A Study on Lamb Wave Propagation in Elastic Continua

A Thesis

submitted in partial fulfillment of the requirements for the award of the degree of

Master of Science

in

Mathematics and Computing

by

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Candidate Declaration

I hereby certify that the work, which is being presented in the thesis, entitled **A Study on Lamb Wave Propagation in Elastic Continua**, in partial fulfillment of the requirements for the award of the degree of **Master of Science** in Mathematics and Computing, and submitted to the Thapar Institution of Engineering and Technology is an authentic record of my own work carried out under the supervision of **Dr.Satish Kumar**. I have also cited the reference about the text(s)/figure(s) from where they have been taken.

The matter presented in this thesis has not been submitted elsewhere for the award of any other degree or diploma from any institution.

Date: 2-7-18

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This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

Date: 2-7-18

Dr.Satish Kumar
Supervisor

Dedicated to Almighty God and my Parents

Abstract

The objective of the thesis is to study the propagation of Lamb waves that travel in elastic plate with free boundaries and also when it is loaded with inviscid liquid. Frequency equations are obtained for the propagation of Lamb waves. The properties of Lamb waves are defined by dispersion curves which are plotted between phase velocities and wave number.

Lamb waves are formed by the interference of multiple reflections at the free surfaces of the plate. In Lamb waves, the particle displacement occurs in both directions that is in direction of wave propagation and perpendicular to the plane of plate. These waves are commonly used in providing information in ultrasonic non destructive testing.

Chapter1 includes the introduction of classical theory of elasticity. The mechanics of continuous elastic bodies is presented in this chapter. The generalised Hooke's Law is discussed with brief explanation of stress-strain curve. The theory of elastic waves and discussion on types of elastic waves like surface and Lamb waves is presented in this chapter. It also contains the basic governing equations of elasticity and expression for the speeds of primary and secondary waves. These expressions are derived using Helmoholtz decomposition. Various applications of Lamb waves are also discussed.

Chapter2 deals with the propagation of Lamb waves in stress free boundaries of elastic plate. In this chapter, the mathematical modeling of problem is done and frequency equations are obtained from the basic governing equation of elasticity in absence of body forces. Dispersion curves are obtained for symmetrical and antisymmetrical modes.

The study carried out in chapter2 is extended to the study of Lamb wave propagation in layered structure in chapter3. Here the plate is sandwiched between the finite layers of inviscid liquid loading. In this chapter frequency equations are derived. The effect of liquid loading are shown graphically. In the absence of the liquid loading the dispersion equations reduced to the equations obtained in the chapter2.

The study carried out in this thesis is the particular case of the article[1] in the absence of couple stress effects.

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Finally, I would like to express my sincere gratitude to my parents for providing me moral support throughout my life. I would like to thank my sister, my friends and my senior scholar Richa Goyal for their love, care and consistent support.

Divjot Kaur

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λ, μ	Elastic constants
ρ	Density
τ_{ij}	Stress tensor
ε_{kl}	Strain tensor
δ_{ij}	Kronecker's delta
ϕ, ψ	Potential functions
$\vec{u}$	Displacement vector
u_{ij}	Displacement components
ω	Frequency
c_1	Longitudinal wave velocity
c_2	Shear wave velocity
λ_L	Bulk modulus for liquid layer
ρ_L	Density of inviscid liquid
ξ	Wave number
c_L	Velocity of sound in liquid
H	Width of the liquid
$u_j^{'}, w_j^{'}$	x and z components of displacement in liquid layers

List of Abbreviations

MW	Mechanical waves
EM	Electromagnetic waves
P waves	Primary waves
S waves	Secondary waves
NDT	Non destructive techinque

Chapter 1

INTRODUCTION

1.1 THEORY OF ELASTICITY

Mechanics is that area of science which is concerned with the behavior of physical bodies when subjected to forces or displacements. It lays the foundation of understanding the world around us that why and how the motion of particles takes place. Solid bodies are divided into two subparts: plastic bodies and elastic bodies. A body is called elastic if it posses the property of recovering its original shape and size when the forces causing the deformation are removed. On removing the deforming forces the body does not return to its original shape and size, is called plastic body. The mechanics of such continuous elastic bodies is known as the mechanics of continuum. This branch of continuum elastic bodies covers a wide range of problems related to elasticity, hydromechanics, plasticity, seismology and electrodynamics. The elastic behavior of materials plays an vital role in engineering field. In the field of civil engineering, the understanding of elastic properties of materials like steel, concrete etc is necessary while designing a building. Mechanical engineering uses elasticity in analysis and design of the machine elements. Application in aerospace engineering includes finding the stresses and fractures in aerostructures. Material engineering uses elasticity to determine stress in crystalline solids. In many problems, the atomistic strength of matter is disregarded and the body is replaced by continuum mathematical model where geometrical points are identified with material points of the body.

The best known constitutive equation for a elastic behavior of solids can be explained in terms of microscopic nature of solid which was discovered by Robert Hooke(1678). Robert Hooke [2], an English physicist(1635-1703 A.D) performed various experiments on springs and establish that the change in the length formed in the body is proportional to the applied force. In 1676, Hooke's gave the law of elasticity known as Hooke's law of elasticity. This Hooke's law of elasticity is an approximation which states that extension of body is directly proportional to the load applied. When forces are applied on a body, it is deformed to smaller or larger extent depending upon the nature of material of body and on the magnitude of deforming force[3]. In many materials the deformation may not be noticeable virtually but it is there. Restoring force developed in the body when body is subjected to a deforming force. Thus the deforming force which is equal in magnitude but is opposite in the direction. This restoring force on a specified area is known as stress, while the amount of deformation is known as strain. The elasticity of the materials is described by stress-strain curve.

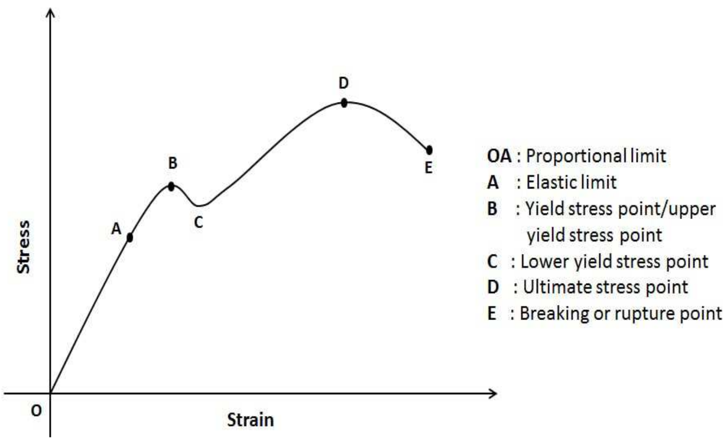

Figure 1.1: Stress-strain curve

The stress and strain are proportional to each other for small deformations and is called the Hooke's law. From the fig (1.1), we see that in region between O and A, Hooke's law is obeyed. Hooke's law is an empirical law and is found to be true for most of the materials. However, in some materials this linear relationship is not true. In the region from A to B, the stress and strain are not linearly proportional. For point B in the curve is called Yield stress point and corresponding stress is called Yield strength of the material. Further, if we increase the load in the portion of the curve between B and D the body will have the permanent deformation and it does not return to its actual shape[4]. In the stress-strain curve beyond a point D on curve additional strain is produced even by a reduced applied force and breaking occurs at point E. Material is said to be brittle if the ultimate strength and breaking points D and E are close and the material is said to be ductile if they are far from each other. However, for the larger stresses beyond the elastic limit the relation does not remain linear. Even for higher stresses, materials shows plastic properties that means they do not return to their actual shape after the stress is removed. For materials like rubber known as elastomers,with the stress the gradient of stress-strain curve increases which means it will be more difficult to stretch the rubber further, while for most metals the gradient of stress-strain curve decreases at very large stresses, which means it is easier to stretch them.

