SIMILARITY MEASURE OF DIFFERENT TYPES OF FUZZY SETS

Thesis submitted in partial fulfillment of the requirement for

The award of the degree of

Masters of Science

in

Mathematics and Computing

Submitted by

Gurbinder Kaur

Roll No. 30803006

Under the guidance of

Dr. Amit Kumar

JULY 2010

School of Mathematics and Computer Applications

Thapar University

Patiala-147004 (PUNJAB)

INDIA

Dedicated

To

God, My Parents and My Supervisor

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Similarity measure of different types of fuzzy sets**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Amit Kumar**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Gurbinder Kaur
Gurbinder Kaur
Reg. No. 30803006

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

06/07/10
(Dr. Amit Kumar)
Supervisor
Assistant professor
SMCA
Thapar University, Patiala

Countersigned by:

15/7
Dr. S.S. Bhatia
(Professor and Head)
SMCA
Thapar University, Patiala

23/7
Dr. R.K. Sharma
Dean of Academic Affairs
Thapar University
Patiala

ACKNOWLEDGEMENT

First of all, I would like to thank the Almighty for granting perseverance. I would like to express my gratitude to **Dr. Amit Kumar, Assistant Professor, SMCA, Thapar University, Patiala,** for their patient guidance and support throughout this work. I was truly very fortunate to have the opportunity to work under him as a student. It was both an honor and a privilege to work with him. He also provides help in technical writing and presentation style and I found this guidance to be extremely valuable. I take this opportunity to express my sincere thanks to **Prof. S. S. Bhatia, Head SMCA, Thapar University, Patiala,** for their valuable support and help without which it would not have been possible for me to complete this work.

I am also thankful to **Mr. Pushpinder Singh**, research scholar, SMCA, and all my friends who devoted their valuable time and helped me in all possible ways towards successful completion of this work. I do not find enough words with which I can express my feeling of thanks to the entire faculty, research scholars and staff of SMCA, Thapar University, Patiala, for their help, inspiration and moral support which went a long way in successful completion of my work. I thank all those who have contributed directly or indirectly to this work.

Lastly, and more importantly, I would like to thank my parents for their years of unyielding love and encouragement. They have always wanted the best for me and I admire my parent's determination and sacrifice to put me through college.

Patiala

July 6 , 2010

Gurbinder Kaur

Gurbinder Kaur

ABSTRACT

In today's highly competitive world, all the people compare the things with each other. Similarity measure is the concept which helps us to know how much two things are similar. For this we, firstly calculate the degree of similarity. Higher is the degree of similarity between two things, they are more similar to each other.

In this thesis, similarity measure between different types of fuzzy numbers is calculated.

The chapter-wise summary of the thesis is as follows:

Chapter 1 is introductory in nature. This chapter includes basics, and concepts used throughout the work.

Chapter 2 presents brief review of the work done in the area of fuzzy similarity measure problem.

In **Chapter 3** we studied the similarity measure between generalized fuzzy numbers. To illustrate the presented method a numerical example is solved and also some properties of similarity measure of generalized fuzzy numbers are proved.

In **Chapter 4** Similarity measure between interval-valued fuzzy sets is studied. The presented method is illustrated by solving a numerical example and some properties are proved.

Chapter 5 In this chapter similarity measure between intuitionistic fuzzy sets has been studied. Also in this chapter we give the shortcomings of some already proposed methods and to overcome those shortcomings other methods are presented. The presented method is illustrated by solving a numerical example and some properties are proved.

LIST OF CONTENTS

Chapter 1

Chapter 2

Chapter 3

Chapter 4

Chapter 5

Chapter 1

INTRODUCTION

The concept of similarity is fundamentally important in almost every scientific field. For example, in mathematics, geometric methods for assessing similarity are used in studies of congruence and homothetic as well as in allied fields such as trigonometry. Topological methods are applied in fields such as semantics. Graph theory is widely used for assessing similarities in taxonomy. Fuzzy set theory has also developed its own measures of similarity, which find application in areas such as management, medicine and meteorology. An important problem in molecular biology is to measure the sequence similarity of pairs of proteins.

The degree to which people perceive two things as similar fundamentally affects their rational thought and behavior. Negotiations between politicians or corporate executives may be viewed as a process of data collection and assessment of the similarity of hypothesized and real motivators. The appreciation of a fine fragrance can be understood in the same way. Similarity is a core element in achieving an understanding of variables that motivate behavior and mediate affect.

