On Inter-Relationships of Different Automorphism Groups of Finite Groups

Thesis

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Declaration of Authorship

I hereby declare that the work which is being presented in this thesis entitled *"On Inter-Relationships of Different Automorphism Groups of Finite Groups"* submitted by me, for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Deepak Gumber, Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or Abroad and that the ideas and references cited herein have been duly acknowledged.

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CERTIFICATE

This is to certify that the thesis *"On Inter-Relationships of Different Automorphism Groups of Finite Groups"* which is submitted by Mr. Mandeep Singh, in fulfillment of the requirement for the award of the degree of *Doctor of Philosophy* in the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is a record of the candidate's own independent and original research work carried out by him under my supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any University or Institute for the award of any degree.

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Mandeep Singh

January, 2020 (Mandeep Singh)

Dedicated
To
my
Family

Abstract

Let G be an arbitrary group and let $\mathrm{Aut}(G)$ denote the full automorphism group of G. An automorphism α of G is called a class-preserving automorphism if for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1} x g_x$; and is called an inner automorphism if for all $x \in G$, there exists a fix element $g \in G$ such that $\alpha(x) = g^{-1} x g$. The group $\mathrm{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\mathrm{Aut}_c(G)$ of all class-preserving automorphisms of G. An automorphism α of G is called an IA-automorphism if $x^{-1}\alpha(x) \in G'$ for all $x \in G$. Let $\mathrm{IA}(G)$ denote the group of all IA-automorphisms of G and let $C_{\mathrm{IA}(G)}(Z(G))$ denote the group of all IA-automorphisms of G fixing the center $Z(G)$ of G elementwise. An automorphism α of G is called a central automorphism if it commutes with all inner automorphisms of G; or equivalently $g^{-1}\alpha(g) \in Z(G)$, the center of G, for all $g \in G$. The group of all central automorphisms of G is denoted as $\mathrm{Aut_z}(G)$.

Following Hegarty [38], we analogously call an automorphism α an absolute central automorphism if $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$, where $L(G)$ is the absolute center of G. Let $\mathrm{Var}(G)$ and $C_{\mathrm{Var}(G)}(Z(G))$ respectively denote the group of all absolute central automorphisms of G and absolute central automorphisms of G fixing the center $Z(G)$ of G elementwise.

An automorphism α of a group G is called a commuting automorphism if each element x in G commutes with its image $\alpha(x)$ under α. Let $A(G)$ denote the set of all commuting automorphisms of G. Observe that $\mathrm{Aut_z}(G)$ is contained in $A(G)$. A group G is called an $A(G)$-group if the set $A(G)$ is a subgroup of $\mathrm{Aut}(G)$.

In this thesis, we mainly study the structure of $C_{\mathrm{Var}(G)}(Z(G))$, $\mathrm{Aut_z}(G)$ and $A(G)$. We find necessary and sufficient conditions on a finite non-abelian p-group G such that $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$ and $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Inn}(G)$. We also find necessary and sufficient conditions on a finite purely non-abelian p-group G such that $\mathrm{Var}(G) = \mathrm{Aut_z}(G)$ and $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Aut_z}(G)$. We obtain some conditions on a finite non-abelian p-group G such that $A(G)$ is a subgroup of $\mathrm{Aut}(G)$.

Chapter 1 contains the introductory part and some basic definitions. In chapter 2, we give necessary and sufficient conditions for a finite non-abelian p-group G such that $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$. We also give necessary and sufficient conditions for a finite non-abelian p-group G such that $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Inn}(G)$. Finally, we give three Gap algorithms. Algorithm 1 can be used to find the absolute center of a group G. Algorithm 2 checks whether an automorphism α of a group G is absolute central or not. Algorithm 3 can be used to find the size of all absolute central automorphisms of G.

Notice that $\mathrm{Var}(G)$ is a normal subgroup of $\mathrm{Aut}_z(G)$ and if $L(G) = Z(G)$, then $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$. A natural question which arises here is that if $L(G) < Z(G)$, then under what conditions $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$? In chapter 3, we give necessary and sufficient conditions for a finite purely non-abelian p-group G such that $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$. We also obtain necessary and sufficient conditions for a finite purely non-abelian p-group G such that $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Aut}_z(G)$. We give an example of a purely non-abelian group which satisfies the hypothesis of our first theorem.

Let $\mathrm{Aut}^{\Phi}(G)$ denote the group of all automorphisms α of G such that $x^{-1}\alpha(x) \in \Phi(G)$, the Frattini subgroup of G, for all $x \in G$. In [58], Muller using cohomological methods proved that if G is a finite p-group which is neither elementary abelian nor extraspecial, then $\mathrm{Aut}^{\Phi}(G)/\mathrm{Inn}(G)$ is a nontrivial normal p-subgroup of the group of outer(non-inner) automorphisms of G; or equivalently, if $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$, then G is either elementary abelian or extraspecial. In chapter 4, we obtain an alternate proof of the main result of Muller [58].

A finite p-group G is called Frattinian if $Z(M) \neq Z(G)$ for all maximal subgroups M of G. A Frattinian p-group G satisfying $C_G(Z(\Phi(G))) = \Phi(G)$ is called strongly Frattinian. In chapter 5, we study finite p-groups G for which G is an $A(G)$-group and prove three theorems. In the first theorem, we prove that if G is a finite non-abelian p-group, p an odd prime, such that $\mathrm{IA}(G) = \mathrm{Inn}(G)$ and $|G/G'| = p^2$, then G is an $A(G)$-group. In the second theorem, we prove that if G is a finite non-abelian p-group of coclass 3, where p is an odd prime, and G is strongly Frattinian, then G

is an $A(G)$-group. In the third theorem, we prove that if G be a finite non-abelian p-group and M_1, M_2 are any two distinct maximal abelain subgroups of G such that $M_1 \cap M_2 = Z(G)$, then G is an $A(G)$-group.

List of Research Papers

(1) M. Singh, *On the equality of certain subgroups of the automorphism groups of finite p-groups*, Mathematical Notes, **106(2)** (2019), 313-315.

(2) M. Singh and D. Gumber, *On the coincidence of the central and absolute central automorphism groups of finite p-groups*, Mathematical Notes, **107(5)** (2020), 863-866.

(3) M. Singh, *A note on p-automorphisms of finite p-groups* (Communicated).

(4) M. Singh, *On commuting automorphisms of finite p-groups* (Communicated).

Contents

CHAPTER 1

Introduction and Basics

1.1 — Introduction

Let G be an arbitrary group and let G', $Z(G)$ and $\Phi(G)$ respectively denote the commutator subgroup, the center and the Frattini subgroup of G. Let $\mathrm{Aut}(G)$ denote the full automorphism group of G. Automorphism groups of various groups are being studied since the very beginning of the subject. For a given group G, determining its automorphism group $\mathrm{Aut}(G)$, its various subgroups or relations between them are major problems in group theory. Various authors have done work on these problems (see for example [2, 4, 14, 15, 20, 36, 47, 49, 54, 60, 71, 80, 88]. For more details, one can see the survey articles by Helleloid [42] and Yadav [86].

An automorphism α of G is called a class-preserving automorphism if for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1} x g_x$; and is called an inner automorphism if for all $x \in G$, there exists a fix element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group $\mathrm{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\mathrm{Aut}_c(G)$ of all class-preserving automorphisms of G. An automorphism φ of G is called a central automorphism if it commutes with all inner automorphisms of G; or equivalently $g^{-1}\varphi(g) \in Z(G)$ for all $g \in G$. The group of all central automorphisms of G is denoted as $\mathrm{Aut}_z(G)$ and the group of all central automorphisms of G fixing $Z(G)$ elementwise is denoted by $C_{\mathrm{Aut}_z(G)}(Z(G))$.

Interest in the equality of two different automorphism groups dates back to 1908, when Hilton [44, p. 233] asked the following question:

Whether a non-abelian group can have an abelian group of isomorphisms (automorphisms), that is, when $\mathrm{Aut}(G) = \mathrm{Aut_z}(G)$***?***

An affirmative answer to this question was given by Miller [54] in 1913. He constructed a non-abelian group G of order 64 and generated by three elements for which $\mathrm{Aut}(G)$ is abelian of order 128. Miller's group of order 64 is the smallest non-abelian group with an abelian automorphism group. In 1975, Jonah and Konvisser [47] constructed a group G of order p^8 and generated by four elements such that $\mathrm{Aut}(G)$ is abelian. In 1982, Struik [80] also gave such an example. In 1987, Curran [23] gave a method for constructing further examples of non-abelian 2-groups which have abelian automorphism groups. In 1995, Morigi [57] raised the following question:

What is the minimal number of generators for a non-abelian p***-group having an abelian automorphism group, for*** p ***an odd prime?***

He himself settled this question and proved that the minimal number of generators for a p-group having an abelian automorphism group is four, for p an odd prime. In 1998, Ban and Yu [15] proved that there is no group G such that $\mathrm{Aut}(G)$ is an abelian p-group of order $\leq p^{11}$, where $p > 2$.

In 1911 Burnside [19, Note B] also posed the following question:

Does there exist a finite group G such that G has a non-inner class preserving automorphism?

In 1913, Burnside [20] himself gave a positive answer to his question. He constructed a group G of order p^6 isomorphic to the group W consisting of all 3×3 matrices of the form

$$M = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}$$

with x, y, z in the field $\mathbb{F}_{p^2}$ of p^2 elements, where p is an odd prime. For this group G, $\mathrm{Aut}_c(G) \neq \mathrm{Inn}(G)$. In 1947, Wall [82] constructed some smaller and simpler groups G for which $\mathrm{Aut}_c(G) \neq \mathrm{Inn}(G)$. The smallest group constructed by Wall is of order 32. In 1968, Sah [68] explored many basic properties of $\mathrm{Aut}_c(G)$ for a finite group G. It is a well known result of Gaschutz [33] that every finite p-group has a non-inner automorphism. Therefore, it becomes interesting topic to study when two different automorphism groups of a group are equal or isomorphic.

1.1.1

Following Bachmuth [14], we call an automorphism of G an IA-automorphism if it induces the identity automorphism on the abelianized group G/G'. Let $\mathrm{IA}(G)$ denote the group of all IA-automorphisms of G and let $C_{\mathrm{IA}(G)}(Z(G))$ denote its subgroup consisting of those IA-automorphisms which fix $Z(G)$ elementwise. For free metabelian groups G of rank 2, Bachmuth [14] in 1965, proved that $\mathrm{IA}(G) = \mathrm{Inn}(G)$. A group G is called semicomplete if $\mathrm{IA}(G) = \mathrm{Inn}(G)$. In 1969, Andreadakis [4] proved that a free product $A * B$ of two non-trivial groups is semicomplete if and only if both A and B are abelian. In 1981, Gupta [36] proved that if G is a

2-generator metabelian group, then $\mathrm{IA}(G)$ is always a metabelian group. In 2002, Panagopoulos [61], studied the semicompleteness of the direct product $G = A \times B$ of two groups A and B in relation to the semicompleteness of its direct factors. In 2011, Attar [7, Theorem 2.1] proved that if G is a finite p-group of class 2, then $\mathrm{IA}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic and $\mathrm{IA}(G) = C_{\mathrm{IA}(G)}(Z(G))$. In 2014, Singh, Gumber and Kalra [77] classified finitely generated nilpotent groups of class 2 for which $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$ and $C_{\mathrm{IA}(G)}(Z(G)) \simeq \mathrm{Inn}(G)$. In particular, they classified all finite nilpotent groups G of class 2 for which (i) $\mathrm{IA}(G) = \mathrm{Inn}(G)$ and (ii) $C_{\mathrm{IA}(G)}(Z(G)) = \mathrm{Inn}(G)$.

1.1.2

Let G be any group. For $g \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[g, \alpha] = g^{-1}\alpha(g)$ is called the autocommutator of g and α. Inductively, define $[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n]$, where $\alpha_i \in \mathrm{Aut}(G)$. Observe that

$$Z(G) = \{g \in G \mid \alpha(g) = g, \text{ for all } \alpha \in \mathrm{Inn}(G)\},$$

and

$$G' = \langle g^{-1}\alpha(g) \mid g \in G, \alpha \in \mathrm{Inn}(G)\rangle.$$

In an analogous manner, Hegarty [38] defined the absolute center $L(G)$ of G as

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

Let $L_1(G) = L(G)$, and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)\}.$$

The autocommutator subgroup G^* of G is defined as

$$G^* = \langle [g, \alpha] \mid g \in G, \alpha \in \mathrm{Aut}(G) \rangle.$$

It is easy to see that $L_n(G) \leq Z_n(G)$, the nth term of the upper central series of G, for all $n \geq 1$ and $G' \leq G^*$. A group G is called autonilpoent of class at most n if $L_n(G) = G$ for some natural number n. Analogous to central automorphisms group $\mathrm{Aut_z}(G)$, Hegarty [38] in 1994, defined absolute central automorphism group. He called an automorphism α of G an absolute central automorphism if $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$. The set of all absolute central automorphisms forms a normal subgroup of $\mathrm{Aut}(G)$ and is denoted by $\mathrm{Var}(G)$. In 2010, Moghaddam and Safa [55] obtained some results about the nature of absolute central automorphisms. In 2015, Nasrabadi and Farimani [59] proved that if G is a finite autonilpotent p-group of class 2, then $\mathrm{Var}(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic. In 2015, Singh and Gumber [78] generalized this result to a finitely generated group. In 2017, Attar [10] characterized finite non-abelian p-group G of arbitrary class for which $\mathrm{Var}(G) = \mathrm{Inn}(G)$. In 2016, Farimani and Nasrabadi [30] gave the necessary and sufficient conditions for a finite non-abelian p-group G such that $\mathrm{Var}(G) = C_{\mathrm{Var}(G)}(Z(G))$. In 2017, Hajizadeh and Nasrabadi [37] also obtained the necessary and sufficient conditions for a finite non-abelian autonilpotent p-group G such that each absolute central automorphism of G fixes the center elementwise.

In chapter 2, we give necessary and sufficient conditions for a finite non-abelian p-group G such that $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$. We also obtain necessary and sufficient conditions for a finite non-abelian p-group G such that $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Inn}(G)$. In Section 2.3, we give three Gap algorithms. Algorithm 1 can be used to find the absolute center of a group G. Algorithm 2 checks whether an automorphism α of a group G is absolute central or not. Algorithm 3 can be used to find the size of the group of absolute central automorphisms of G. The results of this Chapter have appeared in [73].

1.1.3

An automorphism φ of G is called a central automorphism if it commutes with all inner automorphisms of G; or equivalently $g^{-1}\varphi(g) \in Z(G)$, the center of G, for all $g \in G$. Let $\mathrm{Aut_z}(G)$ denote the group of all central automorphisms of G and $C_{\mathrm{Aut}_z(G)}(Z(G))$ denote the group of all those central automorphisms which fix the center of G elementwise. The central automorphism group can be as large as possible when all automorphisms are central, that means when $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$, and can be as small as possible when $\mathrm{Aut_z}(G) = \mathrm{Z}(\mathrm{Inn}(G))$. If G is abelian, then $\mathrm{Inn}(G)$ is trivial and hence $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$. If G is non-abelian and $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$, then since $\mathrm{Inn}(G)$ is abelian, G is a nilpotent group of class 2. Non-abelian p-groups G for which all automorphisms are central have been well studied. If $\mathrm{Aut}(G)$ is abelian, then necessarily $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$. Various authors have considered this situation (for example see [15, 29, 39, 40, 41, 45, 47, 56, 57]). If $\mathrm{Aut}(G)$ is non-abelian, even then all automorphisms may be central and this case has been explored, for example, in [22, 34, 48, 52].