1.2 VECTOR OPERATIONS IN INDEX NOTATIONS

To study generalised Hooke's Law we need certain tensorial notations. Following are the standard notations for some important mathematical operations.
Gradient of scalar

$$\nabla\phi = \phi_{,i}$$

Divergence of vector,

$$\nabla . A = A_{i,i}$$

where A is some vector.
Curl of vector,

$$\nabla \times A = \varepsilon_{ijk} A_{k,j}$$

We note that derivative notation where differentiation with respect to a variable will be indicated by a comma followed by index . Thus

$$\phi_{,i} = \frac{\partial \phi}{\partial x_i}$$

$$A_{i,j} = \frac{\partial A_i}{\partial x_j}$$

1.3 GENERALIZED HOOKE'S LAW

Cauchy in 1822 generalized Hooke's law for the deformation of elastic solids. According to Cauchy,"Each component of stress at any point of elastic body is linear function of the components of strain at the point." The deformation of a solid body induces stresses within.
Hooke's law is defined in the terms of stress-strain relationship where stress is defined as the quantity that is proportional to the force causing the deformation and strain is defined as the measure of degree of deformation. Stress is further divided into two components: normal stress and shearing stress. Normal stress has been defined as the deforming force acting per unit area normal to the surface of the body and Shearing stress has been defined as the deforming force acting per unit area tangential to the surface.
In three dimensional axes of cubic body, stress can be categorized into three parts that

is normal stress and shearing stress which itself can be divided into two components parallel to the direction of the two coordinates. Normal stress has been defined as the force acting perpendicular to the surface and tangential stress has been defined as the force acting parallel to body. There are nine components of stress acts on face of cube. These are $(\tau_{xx}, \tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yy}, \tau_{yz}, \tau_{zx}, \tau_{zy}, \tau_{zz})$ where the first suffix assign to the normal to the plane on which shear stress acts and second suffix assign to the direction of the shear on the plane. In the same way strain is resolved into two components that is longitudinal and shearing strain. The longitudinal strain defined as the change in length per unit original length and shear strain defined when change takes place in the shape of body. these are $(\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yx}, \varepsilon_{yy}, \varepsilon_{yz}, \varepsilon_{zx}, \varepsilon_{zy}, \varepsilon_{zz})$

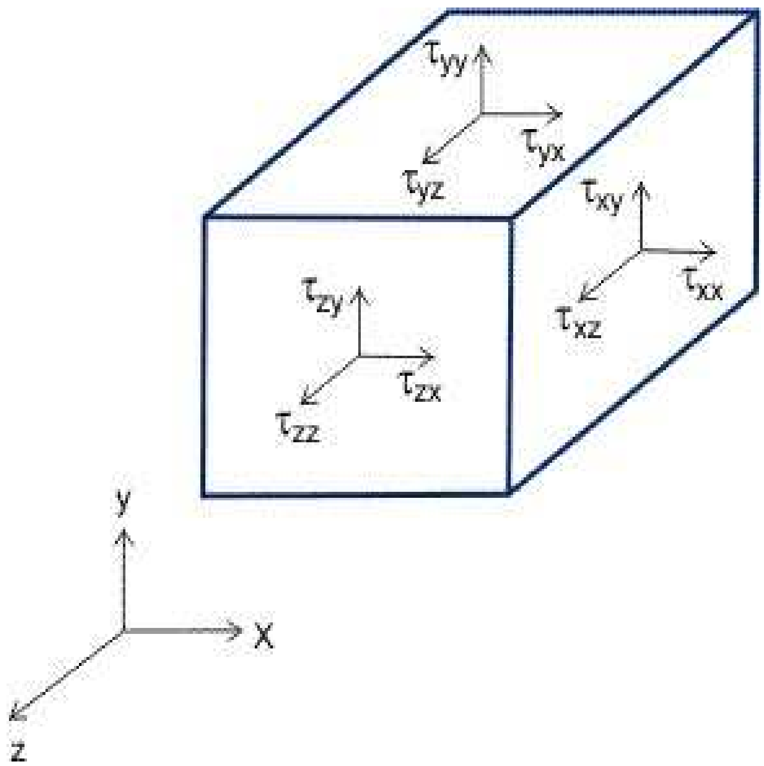

Figure 1.2: Components of stress tensor of elasticity

The theory of elasticity serves as an excellent model for studying the mechanical behaviour of a wide variety of solid materials.
Mathematically,the Cauchy's formulation of generalized Hooke's law is of the form:
$\tau_{ij} = c_{ijkl}\varepsilon_{kl}$, where i,j,k,l=1,2,3.
(τ_{ij}) is the stress tensor and (ε_{kl}) is the strain tensor. Both the stress and strain tensors have nine components and (c_{ijkl}) is the fourth order coefficient and these are called elastic constants or moduli. There are 81 elastic constants. Taking the symmetry of stress and strain components, and by imposing the symmetry conditions elastic constants are

reduced from 81 to 36. Taking the different types of symmetry, elastic constants are reduced to 2 in case of isotropic. The body is said to be isotropic if its elastic properties are independent of the orientation of the coordinate axes. The two elastic constants are expressed as λ and μ and are they are known as Lame's constants.
The constitutive equation for classical isotropic elasticity[5] is as given with two lame constants.

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

where u_i is the displacement vector of the material.
δ_{ij} and τ_{ij} are the Kronecker's delta and stress tensor respectively.

1.4 PROPAGATION OF ELASTIC WAVES

A wave can be defined as the disturbance traveling through a vacuum or medium by transferring energy from one point to another. Wave is the phenomena in which energy is propagated to distinct points. The waves which transmit energy and does not require any medium and travel through the vacuum are electromagnetic waves (EM) waves e.g light waves whereas the waves requiring medium are mechanical waves (MW) e.g sound waves, seismic waves etc. The wave that travels through the medium may experience some local oscillations, but the particles in the medium do not travel with the wave.
Depending on the nature of elastic medium and boundary conditions, wave propagation can be divided into following three types:
1) Waves in infinite media
2) Surface waves
3) Waves in plate

1.4.1 WAVES IN INFINITE MEDIA

For this type of waves in infinite media the boundary interaction are not possible. The equation of motion for a homogeneous isotropic elastic solid without body forces is given as

$$(\lambda + \mu)\nabla(\nabla.\vec{u}) + \mu\nabla^2\vec{u} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \tag{1.1}$$

where,
$\vec{u} = (u_1, u_2, u_3)$ is displacement vector.
λ,μ are Lame's constants and ρ is density.
In the two dimensional problems , we consider
$\vec{u} = (u_1, 0, u_3)$ and $\vec{\psi} = (0, -\psi, 0)$

$$\frac{\partial}{\partial y} = 0 \tag{1.2}$$

To obtain the simple set of equations we use Helmholtz decomposition.
Thus, vector has been decomposed in terms of scalars and vector potentials,

$$\vec{u} = \nabla\phi + \nabla \times \vec{\psi} \tag{1.3}$$

where, ϕ and ψ are scalar and vector potentials respectively. It is the resolution of a vector field into the gradient of a scalar and the curl of zero-divergence and it is called Helmholtz Theorem. The necessary condition $\nabla.\vec{\psi} = 0$ is provided to uniquely determine the all components of u.
Putting equation (1.3) in equation (1.1) and solving for each we obtain,

$$(\lambda + 2\mu)\nabla^2\phi = \rho\ddot{\phi}$$

if we take,

$$c_1 \quad = \quad \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$$

$$\nabla^2\phi = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \tag{1.4}$$

similarly,

$$\nabla^2\psi = \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} \tag{1.5}$$

where

$$c_2 = \sqrt{\frac{\mu}{\rho}}$$

Hence in the interior of elastic solid the waves may propagate at two different velocities. c_1 and c_2 are the wave velocity of longitudinal and shear waves.

