}$$

Further, we can write,

$$\{l(i) \in I_r : i \in \mathbb{N}\} = \{l(i) \in I_r : \overline{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \cup \{l(i) \in I_r : \overline{d}(\Delta^m X_{l(i)}, Y_0) \geq \epsilon\},$$

for which we have

$$\begin{aligned}\limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{l(i) \in I_r : i \in \mathbb{N}\}\}| &= \limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\}| \\ &\quad + \limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(\Delta^m X_{l(i)}, Y_0) \geq \epsilon\}|\end{aligned}$$

Since $(X_{l(i)})$ is non-thin, so by the use of (3.4.4), we have

$$\limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{l(i) \in I_r : \overline{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\}| > 0 \tag{3.4.5}$$

Since $S_\theta - \lim_k \Delta^m X_k = X_0$, so for each $\epsilon > 0$

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r} |\{k \in I_r : \overline{d}(\Delta^m X_k, X_0) \geq \epsilon\}| = 0; \tag{3.4.6}$$

and therefore we can write

$$\limsup_{r\longrightarrow\infty} \frac{1}{h_r} |\{k \in I_r : \overline{d}(\Delta^m X_k, X_0) < \epsilon\}| > 0. \tag{3.4.7}$$

Furthermore, for $0 < 2\epsilon < \overline{d}(X_0, Y_0)$,

$$\{l(i) \in I_r : \overline{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \cap \{k \in I_r : \overline{d}(\Delta^m X_k, X_0) < \epsilon\} = \emptyset;$$

which immediately gives the containment

$$\{l(i) \in I_r : \overline{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \subset \{k \in I_r : \overline{d}(\Delta^m X_k, X_0) \geq \epsilon\};$$

for which

$$\limsup_{r\longrightarrow\infty}\frac{1}{h_r}|\{l(i)\in I_r:\overline{d}(\Delta^m X_{l(i)},Y_0)<\epsilon\}|\leq\limsup_{r\longrightarrow\infty}\frac{1}{h_r}|\{k\in I_r:\overline{d}(\Delta^m X_k,X_0)\geq\epsilon\}|=0. \tag{3.4.8}$$

As the left hand side of (3.4.8) can not be negative, so we must have

$$\limsup_{r\longrightarrow\infty}\frac{1}{h_r}|\{l(i)\in I_r:\overline{d}(\Delta^m X_{l(i)},Y_0)<\epsilon\}|=0.$$

This contradicts (3.4.5). Hence, $\Lambda_{S_\theta}(\Delta^m X)=\{X_0\}$.

Part(ii): Let Z_0 be a l.s.c.p. of the generalized difference sequence $(\Delta^m X_k)$ different from X_0 i.e., $\Gamma_{S_\theta}(\Delta^m X)=\{X_0,Z_0\}$ where $X_0\neq Z_0$. Choose ϵ such that $0<\epsilon<\frac{\overline{d}(X_0,Z_0)}{2}$. Since Z_0 is a l.s.c.p of $(\Delta^m X_k)$ so for each $\epsilon>$, we have

$$\limsup_{r\longrightarrow\infty}\frac{1}{h_r}|\{k\in I_r:\overline{d}(\Delta^m X_k,Z_0)<\epsilon\}|>0. \tag{3.4.9}$$

Since

$$\{k\in I_r:\overline{d}(\Delta^m X_k,X_0)<\epsilon\}\cap\{k\in I_r:\overline{d}(\Delta^m X_k,Z_0)<\epsilon\}=\emptyset$$

for every $0<\epsilon<\frac{d(X_0,Z_0)}{2}$, it follows that

$$\{k\in I_r:\overline{d}(\Delta^m X_k,X_0)\geq\epsilon\}\supseteq\{k\in I_r:\overline{d}(\Delta^m X_k,Z_0)<\epsilon\}.$$

Therefore, by the use of (3.4.9) we have

$$\limsup_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:\overline{d}(\Delta^m X_k,X_0)\geq\epsilon\}|\geq\limsup_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:\overline{d}(\Delta^m X_k,Z_0)<\epsilon\}|>0,$$

which is impossible as by (3.4.6)

$$\limsup_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:\overline{d}(\Delta^m X_k,X_0)\geq\epsilon\}|=0,$$

which is a contradiction. Hence, $\Gamma_{S_\theta}(\Delta^m X)=\{X_0\}$. $\square$

Theorem 3.4.5. *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Then, we have the following.*

(i) if $\liminf_r q_r > 1$*, then* $\Lambda_{S_\theta}(\Delta^m X) \subset \Lambda_S(\Delta^m X)$.

(ii) if $\limsup_r q_r < \infty$*, then* $\Lambda_S(\Delta^m X) \subset \Lambda_{S_\theta}(\Delta^m X)$.

(iii) if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$*, then* $\Lambda_S(\Delta^m X) = \Lambda_{S_\theta}(\Delta^m X)$.

Proof. (i) Suppose $\liminf_r q_r > 1$. Then, there exists a $\delta > 0$ such that $q_r > 1 + \delta$ for sufficient large r. This implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

Assume that $X_0 \in \Lambda_{S_\theta}(\Delta^m X)$, then there exists a θ-nonthin subsequence $(X_{k(j)})$ of (X_k) that is Δ^m-convergent to X_0 and

$$\limsup_r \frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| = d > 0 \tag{3.4.10}$$

Now

$$\begin{aligned} \frac{1}{k_r}|\{k(j) \leq k_r : j \in \mathbb{N}\}| &\geq \frac{1}{k_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| \\ &= \frac{h_r}{k_r}.\frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}| \\ &\geq \left(\frac{\delta}{1+\delta}\right).\frac{1}{h_r}|\{k(j) \in I_r : j \in \mathbb{N}\}|; \end{aligned}$$

it follows by (3.4.10) that

$$\limsup_r \frac{1}{k_r}|\{k(j) \leq k_r : j \in \mathbb{N}\}| > 0.$$

Since $(X_{k(j)})$ is already Δ^m-convergent to X_0, so we have $X_0 \in \Lambda_S(\Delta^m X)$. Hence $\Lambda_{S_\theta}(\Delta^m X) \subset \Lambda_S(\Delta^m X)$.

(ii) If $\limsup_r q_r < \infty$, then there exist a real number H such that $q_r < H$ for all r. Without loss of generality, we can assume $H > 1$ (as otherwise $k_r < k_{r-1}$). Now for all r, $\frac{h_r}{k_{r-1}} = \frac{k_r - k_{r-1}}{k_{r-1}} = q_r - 1 \leq H - 1$. Let $X_0 \in \Lambda_S(\Delta^m X)$, then there is a set $K = \{k(j) : j \in \mathbb{N}\}$ with $\delta(K) \neq 0$ and $\lim_j \Delta^m X_{k(j)} = X_0$. Let $N_r = |\{k \in I_r : k \in K\}| = |K \cap I_r|$ *and* $t_r = \frac{N_r}{h_r}$. For any integer n satisfying $k_{r-1} < n \leq k_r$, we can write

$$\begin{aligned}
\frac{1}{n}|\{k(j) \leq n : j \in \mathbb{N}\}| &\leq \frac{1}{k_{r-1}}|\{k(j) \leq k_r : j \in \mathbb{N}\}| \\
&= \frac{1}{k_{r-1}}\{N_1 + N_2 + \ldots + N_r\} \\
&= \frac{1}{k_{r-1}}\{t_1 h_1 + t_2 h_2 + \ldots t_r h_r\} \\
&= \frac{1}{\sum_{i=1}^{r-1} h_i}\sum_{i=1}^{r-1} h_i t_i + \frac{h_r}{k_{r-1}} t_r \\
&< \frac{1}{\sum_{i=1}^{r-1} h_i}\sum_{i=1}^{r-1} h_i t_i + (H-1)t_r.
\end{aligned}$$

Suppose $t_r \to 0$ as $r \to \infty$. Since θ is a lacunary sequence and the first part on the right side of above expression is a regular weighted mean transform of the sequence $t = (t_r)$, and therefore it also tends to zero as $r \to \infty$. Since $n \to \infty$ as $r \to \infty$, it follows that $\delta(K) = 0$ which is a contradiction as $\delta(K) \neq 0$. Thus, $\lim_{r\to\infty} t_r \neq 0$ and therefore $X_0 \in \Lambda_{S_\theta}(\Delta^m X)$. Hence $\Lambda_S(\Delta^m X) \subset \Lambda_{S_\theta}(\Delta^m X)$.

(iii) This is an immediate consequence of (i) and (ii). □

Theorem 3.4.6. *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Then we have the following.*

(i) if $\liminf_r q_r > 1$, *then* $\Gamma_{S_\theta}(\Delta^m X) \subset \Gamma_S(\Delta^m X)$.

(ii) if $\limsup_r q_r < \infty$, *then* $\Gamma_S(\Delta^m X) \subset \Gamma_{S_\theta}(\Delta^m X)$.

(iii) if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, *then* $\Gamma_S(\Delta^m X) = \Gamma_{S_\theta}(\Delta^m X)$.

Proof. The proof of this theorem can be obtained on the similar lines as that of the above theorem and is therefore omitted here. □

Theorem 3.4.7. *For any lacunary refinement θ' of a lacunary sequence $\theta = (k_r)$, $\Gamma_{S_\theta}(\Delta^m X) \subseteq \Gamma_{S_{\theta'}}(\Delta^m X)$ and $\Lambda_{S_\theta}(\Delta^m X) \subseteq \Lambda_{S_{\theta'}}(\Delta^m X)$.*

Proof. Suppose each I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{v(r)}$ of θ' so that $k_{r-1} < k'_{r,1} < k'_{r,2} < \ldots < k'_{r,v(r)} = k_r$, where $I'_r = (k'_{r,i-1}, k'_{r,i}]$. Note that for all r, $v(r) \geq 1$. Let $(I^*_j)_{j=1}^{\infty}$ be the sequence of abutting intervals $I'_{r,i}$ ordered by increasing right end points. Let $Y_0 \in \Gamma_{S_\theta}(\Delta^m X)$, then for each $\epsilon > 0$,

$$\limsup_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : \overline{d}(\Delta^m X_k, Y_0) < \epsilon\}| > 0. \tag{3.4.11}$$

As before, write $h'_{r,i} = k'_{r,i} - k'_{r,i-1}$ and $h'_{r,1} = k'_{r,1} - k'_{r-1}$. Now for each $\epsilon > 0$, we can write

$$\begin{aligned}\frac{1}{h_r}|\{k \in I_r : \overline{d}(\Delta^m X_k, Y_0) < \epsilon\}| &= \frac{1}{h_r}\sum_{I^*_j \subseteq I_r} h^*_j \frac{1}{h^*_j}|\{k \in I^*_j : \overline{d}(\Delta^m X_k, Y_0) < \epsilon\}| \\ &= \frac{1}{h_r}\sum_{I^*_j \subseteq I_r} h^*_j (C_{\theta'}\chi_K)_j\end{aligned}$$

where χ_K is the characteristics function of the set $K = \{k \in I^*_j : \overline{d}(\Delta^m X_k, Y_0) < \epsilon\}$ and $(C_{\theta'}\chi_K)_j = \frac{|K \cap I^*_j|}{h^*_j}$. Suppose $\lim_{j\to\infty}(C_{\theta'}\chi_K)_j = 0$. Then the right side of above expression is a regular weighted mean transform of $(C_{\theta'}\chi_K)_j$ and therefore tends to zero as $j \to \infty$ which contradicts (3.4.11). Thus $\lim_{j\to\infty}(C_{\theta'}\chi_K)_j \neq 0$, which shows that $Y_0 \in \Gamma_{S_{\theta'}}(\Delta^m X)$. Hence $\Gamma_{S_\theta}(\Delta^m X) \subseteq \Gamma_{S_{\theta'}}(\Delta^m X)$.

Similarly, we can prove $\Lambda_{S_\theta}(\Delta^m X) \subseteq \Lambda_{S_{\theta'}}(\Delta^m X)$. $\square$

Chapter 4

On Generalized Core of Sequences of Fuzzy Numbers

4.1 Introduction

Following the concept of statistical convergence and statistical cluster points, Fridy and Orhan [53] presented natural definitions of the concepts of statistical limit superior and statistical limit inferior for a number sequence $x = (x_k)$. They also developed statistical analogues of Knopp's Core Theorem and some properties of the ordinary limit superior and inferior.

"*If $x = (x_k)$ is a statistically bounded sequence, then the statistical core of x is the closed interval $[s - \liminf x, s - \limsup x]$. In case x is not statistically bounded, $s-core\{x\}$ is defined accordingly as either $[s-\liminf x, +\infty]$, $(-\infty, +\infty)$ or $(-\infty, s-\limsup x]$.*"

Following this Aytar et al. [13] extended the concepts of statistical limit superior and limit inferior for statistically bounded sequences of fuzzy numbers and gave some

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fuzzy analogues of their properties for sequences of real numbers. He also observed that many properties which holds true for real sequences may not be hold for sequences of fuzzy numbers for instance the result "$s - \liminf x = s - \limsup x = x_0$ then $s - \lim x = x_0$", which is important for sequences of real numbers $x = (x_k)$, may not be valid for sequences of fuzzy numbers.

Motivated by works of these authors, we aim in this chapter to extend the notions of S_θ-limit and S_θ-cluster points defined in Chapter-3 to their limit inferior and limit superior for sequences of fuzzy numbers. Later, these definitions have been used to define the concept of S_θ-core for a sequence of fuzzy numbers.

The contents of this chapter have been divided into four sections. Section 4.2 deals with the preliminaries related to this chapter. In Section 4.3, Certain properties of S_θ-limit inferior and S_θ-limit superior have been explored and used to define a new concept of S_θ-core for these kind of sequences. Relations between S_θ-core{X} and S-core{X} have also been obtained in this section. Subsequently, in the last section, some new extensions of above notions have been developed by using the generalized difference operator Δ^m.