In 2001, Curran and McCaughan [24] gave necessary and sufficient conditions for a finite p-group G such that $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$. They proved that for any finite p-group G, $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. In 2007, Attar [5] established that for any finite p-group G, $C_{\mathrm{Aut}_z(G)}(Z(G)) = \mathrm{Inn}(G)$ if and only if G is abelian or nilpotency class of G is 2 and $Z(G)$ is cyclic. In 2013, Azhdari and Malayeri [13, Theorem 2.3] generalized the result of Attar [5] and proved that if G is a finitely generated nilpotent group of class 2, then $C_{\mathrm{Aut}_z(G)}(Z(G)) \simeq \mathrm{Inn}(G)$ if and only if $Z(G)$ is infinite cyclic or $Z(G) \simeq C_m \times H \times \mathbb{Z}^r$, where $C_m \simeq \prod_{p\in\pi(G/Z(G))} Z(G)_p$, $H \simeq \prod_{p\notin\pi(G/Z(G))} Z(G)_p$, $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ is of finite exponent dividing m. In 2009, Yadav [85] gave necessary and sufficient conditions on a finite p-group G of nilpotency class 2 such that $\mathrm{Aut_z}(G) = C_{\mathrm{Aut}_z(G)}(Z(G))$. In 2011, Jafari [46] and in 2012, Attar [8] generalized this result. They have given necessary and sufficient conditions on a finite p-group G of arbitrary nilpotence class such that each central automorphism of G fixes the center elementwise.

Consider now the opposite extreme, the case in which we are specially interested, when the central automorphism group is as small as possible. The bound is $\mathrm{Z}(\mathrm{Inn}(G))$, because it is always contained in $\mathrm{Aut_z}(G)$. In 2004, Curran [25] studied finite p-groups G for which $\mathrm{Aut_z}(G) = \mathrm{Z}(\mathrm{Inn}(G))$. He proved that for any finite p-group G, if $\mathrm{Aut_z}(G) = \mathrm{Z}(\mathrm{Inn}(G))$, then $Z(G) \leq G'$ and $\mathrm{Z}(\mathrm{Inn}(G))$ must not be cyclic. It follows that if G is a finite p-group of maximal class, then $\mathrm{Aut_z}(G) \neq \mathrm{Z}(\mathrm{Inn}(G))$. Observe that if G is of nilpotence class 2, then $\mathrm{Z}(\mathrm{Inn}(G)) = \mathrm{Inn}(G)$ and thus, by the above mentioned result of Curran and McCaughan [24], $\mathrm{Aut_z}(G) = \mathrm{Z}(\mathrm{Inn}(G))$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. Therefore, to characterize all finite p-groups G for which $\mathrm{Aut_z}(G) = \mathrm{Z}(\mathrm{Inn}(G))$, one can assume that nilpotence class of G is bigger than 2 and G is not of maximal class. Of course, in such a case $|G| \geq p^5$. In 2013, Sharma and Gumber [71] characterized all finite p-groups G of order p^5, where p is any prime and of order p^6 for an odd prime p such that the center of the inner automorphism group of G is equal to the group of central automorphisms of G.

Note that for a finite p group G of class 2, we have

$$\mathrm{Inn}(G) \le \mathrm{Aut}_c(G) \le \mathrm{IA}(G) \le \mathrm{Aut}_{\mathrm{z}}(G) \le \mathrm{Aut}(G).$$

In 2013, Yadav [87] obtained certain results on a finite p-group G for which $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}_c(G)$. In 2015, Attar [5] has given a characterization for finite p-group G of nilpotency class 2 such that $\mathrm{Aut}_c(G) = \mathrm{IA}(G)$. In 2017, Attar [10] gave the necessary and sufficient conditions for a finite p-group G of class 2 such that $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{IA}(G)$.

Notice that $\mathrm{Var}(G) \le \mathrm{Aut}_z(G)$ and if $L(G) = Z(G)$, then $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$. A natural question which arises here is: under what conditions $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$ when $L(G) < Z(G)$? In chapter 3, we prove two theorems. In first theorem, we give necessary and sufficient conditions for a finite purely non-abelian p-group G such that $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$, and in second theorem, we obtain necessary and sufficient conditions for a finite purely non-abelian p-group G such that $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Aut}_z(G)$. As an application of our first result, we give an example of a purely non-abelian group G of order 128 for which $L(G) < Z(G)$ and $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$. The results of this chapter have appeared in [74].

Let $\mathrm{Aut}^{\Phi}(G) = \{\alpha \in \mathrm{Aut}(G) \mid x^{-1}\alpha(x) \in \Phi(G), \forall\, x \in G\}$. In chapter 4, we give an alternate proof of the result of Attar [6], and as a consequence, we obtain an alternate proof of the following main theorem of Muller [58]: If G is a finite p-group which is neither elementary abelian nor extraspecial, then $\mathrm{Aut}^{\Phi}(G)/\mathrm{Inn}(G)$ is a nontrivial normal p-subgroup of the group of outer(non-inner) automorphisms of G.

The results of this chapter would appear in [75].

1.1.4

An automorphism α of a group G is called a commuting automorphism if each element x in G commutes with its image $\alpha(x)$ under α. Let $A(G)$ denote the set of all commuting automorphisms of G. A group is said to be an $A(G)$ group if the set of all commuting automorphisms of G forms a subgroup of $\mathrm{Aut}(G)$. Commuting maps like derivations and automorphisms were first studied for various classes of rings [17, 28, 51, 66]. In the year 1984, Herstein [43] proposed the following problem to American Mathematical Monthly: If G is a simple non-abelian group, then prove that $A(G) = 1$. In 1986, Laffey [50] observed that $A(G) = 1$ provided G has no nontrivial abelian normal subgroup, while Pettet [62] proved that $A(G) = 1$ if $Z(G) = 1$ and $G' = G$. Deaconescu, Silberberg and Walls [27] in 2002, raised the following questions about $\mathrm{A}(G)$:

1. Is it true that the set $\mathrm{A}(G)$ is always a subgroup of $\mathrm{Aut}(G)$?

2. What conditions on G imply the equality $\mathrm{A}(G) = \mathrm{Aut_z}(G)$?

3. Is it true that $\mathrm{A}(G) = 1$ if and only if $\mathrm{Aut_z}(G) = 1$?

Regarding to question 1, Deaconescu *et al.* gave an example of a group of order 2^5 in which $\mathrm{A}(G)$ doesn't form a subgroup. In 2013, Vosooghpour and Malayeri [81] showed that minimum order of a non-$A(G)$ p-group is p^5. In 2013, Fouladi and Orfi [31] proved that if G is either a finite AC-group or a p-group of maximal class or a metacyclic p-group, then G is an $A(G)$-group. In 2015, Rai [64] gave some sufficient conditions on a finite p-group G such that $A(G)$ is a subgroup of $\mathrm{Aut}(G)$ and, as a consequence, proved that in a finite p-group G of co-class 2, where p is an odd prime, $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. In 2016, Singh and Gumber [79] gave very elementary and short proofs of main results of Rai and obtained some other related results. Recently in 2019, Shahrabi, Malayeri and Vosooghpour [70], and Rai [65] have given some conditions on a group G for which G is an $A(G)$-group.

In this chapter 5, we prove three theorems. In the first theorem, we prove that if G is a finite non-abelian p-group, p an odd prime, such that $\mathrm{IA}(G) = \mathrm{Inn}(G)$ and $|G/G'| = p^2$, then G is an $A(G)$-group. In the second theorem, we prove that if G is a finite non-abelian p-group of coclass 3, where p is an odd prime, and G is strongly Frattinian, then G is an $A(G)$-group. In the third theorem, we prove that if G be a finite non-abelian p-group and M_1 and M_2 are any two distinct maximal abelain subgroups of G such that $M_1 \cap M_2 = Z(G)$, then G is an $A(G)$-group. The results of this chapter would appear in [76].

1.2 — Basics

In this section, we give a quick review of some of the basic facts of group theory that would be used in the foregoing chapters. The definitions and proofs of results presented here can be found in any standard book on group theory. We, of course, suppose a familiarity of more basic group theoretic terms and concepts like abelian, cyclic, coset, normal subgroup, factor or quotient group, direct product et cetera.

Let G be an arbitrary group and X be a subset of G. The intersection of the family of subgroups of G which contain X is a subgroup of G and is denoted by $\langle X \rangle$. In other words, $\langle X \rangle$ is the smallest subgroup of G which contains X. The subgroup $\langle X \rangle$ is called the subgroup generated by X. If X is non-empty, then $\langle X \rangle$ contains every finite product of the type

$$x_1^{m_1}x_2^{m_2}\dots x_r^{m_r},\ \ r\geq 1,\ x_i\in X,\ m_i\in\mathbb{Z},$$

and conversely all such products form a subgroup of G containing X. It follows that $\langle X\rangle$ consists of all such products. A cyclic group is thus generated by a single element. We shall denote a cyclic group of order m by C_m. The rank of a group G is the smallest cardinality of a generating set of G and is denoted by $d(G)$. That is,

$$d(G) = min\{|X| : X\subseteq G, \langle X\rangle = G\}.$$

The least common multiple of the orders of the elements of a finite group G is called the exponent of G and is denoted by $exp(G)$.

The commutator of two elements $a, b\in G$ is the element $[a, b] = a^{-1}b^{-1}ab$ of G and the commutator subgroup or the derived subgroup G' of G is the subgroup of G generated by all commutators of G. That is,

$$G' = \langle [a, b] : a, b\in G\rangle.$$

It is easy to see that G' is a normal subgroup of G. Observe that G/G' is abelian and if G/H is abelian for a normal subgroup H of G, then $G'\subseteq H$. If X and Y are two subsets of G, then we define $[X, Y] = \langle [x, y] | x\in X, y\in Y\rangle$. Thus $[X, Y]$ is always a subgroup of G. If X and Y are both normal subgroups of G, then $[X, Y]$ is a normal subgroup of G. For $x\in G$, $[x, G]$ denotes the set of all commutators $[x, g]$, where $g\in G$. By $K(G)$ we denote the set of all commutators of G. The higher order commutator of $x_1, x_2, ..., x_k$ is defined inductively as

$$[x_1, x_2, ..., x_k] = [[x_1, x_2, ..., x_{k-1}, x_k]].$$

The followings are well known commutator identities

$$[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$$

and

$$[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$$

where $x, y, z \in G$ and will be frequently used in the thesis without any reference.

Given a group G and a subgroup H of G, a series from H to G is a finite sequence

$$H = G_0 \leq G_1 \leq \cdots \leq G_n = G \tag{1.1}$$

of subgroups of G where each G_i is a subgroup of its successor. If $H = 1$, we say that (1.1) is a series for G. The subgroups G_i in this series are called terms. The length of the series is the number of terms excluding G itself. The series (1.1) is called proper if no two of the terms are equal, that is, $G_i < G_{i+1}$ for $i = 0, 1, ..., n-1$. The series (1.1) is called subnormal series if each G_i is a normal subgroup of its successor, and is called normal series if each G_i is a normal subgroup of G. The quotient groups G_{i+1}/G_i are called the factor groups of the series. The normal series above is called a central series if for each i, $G_{i+1}/G_i \leq Z(G/G_i)$. Let $Z_0 = 1$ and let $Z_{i+1}/Z_i = Z(G/Z_i)$ for $i \geq 0$. Observe that Z_1 is the center of G and Z_{i+1}/Z_i, being the center of G/Z_i, is normal in G/Z_i and hence Z_{i+1} is normal in G for all $i \geq 0$. It follows that the series

$$1 = Z_0 \leq Z_1 \leq Z_2 \leq \cdots$$

is a central series of G. The subgroup Z_i is called the i-th center and this series is called the upper central series of G.

We define subgroups $\gamma_i(G)$, $i \geq 1$, of G by setting

$$\gamma_1(G) = G, \;\; \gamma_{i+1}(G) = [\gamma_i(G), G].$$

Observe that $\gamma_2(G) = G'$, each $\gamma_i(G)$ is normal in G and $\gamma_{i+1}(G) \leq \gamma_i(G)$. The series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \gamma_n(G) \geq \cdots$$

is called the lower central series of G. If the lower central series of a group G terminates in a finite number of steps at 1, and if c is the least natural number such that $\gamma_{c+1}(G) = 1$, then G is called a nilpotent group of class c. The class of a nilpotent group is denoted by $cl(G)$. Observe that if $cl(G) = 2$, then $G' \leq Z(G)$.

A maximal subgroup of G is a proper subgroup M such that there is no subgroup H of G with $M < H < G$. The intersection of all the maximal subgroups of G is the Frattini subgroup $\Phi(G)$ of G. If G has no maximal subgroup, then we set $\Phi(G) = G$. An element g of G is called a non-generator of G if whenever $G = \langle g, X\rangle$, then $G = \langle X\rangle$ where X is a subset of G. An interesting property of $\Phi(G)$ is that it is exactly the set of all non-generators of G.

Two elements a and b of G are called conjugate if there exists an element g of G such that $b = g^{-1}ag$. It is easily seen that "conjugacy" is an equivalence relation on G and therefore it partitions G into equivalence classes. The equivalence class that contains the element a of G is called the conjugacy class of a and is denoted as a^G. That is,

$$a^G = \{g^{-1}ag : g \in G\}.$$

Let H be a non-empty subset of G. The set of elements of G which commute with every element of H is called the centralizer of H in G, and is denoted as $C_G(H)$.

That is,

$$C_G(H) = \{g \in G : hg = gh \ \forall h \in H\}.$$

It is easy to see that $C_G(H)$ is a subgroup of G. If $H = \{h\}$ is singleton, then $C_G(\{h\})$ is simply denoted as $C_G(h)$.

A finite group G is called a purely non-abelian group if it has no non-trivial abelian direct factor. If $Z(G)$ is cyclic or if $Z(G) \leq \Phi(G)$, then G is purely non-abelian. Let G and H be two groups, and let $\alpha : G \longrightarrow H$ be a map. Then α is called a homomorphism from G to H if for all $g_1, g_2 \in G$,

$$\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2).$$

Let $\alpha : G \longrightarrow H$ be a homomorphism. Then

(*i*) α is called the trivial homomorphism if $\alpha(g) = 1$ for all $g \in G_1$.

(*ii*) α is called a monomorphism if it is injective.

(*iii*) α is called an epimorphism if it is surjective.

(*iv*) α is called an isomorphism if it is bijective.

(*v*) α is called an endomorphism if it is a homomorphism of G to itself.

(*vi*) α is called an automorphism if it is an isomorphism of G to itself.

The set of all automorphisms of G is a group under the usual operation of compositions of mappings. We call this group the full automorphism group of G and denote it by $\mathrm{Aut}(G)$.

Let A be an abelian group and let $\mathrm{Hom}(G, A)$ denote the set of all homomorphisms of G into A. For $f, g \in \mathrm{Hom}(G, A)$, define $fg(x) = f(x)g(x)$. Then

$\mathrm{Hom}(G, A)$ becomes an abelian group under this operation. If A, B, C are all finite abelian groups, then $\mathrm{Hom}(A, B \times C) \simeq \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)$ and $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, A)$. Also, $\mathrm{Hom}(C_m, C_n) \simeq C_d$, where $d = \gcd(m, n)$.

For a fix prime number p, a group G is called a p-group if order of every element of G is a power of p. If G is finite, then G is a p-group if and only if $|G|$, the order of G, is a power of p.