LONGITUDINAL WAVES(Primary waves/P-waves/Compressional waves)
Longitudinal waves are that waves which travel parallel in direction of motion. Longitudinal waves are also known as P-waves. P-waves can travel through fluids such as volcanic magma or the water of the oceans and solids, such as granite, mountains. The P-waves have appear in seismology and they are also known as push-pull waves. The P-waves are recorded by seismographs and are produced by earthquakes. Longitudinal means that they transfer energy through compression and are also known as compressional waves. The direction in which wave is moving, the particles moves in same direction, which is the direction that the energy is traveling in, and it is sometimes known as 'direction of wave propagation'. Primary waves are alternatively compressional and extensional, and cause the rocks they pass through to change in volume. Sound wave is one of the example of longitudinal waves.

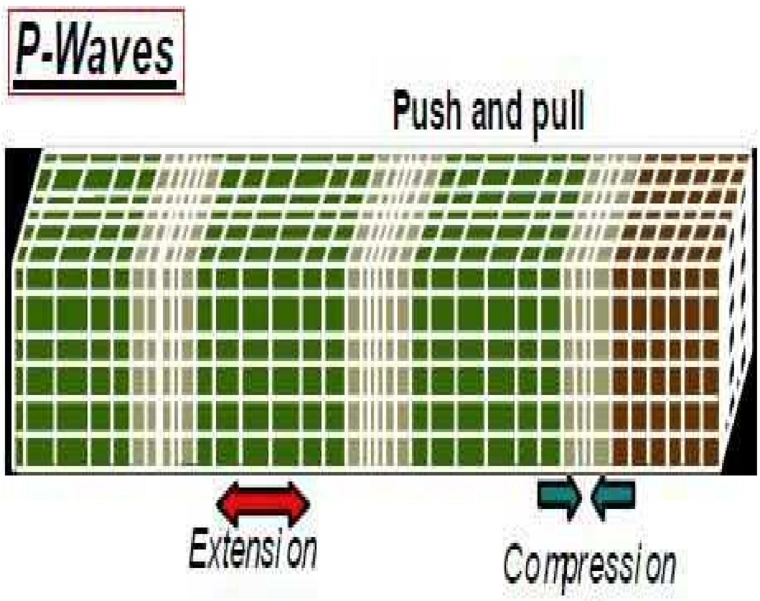

Figure 1.3: Primary Waves

TRANSVERSE WAVES(Secondary waves /S-waves /Shear waves)
Transverse waves are that which travel normal to the direction of motion. Transverse are also known as S-waves. S-waves cannot travel through air or water, they can propagate through solids only. These waves are sometimes referred as shear waves because they are not able to change the volume of the material that they pass through. But they have a larger amplitude (this is the height of a wave, measured from the highest

point to the middle line). They are transverse waves, which means they transfer energy perpendicular (at right angles) to the direction of the wave - like a rope being shaken up and down. S-waves which are known as secondary waves are slower than the primary waves. They are more destructive in the case of an earthquake because of large amplitude.

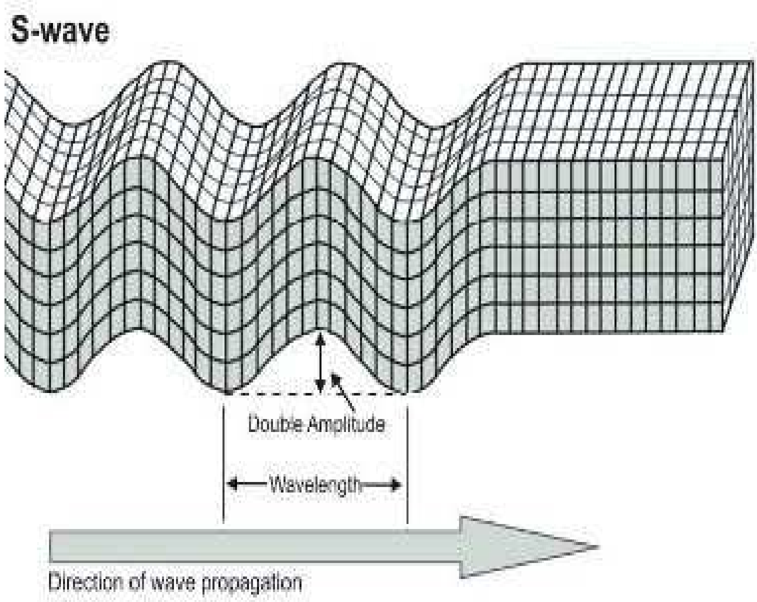

Figure 1.4: Secondary Waves

Difference between longitudinal waves and transverse waves

The differences in longitudinal and transverse waves is in wave sizes,wave speeds,travel capability and type of wave. Longitudinal waves which are known as P-waves are first recorded by seismograph in the any disturbance and travel faster than S-waves. Due to difference in speeds it helps geologists to determine the location of the earthquake. The difference between P-waves and S-waves is of their travel capability. Through any kind of the material P-waves can travel, whether it is solid,liquid or gas. On other side, S-waves can only travel through solids and not by liquids and gases. Therefore, S-waves does not change the volume of material they pass through. This difference helps the Geologist to determine the internal structure of the Earth, that the Earth's outer core is liquid[6]. Other difference in Primary wave and Secondary wave is that Secondary wave is generally of larger amplitude than P-wave. Thus S-waves causes much of the damage in an earthquake as the particles moves up and down, they move the Earth with greater force which results in shaking the surface of the Earth. P-waves are significantly smaller and does not cause much damage.

1.4.2 SURFACE WAVES

The second type of waves are surface waves which are studied in semi infinite media. Surface wave is a mechanical wave that propagates along the interface. Waves in semi-infinite media have been long time standing interest in seismology. They are most destructive type of seismic waves. Problems in ultrasonic, delay lines, blast and impact have also led to analysis of waves in half space. In surface waves [7], the waves that could propagate along the surfaces, such that motion of the wave decayed exponentially with the distance is called Rayleigh wave. The other type of surface wave is Love wave. The circular shearing of the ground is caused by Love waves. These waves are named after A.E.H.Love(1911)[8], British mathematician who created the mathematical model of the waves. During an earthquake, the horizontal shifting of the earth is caused by Love waves. Love waves travels with a lower speed than Primary/Secondary waves, but are faster than Rayleigh waves.