Not surprisingly, similarity has also played a fundamentally important role in psychological experiments and theories. For example, in many experiments people are asked to make direct or indirect judgments about the similarity of pairs of objects. A variety of experimental techniques are used in these studies, but the most common are to ask subjects whether the objects are the same or different, or to ask them to produce a number, between say 1 and 7, that matches their feelings about how similar the objects appear (e.g., with 1 meaning very dissimilar and 7 meaning very similar). The concept of similarity also plays a crucial but less direct role in the modeling of many other psychological tasks. This is especially true in theories of the recognition, identification, and categorization of objects, where a common assumption is that the greater the similarity between a pair of objects, the more likely one will be confused with the other. Similarity also plays a key role in the modeling of preference and liking for products or brands, as well as motivations for product consumption. A common assumption here is that when evaluating a product, people imagine an ideal and then judge the similarity of the offered product to this ideal [17].

1.1 Some general properties of similarity measure

1) A similarity measure can represent the similarity between two documents, two queries, or one document and one query.

2) It is possible to rank the retrieved documents in the order of presumed importance.

3) A similarity measure is a function which computes the degree of similarity between pair of text objects.

4) There are a large number of similarity measures proposed in the literature, because the best similarity doesn’t exist(yet!).All similarity measures should map to the range $[-1,1]$ or $[0,1]$.

5) 0 or -1 shows minimum similarity. (incompatible similarity)

6) 1 shows maximum similarity. (absolute similarity)

1.2 Models of similarity measure

There are three models of similarity measure:

1) Distance based similarity measure

2) Feature based similarity measure

3) Probabilistic similarity measure

1.2.1 Distance based similarity measure

One of the oldest and most influential theoretical assumptions is that perceived similarity is inversely related to psychological distance. A fundamental assumption of many psychological theories is that percepts that are close together will be perceived as similar and percepts that are far apart will be perceived as dissimilar.

1.2.1.1 Multi-dimensional scaling

Multi-dimensional scaling is a technique that uses similarity judgments (or some other proximity measure) to produce a psychological space in which similarity is inversely related to distance. The output of multi-dimensional scaling computer program is a set of coordinates for each stimulus in some hypothetical psychological space. Within this space, stimuli that were judged by subjects to be similar are close together.

The two most popular distance measures in multi-dimensional scaling are Euclidean distance (i.e., as the crow flies distance) and city-block distance. If the coordinates of some stimulus A in an n-dimensional psychological space are $\left(x_{A_1}, x_{A_2}, x_{A_3}, \ldots, x_{A_n}\right)$ then the Euclidean distance from stimulus A to some other stimulus B is,

$$d\left(A,B\right)=\sqrt{\sum_{i=1}^{n}\left(x_{A_i}-x_{B_i}\right)^2}$$

When $n=2$, this equation reduces to the familiar formula from the Pythagorean theorem. The city-block distance between these two stimuli is defined as,

$$d\left(A,B\right)=\sum_{i=1}^{n}\left|x_{A_i}-x_{B_i}\right|$$

Shepard [44] proposed as a universal law that distance and perceived similarity are related via an exponential function

$$S\left(A,B\right)=e^{-d\left(A,B\right)}$$

He further proposed that this exponential function describes the probability that two stimuli fall in a region of stimulus space associated with the same response, which he called a consequential region. Shepard noted some failures of the exponential function in the case of confusable objects, although it was later shown that these problems can be resolved by treating percepts as probabilistic and applying Shepard's [44] similarity function at the moment of decision-making [18].

All types of distance obey certain properties, called the distance axioms. For example, the distance from point A to point B must be equal to the distance from point B to point A. If similarity is inversely related to psychological distance, then perceived similarity must also obey these axioms. Considerable empirical effort therefore, has been spent testing the validity of the distance axioms. The four distance axioms are,

1) Equal self-similarity: $d(A,A)=d(B,B)$ for all points. Therefore, $S(A,A)=S(B,B)$ for all stimuli A and B.

2) Minimality: $d(A,B)>d(A,A)$ for all points A and B. Therefore, $S(A,B)<S(A,A)$ for all stimuli A and B.

3) Symmetry: $d(A,B)=d(B,A)$ for all points A and B. Therefore, $S(A,B)=S(B,A)$ for all stimuli A and B.

4) Triangle inequality: $d(A,B)+d(B,C)>d(A,C)$ for all points A,B and C. Therefore, the dissimilarities among any set of three stimuli should satisfy this same condition. The triangle inequality also implies that if stimuli A and B are similar and stimuli B and C are similar, then stimuli A and C must also be similar.

1.2.2 Feature based similarity measure

Partly in response to empirical evidence against the distance axioms, Tversky [50] presented that perceived similarity is the result of a feature-matching process that differentially weights common and distinct stimulus features. Let $g\left(A\cap B\right)$ denote the salience of the features that are common to stimuli A and B and let $g\left(A-B\right)$ denote the salience of the features that are unique to stimulus A. Then Tversky's [50] feature contrast model proposes that the similarity of stimulus A to stimulus B is equal to

$$S\left(A,B\right)=\alpha\, g\left(A\cap B\right)-\beta\, g\left(A-B\right)-\gamma\, g\left(B-A\right)$$

Where α,β and γ are constants that might vary across individuals, context, and instructions. According to this model, features in common increase similarity, whereas features that are unique to one stimulus decrease similarity. One advantage of the feature contrast model is that it can account for violations in any of the distance.

