Integrability and L^1-convergence of certain Trigonometric Series

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Under the Guidance of

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Certificate

It is certified that the work contained in this thesis entitled **"Integrability and L^1−convergence of certain Trigonometric Series"** in partial fulfillment of the requirements for the award of degree of Master of Science in Mathematics and Computing to the School of Mathematics, Thapar University, Patiala is an authentic record of my own work studied under the supervision of Dr.Jatinderdeep Kaur.
The matter embodied in this thesis has not been submitted by me for the award of any other degree of this or any other University/Institute.

Shruti
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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

The present dissertation entitled, **"Integrability and L^1-convergence of Certain Trigonometric Series"**, contains a brief account of investigations carried out by me on L^1-convergence of Trigonometric Cosine Series under the supervision of **Dr. Jatinderdeep Kaur**, Assistant Professor, School of Mathematics and Computer Applications, Thapar University, Patiala.

The work presented in this dissertation has been divided into four chapters. The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in sequel, we have presented some known results. The purpose of chapter II is to study the integrability and L^1-convergence of Fourier cosine series. In chapter III, we studied the necessary and sufficient condition for the L^1-convergence of Rees-Stanojevic cosine sums.
In chapter IV, we have studied the L^1-convergence of Ram-Kumari modified cosine sum.
Towards the end, references of various publications cited in the present dissertation have been reported.

Table of Contents

Chapter 1

Introduction

The present thesis embodies certain results studied by the author on **"Integrability and L^1-convergence of Certain Trigonometric Series"**. It is well known that *if a trigonometric series converges in L^1-metric to a function $f \in L^1(T)$, then it is a Fourier series of the function f*. But the converse of this result does not hold good in L^1-metric *i.e.* there are many Fourier series which are not convergent in L^1-metric. In this concern, in 1932, Riesz{[4], Vol II, ChVIII article 22} gave a counter example $\left(\sum_{n=2}^{\infty} \frac{\cos nx}{\log n} \ is\ a\ Fourier\ series,\ but\ it\ doesn't\ converge\ in\ L^1 - metric\right)$. This has encouraged various researchers to carry the research on the topic *" Integrability and L^1-convergence of Certain Trigonometric Series "*.

Integrability and L^1-convergence of Fourier cosine series have been studied by number of authors. The work on this topic was initiated by Young, W.H.[38] in 1913 and Kolmogorov, A.N.[21] in 1923 by taking the classes of convex sequences $(\Delta^2 a_n \geq 0)$ and quasi convex sequences $(\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty)$ respectively.

In 1973, Teljakovskii, S.A. [33] studied another class S which was introduced by Sidon [28] in 1939 for L^1-convergence of certain cosine series. The results obtained by these authors were further generalized and extended by Hardy, G.H. and Littlewood, J.E.[13], Kano, T.[20], Garrett, J.W. and Stanojevic, C.V.([9],[10]), Ram, B.([24],[25]), Singh, N. and Sharma, K.M.([29],[30],[31]), Bojanic, R. and Stanojevic, C.V. [2], Chen, C.P. [7], Bala, R. and Ram, B.[3], Moricz, F. [23], Bhatia, S.S. and Ram, B.[6], Tomovski, Z.([34],[35],[36],[37]), Hooda, N. and Ram, B.[14], Kaur, K., Bhatia, S.S. and Ram, B.[15], Kaur, J. and Bhatia, S.S.([16],[17],[18]) and others by considering various generalizations of classes of sequences mentioned above for one dimensional trigonometric cosine and sine series.

During reviewing the literature, We found that many authors introduced modified trigonometric sums *"as these sums approximate their limits better than the classical trigonometric sums, since these sums converge in L^1-metric to the sum of trigonometric series whereas the classical series itself may not."* In this concern, various authors like Rees, C.S. and Stanojevic, C.V.[27], Chen, C.P.[8], Kumari, S. and Ram, B.[22], Ram, B. and Kumari, S.[26], Hooda, N. and Ram, B.[14], Kaur, K., Bhatia, S.S. and Ram, B.[19], Kaur, J. and Bhatia, S.S.([17],[18]) have introduced various new modified trigonometric cosine and sine sums and have studied their L^1-convergence under different classes of coefficient sequences.

To provide sufficient background for later chapters, a summary of basic concepts, techniques and a brief chapter wise resume of the results contained in the dissertation has been given in this introductory chapter. However, some of the definitions and notations will be repeated occasionally in chapters for the sake of convenience.

1.1 Definitions and Notations

Let $\{a_n\}$ be a sequence. Then we write

$$\Delta a_n = a_n - a_{n+1}$$

$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Definition 1.1.1. Null sequence A sequence $\{a_n\}$ is null sequence if

$$a_n \to 0 \ \ as \ \ n \to \infty.$$

Definition 1.1.2. Trigonometric series[4] A trigonometric series is a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx);$$

where $a_0, a_n{}'s, b_n{}'s$ are constants called coefficients of the series.

Example: $\sum_{n=1}^{\infty} \frac{\sin nx}{\log n}$ is a trigonometric sine series.

Definition 1.1.3. Piecewise continuous function A function f is piecewise continuous on the interval $[a, b]$ if

(i) The interval $[a, b]$ can be broken into finite number of subintervals

$$a = t_0 < t_1 < t_2 < ... < t_n = b,$$

such that f is continuous in each subinterval (t_i, t_{i+1}), for $i = 0, 1, 2, ..., n - 1$.