1.4.3 WAVES IN PLATE-LAMB WAVES

Horace Lamb[9] developed the theory of Lamb waves to describe the characteristics of wave propagation in plate. Lamb derived the dispersion relation for different waves traveling across the plane of a free plate. The frequency equations of Lamb waves in infinite plates have been known for some time. In Lamb waves the motion of the particle is elliptical with its x and z components depending upon the width inside plate. Particles moves in two different ways when Lamb waves are transmitted. It is symmetrical Lamb wave if the particle motion is symmetric to the mid surface. It is said to be antisymmetric Lamb wave if the particle motion with respect at mid surface is anti symmetric. Lamb waves are produced in a plate with free boundaries with infinite number of modes[10]. Osborne and Hart[11] investigated Lamb waves activated in the steel plate structures in underwater explosions. Gazis[12] gave the measurable details about Lamb waves.

1.5 APPLICATIONS OF LAMB WAVES

The important key of Lamb waves is that the excitation of single mode in a non-dispersive region[13]. Firestone and Ling([14],[15]) used Lamb waves for damage detection using non destructive techniques in metal sheets. This technique is also used for the calculation of elastic constants of elastic isotropic plates using ultrasonic Lamb waves. These waves travel long distances along plates and shells so they are quick in evaluation of large structures and they might also be helpful for the localized inspection, especially in thin structures[16]. Lee and Yoon[17] suggested the use of Lamb waves for ultrasonic bone assessment. Various experimental investigations on Lamb waves were made by Frederick and Worlton[18] as they correlated the theory with experimental results and also derived the dispersion equations considering phase velocity,frequency and thickness of the plate.

Chapter 2

LAMB WAVE PROPAGATION IN HOMOGENEOUS ISOTROPIC ELASTIC PLATE

2.1 INTRODUCTION

Horace Lamb[9] studied the Lamb wave propagation in the thin elastic plate and is also known as plate waves. Lamb waves are further categorized into symmetric and antisymmetric modes. In symmetric modes, the average displacement over the thickness of the plate has been in longitudinal direction which is also known as longitudinal modes whereas in the antisymmetric modes which is also called flexural modes, the average displacement has been in transverse direction. The Lamb waves are complex vibrational waves that travel to the surface through the thickness of material. Dispersion curves plays an important for detecting damage in composite space as a part of NDT. The propagation of Lamb waves rely on the density and elastic properties. The properties of Lamb waves are defined in terms of dispersion curves which are plots of phase velocities versus wave number-thickness obtained by solving dispersion relation of Lamb wave.

2.2 FORMULATION OF THE PROBLEM

Considering the infinite plate of finite thickness 2d in a homogeneous isotropic elastic solid space. Assume a Cartesian coordinate system whose x-axis is along the surface of the plate and z-axis is pointing vertically downwards in positive direction. The surface is taken to be stress free.

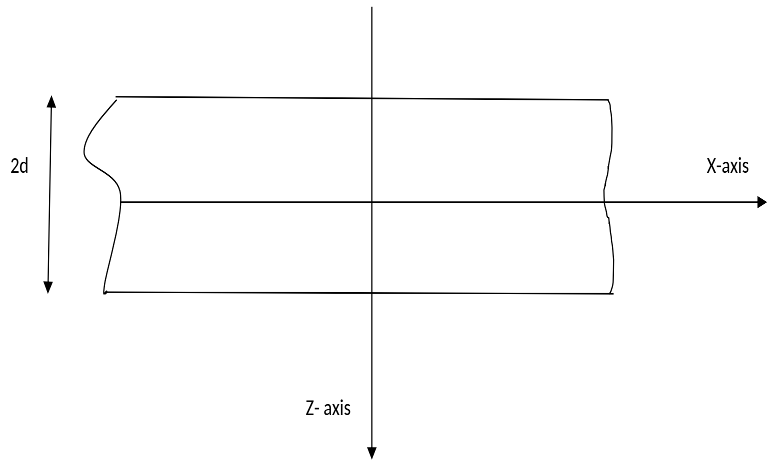

Figure 2.1: Geometry of the problem

The basic governing equation of elastic stress in the absence of body forces:

$$(\lambda + \mu)\nabla(\nabla.\vec{u}) + \mu\nabla^2\vec{u} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \tag{2.1}$$

The constitutive relation is given as

$$\tau_{ij} = \lambda u_{r,r}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \tag{2.2}$$

where,
$\vec{u} = (u_1, u_2, u_3)$ is the displacement vector $(i, j, r = 1, 2, 3)$.
λ,μ are Lame's constants , ρ is density.
τ_{ij} is the stress tensor.
δ_{ij} is the Kronecker's delta.

$$\delta_{ij} = \begin{cases} 1, \text{if i=j} \\ 0, if i \neq j \end{cases}$$

In the two dimensional problems , we take
$\vec{u} = (u_x, 0, u_z)$
By using the Helmholtz decomposition;

$$\vec{u} = \nabla\phi + \nabla \times \vec{\psi} \tag{2.3}$$

Thus using equation (2.3) we get the displacement components as;

$$u_x = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z} \tag{2.4}$$

$$u_y = 0 \tag{2.5}$$

$$u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{2.6}$$

where ϕ and ψ are scalar and vector potential functions respectively representing the longitudinal and transverse waves.
On putting the equations (2.4) ,(2.5),(2.6)in equation (2.1), we get;

$$\nabla^2 \phi = \frac{1}{c_1^2}\frac{\partial^2 \phi}{\partial t^2} \tag{2.7}$$

$$\nabla^2 \psi = \frac{1}{c_2^2}\frac{\partial^2 \psi}{\partial t^2} \tag{2.8}$$

where,

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}$$

$$c_2^2 = \frac{\mu}{\rho}$$

Using equations (2.4) and (2.6) the stress displacement equations will be written in the form of potential functions as:

$$\tau_{xx} = \lambda\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) + 2\mu\left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z \partial x}\right) \tag{2.9}$$

$$\tau_{zz} = \lambda\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) + 2\mu\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z \partial x}\right) \tag{2.10}$$

$$\tau_{xz} = \mu\left(2\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}\right) \tag{2.11}$$

2.3 BOUNDARY CONDITIONS

The boundary conditions are satisfied at free $z = \pm d$ are
1) Normal stress is zero i.e.

$$\tau_{zz} = 0$$

2) Shear stress is zero i.e.

$$\tau_{xz} = 0$$

2.4 SOLUTION OF THE PROBLEM

For the above problem we take the solution of the form

$$\phi = \phi(z)e^{i\xi(x-ct)} \tag{2.12}$$

$$\psi = \psi(z)e^{i\xi(x-ct)} \tag{2.13}$$

where,
c is non dimensional phase velocity, ξ is the wave number.
Now taking the above solution of (2.12) in the equation (2.7), we get

$$\frac{d^2\phi(z)}{dz^2} + \left(\frac{\xi^2 c^2}{c_1^2} - \xi^2\right)\phi(z) = 0$$
$$\frac{d^2\phi(z)}{dz^2} + \alpha^2\phi(z) = 0$$

where

$$\alpha^2 = \xi^2\left(\frac{c^2}{c_1^2} - 1\right)$$

Therefore,

$$\phi(z) = A_1 \sin\alpha z + A_2 \cos\alpha z$$

Putting the above solution in equation (2.12), we get

$$\phi = (A_1 \sin\alpha z + A_2 \cos\alpha z)e^{i\xi(x-ct)} \tag{2.14}$$

where A_1 and A_2 are arbitrary constants
Similarly putting the solution of (2.13) in the equation (2.8), We get

