GEOMETRY OF CR-SUBMANIFOLDS

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Submitted by

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DEDICATED

TO

GOD, MY PARENTS AND MY SUPERVISORS

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Geometry of CR-submanifolds**" in partial fulfillment of the requirements for the award of degree of **Master of Science**, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. M.A. Khan** and **Dr. Rajesh Kumar Gupta**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

(Manjot Kaur)

This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

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PREFACE

To study the geometry of a manifold, it is more convenient to first embed into a known manifold and then study its geometry. This approach gave impetus to study of submanifold which later developed into an independent and fascinating topic of study. The submanifold of an almost Hermitian manifold presents an interesting geometric study as its almost complex structure transform a vector into a vector perpendicular to it. Thus in turn, gives rise to two types of submanifold, namely invariant and anti-invariant submanifolds [1]. These are also known as holomorphic and totally real submanifold. These submanifolds were extensively studied by many differential geometrs. A Bejancu [1] in 1978 introduced the notion of CR-submanifolds of a Kaehler manifold which generalize the holomorphic and totally real submanifolds, after that B.Y. Chen [3] studied CR- submanifold of Kaehler manifold.

Chapter I is introductory and serves. The purpose of developing the basic concepts keeping in view the pre-requisities of the subsequent chapters.

Chapter II is the first technical chapter. Which deals with the CR-submanifold of a Kaehler manifold. It contains the results obtained by B.Y. Chen in his paper "CR-submanifold of a Kaehler manifold I" [3]. In this paper B.Y. Chen studied integrability conditions of the canonical distributions as well as the conditions for, their leaves to be totally geodesic.

Chapter III deals CR-submanifold of nearly Kaehler manifold which is the generalization of CR-submanifold, in more generalized setting namely nearly Kaehler, the results in this chapter are due to K.A. Khan et. al. [5].

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CHAPTER I

INTRODUCTION

The purpose of this chapter is to introduce basic concepts, preliminary notions and some fundamental results which we require for in development of the subject in the present dissertation thus we have given a brief resume of some of the results in the geometry of almost Hermitian manifolds and the allied structures and the geometry of submanifolds of their manifolds. Much though all the results are readily available in review articles and same in standard books e.g., Nomizu and Kobyashi [4], B.Y. Chen [2]. In this chapter we have also fix up our notations and terminology.

Topological Manifold(1.1.1): A topological manifold $\overline{M}$ of n-dimension is a Hausdorff topological space which is locally Euclidean i.e., $\forall\ x \in \overline{M}\ \exists$ a neighborhood $U\ \forall\ x \in \overline{M}$, and a homeomorphism ϕ of U onto an open set in R^n. The dimension R^n is called the dimension of manifold.

Chart(1.1.2): Let $\overline{M}$ be a set . A chart on $\overline{M}$ is a pair (ϕ, U) consisting of a subset U of $\overline{M}$ and a 1-1 map ϕ of U onto an open set in R^n.

Two charts (ϕ_1, U_1) and (ϕ_2, U_2) are said to be C^r- related, if either $U_1 \cap U_2 = \phi$ or $\phi_1 o \phi_2^{-1}$ and $\phi_2 o \phi_1^{-1}$ are C^r functions.

Atlas(1.1.3): The collection of all charts is called atlas.

Differentiable Manifold(1.1.4): Let (U_i, ϕ_i) and (U_j, ϕ_j) be two charts on $\overline{M}$ such that $U_i \cap U_j \neq \phi$. Any point $p \in U_i \cap U_j$ shall have two local coordinate system, $\phi_i(p) = (x^1, \ldots\ldots\ldots x^n)$, $\phi_j(p) = (y^1, \ldots\ldots\ldots y^n)$ say the two coordinate system are related as follows. We require that the mapping $\phi_j o \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ be differentiable of class C^k. Then we say that the atlas on $\overline{M}$ is a differentiable of class C^k. If $\alpha(U_i, \phi_i) y_i \in F$ is a maximal family of charts. $\overline{M}$ equipped with a maximal differentiable structure of class C^k is then called differentiable manifold of class C^k of

$\phi_j o \phi_i^{-1}$ is C^∞ function then $\overline{M}$ is called smooth manifold.

(i) $S^1 = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\}$ is a one – dimensional differentiable manifold.

(ii) S^2 (2 dimension sphere) is a differentiable manifold.

(iii) S^n is n - dimension differentiable manifold.

(iv) $R, R^2........R^n$ are differentiable manifold.

Tangent vector(1.1.5): A tangent vector X_m at $m \in \overline{M}$ is a real valued function on $C^\infty(m)$

i.e.,

$$X_m : C^\infty(m) \to IR \text{ such that}$$

(a) $X_m(af + bg) = aX_m f + bX_m g \qquad a, b \in IR$.

(b) $X_m(fg) = f(m)X_m g + g(m)X_m f$

where $f, g \in C^\infty(m)$ and $a, b \in IR$.

Tangent space(1.1.6): A tangent space $T_m\overline{M}$ to $\overline{M}$ at a point $m \in \overline{M}$ is the set of all tangent vectors of $m \in \overline{M}$.

Vector field(1.1.7): Vector field X on a manifold $\overline{M}$ is a smooth mapping which to each $m \in \overline{M}$ assigns a tangent vector $X_m \in T_m(\overline{M})$. Locally we can express $X = \xi^i \dfrac{\partial}{\partial x^i}$ where ξ^i are differentiable functions, moreover, for $f \in C^\infty(\overline{M})$. $Xf \in C^\infty(\overline{M})$ i.e.

$$Xf(m) = X_m f \qquad m \in \overline{M}.$$

Alternatively, we can define a vector field X on $\overline{M}$; a derivation of the algebra $C^\infty(m)$ i.e.,

$$X : C^\infty(\overline{M}) \to C^\infty(\overline{M}) \text{ such that}$$

(a) $X(af + bg) = aXf + bXg$

(b) $X(fg) = f(Xg) + g(Xf)$

for all $f, g \in C^\infty(\overline{M})$ and $a, b \in IR$.

Set of all vector fields is denoted by $D^{'}(\overline{M})$.

Sub-Manifold(1.1.8): If $\overline{M}$ and N satisfy the following two conditions, then N is called a submanifold of $\overline{M}$.

1. N is a subset of $\overline{M}$.
2. The identity map $i: N \to \overline{M}$ is an imbedding of N into $\overline{M}$.

For example $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is the one-dimensional submanifold of R^2.

Riemannian Manifold(1.1.9): If on a C^{∞} manifold $\overline{M}$, there exist a metric g on $T\overline{M}$ satisfying the following conditions

(a) g is (0,2) tensor. i.e., g is bilinear on $T\overline{M}$.

(b) $g(X,Y) = g(Y,X) \quad \forall\ X,Y \in T\overline{M}.$

(c) $g(X,Y) = 0 \quad \forall\ X \in T\overline{M} \Rightarrow Y = 0$

(d) $g(X,X) \geq 0,\ X \neq 0$

then g is called Riememannian metric and $(\overline{M}, g)$ is called Riemannian manifold.

Structures on manifold

Almost complex structure(1.1.10): An almost complex structure on a real differentiable manifold $\overline{M}$ is a tensor field J which is at every point $P \in \overline{M}$, an endomorphism of the tangent space $T_P(\overline{M})$, such that $J^2 = -I$ where I denotes the identity transformation. A manifold with a fixed almost complex structure is called an almost complex manifold.

On an almost complex manifold there always exist a Riemannian metric g consistent with the almost complex structure J i.e; satisfying

$$g(JU, JV) = g(U,V) \qquad \forall U,V \in T(\overline{M})$$

and here g is called Hermitian metric and an almost complex manifold with a Hermitian metric is called an almost Hermitian manifold.

Now we have an example of almost Hermitian manifold, an almost complex structure on R^2 is defined as

$$J : R^2 \to R^2$$
$$J(x, y) = (-y, x)$$
$$J(1,0) = (0,1)$$
$$J(0,1) = (-1,0)$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Hence R^2 is an almost complex manifold and J is almost complex structure.

Similarly, we can find it's examples in R^4, R^6 ……..

