# ON THE PROBABILITY THAT AN AUTOMORPHISM FIXES A GROUP ELEMENT

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# Certificate

I hereby declare that the dissertation entitled "On The Probability That an Automorphism Fixes a Group Element" is an authentic record of my work carried out as requirements for the award of the degree of Master of Science in Mathematics and Computing at Thapar Institute of Engineering and Technology, Patiala under the supervision of Dr. Deepak Gumber(Professor, School of Mathematics).
*No part of the matter embodied in this report has been submitted to any other university or institute for the award of any degree.*

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*This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.*

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# Abstract

Let $G$ be a finite group. Denote by $P(G)$ the probability that two elements of $G$, selected randomly and with replacement, commute, and by $P_A(G)$ the probability that a randomly chosen automorphism of $G$ fixes a randomly chosen element of $G$. It is known that $P_A(G) \leq 5/8$ for any nonabelian group. We show that $P_A(G) \leq 5/8$ for any noncyclic group, and then prove that this bound is not sharp. We also prove that $P_A(G) = P(G)$ if and only if $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$, where $\mathrm{Aut}_c(G)$ denotes the group of all conjugacy class-preserving automorphisms of $G$. This gives another prespective on a question of Avinoam Mann.

# Contents

# Chapter 1

# Introduction

Let $G$ be a finite group. The probability that two elements of $G$, selected randomly and with replacement, commute will be denoted by $P(G)$ throughout the dissertation. It is called commutativity degree of $G$. Let

$$D := \{(x, y) \in G \times G \mid xy = yx\}$$

and let $k(G)$ denote the number of distinct conjugacy classes of $G$. Gustafson [5] showed that

$$P(G) = \frac{|D|}{|G|^2} = \frac{k(G)}{|G|}.$$

The set of all automorphisms of $G$ is denoted by $\mathrm{Aut}(G)$. An automorphism $\beta$ of $G$ is said to be an inner automorphism if for all $g \in G$, there exists a fix $'a' \in G$ such that $\beta(g) = a^{-1}ga$. The set of all inner automorphisms is denoted by $\mathrm{Inn}(G)$. The conjugacy class of an element $x$ of $G$ is defined as the set $C(x) := \{\alpha(x) \mid \alpha \in \mathrm{Inn}(G)\}$ of images of $x$ under all inner automorphisms. Let

$$F(x) := \{\alpha(x) \mid \alpha \in \mathrm{Aut}(G)\}$$

denote the set of all images of $x$ under all the automorphisms of $G$. $F(x)$ is called the fusion class of $x$ in $G$. The probability that a randomly chosen element from $\mathrm{Aut}(G)$ fixes a randomly chosen element of $G$ is denoted by $P_A(G)$ and is called the autocommutativity degree of $G$. Let

$$E := \{(x, \alpha) \mid \alpha(x) = x, \alpha \in \mathrm{Aut}(G), x \in G\}$$

and let $f(G)$ denote the number of distinct fusion classes of $G$. Sherman [9] showed that

$$P_A(G) = \frac{|E|}{|G||\operatorname{Aut}(G)|} = \frac{f(G)}{|G|}.$$

In 1973, Gustafson [5] proved that $P(G) \leq 5/8$ for any finite nonabelian group. The bound is sharp, for example, for $G = Q_8$, the quaternion group of order 8.

For every group $G$, $P_A(G) \leq P(G)$, and $P_A(G) \leq 5/8$ for any finite nonabelian group $G$. In 1975, Sherman [9] studied $P_A(G)$ for finite abelian groups. He proved that $P_A(G) = 1$ if and only if $G = C_2$, and that $P_A(G) \leq 3/4$ if $G \neq C_2$. It follows that there does not exist any finite abelian group such that $3/4 < P_A(G) < 1$. Sherman [9] also observed that the bound $P_A(G) \leq 3/4$ is sharp as $P_A(C_4) = 3/4$. Some questions arising from these observation are: What are the finite abelian groups with $5/8 < P_A(G) \leq 3/4$ and what are the finite groups with $P_A(G) = 5/8$? In this dissertation we have tried to answer these questions. More precisely,

1. We prove that $C_4$ is the only finite abelian group with $P_A(G) = 3/4$ and $C_6$ and $C_3$ are the only finite abelian groups with $5/8 < P_A(G) < 3/4$.
2. We prove that there does not exist any finite group with $P_A(G) = 5/8$.

Another natural question that arises from the observation $P_A(G) \leq P(G)$ is : What are the finite groups for which $P_A(G) = P(G)$? An attempt to answer this question leads us to a very surprising observation.

Gashch*ü*tz [3] proved that any finite $p-$group of order at least $p^2$ admits a noninner automorphism of order a power of $p$. Motivated by this, in 1999, Avinoam Mann [8] posed the following question.

**Question** [9, Question 10]: Do all $p-$groups have automorphisms that are not class-preserving? If the answer is no, which are the groups that have only class-preserving automorphisms?

The answer to the first part of question is negative. For example, in [**5,6**] finite nilpotent groups $G$ of nilpotency class 2 and 3 are constructed for which $\operatorname{Aut}(G) = \operatorname{Aut}_c(G)$, $\operatorname{Aut}_c(G)$ denotes the group of (conjugacy) class-preserving automorphisms of $G$. Therefore, only the second part of the question $-$ find finite $p$-groups $G$ for which $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)-$is now relevant. In chapter 4, we prove that the problem of finding finite groups for which $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ is equivalent to finding finite groups for which $P_A(G) = P(G)$.

All the new results proved in this dissertation will appear in [4].

# Chapter 2

# Definitions and Preliminaries

**Definition 2.1.** (**Relation**) : Let $A$ and $B$ be two non-empty sets. A relation $R$ from $A$ to $B$ is a subset of Cartesian product $A \times B$. That is

$$R \subseteq (A \times B)$$

A relation from $A$ to $A$ is simply called a relation on $A$.

**Example 2.2.** Let $A$=$\{1, 2, 3\}$ and $B$ =$\{$4,5,6$\}$. Then $R_1$=$\{$(1,4),(2,5)$\}$ is a relation between $A$ and $B$ but $R_2$ =$\{$(1,4),(2,5),(4,4)$\}$ is not a relation between $A$ and $B$.

**Definition 2.3.** (**Equivalence relation**) : A relation $R$ on a non-empty set $A$ is said to be an equivalence relation if

$R$ is reflexive: $(a, a) \in R$ for all $a \in A$

$R$ is symmetric: if $(a, b) \in R,$ then $(b, a) \in R.$

$R$ is transitive: if $(a, b) \in R$ and $(b, c) \in R,$ then $(a, c) \in R.$

**Definition 2.4.** (**Cyclic group**) : A Group $G$ is called cyclic if there exists $c \in G$, such that every element of $G$ can be written as some power of $c$. The element $c$ is called a generator of $G$.

**Example 2.5.** $\mathbb{Z}_n$ is a cyclic group of order $n$.

**Example 2.6.** Klein four group $i.e$ $K_4$= $\{a, b \mid a^2 = b^2 = (ab)^2 = 1\}$ is not a cyclic group .

**Definition 2.7.** (**Homomorphism**) : Let $(G, \star)$ and $(G^{'}, \circ)$ be two groups. A map $f : G \to G^{'}$ is called a homomorphism if

$$f(a \star b) = f(a) \circ f(b) \ \ \forall a, b \in G.$$

**Example 2.8.** A map $f : (\mathbb{Z}, +) \longrightarrow (2\mathbb{Z}, +)$ such that $f(x) = 2x$ is a homomorphism.

**Definition 2.9.** (**Isomorphism**) : Let $(G, \star)$ and $(G^{'}, \circ)$ be two groups. A homomorphism $f : G \longrightarrow G'$ is called an isomorphism if it is one-one and onto.

**Definition 2.10.** (**Isomorphic**) : Two groups $(G, \star)$ and $(G^{'}, \circ)$ are called isomorphic if there exists an isomorphism $f : G \longrightarrow G'$. If $G$ is isomorphic to $G'$, then it is denoted as $G \simeq G'$.

**Example 2.11.** Klein 4-group is isomporphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**Definition 2.12.** (**Automorphism**) : An isomorphism of a group $G$ to itself is called an automorphism. The set of all automorphisms is denoted by Aut$(G)$.