$$\frac{d^2\psi(z)}{dz^2} + \left(\frac{\xi^2 c^2}{c_2^2} - \xi^2\right)\psi(z) = 0$$
$$\frac{d^2\psi(z)}{dz^2} + \beta^2\psi(z) = 0$$

where

$$\beta^2 = \xi^2\left(\frac{c^2}{c_2^2} - 1\right)$$

Therefore,

$$\psi(z) = A_3 \sin\beta z + A_4 \cos\beta z$$

Putting the above solution in equation (2.13), we get

$$\psi = (A_3 \sin\beta z + A_4 \cos\beta z)e^{i\xi(x-ct)} \tag{2.15}$$

where A_3 and A_4 are arbitrary constants.
The resulting potentials and displacements are:

$$\phi = (A_1 \sin\alpha z + A_2 \cos\alpha z)e^{i\xi(x-ct)}$$

$$\psi = (A_3 \sin\beta z + A_4 \cos\beta z)e^{i\xi(x-ct)}$$

$$u_x = i(\xi(A_1 \sin\alpha z + A_2 \cos\alpha z) - \beta(A_3 \cos\beta z - A_4 \sin\beta z))e^{i\xi(x-ct)} \tag{2.16}$$

$$u_z = (\alpha(A_1 \cos\alpha z - A_2 \sin\alpha z) + i\xi(A_3 \sin\beta z + A_4 \cos\beta z))e^{i\xi(x-ct)} \tag{2.17}$$

The stresses in terms of the potentials have been given by equations (2.9) - (2.11). Substituting equation (2.14) and (2.15) in these equations we get;

$$\tau_{zz} = ((-\lambda(\xi^2+\alpha^2) - 2\mu\alpha^2)(A_1 \sin\alpha z + A_2 \cos\alpha z) + 2i\mu\xi\beta(A_3 \cos\beta z - A_4 \sin\beta z))e^{i\xi(x-ct)} \tag{2.18}$$

$$\tau_{xz} = (2i\xi\alpha\mu(A_1 \cos\alpha z - A_2 \sin\alpha z) + \mu(\beta^2 - \xi^2)(A_3 \sin\beta z + A_4 \cos\beta z))e^{i\xi(x-ct)} \tag{2.19}$$

We note that the displacement equations (2.16) and (2.17) contain symmetric and anti-symmetric components. Thus for u_x, we see that the A_2 and A_3 terms gives symmetric displacements, while the A_1 and A_4 terms give antisymmetric displacements with respect to z=0. For u_z, we see that the A_2 and A_3 terms gives symmetric displacements, while the A_1 and A_4 terms give antisymmetric displacements with respect to z=0 respectively.

2.5 DERIVATION OF FREQUENCY EQUATION

Taking the boundary conditions at surface z=+d of the plate, we obtain a system of linear equations

symmetric modes:

Applying boundary conditions to equations (2.18) and (2.19) and letting $A_1 = A_4 = 0$, we obtain reduced equations as:

$$(\xi^2 - \beta^2)A_2 \cos \alpha d - 2\beta\xi A_3 \cos \beta d = 0 \tag{2.20}$$

$$2\xi\alpha A_2 \sin \alpha d + (\xi^2 - \beta^2)A_3 \sin \beta d = 0 \tag{2.21}$$

For symmetric modes, the equation will have non trivial solution if the determinant of coefficients of reduced equations (2.20) and (2.21) vanishes i.e.

$$\begin{vmatrix} (\xi^2 - \beta^2) \cos \alpha d & -2\beta\xi \cos \beta d \\ 2\xi\alpha \sin \alpha d & (\xi^2 - \beta^2) \sin \beta d \end{vmatrix} = 0$$

which gives the Rayleigh-Lamb frequency equation for the propagation of symmetric waves in plate as:

$$\frac{\tan \beta d}{\tan \alpha d} = \frac{-4\alpha\beta\xi^2}{(\xi^2 - \beta^2)^2} \tag{2.22}$$

antisymmetric modes:

Apply boundary conditions to equations (2.18) and (2.19) and letting $A_2 = A_3 = 0$, we obtain reduced equations as:

$$\left(\xi^2 - \beta^2\right)A_1 \sin \alpha d + 2\beta\xi A_4 \sin \beta d = 0 \tag{2.23}$$

$$2\xi\alpha A_1 \cos \alpha d - \left(\xi^2 - \beta^2\right)A_4 \cos \beta d = 0 \tag{2.24}$$

Similarly for antisymmetric modes, the equation will have non trivial solution if the determinant of coefficients from equations (2.23) and (2.24) vanishes i.e,

$$\begin{vmatrix} (\xi^2 - \beta^2) \sin \alpha d & 2\beta\xi \sin \beta d \\ 2\xi\alpha \cos \alpha d & -(\xi^2 - \beta^2) \cos \beta d \end{vmatrix} = 0$$

which gives the Rayleigh-Lamb frequency equation for the propagation of antisymmetric waves in plate.

$$\frac{\tan \beta d}{\tan \alpha d} = \frac{-(\xi^2 - \beta^2)^2}{4\alpha\beta\xi^2} \tag{2.25}$$

Thus, we combine the frequency equations (2.22) and (2.25) for symmetric and antisymmetric waves into single equation given as:

$$\frac{\tan \beta d}{\tan \alpha d} = \left(\frac{-4\alpha\beta\xi^2}{(\xi^2 - \beta^2)^2} \right)^{\pm 1} \tag{2.26}$$

where;

+1 is for symmetric mode and -1 is for antisymmetric mode.

$$\alpha^2 = \frac{\omega^2}{c_1^2} - \xi^2$$

and

$$\beta^2 = \frac{\omega^2}{c_2^2} - \xi^2$$

These equations are same as obtained and discussed in textbook by Karl F.Graff [19]

2.6 NUMERICAL RESULTS AND DISCUSSION

For the graphical representation of phase velocity with respect to wave number in the case of Lamb waves propagating in elastic plate we take the values of longitudinal and shear velocities as $c_1 = 5890m/s$ and $c_2 = 3260m/s$ respectively. The density of the steel plate i.e $7850kg/m^3$. The x-axis shows the product of wave number ξ and the plate thickness d. The y-axis shows the phase velocity c/c_2.

Lamb wave exhibits velocity dispersion i.e velocity depends upon frequency/wave number and on the material properties as well. It is observed that unlike an infinite medium where only two modes exist travelling at unique velocities(in graph only two modes are shown), the plate support infinite number of Lamb waves as well whose velocities depend upon the relationship between wave number and plate thickness. It is observed that zero order symmetric mode exist at every wave number with constant velocity whereas the higher order modes are more dispersive and the phase velocity decreases with the increase in wave number.

In comparison to symmetric mode the zero mode of antisymmetric mode in fig (2.3) is highly dispersive at the lower range of wave number and converges towards constant value for greater values of wave number. Again this mode exist for every wave number whereas the higher modes gets activated after a specified value of dimensionless wave number.