CHAPTER 2

On the equality of certain subgroups of the automorphism group of finite p-groups[1]

2.1 — Introduction

For any group G, Hegarty [38] considered the following definitions of $Z(G)$ and G':

$$Z(G) = \{g \in G \mid \alpha(g) = g, \forall \alpha \in \mathrm{Inn}(G)\}$$

$$G' = \langle g^{-1}\alpha(g) \mid g \in G, \alpha \in \mathrm{Inn}(G)\rangle,$$

and analogously defined the subgroups $L(G)$ and G^* of G as follows:

$$L(G) = \{g \in G \mid \alpha(g) = g, \forall \alpha \in \mathrm{Aut}(G)\}$$

$$G^* = \langle g^{-1}\alpha(g) \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle.$$

He called $L(G)$ the absolute centre of G and G^* the auto-commutator subgroup of G. Observe that $L(G) \leq Z(G)$ and $G' \leq G^*$. An automorphism α of G is called a central automorphism if $x^{-1}\alpha(x) \in Z(G)$ for all $x \in G$. The central automorphism fix the commutator subgroup G' of G elementwise and form a normal subgroup $\mathrm{Aut}_z(G)$ of $\mathrm{Aut}(G)$. Hegarty [38] analogously defined the absolute central automorphism of G

[1]The content of this chapter is published in Mathematical Notes

as follows:

An automorphism α of G is called an absolute central automorphism if it induces the identity automorphism on $G/L(G)$, or equivalently, $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$. Let $\mathrm{Var}(G)$ denote the group of all absolute central automorphisms of G, and let $C_{\mathrm{Var}(G)}(Z(G))$ denote the group of all absolute central automorphisms of G fixing $Z(G)$ elementwise. Clearly $\mathrm{Var}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ contained in $\mathrm{Aut}_z(G)$.

In 2010, Moghaddam and Safa [55] obtained some results about the nature of absolute central automorphisms. In 2015, Nasrabadi and Farimani [59] proved that if G is a finite autonilpotent p-group of class 2, then $\mathrm{Var}(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic. In 2015, Singh and Gumber [78] gave the necessary and sufficient conditions for a finitely generated non-abelian p-group G with $G' \leq Z(G)$ such that $\mathrm{Var}(G) \simeq \mathrm{Inn}(G)$ and, as a consequence, they obtained the necessary and sufficient conditions on a finite p-group G such that $\mathrm{Var}(G) = \mathrm{Inn}(G)$. In 2017, Attar [10] characterized finite non-abelian p-group G of arbitrary class for which $\mathrm{Var}(G) = \mathrm{Inn}(G)$. In 2016, Farimani and Nasrabadi [30] gave the necessary and sufficient conditions for a finite non-abelian p-group G such that $\mathrm{Var}(G) = C_{\mathrm{Var}(G)}(Z(G))$. They proved that if G is a finite non-abelian p-group, then $\mathrm{Var}(G) = C_{\mathrm{Var}(G)}(Z(G))$ if and only if $Z(G)G' \subseteq G'L(G)G^{p^n}$, where $p^n = \exp(L(G))$. In 2017, Hajizadeh and Nasrabadi [37] also obtained the necessary and sufficient conditions for a finite non-abelian autonilpotent p-group G such that each absolute central automorphism of G fixes the center elementwise.

An automorphism α of G is called an IA-automorphism if $x^{-1}\alpha(x) \in G'$ for all $x \in G$. Let $\mathrm{IA}(G)$ denote the group of all IA-automorphisms of G, and let $C_{\mathrm{IA}(G)}(Z(G))$ denote the group of all IA-automorphisms of G fixing $Z(G)$ element-wise. A group G is called semicomplete if $\mathrm{IA}(G) = \mathrm{Inn}(G)$. Many mathematicians, such as Bachmuth [14], Andreadakis [4], Gupta [36], Panagopoulos [61] and Attar [7] investigated semicomplete groups as explained in the introduction of chapter 1. Rai [63] gave the necessary and sufficient conditions for finite p-group G such that $C_{\mathrm{IA}(G)}(Z(G)) = \mathrm{Aut}_z(G)$. He also gave necessary and sufficient conditions for a finite p-group G of nilpotency class 2 such that $C_{\mathrm{IA}(G)}(Z(G)) = \mathrm{Inn}(G)$.

In this chapter, we give necessary and sufficient conditions for a finite p-group G such that $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$. We also obtain necessary and sufficient conditions for a finite p-group G such that $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Inn}(G)$. We also give three Gap algorithms. Algorithm 1 can be used to find the absolute center of a group G. Algorithm 2 checks whether an automorphism α of a group G is absolute central or not. Algorithm 3 can be used to find the size of $\mathrm{Var}(G)$.

2.2 — Main Results

Let $\mathrm{Hom}(G, A)$ denote the group of all homomorphisms of G into an abelian group A. The following two well-known lemmas will be used very frequently without further referring.

Lemma 2.2.1 *Let A, B and C be finite abelian groups. Then*

(i) $\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C)$, *and*

(ii) $\mathrm{Hom}(A, B \times C) \simeq \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)$.

Lemma 2.2.2 $\mathrm{Hom}(C_n, C_m) \simeq C_d$, *where C_i denotes the cyclic group of order i and d is the greatest common divisor of n and m.*

The following lemma is also well known in literature and can be proved using elementary arguments.

Lemma 2.2.3 *Let A and B be finite abelian p-groups such that $exp(A)\ |exp(B)$, then $\mathrm{Hom}(A, B) \simeq A$ if and only if B is cyclic.*

Lemma 2.2.4 ([57, Lemma 0.4]) *Let G be a p-group of class 2. Then*

(*i*) $G' \leq Z(G)$

(*ii*) $\exp(G') = \exp(G/Z(G)$.

Let H and K be two normal subgroups of G. By $\mathrm{Aut}^H(G)$, we denote the subgroup of $\mathrm{Aut}(G)$ consisting of all the automorphisms which centralize G/H and by $\mathrm{Aut}_K(G)$, we denote the subgroup of $\mathrm{Aut}(G)$ consisting of all the automorphisms which centralize K. We denote the intersection $\mathrm{Aut}^H(G) \cap \mathrm{Aut}_K(G)$ by $\mathrm{Aut}_K^H(G)$. The following lemma is a little modification of arguments of [3, Lemma 3].

Lemma 2.2.5 *Let G be any group and K be a central subgroup of G contained in a normal subgroup H of G. Then* $\mathrm{Aut}_H^K(G) \simeq \mathrm{Hom}(G/H, K)$.

Proof. For any $\mu \in \mathrm{Aut}_H^K(G)$, define the map $\psi_\mu : G/H \longrightarrow K$ as $\psi_\mu(gH) =$

$g^{-1}\mu(g)$. Now

$$\begin{aligned}\psi_\mu((g_1H)(g_2H)) &= \psi_\mu(g_1g_2H)\\ &= (g_1g_2)^{-1}\mu(g_1g_2)\\ &= g_2^{-1}g_1^{-1}\mu(g_1)\mu(g_2)\\ &= (g_1^{-1}\mu(g_1))(g_2^{-1}\mu(g_2))\\ &= \psi_\mu(g_1H)\psi_\mu(g_2H).\end{aligned}$$

Thus ψ_μ is a homomorphism. Define the map

$$\psi : \mathrm{Aut}_H^K(G) \longrightarrow \mathrm{Hom}(G/H, K)$$

as $\psi(\mu) = \psi_\mu$. Let $\mu_1, \mu_2 \in \mathrm{Aut}_H^K(G)$. Then $g^{-1}\mu_2(g) \in K$ for all $g \in G$ and $\mu_1(g^{-1}\mu_2(g)) = g^{-1}\mu_2(g)$. Therefore we have

$$\begin{aligned}\psi_{\mu_1\mu_2}(gH) &= g^{-1}\mu_1\mu_2(g)\\ &= g^{-1}\mu_1(\mu_2(g))\\ &= g^{-1}\mu_1(gg^{-1}\mu_2(g))\\ &= g^{-1}\mu_1(g)\mu_1(g^{-1}\mu_2(g))\\ &= g^{-1}\mu_1(g)g^{-1}\mu_2(g)\\ &= \psi_{\mu_1}(gH)\psi_{\mu_2}(gH).\end{aligned}$$

Thus ψ is a homomorphism. Now let $\psi(\mu_1) = \psi(\mu_2)$, then $g^{-1}\mu_1(g) = g^{-1}\mu_2(g)$, for all $g \in G$, which implies that $g_1 = g_2$ and thus ψ is a monomorphism. For any $\tau \in \mathrm{Hom}(G/H, K)$, define the map $\mu : G \longrightarrow G$ as $\mu(g) = g\tau(gH)$ for all $g \in G$.

Since $K \leq Z(G)$,

$$\begin{aligned} \mu(g_1g_2) &= g_1g_2\tau(g_1g_2H) \\ &= g_1g_2\tau((g_1H)(g_2H)) \\ &= g_1g_2\tau(g_1H)\tau(g_2H) \\ &= (g_1\tau(g_1H))(g_2\tau(g_2H)) \\ &= \mu(g_1)\mu(g_2). \end{aligned}$$

Now let $g \in \operatorname{Ker}\mu$, then $\mu(g) = 1$ which implies that $g^{-1} = \tau(gH) \in K \leq H$. Thus $1 = \mu(g) = g\tau(gH) = g\tau(H) = g$. Thus μ is a monomorphism. Next we prove that μ is onto. Observe that $Im\tau \subseteq Im\mu$, since if $y \in Im\tau$ then for some $g \in G$, $y = \tau(gH) \in K \leq H$. Now $\mu(y) = y\tau(yH) = y$ and so $y \in Im\mu$. It follows that for every element $g \in G$, we have $g = \mu(g)(\tau(gH))^{-1} \in Im\mu$, which implies that $G \subseteq Im\mu$ and therefore μ is onto. Thus μ is an automorphism of G inducing identity on both H and G/K and $\psi(\mu) = \tau$. Hence ψ is onto as well. □

Let G be a non-abelian finite p-group such that $G' \leq L(G)$. Let

$$\begin{aligned} G/Z(G) &\simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_l}}, \\ G' &\simeq C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_m}}, \text{ and} \\ L(G) &\simeq C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_n}} \end{aligned}$$

be the cyclic decompositions of respective abelian groups, where $a_i \geq a_{i+1}$, $b_i \geq b_{i+1}$ and $c_i \geq c_{i+1}$ are positive integers. Since G' is a subgroup of $L(G)$, $m \leq n$ and $b_j \leq c_j$ for all j, $1 \leq j \leq m$.

Theorem 2.2.6 *Let G be a finite non-abelian p-group. Then $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$ if and only if $G' = L(G)$ or $G' < L(G)$, $m = n$ and $a_1 = b_s$, where s is the largest integer between 1 and m such that $b_s < c_s$.*

Proof. Observe that if $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$, then for any commmutator $[a,b] \in G'$, $[a,b] = a^{-1}b^{-1}ab = a^{-1}I_b(a) \in L(G)$, where $I_b(a)$ is the inner automorphism of G induced by b, and thus $G' \leq L(G)$. Therefore, by Lemma 2.2.5 we have $C_{\mathrm{IA}(G)}(Z(G)) = \mathrm{Aut}^{G'}_{Z(G)}(G) \simeq \mathrm{Hom}(G/Z(G), G')$ and $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Aut}^{L(G)}_{Z(G)}(G) \simeq \mathrm{Hom}(G/Z(G), L(G))$. Then $|C_{\mathrm{IA}(G)}(Z(G))| = |\,\mathrm{Hom}(G/Z(G), G')|$ and $|C_{\mathrm{Var}(G)}(Z(G))| = |\,\mathrm{Hom}(G/Z(G), L(G))|$. First suppose that $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$ and $G' < L(G)$. Then

$$|\,\mathrm{Hom}(G/Z(G), G')| = |\,\mathrm{Hom}(G/Z(G), L(G))|.$$

It follows that

$$\prod_{i=1}^{l}\prod_{j=1}^{m} p^{\min\{a_i,b_j\}} = \prod_{i=1}^{l}\prod_{j=1}^{n} p^{\min\{a_i,c_j\}}.$$

Now $m \leq n$ and $b_j \leq c_j$ for each j, $1 \leq j \leq m$, therefore $\min\{a_i, b_j\} \leq \min\{a_i, c_j\}$ for all i, $1 \leq i \leq l$ and for all j, $1 \leq j \leq m$. If $m < n$, then

$$\begin{aligned}
|\,\mathrm{Hom}(G/Z(G), G')| &= |\,\mathrm{Hom}(G/Z(G), C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_m}})| \\
&\leq |\,\mathrm{Hom}(G/Z(G), C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_m}})| \\
&< |\,\mathrm{Hom}(G/Z(G), C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_m}})| \\
&\qquad |\,\mathrm{Hom}(G/Z(G), C_{p^{c_{m+1}}} \times C_{p^{c_{m+2}}} \times \cdots \times C_{p^{c_n}})| \\
&= |\,\mathrm{Hom}(G/Z(G), C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_n}})| \\
&= |\,\mathrm{Hom}(G/Z(G), (L(G))|,
\end{aligned}$$

which is not so. Thus $m = n$ and $\min\{a_i, b_j\} = \min\{a_i, c_j\}$ for all i, $1 \leq i \leq l$ and for all j, $1 \leq j \leq m$. Since $G' < L(G)$, there exists some j between 1 and m such that $b_j < c_j$. Let s be the largest integer between 1 and m such that $b_s < c_s$.

If $a_1 > b_s$, then $b_s = \min\{a_1, b_s\} = \min\{a_1, c_s\} > b_s$, a contradiction. Therefore $a_1 \leq b_s$. Since $\mathrm{cl}(G) = 2$, $a_1 = \exp(G/Z(G)) = \exp(G') = b_1$ by Lemma 2.2.4. Therefore $a_1 \leq b_s \leq b_{s-1} \leq ... \leq b_1 = a_1$. Thus $a_1 = b_s$.

Conversely, if $G' = L(G)$, then $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$. Suppose that $G' < L(G)$, $m = n$ and $a_1 = b_s$, where s is the largest integer between 1 and m such that $b_s < c_s$.

Now

$$|C_{\mathrm{IA}(G)}(Z(G))| = |\mathrm{Hom}(G/Z(G), G')| = \prod_{i=1}^{l}\prod_{j=1}^{m} p^{\min\{a_i,b_j\}},$$

and

$$|C_{\mathrm{Var}(G)}(Z(G))| = |\mathrm{Hom}(G/Z(G), L(G))| = \prod_{i=1}^{l}\prod_{j=1}^{n} p^{\min\{a_i,c_j\}}.$$

Observe that $a_i \leq b_j \leq c_j$ for all i, $1 \leq i \leq l$ and for all j, $1 \leq j \leq s$, and $b_j = c_j$ for all j, $s+1 \leq j \leq m$. It follows that

$$|C_{\mathrm{IA}(G)}(Z(G))| = \prod_{i=1}^{l}\prod_{j=1}^{m} p^{\min\{a_i,b_j\}} = p^{s(a_1+a_2+\cdots+a_l)} \prod_{i=1}^{l}\prod_{j=s+1}^{m} p^{\min\{a_i,b_j\}},$$

and

$$|C_{\mathrm{Var}(G)}(Z(G))| = \prod_{i=1}^{l}\prod_{j=1}^{m} p^{\min\{a_i,c_j\}} = p^{s(a_1+a_2+\cdots+a_l)} \prod_{i=1}^{l}\prod_{j=s+1}^{m} p^{\min\{a_i,b_j\}}.$$

Thus $|C_{\mathrm{IA}(G)}(Z(G))| = |C_{\mathrm{Var}(G)}(Z(G))|$ and hence $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$, because $C_{\mathrm{IA}(G)}(Z(G))$ is a subgroup of $C_{\mathrm{Var}(G)}(Z(G))$. □

A non-abelian group G of order p^n is of maximal class if $\mathrm{cl}(G) = n - 1$.

The following corollaries are the consequence of above theorem.

