ON COMBINATORICS OF q–SERIES AND OVERPARTITIONS

A thesis

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by

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Declaration

I hereby declare that thesis entitled **"On Combinatorics of q-series and overpartitions"**, submitted for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Meenakshi Rana, Associate Professor at School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or abroad and that the ideas and references cited herein have been duly acknowledged.

Vasudha

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Certificate

This is to certify that the work, which is being presented in the thesis, entitled **"On Combinatorics of q–series and Overpartitions"** which is being submitted by Ms. Vasudha, in fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is a record of the candidate's own independent and original research work carried out under my supervision. The matter embodied in this thesis has not been submitted in part or full to any university or institute for the award of a degree.

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.........dedicated to my loving little ones

Raghav and Vaishnavi

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Abstract

In this thesis, we study the combinatorial interpretations of various q–series identities listed in Chu–Zhang and Slater's compendium (W. Chu and W. Zhang, Bilateral bailey lemma and Rogers—Ramanujan identities, *Advances in Applied Mathematics,* 42(3):358–391, 2009 and L.J. Slater, Further identities of the Rogers–Ramanujan type, *The Proceedings of the London Mathematical Society,* 2(1):147– 167, 1952). We employ various combinatorial tools such as $(n+t)$–color overpartitions, three-line arrays, split part $(n+t)$–color partitions, 2–color F–partitions and lattice paths to establish the combinatorial interpretations of many q–series. Further we provide bijections between mentioned combinatorial tools.
We then find the combinatorial interpretations of q–series identities involving double sums series related to moduli $4k$ and $4k+2$ using $(n+t)$–color partitions.
In addition to above, we use the combinatorial tools of color partitions, split color partitions and signed partitions notion to define signed color partitions. This tool is used to provide the combinatorial interpretations of 100 q–series identities listed in Chu–Zhang and Slater's compendium. Furthermore, by employing the same tool with attaching weight, we interpret four more q–series identities.
Finally, in this thesis, we study the arithmetic properties of some q–series identities. We find several congruences for the coefficients of power of q that are in arithmetic progressions modulo powers of 2 and 3.

List of Publications

1. V. Gupta and M. Rana. On some combinatorics of Rogers–Ramanujan type identities using signed color partitions. *Current trends in mathematical analysis and its interdisciplinary applications,* 101–118, Birkhäuser/Springer, 2019.

2. V. Gupta, M. Rana, and S. Sharma. On weighted signed color partitions, *Proceedings-Mathematical Sciences,* 130(1)1–10, 2020.

3. V. Gupta and M. Rana. Rogers–Ramanujan type identities for $(n+t)$–color overpartitions. *Journal of Ramanujan Society of Mathematics and Mathematical Sciences,* 8(2):01–16, 2021.

4. V. Gupta and M. Rana. Some Congruences for the Coefficients of Rogers–Ramanujan Type Identities, *Mathematics,* 10(19):3582, 2022.

5. V. Gupta and M. Rana. On some combinatorial interpretations for Rogers–Ramanujan type identities. (Accepted in South East Asian Journal of Mathematics and Mathematical Sciences.

5. V. Gupta and M. Rana. Rogers–Ramanujan type identities related to moduli $4k$ and $4k+2$. (Communicated)

6. V. Gupta and M. Rana. Rogers–Ramanujan type identities and three-line array. (Communicated)

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Chapter 1

Introduction

Combinatorics is a very profound area in modern mathematics. Despite the fact that combinatorial mathematics has been studied since Euler's time, this field only came into existence in later years. It has been noted that there is a deep connection between combinatorics and other major branches of mathematics. One of the most fundamental and significant aspects of combinatorics is "Enumeration." The classical problems of enumerative combinatorics involve computing partitions of various kinds, in which an object is broken down into smaller objects in various ways. In this thesis, we will be studying partitions of natural numbers. The study of partitions was initiated by Leibniz and was apparently the first development in partition theory. However, the most significant contributions to the theory of partitions are due to the great mathematician Euler.

1.1 Preliminaries

In this chapter, we will provide basic definitions and notations that are required in the subsequent chapters. Let us begin with the definition of a partition:

Definition 1.1.1 *A partition of a positive number α is defined to be a sequence of positive integers $(m_1, m_2, \ldots, m_r)$, where*

$$m_1 + m_2 + \ldots + m_r = \alpha,$$

and $m_k \leq m_{k+1}$ $\forall$ $k = 1, 2, \ldots, r-1$. The summand m_k is known as a part of α. Here, $p(\alpha)$ is considered as the number of partitions of α. By convention, 0 has only one partition called empty partition, that is, $p(0) = 1$ and $p(\alpha) = 0$ for negative α.

Remark 1.1.1 *Throughout this thesis, we will write the partition as $(m_r, m_{r-1}, \ldots, m_1)$ instead of $(m_1, m_2, \ldots, m_r)$, where parts are in non increasing order. Now, in the partition $(m_r, m_{r-1}, \ldots, m_1)$, we note m_1 is the smallest part and m_r is the largest part.*

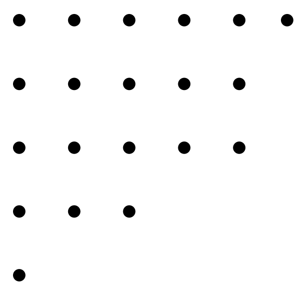
Example 1.1.1 *There are 5 partitions of 4, hence $p(4) = 5$, and the relevant partitions are as follows:*

$$(4), \quad (3\ 1), \quad (2\ 2), \quad (2\ 1\ 1), \quad (1\ 1\ 1\ 1).$$

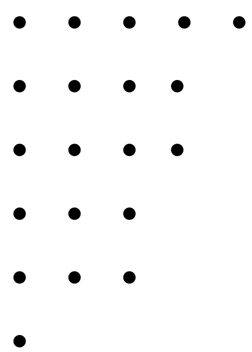
A partition of α can be represented graphically by an array of dots or nodes, known as Ferrers graph.

Definition 1.1.2 *The Ferrers graph for a partition* $(m_r \ m_{r-1} \ \ldots \ m_1)$ *of* α *is the set of arrays of dots or nodes aligned at left and equal distance such that the array of dots having* m_r *dots in the top row,* m_{r-1} *in the next row and so on down to* m_1 *in the last row.*

Example 1.1.2 *For the partition* $(6 \ 5 \ 5 \ 3 \ 1)$ *of* $\alpha = 20$*, the Ferrers graph is as follows:*



If we read above Ferrers graph column wise, it displays a new partition $(5 \ 4 \ 4 \ 3 \ 3 \ 1)$ *of* 20 *as follows, which is the conjugate of the given partition.*



Remark 1.1.2 *If a partition has no change on interchanging the rows and the columns, such partition is known as self-conjugate partition. For instance, the self conjugate partition of* 10 *is* $(4 \ 3 \ 2 \ 1)$.

Another representation of ordinary partition is due to Frobenius in [44] for understanding the representation theory of groups.

Definition 1.1.3 *In* [44], *Frobenius representation of an ordinary partitions is a two rowed array*

$$\begin{pmatrix} a_r & a_{r-1} & \cdots & a_1 \\ b_r & b_{r-1} & \cdots & b_1 \end{pmatrix}$$

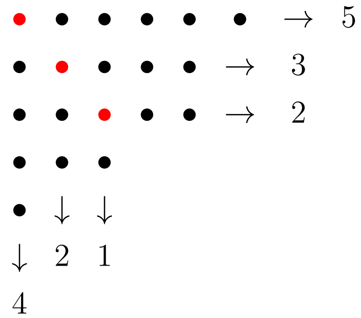
of non negative integers arranged so that each array's components are unique and arranged in decreasing order. The representation described above is referred to as a Frobenius rep-

resentation or a way to express an ordinary partition of α *if*

$$\alpha = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

A bijection of an ordinary partition to Frobenius representation is explained in the following example through Ferrers graph.

Example 1.1.3 *Here, we take a partition* $(6\ 5\ 5\ 3\ 1)$ *of* $\alpha = 20$, *the Ferrers graph of this partition is given below;*



There are three diagonal dots in above Ferrers graph, hence $r = 3$. *Horizontal dots right to the diagonal dots are* $a_1 = 5$, $a_2 = 3$, $a_3 = 2$ *respectively. Vertical dots below the diagonal dots are* $b_1 = 4$, $b_2 = 2$, $b_3 = 1$ *respectively. Thus the Frobenius symbol becomes*

$$\begin{pmatrix} 5 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix}$$

and is associated with ordinary partition $(6\ 5\ 5\ 3\ 1)$ *of* $\alpha = 20$. *So this example shows the bijection between ordinary partition and its corresponding Frobenius symbol.*
In general, there is no explicit formula for determining the value of $p(\alpha)$ instead there are systematic methods to find out the value of $p(\alpha)$ for particular values of α. Hardy and Ramanujan in 1918 broughtforth a famous "asymptotic formula" for $p(\alpha)$ in [49]. The simplest special case of their result is the assertion that as $\alpha \to \infty$

$$p(\alpha) \sim \frac{1}{4\alpha\sqrt{3}} e^{\pi\sqrt{\frac{2\alpha}{3}}}.$$

And using the generating function of partitions, it is easy to codify all the numbers $p(\alpha)$. We note $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $\ldots$, $p(50) = 204226$, $\ldots$, $p(100) = 190569292$, $\ldots$. Euler used generating function to enumerate $p(\alpha)$ as

$$\begin{aligned} \sum_{\alpha=0}^{\infty} p(\alpha) q^{\alpha} &= 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots \\ &= \prod_{k=1}^{\infty} \frac{1}{(1-q^k)} = \frac{1}{(q;q)_\infty}, \end{aligned}$$

where $(x;q)_\infty$ is a q–Pochhammer symbol and $(x;q)_\infty = \prod_{k=1}^{\infty}(1-xq^k)$, also $(x;q)_\alpha = \prod_{k=1}^{\alpha-1}(1-xq^k)$, where '$x$' and '$q$' are complex numbers with, $|q|<1$. We also note that

$$\begin{aligned} &[x_1, x_2, \ldots, x_k; q]_\alpha = (x_1;q)_\alpha (x_2;q)_\alpha \cdots (x_k;q)_\alpha, \\ \text{and} \qquad &[x_1, x_2, \ldots, x_k; q]_\infty = (x_1;q)_\infty (x_2;q)_\infty \cdots (x_k;q)_\infty. \end{aligned}$$

Now the question arises whether generating function can provide any interesting result about the numbers $p(\alpha)$. The answer for this question was derived through a simple manipulation of generating functions by Euler. He proved numerous results using the generating functions in [43], for instance see Theorem 1.1.3 below:

Theorem 1.1.3 *For $\alpha > 0$, the number of partitions of α with distinct parts is same as the number of partitions of α with odd parts. That is,*

$$\prod_{k=1}^{\infty}(1+q^k) = \prod_{k=1}^{\infty}\frac{1}{(1-q^{2k-1})}.$$

Remark 1.1.4 *In general, the generating function including an expression of the form $(x;q)_\alpha$ or $(x;q)_\infty$ is called q–series.*

Also, the study of other partition theoretic functions such as smallest part function, largest part function is an intense area for researchers, for details see [24, 42]

1.2 Rogers–Ramanujan Identities

There are several findings in partition theory that are similar to Euler's work, which show that two sets of partitions have the same number of elements. For instance, the identities given below associate a different set of partitions for the left-hand side and right-hand side, respectively. The different sets of partitions associated with the following identities can be seen in Theorem 1.2.1–1.2.2, respectively. These identities are called "sum-product identities" or "q-series identities" in partition theory.

$$\sum_{\alpha=0}^{\infty}\frac{q^{\alpha^2}}{(q;q)_\alpha} = \prod_{\alpha=1}^{\infty}(1-q^{5\alpha-1})^{-1}(1-q^{5\alpha-4})^{-1}, \tag{1.2.1}$$

$$\sum_{\alpha=0}^{\infty}\frac{q^{\alpha^2+\alpha}}{(q;q)_\alpha} = \prod_{\alpha=1}^{\infty}(1-q^{5\alpha-2})^{-1}(1-q^{5\alpha-3})^{-1}. \tag{1.2.2}$$

They were initially found in 1894 by Rogers [72], and then again in 1913 by Ramanujan. A correspondence followed between Ramanujan and Rogers. In 1919, Ramanujan published an article in which he provided two proofs: the first by Ramanujan himself and the second by Rogers [65]. There was also a note by Hardy, and hence these are known as Rogers-Ramanujan identities. A lot of literature on these identities can be found in [20, 22, 67]. The combinatorial interpretations of (1.2.1)–(1.2.2) are due to MacMahon [60] and are as follows:

Theorem 1.2.1 *The number of partitions of α into parts with minimal difference 2 is same as the number of partitions of α into parts that are $\equiv \pm 1 \pmod 5$.*

Theorem 1.2.2 *The number of partitions of α with minimal part 2 and minimal difference 2 is same as the number of partitions of α into parts that are $\equiv \pm 2 \pmod 5$.*

Recently, Afsharijoo [1] added a new companion to the above q–series identities. This new companion counts partitions with restrictions on even and odd parts. Bailey [28] systematically explored several q–series identities. Additionally, Slater [79] provided a list of 130 q–series identities. Furthermore, Chu and Zhang [35] found many q–series identities using certain transformations. The q–series identities have been studied in different contexts. For instance, the hard hexagon model in statistical mechanics, a specific instance of a solvable family of hard-square-type models, naturally incorporates many q–series identities. Baxter [30] explained that a number of q–series identities occur in the determination of sub-lattice densities and order parameters. Kedem et al. [53] believed that the q–series identities represent the partition function of a physical system with quasiparticles that adhere to specific exclusion statistics. The relationship between q–series identities and fractional statistics is developed by these exclusion statistics, which are related to fractional statistics. Furthermore, the combinatorial interpretations of many q–series identities were studied using different combinatorial tools, and are available in [3, 14, 68]. In order to study more about q–series identities, $(n+t)$–color partition was introduced by Agarwal and Andrews [10].

Definition 1.2.1 *For any integer $t \geq 0$, an $(n+t)$–color partition, is a partition in which a part of size n can appear in $(n+t)$–colors as $n_1, n_2, \ldots, n_{n+t}$. The parts are arranged first by size and then by color. If t is positive integer, the partition may have the part of size 0 but only one copy of zero '0_t' is allowed. For $t = 0$, $(n+t)$–color partitions are simply called as n–color partitions.*
The generating function for n–color partition of α, denoted by $p(\alpha)$, is given by:

$$1 + \sum_{\alpha=1}^{\infty} p(\alpha) q^{\alpha} = \prod_{k=1}^{\infty} (1 - q^k)^{-k}$$

Definition 1.2.2 *The weighted difference of two parts $(m_k)_{x_k}, (m_l)_{x_l}$, $m_k \geq m_l$ in an $(n+t)$–color partition $(m_r)_{x_r} + (m_{r-1})_{x_{r-1}} + \ldots + (m_1)_{x_1}$ such that $(m_r)_{x_r} \geq (m_{r-1})_{x_{r-1}} \geq$*

$\ldots \geq (m_1)_{x_1}$, *is* $m_k - m_l - x_k - x_l$ *and denoted by* $\big(((m_k)_{x_k} - (m_l)_{x_l})\big)$. *For convenience, we use* δ_k *to denote the weighted difference throughout this thesis, where* $\delta_k = \big(((m_k)x_k - (m_{k-1})x_{k-1})\big)$.

Example 1.2.1 *For* $\alpha = 2$ *and* $t = 2$, *the* $(n+2)$*–color partitions are as follows, where only one copy of zero is allowed, that is* 0_2.

(2_4)	$(2_4\ 0_2)$	$(1_3\ 1_3)$	$(1_2\ 1_1)$	$(1_3\ 1_1\ 0_2)$
(2_3)	$(2_3\ 0_2)$	$(1_3\ 1_2)$	$(1_1\ 1_1)$	$(1_2\ 1_2\ 0_2)$
(2_2)	$(2_2\ 0_2)$	$(1_3\ 1_1)$	$(1_3\ 1_3\ 0_2)$	$(1_2\ 1_1\ 0_2)$
(2_1)	$(2_1\ 0_2)$	$(1_2\ 1_2)$	$(1_3\ 1_2\ 0_2)$	$(1_1\ 1_1\ 0_2)$

Several mathematicians have provided interpretations of certain q–series identities such as (1.2.1) and (1.2.2) in partition theory. To exemplify, see [2, 36, 45–47, 50, 83, 84]. In 1987, Agarwal [2] interpreted the following two identities listed in [79], as the Identity no. 46 and 61 respectively:

$$\sum_{\alpha=0}^{\infty} \frac{q^{\frac{\alpha(3\alpha-1)}{2}}}{(q;q)_\alpha (q;q^2)_\alpha} = \frac{1}{(q;q)_\infty} \prod_{\alpha=1}^{\infty} (1-q^{10\alpha})(1-q^{10\alpha-4})(1-q^{10\alpha-6}), \tag{1.2.3}$$

$$\sum_{\alpha=0}^{\infty} \frac{q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_\alpha} = \frac{1}{(q;q)_\infty} \prod_{\alpha=1}^{\infty} (1-q^{14\alpha})(1-q^{14\alpha-6})(1-q^{14\alpha-8}). \tag{1.2.4}$$

The combinatorial interpretations of (1.2.3)–(1.2.4) using n–color partitions are as follows:

Theorem 1.2.3 *The number of* n*–color partitions of* α *such that each pair of parts has a positive weighted difference equals the number of ordinary partitions of* α *into parts* $\not\equiv 0, \pm 4 (\text{mod } 10)$.

Theorem 1.2.4 *The number of* n*–color partitions of* α *such that each pair of parts has a non negative weighted difference equals the number of ordinary partitions of* α *into parts* $\not\equiv 0, \pm 6 (\text{mod } 14)$.

Combinatorial interpretations of various q–series identities have been provided in terms of $(n+t)$–color partitions (refer. [3–5, 15, 48, 69]). Further in 2014, $(n+t)$–color partitions have been extended to 'split $(n+t)$–color partitions' by Agarwal and Sood in [16].

Definition 1.2.3 *A split* $(n+t)$*–color partition is defined as an* $(n+t)$*–color partition of* α (≥ 0), *in which the color* k *of the part* m_k *can be splitted into two parts say, green part denoted by 'g' and red part denoted by 'r', such that* $k = g + r$, *where* $1 \leq g \leq k$, $0 \leq r \leq k-1$.

Example 1.2.2 *For $\alpha = 2$, there are four split n–color partitions shown as:*

$$(2_2), \quad (2_{1+1}), \quad (2_1), \quad (1_1\ 1_1).$$

Remark 1.2.5 *The above example illustrates that whenever the red part is 0, it will not be counted separately. This means that 2_{g+0} is the same as 2_g.*

1.3 Overpartitions

The notion of overpartitions was originated by Corteel and Lovejoy in [37]. They conducted a comprehensive study of overpartitions and analyzed various q–series identities.

Definition 1.3.1 *An overpartition of α is defined as a partition in which the first or the last occurrence of each non-identical number may be overlined. Let $\overline{\mathrm{p}}(\alpha)$ denote the number of overpartitions of an integer α. Conventionally, $\overline{\mathrm{p}}(0) = 1$, and the generating function is given by:*

$$\sum_{\alpha=0}^{\infty} \overline{p}(\alpha) q^{\alpha} = \prod_{\alpha=1}^{\infty} \frac{1+q^{\alpha}}{1-q^{\alpha}} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \dots,$$

which can be accepted as a convolution product between partitions into non identical parts (the overlined terms) and ordinary partitions (the rest).

Example 1.3.1 *There are fourteen overpartitions of 4, hence $\overline{\mathrm{p}}(4) = 14$ given as follows:*

$$(4), (\overline{4}), (3\ 1), (3\ \overline{1}), (\overline{3}\ 1), (\overline{3}\ \overline{1}), (2\ 2), (2\ \overline{2}), (2\ 1\ 1),$$

$$(2\ 1\ \overline{1}), (\overline{2}\ 1\ 1), (\overline{2}\ 1\ \overline{1}), (1\ 1\ 1\ 1), (1\ 1\ 1\ \overline{1}).$$

Several q–series identities have been interpreted combinatorially using overpartitions and possess many analogous properties as ordinary partitions, refer [37–39, 41, 55–58].

Chapter 2

q–series identities for $(n+t)$–color overpartitions

2.1 Introduction

In the mid-1990s, Andrews discussed a number of q–series whose coefficients are determined by twisted divisor functions. Using the same concept, Lovejoy and Mallet derived some similar q–series in [59]. To interpret these q–series combinatorially, they used n–color overpartitions, which is an extension of overpartitions and analogous to n–color partitions. Some of these identities are firmly related to the Rogers–Ramanujan type identities. Furthermore, in [61], Mallet extended the n–color overpartitions to $(n+t)$–color overpartitions and used this tool to interpret generalized multiple series that have been studied by Agarwal, Andrews and Bressoud in some other context (refer [11]). Motivated from above combinatorial work, the purpose of the current chapter is to study many q–series identities using $(n+t)$–color overpartitions.

Definition 2.1.1 [59] *The n–color overpartition is an n–color partition in which the final occurrence of a part m_k can be overlined. The generating function for n–color overpartition is*

$$\sum_{\alpha=1}^{\infty} \overline{\mathrm{p}}(\alpha)q^{\alpha} = \prod_{\alpha=1}^{\infty} \frac{(1+q^{\alpha})^{\alpha}}{(1-q^{\alpha})^{\alpha}} = 1 + 2q + 6q^2 + 16q^3 + 38q^4 + 88q^5 + \dots .$$

Example 2.1.1 *For $\alpha = 4$, $\overline{\mathrm{p}}(4) = 38$, the relevant n–color overpartitions are:*

(4_4) $(\overline{4}_4)$ $(3_3\ 1_1)$ $(3_2\ 1_1)$ $(3_1\ 1_1)$ $(2_2\ 2_2)$ $(\overline{2}_2\ 2_1)$ $(2_2\ 1_1\ 1_1)$ $(2_1\ 1_1\ 1_1)$ $(1_1\ 1_1\ 1_1\ 1_1)$
(4_3) $(\overline{4}_3)$ $(\overline{3}_3\ 1_1)$ $(\overline{3}_2\ 1_1)$ $(\overline{3}_1\ 1_1)$ $(2_2\ \overline{2}_2)$ $(\overline{2}_2\ \overline{2}_1)$ $(2_2\ 1_1\ \overline{1}_1)$ $(2_1\ 1_1\ \overline{1}_1)$ $(1_1\ 1_1\ 1_1\ \overline{1}_1)$
(4_2) $(\overline{4}_2)$ $(3_3\ \overline{1}_1)$ $(3_2\ \overline{1}_1)$ $(3_1\ \overline{1}_1)$ $(2_2\ 2_1)$ $(2_1\ 2_1)$ $(\overline{2}_2\ 1_1\ 1_1)$ $(\overline{2}_1\ 1_1\ 1_1)$
(4_1) $(\overline{4}_1)$ $(\overline{3}_3\ \overline{1}_1)$ $(\overline{3}_2\ \overline{1}_1)$ $(\overline{3}_1\ \overline{1}_1)$ $(2_2\ \overline{2}_1)$ $(2_1\ \overline{2}_1)$ $(\overline{2}_2\ 1_1\ \overline{1}_1)$ $(\overline{2}_1\ 1_1\ \overline{1}_1)$.

The weighted difference of any two parts $(m_k)_{x_k}$ and $(m_l)_{x_l}$ in n–color overpartitions is defined in the same way as it was defined in n–color partition, as discussed in Chapter

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1. In $(n+t)$–color overpartitions, the parts of size n can appear in $(n+t)$, where $t \geq 0$, different colors. Also zero appears only once, either 0_x or $\overline{0}_x$ may be used. In this Chapter we study q–series identities given in Table 2.1.

Table 2.1: q–series identities.

Sr No.	**Function**	**Sum Side**	=	**Product Side**
1.	$h_1(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q;q)_{2\alpha}}$	=	$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^2,-q^4,q^6;q^6]_\infty$
2.	$h_2(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1}(q^2;q^2)_\alpha}$	=	$\frac{1}{(q;q)_\infty}[q^4,q^8,q^{12};q^{12}]_\infty$
3.	$h_3(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha+1}(q^2;q^2)_\alpha}$	=	$\frac{1}{(q;q)_\infty}[q^2,q^{10},q^{12};q^{12}]_\infty$
4.	$h_4(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha(q;q^2)_\alpha q^{\alpha^2}}{(-q;q^2)_\alpha(q^4;q^4)_\alpha}$	=	$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^2,-q^3,q^5;q^5]_\infty$
5.	$h_5(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha(q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(-q;q^2)_\alpha(q^4;q^4)_\alpha}$	=	$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[-q,-q^4,q^5;q^5]_\infty$
6.	$h_6(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha(q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(-q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	=	$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^5,-q^5,q^5;q^5]_\infty$
7.	$h_7(q)$	$\sum_{\alpha=1}^{\infty} \frac{(-q^2;q^2)_{\alpha-1} q^{\alpha^2}}{(q;q)_{2\alpha}}$	=	$\frac{[q^2,q^{14},q^{16};q^{16}]_\infty[q^{12},q^{20};q^{32}]_\infty}{(q;q)_\infty}$
8.	$h_8(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$	=	$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[-q^3,-q^3,q^6;q^6]_\infty$
9.	$h_9(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q^2;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	=	$\frac{(-q^2,q^2)_\infty}{(q^2;q^2)_\infty}[-q,-q^5,q^6;q^6]_\infty$
10.	$h_{10}(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^4)_\alpha q^{\alpha^2}}{(q;q^2)_\alpha(q^4;q^4)_\alpha}$	=	$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^3,-q,-q^4;-q^4]_\infty$
11.	$h_{11}(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^4)_\alpha q^{\alpha(\alpha+2)}}{(q;q^2)_\alpha(q^4;q^4)_\alpha}$	=	$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q,-q^3,-q^4;-q^4]_\infty$
12.	$h_{12}(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q^2)_\alpha(q;q)_\alpha}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^3,q^3,q^6;q^6]_\infty$
13.	$h_{13}(q)$	$\sum_{\alpha=1}^{\infty} \frac{(-q;q)_{\alpha-1} q^{\alpha^2}}{(q;q^2)_\alpha(q;q)_\alpha}$	=	$\frac{[-q^5,-q^7,q^{12};q^{12}]_\infty}{(q;q)_\infty}$
14.	$h_{14}(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q;q)_\alpha q^{\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1}(q;q)_\alpha}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q,q^5,q^6;q^6]_\infty$

2.2 Combinatorial interpretations using $(n+t)$–color overpartitions

In this section, we present combinatorial interpretations of fourteen q–series identities with $(n+t)$–color overpartitions. These identities are listed in the Table 2.1. The first identity appears in [79] as Identity No. 29 and remaining identities in [35] as Identity No. 104, 102, 29, 27, 25, 195, 45, 46, 11, 12, 37, 106, 40, respectively. The sum side in a q–series identity represents generating function for $M_l(\alpha)$, which counts partitions in terms of $(n+t)$–color overpartitions, where $1 \leq l \leq 14$. The product side represents the generating function for $N_l(\alpha)$, $1 \leq l \leq 14$, which counts ordinary partitions. This lead to two-way combinatorial interpretations satisfying,

$$h_l(q) = \sum_{\alpha=0}^{\infty} M_l(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} N_l(\alpha) q^{\alpha}, \quad 1 \leq l \leq 14. \tag{2.2.1}$$

We now provide the combinatorial interpretations using n–color overpartitions for $h_l(q)$ where $1 \leq l \leq 14$ in Theorem 2.2.2–2.2.24, respectively.

Remark 2.2.1 *Throughout this section, we consider*

$$h_l(w,q) = \sum_{\alpha=0}^{\infty} \sum_{r=0}^{\infty} M_l(r,\alpha) w^r q^{\alpha}, \tag{2.2.2}$$

for $|q| < 1$, $|w| < |q|^{-1}$ *and* $l = 1$–14. $M_l(r,\alpha)$ *represents the partitions with identical conditions of* $M_l(\alpha)$ *into* r *parts.*

Theorem 2.2.2 *Let* $M_1(\alpha)$ *represent the number of* n*–color overpartitions of* α *satisfying*

(2.2.2.a) $m_k \equiv x_k \pmod{2}$, $\forall\ k$,

(2.2.2.b) *if* $x_k = 1$ *then the occurrence of* m_k *is not overlined.*

Let $N_1(\alpha)$ *is the number of partitions of* α *such that the odd parts are distinct, the even parts are* $\equiv \pm 2 \pmod{6}$ *and two copies of the parts* $\equiv \pm 2 \pmod{12}$ *are allowed. Then*

$$M_1(\alpha) = N_1(\alpha), \ \forall\, \alpha \geq 0.$$

Example 2.2.1 *We note* $M_1(8) = N_1(8) = 18$, *the relevant* n*–color overpartitions of* $M_1(8)$ *are*

$$(8_8), (\overline{8}_8), (8_6), (\overline{8}_6), (8_4), (\overline{8}_4), (8_2), (\overline{8}_2), (7_5\ 1_1), (\overline{7}_5\ 1_1), (7_3\ 1_1),$$

$$(\overline{7}_3\ 1_1), (7_1\ 1_1), (6_2\ 2_2), (6_2\ \overline{2}_2), (\overline{6}_2\ 2_2), (\overline{6}_2\ \overline{2}_2), (5_1\ 3_1).$$

And the partitions corresponding to $N_1(8)$ *are*

(8),	$(4\ 4)$,	$(4\ 3\ 1)$,	$(4\ 2_2\ 2_2)$,	$(3\ 2_1\ 2_2\ 1)$,	$(2_1\ 2_1\ 2_2\ 2_2)$,
$(7\ 1)$,	$(5\ 2_1\ 1)$,	$(4\ 2_1\ 2_1)$,	$(3\ 2_1\ 2_1\ 1)$,	$(2_1\ 2_1\ 2_1\ 2_1)$,	$(2_1\ 2_2\ 2_2\ 2_2)$,
$(5\ 3)$,	$(5\ 2_2\ 1)$,	$(4\ 2_1\ 2_2)$,	$(3\ 2_2\ 2_2\ 1)$,	$(2_1\ 2_1\ 2_1\ 2_2)$,	$(2_2\ 2_2\ 2_2\ 2_2)$.

Proof of Theorem 2.2.2. As considered above, split the partitions enumerated by

$M_1(r,\alpha)$ into four classes. The first class contains the partitions in which $(m_1)_{x_1} = 1_1$. Removing 1_1 and subtract 2 from all the remaining parts, it will not disturb the inequalities between the parts and transformed partitions will be enumerated by $M_1(r-1,\alpha-2r+1)$. Partitions in the second class contain in which $(m_1)_{x_1} = \overline{2}_2$. By removing $\overline{2}_2$ the least part and then subtract 4 from all the remaining parts. The transformed partitions will be enumerated by $M_1(r-1,\alpha-4r+2)$. Third class contains the partitions that do not have $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ or $(\overline{x}_k)_{x_k}$. Transform the partitions by subtracting 2 from all the parts. The transformed partitions are enumerated by $M_1(r,\alpha-2r)$. In the last class the partition involves $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ $(x \geq 2)$ or $(\overline{x}_k)_{x_k}$ $(x \geq 3)$. Transformed by replacing $(x_k)_{x_k}$ by $(x_k-1)_{x_k-1}$ or $(\overline{x}_k)_{x_k}$ by $(\overline{x}_k-1)_{x_k-1}$ and subtract 2 from the remaining parts. This will result in partitions enumerated by $M_1(r,\alpha-2r+1)$. It should be noted here that we are obtaining those partitions of $\alpha-2r+1$ which have a part of the form $(x_k)_{x_k}$ and $(\overline{x}_k)_{x_k}$ so the number of partitions in this class are enumerated by $M_1(r,\alpha-2r+1) - M_1(r,\alpha-4r+1)$. The above transformations cannot affect the subscripts of corresponding parts. Thus the transformed partitions are enumerated by $M_1(r,\alpha-2r+1) - M_1(r,\alpha-4r+1)$. Hence we get the following recurrence formula for $M_1(r,\alpha)$:

$$\begin{aligned} M_1(r,\alpha) =& M_1(r-1,\alpha-2r+1) + M_1(r-1,\alpha-4r+2) + M_1(r,\alpha-2r) \\ &+ M_1(r,\alpha-2r+1) - M_1(r,\alpha-4r+1), \end{aligned} \tag{2.2.3}$$

where $M_1(0,0) = 1$ and $M_1(r,\alpha) = 0$ for $\alpha < 0$. Substitute $M_1(r,\alpha)$ from (2.2.3) in (2.2.2), we get q–functional equation

$$\begin{aligned} h_1(w,q) =& wqh_1(wq^2,q) + wq^2h_1(wq^4,q) + h_1(wq^2,q) \\ &+ q^{-1}h_1(wq^2,q) - q^{-1}h_1(wq^4,q). \end{aligned} \tag{2.2.4}$$

Setting

$$h_1(w,q) = \sum_{\alpha=0}^{\infty} z_1(\alpha,q)w^\alpha. \tag{2.2.5}$$

Using (2.2.4) in (2.2.5) and then examining the coefficients of w^α, we get

$$z_1(\alpha,q) = \frac{q^{2\alpha-1}(1+q^{2\alpha-1})}{(1-q^{2\alpha})(1-q^{2\alpha-1})} z_1(\alpha-1,q). \tag{2.2.6}$$

Iterating (2.2.6) α times and note that $z_1(0,q) = 1$, we find

$$z_1(\alpha,q) = \frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q^2;q^2)_\alpha (q;q^2)_\alpha}. \tag{2.2.7}$$

Therefore,

$$h_1(w,q) = \sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q^2;q^2)_\alpha (q;q^2)_\alpha} w^\alpha = M_1(w,q), \tag{2.2.8}$$

and

$$\begin{aligned}\sum_{\alpha=0}^{\infty} M_1(\alpha) q^\alpha &= \sum_{\alpha=0}^{\infty}\left(\sum_{r=0}^{\infty} M_1(r,\alpha)\right) q^\alpha \\ &= h_1(1,q) \\ &= h_1(q).\end{aligned}$$

On the right side of $h_1(q)$ is

$$\begin{aligned}\sum_{\alpha=0}^{\infty} N_1(\alpha) q^\alpha =& \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^2,-q^4,q^6;q^6]_\infty, \\ =& \frac{(-q;q^2)_\infty (q^4;q^{12})_\infty (q^6;q^{12})_\infty (q^8;q^{12})_\infty (q^{12};q^{12})_\infty}{(q^2;q^{12})_\infty^2 (q^4;q^{12})_\infty^2 (q^6;q^{12})_\infty (q^8;q^{12})_\infty^2 (q^{10};q^{12})_\infty^2 (q^{12};q^{12})_\infty},\end{aligned}$$

and we know that $(q^2;q^{12})_\infty (q^4;q^{12})_\infty (q^8;q^{12})_\infty (q^{10};q^{12})_\infty = (q^2;q^6)_\infty (q^4;q^6)_\infty$

Remark 2.2.3 *We will only outline the necessary steps for the remaining theorems in this section to reach upto recurrence relations, since the remaining steps are similar as in the proof of* Theorem 2.2.2.

Theorem 2.2.4 *Let $M_2(\alpha)$ represent the number of $(n+1)$–color overpartitions of α satisfying* (2.2.2.b)*, with*

(2.2.4.a) $x_1 = m_1 + 1$,

(2.2.4.b) $(m_1)_{x_1}$ *is not overlined,*

(2.2.4.c) $m_k - x_k \equiv 1 \pmod 2$, $\forall\ k$.

Let $N_2(\alpha)$ is the number of partitions of α in which the parts are $\not\equiv 0, \pm 4 \pmod{12}$. Then

$$M_2(\alpha) = N_2(\alpha),\ \ \forall\, \alpha \geq 0.$$

Proof. $M_2(r,\alpha)$ represent the $(n+1)$–color overpartitions enumerated by $M_2(\alpha)$ of α into r parts. The partitions enumerated by $M_2(r,\alpha)$ split into two classes. The first class contains the partitions in which $(m_1)_{x_1}$ is of the form 0_1. The second class contains the partitions in which $(m_1)_{x_1}$ is of the form $(x_k)_{x_k+1}$, $x_k > 0$. After the transformation, the partitions in the first class are enumerated by $M_1(r-1, \alpha-r+1)$ and in the second class are enumerated by $M_2(r, \alpha - 2r + 1)$. We get the following recurrence relation as:

$$M_2(r, \alpha) = M_1(r - 1, \alpha - r + 1) + M_2(r, \alpha - 2r + 1).$$

Theorem 2.2.5 *Let $M_3(\alpha)$ represent the number of $(n+2)$–color overpartitions of α satisfying* (2.2.2.a)–(2.2.2.b), (2.2.4.b) *and $x_1 = m_1 + 2$.*
Let $N_3(\alpha)$ is the number of partitions of α in which the parts are $\not\equiv \pm 2 \pmod{12}$. Then

$$M_3(\alpha) = N_3(\alpha), \ \forall \, \alpha \geq 0.$$

Proof. The proof can be developed in the similar manner of the proof of Theorem 2.2.4, where the recurrence relation for the enumerator $M_3(r, \alpha)$ is

$$M_3(r, \alpha) = M_1(r - 1, \alpha - 2r + 2) + M_3(r, \alpha - 2r + 1).$$

Theorem 2.2.6 *Let $M_4(\alpha)$ represent the number of n–color overpartitions of α satisfying the following conditions along with* (2.2.2.b)

(2.2.6.a) $m_1 - x_1 \equiv 0 \pmod 4$ *and* $m_k - x_k \equiv 0 \pmod 2$, *for* $k \geq 2$,

(2.2.6.b) $\delta_k \geq 0$ *and* $\equiv 0 \pmod 4$, $\forall \ k > 1$.

Let $N_4(\alpha) = \sum_{i=0}^{\alpha} X_4(\alpha - i)Y_4(i)$, $X_4(\alpha)$ is the number of partitions of α into distinct parts that are either $\equiv 5 \pmod{10}$ or $\equiv \pm 1 \pmod{10}$ with two copies of parts that are $\equiv 5 \pmod{10}$ allowed and one copy of parts that are $\equiv \pm 1 \pmod{10}$ allowed and $Y_4(\alpha)$ is the number of partitions of α into parts that are $\equiv \pm 2 \pmod{10}$ with two copies of each part allowed. Then

$$M_4(\alpha) = N_4(\alpha), \ \forall \, \alpha \geq 0.$$

Proof. Split the partitions enumerated by $M_4(r, \alpha)$ into four classes:
Class (i) contains $(m_1)_{x_1}$ of the form 1_1,
Class (ii) contains $(m_1)_{x_1}$ of the form $\overline{2}_2$,
Class (iii) do not contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ or $(\overline{x}_k)_{x_k}$,
Class (iv) contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ $(x \geq 2)$ and $(\overline{x}_k)_{x_k}$ $(x \geq 3)$.
Proceeding as in Theorem 2.2.2, the following recurrence relation can easily be obtained for $M_4(r, \alpha)$,

$$\begin{aligned} M_4(r, \alpha) =& M_4(r - 1, \alpha - 2r + 1) + M_4(r - 1, \alpha - 4r + 2) + M_4(r, \alpha - 4r) \\ &+ M_4(r, \alpha - 2r + 1) - M_4(r, \alpha - 6r + 1). \end{aligned}$$

Theorem 2.2.7 *Let $M_5(\alpha)$ represent the number of $n-$color overpartitions of α satisfying* (2.2.2.b), (2.2.6.b) *and*

(2.2.7.a) $m_1 \geq 3$,

(2.2.7.b) $m_1 - x_1 \equiv 2 \pmod 4$ *and* $m_k - x_k \equiv 0 \pmod 2$, *for* $k \geq 2$.