(ii) The function f has jump discontinuity at t_i, thus

$$|\lim_{t \to t_i^+} f(t)| < \infty, \quad i = 0, 1, 2, ..., n - 1;$$

$$|\lim_{t \to t_i^-} f(t)| < \infty, \quad i = 1, 2, 3, ..., n;$$

.

Definition 1.1.4. Periodic function A function f is said to be periodic with period P *(P being non zero constant)* if we have

$$f(x + P) = f(x);$$

for all values of x in the domain. If there exists a least positive constant P with this property, it is called the *fundamental period (also primitive period, basic period, or prime period)* of the function. A function with period P will repeat on intervals of length P and these intervals are referred to as *periods*.
Example: The *sine* function is periodic with period 2π, since

$$\sin(x + 2\pi) = \sin x$$

for all values of x. This function repeats on intervals of length 2π. Here 2π is the primitive period of sine function.

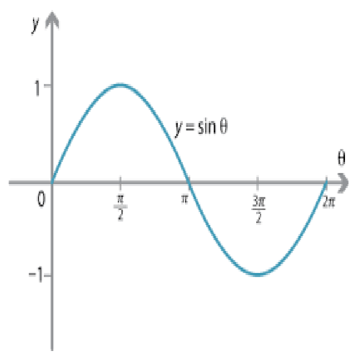

Figure 1.1: Graph of sine function

Definition 1.1.5. Fourier series A series of the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad x \in (\alpha, \alpha + 2\pi); \alpha > 0$$

is called Fourier series.
where

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int\limits_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int\limits_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx$$

where $a_0, a_n, b_n (n = 1, 2, 3, ...)$ are constants independent of x and are called Fourier coefficients.

Remark

Every Fourier series is a trigonometric series. But converse need not be true. As $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{\log n}$ is a trigonometric sine series but not a Fourier sine series.

Definition 1.1.6. O-o Relation[4] Let u_n and v_n be two sequences of real numbers. Then u_n be of order v_n

$$i.e.\ u_n = o(v_n) \ \ if \quad \lim_{n\to\infty} \frac{u_n}{v_n} \to 0, \ as \ n \to \infty$$

and if $\frac{u_n}{v_n}$ is bounded, then

$$u_n = O(v_n).$$

Example: Consider,

$$u_n = \frac{1}{n} \ \ and \ \ v_n = 1$$

It can be seen that

$$\frac{u_n}{v_n} \to 0 \ \ as \ \ n \to \infty$$

therefore,

$$u_n = o(1) \ \ as \ \ n \to \infty$$

Also,

$$0 \le u_n \le 1 \ \ \forall \ n \ \Rightarrow u_n = O(1).$$

Definition 1.1.7. L^1-**Convergence** Let $\{f_n\}$ be a sequence of integrable functions. Then f_n is said to be convergent in L^1-norm if for $\epsilon > 0$, there exists a positive integer N s.t.

$$\int_{-\pi}^{\pi} |f_n(x) - f(x)|dx < \epsilon \;\; \forall \; n \geq N.$$

Definition 1.1.8. Abel's transformation([4], Vol.I, p.1) If $a_0, a_1, a_2, ...$ and $v_0, v_1, v_2, ...$ are any real numbers and assume that

$$V_n = v_0 + v_1 + v_2 + ... + v_n.$$

Then, $\sum\limits_{m=0}^{n} a_m v_m = \sum\limits_{m=0}^{n-1} \Delta a_m V_m + a_n V_n$

where $\Delta a_m = a_m - a_{m+1}$

Definition 1.1.9. Dirichlet's kernel([4], Vol.I, p.85) The Dirichlet's kernel $D_n(t)$ is defined by

$$D_n(t) = \frac{1}{2} + \cos t + \cos 2t + \cdots + \cos nt$$

Multiply both sides by $2\sin\frac{t}{2}$,

$$2\sin\frac{t}{2}D_n(t) = \sin\frac{t}{2} + 2\sin\frac{t}{2}\cos t + \cdots + 2\sin\frac{t}{2}\cos nt$$

On solving,

$$2\sin\frac{t}{2}D_n(t) = \sin\left(n + \frac{1}{2}\right)t$$

$$D_n(t) = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}}.$$

Moreover,

If $t \neq 0(mod\ 2\pi)$, then

$$|D_n(t)| \leq \frac{\pi}{2t}, \quad for \quad 0 < |t| \leq \pi$$

Also, the uniform estimate is

$$|D_n(t)| \leq n + \frac{1}{2}, \quad for\ any\ t$$

and the estimate

$$L_n = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} |D_n(t)| dt \approx \frac{4}{\pi^2} \log n$$

for Lebesgue constant.

Definition 1.1.10. Conjugate Dirichlet's kernel([4], Vol.I, p.86) The conjugate Dirichlet kernel is defined by

$$\widetilde{D_n}(t) = \sin t + \sin 2t + \cdots + \sin nt$$

Multiply both sides by $2 \sin \frac{t}{2}$

$$2 \sin \frac{t}{2} \widetilde{D_n}(t) = 2 \sin \frac{t}{2} \sin t + 2 \sin \frac{t}{2} \sin 2t + \cdots + 2 \sin \frac{t}{2} \sin nt$$

On solving, we get

$$2 \sin \frac{t}{2} \widetilde{D_n}(t) = \cos \left(\frac{t}{2} \right) - \cos \left(n + \frac{1}{2} \right) t$$

$$\widetilde{D_n}(t) = \frac{\cos(\frac{t}{2}) - \cos(n + \frac{1}{2})(t)}{2 \sin(\frac{t}{2})}$$