In general we have example of almost complex structure on R^{2n} i.e.,

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$$

so that (R^{2n} , J) is an almost complex manifold and obviously the Riemannian metric on R^{2n} is Hermitian and hence (R^{2n} , J , g) is almost Hermitian manifold.

Nijenhuis Tensor(1.1.11):

The Nijenhuis tensor S of J is defined as

$$S(U,V) = [U,V] + J[JU,V] + J[U,JV] - [JU,JV] \qquad (1.1.1)$$

it is easy to verify that S satisfies.

$$S(JU,V) = S(U,JV) = -JS(U,V) \qquad (1.1.2)$$

Proof:

$$S(JU,V) = [JU,V] + J[J^2U,V] + J[JU,JV] - [J^2U,JV]$$
$$= [JU,V] - J[U,V] + J[JU,JV] + [U,JV] \qquad (1.1.3)$$
$$S(U,JV) = [U,JV] + J[JU,JV] + J[U,J^2V] - [JU,J^2V]$$
$$= [U,JV] + J[JU,JV] - J[U,V] + [JU,V] \qquad (1.1.4)$$

from (1.1.3) and (1.1.4), we get

$$S(JU,V) = S(U,JV)$$

now we have to prove that $-JS(U,V) = S(U,JV) = S(JU,V)$

$$-JS(U,V) = -J\{[U,V] + J[JU,V] + J[U,JV] - [JU,JV]\}$$

$$= -J\{\nabla_U V - \nabla_V U + J\nabla_{JU} V - J\nabla_V JU + J\nabla_U JV - J\nabla_{JV} U - \nabla_{JU} JV + \nabla_{JV} JU\}$$

$$= [U,JV] + J[JU,JV] - J[U,V] + [JU,V]$$

$$= S(U,JV) = S(JU,V)$$

It is well known that vanishing of $S(U,V)$ is necessary and sufficient condition for an almost complex manifold to be a complex manifold.

Now, if we extend the Riemannian connexion to be a derivative on the tensor algebra of $\overline{M}$, $\overline{\nabla}$ of $\overline{M}$,then we have the formulae.

$$(\overline{\nabla}_U J)V = \overline{\nabla}_U JV - J\overline{\nabla}_U V \qquad (1.1.5)$$

Kaehler metric(1.1.12): A Hermitian metric on an almost complex manifold is called a **Kaehler metric** if the fundamental 2-form is closed i.e., $(\overline{\nabla}_U J)V = 0$. A complex manifold equipped with a Kaehler metric is said to be a Kaehler manifold. In other words, an almost complex manifold $\overline{M}$ is Kaehler manifold if

$$(\overline{\nabla}_U J)V = 0, \qquad \forall\ U,V \in T\overline{M}. \qquad (1.1.6)$$

In this case the connextion $\overline{\nabla}$ on $\overline{M}$ is said to be **Kaehlerian connextion.**

Nearly Kaehler(1.1.13): A Hermitian manifold $\overline{M}$ is said to be nearly Kaehler if

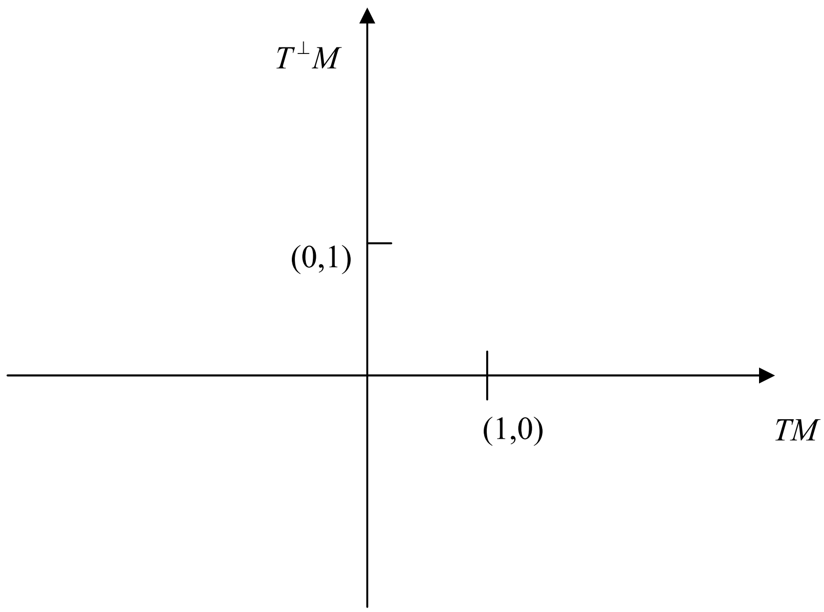
$$(\overline{\nabla}_U J)V + (\overline{\nabla}_V J)U = 0, \qquad \forall\ U,V \in T\overline{M} \qquad (1.1.7)$$

Submanifold theory(1.2)

Let M be the submanifold of manifold $\overline{M}$, then the tangent vectors of $T\overline{M}$ which are normal to TM form the normal bundle $T^{\perp}M$ of M. Hence the tangent space $T\overline{M}$ of $\overline{M}$ admits the following decomposition

$$T\overline{M} = TM \oplus T^{\perp}M$$

For example: In 2-dimensional manifold R^2.



(Figure 1.1)

here $TM = X$-axis $= \{(x,0) \mid x \in R\}$ and $T^{\perp}M = Y$-axis $= \{(0,y) \mid y \in R\}$.

Now if we take any element $(1,1)$ then we can write it as

$$(1,1) = (1,0) + (0,1)$$

where $(1,0)$ in TM and $(0,1)$ in $T^{\perp}M$

i.e; $$(1,1) = (1,0) + (0,1) \in TM \oplus T^{\perp}M .$$

The Riemannian connexion $\overline{\nabla}$ of $\overline{M}$ induces canonically the connexion ∇ and D on TM and on the normal bundle $T^{\perp}M$ respectively governed by the Gaues and Weingarten formulae v.i.z.

Guass formula:

$$\overline{\nabla}_U V = \nabla_U V + h(U,V) \tag{1.2.1}$$

Weingarten formula:

$$\overline{\nabla}_U \xi = -A_{\xi} U + D_U \xi \tag{1.2.2}$$

where U, V are tangent vector field on M and $\xi \in T^{\perp}M$, h and A_{ξ} are second fundamental forms,

$$A_\xi : T^\perp M \times TM \to TM$$

$$g(h(U,V),\xi) = g(A_\xi U,V) \qquad (1.2.3)$$

Totally geodesic submanifold(1.2.1)[2]: A submanifold for which the second fundamental form h is identically zero is called a totally geodesic submanifold.

Totally umbilical(1.2.2)[2]: A submanifold is called totally umbilical if its second fundamental form h satiafies

$$h(U,V) = g(u,v)H$$

Where H is called the mean curvature vector and defined as

$$H = \frac{1}{n}\sum_{i,j}^{n} h(e_i, e_j)$$

CR-submanifolds(1.2.3)[2]: On an almost Hermition manifold $\overline{M}$,

$$g(JU,JV) = g(U,V) \qquad (1.2.4)$$

for all vector fields U,V on $\overline{M}$. In other words,

$$g(JU,U) = 0$$

i.e., $JU \perp U$ for each vector field U on $\overline{M}$. Hence for a submanifold M of $\overline{M}$ if $U \in T_P M, JU$ may or may not belong to $T_P M$. Thus the action of the almost complex structure J on the tangent vectors of the submanifold of the almost Hermitian manifold gives rise to its classification into invariant and anti-invariant submanifold. These sub-manifolds are defined as follows.

Invariant submanifold(1.2.4)[2]: A submanifold M of an almost Hermitian manifold $\overline{M}$ is said to be **invariant** (or **holomorphic)** if

$$J(T_P M) = T_P M$$

for all $P \in M$.

Anti-Invariant (Totally real)(1.2.5)[2]: A submanifold M of an almost Hermitian manifold $\overline{M}$ is said to be anti-invariant if

$$J(T_p M) \subseteq T_p^{\perp} M \qquad \forall \;\; p \in M.$$

A new class of submanifold of an almost Hermitian manifolds of which the above classes namely invariant and anti-invariant (totally real) submanifolds are particular cases. This class of manifolds named as CR-submanifold.