Let $N_5(\alpha) = \sum_{i=0}^{\alpha} X_5(\alpha - i)Y_5(i)$, *where* $X_5(\alpha)$ *is the number of partitions of* α *into distinct parts that are* $\equiv 5 \pmod{10}$ *or* $\equiv \pm 3 \pmod{10}$ *with two copies of parts that are* $\equiv 5 \pmod{10}$ *allowed and one copy of parts that are* $\equiv \pm 3 \pmod{10}$ *allowed and* $Y_5(\alpha)$ *is the number of partitions of* α *into parts that are* $\equiv \pm 4 \pmod{10}$ *with two copies of each part allowed. Then*

$$M_5(\alpha) = N_5(\alpha), \ \forall \, \alpha \geq 0.$$

Proof. For the proof of Theorem 2.2.7, proceed as in Theorem 2.2.6 and get following recurrence relation for $M_5(r, \alpha)$,

$$\begin{aligned} M_5(r, \alpha) =& M_5(r-1, \alpha - 2r - 1) + M_5(r-1, \alpha - 4r) + M_5(r, \alpha - 4r) \\ &+ M_5(r, \alpha - 2r + 1) - M_5(r, \alpha - 6r + 1). \end{aligned}$$

Theorem 2.2.8 *Let $M_6(\alpha)$ represent the number of $(n+2)-$color overpartitions of α satisfying* (2.2.2.a), (2.2.2.b), (2.2.4.b), (2.2.6.b) *and* $x_1 = m_1 + 2$.
Let $N_6(\alpha) = \sum_{i=0}^{\alpha} X_6(\alpha - i)Y_6(i)$, *where* $X_6(\alpha)$ *is the number of partitions of* α *into distinct parts in which the parts are* $\equiv \pm 1, \pm 3 \pmod{10}$, *and* $Y_6(\alpha)$ *is the number of partitions of* α *in which the parts are* $\equiv \pm 2, \pm 4 \pmod{10}$. *Then*

$$M_6(\alpha) = N_6(\alpha), \ \forall \, \alpha \geq 0.$$

Proof. Proceeding as Theorem 2.2.5 easily obtain the proof of Theorem 2.2.8 and corresponding recurrence relation for $M_6(r, \alpha)$ is as follows,

$$M_6(r, \alpha) = M_5(r-1, \alpha - 2r + 2) + M_6(r, \alpha - 2r + 1).$$

Theorem 2.2.9 *Let $M_7(\alpha)$ represent the number of $n-$color overpartitions of α satisfying*

(2.2.9.a) $(m_1)_{x_1}$ *is not overlined,*

(2.2.9.b) $m_k - x_k \equiv 0 \pmod 2$, $\forall \ k$,

(2.2.9.c) $\delta_k \geq 0$ *and* $\equiv 0 \pmod 2 \ \forall \ k > 1$. *For* $\delta_k = 0$, m_k *is not overlined.*

Let $N_7(\alpha)$ *is the number of partitions of* α *in which the parts are* $\not\equiv \pm 2, \pm 12, \pm 14, 16 \pmod{32}$.

Then

$$M_7(\alpha) = N_7(\alpha), \ \forall\, \alpha \geq 1.$$

Proof. To obtain the recurrence relation for the enumerator $M_7(r,\alpha)$, we consider the following q–series:

$$b_1(q) = \sum_{\alpha=0}^{\infty} W_1(\alpha) q^\alpha = \sum_{\alpha=0}^{\infty} \frac{(-q^2;q^2)_\alpha q^{\alpha^2}}{(q;q)_{2\alpha}}. \tag{2.2.9}$$

The interpretation of (2.2.9) in terms of n–color overpartition is given in following lemma.

Lemma 2.2.10 *For* $\alpha \geq 0$, *let* $W_1(\alpha)$ *represent the number of* n*–color overpartitions of* α *satisfying* (2.2.9.b), (2.2.9.c) *and if* $m_1 = x_1$ *then* $(m_1)_{x_1}$ *is not overlined.*

Example 2.2.2 *Considering few terms in the expansion for*

$$\sum_{\alpha=0}^{\infty} \frac{(-q^2;q^2)_\alpha q^{\alpha^2}}{(q;q)_{2\alpha}} = 1 + q + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 11q^7 + 15q^8 + 20q^9 + \dots.$$

We see that, for $\alpha = 8$, *fifteen partitions satisfying the conditions of* $W_1(\alpha)$ *are*

$$(8_8), (8_6), (\overline{8}_6), (8_4), (\overline{8}_4), (8_2), (\overline{8}_2), (7_5\ 1_1), (7_3\ 1_1), (\overline{7}_3\ 1_1), (7_1\ 1_1),$$

$$(\overline{7}_1\ 1_1), (6_2\ 2_2), (5_1\ 3_1), (5_1\ \overline{3}_1).$$

Proof of Lemma 2.2.10. Let $W_1(r,\alpha)$ denotes the number of n–color overpartition of α enumerated by $W_1(\alpha)$ into r parts. Divide the partitions into four classes. First class has partitions which do not involve $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ or $(\overline{x_k})_{x_k-2}$. Subtract 2 from all the parts and we get transformed partition enumerated by $W_1(r,\alpha-2r)$. The second class has the partitions in which $(m_1)_{x_1} = 1_1$. Remove 1_1 part, subtract 2 from all the remaining parts. Corresponding transformed partition is enumerated by $W_1(r-1,\alpha-2r+1)$. The next class has the partition in which $(m_1)_{x_1} = \overline{3}_1$. Delete $\overline{3}_1$ and subtract 4 from the remaining parts. The transformed partition is enumerated by $W_1(r-1,\alpha-4r+1)$. The last class has the partition which involve $(m_1)_{x_1}$ of the form $(x_k)_{x_k}(x \geq 2)$ and $(\overline{x_k})_{x_k-2}(x \geq 4)$. Replace $(x_k)_{x_k}$ by $(x_k-1)_{x_k-1}$ and $(\overline{x_k})_{x_k-2}$ by $(\overline{x_k-1})_{x_k-3}$ and subtract 2 from the remaining parts. This will result in partitions enumerated by $W_1(r,\alpha-2r+1)$. It should be noted here that we are obtaining only those partitions of $\alpha-2r+1$ which involve a least part of the type $(x_k)_{x_k}$ and $(\overline{x_k})_{x_k-2}$ and any other repeated part is added corresponding so the number of partitions in last class enumerated by $W_1(r,\alpha-2r+1) - W_1(r,\alpha-4r+1)$. The transformed classes are reversible. There is one to one correspondence between the classes enumerated by $W_1(r,\alpha)$ and those by

$$W_1(r,\alpha) = W_1(r,\alpha-2r) + W_1(r-1,\alpha-2r+1) + W_1(r-1,\alpha-4r+1) \\ + W_1(r,\alpha-2r+1) - W_1(r,\alpha-4r+1), \tag{2.2.10}$$

where $W_1(0,0) = 1$ and $W_1(r,\alpha) = 0$ for $\alpha < 0$. For $|q| < 1$ and $|w| < |q|^{-1}$ and let $g_1(w,q)$ be defined by

$$g_1(w,q) = \sum_{\alpha=0}^{\infty}\sum_{r=0}^{\infty} W_1(r,\alpha)w^r q^\alpha. \tag{2.2.11}$$

Now, substituting (2.2.11) in (2.2.10) and we get the q–functional equation as,

$$g_1(w,q) = g_1(wq^2,q) + wqg_1(wq^2,q) + wq^3g_1(wq^4,q) \\ + q^{-1}g_1(wq^2,q) - q^{-1}g_1(wq^4,q).$$

Follows the similar steps as in the proof of Theorem 2.2.2, we get the desired result.

Proof of Theorem 2.2.9. Now we shall prove

$$\sum_{\alpha=1}^{\infty} M_7(\alpha)q^\alpha = \sum_{\alpha=1}^{\infty} \frac{(-q^2;q^2)_{\alpha-1}q^{\alpha^2}}{(q;q)_{2\alpha}}.$$

Split the partitions enumerated by $M_7(r,\alpha)$ into four classes. In the first class partitions contains $(m_1)_{x_1} = 1_1$. Remove 1_1 and subtract 2 from all the remaining parts. Transformed partitions are enumerated by $W_1(r-1,\alpha-2r+1)$. The second class contain the partitions which have $(m_1)_{x_1} = 3_1$. By removing 3_1 and subtracting 4 from all the remaining parts. We get the partition enumerated by $W_1(r-1,\alpha-4r+1)$. The third class contains the partitions which do not have $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ and $(x_k)_{x_k-2}$. Subtracting 4 from all the parts and we get the partitions enumerated by $M_7(r,\alpha-4r)$. The last class contains the partitions $(m_1)_{x_1}$ of the form $(x_k)_{x_k}(x \geq 2)$ and $(x_k)_{x_k-2}(x \geq 4)$. Replace $(x_k)_{x_k}$ into $(x_k-1)_{x_k-1}$ and $(x_k)_{x_k-2}$ into $(x_k-1)_{x_k-3}$ after that subtracting 2 from remaining parts and get partitions which are enumerated by $M_7(r,\alpha-2r+1) - M_7(r,\alpha-6r+1)$. Thus the recurrence relation for $M_7(r,\alpha)$ is given by

$$M_7(r,\alpha) = W_1(r-1,\alpha-2r+1) + W_1(r-1,\alpha-4r+1) \\ + M_7(r,\alpha-4r) + M_7(r,\alpha-2r+1) - M_7(r,\alpha-6r+1),$$

and the corresponding q–functional equation is

$$h_7(w,q) = wqg_1(wq^2,q) + wq^3g_1(wq^4,q) + h_7(wq^4,q) + q^{-1}h_7(wq^2,q) - q^{-1}h_7(wq^6,q).$$

Proceeding in the similar way as in the proof of Theorem 2.2.2, we get the desired result.

For the combinatorial interpretation of sum side of $h_8(q)$, we write

$$\sum_{\alpha=0}^{\infty}\frac{(-1;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}} = 1 + 2\sum_{\alpha=1}^{\infty}\frac{(-q^2;q^2)_{\alpha-1}q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}} = 1 + 2\sum_{\alpha=1}^{\infty}\hat{M}_8(\alpha)q^\alpha, \tag{2.2.12}$$

where $\hat{h}_8(q) = \sum_{\alpha=1}^{\infty}\hat{M}_8(\alpha)q^\alpha = \sum_{\alpha=1}^{\infty}\frac{(-q^2;q^2)_{\alpha-1}q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$. Now we give the combinatorial interpretation of $h_8(q)$ in terms of $\hat{h}_8(q)$ in the following theorem.

Theorem 2.2.11 *Let $\hat{M}_8(\alpha)$ represent the number of n–color overpartitions of α satisfying* (2.2.9.a), (2.2.9.b), m_1, $x_1 > 1$, *and* $\delta_k \geq -2$ *and* $\equiv 0 \pmod 2$ $\forall\ k > 1$. *For* $\delta_k = -2$, m_k *is not overlined.*
Let $N_8(\alpha) = \sum_{i=0}^{\alpha} X_8(\alpha - i)Y_8(i)$, *where* $X_8(\alpha)$ *is the number of partitions of* α *in which the parts are* $\equiv \pm 2, \pm 3, \pm 4 \pmod{12}$, *and* $Y_8(\alpha)$ *is the number of partitions of* α *into distinct parts that are* $\equiv \pm 2, \pm 3, \pm 4 \pmod{12}$. *Then*

$$M_8(\alpha) = 2\hat{M}_8(\alpha) = N_8(\alpha),\ \ \forall\ \alpha \geq 1.$$

Example 2.2.3 *The n–color overpartitions corresponding to $\hat{M}_8(6)$ are* 6_6, 6_4, 6_2, $(4_2\ 2_2)$. *Hence* $\hat{M}_8(6) = 4$ *and* $M_8(6) = 8$.

Proof of Theorem 2.2.11. To obtain the recurrence relation for the enumerator $\hat{M}_8(r,\alpha)$ where $\hat{M}_8(r,\alpha)$ represent the partitions enumerated by $\hat{M}_8(\alpha)$ into r parts, we consider the following q–series:

$$b_2(q) = \sum_{\alpha=0}^{\infty}W_2(\alpha)q^\alpha = \sum_{\alpha=0}^{\infty}\frac{(-q^2;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}. \tag{2.2.13}$$

The combinatorial interpretation of (2.2.13) is given in Lemma 2.2.12.

Lemma 2.2.12 *For $\alpha \geq 0$, let $W_2(\alpha)$ represent the number of n–color overpartitions satisfying*

(2.2.12.a) $m_k \equiv x_k \pmod 2$, $\forall\ k$,

(2.2.12.b) $m_1,\ x_1 \geq 2$, *and if* $m_1 = x_1$ *then* m_1 *is not overlined,*

(2.2.12.c) $\delta_k \geq -2$ *and* $\equiv 0 \pmod 2\ \forall\ k$. *For* $\delta_k = -2$, m_k *is not overlined.*

Proof of Lemma 2.2.12.
Let $W_2(r, \alpha)$ denote the partitions enumerated by $W_2(r)$ into r parts. Here we shall split the partitions enumerated by $W_2(r, \alpha)$ into four classes:
Class (i) contains $(m_1)_{x_1} = 2_2$,
Class (ii) contains $(m_1)_{x_1} = \overline{4}_2$,
Class (iii) do not contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ or $(\overline{x}_k)_{x_k-2}$,
Class (iv) contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ $(x > 2)$ and $(\overline{x}_k)_{x_k-2}$ $(x > 4)$.
Proceeding as Lemma 2.2.10, one can easily obtain the following recurrence relation,

$$\begin{aligned} W_2(r, \alpha) =& W_2(r-1, \alpha-2r) + W_2(r-1, \alpha-4r) + W_2(r, \alpha-2r) \\ &+ W_2(r, \alpha-2r+1) - W_2(r, \alpha-4r+1). \end{aligned} \tag{2.2.14}$$

Now we find the recurrence relations for $\hat{M}_8(r, \alpha)$:

$$\begin{aligned} \hat{M}_8(r, \alpha) =& W_2(r-1, \alpha-2r) + W_2(r-1, \alpha-4r) + \hat{M}_8(r, \alpha-4r) \\ &+ \hat{M}_8(r, \alpha-2r+1) - \hat{M}_8(r, \alpha-6r+1). \end{aligned}$$

Theorem 2.2.13 *Let* $M_9(\alpha)$ *represent the number of* $(n+1)$*–color overpartitions of* α *satisfying* (2.2.9.c) *along with:*

(2.2.13.a) $x_1 = m_1 + 1$,

(2.2.13.b) $(m_1)_{x_1}$ *is not overlined,*

(2.2.13.c) $m_k - x_k \equiv 1 \pmod 2,\ \forall\ k$.

Let $N_9(\alpha) = \sum_{i=0}^{\alpha} X_9(\alpha - i) Y_9(i)$, *where* $X_9(\alpha)$ *is the number of partitions of* α *in which the parts are* $\equiv 2,\ 4 \pmod 6$, *and* $Y_9(\alpha)$ *is the number of partitions of* α *into distinct parts that are* $\equiv 0, \pm 1, \pm 2 \pmod 6$. *Then*

$$M_9(\alpha) = N_9(\alpha),\ \ \forall\ \alpha \geq 0.$$

Proof. *The recurrence relation for* $M_9(r, \alpha)$ *is*

$$M_9(r, \alpha) = W_2(r-1, \alpha-r+1) + M_9(r, \alpha-2r+1).$$

Remark 2.2.14 *In the following theorem, we used similar argument as given in* (2.2.12) *and letting*

$$\hat{h}_{10}(q) = \sum_{\alpha=1}^{\infty} \hat{M}_{10}(\alpha) q^{\alpha} = \sum_{\alpha=1}^{\infty} \frac{(-q^4;q^4)_{\alpha-1} q^{\alpha^2}}{(q;q^2)_{\alpha}(q^4,q^4)_{\alpha}}.$$

Theorem 2.2.15 *Let $\hat{M}_{10}(\alpha)$ represent the number of n–color overpartitions of α satisfying*

(2.2.15.a) $m_1 \equiv x_1 \pmod 4$,

(2.2.15.b) $(m_1)_{x_1}$ *is not overlined,*

(2.2.15.c) $\delta_k \geq 0$ *and* $\equiv 0 \pmod 4$ $\forall\ k > 1$. *For $\delta_k = 0$ then m_k is not overlined.*

Let $N_{10}(\alpha) = \sum_{i=0}^{\alpha} X_{10}(\alpha - i) Y_{10}(i)$, where $X_{10}(\alpha)$ is the number of partitions of α in which the parts are $\equiv \pm 1, 4 \pmod 8$, and $Y_{10}(\alpha)$ is the number of partitions of α in which the parts are $\equiv \pm 1, 4 \pmod 8$. Then

$$M_{10}(\alpha) = 2\hat{M}_{10}(\alpha) = N_{10}(\alpha), \ \ \forall\, \alpha \geq 1.$$

Proof. To obtain the recurrence relation for the enumerator $\hat{M}_{10}(r,\alpha)$, where $\hat{M}_{10}(r,\alpha)$ is $\hat{M}_{10}(\alpha)$ into r parts, we consider the following q–series:

$$b_3(q) = \sum_{\alpha=0}^{\infty} W_3(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} \frac{(-q^4;q^4)_{\alpha} q^{\alpha^2}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}. \tag{2.2.15}$$

The combinatorial interpretation of (2.2.15) is given in Lemma 2.2.16.

Lemma 2.2.16 *For $\alpha \geq 0$, let $W_3(\alpha)$ represent the number of n–color overpartitions satisfying* (2.2.15.a), (2.2.15.c) *and if $m_1 = x_1$ then m_1 is not overlined.*

Proof of Lemma 2.2.16

Let $W_3(r,\alpha)$ denote the partitions enumerated by $W_3(r)$ into r parts. We shall split the partitions enumerated by $W_3(r,\alpha)$ into four classes:
Class (i) contains $(m_1)_{x_1}$ of the form 1_1,
Class (ii) contains $(m_1)_{x_1}$ of the form $\overline{5}_1$,
Class (iii) do not contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ or $(\overline{x}_k)_{x_k-4}$,
Class (iv) contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ $(x \geq 2)$ and $(\overline{x}_k)_{x_k-4}$ $(x \geq 6)$.
Proceeding as Lemma 2.2.10, one can easily obtain the following recurrence relation,

$$\begin{aligned} W_3(r,\alpha) =& W_3(r-1,\alpha-2r+1) + W_3(r-1,\alpha-6r+1) + W_3(r,\alpha-4r) \\ &+ W_3(r,\alpha-2r+1) - W_3(r,\alpha-6r+1). \end{aligned} \tag{2.2.16}$$

Now we find the recurrence relations for $\hat{M}_{10}(r,\alpha)$,

$$\begin{aligned}\hat{M}_{10}(r,\alpha) =& W_3(r-1,\alpha-2r+1)+W_3(r-1,\alpha-6r+1)+\hat{M}_{10}(r,\alpha-8r)\\ &+\hat{M}_{10}(r,\alpha-2r+1)-\hat{M}_{10}(r,\alpha-10r+1).\end{aligned}$$

Remark 2.2.17 *In the following theorem, we used similar argument as given in* (2.2.12) *and letting*

$$\hat{h}_{11}(q)=\sum_{\alpha=1}^{\infty}\hat{M}_{11}(\alpha)q^{\alpha}=\sum_{\alpha=1}^{\infty}\frac{(-q^4;q^4)_{\alpha-1}q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha}(q^4,q^4)_{\alpha}}.$$

Theorem 2.2.18 *Let $\hat{M}_{11}(\alpha)$ represent the number of n–color overpartitions of α satisfying* (2.2.15.c), $m_1>2$ *with* $m_1-x_1\equiv 2 \pmod 4$ *and* $(m_1)_{x_1}$ *is not overlined.*
Let $N_{11}(\alpha)=\sum_{i=0}^{\alpha}X_{11}(\alpha-i)Y_{11}(i)$, where $X_{11}(\alpha)$ is the number of partitions of α in which the parts are $\equiv \pm 3,4 \pmod 8$, and $Y_{11}(\alpha)$ is the number of partitions of α into distinct parts are $\equiv \pm 3,4 \pmod 8$. Then

$$M_{11}(\alpha)=2\hat{M}_{11}(\alpha)=N_{11}(\alpha),\ \forall\,\alpha\geq 1.$$

Proof. To obtain the recurrence relation for the enumerator $\hat{M}_{11}(r,\alpha)$, where $\hat{M}_{11}(r,\alpha)$ is $\hat{M}_{11}(\alpha)$ into r parts, we consider the following q–series:

$$b_4(q)=\sum_{\alpha=0}^{\infty}W_4(\alpha)q^{\alpha}=\sum_{\alpha=0}^{\infty}\frac{(-q^4;q^4)_{\alpha}q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}. \tag{2.2.17}$$

The combinatorial interpretation of (2.2.17) is given in Lemma 2.2.19.

Lemma 2.2.19 *For $\alpha\geq 0$, let $W_4(\alpha)$ represent the number of n–color overpartitions satisfying* (2.2.15.a), (2.2.15.c), $m_1\geq 3$, $m_1-x_1\equiv 2 \pmod 4$ *and if $x_1=m_1-2$ then m_1 is not overlined.*

Proof of Lemma 2.2.19

Let $W_4(r,\alpha)$ denote the partitions enumerated by $W_4(r)$ into r parts. We shall split the partitions enumerated by $W_4(r,\alpha)$ into four classes:
Class (i) contains $(m_1)_{x_1}$ of the form 3_1,
Class (ii) contains $(m_1)_{x_1}$ of the form $\overline{7}_1$,
Class (iii) do not contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k-2}$ or $(\overline{x}_k)_{x_k-6}$,
Class (iv) contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k-2}$ $(x\geq 4)$ and $(\overline{x}_k)_{x_k-6}$ $(x\geq 8)$.

The recurrence relation for $W_4(r,\alpha)$ is:

$$\begin{aligned}W_4(r,\alpha) =& W_4(r-1,\alpha-2r-1)+W_4(r-1,\alpha-6r-1)+W_4(r,\alpha-4r)\\ &+W_4(r,\alpha-2r+1)-W_4(r,\alpha-6r+1).\end{aligned}$$

And the recurrence relations for $\hat{M}_{11}(r,\alpha)$ is:

$$\begin{aligned}\hat{M}_{11}(r,\alpha) =& W_4(r-1,\alpha-2r-1)+W_4(r-1,\alpha-6r-1)+\hat{M}_{11}(r,\alpha-8r)\\ &+\hat{M}_{11}(r,\alpha-2r+1)-\hat{M}_{11}(r,\alpha-10r+1).\end{aligned}$$

Remark 2.2.20 *In the following theorem, we used similar argument as given in* (2.2.12) *and letting*

$$\hat{h}_{12}(q)=\sum_{\alpha=1}^{\infty}\hat{M}_{12}(\alpha)q^{\alpha}=\sum_{\alpha=1}^{\infty}\frac{(-q;q)_{\alpha-1}q^{\alpha^2}}{(q;q^2)_{\alpha}(q,q)_{\alpha}}.$$

Theorem 2.2.21 *Let $\hat{M}_{12}(\alpha)$ represent the number of n–color overpartitions of α satisfying*

(2.2.21.a) $(m_1)_{x_1}$ *is not overlined,*

(2.2.21.b) $\delta_k \geq 0\ \forall\ k>1$. *For $\delta_k=0$ then m_k is not overlined.*

Let $N_{12}(\alpha)=\sum_{i=0}^{\alpha}X_{12}(\alpha-i)Y_{12}(i)$, where $X_{12}(\alpha)$ is the number of partitions of α in which the parts are $\equiv \pm1,\pm2 \pmod 6$, and $Y_{12}(\alpha)$ is the number of partitions of α into distinct parts are $\equiv \pm1,\pm2 \pmod 6$. Then

$$M_{12}(\alpha)=2\hat{M}_{12}(\alpha)=N_{12}(\alpha),\ \ \forall\,\alpha\geq 1.$$

Proof. To obtain the recurrence relation for the enumerator $\hat{M}_{12}(r,\alpha)$, where $\hat{M}_{12}(r,\alpha)$ is $\hat{M}_{12}(\alpha)$ into r parts, we consider the following q–series:

$$b_5(q)=\sum_{\alpha=0}^{\infty}W_5(\alpha)q^{\alpha}=\sum_{\alpha=0}^{\infty}\frac{(-q;q)_{\alpha}q^{\alpha^2}}{(q;q^2)_{\alpha}(q;q)_{\alpha}}. \tag{2.2.18}$$

The combinatorial interpretation of (2.2.18) is given in Lemma 2.2.22.

Lemma 2.2.22 *For $\alpha\geq 0$, let $W_5(\alpha)$ represent the number of n–color overpartitions satisfying* (2.2.21.b) *and if $m_1=x_1$ then m_1 is not overlined.*

Proof of Lemma 2.2.22

We shall split the partitions enumerated by $W_5(r,\alpha)$ into four classes:

Class (i) contains $(m_1)_{x_1}$ of the form 1_1,
Class (ii) contains $(m_1)_{x_1}$ of the form $\overline{2}_1$,
Class (iii) do not contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ or $(\overline{x}_k)_{x_k-1}$,
Class (iv) contains $(m_1)_{x_1}$ of the form $(x_k)_{x_k}$ $(x \geq 2)$ and $(\overline{x}_k)_{x_k-1}$ $(x \geq 3)$.
The recurrence relation for $W_5(r, \alpha)$, where $W_5(r, \alpha)$ represent $W_5(\alpha)$ into r columns is:

$$\begin{aligned} W_5(r, \alpha) =& W_5(r-1, \alpha-2r+1) + W_5(r-1, \alpha-3r+1) + W_5(r, \alpha-r) \\ &+ W_5(r, \alpha-2r+1) - W_5(r, \alpha-3r+1). \end{aligned}$$

And the recurrence relation for $\hat{M}_{12}(r, \alpha)$ is,

$$\begin{aligned} \hat{M}_{12}(r, \alpha) =& W_5(r-1, \alpha-2r+1) + W_5(r-1, \alpha-3r+1) + \hat{M}_{12}(r, \alpha-2r) \\ &+ \hat{M}_{12}(r, \alpha-2r+1) - \hat{M}_{12}(r, \alpha-4r+1). \end{aligned}$$

Theorem 2.2.23 *Let $M_{13}(\alpha)$ represent the number of n–color overpartitions of α satisfying all the conditions of $\hat{M}_{12}(\alpha)$ defined in* Theorem 2.2.21.
Let $N_{13}(\alpha) = \sum_{i=0}^{\alpha} X_{13}(\alpha - i)Y_{13}(i)$, where $X_{13}(\alpha)$ is the number of unrestricted partitions of α, and $Y_{13}(\alpha)$ is the number of partitions of α in which the parts are $\equiv \pm 5 \pmod{12}$. Then

$$M_{13}(\alpha) = N_{13}(\alpha), \ \forall \, \alpha \geq 1.$$

Proof. *As $\hat{M}_{12}(\alpha) = M_{13}(\alpha)$, so the recurrence relations for the enumerators $\hat{M}_{12}(\alpha)$ and $M_{13}(\alpha)$ are same.*

Theorem 2.2.24 *For $\alpha \geq 0$, let $M_{14}(\alpha)$ represent the number of $(n+1)$–color overpartitions of α satisfying* (2.2.21.b),

(2.2.24.a) $x_1 = m_1 + 1$,

(2.2.24.b) $(m_1)_{x_1}$ *is not overlined,*

(2.2.24.c) $m_k - x_k \equiv 1 \pmod{2}$, $\forall \, k$.

Let $N_{14}(\alpha) = \sum_{i=0}^{\alpha} X_{14}(\alpha - i)Y_{14}(i)$, where $X_{14}(\alpha)$ is the number of partitions of α in which the parts are $\equiv \pm 2, 3 \pmod{6}$, and $Y_{14}(\alpha)$ is the number of partitions of α into distinct parts. Then

$$M_{14}(\alpha) = N_{14}(\alpha), \ \forall \, \alpha \geq 0.$$

Proof. The recurrence relation for the enumerator $M_{14}(r, \alpha)$ is

$$M_{14}(r, \alpha) = W_5(r-1, \alpha - r + 1) + M_{14}(r, \alpha - 2r + 1).$$

2.3 Conculsion

In this chapter, we have presented combinatorial interpretations of fourteen q–series identities in terms of $(n+t)$–color overpartitions. In the next chapter, we will extend some of the results by providing interpretations using other combinatorial tools, such as three-line arrays, split $(n+t)$–color partitions, 2–color F–partitions, and lattice paths.

Chapter 3

Combinatorics and bijections for certain q–series identities

3.1 Introduction

In [74], Santos et al. introduced the concept of a two-line array. For a positive integer α, let

$$\begin{pmatrix} \lambda_r & \lambda_{r-1} & \dots & \lambda_1 \\ \mu_r & \mu_{r-1} & \dots & \mu_1 \end{pmatrix},$$

where λ_k, $\mu_k \geq 0$ for $1 \leq k \leq r$, be a two-line array such that

$$\alpha = \sum_{k=1}^{r} \lambda_k + \sum_{k=1}^{r} \mu_k.$$

Using the above two-line array representation and by imposing certain restrictions on λ_k and μ_k, Santos and his collaborators interpreted many identities from Slater's list [79], including the Rogers–Ramanujan identities and Lebesgue's partition identities in [74]. They also presented several bijective proofs for partition identities. This work is further related to three-quadrant Ferrers graphs in [32]. Continuing with the above work, Brietzke et al. [33] found combinatorial interpretations for a number of mock theta functions by considering them as q–series and using a two-line array. Some of these functions were already interpreted using $(n+t)$–color partitions. In [18], Alegri introduced a third row of τ_k in a line array representation, such that $\alpha = \sum_k \lambda_k + \sum_k \mu_k + \sum_k \tau_k$, to construct a correspondence between three-line arrays and overpartitions. In [19], a bijection from certain classes of plane partitions to overpartitions and unrestricted partitions was presented.

The contents of this chapter are communicated in two separate journals. The contents presented in Section 3.3 and Section 3.4 accepted in South East Asian Journal of Mathematics and Mathematical Sciences.

3.2 Three-line arrays and $(n+t)$–color overpartitions

In this section, we present combinatorial interpretations of fourteen q–series identities given in Table 2.1 with three-line arrays. We proceed constructively to obtain three-line array interpretations and translate these results into $(n+t)$–color overpartitions discussed in Chapter 2 using certain bijections. Based on these bijections, we classify the fourteen identities into five groups: Group 1 contains 2, Group 2 contains 4, Group 3 contains 3, Group 4 contains 2, and Group 5 contains 3 q–series identities. Additionally, we provide an alternate proof of the sum side of all fourteen q–series identities given in Table 2.1 involving three-line arrays using a classical approach.

Group 1

The sum side in the q–series identity represents generating functions for $P_l(\alpha)$, where $1 \leq l \leq 14$, which count partitions in terms of three-line arrays. The generating functions $M_l(\alpha)$ and $N_l(\alpha)$ are defined in Chapter 2. These lead to 3-way combinatorial interpretations that satisfy

$$\sum_{\alpha=0}^{\infty} P_l(\alpha)q^{\alpha} = \sum_{\alpha=0}^{\infty} M_l(\alpha)q^{\alpha} = \sum_{\alpha=0}^{\infty} N_l(\alpha)q^{\alpha}, \quad 1 \leq l \leq 14. \tag{3.2.1}$$

For Group 1, we construct a bijection for $h_1(q)$ and $h_4(q)$ given in the Table 2.1, between three-line arrays and n–color overpartitions. Let $(m_k)_{x_k}$ and $(m_{k-1})_{x_{k-1}}$ be two consecutive parts of an n–color overpartition. Then the corresponding column $\begin{pmatrix} \lambda_k \\ \mu_k \\ \tau_k \end{pmatrix}$ in the three-line array is given by

$$\phi : (m_k)_{x_k} \to \begin{cases} \begin{pmatrix} m_{k-1}+x_{k-1}+1 \\ x_k-1 \\ \delta_k \end{pmatrix} & \text{if } m_k \text{ is not overlined,} \\ \\ \begin{pmatrix} m_{k-1}+x_{k-1}+2 \\ x_k-2 \\ \delta_k \end{pmatrix} & \text{if } m_k \text{ is overlined.} \end{cases} \tag{3.2.2}$$

In the reverse implication, let $\begin{pmatrix} \lambda_k \\ \mu_k \\ \tau_k \end{pmatrix}$ be any column in the three-line array. Then the corresponding part in an n–color overpartition, $(m_k)_{x_k}$, is given by

$$\phi^{-1} : \begin{pmatrix} \lambda_k \\ \mu_k \\ \tau_k \end{pmatrix} \to \begin{cases} (\overline{\lambda_k+\mu_k+\tau_k})_{\mu_k+2}, & \text{if } \lambda_k \equiv\ 0 \pmod 2, \\ \\ (\lambda_k+\mu_k+\tau_k)_{\mu_k+1}, & \text{if } \lambda_k \not\equiv\ 0 \pmod 2. \end{cases} \tag{3.2.3}$$

Remark 3.2.1 *If there is any further change in the mapping, we mention this at their respective place.*

We now provide the combinatorial interpretations in terms of three-line arrays for sum side of $h_1(q)$ and $h_4(q)$ in Theorem 3.2.2 and Theorem 3.2.3 respectively.

Theorem 3.2.2 *Let $P_1(\alpha)$ represent the number of three-line arrays into r columns satisfying*

(3.2.2.a) $\lambda_1 \in \{1, 2\}$,

(3.2.2.b) $\tau_k \equiv 0 \pmod 2$, $\forall\ k$

$$\textbf{(3.2.2.c)}\ \ \lambda_k = \begin{cases} 2 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ are odd },\\ 3 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ have opposite parity},\\ 4 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ are even}, \end{cases}$$

where $2 \leq k \leq r$.

Then

$$P_1(\alpha) = M_1(\alpha) = N_1(\alpha),\ \ \forall\ \alpha \geq 0.$$

Proof We begin by expanding sum side of $h_1(q)$

$$\begin{aligned} h_1(q) &= \sum_{r=0}^{\infty} \frac{(-q;q^2)_r q^{r^2}}{(q;q)_{2r}} \qquad (3.2.4)\\ &= \sum_{r=0}^{\infty} \frac{(1+q)(1+q^3)\cdots(1+q^{2r-1})q^{1+3+\ldots+(2r-1)}}{(1-q)(1-q^3)\cdots(1-q^{2r-1})(1-q^2)(1-q^4)\cdots(1-q^{2r})}. \end{aligned}$$

Clearly the factor $q^{r^2} = q^{1+3+5\ldots+(2r-1)}$ in the above expression generates the partition into r odd parts. It corresponds to the following three-line array

$$\begin{pmatrix} 2r-1 & \ldots & 3 & 1\\ 0 & \ldots & 0 & 0\\ 0 & \ldots & 0 & 0 \end{pmatrix}.$$

The factor $(-q;q^2)_r$ generates the partition into distinct odd parts $\leq 2r-1$, say, $1 \cdot a_r$, $3 \cdot a_{r-1}$, $\ldots$, $(2r-1) \cdot a_1$, where $a_i \in \{0, 1\}$, $\forall\ i$. Thus the transformed three-line array becomes

$$\begin{pmatrix} 2r-1+a_r+\sum_{i=1}^{r-1} 2a_i & \dots & 3+a_2+2a_1 & 1+a_1 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

The factor $(q;q^2)_r^{-1}$ generates the partition into odd parts $\leq 2r-1$, say, $1 \cdot b_r$, $3 \cdot b_{r-1}$, $\dots$, $(2r-1) \cdot b_1$, where $b_i \geq 0$, $\forall\ i$. The three-line array transforms to

$$\begin{pmatrix} 2r-1+a_r+\sum_{i=1}^{r-1} 2(a_i+b_i) & \dots & 3+a_2+2a_1+2b_1 & 1+a_1 \\ b_r & \dots & b_2 & b_1 \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

The factor $(q^2;q^2)_r^{-1}$ generates the partition into even parts $\leq 2r$, say, $2 \cdot c_r$, $4 \cdot c_{r-1}$, $\dots$, $2r \cdot c_1$, where $c_i \geq 0$, $\forall\ i$. Hence, the three-line array transforms to

$$\begin{pmatrix} 2r-1+a_r+\sum_{i=1}^{r-1} 2(a_i+b_i+c_i) & \dots & 3+a_2+2a_1+2b_1+2c_1 & 1+a_1 \\ b_r & \dots & b_2 & b_1 \\ 2c_r & \dots & 2c_2 & 2c_1 \end{pmatrix}$$

and

$$\alpha = 1 \cdot (a_r+b_r+1)+3 \cdot (a_{r-1}+b_{r-1}+1)+\dots+(2r-1) \cdot (a_1+b_1+1)+2c_r+4c_{r-1}+\dots+2rc_1.$$

Therefore, $P_1(\alpha)$ enumerates the three-line arrays with $\mu_k = b_k$, $\tau_k = 2c_k\ \forall\ k$ and satisfying (3.2.2.a)–(3.2.2.c).

Example 3.2.1 *For $\alpha = 6$, the three-line arrays and n–color overpartitions satisfying* Theorem 3.2.2 *and* Theorem 2.2.2 *are listed in* Table 3.1:

Theorem 3.2.3 *Let $P_4(\alpha)$ represent the number of three-line arrays into r columns satisfying* (3.2.2.a), (3.2.2.c), *and* $\tau_k \equiv 0 \pmod 4\ \forall\ k$. *Then*

$$P_4(\alpha) = M_4(\alpha) = N_4(\alpha)\ \ \forall\ \alpha \geq 0.$$

Group 2

In this group, the bijection used for the sum side of $h_2(q)$, $h_3(q)$, $h_5(q)$, and $h_6(q)$ to establish the connection between three-line arrays and $(n+t)$–color overpartitions is the same as the one defined in Group 1. The combinatorial interpretations in terms of three-line arrays of the sum side of $h_2(q)$, $h_3(q)$, $h_5(q)$, and $h_6(q)$ are given in Theorems 3.2.4–3.2.10, respectively.

Table 3.1: $P_1(\alpha) = M_1(\alpha) = N_1(\alpha)$ for $\alpha = 6$

$P_1(6)$	$M_1(6)$	$N_1(6)$	$P_1(6)$	$M_1(6)$	$N_1(6)$
$\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$	(6_6)	$(5\ 1)$	$\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$	$(\overline{6}_6)$	$(4\ 2_1)$
$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$	(6_4)	$(4\ 2_2)$	$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$	$(\overline{6}_4)$	$(3\ 2_1\ 1)$
$\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$	(6_2)	$(3\ 2_2\ 1)$	$\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$	$(\overline{6}_2)$	$(2_1\ 2_1\ 2_1)$
$\begin{pmatrix} 3 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$	$(5_3\ 1_1)$	$(2_1\ 2_1\ 2_2)$	$\begin{pmatrix} 4 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\overline{5}_3\ 1_1)$	$(2_1\ 2_2\ 2_2)$
$\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}$	$(5_1\ 1_1)$	$(2_2\ 2_2\ 2_2)$			

Theorem 3.2.4 *Let $P_2(\alpha)$ represent the number of three-line arrays into $r+1$ columns satisfying*

(3.2.4.a) $\lambda_1 = 0 = \tau_1$,

(3.2.4.b) $\lambda_2 \geq 2$,

(3.2.4.c) $$\lambda_k = \begin{cases} 2 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ are even ,} \\ 3 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ have opposite parity,} \\ 4 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ are odd,} \end{cases}$$
where $3 \leq k \leq r+1$,

(3.2.4.d) $\tau_k \equiv 0 \pmod 2$, $\forall\ k$.