4.2 Preliminaries

Before we start this section, we recall some concepts given in papers of Aytar [13] and Aytar et al. [12].

A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically bounded from above if there exists a fuzzy number μ (called the statistical upper bound) such that

$\delta(\{k \in \mathbb{N} : X_k > \mu\} \cup \{k \in \mathbb{N} : X_k \nsim \mu\}) = 0.$

Similarly, $X = (X_k)$ is said to be statistically bounded from below provided that there exists a fuzzy number ν (called the statistical lower bound) such that

$\delta(\{k \in \mathbb{N} : X_k < \nu\} \cup \{k \in \mathbb{N} : X_k \nsim \nu\}) = 0.$

If the sequence $X = (X_k)$ is both statistically bounded from above and statistically bounded from below then it is called statistically bounded.

Given a sequence $X = (X_k)$, let us define:

$$A_X = \{\mu \in L(R) : \delta(\{k \in \mathbb{N} : X_k < \mu\}) \neq 0\};$$
$$\overline{A}_X = \{\mu \in L(R) : \delta(\{k \in \mathbb{N} : X_k > \mu\}) = 1\};$$
$$B_X = \{\mu \in L(R) : \delta(\{k \in \mathbb{N} : X_k > \mu\}) \neq 0\};$$
$$\overline{B}_X = \{\mu \in L(R) : \delta(\{k \in \mathbb{N} : X_k < \mu\}) = 1\}.$$

Definition 4.2.1. [13] If $X = (X_k)$ be a statistically bounded sequence of fuzzy numbers, then the statistical limit inferior of $X = (X_k)$ is given by

$$S - \liminf X = \inf A_X.$$

Also, the statistical limit superior is defined as

$$S - \limsup X = \sup B_X.$$

4.3 S_θ-limit inferior and S_θ-limit superior

Let, $X = (X_k)$ be a sequence of fuzzy numbers and $\theta = (k_n)$ be any lacunary sequence. Define the sets:

$$M_X = \{\mu \in L(\mathbb{R}) : \delta_\theta(\{k \in I_n : X_k < \mu\}) \neq 0\};$$
$$\overline{M}_X = \{\mu \in L(\mathbb{R}) : \delta_\theta(\{k \in I_n : X_k > \mu\}) = 1\};$$
$$N_X = \{\mu \in L(\mathbb{R}) : \delta_\theta(\{k \in I_n : X_k > \mu\}) \neq 0\};$$
$$\overline{N}_X = \{\mu \in L(\mathbb{R}) : \delta_\theta(\{k \in I_n : X_k < \mu\}) = 1\}.$$

We use sets M_X and N_X to define the concepts S_θ-limit inferior and S_θ-limit superior for a sequence $X = (X_k)$ as follows:

Definition 4.3.1. Let $\theta = (k_n)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. The lacunary statistical limit superior or $S_\theta - \limsup X$ of $X = (X_k)$ is defined by

$$S_\theta - \limsup X = \begin{cases} \sup N_X, & \text{if } N_X \neq \phi, \\ -\overline{\infty}, & \text{otherwise,} \end{cases}$$

and the lacunary statistical limit inferior or $S_\theta - \liminf X$ is defined by

$$S_\theta - \liminf X = \begin{cases} \inf M_X, & \text{if } M_X \neq \phi, \\ \overline{\infty}, & \text{otherwise.} \end{cases}$$

Note that $S_\theta - \limsup X = \inf \overline{N}_X$ and $S_\theta - \liminf X = \sup \overline{M}_X$. A simple example will help to illustrate the concepts just defined.

Example 4.3.1. Define a sequence $X = (X_k)$ as follows,

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} x-(k-1) & \text{if } x \in [k-1,k] \\ (k+1)-x & \text{if } x \in (k,k+1] \\ 0 & \text{otherwise} \end{array}\right\}, & \text{if } k \in [k_r - \sqrt{h_r}+1,\ k_r], \\ \left.\begin{array}{ll} \overline{1} & \text{if } k \text{ is odd} \\ \overline{0} & \text{if } k \text{ is even} \end{array}\right\}, & \text{if } k \notin [k_r - \sqrt{h_r}+1,\ k_r]. \end{cases}$$

We can easily see that $M_X = (\overline{0}, +\overline{\infty})$ and $N_X = (-\overline{\infty}, \overline{1})$ and therefore inf $M_X = \overline{0}$ and sup $N_X = \overline{1}$. Hence, $S_\theta - \liminf X = \overline{0}$ and $S_\theta - \limsup X = \overline{1}$.

We, now develop certain properties of $S_\theta-$limit inferior and $S_\theta-$limit superior for sequences of fuzzy numbers.

Theorem 4.3.1. *Let $\theta = (k_n)$ be a lacunary sequence and $X = (X_k)$ be a sequence* *of fuzzy numbers. Then* $S_\theta - \limsup X = \beta$ *if for every* $\epsilon > 0$,

$$\delta_\theta(\{k \in I_n : X_k > \beta \ominus \bar{\epsilon}\} \cup \{k \in I_n : X_k \nsim \beta \ominus \bar{\epsilon}\}) \neq 0$$

and

$$\delta_\theta(\{k \in I_n : X_k > \beta \oplus \bar{\epsilon}\}) = 0.$$

Proof. Let us assume that there exists $\epsilon > 0$ such that $\delta_\theta(\{k \in I_n : X_k > \beta \oplus \bar{\epsilon}\}) \neq 0$. This means that $\beta \oplus \bar{\epsilon} \in N_X$, which is a contradiction as $\beta = \sup N_X$.

Now, we need to prove that

$\delta_\theta(\{k \in I_n : X_k > \beta \ominus \bar{\epsilon}\} \cup \{k \in I_n : X_k \nsim \beta \ominus \bar{\epsilon}\}) \neq 0$

But, on the contrary, we assume that

$$\delta_\theta(\{k \in I_n : X_k > \beta \ominus \bar{\epsilon}\}) = 0 \text{ and } \delta_\theta(\{k \in I_n : X_k \nsim \beta \ominus \bar{\epsilon}\}) = 0. \tag{4.3.1}$$

For each $k \in I_n$, there are only three possibilities that either $X_k > \beta \ominus \bar{\epsilon}$, $X_k \nsim \beta \ominus \bar{\epsilon}$ or $X_k \leq \beta \ominus \bar{\epsilon}$. Thus, from (4.3.1), we have $\delta_\theta(\{k \in I_n : X_k \leq \beta \ominus \bar{\epsilon}\}) = 1$. This implies that $\beta \ominus \bar{\epsilon} \in \overline{N}_X$, which is not possible as $\beta = \inf \overline{N}_X$. □

Theorem 4.3.2. *Let* $\theta = (k_n)$ *be a lacunary sequence and* $X = (X_k)$ *be a sequence of fuzzy numbers. Then* $S_\theta - \liminf X = \gamma$ *if for* $\epsilon > 0$,

$$\delta_\theta(\{k \in I_n : X_k < \gamma \oplus \bar{\epsilon}\} \cup \{k \in I_n : X_k \nsim \gamma \oplus \bar{\epsilon}\}) \neq 0$$

and

$$\delta_\theta(\{k \in I_n : X_k < \gamma \ominus \bar{\epsilon}\}) = 0.$$

Proof. This theorem can be proved on the similar lines as the Theorem 4.3.1. □

Theorem 4.3.3. *Let* $\theta = (k_n)$ *be a lacunary sequence and* $X = (X_k)$ *be the sequence of fuzzy numbers. Then* $S_\theta - \liminf X \leq S_\theta - \limsup X$.

Proof. From the definition of S_θ-limit inferior, we have $S_\theta - \liminf X = \sup \overline{M}_X$ and also from the definitions of the sets $\overline{M}_X$ and N_X, we see that $\overline{M}_X \subseteq N_X$. Hence, we get $\sup \overline{M}_X \leq \sup N_X$ i.e. $S_\theta - \liminf X \leq S_\theta - \limsup X$. □

Definition 4.3.2. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary statistically bounded from above if there exists a fuzzy number μ such that

$\delta_\theta(\{k \in I_n : X_k > \mu\} \cup \{k \in I_n : X_k \nsim \mu\}) = 0.$

Similarly, (X_k) is said to be lacunary statistically bounded from below provided that there exists a fuzzy number ν such that

$\delta_\theta(\{k \in I_n : X_k < \mu\} \cup \{k \in I_n : X_k \nsim \mu\}) = 0.$

If the sequence (X_k) is both lacunary statistically bounded from above and below then it is called lacunary statistically bounded or S_θ-bounded.

Theorem 4.3.4. *A lacunary statistical bounded sequence $X = (X_k)$ of fuzzy numbers is lacunary statistically convergent if and only if, both, $S_\theta - \liminf X_k$ and $S_\theta - \limsup X_k$ coincide with the lacunary statistical limit of the sequence.*

Proof. Let the sequence $X = (X_k)$ is lacunary statistical convergent to X_0 and also $S_\theta - \liminf X = \beta$ and $S_\theta - \limsup X = \gamma$. Then for $\epsilon > 0$, we have

$$\delta_\theta(\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}) = 0.$$

By the definition of $\overline{d}$

$$\delta_\theta(\{k \in I_n : \sup_{\alpha\in[0,1]} d(X_k^\alpha, X_0^\alpha) \geq \epsilon\}) = 0;$$
$$\delta_\theta(\{k \in I_n : \sup_{\alpha\in[0,1]} d(X_k^\alpha, X_0^\alpha) < \epsilon\}) = 1.$$

That is, $\sup_{\alpha\in[0,1]} d(X_k^\alpha, X_0^\alpha) < \epsilon$ for all $k \in I_n$ except some finite number of terms or $\sup_{\alpha\in[0,1]} |\underline{X_k^\alpha} - \underline{X_0^\alpha}| < \epsilon$ and $\sup_{\alpha\in[0,1]} |\overline{X_k^\alpha} - \overline{X_0^\alpha}| < \epsilon$ for all $k \in I_n$ except some finite number of terms. Then, we have

$$X_0 \ominus \overline{\epsilon} < X_k < X_0 \oplus \overline{\epsilon}$$

for all $k \in I_n$ except some finite number of terms. This implies $\delta_\theta(\{k \in I_n : X_k \geq X_0 \oplus \overline{\epsilon}\}) = 0$ and $\delta_\theta(\{k \in I_n : X_k \leq X_0 \ominus \overline{\epsilon}\}) = 0$. This means that $\gamma \leq X_0$ and $\beta \geq X_0$. But by Theorem 4.3.3, $\beta \leq \gamma$. Hence $\beta = \gamma = X_0$.

Conversely. Let us assume that $\beta = \gamma = X_0$. Then Theorem 4.3.1 and Theorem 4.3.2 imply that $\delta_\theta(\{k \in I_n : X_k > X_0 \oplus \frac{\overline{\epsilon}}{2}\}) = 0$ and $\delta_\theta(\{k \in I_n : X_k < X_0 \ominus \frac{\overline{\epsilon}}{2}\}) = 0$. Thus, $\delta_\theta(\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}) = 0$. Hence $S_\theta - \lim_k X_k = X_0$. □

The proofs of the following two Theorems are easy to obtain and so we omit here.

Theorem 4.3.5. *Let $X = (X_k)$ be a sequence of fuzzy numbers. Then*

(i) $S_\theta - \liminf(-X) = -(S_\theta - \limsup X)$

(ii) $S_\theta - \limsup(-X) = -(S_\theta - \liminf X)$.

Theorem 4.3.6. *If $X = (X_k)$ and $Y = (Y_k)$ be two lacunary statistical bounded sequences of fuzzy numbers, then*

(i) $S_\theta - \liminf X_k \oplus\ S_\theta - \lim\ \inf\ Y_k \leq S_\theta - \lim\ \inf\,(X_k \oplus Y_k)$,

(ii) $S_\theta - \lim\ \inf\,(X_k \oplus Y_k) \leq S_\theta - \lim\ \inf\ X_k\ \oplus\ S_\theta - \limsup Y_k$,

(iii) $S_\theta - \liminf X_k \oplus\ S_\theta - \lim\ \sup\ Y_k \leq S_\theta - \lim\ \sup\,(X_k \oplus Y_k)$,

(iv) $S_\theta - \lim\ \sup\,(X_k \oplus Y_k) \leq S_\theta - \lim\ \sup\ X_k\ \oplus\ S_\theta - \limsup Y_k$.

We, now, define $S-$core and $S_\theta-$core for sequences of fuzzy numbers and obtain their relationships.

Definition 4.3.3. The statistical core of a statistically bounded sequence $X = (X_k)$ of fuzzy numbers is the closed interval $[S - \liminf X, S - \limsup X]$. If X is not a statistically bounded sequence of fuzzy numbers, then the statistical core of X is defined as either $[S-\liminf X, +\overline{\infty}]$, $[-\overline{\infty}, +\overline{\infty}]$ or $[-\overline{\infty}, S-\limsup X]$ accordingly as X is not statistically bounded above, statistically bounded and statistically bounded below.

Definition 4.3.4. The lacunary statistical core of a lacunary statistically bounded sequence $X = (X_k)$ of fuzzy numbers is the closed interval $[S_\theta - \liminf X, S_\theta - \limsup X]$. If X is not a lacunary statistically bounded sequence of fuzzy numbers, then the lacunary statistical core is defined as either $[S_\theta - \lim\ \inf\ X, +\overline{\infty}]$, $[-\overline{\infty}, +\overline{\infty}]$ or $[-\overline{\infty}, S_\theta - \lim\ \sup\ X]$ accordingly as X is not lacunary statistically bounded above, lacunary statistically bounded and lacunary statistically bounded below.

Theorem 4.3.7. *For any lacunary sequence θ, $S_\theta - core\{X\} \subseteq S - core\{X\}$ if and only if* $\liminf_n q_n > 1$.