Thus CR-submanifold provides a single setting to study the invariant and anti-invariant submanifolds of an almost Hermitian manifold.

CR-submanifold(1.2.6)[3]: A Riemannian submanifold is said to be a CR-submanifold of an almost Hermitian manifold $\overline{M}$ if there exists on M a C^{∞}- holomorphic distribution D such that its orthogonal complementary distribution $D^{\perp}$ is totally real i.e., $JD^{\perp} \subseteq T_p^{\perp} M$ for all $p \in M$.

(Clearly every real hypersurface M of an almost Hermitian manifold is a CR-submanifold if dim(M)>1.

Now we have an example of CR-submanifold.

$$J : R^4 \rightarrow R^4$$

$$J(x, y, z, t) = (-y, x, -t, z)$$

$$J(x, y, 0, 0) = (-y, x, 0, 0)$$

$$D = xy \text{ plane in } R^4.$$

$$D^{\perp} = z\text{-axis in } R^4$$

$$R^3 = D \oplus D^{\perp}$$

R^3 is a CR-submanifold.

Remark: We observe from the above definition that the dimension of D is always even and $JD^{\perp}$ being a sub bundle of $T^{\perp} M$, the normal bundle splits as,

$$T^{\perp} M = JD^{\perp} \oplus \upsilon$$

where υ is the complement of $JD^{\perp}$ in $T^{\perp}M$ and it is easy to verify that υ is invariant under J.

Proper CR-submanifold(1.2.7)[3]: A CR-submanifold M is said to be **proper** if neither D nor $D^{\perp} = \{0\}$. Obviously if $D = \{0\}$ then M is totally real submanifold and if $D^{\perp} = \{0\}$, then M is a holomorphic submanifold.

Anti-holomorphic(1.2.8)[3]: A CR-submanifold is called anti-holomorphic submanifold if

$$JD_p^{\perp} = T_p^{\perp}M \qquad \forall \ p \in M.$$

CR-Product(1.3)[3]: A CR- submanifold M is called a CR-product if it is locally a Riemannian product of a holomorphic submanifold M^T and a totally real submanifold $M^{\perp}$.

For a CR-product submanifold, the leaves of D and $D^{\perp}$ are totally geodesic in M and vice-versa. We know that the leaves of a distribution D on a manifold M are totally geodesic in M if and only if $\nabla_X Y \in D$ for all $X, Y \in D$. Thus in the setting of CR-submanifold of an almost Hermitian manifold, the leaves of D are totally geodesic in M if and only if

$$\nabla_X Y \in D \tag{1.3.1}$$

for all $X, Y \in D$. Which is equivalent to the conditions

$$\nabla_X W \in D^{\perp} \tag{1.3.2}$$

for $X \in D$ and $Z, W \in D^{\perp}$. Similarly for the geodesicness of the leaves of $D^{\perp}$, the conditions

$$\nabla_Z W \in D^{\perp} \tag{1.3.3}$$

and,

$$\nabla_Z X \in D \tag{1.3.4}$$

for $X \in D$ and $Z, W \in D^{\perp}$ are equivalent.

A CR-submanifold is a CR-product if and only if the leaves of D and $D^{\perp}$ are totally geodesic in M. Hence combining (1.3.1) and (1.3.4), we conclude that a CR-submanifold of an almost Hermitian manifold is a CR-product if and only if

$$\nabla_U X \in D \tag{1.3.5}$$

for all $U \in TM$ and $X \in D$. Similarly, combining (1.3.3) and (1.3.4), it is concluded that a CR-submanifold is a CR-product if and only if

$$\nabla_U Z \in D^{\perp} \tag{1.3.6}$$

for all $Z \in D^{\perp}$.Condition (1.3.5) and (1.3.6) are equivalent because

$$g(\nabla_U X, Z) = 0 \Leftrightarrow g(X, \nabla_U Z) = 0.$$

To prove (1.3.1) is equivalent to (1.3.2).

Let $\nabla_X Y \in D \qquad \forall\, X, Y \in D$

$$\nabla_X W \in D^{\perp} \qquad \forall\; X \in D \,\&\, W \in D^{\perp}.$$

Let $Z \in D^{\perp}$

$$g(\nabla_X Y, Z) = 0.$$

We know that $\nabla_X (g(Z,W)) = (\nabla_X g)(Z,W) + g(\nabla_X Z, W) + g(Z, \nabla_X W)$

therefore, $\qquad \nabla_X (g(Y,Z)) - g(Y, \nabla_X Z) - (\nabla_X g)(Y,Z) = 0$

$$0 - g(Y, \nabla_X Z) - 0 = 0$$

$$g(Y, \nabla_X Z) = 0$$

i.e., $\qquad \nabla_X Z \in D^{\perp} \quad \forall X \in D \,\&\, Z \in D^{\perp}$

so, (1.3.1) and (1.3.2) are equivalent.

Now to prove (1.3.3) and (1.3.4) are equivalent take $\nabla_Z W \in D^{\perp}$.

To prove $\nabla_Z X \in D$

$\nabla_Z W \in D^{\perp}$, let $X \in D$.

$$g(\nabla_Z W, X) = 0$$

$$\nabla_Z (g(W,X)) - g(W, \nabla_Z X) - (\nabla_X g)(W,X) = 0$$

$$0 - g(W, \nabla_Z X) - 0 = 0$$

$$g(W, \nabla_Z X) = 0$$

therefore, $\nabla_Z X \in D$

hence proved.

Now to prove

$$g(\nabla_U X, Z) = 0 \Leftrightarrow g(X, \nabla_U Z) = 0 \quad \forall U \in TM, \nabla_U X \in D, X \in D, Z \in D^{\perp}.$$

Let $g(\nabla_U X, Z) = 0$

$$\nabla_U (g(X,Z)) - g(X, \nabla_U Z) - (\nabla_U g)(Z, W) = 0$$

$$0 - g(X, \nabla_U Z) - 0 = 0$$

$$g(X, \nabla_U Z) = 0.$$

Now let $g(X, \nabla_U Z) = 0$

$$\nabla_U (g(X,Z)) - g(\nabla_U X, Z) - (\nabla_U g)(X, Z) = 0$$

$$0 - g(\nabla_U X, Z) - 0 = 0$$

$g(\nabla_U X, Z) = 0$, hence proved.

Mixed foliate(1.3.1)[3]: A CR-submanifold M in a Kaehler manifold is said to be mixed foliate if

(1) D is integrable.

(2) $h(D, D^{\perp}) = 0$.

For any vector field U tangent to M, we put

$$JU = PU + FU \tag{1.3.7}$$

where PU and FU are the tangential and normal components of JU respectively.

$$P : T(M) \to D, F : T^{\perp}(M) \to JD^{\perp}.$$

It is easy to observe that $PU \in D$ and $FU \in JD^{\perp}$.

Similarly, for any vector ξ normal to M, if we put

$$J\xi = t\xi + f\xi \tag{1.3.8}$$

with $t\xi$ and $f\xi$ as tangential and normal components of $J\xi$ respectively.

Some observations:

(1) For any $U \in TM, PU \in D$ and $FU \in JD^{\perp}$.

(2) For any $\xi \in T^{\perp}M, t\xi \in D^{\perp}$ and $f\xi \in \upsilon$.

Verification:

(1) For $U \in TM$ and $Z \in D^{\perp}$.

$$g(PU, Z) = g(JU, Z)$$

$$g(PU, Z) = -g(U, JZ)$$

as $JZ \in T^{\perp}M$

$$g(PU, Z) = 0$$

so, $$PU \in D.$$

Let $\xi \in \upsilon$.

$$g(FU, \xi) = g(JU, \xi)$$

$$g(FU, \xi) = -g(U, J\xi)$$

as $J\xi \in \upsilon$

$$g(FU, \xi) = 0$$

so, $$FU \in JD^{\perp}.$$

(2) For $\xi \in T^{\perp}M$ and $X \in D$.

$$g(t\xi, X) = g(J\xi, X)$$

$$g(t\xi, X) = -g(\xi, JX)$$

$$g(t\xi, X) = 0$$

so, $$t\xi \in D^{\perp}.$$

Now $\xi \in D^{\perp}$, let $Y \in JD^{\perp}$.

$$g(Y, f\xi) = g(Y, J\xi)$$

$$g(Y, f\xi) = -g(JY, \xi)$$

$$g(Y, f\xi) = 0$$

so, $$f\xi \in \upsilon$$

CHAPTER II

CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD

This chapter is devoted to the study of CR-submanifold of Kaehler manifold. All the results of this chapter are due to B.Y. Chen [3].