1.2.3 Probabilistic similarity measure

All similarity measures considered so far assume that repeated presentation of the same stimulus always elicits the exact same percept – that is, they assume that the percept is deterministic. But many theorists have argued that the information that forms a percept varies over time, and thus that percepts are probabilistic. This is consistent with personal experience concerning the taste of products, views on political issues, or opinions about people. Biological processes involved in generating percepts, the chemical and physical variation associated with stimuli, and limitations on our ability to know the information state absolutely all favor models that assume probabilistic percepts.

Many probabilistic models have been proposed. In general these models all make two assumptions that were inspired by Thurstone [48] and by signal detection theory

Tanner and Swets [47] the percept elicited by a stimulus varies probabilistically over repeated exposures to that stimulus) there is a well-defined decision rule that describes how a response is selected for any momentary value of the percept. Some probabilistic models retain the assumption that similarity is inversely related to psychological distance by Ennis and Johnson [19]. Other models rely instead on the signal-detection notion of a decision bound by Ashby and Perrin [1]. Assuming the percept is probabilistic fundamentally changes the predictions of these models. For example, all of the models that assume a probabilistic percept can account for violations in at least some of the distance axioms.

1.3 Fuzzy set theory

In this section some basic definitions of fuzzy set theory are presented [26].

Definition 1.1

A crisp set or a classical set A is defined as a collection of distinct and distinguishable objects. The objects are called elements of A. A crisp set A, defined on the universal set X, can also be represented by $A = \{(x, \mu_A(x)); x \in X\}$ where $\mu_A : X \to \{0,1\}$ is called characteristic function defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 1.2

The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1

to each member in X . This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range $[0,1]$ i.e., $\mu_{\tilde{A}} : X \to [0,1]$. The assigned values indicate the membership grade of the element in the set A.

The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \left\{(x, \mu_{\tilde{A}}(x)); x \in X\right\}$ defined by $\mu_{\tilde{A}}$ for each $x \in X$ is called a fuzzy set. $\mu_{\tilde{A}}(x)$ is the degree of membership of x in $\tilde{A}$. The closer the value of $\mu_{\tilde{A}}(x)$ is to 1, the more x belongs to $\tilde{A}$.

Definition 1.3

Let $\tilde{A}$ be a fuzzy set and α be a real number in the interval $\left[0,1\right]$. The crisp set A_α defined by $A_\alpha = \left\{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\right\}$ is called $\alpha-$ cut of $\tilde{A}$.

The crisp set $A_{\alpha^+} = \left\{x \in X : \mu_{\tilde{A}}(x) > \alpha\right\}$ is called strong $\alpha-$ cut of $\tilde{A}$.

Definition 1.4

A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be a fuzzy number if its membership function has the following characteristics:

1) $\mu_{\tilde{A}} : R \to [0,1]$ is continuous.

2) $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, c] \cup [d, \infty)$.

3) Its strictly increasing on $[c, a]$ and strictly decreasing on $[b, d]$.

4) $\mu_{\tilde{A}}(x) = 1$ for all $x \in [a,b]$.

Definition 1.5

The support of a fuzzy set $\tilde{A}$ is the crisp subset of X and is presented as:

$$\operatorname{supp}(\tilde{A}) = \left\{ x \in X \mid \mu_{\tilde{A}}(x) > 0 \right\}.$$

Definition 1.6

A fuzzy number $\tilde{A} = (a,b,c)$ is called a triangular fuzzy number if its membership function $\mu_{\tilde{A}}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{x-c}{b-c}, & b \le x \le c \end{cases}$$

The triangular fuzzy number $\tilde{A}$ has the shape of a triangle as shown in Fig. 1.1 given below:

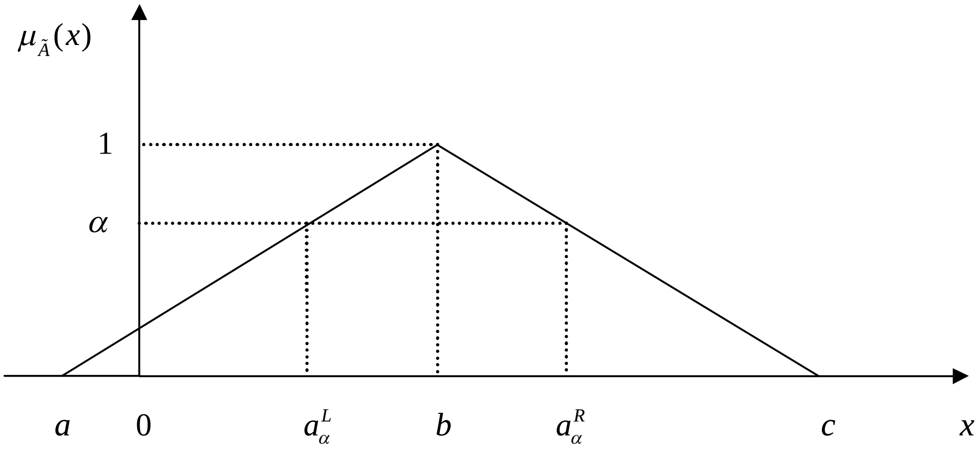

Fig. 1.1 A Triangular fuzzy number $\tilde{A} = (a,b,c)$

Further, the α - cut of the triangular fuzzy number $\tilde{A}=(a,b,c)$ is the closed interval

$$A_\alpha=\left[a_\alpha^L,a_\alpha^R\right]=\left[a+(b-a)\alpha\ ,\ c+(b-c)\alpha\right],\ \alpha\in(0,1].$$

Definition 1.7

A fuzzy number $\tilde{A}=(a,b,c,d)$ is called a trapezoidal fuzzy number if its membership function is given by:

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-a}{b-a}, & a\le x\le b\\ 1, & b\le x\le c\\ \dfrac{x-d}{c-d}, & c\le x\le d\end{cases}$$

The trapezoidal fuzzy number $\tilde{A}$ has the shape of a trapezoid as shown in the Fig. 1.2 given below:

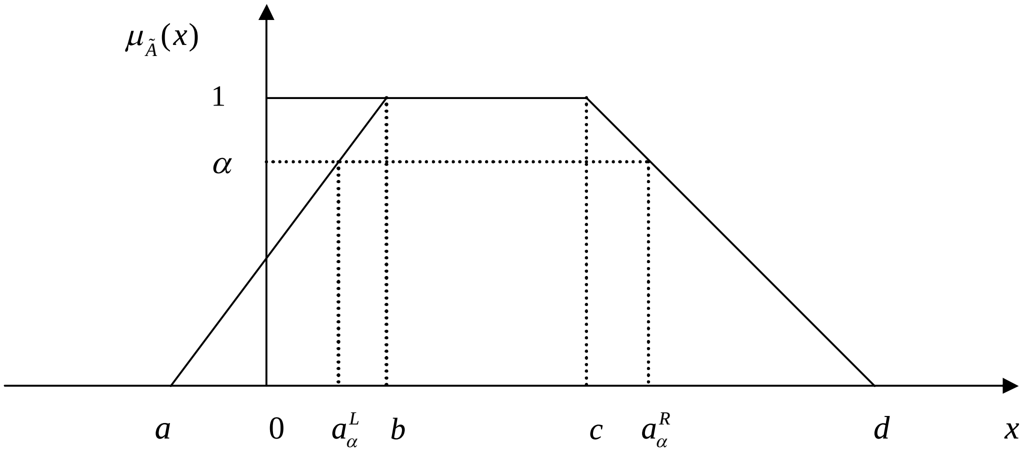

Fig. 1. 2 A Trapezoidal fuzzy number $\tilde{A}=(a,b,c,d)$

Further, the α - cut of the trapezoidal fuzzy number $\tilde{A}=(a,b,c,d)$ is the closed interval

$$A_\alpha=\left[a_\alpha^L,a_\alpha^R\right]=\left[a+(b-a)\alpha\ ,\ d+(c-d)\alpha\right],\ \alpha\in(0,1].$$

1.4 Fuzzy similarity measure

Fuzzy similarity measure is a significant conception in fuzzy set theory. Fuzzy similarity measure describes the similarity degree between two fuzzy subsets.

Let $F(U)$ denote the fuzzy set with domain U. For $\tilde{A} \in F(U)$ and $u \in U$, $\mu_{\tilde{A}}(u)$ is defined as the membership function of u for $\tilde{A}$ and $\tilde{A}^c$ denote the supplementary set of $\tilde{A}$, which is given by $\mu_{\tilde{A}^c}(u) = 1 - \mu_{\tilde{A}}(u)$. The general definition of fuzzy similarity measure is as follows,

Definition 1.8 [37], A real function $S : F(U) \times F(U) \to [0,1]$ is defined as the fuzzy similarity measure of $F(U)$, if S satisfies following properties

1) $\forall \tilde{A}, \tilde{B} \in F(U),\ S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A})$