Proof. If $\liminf_n q_n > 1$, then we can find a $\delta > 0$ such that $q_n \geq 1 + \delta$ for sufficiently large n which further implies that $\frac{h_n}{k_n} \geq \frac{\delta}{1+\delta}$.
For a sequence $X = (X_k)$ of fuzzy numbers and $\mu \in L(\mathbb{R})$,

$$\{k \leq n : X_k > \mu\} \supset \{k \in I_n : X_k > \mu\}.$$

Now, we have

$$\begin{aligned}\frac{1}{k_n}|\{k \leq k_n : X_k > \mu\}| &\geq \frac{1}{k_n}|\{k \in I_n : X_k > \mu\}| \\ &= \left(\frac{h_n}{k_n}\right).\frac{1}{h_n}|\{k \in I_n : X_k > \mu\}|\ .\end{aligned}$$

This implies that $S_\theta - \limsup X_k \leq S - \limsup X_k$ if and only if $\liminf \frac{h_n}{k_n} = \liminf(q_n - 1) > 0$, that is, $\liminf_n q_n > 1$. With the similar argument, we can prove that $S - \liminf X_k \leq S_\theta - \liminf X_k$ if and only if $\liminf_n q_n > 1$. Hence, we can conclude that $S_\theta - \text{core}\{X\} \subseteq S - \text{core}\{X\}$ if and only if $\liminf_n q_n > 1$. $\square$

Theorem 4.3.8. *For any lacunary sequence θ, $S - core\{X\} \subseteq S_\theta - core\{X\}$ if and only if* $\limsup_n q_n < \infty$.

Proof. If $\limsup q_n < \infty$, then there is an H such that $q_n < H$ for all n. We may assume $H > 1$ as $\theta = (k_n)$ is a lacunary sequence. Also for each $n \in \mathbb{N}$, we have

$$\frac{h_n}{k_{n-1}} = \frac{k_n - k_{n-1}}{k_{n-1}} = q_n - 1 < H - 1.$$

For a sequence $X = (X_k)$ of fuzzy numbers and $\mu \in L(\mathbb{R})$, define $N_n = |\{k \in I_n : X_k > \mu\}|$ and $t_n = \frac{N_n}{h_n}$, then for any integer n satisfying $k_{n-1} < n \leq k_n$; we can write

$$\begin{aligned}
\frac{1}{n}|\{k \leq n : X_k > \mu\}| \leq \frac{1}{k_{n-1}}|\{k \leq k_n : X_k > \mu\}| \\
&= \frac{1}{k_{n-1}}\{N_1 + N_2 + \ldots + N_n\} \\
&= \frac{1}{k_{n-1}}\{t_1h_1 + t_2h_2 + \ldots t_nh_n\} \\
&= \frac{1}{\sum_{i=1}^{n-1} h_i}\sum_{i=1}^{n-1} h_it_i + \frac{h_n}{k_{n-1}}t_n \\
&< \frac{1}{\sum_{i=1}^{n-1} h_i}\sum_{i=1}^{n-1} h_it_i + (H-1)t_n.
\end{aligned}$$

Suppose $t_n \to 0$ as $r \to \infty$. Since θ is lacunary and the first part on the right side of the above expression is a regular weighted mean transform of the sequence $t = (t_n)$, and therefore it, too tends to zero as $n \to \infty$. Consequently, we get $\delta(\{k \in I_n : X_k > \mu\}) = 0$, which is not possible. Thus $\lim_n t_n \neq 0$ and therefore $S - \limsup X_k \leq S_\theta - \limsup X_k$. Similarly, we can prove that $S_\theta - \liminf X_k \leq S - \liminf X_k$ if and only if $\limsup_n q_n < \infty$. Hence, we can conclude that $S_\theta - \text{core}\{X\} \subseteq S - \text{core}\{X\}$ if and only if $\limsup_n q_n < \infty$. $\square$

4.4 $S_\theta-$limit inferior and $S_\theta-$limit superior for generalized difference sequences $(\Delta^m X_k)$

The aim of this section is to present stronger generalizations and extensions of several definitions and results obtained in the last section. We start by defining $S_\theta-$limit inferior and $S_\theta-$superior for generalized difference sequences $(\Delta^m X_k)$ of fuzzy numbers.

For any sequence $X = (X_k)$ of fuzzy numbers and the generalized difference operator Δ^m, we define the sets:

$$M_{\Delta^m} = \{\mu \in L(\mathbb{R}) : \delta_\theta(\{k \in I_n : \Delta^m X_k < \mu\}) \neq 0\};$$

$$\overline{M}_{\Delta^m} = \{\mu \in L(\mathbb{R}) : \delta_\theta(\{k \in I_n : \Delta^m X_k > \mu\}) = 1\};$$

$$N_{\Delta^m} = \{\mu \in L(\mathbb{R}) : \delta_\theta(\{k \in I_n : \Delta^m X_k > \mu\}) \neq 0\};$$

$$\overline{N}_{\Delta^m} = \{\mu \in L(\mathbb{R}) : \delta_\theta(\{k \in I_n : \Delta^m X_k < \mu\}) = 1\}.$$

Definition 4.4.1. Let $\theta = (k_n)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. The lacunary statistical limit superior or $S_\theta - \limsup \Delta^m X$ of the generalized difference sequence $(\Delta^m X_k)$ is defined by

$$S_\theta - \limsup \Delta^m X = \begin{cases} \sup N_{\Delta^m}, & \text{if } N_{\Delta^m} \neq \phi, \\ -\overline{\infty}, & \text{otherwise,} \end{cases}$$

and the lacunary statistical limit inferior or $S_\theta - \liminf \Delta^m X$ is defined by

$$S_\theta - \liminf \Delta^m X = \begin{cases} \inf M_{\Delta^m}, & \text{if } M_{\Delta^m} \neq \phi, \\ \overline{\infty}, & \text{otherwise.} \end{cases}$$

Note that $S_\theta - \limsup \Delta^m X = \inf \overline{N}_{\Delta^m}$ and $S_\theta - \liminf \Delta^m X = \sup \overline{M}_{\Delta^m}$. A simple example will help to illustrate the concepts just defined.

Example 4.4.1. Let $\theta = (k_r)$ be a lacunary sequence. We define a sequence of fuzzy numbers $X = (X_k)$ as follows. For $x \in \mathbb{R}$, define

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} x-k+1, & \text{if } k-1 \leq x \leq k \\ -x+k+1, & \text{if } k < x \leq k+1 \\ 0, & \text{otherwise} \end{array}\right\}, & \text{if } \quad k \in [k_r - [\sqrt{h_r}] + 1, k_r]; r \in \mathbb{N} \\ \left.\begin{array}{ll} \gamma_1, & \text{if } k \text{ is even} \\ \gamma_2, & \text{if } k \text{ is odd} \end{array}\right\}, & \text{if } \quad k \notin [k_r - [\sqrt{h_r}] + 1, k_r]. \end{cases}$$

where the fuzzy numbers γ_1 and γ_2 are given by

$$\gamma_1 = \begin{cases} x-5, & \text{if } 5 \leq x \leq 6, \\ 7-x & \text{if } 6 < x \leq 7, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\gamma_2 = \begin{cases} x+7, & \text{if } -7 \leq x \leq -6, \\ -5-x & \text{if } -6 < x \leq -5, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we obtain

$$[X_k]^\alpha = \begin{cases} [k-1+\alpha, k+1-\alpha], & \text{if } k \in [k_r - [\sqrt{h_r}] + 1, k_r] \\ [5+\alpha, 7-\alpha], & \text{if } k \notin [k_r - [\sqrt{h_r}] + 1, k_r] \text{ and } k \text{ is even,} \\ [-7+\alpha, -5-\alpha], & \text{if } k \notin [k_r - [\sqrt{h_r}] + 1, k_r] \text{ and } k \text{ is odd,} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [-3+\alpha, 1-2\alpha], & \text{if both } k, k+1 \in [k_r - [\sqrt{h_r}]+1, k_r] \\ [k-8+2\alpha, k-4-2\alpha] & \text{if } k \in [k_r - [\sqrt{h_r}]+1, k_r] \text{ but not } k+1 \text{ and } k \text{ is odd,} \\ [k+4+2\alpha, k+8-2\alpha] & \text{if } k \in [k_r - [\sqrt{h_r}]+1, k_r] \text{ but not } k+1 \text{ and } k \text{ is even,} \\ [-k-9+2\alpha, -k-5-2\alpha] & \text{if } k+1 \in [k_r - [\sqrt{h_r}]+1, k_r] \text{ but not } k \text{ and } k \text{ is odd,} \\ [-k+3+2\alpha, -k+7-2\alpha] & \text{if } k+1 \in [k_r - [\sqrt{h_r}]+1, k_r] \text{ but not } k \text{ and } k \text{ is even,} \\ [-14+2\alpha, -10-2\alpha] & \text{if both } k, k+1 \notin [k_r - [\sqrt{h_r}]+1, k_r] \text{ and } k \text{ is odd,} \\ [10+2\alpha, 14-2\alpha] & \text{if both } k, k+1 \notin [k_r - [\sqrt{h_r}]+1, k_r] \text{ and } k \text{ is even,} \end{cases}$$

Clearly, $M_\Delta = (\eta_1, \overline{\infty})$ and $N_\Delta = (-\overline{\infty}, \eta_2)$ for the sequences X_k, where

$$[\eta_1]^\alpha = [-2(7-\alpha), -2(5+\alpha)] \quad \text{and} \quad [\eta_2]^\alpha = [2(5+\alpha), 2(7-\alpha)].$$

Therefore, S_θ- $\liminf \Delta X = \eta_1$ and S_θ^Δ- $\limsup \Delta X = \eta_2$. While taking the difference of order m, we get $M_{\Delta^m} = (\eta_3, \overline{\infty})$ and $N_{\Delta^m} = (-\overline{\infty}, \eta_4)$, where

$$[\eta_3]^\alpha = [-2^m(7-\alpha), -2^m(5+\alpha)] \text{ and } [\eta_4]^\alpha = [2^m(5+\alpha), 2^m(7-\alpha)].$$

Hence $S_\theta - \liminf \Delta^m X = \eta_3$ and $S_\theta - \limsup \Delta^m X = \eta_4$. □

Theorem 4.4.1. *Let $\theta = (k_n)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Then $S_\theta - \limsup \Delta^m X = \beta$, if for every $\epsilon > 0$,*

$$\delta_\theta(\{k \in I_n : \Delta^m X_k > \beta \ominus \bar{\epsilon}\} \cup \{k \in I_n : \Delta^m X_k \nsim \beta \ominus \bar{\epsilon}\}) \neq 0$$

and

$$\delta_\theta(\{k \in I_n : \Delta^m X_k > \beta \oplus \bar{\epsilon}\}) = 0.$$

Proof. Let us assume that there exists $\epsilon > 0$ such that $\delta_\theta(\{k \in I_n : \Delta^m X_k > \beta \oplus \bar{\epsilon}\}) \neq 0$. This means that $\beta \oplus \bar{\epsilon} \in N_{\Delta^m}$, which is a contradiction as $\beta = \sup N_{\Delta^m}$.

We, now need to prove that

$$\delta_\theta(\{k \in I_n : \Delta^m X_k > \beta \ominus \overline{\epsilon}\} \cup \{k \in I_n : \Delta^m X_k \nsim \beta \ominus \overline{\epsilon}\}) \neq 0$$

But, on contrary, we assume that

$$\delta_\theta(\{k \in I_n : \Delta^m X_k > \beta \ominus \overline{\epsilon}\}) = 0 \text{ and } \delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \beta \ominus \overline{\epsilon}\}) = 0. \quad (4.4.1)$$

For each $k \in I_n$, there are only three possibilities that either $\Delta^m X_k > \beta \ominus \overline{\epsilon}$, $\Delta^m X_k \nsim \beta \ominus \overline{\epsilon}$ or $\Delta^m X_k \leq \beta \ominus \overline{\epsilon}$. Thus, from (4.4.1), we have $\delta_\theta(\{k \in I_n : \Delta^m X_k \leq \beta \ominus \overline{\epsilon}\}) = 1$. This implies that $\beta \ominus \overline{\epsilon} \in \overline{N}_{\Delta^m}$, which is not possible as $\beta = \inf \overline{N}_{\Delta^m}$. This completes the proof. □

Dual statement of the above theorem is given as under:

Theorem 4.4.2. *Let $\theta = (k_n)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Then $S_\theta - \liminf \Delta^m X = \gamma$ if for every $\epsilon > 0$,*

$$\delta_\theta(\{k \in I_n : \Delta^m X_k < \gamma \oplus \overline{\epsilon}\} \cup \{k \in I_n : \Delta^m X_k \nsim \gamma \oplus \overline{\epsilon}\}) \neq 0$$

and

$$\delta_\theta(\{k \in I_n : \Delta^m X_k < \gamma \ominus \overline{\epsilon}\}) = 0.$$

Following example shows that the $S_\theta - \liminf \Delta^m X$ or $S_\theta - \limsup \Delta^m X$ may not be members of the set $\Gamma_{S_\theta}(\Delta^m X)$ of all cluster points of the generalized difference sequences $(\Delta^m X)$ of fuzzy numbers.