(2.1) Some basic results:

Result I: Let M be a CR-submanifold of Kaehler manifold then

$$J\nabla_U Z + Jh(U,Z) = -A_{JZ}U + D_U JZ.$$

For any $U \in TM$ and $Z \in D^{\perp}$.

Proof:

$$\overline{\nabla}_U JZ = (\overline{\nabla}_U J)Z + J(\overline{\nabla}_U Z)$$

now using Gauss and Weingarten formulae, we get

$$-A_{JZ}U + D_U JZ = J(\nabla_U Z + h(U,Z))$$

$$J\nabla_U Z + Jh(U,Z) = -A_{JZ}U + D_U JZ \tag{2.1.1}$$

Result II: $h(X,Y) = h(Y,X)$ for all $X, Y \in D.$

Proof: By Gauss formula, we obtain

$$\begin{aligned} h(X,Y) - h(Y,X) &= \overline{\nabla}_X Y - \nabla_X Y - \overline{\nabla}_Y X + \nabla_Y X \\ &= \overline{\nabla}_X Y - \overline{\nabla}_Y X + \nabla_Y X - \nabla_X Y \\ &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - (\nabla_X Y - \nabla_Y X) \\ &= [X,Y] - [X,Y] = 0 \end{aligned}$$

$$h(X,Y) - h(Y,X) = 0$$

i.e., $$h(X,Y) = h(Y,X)$$

hence prove the result.

Lemma(2.1.1)[3]: Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$ then

$$g(\nabla_U Z, X) = g(JA_{JZ}U, X) \tag{2.1.2}$$

$$A_{JZ}W = A_{JW}Z \tag{2.1.3}$$

$$A_{J\xi}X = -A_{\xi}JX \tag{2.1.4}$$

for all $X \in D; Z, W \in D^{\perp}; U \in TM$ and $\xi \in \upsilon$.

Proof: For equation (2.1.2), we have

$$J\nabla_U Z = -A_{JZ}U + D_U JZ - Jh(U,Z) \qquad \text{from (2.1.1)}$$

$$J^2\nabla_U Z = -JA_{JZ}U + JD_U JZ - J^2 h(U,Z)$$

$$-\nabla_U Z = -JA_{JZ}U + JD_U JZ + h(U,Z)$$

$$\nabla_U Z = JA_{JZ}U - JD_U JZ - h(U,Z)$$

$$g(\nabla_U Z, X) = g(JA_{JZ}U - JD_U JZ - h(U,Z), X)$$

$$= g(JA_{JZ}U, X) - g(JD_U JZ, X) - g(h(U,Z), X)$$

the last two terms in the right hand side vanish as the corresponding vector fields are perpendicular. Thus we get

$$g(\nabla_U Z, X) = g(JA_{JZ}U, X)$$

for all $X \in D$, this verifies equation (2.1.2).

Now for equation (2.1.3), we have

$$g(A_{JZ}W, U) = g(h(U,W), JZ)$$

$$= -g(Jh(U,W), Z)$$

using Gauss formula, we get

$$= -g(J(\bar{\nabla}_U W - \nabla_U W), Z)$$

$$= -g((J\bar{\nabla}_U W, Z) + g(J\nabla_U W, Z)$$

$$\left(J\bar{\nabla}_U W = -(\bar{\nabla}_U J)W + \bar{\nabla}_U JW = 0 + \bar{\nabla}_U JW = \bar{\nabla}_U JW\right)$$

$$= -g(\bar{\nabla}_U JW, Z) + g(J\nabla_U W, Z)$$

using Weingarten formula, we get

$$= -g(-A_{JW}U + D_U JW, Z) + g(J\nabla_U W, Z)$$

$$= g(A_{JW}U, Z) - g(D_U JW, Z) + g(J\nabla_U W, Z)$$

the last two terms in the right hand side vanish as the corresponding vector fields are perpendicular. Thus we get

$$g(A_{JZ}W,U) = g(A_{JW}Z,U)$$

for all $U \in TM$. This verifies equation (2.1.3).

For equation (2.1.4), we have

$$g(A_\xi JX,U) = g(h(JX,U),\xi) = g(h(U,JX),\xi)$$

$$= g(\overline{\nabla}_U JX - \nabla_U JX,\xi)$$

$$= g(\overline{\nabla}_U JX,\xi)$$

$$= g((\overline{\nabla}_U J)X + J\overline{\nabla}_U X,\xi)$$

since $\overline{M}$ is a Kaehler manifold and so $(\overline{\nabla}_U J)X = 0$.

$$= g(J\overline{\nabla}_U X,\xi)$$

$$= g(J\nabla_U X + Jh(U,X),\xi)$$

$$= g(Jh(U,X),\xi)$$

$$= -g(h(U,X),J\xi)$$

$$= -g(A_{J\xi}X,U)$$

$$\Rightarrow \qquad A_\xi JX = -A_{J\xi}X$$

hence, the Lemma is proved completely.

Lemma(2.2.2)[3]: Let M be a CR- submanifold of a Kaehler manifold $\overline{M}$. Then for any $Z,W \in D$, we have

$$D_W JZ - D_Z JW \in JD^\perp \qquad (2.1.5)$$

Proof: For $Z \in D^\perp$ and $\xi \in \upsilon$,by Weingarten formula, we have

$$\overline{\nabla}_Z J\xi = -A_{J\xi}Z + D_Z J\xi$$

$$g(\overline{\nabla}_Z J\xi,W) = g(-A_{J\xi}Z + D_Z J\xi,W)$$

$$= g(-A_{J\xi}Z,W) + g(D_Z J\xi,W)$$

$$= g(-A_{J\xi}Z,W)$$

$$g(\overline{\nabla}_Z J\xi, W) = g(-A_{J\xi}Z, W)$$

$$g(A_{J\xi}Z, W) = -g(\overline{\nabla}_Z J\xi, W)$$

from (1.1.5) and (1.1.6), we get

$$= -g(J\overline{\nabla}_Z \xi, W)$$

$$= g(\overline{\nabla}_Z \xi, JW)$$

$$= -g(\xi, \overline{\nabla}_Z JW)$$

$$g(\xi, D_Z JW) = -g(A_{J\xi}Z, W) \qquad (2.1.6)$$

thus using(2.1.6), we get

$$g(\xi, D_W JZ - D_Z JW) = g(\xi, D_W JZ) - g(\xi, D_Z JW)$$

$$= g(A_{J\xi}W, Z) - g(A_{J\xi}Z, W)$$

$$= g(A_{J\xi}W, Z) - g(h(Z, W), J\xi)$$

$$= g(A_{J\xi}W, Z) - g(h(W, Z), J\xi)$$

$$= g(A_{J\xi}W, Z) - g(A_{J\xi}W, Z)$$

$$= 0$$

Since this is true for all $\xi \in \upsilon$ and $Z, W \in D$, relation (2.1.5) holds.

Lemma(2.2.3)[3]: Let M be CR- submanifold of a Kaehler manifold $\overline{M}$. Then D is integrable iff

$$g(h(X, JY), JZ) = g(h(JX, Y), JZ)$$

for any vector field $X, Y \in D$ and $Z \in D^{\perp}$.