It has properties

(i) $\widetilde{D_n}(t) \leq \frac{1}{\sin \frac{t}{2}}$

(ii) $\widetilde{L_n} = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} |\widetilde{D_n}(t)| dt \approx \log n$

Definition 1.1.11. Fejer's kernel([1], p.17.10) The Fejer's kernel is defined as

$$K_n(t) = \frac{1}{n+1} \sum_{j=0}^{n} D_j(t)$$

It has properties

(*i*) $K_n(t) \geq 0 \quad , \forall\ t$

(*ii*) $\frac{1}{\pi} \int\limits_{-\pi}^{\pi} K_n(t) dt = 1$

(*iii*) $K_n(t) \leq n+1 \quad \forall\ n.$

Definition 1.1.12. Conjugate Fejer kernel([1], p.17.10) The Conjugate Fejer Kernel is defined as

$$\widetilde{K_n}(t) = \frac{1}{n+1} \sum_{j=0}^{n} \widetilde{D_j}(t)$$

It has properties

$(i)\widetilde{K_n}(t) > 0,\ \ 0 < t < \pi,\ \ n = 1, 2, 3, ...$

$(ii)|\widetilde{K_n}(t)| < \frac{n}{2} \quad \forall\ n.$

Definition 1.1.13. Convex sequence([4], Vol.I, p.4) A sequence $\{a_n\}$ $(n = 0, 1, 2, 3...)$ is called *convex* if,

$$\Delta^2 a_n \geq 0$$

Definition 1.1.14. Quasi convex sequence[4] A sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty$$

Definition 1.1.15. Sequence of bounded variation([4], Vol. I, p.3) A sequence $\{a_n\}$ is said to be of bounded variation if

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty.$$

We denote the class of all null sequences of bounded variation as BV.

Definition 1.1.16. Class S[28] A sequence $\{a_n\}$ belongs to S if $a_n \to 0$ *as* $n \to \infty$ and there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all n.

The class S is usually called as Sidon-Teljakovskii class. Since, firstly, Teljakovskii expressed Sidon's conditions[28] in a succinct equivalent form and secondly, he showed that the class S is also a class of L^1-convergence.

Obviously, $S \subset BV$. Further, letting $A_n = \sum_{k=n}^{\infty} |\Delta^2 a_k|$

We observe that every quasi-convex null sequence satisfies the class S.

Let BV be the class of all null sequences of bounded variation. It is easily seen that

$$S \subset BV$$

Definition 1.1.17. The Class C of Garrett and Stanojevic[11] A null sequence $\{a_n\}$ belongs to the class C if for every $\epsilon > 0$, there exist a $\delta(\epsilon) > 0$, independent of n, and such that

$$C_n(\delta) = \int_0^{\delta} |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)| dx < \epsilon$$

for all $n \geq 0$, where D_k is the Dirichlet kernel.

Further in 1978, Fomin introduced a new class of coefficient sequence as follows:

Definition 1.1.18. Class F_p[12] A null sequence $\{a_n\}$ belongs to the class F_p , if for some $p > 1$,

$$\sum_{k=1}^{\infty} \left(\frac{\sum_{n=k}^{\infty} |\Delta a_n|^p}{k} \right)^{\frac{1}{p}} < \infty.$$

The class F_p is wider when p is closer to 1. Hence, Without loss of generality, we may assume that $1 < p \leq 2$. It can also be shown that

$$S \subset F_p \subset BV$$

In 1981, Stanojevic, C.V. gave more generalized classes of coefficient sequences as:

Definition 1.1.19. Class C_p[32] A null sequence $\{a_n\}$ belongs to the class C_p if for some $1 < p \leq 2$,

$$n\left(\frac{\sum\limits_{n=k}^{\infty}|\Delta a_n|^p}{k}\right)^{\frac{1}{p}} = o(1), k \to \infty$$

Definition 1.1.20. Class C_p^*[32] A null sequence $\{a_n\}$ belongs to the class C_p^* , if for some $1 < p \leq 2$,

$$\sum_{n=1}^{\infty} n^{p-1}|\Delta a_n|^p < \infty$$

Definition 1.1.21. Class P[32] A null sequence $\{a_n\}$ belongs to the class P if

$$\frac{1}{k}\sum_{n=1}^{k} n|\Delta a_n| = o(1), k \to \infty$$

1.2 Well Known Results

The following results about the behavior of cosine series are well known in literature

Theorem 1.2.1. *([4], [21], [38]) If $\{a_k\}$ is a quasi-convex null sequence, then*

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^1[0, \pi]. \tag{1.2.1}$$

In 1968, Kano[20] generalized Theorem 1.2.1 in the following form:

Theorem 1.2.2. *If $\{a_k\}$ is a null sequence such that*

$$\sum_{k=1}^{\infty} k^2|\Delta^2\left(\frac{a_k}{k}\right)| < \infty, \tag{1.2.2}$$

then (1.2.1) is a Fourier cosine series or equivalently represent integrable functions.

Concerning the integrability of trigonometric series belonging to the class S, Teljakovskii [33] established the following theorems:

Theorem 1.2.3. *Let the cosine series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

belongs to the class S. Then it is a Fourier series and the following relation holds:

$$\int_0^{\pi} |\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx| dx \leq C \sum_{k=0}^{\infty} A_k,$$

where C is an absolute constant.