**Theorem 2.13.** *For any group $G$, the set* $\operatorname{Aut}(G)$ *is a group under composition of functions.*

*Proof.* Let $\operatorname{Aut}(G)$ be the set of all automorphisms of $G$. *i.e*, $\operatorname{Aut}(G) = \{f : f \text{ is an automorphism of } G\}$.
The identity function $I : G \mapsto G$ defined as $I(a) = a$ for all $a \in G$ is one-one and onto. We need to show that $I$ is an automorphism.
Let $a, b \in G$. Then trivially

$$I(ab) = ab = I(a)I(b).$$

Therefore, $I \in \operatorname{Aut}(G)$. Thus $\operatorname{Aut}(G) \neq \phi$.
*To prove* : We need to prove that $(\operatorname{Aut}(G), .)$ forms a group.

1)**Closure Law** : Let $h$, $g \in \operatorname{Aut}(G)$ and $f = hg$. Now, $f$ is a bijective map on $G$ as composition of two bijective maps is bijective.
We need to show that $f(ab) = f(a)f(b)$ for all $a, b \in G$.
Let $a, b \in G$. Then
$f(ab) = hg(ab) = h(g(ab)) = h(g(a)g(b)) = h(g(a))h(g(b)) = f(a)f(b)$.
Thus, $f$ is an automorphism.

2)**Associative Law** : Since composition of functions is associative in general, that is, $\forall f, g$ and $h \in \operatorname{Aut}(G)$, $(fg)h = f(gh)$, hence $\operatorname{Aut}(G)$ is associative under composition of automorphisms.

3)**Existence of identity** : The identity function $I$ shown above is the identity of $\operatorname{Aut}(G)$.

4)<u>**Existence of Inverse**</u> : Let $h \in \mathrm{Aut}(G)$. Then $h$ is one-one and onto map. So $h^{-1}$ exists and is bijective.

We need to prove that $h^{-1}$ is an automorphism.

Since $h^{-1}$ is onto, therefore for $a, b \in G$ there exists $x, y \in G$ such that

$$h^{-1}(a) = x \text{ and } h^{-1}(b) = y.$$

$$\Rightarrow h(x) = a \text{ and } h(y) = b.$$

Therefore,

$$h(xy) = h(x)h(y) = ab$$

$$\Rightarrow h^{-1}(ab) = xy = h^{-1}(a)h^{-1}(b).$$

Thus, $h^{-1}(ab) = h^{-1}(a)h^{-1}(b)$ for all $a, b \in G$. This implies $h^{-1} \in \mathrm{Aut}(G)$. Clearly $hh^{-1} = I = h^{-1}h$. So each element of $\mathrm{Aut}(G)$ possesses an inverse. Hence $\mathrm{Aut}(G)$ is a group under composition of functions. □

**Definition 2.14.** (**Inner Automorphism**) : Let $G$ be a group and $a$ be an element of $G$. The automorphism $h_a : G \to G$ defined as

$$h_a(x) = axa^{-1}$$

is called an inner automorphism of $G$. The set of all inner automorphisms of $G$ is denoted by $\mathrm{Inn}(G)$.

**Lemma 2.15.** $\mathrm{Inn}(G)$ *is a subgroup of* $\mathrm{Aut}(G)$.

*Proof.* We know that $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$. It is easy to see that $\mathrm{Inn}(G)$ is non empty as the identity automorphism, $I \in \mathrm{Inn}(G)$.

Therefore, to show that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$, we need to prove that:

1. $(h_a \circ h_b) \in \mathrm{Inn}(G)$ for all $h_a, h_b \in \mathrm{Inn}(G)$.
2. $h_{a^{-1}} \in \mathrm{Inn}(G)$ for all $h_a \in \mathrm{Inn}(G)$.

Let $h_a, h_b \in \mathrm{Inn}(G)$. Then $h_a(x) = axa^{-1}$ and $h_b(x) = bxb^{-1}$.

We need to show that $(h_a \circ h_b) = h_{ab} \in \mathrm{Inn}(G)$.

Consider,

$$(h_a \circ h_b)(x) = h_a(h_b(x)) = h_a(bxb^{-1}) = abxb^{-1}a^{-1} = (ab)x(ab)^{-1} = h_{ab}(x).$$

Hence $(h_a \circ h_b) = h_{ab} \in \mathrm{Inn}(G)$.

Now, we need to show that for every $h_a \in \mathrm{Inn}(G)$ there exists ${h_a}^{-1} \in \mathrm{Inn}(G)$.

Let $a,\ x$ be any two elements of $G$. Then consider

$$(h_a \circ h_{a^{-1}})(x) = h_a(h_{a^{-1}}(x)) = h_a(a^{-1}xa) = aa^{-1}xaa^{-1} = x = I(x)$$

and

$$(h_{a^{-1}} \circ h_a)(x) = h_{a^{-1}}(h_a(x)) = h_{a^{-1}}(axa^{-1}) = a^{-1}axa^{-1}a = x = I.$$

and hence $h_{a^{-1}} \in \mathrm{Inn}(G)$.

Hence $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$. □

**Definition 2.16.** (**Centre of a group**) : The center of a group $G$ is the collection of all those elements of $G$ which commute with every element of $G$. It is denoted by $Z(G)$. That is,

$$Z(G) = \{g \mid ag = ga \text{ for all } a \in G\}.$$

**Lemma 2.17.** *$Z(G)$ is a subgroup of $G$.*

*Proof.* It is clear that $Z(G)$ is nonempty as $e \in Z(G)$. Therefore to show that $Z(G)$ is a subgroup of $G$, we need to show that:

1. $g_1g_2 \in Z(G)$ for all $g_1, g_2 \in Z(G)$.
2. $g_1^{-1} \in Z(G)$ for all $g_1 \in Z(G)$.

Let $g_1, g_2 \in Z(G)$ then $g_1a = ag_1$ and $g_2a = ag_2 \ \ \forall a \in G$.
We need to show that $g_1g_2 \in Z(G)$.
Consider,

$$a(g_1g_2) = g_1ag_2 = (g_1g_2)a$$

This implies $g_1g_2 \in Z(G)$.
Now, we need to show that for every $g_1 \in Z(G)$ there exist $g_1^{-1} \in Z(G)$.
Consider,

$$g_1^{-1}(g_1a)g_1^{-1} = g_1^{-1}(ag_1)g_1^{-1}$$

$$\Rightarrow (g_1^{-1}g_1)ag_1^{-1} = g_1^{-1}a(g_1g_1^{-1})$$

$$\Rightarrow ag_1^{-1} = g_1^{-1}a$$

This implies $g_1^{-1} \in Z(G)$.
This shows $Z(G)$ is a subgroup of $G$. □

**Definition 2.18.** (**Absolute Centre of a Group** ): Let $G$ be a group. The absolute centre of $G$ consists of those elements of $G$ that are fixed by all automorphisms of $G$. Absolute centre of a group is denoted by $L(G)$.
Symbolically,

$$L(G) = \{x \mid \alpha(x) = x \ \text{ for all } \alpha \in \mathrm{Aut}(G)\}$$

**Example 2.19.** : Absolute centre of $C_2 \times C_2$ contains only identity element.

*Proof.* : Elements of $C_2 \times C_2$ are $\{1, x, y, xy\}$.
All automorphisms of $C_2 \times C_2$ are given by :

1. $\beta_1(x) = x, \quad \beta_1(y) = y$
2. $\beta_2(x) = xy, \quad \beta_2(y) = y$
3. $\beta_3(x) = y, \quad \beta_3(y) = x$
4. $\beta_4(x) = xy, \quad \beta_4(y) = x$
5. $\beta_5(x) = x, \quad \beta_5(y) = xy$
6. $\beta_6(x) = y, \quad \beta_6(y) = xy$

Clearly, no non-identity element of $C_2 \times C_2$ is fixed by all automorphisms. Therefore,

$$L(C_2 \times C_2) = \{1\}$$

□

**Example 2.20.** Absolute centre of the group $C_6$ contains two elements.

*Proof.* Elements of $C_6$ are $\{1, x, x^2, x^3, x^4, x^5\}$.
All automorphisms of $C_6$ are:

1. $\beta_1(x) = (x)$
2. $\beta_2(x) = (x^5)$

Hence,

$$L(C_6)=\{1, x^3\}.$$

$\square$

**Lemma 2.21.** *Let $G$ be a group. Then $L(G)$ is a subgroup of $Z(G)$.*

*Proof.* $L(G) = \{x \mid \alpha(x) = x \text{ for all } \alpha \in \mathrm{Aut}(G)\}$.
It is easy to see that $L(G)$ is non empty as $e \in L(G)$. Therefore to prove $L(G)$ is a subgroup of $Z(G)$, we need to prove:

1. $ab \in L(G) \ \ \forall a, b \in L(G)$.