Proof: For any Kaehler manifold $\overline{M}$, we have $\overline{\nabla}J = 0$. If M is a CR- submanifold in $\overline{M}$, then from (2.1.1)

$$J\nabla_U Z + Jh(U, Z) = -A_{JZ}U + D_U JZ$$

for any $U \in TM$ and $Z \in D^{\perp}$, taking Riemannian product with JY (with $Y \in D$) in both sides of above equation and using equation (1.2.4), we get

$$g(J\nabla_U Z, JY) = g(-A_{JZ}U + D_U JZ - Jh(U, Z), JY)$$

$$= g(-A_{JZ}U, JY) + g(D_U JZ, JY) - g(Jh(U, Z), JY)$$

$$= g(-A_{JZ}U, JY)$$

$$= -g(A_{JZ}U, JY)$$

$$= -g(h(U, JY), JZ)$$

$$g(\nabla_U Z, Y) = -g(h(U, JY), JZ)$$

$$-g(Z, \nabla_U Y) = -g(h(U, JY), JZ)$$

$$g(Z, \nabla_U Y) = g(h(U, JY), JZ) \tag{2.1.7}$$

put $U = X$ in (2.1.7), we get

$$g(Z, \nabla_X Y) = g(h(X, JY), JZ) \tag{2.1.8}$$

put $U = Y$ and $Y = X$ in (2.1.7), we get

$$g(Z, \nabla_Y X) = g(h(Y, JX), JZ) \tag{2.1.9}$$

now subtracting (2.1.8) and (2.1.9), we get

$$g(Z, \nabla_X Y) - g(Z, \nabla_Y X) = g(h(X, JY), JZ) - g(h(Y, JX), JZ)$$

$$g(Z, [X, Y]) = g(h(X, JY) - (h(Y, JX), JZ)$$

If $g(h(X, JY), JZ) = g(h(Y, JX), JZ)$, then

$$g(Z, [X, Y]) = 0$$

$\Rightarrow$ $[X, Y] \in D$, hence D is integrable.

Conversely, let D is integrable, therefore $[X, Y] \in D$ then

$$g(Z, [X, Y]) = 0$$

$$g(h(X, JY), JZ) = g(h(Y, JX), JZ),$$

hence proved the Lemma.

Lemma(2.2.4)[3]: For a CR- submanifold M in a Kaehler manifold $\overline{M}$, the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M iff

$$g(h(D, D^{\perp}), JD^{\perp}) = 0 \tag{2.1.10}$$

Proof: For any $Z, W \in D^{\perp}, X \in D$,

$$g(\nabla_Z W, JX) = g(\overline{\nabla}_Z W - h(Z, W), JX)$$

$$= g(\overline{\nabla}_Z W, JX) - g(h(Z, W), JX)$$

$$= g(\overline{\nabla}_Z W, JX)$$

$$= -g(J\overline{\nabla}_Z W, X)$$

$$= -g(\overline{\nabla}_Z JW + (\overline{\nabla}_Z J)W, X)$$

$$= -g(\overline{\nabla}_Z JW, X)$$

$$= -g(-A_{JW}Z + D_Z JW, X)$$

$$= -g(-A_{JW}Z, X) - g(D_Z JW, X)$$

$$= g(A_{JW}Z, X) - g(D_Z JW, X)$$

$$= g(A_{JW}Z, X)$$

$$= g(h(Z, X), JW)$$

$$g(\nabla_Z W, JX) = g(h(Z, X), JW)$$

or, $$g(\nabla_Z W, JX) = g(h(D, D^{\perp}), JD^{\perp}) \qquad (2.1.11)$$

Now if we take $g(h(D, D^{\perp}), JD^{\perp}) = 0$

from equation (2.1.11), we get

$$g(\nabla_Z W, JX) = 0$$

$\Rightarrow$ $$\nabla_Z W \in D^{\perp}$$

$\Rightarrow D^{\perp}$ is totally geodesic.

Conversely, let $D^{\perp}$ is totally geodesic,

$\Rightarrow$ $$\nabla_Z W \in D^{\perp}$$

$\Rightarrow$ $$g(\nabla_Z W, JX) = 0$$

from equation (2.1.11), we get

$$g(h(X, Z), JZ) = 0$$

or, $$g(h(D, D^{\perp}), JD^{\perp}) = 0$$

which proves the Lemma.

Lemma(2.2.5)[3]: The leaves of the holomorphic distribution D on a CR-submanifold M of a Kaehler manifold $\overline{M}$ are totally geodesic in M iff

$$g(h(D, D), JD^{\perp}) = 0 \qquad (2.1.12)$$

Proof: For any $X, Y \in D$ and $Z \in D^{\perp}$.

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y - h(X,Y), Z)$$

$$= g(\overline{\nabla}_X Y, Z) - g(h(X,Y), Z)$$

$$= g(\overline{\nabla}_X Y, Z)$$

$$= g(J\overline{\nabla}_X Y, JZ)$$

$$= g(\overline{\nabla}_X JY - (\overline{\nabla}_X J)Y, JZ)$$

$$= g(\overline{\nabla}_X JY, JZ)$$

$$= g(\nabla_X JY + h(X, JY), JZ)$$

$$= g(\nabla_X JY, JZ) + g(h(X, JY), JZ)$$

$$= g(h(X, JY), JZ)$$

$$g(\nabla_X Y, Z) = g(h(X,Y), JZ) \tag{2.1.13}$$

now if we take $g(h(D,D), JD^{\perp}) = 0$, then

from equation (2.1.13), we get

$$g(\nabla_X Y, Z) = 0$$

$\Rightarrow$ $$\nabla_X Y \in D$$

$\Rightarrow D$ is totally geodesic.

Conversely, let D is totally geodesic.

$\Rightarrow$ $$\nabla_X Y \in D$$

$\Rightarrow$ $$g(\nabla_X Y, Z) = 0$$

from equation (2.1.13), we get

$$g(h(X,Y), JZ) = 0$$

or, $$g(h(D,D), JD^{\perp}) = 0,$$

this proves the Lemma.

Lemma(2.2.6)[3]: If (2.1.10) holds and D is integrable then for any $X \in D$ and $\xi \in JD^{\perp}$.

$$A_{\xi} JX = -JA_{\xi} X \tag{2.1.14}$$

Proof: As D is integrable, by Lemma (2.2.3)

$$g(h(X,JY),JZ)=g(h(JX,Y),JZ)$$

for $X,Y\in D$ and $Z\in D^{\perp}$,

$$g(A_{JZ}X,JY)=g(A_{JZ}JX,Y)$$

$$-g(JA_{JZ}X,Y)=g(A_{JZ}JX,Y)$$

$$g(A_{JZ}JX+JA_{JZ}X,Y)=0$$

now $g(A_{JD^{\perp}}D,D^{\perp})=0$ by (2.1.10), the equation yields,

$$A_{JZ}JX+JA_{JZ}X=0$$

$$A_{JZ}JX=-JA_{JZ}X$$

or,

$$A_{\xi}JX=-JA_{\xi}X$$

for all X in D and ξ in $JD^{\perp}$, this proves the Lemma.

(2.2) CR-Product in Kaehler manifold

Definition (2.2.1)[3]: A CR-submanifold M of a Kaehler manifold is called a CR-product if it is locally the Riemannian product of a holomorphic submanifold $M^{\perp}$ and a totally real submanifold $M^{\perp}$ of M.

Proposition(2.2.1)[3]:A CR- submanifold with integrable distribution, is a CR- product iff

$$A_{JD^{\perp}}D=0$$

Proof: By Lemma (2 .2.4) and (2.2.5), we get

$$g(h(D,D^{\perp}),JD^{\perp})=0$$

and,

$$g(h(D,D),JD^{\perp})=0$$

$$g(A_{JD^{\perp}}D,D^{\perp})=0$$

and,

$$g(A_{JD^{\perp}}D,D)=0$$

$$A_{JD^{\perp}}D=0$$

Theorem (2.2.1)[3] : A CR- submanifold M of a Kaehler manifold $\overline{M}$ is a CR- product iff $\nabla P = 0$.