2) If $\tilde{A} \in F(U)$, then $S(\tilde{A}, \tilde{A}^c) = 0$

3) $\forall \tilde{A} \in F(U),\ S(\tilde{A}, \tilde{A}) = 1$

4) $\forall \tilde{A}, \tilde{B}, \text{and}\, \tilde{C} \in F(U), \text{if } \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}, \text{then } S(\tilde{A}, \tilde{B}) \geq S(\tilde{A}, \tilde{C}), S(\tilde{B}, \tilde{C}) \geq S(\tilde{A}, \tilde{C})$

Definition 1.9 [39] A real function $S : F(U) \times F(U) \to [0,1]$ is defined as the fuzzy similarity measure of $F(U)$, if S satisfies following properties,

1) $\forall \tilde{A}, \tilde{B} \in F(U),\ S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A})$

2) If $\mu_{\tilde{A}}(u) \in \{0,1\}$, then $S\left(\tilde{A}, \tilde{A}^c\right) = 0$

3) $\forall \tilde{A} \in F(U), S\left(\tilde{A}, \tilde{A}\right) = 1$

4) $\forall \tilde{A}, \tilde{B}, \text{and}\, \tilde{C} \in F(U), \text{if } \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}, \text{then}\, S\left(\tilde{A}, \tilde{C}\right) \leq \text{minimum}\left(S\left(\tilde{A}, \tilde{B}\right), S\left(\tilde{B}, \tilde{C}\right)\right)$

1.5 Some types of distance based similarity measure

The most obvious way of calculating similarity of fuzzy sets is based on their distance. There are more approaches on how the relation between the two notions in form of a function can be expressed. Two of them are presented below

1.5.1 Non α-cut based similarity measures

There are many useful distance definitions of fuzzy sets in the literature. The simplest one is the Disconsistency Measure of the fuzzy sets $\tilde{A}$ and $\tilde{B}$

$$S_D = 1 - \sup \mu_{\tilde{A} \cap \tilde{B}}(x) \tag{1.1}$$

where $\tilde{A} \cap \tilde{B}$ is the min *t*-norm, $\mu_{\tilde{A} \cap \tilde{B}}(x) = \text{Minimum}\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\} \forall x \in X$. It is basically the same measure as used in the min-max composition. The disconsistency measure is one crisp value in range of [0, 1].

In the followings, some distance measures, which are used for expressing the similarity of trapezoidal shaped fuzzy sets (or fuzzy sets have membership functions can be traced back to a trapezoid form) will be presented.

In case of trapezoidal shaped fuzzy sets, the fuzzy set can be characterized by a vector of four values, by the upper and lower endpoints of the core and support e.g.

$X = \left(x_1, x_2, x_3, x_4\right)$

In this case the similarity between sets $\tilde{A}$ and $\tilde{B}$ can be described by the formula (1.2) presented by Chen [9].

$$S\left(\tilde{A},\tilde{B}\right) = 1 - \frac{\sum_{i=1}^{4}\left|a_i - b_i\right|}{4} \tag{1.2}$$

If the universes of the fuzzy sets are normalized, then $S\left(\tilde{A},\tilde{B}\right) \in \left[0,1\right]$. The advantage of (1.2) is its simplicity and low computational complexity. However, its drawback is that it can easily lead to the same grade of similarity in case of different shapes, too.

1.5.2 α-cut based similarity measures

Most of the distance definitions are based on the α-cuts of the two fuzzy sets, for example:

Hausdorff Measure (∞):

$$HM_{\infty}\left(\tilde{A},\tilde{B}\right) = \underset{\alpha \geq 0}{\text{Supremum}}\ HM\left(\tilde{A}_{\alpha},\tilde{B}_{\alpha}\right) \tag{1.3}$$

Hausdorff Measure (*):

$$HM_{*}\left(\tilde{A},\tilde{B}\right) = HM\left(\tilde{A}_{1},\tilde{B}_{1}\right) \tag{1.4}$$

where

$$HM\left(\tilde{A},\tilde{B}\right) = \text{Maximum}\left\{\underset{\mathrm{a}\in\tilde{\mathrm{A}}}{\text{Supremum}}\ \underset{b\in\tilde{B}}{\text{Infimum}}\, d\left(a,b\right), \underset{a\in\tilde{B}}{\text{Supremum}}\ \underset{b\in\tilde{A}}{\text{Infimum}}\, d\left(a,b\right)\right\} \tag{1.5}$$

and $d\left(a,b\right)$ is the Euclidean distance.

Kaufmann and Gupta Measure (∞):

$$\Delta_{\infty}\left(\tilde{A},\tilde{B}\right) = \underset{\alpha \geq 0}{\text{Supremum}}\,\Delta\left(\tilde{A}_{\alpha},\tilde{B}_{\alpha}\right) \tag{1.6}$$

Kaufmann and Gupta Measure (*):

$$\Delta_*\left(\tilde{A},\tilde{B}\right)=\Delta\left(\tilde{A}_1,\tilde{B}_1\right) \tag{1.7}$$

where

$$\Delta\left(\tilde{A}_\alpha,\tilde{B}_\alpha\right)=\frac{\left(|a_1-b_1|+|a_2-b_2|\right)}{2\left(\beta_2-\beta_1\right)} \tag{1.8}$$

and $\left[a_1,a_2\right],\left[b_1,b_2\right]$ are the supports of $\tilde{A}_\alpha,\tilde{B}_\alpha$, respectively $\left[\beta_1,\beta_2\right]$ is the support of both $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$, $\alpha\in\left[0,1\right]$.