We observe that Theorem 1.2.1 and Theorem 1.2.2 provide just only the sufficient conditions for the integrability of cosine series. Rees and Stanojevic [27] showed that $\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$ is a necessary and sufficient condition for $L^1[0, \pi]$ integrability but for a different type of cosine sums. They proved the following results:

Theorem 1.2.4. *Let $b_k = \frac{a_k}{k} \downarrow 0$. Then*

$$g(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left[\frac{b_k}{2} + \left(\sum_{j=k}^{n} b_j \right) \cos kx \right]$$

exists for $x \in (0, \pi]$ and $g \in L^1[0, \pi]$ if and only if $\sum_{k=1}^{\infty} b_k < \infty$.

Theorem 1.2.5. *Let $(k+1)|\Delta^2 a_k| \downarrow 0$. Then*

$$h(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left[\frac{1}{2}(k+1)|\Delta^2 a_k| + \left(\sum_{j=k}^{n} (j+1)|\Delta^2 a_j| \right) \cos kx \right]$$

exists for $x \in (0, \pi]$ and $h \in L^1[0, \pi]$ if and only if $\{a_k\}$ is quasi-convex.

In Chapter II, L^1-convergence of trigonometric cosine series with class S and class S_r have been studied.
The objective of Chapter III is to study the necessary and sufficient condition of Rees-Stanojevic Modified Cosine Sum under different classes of coefficient sequences.
The aim of chapter IV is to study the L^1-convergence of Ram-Kumari Modified cosine sum under the class $BV \cap C$.

Chapter 2

On Integrability and L^1-convergence of Fourier Cosine Series

2.1 Introduction

The objective of this chapter is to study the integrability and L^1-convergence of Fourier Cosine series under the classes S and S_r of coefficient sequences defined as follows:

Consider the cosine trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{2.1.1}$$

Let the partial sum of (2.1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

Definition 2.1.1. A null sequence $\{a_n\}$ is said to belong to class S, if

$$a_n \to 0 \ \ as \ n \to \infty \tag{2.1.2}$$

and there exists numbers A_n s.t.

$$A_n \downarrow 0, \qquad \sum_{n=0}^{\infty} A_n < \infty \tag{2.1.3}$$

$$and \;\; |\triangle a_n| \leq A_n \qquad \forall n \tag{2.1.4}$$

It can be shown by (2.1.3) that

$$nA_n \to 0 \;\; as \; n \to \infty. \tag{2.1.5}$$

Definition 2.1.2. A null sequence $\{a_k\}$ is said to belong to class S_r ; $r = 0, 1, 2, ...$ if there exists a sequence $\{A_k\}$ such that

$$A_k \downarrow 0, k \to \infty, \tag{2.1.6}$$

$$\sum_{k=0}^{\infty} k^r A_k < \infty, \quad r = 0, 1, 2, ... \tag{2.1.7}$$

and

$$|\Delta a_k| \leq A_k, \quad \forall\ k. \tag{2.1.8}$$

Clearly, $S_{r+1} \subset S_r, \ \forall\ r = 0, 1, 2, 3, ...$ where class $S_0 =$ class S
Note that by $A_k \downarrow 0, k \to \infty$ *and* $\sum\limits_{k=0}^{\infty} k^r A_k < \infty$, we have

$$k^{r+1} A_k = o(1), k \to \infty.$$

For $r = 0$, this class reduces to class S.

2.2 Lemmas

Lemma 2.2.1. *[33] Let the numbers $\alpha_i, \ i = 0, 1, 2, 3, \cdots, n$ satisfy the condition $|\alpha_i| < 1$. Then the inequality*

$$\int_0^{\pi} |\sum_{i=0}^{n} \alpha_i \frac{\sin{(i + \frac{1}{2})x}}{2 \sin{\frac{x}{2}}}| dx \leq C(n+1)$$

holds, where C is an absolute constant.

Proof. This relation was proved by Sidon with the aid of theorem about the force of symmetry of the Fourier series.
Fomin G.A. established the more general hypothesis. For the sake of completeness, the proof of this relation is given below:

As we know $D_i(x) = \frac{\sin\left(i+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$
and Dirichlet Kernel is bounded, therefore

$$\left|\frac{\sin\left(i+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right| \leq A$$

Now,

$$\int_0^\pi \left|\sum_{i=0}^{n} \alpha_i \frac{\sin(i+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| dx$$

$$\leq \int_0^\pi |A\sum_{i=0}^{n}(1)|dx$$

$$\leq A(n+1)\pi$$

$$= C(n+1).$$

□

Lemma 2.2.2. *Let $\{a_k\}$ be a sequence of real numbers such that $|a_k| \leq 1 \ \ \forall\ k$. Then there exists a constant $C > 0$ such that for any $n \geq 0$ and $r = 0, 1, 2, 3....$*

$$\int_0^\pi |\sum_{k=0}^{n} a_k {D_k}^r(x)|dx \leq C(n+1)^{r+1}.$$

Proof. We note that $\sum\limits_{k=0}^{n} a_k D_k(x)$ is a cosine trigonometric polynomial of order n. Apply first Berstein's inequality([5], Vol. II, p.11) and then using the above stated lemma, we have

$$\int_0^\pi |\sum_{k=0}^{n} a_k {D_k}^r(x)|dx \leq (n+1)^r \int_0^\pi |\sum_{k=0}^{n} a_k D_k(x)|dx \leq C(n+1)^{r+1},$$

where $C > 0$. □

2.3 Main Results

Considering the integrability of trigonometric cosine series whose coefficients satisfies condition S, the following theorem can be proved:

Theorem 2.3.1. *[33] Assume that the coefficients of the series* (1) *satisfies the condition S . Then the series (2.1.1) is a Fourier series and the following result holds:*

$$\int_0^\pi |\frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx| dx \leq C \sum_{n=0}^\infty A_n$$

where C is an absolute constant.