Proof: As $\overline{M}$ is Kaehler manifold,

$$\overline{\nabla}_U JV = J\overline{\nabla}_U V$$

now we know that $JV = PV + FV$ and by Gauss formula, we get

$$\overline{\nabla}_U (PV + FV) = J(\nabla_U V + h(U,V))$$

$$\overline{\nabla}_U PV + \overline{\nabla}_U FV = J\nabla_U V + Jh(U,V)$$

now applying Gauss and Weingarten formulae, we get

$$\nabla_U PV + h(U,PV) - A_{FV}U + D_U FV = P\nabla_U V + F\nabla_U V + th(U,V) + fh(U,V)$$

now comparing tangential parts in both sides of the above equation, we get

$$\nabla_U PV = P\nabla_U V + A_{FV}U + th(U,V)$$

$$(\nabla_U P)V + P\nabla_U V = P\nabla_U V + A_{FV}U + th(U,V)$$

$$(\nabla_U P)V = A_{FV}U + th(U,V).$$

If P is parallel (i.e., $\nabla P = 0$) then from the above equation, we get

$$th(U,V) = -A_{FV}U$$

for any vector $U, V \in TM$. In particuler if $X \in D$, then $FX = 0$.

Putting $V = X$.

$$th(U,X) = 0$$

$$g(th(U,X),Z) = 0, \quad \text{for } Z \in D^{\perp}$$

$$g(Jh(U,X),Z) = 0$$

$$g(h(U,X),JZ) = 0$$

$$g(A_{JZ}X,U) = 0$$

$$\Rightarrow \qquad A_{JZ}X = 0$$

for any $Z \in D^{\perp}$ and $X \in D$.Thus by Lemma (2.2.3) and (2.2.5), D is integrable and the leaves are totally geodesic in M. Similarly on using Lemma (2.2.4) leaves of $D^{\perp}$ are totally geodesic in M, thus M is a CR- product.

Conversely, if M is CR- product then

$$g(\nabla_X Y, Z) = 0 \text{ and } g(\nabla_Z W, Y) = 0$$

for all X,Y in D and Z,W in $D^{\perp}$.

$$\nabla_X Y \in D \text{ and } \nabla_Z Y \in D$$

i.e.,
$$\nabla_U Y \in D$$

for all $U \in TM$ and $Y \in D$,

$$Jh(U,Y) = J(\overline{\nabla}_U Y - \nabla_U V)$$

$$= J\overline{\nabla}_U Y - J\nabla_U V$$

$$= \overline{\nabla}_U JY - \nabla_U JV$$

$$= h(U,JY)$$

from the tangential part, we get

$$(\nabla_U P)Y = th(U,Y) + A_{FY}U$$

$$(\nabla_U P)Y = 0.$$

Similarly as $\nabla_U Z \in D^{\perp}$, for any $Z \in D^{\perp}$ and $U \in TM$, we may prove

$$(\nabla_U P)Z = 0$$

so, if M is CR- product, then

$$\nabla_U P = 0.$$

This proves the theorem completely.

CR-submanifold with $\nabla F = 0$.

On a CR-submanifold M in a Kaehler manifold $\overline{M}$, there is a canonical normal bundle value 1-form F on TM and a tangent bundle value 1-form t on $T^{\perp}M$. In this section, we shall classify CR-submanifold with parallel F (on t).

Lemma(2.2.7)[3]: For any vector field U,V tangent to M and ξ normal to M, We have

$$(\nabla_U t)\xi = A_{f\xi}U - PA_{\xi}U \tag{2.2.1}$$

$$(\nabla_U f)\xi = -FA_{\xi}U - h(U,t\xi) \tag{2.2.2}$$

$$(\nabla_U F)V = fh(U,V) - h(U,PV) \tag{2.2.3}$$

Proof: Using Gauss and Weingarten formulae, we get

$$\begin{aligned}\overline{\nabla}_U J\xi &= \overline{\nabla}_U (t\xi + f\xi) \\ &= \overline{\nabla}_U t\xi + \overline{\nabla}_U f\xi \\ &= \nabla_U t\xi + h(U,t\xi) - A_{f\xi}U + D_U f\xi \qquad (2.2.4)\end{aligned}$$

In above equation we have also made use of equation (1.3.8). Again using equation (1.3.7), the left hand side of above equation simplifies to

$$\begin{aligned}\overline{\nabla}_U J\xi &= J\overline{\nabla}_U \xi \\ &= J(-A_\xi U + D_U \xi) \\ &= -JA_\xi U + JD_U \xi \\ &= -PA_\xi U - FA_\xi U + tD_U \xi + fD_U \xi \qquad (2.2.5)\end{aligned}$$

equation (2.2.4) and (2.2.5), we get

$$-PA_\xi U - FA_\xi U + tD_U \xi + fD_U \xi = \nabla_U t\xi + h(U,t\xi) - A_{f\xi}U + D_U f\xi$$

comparing the tangential and normal components of both sides in the above equation, we get

$$\nabla_U t\xi - tD_U \xi = A_{f\xi}U - PA_\xi U \qquad (2.2.6)$$

$$D_U f\xi - fD_U \xi = -FA_\xi U - h(U,t\xi) \qquad (2.2.7)$$

i.e.,

$$(\nabla_U t)\xi = A_{f\xi}U - PA_\xi U$$

$$(\nabla_U f)\xi = -FA_\xi U - h(U,t\xi).$$

This proves the equation (2.2.1) and (2.2.2).

Now for any U, V tangent to M. Gauss and Weingarten equation yield,

$$\overline{\nabla}_U JV = \overline{\nabla}_U (PV + FV)$$

$$\overline{\nabla}_U JV = \overline{\nabla}_U PV + \overline{\nabla}_U FV$$

$$\overline{\nabla}_U JV = \nabla_U PV + h(U,PV) - A_{FV}U + D_U FV \qquad (2.2.8)$$

the left hand side on using (1.3.7),(1.3.8) and the fact that $\overline{M}$ is Kaehler. We get

$$\begin{aligned}\overline{\nabla}_U JV &= J\overline{\nabla}_U V \\ &= J(\nabla_U V + h(U,V)) \\ &= J\nabla_U V + Jh(U,V)\end{aligned}$$

$$\overline{\nabla}_U JV = P\nabla_U V + F\nabla_U V + th(U,V) + fh(U,V) \tag{2.2.9}$$

from (2.2.8) and (2.2.9), we get

$$P\nabla_U V + F\nabla_U V + th(U,V) + fh(U,V) = \nabla_U PV + h(U,PV) - A_{FV}U + D_U FV$$

thus, on comparing the normal parts, we get

$$D_U FV - F\nabla_U V = fh(U,V) - h(U,PV)$$

i.e., $$(\nabla_U F)V = fh(U,V) - h(U,PV).$$

This proves the equation (2.2.3).

Hence proves the Lemma.

Proposition(2.2.2)[3]: Let M be a CR-submanifold of a Kaehler manifold $\overline{M}$. Then $\nabla F = 0$ iff $\nabla t = 0$.

Proof: First take $\nabla t = 0$, we have to show $\nabla F = 0$.

From Lemma (2.2.7), we see that if $\nabla t = 0$ for any U,V tangent to M and ξ normal to M, then

$$A_{f\xi}U - PA_\xi U = 0$$

$$g(A_{f\xi}U,V) = g(PA_\xi U,V)$$

$$g(h(U,V), f\xi) = -g(A_\xi U, PV)$$

$$g(h(U,V), f\xi) = -g(h(U,PV),\xi)$$

$$-g(fh(U,V),\xi) = -g(h(U,PV),\xi)$$

$$fh(U,V) = h(U,PV)$$

$$fh(U,V) - h(U,PV) = 0$$

therefore by (2.2.3), we get

$$(\nabla_U F) = 0$$

i.e., $$\nabla F = 0$$

Conversely, let $\nabla F = 0$, we have to show $\nabla t = 0$.

From Lemma (2.2.7), we see that if $\nabla F = 0$ for any U,V tangent to M and ξ normal to M, then

$$h(U,PV) = fh(U,V)$$

$$g(h(U, PV), \xi) = g(fh(U, V), \xi)$$

$$g(h(U, PV), \xi) = -g(h(U, V), f\xi)$$

$$g(A_\xi U, PV) = -g(A_{f\xi} U, V)$$

$$-g(PA_\xi U, V) = -g(A_{f\xi} U, V)$$

$$g(PA_\xi U, V) = g(A_{f\xi} U, V)$$

i.e., $$PA_\xi U = A_{f\xi} U$$

$$A_{f\xi} U - PA_\xi U = 0$$

therefore by equation (2.2.1), we get

$$(\nabla_U t)\xi = 0$$

i.e., $$\nabla t = 0 .$$

This proves the proposition.