Example 4.4.2. For $x \in \mathbb{R}$, define

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} \frac{x-3}{5} & \text{if } x \in [3,8], \\ 9-x & \text{if } x \in (8,9], \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k \text{ is odd} \\ \left.\begin{array}{ll} x-6 & \text{if } x \in [6,7], \\ 8-x & \text{if } x \in (7,8], \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k \text{ is even.} \end{cases}$$

Then, for $\alpha \in (0,\ 1]$, we obtain

$$[X_k]^\alpha = \begin{cases} [3+5\alpha,\ 9-\alpha], & \text{if } k \text{ is odd} \\ [6+\alpha,\ 8-\alpha] & \text{otherwise.} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [-5+6\alpha, 3-2\alpha], & \text{if } k \text{ is odd} \\ [-3+2\alpha, 5-6\alpha], & \text{otherwise.} \end{cases}$$

Thus, for $m = 1$, it is clear that for the sequence ΔX_k, $\Gamma_{S_\theta}(\Delta(X)) = \{\mu_1,\ \mu_2\}$, where $[\mu_1]^\alpha = [-3+\alpha, 1-2\alpha]$, $[\mu_2]^\alpha = [-2+2\alpha, 2-2\alpha]$ are α-level sets for the fuzzy numbers

$$\mu_1(x) = \begin{cases} \frac{x+5}{6}, & \text{if } x \in [-5,1], \\ \frac{3-x}{2} & \text{if } x \in (1,3], \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\mu_2(x) = \begin{cases} \frac{x+3}{2}, & \text{if } x \in [-3,-1], \\ \frac{5-x}{6} & \text{if } x \in (-1,5], \\ 0 & \text{otherwise.} \end{cases}$$

It follows

$$S_\theta - \liminf \Delta^m X = \begin{cases} \frac{x+5}{6}, & \text{if } x \in [-5,-2], \\ \frac{x+3}{2}, & \text{if } x \in (-2,-1], \\ \frac{5-x}{6} & \text{if } x \in (-1,2], \\ \frac{3-x}{2} & \text{if } x \in (2,3], \\ 0 & \text{otherwise.} \end{cases}$$

but neither μ_1 is the $S_\theta - \liminf \Delta^m X$ nor μ_2 □

Definition 4.4.2. For a sequence $X = (X_k)$ of fuzzy numbers, the generalized difference sequence $(\Delta^m X_k)$ is said to be lacunary statistically bounded from above if there exists a fuzzy number μ such that

$\delta_\theta(\{k \in I_n : \Delta^m X_k > \mu\} \cup \{k \in I_n : \Delta^m X_k \nsim \mu\}) = 0.$

Similarly, $(\Delta^m X_k)$ is said to be lacunary statistically bounded from below provided that there exists a fuzzy number ν such that

$\delta_\theta(\{k \in I_n : \Delta^m X_k < \mu\} \cup \{k \in I_n : \Delta^m X_k \nsim \mu\}) = 0.$

If the sequence $(\Delta^m X_k)$ is both lacunary statistically bounded from above and below then it is called lacunary statistically bounded or S_θ-bounded.

Theorem 4.4.3. *Let, the generalized difference sequence $(\Delta^m X_k)$ is $S_\theta-$bounded. Let μ is the S_θ-limit superior of $(\Delta^m X_k)$ and for each ϵ,*

$$\delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \mu \oplus \overline{\epsilon}\}) = 0 \text{ and } \delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \mu \ominus \overline{\epsilon}\}) = 0. \quad (4.4.2)$$

Then μ is a S_θ-cluster point of $(\Delta^m X_k)$.

Proof. Since μ is the S_θ-limit superior of $(\Delta^m X_k)$ then for each ϵ,

$\delta_\theta(\{k \in I_n : \Delta^m X_k > \mu \ominus \overline{\epsilon}\}) \neq 0.$

Also form Theorem 4.4.1, we have

$\delta_\theta(\{k \in I_n : \Delta^m X_k > \mu \oplus \bar{\epsilon}\}) = 0.$

This implies

$\delta_\theta(\{k \in I_n : \Delta^m X_k \leq \mu \oplus \bar{\epsilon}\}) = 1.$

Also it is assumed that $\delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \mu \oplus \bar{\epsilon}\}) = 0$. We can conclude that

$\delta_\theta(\{k \in I_n : \mu \ominus \bar{\epsilon} < \Delta^m X_k \leq \mu \oplus \bar{\epsilon}\}) \neq 0$

Thus we can write

$\delta_\theta(\{k \in I_n : \overline{d}(\Delta^m X_k, \mu) \leq \bar{\epsilon}\}) < 0.$

This completes the proof. $\square$

Similarly, we can give a condition for which lacunary statistical limit inferior can be a lacunary statistical cluster point of the generalized difference sequence $\Delta^m X_k$.

Theorem 4.4.4. *Let, the generalized difference sequence* $(\Delta^m X_k)$ *is* $S_\theta-$*bounded. Let* ν *is the* S_θ*-limit inferior of* $(\Delta^m X_k)$ *and for each* ϵ*,*

$$\delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \nu \oplus \bar{\epsilon}\}) = 0 \text{ and } \delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \nu \ominus \bar{\epsilon}\}) = 0. \quad (4.4.3)$$

Then ν *is a* S_θ*-cluster point of* $(\Delta^m X_k)$*.*

Theorem 4.4.5. *Let, the generalized difference sequence* $(\Delta^m X_k)$ *of fuzzy numbers be* S_θ*-bounded and* μ *be a* S_θ*-cluster point of* $(\Delta^m X_k)$*, then*

$$S_\theta - \liminf \Delta^m X \leq \mu \leq S_\theta - \limsup \Delta^m X.$$

Proof. Firstly, we shall prove that $\mu \leq S_\theta - \limsup \Delta^m X$. Since μ is a S_θ-cluster point of $(\Delta^m X_k)$, therefore for $\epsilon > 0$

$\delta_\theta(\{k \in I_n : \overline{d}(\Delta^m X_k, \mu) < \epsilon\}) \neq 0.$

That is

$\delta_\theta(\{k \in I_n : \mu \ominus \bar{\epsilon} < \Delta^m X_k < \mu \oplus \bar{\epsilon}\}) \neq 0.$

This implies that

$0 \neq \delta_\theta(\{k \in I_n : \mu \ominus \bar{\epsilon} < \Delta^m X_k < \mu \oplus \bar{\epsilon}\}) \leq \delta_\theta(\{k \in I_n : \mu \ominus \bar{\epsilon} < \Delta^m X_k\}).$

By the definition of N_{Δ^m}, clearly $\mu \ominus \bar{\epsilon} \in N_{\Delta^m}$ and therefore we have $\mu \ominus \bar{\epsilon} \leq \sup N_{\Delta^m}$ or we can write that $\mu \leq \sup N_{\Delta^m} = S_\theta - \limsup \Delta^m X$. Similarly we can prove that $S_\theta - \liminf \Delta^m X \leq \mu$, which completes the proof. □

Combining Theorems 4.4.3 and 4.4.5, we get the following conditions which state that $S_\theta - \limsup \Delta^m X$ and $S_\theta - \liminf \Delta^m X$ are the greatest and least statistical cluster points of $\Delta^m X$.

Corollary 4.4.6. *Let, the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers be S_θ-bounded and μ is the S_θ-limit superior of $(\Delta^m X_k)$ and for each $\epsilon > 0$,*

$$\delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \mu \oplus \bar{\epsilon}\}) = 0 \text{ and } \delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \mu \ominus \bar{\epsilon}\}) = 0. \quad (4.4.4)$$

Then μ is the greatest S_θ-cluster point of $(\Delta^m X_k)$.

Corollary 4.4.7. *Let, the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers be S_θ-bounded and ν is the S_θ-limit inferior of X and for each $\epsilon > 0$,*

$$\delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \nu \oplus \bar{\epsilon}\}) = 0 \text{ and } \delta_\theta(\{k \in I_n : \Delta^m X_k \nsim \nu \ominus \bar{\epsilon}\}) = 0. \quad (4.4.5)$$

Then ν is the least S_θ-cluster point of $(\Delta^m X_k)$.

Theorem 4.4.8. *Let $\theta = (k_n)$ be a lacunary sequence and $(\Delta^m X_k)$ be a generalized difference sequence of fuzzy numbers. Then*

$$S_\theta - \liminf \Delta^m X \leq S_\theta - \limsup \Delta^m X.$$

Proof. From the definition of S_θ-limit inferior of $(\Delta^m X_k)$, we have $S_\theta - \liminf \Delta^m X = \sup \overline{M}_{\Delta^m}$ and also from the definitions of the sets $\overline{M}_{\Delta^m}$ and N_{Δ^m}, we see that $\overline{M}_{\Delta^m} \subseteq N_{\Delta^m}$. Hence, we get $\sup \overline{M}_{\Delta^m} \leq \sup N_{\Delta^m}$ i.e. $S_\theta - \liminf \Delta^m X \leq S_\theta - \limsup \Delta^m X$. □

Theorem 4.4.9. *A S_θ-bounded sequence $(\Delta^m X_k)$ of fuzzy numbers is lacunary Δ^m-statistically convergent if and only if,*

$$S_\theta - \liminf \Delta^m X = S_\theta - \limsup \Delta^m X.$$

Proof. Let the sequence $X = (X_k)$ is lacunary Δ^m-statistical convergent to X_0 and also $S_\theta - \liminf \Delta^m X = \beta$ and $S_\theta - \limsup \Delta^m X = \gamma$. Then for $\epsilon > 0$, we have

$$\delta_\theta(\{k \in I_n : \overline{d}(\Delta^m X_k, X_0) \geq \epsilon\}) = 0.$$

By the definition of $\overline{d}$

$$\delta_\theta(\{k \in I_n : \sup_{\alpha\in[0,1]} d(\Delta^m X_k^\alpha, X_0^\alpha) \geq \epsilon\}) = 0;$$
$$\delta_\theta(\{k \in I_n : \sup_{\alpha\in[0,1]} d(\Delta^m X_k^\alpha, X_0^\alpha) < \epsilon\}) = 1.$$

That is, $\sup_{\alpha\in[0,1]} d(\Delta^m X_k^\alpha, X_0^\alpha) < \epsilon$ for all $k \in I_n$ except some finite number of terms or $\sup_{\alpha\in[0,1]} |\Delta^m X_{k\,1}^\alpha - X_{0\,1}^\alpha| < \epsilon$ and $\sup_{\alpha\in[0,1]} |\Delta^m X_{k\,2}^\alpha - X_{0\,2}^\alpha| < \epsilon$ for all $k \in I_n$ except some finite number of terms. Then, we have

$$X_0 \ominus \overline{\epsilon} < \Delta^m X_k < X_0 \oplus \overline{\epsilon}$$

for all $k \in I_n$ except some finite number of terms. This implies $\delta_\theta(\{k \in I_n : \Delta^m X_k \geq X_0 \oplus \overline{\epsilon}\}) = 0$ and $\delta_\theta(\{k \in I_n : \Delta^m X_k \leq X_0 \ominus \overline{\epsilon}\}) = 0$. This means that $\gamma \leq X_0$ and $\beta \geq X_0$. But by Theorem 4.4.8 $\beta \leq \gamma$. Hence $\beta = \gamma = X_0$.

Conversely. Let us assume that $\beta = \gamma = X_0$. Then, Theorem 4.4.1 and Theorem 4.4.2 imply that $\delta_\theta(\{k \in I_n : \Delta^m X_k > X_0 \oplus \frac{\epsilon}{2}\}) = 0$ and $\delta_\theta(\{k \in I_n : \Delta^m X_k < X_0 \ominus \frac{\epsilon}{2}\}) = 0$. Thus, $\delta_\theta(\{k \in I_n : \overline{d}(\Delta^m X_k, X_0) \geq \epsilon\}) = 0$. Hence $S_\theta - \lim_k X_k = X_0$. $\square$

The proofs of the following two Theorems are easy to obtain and so we omit them.

Theorem 4.4.10. *Let $X = (X_k)$ be a sequence of fuzzy numbers. Then*

(i) $S_\theta - \liminf(-\Delta^m X) = -(S_\theta - \limsup \Delta^m X)$

(ii) $S_\theta - \limsup(-\Delta^m X) = -(S_\theta - \liminf \Delta^m X)$.

Theorem 4.4.11. *If $X = (X_k)$ and $Y = (Y_k)$ be two lacunary statistical bounded sequences of fuzzy numbers, then*

(i) $S_\theta - \liminf \Delta^m X_k \oplus S_\theta - \liminf \Delta^m Y_k \leq S_\theta - \liminf(\Delta^m X_k \oplus \Delta^m Y_k)$,

(ii) $S_\theta - \liminf(\Delta^m X_k \oplus \Delta^m Y_k) \leq S_\theta - \liminf \Delta^m X_k \oplus S_\theta - \limsup \Delta^m Y_k$,

(iii) $S_\theta - \liminf \Delta^m X_k \oplus S_\theta - \limsup \Delta^m Y_k \leq S_\theta - \limsup(\Delta^m X_k \oplus \Delta^m Y_k)$,

(iv) $S_\theta - \limsup(\Delta^m X_k \oplus \Delta^m Y_k) \leq S_\theta - \limsup \Delta^m X_k \oplus S_\theta - \limsup \Delta^m Y_k$.

Chapter 5

On $S_\lambda-$limit inferior and $S_\lambda-$limit superior

5.1 Introduction

Many generalizations of statistical convergence have appeared in literature in the past. One amongst them is introduced by Mursaleen [83] by using (V, λ)-summability called as λ-statistical convergence, where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. Mursaleen [83] also studied some relationships of λ-statistical convergence with the well known methods of Cesaro summability and (V, λ)-summability. These ideas have further been extended by Savaş [97] on fuzzy number sequences. This motivated Tuncer and Benli [107] to define the concepts of λ-statistical limit and λ-statistical cluster point for fuzzy number sequences.

In this chapter, we continue to extend their pioneer work and use it to define the λ-statistical limit inferior and limit superior for sequences of fuzzy numbers. The

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contents of this chapter have been divided in four sections. In section 5.2, a brief information about the concepts of λ-statistical convergence and λ-statistical limit and cluster points for sequences of fuzzy numbers has been given. In section 5.3, the notion of λ-statistical limit inferior and limit superior for fuzzy numbers sequences have been introduced. Further, in this section some sufficient conditions have been presented to prove that λ-statistical limit inferior and limit superior can be a λ-statistical cluster point of the sequence $X = (X_k)$ of fuzzy numbers. Section 5.4, further explores these ideas on sequences of fuzzy numbers in a more generalized way for λ-statistical convergence of order β, where $\beta \in (0, 1]$ be any real number.

5.2 Preliminaries

This section starts with a gentle introduction and background of the paper by Savaş [97], who introduced the concept of λ-statistical convergence for sequences of fuzzy numbers.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$ and the interval $I_n = [n - \lambda_n + 1, n]$. The set of all such sequences will be denoted by Ω.