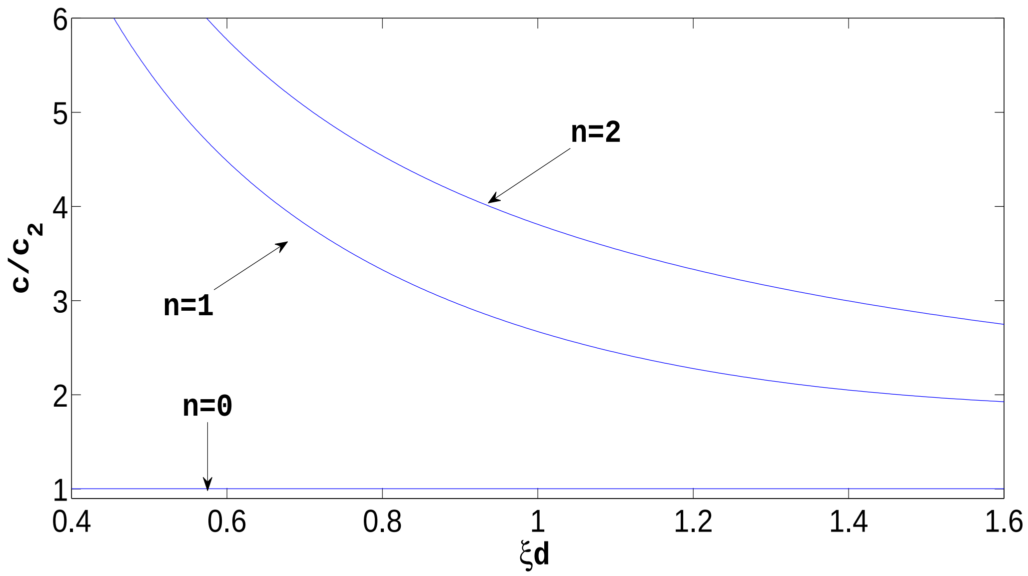

Figure 2.2: Dispersion curves for the symmetric modes with the variation of phase velocity with wave number

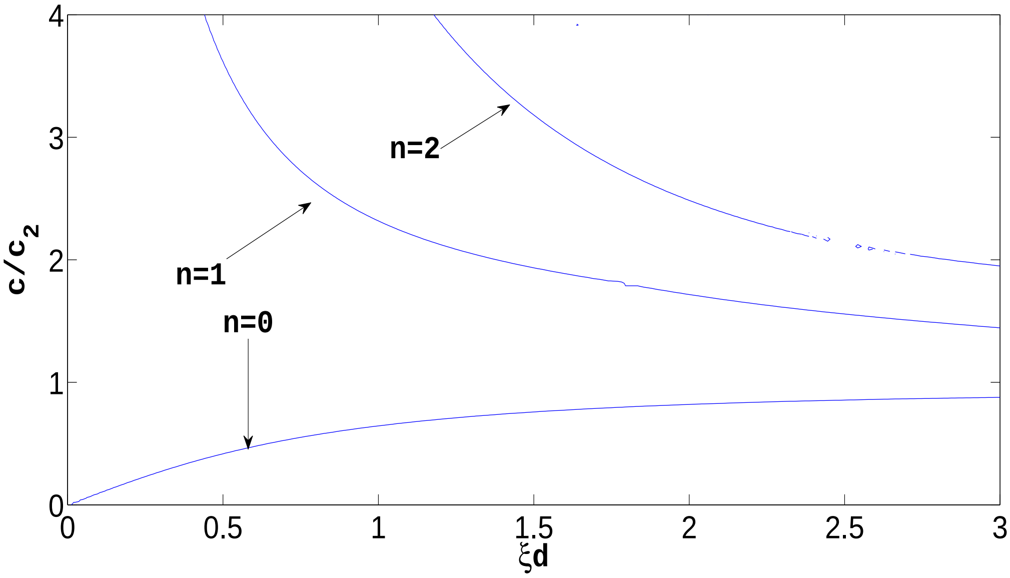

Figure 2.3: Dispersion curves for the antisymmetric modes with the variation of phase velocity with wave number

2.7 CONCLUSION

The symmetrical and antisymmetric zero-order modes have special importance. The symmetric zero-order mode is constant and at low frequency/wave number it is called extensional mode whereas for the antisymmetric zero-order mode is highly dispersive and at low frequency/wave number it is known as ”flexural mode”. These terms describes nature of the motion of particle and the elastic stiffness that governs the velocities of propagation. These two modes are very significant because they exist at all frequencies and they have greater energy than the higher-order modes in many practical conditions. These modes are comparatively easily identified, it is used as a technique of nondestructive testing using Lamb waves.

Chapter 3

PROPAGATION OF LAMB WAVE IN AN ELASTIC PLATE SANDWICHED BETWEEN FINITE LAYERS OF INVISCID LIQUID

3.1 INTRODUCTION

In this chapter, the work carried out in chapter2 is extended to include the study of lamb waves propagation in homogeneous elastic plate loaded with the inviscid liquid. Lamb waves propagate in a traction free plates, however the model containing an elastic plate in contact with fluid has practical application in NDT(non destructive testing) and biomedical fields. Lamb waves also have wide applications aerospace and industrial fields. For the detection of cracks, corrosion these waves are highly useful. It is also used in non destructive testing techniques in other materials. Schoch[20] also examined the various effects of inviscid loaded liquid on plates of propagation of Lamb waves. He also derived the dispersion relation for Leaky Lamb waves for an isotropic plates. According to him, only some frequency velocity pairs can propagate through it. Wu and Zhu [21] studied the propagation of Lamb waves in plates loaded with inviscid liquid layers on both sides. Further, Wu and Zhu[22] reconstructed the dispersion curves of Lamb waves of plate sandwiched between the inviscid liquid. Here, we will derive the frequency equation for propagation of Lamb wave in elastic plate sandwiched between the finite layers of inviscid liquid loading.

3.2 FORMULATION OF THE PROBLEM

Taking an infinite homogeneous elastic plate of width 2d. The top and bottom of plate is bordered with inviscid liquid layer of thickness H. Consider the origin (x, y, z) of coordinate system in middle of the plate. X-axis choose to be along the surface of medium whereas z-axis is downwards in positive direction. We assume that the solutions are explicitly independent of y coordinate.

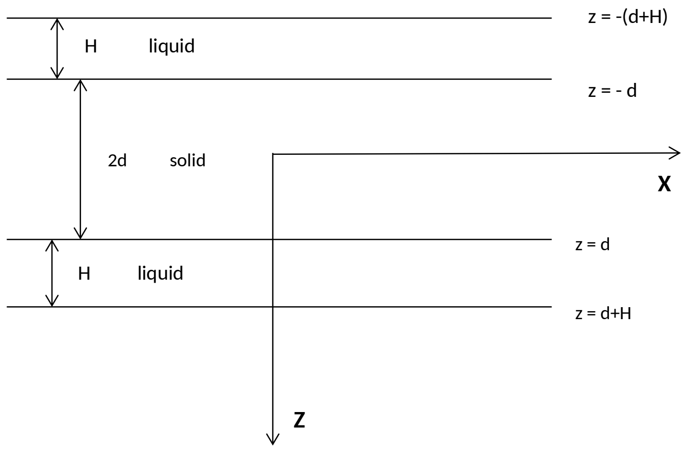

Figure 3.1: Geometry of the problem

The basic governing equation of motion and constitutive relation of classical theory of elasticity for an isotropic medium in the absence of body forces are given in Chapter-2 The basic governing equation in the case of liquid medium is of form,