CHAPTER III

CR-SUBMANIFOLD OF ALMOST HERMITIAN MANIFOLDS

(3.1) Integrability conditions of distribution on a CR- submanifold of Hermitian manifold.

Let $\overline{M}$ be almost Hermitian manifold and M be a CR- submanifold of $\overline{M}$. Then for $U,V \in TM$.

$$(\overline{\nabla}_U J)V = \overline{\nabla}_U JV - J\overline{\nabla}_U V \tag{3.1.1}$$

making use of equation (1.2.1),(1.2.2),(1.3.7) and (1.3.8) the above equation can be simplified as

$$\begin{aligned}(\overline{\nabla}_U J)V &= \overline{\nabla}_U JV - J\overline{\nabla}_U V \\ &= \overline{\nabla}_U (PV + FV) - J(\nabla_U V + h(U,V) \\ &= \overline{\nabla}_U PV + \overline{\nabla}_U FV - J\nabla_U V - Jh(U,V) \\ &= \nabla_U PV + h(U,PV) - A_{FV}U + D_U FV - P\nabla_U V \\ &\quad - F\nabla_U V - th(U,V) - fh(U,V)\end{aligned}$$

now comparing tangent and normal part of $(\overline{\nabla}_U J)V$ and denoting by $P(U,V)$ and $Q(U,V)$ respectively.

$$\begin{aligned}P(U,V) &= \nabla_U PV - A_{FV}U - P\nabla_U V - th(U,V) \\ &= \nabla_U PV - P\nabla_U V - A_{FV}U - th(U,V) \\ &= (\nabla_U P)V - A_{FV}U - th(U,V)\end{aligned}$$

$$P(U,V) = (\nabla_U P)V - A_{FV}U - th(U,V) \tag{3.1.2}$$

$$\begin{aligned}Q(U,V) &= h(U,PV) + D_U FV - F\nabla_U V - fh(U,V) \\ &= (\nabla_U F)V + h(U,PV) - fh(U,V)\end{aligned}$$

$$Q(U,V) = (\nabla_U F)V + h(U,PV) - fh(U,V) \tag{3.1.3}$$

now take $\xi \in T^{\perp}M$ and $U \in TM$ then

$$(\overline{\nabla}_U J)\xi = \overline{\nabla}_U J\xi - J\overline{\nabla}_U \xi$$

$$= \overline{\nabla}_U (t\xi + f\xi) - J(-A_\xi U + D_U \xi)$$

$$= \overline{\nabla}_U t\xi + \overline{\nabla}_U f\xi + JA_\xi U - JD_U \xi$$

$$= \nabla_U t\xi + h(t\xi, U) - A_{f\xi} U + D_U f\xi + PA_\xi U$$

$$+ FA_\xi U - tD_U \xi - fD_U \xi$$

now comparing tangent and normal part of $(\overline{\nabla}_U J)\xi$ and denoting by $P(U,\xi)$ and $Q(U,\xi)$ respectively.

$$P(U,\xi) = \nabla_U t\xi - A_{f\xi} U + PA_\xi U - tD_U \xi$$

$$= (\nabla_U t)\xi - A_{f\xi} U + PA_\xi U$$

$$P(U,\xi) = (\nabla_U t)\xi - A_{f\xi} U + PA_\xi U \tag{3.1.4}$$

$$Q(U,\xi) = h(t\xi, U) + D_U f\xi + FA_\xi U - fD_U \xi$$

$$= (\nabla_U f)\xi + h(t\xi, U) + FA_\xi U$$

$$Q(U,\xi) = (\nabla_U f)\xi + h(t\xi, U) + FA_\xi U \tag{3.1.5}$$

The following properties of P and Q are used in our subsequent discussion.

(P_1) $P(U+V,W) = P(U,W) + P(V,W)$

$Q(U+V,W) = Q(U,W) + Q(V,W)$

(P_2) $P(U,V+W) = P(U,V) + P(U,W)$

$Q(U,V+W) = Q(U,V) + Q(U,W)$

(P_3) $g(P(U,V),W) = -g(V,P(U,W))$

(P_4) $g(Q(U,V),\xi) = -g(V,Q(U,\xi))$

(P_5) $P(U,JV) + Q(U,JV) = -J((P(U,V) + Q(U,V))$

for all U,V and W in TM and ξ in $T^\perp M$. We may now use P and Q to obtain the following integrability conditions of the distribution D and $D^\perp$.

Proposition(3.1.1)[5]: Let M be a CR- submanifold of an almost Hermitian manifold $\overline{M}$. Then the holomorphic distribution D is integrable if and only if.

$$Q(X,Y)-Q(Y,X)=h(X,JY)-h(JX,Y) \text{ for each } X,Y\in D \tag{3.1.6}$$

Proof: For $\xi \in T^{\perp}M$, we have

$$g(\overline{\nabla}_X JY-\overline{\nabla}_Y JX,\xi)=g(\nabla_X Y+h(X,JY)-\nabla_Y JX-h(JX,Y),\xi)$$
$$=g(h(X,JY)-h(JX,Y),\xi)$$

or, $$g((\overline{\nabla}_X J)Y+J\overline{\nabla}_X Y-(\overline{\nabla}_Y J)X-J\overline{\nabla}_Y X,\xi)=g(h(X,JY)-h(JX,Y),\xi)$$

now as we know that $(\overline{\nabla}_X J)Y=P(X,Y)+Q(X,Y)$

$$g(P(X,Y)+Q(X,Y)+J\overline{\nabla}_X Y-P(Y,X)-Q(Y,X)-J\overline{\nabla}_Y X,\xi)$$
$$=g(h(X,JY)-h(JX,Y),\xi)$$

$$g(Q(X,Y)+J\overline{\nabla}_X Y-Q(Y,X)-J\overline{\nabla}_Y X,\xi)=g(h(X,JY)-h(JX,Y),\xi)$$

$$g(Q(X,Y)-Q(Y,X)+J\overline{\nabla}_X Y-J\overline{\nabla}_Y X,\xi)=g(h(X,JY)-h(JX,Y),\xi)$$

$$g(Q(X,Y)-Q(Y,X)+J(\overline{\nabla}_X Y-\overline{\nabla}_Y X),\xi)=g(h(X,JY)-h(JX,Y),\xi)$$

using (1.3.7), we get

$$g(Q(X,Y)-Q(Y,X)+P(\overline{\nabla}_X Y-\overline{\nabla}_Y X)+F(\overline{\nabla}_X Y-\overline{\nabla}_Y X),\xi)$$
$$=g(h(X,JY)-h(JX,Y),\xi)$$

$$g(Q(X,Y)-Q(Y,X)+F(\overline{\nabla}_X Y-\overline{\nabla}_Y X),\xi)=g(h(X,JY)-h(JX,Y),\xi)$$

we know that $[X,Y]=\overline{\nabla}_X Y-\overline{\nabla}_Y X$

$$g(Q(X,Y)-Q(Y,X)+F[X,Y],\xi)=g(h(X,JY)-h(JX,Y),\xi)$$

$$g(F[X,Y],\xi)=g(h(X,JY)-h(JX,Y),\xi)-g(Q(X,Y)-Q(Y,X),\xi)$$

$$g(F[X,Y],\xi)=g(h(X,JY)-h(JX,Y)-Q(X,Y)+Q(Y,X),\xi) \tag{3.1.7}$$

if we take D is integrable, then we know for all $X,Y\in D,[X,Y]\in D$.