Then

$$P_2(\alpha) = M_2(\alpha) = N_2(\alpha)\ \ \forall\ \alpha \geq 0.$$

Remark 3.2.5 *Here we observe that the mapping used for the first part is* $\phi : (m_1)_{x_1} \to \begin{pmatrix} 0 \\ x_1 - 1 \\ 0 \end{pmatrix}$ *and the inverse mapping is* $\phi^{-1} : \begin{pmatrix} 0 \\ \mu_1 \\ 0 \end{pmatrix} \to (\mu_1)_{\mu_1+1}$.

The remaining r parts are same as defined in (3.2.2) *and*

$$\phi^{-1} : \begin{pmatrix} \lambda_k \\ \mu_k \\ \tau_k \end{pmatrix} \to \begin{cases} (\overline{\lambda_k + \mu_k + \tau_k})_{\mu_k+2}, & \text{if } \lambda_k \equiv\ 0 \pmod 2, \\ \\ (\lambda_k + \mu_k + \tau_k)_{\mu_k+1}, & \text{if } \lambda_k \not\equiv\ 0 \pmod 2. \end{cases}$$

Example 3.2.2 *For $\alpha = 6$, $P_2(6) = 9 = M_2(6) = N_2(6)$.*

The three-line arrays corresponding to $P_2(6)$ *satisfying* Theorem 3.2.4 *are*

$$\begin{pmatrix}0\\6\\0\end{pmatrix},\begin{pmatrix}2&0\\4&0\\0&0\end{pmatrix},\begin{pmatrix}2&0\\2&0\\2&0\end{pmatrix},\begin{pmatrix}2&0\\0&0\\4&0\end{pmatrix},\begin{pmatrix}4&0\\1&1\\0&0\end{pmatrix},\begin{pmatrix}3&0\\3&0\\0&0\end{pmatrix},\begin{pmatrix}3&0\\1&0\\2&0\end{pmatrix},\begin{pmatrix}3&0\\2&1\\0&0\end{pmatrix},\begin{pmatrix}4&2&0\\0&0&0\\0&0&0\end{pmatrix}.$$

The $(n+1)$*–color overpartitions corresponding to* $M_2(6)$ *satisfying* Theorem 2.2.4 *are*
(6_7), $(6_5\ 0_1)$, $(6_3\ 0_1)$, $(6_1\ 0_1)$, $(5_2\ 1_2)$, $(\overline{6}_5\ 0_1)$, $(\overline{6}_3\ 0_1)$, $(\overline{5}_2\ 1_2)$, $(4_1\ 2_1\ 0_1)$.
The partitions corresponding to $N_2(6)$ *are*
(6), $(5\ 1)$, $(3\ 3)$, $(3\ 2\ 1)$, $(3\ 1\ 1\ 1)$, $(2\ 2\ 2)$, $(2\ 2\ 1\ 1)$, $(2\ 1\ 1\ 1\ 1)$, $(1\ 1\ 1\ 1\ 1\ 1)$.

Theorem 3.2.6 *Let* $P_3(\alpha)$ *represent the number of three-line arrays into* $r+1$ *columns satisfying* (3.2.4.a) *and* (3.2.4.d) *along with:*

(3.2.6.a) $\lambda_2 \geq 3$,

(3.2.6.b) $\lambda_k = \begin{cases} 2+\lambda_{k-1}+2\mu_{k-1}+\tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ are odd },\\ 3+\lambda_{k-1}+2\mu_{k-1}+\tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ have opposite parity,}\\ 4+\lambda_{k-1}+2\mu_{k-1}+\tau_{k-1} & \text{if } \lambda_k \textit{ and } \lambda_{k-1} \textit{ are even,}\end{cases}$
where $3 \leq k \leq r+1$.

Then

$$P_3(\alpha) = M_3(\alpha) = N_3(\alpha)\ \forall\ \alpha \geq 0.$$

Remark 3.2.7 *Here we observe that the mapping used for the first part is* $\phi : (m_1)_{x_1} \rightarrow \begin{pmatrix}0\\x_1-2\\0\end{pmatrix}$ *and the inverse mapping is* $\phi^{-1} : \begin{pmatrix}0\\\mu_1\\0\end{pmatrix} \rightarrow (\mu_1)_{\mu_1+2}$. *And the map for the next* r *parts are same as defined in* (3.2.2)*–*(3.2.3).

Theorem 3.2.8 *Let* $P_5(\alpha)$ *represent the number of three-line arrays into* r *columns satisfying* (3.2.6.b) *for* $2 \leq k \leq r$, *along with:*

(3.2.8.a) $\lambda_1 \in \{3, 4\}$,

(3.2.8.b) $\tau_k \equiv 0 \pmod 4$, $\forall\ k$.

Then

$$P_5(\alpha) = M_5(\alpha) = N_5(\alpha)\ \forall\ \alpha \geq 0.$$

Remark 3.2.9 *In above theorem the mapping used for the first part is*

$$\phi : (m_1)_{x_1} \rightarrow \begin{cases} \begin{pmatrix} 3 \\ x_1 - 1 \\ m_1 - x_1 - 2 \end{pmatrix} & \text{if } m_1 \text{ is not overlined,} \\ \\ \begin{pmatrix} 4 \\ x_1 - 2 \\ m_1 - x_1 - 2 \end{pmatrix} & \text{if } m_1 \text{ is overlined.} \end{cases}$$

And for the next $r-1$ *parts are same as defined in* (3.2.2)–(3.2.3).

Theorem 3.2.10 *Let* $P_6(\alpha)$ *represent the number of three-line arrays into* $r+1$ *columns satisfying* (3.2.4.a), (3.2.6.a), (3.2.6.b), *and* (3.2.8.b). *Then*

$$P_6(\alpha) = M_6(\alpha) = N_6(\alpha) \ \forall \ \alpha \geq 0.$$

Group 3

The bijection between three-line arrays and $(n+t)$–color overpartitions of the sum side for $h_7(q)$–$h_9(q)$ is given by

$$\phi : (m_k)_{x_k} \rightarrow \begin{cases} \begin{pmatrix} m_{k-1} + x_{k-1} + 1 \\ x_k - 1 \\ \delta_k \end{pmatrix} & \text{if } m_k \text{ is not overlined,} \\ \\ \begin{pmatrix} m_{k-1} + x_{k-1} + 3 \\ x_k - 1 \\ \delta_k - 2 \end{pmatrix} & \text{if } m_k \text{ is overlined,} \end{cases} \tag{3.2.5}$$

and

$$\phi^{-1} : \begin{pmatrix} \lambda_k \\ \mu_k \\ \tau_k \end{pmatrix} \rightarrow \begin{cases} (\lambda_k + \mu_k + \tau_k)_{\mu_k+1} & \text{if } \ \lambda_k = 2 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1}, \\ \overline{(\lambda_k + \mu_k + \tau_k)}_{\mu_k+1} & \text{if } \lambda_k = 4 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1}. \end{cases} \tag{3.2.6}$$

The combinatorial interpretations in terms of three-line arrays of sum side of $h_7(q)$–$h_9(q)$ are given in Theorems 3.2.11–3.2.14 respectively.

Theorem 3.2.11 *Let* $P_7(\alpha)$ *represent the number of three-line arrays into* r *columns satisfying*

(3.2.11.a) $\lambda_1 = 1$,

(3.2.11.b) $\lambda_k \in \{2 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1},\ 4 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1}\}$, *for* $2 \leq k \leq r$,

(3.2.11.c) $\tau_k \equiv 0 \pmod 2$, $\forall\ k$.

Then

$$P_7(\alpha) = M_7(\alpha) = N_7(\alpha) \ \forall \ \alpha \geq 0.$$

For the combinatorial interpretation of sum side of $h_8(q)$ in terms of three-line array, here we again use $\hat{h}_8(q)$, which is already defined in Chapter 2 and we rewrite as

$$\hat{h}_8(q) = \sum_{\alpha=1}^{\infty} \hat{P}_8(\alpha) q^{\alpha} = \sum_{r=1}^{\infty} \frac{(-q^2;q^2)_{r-1} q^{r(r+1)}}{(q;q)_{2r}}.$$

Now we give the combinatorial interpretation of $h_8(q)$ in terms of $\hat{h}_8(q)$ in the following theorem.

Theorem 3.2.12 *Let $\hat{P}_8(\alpha)$ represent the number of three-line arrays into r columns satisfying* (3.2.11.*b*), (3.2.11.*c*) *and $\lambda_1 = 2$. Then*

$$P_8(\alpha) = 2\hat{P}_8(\alpha) = M_8(\alpha) = 2\hat{M}_8(\alpha) = N_8(\alpha) \ \forall \ \alpha \geq 1.$$

Remark 3.2.13 *Here for the first part, we use the following mapping:*

$$\phi : (m_1)_{x_1} \rightarrow \begin{pmatrix} 2 \\ x_1 - 1 \\ m_1 - x_1 - 1 \end{pmatrix}.$$

And for the next $r-1$ parts are same as defined in (3.2.5)–(3.2.6).

Example 3.2.3 *For $\alpha = 6$, the three-line arrays corresponding to $\hat{P}_8(6)$ satisfying* Theorem 3.2.12 *are*

$$\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

And the n–color overpartitions corresponding to $\hat{M}_8(6)$ satisfying Theorem 2.2.11 *are*

$$(6_6), \ (6_4), \ (6_2), \ (4_2 \ 2_2).$$

Hence $\hat{P}_8(6) = \hat{M}_8(6) = 4$ and $P_8(6) = M_8(6) = 8$.

Theorem 3.2.14 *Let $P_9(\alpha)$ represent the number of three-line arrays into $r+1$ columns satisfying* (3.2.11.b) *and* (3.2.11.c) *along with:*

(3.2.14.a) $\lambda_1 = 0 = \tau_1$,

(3.2.14.b) $\lambda_2 \geq 2$.

Then

$$P_9(\alpha) = M_9(\alpha) = N_9(\alpha) \ \forall \ \alpha \geq 0.$$

Remark 3.2.15 *Here for the first part, the mapping is defined as*

$$\phi : (m_1)_{x_1} \to \begin{pmatrix} 0 \\ x_1 - 1 \\ 0 \end{pmatrix},\ \phi^{-1} : \begin{pmatrix} 0 \\ \mu_1 \\ 0 \end{pmatrix} \to (\mu_1)_{\mu_1+1}.$$

Group 4

To establish the bijection between three-line arrays and n–color overpartitions of sum side of $h_{10}(q)$ and $h_{11}(q)$, we use

$$\phi : (m_k)_{x_k} \to \begin{cases} \begin{pmatrix} m_{k-1} + x_{k-1} + 1 \\ x_k - 1 \\ \delta_k \end{pmatrix} & \text{if } m_k \text{ is not overlined,} \\ \\ \begin{pmatrix} m_{k-1} + x_{k-1} + 5 \\ x_k - 1 \\ \delta_k - 4 \end{pmatrix} & \text{if } m_k \text{ is overlined,} \end{cases} \tag{3.2.7}$$

and

$$\phi^{-1} : \begin{pmatrix} \lambda_k \\ \mu_k \\ \tau_k \end{pmatrix} \to \begin{cases} (\lambda_k + \mu_k + \tau_k)_{\mu_k+1} & \text{if } \lambda_k = 2 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1}, \\ (\overline{\lambda_k + \mu_k + \tau_k})_{\tau_k+1} & \text{if } \lambda_k = 6 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1}. \end{cases} \tag{3.2.8}$$

The combinatorial interpretations in terms of three-line arrays of sum side of $h_{10}(q)$ and $h_{11}(q)$ are given in Theorem 3.2.16 and Theorem 3.2.18 respectively.

Theorem 3.2.16 *Let $\hat{P}_{10}(\alpha)$ represent the number of three-line arrays into r columns satisfying*

(3.2.16.a) $\lambda_1 = 1$,

(3.2.16.b) $\lambda_k \in \{2 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1}, 6 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1}\}$, *for* $2 \le k \le r$,

(3.2.16.c) $\tau_k \equiv 0 \pmod 4$, $\forall\ k$.

Then

$$P_{10}(\alpha) = 2\hat{P}_{10}(\alpha) = M_{10}(\alpha) = 2\hat{M}_{10}(\alpha) = N_{10}(\alpha)\ \forall\ \alpha \ge 1.$$

Remark 3.2.17 *In the above theorem, we used similar argument as given in* Theorem 2.2.15 *and* Remark 2.2.14 *and letting* $\sum_{\alpha=1}^{\infty} \hat{P}_{10}(\alpha) q^{\alpha} = \hat{h}_{10}(q)$.

Theorem 3.2.18 *Let $\hat{P}_{11}(\alpha)$ represent the number of three-line arrays into r columns satisfying $\lambda_1 = 3$ and* (3.2.16.b)–(3.2.16.c). *Then*

$$P_{11}(\alpha) = 2\hat{P}_{11}(\alpha) = M_{11}(\alpha) = 2\hat{M}_{11}(\alpha) = N_{11}(\alpha)\ \forall\ \alpha \ge 1.$$

Remark 3.2.19 *In the above theorem, we used similar argument as given in* Theorem 2.2.18 *and* Remark 2.2.17 *and letting* $\sum_{\alpha=1}^{\infty} \hat{P}_{11}(\alpha)q^{\alpha} = \hat{h}_{11}(q)$.
The mapping for the first part is $\phi : (m_1)_{x_1} \to \begin{pmatrix} 3 \\ x_1 - 1 \\ m_1 - x_1 - 2 \end{pmatrix}$. *And the remaining mapping is same as in* (3.2.7)–(3.2.8).

Group 5

To establish the bijection between three-line arrays and $(n+t)$–color overpartitions for the sum side of $h_{12}(q)$–$h_{14}(q)$, we use

$$\phi : (m_k)_{x_k} \to \begin{cases} \begin{pmatrix} m_{k-1} + x_{k-1} + 1 \\ x_k - 1 \\ \delta_k \end{pmatrix} & \text{if } m_k \text{ is not overlined,} \\ \\ \begin{pmatrix} m_{k-1} + x_{k-1} + 2 \\ x_k - 1 \\ \delta_k - 1 \end{pmatrix} & \text{if } m_k \text{ is overlined,} \end{cases}$$

and

$$\phi^{-1} : \begin{pmatrix} \lambda_k \\ \mu_k \\ \tau_k \end{pmatrix} \to \begin{cases} (\lambda_k + \mu_k + \tau_k)_{\mu_k+1} & \text{if } \lambda_k = 2 + \lambda_{k-1} + \mu_{k-1} + \tau_{k-1}, \\ (\overline{\lambda_k + \mu_k + \tau_k})_{\mu_k+2} & \text{if } \lambda_k = 3 + \lambda_{k-1} + \mu_{k-1} + \tau_{k-1}. \end{cases}$$

The combinatorial interpretations in terms of three-line arrays of sum side of $h_{12}(q)$–$h_{14}(q)$ are given in Theorems 3.2.20–3.2.23 respectively.

Theorem 3.2.20 *Let* $\hat{P}_{12}(\alpha)$ *represent the number of three-line arrays into* r *columns satisfying*

(3.2.20.a) $\lambda_1 = 1$,

(3.2.20.b) $\lambda_k \in \{2 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1},\ 3 + \lambda_{k-1} + 2\mu_{k-1} + \tau_{k-1}\}$ *for* $2 \le k \le r$.

Then

$$P_{12}(\alpha) = 2\hat{P}_{12}(\alpha) = M_{12}(\alpha) = 2\hat{M}_{12}(\alpha) = N_{12}(\alpha)\ \forall\ \alpha \ge 1.$$

Remark 3.2.21 *In the above theorem, we used similar argument as given in* Theorem 2.2.21 *and* Remark 2.2.20, *letting* $\sum_{\alpha=1}^{\infty} \hat{P}_{12}(\alpha)q^{\alpha} = \hat{h}_{12}(q)$.

Theorem 3.2.22 *Let* $P_{13}(\alpha)$ *represent the number of three-line arrays into* r *columns satisfying all the conditions of* $\hat{P}_{12}(\alpha)$ *defined in* Theorem 3.2.20. *Then*

$$P_{13}(\alpha) = M_{13}(\alpha) = N_{13}(\alpha)\ \forall\ \alpha \ge 1.$$

Theorem 3.2.23 *Let* $P_{14}(\alpha)$ *represent the number of three-line arrays into* $r+1$ *columns* *satisfying* (3.2.20.b) *along with:*

(3.2.23.a) $\lambda_1 = 0 = \tau_1$,

(3.2.23.b) $\lambda_2 \geq 2$.

Then

$$P_{14}(\alpha) = M_{14}(\alpha) = N_{14}(\alpha) \ \forall \ \alpha \geq 0.$$

The Proof of Theorems 3.2.3–3.2.23 can be supplied on lines of Theorem 3.2.2, hence omitted.

3.2.1 Alternative Proofs

In this section we provide an alternate proof for arrays enumerated by $P_1(\alpha)$, and for the remaining $P_k(\alpha)$ where $2 \leq k \leq 14$, we provide only classes which are obtained by splitting the partitions enumerated by $P_k(r, \alpha)$ in Table 3.2. Throughout this section, if $P_k(\alpha)$ denote the three-line arrays with some conditions in any number of columns, then $P_k(r, \alpha)$ will denote the three-line arrays with same conditions into r columns. In the proofs we follow the method of proof of Theorem 2.2.2.

Sketch Proof of Theorem 3.2.2. Split the arrays enumerated by $P_1(r, \alpha)$ into the following four classes:

(i) those arrays in which first column is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

(ii) those arrays in which first column is $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$,

(iii) those arrays in which $\tau_1 \neq 0$,

(iv) those arrays in which $\tau_1 = 0$, and $\mu_1 \neq 0$.

Transform the arrays of class (i), by eliminating the first column $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ of the array and subtracting 2 from all λ_k, keeping μ_k and τ_k as same for $2 \leq k \leq r$. We see that transformed arrays are enumerated by $P_1(r-1, \alpha - 2r + 1)$. In class (ii), deleting first column $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ of the array and subtracting 4 from all λ_k, keeping μ_k and τ_k as same for $2 \leq k \leq r$. The transformed arrays are enumerated by $P_1(r-1, \alpha-4r+2)$. In class (iii), subtracting 2 from

λ_k, $2 \le k \le r$, keeping μ_k $\forall$ k as same and subtract 2 from τ_1. Remaining τ_k, $2 \le k \le r$ remains same. The transformed arrays are enumerated by $P_1(r, \alpha - 2r)$.
Finally, we transform the arrays of class (iv), by subtracting 2 from λ_k, $2 \le k \le r$ and subtracting 1 from μ_1 and remaining μ_k for $2 \le k \le r$ and τ_k for $\forall$ k are same, we see that transformed arrays are enumerated by $P_1(r, \alpha - 2r + 1)$ having the first column as $\begin{pmatrix} \lambda_k \\ \mu_k \\ 0 \end{pmatrix}$, $\mu_k \neq 0$. Thus number of arrays in class (iv) are obtained by subtracting the number of arrays which are enumerated by $P_1(r, \alpha - 2r + 1)$ with the first column as $\begin{pmatrix} \lambda_1 \\ \mu_1 \\ \tau_1 \end{pmatrix}$ where $\tau_1 \neq 0$ from $P_1(r, \alpha - 2r + 1)$. Thus the transformed arrays are enumerated by $P_1(r, \alpha - 2r + 1) - P_1(r, \alpha - 4r + 1)$. Hence we get the following recurrence formula for $P_1(r, \alpha)$:

$$\begin{aligned} P_1(r, \alpha) &= P_1(r-1, \alpha - 2r + 1) + P_1(r-1, \alpha - 4r + 2) + P_1(r, \alpha - 2r) \\ &\quad + P_1(r, \alpha - 2r + 1) - P_1(r, \alpha - 4r + 1), \end{aligned} \tag{3.2.9}$$

where $P_1(0, 0) = 1$ and $P_1(r, \alpha) = 0$ for $\alpha < 0$.
Hence we get the recurrence relation for $P_1(r, \alpha)$, which is same as the recurrence relation for $M_1(r, \alpha)$, so the remaining proof of this theorem can be constructed on the similar line as the proof of Theorem 2.2.2.

The proof of the Theorem 3.2.3–Theorem 3.2.23 follows the same steps as done for the proof of Theorem 2.2.4–Theorem 2.2.24. Only the distribution of the classes varies, so in Table 3.2, we provide the classes and the corresponding enumerator. These classes depends on first column. For $P_k(\alpha)$, $k = 2, 3, 4, 6, 9$ and 14 we have only two classes given in the following table, and for the remaining enumerators we have four classes: first and second classes are given in the table, third class has those arrays in which $\tau_1 \neq 0$, and fourth class has those arrays in which $\tau_1 = 0$, and $\mu_1 \neq 0$.

Remark 3.2.24 *To obtain the recurrence relation for the enumerator $P_7(r, \alpha)$, we consider q–series, $b_1(q) = \sum_{\alpha=0}^{\infty} V_1(\alpha) q^{\alpha} = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n}}$, where $V_1(r, \alpha)$ represents the number of the three-line arrays enumerated by $V_1(\alpha)$ of α into r columns. Continuing this*

$$b_k(q) = \sum_{\alpha=0}^{\infty} V_k(\alpha) q^{\alpha}$$

where $k = 2$–5.

Table 3.2: Distribution of classes

Enumerator	class 1	class 2	Enumerator	class 1	class 2
$P_2(\alpha)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\mu_1\\0\end{pmatrix}$	$\hat{P}_{11}(\alpha)$	$\begin{pmatrix}3\\0\\0\end{pmatrix}$	$\begin{pmatrix}3\\0\\4\end{pmatrix}$
$P_3(\alpha)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\mu_1\\0\end{pmatrix}$	$\hat{P}_{12}(\alpha)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$
$P_4(\alpha)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}2\\0\\0\end{pmatrix}$	$P_{13}(\alpha)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$
$P_5(\alpha)$	$\begin{pmatrix}3\\0\\0\end{pmatrix}$	$\begin{pmatrix}4\\0\\0\end{pmatrix}$	$P_{14}(\alpha)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\mu_1\\0\end{pmatrix}$
$P_6(\alpha)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\mu_1\\0\end{pmatrix}$	$V_1(\alpha)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}3\\0\\0\end{pmatrix}$
$P_7(\alpha)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\2\end{pmatrix}$	$V_2(\alpha)$	$\begin{pmatrix}2\\0\\0\end{pmatrix}$	$\begin{pmatrix}4\\0\\0\end{pmatrix}$
$\hat{P}_8(\alpha)$	$\begin{pmatrix}2\\0\\0\end{pmatrix}$	$\begin{pmatrix}2\\0\\2\end{pmatrix}$	$V_3(\alpha)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}5\\0\\0\end{pmatrix}$
$P_9(\alpha)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\mu_1\\0\end{pmatrix}$	$V_4(\alpha)$	$\begin{pmatrix}3\\0\\0\end{pmatrix}$	$\begin{pmatrix}7\\0\\0\end{pmatrix}$
$P_{10}(\alpha)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\4\end{pmatrix}$	$V_5(\alpha)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}2\\0\\0\end{pmatrix}$

3.3 Split part $(n+t)$–color partitions

In this section, we give another new combinatorial interprtation for $h_l(q)$, where $7 \leq l \leq 14$ using split part $(n+t)$–color partitions. In 2014, Agarwal and Sood [16] interpreted two eighth order, 'mock theta functions' of Gordon and McIntosh in terms of split $(n+t)$–color partitions. Inspired from their work we introduce split part $(n+t)$–color partition, defined as:

Definition 3.3.1 *The split part $(n+t)$–color partition is $(n+t)$–color partition in which the part splits into two parts. Consider a part $(m_k)_{x_k}$ from the partition and we split m_k into two parts as $(m_k)_{x_k} = (m_k^{'} + m_k^{''})_{x_k}$, $1 \leq m_k^{'} \leq m_k$ and $0 \leq m_k^{''} \leq m_k - 1$.*

Example 3.3.1 *For $\alpha = 3$, the split part n–color partitions are:*

$$(3_3),\ (2+1)_3,\ (1+2)_3,\ (3_2),\ (2+1)_2,\ (1+2)_2,\ (3_1),\ (2+1)_1,\ (1+2)_1,\ (2_2\ 1_1),$$

$$((1+1)_2\ 1_1),\ (2_1\ 1_1),\ ((1+1)_1\ 1_1),\ (1_1\ 1_1\ 1_1).$$

The sum side of $h_l(q)$, $7 \leq l \leq 14$ represents generating functions for $Q_l(\alpha)$, where $7 \leq l \leq 14$, which count partitions in terms of split part $(n+t)$–color partitions. These lead to 4-way combinatorial interpretations satisfying

$$\sum_{\alpha=0}^{\infty} Q_l(\alpha)q^\alpha = \sum_{\alpha=0}^{\infty} P_l(\alpha)q^\alpha = \sum_{\alpha=0}^{\infty} M_l(\alpha)q^\alpha = \sum_{\alpha=0}^{\infty} N_l(\alpha)q^\alpha,\ \ 7 \leq l \leq 14,\ \ \alpha \geq 0.$$

Now we provide the combinatorial interpretations of $Q_l(\alpha)$, where $7 \leq l \leq 14$ in terms of split part $(n+t)$–color partition given in Theorem 3.3.1–Theorem 3.3.12

Theorem 3.3.1 *Let $Q_7(\alpha)$ enumerate the number of split part n–color partitions of α such that*

(3.3.1.a) $m_k \equiv x_k \pmod 2\ \forall\ k$,

(3.3.1.b) *m_1 should not be splitted,*

(3.3.1.c) *$\delta_k \geq 0$, and $\equiv 0 \pmod 2$, $\forall\ k > 1$ if $\delta_k = 0$ then the part m_k should not be splitted otherwise $(m_k)_{x_k} = (m_k' + m_k'')_{x_k}$ where $m_k' = m_{k-1} + x_k + x_{k-1}$ and $m_k'' = m_k - m_k' \equiv 0 \pmod 2$.*

Then

$$Q_7(\alpha) = P_7(\alpha) = M_7(\alpha) = N_7(\alpha)\ \ \forall\ \alpha \geq 0.$$

Theorem 3.3.2 *Let $\hat{Q}_8(\alpha)$ enumerate the number of split part n–color partitions of α satisfying* (3.3.1.a), (3.3.1.b), *m_k, $x_k > 1$ and*

(3.3.2.a) *$\delta_k \geq -2$ and $\equiv 0 \pmod 2$ $\forall\ k > 1$,*

(3.3.2.b) *if $\delta_k = -2$ then the part m_k should not be splitted otherwise $(m_k)_{x_k} = (m_k' + m_k'')_{x_i}$,*

(3.3.2.c) *$m_k' = m_{k-1} + x_{k-1} + x_k - 2$ and $m_k'' = m_k - m_k' \equiv 0 \pmod 2$ for $\delta_k = 0$, $m_k' = m_{k-1} + x_{k-1} + x_k$ and $m_k'' = m_k - m_k' \equiv 0 \pmod 2$ for $\delta_k > 0$.*

Then

$$Q_8(\alpha) = 2\hat{Q}_8(\alpha) = P_8(\alpha) = M_8(\alpha) = N_8(\alpha)\ \ \forall\ \alpha \geq 0.$$

Remark 3.3.3 *For the combinatorial interpretation of sum side of $h_8(q)$ in terms of split part n–color partitions, here we use $\hat{h}_8(q)$, in* (2.2.12) *(as defined in* Chapter 2*) and we write as $\hat{h}_8(q) = \sum_{\alpha=1}^{\infty} \hat{Q}_8(\alpha)q^\alpha$.*

Theorem 3.3.4 *Let $Q_9(\alpha)$ enumerate the number of split part $(n+1)$–color partitions of α satisfying* (3.3.1.c) *and*

(3.3.4.a) $x_1 = m_1 + 1$, *and m_1 should not be splitted,*

(3.3.4.b) $m_k - x_k \equiv 1 \pmod 2$, $\forall\ k$.

Then

$$Q_9(\alpha) = P_9(\alpha) = M_9(\alpha) = N_9(\alpha) \quad \forall\ \alpha \geq 0.$$

Theorem 3.3.5 *Let $\hat{Q}_{10}(\alpha)$ enumerate the number of split part n–color partitions of α satisfying*

(3.3.5.a) $m_1 \equiv x_1 \pmod 4$, *m_1 should not be splitted.*

(3.3.5.b) $\delta_k \geq 0$ *and* $\equiv 0 \pmod 4$ $\forall\ k > 1$, *if $\delta_k = 0$ then the part m_k should not be splitted otherwise $(m_k)_{x_k} = (m_k^{'} + m_k^{''})_{x_k}$ where $m_k^{'} = m_{k-1} + x_{k-1} + x_k$ and $m_k^{''} = m_k - m_k^{'} \equiv 0 \pmod 4$.*

Then

$$Q_{10}(\alpha) = 2\hat{Q}_{10}(\alpha) = P_{10}(\alpha) = M_{10}(\alpha) = N_{10}(\alpha) \quad \forall\ \alpha \geq 0.$$

Remark 3.3.6 *In the above theorem, we used similar argument as given in* Remark 2.2.14 *and letting $\sum_{\alpha=1}^{\infty} \hat{Q}_{10}(\alpha) q^{\alpha} = \hat{h}_{10}(q)$.*

Theorem 3.3.7 *Let $\hat{Q}_{11}(\alpha)$ enumerate the number of split part n–color partitions of α satisfying* (3.3.5.b), *$m_1 - x_1 \equiv 2 \pmod 4$ $\forall\ k$, $m_1 \geq 3$ and m_1 should not be splitted. Then*

$$Q_{11}(\alpha) = 2\hat{Q}_{11}(\alpha) = P_{11}(\alpha) = M_{11}(\alpha) = N_{11}(\alpha) \quad \forall\ \alpha \geq 0.$$

Remark 3.3.8 *In the above theorem, we used similar argument as given in* Remark 2.2.17 *and letting $\sum_{\alpha=1}^{\infty} \hat{Q}_{11}(\alpha) q^{\alpha} = \hat{h}_{11}(q)$.*

Theorem 3.3.9 *Let $\hat{Q}_{12}(\alpha)$ enumerate the number of split part n–color partitions of α satisfying*

(3.3.9.a) *m_1 should not be splitted,*

(3.3.9.b) $\delta_k \geq 0$ $\forall\ k > 1$, *if $\delta_k = 0$ then the part m_k should not be splitted otherwise $(m_k)_{x_k} = (m_k^{'} + m_k^{''})_{x_k}$ where $m_k^{'} = m_{k-1} + x_{k-1} + x_k$ and $m_k^{''} = m_k - m_k^{'}$.*

Then

$$Q_{12}(\alpha) = 2\hat{Q}_{12}(\alpha) = P_{12}(\alpha) = M_{12}(\alpha) = N_{12}(\alpha) \quad \forall\ \alpha \geq 1.$$

Remark 3.3.10 *In the above theorem, we used similar argument as given in* Remark 2.2.20, *letting $\sum_{\alpha=1}^{\infty} \hat{Q}_{12}(\alpha) q^{\alpha} = \hat{h}_{12}(q)$.*

Theorem 3.3.11 *Let Q_{13} enumerate the number of split part n–color partitions of α satisfying all the conditions of* Theorem 3.3.9. *Then*

$$Q_{13}(\alpha) = 2\hat{Q}_{13}(\alpha) = P_{13}(\alpha) = M_{13}(\alpha) = N_{13}(\alpha) \ \ \forall \ \alpha \geq 1.$$

Theorem 3.3.12 *Let $Q_{14}(\alpha)$ enumerate the number of split part $(n+1)$–color partitions of α satisfying* (3.3.4.a), (3.3.4.b) *and* (3.3.9.b). *Then*

$$Q_{14}(\alpha) = P_{14}(\alpha) = M_{14}(\alpha) = N_{14}(\alpha) \ \forall \ \ \alpha \geq 0.$$

3.3.1 Sketch Proof of Theorem 3.3.1

To obtain the proof of Theorem 3.3.1, we consider the following q–series,

$$\sum_{\alpha=0}^{\infty} U_1(\alpha) q^{\alpha} = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n}}. \tag{3.3.1}$$

The interpretation of (3.3.1) in terms of split part $(n+t)$–color partitions is given in Lemma 3.3.13. Let $U_1(r,\alpha)$ and $Q_7(r,\alpha)$, denote the number of partitions enumerated by $U_1(\alpha)$ and $Q_7(\alpha)$ into r parts.

Lemma 3.3.13 *Let $U_1(\alpha)$ enumerate the number of split part n–color partitions of α satisfying* (3.3.1.a), (3.3.1.c) *and for $m_1 \neq x_1$ then it should be splitted as $(m_1^{'} + m_1^{''})_{x_1}$ where $m_1^{'} = x_1$ and $m_1^{''} \equiv$* 0(mod 2). *Then* (3.3.1) *holds.*

We illustrate the Lemma 3.3.13 with following example:

Example 3.3.2 *For $\alpha = 8$, there are fifteen split part n–color partitions for $U_1(8)$,*

$$(8_8),\ (8_6),\ (6+2)_6,\ (8_4),\ (4+4)_4,\ (8_2),\ (2+6)_2,\ (7_5\ 1_1),\ (7_3\ 1_1),\ ((5+2)_3\ 1_1),$$
$$(7_1\ 1_1),\ ((3+4)_1\ 1_1),\ (6_2\ 2_2),\ (5_1\ 3_1),\ (5_1\ (1+2)_1).$$

Proof of Lemma 3.3.13. Divide the partitions into four classes:

(i) $(m_1)_{x_1}$ is not of the form $(m_k)_{m_k}$ or $(m_k^{'}+2)_{m_k^{'}}$,

(ii) $(m_1)_{x_1}$ is of the form 1_1,

(iii) $(m_1)_{x_1}$ is of the form $(1+2)_1$,

(iv) $(m_1)_{x_1}$ is of the form $(m_k)_{m_k}$ $(m \geq 2)$ or $(m_k^{'}+2)_{m_k^{'}}$ $(m^{'} \geq 2)$.

Transform the partitions which lie in class (i) by subtracting 2 from all the parts. For part of the form $(m_k^{'}+m_k^{''})_{x_k}$, subtract 2 from $m_k^{''}$. The transformed partitions are enumerated by $U_1(r, \alpha - 2r)$. In class (ii) remove the least part 1_1 and subtract 2 from the remaining parts. If part of the form $(m_k^{'}+m_k^{''})_{x_k}$, subtract 2 from $m_k^{'}$ and we get partitions enumerated by $U_1(r-1, \alpha-2r+1)$. Next, in class (iii) remove the least part $(1+2)_1$ and then subtract 4 from all the remaining parts or from $m_k^{'}$ for the part in splitted form. The transformed partitions are enumerated by $U_1(r-1, \alpha-4r+1)$. In the last class (iv), replace the least part $(m_1)_{m_1}$ by $(m_1-1)_{m_1-1}$ and $(m_1^{'}+2)_{m_1^{'}}$ by $\big((m_1^{'}-1)+2\big)_{m_1^{'}-1}$ and subtract 2 from the remaining parts. If part of the form $(m_k^{'}+m_k^{''})_{x_k}$, subtract 2 from $m_k^{'}$. This will result in partitions enumerated by $U_1(r, \alpha-2r+1)$. It should be noted here that we are obtaining only those partitions of $\alpha-2r+1$ which involve a least part of the type $(m_1)_{m_1}$ or $(m_1^{'}+2)_{m_1^{'}}$. So the number of partitions in class (iv) are enumerated by $U_1(r, \alpha-2r+1) - U_1(r, \alpha-4r+1)$.

The above transformations are clearly reversible and so establish a bijection between the partitions enumerated by $U_1(r, \alpha)$ and which are enumerated by $U_1(r, \alpha-2r) + U_1(r-1, \alpha-2r+1) + U_1(r-1, \alpha-4r+1) + U_1(r, \alpha-2r+1) - U_1(r, \alpha-4r+1)$.

This leads to the recurrence relation

$$\begin{aligned} U_1(r, \alpha) =& U_1(r, \alpha-2r) + U_1(r-1, \alpha-2r+1) + U_1(r-1, \alpha-4r+1) \\ &+ U_1(r, \alpha-2r+1) - U_1(r, \alpha-4r+1). \end{aligned}$$

The recurrence relation for $U_1(r, \alpha)$ is same as for $W_1(r, \alpha)$, so the remaining proof can be elaborated from Lemma 2.2.10.

Now we give a concise proof of Theorem 3.3.1

Split the partitions enumerated by $Q_7(r, \alpha)$ into four classes. In the first class, partitions contains $(m_1)_{x_1} = 1_1$. Remove $(m_1)_{x_1}$ and subtract 2 from all the remaining parts. The parts of the form $(m_k^{'}+m_k^{''})_{x_k}$, subtract 2 from $m_k^{'}$ so the transformed partitions are enumerated by $U_1(r-1, \alpha-2r+1)$. Second class contains the partitions with $(m_1)_{x_1} = 3_1$. By removing the least part and subtract 4 from all the remaining parts. The parts of the form $(m_k^{'}+m_k^{''})_{x_k}$ subtract 4 from $m_k^{'}$, we get the partition enumerated by $U_1(r-1, \alpha-4r+1)$. The partition in the third class contains the partitions which do not have $(m_k)_{m_k}$ and $(m_k)_{m_k-2}$ as the least part. Subtract 4 from all the parts and from $m_k^{'}$ for parts in the splitted form. Thus, we get the partitions enumerated by $Q_7(r, \alpha-4r)$. The last class contains the partitions in which $(m_1)_{m_1}$ or $(m_1)_{m_1-2}$ as the least part. Replace $(m_1)_{m_1}$ by $(m_1-1)_{m_1-1}$ and $(m_1)_{m_1-2}$ by $(m_1-1)_{m_1-3}$, then subtract 2 from remaining parts and from $m_k^{'}$ in the splitted part, that is enumerated by $Q_7(r, \alpha-2r+1) - Q_7(r, \alpha-6r+1)$. Thus the recurrence relation for $Q_7(r, \alpha)$ is given by

$$\begin{aligned} Q_7(r, \alpha) =& U_1(r-1, \alpha-2r+1) + U_1(r-1, \alpha-4r+1) \\ &+ Q_7(r, \alpha-4r) + Q_7(r, \alpha-2r+1) - Q_7(r, \alpha-6r+1), \end{aligned}$$

Proceeding in the same manner as in the proof of Lemma 3.3.13, we get the desired result.

Remark 3.3.14 *We can naturally connect the interpretations of $h_7(q)$ in terms of n–color overpartitions and split part n–color partitions. Let $(m_k)_{x_k}$ and $(m_{k-1})_{x_{k-1}}$ be k^{th} and $(k-1)^{th}$ parts of a n–color overpartition, then the corresponding k^{th} part of split part n–color partition be $(m_k^{'} + m_k^{''})_{x_k}$, that is given by*

$$\phi : (m_k)_{x_k} \rightarrow \begin{cases} (m_k)_{x_k} & \text{if } m_k \text{ is not overlined,} \\ \\ (m_k^{'} + m_k^{''})_{x_k} & \text{if } m_k \text{ is overlined,} \end{cases} \tag{3.3.2}$$

where $m_k^{'} = m_{k-1} + x_{k-1} + x_k$ and $m_k^{''} = m_k - m_k^{'}$.

In the reverse implication, let $(m_k^{'} + m_k^{''})_{x_k}$ be k^{th} part of split part n–color partitions then the corresponding part in an n–color overpartition $(m_k)_{x_k}$, is given by

$$\phi^{-1} : (m_k^{'} + m_k^{''})_{x_k} \rightarrow \begin{cases} (m_k^{'})_{x_k} & \text{if } m_k^{''} = 0, \\ \\ (\overline{m_k^{'} + m_k^{''}})_{x_k} & \text{if } m_k^{''} \neq 0. \end{cases} \tag{3.3.3}$$

We get $Q_7(\alpha) = P_7(\alpha) = M_7(\alpha) = N_7(\alpha)\ \forall\ \alpha \geq 0$.