Definition 5.2.1. [97] A sequence $X = (X_k)$ of fuzzy numbers is said to be λ-statistically convergent to a fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Briefly denoted as $S_\lambda\text{-}\lim X_k = X_0$.

Let $S_\lambda(X)$ denotes the set of all statistically convergent sequences of fuzzy numbers.

Using the above concepts of S_λ-convergence, Tuncer and Benli [107] defined the concetps of λ-statistical limit point and λ-statistical cluster point of a sequence of fuzzy numbers by first introducing the notions of λ-thin and λ-nonthin subsequences.

Let $\lambda = (\lambda_n)$ be a lambda sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Let $(X_{k(j)})$ be a subsequence of $X = (X_k)$ and $K = \{k(j) : j \in \mathbb{N}\}$. Denote $(X_{k(j)})$ by $(X)_K$. If

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k(j) \in I_n : j \in \mathbb{N}\}| = 0,$$

then (X_K) is called a $\lambda-$thin subsequence. On the other hand, (X_K) is a $\lambda-$nonthin subsequence of X provided that

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} |\{k(j) \in I_n : j \in \mathbb{N}\}| > 0.$$

Definition 5.2.2. [107] Let $\lambda = (\lambda_n)$ be a lambda sequence. A fuzzy number X_0 is said to be a $\lambda-$statistical limit point or $S_\lambda-$limit point of a sequence of fuzzy numbers $X = (X_k)$ if there exists a $\lambda-$nonthin subsequence of X that converges to X_0.

Let $\Lambda^F_{S_\lambda}(X)$ denotes the set of all $S_\lambda-$limit points of the fuzzy number sequence $X = (X_k)$.

Definition 5.2.3. [107] Let $\lambda = (\lambda_n)$ be a lambda sequence. A fuzzy number Y_0 is said to be a $\lambda-$statistical cluster point or $S_\lambda-$cluster point of a sequence of fuzzy numbers $X = (X_k)$ if for each $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let $\Gamma^F_{S_\lambda}(X)$ denotes the set of all $S_\lambda-$cluster points of the sequence $X = (X_k)$ of fuzzy numbers.

5.3 λ-statistical limit inferior and limit superior

In this section, we introduce the definitions of λ-statistical limit superior and limit inferior for sequences of fuzzy numbers.

For a non-decreasing sequence $\lambda = (\lambda_n)$ of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$, we define the following sets:

$$M_X = \{\mu \in L(\mathbb{R}) : \delta_\lambda(\{k \in I_n : X_k < \mu\}) \neq 0\};$$
$$\overline{M}_X = \{\mu \in L(\mathbb{R}) : \delta_\lambda(\{k \in I_n : X_k > \mu\}) = 1\};$$
$$N_X = \{\mu \in L(\mathbb{R}) : \delta_\lambda(\{k \in I_n : X_k > \mu\}) \neq 0\};$$
$$\overline{N}_X = \{\mu \in L(\mathbb{R}) : \delta_\lambda(\{k \in I_n : X_k < \mu\}) = 1\}.$$

to introduce the concepts of S_λ-limit inferior and S_λ-limit superior for a sequence $X = (X_k)$ of fuzzy numbers.

Definition 5.3.1. Let $X = (X_k)$ be a sequence of fuzzy numbers. The $\lambda-$statistical limit superior or $S_\lambda - \limsup X$ of $X = (X_k)$ is defined by

$$S_\lambda - \limsup X = \begin{cases} \sup N_X, & \text{if } N_X \neq \phi, \\ -\overline{\infty}, & \text{otherwise,} \end{cases}$$

and the $\lambda-$statistical limit inferior or $S_\lambda - \liminf X$ is defined by

$$S_\lambda - \liminf X = \begin{cases} \inf M_X, & \text{if } M_X \neq \phi, \\ \overline{\infty}, & \text{otherwise.} \end{cases}$$

Note that $S_\lambda - \limsup X = \inf \overline{N}_X$ and $S_\lambda - \liminf X = \sup \overline{M}_X$.

Theorem 5.3.1. *Let $X = (X_k)$ be a sequence of fuzzy numbers. Then, for a finite fuzzy number β, $S_\lambda - \limsup X = \beta$ if for every $\epsilon > 0$,*

$$\delta_\lambda(\{k \in I_n : X_k > \beta \ominus \overline{\epsilon}\} \cup \{k \in I_n : X_k \nsim \beta \ominus \overline{\epsilon}\}) \neq 0$$

and

$$\delta_\lambda(\{k \in I_n : X_k > \beta \oplus \overline{\epsilon}\}) = 0.$$

Proof. Let us assume that there exists $\epsilon > 0$ such that $\delta_\lambda(\{k \in I_n : X_k > \beta \oplus \overline{\epsilon}\}) \neq 0$. This means that $\beta \oplus \overline{\epsilon} \in N_X$, which is a contradiction as $\beta = \sup N_X$.

Now, we need to prove that

$$\delta_\lambda(\{k \in I_n : X_k > \beta \ominus \overline{\epsilon}\} \cup \{k \in I_n : X_k \not\sim \beta \ominus \overline{\epsilon}\}) \neq 0$$

Let us assume that

$$\delta_\lambda(\{k \in I_n : X_k > \beta \ominus \overline{\epsilon}\}) = 0 \text{ and } \delta_\lambda(\{k \in I_n : X_k \not\sim \beta \ominus \overline{\epsilon}\}) = 0. \tag{5.3.1}$$

For each $k \in I_n$, there are only three possibilities that either $X_k > \beta \ominus \overline{\epsilon}$, $X_k \not\sim \beta \ominus \overline{\epsilon}$ or $X_k \leq \beta \ominus \overline{\epsilon}$. Thus, from (5.3.1), we have $\delta_\lambda(\{k \in I_n : X_k \leq \beta \ominus \overline{\epsilon}\}) = 1$. This implies that $\beta \ominus \overline{\epsilon} \in \overline{N}_X$, which is not possible as $\beta = \inf \overline{N}_X$. □

The dual statement of Theorem 5.3.1 for $S_\lambda - \liminf X$ may be given as under:

Theorem 5.3.2. *Let $X = (X_k)$ be a sequence of fuzzy numbers. Then, for a finite fuzzy number γ, $S_\lambda - \liminf X = \gamma$ if for every $\epsilon > 0$,*

$$\delta_\lambda(\{k \in I_n : X_k < \gamma \oplus \overline{\epsilon}\} \cup \{k \in I_n : X_k \not\sim \gamma \oplus \overline{\epsilon}\}) \neq 0$$

and

$$\delta_\lambda(\{k \in I_n : X_k < \gamma \ominus \overline{\epsilon}\}) = 0.$$

The following example shows that the $S_\lambda - \liminf$ or $S_\lambda - \liminf$ may not be $\lambda-$statistical cluster points for a sequence of fuzzy numbers.

Example 5.3.1. Let $X = (X_K)$ be a sequence of fuzzy number defined as

$$X_k(x) = \begin{cases} \nu(x), & \text{if } k \text{ is an odd number,} \\ \mu(x), & \text{otherwise,} \end{cases}$$

where

$$\nu(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 3) \cup (9, \infty), \\ \frac{x-3}{5}, & \text{if } x \in [3, 8], \\ 9 - x, & \text{if } x \in (8, 9]. \end{cases}$$

and

$$\mu(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 6) \cup (8, \infty), \\ x - 6, & \text{if } x \in [6, 7], \\ 8 - x, & \text{if } x \in (7, 8]. \end{cases}$$

It follows

$$S_\lambda - \liminf X = \begin{cases} 0, & \text{if } x \in (-\infty, 3) \cup (8, \infty), \\ \frac{x-3}{5} & \text{if } x \in [3, \frac{27}{4}], \\ x - 6, & \text{if } x \in (\frac{27}{4}, 7], \\ 8 - x, & \text{otherwise.} \end{cases}$$

Hence, $\Gamma^F_{S_\lambda} = \{\nu, \mu\}$ but neither ν is the λ-statistical limit inferior of the fuzzy number sequence X nor μ.

We, now present some sufficient conditions for proving that the $S_\lambda - \liminf$ or $S_\lambda - \liminf$ can be a λ-statistical cluster point, with the help of following theorems. But, before that we define $\lambda-$statistically bounded sequences of fuzzy numbers.

Definition 5.3.2. A sequence $X = (X_k)$ of fuzzy numbers is said to be $\lambda-$statistically bounded from above if there exists a fuzzy number μ such that

$\delta_\lambda(\{k \in I_n : X_k > \mu\} \cup \{k \in I_n : X_k \nsim \mu\}) = 0.$

Similarly, $X = (X_k)$ is said to be $\lambda-$statistically bounded from below provided that there exists a fuzzy number ν such that

$\delta_\lambda(\{k \in I_n : X_k < \mu\} \cup \{k \in I_n : X_k \nsim \mu\}) = 0.$

If the sequence $X = (X_k)$ is both $\lambda-$statistically bounded from above as well as below, then it is $\lambda-$statistically bounded.

Theorem 5.3.3. *Let $X = (X_k)$ be a $S_\lambda-$bounded sequence of fuzzy numbers. Let μ is the λ-statistical limit superior of X and for each $\epsilon > 0$,*

$$\delta_\lambda(\{k \in I_n : X_k \nsim \mu \oplus \bar{\epsilon}\}) = 0 \text{ and } \delta_\lambda(\{k \in I_n : X_k \nsim \mu \ominus \bar{\epsilon}\}) = 0. \tag{5.3.2}$$

Then μ is a $\lambda-$statistical cluster point of X.

Proof. Since μ is the λ-statistical limit superior of X then for each ϵ,

$$\delta_\lambda(\{k \in I_n : X_k > \mu \ominus \bar{\epsilon}\}) \neq 0.$$

Also, from Theorem 5.3.1, we have

$$\delta_\lambda(\{k \in I_n : X_k > \mu \oplus \bar{\epsilon}\}) = 0.$$

This implies that

$$\delta_\lambda(\{k \in I_n : X_k \leq \mu \oplus \bar{\epsilon}\}) = 1.$$

Together with equation (5.3.1), we can conclude that

$$\delta_\lambda(\{k \in I_n : \mu \ominus \bar{\epsilon} < X_k \leq \mu \oplus \bar{\epsilon}\}) \neq 0.$$

Thus we can write

$$\delta_\lambda(\{k \in I_n : \bar{d}(X_k, \mu) \leq \bar{\epsilon}\}) < 0.$$

This completes the proof. □

The dual statement of Theorem 5.3.3 for λ-statistical limit inferior may be given as follows:

Theorem 5.3.4. *Let $X = (X_k)$ be a $S_\lambda-$bounded sequence of fuzzy numbers. Let ν is the $\lambda-$statistical limit inferior of X and for each ϵ,*

$$\delta_\lambda(\{k \in I_n : X_k \nsim \nu \oplus \overline{\epsilon}\}) = 0 \text{ and } \delta_\lambda(\{k \in I_n : X_k \nsim \nu \ominus \overline{\epsilon}\}) = 0. \tag{5.3.3}$$

Then ν is a $\lambda-$statistical cluster point of X.

Theorem 5.3.5. *Let $X = (X_k)$ be $\lambda-$statistically bounded sequences of fuzzy numbers and μ be a $\lambda-$statistical cluster point of X, then*

$$S_\lambda - \liminf X \leq \mu \leq S_\lambda - \limsup X.$$

Proof. First, we prove that $\mu \leq S_\lambda - \limsup X$. Since μ is a $\lambda-$statistical cluster point of X, then for $\epsilon > 0$

$$\delta_\lambda(\{k \in I_n : \overline{d}(X_k, \mu) < \epsilon\}) \neq 0. \tag{5.3.4}$$

i.e.,

$$\delta_\lambda(\{k \in I_n : \mu \ominus \overline{\epsilon} < X_k < \mu \oplus \overline{\epsilon}\}) \neq 0. \tag{5.3.5}$$

This implies that

$$0 \neq \delta_\lambda(\{k \in I_n : \mu \ominus \overline{\epsilon} < X_k < \mu \oplus \overline{\epsilon}\}) \leq \delta_\lambda(\{k \in I_n : \mu \ominus \overline{\epsilon} < X_k\}).$$

By the definition of N_X, clearly $\mu \ominus \overline{\epsilon} \in N_X$ and therefore, we have $\mu \ominus \overline{\epsilon} \leq \sup N_X$ or we can write that $\mu \leq \sup N_X = S_\lambda - \limsup X$. Similarly, we can prove that $S_\lambda - \liminf X \leq \mu$, which completes the proof. □

Combining Theorems 5.3.3 and 5.3.5, we get the following conditions which shows that $S_\lambda - \limsup X$ and $S_\lambda - \liminf X$ are the greatest and least statistical cluster points of X.

Corollary 5.3.6. *Let $X = (X_k)$ be a $S_\lambda-$bounded sequence of fuzzy numbers. Let μ is the λ-statistical limit superior of X and for each $\epsilon > 0$,*

$$\delta_\lambda(\{k \in I_n : X_k \nsim \mu \oplus \bar{\epsilon}\}) = 0 \text{ and } \delta_\lambda(\{k \in I_n : X_k \nsim \mu \ominus \bar{\epsilon}\}) = 0. \tag{5.3.6}$$

Then μ is the greatest $\lambda-$statistical cluster point of X.

Corollary 5.3.7. *Let $X = (X_k)$ be a $S_\lambda-$bounded sequence of fuzzy numbers. Let ν is the $\lambda-$statistical limit inferior of X and for each $\epsilon > 0$,*

$$\delta_\lambda(\{k \in I_n : X_k \nsim \nu \oplus \bar{\epsilon}\}) = 0 \text{ and } \delta_\lambda(\{k \in I_n : X_k \nsim \nu \ominus \bar{\epsilon}\}) = 0. \tag{5.3.7}$$

Then ν is the least $\lambda-$statistical cluster point of X.