Corollary 2.2.7 *Let G be a finite non-abelian p-group of maximal class. Then $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$ if and only if $G' = L(G) = Z(G)$.*

Corollary 2.2.8 *Let G be a finite non-abelian p-group such that $\exp(L(G)) = p$. Then $C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$ if and only if $G' = L(G)$.*

Theorem 2.2.9 *Let G be a finite non-abelian p-group. Then $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Inn}(G)$ if and only if $G' \le L(G)$ and $L(G)$ is cyclic.*

Proof. First suppose that $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Inn}(G)$. Let $g \in G$, then the inner automorphism θ_g induced by g is absolute central automorphism and $[x,g] = x^{-1}g^{-1}xg = x^{-1}\theta_g(x) \in L(G),\ x \in G$. It follows that $G' \le L(G)$. Therefore, the exponent of $L(G)$ is greater than or equal to the exponent of G'. Let the exponent of $L(G)$ be p^e. If possible, suppose that $L(G)$ is not cyclic. Then $L(G) = C_{p^e} \times M$, where C_{p^e} is a cyclic subgroup of order p^e and M is some non-trivial proper subgroup of $L(G)$. Now by Lemma 2.2.3, we have

$$\begin{aligned}
|\operatorname{Hom}(G/Z(G), L(G))| &= |\operatorname{Hom}(G/Z(G), C_{p^e} \times M)| \\
&= |\operatorname{Hom}(G/Z(G), C_{p^e}) \times \operatorname{Hom}(G/Z(G), M)| \\
&= |\operatorname{Hom}(G/Z(G), C_{p^e})||\operatorname{Hom}(G/Z(G), M)| \\
&> |G/Z(G)| \\
&= |\operatorname{Inn}(G)|.
\end{aligned}$$

Thus $|C_{\mathrm{Var}(G)}(Z(G))| = |\operatorname{Hom}(G/Z(G), L(G))| > |\operatorname{Inn}(G)|$. This contradicts the given hypothesis. Hence $L(G)$ must be cyclic.

Conversely, suppose that $G' \le L(G)$ and $L(G)$ is cyclic. Then G is of class 2 because $L(G) \le Z(G)$. Therefore, by Lemma 2.2.4, we have $\exp(G/Z(G)) = \exp(G') = |G'|$. Thus $\exp(G/Z(G))$ divides $\exp|L(G)|$. It follows from Lemma 2.2.3

that

$$|C_{\mathrm{Var}(G)}(Z(G))| = |\operatorname{Hom}(G/Z(G), L(G))| = |G/Z(G)| = |\operatorname{Inn}(G)|.$$

Since $G' \leq L(G)$, $\operatorname{Inn}(G) \leqslant C_{\mathrm{Var}(G)}(Z(G))$ and hence $C_{\mathrm{Var}(G)}(Z(G)) = \operatorname{Inn}(G)$.

□

Example 2.2.10 Let

$$G = \langle x, y | x^8 = y^2 = 1, y^{-1}xy = x^5 \rangle$$

be a group of order 16. Observe that $\mathrm{cl}(G) = 2$, $\Phi(G) = Z(G) = \langle x^2 \rangle \simeq C_4$ and $G' = L(G) \simeq C_2$. Therefore $C_{\mathrm{Var}(G)}(Z(G)) \simeq \operatorname{Hom}(G/Z(G), L(G)) \simeq \operatorname{Hom}(C_2 \times C_2, C_2) \simeq C_2 \times C_2$ and thus $|C_{\mathrm{Var}(G)}(Z(G))| = 4$. Also $|\operatorname{Inn}(G)| = |G/Z(G)| = 4$. Since $\operatorname{Inn}(G) \leq C_{\mathrm{IA}(G)}(Z(G)) = C_{\mathrm{Var}(G)}(Z(G))$, $C_{\mathrm{Var}(G)}(Z(G)) = \operatorname{Inn}(G)$.

2.3 — Algorithms

In this section, we give three Gap algorithms. Algorithm 1 can be used to find the absolute center of a group G. Algorithm 2 checks whether an automorphism α of a group G is absolute central or not. Algorithm 3 can be used to find the size of all absolute central automorphisms of G.

Algorithm 1 To find the absolute center of a group.

```
autocenter:=function(G)
local z,au,N,x;
z:=Centre(G);
au:=AutomorphismGroup(G);
N:=TrivialSubgroup(G);
for x in Elements(z)
if not x in N and ForAll(GeneratorsOfGroup(au),a → ImagesRepresentative(a,x)=x)
N:=ClosureGroup(N,x);
fi;
od;
return N;
end;
```

Algorithm 2 To check wether α is absolute central or not.

```
IsAlmostCentral:=function(G)
local L;
result:=[ ];
L:=autocenter(G);
aut:=AutomorphismGroup(G);
for a in aut do
result:=ForAll(GeneratorsOfGroup(G),g→g^-1*ImagesRepresentative(a,g) in L);
if result = true then
fi;
od;
end;
```

Algorithm 3 To calculate the size of group of absolue central automorphisms.

```
IsAlmostCentral:=function(G)
local L,count;
result:=[ ];
count:=0;
L:=autocenter(G);
aut:=AutomorphismGroup(G);
for a in aut do
result:=ForAll(GeneratorsOfGroup(G),g→g^-1*ImagesRepresentative(a,g) in L);
if result = true then
count:=count+1;
fi;
od;
Print(count);
end;
```

CHAPTER 3

On equality of central and absolute central automorphism groups of finite p-groups[1]

3.1 — Introduction

In 2001, Curran and McCaughan [24] characterized finite p-groups G for which $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$. They proved that if G is a finite p-group, then $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. In 2009, Yadav [85] gave necessary and sufficient conditions on a finite p-group G of nilpotency class 2 such that $\mathrm{Aut_z}(G) = C_{\mathrm{Aut}_z(G)}(Z(G))$. In 2011, Jafari [46] generalized this result of Yadav. He proved that if G is a finite p-group, then $\mathrm{Aut_z}(G) = C_{\mathrm{Aut_z}(G)}(Z(G))$ if and only if $Z(G)G' \subseteq G^{p^n}G'$ where $\exp(Z(G)) = p^n$. In 2012, Attar [8] also generalized this result of Yadav. He gave necessary and sufficient conditions on a finite p-group G of arbitrary nilpotence class such that each central automorphism of G fixes the center of G elementwise.

In 2004, Curran [25] studied finite p-groups G for which $\mathrm{Aut_z}(G) = \mathrm{Z}(\mathrm{Inn}(G))$. He proved that for any finite p-group G, if $\mathrm{Aut_z}(G) = \mathrm{Z}(\mathrm{Inn}(G))$, then $Z(G) \leq G'$

[1]The content of this chapter is published in Mathematical Notes

and Z(Inn(G)) must not be cyclic. In 2013, Sharma and Gumber [71] characterized all finite p-groups G of order p^5, where p is any prime and of order p^6 for an odd prime p such that the center of the inner automorphism group of G is equal to the group of central automorphisms of G. In 2015, Gumber and Kalra [35] gave necessary and sufficient conditions on G for which $\mathrm{Aut_z}(G) = \mathrm{Z}(\mathrm{Inn}(G))$ in the case when Z(G) is cyclic. They also characterized finite p-groups G of co-class up to 4 and groups of order up to p^7 for which $\mathrm{Aut_z}(G)$ is minimal. In 2018, Sharma, Kalra and Gumber [72] characterized all finite p-groups of order p^7, where p is a prime such that $C_{\mathrm{Aut_z}(G)}(Z(G)) = \mathrm{Z}(\mathrm{Inn}(G)) < \mathrm{Aut_z}(G)$.

In 2013, Yadav [87, Theorem A] gave necessary and sufficient conditions on a finite p-group G of class 2 such that $\mathrm{Aut}_c(G) = \mathrm{Aut_z}(G)$. He also proved that, if G is a finite p-group of class 2 such that $\mathrm{Aut}_c(G) = \mathrm{Aut_z}(G)$, then $d(G)$ is even. In 2013, Kalra and Gumber [49] proved that if G is a finite non-abelian p-group, then $\mathrm{Aut}_c(G) = \mathrm{Aut_z}(G)$ if and only if $\mathrm{Aut}_c(G) \simeq \mathrm{Hom}(G/Z(G), G')$ and $G' = Z(G)$. As a consequence of this result, they obtained an easy and short proof of the main result of Curran and McCaughan [24] which states that: If G is a finite p-group, then $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. They also classified all finite p-groups G such that $\mathrm{Aut}_c(G) = \mathrm{Aut_z}(G)$ when $Z(G)$ is cyclic or elementary abelian. And consequently, they characterized all finite p-groups of order $p^n (n \leq 7)$ such that $\mathrm{Aut}_c(G) = \mathrm{Aut_z}(G)$. In 2017, Attar [10] gave the necessary and sufficient conditions for a finite p-group G of class 2 such that $\mathrm{Aut_z}(G) = \mathrm{IA}(G)$.

In the recent past, there has been some interest on the equality of Var(G) with the subgroups of Aut(G). See for example ([37],[59],[78]). Notice that $\mathrm{Var}(G) \leq$

$\mathrm{Aut}_z(G)$ and if $L(G) = Z(G)$, then $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$. A natural question which arises here is as follow:

If $L(G) < Z(G)$***, then under what conditions*** $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$?

In this chapter, we answer this question. We give necessary and sufficient conditions for a finite purely non-abelian p-group G such that $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$. We also obtain necessary and sufficient conditions for a finite purely non-abelian p-group G such that $C_{\mathrm{Var}(G)}(Z(G)) = \mathrm{Aut}_z(G)$. We also give an example of a purely non-abelian group of order 128 that satisfies the hypothesis of our first theorem.

3.2 — Main Results

A non-abelian group G is called purely non-abelian if it has no nontrivial abelian direct factor. Observe that if $Z(G) \leq \Phi(G)$, then G is purely non-abelian.

Lemma 3.2.1 ([1, Theorem 1]) *If G is a purely non-abelian group, then there is a one-to-one correspondence between* $\mathrm{Aut}_z(G)$ *and* $\mathrm{Hom}(G/G', Z(G))$.

Lemma 3.2.2 ([24, Lemma D]) *Let A, B, C and D be finite abelian p-groups such that A is isomorphic to a proper subgroup of B and C is isomorphic to a proper subgroup of D. Then* $|\mathrm{Hom}(A, C)| < |\mathrm{Hom}(B, D)|$.

Lemma 3.2.3 ([55, Lemma 1]) *If G is an arbitrary group, then* $\mathrm{Var}(G)$ *acts trivially on the subgroup $E(G)$ of G.*

Proof. Let $f \in \mathrm{Var}(G)$ then $x^{-1}f(x) \in L(G)$ for all $x \in G$ and so $f(x) = xy$, for some $y \in L(G)$. Now, let $g \in C_{\mathrm{Aut}(G)}(\mathrm{Var}(G))$. Since g commute with f, therefore we have

$$\begin{aligned} f([x,g]) &= f(x^{-1}g(x)) = f(x)^{-1}f(g(x)) \\ &= f(x)^{-1}(fg)(x) = (xy)^{-1}(gf)(x) \\ &= y^{-1}x^{-1}g(f(x)) = y^{-1}x^{-1}g(xy) \\ &= y^{-1}x^{-1}g(x)g(y) = y^{-1}x^{-1}g(x)y \\ &= x^{-1}g(x) = [x,g]. \end{aligned}$$

□

Lemma 3.2.4 ([55, Theorem C]) *Let G be a purely non-abelian finite group, then* $\mathrm{Var}(G) \simeq \mathrm{Hom}(G, L(G))$.

Proof. Consider the map $\sigma : \mathrm{Var}(G) \longrightarrow \mathrm{Hom}(G, L(G))$ defined by

$$\sigma(f) = \sigma_f,$$

where $\sigma_f : G \longrightarrow L(G)$ is defined by $\sigma_f(g) = g^{-1}f(g)$. Clearly, σ_f is a well defined homomorphism and hence $\sigma_f \in \mathrm{Hom}(G, L(G))$. One can check that the map σ is well defined and monomorphism. It is surjective, for if $h \in \mathrm{Hom}(G, L(G))$ then the map $f : G \longrightarrow G$ given by $f(g) = gh(g)$ is an endomorphism of G, and also $g^{-1}f(g) = h(g) \in L(G) \leq Z(G)$, which implies that f is a central endomorphism and hence f is a normal endomorphism, that is f commutes with every inner automorphism of G. Clearly, the finite group G satisfies the maximum and minimum conditions properties for its normal subgroups. Now, since f is a normal endomorphism, it implies that $Im(f)$ is normal in G. It is easy to see that f^n is also a normal endomorphism and hence Imf^n is a normal subgroup of G, for all $n \geq 1$. Thus the following two series terminate.

$$\mathrm{Ker}f \leq \mathrm{Ker}f^2 \leq \cdots,$$

$$Imf \geq Imf^2 \geq \cdots.$$

Hence, there exist $r \in N$ such that

$$\mathrm{Ker}f^r = \mathrm{Ker}f^{r+1} = \cdots = I,$$

$$Imf^r = Imf^{r+1} = \cdots = K.$$

Now, we show that $G = IK$. For every element $g \in G$, $f^r(g) \in Imf^r = Imf^{2r}$ and so $f^r(g) = f^{2r}(h)$, for some $h \in G$. Therefore $f^r(g) = f^r(f^r(h))$ which implies that $(f^r(h))^{-1}g \in \mathrm{Ker}f^r = K$, that is $g \in IK$ and hence $G = IK$. Clearly, $I \cap K = \langle 1 \rangle$ and therefore $G = I \times K$, that is to say

$$G = \mathrm{Ker}f^r \times Imf^r.$$

Clearly, $\mathrm{Ker}f^r$ is abelian, as $\mathrm{Ker}f^n \leq Z(G)$, for all $n \in N$. By the assumption, G is purely non-abelian and hence $\mathrm{Ker}f^r = \langle 1 \rangle$, which implies that $\mathrm{Ker}f = \langle 1 \rangle$. This shows that f is injective and $Imf^r = G$, which shows that $Imf = G$, that is f is surjective and hence $f \in \mathrm{Var}(G)$. Clearly $\sigma(f) = \sigma_f = h$, which implies that σ is surjective and so

$$\mathrm{Var}(G) \simeq \mathrm{Hom}(G, L(G)).$$

□

Let

$$C_{\mathrm{Aut}(G)}(\mathrm{Var}(G)) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha\beta = \beta\alpha, \forall \beta \in \mathrm{Var}(G)\}$$

denote the centralizer of $\mathrm{Var}(G)$ in $\mathrm{Aut}(G)$. In [55], Moghaddam and Safa defined

$$E(G) = [G, C_{\mathrm{Aut}(G)}(\mathrm{Var}(G))] = \langle g^{-1}\alpha(g) \mid g \in G, \alpha \in C_{\mathrm{Aut}(G)}(\mathrm{Var}(G))\rangle.$$

One can easily see that $E(G)$ is a characteristic subgroup of G containing the derived group $G' = [G, \mathrm{Inn}(G)]$, and each absolute central automorphism fixes $E(G)$ elementwise by Lemma 3.2.3.

Let

$$\begin{array}{rcl} G/E(G) & \simeq & C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}, \\ G/G' & \simeq & C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_l}}, \\ L(G) & \simeq & C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_m}}, \text{ and} \\ Z(G) & \simeq & C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_n}} \end{array}$$

be the cyclic decompositions of respective abelian groups, where $a_i \geq a_{i+1}$, $b_i \geq b_{i+1}$, $c_i \geq c_{i+1}$ and $d_i \geq d_{i+1}$ are positive integers. Since $G/E(G)$ is a quotient group of G/G', $k \leq l$ and $a_i \leq b_i$ for all i, $1 \leq i \leq k$.