2. $a^{-1} \in L(G) \ \ \forall a \in L(G)$.

Let $a, b \in L(G)$. Therefore, $\alpha(a) = a$ and $\alpha(b) = b$
We have to show that $ab \in L(G)$.
Consider,

$$\alpha(ab) = \alpha(a)\alpha(b) = ab$$

This implies, $ab \in L(G)$.
Now we need to show that $a^{-1} \in L(G)$ that is $\alpha(a^{-1}) = a^{-1} \ \ \forall \alpha \in \mathrm{Aut}(G)$
Consider,

$$\alpha(a^{-1}) = (\alpha(a))^{-1} = (a)^{-1} = a^{-1}$$

Hence, $L(G)$ is a subgroup of $Z(G)$. $\square$

**Definition 2.22.** (**Conjugate elements**): Let $a, b$ be two elements of group $G$. Then $a$ is said to be conjugate of $b$ if there exist some $c \in G$ such that

$$a = c^{-1}bc.$$

**Lemma 2.23.** *The relation of conjugacy on a group $G$ is an equivalence relation.*

*Proof.* 1) $Reflexivity$ : As $x = x^{-1}xx$ So, $x \sim x$

2) $Symmetry$ : $Let\ x \sim y \Rightarrow x = a^{-1}ya$, for some $a \in G$

$\Rightarrow y = axa^{-1} = b^{-1}xb$, where $b = a^{-1}$

$\Rightarrow y \sim x$

3) $Transitivity$ : $Let\ x \sim y$ and $y \sim z$.

So, there exist $a, b \in G$ such that

$\Rightarrow x = a^{-1}ya,\ y = b^{-1}zb$

$\Rightarrow x = a^{-1}(b^{-1}zb)a = (ba)^{-1}z(ba) = c^{-1}zc$ , where $c = ba$

$\Rightarrow x \sim z$

This proves that the relation of conjugacy on a group $G$ is an equivalence relation. □

**Definition 2.24.** (**Conjugacy Class**) : We have seen that the relation of conjugacy on the elements of $G$ is a equivalence relation and it partitions $G$ into equivalence classes. Equivalence class of an element $b \in G$ is defined as:

$$Cl(b) = \{a \in G \mid \text{there exists } g \in G \text{ with } a = g^{-1}bg\}$$

and is called the conjugacy class of $b$.

**Definition 2.25.** (**Centralizer**) : The centralizer of a non-empty subgroup $K$ of a group $G$ consists of those elements of $G$ which commute with every element of $K$. Symbolically,

$$C_G(K) = \{a \in G \mid ak = ka \text{ for all } k \in K\}$$

It is easy to see that $C_G(K)$ is a subgroup of $G$.

The centralizer of an element $a$ of $G$ is the set of elements of $G$ which commute with $a$. Symbolically,

$$C_G(a) = \{x \in G \mid ax = xa\}$$

*Note* : The number of conjugates of $a$ in a group $G = [G : C_G(a)]$.

**Definition 2.26.** (**Class equation**) : Since $G$ is paritioned into disjoint equivalence classes, therefore

$$G = Z(G) \cup Cl(x_i), \quad 1 \leq i \leq t,$$

where each $Cl(x_i)$ is a non-trival conjugacy class of $G$. Thus

$$|G| = |Z(G)| + \sum_{i=1}^{t} |Cl(x_i)|.$$

This equation is called the class equation of $G$.

**Definition 2.27.** (**Action**) : Let $G$ be a group and $H$ be any set. A map $f : H \times G \to G$ such that $(h, g) \mapsto h^g$ is called an action of $G$ on $H$ if

1. $(h^{g_1})^{g_2} = h^{g_1 g_2}$ for all $h \in H$ and $g_1, g_2 \in G$.
2. $h^1 = h$ for all $h \in H$.

   If there exist a action from $G$ on $H$ then we say that $G$ acts on $H$.

**Definition 2.28.** (**Orbit**) : Let $G$ be a group and let $G$ acts on a set $H$. Then the orbit of $h \in H$ consists of those elements of $H$ which are obtained by applying elements of $G$ to $h$. Symbolically,

$$h^G = \{h^{g_1} \mid g_1 \in G\} \subseteq H.$$

**Definition 2.29.** (**Stabiliser**) : Let $G$ acts on a set $H$ and let $h \in H$. Then stabiliser of $h$ in $G$ consists of those elements of $G$ that fix $h$. That is

$$G_h = \{g \in G \mid h^g = h\}.$$

**Theorem 2.30.** *(**Orbit Stabilizer Theorem**) : Let $G$ be a group acting on set $H$ and let $a \in H$. Then*

$$|h^G| = |G : G_h|.$$

*Proof.* Let us define map

$$f : G_h a \longrightarrow h^a$$

where $G_h a$ is the set of cosets of stabiliser $G_h$ in $G$ and $h^a$ is the orbit of $h$.
*To show* : We need to show that $f$ is a bijection.
First we will check it is well defined.
Suppose $G_h a = G_h b$ for some $a$ and $b$. Then $ab^{-1} \in G_h$. So

$$h^{ab^{-1}} = h.$$

Apply $b$

$$h^{ab^{-1}b} = h^b,$$

Also

$$h^{a(b^{-1}b)} = h^a.$$

This implies that

$$h^a = h^b.$$

Hence $f$ is well-defined.
Now we need to show that $f$ is one-one. Suppose that

$$f(G_h a) = f(G_h b) \text{ for some } a, b \in G.$$

Then

$$h^a = h^b.$$

Applying $b^{-1}$

$$h^{ab^{-1}} = h^{bb^{-1}} = h^1 = h,$$

so $ab^{-1} \in G_h$ and hence $G_h a = G_h b$. Thus $f$ is one-to-one.

Finally, we need to prove that $f$ is onto.

If $\beta \in h^G$ then there exists $a \in G$ such that $\beta = h^a$. Hence $f(G_h a) = \beta$. So $f$ is onto.

Hence $f : G_h a \longrightarrow h^a$ defines a bijective map from the set of cosets of $G_h a$ to orbit of $h$.

Thus $G_h a \simeq h^a$ or we can say that $G/G_h \simeq h^a$.

Hence

$$|h^a| = |G : G_h|.$$

This proves the result. □

**Definition 2.31.** (**Fused elements**) : Let $a, b$ be two elements of a group $G$. Then $a$ is fused to $b$ if there exists some $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a) = b$. The relation of fusion is given by

$$R = \{(a, b) \mid \exists \alpha \in \mathrm{Aut}(G),\ \alpha(a) = b\}$$

**Definition 2.32.** (**Fusion class**) : Let $a$ be an element of a group $G$. The fusion class of $a$ consists of images of $a$ under all automorphisms of $G$. The fusion class of $a$ is denoted by $F(a)$. So

$$F(a) = \{\alpha(a) \ \mid \ \alpha \in \mathrm{Aut}(G)\}$$

**Example 2.33.** Two fusion classes of $C_3$ are $F(0) = \{0\}$ and $F(1) = \{1, 2\}$.

**Lemma 2.34.** *The relation $R$ of fusion on a group $G$ is an equivalence relation.*

*Proof.* 1) *Reflexivity* : For any $a \in G, I(a) = a$, where $I$ is an identity automorphism.
Therefore $(a, a) \in R$
2) *Symmetry* : Let $(a, b) \in R$ . Then $\alpha(a) = b$ for some $\alpha \in \mathrm{Aut}(G)$
$\Rightarrow a = \alpha^{-1}(b)$ $[\alpha \in \mathrm{Aut}(G) \Rightarrow \alpha^{-1} \in \mathrm{Aut}(G)]$
$\Rightarrow (b, a) \in R$
3) *Transitivity* : Let $(a, b) \in R, (b, c) \in R$.
$\Rightarrow \alpha(a) = b, \beta(b) = c$ for some $\alpha, \beta \in \mathrm{Aut}(G)$.
$\Rightarrow \beta(\alpha(a)) = c$
$\Rightarrow \beta \circ \alpha(a) = c$ $[\beta, \alpha \in \mathrm{Aut}(G) \Rightarrow \beta \circ \alpha \in \mathrm{Aut}(G)]$.
This proves that the relation of fusion on a group $G$ is an equivalence relation.