$\Rightarrow$ $$F[X,Y]=0.$$

Therefore, if D is integrable, then $g(F[X,Y],\xi)=0$, from equation (3.1.7)

$$g(h(X,JY)-h(JX,Y)-Q(X,Y)+Q(Y,X),\xi)=0$$

$$h(X,JY)-h(JX,Y)-Q(X,Y)+Q(Y,X)=0$$

therefore $$Q(X,Y)-Q(Y,X)=h(X,JY)-h(JX,Y),$$

on the other way when we take

$$Q(X,Y)-Q(Y,X)=h(X,JY)-h(JX,Y)$$

$$Q(X,Y)-Q(Y,X)-h(X,JY)+h(JX,Y)=0$$

therefore, from equation (3.1.7), we get

$$g(F[X,Y],\xi)=0$$

thus, $F[X,Y]=0$

so, $[X,Y]\in D$.

Hence D is integrable, this proves the proposition.

Proposition (3.1.2)[5]: Let M be a CR- submanifold of an almost Hermitian manifold $\overline{M}$. Then the totally real distribution $D^{\perp}$ is integrable if and only if.

$$P(Z,W)-P(W,Z)=A_{JZ}W-A_{JW}Z \tag{3.1.8}$$

for all $Z,W\in D^{\perp}$.

Proof: For any $U\in TM$, we may write

$$g(J[Z,W],U)=g(J\overline{\nabla}_Z W-J\overline{\nabla}_W Z,U)$$

now using equation (3.1.1), right hand side will be

$$=g(\overline{\nabla}_Z JW-(\overline{\nabla}_Z J)W-\overline{\nabla}_W JZ+(\overline{\nabla}_W J)Z,U)$$

now by (3.1) we get,

$$=g(\overline{\nabla}_Z JW-\overline{\nabla}_W JZ+P(W,Z)+Q(W,Z)-P(Z,W)-Q(Z,W),U)$$

$$g(J[Z,W],U)=g(\overline{\nabla}_Z JW-\overline{\nabla}_W JZ+P(W,Z)-P(Z,W),U) \tag{3.1.9}$$

first take $D^{\perp}$ is integrable.

Therefore, for all $Z,W\in D^{\perp},[Z,W]\in D^{\perp}$ and $J[Z,W]\in T^{\perp}M$.

Now for $U\in TM$.

$$g(J[Z,W],U)=0$$

from equation (3.1.9), we get

$$g(\overline{\nabla}_Z JW-\overline{\nabla}_W JZ+P(W,Z)-P(Z,W),U)=0$$

now, using Weingarten formula, above equation will be come

$$g(-A_{JW}Z+D_Z JW+A_{JZ}W-D_W JZ+P(W,Z)-P(Z,W),U)=0$$

i.e., $$g(-A_{JW}Z+A_{JZ}W+P(W,Z)-P(Z,W),U)=0$$

as $D^{\perp}$ is integrable.

$$P(Z,W)-P(W,Z)=A_{JZ}W-A_{JW}Z$$

this proves the one way.

Now conversely take $P(Z,W)-P(W,Z)=A_{JZ}W-A_{JW}Z$, we have to prove $D^{\perp}$ is integrable.

Take $P(Z,W)-P(W,Z)=A_{JZ}W-A_{JW}Z$ and using equation (3.1.9), we get

$$g(J[Z,W],U)=0$$

thus $J[Z,W]\in T^{\perp}M$ as $U\in TM$.

so, $$[Z,W]\in D^{\perp}.$$

Therefore $D^{\perp}$ is integrable, this completes the proof.

Proposition (3.1.3)[5]: Let the holomorphic distribution D on a CR- sub-manifold of an almost Hermitian manifold $\overline{M}$ be integrable. Then its leaves are totally geodesic in M if any of the following equivalent conditions hold.

(i) $P(X,Y)+th(X,Y)\in D$

(ii) $P(X,Z)+A_{JZ}X\in D^{\perp}$

(iii) $Q(X,Y)-h(X,JY)\in \upsilon$

for all $X,Y\in D$ and $Z\in D^{\perp}$.

Proof:

$$P(X,Y)+th(X,Y)\in D$$

by using equation (3.1.2), we get

$$g(P(X,Y),Z)=g((\nabla_X P)Y-A_{FY}X-th(X,Y),Z)$$
$$=-g(th(X,Y),Z)$$
$$g(P(X,Y)+th(X,Y),Z)=0$$

therefore $P(X,Y)+th(X,Y)\in D$ as $Z\in D^{\perp}$.

Now we have to prove (i) is equivalent to (ii)

$$g(P(X,Y)+th(X,Y),Z)=0$$
$$g(P(X,Y),Z)+g(th(X,Y),Z)=0$$

on using property (P_3), so we have

$$-g(Y,P(X,Z))+g(Jh(X,Y),Z)=0$$
$$g(Y,P(X,Z))+g(h(X,Y),JZ)=0$$
$$g(P(X,Z),Y)+g(A_{JZ}X,Y)=0$$
$$g(P(X,Z)+A_{JZ}X,Y)=0$$

for all $X,Y\in D$ and $Z\in D^{\perp}$. Thus

$$P(X,Z)+A_{JZ}X\in D^{\perp}.$$

This prove the equivalence of (i) and (ii).

again from part (i)

$$g(P(X,Y)+th(X,Y),Z)=0$$

for all $X,Y\in D$ and $Z\in D^{\perp}$, now as

$$Q(X,Y)\in T^{\perp}M \text{ and } g(th(X,Y),Z)=g(Jh(X,Y),Z),$$

the equation $g(P(X,Y)+th(X,Y),Z)=0$ is equivalent to

$$g(P(X,Y)+Q(X,Y)+Jh(X,Y),Z)=0$$

i.e.,
$$g(J(P(X,Y)+Q(X,Y))-h(X,Y),JZ)=0$$

now using the fact that $P(X,Y)\in TM$ and using property (P_5), the above equation gives

$$-g(P(X,JY)+Q(X,JY),JZ)-g(h(X,Y),JZ)=0$$
$$g(-Q(X,JY)-h(X,Y),JZ)=0$$

replacing Y by JY, we obtain

$$g(Q(X,Y)-h(X,JY),JZ)=0$$

or,
$$Q(X,Y)-h(X,JY)\in\upsilon.$$

This show, that (i) and (iii) are equivalent and the proof of the proposition is complete.

Proposition (3.1.4)[5]: If the totally real distribution $D^{\perp}$ on a CR- submanifold M of an almost Hermitian manifold $\overline{M}$ is integrable, then its leaves are totally geodesic in M if and only if any of the following equivalent conditions hold.

(i) $P(W,Z)+A_{JZ}W\in D^{\perp}$

(ii) $P(Z,X)+th(X,Z)\in D$

(iii) $Q(W,X)-h(W,JX)\in \upsilon$

for all $X\in D$ and $Z,W\in D^{\perp}$.

Proof: As $th(Z,W)\in D^{\perp}$ and the leaves of $D^{\perp}$ are totally geodesic in M if and only if

$$(\nabla_Z P)W\in D^{\perp}$$

$$P(W,Z)+A_{JZ}W\in D^{\perp}$$

$$g(P(W,Z)+A_{JZ}W,X)=0$$

for all $X\in D$ and $Z,W\in D^{\perp}$.

$$g(P(W,Z),X)+g(A_{JZ}W,X)=0$$

$$-g(Z,P(W,X))+g(h(X,W),JZ)=0$$

$$-g(Z,P(W,X))-g(th(X,W),Z)=0$$

$$g(P(W,X)+th(X,W),Z)=0,$$

therefore, $P(W,X)+th(X,W)\in D$

this proves the equivalence of (i) and (ii) .Similarly using property (P_3) and (P_4), the equivalence of (iii) and (i) can be established.

From above two propositions we have following Theorem.

Theorem(3.1.1)[5]: For a CR-submanifold of an almost Hermitian manifold $\overline{M}$, the following are equivalent.

(i) M is a CR-product.

(ii) $P(U,X)+th(U,X)\in D$

(iii) $P(U,Z)+A_{JZ}U\in D^{\perp}$

(iv) $Q(U,X)-h(U,JX)\in \upsilon$

for all $X\in D, Z\in D^{\perp}$ and $U\in TM$.

Proof: Recalling, that a CR- submanifold M is a CR-product with leaves of both the distribution D and $D^{\perp}$ totally geodesic in M, we may established this theorem by making use of the proposition (3.1.3) and (3.1.4) and Properties of P and Q.

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