$$\lambda_L \nabla(\nabla.\vec{u}_L) = \rho_L \ddot{\vec{u}}_L \tag{3.1}$$

$$\tau^L_{ij} = \lambda_L \rho_{ij} u^L_{k,k} \tag{3.2}$$

λ_L is Bulk modulus and ρ_L is density of liquid and τ^L_{ij} is stress tensor for liquid medium . By using the Helmholtz decomposition

$$u'_j = \nabla\phi_j + \nabla \times \vec{\psi_j} \tag{3.3}$$

j=1 is for bottom layer of inviscid liquid
j=2 is for top layer inviscid liquid

$$u'_1 = \frac{\partial \phi_1}{\partial x} - \frac{\partial \psi_1}{\partial z} \tag{3.4}$$

$$u'_2 = \frac{\partial \phi_2}{\partial x} - \frac{\partial \psi_2}{\partial z} \tag{3.5}$$

$$w'_1 = \frac{\partial \phi_1}{\partial z} + \frac{\partial \psi_1}{\partial x} \tag{3.6}$$

$$w_2^{'} = \frac{\partial \phi_2}{\partial z} + \frac{\partial \psi_2}{\partial x} \tag{3.7}$$

where ϕ_1 and ψ_1 are scalar and vector potential for bottom liquid layer. ϕ_2 and ψ_2 are scalar and vector potential for top liquid layer.
$u_1^{'}$, $u_2^{'}$, $w_1^{'}$ and $w_2^{'}$ are respectively x and z components of the particle displacement in respective layers of the liquid.
Also the inviscid liquid does not support the shear motion, hence $\psi_j = 0$, j=1,2.
The potential function equation for layers of the liquid:-

$$\nabla^2 \phi_j = \frac{\rho_L}{\lambda_L} \ddot{\phi}_L$$

Therefore,

$$\nabla^2 \phi_j = \frac{1}{c_L^2} \ddot{\phi}_L \tag{3.8}$$

$$c_L^2 = \frac{\lambda_L}{\rho_L}$$

The stress displacement equation will be written in form of potential vectors as:

$$\tau_{zz}^{L_j} = \lambda_L \left(\frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_j}{\partial z^2} \right) \tag{3.9}$$

where,
$j = 1$ for the bottom layer of inviscid liquid.
$j = 2$ for the top layer of inviscid liquid.

3.3 BOUNDARY CONDITIONS

The following boundary conditions are to be satisfied at solid liquid interface at z=+d.
1) The normal component of stress of the plate is equal to the normal component of stress of liquid i.e.

$$\tau_{zz}^{S} = \tau_{zz}^{L} \tag{3.10}$$

2) Liquid does not support the shear component so, the tangential component of the stress in the plate should be zero i.e.

$$\tau_{zx}^{S} = 0 \tag{3.11}$$

3) The displacement of vertical z-axis of solid plate is equal to the vertical displacement of liquid space i.e.

$$w^S = w^L \tag{3.12}$$

3.4 SOLUTION OF THE PROBLEM

Considering the solution of the equation (3.8) be of the form

$$\phi_j = \phi_j(z) e^{i\xi(x-ct)} \tag{3.13}$$

$$\frac{d^2\phi_j(z)}{dz^2} + \left(\frac{\xi^2 c^2}{c_L^2} - \xi^2\right)\phi_j = 0$$

$$\frac{d^2\phi_j}{dz^2} + \delta^2\phi_j = 0$$

where,

$$\delta^2 = \xi^2\left(\frac{c^2}{c_L^2} - 1\right)$$

Therefore,

$$\phi_1 = A_5 \sin\delta(z - (d+H)) e^{i\xi(x-ct)}, d < z < d+H \tag{3.14}$$

$$\phi_2 = A_6 \sin\delta(z + (d+H)) e^{i\xi(x-ct)}, -(d+H) < z < -d \tag{3.15}$$

By using the solutions of equations (3.14) and (3.15) and $\psi_1, \psi_2 = 0$, we get $w_1^{'}$ and $w_2^{'}$ as the x and z components of particle displacement in the bottom and top layers of liquid respectively from the equations (3.6) and (3.7) as:

$$w_1^{'} = \delta A_5 \cos\delta(z - (d+H)) e^{i\xi(x-ct)} \tag{3.16}$$

$$w_2^{'} = \delta A_6 \cos\delta(z + (d+H)) e^{i\xi(x-ct)} \tag{3.17}$$

where,
$w_1^{'}$ is particle displacement component for the bottom layer.

w_2' is particle displacement component for the top layer.
The stress equation in terms of potential has been written in equation (3.9) now substituting the solution from equation (3.14) and (3.15), we obtain it as:

$$\tau_{zz}^{L_1} = -\lambda_L\big[\big(\xi^2+\delta^2\big)A_5\sin\delta(z-(d+H))\big]e^{i\xi(x-ct)} \tag{3.18}$$

$$\tau_{zz}^{L_2} = -\lambda_L\big[\big(\xi^2+\delta^2\big)A_6\sin\delta(z+(d+H))\big]e^{i\xi(x-ct)} \tag{3.19}$$

3.5 DERIVATION OF FREQUENCY EQUATION

Imposing the boundary condition at the solid-liquid interfaces z=+d and using the equations (3.19), (3.20) of stresses of solid plate from chapter 2, we get:

$$\big(\mu(\xi^2-\beta^2)(A_1\sin\alpha d+A_2\cos\alpha d)-2\mu\xi\beta(A_3\cos\beta d-A_4\sin\beta d)-\lambda_L\big(\xi^2+\delta^2\big)A_5\sin(\delta H)\big)=0 \tag{3.20}$$

$$(2i\xi\alpha\mu(A_1\cos\alpha z-A_2\sin\alpha z)+\mu(\beta^2-\xi^2)(A_3\sin\beta z+A_4\cos\beta z))=0 \tag{3.21}$$

$$\delta A_5\cos\delta H+\alpha A_2\sin\alpha d-\alpha A_1\cos\alpha d-i\xi A_3\sin\beta-i\xi A_4\cos\beta d=0 \tag{3.22}$$

Considering the case of symmetric modes:
Taking $A_1 = A_4 = 0$ in the boundary equations and the equation will have non trivial solution if the determinant of coefficients of reduced equations vanishes as:

$$\begin{vmatrix} \mu(\xi^2-\beta^2)\cos\alpha d & 2i\mu\beta\xi\cos\beta d & -\lambda_L(\xi^2+\delta^2)\sin\delta H \\ -2i\xi\alpha\sin\alpha d & (\xi^2-\beta^2)\sin\beta d & 0 \\ \alpha\sin\alpha d & -i\xi\sin\beta d & \delta\cos\delta H \end{vmatrix} = 0$$

which gives the Rayleigh-Lamb frequency equation for the propagation of symmetric waves in inviscid liquid.
On expanding the above determinant we obtain the equation as:

$$\mu\delta(\beta^2-\xi^2)^2\tan\beta d+4\alpha\beta\delta\mu\xi^2\tan\alpha d-\lambda_L\alpha(\xi^2+\delta^2)(2\xi^2+(\beta^2-\xi^2))\tan\delta H=0 \tag{3.23}$$

Putting $\lambda_L = 0$ we get the frequency equation,

$$\frac{\tan\beta d}{\tan\alpha d} = \frac{-4\alpha\beta\xi^2}{(\xi^2-\beta^2)^2} \tag{3.24}$$