Theorem 3.2.5 *Let G be a finite purely non-abelian p-group. Then* $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$ *if and only if either $L(G) = Z(G)$ or $G' = E(G)$, $m = n$ and $a_1 \leq c_s$, where s is the largest integer between 1 and m such that $c_s < d_s$.*

Proof. Suppose that $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$ and $L(G) < Z(G)$. Notice that $L(G)$ is non-trivial, because if $L(G) = 1$, then $\mathrm{Var}(G)$ and hence $\mathrm{Aut}_z(G)$ is trivial, which is not possible for a finite p-group. Since G is purely non-abelian, $|\mathrm{Aut}_z(G)| = |\mathrm{Hom}(G/G', Z(G))|$ by Lemma 3.2.1. Also, since $\mathrm{Var}(G)$ fixes $E(G)$ elementwise by Lemma 3.2.3, it follows from Lemma 3.2.4 that $|\mathrm{Var}(G)| = |\mathrm{Hom}(G/E(G), L(G))|$. If $G/E(G)$ is a proper quotient group of G/G', then $|\mathrm{Hom}(G/E(G), L(G))| < |\mathrm{Hom}(G/G', Z(G))|$ by Lemma 3.2.2, which is not so. Thus $G/E(G) = G/G'$.

It follows that $G' = E(G)$, because $G' \leq E(G)$. Therefore,

$$|\operatorname{Hom}(G/E(G), L(G))| = |\operatorname{Hom}(G/E(G), Z(G))|,$$

and hence

$$\prod_{i=1}^{k}\prod_{j=1}^{m} p^{\min\{a_i,c_j\}} = \prod_{i=1}^{k}\prod_{j=1}^{n} p^{\min\{a_i,d_j\}}.$$

Since $m \leq n$ and $c_j \leq d_j$ for each j, $\min\{a_i, c_j\} \leq \min\{a_i, d_j\}$ for all i and j. If $m < n$, then $|\operatorname{Hom}(G/E(G), L(G))| < |\operatorname{Hom}(G/E(G), Z(G))|$, which is not so. Thus $m = n$ and hence $\min\{a_i, c_j\} = \min\{a_i, d_j\}$ for all i and j. Since $L(G) < Z(G)$, there exists some j between 1 and m such that $c_j < d_j$. Let s be the largest integer between 1 and m such that $c_s < d_s$. If $a_1 > c_s$, then $c_s = \min\{a_1, c_s\} = \min\{a_1, d_s\}$, which is not possible. Therefore $a_1 \leq c_s$.

Conversely, if $L(G) = Z(G)$, then $\operatorname{Var}(G) = \operatorname{Aut}_z(G)$. Suppose that $G' = E(G)$, $m = n$ and $a_1 \leq c_s$, where s is the largest integer between 1 and m such that $c_s < d_s$. Since G is purely non-abelian,

$$|\operatorname{Var}(G)| = |\operatorname{Hom}(G/E(G), L(G))| = \prod_{i=1}^{k}\prod_{j=1}^{m} p^{\min\{a_i,c_j\}}$$

and

$$|\operatorname{Aut}_z(G)| = |\operatorname{Hom}(G/G', Z(G))| = |\operatorname{Hom}(G/E(G), Z(G))| = \prod_{i=1}^{k}\prod_{j=1}^{n} p^{\min\{a_i,d_j\}}.$$

Observe that $a_i \leq a_1 \leq c_s < d_s$ for all i, $1 \leq i \leq k$ and $c_j = d_j$ for all j, $s+1 \leq j \leq m$. It follows that

$$|\operatorname{Var}(G)| = \prod_{i=1}^{k}\prod_{j=1}^{m} p^{\min\{a_i,c_j\}} = p^{s(a_1+a_2+\cdots+a_k)} \prod_{i=1}^{k}\prod_{j=s+1}^{m} p^{\min\{a_i,c_j\}}$$

and

$$|\operatorname{Aut}_z(G)| = \prod_{i=1}^{k}\prod_{j=1}^{n} p^{\min\{a_i,d_j\}} = p^{s(a_1+a_2+\cdots+a_k)}\prod_{i=1}^{k}\prod_{j=s+1}^{m} p^{\min\{a_i,c_j\}}.$$

It follows that $|\operatorname{Var}(G)| = |\operatorname{Aut}_z(G)|$ and hence $\operatorname{Var}(G) = \operatorname{Aut}_z(G)$. □

We next find necessary and sufficient conditions for a finite purely non-abelian p-group G such that $C_{\operatorname{Var}(G)}(Z(G)) = \operatorname{Aut}_z(G)$. Observe that if $L(G) = Z(G)$, then $\operatorname{Aut}_z(G) = \operatorname{Var}(G)$. In 2017, Hajizadeh and Nasrabadi [37] obtained the necessary and sufficient conditions for G such that $C_{\operatorname{Var}(G)}(Z(G)) = \operatorname{Var}(G)$. We, therefore, restrict to the case when $L(G) < Z(G)$. Let

$$G/E(G)Z(G) \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}},$$

and let G/G', $L(G)$ and $Z(G)$ be with the same cyclic decompositions as above.

Theorem 3.2.6 *Let G be a finite purely non-abelian p-group such that $L(G) < Z(G)$. Then $C_{\operatorname{Var}(G)}(Z(G)) = \operatorname{Aut}_z(G)$ if and only if $G' = E(G)Z(G)$, $m = n$ and $a_1 \leq c_s$, where s is the largest integer between 1 and m such that $c_s < d_s$.*

Proof. The only if part can be easily proved as in Theorem 3.2.5. Suppose that $C_{\operatorname{Var}(G)}(Z(G)) = \operatorname{Aut}_z(G)$. As observed in Theorem 3.2.5, $L(G)$ is non-trivial. Since G is purely non-abelian, $|\operatorname{Aut}_z(G)| = |\operatorname{Hom}(G/G', Z(G))|$ by Lemma 3.2.1. Also, $|C_{\operatorname{Var}(G)}(Z(G))| = |\operatorname{Hom}(G/E(G)Z(G), L(G))|$ by [55, Prop. 2]. If $G/E(G)Z(G)$ is a proper quotient group of G/G', then by Lemma 3.2.2 $|\operatorname{Hom}(G/E(G)Z(G), L(G))| < |\operatorname{Hom}(G/G', Z(G))|$, which is not so. Thus $G/E(G)Z(G) = G/G'$ and hence $G' = E(G)Z(G)$, because $G' \leq E(G)$. It follows that

$$|\operatorname{Hom}(G/E(G)Z(G), L(G))| = |\operatorname{Hom}(G/E(G)Z(G), Z(G))|.$$

Using similar arguments, as in the proof of Theorem 3.2.5, we can now prove that $m = n$ and $a_1 \leq c_s$, where s is the largest integer between 1 and m such that $c_s < d_s$. □

We give an example of a group that satisfies the hypothesis of Theorem 3.2.5. Consider $G = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7 \rangle$ with the relations:

$f_1^2 = f_2^2 = f_3^2 = f_5^2 = f_6^2 = f_7^2 = 1, f_4^2 = f_1, [f_2, f_1] = f_4, [f_3, f_1] = f_5, [f_4, f_1] = [f_6, f_1] = f_7, [f_5, f_1] = [f_7, f_1] = 1, [f_3, f_2] = f_6, [f_4, f_2] = f_7, [f_5, f_2] = f_7, [f_6, f_2] = [f_7, f_2] = [f_4, f_3] = [f_5, f_3] = [f_6, f_3] = [f_7, f_3] = [f_5, f_4] = [f_6, f_4] = [f_7, f_4] = [f_6, f_5] = [f_7, f_5] = [f_7, f_6] = 1.$

This group is the group number 742 in the GAP Library of groups of order 128. In this group, $L(G) = \langle f_7 \rangle$, $Z(G) = \langle f_4 f_5 f_6, f_7 \rangle$ and $\Phi(G) = \langle f_4, f_5, f_6, f_7 \rangle$. Also $|L(G)| = 2$, $|Z(G)| = 4$ and $|\Phi(G)| = 16$. Therefore $L(G) < Z(G) < \Phi(G)$ and thus G is a purely non-abelian group.

By using the following commands in GAP:

$A := AutomorphismGroup(G);$
$I := InnerAutomorphismsAutomorphismGroup(A);$
$C := Centraliser(A, I);$
$Elements(C);$

We have the following 8 central automorphisms:

$\alpha_1(f_2f_3f_5f_7) = f_2f_3f_5f_7, \alpha_1(f_1f_4f_6) = f_1f_4f_6f_7, \alpha_1(f_2f_3f_4f_5f_6) = f_2f_3f_4f_5f_6, \alpha_1(f_1f_3f_5f_6) = f_1f_3f_5f_6$

$\alpha_2(f_2f_3f_5f_7) = f_2f_3f_5f_7, \alpha_2(f_1f_4f_6) = f_1f_4f_6f_7, \alpha_2(f_2f_3f_4f_5f_6) = f_2f_3f_4f_5f_6, \alpha_2(f_1f_3f_5f_6) = f_1f_3f_5f_6f_7$

$\alpha_3(f_2f_3f_5f_7) = f_2f_3f_5, \alpha_3(f_1f_4f_6) = f_1f_4f_6, \alpha_3(f_2f_3f_4f_5f_6) = f_2f_3f_4f_5f_6f_7, \alpha_3(f_1f_3f_5f_6) = f_1f_3f_5f_6$

$\alpha_4(f_2f_3f_5f_7) = f_2f_3f_5, \alpha_4(f_1f_4f_6) = f_1f_4f_6, \alpha_4(f_2f_3f_4f_5f_6) = f_2f_3f_4f_5f_6f_7, \alpha_4(f_1f_3f_5f_6) = f_1f_3f_5f_6f_7$

$\alpha_5(f_2f_3f_5f_7) = f_2f_3f_5, \alpha_5(f_1f_4f_6) = f_1f_4f_6f_7, \alpha_5(f_2f_3f_4f_5f_6) = f_2f_3f_4f_5f_6f_7, \alpha_5(f_1f_3f_5f_6) = f_1f_3f_5f_6$

$\alpha_6(f_2f_3f_5f_7) = f_2f_3f_5, \alpha_6(f_1f_4f_6) = f_1f_4f_6f_7, \alpha_6(f_2f_3f_4f_5f_6) = f_2f_3f_4f_5f_6f_7, \alpha_6(f_1f_3f_5f_6) = f_1f_3f_5f_6f_7$

$\alpha_7(f_2f_3f_5f_7) = f_2f_3f_5f_7, \alpha_7(f_1f_4f_6) = f_1f_4f_6, \alpha_7(f_2f_3f_4f_5f_6) = f_2f_3f_4f_5f_6, \alpha_7(f_1f_3f_5f_6) = f_1f_3f_5f_6$

$\alpha_8(f_2f_3f_5f_7) = f_2f_3f_5f_7, \alpha_8(f_1f_4f_6) = f_1f_4f_6, \alpha_8(f_2f_3f_4f_5f_6) = f_2f_3f_4f_5f_6, \alpha_8(f_1f_3f_5f_6) = f_1f_3f_5f_6f_7$

It is easy to see that

$$\begin{aligned}
(f_1f_3f_4)^{-1}\alpha_1(f_1f_3f_4) &= f_4^{-1}f_3^{-1}f_1^{-1}f_1f_3 \\
&= f_4^{-1} = f_4 \in L(G), \\
(f_1f_2f_5)^{-1}\alpha_1(f_1f_2f_5) &= f_5^{-1}f_2^{-1}f_1^{-1}f_1f_2f_5 \\
&= 1 \in L(G), \text{ and} \\
f_2^{-1}\alpha_1(f_2) &= f_2^{-1}f_2f_5 \\
&= f_5 \in L(G)
\end{aligned}$$

Thus $\alpha_1 \in \mathrm{Var}(G)$. Similarly, we can show that other 7 central automorphisms belong to $\mathrm{Var}(G)$. It follows that $\mathrm{Aut}_z(G) \leq \mathrm{Var}(G)$, and since $\mathrm{Var}(G) \leq \mathrm{Aut}_z(G)$, therefore $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$.

The following example shows that there exists finite non-abelian p-groups with $L(G) < Z(G)$ which are not purely non-abelian but $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$. Consider

$G = \langle f_1, f_2, f_3, f_4, f_5 \rangle$ with the relations:

$f_2^2 = f_4^2 = f_5^2 = 1, f_1^2 = f_3^2 = f_5, [f_2, f_1] = f_4, [f_3, f_1] = [f_4, f_1] = [f_5, f_1] = [f_3, f_2] = [f_4, f_2] = [f_5, f_2] = [f_4, f_3] = [f_5, f_3] = [f_5, f_4] = 1.$

This group is the group number 25 in the GAP Library of groups of order 32. In this group, $L(G) = \langle f_4, f_5 \rangle$ and $Z(G) = \langle f_3, f_4, f_5 \rangle$. Also $|L(G)| = 4$ and $|Z(G)| = 8$. Therefore $L(G) < Z(G)$. Observe that $G = C_4 \times D_8$, and therefore G is not a purely non-abelian group.

By using the following commands in GAP:

$A := AutomorphismGroup(G);$
$I := InnerAutomorphismsAutomorphismGroup(A);$
$C := Centraliser(A, I);$
$Elements(C);$

We have the following 64 central automorphisms:

$\alpha_1(f_1f_3f_4) = f_1f_3, \alpha_1(f_1f_2f_5) = f_1f_2f_5, \alpha_1(f_2) = f_2f_5$

$\alpha_2(f_1f_3f_4) = f_1f_3f_4, \alpha_2(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_2(f_2) = f_2f_4f_5$

$\alpha_3(f_1f_3f_4) = f_1f_3f_5, \alpha_3(f_1f_2f_5) = f_1f_2, \alpha_3(f_2) = f_2$

$\alpha_4(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_4(f_1f_2f_5) = f_1f_2f_4, \alpha_4(f_2) = f_2f_4$

$\alpha_5(f_1f_3f_4) = f_1f_3, \alpha_5(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_5(f_2) = f_2f_5$

$\alpha_6(f_1f_3f_4) = f_1f_3f_4, \alpha_6(f_1f_2f_5) = f_1f_2f_5, \alpha_6(f_2) = f_2f_4f_5$

$\alpha_7(f_1f_3f_4) = f_1f_3f_5, \alpha_7(f_1f_2f_5) = f_1f_2f_4, \alpha_7(f_2) = f_2$

$\alpha_8(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_8(f_1f_2f_5) = f_1f_2, \alpha_8(f_2) = f_2f_4$

$\alpha_9(f_1f_3f_4) = f_1f_3, \alpha_9(f_1f_2f_5) = f_1f_2, \alpha_9(f_2) = f_2f_5$

$\alpha_{10}(f_1f_3f_4) = f_1f_3f_4, \alpha_{10}(f_1f_2f_5) = f_1f_2f_4, \alpha_{10}(f_2) = f_2f_4f_5$

$\alpha_{11}(f_1f_3f_4) = f_1f_3f_5, \alpha_{11}(f_1f_2f_5) = f_1f_2f_5, \alpha_{11}(f_2) = f_2$

$\alpha_{12}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{12}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{12}(f_2) = f_2f_4$

$\alpha_{13}(f_1f_3f_4) = f_1f_3, \alpha_{13}(f_1f_2f_5) = f_1f_2f_4, \alpha_{13}(f_2) = f_2f_5$

$\alpha_{14}(f_1f_3f_4) = f_1f_3f_4, \alpha_{14}(f_1f_2f_5) = f_1f_2, \alpha_{14}(f_2) = f_2f_4f_5$

$\alpha_{15}(f_1f_3f_4) = f_1f_3f_5, \alpha_{15}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{15}(f_2) = f_2$

$\alpha_{16}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{16}(f_1f_2f_5) = f_1f_2f_5, \alpha_{16}(f_2) = f_2f_4$

$\alpha_{17}(f_1f_3f_4) = f_1f_3, \alpha_{17}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{17}(f_2) = f_2f_4f_5$

$\alpha_{18}(f_1f_3f_4) = f_1f_3f_4, \alpha_{18}(f_1f_2f_5) = f_1f_2f_5, \alpha_{18}(f_2) = f_2f_5$