Here we are omitting the proofs of the remaining Theorems 3.3.2–3.3.12, one can easily elaborate on the lines of Theorem 3.3.1.

3.4 2–color F–partitions

In the sequel to previous sections, we now give a connection for the sum side of $h_l(q)$ where $7 \leq l \leq 11$ in terms of $(n+t)$–color overpartitions and 2–color F–partitions. $cF_l(\alpha)$ for $7 \leq l \leq 11$, count partitions in terms of 2–color F–partition. These lead to 5-way combinatorial interpretations satisfying

$$\sum_{\alpha=0}^{\infty} cF_l(\alpha) = \sum_{\alpha=0}^{\infty} Q_l(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} P_l(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} M_l(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} N_l(\alpha) q^{\alpha}, \quad 7 \leq l \leq 11. \tag{3.4.1}$$

We define F–partition in Chapter 1. Now we recall the definition of n–color F–partitions.

Definition 3.4.1 [21] *The n–color F–partitions is the color F–partitions with n copies* *of the non negative integer 'x' with color 'i', so*

$$x_i : 0 \le x \le i-1,\ 1 \le i \le x,$$

and $x_i \ne x'_{i'}$, unless $x = x'$ and $i = i'$. There is a strict decrease among the parts along the rows and the parts follow the order

$$0_1 < 0_2 < \cdots < 1_1 < 1_2 < \cdots < 2_1 < 2_2 < \cdots < 3_1 < 3_2 < \cdots .$$

Consider colored F–partitions of α in which the parts in either row appear from n copies and are distinct. Let $cF(\alpha)$ denote the number of all such partitions.

Example 3.4.1 *For $\alpha = 2$, the 2–color F–partitions enumerated by $cF(2)$ are*

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}.$$

Remark 3.4.1 *In the main results, we use 2–color F–partitions in which first and second row entries of each column have identical subscripts. For the notation purpose 2–color F–partitions with each column having same subscript array as follows:*

$$\begin{pmatrix} (a_r)_{i_r} & (a_{r-1})_{i_{r-1}} & \cdots & (a_1)_{i_1} \\ (b_r)_{i_r} & (b_{r-1})_{i_{r-1}} & \cdots & (b_1)_{i_1} \end{pmatrix}$$

where $a_1 < a_2 < \cdots < a_r$; $b_1 < b_2 < \cdots < b_r$ and $i_k = 1$ or 2, for $1 \le k \le r$.

Now we present the combinatorial interpretations of $h_l(q)$ where $7 \le l \le 11$ using 2–color F–partitions generated by $cF_l(\alpha)$, $7 \le l \le 11$ given in Theorem 3.4.2–Theorem 3.4.6.

Theorem 3.4.2 *Let $cF_7(\alpha)$ enumerate the number of 2–color F–partitions of α such that*

(3.4.2.a) *for each array $\begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix}$, $a_k - b_k \ge 0$,*

(3.4.2.b) *$i_1 = 1$,*

(3.4.2.c) *for two consecutive arrays $\begin{pmatrix} (a_k)_{i_k} & (a_{k-1})_{i_{k-1}} \\ (b_k)_{i_k} & (b_{k-1})_{i_{k-1}} \end{pmatrix}$, $b_k - a_{k-1} \ge 1$ and for $b_k - a_{k-1} = 1$, then $i_k = 1$.*

Then

$$cF_7(\alpha) = Q_7(\alpha) = P_7(\alpha) = M_7(\alpha) = N_7(\alpha) \ \ \forall\ \alpha \ge 1.$$

Example 3.4.2 *For $\alpha = 6$, corresponding to $cF_7(6)$, the relevant partitions are*

$$\begin{pmatrix} 3_1 \\ 2_1 \end{pmatrix}, \begin{pmatrix} 4_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 5_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 3_1 & 0_1 \\ 1_1 & 0_1 \end{pmatrix}, \begin{pmatrix} 2_1 & 0_1 \\ 2_1 & 0_1 \end{pmatrix}, \begin{pmatrix} 2_2 & 0_1 \\ 2_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 2_1 & 0_1 \\ 1_1 & 0_1 \end{pmatrix}.$$

Theorem 3.4.3 *As follows* (2.2.12), *let $c\hat{F}_8(\alpha)$ enumerate the number of 2–color F–partitions of α such that*

(3.4.3.a) *for each array* $\begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix}$, $a_k - b_k \geq 1$,

(3.4.3.b) $i_1 = 1$,

(3.4.3.c) *for two consecutive arrays* $\begin{pmatrix} (a_k)_{i_k} & (a_{k-1})_{i_{k-1}} \\ (b_k)_{i_k} & (b_{k-1})_{i_{k-1}} \end{pmatrix}$, $b_k - a_{k-1} \geq 0$ *and for* $b_k - a_{k-1} = 0$, $i_k = 1$.

Then

$$cF_8(\alpha) = 2c\hat{F}_8(\alpha) = Q_8(\alpha) = P_8(\alpha) = M_8(\alpha) = N_8(\alpha) \quad \forall\ \alpha \geq 1.$$

Theorem 3.4.4 *Let $cF_9(\alpha)$ enumerate the number of 2–color F–partitions of α such that*

(3.4.4.a) *for each array* $\begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix}$, $b_k \leq a_k + 1$,

(3.4.4.b) $b_1 = 0$, $b_k \geq 1$ *for* $2 \leq k \leq r$ *and* $i_1 = 1$,

(3.4.4.c) *for two consecutive arrays* $\begin{pmatrix} (a_k)_{i_k} & (a_{k-1})_{i_{k-1}} \\ (b_k)_{i_k} & (b_{k-1})_{i_{k-1}} \end{pmatrix}$, $b_k - a_{k-1} \geq 2$ *and for* $b_k - a_{k-1} = 2$, $i_k = 1$.

Then

$$cF_9(\alpha) = Q_9(\alpha) = P_9(\alpha) = M_9(\alpha) = N_9(\alpha) \quad \forall\ \alpha \geq 0.$$

Theorem 3.4.5 *As follows* Remark 2.2.14, *let $c\hat{F}_{10}(\alpha)$ enumerate the number of 2–color F–partitions of α such that*

(3.4.5.a) *for each array* $\begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix}$, $a_k - b_k \geq 0$,

(3.4.5.b) $b_1 = 0 \pmod 2$, *and* $l_1 = 1$,

(3.4.5.c) *for two consecutive arrays* $\begin{pmatrix} (a_k)_{i_k} & (a_{k-1})_{i_{k-1}} \\ (b_k)_{i_k} & (b_{k-1})_{i_{k-1}} \end{pmatrix}$, $b_k - a_{k-1} \geq 1$, $b_k \not\equiv a_{k-1} \pmod 2$ *and for* $b_k - a_{k-1} = 1$, $i_k = 1$.

Then

$$cF_{10}(\alpha) = 2c\hat{F}_{10}(\alpha) = Q_{10}(\alpha) = P_{10}(\alpha) = M_{10}(\alpha) = N_{10}(\alpha) \ \ \forall \ \alpha \geq 0.$$

Theorem 3.4.6 *As follows* Remark 2.2.17, *let* $c\hat{F}_{11}(\alpha)$ *enumerate the number of* 2*–color* F*–partitions of* α *such that*

(3.4.6.a) *for each array* $\begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix}$, $a_k - b_k \geq 0$,

(3.4.6.b) $a_1 + b_1 \geq 2$, $b_1 \equiv 1 \pmod 2$ *and* $l_1 = 1$,

(3.4.6.c) *for two consecutive arrays* $\begin{pmatrix} (a_k)_{i_k} & (a_{k-1})_{i_{k-1}} \\ (b_k)_{i_k} & (b_{k-1})_{i_{k-1}} \end{pmatrix}$, $b_k - a_{k-1} \geq 1$, $b_k - a_{k-1} \equiv 1 \pmod 2$ *and for* $b_k - a_{k-1} = 1$, $i_k = 1$.

Then

$$cF_{11}(\alpha) = 2c\hat{F}_{11}(\alpha) = Q_{11}(\alpha) = P_{11}(\alpha) = M_{11}(\alpha) = N_{11}(\alpha) \ \ \forall \ \alpha \geq 0.$$

3.4.1 Proofs of Theorems 3.4.2–3.4.6

Proof of Theorem 3.4.2. We construct a bijection between the 2–color F–partitions enumerated by $cF_7(\alpha)$ and n–color overpartitions enumerated by $M_7(\alpha)$. To achieve this, we define a map ϕ from each column $\begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix}$ of the 2–color F–partition enumerated by $cF_7(\alpha)$ to a single part $(m_k)_{x_k}$ or $(\overline{m_k})_{x_k}$ of n–color overpartition enumerated by $M_7(\alpha)$. The mapping ϕ is

$$\phi : \begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix} \rightarrow \begin{cases} (a_k + b_k + 1)_{a_k - b_k + 1} & \text{if } i_k = 1, \\ \overline{(a_k + b_k + 1)}_{a_k - b_k + 1} & \text{if } i_k = 2. \end{cases} \tag{3.4.2}$$

Now suppose we have

$$\phi : \begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix} \rightarrow \begin{cases} (m_k)_{x_k} & \text{if } i_k = 1, \\ (\overline{m_k})_{x_k} & \text{if } i_k = 2, \end{cases} \quad \text{and} \quad \phi : \begin{pmatrix} (a_{k-1})_{i_{k-1}} \\ (b_{k-1})_{i_{k-1}} \end{pmatrix} \rightarrow \begin{cases} (m_{k-1})_{x_{k-1}} & \text{if } i_k = 1, \\ (\overline{m_{k-1}})_{x_{k-1}} & \text{if } i_k = 2. \end{cases}$$

Then the weighted difference for two parts $(m_k)_{x_k}$ and $(m_{k-1})_{x_{k-1}}$ is given by

$$\delta_k = m_k - m_{k-1} - x_k - x_{k-1} = 2(b_k - a_{k-1} - 1). \tag{3.4.3}$$

$$\text{Also,} \ \ m_k - x_k = (a_k + b_k + 1) - (a_k - b_k + 1) = 2b_k, \tag{3.4.4}$$

which imply $m_k - x_k \equiv 0 \pmod 2$.

Using (3.4.3), (3.4.4) and the given conditions (2.2.9.b)–(2.2.9.c) we get the desired conditions (3.4.2.a)–(3.4.2.c). To evaluate the reverse implications, we consider the inverse images of two consecutive parts $(m_k)_{x_k}$ or $(\overline{m}_k)_{x_k}$, $(m_{k-1})_{x_{k-1}}$ or $(\overline{m}_{k-1})_{x_{k-1}}$ of n–color overpartition enumerated by $M_7(\alpha)$ as:

$$\phi^{-1} : (m_k)_{x_k} = \begin{pmatrix} (\frac{m_k+x_k-2}{2})_1 \\ (\frac{m_k-x_k}{2})_1 \end{pmatrix} \quad \text{or} \quad \phi^{-1} : (\overline{m}_k)_{x_k} = \begin{pmatrix} (\frac{m_k+x_k-2}{2})_2 \\ (\frac{m_k-x_k}{2})_2 \end{pmatrix},$$

$$\text{and } \phi^{-1} : (m_{k-1})_{x_{k-1}} = \begin{pmatrix} (\frac{m_{k-1}+x_{k-1}-2}{2})_1 \\ (\frac{m_{k-1}-x_{k-1}}{2})_1 \end{pmatrix} \quad \text{or} \quad \phi^{-1} : (\overline{m}_{k-1})_{x_{k-1}} = \begin{pmatrix} (\frac{m_{k-1}+x_{k-1}-2}{2})_2 \\ (\frac{m_{k-1}-x_{k-1}}{2})_2 \end{pmatrix}.$$

So, $m_k - x_k = 2b_k$, $m_{k-1} + x_{k-1} = 2a_{k-1} + 2$, and hence

$$\begin{aligned} \delta_k &= (a_k + b_k + 1) - (a_{k-1} + b_{k-1} + 1) - (a_k - b_k + 1) - (a_{k-1} - b_{k-1} + 1), \\ &= 2(b_k - a_{k-1} - 1). \end{aligned} \tag{3.4.5}$$

$$b_k - a_{k-1} = \frac{m_k - m_{k-1} - x_k - x_{k-1} + 2}{2} - \frac{m_k - x_k}{2} = x_k - 1. \tag{3.4.6}$$

From (3.4.5), (3.4.6) and the conditions (3.4.2.a)–(3.4.2.c), we easily get (2.2.9.b)–(2.2.7.c).

Sketch proofs of the bijection of $c\hat{F}_k(\alpha)$ and $\hat{M}_k(\alpha)$ for $8 \leq k \leq 11$.

- For $k = 8, 10$ and 11 the proofs can be supplied on the line of Theorem 3.4.2.
- For $k = 9$ the map ϕ is given by

$$\phi : \begin{pmatrix} (a_k)_{i_k} \\ (b_k)_{i_k} \end{pmatrix} \to \begin{cases} (a_k + b_k + 1)_{a_k - b_k + 2} & \text{if } i_k = 1 \\ \overline{(a_k + b_k + 1)}_{a_k - b_k + 2} & \text{if } i_k = 2 \end{cases}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : (m_k)_{x_k} \to \begin{pmatrix} (\frac{m_k+x_k-3}{2})_1 \\ (\frac{m_k-x_k+1}{2})_1 \end{pmatrix},$$

$$\phi^{-1} : \overline{(m_k)}_{x_k} \to \begin{pmatrix} (\frac{m_k+x_k-3}{1})_2 \\ (\frac{m_k-x_k+1}{2})_2 \end{pmatrix},$$

where $m_k - x_k \equiv 1 \pmod 2$. Here also the phantom column $\begin{pmatrix} -1_1 \\ 0_1 \end{pmatrix}$ appears in the 2–color F–partition corresponding to the part 0_1 in the $(n+1)$–color overpartition. Since the part 0_1 does not have any contribution towards the total value of α therefore the corresponding column in the colored F–partitions is eliminated. One can easily obtain the results.

3.5 Lattice paths

The lattice paths introduced by Agarwal and Bressoud in [12] are the paths considered in the positive cartesian coordinate system. This specific terminology is used to interpret certain multiple summation q–series in [6,8,9,13]. While Corteel and Mallet were studying overpartitions, they modified the terminology of these path in [39]. The definition of lattice paths with the modified terminology are given below:

Definition 3.5.1 [39] *The path starts on the Y-axis, end on the X-axis with the following steps are considered:*
North-East NE ($\nearrow$)*: from* $(a, b) \to (a + 1, b + 1)$,
South-East SE ($\searrow$)*: from* $(a, b) \to (a + 1, b - 1)$, *where* $b > 0$,
South S ($\downarrow$)*: from* $(a, b) \to (a, b - 1)$, *where* $b \geq 1$ *(S step can only occur after a NE step),*
Horizontal H: $(a, 0) \to (a + 1, 0)$. *All our lattice paths are either empty or terminate with a southeast step from* $(a, 1)$ *to* $(a + 1, 0)$ *or south step from* $(a, 1)$ *to* $(a, 0)$.

In describing the lattice paths, following terminology is used:
Peak is a vertex preceded by a North-East step (or located at the beginning of the peak) and followed by a South step (in which case it is called NES peak) or by a South-East step (in which case it is called NESE peak).
Valley is a vertex preceded by a S step or SE step and followed by a NE step. Note that a S step or SE step followed by H step followed by a NE step does not constitute a valley.
Mountain is a section of the path which starts on either the X-axis or Y-axis, which ends on the X-axis and which does not touch the X-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.
Plain is a section of the path consisting only H steps which starts on the Y-axis or at a vertex preceded by a SE step or at a vertex preceded by a S step and ends at a vertex followed by a NE step.
Height of a vertex is its Y-coordinate.
Weight of a vertex is its X-coordinate.
Weight of a lattice path is the sum of the weights of its peaks.

Example 3.5.1 *The* Figure 3.1 *has five peaks, three valleys, three mountains and two plains. In this figure there is one peak of height* 3*, two peaks of height* 4*, two peaks of height* 1 *and the weight of this path is*

$$0 + 4 + 6 + 13 + 15 = 38.$$

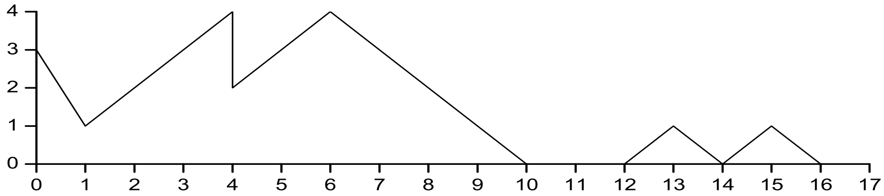
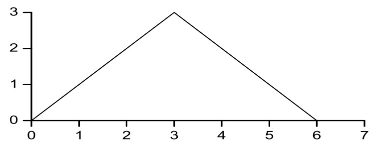
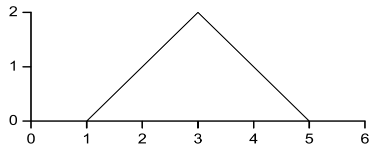
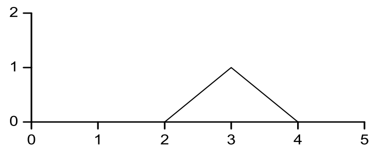
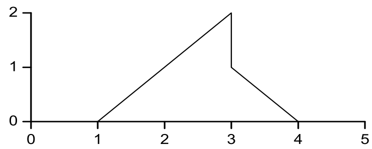


Figure 3.1: Lattice Path.

Remark 3.5.1 *In* [39], *Corteel and Mallet allow the SE steps only after NE steps. To explain these steps, here we give an example. Using NE, SE and S steps, we have lattice paths for* $\alpha = 3$ *in* Table 3.3.

Table 3.3: lattice paths for $\alpha = 3$

Fig.1		Fig.2	
Fig.3		Fig.4	

We only consider lattice paths that satisfy the corresponding conditions given in the theorems. In this section, we provide interpretations for $h_l(q)$, where $7 \leq l \leq 11$, in terms of lattice paths. The functions $R_l(\alpha)$, for $7 \leq l \leq 11$, count partitions in terms of these lattice paths. We also demonstrate a connection between $(n+t)$–color overpartitions and lattice paths, which leads to 6-way combinatorial interpretations that satisfy

$$\sum_{\alpha=0}^{\infty} R_l(\alpha) = \sum_{\alpha=0}^{\infty} cF_l(\alpha) = \sum_{\alpha=0}^{\infty} Q_l(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} P_l(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} M_l(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} N_l(\alpha) q^{\alpha}, \tag{3.5.1}$$

for $7 \leq l \leq 11$. Now the interpretations of $h_l(q)$ in terms of lattice paths generated by $R_l(\alpha)$, where $7 \leq l \leq 11$ given in Theorem 3.5.2–Theorem 3.5.6

Theorem 3.5.2 *Let* $R_7(\alpha)$ *be the number of lattice paths of weight* α *starts at* $(0, 0)$, *have no valley above height* 0, *no plain with odd length, NENESS steps permitted only if there* *is already at least one NE step in the peak. But the first peak lacks such a step. Then,*

$$R_7(\alpha) = cF_7(\alpha) = Q_7(\alpha) = P_7(\alpha) = M_7(\alpha) = N_7(\alpha) \;\; \forall\; \alpha \geq 1.$$

Proof In $\frac{(-q^2;q^2)_{r-1}q^{r^2}}{(q;q)_{2r}}$, the factor q^{r^2} generates the lattice path with r NESE peaks with height 1, initiating at $(0,0)$ and ending at $(2r,0)$. For $r=3$, the path is shown in Figure 3.2.

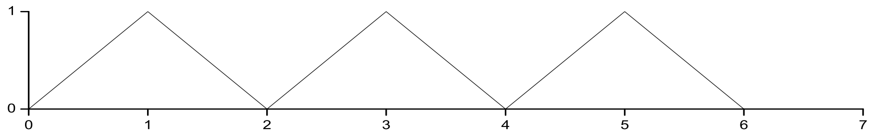


Figure 3.2: Lattice Path.

The factor $(q^2;q^2)_r^{-1}$ generates r even non negative numbers $\leq 2r$, say, $a_r \times 2$, $a_{r-1} \times 4$, $\ldots$, $a_1 \times 2r$, $a_k \geq 0$, $\forall\; k$. This factor transforms the path by increasing the weight of the k^{th} peak by $2a_k$. The path of Figure 3.2 changes to the path shown in Figure 3.3 for $a_1 = 1, a_2 = 0, a_3 = 1$.

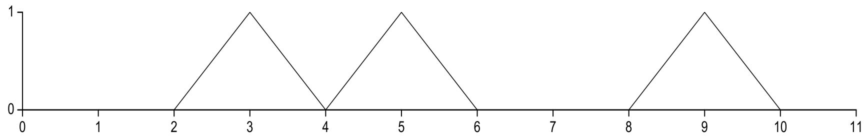


Figure 3.3: Transformed lattice path for $a_1 = 1, a_2 = 0, a_3 = 1$.

The factor $(q;q^2)_r^{-1}$ generates r odd non negative numbers $\leq 2r-1$, say $b_r \times 1$, $b_{r-1} \times 3$, $\ldots$, $b_1 \times (2r-1)$, $b_k \geq 0$, $\forall\; k$. This factor contributes to rise in the height of the k^{th} peak by b_k. With the increase of height of given peak by one unit it results in increase of its weight by one unit and the weight of each subsequent peak increases by two units. The path in Figure 3.3 changes to the path shown in Figure 3.4 for $b_1 = 1, b_2 = 1, b_3 = 0$.

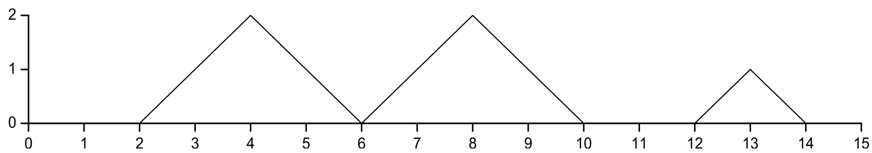


Figure 3.4: Transformed lattice path for $b_1 = 1, b_2 = 1, b_3 = 0$.

The factor $(-q^2;q^2)_{r-1}$ generates $r-1$ non negative multiples of $2, 4, \ldots, 2r-2$, say $c_r \times 2$, $c_{r-1} \times 4$, $\ldots$, $c_2 \times (2r-2)$, $c_k = 0$ or $1\; \forall\; k$. It generates NENESS peaks of height two each in some or all the peaks except the first peak. This factor changes the path by raising the weight of the k^{th} peak by 2 units, height remains same. The Figure 3.4 changes to the path as depicted in Figure 3.5 for $c_1 = 0, c_2 = 1, c_3 = 1$. Every path enumerated by $R_7(\alpha)$ is uniquely generated in this manner.

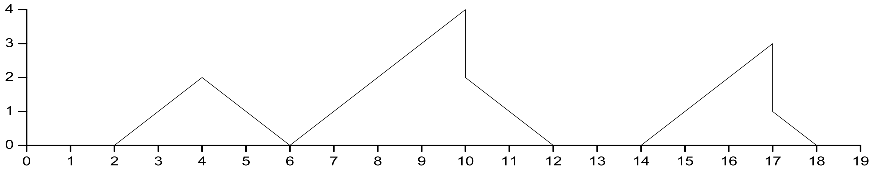


Figure 3.5: Transformed lattice path for $c_1 = 0, c_2 = 1, c_3 = 1$.

Now, we shall prove that $R_7(\alpha) = M_7(\alpha)$. We establish a bijection between the n–color overpartitions enumerated by $M_7(\alpha)$ and the lattice paths enumerated by $R_7(\alpha)$. To achieve this, each path is encoded as the sequence of the weights of the peaks with each weight subscripted by the height of the corresponding peak. Thus we denote $(k-1)^{th}$ and $(k)^{th}$ peaks by A_x and B_y $(B \geq A)$, respectively, then

$$\begin{aligned} A &= (2k-3) + a_{k-1} + 2(b_1 + b_2 + \ldots + b_{k-2}) + b_{k-1} + c_{k-1}, \\ x &= 1 + b_{k-1}, \\ B &= (2k-1) + a_k + 2(b_1 + b_2 + \ldots + b_{k-1}) + b_k + c_k, \\ y &= 1 + b_k. \end{aligned}$$

The weighted difference of these two parts is

$$\big((B_y - A_x)\big) = (a_k - a_{k-1}) + (c_k - c_{k-1}) \geq 0,$$

and $\equiv 0 \pmod 2$. The peak which has NENESS path corresponds to overline part. In the weighted difference if $c_k = 0$ (resp. 1) and $c_{k-1} = 1$ (resp. 0) then B_y corresponds to overline part. Further if we consider the part B_y of n–color overpartition, we find that the parity of both B and y is evaluated by b_k. If b_k is even (resp. odd) then both B and y are odd (resp. even). This proves that the parts and their corresponding subscripts have the same parity. The least part is not overlined as there is no NENESS path in the first peak. Since the lengths of the plains are given in terms of a_k (≥ 0), $1 \leq k \leq r$, which are even numbers, the length of plains, if any, are even.
Consider two parts of a partition enumerated by $R_1(\alpha)$ in order to demonstrate the reverse implication, say C_u and D_v. (Note that there is no need to consider overlined part.) Let $A_1 = (C, u)$ and $A_2 = (D, v)$ be the corresponding peaks in the related lattice path. If there is a plain between A_1 and A_2, its length would be $D - C - v - u$ which is the weighted

difference between C_u and D_v and is therefore non negative and even. Thus, we deduce that the corresponding path may have plains of even length. Next, we establish through contradiction that no valley can exist above height 0. Assume that between the peaks A_1 and A_2, there is a valley V of height k ($k > 0$). In this case, there is fall of $u-k$ from A_1 to V and rise of $v-k$ from V to A_2 in Figure 3.6. This implies that $D = C+(u-k)+(v-k)$, or $D-C-u-v = -2k$. But since the weighted difference ≥ 0, $k = 0$. This completes the bijection.

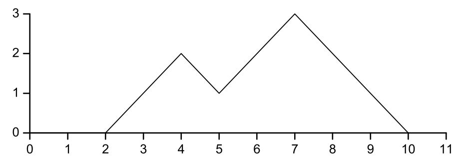
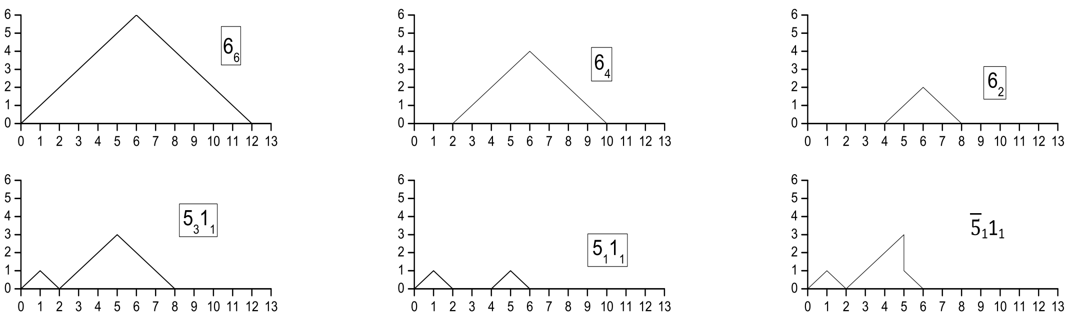


Figure 3.6: Lattice Path.

Example 3.5.2 *For $\alpha = 6$, the lattice paths and n–color overpartitions satisfying* Theorem 3.5.2 *and* Theorem 2.2.9 *are shown in following figure.*



Theorem 3.5.3 *As follow* (2.2.12)*, Let $\hat{R}_8(\alpha)$ be the number of lattice paths of weight α starts at* $(0,0)$*, have no valley above height* 0*, no plain with odd length except at the beginning of the path where a plain of odd length always exists. The NENESS steps permitted only if there is already at least one NE step in the peak. But the first peak lacks such a step. Then,*

$$R_8(\alpha) = 2\hat{R}_8(\alpha) = cF_8(\alpha) = Q_8(\alpha) = P_8(\alpha) = M_8(\alpha) = N_8(\alpha) \;\; \forall\ \alpha \geq 1.$$

Sketch proof of Theorem 3.5.3. In $\frac{(-q^2;q^2)_{r-1}q^{r(r+1)}}{(q;q)_{2r}}$, the proof can be supplied on the line of Theorem 3.5.2, only the extra factor q^r puts a horizontal step in front of the first peak. This adds one unit to the weight of each peak, so the weight of the lattice path is raised by r units. If (A, x) and (B, y) are $(k-1)^{th}$ and $(k)^{th}$ peak respectively, of lattice paths, then they become

$$\begin{aligned} A &= (2k-1) + a_{k-1} + 2(b_1 + b_2 + \ldots + b_{k-2}) + b_{k-1} + c_{k-1}, \\ x &= 1 + b_{k-1}, \\ B &= (2k+1) + a_k + 2(b_1 + b_2 + \ldots + b_{k-1}) + b_k + c_k, \\ y &= 1 + b_k. \end{aligned}$$

Theorem 3.5.4 *Let $R_9(\alpha)$ be the number of lattice paths of weight α starts at $(0, 1)$, have no valley above height 0, no plain with odd length, NENESS steps permitted only if there is already at least one NE step in the peak. But the first peak lacks such a step. Then,*

$$R_9(\alpha) = cF_9(\alpha) = Q_9(\alpha) = P_9(\alpha) = M_9(\alpha) = N_9(\alpha) \ \ \forall \ \alpha \geq 0.$$

Sketch proof of Theorem 3.5.4. In $\frac{(-q^2;q^2)_r q^{r(r+1)}}{(q;q)_{2r+1}}$, the proof can be supplied on the line of Theorem 3.5.2, only the extra factor q^r puts a SE step from $(0, 1)$ to $(1, 0)$ at the front of lattice path. The extra factor of $(1-q^{2r+1})^{-1}$ provides a non negative multiple of $2r+1$ say $b_0 \times (2r + 1)$. This is expressed by raising the first peak by the height $b_0 + 1$. So if we denote the $(k-1)^{th}$ and $(k)^{th}$ peaks of Theorem 3.5.2 now it transforms into $(k)^{th}$ and $(k+1)^{th}$ peaks respectively. The first part is $(b_0)_0 + 1$, which is of the form $(x_0)_{x_0+1}$ and shows that we are using $(n + 1)$ copies of n.

Theorem 3.5.5 *As follow* Remark 2.2.14*, Let $\hat{R}_{10}(\alpha)$ be the number of lattice paths of weight α starts at $(0, 0)$, have no valley above height 0, the length of the plains, if any, is* $\equiv 0 \pmod 4$*, NENENENESSSS steps permitted only if there is already at least one NE step in the peak. But the first peak lacks such a step. Then,*

$$R_{10}(\alpha) = 2\hat{R}_{10}(\alpha) = cF_{10}(\alpha) = Q_{10}(\alpha) = P_{10}(\alpha) = M_{10}(\alpha) = N_{10}(\alpha) \ \ \forall \ \alpha \geq 1.$$

Sketch proof of Theorem 3.5.5. The factor $(q^4; q^4)_r^{-1}$ generates r non negative multiple of 4 say, $a_r \times 4$, $a_{r-1} \times 16$, $\ldots$, $a_1 \times 4r$, $a_k \geq 0$, $\forall\ k$. This factor changes the path by raising the weight of the k^{th} peak by $4a_k$ units. And the factor $(-q^4; q^4)_{r-1}$ contributes $r-1$ non negative multiples of 4 say $c_r \times 4$, $c_{r-1} \times 16$, $\ldots$, $c_2 \times (4r-4)$, $c_k = 0$ or $1\ \forall\ k$. It generates NENENENESSSS peaks of height two each in some or all the peaks except the first peak. This factor changes the path by raising the weight of the k^{th} peak by 4 units, height remains same. For $r = 2$, $a_1 = 1, a_2 = 0$ and $c_1 = 0, c_2 = 1$, the lattice path as depicted in Figure 3.7.

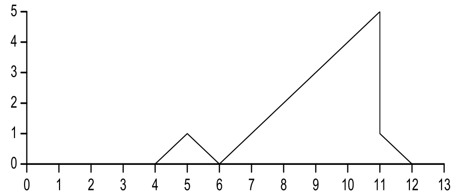


Figure 3.7: Lattice Path.

Theorem 3.5.6 *As follow* Remark 2.2.17, *Let* $\hat{R}_{11}(\alpha)$ *be the number of lattice paths of weight* α *starts at* $(0,0)$, *have no valley above height* 0, *there is a plain of length* $\equiv 2 \pmod 4$ *in the beginning of the path and the length of the other plains, if any, is* $\equiv 0 \pmod 4$, *NENENENENESSSS steps permitted only if there is already at least one NE step in the peak. But the first peak lacks such a step. Then,*

$$R_{11}(\alpha) = 2\hat{R}_{11}(\alpha) = cF_{11}(\alpha) = Q_{11}(\alpha) = P_{11}(\alpha) = M_{11}(\alpha) = N_{11}(\alpha) \ \ \forall\ \alpha \geq 1.$$

On the lines of the aforementioned proofs, the proof of Theorem 3.5.6 can be simply developed.

3.6 Conclusion

The main objective of this chapter is to expand upon the interpretations of the fourteen q–series identities presented in Chapter 2 by utilizing various combinatorial tools. We establish bijections for several q–series identities and develop direct interpretations using the classical method with three-line arrays. Additionally, we establish connections between eight out of these fourteen q–series identities and the newly introduced tool of split part $(n+t)$–color partitions. Furthermore, we demonstrate connections between five out of these fourteen q–series identities and 2–color F–partitions and lattice paths.

Chapter 4

q–series identities related to moduli $4k$ and $4k+2$

4.1 Introduction

Bailey found a number of q–series identities in [27, 28]. His idea was extensively used by Slater to find 130 such identities, which are cataloged in [79]. Some q–series identities are studied in [27, 28, 52, 71, 79] using various approaches. Within the last few decades, some q–series have been extended to multiple series, and the results of infinite families of q–series identities can be seen in [21, 85].

Using Bailey's idea and the q–hypergeometric transformations of Verma and Jain in [85], Sills in [76, 77] discovered some new double-sum identities. In this chapter, we provide combinatorial interpretations of Sills' double-sum identities given in [77] with identity numbers A.7, A.6, and A.4, using $(n+t)$–color partitions, respectively.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{q^{s^2+2r^2}}{(q;q^2)_s(q^2;q^2)_r(q;q)_{s-2r}}=\frac{[q^8,q^{10},q^{18};q^{18}]_\infty}{(q;q)_\infty}, \tag{4.1.1}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{q^{s^2+2r^2+2r}}{(q;q^2)_s(q^2;q^2)_r(q;q)_{s-2r}}=\frac{[q^6,q^{12},q^{18};q^{18}]_\infty}{(q;q)_\infty}, \tag{4.1.2}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{q^{s^2+2s+2r^2+2r}}{(q;q^2)_{s+1}(q^2;q^2)_r(q;q)_{s-2r}}=\frac{[q^2,q^{16},q^{18};q^{18}]_\infty}{(q;q)_\infty}. \tag{4.1.3}$$

Verma and Jain in [85] gave the analytic generalization of (4.1.1) and (4.1.3) respectively,

The contents of this chapter are communicated for publication.

in the following form

$$\sum_{n=0}^{\infty}\sum_{r_1,\ldots,r_{p-3}}^{\infty}\frac{q^{2(M_1^2+\ldots+M_{p-3}^2)+4M_{p-3}^3+n^2+4nM_{p-3}}}{(q;q)_n(q^2;q^2)_{r_1}\cdots(q^2;q^2)_{r_{p-3}}(q;q^2)_{n+2M_{p-3}}}=\prod_{n=0,\pm 2p(mod\ 4p+2)}(1-q^n)^{-1} \tag{4.1.4}$$

$$\sum_{n=0}^{\infty}\sum_{r_1,\ldots,r_{p-3}}^{\infty}\frac{q^{2(M_1^2+\ldots+M_{p-3}^2)+4M_{p-3}^3+n^2+4nM_{p-3}}}{(q;q)_n(q^2;q^2)_{r_1}\cdots(q^2;q^2)_{r_{p-3}}}.\frac{q^{2(n+3M_{p-3}+M_1+\ldots M_{p-4})}}{(q;q^2)_{n+2M_{p-3}}}$$
$$=\prod_{n\neq 0,\pm 2(mod\ 4p+2)}(1-q^n)^{-1}. \tag{4.1.5}$$

Remark 4.1.1 *We note* $M_i = r_1 + r_2 + \ldots + r_i$ *and* $M_{-1} = M_0 = 0$. *The identities* (4.1.1) *and* (4.1.3) *are special cases of* (4.1.4) *and* (4.1.5), *respectively for* $p = 4$ *and* $n + 2r = n_1$.

4.2 Main Results

In this section to provide the combinatorial interpretations of (4.1.1)–(4.1.3), we employ the tool $(n+t)$–color partitions in Theorems 4.2.1–4.2.3 respectively. The summation side of (4.1.1)–(4.1.3) are enumerated by $X_l(\alpha)$ and the product side is enumerated by $Y_l(\alpha)$, where $\alpha \geq 0$ and $1 \leq l \leq 3$.

Theorem 4.2.1 *Let* $P_1(r,\alpha)$ *count the number of* n*–color partitions of* α *into* r *parts satisfying*

(4.2.1.a) m_k *is even and* $x_k = 2$, $\forall\ k$

(4.2.1.b) $\delta_k \geq 0$, *and* $\equiv 0 \pmod 2$ $\forall\ k$

and $Q_1(r,s,\alpha)$, $r \geq 0$ *count the number of* n*–color partitions of* α *into* s *parts satisfying*

(4.2.1.c) $m_1 = x_1$, *for* $r > 0$

(4.2.1.d) $\delta_k = \begin{cases} 0 & \text{if } 1 \leq k \leq s-2r-1, \\ \geq 0 & \text{if } s-2r \leq k \leq s. \end{cases}$

And, let $X_1(\alpha) = \sum_{r=0}^{\infty}\sum_{j=0}^{\alpha} P_1(r,j)Q_1(r,\alpha-j)$. *Then*

$$\sum_{\alpha=0}^{\infty} X_1(\alpha)q^{\alpha} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{q^{s^2+2r^2}}{(q;q^2)_s(q^2;q^2)_r(q;q)_{s-2r}}. \tag{4.2.1}$$

Let $Y_1(\alpha)$ *count the number of partitions of* α *into parts* $\not\equiv 0, \pm 8 \pmod{18}$. *Then,*

$$X_1(\alpha) = Y_1(\alpha)\ \forall\ \alpha \geq 0$$
$$\textit{where}\ \sum_{\alpha=0}^{\infty} Y_1(\alpha) q^{\alpha} = \frac{[q^8, q^{10}, q^{18}; q^{18}]_\infty}{(q;q)_\infty}.$$

Theorem 4.2.2 *Let* $P_2(r, \alpha)$ *count the number of* n*–color partitions of* α *into* r *parts satisfying all the conditions of* $P_1(r, \alpha)$ *with* $m_k \geq 4\ \forall\ k$. $Q_2(r, s, \alpha)$ *counted same as* $Q_1(r, s, \alpha)$.
And, let $X_2(\alpha) = \sum\limits_{r=0}^{\infty} \sum\limits_{j=0}^{\alpha} P_2(r, j) Q_2(k, \alpha - j)$. *Then*

$$\sum_{\alpha=0}^{\infty} X_2(\alpha) q^{\alpha} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{q^{s^2+2r^2+2r}}{(q;q^2)_s (q^2;q^2)_r (q;q)_{s-2r}}. \tag{4.2.2}$$

Let $Y_2(\alpha)$ *count the number of partitions of* α *into parts* $\not\equiv 0, \pm 6 \pmod{18}$. *Then,*

$$X_2(\alpha) = Y_2(\alpha)\ \forall\ \alpha \geq 0$$
$$\textit{where}\ \sum_{\alpha=0}^{\infty} Y_2(\alpha) q^{\alpha} = \frac{[q^6, q^{12}, q^{18}; q^{18}]_\infty}{(q;q)_\infty}.$$

Theorem 4.2.3 $P_3(r, \alpha)$ *count the number of* n*–color partitions same as* $P_2(r, \alpha)$ *and* $Q_3(r, s, \alpha)$ *count the number of* $(n+2)$*–color partitions of* α *into* s *parts with subscript* $x_1 = m_1 + 2$ *and* (4.2.1.d).
And, let $X_3(\alpha) = \sum\limits_{r=0}^{\infty} \sum\limits_{j=0}^{\alpha} P_3(r, j) Q_3(r, \alpha - j)$. *Then*

$$\sum_{\alpha=0}^{\infty} X_3(\alpha) q^{\alpha} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{q^{s^2+2s+2r^2+2r}}{(q;q^2)_{s+1} (q^2;q^2)_r (q;q)_{s-2r}}. \tag{4.2.3}$$

Let $Y_3(\alpha)$ *count the number of partitions of* α *into parts* $\not\equiv 0, \pm 2 \pmod{18}$. *Then,*

$$X_3(\alpha) = Y_3(\alpha)\ \forall\ \alpha \geq 0$$
$$\textit{where}\ \sum_{\alpha=0}^{\infty} Y_3(\alpha) q^{\alpha} = \frac{[q^2, q^{16}, q^{18}; q^{18}]_\infty}{(q;q)_\infty}.$$

We illustrate Theorem 4.2.1 with the help of following example.