Both the Hausdorff Measure and Kaufmann and Gupta Measure are a crisp value in range of [0, ∞].

1.6 Applications of fuzzy similarity measure

1) Grid resource scheduling based on similarity measure [39] The approach is a family of fuzzy similarity measures, and any common measures can be obtained easily only by specifying a parameter. The resource scheduling of Grid is just based on it. Then, a formal model of Grid resources with three layers is proposed for effectively and efficiently organizing and accessing all kinds of heterogeneous resources.

2) Color image retrieval and classification [3] As a hotspot in image processing color image retrieval and classification are very important in the field of image processing. A method is proposed which is in token of color characteristic of one image using hue and there by is used to calculate similarity between two pictures.

3) Content based image retrieval [33] Content-based image retrieval (CBIR) aims at efficient retrieval of relevant images from large image database based on automatically derived imagery features. Many fuzzy methods have been applied to the content-based image retrieval (CBIR) to retrieve the similar images according to the similarity of fuzzy sets.

4) A matching pursuit for fuzzy clustering and classification of signals [40] Matching pursuits is a well known technique for signal representation and has also been used as a feature extractor for some classification systems. A matching pursuits based similarity measure that uses only the dictionary, coefficients and residual information provided by the matching pursuit algorithm while comparing two signals. Hence it is easily applicable to a variety of problems.

5) Pattern recognition [22] Pattern recognition is an important issue in Information Science and Artificial Intelligence (AI). Similarity measure is the basic theoretical foundation in pattern recognition of fuzzy information.

6) In fingerprint matching [15] Fuzzy similarity measure is used to develop a algorithm for distorted fingerprints matching, which is based on normalized similarity measure.

7) Risk analysis [14] Fuzzy similarity measure is also used in fuzzy risk analysis problem. Similarity measure is the basic theoretical foundation in pattern recognition of fuzzy information.

8) In music copyright system [28] Fuzzy similarity measure is also used in music copyright system for music phoneme segmentation.

Chapter 2

LITERATURE REVIEW

The theory of fuzzy sets, proposed by Zadeh [54], has gained successful applications in various fields. Measures of similarity between fuzzy sets, as an important content in fuzzy mathematics, have gained attention from researchers for their wide applications in real world. Based on similarity measures that are very useful in some areas, such as pattern recognition, machine learning, decision making and market prediction, many measures of similarity between fuzzy sets have been proposed and researched in recent years.

Zwick et al. [55] reviewed 19 similarity measures of fuzzy sets and compared their performance in an experiment.

Turksen and Zhong [49] used similarity measure of fuzzy sets to proposed an Approximation Analogical reasoning schema (AARS) which exhibits the advantage of fuzzy set theory and analogical reasoning in expert system.

Bilgic and Turksen [4] proposed the search process to fire rules in Pattern Matching Inferencing using similarity measure. Also, they proposed two new search strategies using the properties of similarity measure.

Pappies and Karacatilides [41] presented and compared the properties of several measures of similarity of fuzzy values. The measures examined include the measure based on the union and intersection, the one based on the maximum difference and the one based on the differences as well as the sum of corresponding grades of membership. It is shown that several properties are common to all measures. However, some properties do not hold for all of them.

Hyung et al. [25] proposed two similarity measure, one for the similarity between fuzzy sets and the other between elements in fuzzy sets. With example, it is shown that the proposed measures can be used in the behavior analysis in an organization.

Chen et al. [16] extended the work of Pappis and Karacapilidis [41] to present and compare the properties of several measures of similarity of fuzzy values. The measures examined based on the geometric model, the set-theoretic approach, and the matching function *S*. It is shown that several properties are common to all measures and some properties do not hold for all of them.

Chen [7] proposed two similarity measures for measuring the degree of similarity between vague sets. The proposed measures can provide a useful way for measuring the degree of similarity between vague sets.

Wang [51] proposed two new similarity measures to represent the similarity measure between fuzzy sets and between elements, respectively. These similarity measures can be computed easily and express the confidable similarity relation apparently. Some examples are illustrated for the comparison between these measures and other previous measures.

Chen [9] presented some similarity measures between vague sets and between elements. An example is also presented to illustrate the application of the proposed similarity measures in handling behavior analysis problems. The proposed method can provide a useful way in handling the behavior analysis problems.