$\alpha_{19}(f_1f_3f_4) = f_1f_3f_5, \alpha_{19}(f_1f_2f_5) = f_1f_2f_4, \alpha_{19}(f_2) = f_2f_4$

$\alpha_{20}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{20}(f_1f_2f_5) = f_1f_2, \alpha_{20}(f_2) = f_2$

$\alpha_{21}(f_1f_3f_4) = f_1f_3, \alpha_{21}(f_1f_2f_5) = f_1f_2f_5, \alpha_{21}(f_2) = f_2f_4f_5$

$\alpha_{22}(f_1f_3f_4) = f_1f_3f_4, \alpha_{22}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{22}(f_2) = f_2f_5$

$\alpha_{23}(f_1f_3f_4) = f_1f_3f_5, \alpha_{23}(f_1f_2f_5) = f_1f_2, \alpha_{23}(f_2) = f_2f_4$

$\alpha_{24}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{24}(f_1f_2f_5) = f_1f_2f_4, \alpha_{24}(f_2) = f_2$

$\alpha_{25}(f_1f_3f_4) = f_1f_3, \alpha_{25}(f_1f_2f_5) = f_1f_2f_4, \alpha_{25}(f_2) = f_2f_4f_5$

$\alpha_{26}(f_1f_3f_4) = f_1f_3f_4, \alpha_{26}(f_1f_2f_5) = f_1f_2, \alpha_{26}(f_2) = f_2f_5$

$\alpha_{27}(f_1f_3f_4) = f_1f_3f_5, \alpha_{27}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{27}(f_2) = f_2f_4$

$\alpha_{28}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{28}(f_1f_2f_5) = f_1f_2f_5, \alpha_{28}(f_2) = f_2$

$\alpha_{29}(f_1f_3f_4) = f_1f_3, \alpha_{29}(f_1f_2f_5) = f_1f_2, \alpha_{29}(f_2) = f_2f_4f_5$

$\alpha_{30}(f_1f_3f_4) = f_1f_3f_4, \alpha_{30}(f_1f_2f_5) = f_1f_2f_4, \alpha_{30}(f_2) = f_2f_5$

$\alpha_{31}(f_1f_3f_4) = f_1f_3f_5, \alpha_{31}(f_1f_2f_5) = f_1f_2f_5, \alpha_{31}(f_2) = f_2f_4$

$\alpha_{32}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{32}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{32}(f_2) = f_2$

$\alpha_{33}(f_1f_3f_4) = f_1f_3, \alpha_{33}(f_1f_2f_5) = f_1f_2, \alpha_{33}(f_2) = f_2$

$\alpha_{34}(f_1f_3f_4) = f_1f_3f_4, \alpha_{34}(f_1f_2f_5) = f_1f_2f_4, \alpha_{34}(f_2) = f_2f_4$

$\alpha_{35}(f_1f_3f_4) = f_1f_3f_5, \alpha_{35}(f_1f_2f_5) = f_1f_2f_5, \alpha_{35}(f_2) = f_2f_5$

$\alpha_{36}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{36}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{36}(f_2) = f_2f_4f_5$

$\alpha_{37}(f_1f_3f_4) = f_1f_3, \alpha_{37}(f_1f_2f_5) = f_1f_2f_4, \alpha_{37}(f_2) = f_2$

$\alpha_{38}(f_1f_3f_4) = f_1f_3f_4, \alpha_{38}(f_1f_2f_5) = f_1f_2, \alpha_{38}(f_2) = f_2f_4$

$\alpha_{39}(f_1f_3f_4) = f_1f_3f_5, \alpha_{39}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{39}(f_2) = f_2f_5$

$\alpha_{40}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{40}(f_1f_2f_5) = f_1f_2f_5, \alpha_{40}(f_2) = f_2f_4f_5$

$\alpha_{41}(f_1f_3f_4) = f_1f_3, \alpha_{41}(f_1f_2f_5) = f_1f_2f_5, \alpha_{41}(f_2) = f_2$

$\alpha_{42}(f_1f_3f_4) = f_1f_3f_4, \alpha_{42}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{42}(f_2) = f_2f_5$

$\alpha_{43}(f_1f_3f_4) = f_1f_3f_5, \alpha_{43}(f_1f_2f_5) = f_1f_2, \alpha_{43}(f_2) = f_2f_5$

$\alpha_{44}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{44}(f_1f_2f_5) = f_1f_2f_4, \alpha_{44}(f_2) = f_2f_4f_5$

$\alpha_{45}(f_1f_3f_4) = f_1f_3, \alpha_{45}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{45}(f_2) = f_2$

$\alpha_{46}(f_1f_3f_4) = f_1f_3f_4, \alpha_{46}(f_1f_2f_5) = f_1f_2f_5, \alpha_{46}(f_2) = f_2f_4$

$\alpha_{47}(f_1f_3f_4) = f_1f_3f_5, \alpha_{47}(f_1f_2f_5) = f_1f_2f_4, \alpha_{47}(f_2) = f_2f_5$

$\alpha_{48}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{48}(f_1f_2f_5) = f_1f_2, \alpha_{48}(f_2) = f_2f_4f_5$

$\alpha_{49}(f_1f_3f_4) = f_1f_3, \alpha_{49}(f_1f_2f_5) = f_1f_2f_4, \alpha_{49}(f_2) = f_2f_4$

$\alpha_{50}(f_1f_3f_4) = f_1f_3f_4, \alpha_{50}(f_1f_2f_5) = f_1f_2, \alpha_{50}(f_2) = f_2$

$\alpha_{51}(f_1f_3f_4) = f_1f_3f_5, \alpha_{51}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{51}(f_2) = f_2f_4f_5$

$\alpha_{52}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{52}(f_1f_2f_5) = f_1f_2f_5, \alpha_{52}(f_2) = f_2f_5$

$\alpha_{53}(f_1f_3f_4) = f_1f_3, \alpha_{53}(f_1f_2f_5) = f_1f_2, \alpha_{53}(f_2) = f_2f_4$

$\alpha_{54}(f_1f_3f_4) = f_1f_3f_4, \alpha_{54}(f_1f_2f_5) = f_1f_2f_4, \alpha_{54}(f_2) = f_2$

$\alpha_{55}(f_1f_3f_4) = f_1f_3f_5, \alpha_{55}(f_1f_2f_5) = f_1f_2f_5, \alpha_{55}(f_2) = f_2f_4f_5$

$\alpha_{56}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{56}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{56}(f_2) = f_2f_5$

$\alpha_{57}(f_1f_3f_4) = f_1f_3, \alpha_{57}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{57}(f_2) = f_2f_4$

$\alpha_{58}(f_1f_3f_4) = f_1f_3f_4, \alpha_{58}(f_1f_2f_5) = f_1f_2f_5, \alpha_{58}(f_2) = f_2$

$\alpha_{59}(f_1f_3f_4) = f_1f_3f_5, \alpha_{59}(f_1f_2f_5) = f_1f_2f_4, \alpha_{59}(f_2) = f_2f_4f_5$

$\alpha_{60}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{60}(f_1f_2f_5) = f_1f_2, \alpha_{60}(f_2) = f_2f_5$

$\alpha_{61}(f_1f_3f_4) = f_1f_3, \alpha_{61}(f_1f_2f_5) = f_1f_2f_5, \alpha_{61}(f_2) = f_2f_4$

$\alpha_{62}(f_1f_3f_4) = f_1f_3f_4, \alpha_{62}(f_1f_2f_5) = f_1f_2f_4f_5, \alpha_{62}(f_2) = f_2$

$\alpha_{63}(f_1f_3f_4) = f_1f_3f_5, \alpha_{63}(f_1f_2f_5) = f_1f_2, \alpha_{63}(f_2) = f_2f_4f_5$

$\alpha_{64}(f_1f_3f_4) = f_1f_3f_4f_5, \alpha_{64}(f_1f_2f_5) = f_1f_2f_4, \alpha_{64}(f_2) = f_2f_5$

It is easy to see that

$$\begin{aligned}
(f_1f_3f_4)^{-1}\alpha_1(f_1f_3f_4) &= f_4^{-1}f_3^{-1}f_1^{-1}f_1f_3 \\
&= f_4^{-1} = f_4 \in L(G), \\
(f_1f_2f_5)^{-1}\alpha_1(f_1f_2f_5) &= f_5^{-1}f_2^{-1}f_1^{-1}f_1f_2f_5 \\
&= 1 \in L(G), \text{ and} \\
f_2^{-1}\alpha_1(f_2) &= f_2^{-1}f_2f_5 \\
&= f_5 \in L(G)
\end{aligned}$$

Thus $\alpha_1 \in \mathrm{Var}(G)$. Similarly, we can show that other 63 central automorphisms belong to $\mathrm{Var}(G)$. It follows that $\mathrm{Aut}_z(G) \leq \mathrm{Var}(G)$, and since $\mathrm{Var}(G) \leq \mathrm{Aut}_z(G)$, therefore $\mathrm{Var}(G) = \mathrm{Aut}_z(G)$.

CHAPTER 4

A note on p-automorphisms of finite p-groups

4.1 — Introduction

For a finite p-group G, $\Omega_1(G)$ denotes the subgroup generated by elements of order p. A finite p-group G is called extraspecial if $G' = Z(G) \simeq C_p$. A central automorphism of a group is an automorphism that commutes with all inner automorphisms. The central automorphisms fix the commutator subgroup G' of G elementwise and form a normal subgroup $\mathrm{Aut_z}(G)$ of $\mathrm{Aut}(G)$. An automorphism α of G is called an absolute central automorphism if $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$, where $L(G)$ denote the absolute center of G. Let $\mathrm{Var}(G)$ denote the group of all absolute central automorphisms of G. An automorphism α of G is called an IA-automorphism if $x^{-1}\alpha(x) \in G'$ for all $x \in G$. Let $\mathrm{IA}(G)$ denote the group of all IA-automorphisms of G and let $C_{\mathrm{IA}(G)}(Z(G))$ denote its subgroup consisting of those IA-automorphisms which fix $Z(G)$ elementwise. Observe that if G is a finite p-group of class 2, then we have the following sequence of subgroups

$$\mathrm{Inn}(G) \leq \mathrm{Aut}_c(G) \leq \mathrm{IA}(G) \leq \mathrm{Aut_z}(G) \leq \mathrm{Aut}(G).$$

It is interesting to find as to when an automorphism of a group is an inner automorphism. A group G is called semicomplete if $\mathrm{IA}(G) = \mathrm{Inn}(G)$. In 1965, Bachmuth [14] proved that if G is a free metabelian group of rank 2, then G is semicomplete. In 1969, Andreadakis [4] proved that a free product $A * B$ of two non-trivial groups is semicomplete if and only if both A and B are abelian. In 2002, Panagopoulos [61], studied the semicompleteness of the direct product $G = A \times B$ of two groups A and B in relation to the semicompleteness of its direct factors. In 2011, Attar [7, Theorem 2.1] has proved that if G is a finite p-group of class 2, then $\mathrm{IA}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic and $\mathrm{IA}(G) = C_{\mathrm{IA}(G)}(Z(G))$. In 2014, Singh, Gumber and Kalra [77] classified finitely generated nilpotent groups of class 2 for which $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$ and $C_{\mathrm{IA}(G)}(Z(G)) \simeq \mathrm{Inn}(G)$. In particular, they classified all finite nilpotent groups G of class 2 for which (i) $\mathrm{IA}(G) = \mathrm{Inn}(G)$ and (ii) $C_{\mathrm{IA}(G)}(Z(G)) = \mathrm{Inn}(G)$. In 2014, Rai [63] also gave the necessary and sufficient conditions for a finite p-group G of nilpotency class 2 such that $C_{\mathrm{IA}(G)}(Z(G)) = \mathrm{Inn}(G)$.

In 2001, Curran and McCaughan [24] characterized finite p-groups G for which $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$. In 2015, Nasrabadi and Farimani [59] gave the necessary and sufficient conditions for a finite autonilpotent p-group class 2 such that all absolute central automorphisms are inner. In 2015, Singh and Gumber [78] generalized the result of Nasrabadi and Farimani to arbitrary finite p-groups. In 2017, Attar [10] characterized finite non-abelian p-group G of arbitrary class for which $\mathrm{Var}(G) = \mathrm{Inn}(G)$. Let

$$\mathrm{Aut}^{\Phi}(G) = \{\alpha \in \mathrm{Aut}(G) \mid x^{-1}\alpha(x) \in \Phi(G), \forall x \in G\}.$$

Clearly $\mathrm{Aut}^{\Phi}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$. By a well-known theorem of P. Hall the group $\mathrm{Aut}^{\Phi}(G)$ is a p-group. In 1979, Muller [58], using cohomological methods, proved that if G is a finite p-group, then $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$ if and only if G is elementary abelian or extraspecial. Let $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))$ be the group of all $\mathrm{Aut}^{\Phi}(G)$ automorphisms of G fixing $Z(\Phi(G))$ elementwise. In 2009, Attar [6] using the presentations of Schmid [69] proved that if G is a finite non-abelian p-group, then $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)) = \mathrm{Inn}(G)$ if and only if $\Phi(G) = Z(G)$ and $Z(G)$ is cyclic. Here, we obtain alternate and easy proof of the result of Attar without using the presentations of Schmid, and as a consequence we obtain the alternate proof of the main result of Muller [58] .

4.2 — Main results.

We will use the following well known commutator identity without any further reference:

$$[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z].$$

In the following theorem, we shall give an alternate proof of the result of [6].

Theorem 4.2.1 *Let G be a finite non-abelian p-group such that* $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) = \mathrm{Inn}(G)$ *if and only if* $\Phi(G) = Z(G)$ *and* $Z(G)$ *is cyclic.*

Proof. Let $\Phi(G) = Z(G)$ and $Z(G)$ is cyclic. Then $\mathrm{Aut}^{\Phi}(G) = \mathrm{Aut_z}(G)$ and G is of class 2. It follows from Lemma 2.2.3 and 2.2.4 that

$$\begin{aligned} C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)) &= C_{\mathrm{Aut_z}(G)}(Z(G)) = \mathrm{Aut}_{Z(G)}^{Z(G)}(G) \\ &\simeq \mathrm{Hom}(G/Z(G), Z(G)) \simeq G/Z(G) = \mathrm{Inn}(G), \end{aligned}$$

and therefore $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) = \mathrm{Inn}(G)$.

Conversely suppose that $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) = \mathrm{Inn}(G)$. We first prove that $Z(G) \leq$

$\Phi(G)$. Suppose on the contrary that $Z(G) \nleq M$ for some maximal subgroup M of G. Take an element $g \in Z(G)\backslash M$ and an element z of order p in $Z(G) \cap \Phi(G)$. Define the map $\alpha : G \longrightarrow G$ by $\alpha(mg^k) = mg^k z^k$ for every $m \in M$ and $k \in \{0, 1, ..., p-1\}$. It is easy to check that α is an automorphism of G. Since $(mg^k)^{-1}\alpha(mg^k) = z^k \in \Phi(G)$ for all $1 \leq k \leq p-1$, α which fixes $G/\Phi(G)$. Also since α fixes M, therefore α fixes $Z(\Phi(G))$. By hypothesis α must be an inner automorphism of G. Let $\alpha = \theta_a$ for some $a \in G$. Since $g \in Z(G)$, we have $gz = \alpha(g) = \theta_a(g) = a^{-1}ga = g$, thus $z = 1$, which is a contradiction. Thus $Z(G) \leq \Phi(G)$.