Example 4.2.1 *For* $\alpha = 8$, $P_1(r, \alpha)$ *and* $Q_1(r, s, \alpha)$ *as counted in* Theorem 4.2.1 *are given in* Table 4.1 *and* 4.2 *respectively.* $P(1, 2) = 1$, *the relevant partition is* (2_2); $P(1, 4) = 1$,

Table 4.1: $P_1(r, 8)$

$r \backslash \alpha$	0	2	4	6	8
0	1	0	0	0	0
1	0	1	1	1	1
2	0	0	0	0	1
≥ 3	0	0	0	0	0

the relevant partition is (4_2); $P(1,6) = 1$, *the relevant partition is* (6_2); $P(1,8) = 1$, *the relevant partition is* (8_2) *and* $P(2,8) = 1$ *the relevant partition is* $(6_2\ 2_2)$.
$Q_1(0,1,8) = 8$, *where the relevant partitions are* $(8_1),\ (8_2),\ (8_3),\ (8_4),\ (8_5),\ (8_6),\ (8_7),\ (8_8)$.

Table 4.2: $Q_1(r, s, 8)$

$s \backslash r$	0	1	≥ 2
0	0	0	0
1	8	0	0
2	11	2	0
≥ 3	0	0	0

$Q_1(0,2,8) = 11$, *the relevant partitions are* $(7_5\ 1_1),\ (7_4\ 1_1),\ (7_3\ 1_1),\ (7_2\ 1_1),\ (7_1\ 1_1),\ (6_3\ 2_1),\ (6_2\ 2_1),\ (6_1\ 2_1),\ (5_2\ 2_2),\ (5_1\ 2_2),\ (5_1\ 3_1)$.
$Q_1(1,2,8) = 2$, *the relevant partitions are* $(7_5\ 1_1),\ (6_2\ 2_2)$.
Hence,

$$Q_1(0,8) = \sum_{m=0}^{8} Q_1(0,m,8) = 19,$$

and so $Q_1(r,\alpha)$, *for* $1 \leq r, \alpha \leq 8$, *is given in the* Table 4.3.

Table 4.3: $Q_1(r, \alpha)$

$r \backslash \alpha$	0	1	2	3	4	5	6	7	8
0	1	1	2	3	5	7	10	15	19
1	-	-	0	0	1	1	1	2	2
≥ 2	0	0	0	0	0	0	0	0	0

$$\textit{Therefore,}\quad X_1(8) = \sum_{r=0}^{8}\sum_{\alpha=0}^{8} P_1(r,\alpha)Q_1(r,8-\alpha)$$
$$\begin{aligned}&=(19+0+0+0+0+0+0+0)+(0+0+0+0+0+0+0+0)\\&\quad+(0+1+0+0+0+0+0+0)+(0+0+0+0+0+0+0+0)\\&\quad+(0+1+0+0+0+0+0+0)+(0+0+0+0+0+0+0+0)\\&\quad+(0+0+0+0+0+0+0+0)+(0+0+0+0+0+0+0+0)\\&\quad=21\end{aligned}$$

and partitions counted by $Y_1(8)$ *are:*
(7 1), (6 2), (6 1 1), (5 3), (5 2 1), (5 1 1 1), (4 4), (4 3 1), (4 2 2), (4 2 1 1), (4 1 1 1 1), (3 3 2), (3 3 1 1), (3 2 2 1), (3 2 1 1 1), (3 1 1 1 1 1), (2 2 2 2), (2 2 2 1 1), (2 2 1 1 1 1), (2 1 1 1 1 1 1), (1 1 1 1 1 1 1 1).

4.2.1 Proofs

Here we provide the complete proof of Theorem 4.2.1 and the proofs of Theorems 4.2.2–4.2.3 follow similar arguments, hence we delete repetitive proofs.

Proof of Theorem 4.2.1 We have

$$\sum_{\alpha=0}^{\infty} P_1(r,\alpha)q^{\alpha} = \frac{q^{2r^2}}{(q^2;q^2)_r}. \tag{4.2.4}$$

In $\frac{q^{2r^2}}{(q^2;q^2)_r}$, the factor $q^{2r^2} = q^{2((2r-1)+(2r-3)+\ldots+3+1)}$ generates an n–color partition into r parts of the form

$$(4r-2)_2 + (4r-6)_2 + \ldots + 2_2. \tag{4.2.5}$$

The factor $\frac{1}{(q^2;q^2)_r}$ generates r non-negative parts which are $\equiv 0(\text{mod } 2)$, say, $2\cdot a_1, 4\cdot a_2, \ldots, 2r\cdot a_r$ such that (4.2.5) transforms to

$$(4r-2+a_1+\ldots+a_r)_2 + (4r-6+a_1+\ldots+a_{r-1})_2 + \ldots + (2+a_1)_2. \tag{4.2.6}$$

Hence the parts are even with subscript 2, $\delta_k \geq 0\ \forall\ k$.
Next we prove,

$$\sum_{\alpha=0}^{\infty} Q_1(r,s,\alpha)q^{\alpha} = \frac{q^{s^2}}{(q;q^2)_s(q;q)_{s-2r}}.$$

The factor $q^{s^2} = q^{(2s-1)+(2s-3)+\ldots+3+1}$ generates an n–color partition into s parts of the form

$$(2s-1)_1 + (2s-3)_1 + \ldots + 3_1 + 1_1. \tag{4.2.7}$$

The factor $\frac{1}{(q;q^2)_s}$ generates s non-negative odd parts, say, $1 \cdot b_s, 3 \cdot b_{s-1}, \ldots, (2s-1) \cdot b_1$. It transforms the partition (4.2.7) into

$$\Big((2s-1) + 2b_1 + 2b_2 + \ldots + 2b_{s-1} + b_s\Big)_{(1+b_s)} + \Big((2s-3) + 2b_1 + 2b_2 + \ldots + 2b_{s-2} + b_{s-1}\Big)_{(1+b_{s-1})} + \ldots + (3 + 2b_1 + b_2)_{(1+b_2)} + (1 + b_1)_{(1+b_1)}. \tag{4.2.8}$$

The factor $\frac{1}{(q;q)_{s-2r}}$ depends on the value of r. Here, we discuss the cases separately for $r = 0$, $0 < r \leq [\frac{s}{2}]$ and $r > [\frac{s}{2}]$.

For the last case, $r > [\frac{s}{2}]$, the factor $(q;q)_{r-2s} = 0$ and hence $\frac{1}{(q;q)_{s-2r}}$ will not make any addition in counting of the relevent partitions.

For $r = 0$, $\frac{1}{(q;q)_{s-2r}}$ reduces to $\frac{1}{(q;q)_s}$ which generates s non negative parts, $1 \times c_s, 2 \times c_{s-1}, \cdots, s \times c_1$. So, (4.2.8) becomes

$$\begin{aligned}&\Big((2s-1) + 2b_1 + \ldots + 2b_{s-1} + b_s + c_1 + \ldots + c_s\Big)_{(1+b_s)} \\ &+ \Big((2s-3) + 2b_1 + \ldots + 2b_{s-2} + b_{s-1} + c_1 + \cdots + c_{s-1}\Big)_{(1+b_{s-1})} + \ldots \\ &+ (3 + 2b_1 + b_2 + c_1 + c_2)_{(1+b_2)} + (1 + b_1 + c_1)_{(1+b_1)}.\end{aligned} \tag{4.2.9}$$

Clearly the weighted difference between any two consecutive parts is $\delta_k = c_k \geq 0 \; \forall \; k$.
For $0 < r \leq [\frac{s}{2}]$, $\frac{1}{(q;q)_{s-2r}}$ generates $(s-2r)$ non negative parts, say, $1 \cdot c_{s-2r}, 2 \cdot c_{s-2r-1}, \ldots, (s-2r) \cdot c_1$. For this case, (4.2.8) translates to

$$\begin{aligned}&\Big((2s-1) + 2b_1 + \ldots + 2b_{s-1} + b_s + c_1 + \ldots + c_{s-2r}\Big)_{(1+b_s)} + \ldots \\ &+ \Big((2(s-2r+1)-1) + 2b_s + \ldots + 2b_{s-2r} + l_{s-2r+1} + c_{s-2r} + c_{s-2r+1}\Big)_{(1+b_{s-2r+1})} \\ &+ \Big((2(s-2r)-1) + 2b_s + \ldots + 2b_{s-2r-1} + l_{s-2r} + \; c_{s-2r}\Big)_{(1+l_{s-2r})} \\ &+ \Big((2(s-2r-1)-1) + 2b_s + \ldots + 2b_{s-2r-2} + b_{s-2r-1}\Big)_{(1+b_{s-2r-1})} \\ &+ \cdots + (3 + 2b_1 + b_2)_{(1+b_2)} + (1 + b_1)_{(1+b_1)}.\end{aligned} \tag{4.2.10}$$

Using (4.2.10), $\delta_k = 0$, if $1 \le k \le s-2r-1$, and $\delta_k = c_k \ge 0$, if $s-2r \le k \le s$. Now

$$\sum_{\alpha=0}^{\infty} Q_1(r,s,\alpha)q^\alpha \quad = \quad \frac{q^{r^2}}{(q;q^2)_s(q;q)_{s-2r}}, \tag{4.2.11}$$

and

$$\sum_{\alpha=0}^{\infty} Q_1(r,\alpha)q^\alpha = \sum_{\alpha=0}^{\infty}\sum_{s=0}^{\infty} Q_1(r,s,\alpha)q^\alpha = \sum_{s=0}^{\infty} \frac{q^{s^2}}{(q;q^2)_s(q;q)_{s-2r}}.$$

Also,

$$\begin{aligned}
X_1(\alpha) &= \sum_{r,j} P_1(r,j)Q_1(r,\alpha-j), \\
\sum_{\alpha=0}^{\infty} X_1(\alpha)q^\alpha &= \sum_{r,j,\alpha} P_1(r,j)Q_1(r,\alpha-j)q^\alpha \\
&= \sum_{r,j,\alpha} P_1(r,j)Q_1(r,\alpha)q^{\alpha+j} \\
&= \sum_{r,j,\alpha} P_1(r,j)q^j Q_1(r,\alpha)q^\alpha \\
&= \sum_{r=0}^{\infty}(\sum_{j=0}^{\infty} P_1(r,j)q^j)(\sum_{\alpha=0}^{\infty} Q_1(r,\alpha)q^\alpha) \\
&= \sum_{r=0}^{\infty} \frac{q^{2r^2}}{(q^2;q^2)_r} \sum_{s=0}^{\infty} \frac{q^{s^2}}{(q;q^2)_s(q;q)_{s-2r}} \\
&= \sum_{r,s=0}^{\infty} \frac{q^{s^2+2r^2}}{(q;q^2)_s(q;q)_{s-2r}(q^2;q^2)_r}.
\end{aligned}$$

4.3 Conclusion

The results discussed in this chapter are sufficient to interpreting the following two identities (A.14)–(A.15) of [77] in terms of $(n+t)$–color partitions.

$$\sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{\nu^2+4k^2+4k}}{(q;q^2)_\nu(q^4;q^4)_k(q;q)_{\nu-2k}} = \frac{(q^8,q^{20},q^{28};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{4.3.1}$$

$$\sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{\nu^2+4k^2}}{(q;q^2)_\nu(q^4;q^4)_k(q;q)_{\nu-2k}} = \frac{(q^{12},q^6,q^{28};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{4.3.2}$$

Here, we have explored double-sum q–series identities using colored partitions. An obvious question arises: Is there a possibility of establishing bijection between the $(n+t)$–color partitions and some other combinatorial tools for double sum q–series identities as analogues to bijections available for single sum q–series identities between different combinatorial objects, say lattice paths and F–partitions refer to [7, 75, 80].

Chapter 5

q–series identities for signed color partitions

5.1 Introduction

Euler included a result in chapter titled "De Partitio Numerorum", which is given in [43]: every positive integer is uniquely represented as the sum or difference of distinct power of 3. He wrote this in terms of generating function:

$$\sum_{\alpha=-\infty}^{\infty} x^{\alpha} = \prod_{\alpha=0}^{\infty}(x^{-3^{\alpha}} + x^{3^{\alpha}} + 1), \tag{5.1.1}$$

which converges for no values of x. But in [23], Andrews treated (5.1.1) as an identity in formal Laurent series. He showed a great interest in this section and asked the question, "Why have we thought so little about partition generating functions in which some of the partitions might have some negative parts ?" Andrews explored Euler's eye-catching identity (5.1.1) and find some new and appealing results. He referred to called such partitions as "Signed Partitions" in which parts may appear with + or − sign.

Definition 5.1.1 *In* [62]*, a signed partition* Φ *of an integer* α*, denoted by* $\Phi \dashv \alpha$*, is a partition pair* (ϕ_1, ϕ_2)*, where*

$$\alpha = \phi_1 + \phi_2,$$

ϕ_1 *(resp.* ϕ_2*) is the positive (resp. negative) subpartition of* Φ *and* $\phi_1^{l(\phi_1)}, \ldots, \phi_1^2, , \phi_1^1$ *(resp.* $\phi_2^{l(\phi_2)}, \ldots, \phi_2^2, \phi_2^1$*) are the positive (resp. negative) parts of* Φ.

Remark 5.1.1 *There are infinitely many unrestricted signed partitions of an integer. However, we are particularly interested in finite sets of signed partitions, which can be obtained by imposing some suitable restrictions on the parts.*

Example 5.1.1 *Let* $\phi_1 = 6 + 3 + 3$ *and* $\phi_2 = -3 - 2 - 1$ *then* $\Phi = (6 + 3 + 3, -3 - 2 - 1)$ *is a signed partition of* 6.

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Signed partitions naturally fit in the interpretation of many classical q–series identities. For instance, see [23] for interpretations of Göllnitz–Gordon identities using signed partitions. In [78], Sills provided a bijection between ordinary and signed partitions for the Göllnitz–Gordon identity. In [54], Keith explored four combinatorial theorems by presenting bijections between restricted signed partitions and ordinary partitions. He also studied the behavior of signed partitions of zero in arithmetic progression. In [62], McLaughlin and Sills provided interpretations of q–series identities belonging to the family of mod 36 with the missing member of the family in the same flavor.
Inspired by their work, we define signed color partitions, which have their roots in colored partitions introduced in Chapter 1 to interpret 100 q–series identities from Chu-Zhang's Compendium [35] and Slater's Compendium [79].

Definition 5.1.2 *A signed color partition Θ is a signed partition pair (θ_1, θ_2) where θ_1 and θ_2 are the $(n+t)$–color partitions, where $\theta_1 = \big((\theta_1^{l(\theta_1)})_{x_{l(\theta_1)}}, (\theta_1^2)_{x_2}, (\theta_1^1)_{x_1}\big)$ and $\theta_2 = \big((\theta_2^{l(\theta_1)})_{y_{l(\theta_1)}}, \ldots, (\theta_2^2)_{y_2}, (\theta_2^1)_{y_1}\big)$*

Example 5.1.2 $\Theta = (9_3 + 4_2 + 1_1, -6_2 - 3_1 - 1_1)$ *is an example of signed color partitions of* 4 *where,* $\theta_1 = 9_3 + 4_2 + 1_1$ *and* $\theta_2 = -6_2 - 3_1 - 1_1$. *We may write* $\theta_1 = (9_3, 4_2, 1_1)$ *and* $\theta_2 = (6_2, 3_1, 1_1)$, *here* θ_1 *is positive part and* θ_2 *is negative part of signed partition* Θ.

Remark 5.1.2 *The color x_k for the k^{th} part can be split in connection with split color partitions. If all the parts come with only one color it is signed partitions (as defined earlier) otherwise it becomes a signed color partitions.*

In this chapter we provide interpretations of the sum side of 100 q–series identities that appear in [35,79], using signed color partitions. These 100 q–series and their combinatorial interpretations are given in Table 5.1 in concise form. The proofs of combinatorial interpretations are given in the next section. To ease the reading of the Table 5.1, we consider an example.

Consider a q–series identity that appear in [35] and [79] as Identity no. 1 and 6 respectively.

$$\sum_{\alpha=0}^{\infty} \frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_\alpha} = \frac{[-q, -q^2, q^3; q^3]_\infty}{(q;q)_\infty}.$$

In Table 5.1 we have provided the interpretation for sum side of above q–identity and similar others using signed partitions.

Column 2: $f_i(q)$ correspond to count of signed partitions or signed color partitions for the sum side, that is, $f_1(q) = \sum_{\alpha=0}^{\infty} \frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_\alpha}$.

Column 3: θ_1 correspond to count of positive part of signed partitions, due to the q–series $\sum_{\alpha=1}^{\infty} \frac{q^{\alpha \frac{(3\alpha-1)}{2}}}{(q;q)_\alpha (q;q^2)_\alpha}$, as we note

$$f_1(q) = \sum_{\alpha=0}^{\infty} \frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_\alpha} = 1 + 2\sum_{\alpha=1}^{\infty} \frac{q^{\alpha \frac{(3\alpha-1)}{2}}}{(q;q)_\alpha (q;q^2)_\alpha} \prod_{i=1}^{\alpha-1} (1 - q^{-i}).$$

Column 4: This column provide the combinatorial interpretations relevant to θ_1.

Column 5: θ_2 corresponds to count of negative part of signed partitions or signed color partitions, due to $\prod_{i=1}^{\alpha-1} (1 + q^{-i})$.

Column 6: This column provide the combinatorial interpretation relevant to θ_2.

Column 7: Here we give the identity number reference(s) for each q–series identity from its cited paper.

Table 5.1

Sr. no.	$f_i(q)$	θ_1	Interpretation	θ_2	Interpretation	Identity No.
1.	$\frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\frac{q^{\alpha(3\alpha-1)/2}}{(q;q)_\alpha (q;q^2)_\alpha}$	$m_1 - x_1 \equiv 1 (\text{mod } 2)$, $\delta_k \geq 1$	$\prod_{i=1}^{\alpha-1}(1+q^{-i})$	Distinct parts $< \nu$	1 [35], 6 [79]
2.	$\frac{(-q;q)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$\frac{q^{3\alpha(\alpha+1)/2}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$(n+1)$ copies of n, least part is $(x_1)_{(x_1)+1}$, $\delta_k > 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	3 [35]
3.	$\frac{(q;q^2)_\alpha q^{2\alpha(\alpha+1)}}{(-q;q^2)_{\alpha+1} (q^4,q^4)_\alpha}$	$\frac{q^{3\alpha^2+2\alpha}}{(q;q^2)_{\alpha+1} (q^4;q^4)_\alpha}$	$(n+2)$ copies of n, $m_k \equiv x_k (\text{mod } 2)$, $x_1 > 2$, least part is $(x_1)_{x_1+2}$, $\delta_k \geq 0$, $\equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	5 [35], 27 [79]
4.	$\frac{(q;q^2)_\alpha q^{2\alpha^2}}{(-q;q^2)_\alpha (q^4,q^4)_\alpha}$	$\frac{q^{3\alpha^2}}{(q;q^2)_\alpha (q^4;q^4)_\alpha}$	$m_k \equiv x_k (\text{mod } 2)$, $x_1 > 2$, $m_1 \equiv x_1 (\text{mod } 4)$, $\delta_k \geq 0$, $\equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	6 [35]
5.	$\frac{(-q;q)_\alpha q^{\frac{\alpha(\alpha+1)}{2}}}{(q;q)_\alpha}$	$\frac{q^{\alpha(\alpha+1)}}{(q;q)_\alpha}$	$m_k > 1$, $x_k = 1, \delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct part $\leq \nu$	7 [35], 8 [79]
6.	$\frac{(-q;q)_{\alpha+1} q^{\frac{\alpha(\alpha+1)}{2}}}{(q;q)_\alpha}$	$\frac{q^{(\alpha+1)^2}}{(q;q)_\alpha}$	$x_k = 1, 1_1$ must be a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha+1}(1+q^{-i})$	Distinct part $\leq \nu$	8 [35]

7.	$\frac{(-1;q)_{2\alpha}q^{\alpha}}{(q;q)_{2\alpha}}$	$\frac{q^{2\alpha^2}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k (\mathrm{mod}\ 2)$, $x_1 > 1$, $\delta_k \geq 0$	$\prod_{i=1}^{2\alpha-1}(1+q^{-i})$	Distinct parts $\leq 2\nu - 1$	9 [35]
8.	$\frac{(-q;q)_{2\alpha}q^{\alpha}}{(q;q)_{2\alpha+1}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+2)$ copies of n, $m_k \equiv x_k (\mathrm{mod}\ 2)$, least part is $(x_1)_{x_1+2}$, $x_1 > 1$, $\delta_k \geq 2$ or 0	$\prod_{i=1}^{2\alpha}(1+q^{-i})$	Distinct parts $< 2\nu - 2$	10 [35]
9.	$\frac{(-1;q^4)_{\alpha}q^{\alpha^2}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$	$\frac{q^{3\alpha^2-2\alpha}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$	$m_k \equiv x_k (\mathrm{mod}\ 2)$, $m_1 \equiv x (\mathrm{mod}\ 4)$, $\delta_k \geq 4, \equiv 0 (\mathrm{mod}\ 4)$	$\prod_{i=1}^{\alpha-1}(1+q^{-4i})$	Distinct parts, $\equiv 0 (\mathrm{mod}\ 4)$, $\leq 4\nu - 4$	11 [35], 66 [79]
10.	$\frac{(-1;q^4)_{\alpha}q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$	$\frac{q^{3\alpha^2}}{(q;q^2)_{\alpha}(q^4;q^4)_{\alpha}}$	$m_k \equiv x_k (\mathrm{mod}\ 2)$, $x_1 > 2$, $m_1 \equiv x_1 (\mathrm{mod}\ 4)$, $\delta_k \geq 0, \equiv 0 (\mathrm{mod}\ 4)$	$\prod_{i=1}^{\alpha-1}(1+q^{-4i})$	Distinct parts, $\equiv 0 (\mathrm{mod}\ 4)$, $\leq 4\nu - 4$	12 [35], 67 [79]
11.	$\frac{(-q^4;q^4)_{\alpha}q^{\alpha^2}}{(q;q^2)_{\alpha+1}(q^4;q^4)_{\alpha}}$	$\frac{q^{3\alpha^2+2\alpha}}{(q;q^2)_{\alpha+1}(q^4;q^4)_{\alpha}}$	$(n+2)$ copies of n, $m_k \equiv x_k (\mathrm{mod}\ 2)$, $(m_1)_{x_1} = (x_1)_{x_1+2}$, $\delta_k \geq 0, \equiv 0 (\mathrm{mod}\ 4)$	$\prod_{i=1}^{\alpha}(1+q^{-4i})$	Distinct parts, $\equiv 0 (\mathrm{mod}\ 4)$, $\leq 4\nu - 4$	13 [35]

12.	$\frac{(-q;q^2)^2_\alpha q^{\alpha^2}}{(q;q^2)_\alpha (q^4;q^4)_\alpha}$	$\frac{(-q;q^2)_\alpha q^{2\alpha^2}}{(q;q^2)_\alpha (q^4;q^4)_\alpha}$	$m_k \equiv x_k (\text{mod } 2)$, $x_1 > 1$, $m_1 \equiv x_1 (\text{mod } 4)$, parts are m_{g+r} $g > 1, r = 0, 1$, $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	14 [35]
13.	$\frac{(-q;q^2)^2_\alpha q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha+1} (q^4;q^4)_\alpha}$	$\frac{(-q;q^2)_\alpha q^{2\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1} (q^4;q^4)_\alpha}$	$(n+1)$ copies of n, $m_k \equiv x_k + 1 (\text{mod } 2)$, least part is $(x_1)_{x_1+1}$, $g > 0, r = 0, 1$, $\delta_k \geq 1, \equiv 2 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	16 [35]
14.	$\frac{(-1)^\alpha (q;q^2)_\alpha q^{3\alpha^2-2\alpha}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{2\alpha(2\alpha-1)}}{(q^2;q^2)_{2\alpha}}$	m_k and x_k are even, $m_1 \equiv x_1 (\text{mod } 4)$, $\delta_k \geq 4, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	22 [35], 15 [79]
15.	$\frac{(-1)^\alpha (q;q^2)_\alpha q^{3\alpha^2}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{4\alpha^2}}{(q^2;q^2)_{2\alpha}}$	m_k and x_k are even, ≥ 4, $m_1 \equiv x_1 (\text{mod } 4)$, $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	24 [35], 19 [79]

16.	$\frac{(-1)^\alpha (q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(-q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$,	$(n+2)$ copies of n, $m_k \equiv x_k (\text{mod } 2)$, $x_1 > 1$, least part is $(x_1)_{x_1+2}$, $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	25 [35],
17.	$\frac{(-1)^\alpha (q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(-q;q^2)_{\alpha}(q^4;q^4)_\alpha}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q^2)_{\alpha}(q^4;q^4)_\alpha}$,	$m_k \equiv x_k (\text{mod } 2)$, $x_1 > 3$, $m_1 \equiv x_1 (\text{mod } 4)$, $\delta_k \geq -4, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	27 [35]
18.	$\frac{(-1)^\alpha (q;q^2)_\alpha q^{\alpha^2}}{(-q;q^2)_{\alpha}(q^4;q^4)_\alpha}$	$\frac{q^{2\alpha^2}}{(q;q^2)_{\alpha}(q^4;q^4)_\alpha}$,	$m_k \equiv x_k (\text{mod } 2)$, $m, x > 1$, $m_1 \equiv x_1 (\text{mod } 4)$, $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	29 [35], 21 [79]
19.	$\frac{(-q^2;q^2)_\alpha q^{\alpha(\alpha+1)}}{(-q;q)_{2\alpha}(q^2;q^2)_\alpha}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q^2)_{\alpha}(q^4;q^4)_\alpha}$,	$m_k \equiv x_k (\text{mod } 2)$, $x_1 > 3$, $m_1 \equiv x_1 (\text{mod } 4)$, $\delta_k \geq -4, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu$	32 [35]
20.	$\frac{(q;q^2)_\alpha^2 q^{2\alpha^2}}{(q^2;q^2)_{2\alpha}}$	$\frac{(-q;q^2)_\alpha q^{3\alpha^2}}{(q^2;q^2)_{2\alpha}}$	m_k and x_k are odd, $m_1, x_1 > 2$, $m_1 \equiv x_1 (\text{mod } 4)$, parts are m_{g+r} $g \geq 3$ and odd, $r = 0, 1$, $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	33 [35]

21.	$\frac{(-q;q^2)_\alpha q^{2\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	$\frac{q^{3\alpha^2+2\alpha}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$,	$(n+2)$ copies of n, $m_k \equiv x_k(\text{mod } 2)$, $x_1 \geq 3$, least part is $(x_1)_{x_1+2}$, $\delta_k \geq 0, \equiv 0(\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	34 [35], 27 [79]
22.	$\frac{(-1;q)_{2\alpha} q^\alpha}{(q^2;q^2)_\alpha}$	$\frac{q^{2\alpha^2}}{(q^2;q^2)_\alpha}$,	m_k is even and $x_k = 2$, $\delta_k \geq 0$	$\prod_{i=1}^{2\alpha-1}(1+q^{-i})$	Distinct parts $\leq 2\nu - 1$	36 [35], 24,30 [79]
23.	$\frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\frac{q^{\frac{\alpha(3\alpha-1)}{2}}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\delta_k \geq 1$	$\prod_{i=1}^{\alpha-1}(1+q^{-i})$	Distinct parts $< \nu$	37 [35]
24.	$\frac{(-q;q)_\alpha q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$\frac{q^{\frac{\alpha(3\alpha+1)}{2}}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$n+1$ copies of n, least part is $(x_1)_{x_1+1}$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	38 [35], 26 [79]
25.	$\frac{(-q;q)_{2\alpha} q^\alpha}{(q^2;q^2)_\alpha}$	$\frac{q^{2\alpha(\alpha+1)}}{(q^2;q^2)_\alpha}$	m_k is even and $m_1 \geq 4, x_1 = 2$, $\delta_k \geq 0$	$\prod_{i=1}^{2\alpha}(1+q^{-i})$	Distinct parts $\leq 2\nu$	39 [35]
26.	$\frac{(-q;q)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$\frac{q^{\frac{3\alpha(\alpha+1)}{2}}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$(n+1)$ copies of n, least part is $(x_1)_{x_1+1}$ $\delta_k > 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	40 [35], 22 [79]
27.	$\frac{(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q;q)_{2\alpha+1}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$,	$(n+2)$ copies of n, $m_k \equiv x_k(\text{mod } 2)$, $x_1 > 1$,least part is $(x_1)_{x_1+2}$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	41 [35]

28.	$\frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q^4;q^4)_\alpha}$	$\frac{q^{2\alpha^2}}{(q^4;q^4)_\alpha}$,	m_k is even and $x_k = 2$, $m_1 \equiv x_1 (\text{mod } 4)$, $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	42 [35], 25 [79]
29.	$\frac{(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q^4;q^4)_\alpha}$	$\frac{q^{2\alpha(\alpha+1)}}{(q^4;q^4)_\alpha}$,	$m_k \equiv 0 (\text{mod } 4)$, $x_k = 2$, $m_k \equiv x_k + 2 (\text{mod } 4)$, $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	43 [35]
30.	$\frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q;q)_{2\alpha}}$	$\frac{q^{2\alpha^2}}{(q;q)_{2\alpha}}$,	$m_k \equiv x_k (\text{mod } 2)$, $x_1 > 1$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	44 [35], 29 [79]
31.	$\frac{(-1;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$	$\frac{q^{2\alpha^2}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k (\text{mod } 2)$, $x_1 > 1$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	45 [35]
32.	$\frac{(-q^2;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+2)$ copies of n, $m_k \equiv x_k (\text{mod } 2)$, $x_1 > 1$, least part is $(x_1)_{x_1+1}$, $\delta_k \geq 0$, even	$\prod_{i=1}^{\alpha}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	46 [35], 28 [79]
33.	$\frac{(q;q^2)_\alpha q^{2\alpha^2}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{3\alpha^2}}{(q^2;q^2)_{2\alpha}}$,	$m_k, x_k > 2$ are odd, $m_1 \equiv x_1 (\text{mod } 4)$ $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	51 [35], 33 [79]

34.	$\frac{(q;q^2)_\alpha q^{2\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{3\alpha^2+2\alpha}}{(q^2;q^2)_{2\alpha}}$,	m_k, x_k are odd, $m_k \geq 5, x_1 > 2$, $m_k \equiv x_k + 2 \pmod 4$ $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	52 [35], 32 [79]
35.	$\frac{(q;q^2)_{\alpha+1} q^{2\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha+1}}$	$\frac{q^{3\alpha^2-2\alpha}}{(q^2;q^2)_{2\alpha-1}}$,	$m_k - x_k \equiv 1 \pmod 2$, least part is $(x_1)_{x_1}$, $\delta_k \geq 4, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	53 [35], 31 [79]
36.	$\frac{(-q;q^2)_\alpha q^{2\alpha+\binom{\alpha}{2}}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$\frac{q^{\frac{3\alpha(\alpha+1)}{2}}}{(q;q)_\alpha (q;q^2)\alpha+1}$	$(n+2)$–copies of n, least part is $(x_1)_{x_1+2}$, $\delta_k \geq 1$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	61 [35], 35, 106 [79]
37.	$\frac{(-q^2;q^2)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$\frac{q^{\frac{3\alpha(\alpha+1)}{2}}}{(q;q)_\alpha (q;q^2)\alpha+1}$	$(n+2)$–copies, least part is $(x_1)_{x_1+2}$, $\delta_k \geq 1$	$\prod_{i=1}^{\alpha}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	63 [35]
38.	$\frac{(-q;q^2)_\alpha q^{\binom{\alpha}{2}}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\frac{q^{\frac{3\alpha(\alpha-1)}{2}}}{(q;q)_\alpha (q;q^2)\alpha}$	$\delta_k > 0$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	65 [35]
39.	$\frac{(-q;q^2)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$\frac{q^{\frac{\alpha(3\alpha+1)}{2}}}{(q;q)_\alpha (q;q^2)\alpha+1}$	$(n+1)$–copies of n, least part is $(x_1)_{x_1+1}$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	67 [35], 37, 105 [79]
40.	$\frac{(-1;q^2)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\frac{q^{\frac{\alpha(3\alpha-1)}{2}}}{(q;q)_\alpha (q;q^2)\alpha}$	$\delta_k > 0$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct even parts $\leq 2\nu - 2$	70 [35]

41.	$\frac{(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q^2;q^2)_\alpha}$	$\frac{q^{\frac{2\alpha(\alpha+1)}{2}}}{(q^2;q^2)\alpha}$	m_k, x_k are even and $m_1 \geq 4, x_1 = 2$, $\delta_k \geq 0$, even	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	73 [35], 34 [79]
42.	$\frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q^2;q^2)_\alpha}$	$\frac{q^{2\alpha^2}}{(q^2;q^2)\alpha}$	m_k, x_k are even and $x_1 = 2$, $\delta_k \geq 0$, even	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	74 [35], 36 [79]
43.	$\frac{(-q;q)_{4\alpha} q^{2\alpha^2}}{(-q^4;q^4)_\alpha (q^4;q^4)_{2\alpha}}$	$\frac{q^{6\alpha^2}}{(q^8;q^8)\alpha(q^2;q^4)_\alpha}$	m_k, x_k are even and $m_1, x_1 \geq 6$, $m_1 \equiv x_1 (\mathrm{mod}\ 8)$, $\delta_k \geq 0, \equiv 0 (\mathrm{mod}\ 8)$	$\prod_{i=1}^{2\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 4\nu$	75 [35]
44.	$\frac{(-q;q^2)_{2\alpha} q^{2\alpha(\alpha+2)}}{(q^2;q^2)_{2\alpha+1}(-q^4;q^4)_\alpha}$	$\frac{q^{6\alpha^2+4\alpha}}{(q^8;q^8)_\alpha (q^2;q^4)_{\alpha+1}}$	$(n+4)$–copies of n, m_k, x_k is even, least part is $(x_1)_{x_1+4}$, $\delta_k \geq 0, \equiv 0 (\mathrm{mod}\ 8)$	$\prod_{i=1}^{2\alpha}(1+q^{-(2i-1)})$	Distinct odd part $\leq 4\nu - 5$	77 [35]
45.	$\frac{(-q^2;q^2)_\alpha(-q^3;q^6)_\alpha q^{\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha+1}(-q;q^2)_\alpha}$	$\frac{(q^3;q^6)_\alpha q^{\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha+1}(q;q^2)_\alpha}$	$(n+2)$–copies of n, $m_k \equiv x_k (\mathrm{mod}\ 2)$, x_k is even, least part is $(x_1)_{x_1+2}$, parts are m_{g+r} $g \geq 0, r = 0, 2$, $\delta_k \geq 0, \equiv 0 (\mathrm{mod}\ 4)$	$\prod_{i=1}^{\alpha}(1+q^{-2i})$	Distinct even parts $\leq 2\nu - 2$	92 [35]

46.	$\frac{(-q;q)_\alpha q^{3\binom{\alpha+1}{2}}}{(q;q)_{2\alpha+1}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k (\text{mod } 2)$, $m_1 > 1$, $(x_1)_{x_1+2}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	93 [35], 44 [79]
47.	$\frac{(-q;q)_\alpha q^{\alpha^2+\binom{\alpha}{2}}}{(q;q)_{2\alpha}}$	$\frac{q^{2\alpha^2}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k (\text{mod } 2)$, $m_1 > 1$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $\leq \nu$	94 [35], 46 [79]
48.	$\frac{(-q;q)_\alpha q^{\alpha^2+\binom{\alpha+1}{2}}}{(q;q)_{2\alpha+1}}$	$\frac{q^{\alpha(2\alpha+1)}}{(q;q)_{2\alpha+1}}$	$m_k \equiv x_k (\text{mod } 2)$, $m_1 \geq 3$, $\delta_k \geq 2$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $\leq \nu$	95 [35]
49.	$\frac{(-q;q)_\alpha q^{2\alpha+\binom{\alpha}{2}}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$\frac{q^{\alpha(\alpha+2)}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$(n+2)$–copies of n, $(x_1)_{x_1+2}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	96 [35], 43 [79]
50.	$\frac{(-q;q)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$\frac{q^{\alpha(\alpha+1)}}{(q;q)_\alpha (q;q^2)_{\alpha+1}}$	$(n+1)$–copies of n, $(x_1)_{x_1+1}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	97 [35], 45 [79]
51.	$\frac{(-1;q)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\frac{q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\delta_k \geq 0$	$\prod_{i=1}^{\alpha-1}(1+q^{-i})$	Distinct parts $< \nu$	98 [35]
52.	$\frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q;q)_{2\alpha}}$	$\frac{q^{2\alpha^2}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k (\text{mod } 2)$, $x_1 > 1$, $\delta_k \geq 0$ or 2	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	99 [35]

53.	$\frac{(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q;q)_{2\alpha+1}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k (\text{mod } 2)$, $m_1 > 1$, $(x_1)_{x_1+2}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	102 [35], 50 [79]
54.	$\frac{(-q^2;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k (\text{mod } 2)$, $m_1 > 1$, $(x_1)_{x_1+2}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	103 [35]
55.	$\frac{(-q;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$\frac{q^{\alpha(2\alpha+1)}}{(q;q)_{2\alpha+1}}$	$m_k \equiv x_k (\text{mod } 2)$, $\delta_k \geq 2$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1})$	Distinct odd parts $< 2\nu$	104 [35], (11,15,64) [79]
56.	$\frac{(-q;q)_{\alpha-1} q^{\alpha^2}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\frac{q^{\frac{\alpha(3\alpha-1)}{2}}}{(q;q)_\alpha (q;q^2)_\alpha}$	$\delta_k \geq 1$	$\prod_{i=1}^{\alpha-1}(1+q^{-i})$	Distinct parts $< \nu$	106 [35], 58 [79]
57.	$\frac{(q;q^2)_{2\alpha} q^{4\alpha^2}}{(q^4;q^4)_{2\alpha}}$	$\frac{q^{8\alpha^2}}{(q^4;q^4)_{2\alpha}}$	$m_k \equiv x_k (\text{mod } 2)$, $m_1, x_1 \geq 8$, $m_1 \equiv x_1 (\text{mod } 8)$ $\delta_k \geq 0, \equiv 0 (\text{mod } 8)$	$\prod_{i=1}^{2\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu + 2$	109 [35], 53 [79]
58.	$\frac{(-q^2;q^2)_\alpha q^{\alpha}}{(q;q)_{2\alpha+1}}$	$\frac{q^{\alpha(\alpha+2)}}{(q;q)_{2\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k (\text{mod } 2)$, $(x_1)_{x_1+2}$ is a part, $\delta_i \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	110 [35]