Santini and Jain [43] developed a similarity measure, based on fuzzy logic, that exhibits several features that match experimental findings in humans. The model is dubbed Fuzzy Feature Contrast (FFC) and is an extension to a more general domain of the Feature Contrast model due to Tversky. They showed how the FFC model can be used to model similarity assessment from fuzzy judgment of properties, and addressed the use of fuzzy measures to deal with dependencies among the properties.

Li and Cheng [32] presented a new method for fuzzy risk analysis based on similarity measures of generalized fuzzy numbers. Firstly, they presented a method called the simple center of gravity method to calculate the center-of-gravity points of generalized fuzzy numbers. There are many concepts of measures of similarity between fuzzy sets have been proposed in the literature, those measures cannot deal with the similarity measures between intuitionistic fuzzy sets. To overcome this shortcoming they introduce the definition of the degree of similarity between IFSs is introduced. Then, they proposed several new similarity measures between intuitionistic fuzzy sets. And applied the proposed similarity measures of intuitionistic fuzzy sets to pattern recognitions.

Chen and Chen [11] used the simple center of gravity method to proposed a new method to measure the degree of similarity between generalized fuzzy numbers. The proposed similarity measure used the simple center of gravity method to calculate the center-of-gravity points of trapezoidal or triangular generalized fuzzy numbers and then to calculate the degree of similarity between generalized fuzzy numbers. They also proved some properties of the proposed similarity measure and used an example to compare the proposed method with the existing similarity measures. The proposed similarity measure can overcome the drawbacks of the existing methods. They also applied the proposed similarity measure to develop a new method to deal with fuzzy risk analysis problems.

Liu [38], pointed out that the similarity measure, proposed by Li and Cheng [32] are not reasonable in some cases. To overcome this shortcoming Liu [38] proposed some new methods for measuring the degree of similarity between intuitionistic fuzzy sets and between elements and also discussed their properties. By comparing Liu [38] showed that the proposed methods turned out to be more reasonable. Finally, Liu [38] used the same examples given by Li and Cheng [32] to show the application of the new measure methods in pattern recognition.

Chen and Chen [11] applied the center-of-gravity based similarity measures of generalized fuzzy numbers to deal with fuzzy risk analysis problems. However, when two different generalized fuzzy numbers have the same center-of-gravity points, the Chen's method cannot correctly determine the fuzzy risk. To overcome above shortcoming, a new hierarchical fuzzy risk assessment method is proposed by Liao et al. [36]. In the proposed method, a fuzzy weighted average method is presented to calculate the fuzzy risk value on the basis of the hierarchical security structures, and the similarity measure of generalized fuzzy numbers is used to create a complete linguistic approximation model for the network security risks. Finally, two examples demonstrate the feasibility and validity of the proposed method.

Li et al. [34], compared and summarized the existing similarity measures between intuitionistic fuzzy sets vague sets by their counterintuitive examples in pattern recognition, and demonstrated The positive aspects of each similarity measure, along with counter cases and discussed the conditions under which each may not work as desired.

Chen and Chen [12] proposed a new similarity measure between interval-valued fuzzy numbers. The proposed similarity measure considers the similarity of the gravities on the X-axis between upper fuzzy numbers, the difference of the spreads between upper fuzzy numbers, the heights of the upper fuzzy numbers, the degree of similarity on the X-axis between interval-valued fuzzy numbers, and the gravities on the Y-axis between interval-valued fuzzy numbers. They also proved three properties of the proposed similarity measure between interval-valued fuzzy numbers.

Chen and Chen [13] presented a new method for handling fuzzy risk analysis problems based on measures of similarity between interval-valued fuzzy numbers. They proposed a similarity measure to calculate the degree of similarity between interval-valued fuzzy numbers. The proposed similarity measure uses the concept of geometry to calculate the center-of-gravity points of the lower fuzzy numbers and the upper fuzzy numbers of interval-valued fuzzy numbers, respectively, to calculate the degree of similarity between interval-valued fuzzy numbers. They also proved some properties of the proposed similarity measure. By comparing the results they showed that the proposed method is more flexible and more intelligent than the methods presented in Chen and Chen [11].

Sanguansat and Chen [42] presented a new similarity measure between interval-valued fuzzy numbers. It combines the concepts of geometric distance, the perimeters and the spreads of differences between interval-valued fuzzy numbers on both the X-axis and the Y-axis. The proposed method can overcome the drawbacks of the existing similarity measures. Finally, based on the proposed similarity measure between interval-valued fuzzy numbers, they presented a new fuzzy risk analysis algorithm for dealing with fuzzy risk analysis problems.