Now we prove that $\Phi(G) \leq Z(G)$. We first claim that $Z(\Phi(G)) \leq Z(G)$. If possible, suppose that $Z(\Phi(G)) \nleq Z(G)$. So there exist an element $x \in Z(\Phi(G))$ such that $x \notin Z(G)$ and therefore $xa \neq ax$ for some $a \in G$. Thus there exist an inner automorphism f_a such that $f_a(x) \neq x$. Hence $f_a \notin C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))$, which is a contradiction to the hypothesis. Thus $Z(\Phi(G)) \leq Z(G)$. Let $\overline{G} = G/Z(G)$ and $Z_2^*(G)/Z(G) = \Omega_1(Z(\overline{G}))$. Then $Z_2^*(G) = \{a \in Z_2(G) \mid a^p \in Z(G)\}$. Observe that $[Z_2^*(G), \Phi(G)] = 1$, since for all $a \in Z_2^*(G)$, $b \in G'$, $g^p \in G^p$, we have

$$\begin{aligned} [a, bg^p] &= [a, g^p][a, b][a, b, g^p] \\ &= [a, g^p] = [a, g]^p \\ &= [a^p, g] = 1. \end{aligned}$$

In particular, $Z_2^*(G) \cap \Phi(G) \leq Z(\Phi(G))$, and so by hypothesis, $Z_2^*(G) \cap \Phi(G) \leq Z(G)$. It follows that $\overline{Z_2^*(G) \cap \Phi(G)} \leq \overline{Z(G)} = 1$ and therefore $\overline{Z_2^*(G)} \cap \overline{\Phi(G)} = 1$. Then $\Omega_1(Z(\overline{G})) \cap \overline{\Phi(G)} = 1$. Since $\overline{\Phi(G)}$ is a normal subgroup of $\overline{G}$ and $\overline{G}$ is a p-group, this implies that $\overline{\Phi(G)} = 1$. Therefore $\Phi(G) \leq Z(G)$. Thus $Z(G) = \Phi(G)$.

It follows from the hypothesis that

$$G/Z(G) = \mathrm{Inn}(G) = C_{\mathrm{Aut_z}(G)}(Z(G)) = \mathrm{Aut}_{Z(G)}^{Z(G)}(G) \simeq \mathrm{Hom}(G/Z(G), Z(G)),$$

and hence $Z(G)$ is cyclic by Lemma 2.2.3. □

A finite p-group G is called Frattinian if $Z(M) \neq Z(G)$ for all maximal subgroups M of G. A Frattinian p-group G satisfying $C_G(Z(\Phi(G))) = \Phi(G)$ is called strongly Frattinian.

Lemma 4.2.2 *Let A and B be finite p-groups. If $[a, b] = 1$ for all $a \in A$ and for all $b \in B$, then $\Phi(AB) = \Phi(A)\Phi(B)$.*

Proof. Since $[a, b] = 1$ for all $a \in A$ and for all $b \in B$, we have $B'A^p = A^pB'$ and

$$(AB)' = \langle [x, y] \mid x, y \in AB \rangle = \langle [a, b] \mid a, b \in A \quad or \quad a, b \in B \rangle = A'B'$$

and therefore $\Phi(AB) = (AB)'(AB)^p = A'B'A^pB^p = \Phi(A)\Phi(B)$. □

Lemma 4.2.3 ([69, Proposition 3]) *If G is a strongly Frattinian p-group, the normal p-subgroup of* $\mathrm{Aut}(G)$ *consisting of all automorphisms centralizing $G/\Phi(G)$ and $Z(\Phi(G))$ is not inner.*

In the following theorem, we will use the arguments of [6].

Theorem 4.2.4 *Let G be a finite non-abelian p-group. Then $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) \not< \mathrm{Inn}(G)$.*

Proof. Suppose if possible that $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) < \mathrm{Inn}(G)$. As observed in Theorem 4.2.1 $Z(G) \leq \Phi(G)$. Now we prove that $Z(G) < Z(M)$ for every maximal subgroup M of G. Suppose for a contradiction that $Z(G) = Z(M)$ for some maximal subgroup M of G. Let $g \in G \setminus M$ and z is an element of order p in $Z(G)$. Then the map β defined on G by $\beta(mg^k) = mg^kz^k$ for every $m \in M$ and $k \in \{0, 1, ..., p-1\}$ belongs to $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))$. By assumption, $\beta = \gamma_a$ for some $a \in G$. Therefore $a \in C_G(M) = Z(M) = Z(G)$, which is a contradiction to the hypothesis. Thus $Z(G) \neq Z(M)$ for all maximal subgroups M of G. Hence, by [69], G has one the following forms:

(a) $G = E_1E_2 \cdots E_s$ is the central product of non-abelian p-groups E_i, where $[E_i, E_j] = 1$ for all $i \neq j$, $|E_i| = p^2|Z(G)|$, and $Z(G) = Z(E_i)$ for all $1 \leq i \leq s$; or

(b) $G = EF$, where $E = C_G(F)$, $C_F(Z(\Phi(F)) = \Phi(F)$, $\Phi(E) \leq Z(G)$, and both E, F are Frattinian p-groups.

Moreover, in case (b) either $E = Z(G)$(and therefore $G = F$), or E is a central product as in case (a). If the group G is as in case (a), by Lemma 4.2.2,

$$\Phi(G) = \Phi(E_1)\Phi(E_2) \cdots \Phi(E_s) = Z(E_1)Z(E_2) \cdots Z(E_s) \leq Z(G)$$

because E_i are Redei p-groups. Thus $Z(G) = \Phi(G)$. and therefore $C_{\mathrm{Aut}_z(G)}(Z(G)) < \mathrm{Inn}(G)$. Since G is of class 2, $\mathrm{Inn}(G) \leq C_{\mathrm{Aut}_z(G)}(Z(G)) < \mathrm{Inn}(G)$, which is a contradiction. Suppose G satisfies case (b). If $G = F$, then $C_G(Z(\Phi(G)) = \Phi(G)$ and therefore G is strongly Frattinian, which is not possible by Lemma 4.2.3. Let $G \neq F$. Therefore E is a central product as in case (a) and so $\Phi(E) = Z(E)$. Since $G = EF$, by Lemma 4.2.2, $\Phi(G) = \Phi(E)\Phi(F) = Z(E)\Phi(F)$ and hence $Z(\Phi(G)) \leq Z(E)Z(\Phi(F))$. Since $C_F(Z(\Phi(F)) = \Phi(F)$, F is strongly Frattinian.

By Lemma 4.2.3, there exists $\alpha \in C_{\mathrm{Aut}^{\Phi}(F)}(Z(\Phi(F))) \setminus \mathrm{Inn}(F)$. Since $E = C_G(F)$ and $C_F(Z(\Phi(F))) = \Phi(F)$, $E \cap F \leq Z(\Phi(F))$. Now let σ be the map on G defined by $\sigma(ab) = a\alpha(b)$ for every $a \in E$ and $b \in F$. We show that σ is well defined. Let $ab, cd \in EF$ such that $\sigma(ab) = a\alpha(b)$ and $\sigma(cd) = c\alpha(d)$. Let $ab = cd$. Since $E \cap F \leq Z(\Phi(F))$ and α fixes $Z(\Phi(F))$ elementwise, we get $\sigma(ab) = \sigma(cd)$. Therefore σ is well defined. Since $Z(\Phi(G)) \leq Z(E)Z(\Phi(F))$, it is easy to check that $\sigma \in C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G)))$ and so it is an inner automorphism of G. It implies that α is an inner automorphism of F, which is not possible. Therefore our supposition is wrong. Hence $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) \not\leq \mathrm{Inn}(G)$. □

Lemma 4.2.5 ([24, Corollary 1]) *If G is a finite p-group, then $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.*

In the next theorem, we give an alternate proof the main result of [58].

Theorem 4.2.6 *If G is a finite p-group which is neither elementary abelian nor extraspecial, then Then $\mathrm{Aut}^{\Phi}(G)/\mathrm{Inn}(G)$ is a nontrivial normal p-subgroup of the group of outer automorphisms of G.*

Proof. If G is an abelian group, then the result can be proved easily. So let G be a finite non-abelian p-group which is not extraspecial. Suppose if possible that $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$. Then $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) \leq \mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$ and therefore $C_{\mathrm{Aut}^{\Phi}(G)}(Z(\Phi(G))) = \mathrm{Inn}(G)$ by Theorem 4.2.4. It follows from Theorem 4.2.1 that $\Phi(G) = Z(G)$ and $Z(G)$ is cyclic. Therefore $\mathrm{Aut}_z(G) = \mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$ and thus by Lemma 4.2.5, $G' = Z(G)$ and $Z(G)$ is cyclic. Also since G is of class 2, $\exp(G') = \exp(G/Z(G)) = p$ by Lemma 2.2.4. Thus G is an extraspecial group, which is a contradiction. □

CHAPTER 5

On commuting automorphisms of some p-groups

5.1 — Introduction

An automorphism α of G is called a commuting automorphism if every element of G commutes with it's image under α or $\alpha(x)x = x\alpha(x)$ for all $x \in G$. Let $A(G)$ denote the set of all commuting automorphisms of G. Clearly $\mathrm{Aut_z}(G)$ is always contained in $\mathrm{A}(G)$. In 1984, Herstein [43] proposed the following problem to American Mathematical Monthly:

If** G **is a simple non-abelian group, then prove that** $A(G) = 1$**.

Laffey and Pettet answered to this problem. Laffey [50] proved that if G has no non-trivial abelian normal subgroup, then $A(G) = 1$. Pettet (see [62]), in his personal communication, observed that if $Z(G) = 1$ and $G' = G$, then $A(G) = 1$. Let $E_2(G) = \{g \in G \mid [g, x, x] = 1\ \forall x \in G\}$ denote the set of right 2-Engel elements of G. In 2001, Deaconescu and Walls [26] have shown that there is a close connection between the right 2-Engel elements and the set of commuting automorphisms of a group. In 2002, Deaconescu, Silberberg and Walls [27] proved some interesting properties of commuting automorphisms and raised the natural question: Is it true that the set $A(G)$ is always a subgroup of $\mathrm{Aut}(G)$? They themself constructed the example of a finite non-abelian 2-group of order 32 in which $A(G)$ doesn't form a subgroup:

$G = \langle a, b, c, d \mid a^4 = b^2 = d^2 = 1, a^2 = c^2, [a, b] = [c, d] = a^2, [a, c] = [a, d] = [b, c] = [b, d] = 1\rangle$.

A group G is called an $A(G)$-group if the set

$$A(G) = \{\alpha \in \mathrm{Aut}(G) : \alpha(x)x = x\alpha(x),\ \forall x \in G\}$$

forms a subgroup of $\mathrm{Aut}(G)$. Vosooghpour and Malayeri [81] proved that if G is a finite p-group of order p^n where $n \leq 4$, then G is an $A(G)$-group. They also proved that if p is a prime number and $n \geqslant 5$, then there exists a non-$A(G)$ group of order p^n. In 2013, Vosooghpour, Kargarian and Malayeri [83] obtained the structure of $A(G)$, where G is a non-abelian p-group of order p^n with cyclic maximal subgroup. Fouladi and Orfi [32] proved that if G is either a finite AC-group or a p-group of maximal class or a metacyclic p-group, then G is an $A(G)$-group. In 2013, Basri, Sarmin, Ali and Beuerle [16] developed GAP Codes for finite non-abelian metacyclic p-groups, where p is an odd prime, which help to perform computations on its various characteristics.

In 2015, Rai [64] gave some sufficient conditions on a finite p-group G such that $A(G)$ is a subgroup of $\mathrm{Aut}(G)$ and, as a consequence, proved that in a finite p-group G of co-class 2, where p is an odd prime, $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. In 2016, Singh and Gumber [79] gave very elementary and short proofs of main results of Rai and obtained other related results. In 2019, Shahrabi, Malayeri and Vosooghpour [70] proved that a finite 2-group G of almost maximal class is an

$A(G)$-group. Recently in 2019, Rai [65] proved that the direct product of two finite $A(G)$-groups is also an $A(G)$-group. He also proved that $GL(n,q)$ for $n = 3$ or $q > n$, $PSL(2,q)$ and ZM-groups are $A(G)$-groups.

Observe that if G is a non-abelian 2-group and $|G/G'| = 4$, then by [18, Proposition 1.6], G is of maximal class and therefore G is an $A(G)$-group by Fouladi and Orfi [32]. A natural question which arise here is that if for any group G, $|G/G'| = p^2$, where p is an odd prime, then under what conditions G is an $A(G)$-group. We answer this question in this chapter. Infact, in this chapter, we prove three theorems. In the first theorem, we prove that if G is a finite non-abelian p-group, p an odd prime, such that $\mathrm{IA}(G) = \mathrm{Inn}(G)$ and $|G/G'| = p^2$, then G is an $A(G)$-group. In the second theorem, we prove that if G is a finite non-abelian p-group of coclass 3, where p is an odd prime, and is strongly Frattinian, then G is an $A(G)$-group. In the third theorem, we prove that if G is a finite non-abelian p-group and, M_1, M_2 are any two distinct maximal abelain subgroups of G such that $M_1 \cap M_2 = Z(G)$, then G is an $A(G)$-group.

5.2 — Main Results

Lemma 5.2.1 (Laffey) *If φ is a commuting automorphism of a group G, then $[\varphi(x), y] = [x, \varphi(y)]$ for all $x, y \in G$.*

Proof. Observe that $[\varphi(x), y] = \varphi(x)^{-1}y^{-1}\varphi(x)y = x^{-1}\varphi(y)^{-1}\varphi(y)x\varphi(x)^{-1}y^{-1}\varphi(x)y = x^{-1}\varphi(y)^{-1}\varphi(y)\varphi(x)^{-1}xy^{-1}\varphi(x)y = x^{-1}\varphi(y)^{-1}xy^{-1}\varphi(y)\varphi(x)^{-1}\varphi(x)y = x^{-1}\varphi(y)^{-1}xy^{-1}$ $\varphi(y)y = [x, \varphi(y)]$. □

Proposition 5.2.2 (Pettet) *Let G be a group, let $\alpha \in A(G)$, and let $y \in G$. Then*

$[y, \alpha] \in C_G(G')$.

Proof. Let $x, y, z \in G$. Then by Lemma 5.2.1,

$$\begin{aligned} [\alpha(xy), z] &= [\alpha(x)\alpha(y), z] \\ &= [\alpha(x), z][\alpha(x), z, \alpha(y)][\alpha(y), z] \\ &= [x, \alpha(z)][\alpha(x), z, \alpha(y)][y, \alpha(z)]. \end{aligned}$$

Also $[xy, \alpha(z)] = [x, \alpha(z)][x, \alpha(z), y][y, \alpha(z)]$. Since $[\alpha(xy), z] = [xy, \alpha(z)]$ by Lemma 5.2.1, $[x, \alpha(z)][\alpha(x), z, \alpha(y)][y, \alpha(z)] = [x, \alpha(z)][x, \alpha(z), y][y, \alpha(z)]$. Therefore $[\alpha(x), z, \alpha(y)] = [x, \alpha(z), y]$. Thus

$$\begin{aligned} [\alpha(x), z, \alpha(y)] = [\alpha(x), z, y] &\implies [[\alpha(x), z], \alpha(y)] = [[\alpha(x), z], y] \\ &\implies (\alpha(y))^{-1}[\alpha(x), z]\alpha(y) = y^{-1}[\alpha(x), z]y \\ &\implies y^{-1}\alpha(y) \in C_G([\alpha(x), z]) \\ &\implies y^{-1}\alpha(y) \in C_G(G'). \end{aligned}$$

Hence $[y, \alpha] \in C_G(G')$ for all $y \in G$ and for all $\alpha \in A(G)$. □

Proposition 5.2.3 ([27, Lemma 2.4(vi)]) *Let G be a group. Then $\alpha\beta \in A(G)$ if and only if $[\alpha(x), \beta(x)] = 1$ for all $x \in G$.*

Proof. Observe that $\alpha\beta \in A(G)$ if and only if $[x, \alpha\beta(x)] = 1$ for all $x \in G$ if and only if $[\alpha(x), \beta(x)] = 1$ for all $x \in G$ by Lemma 5.2.1. □

Proposition 5.2.4 ([27, Lemma 2.4(viii)]) *Let G be a group and let $\alpha \in A(G)$. Then $\alpha^2 \in \mathrm{Aut_z}(G)$ if and only if $G' \leq C_G(\alpha)$.*

Proof. Let $\alpha^2 \in \mathrm{Aut_z}(G)$. Then for all $y \in G$, $\alpha^2(y) = yz$ for some $z \in Z(G)$. Let $x, y \in G$. Then by Lemma 5.2.1,

$$\begin{aligned}\alpha^2 \in \mathrm{Aut_z}(G) &\iff [x,y] = [x, \alpha^2(y)] \\ &\iff [x,y] = [\alpha(x), \alpha(y)] = \alpha([x,y]) \\ &\iff G' \leq C_G(\alpha).\end{aligned}$$

□

Let $E_2(G) = \{g \in G \mid [g,x,x] = 1 \ \forall x \in G\}$ denote the set of right 2-Engel elements of G.