5.4 S_{λ^β}-convergence

Gadjiev and Orhan [54] defined statistical convergence of order α for a number sequence, where $\alpha \in (0, 1]$ be any real number. Colak and Baktas [27] also expanded the concept and defined λ-statistical convergence of order α for a number sequence. Altinok [7] introduced a more generalized summability method for sequences of fuzzy numbers called as $S_{\lambda^\beta}-$convergence. In this section, we aim to investigate further characterization of this summability method and define $S_{\lambda^\beta}-$limit points, $S_{\lambda^\beta}-$cluster points, $S_{\lambda^\beta}-$limit inferior and $S_{\lambda^\beta}-$limit superior of sequences of fuzzy numbers.

Definition 5.4.1. [7] Let β be any real number such that $\beta \in (0,\, 1]$. The λ^β-density of a set $A \subseteq N$ is denoted as $\delta_{\lambda^\beta}(A)$ and is defined by

$$\delta_{\lambda^\beta}(A) = \lim_{n\to\infty} \frac{1}{\lambda^\beta} |\{k \in I_n : k \in A\}|,$$

if the limit exists.

Definition 5.4.2. [7] Let $\lambda \in \Omega$ and $\beta \in (0,\, 1]$ is given. A sequence $X = (X_k)$ of fuzzy numbers is said to be λ-statistically convergent of order β to a fuzzy number X_0, if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda^\beta} |\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Symbolically, it is written as $S_{\lambda^\beta}\text{-}\lim X_k = X_0$.

Let $S_{\lambda^\beta}(X)$ denotes the set of all S_{λ^β}-convergent sequences of fuzzy numbers.

Remarks 5.4.1. 1. For the choice of $\beta = 1$, the λ-statistical convergence of order β reduces to λ-statistical convergence.

2. The λ-statistical convergence of order β is well defined for $\beta \in (0, 1]$ but is not well defined for $\beta > 1$. This case can be seen in the following example.

Example 5.4.1. Let $X = (X_K)$ be a sequence of fuzzy number defined as

$$X_k(x) = \begin{cases} \nu(x), & \text{if } k \text{ is an odd number,} \\ \mu(x), & \text{otherwise,} \end{cases}$$

where

$$\nu(x) = \begin{cases} x - 7, & \text{if } x \in [7, 8], \\ 8 - x, & \text{if } x \in (8, 9], \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mu(x) = \begin{cases} x - 4, & \text{if } x \in [4, 5], \\ 6 - x, & \text{if } x \in (5, 6] \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $\alpha \in (0,\, 1]$, the α-level sets of X_k are

$$[X_k]^\alpha = \begin{cases} [7 + \alpha,\, 9 - \alpha], & \text{if } k \text{ is odd} \\ [4 + \alpha,\, 6 - \alpha], & \text{otherwise.} \end{cases}$$

Hence, for $\beta > 1$ the sequence (X_k) is λ-statistical convergence of order β to η_1 and η_2 as well, since

$$\lim_{n\to\infty} \frac{1}{\lambda^\beta} \left| \left\{ k \in I_n : \overline{d}(X_k, \eta_1) \geq \epsilon \right\} \right| = \lim_{n\to\infty} \frac{\lambda_n}{2\lambda_n^\beta} = 0$$

$$\lim_{n\to\infty} \frac{1}{\lambda^\beta} \left| \left\{ k \in I_n : \overline{d}(X_k, \eta_2) \geq \epsilon \right\} \right| = \lim_{n\to\infty} \frac{\lambda_n}{2\lambda_n^\beta} = 0$$

where $[\eta_1]^\alpha = [7+\alpha,\, 9-\alpha]$ and $[\eta_2]^\alpha = [4+\alpha, 6-\alpha]$. Hence, $S_{\lambda^\beta}\text{-}\lim X_k$ is not unique. $\square$

The main results of this section are the following theorems:

Theorem 5.4.1. *For $0 < \beta \leq \gamma \leq 1$, $S_{\lambda^\beta}(X) \subseteq S_{\lambda^\gamma}(X)$ and the inclusion is strict.*

Proof. Let $0 < \beta \leq \gamma \leq 1$, then obliviously $S_{\lambda^\beta}(X) \subseteq S_{\lambda^\gamma}(X)$ as

$$\frac{1}{\lambda^\gamma} |\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}| \leq \frac{1}{\lambda^\beta} |\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}|.$$

To prove the strictness of the inclusion, let us consider the following counter example:

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} x-2 & \text{if } x \in [2,3], \\ 4-x & \text{if } x \in (3,4], \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k = n^3 \\ \left.\begin{array}{ll} x-6 & \text{if } x \in [6,7], \\ 8-x & \text{if } x \in (7,8], \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k \neq n^3. \end{cases}$$

Then, for $\alpha \in (0,\, 1]$,

$$[X_k]^\alpha = \begin{cases} [2+\alpha,\, 4-\alpha], & \text{if } k = n^3 \\ [6+\alpha,\, 8-\alpha] & \text{if } k \neq n^3. \end{cases}$$

For $\gamma \in (\frac{1}{3},\, 1]$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda^\gamma} |\{k \in I_n : \overline{d}(X_k, \zeta) \geq \epsilon\}| = \lim_{n\to\infty} \frac{\sqrt[3]{\lambda_n - 1}}{\lambda_n^\gamma} = 0$$

where $\zeta = [-2 + 2\alpha,\, 2 - 2\alpha]$. This means, the sequence (X_k) is S_{λ^β}-convergent of order γ but is not S_{λ^β}-convergent of order β, for $\beta \in (0,\, \frac{1}{3}]$. $\square$

Theorem 5.4.2. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Then, for*

$$0 < \beta \leq \gamma \leq 1 \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0,$$

(i) $S_{\mu^\gamma}(X) \subseteq S_{\lambda^\beta}(X)$,

(ii) $S_{\mu^\beta}(X) \subseteq S_{\lambda^\beta}(X)$,

(iii) $S_{\mu}(X) \subseteq S_{\lambda^\beta}(X)$.

Proof. (i) Suppose, $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$ for the sequences $\lambda,\, \mu \in \Omega$ such that $\lambda_n \leq \mu_n$, for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Assume that $S_{\mu^\gamma}\text{-}\lim_k X_k = X_0$. Now, for $\epsilon > 0$, we have

$$\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\} \subset \left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\},$$

where $J_n = [n - \mu_n + 1,\, n]$.

Now, we can write

$$\frac{1}{\mu_n^\gamma} \left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \leq \frac{1}{\mu_n^\gamma} \left|\left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right|$$

or

$$\frac{\lambda_n^\beta}{\mu_n^\gamma} . \frac{1}{\lambda_n^\beta} \left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \leq \frac{1}{\mu_n^\gamma} \left|\left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right|.$$

Since S_{μ^γ}- $\lim_k X_k = X_0$ and $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$, taking the limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta} \left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| = 0$$

which completes the proof.

The results (ii) and (iii) are the immediate consequences of (i). □

Theorem 5.4.3. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Then, for*

$$0 < \beta \leq \gamma \leq 1, \quad \lim_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1,$$

(i) $S_{\lambda^\beta}(X) = S_{\mu^\gamma}(X)$,

(ii) $S_{\lambda^\beta}(X) = S_{\mu^\beta}(X)$,

(iii) $S_{\lambda^\beta}(X) = S_{\mu}(X)$.

Proof. (i) Suppose $\lim_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1$ for the sequences λ, $\mu \in \Omega$ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Assume that $(X_k) \in S_{\lambda^\beta}(X)$. Since $I_n \subset J_n$, then for $\epsilon > 0$, we may write

$$\left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\} \supset \left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}$$

Thus, we obtain

$$\begin{aligned}
\frac{1}{\mu_n^\gamma} \left|\left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| &= \frac{1}{\mu_n^\gamma} \left|\left\{n - \mu_n + 1 \leq k \leq n - \lambda_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \\
&\quad + \frac{1}{\mu_n^\gamma} \left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \\
&\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^\gamma}\right) + \frac{1}{\mu_n^\gamma} \left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \\
&\leq \left(\frac{\mu_n - \lambda_n^\beta}{\mu_n^\gamma}\right) + \frac{1}{\mu_n^\gamma} \left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \\
&\leq \left(\frac{\mu_n}{\mu_n^\gamma} - \frac{\lambda_n^\beta}{\mu_n^\gamma}\right) + \frac{1}{\lambda_n^\beta} \left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right|
\end{aligned}$$

for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. The right hand side of the above inequality tends to 0 as $n \to \infty$. Hence, $(X_k) \in S_{\mu^\gamma}(X)$ and therefore $S_{\lambda^\beta}(X) \subseteq S_{\mu^\gamma}(X)$. Now, the use of part (i) of Theorem 5.3.2, immediately implies that $S_{\lambda^\beta}(X) = S_{\mu^\gamma}(X)$.

It is very easy to prove (ii), (iii) and hence omitted. $\square$

5.4.1 S_{λ^β}-limit and S_{λ^β}-cluster points

In this section, we define S_{λ^β}-statistical limit and S_{λ^β}-statistical cluster points of sequences of fuzzy numbers and prove some inclusion results. For this purpose, we need to introduce the following concepts of λ-thin and λ-nonthin subsequences of order β for a sequence of fuzzy numbers.

Definition 5.4.3. Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. A subsequence $(X)_K$, where $K = \{k(j) : j \in \mathbb{N}\}$, of a fuzzy number sequence $X = (X_k)$ is a λ^β-thin subsequence if

$$\lim_{r\to\infty} \frac{1}{\lambda_n^\beta} |\{k(j) \in I_r : j \in \mathbb{N}\}| = 0,$$

On the other hand, $(X)_K$ is a λ^β-nonthin subsequence of (X_k) if

$$\limsup_{r\to\infty} \frac{1}{\lambda_n^\beta} |\{k(j) \in I_r : j \in \mathbb{N}\}| > 0.$$

.

Definition 5.4.4. Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. A fuzzy number X_0 is said to be a S_{λ^β}-limit point of the sequence (X) of fuzzy numbers, provided that there is a λ^β-nonthin subsequence of (X_k), that is convergent to X_0.

Let $\Lambda^F_{S^\beta_\lambda}$ denotes the set of all S_{λ^β}-limit point of the sequence (X_k) of fuzzy numbers.

Definition 5.4.5. Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. A fuzzy number Y_0 is said to be a S_{λ^β}-cluster point of the sequence (X_k) of fuzzy numbers, provided that for each $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k \in I_n : \overline{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let $\Gamma^F_{S^\beta_\lambda}$ denotes the set of all S_{λ^β}-cluster point of the sequence (X_k) of fuzzy numbers.

Theorem 5.4.4. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Then, for*

$$0 < \beta \leq \gamma \leq 1 \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0,$$

(i) $\Lambda^F_{S^\gamma_\mu} \supseteq \Lambda^F_{S^\beta_\lambda}$,

(ii) $\Lambda^F_{S^\beta_\mu} \supseteq \Lambda^F_{S^\beta_\lambda}$,

(iii) $\Lambda^F_{S_\mu} \supseteq \Lambda^F_{S^\beta_\lambda}$.

Proof. (i) Suppose, $0 < \beta \leq \gamma \leq 1$ and $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$. Assume that $X_0 \in \Lambda^F_{S^\beta_\lambda}$, then there is λ^β-nonthin subsequence $(X_{k(j)})$ of (X_k) that is convergent to X_0 and

$$\limsup_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k(j) \in I_n : j \in \mathbb{N}\}| = d > 0. \tag{5.4.1}$$

Now for $\epsilon > 0$, we have

$$\{k(j) \in J_n : j \in \mathbb{N}\} \supset \{k(j) \in I_n : j \in \mathbb{N}\},$$

where $J_n = [n - \mu_n + 1,\, n]$. Since,

$$\frac{1}{\mu_n^\gamma} |\{k(j) \in J_n : j \in \mathbb{N}\}| \geq \frac{\lambda_n^\beta}{\mu_n^\gamma} . \frac{1}{\lambda_n^\beta} |\{k(j) \in I_n : j \in \mathbb{N}\}|,$$

it follows from (5.4.1) that

$$\limsup_{n\to\infty} \frac{1}{\mu_n^\gamma} |\{k(j) \in J_n : j \in \mathbb{N}\}| > 0.$$

Since $(X_{k(j)})$ is already convergent to X_0, so we have $X_0 \in \Lambda_{S_\mu^\gamma}$. Hence $\Lambda^F_{S_\mu^\gamma} \supseteq \Lambda^F_{S_\lambda^\beta}$.

The results (ii) and (iii) are the immediate consequences of (i). $\square$

Theorem 5.4.5. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Then, for*

$$0 < \beta \leq \gamma \leq 1 \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0,$$

(i) $\Gamma^F_{S_\mu^\gamma} \supseteq \Gamma^F_{S_\lambda^\beta}$,

(ii) $\Gamma^F_{S_\mu^\beta} \supseteq \Gamma^F_{S_\lambda^\beta}$,

(iii) $\Gamma^F_{S_\mu} \supseteq \Gamma^F_{S_\lambda^\beta}$.

Proof. The proof of this theorem can be obtain on the similar lines as that of Theorem 5.4.4. $\square$

Theorem 5.4.6. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Then, for*

$$0 < \beta \leq \gamma \leq 1, \quad \limsup_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1,$$

(i) $\Lambda^F_{S_\mu^\gamma} = \Lambda^F_{S_\lambda^\beta}$,

(ii) $\Lambda^F_{S_\mu^\beta} = \Lambda^F_{S_\lambda^\beta}$,

(iii) $\Lambda^F_{S_\mu} = \Lambda^F_{S_\lambda^\beta}$.