Proof. Given: The coefficients of the series satisfies condition S Therefore,

$$\sum_{n=0}^\infty |\Delta a_n| < \infty$$

hence, it is sufficient to prove the convergence of

$$I = \int_0^\pi |\frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx| dx$$

We have

$$\begin{aligned} I &= \int_0^\pi |\sum_{n=0}^\infty \Delta a_n \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}| dx \\ I &= \int_0^\pi |\sum_{n=0}^\infty A_n \frac{\Delta a_n}{A_n} \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}| dx \end{aligned}$$

Applying Abel's transformation, and using (2.1.4), we get

$$\begin{aligned} I &= \int_0^\pi |\sum_{n=0}^\infty (A_n - A_{n+1}) \sum_{i=0}^n \frac{\Delta a_i}{A_i} \frac{\sin(i+\frac{1}{2})x}{2\sin\frac{x}{2}}| dx \\ &\leq \sum_{n=0}^\infty (A_n - A_{n+1}) \int_0^\pi |\sum_{i=0}^n \frac{\Delta a_i}{A_i} \frac{\sin(i+\frac{1}{2})x}{2\sin\frac{x}{2}}| dx \end{aligned}$$

from this and by using the Lemma 2.2.1, we get

$$\begin{aligned}
I &\leq C\sum_{n=0}^{\infty}(A_n - A_{n+1})(n+1) \\
&= C\sum_{n=0}^{\infty}[n(A_n - A_{n+1}) + 1(A_n + A_{n+1})] \\
&= C([0 + (A_1 - A_2) + 2(A_2 - A_3) + 3(A_3 - A_4) + \cdots] + \\
&\qquad [(A_0 - A_1) + (A_1 - A_2) + (A_2 - A_3) + \cdots]) \\
&= C[A_0 + A_1 + A_2 + A_3 + \cdots] \\
&= C\sum_{n=0}^{\infty} A_n
\end{aligned}$$

. □

Theorem 2.3.2. *If (2.1.1) belongs to class* S*, then* $||S_n - f||_{L^1} = o(1)$ *as* $n \to \infty$ *iff* $a_{n+1}\log n \to 0$ *as* $n \to \infty$.

Proof. First consider,

$$||f - S_n|| = ||\sum_{k=n+1}^{\infty} a_k \cos kx||$$

On applying Abel's Transformation, we get

$$\begin{aligned}
||f - S_n|| &= ||\sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1}D_n(x)|| \\
&\leq ||\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)|| + ||a_{n+1}D_n(x)|| \\
&\leq \sum_{k=n+1}^{\infty} \Delta a_k \int_0^{\pi} |D_k(x)|dx + |a_{n+1}| \int_0^{\pi} |D_n(x)|dx \\
&\leq \sum_{k=n+1}^{\infty} kA_k + |a_{n+1}|\log n
\end{aligned}$$

By (2.1.5) and the given hypothesis, $||S_n - f||_{L^1} = o(1)$ *as* $n \to \infty$ iff $a_{n+1} \log n \to 0$ as $n \to \infty$.

$\square$

Theorem 2.3.3. *If (2.1.1) belongs to class* S_r*, then* $||f^r - {S_n}^r||_{L^1} = o(1)$*,* $n \to \infty$ *iff* $a_{n+1} n^r \log n = o(1)$ *as* $n \to \infty$*.*

Proof. Consider

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{2.3.1}$$

Taking r^{th} derivative to both sides, we get

$$f^r(x) = \sum_{k=1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)$$

and

$${S_n}^r(x) = \sum_{k=0}^{n} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right);$$

where ${S_n}^r(x)$ is the r^{th} derivative of n^{th} partial sum of (2.3.1).

Now consider,

$$||f^r - {S_n}^r|| = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)$$

Apply Abel's transformation, we get

$$\begin{aligned}
&= \ ||\sum_{k=n+1}^{\infty} \Delta a_k {D_k}^r(x) - a_{n+1} {D_n}^r(x)|| \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k {D_k}^r(x) - a_{n+1} {D_n}^r(x) \right| dx \\
&\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k {D_k}^r(x) \right| dx + \int_0^{\pi} |a_{n+1} {D_n}^r(x)| \, dx \\
&= O(\sum_{k=n+1}^{\infty} k^r A_k) + o(n^r |a_{n+1}| \log n) \\
&= o(1) + o(n^r |a_{n+1}| \log n)
\end{aligned}$$

So,$||f^r - {S_n}^r||_{L^1} = o(1)$, $n \to \infty$ iff $a_{n+1} n^r \log n = o(1)$ as $n \to \infty$. $\square$

Remark: For $r = 0$, the above result reduces to theorem 2.1.4.

Chapter 3

Necessary and Sufficient Condition for L^1-Convergence of Rees Stanojevic Cosine Sums

3.1 Introduction

Consider the cosine trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{3.1.1}$$

Let the partial sum of (3.1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$
Rees and Stanojevic [27] introduced modified cosine sums as

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_j)\cos kx$$

These modified cosine sums approximate their limits better than the classical cosine series as they converge in L^1-metric to the sum of the cosine series whereas the classical cosine series itself may not. The aim of this chapter is to study the Necessary and Sufficient condition of L^1-convergence of Rees and Stanojevic Modified Cosine Sum under different classes of coefficient sequences.