□

**Definition 2.35.** (**Fusion class equation**) :Since $G$ is paritioned into disjoint fusion classes, therefore

$$G = L(G) \cup F(a_i), \quad 0 \leq i \leq t,$$

where each $F(a_i)$ is a non-trival fusion class of $G$. Thus

$$|G| = |L(G)| + \sum_{i=1}^{t} |F(a_i)|.$$

This equation is called the fusion class equation of $G$.

**Definition 2.36.** (**Commutativity degree** ) : For a finite group $G$, the probability that two elements of $G$, selected randomly and with replacement, commute is denoted by $P(G)$ and is called the commutativity degree of $G$. Symbolically,

$$P(G) = \frac{|D|}{|G|^2},$$

where $D := \{(a, b) \in G \times G \mid ab = ba\}$.

**Lemma 2.37.** *[5] The commutativity degree for a finite group $G$ is*

$$P(G) = \frac{k(G)}{|G|},$$

*where $k(G)$ denotes the number of distinct conjugacy classes of $G$.*

*Proof.* Let $Cl(a_1), Cl(a_2), \ldots, Cl(a_k)$ be the distinct conjugacy classes of $G$. Then $k(G) = k$. By definition,

$$P(G) = \frac{|D|}{|G|^2},$$

where $D := \{(a, b) \in G \times G \mid ab = ba\}$.

Note that for $a \in G$, $(a, b) \in D$ if and only if $b \in C_G(a)$. Thus

$$|D| = \sum_{a \in G} |C_G(a)|.$$

It follows that

$$\begin{aligned} |D| &= \sum_{a \in G} |C_G(a)| \\ &= \sum_{j=1}^{k} |[Cl(a_j)]||C_G(a_j)| \\ &= \sum_{j=1}^{k} |[G : C_G(a_j)]||C_G(a_j)| \\ &= \sum_{j=1}^{k} |G| \\ &= k|G| \end{aligned}$$

Hence,

$$P(G) = \frac{|D|}{|G|^2} = \frac{k|G|}{|G|^2} = \frac{k(G)}{|G|}.$$

□

**Lemma 2.38.** *If $G$ is any finite nonabelian group, then $P(G) \leq 5/8$ and this bound is sharp.*

*Proof.* The class equation of $G$ is

$$|G| = |Z| + |A_1| + |A_2| + \cdots + |A_t|$$

where $A_1$ , $A_2$ , $\ldots, A_t$ are non-trival conjugacy classes and $Z$ is the centre of $G$. We know that each non-trival conjugacy class contains atleast two elements $i.e$ $|A_j| \geq 2$ for $j = 1, 2, \ldots, t$.

Therfore

$$\begin{aligned} |G| &\geq |Z| + 2t \\ \Rightarrow \frac{|G| - |Z|}{2} &\geq t. \end{aligned} \tag{2.1}$$

As

$$k(G) = t + |Z| \tag{2.2}$$

From equation 2.1 and 2.2, we get

$$k(G) = t + |Z| \leq \frac{|G| - |Z|}{2} + |Z| = \frac{|G| + |Z|}{2}. \tag{2.3}$$

Since $G$ is non-abelian, $G/Z$ is non-cyclic. Thus $|G/Z| \geq 4$ and hence $|Z| \leq |G|/4$. It follows from equation 2.3 that

$$k(G) \leq \frac{|G|}{2} + \frac{|G|}{8} = \frac{5|G|}{8},$$

and hence

$$P(G) = \frac{k(G)}{|G|} \leq \frac{5}{8}.$$

Now, we need to show that this bound is sharp. Let us take $G \simeq Q_8$, the quaternion group of order 8.

Since $Q_8 = \{i, j, k \mid i^2 = j^2 = k^2 = ijk = -1\}$. We can write the elements of $Q_8$ as $\{1, -1, i, -i, j, -j, k, -k\}$.

Conjugacy classes of $Q_8$ are:

- $C(1) = \{1\}$
- $C(-1) = \{-1\}$
- $C(i) = \{i, -i\}$
- $C(j) = \{j, -j\}$
- $C(k) = \{k, -k\}$.

Hence $k(G) = 5$.

Therfore,

$$P(G) = \frac{k(G)}{|G|} = \frac{5}{8}.$$

□

**Definition 2.39.** Let $G$ be a finite group. For $a \in G$,

$$C_{\mathrm{Aut}(G)}(a) = \{\alpha \in \mathrm{Aut}(G) |\ \alpha(a) = a\}.$$

**Definition 2.40.** (**Autocommutativity degree** ): The probability that a randomly chosen element from $\mathrm{Aut}(G)$ fixes a randomly chosen element of

$G$ is denoted by $P_A(G)$ and is called the autocommutativity degree of $G$. Symbolically,

$$P_A(G) = \frac{|E|}{|G||\operatorname{Aut}(G)|},$$

where $E := \{(x, \alpha) \mid \alpha(x) = x, \alpha \in \operatorname{Aut}(G), x \in G\}$. The concept of autocommutativity degree is due to sherman [9].

**Lemma 2.41.** *[9]. The autocommutative degree of $G$ is given by,*

$$P_A(G) = \frac{f(G)}{|G|},$$

*where $f(G)$ denotes the number of distinct fusion classes of $G$.*

*Proof.* Let $\{F(a_1), F(a_2), \ \dots \ , F(a_t)\}$ be the set of disjoint fusion classes of $G$. Then $f(G) = t$. By definition,

$$P_A(G) = \frac{|E|}{|\operatorname{Aut}(G)||G|}.$$

Observe that

$$\begin{aligned}
|E| &= \sum_{a \in G} |C_{\operatorname{Aut}(G)}(a)| \\
&= \sum_{i=1}^{t} |C_{\operatorname{Aut}(G)}(a_i)||F(a_i)| \\
&= \sum_{i=1}^{t} |C_{\operatorname{Aut}(G)}(a_i)||[\operatorname{Aut}(G) : C_{\operatorname{Aut}(G)}(a_i)]| \\
&= \sum_{i=1}^{t} |\operatorname{Aut}(G)| \\
&= k.|\operatorname{Aut}(G)|
\end{aligned}$$

Therfore,

$$P_A(G) = \frac{k.|\operatorname{Aut}(G)|}{|G||\operatorname{Aut}(G)|} = \frac{f(G)}{|G|}.$$

□

**Lemma 2.42.** *$P_A(G) = 1$ if and only if $G \simeq C_2$.*

*Proof.* First suppose that $G \simeq C_2 = \{\overline{0}, \overline{1}\}$.

We need to show that $P_A(G) = 1$.

Fusion classes of $C_2$ are $F(\overline{0}) = \{\overline{0}\}$ and $F(\overline{1}) = \{\overline{1}\}$.

Therefore $f(G) = 2$ and

$$P_A(G) = \frac{f(G)}{|G|} = \frac{2}{2} = 1.$$

Conversely, let us suppose that $P_A(G) = 1$.

To prove that $G \simeq C_2$, we need to show that $G$ is an elementary abelian 2-group.

As $P_A(G) = 1$, therefore $f(G) = |G|$. The automorphism $a \mapsto -a$ must be the identity map. This implies that

$$a = -a \quad \Rightarrow 2a = 0$$

Hence $G$ is an elementary abelian 2-group. Then by [9, Para 1 P.263]

$$P_A(G) = \frac{2}{2^k} = 1 \quad \text{where } |G| = 2^k$$

This implies $k = 1$. That is, $G = C_2$. □

**Lemma 2.43.** *$P_A(G) \leq 3/4$ if $G \neq C_2$ and the bound is sharp.*

*Proof.* The fusion class equation of $G$ is

$$|G| = |L(G)| + \sum_{i=1}^{r} |F_i|,$$

where $F_1, F_2, \ldots, F_r$ are non-trival fusion classes and $L(G)$ is the absolute center of $G$. We know that each non-trival fusion class contains at least two elements *i.e* $|F_i| \geq 2$ for $i = 1, 2, \ldots, r$. Therfore

$$|G| \geq |L(G)| + 2r$$

$$\Rightarrow \frac{|G| - |L(G)|}{2} \geq r \tag{2.4}$$

As

$$f(G) = r + |L(G)| \tag{2.5}$$

from equation 2.4 and 2.5, we get

$$f(G) = r + |L(G)| \leq \frac{|G| - |L(G)|}{2} + |L(G)| = \frac{|G| + |L(G)|}{2}. \tag{2.6}$$

If $G \neq C_2$, then $L(G) \neq G$ and therefore $[G : L(G)] \geq 2$ *i.e* $|L(G)| \leq |G|/2$. Therefore

$$f(G) \leq \frac{|G| + |L(G)|}{2} \leq \frac{|G|}{2} + \frac{|G|}{4} = \frac{3}{4}|G|,$$

and hence

$$P_A(G) = \frac{f(G)}{|G|} \leq \frac{3}{4}.$$

Now, we need to show that this bound is sharp. Let us take $G \simeq C_4$.