Proof. (i) Suppose $\limsup_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1$ and $\limsup_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1$ for the sequences λ, $\mu \in \Omega$ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Let $X_0 \in \Lambda^F_{S_\mu^\gamma}$, then there is μ^γ-nonthin subsequence $(X_{k(j)})$ of (X_k) that is convergent to X_0 and

$$\limsup_{n\to\infty} \frac{1}{\mu_n^\gamma} |\{k(j) \in J_n : j \in \mathbb{N}\}| = d > 0, \tag{5.4.2}$$

where $J_n = [n - \mu_n + 1, n]$. Since $I_n \subset J_n$, then for $\epsilon > 0$, we may write

$$\{k(j) \in J_n : j \in \mathbb{N}\} \supset \{k(j) \in I_n : j \in \mathbb{N}\}.$$

Now, following the similar lines of proof as that of Theorem 5.4.2 (i), we obtain

$$\frac{1}{\mu_n^\gamma} |\{k(j) \in J_n : j \in \mathbb{N}\}| \leq \left(\frac{\mu_n}{\mu_n^\gamma} - \frac{\lambda_n^\beta}{\mu_n^\gamma}\right) + \frac{1}{\lambda_n^\beta} |\{k(j) \in I_n : j \in \mathbb{N}\}|$$

for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. The equation (5.4.2) implies that $X_0 \in \Lambda^F_{S^\beta_\lambda}$. Hence, $\Lambda^F_{S^\gamma_\mu} \subseteq \Lambda^F_{S^\beta_\lambda}$. Now, the use of part (i) of Theorem 5.4.4, immediately implies that $\Lambda^F_{S^\gamma_\mu} = \Lambda^F_{S^\beta_\lambda}$.

The proofs of part (ii) and (iii) are easy to prove and hence are omitted. $\square$

Theorem 5.4.7. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Then, for*

$$0 < \beta \leq \gamma \leq 1, \quad \limsup_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1,$$

(i) $\Gamma^F_{S^\gamma_\mu} = \Gamma^F_{S^\beta_\lambda}$,

(ii) $\Gamma^F_{S^\beta_\mu} = \Gamma^F_{S^\beta_\lambda}$,

(iii) $\Gamma^F_{S_\mu} = \Gamma^F_{S^\beta_\lambda}$.

Proof. Proof of this theorem follows the similar lines as that of Theorem 5.4.6. $\square$

5.4.2 $S_{\lambda^\beta}-$limit inferior and $S_{\lambda^\beta}-$limit superior

In this section, we introduce the notions of $S_{\lambda^\beta}-$limit inferior and $S_{\lambda^\beta}-$limit superior for sequences of fuzzy numbers.

Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. For a sequence $X = (X_k)$ of fuzzy numbers, let us define the sets:

$$M = \{\mu \in L(\mathbb{R}) : \delta_{\lambda^\beta}(\{k \in I_n : X_k < \mu\}) \neq 0\};$$
$$\overline{M} = \{\mu \in L(\mathbb{R}) : \delta_{\lambda^\beta}(\{k \in I_n : X_k > \mu\}) = 1\};$$
$$N = \{\mu \in L(\mathbb{R}) : \delta_{\lambda^\beta}(\{k \in I_n : X_k > \mu\}) \neq 0\};$$
$$\overline{N} = \{\mu \in L(\mathbb{R}) : \delta_{\lambda^\beta}(\{k \in I_n : X_k < \mu\}) = 1\}.$$

Definition 5.4.6. Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. The S_{λ^β}-limit superior of (X_k) is defined by

$$S_{\lambda^\beta}\text{-}\limsup X = \begin{cases} \sup N, & \text{if } N \neq \phi, \\ -\overline{\infty}, & \text{otherwise.} \end{cases}$$

Similarly, the S_{λ^β}-limit inferior is defined by

$$S_{\lambda^\beta}\text{-}\liminf X = \begin{cases} \inf M, & \text{if } M \neq \phi, \\ \overline{\infty}, & \text{otherwise.} \end{cases}$$

Note that $S_{\lambda^\beta} - \limsup X = \inf \overline{N}$ and $S_{\lambda^\beta} - \liminf X = \sup \overline{M}$. The concepts defined as above can be illustrated with help of the following example:

Example 5.4.2. Let $\lambda = (\lambda_n) \in \Omega$. We define a sequence of fuzzy numbers $X = (X_k)$ as follows. For $x \in \mathbb{R}$, define

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} x - k + 1, & \text{if } k - 1 \leq x \leq k \\ -x + k + 1, & \text{if } k < x \leq k + 1 \\ 0, & \text{otherwise} \end{array}\right\}, & \text{if} \quad k = n^2 \\ \left.\begin{array}{ll} \vartheta_1, & \text{if } k \text{ is even} \\ \vartheta_2, & \text{if } k \text{ is odd} \end{array}\right\}, & \text{if} \quad k \neq n^2. \end{cases}$$

where the fuzzy numbers ϑ_1 and ϑ_2 are given by

$$\vartheta_1 = \begin{cases} x-5, & \text{if } 5 \leq x \leq 6, \\ 7-x, & \text{if } 6 < x \leq 7, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\vartheta_2 = \begin{cases} x+7, & \text{if } -7 \leq x \leq -6, \\ -5-x, & \text{if } -6 < x \leq -5, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we obtain

$$[X_k]^\alpha = \begin{cases} [k-1+\alpha, k+1-\alpha], & \text{if } k = n^2 \\ [5+\alpha, 7-\alpha], & \text{if } k \neq n^2 \text{ and } k \text{ is even,} \\ [-7+\alpha, -5-\alpha], & \text{if } k \neq n^2 \text{ and } k \text{ is odd.} \end{cases}$$

Clearly,

$M = (\eta_1, \overline{\infty})$ and $N = (-\overline{\infty}, \eta_2)$

for the sequences (X_k), where

$[\eta_1]^\alpha = [-7+\alpha, -5-\alpha]$ and $[\eta_2]^\alpha = [5+\alpha, 7-\alpha]$.

Therefore, S_{λ^β}- $\liminf X = \eta_1$ and S_{λ^β}- $\limsup X = \eta_2$, only for $\beta \in (\frac{1}{2}, 1]$. $\square$

Theorem 5.4.8. *Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. Then, for a sequence $X = (X_k)$ of fuzzy numbers, S_{λ^β}- $\limsup X = \eta$ if and only if, for each ϵ*

$$\delta_{\lambda^\beta}(\{k \in I_n : X_k > \eta \ominus \overline{\epsilon}\} \cup \{k \in I_n : X_k \nsim \eta \ominus \overline{\epsilon}\}) \neq 0$$

and

$$\delta_{\lambda^\beta}(\{k \in I_n : X_k > \eta \oplus \overline{\epsilon}\}) = 0.$$

Proof of this theorem can be obtain on similar lines as that of the Theorem 5.3.1 and so is omitted here. Similar statement for S_{λ^β}- $\liminf X$ can be given as follows:

Theorem 5.4.9. *Let* $\lambda = (\lambda_n) \in \Omega$ *and* $\beta \in (0, 1]$. *Then, for a sequence* $X = (X_k)$ *of fuzzy numbers,* S_{λ^β}-$\liminf X = \xi$ *if and only if, for each* ϵ

$$\delta_{\lambda^\beta}(\{k \in I_n : X_k < \xi \oplus \overline{\epsilon}\} \cup \{k \in I_n : X_k \nsim \xi \oplus \overline{\epsilon}\}) \neq 0$$

and

$$\delta_{\lambda^\beta}(\{k \in I_n : X_k < \xi \ominus \overline{\epsilon}\}) = 0.$$

Theorem 5.4.10. *Let* $\lambda = (\lambda_n) \in \Omega$ *and* $\beta \in (0, 1]$. *Then, for a sequence* $X = (X_k)$ *of fuzzy numbers*

$$S_{\lambda^\beta}\text{-}\liminf X \leq S_{\lambda^\beta}\text{-}\limsup X.$$

Proof. From the definition of S_{λ^β}-limit inferior of (X_k), we have

$$S_{\lambda^\beta} - \liminf X = \sup \overline{M}$$

and also from the definitions of the sets $\overline{M}$ and N, we see that $\overline{M} \subseteq N$.
Hence, we get

$$\sup \overline{M} \leq \sup N,$$

which implies that

$$S_{\lambda^\beta} - \liminf X \leq S_{\lambda^\beta} - \limsup X.$$

□

Chapter 6

Generalized sequential convergence in fuzzy neighborhood spaces

6.1 Introduction

In past years, much work have been published on the sequential convergence in fuzzy topological spaces and has appeared with variety in literature. For instance, Lowen [74] developed a theory of convergence in fuzzy topological spaces based on the idea of prefilters and used it later in his work [75] to introduce fuzzy neighborhood spaces (FNS), a subcategory of fuzzy topological spaces (FTS). Lowen proved some basic properties of this subcategory. Subsequently, Wuyts and Lowen [111] introduced and studied a number of separation properties of fuzzy neighborhood spaces, all of which were extensions of (T_0), (T_1), and (T_2) in topological spaces. Khan and Ahsanullah [62] introduced a quite different notions of sequential convergence which they called α-convergence in fuzzy neighborhood spaces and also studied convergent sequences and Cauchy sequences in the light of fuzzy (probabilistic) pseudometric spaces.

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In this chapter, we generalize the concept of sequential convergence by introducing the notions of α-statistical convergence and statistical convergence for sequences in fuzzy neighborhood spaces which turn out to be a very useful subcategory of fuzzy topological spaces. The contents of this chapter have been divided in three sections. Section 6.2 deals with the preliminaries related to fuzzy neighborhood spaces. In section 6.3, we define the notions of α-statistical convergence and statistical convergence in fuzzy neighborhood spaces. Some properties of α-statistical convergence and statistical convergence in these spaces have been developed in this section.

6.2 Preliminaries

In this section, apart from setting up the notations and terminologies to be used in the sequel, some known results interrelated with the work done in the present chapter have also been presented.

Let $I = [0, 1]$, $I_0 = (0, 1]$, $I_1 = [0, 1)$ and $I_{0,1} = (0, 1)$.

For any set X, let I^X denote the set of all fuzzy subsets of X. A member ν of I^X is contained in a member μ of I^X denoted $\nu \leq \mu$ if and only if $\nu(x) \leq \mu(x)$, for every $x \in X$. Let ν, $\mu \in I^X$.

Define the following fuzzy sets:

(1) $\nu \wedge \mu \in I^X$ by $(\nu \wedge \mu)(x) = \min\{\nu(x), \mu(x)\}$ for every $x \in X$ (intersection).

(2) $\nu \vee \mu \in I^X$ by $(\nu \vee \mu)(x) = \max\{\nu(x), \mu(x)\}$ for every $x \in X$ (union).

(3) $\nu^C \in I^X$ by $\nu^C(x) = 1 - \nu(x)$ for every $x \in X$.

If X is a set, then a non-empty collection $\Sigma(x) \subseteq I^X$ is called a prefilter (on X) iff:

(1) $0 \notin \Sigma(x)$,

(2) $\nu \in \Sigma(x), \mu \in \Sigma(x) \Rightarrow \nu \wedge \mu \in \Sigma(x)$,

(3) $\nu \in \Sigma(x), \nu \leq \mu \Rightarrow \mu \in \Sigma(x)$.

If X is a set, then a non-empty collection $\Delta(x) \subseteq I^X$ is called a prefilter base (on X)

iff:

(1) $0 \notin \Delta(x)$,

(2) $\nu \in \Delta(x), \mu \in \Delta(x) \Rightarrow \exists \lambda \in \Delta(x) : \lambda \leq \nu \wedge \mu$.

Based on the idea of prefilters, Lowen [75] introduced a subcategory of fuzzy topological space and called it fuzzy neighborhood space.

Definition 6.2.1. [75] A family of prefilters $\Sigma = (\Sigma(x))_{x \in X}$ in a set X is called a fuzzy neighborhood system on X if and only if the following are true:

(i) $\forall\, x \in X,\ \forall\, \nu \in \Sigma(x) : \nu(x) = 1$;

(ii) $\forall\, x \in X :\ \Sigma(x)^{\sim} = \Sigma(x)$;

(iii) $\forall\, x \in X,\, \nu \in \Sigma(x),\, \forall\, \epsilon \in I_0,\, \exists(\nu_z^{\epsilon})_{z \in X} \in \prod_{z \in X} \Sigma(z) : \forall\, y, z \in X : \nu_x(z) \wedge \nu_z(y) \leq \nu(y) + \epsilon$,

where $\sim$ denotes the saturation operation as given by

$\Sigma^{\sim} = \{\nu : \forall\, \epsilon > 0, \exists \nu_\epsilon \in \Sigma : \nu_\epsilon - \epsilon \leq \nu\}$.

Given a fuzzy neighborhood system Σ on X; one can obtain a fuzzy topology $t(\Sigma)$, and the pair $(X, t(\Sigma))$ is known as fuzzy neighborhood space.

While studying the notion of fuzzy neighborhood space in [75] and [111], Lowen obtained following important results.

Theorem 6.2.1. *[75] If (X, τ) is a topological space, then the associated fuzzy topological space $(X, \omega(\tau))$ has as fuzzy neighborhood system the family $\omega(\mathcal{U}(x)_{x \in X})$, Where $\mathcal{U}(x)$ is the neighborhood system of τ and where for all $x \in X$*

$\omega(\mathcal{U}(x)) = \{\nu \in I^X, for\, each \quad \epsilon \in I_1, \nu^{-1}(0,1] \in \mathcal{U}(x)\}$.

Theorem 6.2.2. *[75] If $\Sigma = (\Sigma(x))_{x \in X}$ is a fuzzy neighborhood system on X, then $\Delta = (\Delta(x))_{x \in X}$ is a basis for Σ if and only if for all $x \in X$, $\Delta(x)$ is a prefilterbasis and for all $\nu \in \Sigma(x)$ and for all $\epsilon \in I_0$ there exists $\theta_\epsilon \in \Delta(x)$ such that $\theta_\epsilon - \epsilon \leq \nu$.*

Theorem 6.2.3. *[75] If $(X, t(\Sigma))$ is a fuzzy neighborhood space and $\alpha \in I_1$, then the level topological space $(X, l_\alpha(t(\Sigma)))$ has a neighborhood system the family $(l_\alpha(\Sigma(x)))_{x\in X}$ where for all $x \in X$,*

$l_\alpha(\Sigma(x)) = \{\nu^{-1}(\beta, 1] : \nu \in \Sigma(x), \beta \in [0, 1-\alpha)\}.$

Definition 6.2.2. [111] A fuzzy neighborhood space $(X, t(\Sigma))$ is said to be NT_2 if and only if for all $x \neq y \in X$ and for all $\epsilon \in I_0$ there exist $\nu_x \in \Sigma(x)$ and $\nu_y \in \Sigma(y)$ such that $\nu_x \wedge \nu_y < \epsilon$.