Proposition 5.2.5 ([27, Proposition 2.5]) *Let G be a group. Then $E_2(G)$ is a subgroup of G and $E_2(G)' \leq Z_2(G)$.*

Proposition 5.2.6 ([27, Lemma 2.6(i)]) *Let G be a group and let $\alpha \in A(G)$. then $[G, \alpha] \leq E_2(G)$.*

Proof. Let $x, y \in G$. Then by Lemma 5.2.1 and Proposition 5.2.2,

$$\begin{aligned}[x^{-1}\alpha(x), y] &= [x^{-1}, y][x^{-1}, y, \alpha(x)][\alpha(x), y] \\ &= [x^{-1}, y][[x^{-1}, y], (x^{-1}\alpha(x))x][x, \alpha(y)] \\ &= [x^{-1}, y][[x^{-1}, y], x][[x^{-1}, y], x^{-1}\alpha(x)][[x^{-1}, y], x^{-1}\alpha(x), x][x, \alpha(y)] \\ &= [x^{-1}, y][x^{-1}, y, x][x, \alpha(y)] \\ &= y^{-1}x^{-1}y(\alpha(y))^{-1}x\alpha(y) \\ &= y^{-1}x^{-1}(\alpha(y))^{-1}yxy^{-1}\alpha(y)y \\ &= y^{-1}[x, y^{-1}\alpha(y)]y.\end{aligned}$$

Therefore

$$\begin{aligned}[y, x^{-1}\alpha(x)] &= ([x^{-1}\alpha(x), y])^{-1} \\ &= (y^{-1}[x, y^{-1}\alpha(y)]y)^{-1} \\ &= y^{-1}([x, y^{-1}\alpha(y)])^{-1}y \\ &= y^{-1}([y^{-1}\alpha(y), x])y \\ &= y^{-1}(x^{-1}[y, x^{-1}\alpha(x)]x)y \\ &= (xy)^{-1}[y, x^{-1}\alpha(x)](xy).\end{aligned}$$

It implies that $[y, x^{-1}\alpha(x)] \in C_G(xy)$ for all $x, y \in G$. Now, replacing y by $x^{-1}y$ implies that $[x^{-1}y, x^{-1}\alpha(x)] \in C_G(y)$. It follows that $[y, x^{-1}\alpha(x)] \in C_G(y)$. Thus $[x^{-1}\alpha(x), y, y] = 1$ for all $y \in G$, and hence $[G, \alpha] \leq E_2(G)$. □

Proposition 5.2.7 ([27, Theorem 1.4]) *If G is a group and $\alpha \in A(G)$, then $[G^2, \alpha] \subseteq Z_2(G)$.*

Proof. Let $x \in E_2(G) \cap C_G(G')$ such that $I_x \in \mathrm{Inn}(G)$. Then for all $y \in G$,

$$\begin{aligned}[I_x(y), y] &= [(x^{-1}y)x, y] \\ &= [x^{-1}y, y][x^{-1}y, y, x][x, y] \\ &= [x^{-1}, y][x^{-1}, y, y][x, y] \\ &= [x^{-1}, y][x, y] \\ &= [x^{-1}, y]x^{-1}y^{-1}xy \\ &= x^{-1}[x^{-1}, y]y^{-1}xy \\ &= 1.\end{aligned}$$

Thus $I_x \in A(G)$. Since $x \in C_G(G')$, I_x fixes G' element-wise, and therefore $I_x^2 \in \mathrm{Aut_z}(G)$ by Proposition 5.2.4. Let $I_x^2(g) = gz$ for some $z \in Z(G)$. Then

$$\begin{aligned} I_{x^2}(g) = gz &\implies g^{-1}x^{-2}gx^2 = z \\ &\implies x^2 \in Z_2(G). \end{aligned}$$

Thus $(E_2(G) \cap C_G(G'))/Z_2(G)$ is an elementary abelian 2-group by Proposition 5.2.5. Since $[G, \alpha] \subseteq E_2(G) \cap C_G(G')$ by Proposition 5.2.2 and Proposition 5.2.6, $[G, \alpha]^2 \subseteq Z_2(G)$, and hence $[G^2, \alpha] \subseteq Z_2(G)$. □

Proposition 5.2.8 ([64, Lemma 3.2]) *Let p be an odd prime and let G be a finite p-group such that $Z_2(G)$ is abelian. Then G is an $A(G)$-group.*

Proof. Since p is an odd prime, $G^2 = G$. Let $\alpha, \beta \in A(G)$ and $x \in G$. It follows from Proposition 5.2.7 that $x^{-1}\alpha(x), x^{-1}\beta(x) \in Z_2(G)$. Let $\alpha(x) = xg_1$ and $\beta(x) = xg_2$, where $g_1, g_2 \in Z_2(G)$. Observe that since $\alpha, \beta \in A(G)$, $g_1, g_2 \in C_G(x)$. Thus $[\alpha(x), \beta(x)] = [xg_1, xg_2] = 1$. By Lemma 5.2.3, $\alpha\beta \in A(G)$. Since $A(G)$ is closed under powers and G is finite we also have $\alpha^{-1} \in A(G)$. Thus $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. □

A finite non-abelian group G is called purely non-abelian if it has no nontrivial abelian direct factor. Observe that if $Z(G) \leq \Phi(G)$ or $Z(G)$ is cyclic, then G is purely non-abelian.

Lemma 5.2.9 *Let G be a finite non-abelian p-group, p an odd prime such that G/G' is elementary abelian and G is strongly Frattinian. Then G is an $A(G)$-group.*

Proof. Since G/G' is elementary abelian, $G' = \Phi(G)$. Also since $[Z_2(G), G'] = 1$

and G is strongly Frattinian, $Z_2(G) \leq C_G(G') = C_G(\Phi(G)) = Z(\Phi(G))$. Therefore $Z_2(G)$ is abelian and thus G is an $A(G)$-group by Proposition 5.2.8. □

Theorem 5.2.10 *Let G be a finite non-abelian p-group, p an odd prime such that* $\mathrm{IA}(G) = \mathrm{Inn}(G)$ *and* $|G/G'| = p^2$. *Then G is an $A(G)$-group.*

Proof. Since G/G' is elementary abelian, $G' = \Phi(G)$. We first prove that $Z(G) \leq \Phi(G)$. Suppose on the contrary that $Z(G) \nleq M$ for some maximal subgroup M of G. Take an element $g \in Z(G) \setminus M$ and an element z of order p in $Z(G) \cap G'$. The map α defined on G by $\alpha(mg^k) = mg^k z^k$ for every $m \in M$ and $k \in \{0, 1, ..., p-1\}$ is an IA-automorphism. By hypothesis α must be an inner automorphism of G. Let $\alpha = f_a$ for some $a \in G$. Since $g \in Z(G)$, we have $gz = \alpha(g) = f_a(g) = a^{-1}ga = g$, thus $z = 1$, which is a contradiction. Thus $Z(G) \leq \Phi(G)$. By using the arguments as in Theorem 4.2.4, we get that G has one the following forms:

(*i*) $G = E_1E_2 \cdots E_s$ is the central product of non-abelian p-groups E_i, where $[E_i, E_j] = 1$ for all $i \neq j$, $|E_i| = p^2|Z(G)|$ and $Z(G) = Z(E_i)$ for all $1 \leq i \leq s$; or

(*ii*) $G = EF$, where $E = C_G(F)$, $C_F(Z(\Phi(F)) = \Phi(F)$, $\Phi(E) \leq Z(G)$, and both E, F are Frattinian p-groups. Moreover, in case (b) either $E = Z(G)$(and therefore $G = F$), or E is a central product as in case (a).

If the group G is as in case (a), by Lemma 4.2.2, we have

$$\Phi(G) = \Phi(E_1)\Phi(E_2)...\Phi(E_s) = Z(E_1)Z(E_2)...Z(E_s) \leq Z(G)$$

because E_i are Redei p-groups. Therefore $Z(G) = \Phi(G)$. Therefore $\mathrm{Aut_z}(G) = \mathrm{Aut}^{\Phi}(G) = \mathrm{IA}(G) = \mathrm{Inn}(G)$ and thus by Lemma 4.2.5, $G' = Z(G)$ and $Z(G)$ is cyclic. Also since G is of class 2, $\exp(G') = \exp(G/Z(G)) = p$ by Lemma 2.2.4. Thus $|G'| = p$ and therefore G is an extraspecial p-group of order p^3 and so is an $A(G)$-group. Suppose G satisfies case (b) If $G = F$, then $C_G(Z(\Phi(G)) = \Phi(G)$ and therefore G is strongly Frattinian and thus G is an $A(G)$-group by Lemma 5.2.9. Let $G \neq F$. As observed in Theorem 4.2.4, we arrive at a contradiction, which completes the proof. □

Lemma 5.2.11 ([25],Corollary 3.7, 3.8) *Let G be a finite non-abelian p-group such that* $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$. *Then $Z(G) \leq G'$ and $Z(\mathrm{Inn}(G))$ is not cyclic.*

Theorem 5.2.12 *Let G be a finite non-abelian p-group of coclass* 3, *where p is an odd prime and G is strongly Frattinian. Then G is an $A(G)$-group.*

Proof. Since G is of coclass 3, we have the following choices of $Z(G)$, $Z_2(G)$ and G': $|Z(G)| = p,\ p^2,\ p^3$; $|Z_2(G)| = p^2,\ p^3,\ p^4$ and $|G'| = p^{n-2},\ p^{n-3},\ p^{n-4}$.

(*i*) If $|G'| = p^{n-2}$, then $G/G' \simeq C_p \times C_p$.

(*ii*) If $|G'| = p^{n-3}$, then $G/G' \simeq C_p \times C_p \times C_p$ or $G/G' \simeq C_{p^2} \times C_p$.

(*iii*) If $|G'| = p^{n-4}$, then $G/G' \simeq C_p \times C_p \times C_p \times C_p$ or $G/G' \simeq C_{p^3} \times C_p$ or $G/G' \simeq C_{p^2} \times C_{p^2}$ or $G/G' \simeq C_{p^2} \times C_p \times C_p$.

In (*i*), (*ii*), (*iii*) if G/G' is elementary abelian, then G is an $A(G)$-group by Lemma 5.2.9. If $G/G' \simeq C_{p^2} \times C_p$ or $G/G' \simeq C_{p^3} \times C_p$ or $G/G' \simeq C_{p^2} \times C_{p^2}$, then $d(G) = 2$. It follows [67, Remark 3.2] that $d(Z(G) = 1$ and $Z_2(G)/Z(G)$ is an elementary abelian group of order p^2. Thus $[Z_2(G), \Phi(G)] = 1$. It follows that $Z_2(G) \leq C_G(\Phi(G)) = Z(\Phi(G))$. Therefore $Z_2(G)$ is abelian and thus G

is an $A(G)$ group by Lemma 5.2.8. We are left with $G/G' \simeq C_{p^2} \times C_p \times C_p$. Here $d(G) = 3$. Also since $|Z(G)| = p$ and $|Z_2| = p^4$, $|Z_2(G)/Z(G)| = p^3$. Since $Z(G)$ is cyclic, G is purely non abelian. Therefore by Lemma 3.2.1, we have $|\operatorname{Aut_z}(G)| = |\operatorname{Hom}(G/G', Z(G))| = p^3 = |Z_2(G)/Z(G)|$. Now $Z(\operatorname{Inn}(G))$ is a subgroup of $\operatorname{Aut_z}(G)$, so $\operatorname{Aut_z}(G) = Z(\operatorname{Inn}(G))$. By Lemma 5.2.11, $Z(G) \leq G'$ and $Z_2(G)/Z(G)$ is not cyclic. It follows from Lemma 4.2.5 that $\operatorname{Aut_z}(G) \simeq \operatorname{Hom}(G/G', Z(G)) \simeq Z_2(G)/Z(G)$ and therefore $d(G)d(Z(G)) = d(Z_2(G)/Z(G))$ and hence $d(Z_2(G)/Z(G)) = 3$. Thus $Z_2(G)/Z(G)$ is elementary abelian and thus G is an $A(G)$-group by arguments used above. □

A group G is called an AC-group if the centralizer of every non-central element of G is abelian.

Lemma 5.2.13 ([31, Lemma 2.3]) *If G is an AC group, then $A(G) \leq \operatorname{Aut}(G)$.*

Theorem 5.2.14 *Let G be a finite non-abelian p-group. Let M_1 and M_2 be any two distinct maximal abelain subgroups of G such that $M_1 \cap M_2 = Z(G)$. Then G is an $A(G)$-group.*

Proof. Let $x \in G \setminus Z(G)$. We claim that $C_G(x)$ is abelian. If possible suppose that $C_G(x)$ is non-abelian. Let M be a maximal abelian subgroup of $C_G(x)$. Then $M \neq C_G(x)$ and $Z(G) < Z(G)\langle x\rangle \leq M$. Let $b \in C_G(x) \setminus M$ and let N be a maximal abelian subgroup of $C_G(x)$ containing $\langle b\rangle$ so that $M \neq N$ and N also contains the abelian subgroup $Z(G)\langle x\rangle$. Obviously, M and N are also maximal abelian subgroups of G but $Z(G) < Z(G)\langle x\rangle \leq M \cap N$ which contradicts the hypothesis. Thus $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$. Thus G is an AC-group and therefore by Lemma 5.2.13, G is an $A(G)$-group. □

In 2002, Deaconescu, Silberberg and Wall [27] suggested the following question about the set $A(G)$: Is it true that $A(G) = 1$ if and only if $\mathrm{Aut_z}(G) = 1$? We answer this question in case G is a purely non-abelian finite p-group.

Proposition 5.2.15 *Let G be a purely non-abelian finite p-group. Then $A(G) = 1$ if and only if* $\mathrm{Aut_z}(G) = 1$.

Proof. Let $A(G) = 1$. Since $\mathrm{Aut_z}(G)$ is contained in $A(G)$, $\mathrm{Aut_z}(G) = 1$. Conversely suppose $\mathrm{Aut_z}(G) = 1$. Since G is purely non-abelian, by Lemma 3.2.1

$$|\mathrm{Aut_z}(G)| = |\mathrm{Hom}(G/G', Z(G))| = 1$$

and therefore either $G = G'$ or $Z(G) = 1$. Since G is a p-group, $Z(G) \neq 1$. Therefore $G = G'$. Let $\alpha \in A(G)$, then by Proposition 5.2.2, $x^{-1}\alpha(x) \in C_G(G')$ for all $x \in G$. Therefore $x^{-1}\alpha(x) \in C_G(G) = Z(G)$ for all $x \in G$ and thus $\alpha \in \mathrm{Aut_z}(G)$. Hence $A(G) = \mathrm{Aut_z}(G) = 1$. $\square$

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