59.	$\frac{(-q;q^2)_\alpha q^\alpha}{(q;q)_{2\alpha+1}}$	$\frac{q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+1)$–copies of n, $m \equiv x+1 \pmod 2$, $(x_1)_{x_1+1}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	111 [35]
60.	$\frac{(-1;q^2)_\alpha q^\alpha}{(q;q)_{2\alpha}}$	$\frac{q^{\alpha^2}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k \pmod 2$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	112 [35]
61.	$\frac{(-q;q^2)_\alpha q^{2\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	$\frac{q^{3\alpha^2+2\alpha}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	$(n+2)$–copies of n, $m_k \equiv x_k \pmod 2$, $(x_1)_{x_1+2}$ is a part, $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	113 [35]
62.	$\frac{(-q;q^2)_\alpha q^{2\alpha(\alpha-1)}}{(q;q^2)_\alpha(q^4;q^4)_\alpha}$	$\frac{q^{3\alpha^2-2\alpha}}{(q;q^2)_\alpha(q^4;q^4)_\alpha}$	$m_k \equiv x_k \pmod 2$, $m_1 \equiv x_1 \pmod 4$, $\delta_k \geq 4, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	114 [35]
63.	$\frac{(-q;q^2)_{3\alpha} q^{3\alpha^2}}{(q^6;q^6)_{2\alpha}}$	$\frac{q^{12\alpha^2}}{(q^6;q^6)_{2\alpha}}$	$m_k \equiv x_k \pmod 2$, $m_k, x_k \geq 12$ $m_1 \equiv x_1 \pmod{12}$, $\delta_k \geq 0, \equiv 0 \pmod{12}$	$\prod_{i=1}^{3\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $\leq 6\nu - 1$	116 [35]
64.	$\frac{(-q^2;q^2)_{3\alpha}(-q^3;q^6)_\alpha q^{3\alpha^2}}{(-q^6;q^6)_\alpha(q^6;q^6)_{2\alpha}}$	$\frac{q^{9\alpha^2}}{(q^{12};q^{12})_\alpha(q^3;q^6)_\alpha}$	$m_k \equiv x_k \pmod 2$, $m_1, x_1 \geq 9$, $m_1 - x_1 \equiv 3 \pmod 6$, $\delta_k \geq 0, \equiv 0 \pmod{12}$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct even parts $\not\equiv 0 \pmod 6$ $\leq 6\nu - 2, 6\nu - 4$	119 [35]

65.	$\frac{(-q^2;q^4)_\alpha q^{\alpha(\alpha+2)}}{(q;q)_{2\alpha+1}(-q^2;q^2)_\alpha}$	$\frac{q^{3\alpha^2+2\alpha}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	$(n+2)$ copies of n, $(x_1)_{x_1+2}$ is a part , $m_k \equiv x_k \pmod 2$, $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(4i-2)})$	Distinct part of the form $2 \pmod 4 \leq 4\nu - 6$	127 [35], 70 [79]
66.	$\frac{(-q;q^2)_\alpha^2 q^{\alpha(\alpha+2)}}{(q;q)_{2\alpha+1}(-q^2;q^2)_\alpha}$	$\frac{(-q;q^2)_\alpha q^{2\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	$(n+2)$ copies of n, $m_k - x_k \equiv 0 \pmod 2$, least part $(x_1)_{x_1+2}$, $r = 0, 1$, $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	128 [35]
67.	$\frac{(-q^2;q^2)_{\alpha-1} q^{\alpha}}{(q^2;q^2)_\alpha}$	$\frac{q^{\alpha^2}}{(q^2;q^2)_\alpha}$	m_k, x_k are odd, $m_k \equiv x_k \pmod 4$ $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	129 [35]
68.	$\frac{(-q^2;q^2)_{\alpha-1} q^{\alpha^2}}{(q;q)_{2\alpha}}$	$\frac{q^{\alpha(2\alpha-1)}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k \pmod 2$ $\delta_k \geq 2$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	131 [35], 72 [79]
69.	$\frac{(-q^2;q^4)_\alpha q^{\alpha^2}}{(q;q)_{2\alpha}(-q^2;q^2)_\alpha}$	$\frac{q^{3\alpha^2}}{(q;q^2)_\alpha(q^4;q^4)_\alpha}$	$m_k \equiv x_k \pmod 2$, $x_1 > 2$ $m_1 \equiv x_1 \pmod 4$ $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(4i-2)})$	Distinct parts $\equiv 2 \pmod 4 \leq 4\nu - 2$	132 [35]

70.	$\frac{(q;q^2)_{3\alpha} q^{6\alpha^2}}{(q^6;q^6)_{2\alpha}(q^3;q^6)_\alpha}$	$\frac{q^{12\alpha^2}}{(q^6;q^6)_{2\alpha}}$	$m_k \equiv x_k (\text{mod } 2)$, $m_1, x_1 > 12$, $m_1 \equiv x_1 (\text{mod } 12)$, $\delta_k \geq 0, \equiv 0 (\text{mod } 12)$	$\prod_{i=1}^{\alpha}(1+q^{-(6i-5)})(1+q^{-(6i-1)})$	Distinct odd parts $\not\equiv 3 (\text{mod } 6) \leq 6\nu-5, 6\nu-1$	135 [35]
71.	$\frac{(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$,	$(n+2)$–copies of n, $m_k \equiv x_k (\text{mod } 2)$, $m_1, x_1 > 1$, least part is $(x_1)_{x_1+2}$, $\delta_k \geq 0 \equiv 0 (\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu - 2$	144 [35]
72.	$\frac{(-q;q)_\alpha q^{\alpha^2+\binom{\alpha}{2}}}{(q;q)_{2\alpha}}$	$\frac{q^{2\alpha^2}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k (\text{mod } 2)$, $m_1, x_1 > 1$, $\delta_k \geq 0, 2$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $\leq \nu$	153 [35]
73.	$\frac{(-q;q)_\alpha q^{\alpha^2+\binom{\alpha+1}{2}}}{(q;q)_{2\alpha+1}}$	$\frac{q^{\alpha(2\alpha+1)}}{(q;q)_{2\alpha+1}}$	$m_k \equiv x_k (\text{mod } 2)$, $\delta_k \geq 2$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $\leq \nu$	154 [35], 62 [79]
74.	$\frac{(-q;q)_\alpha q^{3\binom{\alpha+1}{2}}}{(q;q)_{2\alpha+1}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k (\text{mod } 2)$, $m_1, x_1 > 1$, $(x_1)_{x_1+2}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	155 [35], 63 [79]
75.	$\frac{(-q^3;q^3)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_{2\alpha+1}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k (\text{mod } 2)$, $m_1 > 1$, $(x_1)_{x_1+2}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-3i})$	Distinct parts, multiple of 3 $\leq 3\nu - 3$	161 [35]

76.	$\frac{(-q;q)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_{2\alpha}}$	$\frac{q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$	$m_k - x_k \equiv 1 \pmod 2$, $m_1 > 1$, $(x_1)_{x_1+2}$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $\leq \nu$	164 [35], 81 [79]
77.	$\frac{(-q;q)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_{2\alpha+1}}$	$\frac{q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+1)$–copies of n, $m_k - x_k \equiv 1 \pmod 2$, $(x_1)_{x_1+1}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	165 [35], 80 [79]
78.	$\frac{(-q;q)_\alpha q^{2\alpha+\binom{\alpha}{2}}}{(q;q)_{2\alpha+1}}$	$\frac{q^{\alpha(\alpha+2)}}{(q;q)_{2\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k \pmod 2$, $(x_1)_{x_1+2}$ is a part, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	166, 82 [79]
79.	$\frac{(-q;q^2)_\alpha q^{3\alpha^2}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{4\alpha^2}}{(q^2;q^2)_{2\alpha}}$	m_k, x_k is even, $x_1 \geq 4$, $m_1 \equiv x_1 \pmod 4$, $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	178 [35], 100 [79]
80.	$\frac{(-q;q^2)_\alpha q^{3\alpha^2-2\alpha}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{2\alpha(2\alpha-1)}}{(q^2;q^2)_{2\alpha}}$	m_k, x_k is even, $x_1 \geq 2$, $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	179 [35], 95 [79]
81.	$\frac{(-q^3;q^6)_\alpha q^{\alpha^2}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{4\alpha^2}}{(q^2;q^2)_{2\alpha}}$	m_k, x_k are even and ≥ 4, $m_k \equiv x_k \pmod 4$, $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(6i-3)})$	Distinct part $\equiv 3 \pmod 6 \leq 6\nu - 3$	183 [35]

82.	$\frac{(-q;q^2)_\alpha(q^3;q^6)_\alpha q^{\alpha^2}}{(q^2;q^2)_{2\alpha}(q;q^2)_\alpha}$	$\frac{(q^3;q^6)_\alpha q^{2\alpha^2}}{(q^2;q^2)_{2\alpha}(q;q^2)_\alpha}$	$m_k \equiv x_k \pmod 2$, $m_1, x_1 > 1$, x_1 is even, $m_1 \equiv x_1 \pmod 4$, parts are m_{g+r}, $g > 0, r = 0, 2, \;,$ $\delta_k \geq 0, \equiv \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	184 [35]
83.	$\frac{(-q^2;q^2)_\alpha(-q^3;q^6)_\alpha q^{\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha}(-q;q^2)_{\alpha+1}}$	$\frac{(q^3;q^6)_\alpha q^{2\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha}(q;q^2)_{\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k \pmod 2$, $m_1, x_1 > 1$, x_1 is even, least part is $(x_1)_{x_1+2}$, parts are m_{g+r}, $g > 0, r = 0, 2$, $\delta_i \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	189 [35]
84.	$\frac{(-q^2;q^2)_\alpha(-q^3;q^6)_\alpha q^{\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha+1}(-q;q^2)_\alpha}$	$\frac{(q^3;q^6)_\alpha q^{2\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha+1}(q;q^2)_\alpha}$	$(n+2)$–copies of n, $m_k \equiv x_k \pmod 2$, $m_k, x_k > 1$, x_k is even, least part is $(x_k)_{x_k+2}$, parts are m_{g+r}, $g > 0, r = 0, 2$, $\delta_k \geq 0, \equiv 0 \pmod 4$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	190 [35], 107 [79]

85.	$\frac{(-q^2;q^2)_\alpha(-q^3;q^6)_\alpha q^{\alpha(\alpha+3)}}{(q^2;q^2)_{2\alpha+1}(-q;q^2)_\alpha}$	$\frac{(q^3;q^6)_\alpha q^{2\alpha(\alpha+2)}}{(q;q^2)_\alpha(q^2;q^2)_{2\alpha+1}(q;q^2)_\alpha}$	$(n+4)$–copies of n, $m_k \equiv x_k(\text{mod } 2)$, $m_1, x_1 > 1$, x_1 is even, least part is $(x_1)_{x_1+4}$, $r = 0, 2$, $\delta_k \geq 0, \equiv 0(\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	191 [35]
86.	$\frac{(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{2\alpha(\alpha+1)}}{(q^2;q^2)_{2\alpha}}$,	m_1, x_1 are even, $m_1 \geq 4$, $m_1 \equiv x_1 + 2(\text{mod } 4)$, $\delta_k \geq 0, \equiv 0(\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	192 [35], 118 [79]
87.	$\frac{(-q;q^2)_\alpha q^{\alpha^2}}{(q^2;q^2)_{2\alpha}}$	$\frac{q^{2\alpha^2}}{(q^2;q^2)_{2\alpha}}$,	m_k, x_k are even, $a_1, x_1 > 2$, $m_1 \equiv x_1(\text{mod } 4)$, $\delta_k \geq 0, \equiv 0(\text{mod } 4)$	$\prod_{i=1}^{\alpha}(1+q^{-(2i-1)})$	Distinct odd parts $< 2\nu$	193 [35], 117 [79]
88.	$\frac{(-q^2;q^2)_{\alpha-1} q^{\alpha^2}}{(q;q)_{2\alpha}}$	$\frac{q^{\alpha(2\alpha-1)}}{(q;q)_{2\alpha}}$,	$m_k \equiv x_k(\text{mod } 2)$, $\delta_k \geq 2$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	195 [35], 121 [79]
89	$\frac{(-q^2;q^2)_{\alpha-1}(-q;q)_\alpha q^{\frac{\alpha(\alpha+1)}{2}}}{(q;q)_{2\alpha}}$	$\frac{(-q^2;q^2)_{\alpha-1} q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k+1(\text{mod } 2)$, least part $(x_1)_{x_1}$, other are m_{g+r}, $g \geq 1, r = 2$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $\leq \nu$	104 [79]

90.	$\frac{(-q;q^2)_\alpha(-q;q)_\alpha q^{\frac{\alpha(\alpha+3)}{2}}}{(q;q)_{2\alpha+1}}$	$\frac{(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q;q)_{2\alpha+1}}$	$(n+2)$–copies of n, $m_k \equiv x_k(\text{mod } 2)$, least part is $(x_1)_{x_1+2}$ parts are m_{g+r}, $g \geq 1, r = 0, 1$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $< \nu$	106 [79]
91.	$\frac{(-q^2;q^2)_{\alpha-1}(-q;q)_\alpha q^{\frac{\alpha(\alpha-1)}{2}}}{(q;q)_{2\alpha}}$	$\frac{(-q^2;q^2)_{\alpha-1} q^{\alpha^2}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k(\text{mod } 2)$, least part $(x_1)_{x_1}$, other are m_{g+r}, $g \geq 1, r = 2$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts $\leq \nu$	101 [79]
92.	$\frac{(q^4;q^4)_{\alpha-1}(-q;q^2)_\alpha q^{\alpha^2}}{(q^2;q^4)_\alpha(q^2;q^2)_\alpha(q^2;q^2)_{\alpha-1}}$	$\frac{(-q;q^2)_\alpha q^{\alpha(2\alpha-1)}}{(q^2;q^4)_\alpha(q^2;q^2)_\alpha}$	$m_k \equiv x_k(\text{mod } 2)$, parts are m_{g+r}, $g =$ odd, $r = 0, 1$, $\delta_k \geq 2$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	72 [79]
93.	$\frac{(-1;q^4)_\alpha(-q;q^2)_\alpha q^{\alpha(\alpha+2)}}{(q^2;q^2)_{2\alpha}}$	$\frac{(-q;q^2)_\alpha q^{3\alpha^2}}{(q^2;q^2)_{2\alpha}}$	$m_k \equiv x_k(\text{mod } 2)$, $x_1 > 2$, $m_1 \equiv x_1(\text{mod } 4)$, parts are m_{g+r}, $g =$ odd, $r = 0, 1$, $\delta_k \geq 0, \equiv 0(\text{mod } 4)$	$\prod_{i=1}^{\alpha-1}(1+q^{-4i})$	Distinct parts, multiple of 4 $\leq 4\nu - 4$	67 [79]
94.	$\frac{(-q;q^2)_\alpha q^{\alpha(2\alpha-1)}}{(q^2;q^4)_\alpha(q^2;q^2)_\alpha}$	$\frac{q^{3\alpha^2-\alpha}}{(q^2;q^4)_\alpha(q^2;q^2)_\alpha}$,	$m_k, x_k \equiv 0(\text{mod } 2)$, $x_1 \geq 2$, $\delta_k \geq 2$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct odd parts $< 2\nu$	52 [79]

95.	$\frac{(-1;q^2)_\alpha q^{\alpha^2}}{(q;q)_{2\alpha}}$	$\frac{q^{\alpha(2\alpha-1)}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k (\text{mod } 2)$, $\delta_k \geq 2$	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	47 [79]
96.	$\frac{(-1;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$	$\frac{q^{2\alpha^2}}{(q;q)_{2\alpha}}$	$m_k \equiv x_k (\text{mod } 2)$, $x_1 \geq 2$ $\delta_k \geq 2$ or 0	$\prod_{i=1}^{\alpha-1}(1+q^{-2i})$	Distinct parts, multiple of 2 $\leq 2\nu - 2$	48 [79]
97.	$\frac{(-q;q)_\alpha q^{\frac{\alpha(\alpha-1)}{2}}}{(q;q)_\alpha}$	$\frac{q^{\alpha^2}}{(q;q)_\alpha}$	$x_k = 1$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-i})$	Distinct parts, $\leq \nu$	13 [79]
98.	$\frac{(-1;q)_\alpha q^{\frac{\alpha(\alpha+1)}{2}}}{(q;q)_\alpha}$	$\frac{q^{\alpha^2}}{(q;q)_\alpha}$	$x_k = 1$, $\delta_k \geq 0$	$\prod_{i=1}^{\alpha}(1+q^{-(i-1)})$	Distinct parts, $< \nu$	12 [79]
99.	$\frac{(q^3;q^3)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}(q;q)_\alpha}$	$\frac{q^{2\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	$(n+2)$ copies of n, $m_k \equiv x_k (\text{mod } 2)$, least part is $(x_1)_{x_1+2}$, $x_1 > 1$, $\delta_k \geq 2$ or 0	$\prod_{i=1}^{\alpha}(1+q^{-(i)}+q^{-(2i)})$	Distinct parts, $< \nu$, may appear at most twice	147 [35]
100.	$\frac{(-q;q)_{2\alpha} q^{\frac{\alpha(\alpha+1)}{2}}}{(q;q^2)_{\alpha+1}(q^4;q^4)_\alpha}$	$\frac{q^{3\alpha^2-2\alpha}}{(q;q^2)_{\alpha+1}(q^4,q^4)_\alpha}$	$(n+2)$ copies of n, $(x_1)_{x_1+2}$ is a part, $m_k \equiv x_k (\text{mod } 2)$, $\delta_k \geq 0, \equiv 0 (\text{mod } 4)$,	$\prod_{i=1}^{2\alpha}(1+q^{i})$	Distinct parts, $< 2\nu - 2$	11 [79]

5.2 Main results

In this section, we provide the proofs of combinatorial interpretations of some q–series identities listed at Sr. No. 1^{st}, 43^{rd} and 82^{nd} in Table 5.1. The combinatorial interpretations of the remaining q–series identities mentioned in table can be easily obtained from the proofs of Theorem 5.2.1–5.2.3.

Theorem 5.2.1 *For $\alpha \geq 0$, let $A_1(\alpha)$ denote the number of signed color partitions $\Theta = (\theta_1, \theta_2)$ and θ_1 be an n–color partitions such that $\delta_k \geq 1$.*
Let θ_2 be n–color partitions with first copy such that parts are distinct $< \nu$, where ν is the number of distinct part in θ_1. Then

$$f_1(q) = \sum_{\alpha=0}^{\infty} A_1(\alpha) q^{\alpha} = \sum_{\alpha=0}^{\infty} \frac{(-1;q)_{\alpha} q^{\alpha^2}}{(q;q)_{\alpha}(q;q^2)_{\alpha}}.$$

Proof We have

$$\begin{aligned}\sum_{\alpha=0}^{\infty} A_1(\alpha) q^{\alpha} =& \sum_{\alpha=0}^{\infty} \frac{(-1;q)_{\alpha} q^{\alpha^2}}{(q;q)_{\alpha}(q;q^2)_{\alpha}} \\ =& 1 + 2\sum_{\alpha=1}^{\infty} \frac{(-q;q)_{\alpha-1} q^{\alpha^2}}{(q;q)_{\alpha}(q;q^2)_{\alpha}} \\ =& 1 + 2\sum_{\alpha=1}^{\infty} \frac{q^{\alpha^2}}{(q;q)_{\alpha}(q;q^2)_{\alpha}} \prod_{i=1}^{\alpha-1}(1-q^i) \\ =& 1 + 2\sum_{\alpha=1}^{\infty} \frac{q^{\alpha\frac{(3\alpha-1)}{2}}}{(q;q)_{\alpha}(q;q^2)_{\alpha}} \prod_{i=1}^{\alpha-1}(1-q^{-i}).\end{aligned}$$

In above, $\sum_{\alpha=1}^{\infty} \frac{q^{\alpha\frac{(3\alpha-1)}{2}}}{(q;q)_{\alpha}(q;q^2)_{\alpha}}$ and $\prod_{i=1}^{\alpha-1}(1+q^{-i})$ correspond to partitions given by θ_1 and θ_2. Now, $\prod_{i=1}^{\alpha-1}(1+q^{-i})$ denote the number of n–color partitions with first copy of α such that parts are distinct $< \nu$. Also, $\sum_{\alpha=1}^{\infty} \frac{q^{\alpha\frac{(3\alpha-1)}{2}}}{(q;q)_{\alpha}(q;q^2)_{\alpha}}$ enumerate the partition by $B_1(\alpha)$. Let $B_1(r,\alpha)$ denote the number of partitions of α enumerated by $B_1(\alpha)$ into r parts. We split the partition into three classes: (i) those do not contain $(m_1)_{x_1}$ of the form $(x_1)_{x_1}$ (ii) those contain $(m_1)_{x_1}$ of the form 1_1, (iii) those contain $(m_1)_{x_1}$ of the form $(x_1)_{x_1}$ for $(x > 1)$. With some simple transformation, we get the recurrence relation, which is reversible.

$$B_1(r,\alpha) = B_1(r,\alpha-r) + B_1(r-1,\alpha-3r+2) + B_1(r,\alpha-2r+1) - B_1(r,\alpha-3r+1),$$

where $B_1(0,0)=1$ and $B_1(r,\alpha)=0$ for $\alpha<0$.
Further using the fact, $f_1(w,q)=\sum_{\alpha=1}^{\infty}\sum_{r=1}^{\infty}B_1(r,\alpha)w^rq^\alpha$, where $|q|<1, |wq|<1$. We get the q–functional equation

$$f_1(w,q)=f_1(wq,q)+wqf_1(wq^3,q)+q^{-1}f_1(wq^2,q)-q^{-1}f_1(wq^3,q).$$

We can developed the proof as done earlier.

Example 5.2.1 *Consider $\alpha=6$ then $A_1(6)=2\cdot 12=24$, the relevant signed color partitions are*

$$(6_6),\ (6_5),\ (6_4),\ (6_3),\ (6_2),\ (6_1),\ (5_2\ 1_1),\ (5_1\ 1_1),\ (6_3+1_1-1_1),$$

$$(6_2+1_1-1_1),\ (6_1+1_1-1_1),\ (5_1+2_1-1_1).$$

Theorem 5.2.2 *For $\alpha\geq 0$, let $A_{43}(\alpha)$ denote the number of signed color partitions $\Theta=(\theta_1,\theta_2)$ and θ_1 be an n–color partitions such that*

(5.2.2.a) *m_k and x_k are even and $\geq 6\ \forall\ k$,*

(5.2.2.b) *$m_1-x_1\equiv 0 \pmod 8$,*

(5.2.2.c) *$\delta_k\geq 0$ and $\equiv 0 \pmod 8$.*

Let θ_2 be an n–color partitions with first copy such that parts are odd and distinct $<4\nu$, where ν is the number of distinct part in θ_1. Then

$$f_{43}(q)=\sum_{\alpha=0}^{\infty}A_{43}(\alpha)q^\alpha=\sum_{\alpha=0}^{\infty}\frac{(-q;q)_{4\alpha}q^{2\alpha^2}}{(-q^4;q^4)_\alpha(q^4;q^4)_{2\alpha}}.$$

Example 5.2.2 *Consider $\alpha=11$ then $A_{43}(11)=5$, the relevant signed color partitions are*

$$(12_{12}-1_1),\ (14_{14}-3_1),\ (14_6-3_1),\ (18_6+6_6-7_1-5_1-1_1),\ (20_8+6_6-7_1-5_1-3_1).$$

Proof We have

$$\begin{aligned}\sum_{\alpha=0}^{\infty}A_{43}(\alpha)q^\alpha &=\sum_{\alpha=0}^{\infty}\frac{(-q;q)_{4\alpha}q^{2\alpha^2}}{(-q^4;q^4)_\alpha(q^4;q^4)_{2\alpha}}\\ &=\sum_{\alpha=0}^{\infty}\frac{(-q;q^2)_{2\alpha}q^{2\alpha^2}}{(q^8;q^8)_\alpha(q^2;q^4)_\alpha}\end{aligned}$$

$$= \sum_{\alpha=0}^{\infty} \frac{q^{2\alpha^2}}{(q^8;q^8)_\alpha (q^2;q^4)_\alpha} \prod_{i=1}^{2\alpha} (1+q^{(2i-1)})$$
$$= \sum_{\alpha=0}^{\infty} \frac{q^{6\alpha^2}}{(q^8;q^8)_\alpha (q^2;q^4)_\alpha} \prod_{i=1}^{2\alpha} (1+q^{-(2i-1)}).$$

In above, $\sum_{\alpha=0}^{\infty} \frac{q^{6\alpha^2}}{(q^8;q^8)_\alpha (q^2;q^4)_\alpha}$ and $\prod_{i=1}^{2\alpha}(1+q^{-(2i-1)})$ corresponds to partitions given by θ_1 and θ_2 respectively. Now, $\prod_{i=1}^{2\alpha}(1+q^{-(2i-1)})$ denote the number of n–color partition with first copy of α such that parts are odd and distinct $< 4\nu$. Also, $\sum_{\alpha=0}^{\infty} \frac{q^{6\alpha^2}}{(q^8;q^8)_\alpha (q^2;q^4)_\alpha}$ enumerate the partition by $B_{43}(\alpha)$. Let $B_{43}(r,\alpha)$ denote the number of partitions of α enumerated by $B_{43}(\alpha)$ into r parts. We split the partition into three classes: (i) those do not contain $(m_1)_{x_1}$ of the form $(x_1)_{x_1}$, (ii) those contain $(m_1)_{x_1}$ of the form 6_6, (iii) those contain $(m_1)_{x_1}$ of the form $(x_1)_{x_1}(x_1 > 6)$. With some simple transformation we get following the recurrence relation, which is reversible:

$$\begin{aligned} B_{43}(r,\alpha) =& B_{43}(r,\alpha-8r) + B_{43}(r-1,\alpha-12r+6) + B_{43}(r,\alpha-4r+2) \\ &- B_{43}(r,\alpha-12r+2), \end{aligned}$$

where $B_{43}(0,0) = 1$ and $B_{43}(r,\alpha) = 0$ for $\alpha < 0$. Further using the fact, $f_{43}(w,q) = \sum_{\alpha=0}^{\infty}\sum_{r=0}^{\infty} B_{43}(r,\alpha) w^r q^\alpha$, where $|q| < 1, |wq| < 1$. We get the q–functional equation

$$f_{43}(w,q) = f_{43}(wq^8,q) + wq^6 f_{43}(wq^{12},q) + q^{-2} f_{43}(wq^4) - q^2 f_{43}(wq^{12}).$$

We can develop the remaining proof as done earlier.

Theorem 5.2.3 *For $\alpha \geq 0$, let $A_{82}(\alpha)$ denote the number of signed color partitions $\Theta = (\theta_1, \theta_2)$ and θ_1 be the split n–color partitions such that*

(5.2.3.a) $m_k \equiv x_k \pmod 2$ *and* $x_1 > 1$,

(5.2.3.b) $m_1 \equiv x_1 \pmod 4$,

(5.2.3.c) $x = g + r$ *where* $r = \begin{cases} 2 & g \text{ is odd} \\ 0 \text{ or } 2 & g \text{ is even} \end{cases}$

(5.2.3.d) $\delta_k \geq 0$ *and* $\equiv 0 \pmod 4$ $\forall\ k$.

Let θ_2 be an n–color partitions with first copy such that parts are odd and distinct less than

2ν, *where* ν *is the number of distinct parts in* θ_1. *Then*

$$f_{82}(q)=\sum_{\alpha=0}^{\infty}A_{82}(\alpha)q^{\alpha}=\sum_{\alpha=0}^{\infty}\frac{(-q;q^2)_{\alpha}(q^3;q^6)_{\alpha}q^{\alpha^2}}{(q;q^2)_{\alpha}(q^2;q^2)_{2\alpha}}.$$

Remark 5.2.4 *The conditions* (5.2.3.a), (5.2.3.b) *and* (5.2.3.d) *are allowed for the whole subscript* x *irrespective of green* (g) *and red* (r) *parts separately.*

Example 5.2.3 *For* $\alpha=6, A_{82}(\alpha)=8$ *and the relevant signed color partitions are*

(6_6), (6_2), (6_{4+2}), $(7_{5+2}-1_1)$, $(7_{1+2}-1_1)$, $(7_{1+2}+2_2-3_1)$, $(8_4+2_2-3_1-1_1)$, $(8_{2+2}+2_2-3_1-1_1)$.

Proof We have

$$\begin{aligned}\sum_{\alpha=0}^{\infty}A_{82}(\alpha)q^{\alpha} &= \sum_{\alpha=0}^{\infty}\frac{(-q;q^2)_{\alpha}(q^3;q^6)_{\alpha}q^{\alpha^2}}{(q;q^2)_{\alpha}(q^2;q^2)_{2\alpha}}\\ &= \sum_{\alpha=0}^{\infty}\frac{(q^3;q^6)_{\alpha}q^{\alpha^2}}{(q;q^2)_{\alpha}(q^2;q^2)_{2\alpha}}\prod_{i=1}^{\alpha}(1+q^{2i-1})\\ &= \sum_{\alpha=0}^{\infty}\frac{(q^3;q^6)_{\alpha}q^{2\alpha^2}}{(q;q^2)_{\alpha}(q^2;q^2)_{2\alpha}}\prod_{i=1}^{\alpha}(1+q^{-(2i-1)}).\end{aligned}\tag{5.2.1}$$

In above, $\sum\limits_{\alpha=0}^{\infty}\frac{(q^3;q^6)_{\alpha}q^{2\alpha^2}}{(q;q^2)_{\alpha}(q^2;q^2)_{2\alpha}}$ and $\prod\limits_{i=1}^{\alpha}(1+q^{-(2i-1)})$ corresponds to partitions given by θ_1 and θ_2. Now, $\prod\limits_{i=1}^{2\alpha}(1+q^{-(2i-1)})$ denote the number of n–color partitions with first copy of α such that parts are odd and distinct $< 2\nu$. Also, $\sum\limits_{\alpha=0}^{\infty}\frac{(q^3;q^6)_{\alpha}q^{2\alpha^2}}{(q;q^2)_{\alpha}(q^2;q^2)_{2\alpha}}$ enumerate the partition by $B_{82}(\alpha)$. Let $B_{82}(r,\alpha)$ denote the number of partitions of α enumerated by $B_{82}(\alpha)$ into r parts. We split the partition into five classes: (i) those do not contain $(m_1)_{x_1}$ of the form $(x_1)_{x_1}$ and $(x_1)_{\overline{x_1-2}+2}$, (ii) those contain $(m_1)_{x_1}$ of the form 2_2, (iii) those contain $(m_1)_{x_1}$ of the form 3_{1+2}, (iv) those contain $(m_1)_{x_1}$ of the form 4_{2+2}, (v)those contain $(m_1)_{x_1}$ of the form $(x_1)_{x_1}(x_1>2)$ and $(x_1)_{\overline{x_1-2}+2}(x_1\geq 5)$. With some simple transformation, we get the recurrence relation, which is reversible.

$$\begin{aligned}B_{82}(r,\alpha)=&B_{82}(r,\alpha-4r)+B_{82}(r-1,\alpha-4r+2)+B_{82}(r-1,\alpha-6r+3)\\ &+B_{82}(r-1,\alpha-8r+4)+B_{82}(r,\alpha-4r+2)-B_{82}(r,\alpha-8r+2),\end{aligned}$$

where $B_{82}(0,0)=1$ and $B_{82}(r,\alpha)=0$ for $\alpha<0$. Further using the fact, $f_{82}(w,q)=\sum\limits_{\alpha=0}^{\infty}\sum\limits_{r=0}^{\infty}B_{82}(r,\alpha)w^rq^{\alpha}$, where $|q|<1, |wq|<1$. We get the q–functional equation

$$
\begin{aligned}
f_{82}(w,q) = & f_{82}(wq^4,q) + wq^2 f_{82}(wq^4,q) + wq^3 f_{82}(wq^6,q) \\
& + wq^4 f_{82}(wq^8,q) + q^{-2} f_{82}(wq^4,q) - q^{-2} f_{82}(wq^8,q).
\end{aligned}
$$

We can easily elaborate the further proof.

5.3 Conclusion

In this chapter, we combined the concept of n–color partitions and signed partitions to interpret 100 q–series identities (on the sum side) from [35, 79]. Although we discussed various combinatorial interpretations in this chapter, we found some q–series identities that could not be interpreted simply using the previous techniques. In order to answer this difficulty, we introduce weighted signed color partitions in the next chapter.

Chapter 6

q–series identities and weighted signed color partitions

6.1 Introduction

In the previous chapter, we introduced signed color partitions and provided combinatorial interpretations for 100 q–series identities (sum side). This rich collection of combinatorial interpretations of q–series identities allows us to explore signed color partitions. We recall that all 100 q–series, when expanded in q take the form $a_0+a_1q+a_2q^2+\ldots+a_rq^r+\ldots$, where the coefficients $a_k > 0 \; \forall \; k$. However, while providing combinatorial interpretations, we split the series into two parts: the positive part θ_1 and the negative part θ_2, forming the signed partition $\varTheta = (\theta_1, \theta_2)$. Even though θ_2 is negative part, but the overall count corresponding to $\varTheta$ for all 100 q–series, discussed in previous chapter, is positive. Now, consider a q–series having expansion in q of the form $a_0+a_1q+a_2q^2+\ldots+a_rq^r+\ldots$, where coefficients a_k may be positive or negative or zero. In order to provide the combinatorial interpretation, we use the technique introduced by Brietzke et al. [34]. They have interpreted various mock theta functions using two-line arrays [74] which include some negative coefficients in the expansion. To get the interpretations of such mock theta functions, they assigned weights to each two-line array generated by the unsigned version of the mock theta function being interpreted. Inspiring from their work, we give the interpretations of four q–series identities by assigning weights to the signed color partitions generated by the unsigned version of these q–series identities. We will study the following q–series identities appear in [35] as Identity No. 50, 78, 158, 76 respectively.

$$\sum_{\alpha=0}^{\infty} \frac{(q;q^3)_\alpha (q^2;q^3)_{\alpha+1} q^{3\binom{\alpha+1}{2}}}{(q^3;q^3)_\alpha (q^3;q^6)_{\alpha+1}} = \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(q^2;q^2)_\infty, \tag{6.1.1}$$

$$\sum_{\alpha=0}^{\infty}(-1)^\alpha \frac{(q;q^2)_{2\alpha} q^{2\alpha(\alpha+2)}}{(-q^2;q^2)_{2\alpha+1}(q^4;q^4)_\alpha} = \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}[q^8,-q^7,-q^9;q^8]_\infty, \tag{6.1.2}$$

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$$\sum_{\alpha=0}^{\infty}\frac{(-q;q)_\alpha(-1;q^3)_\alpha q^{\binom{\alpha+1}{2}}}{(q;q)_{2\alpha}(-1;q)_\alpha}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q,q^5;q^6]_\infty[q^8,q^4;q^{12}]_\infty, \tag{6.1.3}$$

$$\sum_{\alpha=0}^{\infty}(-1)^\alpha\frac{(q;q)_{4\alpha}q^{2\alpha^2}}{(q^4;q^4)_\alpha(q^4;q^4)_{2\alpha}}=\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}[q^8,-q^3,-q^5;q^8]_\infty. \tag{6.1.4}$$

We study identities (6.1.1)–(6.1.4) by using signed color partitions.

6.2 Main Results

In this section, we firstly provide the signed color partition–theoretic interpretation of (6.1.1) in Theorem 6.2.1. We first rewrite the summation side of (6.1.1) as follows:

$$\begin{aligned}\sum_{\alpha=0}^{\infty}h_1(\alpha)q^\alpha &= \sum_{\alpha=0}^{\infty}\frac{(q;q^3)_\alpha(q^2;q^3)_{\alpha+1}q^{3\binom{\alpha+1}{2}}}{(q^3;q^3)_\alpha(q^3;q^6)_{\alpha+1}}\\ &= \sum_{\alpha=0}^{\infty}\frac{q^{3\binom{\alpha+1}{2}}}{(q^3;q^3)_\alpha(q^3;q^6)_{\alpha+1}}(1-q^{3\alpha+2})\prod_{j=1}^{\alpha}(1-q^{3j-1})(1-q^{3j-2})\\ &= \sum_{\alpha=0}^{\infty}\frac{(-1)^{2\alpha+1}q^{(9\alpha^2+9\alpha+4)/2}}{(q^3;q^3)_\alpha(q^3;q^6)_{\alpha+1}}(1-q^{-3\alpha-2})\prod_{j=1}^{\alpha}(1-q^{-3j+1})(1-q^{-3j+2})\\ &= \sum_{\alpha=0}^{\infty}\Big(\frac{q^{(9\alpha^2+9\alpha+4)/2}}{(q^3;q^3)_\alpha(q^3;q^6)_{\alpha+1}}\Big)\Big((-1)^{2\alpha+1}(1-q^{-3\alpha-2})\prod_{j=1}^{\alpha}(1-q^{-3j+1})(1-q^{-3j+2})\Big).\end{aligned}$$

The factor $\frac{q^{(9\alpha^2+9\alpha+4)/2}}{(q^3;q^3)_\alpha(q^3;q^6)_{\alpha+1}}$ generates the positive parts and $(-1)^{2\alpha+1}(1-q^{-3\alpha-2})\prod_{j=1}^{\alpha}(1-q^{-(3j-2)})(1-q^{-(3j-1)})$ generates the negative parts of the signed color partitions. Let $g_1(\alpha)$ be the generating function for the product side of (6.1.1) then

$$\begin{aligned}\sum_{\alpha=0}^{\infty}g_1(\alpha)q^\alpha &= \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(q^2;q^2)_\infty\\ &= \frac{(q^2;q^4)_\infty(q^4;q^6)_\infty}{(q^3;q^6)^2_\infty}.\end{aligned}$$

Theorem 6.2.1 *Let Θ_1 denote the signed color partition pairs (θ_1,θ_2) and θ_1 be an $(n+5)$–color partition such that*

(6.2.1.a) $m_k \equiv \begin{cases} 2 \pmod 3 & \text{if } k=1\ , \\ 0 \pmod 3 & \text{if } k>1, \end{cases}$

(6.2.1.b) $x_1 \geq 7$ *and* $\equiv 1 \pmod 3$, $x_1 = m_1+5$,

(6.2.1.c) $x_k \geq 9$ *and* $\equiv 0 \pmod 3$ *for* $2 \leq k \leq r$,

(6.2.1.d) $\delta_k \geq -9$ *and* $\equiv 0 \pmod 3$ *for* $2 \leq k \leq r$,

(6.2.1.e) *Each partition is to be counted with weight* $w_{\theta_1} = 1$.

Let θ_2 *be an* n*–color partition such that*

(6.2.1.f) $(m_l)_{x_l} \neq (m_{l-1})_{x_{l-1}}, \ \forall \ l$,

(6.2.1.g) $m_l \equiv 1$ *or* $2 \pmod 3$,
if $m_l \equiv 1 \pmod 3$ *then* $-3r - 2 \leq m_l \leq -2$ *and*
if $m_l \equiv 2 \pmod 3$ *then* $-3r + 2 \leq m_l \leq -1$,

(6.2.1.h) $x_l = 1, \ 1 \leq l \leq r$,

(6.2.1.i) *The weight count for each partition is* $w_{\theta_2} = (-1)^{(number\ of\ parts\ in\ \theta_2 + 1)}$.