Khan and Ahsanullah [62] introduced the notions of α-convergence and convergence in fuzzy neighborhood spaces as follows.

Definition 6.2.3. [62] Let $(X, t(\Sigma))$ be a fuzzy neighborhood space and (x_k) a sequence in X. For given $\alpha \in I_0$, (x_k) is said to be α-convergent to $x \in X$ if and only if for all $\nu \in \Sigma(x)$, there exists $m \in \mathbb{N}$ such that for all $k \geq m$

$\nu(x_k) > 1 - \alpha.$

x is called an α-limit of x_k and indicated as $x_k \rightsquigarrow^\alpha x$.

Definition 6.2.4. [62] In a fuzzy neighborhood space $(X, t(\Sigma))$, a sequence (x_k) in X is called convergent to $x \in X$ if and only if for all $\alpha \in I_0$, $x_k \rightsquigarrow^\alpha x$. We abbreviate it as $x_k \rightsquigarrow x$.

6.3 Statistical convergence in Fuzzy Neighborhood Spaces

In this section, we introduce the notions of α-statistical convergence and statistical convergence of sequences in fuzzy neighborhood spaces.

Definition 6.3.1. Let $(X, t(\Sigma))$ be a fuzzy neighborhood space and $x = (x_k)$ be a sequence in X. For a given $\alpha \in I_0$, (x_k) is said to be α-statistical convergent to $x \in X$, if and only if, for all $\nu \in \Sigma(x)$, we have $\delta\{k \in \mathbb{N} : \nu(x_k) \leq 1 - \alpha\} = 0$ i.e.,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \nu(x_k) \leq 1 - \alpha\}| = 0$$

where, the vertical bars denotes the cardinality of the enclosed set. In this case, x is called an α-statistical limit of (x_k) and is denoted by $S^\alpha - \lim x_k = x$ or $x_k \rightsquigarrow^{S^\alpha} x$.

Definition 6.3.2. Let $(X, t(\Sigma))$ be a fuzzy neighborhood space. A sequence $x = (x_k)$ in X is said to be statistical convergent to $x \in X$, if and only if, for all $\alpha \in I_0$, (x_k) is α-statistical convergent to x. In other words, for each $\alpha \in I_0$ and for all $\nu \in \Sigma(x)$, we have

$$\delta\{k \in \mathbb{N} : \nu(x_k) \leq 1 - \alpha\} = 0. \tag{6.3.1}$$

It is abbreviated as $S-\lim x_k = x$ or $x_k \rightsquigarrow^S x$. Let S denotes the set of all statistically convergent sequences in X.

Remarks 6.3.1. If $S^\alpha - \lim x_k = x$ and $\alpha < \beta \leq 1$, then $S_\beta - \lim x_k = x$. Thus $1-$statistical convergence, which simply says that $\forall \nu \in \Sigma(x)$ we have $\delta\{k \in \mathbb{N} : \nu(x_k) \leq 0\} = 0$, is the strongest of all $\alpha-$statistical convergence.

Theorem 6.3.1. *Let $(X, t(\Sigma))$ be a fuzzy neighborhood space and $\Delta(x)$ a basis for $\Sigma(x)$. Then $S - \lim x_k = x$, if and only if, for all $\theta \in \Delta(x)$ and for all $\alpha \in I_0$, $\delta\{k \in \mathbb{N} : \theta(x_k) \leq 1 - \alpha\} = 0$.*

Proof. First suppose that $S - \lim x_k = x$. Select $\theta \in \Delta(x)$ and $\alpha \in I_0$ arbitrary. Since (x_k) is statistically convergent to x so in particular, we have $x_k \rightsquigarrow^{S^\alpha} x$, which implies that for all $\nu \in \Sigma(x)$ we have

$$\delta\{k \in \mathbb{N} : \nu(x_k) \leq 1 - \alpha\} = 0.$$

Further, the fact $\Delta(x) \subset \Sigma(x)$ implies that

$$\delta\{k \in \mathbb{N} : \theta(x_k) \leq 1 - \alpha\} = 0.$$

Conversely, suppose that for all $\theta \in \Delta(x)$ and for all $\alpha \in I_0$, we have

$$\delta\{k \in \mathbb{N} : \theta(x_k) \leq 1 - \alpha\} = 0. \tag{6.3.2}$$

We need to prove that $S - \lim x_k = x$. Let $\beta \in I_0$ and $\nu \in \Sigma(x)$ be arbitrary. Since, $\Delta(x)$ is a basis for $\Sigma(x)$, so we can a find a $\theta \in \Delta(x)$ such that

$$\theta - \frac{1}{2}\beta \leq \nu.$$

Consequently, density of the set

$$A = \{k \in \mathbb{N} : \theta(x_k) \leq 1 - \frac{1}{2}\beta\}$$

is zero. Let

$$B = \{k \in \mathbb{N} : \nu(x_k) \leq 1 - \beta\}$$

To prove the result, it is sufficient to show that $B \subset A$. If $k_0 \in B$, then we have

$$1 - \beta \geq \nu(x_{k_0}) \geq \theta(x_{k_0}) - \frac{1}{2}\beta$$

i.e. $1 - \frac{1}{2}\beta \geq \theta(x_{k_0})$. This shows that $k_0 \in A$. Hence, $\delta(B) = 0$ as $\delta(A) = 0$. $\square$

Theorem 6.3.2. *Let (X, τ) be a topological space and $x = (x_k)$ be a sequence in X. If $x \in X$ and (x_k) statistically converges to x in (X, τ), then for any $\alpha \in I_0$, (x_k) is α-statistically convergent to x in $(X, \omega(\tau))$, where $(X, \omega(\tau))$ is the associated fuzzy topological space. Conversely, if (x_k) is α-statistically convergent to x in $(X, \omega(\tau))$ for some $\alpha \in I_0$, then (x_k) is statistically converges to x in (X, τ).*

Proof. Suppose, (x_k) is statistical convergent to x in (X, τ). Let, $\alpha \in I_0$ and $\nu \in \omega(\mathcal{U}(x))$, where $\mathcal{U}(x)$ is the neighborhood system at x in (X, τ). Then,

$$\upsilon^{-1}(\epsilon, 1] \in \mathcal{U}(x)$$

for each $\epsilon \in I_1$. In particular, taking $\epsilon = (1 - \alpha)$, we have

$$\nu^{-1}((1 - \alpha, 1]) \in \mathcal{U}(x).$$

Hence, we can find out $m \in \mathbb{N}$ such that for all $k \geq m$,

$$x_k \in \nu^{-1}((1 - \alpha, 1])$$

i.e. $\nu(x_k) > 1 - \alpha$ for all $k \geq m$. Consequently, the set

$$A = \{k \in \mathbb{N} : \nu(x_k) \leq 1 - \alpha\}$$

is finite and therefore $\delta(A) = 0$. Which shows that (x_k) is α-statistically convergent to x in $(X, \omega(\tau))$.

Conversely, Suppose that $S^\alpha - \lim x_k = x$ in $(X, \omega(\tau))$ for $\alpha \in I_0$. We will prove that (x_k) converges statistically to x in (X, τ).

Let $V \in \mathcal{U}(x)$. Then $1_V \in \omega(\mathcal{U}(x))$, since $1_V^{-1}((\epsilon, 1]) = V$ for all $\epsilon \in I_1$. This together with the definition of α-statistical convergence, immediately implies that

$$\delta(\{k \in \mathbb{N} : 1_V(x_k) \leq 1 - \alpha\}) = 0.$$

Thus, we have

$$\delta(\{k \in \mathbb{N} : 1_V(x_k) = 0\}) = \delta(\{k \in \mathbb{N} : x_k \notin V\}) = 0.$$

Hence, (x_k) is statistically converges to x in (X, τ). $\square$

Theorem 6.3.3. *In an NT_2 fuzzy neighborhood space $(X, t(\Sigma))$, a sequence $x = (x_k)$ has a unique α-statistical limit for $\alpha \in I_1$.*

Proof. Suppose there exists $x, y(x \neq y)$ in X such that $x_k \rightsquigarrow^{S^\alpha} x, y_k \rightsquigarrow^{S^\alpha} y$. Since X is NT_2, so for $\alpha \in I_{0,1}$ there exist $\nu_x \in \Sigma(x)$ and $\nu_y \in \Sigma(y)$ such that

$$\nu_x(x_k) \wedge \nu_y(x_k) < 1 - \alpha$$

Since $x_k \rightsquigarrow^{S^\alpha} x$ and $y_k \rightsquigarrow^{S^\alpha} y$, then the sets

$$K_1 = \{k \in \mathbb{N} : \nu_x(x_k) \leq 1 - \alpha\}$$

and

$$K_2 = \{k \in \mathbb{N} : \nu_y(x_k) \leq 1 - \alpha\},$$

have natural density zero i.e. $\delta(K_1) = \delta(K_2) = 0$.

Let $K = K_1^C \cap K_2^C$, then $\delta(K) = 1$. For $k_0 \in K$, $\nu_x(x_{k_0}) > 1 - \alpha$ and $\nu_y(x_{k_0}) > 1 - \alpha$. This implies

$$1 - \alpha > \nu_x(x_{k_0}) \wedge \nu_x(x_{k_0}) \leq 1 - \alpha,$$

which is a contraction. Hence, $x = y$. $\square$

Theorem 6.3.4. *Let $(X, t(\Sigma))$ and $(X', t(\Sigma'))$ be two fuzzy neighborhood spaces, $f : X \to X'$ be continuous at $x \in X$. For $\alpha \in I_0$, if $x_k \rightsquigarrow^{S^\alpha} x$, then $f(x_k) \rightsquigarrow^{S^\alpha} f(x)$.*

Proof. To prove the result, consider $\nu' \in \Sigma'(f(x))$ arbitrary. By continuity of f, we have $f^{-1}(\nu') \in \Sigma(x)$. Since $x_k \rightsquigarrow^{S^\alpha} x$, therefore

$$\delta(A) = \delta(\{k \in \mathbb{N} : f^{-1}(\nu')(x_k) \leq 1 - \alpha\}) = 0$$

or

$$\delta(A^C) = \delta(\{k \in \mathbb{N} : f^{-1}(\nu')(x_k) > 1 - \alpha\}) = 1.$$

To prove the result, it is sufficient to prove that the set

$$B = \{k \in \mathbb{N} : \nu'(f(x_k)) \leq 1 - \alpha\} \subseteq A$$

.Since A^C is an infinite set (as otherwise i.e. if A^C is finite, then $\delta(A^C) = 0$). Let $k_0 \in A^C$, then

$$f^{-1}(\nu')(x_{k_0}) > 1 - \alpha.$$

This implies that

$$\nu'(f(x_{k_0})) > 1 - \alpha,$$

which shows $k_0 \in B^C$.

Hence, $A^C \subseteq B^C$ i.e., $B \subseteq A$ and therefore, we have $\delta(B) = 0$. □

Theorem 6.3.5. *Let $(X, t(\Sigma))$ be a fuzzy neighborhood space. A sequence $x = (x_k)$ in X is statistical convergent to $x \in X$, if and only if, there exists a subset $K \subset \mathbb{N}$ with $\delta(K) = 1$ such that $x_k \rightsquigarrow x$ over K.*

Proof. First suppose that $x = (x_k)$ be statistically convergent to x. Then for each $\alpha \in I_0$ and for all $\nu \in \Sigma(x)$, if we denote

$M = \{k \in \mathbb{N} : \nu(x_k) \leq 1 - \alpha\}$; and $K = \{k \in \mathbb{N} : \nu(x_k) > 1 - \alpha\}$,

then $\delta(M) = 0$ and $\delta(K) = 1$. Moreover $K \cap M = \emptyset$.

Now, we have to show that $(x_k)_{k \in K}$ is convergent to x. Suppose (x_k) is not convergent to x over K. Then, there exists $\alpha_0 \in I_0$ and $\nu_0 \in \Sigma(x)$ such that

$\nu_0(x_k) \leq 1 - \alpha_0$ for infinitely many terms.

Let

$$K_1 = \{k_n \in K : \nu_0(x_{k_n}) \leq 1 - \alpha_0\} \text{ and } M_1 = \{k \in \mathbf{N} : \nu_0(x_k) \leq 1 - \alpha_0\}.$$

Clearly, $\emptyset \neq K_1 \subseteq K$ and also $K_1 \subseteq M_1$.

Further, for $\alpha < \alpha_0, M_1 \subseteq M$, which contradicts the fact that $K \cap M = \emptyset$ for all $\alpha \in I_0$. Hence $(x_k)_{k \in K}$ is convergent to x.

Conversely, Suppose that there exists a subset $K \subset \mathbb{N}$ such that $\delta(K) = 1$ and $x_k \rightsquigarrow x$ over K. Let $\alpha \in I_0$ and $\nu \in \Sigma(x)$ be arbitrary. $x_k \rightsquigarrow x$ over K so, there exists a positive integer m such that

$$\nu(x_{k_i}) > 1 - \alpha \text{ for all } i \geq m.$$

The containment

$$\{k \in \mathbb{N} : \nu(x_k) \leq 1 - \alpha\} \subseteq \mathbb{N} - \{k_{m+1}, k_{m+2}, \ldots\},$$

immediately implies that

$$\delta\{k \in \mathbb{N} : \nu(x_k) \leq 1 - \alpha\} \leq 1 - 1 = 0.$$

This shows that $x = (x_k)$ is statistically convergent to x. $\square$

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