The elements of $C_4$ are $\{0, x, x^2, x^3\}$.

Automorphisms of $C_4$ are given as $\beta_1(x) = x$, and $\beta_2(x) = x^3$.

Thus, Fusion classes of $C_4$ are $F(0) = \{0\}$, $F(x) = \{x, x^3\}$ and $F(x^2) = \{x^2\}$.

Therefore, we get $f(C_4) = 3$ and

$$P_A(C_4) = \frac{3}{4}.$$

□

**Definition 2.44.** (**Commutator**) : Let $x,\ y$ be any two elements of a group $G$. The commutator of $x$ and $y$ is the element $x^{-1}y^{-1}xy$.

**Definition 2.45.** (**Commutator Subgroup**) : The **commutator subgroup** $G'$ of a group $G$ is the subgroup of $G$ generated by the set of all commutators of $G$. That is

$$G' = \langle\ [x, y] \mid x, y \in G\ \rangle.$$

**Definition 2.46.** (***p*-subgroup**) : Let $G$ be a group and $p$ be a prime number. Then a subgroup $K$ of $G$ is said to be a ***p*-subgroup** of $G$ if order of each element of $K$ is some power of $p$.

**Definition 2.47.** (**Sylow *p*-subgroup of a finite group**): Let $G$ be a finite group and $p$ be a prime number such that $p^k$ divides $|G|$ but $p^{k+1}$ does not divide $|G|$. Then any subgroup of $G$ of order $p^k$ is called **Sylow *p*-subgroup of *G*** .

**Definition 2.48.** (**Internal direct Product**) : Let $G$ be a group and $K_1, K_2, \dots K_n$ be the subgroup of $G$. Then $G$ is called internal direct product of $K_1, K_2, \dots K_t$
if

1. $K_i \lhd G$ for all $i = 1, 2, \dots, t$.

2. $G = K_1 K_2 \cdots K_t$.

3. $K_i \cap (K_1 K_2 \cdots, K_i, \cdots, K_t) = 1$ where $K_1 K_2 \cdots, K_i, \cdots, K_t$ means product of all $K_j$ excluding $K_i$.

# Chapter 3

# Bounds on $P_A(G)$

In this section we will discuss about **bounds on** $P_A(G)$. Since we know that the centre $Z(G)$ consists of those elements of $G$ that are fixed by inner automorphisms of $G$. Similarly, $L(G)$, called *the absolute centre of* $G$, consists of those elements of $G$ that are fixed by all automorphisms of $G$. It is ovbious that $L(G)$ is subgroup of $Z(G)$. In this chapter we will find bounds on $P_A(G)$. To prove our results we require some results of [1] and [9].

**Lemma 3.1.** *[1, Theorem 2.2] A finite group* $G$ *is cyclic if and only if* $G/L(G)$ *is cyclic. Moreover, if* $G/L(G) \simeq C_n$*, then either* $G \simeq C_{2n}$ *if* $n$ *is even, or* $G \simeq C_n$ *or* $C_{2n}$ *if* $n$ *is odd.*

**Lemma 3.2.** *[1, Theorem 3.1] Let* $G$ *be a finite group such that* $G/L(G) \simeq C_2 \times C_2$*. Then* $G$ *is isomorphic to one of the following groups:* $C_2 \times C_2$*,* $C_4 \times C_2$*, the dihedral group* $D_8$ *of order 8, the quaternion group* $Q_8$ *of order 8, or the group* $G_1 := \{(x, y) \mid x^4 = y^4 = 1, y^{-1}xy = x^{-1}\}$*.*

**Lemma 3.3.** *[9, Equation(**),p.262] Let* $G \simeq G_1 \times G_2 \times \cdots \times G_r$*, where* $G_i$ *are cyclic Sylow* $p_i-$*subgroups* $G$*. Then*

$$P_A(G) = P_{A_1}(G_1)P_{A_2}(G_2)\cdots P_{A_r}(G_r)$$

**Lemma 3.4.** *[9, p.263, paragraph 1] Let $G$ be a cyclic group. Then $G$ must conatin at least one element of order $p^k$ for $k = 0, 1 \cdots n$. As elements present in same fusion class have equal orders, $G$ has at least $n+1$ fusion classes. Since elements having same orders can be expressed as powers of elements having order prime to $p$, the elements having a particular order constructs a fusion class. Thus*

$$P_A(G) = \frac{n+1}{p^n}.$$

Gustafson [5] proved that $P_A(G) \leq 5/8$ for a finite nonabelian group $G$. We extend this result to a finite noncyclic group.

**Lemma 3.5.** *If $G$ is a finite noncyclic group, then $P_A(G) \leq 5/8$.*

*Proof.* Let $G$ be a finite group and $L(G)$ its absolute centre.

The fusion class equation of $G$ is given by:

$$|G| = L(G) + \sum_{j=1}^{r} |X_j|,$$

where $X_1, X_2, \ldots, X_r$ are the non-trivial fusion classes. We have $|X_j| \geq 2$ for $j = 1, 2, \ldots, r$. It follows that

$$|G| \geq |L(G)| + 2r$$
$$\Rightarrow \frac{|G|}{2} \geq \frac{|L(G)|}{2} + r$$

Adding $|L(G)|/2$ both sides

$$\frac{|G|}{2} + \frac{|L(G)|}{2} \geq |L(G)| + r \tag{3.1}$$

Since $G$ is noncyclic, $G/L(G)$ is not cyclic by Lemma 3.1. Therefore, $|G/L(G)| \geq 4$ and hence $|L(G)| \leq |G|/4$. From Equation 3.1 it follows that

$$f(G) = |L(G)| + r \leq \frac{|G|}{2} + \frac{|L(G)|}{2} \leq \frac{5}{8}|G|.$$

Therefore

$$P_A(G) = \frac{f(G)}{|G|} \leq \frac{5}{8}.$$

□

**Lemma 3.6.** *Let* $G \simeq C_n$. *Then* $P_A(G) \geq 5/8$ *if and only if* $G \simeq C_2, C_3, C_4$ *or* $C_6$.

*Proof.* Firstly suppose that $P_A(G) \geq 5/8$. Suppose $n = {p_1}^{a_1}{p_2}^{a_2}\cdots{p_r}^{a_r}$, where $p_1 < p_2 < \cdots < p_r$ are distinct prime numbers and $a_i$ are positive integers. Let $G \simeq G_1 \times G_2 \times \cdots \times G_r$, where $G_i$ are cyclic Sylow $p_i-$subgroups $G$. Then

$$P_A(G) = P_{A_1}(G_1)P_{A_2}(G_2)\cdots P_{A_r}(G_r) \tag{3.2}$$

by Lemma 3.3, where $A_i$ is the automorphism group of $G_i$. By Lemma 3.4 we have

$$P_A(G) = \frac{n+1}{p^n}$$

Using these concepts we can write equation 3.2 as,

$$P_A(G) = \frac{a_1+1}{{p_1}^{a_1}}\,\frac{a_2+1}{{p_2}^{a_2}}\cdots\frac{a_r+1}{{p_r}^{a_r}}. \tag{3.3}$$

We prove by induction that

$$\frac{n+1}{p^n} \leq \frac{2}{p}$$

for all positive integers $n$. Assume that $n+1 \leq 2p^{n-1}$. Then

$$\frac{n+2}{p^{n+1}} \leq \frac{1+2p^{n-1}}{p^{n+1}} \leq \frac{2p^{n-1}(p-1)+2p^{n-1}}{p^{n+1}} = \frac{2p^n}{p^{n+1}} = \frac{2}{p}.$$

If $r \geq 3$, then, by equation 3.3,

$$P_A(G) \leq \frac{2}{p_1}\frac{2}{p_2}\cdots\frac{2}{p_r} = \frac{2^r}{p_1p_2\ldots p_r} \leq \frac{2.2.2}{2.3.5} < 5/8.$$

If $r = 2$, then $P_A(G) \geq 5/8$ implies that

$$\frac{a_1+1}{{p_1}^{a_1}} \, \frac{a_2+1}{{p_2}^{a_2}} \geq 5/8.$$

This inequality holds if and only if $a_1 = 1, a_2 = 1, p_1 = 2$ and $p_2 = 3$. Therfore, $G \simeq C_2 \times C_3 \simeq C_6$. Finally, if $r = 1$, then $P_A(G) \geq 5/8$ implies that

$$\frac{a_1+1}{{p_1}^{a_1}} \geq 5/8.$$

This inequality holds if and only if either $(i)$ $a_1 = 2$ and $p_1 = 2$, or $(ii)$ $a_1 = 1$ and $p_1 = 2$ or 3. Therfore $G \simeq C_2, C_3, or\ C_4$.