Then

$$\sum_{\alpha=0}^{\infty} \Big(\sum_{\Theta_1 \dashv \alpha} w_{\theta_1} w_{\theta_2} \Big) q^{\alpha} = \sum_{\alpha=0}^{\infty} h_1(\alpha) q^{\alpha}. \tag{6.2.1}$$

Let $g_1(\alpha)$ *be the weighted number of* 2*–colored partitions of* α *with parts* $\equiv \pm 2, 3 \pmod 6$, *where parts* $\equiv \pm 2 \pmod 6$ *are distinct and occur with color* 1. *Each partition is counted with weight* $(-1)^{(number\ of\ parts\ \equiv \pm 2 \pmod 6)}$. *Then*

$$h_1(\alpha) = g_1(\alpha). \quad \forall \ \alpha$$

Example 6.2.1 *We exhibit* Theorem 6.2.1 *for* $\alpha = 9$, *by illustrating*

$$h_1(9) = 8 = g_1(9).$$

The relevant signed color partitions for θ_1 *and* θ_2 *in the first column, their respective weights in second and third columns and their combine weight in fourth column respectively are given in the* Table 6.1.

Table 6.1: Weighted calculations for signed color partitions of $\alpha = 9$.

Relevant Partitions $\Theta_1 : (\theta_1, \theta_2)$	w_{θ_1}	w_{θ_2}	$w_{\theta_1} w_{\theta_2}$
$(9_9 + 2_7, -2_1)$	1	$(-1)^2$	1
$(11_{16}, -2_1)$	1	$(-1)^2$	1
$(12_{12} + 2_7, -5_1)$	1	$(-1)^2$	1
$(12_9 + 2_7, -5_1)$	1	$(-1)^2$	1
$(15_{15} + 2_7, -1_1 - 2_1 - 5_1)$	1	$(-1)^4$	1
$(15_{12} + 2_7, -1_1 - 2_1 - 5_1)$	1	$(-1)^4$	1
$(15_9 + 2_7, -1_1 - 2_1 - 5_1)$	1	$(-1)^4$	1
$(18_9 + 9_9 + 2_7, -1_1 - 2_1 - 4_1 - 5_1 - 8_1)$	1	$(-1)^6$	1

Hence $h_1(9) = \sum_{\Theta_1 \dashv \alpha} w_{\theta_1} w_{\theta_2} = 8$*. And the table below presents the partitions corresponding to* $g_1(\alpha) = 8$ *for* $\alpha = 9$ *and their weight respectively.*

Table 6.2: Relevant partitions corresponding to $g_1(\alpha)$

Partition enumerated by $g_1(\alpha)$	Corresponding weight
(9_1)	1
(9_2)	1
$(4_1 + 3_1 + 2_1)$	$(-1)^2$
$(4_1 + 3_2 + 2_1)$	$(-1)^2$
$(3_2 + 3_2 + 3_2)$	1
$(3_2 + 3_2 + 3_1)$	1
$(3_2 + 3_1 + 3_1)$	1
$(3_1 + 3_1 + 3_1)$	1

Proof To generate $(n+5)$–color partitions corresponding to

$$\frac{q^{(9r^2+9r+4)/2}}{(q^3;q^3)_r(q^3;q^6)_{r+1}},$$

first consider the factor $q^{(9r^2+9r+4)/2}$ which generates $(n+5)$–color partitions of the type $(9r)_9 + (9r-9)_9 \cdots + 18_9 + 9_9 + 2_7$. It corresponds to the following two-line array where first array depicts the parts and second array gives the color of corresponding part:

$$\begin{pmatrix} 9r & 9r-9 & \cdots & 18 & 9 & 2 \\ 9 & 9 & \cdots & 9 & 9 & 7 \end{pmatrix}.$$

The factor $\frac{1}{(q^3;q^3)_r}$ generates the r non negative numbers $\equiv 0 \pmod 3$, say $u_1 \times 3, u_2 \times 6, \cdots, u_r \times 3r$, u_k's are non negative. By introducing this factor, k^{th} part is increased by $3(u_r + u_{r-1} + \cdots + u_{r-k+1})$. It modifies the above two-line array to

$$\begin{pmatrix} 9r+3u_r+\cdots+3u_1 & 9r-9+3u_r+\cdots+3u_2 & \cdots & 18+3u_r+3u_{r-1} & 9+3u_r & 2 \\ 9 & 9 & \cdots & 9 & 9 & 7 \end{pmatrix}.$$

The factor $\frac{1}{(q^3;q^6)_{r+1}}$ generates the $(r+1)$ non negative multiples of $6k-3$, $1 \leq k \leq r+1$, say $v_1 \times 3, v_2 \times 9, \cdots, v_{r+1} \times (6r+3)$, v_k's are non negative. This factor increases the k^{th} part by $6v_{r+1} + 6v_r + \cdots + 6v_{r-k+3} + 3v_{r-k+2}$ and the corresponding subscript by $3v_r - k + 2$. The modified two-line array becomes

$$\begin{pmatrix} 9r+3u_r+\cdots+3u_1 & \cdots & 9+3u_r+6v_{r+1}+3v_r & 2+3v_{r+1} \\ +6v_{r+1}+\cdots+6v_2+3v_1 & & & \\ 9+3v_1 & \cdots & 9+3v_r & 7+3v_{r+1} \end{pmatrix}.$$

For $2 \leq k \leq r+1$, the k^{th} and $(k-1)^{th}$ parts of the $(n+5)$–color partition are

$$\begin{aligned} m_k &= 9k + 3u_r + \cdots + 3u_k + 6v_{r+1} + \cdots + 6v_{k-1} + 3v_k, && (6.2.2) \\ x_k &= 9 + 3v_k, && (6.2.3) \\ m_{k-1} &= 9(k-1) + 3u_r + \cdots + 3u_{k-1} + 6v_{r+1} + \cdots + 6v_{k-2} + 3v_{k-1}, && (6.2.4) \\ x_{k-1} &= 9 + 3v_{k-1}, && (6.2.5) \end{aligned}$$

$$m_1 = 2 + 3v_{r+1}, \qquad (6.2.6)$$

$$\text{and} \quad x_1 = 7 + 3v_{r+1}. \qquad (6.2.7)$$

Thus for $1 \leq k \leq r+1$

$$m_k - m_{k-1} - x_k - x_{k-1} = -9 + 3u_k.$$

So (6.2.1.d) holds.
Clearly (6.2.7) implies (6.2.1.b) and (6.2.3) implies (6.2.1.c) and from (6.2.2) and (6.2.6) we get (6.2.1.a). And the term $(1 + q^{-3\alpha-2}) \prod_{j=1}^{\alpha} (1 - q^{-3j-2})(1 - q^{-3j+1})$ clearly generates n–color partitions satisfying conditions (6.2.1.f)–(6.2.1.i).

Now we will, we interpret (6.1.2) q–series identity using similar combinatorial arguments as done in Theorem 6.2.1. To study the identity (6.1.2), first consider its summation side

$$\sum_{\alpha=0}^{\infty} h_2(\alpha)q^\alpha = \sum_{\alpha=0}^{\infty}(-1)^\alpha \frac{(q;q^2)_{2\alpha}q^{2\alpha(\alpha+2)}}{(-q^2;q^2)_{2\alpha+1}(q^4;q^4)_\alpha}$$
$$= \sum_{\alpha=0}^{\infty}(-1)^\alpha \frac{q^{6\alpha^2+4\alpha}}{(-q^2;q^4)_{\alpha+1}(q^8;q^8)_\alpha}\prod_{j=1}^{2\alpha}(1-q^{-(2j-1)}).$$

As a matter of fact we consider that the factor $(-1)^\alpha \frac{q^{6\alpha^2+4\alpha}}{(-q^2;q^4)_{\alpha+1}(q^8;q^8)_\alpha}$ generates the positive parts and $\prod_{j=1}^{2\alpha}(1-q^{-(2j-1)})$ generate the negative parts of the signed color partitions. Let $g_2(\alpha)$ be the generating function for the product side of (6.1.2) then

$$\sum_{\alpha=0}^{\infty} g_2(\alpha)q^\alpha = \frac{(q^2;q^4)_\infty[q^8,-q^7,-q^9;q^8]_\infty}{(q^4;q^4)_\infty}$$
$$= \frac{(-q^7;q^8)_\infty(-q^9;q^8)_\infty}{(-q^2;q^8)_\infty(-q^6;q^8)_\infty}.$$

Theorem 6.2.2 *Let Θ_2, denote the signed color partition pairs (θ_1,θ_2) and θ_1 be an $(n+4)-$color partition such that*

(6.2.2.a) $m_k, x_k \equiv 0 \pmod 2$, $\forall\ k$

(6.2.2.b) $m_1 = x_1 - 4$,

(6.2.2.c) $\delta_k \equiv 0 \pmod 8$, $\forall\ k$

(6.2.2.d) *The weight count for each partition is* $w_{\theta_1} = \sum_{i=1}^{r+1} x_i - 2r - 2.$

Let θ_2 be an $n-$color partition such that

(6.2.2.e) $(m_l)_{x_l} \neq (m_{l-1})_{x_{l-1}}$, $\forall\ l$

(6.2.2.f) $m_l \equiv 1 \pmod 2$ *and* $x_l = 1$, $\forall\ l$

(6.2.2.g) $m_l < 2r$,

(6.2.2.h) *The weight count for each partition is* $w_{\theta_2} = (-1)^{(number\ of\ parts\ in\ \theta_2)}$.

Then

$$\sum_{\alpha=0}^{\infty}\Big(\sum_{\Theta_2 \dashv \alpha} w_{\theta_1}w_{\theta_2}\Big)q^\alpha = \sum_{\alpha=0}^{\infty} h_2(\alpha). \tag{6.2.8}$$

Let $g_2(\alpha)$ be the weighted number of partitions of α into parts ≥ 2 and $\equiv \pm 2 \pmod 8$ and distinct parts $\equiv \pm 1 \pmod 8$. Each partition is counted with the weight

$(-1)^{(number\ of\ parts\ \equiv\pm2\ (\mathrm{mod}\ 8))}$. *Then*

$$h_2(\alpha) = g_2(\alpha). \quad \forall\ \alpha$$

To get a clear understanding of the Theorem 6.2.2, we look at the following example.

Example 6.2.2 *For* $\alpha = 16$, Table 6.3 *gives relevent signed color partitions for* θ_1 *and* θ_2 *with corresponding weights satisfying the conditions of* Theorem 6.2.2.

Table 6.3: Weight calculation for signed color partitions of $\alpha = 16$

Relevant Partitions $\Theta_2 : (\theta_1, \theta_2)$	w_{θ_1}	w_{θ_2}	$w_{\theta_1}w_{\theta_2}$
$(14_{18}, 0)$	-1	1	-1
$(14_{10} + 0_4, 0)$	-1	1	-1
$(18_{14} + 0_4, -3_1 - 1_1)$	-1	1	-1
$(16_8 + 2_6, -3_1 - 1_1)$	-1	1	-1
$(18_6 + 0_4, -3_1 - 1_1)$	-1	1	-1

From (6.2.8) *we get,* $h_2(16) = \sum_{\Theta_2 \dashv \alpha} w_{\theta_1} w_{\theta_2} = -5$.

Remark 6.2.3 *The proofs of* Theorems 6.2.2, 6.2.4, 6.2.5 *can be established in a same manner as developed for* Theorem 6.2.1.

To study the identity (6.1.3), consider the summation side and after splitting the terms as done for (6.1.1) and (6.1.2) we get,

$$\begin{aligned}\sum_{\alpha=0}^{\infty} h_3(\alpha) q^{\alpha} &= \sum_{\alpha=0}^{\infty} \frac{(-q;q)_{\alpha}(-1;q^3)_{\alpha} q^{\binom{\alpha+1}{2}}}{(q;q)_{2\alpha}(-1;q)_{\alpha}} \\ &= \sum_{\alpha=0}^{\infty} \frac{(-q;q)_{\alpha} q^{\alpha(3\alpha-1)/2}}{(q;q)_{2\alpha}} \prod_{j=1}^{\alpha-1}(1 - q^{-j} + q^{-2j}).\end{aligned}$$

In the last expression, the factor $\frac{(-q;q)_{\alpha} q^{\alpha(3\alpha-1)/2}}{(q;q)_{2\alpha}}$ generate the positive parts and $\prod_{j=1}^{\alpha-1}(1 - q^{-j} + q^{-2j})$ generates the negative parts of the signed color partitions. Let $g_3(\alpha)$ be the generating function for the product side of (6.1.3) then

$$\begin{aligned}\sum_{\alpha=0}^{\infty} g_3(\alpha) q^{\alpha} &= \frac{(-q;q)_{\infty}}{(q;q)_{\infty}} [q^6, q, q^5; q^6]_{\infty} [q^8, q^4; q^{12}]_{\infty} \\ &= \frac{(-q;q)_{\infty}}{(q^2;q^{12})_{\infty}(q^3;q^{12})_{\infty}(q^9;q^{12})_{\infty}(q^{10};q^{12})_{\infty}}\end{aligned}$$

$$= \sum_{\alpha=0}^{\infty} a_3(\alpha)q^\alpha \sum_{\alpha=0}^{\infty} b_3(\alpha)q^\alpha$$
$$= \sum_{\alpha=0}^{\infty}\sum_{k=0}^{\alpha} a_3(\alpha-k)b_3(k)q^\alpha,$$

where $\sum_{\alpha=0}^{\infty} a_3(\alpha)q^\alpha = (-q;q)_\infty$ and $\sum_{\alpha=0}^{\infty} b_3(\alpha)q^\alpha = \frac{1}{(q^2;q^{12})_\infty(q^3;q^{12})_\infty(q^9;q^{12})_\infty(q^{10};q^{12})_\infty}$.

Theorem 6.2.4 *Let Θ_3 denote the signed color partition pairs (θ_1, θ_2) and θ_1 be an n–color partition such that*

(6.2.4.a) $\delta_k \geq 1$,

(6.2.4.b) *Each partition is to be counted with weight $w_{\theta_1} = 1$.*

Let θ_2 be the n–color partition such that

(6.2.4.c) $(m_l)_{x_l} \neq (m_{l-1})_{x_{l-1}}$, $\forall\ l$

(6.2.4.d) $x_l = 1, 2$, $\forall\ l$,

(6.2.4.e) $m_l < r$, $\forall\ l$

(6.2.4.f) *The weight count for each partition is $w_{\theta_2} = (-1)^{(\text{number of distinct parts of } \theta_2)}$.*

Then

$$\sum_{\alpha=0}^{\infty}\Big(\sum_{\Theta_3 \dashv \alpha} w_{\theta_1}w_{\theta_2}\Big)q^\alpha = \sum_{\alpha=0}^{\infty} h_3(\alpha). \tag{6.2.9}$$

Let $a_3(\alpha)$ denote the number of distict partitions of α and $b_3(\alpha)$ denote the number of partitions of α such that parts are $\equiv \pm 2, \pm 3 \pmod{12}$. Then

$$g_3(\alpha) = \sum_{k=0}^{\alpha} a_3(\alpha-k)b_3(k) = h_3(\alpha). \quad \forall\ \alpha$$

For the last the identity (6.1.4), again consider the summation side and splitting the terms as done for other q–series identities we get,

$$\begin{aligned}\sum_{\alpha=0}^{\infty} h_4(\alpha)q^\alpha &= \sum_{\alpha=0}^{\infty}(-1)^\alpha \frac{(q;q)_{4\alpha}q^{2\alpha^2}}{(q^4;q^4)_\alpha(q^4;q^4)_{2\alpha}} \\ &= \sum_{\alpha=0}^{\infty}(-1)^\alpha \frac{q^{6\alpha^2}}{(-q^2;q^4)_\alpha(q^8;q^8)_\alpha}\prod_{j=1}^{\alpha}(1-q^{-(4j-1)})(1-q^{-(4j-3)}).\end{aligned}$$

In the above expression the factor $(-1)^\alpha \frac{q^{6\alpha^2}}{(-q^2;q^4)_\alpha(q^8;q^8)_\alpha}$ generates the positive parts and $\prod_{j=1}^{2\alpha}(1-q^{-(4j-1)})(1-q^{-(4j-3)})$ generate the negative parts of the signed color partition.

If the corresponding product side of (6.1.4) is generated by $g_4(\alpha)$ then

$$\begin{aligned}\sum_{\alpha=0}^{\infty} g_4(\alpha)q^{\alpha} &= \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}[q^8,-q^7,-q^9;q^8]_\infty \\ &= \frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty}{(-q^2;q^8)_\infty(-q^6;q^8)_\infty}.\end{aligned}$$

Theorem 6.2.5 *Let Θ_4 denote the signed color partition pairs (θ_1,θ_2) and θ_1 be an n–color partition such that*

(6.2.5.a) $m_k \equiv x_k \pmod 2,\ \forall\ k$

(6.2.5.b) $x_k \geq 6,\ \forall\ k$

(6.2.5.c) $\delta_k \equiv 0 \pmod 8,\ \forall\ k$

(6.2.5.d) *The weight count for each partition is* $w_{\theta_1} = \sum_{i=1}^{r} x_i - 2r.$

Let θ_2 be an n–color partition such that

(6.2.5.e) $m_{l_{x_l}} \neq m_{l-1_{x_{l-1}}},\ \forall\ l$

(6.2.5.f) $m_l \equiv \pm 1 \pmod 4$ *and* $x_l = 1,\ \forall\ l$

(6.2.5.g) $m_l < 2r,\ \forall\ l,$

(6.2.5.h) *The weight count for each partition is* $w_{\theta_2} = (-1)^{(number\ of\ parts\ of\ \theta_2)}.$

Then

$$\sum_{\alpha=0}^{\infty}\Big(\sum_{\Theta_4 \dashv \alpha} w_{\theta_1}w_{\theta_2}\Big)q^{\alpha} = \sum_{\alpha=0}^{\infty} h_4(\alpha). \tag{6.2.10}$$

Let $g_4(\alpha)$ is the weighted number of partitions of α into parts $\equiv \pm 2 \pmod 8$ and distinct parts $\equiv \pm 3 \pmod 8$. Each partition is counted with the weight $(-1)^{(number\ of\ parts\ \pm\ 2\ (\mathrm{mod}\ 8))}$. Then

$$h_4(\alpha) = g_4(\alpha) \quad .\forall\ \alpha$$

6.3 Conclusion

In current chapter, we have developed the combinatorial interpretations of four q–series identities in terms of signed color partitions by assigning certain weight.The approach of using the signed color partitions is interesting as it helps in providing the combinatorial interpretations of many such identities. In this chapter, we have provided only four such interpretations, and many others can be obtained as application of our work.

Chapter 7

Some congruences for the coefficients of q–series identities

7.1 Introduction

In 1919, Ramanujan [65] gave the beautiful partition congruences for the partition function $p(\alpha)$ as:

$$p(5\alpha + 4) \equiv 0 \pmod 5, \tag{7.1.1}$$

$$p(7\alpha + 5) \equiv 0 \pmod 7, \tag{7.1.2}$$

$$p(11\alpha + 6) \equiv 0 \pmod{11}. \tag{7.1.3}$$

These congruences were generalized and written in the form

$$p(\ell\alpha - \delta_\ell) \equiv 0 \pmod \ell,$$

where $\delta_\ell = (\ell^2 - 1)/24$ and $\ell = 5, 7, 11$. The above congruences were extended for arbitrary powers of $5, 7, 11$ (for more details, see [17]). Ramanujan [65] proved (7.1.1) and (7.1.2) in 1919. Later in 1921 [66], he gave a proof of (7.1.3) by employing different methods. The works of Newman [63], and of Atkin and J. N. O'Brien [25], and of Atkin and H. P. F. Swinnerton-Dyer [26] have shown that there are many other congruences for the partition function. For example, Atkin and O'Brien [25] found that $p(594 \cdot 13\alpha + 111247) \equiv 0 \pmod{13}$. Additionally, many mathematicians were interested in finding the arithmetic properties of some restricted partition functions [29, 64, 70, 73, 81, 82].

In Chapter 5 and 6 we have found the combinatorial interpretations of many q–series identities, using signed color partitions. In the literature, to the best of our knowledge, we have not found any congruences for q–series identities. In this chapter we found the congruences for 17 q–series identities of Chapter 5 and 6 which are listed in Tables 7.1–7.3. We have arranged these q–series identities into three groups: Group 1 contains 10 identities, which are listed in Table 7.1; Group 2 contains 3 identities, which are listed in

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Table 7.2; and Group 3 contains 4 identities, which are listed in Table 7.3.

7.2 Preliminaries

We require the following definitions and lemmas to prove the main results in the next section. For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a,b) = \sum_{m=-\infty}^{\infty} a^{\frac{m(m+1)}{2}} b^{\frac{m(m-1)}{2}}. \tag{7.2.1}$$

Using Jacobi's triple-product identity [31] (entry 19, p. 35), (7.2.1) becomes

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{7.2.2}$$

Throughout this chapter, we use

$$f_k = (q^k;q^k)_\infty,$$

for positive integer k. The special cases of $f(a, b)$ are

$$\varphi(q) = f(q;q) = 1 + 2\sum_{m=1}^{\infty} q^{m^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{7.2.3}$$

$$\psi(q) = f(q;q^3) = \sum_{m=1}^{\infty} q^{\frac{m(m+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}. \tag{7.2.4}$$

In some of the proofs, we also employ Jacobi's identity from [51] as eq (1.7.1):

$$f_1^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{7.2.5}$$

Lemma 7.2.1 *We have*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{7.2.6}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{7.2.7}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{7.2.8}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{7.2.9}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{7.2.10}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{7.2.11}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{7.2.12}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{7.2.13}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{7.2.14}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{7.2.15}$$

Proof Using the two-dissection of $\varphi(q)$ and $\varphi(q^2)$ from [51], Equation (1.9.4) and (1.10.1)), we obtain (7.2.6) and (7.2.8). On replacing q by $-q$ in (7.2.3), we obtain (7.2.7) and (7.2.9). Furthermore, (7.2.10), (7.2.11), (7.2.13), (7.2.14), and (7.2.15) are equations (30.12.3), (22.1.13), (22.1.14), (30.10.4), and (30.10.3), respectively, in [51]. Next, (7.2.12) follows from (7.2.11) by using q instead of $-q$.

Lemma 7.2.2 *We have*

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}, \tag{7.2.16}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{7.2.17}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{7.2.18}$$

$$f_1^3 = f_3 c(q^3) - 3q f_9^3. \tag{7.2.19}$$

where

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

Proof The first identity follows from equations (33.2.1) and (33.2.5) in [51]. The second identity is equivalent to the three-dissection of $\varphi(-q)$ (see [51] Equation (14.3.2)). We obtained (7.2.18) by replacing q with ωq and $\omega^2 q$ and multiplying the two results, where ω is the primitive cube root of unity.

The three-dissection of $\psi(q)$ follows as:

Lemma 7.2.3 *We have*

$$\psi(q) = \frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{7.2.20}$$

Proof Identity (7.2.20) is Equation (14.3.3) of [51].

Lemma 7.2.4 [40] *For any prime $p \geq 5$,*

$$f_1 = \sum_{\substack{k=\frac{-(p-1)}{2} \\ k\neq(\pm p-1)/6}}^{\frac{(p-1)}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{(3p^2+(6k+1)p)}{2}}, -q^{\frac{(3p^2-(6k+1)p)}{2}})$$

$$+ (-1)^{\frac{(\pm p-1)}{6}} q^{\frac{(p^2-1)}{24}} f_{p^2}, \tag{7.2.21}$$

$$\textit{where} \quad \frac{\pm p-1}{6} = \begin{cases} \frac{(p-1)}{6} & \text{if } p \equiv 1 \pmod 6, \\ \frac{(-p-1)}{6} & \text{if } p \equiv -1 \pmod 6. \end{cases} \tag{7.2.22}$$

If $\frac{-p-1}{2} \leq k \leq \frac{p-1}{2}$ and $k \neq \frac{\pm p-1}{2}$, then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

7.3 Main Results

In Tables 7.1–7.3, the sum sides of q–series identities are the generators for the partitions written in the second column, and the product sides of the q–series identities are written in the third column.

Group 1 We now present 10, q–series identities in this group from [35] with Identity Nos. 8, 9, 10, 33, 45, 70, 98, 104, 111, and 112, as shown below.

Table 7.1: q–series identities.

Function	**Sum Side**	=	**Product Side**
$A_1(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q;q)_{\alpha+1} q^{\frac{\alpha(\alpha+1)}{2}}}{(q;q)_\alpha}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^4, q^2, q^2; q^4]_\infty$
$A_2(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q)_{2\alpha} q^{\alpha}}{(q;q)_{2\alpha}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^4, -q^2, -q^2; q^4]_\infty$
$A_3(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q;q)_{2\alpha} q^{\alpha}}{(q;q)_{2\alpha+1}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^4, -q^4, -q^4; q^4]_\infty$
$A_4(q)$	$\sum_{\alpha=0}^{\infty} \frac{(q;q^2)_\alpha^2 q^{2\alpha^2}}{(q^2;q^2)_{2\alpha}}$	=	$\frac{[q^6,q^3,q^3;q^6]_\infty}{(q^2;q^2)_\infty}$
$A_5(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha}}$	=	$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^6, -q^3, -q^3; q^6]_\infty$
$A_6(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^2)_\alpha q^{\frac{\alpha(\alpha+1)}{2}}}{(q;q)_\alpha (q;q^2)_\alpha}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^8, q^4, q^4; q^8]_\infty$

Function	Sum Side	=	Product Side
$A_7(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q)_\alpha q^{\frac{\alpha(\alpha+1)}{2}}}{(q;q^2)_\alpha (q;q)_\alpha}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{10}, q^5, q^5; q^{10}]_\infty$
$A_8(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q^2)_{2\alpha+1}(q^2;q^2)_\alpha}$	=	$\frac{[q^{12}, q^4, q^8; q^{12}]_\infty}{(q;q)_\infty}$
$A_9(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-q;q^2)_\alpha q^\alpha}{(q;q)_{2\alpha+1}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{12}, q^3, q^9; q^{12}]_\infty$
$A_{10}(q)$	$\sum_{\alpha=0}^{\infty} \frac{(-1;q^2)_\alpha q^\alpha}{(q;q)_{2\alpha}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{12}, q^6, q^6; q^{12}]_\infty$

From the binomial theorem, we have

$$f_1^{2^k} \equiv f_2^{2^{k-1}} \pmod{2^k}. \tag{7.3.1}$$

Before stating the main results, we define

$$A_i(q) = \sum_{\alpha=0}^{\infty} a_i(\alpha) q^\alpha.$$

Theorem 7.3.1 *For $\alpha \geq 0$, we have*

$$a_1(4\alpha+2) \equiv 0 \pmod 2, \tag{7.3.2}$$
$$a_1(4\alpha+3) \equiv 0 \pmod 4, \tag{7.3.3}$$
$$a_1(8\alpha+5) \equiv 0 \pmod 8, \tag{7.3.4}$$
$$a_1(8\alpha+6) \equiv 0 \pmod 4, \tag{7.3.5}$$
$$a_1(8\alpha+7) \equiv 0 \pmod{16}, \tag{7.3.6}$$
$$a_1(16\alpha+t) \equiv 0 \pmod 4, \textit{for } t \in \{10, 12\} \tag{7.3.7}$$
$$a_1(32\alpha+20) \equiv 0 \pmod 4, \tag{7.3.8}$$
$$a_1(48\alpha+34) \equiv 0 \pmod 4. \tag{7.3.9}$$

Theorem 7.3.2 *For $\alpha \geq 0$, we have*

$$a_2(4\alpha+3) \equiv 0 \pmod 4, \tag{7.3.10}$$
$$a_2(8\alpha+t) \equiv 0 \pmod 4, \textit{for } t \in \{3, 6\} \tag{7.3.11}$$
$$a_2(8\alpha+7) \equiv 0 \pmod 8, \tag{7.3.12}$$
$$a_2(16\alpha+10) \equiv 0 \pmod 4. \tag{7.3.13}$$

Theorem 7.3.3 *For $\alpha \geq 0$, we have*

$$a_3(3\alpha+2) \equiv 0 \pmod{2}, \tag{7.3.14}$$

$$a_3(4\alpha+2) \equiv 0 \pmod{4}, \tag{7.3.15}$$

$$a_3(4\alpha+3) \equiv 0 \pmod{8}, \tag{7.3.16}$$

$$a_3(12\alpha+t) \equiv 0 \pmod{4}, \textit{for } t \in \{2,3,6,11\}. \tag{7.3.17}$$

Theorem 7.3.4 *For $\alpha \geq 0$, we have*

$$a_4(2\alpha+1) \equiv 0 \pmod{2}, \tag{7.3.18}$$

$$a_4(\alpha) \equiv p(\alpha) \pmod{2}. \tag{7.3.19}$$

Theorem 7.3.5 *For $\alpha \geq 0$, we have*

$$a_5(2\alpha+1) \equiv 0 \pmod{2}, \tag{7.3.20}$$

$$a_5(6\alpha+1) \equiv 0 \pmod{8}, \tag{7.3.21}$$

$$a_5(6\alpha+5) \equiv 0 \pmod{4}, \tag{7.3.22}$$

$$a_5(18\alpha+6) \equiv 0 \pmod{4}, \tag{7.3.23}$$

$$a_5(36\alpha+30) \equiv 0 \pmod{4}, \tag{7.3.24}$$

$$a_5(24\alpha+t) \equiv 0 \pmod{4}, \textit{for } t \in \{9,15,21\} \tag{7.3.25}$$

$$a_5(54\alpha+36) \equiv 0 \pmod{4}. \tag{7.3.26}$$

Theorem 7.3.6 *For $\alpha \geq 0$, we have*

$$a_6(3\alpha+1) \equiv 0 \pmod{2}, \tag{7.3.27}$$

$$a_6(3\alpha+2) \equiv 0 \pmod{4}, \tag{7.3.28}$$

$$a_6(4\alpha+3) \equiv 0 \pmod{8}, \tag{7.3.29}$$

$$a_6(4\alpha+4) \equiv 0 \pmod{4}, \tag{7.3.30}$$

$$a_6(8\alpha+t) \equiv 0 \pmod{4}, \textit{for } t \in \{4,5\} \tag{7.3.31}$$

$$a_6(8\alpha+t) \equiv 0 \pmod{16}, \textit{for } t \in \{6,7\} \tag{7.3.32}$$

$$a_6(9\alpha) \equiv a_6(\alpha) \pmod{8}, \tag{7.3.33}$$

$$a_6(9\alpha+t) \equiv 0 \pmod{8}, \textit{for } t \in \{3,6\} \tag{7.3.34}$$

$$a_6(12\alpha+3) \equiv 0 \pmod{8}, \tag{7.3.35}$$

$$a_6(36\alpha+t) \equiv 0 \pmod{8}, \textit{for } t \in \{21,33\}. \tag{7.3.36}$$

Theorem 7.3.7 *For $\alpha \geq 0$, we have*

$$a_7(9\alpha+t) \equiv 0 \pmod 4, \textit{for } t \in \{3,6\} \tag{7.3.37}$$
$$a_7(15\alpha+t) \equiv 0 \pmod 4, \textit{for } t \in \{2,8,11,14\} \tag{7.3.38}$$
$$a_7(12\alpha+t) \equiv 0 \pmod 4, \textit{for } t \in \{7,10\} \tag{7.3.39}$$
$$a_7(24\alpha+13) \equiv 0 \pmod 4, \tag{7.3.40}$$
$$a_7(48\alpha+t) \equiv 0 \pmod 4, \textit{for } t \in \{28,40\}. \tag{7.3.41}$$

Theorem 7.3.8 *For prime $p \geq 5$*

$$a_8\left(3p^2\alpha + 3pi + \frac{p^2-1}{8}\right) \equiv 0 \pmod 2, \tag{7.3.42}$$

where $i = 1, 2, \cdots, (p-1)$.

Theorem 7.3.9 *For $\alpha \geq 0$, we have*

$$a_8(9\alpha+1) \equiv a_8(\alpha) \pmod 4, \tag{7.3.43}$$
$$a_8(9\alpha+4) \equiv 0 \pmod 4, \tag{7.3.44}$$
$$a_8(9\alpha+7) \equiv 0 \pmod 4. \tag{7.3.45}$$

Theorem 7.3.10 *For $\alpha \geq 0$, we have*

$$a_9(3\alpha+1) \equiv 0 \pmod 2, \tag{7.3.46}$$
$$a_9(3\alpha+2) \equiv 0 \pmod 4, \tag{7.3.47}$$
$$a_9(9\alpha+6) \equiv 0 \pmod 2, \tag{7.3.48}$$
$$a_9(27\alpha+t) \equiv 0 \pmod 2, \textit{for } t \in \{12,21\}. \tag{7.3.49}$$

Theorem 7.3.11 *For $\alpha \geq 0$, we have*

$$a_{10}(8\alpha+t) \equiv 0 \pmod 4, \textit{for } t \in \{3,5,7\}, \tag{7.3.50}$$
$$a_{10}(16\alpha+12) \equiv 0 \pmod 4, \tag{7.3.51}$$
$$a_{10}(24\alpha+17) \equiv 0 \pmod 4, \tag{7.3.52}$$
$$a_{10}(32\alpha+20) \equiv 0 \pmod 4, \tag{7.3.53}$$
$$a_{10}(48\alpha+t) \equiv 0 \pmod 4, \textit{for } t \in \{22,38\}. \tag{7.3.54}$$

Group 2 In this group we have the following q–series identities with Identity No. $1, 36$ and 37 in [35]. These q–series identities have same congruences.

Table 7.2: q–series identities

Function	Sum side	=	Product side
$A_{11}(q)$	$\sum_{\alpha=0}^{\infty}\frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q^2)_{2\alpha}}$	=	$\frac{[q^3,-q,-q^2;q^3]_\infty}{(q;q)_\infty}$
$A_{12}(q)$	$\sum_{\alpha=0}^{\infty}\frac{(-1;q)_{2\alpha} q^{\alpha}}{(q^2;q^2)_{\alpha}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q^3,q^3;q^6]_\infty$
$A_{13}(q)$	$\sum_{\alpha=0}^{\infty}\frac{(-1;q)_\alpha q^{\alpha^2}}{(q;q^2)_{\alpha}(q;q)_\alpha}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q^3,q^3;q^6]_\infty$

Theorem 7.3.12 *For $\alpha \geq 0$ and $i = 11, 12$ and 13 we have*

$$a_i(3\alpha+1) \equiv 0 \pmod{2}, \tag{7.3.55}$$
$$a_i(3\alpha+2) \equiv 0 \pmod{4}, \tag{7.3.56}$$
$$a_i(4\alpha+2) \equiv 0 \pmod{4}, \tag{7.3.57}$$
$$a_i(4\alpha+3) \equiv 0 \pmod{6}, \tag{7.3.58}$$
$$a_i(6\alpha+5) \equiv 0 \pmod{16}, \tag{7.3.59}$$
$$a_i(8\alpha+4) \equiv 0 \pmod{2}, \tag{7.3.60}$$
$$a_i(8\alpha+5) \equiv 0 \pmod{4}, \tag{7.3.61}$$
$$a_i(8\alpha+t) \equiv 0 \pmod{12}, \textit{for } t \in \{6,7\} \tag{7.3.62}$$
$$a_i(24\alpha+14) \equiv 0 \pmod{8}, \tag{7.3.63}$$
$$a_i(24\alpha+20) \equiv 0 \pmod{16}, \tag{7.3.64}$$
$$a_i(32\alpha+24) \equiv 0 \pmod{8}, \tag{7.3.65}$$
$$a_i(40\alpha+t) \equiv 0 \pmod{4}, \textit{for } t \in \{17,33\} \tag{7.3.66}$$
$$a_i(40\alpha+t) \equiv 0 \pmod{12}, \textit{for } t \in \{11,19\} \tag{7.3.67}$$
$$a_i(64\alpha+40) \equiv 0 \pmod{8}. \tag{7.3.68}$$

Group 3 In this group the following q–series identity from [35] with Identity No. 3, 39, 46 and 103 respectively. The identities $A_{14}(q)$, $A_{15}(q)$ and $A_{16}(q)$, $A_{17}(q)$ have same congruences.

Table 7.3: q–series identities

Function	Sum side	=	Product side
$A_{14}(q)$	$\sum_{\alpha=0}^{\infty}\frac{(-q;q)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_\alpha(q;q^2)_{\alpha+1}}$	=	$\frac{[q^3,-q^3,-q^3;q^3]_\infty}{(q;q)_\infty}$
$A_{15}(q)$	$\sum_{\alpha=0}^{\infty}\frac{(-q;q)_{2\alpha}q^{\alpha}}{(q^2;q^2)_\alpha}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q,q^5;q^6]_\infty$
$A_{16}(q)$	$\sum_{\alpha=0}^{\infty}\frac{(-q^2;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2m+1}}$	=	$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^6,-q,-q^5;q^6]_\infty$
$A_{17}(q)$	$\sum_{\alpha=0}^{\infty}\frac{(-q^2;q^2)_\alpha q^{\alpha(\alpha+1)}}{(q;q)_{2\alpha+1}}$	=	$\frac{[q^6,-q,-q^5;q^6]_\infty}{(q;q)_\infty}$

Theorem 7.3.13 *For $\alpha \geq 0$ and $i = 14, 15$, we have*

$$a_i(4\alpha+2) \equiv 0 \pmod{2}, \tag{7.3.69}$$
$$a_i(4\alpha+3) \equiv 0 \pmod{4}, \tag{7.3.70}$$
$$a_i(16\alpha+t) \equiv 0 \pmod{8}, for\ t \in \{11, 15\}. \tag{7.3.71}$$

Theorem 7.3.14 *For $\alpha \geq 0$ and $i = 16, 17$, we have*

$$a_i(4\alpha+1) \equiv a_i(\alpha) \pmod{4}, \tag{7.3.72}$$
$$a_i(8\alpha+4) \equiv 0 \pmod{4}, \tag{7.3.73}$$
$$a_i(8\alpha+6) \equiv 0 \pmod{8}, \tag{7.3.74}$$
$$a_i(16\alpha+t) \equiv 0 \pmod{8}, for\ t \in \{11, 15\}. \tag{7.3.75}$$

7.4 Proofs of Main Results

Proof of Theorem 7.3.1 Consider

$$\sum_{\alpha=0}^{\infty} a_1(\alpha)q^\alpha = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^4,q^2,q^2;q^4]_\infty = \frac{f_2^3}{f_4}\frac{1}{f_1^2}$$

$$\sum_{\alpha=0}^{\infty} a_1(\alpha)q^\alpha = \frac{f_2^3}{f_4}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right).$$

Extracting even and odd terms, we get

$$\sum_{\alpha=0}^{\infty} a_1(2\alpha)q^\alpha = \frac{f_4^5}{f_2 f_8^2}\frac{1}{f_1^2}, \tag{7.4.1}$$

$$\sum_{\alpha=0}^{\infty} a_1(2\alpha+1)q^\alpha = \frac{2f_2f_8^2}{f_4}\frac{1}{f_1^2}. \tag{7.4.2}$$

Substituting (7.2.6) in (7.4.1), on extracting even and odd terms, we get

$$\sum_{\alpha=0}^{\infty} a_1(4\alpha)q^\alpha = \frac{f_2^5f_4^3}{f_1^6f_8^2}, \tag{7.4.3}$$

$$\sum_{\alpha=0}^{\infty} a_1(4\alpha+2)q^\alpha = \frac{2f_2^7f_8^2}{f_1^6f_4^3}. \tag{7.4.4}$$

From (7.4.4), we reach at (7.3.2). Also,

$$\sum_{\alpha=0}^{\infty} a_1(4\alpha+2)q^\alpha \equiv 2\frac{f_2^4f_8^2}{f_4^3} \pmod 4,$$

and we extract odd terms to reach at (7.3.5). On bringing out the even terms from the above equation and using (7.3.1), we have

$$\sum_{\alpha=0}^{\infty} a_1(8\alpha+2)q^\alpha \equiv 2\frac{f_1^4f_4^2}{f_2^3} \equiv 2\frac{f_4^2}{f_2} \pmod 4.$$

Extracting odd terms from above equation to get (7.3.7) for $t=10$ and on extracting even terms, we get

$$\sum_{\alpha=0}^{\infty} a_1(16\alpha+2)q^\alpha \equiv 2\frac{f_2^2}{f_1} \pmod 4.$$

Using (7.2.20) in above equation and extracting the terms involving $q^{3\alpha+2}$, dividing by q^2 and replacing q^3 by q to get (7.3.9).
On substituting (7.2.6) in (7.4.2), extracting even and odd terms, we obtain

$$\sum_{\alpha=0}^{\infty} a_1(4\alpha+1)q^\alpha = 2\frac{f_4^7}{f_2f_8^2}\frac{1}{f_1^4}, \tag{7.4.5}$$

$$\sum_{\alpha=0}^{\infty} a_1(4\alpha+3)q^\alpha = 4f_4f_2f_8^2\frac{1}{f_1^4}. \tag{7.4.6}$$

From (7.4.6), we readily reach at (7.3.3). Putting (7.2.8) in (7.4.5) and (7.4.6), then extracting odd terms from both equations, we get (7.3.4) and (7.3.6) respectively. Consider (7.4.3),

$$\sum_{\alpha=0}^{\infty} a_1(4\alpha)q^\alpha = \frac{f_2^5f_4^3}{f_1^6f_8^2} \equiv \frac{f_2^3f_4^3}{f_8^2}\frac{1}{f_1^2} \pmod 4.$$

Applying (7.2.6) in above relation, extracting odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_1(8\alpha+4)q^\alpha \equiv 2\frac{f_2^5 f_8^2}{f_4^3}\frac{1}{f_1^2} \pmod 4,$$

and again putting (7.2.6), then extracting odd terms gives (7.3.7) for $t = 12$ and extracting even terms gives

$$\sum_{\alpha=0}^{\infty} a_1(16\alpha+4)q^\alpha \equiv 2\frac{f_4^7}{f_2^3 f_8^2} \pmod 4.$$

Extracting the odd terms from the above equation, we reach at (7.3.8).