It is the same result as in equation (2.22) from chapter 2.
Now, considering the case of antisymmetric modes:
Taking $A_2 = A_3 = 0$ in the boundary equations and by applying the condition of the non trivial solution of equation and equating the determinant of coefficients to zero we get:

$$\begin{vmatrix} \mu(\xi^2-\beta^2)\sin\alpha d & -2i\mu\beta\xi\sin\beta d & -\lambda_L(\xi^2+\delta^2)\sin\delta H \\ 2i\xi\alpha\cos\alpha d & (\xi^2-\beta^2)\cos\beta d & 0 \\ -\alpha\cos\alpha d & -i\xi\cos\beta d & \delta\cos\delta H \end{vmatrix} = 0$$

which gives the Rayleigh-Lamb frequency equation for the propagation of antisymmetric waves in inviscid liquid.
On expanding the above determinant we obtain the equation as:

$$\mu\delta(\beta^2-\xi^2)^2\tan\beta d + 4\alpha\beta\delta\mu\xi^2\tan\alpha d - \lambda_L\alpha(\xi^2+\delta^2)(-2\xi^2+(\beta^2-\xi^2))\tan\delta H = 0 \quad (3.25)$$

Putting $\lambda_L = 0$ we get the frequency equation as:

$$\frac{\tan\beta d}{\tan\alpha d} = \frac{-(\xi^2-\beta^2)^2}{4\alpha\beta\xi^2} \quad (3.26)$$

It is the same result as in equation (2.25) from chapter 2.

3.6 NUMERICAL RESULTS AND DISCUSSION

The fluid medium used is inviscid liquid with $c_L = 1500m/s$ and density of liquid is taken as $\rho_L = 1000kg/m^3$.
The Lamb waves propagates in elastic plate (taken as steel plate) sandwiched between finite layers of inviscid liquid, we take values of longitudinal and shear velocities be $c_1 = 5890m/s$ and $c_2 = 3260m/s$. For the graphical representation of phase velocity with respect to wave number, the x-axis is taken as product of wave number ξ and the thickness d. The y-axis is taken as the dimensionless phase velocity. Figs (3.2) and (3.3) are drawn showing dispersion curves for symmetric and antisymmetric modes respectively. Here we take the thickness of the plate-liquid layer as $r = 0.01(H/d)$ (H is thickness of liquid layer and d is the thickness of the plate). It is observed that as the wave number increases, new modes start appearing. It is observed that magnitude of phase velocity of these profiles is high for lower wave number,which decreases at steady rate and becomes constant for higher wave numbers. In comparison to the phase velocity profiles obtained for elastic plate without liquid loading in figs (2.2) and (2.3) in chapter2, the phase velocity profiles gets suppressed due to presence of inviscid liquid loading in figs (3.2) and (3.3).

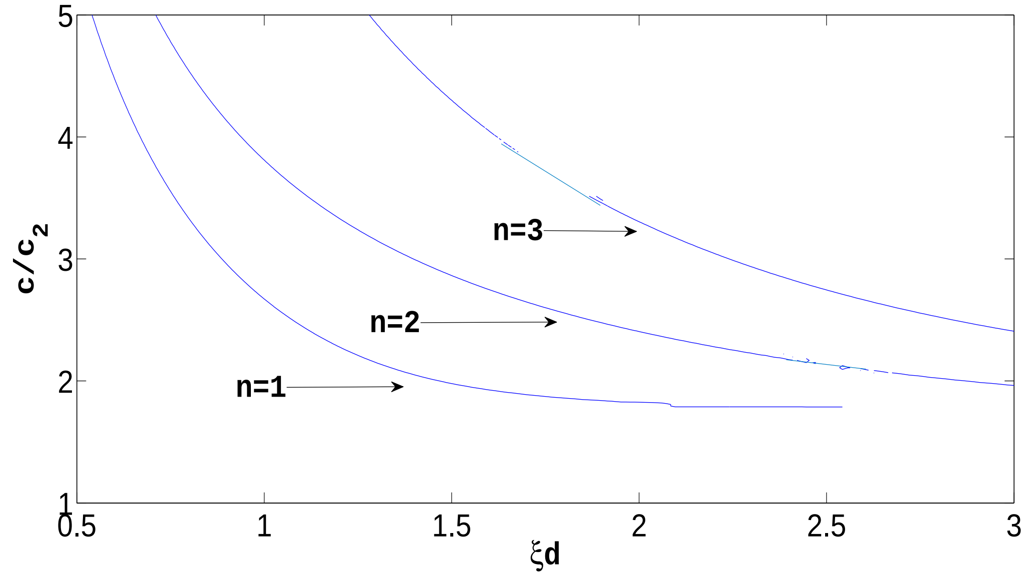

Figure 3.2: Variation of phase velocity profiles for the symmetric modes of Lamb waves in an elastic plate sandwiched between finite layers of inviscid liquid with the with wave number

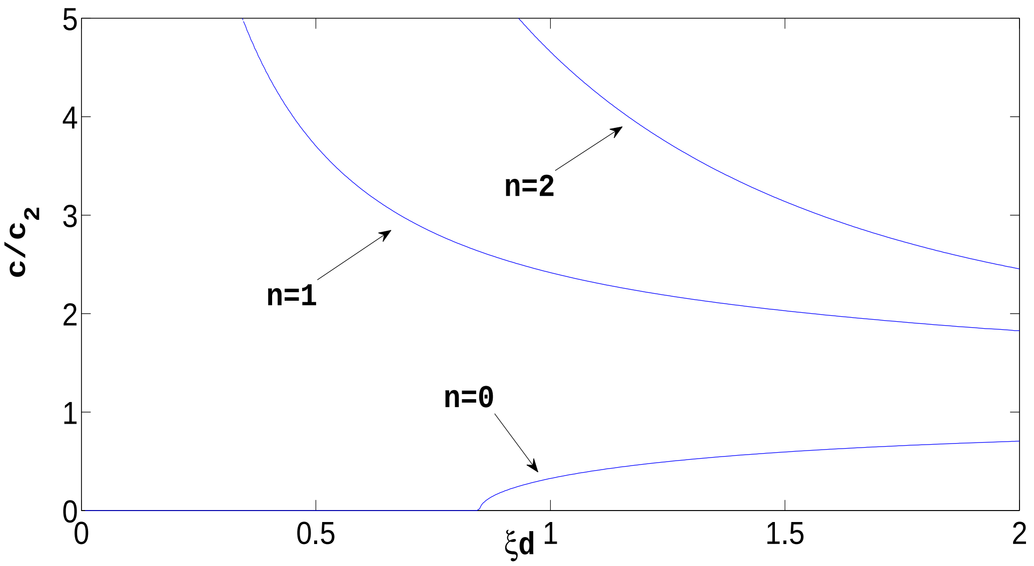

Figure 3.3: Variation of phase velocity profiles for the antisymmetric modes of Lamb waves in an elastic plate sandwiched between finite layers of inviscid liquid with the with wave number

3.7 CONCLUSION

The model of the problem consist of the elastic plate sandwiched between the finite layer of inviscid liquid, it has the vital use in the ultrasonic immersion testing of plates. The effect of the liquid loading has been studied on the propagation of phase velocity versus wave number of the Lamb wave. It is noticed that the phase velocity profiles gets depressed in comparison of the phase velocity profiles obtained in dispersion curves of Lamb waves in elastic plate without liquid loading in chapter2.

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