Conversely, *Case 1* : $G \simeq C_2$=$\{1, a\}$, $|G| = 2$.

$C_2$ has only one automorphism $\alpha_1$ defined by $\alpha_1(a) = a$.

Therefore fusion classes of $C_2$ are $F(1) = \{1\}$ and $F(a) = \{a\}$.

Thus $f(G) = 2$ and

$$P_A(G) = \frac{f(G)}{|G|} = \frac{2}{2} = 1.$$

*Case 2* : $G \simeq C_3 = \{1, a, a^2\}$, $|G| = 3$.

$C_3$ has only two automorphisms $\alpha_1$ and $\alpha_2$ defined by $\alpha_1(a) = a$ and $\alpha_2(a) = a^2$.

Then fusion classes of $C_3$ are : $F(1) = \{1\}$ and $F(a) = \{a, a^2\}$.

Therefore $f(G) = 2$ and

$$P_A(G) = \frac{f(G)}{|G|} = \frac{2}{3}.$$

*Case 3* : $G \simeq C_4 = \{1, a, a^2, a^3\}$, $|G| = 4$.

$C_4$ has only two automorphisms $\alpha_1$ and $\alpha_2$ defined by $\alpha_1(a) = a$ and $\alpha_2(a) = a^3$.

Thus fusion classes of $C_4$ are $F(1) = \{1\}$, $F(a) = \{a, a^3\}$ and $F(a^2) = \{a^2\}$. Hence $f(G) = 3$ and

$$P_A(G) = \frac{f(G)}{|G|} = \frac{3}{4}.$$

*Case 4* : $G \simeq C_6 = \{1, a, a^2, a^3, a^4, a^5\}$, $|G| = 6$.

$C_6$ has only two automorphisms $\alpha_1$ and $\alpha_2$ defined by $\alpha_1(a) = a$ and $\alpha_2(a) = a^5$.

As a result, fusion classes of $C_6$ are $F(1) = \{1\}$, $F(a) = \{a, a^5\}$, $F(a^2) = \{a^2, a^3\}$ and $F(a^3) = \{a^3\}$.

Thus $f(G) = 4$ and

$$P_A(G) = \frac{f(G)}{|G|} = \frac{4}{6} = \frac{2}{3}.$$

□

As a result of Lemma 3.5 and 3.6, we can classify all finite abelian groups with $5/8 < P_A(G) \leq 3/4$ in the following three corollaries.

**Corollary 3.7.** *For any finite abelian group $G$, $P_A(G) = 3/4$ if and only if $G \simeq C_4$.*

*Proof.* $P_A(G) = 3/4 > 5/8$. By Lemma 3.5 it is clear the $G$ cannot be a finite noncyclic group. Thus by Lemma 3.6 $G \simeq C_2, C_3, C_4$ or $C_6$. Among these group $C_4$ is the only group for which $P_A(G) = 3/4$.

Converse can be seen from *Case* 3 of Lemma 3.6. □

**Corollary 3.8.** *For any finite abelian group $G$, $P_A(G) = 2/3$ if and only if $G$ is isomorphic to $C_3$ or $C_6$.*

*Proof.* $P_A(G) = 2/3 > 5/8$. By Lemma 3.5 it is clear the $G$ cannot be a finite noncyclic group. Thus by Lemma 3.6 $G \simeq C_2, C_3, C_4$ or $C_6$. Among these group $C_3$ and $C_6$ are the only group for which $P_A(G) = 2/3$.

Converse can be seen from *Case* 2 and *Case* 4 of Lemma 3.6. $\square$

**Corollary 3.9.** *There does not exist any finite abelian group $G$ with $5/8 < P_A(G) < 2/3$ or $2/3 < P_A(G) < 3/4$.*

*Proof.* $P_A(G) > 5/8$. By Lemma 3.5 it is clear the $G$ cannot be a finite noncyclic group. Thus by Lemma 3.6 $G \simeq C_2, C_3, C_4$ or $C_6$.

For each case $P_A(G) \notin (5/8, 2/3) \cup (2/3, 3/4)$. Hence there does not exist any abelian group $G$ with $5/8 < P_A(G) < 2/3$ or $2/3 < P_A(G) < 3/4$. $\square$

We conclude the section by proving the impossibility of any finite group with $P_A(G) = 5/8$.

**Lemma 3.10.** *Let $G$ be any finite group with $P_A(G) = 5/8$. Then $|G/L(G)| \leq 4$.*

*Proof.* If possible, suppose that $|G/L(G)| > 4$. Then

$$\begin{aligned}\frac{5}{8} &= P_A(G)\\ &= \frac{f(G)}{|G|}\\ &= \frac{|L(G)|}{|G|} + \frac{f(G) - |L(G)|}{|G|}\\ &< \frac{1}{4} + \frac{f(G) - |L(G)|}{|G|}\end{aligned}$$

This implies that

$$|L(G)| < f(G) - \frac{3}{8}|G|. \tag{3.4}$$

It follows from fusion class equation of $G$ that

$$\frac{|G| - |L(G)|}{2} \geq r = f(G) - |L(G)|$$

where $r$ is the number of nontrival fusion classes of $G$. Therefore

$$2f(G) - |G| \leq |L(G)| \tag{3.5}$$

Using equation 3.4 and 3.5

$$2f(G) - |G| < f(G) - \frac{3}{8}|G|.$$

Thus

$$P_A(G) = \frac{f(G)}{|G|} < 5/8$$

This is a contradiction to the assumption, and hence $|G/L(G)| \leq 4$. □

**Lemma 3.11.** *If* $G \simeq C_2 \times C_2$, *then* $P_A(G) = 1/2$.

*Proof.* Let $G \simeq C_2 \times C_2 = \{a, b \mid a^2 = b^2 = 1, ab = ba\}$.

We can write the elements of $C_2 \times C_2$ as $\{1, a, b, ab\}$

All automorphisms of $G$ are defined as

1. $\phi_1(a) = a, \quad \phi_1(b) = b$
2. $\phi_2(a) = ab, \quad \phi_2(b) = b$
3. $\phi_3(a) = b, \quad \phi_3(b) = a$
4. $\phi_4(a) = ab, \quad \phi_4(b) = a$
5. $\phi_5(a) = a, \quad \phi_5(b) = ab$
6. $\phi_6(a) = b, \quad \phi_6(b) = ab$

Thus fusion classes are

- $F(a) = \{a, b, ab\}$

- $F(1) = \{1\}$

Hence $f(G) = 2$ and
$$P_A(G) = \frac{f(G)}{|G|} = \frac{2}{4} = \frac{1}{2}.$$

□

**Lemma 3.12.** *If* $G \simeq C_4 \times C_2$, *then* $P_A(G) = 1/2$.

*Proof.* Let $G \simeq C_4 \times C_2 = \{a, b \mid a^4 = b^2 = 1, ab = ba\}$.