Proof of Theorem 7.3.2

$$\sum_{\alpha=0}^{\infty} a_2(\alpha)q^\alpha = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^4, -q^2, -q^2; q^4]_\infty = \frac{f_4^5}{f_8^2 f_2}\frac{1}{f_1^2}.$$

Using (7.2.6) in above equation, extract even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_2(2\alpha)q^\alpha = \frac{f_2^5 f_4^3}{f_1^6 f_8^2}, \tag{7.4.7}$$

$$\sum_{\alpha=0}^{\infty} a_2(2\alpha+1)q^\alpha = 2\frac{f_2^7 f_8^2}{f_1^6 f_4^3}. \tag{7.4.8}$$

Putting (7.2.6) in (7.4.7) and extracting odd terms, we get

$$\sum_{\alpha=0}^{\infty} a_2(4\alpha+2)q^\alpha \equiv 6\frac{f_4^7}{f_8^2} \pmod 4.$$

Again extracting odd terms from above equation to get (7.3.11) and extracting even terms, we get

$$\sum_{\alpha=0}^{\infty} a_2(8\alpha+2)q^\alpha \equiv \frac{6f_2^7}{f_4^2} \pmod 4.$$

On extracting odd terms, we get (7.3.13).

Consider (7.4.8) and substituting (7.2.6), we have

$$\sum_{\alpha=0}^{\infty} a_2(2\alpha+1)q^\alpha = 2\frac{f_2^7 f_8^2}{f_4^3}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)^3$$

Then extracting odd terms to get (7.3.10) and on taking modulo 8, we have

$$\sum_{\alpha=0}^{\infty} a_2(4\alpha+3)q^\alpha \equiv 4\frac{f_4^{11}}{f_1^8 f_2 f_8^2} \equiv 4\frac{f_4^{11}}{f_2^5 f_4^2} \pmod 8.$$

Extracting odd terms from above, we get (7.3.12).

Proof of Theorem 7.3.3 Consider

$$\sum_{\alpha=0}^{\infty} a_3(\alpha)q^\alpha = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^4,-q^4,-q^4;q^4]_\infty = \frac{f_2f_8^2}{f_4}\frac{1}{f_1^2}. \tag{7.4.9}$$

Substituting the value from (7.2.6), we get

$$\sum_{\alpha=0}^{\infty} a_3(\alpha)q^\alpha = \frac{f_2f_8^2}{f_4}\left(\frac{f_8^5}{f_2^5f_{16}^2} + 2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right).$$

Extracting even and odd terms, we get

$$\sum_{\alpha=0}^{\infty} a_3(2\alpha)q^\alpha = \frac{f_4^7}{f_2f_8^2}\frac{1}{f_1^4}, \tag{7.4.10}$$

$$\sum_{\alpha=0}^{\infty} a_3(2\alpha+1)q^\alpha = 2f_4f_2f_8^2\frac{1}{f_1^4}. \tag{7.4.11}$$

Using (7.2.8) in both (7.4.10) and (7.4.11), then extracting the odd terms from both of them to get (7.3.15) and (7.3.16) respectively. Again from (7.4.9),

$$\sum_{\alpha=0}^{\infty} a_3(\alpha)q^\alpha = \frac{f_2}{f_1^2}\frac{f_8^2}{f_4}.$$

Using (7.2.18) and (7.2.20), we have

$$\sum_{\alpha=0}^{\infty} a_3(\alpha)q^\alpha = \left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q\frac{f_6^3f_9^3}{f_3^7} + 4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right)\left(\frac{f_{24}f_{36}^2}{f_{12}f_{72}} + q^4\frac{f_{72}^2}{f_{36}}\right) \tag{7.4.12}$$

Then extract the terms involving $q^{3\alpha}$ and replacing q^3 by q to get

$$\sum_{\alpha=0}^{\infty} a_3(3\alpha)q^\alpha \equiv \frac{f_3^2f_8f_{12}^2}{f_4f_6f_{24}} \pmod 4. \tag{7.4.13}$$

Extract the terms involving $q^{3\alpha+2}$ from (7.4.12), dividing both sides by q^2 and replacing q^3 by q to get

$$\sum_{\alpha=0}^{\infty} a_3(3\alpha+2)q^\alpha = 4\frac{f_2^2f_6^3f_8f_{12}^2}{f_1^6f_4f_{24}} + 2q\frac{f_2^3f_3^3f_{24}^2}{f_1^7f_{12}}. \tag{7.4.14}$$

From above equation we readily reach at (7.3.14). Now using (7.2.7) in (7.4.13), we have

$$\sum_{\alpha=0}^{\infty} a_3(3\alpha)q^\alpha \equiv \frac{f_8f_{12}^2}{f_4f_6f_{24}}\left(\frac{f_6f_{24}^5}{f_{12}^2f_{48}^2} - 2q^3\frac{f_6f_{48}^2}{f_{24}}\right) \pmod 4.$$

On extracting even and odd terms, we get

$$\sum_{\alpha=0}^{\infty} a_3(6\alpha)q^{\alpha} \equiv \frac{f_4 f_{12}^4}{f_2 f_{24}^2} \pmod 4, \tag{7.4.15}$$

$$\sum_{\alpha=0}^{\infty} a_3(6\alpha+3)q^{\alpha} \equiv 2q\frac{f_4 f_6^2 f_{24}^2}{f_2 f_{12}^2} \pmod 4. \tag{7.4.16}$$

Extracting the odd and even terms from (7.4.15) and (7.4.16) respectively to get (7.3.17) for $t = 3, 6$. From (7.4.14), we have

$$\sum_{\alpha=0}^{\infty} a_3(3\alpha+2)q^{\alpha} \equiv 2q\frac{f_{24}^2}{f_{12}}\frac{f_3^3}{f_1} \pmod 4.$$

Using (7.2.13), then extracting the even and odd terms we get

$$\sum_{\alpha=0}^{\infty} a_3(6\alpha+2)q^{\alpha} \equiv 2q\frac{f_{12}^2 f_6^3}{f_2 f_6} \pmod 4, \tag{7.4.17}$$

$$\sum_{\alpha=0}^{\infty} a_3(6\alpha+5)q^{\alpha} \equiv 2\frac{f_2^2 f_{12}^2}{f_6} \pmod 4. \tag{7.4.18}$$

Extracting even and odd terms from (7.4.17) and (7.4.18) respectively to get (7.3.17) for $t = 2, 11$.

Proof of Theorem 7.3.4

$$\sum_{\alpha=0}^{\infty} a_4(\alpha)q^{\alpha} = \frac{[q^6, q^3, q^3; q^6]_\infty}{(q^2;q^2)_\infty} = f_3^2\frac{1}{f_2 f_6}. \tag{7.4.19}$$

Substituting (7.2.7) in above equation to get

$$\sum_{m=0}^{\infty} a_4(\alpha)q^{\alpha} = \frac{1}{f_2 f_6}\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right),$$

On extracting odd terms we get (7.3.18).

$$\sum_{\alpha=0}^{\infty} a_4(2\alpha)q^{\alpha} = \frac{f_{12}^5}{f_{24}^2}\frac{1}{f_1 f_6^2}. \tag{7.4.20}$$

Reducing (7.4.19) modulo 2, we have

$$\sum_{\alpha=0}^{\infty} a_4(\alpha)q^{\alpha} \equiv \frac{1}{f_1} \pmod 2,$$

and we arrive at (7.3.19).

Proof of Theorem 7.3.5

$$\sum_{\alpha=0}^{\infty} a_5(\alpha)q^{\alpha} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^6,-q^3,-q^3;q^6]_\infty = \frac{f_4 f_6^5}{f_2^2 f_{12}^2}\frac{1}{f_3^2}.$$

Using (7.2.6) in above equation,

$$\sum_{\alpha=0}^{\infty} a_5(\alpha)q^{\alpha} = \frac{f_4 f_6^5}{f_2^2 f_{12}^2}\left(\frac{f_{24}^5}{f_6^5 f_{48}^2} + 2q^3 \frac{f_{12}^2 f_{48}^2}{f_6^5 f_{24}}\right)$$

On extracting the odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_5(2\alpha+1)q^{\alpha} = 2q\frac{f_2 f_{24}^2}{f_1^2 f_{12}}, \tag{7.4.21}$$

which yields (7.3.20). Extracting even terms then using (7.2.18), we extract the terms involving $q^{3\alpha}$, $q^{3\alpha+1}$ and $q^{3\alpha+2}$ terms we get (7.4.22),(7.3.21), (7.3.22) respectively.

$$\sum_{\alpha=0}^{\infty} a_5(6\alpha)q^{\alpha} = \frac{f_2^2 f_4^5 f_3^6}{f_1^8 f_6^3 f_8^2}. \tag{7.4.22}$$

Taking modulo 4,

$$\sum_{\alpha=0}^{\infty} a_5(6\alpha)q^{\alpha} \equiv \frac{f_4 f_3^6}{f_2^2 f_6^3} \pmod 4.$$

Using (7.2.18), we have

$$\sum_{\alpha=0}^{\infty} a_5(6\alpha)q^{\alpha} \equiv \frac{f_3^6}{f_6^3}\left(\frac{f_{12}^4 f_{18}^6}{f_6^8 f_{36}^3} + 2q^2\frac{f_{12}^3 f_{18}^3}{f_6^7} + 4q^4\frac{f_{12}^2 f_{36}^3}{f_6^6}\right) \pmod 4.$$

Extracting the term involving $q^{3\alpha+1}$, $q^{3\alpha+2}$, $q^{3\alpha}$ we have (7.3.23),

$$\sum_{\alpha=0}^{\infty} a_5(18\alpha+12)q^{\alpha} \equiv 2\frac{f_1^6 f_4^3 f_6^3}{f_2^{10}} \equiv 2\frac{f_4^3 f_6^3}{f_2^7} \pmod 4, \tag{7.4.23}$$

$$\sum_{\alpha=0}^{\infty} a_5(18\alpha)q^{\alpha} \equiv \frac{f_1^6 f_4^4 f_6^6}{f_2^{11} f_{12}^3} \equiv \frac{f_1^2 f_6^6}{f_2 f_{12}^3} \pmod 4. \tag{7.4.24}$$

Extracting the odd terms from (7.4.23), we arrive at (7.3.24). Using (7.2.17) in (7.4.24), to get (7.3.26). Then consider (7.4.21) and substituting the values from (7.2.18), to get

(7.3.21), (7.3.22) and

$$\sum_{\alpha=0}^{\infty} a_5(6\alpha+3)q^\alpha \equiv 2\frac{f_2^4 f_3^6 f_8^2}{f_1^8 f_4 f_6^3} \equiv 2\frac{f_8^2}{f_4} \pmod 4.$$

Extracting the terms involving $q^{4\alpha+3}, q^{4\alpha+2}, q^{4\alpha+1}$ to get (7.3.25).

Proof of Theorem 7.3.6

$$\sum_{\alpha=0}^{\infty} a_6(\alpha)q^\alpha = \frac{f_2 f_4^2}{f_8}\frac{1}{f_1^2}. \tag{7.4.25}$$

Using (7.2.6) and extracting even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_6(2\alpha)q^\alpha = \frac{f_2^2 f_4^4}{f_8^2}\frac{1}{f_1^4}, \tag{7.4.26}$$

$$\sum_{\alpha=0}^{\infty} a_6(2\alpha+1)q^\alpha = 2\frac{f_2^4 f_8^2}{f_4^2}\frac{1}{f_1^4}. \tag{7.4.27}$$

Substituting the value from (7.2.6) in (7.4.26), on extracting the even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_6(4\alpha)q^\alpha = \frac{f_2^{18}}{f_4^6 f_1^{12}}, \tag{7.4.28}$$

$$\sum_{\alpha=0}^{\infty} a_6(4\alpha+2)q^\alpha = 4\frac{f_2^6 f_4^2}{f_1^8}. \tag{7.4.29}$$

Taking modulo 4 in (7.4.28),

$$\sum_{\alpha=0}^{\infty} a_6(4\alpha)q^\alpha \equiv \frac{f_2^{12}}{f_4^6} \equiv 1 \pmod 4.$$

This leads to (7.3.30). Consider (7.4.29) and taking modulo 16,

$$\sum_{\alpha=0}^{\infty} a_6(4\alpha+2)q^\alpha \equiv 4f_2^2 f_4^2 \pmod{16}.$$

Extracting odd terms from above equation gives (7.3.32). Consider (7.4.27) and using (7.2.8), extracting even and odd terms we get

$$\sum_{\alpha=0}^{\infty} a_6(4\alpha+1)q^\alpha = 2\frac{f_2^{12}}{f_4^2}\frac{1}{f_1^{10}}, \tag{7.4.30}$$

$$\sum_{\alpha=0}^{\infty} a_6(4\alpha+3)q^{\alpha} = 8\frac{f_4^4}{f_1^6}. \tag{7.4.31}$$

From above, we arrive at (7.3.29). Consider (7.4.30) and using (7.2.6) and (7.2.8), extracting odd terms gives (7.3.31) (for $t = 5$). Similarly, taking modulo 16 in (7.4.31), we obtain

$$\sum_{\alpha=0}^{\infty} a_6(4\alpha+3)q^{\alpha} \equiv 8\frac{f_4^4}{f_2^3} \pmod{16}.$$

Extracting odd terms gives (7.3.32) (for $t = 16$). Consider (7.4.25) and using (7.2.17) and (7.2.18)

$$\sum_{\alpha=0}^{\infty} a_6(\alpha)q^{\alpha} = \left(\frac{f_{36}^2}{f_{72}} - 2q^4\frac{f_{12}f_{72}^2}{f_{24}f_{36}}\right)\left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q\frac{f_6^3f_9^3}{f_3^7} + 4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right).$$

Extracting the terms involving $q^{3\alpha}, q^{3\alpha+1}, q^{3\alpha+2}$ gives (7.4.32), (7.3.27), (7.3.28) respectively.

$$\begin{aligned}\sum_{\alpha=0}^{\infty} a_6(3\alpha)q^{\alpha} &\equiv \frac{f_{12}^2f_2^4f_3^6}{f_{24}f_1^8f_6^3} \pmod{8}, \\ &\equiv \frac{f_{12}^2f_3^6}{f_{24}f_6^3} \pmod{8}.\end{aligned} \tag{7.4.32}$$

Extracting the terms involving $q^{3\alpha+2}, q^{3\alpha+1}, q^{3\alpha}$ to get (7.3.34) and

$$\sum_{\alpha=0}^{\infty} a_6(9\alpha)q^{\alpha} \equiv \frac{f_4^2f_2}{f_8f_1^2} \pmod{8},$$

which proves (7.3.33). Consider (7.4.32) and using (7.2.6), we have

$$\sum_{\alpha=0}^{\infty} a_6(3\alpha)q^{\alpha} \equiv \frac{f_{12}^2f_6}{f_{24}}\left(\frac{f_{24}^5}{f_6^5f_{48}^2} + 2q^3\frac{f_{12}^2f_{48}^2}{f_6^5f_{24}}\right) \pmod{8}.$$

Extracting the odd terms, we obtain

$$\sum_{\alpha=0}^{\infty} a_6(6\alpha+3)q^{\alpha} \equiv 2q\frac{f_6^4f_{24}^2}{f_3^4f_{12}^2} \pmod{8}.$$

Substituting the values from (7.2.8),

$$\sum_{\alpha=0}^{\infty} a_6(6\alpha+3)q^{\alpha} \equiv 2q\frac{f_6^4f_{24}^2}{f_{12}^2}\left(\frac{f_{12}^4}{f_6^{14}f_{24}^4} + 4q^3\frac{f_{12}^2f_{24}^4}{f_6^{10}}\right) \pmod{8}.$$

On extracting the even and odd terms, we get (7.3.35) and

$$\sum_{\alpha=0}^{\infty} a_6(12\alpha+9)q^{\alpha} \equiv 2\frac{f_6^2}{f_3^{10}f_{12}^2} \pmod{8}.$$

Extracting the terms involving $q^{3\alpha+1}, q^{3\alpha+2}$ to get (7.3.36).

Proof of Theorem 7.3.7

$$\sum_{\alpha=0}^{\infty} a_7(\alpha)q^{\alpha} = \frac{f_2}{f_1^2}\frac{f_5^2}{f_{10}}.$$

Using (7.2.17) and (7.2.18), we have

$$\sum_{\alpha=0}^{\infty} a_7(\alpha)q^{\alpha} = \left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)\left(\frac{f_{45}^2}{f_{90}} - 2q^5\frac{f_{15}f_{90}^2}{f_{30}f_{45}}\right).$$

Extracting the terms involving $q^{3\alpha}$, $q^{3\alpha+1}$ and $q^{3\alpha+2}$ and taking modulo 4, we have

$$\sum_{\alpha=0}^{\infty} a_7(3\alpha)q^{\alpha} \equiv \frac{f_3^6 f_{15}^2}{f_6^3 f_{30}} \pmod{4}, \tag{7.4.33}$$

$$\sum_{\alpha=0}^{\infty} a_7(3\alpha+1)q^{\alpha} \equiv 2f_6\frac{f_3}{f_1} \pmod{4}, \tag{7.4.34}$$

$$\sum_{\alpha=0}^{\infty} a_7(3\alpha+2)q^{\alpha} \equiv 2q\frac{f_{15}^3}{f_5} \pmod{4}. \tag{7.4.35}$$

Extracting the terms involving $q^{3\alpha+1}$, $q^{3\alpha+2}$ from (7.4.33) and then dividing by q, q^2 respectively and replacing q^3 by q to get (7.3.37). Consider (7.4.34) and using (7.2.15), on extracting even and odd terms, we get

$$\sum_{\alpha=0}^{\infty} a_7(6\alpha+1)q^{\alpha} \equiv 2\frac{f_8 f_{12}^2}{f_4 f_{24}} \pmod{4}, \tag{7.4.36}$$

$$\sum_{\alpha=0}^{\infty} a_7(6\alpha+4)q^{\alpha} \equiv 2\frac{f_6 f_{24}}{f_1^2 f_{12}} \pmod{4}. \tag{7.4.37}$$

Consider (7.4.36) and extracting even and odd terms, we get (7.4.38) and (7.3.39) (for $t=7$) respectively.

$$\sum_{\alpha=0}^{\infty} a_7(12\alpha+1)q^{\alpha} \equiv 2\frac{f_4 f_6^2}{f_2 f_{12}} \pmod{4}. \tag{7.4.38}$$

Extracting odd terms from above equation gives (7.3.40). From (7.4.37), extracting even and odd terms gives (7.4.39) and (7.3.39) (for $t = 10$) respectively.

$$\sum_{\alpha=0}^{\infty} a_7(12\alpha+4)q^\alpha \equiv 2\frac{f_{12}}{f_6}\frac{f_3}{f_1} \pmod 4. \tag{7.4.39}$$

Using (7.2.15), extracting even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_7(24\alpha+4)q^\alpha \equiv 2\frac{f_8 f_{12}^2}{f_4 f_{24}} \equiv 2f_4 \pmod 4, \tag{7.4.40}$$

$$\sum_{\alpha=0}^{\infty} a_7(24\alpha+16)q^\alpha \equiv 2\frac{f_6 f_{24}}{f_{12}} \pmod 4. \tag{7.4.41}$$

Extracting odd terms from (7.4.40) and (7.4.41), we arrive at (7.3.41). Consider (7.4.35), extracting the terms involving $q^{5\alpha}$, $q^{5\alpha+2}$, $q^{5\alpha+3}$ and $q^{5\alpha+4}$, we get (7.3.38).

Proof of Theorem 7.3.8 and 7.3.9

$$\sum_{\alpha=0}^{\infty} a_8(\alpha)q^\alpha = \frac{[q^{12}, q^4, q^8; q^{12}]_\infty}{(q;q)_\infty} = \frac{f_4}{f_1}.$$

Using (7.2.16) extracting the terms involving $q^{3\alpha}$ and $q^{3\alpha+1}$ we have

$$\sum_{\alpha=0}^{\infty} a_8(3\alpha)q^\alpha = \frac{f_4 f_6^4}{f_1^3 f_{12}^2}, \tag{7.4.42}$$

$$\sum_{\alpha=0}^{\infty} a_8(3\alpha+1)q^\alpha = \frac{f_2^2 f_3^3 f_{12}}{f_1^4 f_6^2}. \tag{7.4.43}$$

Taking modulo 2 in (7.4.42),

$$\sum_{\alpha=0}^{\infty} a_8(3\alpha)q^\alpha \equiv \frac{f_2}{f_1} \equiv f_1 \pmod 2.$$

From Lemma 7.2.4, we have

$$\sum_{\alpha=0}^{\infty} a_8(3\alpha)q^\alpha \equiv \sum_{\substack{k=\frac{-(p-1)}{2} \\ k\neq(\pm p-1)/6}}^{\frac{(p-1)}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{(3p^2+(6k+1)p)}{2}}, -q^{\frac{(3p^2-(6k+1)p)}{2}})$$
$$+ (-1)^{\frac{(\pm p-1)}{6}} q^{\frac{(p^2-1)}{24}} f_{p^2} \pmod 2.$$

Extract the terms involving $q^{p\alpha+(p^2-1)/24}$, dividing by $q^{(p^2-1)/24}$ and replacing $q^{p\alpha}$ by q^α, we get

$$\sum_{\alpha=0}^{\infty} a_8\left(3p\alpha + \frac{p^2-1}{8}\right) q^\alpha \equiv (-1)^{(\pm p-1)/6} f_p \pmod 2.$$

Extracting the terms involving $q^{p\alpha+i}$ for $i = 1, 2, \cdots, (p-1)$,

$$\sum_{\alpha=0}^{\infty} a_8\left(3p^2\alpha + 3pi + \frac{p^2-1}{8}\right) q^\alpha \equiv 0 \pmod{2}.$$

which proves (7.3.42). Taking modulo 4 in (7.4.43), we have

$$\sum_{\alpha=0}^{\infty} a_8(3\alpha+1) q^\alpha \equiv \frac{f_2^2 f_3^3 f_{12}}{f_2^2 f_6^2} \pmod{4}.$$

Extracting the terms involving $q^{3\alpha}$, $q^{3\alpha+1}$ and $q^{3\alpha+2}$ from above, we obtain (7.4.44), (7.3.44) and (7.3.45) respectively.

$$\sum_{\alpha=0}^{\infty} a_8(9\alpha+1) q^\alpha \equiv \frac{f_1^3 f_4}{f_2^2} \equiv \frac{f_4}{f_1} \pmod{4} \tag{7.4.44}$$

From above, it is easy to conclude (7.3.43).

Proof of Theorem 7.3.10

$$\sum_{\alpha=0}^{\infty} a_9(\alpha) q^\alpha = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^{12}, q^3, q^9; q^{12}]_\infty = \frac{f_2 f_3 f_{12}}{f_1^2 f_6}.$$

Using (7.2.18), extracting the terms involving $q^{3\alpha}$, $q^{3\alpha+1}$ and $q^{3\alpha+2}$, we get (7.4.45), (7.3.46) and (7.3.47) respectively.

$$\sum_{\alpha=0}^{\infty} a_9(3\alpha) q^\alpha = \frac{f_2^3 f_3^6 f_4}{f_1^7 f_6^3}. \tag{7.4.45}$$

Taking modulo 2, we have

$$\sum_{\alpha=0}^{\infty} a_9(3\alpha) q^\alpha \equiv \frac{f_4}{f_1} \pmod{2}.$$

Substituting the values fro (7.2.16), extracting the terms involving $q^{3\alpha+2}, q^{3\alpha+1}$ to get (7.3.48) and

$$\sum_{\alpha=0}^{\infty} a_9(9\alpha+3) q^\alpha \equiv f_3^3 \pmod{2}.$$

Extracting the terms involving $q^{3\alpha+1}, q^{3\alpha+2}$ to get (7.3.49).

Proof of Theorem 7.3.11

$$\sum_{\alpha=0}^{\infty} a_{10}(\alpha)q^{\alpha} = \frac{(-q;q)_{\infty}}{(q;q)_{\infty}}[q^{12},q^{6},q^{6};q^{12}]_{\infty} = \frac{f_2 f_6^2}{f_1^2 f_{12}}.$$

Using (7.2.6), extracting the even and odd terms and taking modulo 4, we have

$$\sum_{\alpha=0}^{\infty} a_{10}(2\alpha)q^{\alpha} \equiv \frac{f_4^5}{f_2^2 f_6 f_8^2} \cdot f_3^2 \pmod{4}, \tag{7.4.46}$$

$$\sum_{\alpha=0}^{\infty} a_{10}(2\alpha+1)q^{\alpha} \equiv 2f_4^3 \pmod{4}. \tag{7.4.47}$$

Using (7.2.7) in (7.4.46), we get

$$\sum_{\alpha=0}^{\infty} a_{10}(2\alpha)q^{\alpha} \equiv \frac{f_4^5}{f_2^2 f_6 f_8^2}\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}} - 2q^3 \frac{f_6 f_{48}^2}{f_{24}}\right) \pmod{4}.$$

Extracting even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_{10}(4\alpha)q^{\alpha} \equiv \frac{f_2^5 f_{12}^5}{f_4^2 f_6^2 f_{24}} \frac{1}{f_1^2} \pmod{4}, \tag{7.4.48}$$

$$\sum_{\alpha=0}^{\infty} a_{10}(4\alpha+2)q^{\alpha} \equiv 2q\frac{f_2^4 f_{24}^2}{f_4^2 f_{12}} \pmod{4}. \tag{7.4.49}$$

Substituting (7.2.6) in (7.4.48) and extracting odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_{10}(8\alpha+4)q^{\alpha} \equiv 2\frac{f_6^4 f_8^2}{f_4 f_{12}} \pmod{4}.$$

On extracting odd and even terms, we reach at (7.3.51) and

$$\sum_{\alpha=0}^{\infty} a_{10}(16\alpha+4)q^{\alpha} \equiv 2\frac{f_4^2 f_6}{f_2} \pmod{4},$$

respectively. Extracting odd terms from above equation to get (7.3.53). From (7.4.47), we get (7.3.50) and

$$\sum_{\alpha=0}^{\infty} a_{10}(8\alpha+1)q^{\alpha} \equiv 2f_1^3 \pmod{4}.$$

Using (7.2.19) in above, we arrive at (7.3.52).

Proof of Theorem 7.3.12 For $i = 11, 12, 13$, we consider

$$\sum_{\alpha=0}^{\infty} a_i(\alpha)q^\alpha = \frac{f_2 f_3^2}{f_6 f_1^2}. \tag{7.4.50}$$

Using (7.2.14) in above equation and extracting even terms and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_i(2\alpha)q^\alpha = \frac{f_2^4 f_6^2}{f_4 f_{12}} \frac{1}{f_1^4}, \tag{7.4.51}$$

$$\sum_{\alpha=0}^{\infty} a_i(2\alpha+1)q^\alpha = 2\frac{f_2 f_4 f_{12}}{f_6} \frac{f_3}{f_1^3}. \tag{7.4.52}$$

Using (7.2.8) in (7.4.51), again extracting the even and odd terms, we obtain

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha)q^\alpha \equiv \frac{f_2^8}{f_4^4} \pmod{2}, \tag{7.4.53}$$

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+2)q^\alpha \equiv 4\frac{f_2 f_4^4}{f_6} \pmod{12}. \tag{7.4.54}$$

From (7.4.54), we get (7.3.57). Now on extracting odd terms from (7.4.53) and (7.4.54), we reach at (7.3.60) and (7.3.62) (for $t = 6$) respectively. From (7.4.51), we have

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha)q^\alpha \equiv \frac{f_2^9}{f_6 f_4^4} \frac{f_3^2}{f_1^2} \pmod{8}.$$

Using (7.2.14), extracting even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_i(8\alpha)q^\alpha \equiv \frac{f_6^2}{f_4 f_{12}} f_1^4 \pmod{8}, \tag{7.4.55}$$

$$\sum_{\alpha=0}^{\infty} a_i(8\alpha+4)q^\alpha \equiv 2\frac{f_4 f_{12}}{f_2 f_6}.f_1 f_3 \pmod{8}. \tag{7.4.56}$$

Substituting the values from (7.2.8) in (7.4.55) and extracting even terms and odd terms, we obtain

$$\sum_{\alpha=0}^{\infty} a_i(16\alpha)q^\alpha \equiv \frac{f_3^2 f_2^{10}}{f_2 f_6 f_4^4} \frac{1}{f_1^2} \pmod{8}, \tag{7.4.57}$$

$$\sum_{\alpha=0}^{\infty} a_i(16\alpha+8)q^\alpha \equiv 4\frac{f_4^4}{f_2^2} \pmod{8}. \tag{7.4.58}$$

On extracting odd terms from (7.4.58), we get (7.3.65) and extracting even terms gives us

$$\sum_{\alpha=0}^{\infty} a_i(32\alpha+8)q^{\alpha} \equiv 4f_2^3 \pmod 8.$$

Extracting odd terms gives us (7.3.68). Now consider (7.4.52) and using (7.2.12) and extracting even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+1)q^{\alpha} \equiv 2f_2^3 \pmod 4, \tag{7.4.59}$$

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+3)q^{\alpha} \equiv 6f_6^3 \pmod{12}. \tag{7.4.60}$$

Also, we get (7.3.58). From (7.4.59), extracting the odd terms gives us (7.3.61), while extracting even terms gives

$$\sum_{\alpha=0}^{\infty} a_i(8\alpha+1)q^{\alpha} \equiv 2f_1^3 \pmod 4.$$

According to Jacobi triple product

$$\sum_{\alpha=0}^{\infty} a_i(8\alpha+1)q^{\alpha} \equiv 2\sum_{\alpha=0}^{\infty}(-1)^{\alpha}(2\alpha+1)q^{\alpha(\alpha+1)/2} \pmod 4.$$

As $\alpha(\alpha+1)/2 \not\equiv 2, 4 \pmod 5$, we get (7.3.67). Consider (7.4.60) and extracting the odd terms to get (7.3.62) (for $t=7$), extracting even terms and using Jacobi triple product, we ultimately reach at (7.3.66). Consider, (7.4.50) and using (7.2.18), extracting the terms involving $q^{3\alpha}$, $q^{3\alpha+1}$, $q^{3\alpha+2}$, we obtain

$$\sum_{\alpha=0}^{\infty} a_i(3\alpha)q^{\alpha} \equiv \frac{f_3^4}{f_6^2} \pmod 3, \tag{7.4.61}$$

$$\sum_{\alpha=0}^{\infty} a_i(3\alpha+1)q^{\alpha} \equiv 0 \pmod 2, \tag{7.4.62}$$

$$\sum_{\alpha=0}^{\infty} a_i(3\alpha+2)q^{\alpha} \equiv 0 \pmod 4. \tag{7.4.63}$$

Also,

$$\sum_{\alpha=0}^{\infty} a_i(3\alpha+2)q^{\alpha} \equiv 4\frac{f_6^3}{f_2} \pmod{16}.$$

On extracting odd parts, we get (7.3.59) and on extracting even parts and using (7.2.13), we have

$$\sum_{\alpha=0}^{\infty} a_i(6\alpha+2)q^{\alpha} \equiv 4\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod{16}. \tag{7.4.64}$$

On extracting odd terms, we get

$$\sum_{\alpha=0}^{\infty} a_i(12\alpha+8)q^{\alpha} \equiv 4\frac{f_6^3}{f_2} \pmod{16}.$$

On extracting odd terms, we get (7.3.64). Taking modulo 8 for (7.4.64) and using (7.2.13) and extracting the even terms gives

$$\sum_{\alpha=0}^{\infty} a_i(12\alpha+2)q^{\alpha} \equiv 4f_2^2 \pmod{8}.$$

On extracting odd terms, we reach at (7.3.63).

Proof of Theorem 7.3.13 For $i = 14$, 15, we consider

$$\sum_{\alpha=0}^{\infty} a_i(\alpha)q^{\alpha} = f_6^2 \frac{1}{f_1 f_3}$$

Substituting (7.2.10) in above equation and then extracting even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_i(2\alpha)q^{\alpha} = \frac{f_4^2 f_6^5}{f_2 f_3^2 f_{12}^2} \cdot \frac{1}{f_1^2}, \tag{7.4.65}$$

$$\sum_{\alpha=0}^{\infty} a_i(2\alpha+1)q^{\alpha} = \frac{f_2^5 f_{12}^2}{f_4^2 f_6} \cdot \frac{1}{f_1^4}. \tag{7.4.66}$$

Taking modulo 2 in (7.4.65), we have

$$\sum_{\alpha=0}^{\infty} a_i(2\alpha)q^{\alpha} \equiv \frac{f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{2}.$$

On extracting odd terms from above equation, we readily reach at (7.3.69). Consider (7.4.66) and substituting (7.2.8),

$$\sum_{\alpha=0}^{\infty} a_i(2\alpha+1)q^{\alpha} = \frac{f_2^5 f_{12}^2}{f_4^2 f_6}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right).$$

On extracting odd terms we obtain (7.3.70) and (7.4.67). Similarly extracting even terms from the same, we have

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+3)q^\alpha \equiv 4\frac{f_6^2 f_4^4}{f_2^2}\frac{1}{f_1 f_3} \pmod 8. \tag{7.4.67}$$

Using (7.2.10) in (7.4.67), extracting odd terms, we get Using (7.2.10), on extracting even and odd terms, we get

$$\sum_{\alpha=0}^{\infty} a_i(8\alpha+3)q^\alpha \equiv 4\frac{f_2 f_4^2 f_6^4}{f_{12}^2} \pmod 8,$$
$$\sum_{\alpha=0}^{\infty} a_i(8\alpha+7)q^\alpha \equiv \frac{f_2^6 f_{12}^2}{f_4^2 f_6} \pmod 8.$$

Extracting odd terms from both the above equations to get (7.3.71).

Proof of Theorem 7.3.14 For $i = 16$, 17, we consider

$$\sum_{\alpha=0}^{\infty} a_i(\alpha)q^\alpha = \frac{f_{12}}{f_6}\cdot\frac{f_3}{f_1}.$$

Using (7.2.15) and extracting even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_i(2\alpha)q^\alpha = \frac{f_2 f_8 f_{12}^2}{f_4 f_{24}}\frac{1}{f_1^2}, \tag{7.4.68}$$
$$\sum_{\alpha=0}^{\infty} a_i(2\alpha+1)q^\alpha = \frac{f_4^2 f_6 f_{24}}{f_8 f_{12}}\cdot\frac{1}{f_1^2}. \tag{7.4.69}$$

Using (7.2.6) in (7.4.68), then extracting even and odd terms to get

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha)q^\alpha = \frac{f_4^6 f_6^2}{f_1^4 f_2 f_8^2 f_{12}}, \tag{7.4.70}$$
$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+2)q^\alpha = 2\frac{f_2 f_6^2 f_8^2}{f_{12}}\frac{1}{f_1^4}. \tag{7.4.71}$$

Reducing (7.4.70), modulo 4 and extracting odd terms, we readily reach at (7.3.73). Consider (7.4.69) and using (7.2.6), on extracting even and odd terms, we obtain

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+1)q^\alpha = \frac{f_2^2 f_4^4 f_{12}}{f_1^4 f_6 f_8^2}\frac{f_3}{f_1}, \tag{7.4.72}$$

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+3)q^{\alpha} = 2\frac{f_2^4 f_3 f_8^2 f_{12}}{f_1^5 f_4^2 f_6}. \tag{7.4.73}$$

Taking modulo 4 in (7.4.72),

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+1)q^{\alpha} \equiv \frac{f_{12}}{f_6}\frac{f_3}{f_1} \pmod 4,$$

which implies $a_i(4\alpha+1) \equiv a_i(\alpha) \pmod 4$. Now consider (7.4.71) and using (7.2.8), on extracting odd terms, we get (7.3.74). Taking modulo 8 in (7.4.73), we have

$$\sum_{\alpha=0}^{\infty} a_i(4\alpha+3)q^{\alpha} = 2\frac{f_2^2 f_8^2 f_{12}}{f_4^2 f_6}\frac{f_3}{f_1} \pmod 8.$$

Using (7.2.15) in above and extracting even and odd terms, we have

$$\sum_{\alpha=0}^{\infty} a_i(8\alpha+3)q^{\alpha} \equiv 2\frac{f_4 f_8 f_{12}^2}{f_2 f_{24}} \pmod 8,$$
$$\sum_{\alpha=0}^{\infty} a_i(8\alpha+7)q^{\alpha} \equiv 2\frac{f_4^4 f_6 f_{24}}{f_2^2 f_8 f_{12}} \pmod 8.$$

Extracting odd terms from above equations, to get (7.3.75).

7.5 Conculsion

This chapter includes the study of arithmetic properties for some q–series identities, particularly congruences modulo powers of 2, 3, and 6. The “sum-product identities” have been studied by many mathematicians in various contexts [3, 5, 6, 22]. However, in the literature, to the best of our knowledge, we have not found any congruences for q–series identities. Nonetheless, there are a number of research articles in the literature that study congruences for partition functions. Thus, this chapter adds one more direction to the study of q–series identities. For future work, one could find other arithmetic properties, including parity, divisibility properties, and congruences modulo higher primes for these q–series identities or others available in the literature. Furthermore, one can think of generalizing the congruences shown in this chapter. Moreover, it will be fascinating to prove these congruences using some other techniques, such as the ones based on modular forms.

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