Definition 3.1.1. A null sequence $\{a_k\}$ belongs to the class C if for every $\epsilon > 0$, there exists a $\delta > 0$, independent of n and such that

$$C_n(\delta) = \frac{1}{\pi}\int_0^{\delta} |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| dx < \epsilon \quad \forall n \tag{3.1.2}$$

Definition 3.1.2. Sequence of bounded variation A sequence $\{a_n\}$ is said to be of bounded variation if

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty.$$

The class of all null sequences of bounded variation is denoted as BV.

3.2 Main Results

Theorem 3.2.1. *[11] Let*

$$g_n(x) = S_n(x) - a_{n+1}D_n(x) \tag{3.2.1}$$

where $\{a_k\} \in BV$. *Then prove that* $||g_n - f|| = o(1)$ *as* $n \to \infty$ *if and only if* $\{a_k\} \in$ C.

OR

Let $g_n(x) = S_n(x) - a_{n+1}D_n(x)$
Then g_n *converges to* f *in* L^1-*metric if and only if given* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that* $\int_0^{\delta} \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) < \epsilon$.

Proof. For the 'if' part
Let $\epsilon > 0$. Then there exists $\delta > 0$ $s.t.$

$$\int_0^{\delta} |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)| < \frac{\epsilon}{2} \quad \forall n \geq 0. \tag{3.2.2}$$

Then

$$\begin{aligned}
\int_0^\pi |f-g_n| &= \int_0^\pi |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)|dx \\
&= \int_0^\delta |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)|dx + \int_\delta^\pi |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)|dx \\
&< \frac{\epsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \int_\delta^\pi |D_k(x)|dx \qquad [from\ (3.1.2)] \\
&\leq \frac{\epsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \int_\delta^\pi \csc\frac{x}{2}dx \\
&= \frac{\epsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \left[-2\log|\csc\frac{\delta}{2} - \cot\frac{\delta}{2}|\right] < \epsilon
\end{aligned}$$

because for sufficiently large n since $\sum\limits_{k=0}^{\infty} |\Delta a_k| < \infty$.

Now for the 'only if' part, let $\epsilon > 0$. Then there exist an integer $M \ \ s.t.$

$$\int_0^\pi |f(x) - g_n(x)| < \frac{\epsilon}{2} \ \ if \ \ n \geq M$$

$$i.e. \ \ \int_0^\pi |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| < \frac{\epsilon}{2} \ \ if \ \ n \geq M$$

Now, if $\sum\limits_{k=0}^{M} |\Delta a_k| = 0$, then for $n > M$,

$$\int_0^\pi |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| < \frac{\epsilon}{2} < \epsilon$$

and for $0 \leq n \leq M$,

$$\int_0^\pi |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| = \int_0^\pi |\sum_{k=M+1}^{\infty} \Delta a_k D_k(x)| < \frac{\epsilon}{2} < \epsilon$$

If $\sum_{k=0}^{M} |\Delta a_k| \neq 0$, let $\delta = \frac{\epsilon}{2M} \sum_{k=0}^{M} |\Delta a_k|$

For $n \geq M$

$$\int_0^\delta |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| \leq \int_0^\pi |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| < \frac{\epsilon}{2} < \epsilon$$

For $0 \leq n < M$

$$\begin{aligned}
\int_0^\delta |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| &\leq \int_0^\delta |\sum_{k=n}^{M-1} \Delta a_k D_k(x)| + \int_0^\delta |\sum_{k=M}^{\infty} \Delta a_k D_k(x)| \\
&\leq \int_0^\delta \sum_{k=n}^{M-1} k|\Delta a_k| + \int_0^\pi |\sum_{k=M}^{\infty} \Delta a_k D_k(x)| \\
&< \delta \sum_{k=0}^{M-1} k|\Delta a_k| + \frac{\epsilon}{2} \\
&\leq \delta M \sum_{k=0}^{M-1} |\Delta a_k| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
&\Rightarrow \int_0^\delta |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| < \epsilon
\end{aligned}$$

So given $\epsilon > 0$, there exists a $\delta > 0$ such that $\int_0^\delta |\sum_{k=n}^{\infty} \Delta a_k D_k(x)| < \epsilon \quad \forall\ n \geq 0$ If $\lim_{n\to\infty} \int_0^\pi |f(x) - g_n(x)| = 0$ it is clear that $f \in L^1[0, \pi]$.

$$\int_0^\pi |f(x) \leq \int_0^\pi |f(x) - g_n(x)| + \int_0^\pi |g_n(x)| < \infty.$$

since $g_n(x)$ is a finite cosine sum. So the result holds. $\square$

3.3 Classes of L^1-convergence

In 1978, Fomin extended the class S and introduced a new class of coefficient sequence as follows:

Definition 3.3.1. Class F_p[32] A null sequence $\{a_n\}$ belongs to the class F_p , if for some $p > 1$,

$$\sum_{k=1}^{\infty} \left(\frac{\sum\limits_{n=k}^{\infty} |\Delta a_n|^p}{k} \right)^{\frac{1}{p}} < \infty.$$

Remark: It can be noted that class F_p is wider when p is closer to 1.
In this section, the necessary and sufficient condition for L^1- convergence of cosine series has been proved as follows:

Theorem 3.3.1. *Let* $1 < p \leq 2$. *Then*

$$F_p \subset C \cap BV$$

Proof. From (3.1.2) we have

$$C_n(\delta) \leq C_n(\pi) = C_n.$$

Hence, it suffices to show that from $\{a_k\} \in F_p$, it follows that $C_n = o(1),\ n \to \infty$. Before we proceed, we notice two consequences of $\{a_k\} \in F_p$ *i.e.*

$$f \in L^1(0, \pi), \tag{3.3.1}$$

$$\sum_{k=1}^{\infty} k^{p-1} |\Delta a_k|^p < \infty. \tag{3.3.2}$$

In the rest of the proof we shall show that (3.3.1) and (3.3.2) imply that $C_n = o(1),\ n \to \infty$.
Consider

$$C_{n+1} = \frac{1}{\pi} \int_0^{\pi} | \sum_{k=n+1}^{\infty} \Delta a_k D_k(x)| dx$$

$$= \frac{1}{\pi} \int_0^{\pi} |g_n(x) - f(x)| dx.$$

Because of $\{a_k\} \in BV$, f is the pointwise limit of S_n in $(0, \pi]$, and because of (3.2.1) f is also the pointwise limit of g_n in $(0, \pi]$. The integral C_{n+1} we split in the following way.

$$C_{n+1} = \frac{1}{\pi} \int_0^{\frac{1}{n}} |g_n(x) - f(x)| dx + \frac{1}{\pi} \int_{\frac{1}{n}}^{\pi} |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)| dx \qquad (3.3.3)$$

The first integral in (3.3.3) we estimate as

$$\frac{1}{\pi} \int_0^{\frac{1}{n}} |g_n(x) - f(x)| dx \leq \frac{1}{\pi} \int_0^{\frac{1}{n}} |\sigma_n(x) - f(x)| dx + \frac{1}{\pi} \int_0^{\frac{1}{n}} |g_n(x) - \sigma_n(x)| dx,$$

where $\sigma_n(x)$ is the Fejer's sum of s_n. It can be shown that

$$\frac{1}{\pi} \int_0^{\frac{1}{n}} |\sigma_n(x) - f(x)| dx = O(||\sigma_n - f||), \ n \to \infty.$$

From (3.2.1), we get

$$g_n(x) - \sigma_n(x) = \frac{1}{n+1} \sum_{k=1}^{n} k \Delta a_k D_k(x) - \frac{1}{n+1} \sum_{k=1}^{n} a_k D_k(x).$$

Hence,

$$\frac{1}{\pi} \int_0^{\frac{1}{n}} |g_n(x) - \sigma_n(x)| dx \leq \frac{1}{\pi} \frac{1}{n+1} \sum_{k=1}^{n} k |\Delta a_k| \int_0^{\frac{1}{n}} |D_k(x)| dx + \frac{1}{\pi} \frac{1}{n+1} \sum_{k=1}^{n} |a_k| \int_0^{\frac{1}{n}} |D_k(x)| dx$$

or

$$\frac{1}{\pi} \int_0^{\frac{1}{n}} |g_n(x) - \sigma_n(x)| dx = O\left(\frac{1}{n} \sum_{k=1}^{n} k |\Delta a_k| \right), \quad n \to \infty,$$

where the trivial $o(1)$ term is omitted. Altogether, for the first integral in (3.3.3) we have

$$\frac{1}{\pi} \int_0^{\frac{1}{n}} |g_n(x) - f(x)| dx = O\left(||\sigma_n - f||\right) + O\left(\frac{1}{n} \sum_{k=1}^{n} k |\Delta a_k| \right), \ n \to \infty.$$

The second integral in (3.3.3) , we write as

$$\begin{aligned} I_n &= \frac{1}{\pi}\int_{\frac{1}{n}}^{\pi} |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)| dx \\ &= \frac{1}{\pi}\int_{\frac{1}{n}}^{\pi} \frac{1}{2\sin\frac{x}{2}} \left|\sum_{k=n+1}^{\infty} \Delta a_k \sin\left(k+\frac{1}{2}\right)x\right| dx. \end{aligned}$$

Applying the Holder inequality we get

$$I_n \leq \frac{1}{\pi}\left(\int_{\frac{1}{n}}^{\pi} \frac{dx}{2^p \sin^p\frac{x}{2}}\right)^{\frac{1}{p}} \left(\int_{\frac{1}{n}}^{\pi} \left|\sum_{k=n+1}^{\infty} \Delta a_k \sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$ or

$$I_n \leq A_p((n+1)^{p-1})^{\frac{1}{p}} \left(\int_{\frac{1}{n}}^{\pi} \left|\sum_{k=n+1}^{\infty} \Delta a_k \sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{\frac{1}{q}}, \qquad (3.3.4)$$

(A_p, B_p *and* C_p are absolute constants depending on p).