We can write the elements of $C_4 \times C_2$ as$\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

The automorphisms of $G$ are listed below:

1. $\phi_1(a) = a, \quad \phi_1(b) = b$
2. $\phi_2(a) = a^3, \quad \phi_2(b) = b$
3. $\phi_3(a) = ab, \quad \phi_3(b) = b$
4. $\phi_4(a) = a^3b, \quad \phi_4(b) = b$
5. $\phi_5(a) = a, \quad \phi_5(a, b) = a^2b$
6. $\phi_6(a) = a^3, \quad \phi_6(b) = a^2b$
7. $\phi_7(a) = ab, \quad \phi_7(b) = a^2b$
8. $\phi_8(a) = a^3b, \quad \phi_8(b) = a^2b$

Hence the fusion classes of $G$ are

- $F(a) = \{a, a^3, ab, a^3b\}$
- $F(b) = \{b, a^2b\}$

- $F(a^2) = \{a^2\}$
- $F(1) = \{1\}$

Therfore $f(G) = 4$ and

$$P_A(G) = \frac{f(G)}{|G|} = \frac{4}{8} = \frac{1}{2}.$$

$\square$

**Lemma 3.13.** *If $G \simeq Q_8$, then $P_A(G) = 3/8$.*

*Proof.* Let $G \simeq Q_8 = \{a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1}\}$
The elements of $Q_8$ can be written as$\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$.
All the automorphisms of $Q_8$ are defined by

1. $\phi_1(a) = a, \quad \phi_1(b) = b$
2. $\phi_2(a) = a, \quad \phi_2(b) = ab$
3. $\phi_3(a) = a, \quad \phi_3(b) = a^2b$
4. $\phi_4(a) = a, \quad \phi_4(b) = a^3b$
5. $\phi_5(a) = b, \quad \phi_5(b) = a$
6. $\phi_6(a) = b, \quad \phi_6(b) = a^3$
7. $\phi_7(a) = b, \quad \phi_7(b) = ab$
8. $\phi_8(a) = b, \quad \phi_8(b) = a^3b$
9. $\phi_9(a) = a^3, \quad \phi_9(b) = b$

10. $\phi_{10}(a) = a^3, \quad \phi_{10}(b) = a^2b$
11. $\phi_{11}(a) = a^3, \quad \phi_{11}(b) = a^3b$
12. $\phi_{12}(a) = a^3, \quad \phi_{12}(b) = ab$
13. $\phi_{13}(a) = ab, \quad \phi_{13}(b) = a$
14. $\phi_{14}(a) = ab, \quad \phi_{14}(b) = b$
15. $\phi_{15}(a) = ab, \quad \phi_{15}(b) = a^3$
16. $\phi_{16}(a) = ab, \quad \phi_{16}(b) = a^2b$
17. $\phi_{17}(a) = a^2b, \quad \phi_{17}(b) = a$
18. $\phi_{18}(a) = a^2b, \quad \phi_{18}(b) = a^3$
19. $\phi_{19}(a) = a^2b, \quad \phi_{19}(a) = ab$
20. $\phi_{20}(a) = a^2b, \quad \phi_{20}(b) = a^3b$
21. $\phi_{21}(a) = a^3b, \quad \phi_{21}(b) = a$
22. $\phi_{22}(a) = a^3b, \quad \phi_{22}(b) = b$
23. $\phi_{23}(a) = a^3b, \quad \phi_{23}(b) = a^3$
24. $\phi_{24}(a) = a^3b, \quad \phi_{24}(b) = ab$

Therefore fusion classes of $G$ are

- $F(a) = \{a, a^3, b, ab, a^2b, a^3b\}$
- $F(a^2) = \{a^2\}$

- $F(1) = \{1\}$

Hence $f(G) = 3$ and

$$P_A(G) = \frac{f(G)}{|G|} = \frac{3}{8}.$$

□

**Lemma 3.14.** *If $G \simeq D_8$, then $P_A(G) = 1/2$.*

*Proof.* Let $G \simeq D_8 = \{a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1}\}$

The elements of $D_8$ can be expressed as $\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$

All the automorphisms of $G$ are listed below:

1. $\phi_1(a) = a, \quad \phi_1(b) = b$
2. $\phi_2(a) = a, \quad \phi_2(b) = ab$
3. $\phi_3(a) = a, \quad \phi_3(b) = a^2b$
4. $\phi_4(a) = a, \quad \phi_4(b) = a^3b$
5. $\phi_5(a) = a^3, \quad \phi_5(b) = b$
6. $\phi_6(a) = a^3, \quad \phi_6 b = ab$
7. $\phi_7(a) = a^3, \quad \phi_7(b) = a^2b$
8. $\phi_8(a) = a^3, \quad \phi_8(b) = a^3b$

Thus fusion classes of $G$ are

- $F(a) = \{a, a^3\}$
- $F(b) = \{b, ab, a^2b, a^3b\}$

- $F(a^2) = \{a^2\}$
- $F(1) = \{1\}$

Therfore $f(G) = 4$ and

$$P_A(G) = \frac{f(G)}{|G|} = \frac{4}{8} = \frac{1}{2}.$$

$\square$

**Lemma 3.15.** *If* $G \simeq G_1 = \{a, b \mid a^4 = b^4 = 1, b^{-1}ab = a^{-1}\}$, *then* $P_A(G) = 3/8$.

*Proof.* Let $G \simeq G_1 = \{a, b \mid a^4 = b^4 = 1, b^{-1}ab = a^{-1}\}$. The elements of $G_1$ can be expressed as $\{1, a, a^2, a^3, b, b^2, b^3, ab, ab^2, ab^3, a^2b, a^2b^2, a^2b^3, a^3b, a^3b^2, a^3b^3\}$ All the automorphism of $G$ are:

1. $\phi_1(a) = a, \quad \phi_1(b) = b$
2. $\phi_2(a) = a, \quad \phi_2(b) = b^3$
3. $\phi_3(a) = a, \quad \phi_3(b) = ab$
4. $\phi_4(a) = a, \quad \phi_4(b) = a^2b$
5. $\phi_5(a) = a, \quad \phi_5(a, b) = a^3b$
6. $\phi_6(a) = a, \quad \phi_6(b) = ab^3$
7. $\phi_7(a) = a, \quad \phi_7(b) = a^2b^3$
8. $\phi_8(a, b) = a, \quad \phi_8(a) = a^3b^3$
9. $\phi_9(a) = a^3, \quad \phi_9(b) = b$

10. $\phi_{10}(a) = a^3, \quad \phi_{10}(b) = b^3$

11. $\phi_{11}(a) = a^3, \quad \phi_{11}(b) = ab$

12. $\phi_{12}(a) = a^3, \quad \phi_{12}(b) = a^2b$

13. $\phi_{13}(a) = a^3, \quad \phi_{13}(b) = a^3b$

14. $\phi_{14}(a) = a^3, \quad \phi_{14}(b) = ab^3$

15. $\phi_{15}(a) = a^3, \quad \phi_{15}(b) = a^2b^3$

16. $\phi_{16}(a) = a^3, \quad \phi_{16}(b) = a^3b^3$

17. $\phi_{17}(a) = ab^2, \quad \phi_{17}(b) = b$

18. $\phi_{18}(a) = ab^2, \quad \phi_{18}(b) = b^3$

19. $\phi_{19}(a) = ab^2, \quad \phi_{19}(b) = ab$

20. $\phi_{20}(a) = ab^2, \quad \phi_{20}(b) = a^2b$

21. $\phi_{21}(a) = ab^2, \quad \phi_{21}(b) = a^3b$

22. $\phi_{22}(a) = ab^2, \quad \phi_{22}(b) = ab^3$

23. $\phi_{23}(a) = ab^2, \quad \phi_{23}(b) = a^2b^3$

24. $\phi_{24}(a) = ab^2, \quad \phi_{24}(b) = a^3b^3$

25. $\phi_{25}(a) = a^3b^2, \quad \phi_{25}(b) = b$

26. $\phi_{26}(a) = a^3b^2, \quad \phi_{26}(b) = b^3$

27. $\phi_{27}(a) = a^3b^2, \quad \phi_{27}(b) = ab$

28. $\phi_{28}(a) = a^3b^2, \quad \phi_{28}(b) = a^2b$

29. $\phi_{29}(a) = a^3b^2, \quad \phi_{29}(b) = a^3b$

30. $\phi_{30}(a) = a^3b^2, \quad \phi_{30}(b) = ab^3$

31. $\phi_{31}(a) = a^3b^2, \quad \phi_{31}(b) = a^2b^3$

32. $\phi_{32}(a) = a^3b^2, \quad \phi_{32}(b) = a^3b^3$

So Fusion classes of $G$ can expressed as:

- $F(a) = \{a, a^3, ab^2, a^3b^2\}$
- $F(a^2) = \{a^2\}$
- $F(b) = \{b, b^3, ab, a^2b, a^3b, ab^3, a^2b^3, a^3b^3\}$
- $F(b^2) = \{b^2\}$
- $F(a^2b^2) = \{a^2b^2\}$
- $F(1) = \{1\}$

Therfore, the number of distinct fusion classes $f(G)$ is 6. Hence

$$P_A(G) = \frac{f(G)}{|G|} = \frac{6}{16} = \frac{3}{8}.$$

□

**Theorem 3.16.** *There does not exist any finite group $G$ such that $P_A(G) = 5/8$.*

*Proof.* Let $G$ be a group with $P_A(G) = 5/8$. Then, by Lemma 3.10, $|G/L(G)| \leq 4$. If $|G/L(G)| \leq 3$ then by Lemma 3.1, $G$ is isomorphic to a cyclic group of order $2, 3, 4$ and $6$. Thus $P_A(G)$ cannot be $5/8$ in any of the cases. We therefore suppose that $|G/L(G)| = 4$. Then $G/L(G) \simeq C_4$ or $C_2 \times C_2$.
If $|G/L(G)| \simeq C_4$, then $G \simeq C_8$ by Lemma 3.1. Since $Aut(C_8) \simeq C_2 \times C_2$, $F(x) = \{\alpha(x) \mid \alpha \in C_2 \times C_2 \}$ and thus $f(C_8) = 4$. It follows that $P_A(C_8) = 1/2$. If $|G/L(G)| \simeq C_2 \times C_2$, then by Lemma 3.2, $G$ is isomorphic to one of the groups

1. $C_2 \times C_2$
2. $C_4 \times C_2$
3. $D_8$
4. $Q_8$
5. $G_1 = \{(a, b) \mid a^4 = b^4 = 1, b^{-1}ab = a^{-1}\}$.

Let $G \simeq C_2 \times C_2 = \{a, b \mid a^2 = b^2 = 1, ab = ba\}$.
Then by Lemma 3.11,

$$P_A(G) = \frac{1}{2} \neq \frac{5}{8}.$$

Let $G \simeq C_4 \times C_2 = \{a, b \mid a^4 = b^2 = 1, ab = ba\}$.
Then by Lemma 3.12,

$$P_A(G) = \frac{1}{2} \neq \frac{5}{8}.$$

Let $G \simeq Q_8 = \{a, b \mid a^4 = 1,\ a^2 = b^2,\ bab^{-1} = a^{-1}\}$.

Then by Lemma 3.13,

$$P_A(G) = \frac{3}{8} < \frac{5}{8}.$$

Let $G \simeq D_8 = \{a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1}\}$.

Then by Lemma 3.14,

$$P_A(G) = \frac{1}{2} < \frac{5}{8}.$$

Finally, Let $G \simeq G_1 = \{a, b \mid a^4 = b^4 = 1, b^{-1}ab = b^{-1}\}$.

Then by Lemma 3.15,

$$P_A(G) = \frac{3}{8} < \frac{5}{8}.$$

This proves the result. $\square$

# Chapter 4

# The Question of Mann

In this chapter we will prove the equivalence of the equalities $P_A(G) = P(G)$ and $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$, where $\mathrm{Aut}_c(G)$ denotes the group of all conjugacy class-preserving automorphisms of $G$. We would like to mention here that the equivalence may help to answer the question of Mann because it does help one eliminate certain classes of groups from consideration as shown in Corollaries 4.3 and 4.5
To prove our results we require some results of Cheng [2].

**Lemma 4.1.** *[2, Theorem 3] Suppose $G$ is a finite $p$-group, such that $G' = \langle a \rangle$ is cyclic. Assume that either $p > 2$, or $p = 2$ and $[a, G] \leq \langle a^4 \rangle$. If $\alpha$ is an automorphism of $G$ which centralizes both $G/G'$ and $Z(G)$, then $\alpha$ is inner.*

**Theorem 4.2.** *Let $G$ be a finite group. The following are equivalent:*

1. $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$
2. $P_A(G) = P(G)$
3. $k(G) = f(G)$

*Proof.* (1) We will show that $P_A(G) = P(G)$ if and only if $k(G) = f(G)$.

First suppose that $P_A(G) = P(G)$.

We know that

$$P_A(G) = \frac{f(G)}{|G|} \tag{4.1}$$

where $f(G)$ denotes number of distinct fusion classes of $G$,

and

$$P(G) = \frac{k(G)}{|G|} \tag{4.2}$$

where $k(G)$ denotes the number of distinct conjugacy classes of $G$.

Therefore,

$$\frac{f(G)}{|G|} = \frac{k(G)}{|G|},$$

and hence

$$f(G) = k(G).$$

Conversely, suppose that $f(G) = k(G)$ .

Divide both sides by $|G|$. Then

$$\frac{f(G)}{|G|} = \frac{k(G)}{|G|}.$$

Hence,

$$P_A(G) = P(G).$$

This proves the first result.

(2) Now we will prove that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ if and only if $k(G) = f(G)$.

First suppose that $k(G) = f(G)$.

If for some $x \in G$, $Cl(x)$ is properly contained in $F(x)$, then there exists an element $y \in F(x) \setminus Cl(x)$. This shows that $F(x)$ is union of more than one conjugacy classes and thus

$$f(G) < k(G),$$

which is a contradication to our supposition.

Therefore $Cl(x) = F(x)$ for each $x \in G$.

Let $\alpha \in \mathrm{Aut}(G)$. Then $\alpha(x) \in F(x) = Cl(x)$.

Therefore, $\alpha \in \mathrm{Aut}_c(G)$.

and hence

$$\mathrm{Aut}(G) = \mathrm{Aut}_c(G).$$

Conversely, suppose that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$.

We need to prove that $k(G) = f(G)$.

It is sufficient to prove that each fusion class is a conjugacy class.

For an element $x \in G$, consider a fusion class $F(x)$ and let $y \in F(x)$. Then there exists an automorphism $\alpha$ of $G$ such that $\alpha(x) = y$.

Since $\alpha$ is a class preserving automorphism, $y \in Cl(x)$ and hence

$$F(x) = Cl(x)$$

This proves the result. □

**Corollary 4.3.** *Let $G$ be a finite nonabelian $p$-group with $G' = [G, G]$ cyclic, where $p$ is an odd prime. Then $P_A(G) \neq P(G)$.*

*Proof.* If possible, suppose that $P_A(G) = P(G)$.

Then by Theorem 4.2, we have $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$.

Also, it follows from Lemma 4.1 that every automorphism centralizing $G/G'$ and $Z(G)$ is inner. Thus, $\mathrm{Aut}(G) = \mathrm{Aut}_c(G) = \mathrm{Inn}(G)$.

It is contradiction to famous result of Gashch*ü*tz [3] that every finite $p-$group of order at least $p^2$ has a noninner automorphism.
Therfore, $P_A(G) \neq P(G)$. $\square$

**Lemma 4.4.** *Let $G$ be a finite nonabelian group such that $|G/Z(G)| = 4$. Then $P(G) = 5/8$.*

*Proof.* Let $b_1, b_2 \in G$ such that $b_1b_2 \neq b_2b_1$.
Then $G/Z(G) =\{Z(G), b_1Z(G), b_2Z(G), b_3Z(G)\}$, where $b_3 = b_1b_2$. It is easy see that

$$\begin{aligned}
|D| &= |(x, y) \in G \times G \mid xy = yx| \\
&= |(x, y) \in G \times G \mid x \in Z(G), y \in G| \\
&+ \sum_{i=1}^{3} |(x, y) \in G \times G \mid x \in b_iZ(G), y \in Z(G)| \\
&+ \sum_{i=1}^{3} |(x, y) \in G \times G \mid x \in b_iZ(G), y \in b_iZ(G)| \\
&= |Z(G)||G| + 3|Z(G)||Z(G)| + 3|Z(G)||Z(G)| \\
&= |Z(G)||G| + 6|Z(G)|^2
\end{aligned}$$

Therefore,

$$\begin{aligned}P(G) &= \frac{|D|}{|G|^2}\\ &= \frac{|Z(G)||G| + 6|Z(G)|^2}{|G|^2}\\ &= \frac{|Z(G)||G|}{|G|^2} + 6\frac{|Z(G)|^2}{|G|^2}\\ &= \frac{|Z(G)|}{|G|} + 6|\frac{Z(G)}{G}|^2\\ &= \frac{1}{4} + 6(\frac{1}{16})\\ &= \frac{5}{8}.\end{aligned}$$

Hence, the result is proved. □

**Corollary 4.5.** *Let $G$ be a finite nonabelian group with $G/Z(G) \simeq C_2 \times C_2$. Then $\mathrm{Aut}_c(G) \neq \mathrm{Aut}(G)$.*

*Proof.* Since $G/Z(G) \simeq C_2 \times C_2$. Therefore $|G/Z(G)| = |C_2 \times C_2| = 4$. Hence by Lemma 4.4

$$P(G) = \frac{5}{8}.$$

But there does not exist any finite group $G$ with $P_A(G) = 5/8$ by Theorem 3.16

Thus

$$P(G) \neq P_A(G),$$

and hence by Theorem 4.2

$$\mathrm{Aut}(G) \neq \mathrm{Aut}_c(G).$$

□

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