For $\{a_k\} \in BV$ and fixed n, the sequence $\sum\limits_{k=n+1}^{N} \Delta a_k \sin\left(k+\frac{1}{2}\right)x$ converges uniformly to $\sum\limits_{k=n+1}^{\infty} \Delta a_k \sin\left(k+\frac{1}{2}\right)x,\ as\ N \to \infty$. Thus

$$\left(\int_{\frac{1}{n}}^{\pi} \left|\sum_{k=n+1}^{\infty} \Delta a_k \sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{\frac{1}{q}} = \lim_{n\to\infty} \left(\int_{\frac{1}{n}}^{\pi} \left|\sum_{k=n+1}^{N} \Delta a_k \sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{\frac{1}{q}},$$

Applying the Haussdorff-Young inequality to the last integral we get

$$\left(\int_{\frac{1}{n}}^{\pi} |\sum_{k=n+1}^{N} \Delta a_k \sin\left(k+\frac{1}{2}\right)x|^q dx\right)^{\frac{1}{q}} \leq B_p\left(\sum_{k=n+1}^{N} |\Delta a_k|^p\right)^{\frac{1}{p}}$$

For the integral (3.3.4) we now have the estimate

$$I_n \le C_p((n+1)^{p-1})^{\frac{1}{p}} \left(\sum_{k=n+1}^{\infty} |\Delta a_k|^p \right)^{\frac{1}{p}} \le C_p \left(\sum_{k=n+1}^{\infty} k^{p-1} |\Delta a_k|^p \right)^{\frac{1}{p}} \tag{3.3.5}$$

Combining all estimates we obtain

$$C_{n+1} = O(||\sigma_n - f||) + O\left(\frac{1}{n} \sum_{k=1}^{n} k|\Delta a_k| \right) + O\left(\sum_{k=n+1}^{\infty} k^{p-1} |\Delta a_k|^p \right), \ n \to \infty.$$

However the first term is $o(1)$ because of (3.3.1), the second because of $\{a_k\} \in BV$, and the third because of (3.3.2). Finally

$$C_n = o(1), \quad n \to \infty.$$

This completes the proof of the theorem. □

Remark: For some $1 < p \le 2$, let $\{a_k\} \in$ class F_p. Then (3.1.1) is a Fourier series of some $f \in L^1(0,\pi)$ and $||S_n - f|| = o(1), \quad n \to \infty \quad \Leftrightarrow \quad a_n \log n = o(1), \quad n \to \infty$.

Chapter 4

L^1- Convergence of Ram-Kumari Modified Cosine Sum

4.1 Introduction

Consider the cosine trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{4.1.1}$$

Ram and Kumari [22] introduced new modified cosine sum as

$$k_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \tag{4.1.2}$$

and studied its L^1- convergence under different classes of coefficient sequences. The aim of this chapter is to study the necessary condition for the integrability of (4.1.2)under the class $BV \cap C$ (*already defined in previous chapter*).

4.2 Lemma

Lemma 4.2.1. $||D_n{}^r(x)||_{L^1} = O(n^r \log n), r = 0, 1, 2, \cdots,$ *where* $D_n{}^r(x)$ *represents the* r^{th} *derivative of Dirichlet kernel.*

4.3 Main Result

Theorem 4.3.1. *Let* $\{a_k\} \in C \cap BV$. *Then* $||k_n - f|| = o(1),\ n \to \infty$ *if* $a_n \log n = o(1)\ , n \to \infty$

Proof.

$$\begin{aligned}
k_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx \\
k_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \cdots + \Delta\left(\frac{a_n}{n}\right)\right] \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left[\left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1}\right) + \left(\frac{a_{k+1}}{k+1} - \frac{a_{k+2}}{k+2}\right) + \cdots + \left(\frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right)\right] \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right] \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx \\
k_n(x) &= S_n(x) - \frac{a_{n+1}}{n+1}\widetilde{D_n'}(x)
\end{aligned}$$

where $\widetilde{D_n'}(x)$ is the first order derivative of conjugate Dirichlet's Kernel. Now, consider,

$$\begin{aligned}
\int_0^\pi |f(x) - k_n(x)|dx &= \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x) + \frac{a_{n+1}}{n+1}\widetilde{D_n'}(x)\right| dx \\
&\le \int_0^\pi \left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right|dx + \int_0^\pi \left|\frac{a_{n+1}}{n+1}\widetilde{D_n'}(x)\right|dx \quad (4.3.1) \\
&= I_1 + I_2
\end{aligned}$$

(4.3.2)

Let $\{a_k\} \in C$ then by definition, we have

For $\epsilon > 0$, there exists $\delta > 0$ such that $\int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \frac{\epsilon}{2}$ for all $n \geq 0$. Then

$$\begin{aligned}
I_1 &= \int_0^\pi |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)| dx \\
&= \int_0^\delta |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)| dx + \int_\delta^\pi |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)| dx \\
&< \frac{\epsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \int_\delta^\pi |D_k(x)| dx \quad [from\ 3.1.2] \\
&\leq \frac{\epsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \int_\delta^\pi \csc \frac{x}{2} dx \\
&= \frac{\epsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \left[-2 \log |\csc \frac{\delta}{2} - \cot \frac{\delta}{2}| \right] < \epsilon
\end{aligned}$$

because for sufficiently large n since $\{a_k\} \in BV$

$$\sum_{k=0}^{\infty} |\Delta a_k| < \infty. \tag{4.3.3}$$

By Lemma 4.2.1,

$$\begin{aligned}
I_2 &= \int_0^\pi |\frac{a_{n+1}}{n+1} \widetilde{\acute{D}_n}(x)| dx \leq \int_{-\pi}^{\pi} |\frac{a_{n+1}}{n+1} \widetilde{\acute{D}_n}(x)| dx \\
&= \frac{|a_{n+1}|}{n+1} \int_{-pi}^{\pi} |\widetilde{\acute{D}_n}(x)| dx \\
&= C|a_{n+1}| \int_{-\pi}^{\pi} |\widetilde{\acute{D}_n}(x)| dx \\
&\sim |a_{n+1}| \log n| \qquad (4.3.4) \\
& \qquad (4.3.5)
\end{aligned}$$

So the conclusion of the theorem follows from (4.3.2) and (4.3.3) .

□

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