Wei and Chen [52] presented a new method for fuzzy risk analysis based on similarity measures between generalized fuzzy numbers. Firstly, they presented a new similarity measure between generalized fuzzy numbers. It combines the concepts of geometric distance, the perimeter and the height of generalized fuzzy numbers for calculating the degree of similarity between generalized fuzzy numbers. They also proved some properties of the proposed similarity measure. They made an experiment to use 15 sets of generalized fuzzy numbers to compare the experimental results of the proposed method with the existing similarity measures. The proposed method can overcome the drawbacks of the existing similarity measures. Based on the proposed similarity measure between generalized fuzzy numbers, they presented a new fuzzy risk analysis algorithm for dealing with fuzzy risk analysis problems.

Xu et al. [53] presented an efficient approach for fuzzy risk analysis based on some new arithmetic operators of the trapezoidal fuzzy numbers and propose a new similarity of the trapezoidal fuzzy numbers to deal with fuzzy risk analysis problems. At the same time, they made an experiment to use 30 sets of trapezoidal fuzzy numbers to compare the experimental results of proposed approach with the existing similarity measures. At last, they used an example to illustrate the efficiency of the new approach.

Chapter 3

SIMILARITY MEASURE OF GENERALIZED FUZZY NUMBERS

In this chapter center-of-gravity (COG) method is used to calculate similarity measure of generalized fuzzy numbers. The traditional COG method [10] is very useful to deal with the defuzzification problems and the fuzzy ranking problems by using the COG points. However, there are some drawbacks in the traditional COG method, i.e., it cannot directly calculate the COG point of a crisp interval or a real number, and it is very time-consuming to calculate the COG point. A modified method, called the simple center-of-gravity method (SCGM), is presented to calculate the COG points of fuzzy numbers based on the concepts of plane vectors and linear equations. The proposed SCGM method can overcome the drawbacks of the traditional COG method.

3.1 Preliminaries

In this section, we briefly review the concept of generalized fuzzy numbers and their arithmetic operations, the traditional COG method and some existing similarity measure of fuzzy numbers.

3.1.1 Generalized fuzzy numbers

Chen [6] represented a generalized fuzzy number $\tilde{A}$ as $\tilde{A} = (a, b, c, d; w)$ where $0 < w \leq 1$ and a, b, c and d are real numbers. The generalized fuzzy number $\tilde{A}$ is a fuzzy subset of R, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions.

1) $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to the closed interval [0, 1];

2) $\mu_{\tilde{A}}(x)=0$, where $-\infty < x \leq a$

3) $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a,b]$

4) $\mu_{\tilde{A}}(x)=w$, where $b \leq x \leq c$

5) $\mu_{\tilde{A}}(x)$ is strictly decreasing on $[c,d]$

6) $\mu_{\tilde{A}}(x)=0$, where $d \leq x < \infty$

If $w=1$, then the generalized fuzzy number $\tilde{A}$ is called a normal fuzzy number denoted as $\tilde{A}=(a,b,c,d)$. If $a=b$ and $c=d$, then $\tilde{A}$ is called a crisp interval. If $a=b=c=d$ and $w=1$, then $\tilde{A}$ is called a real number.

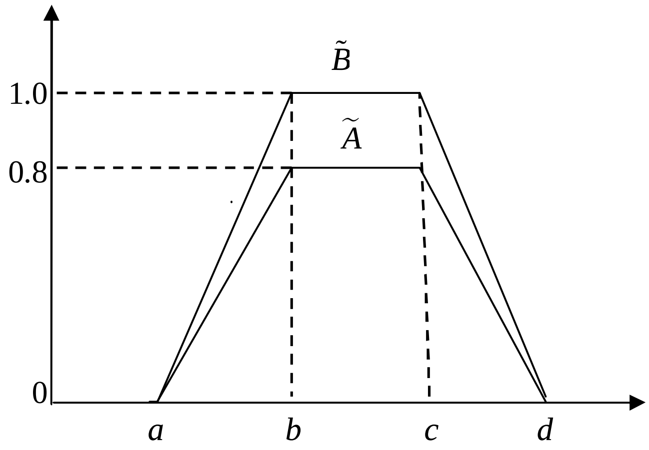

Fig. 3.1 Two generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$

Fig. 3.1 shows two different generalized trapezoidal fuzzy numbers i.e., $\tilde{A}=(a,b,c,d;w_1)$ and $\tilde{B}=(a,b,c,d;w_2)$ which denote two different decisionmakers's opinions. The values w_1 and w_2 represent the degrees of confidence of the opinions of the decisionmakers A and B respectively, where $w_1 = 0.8$ and $w_2 = 1.0$.

3.1.2 Arithmetic operations

The traditional fuzzy arithmetic operations only can deal with normalized fuzzy numbers, they not only change the type of membership function of fuzzy number after arithmetical operations, but also have a drawback of requiring troublesome and tedious arithmetical operations. Chen [6] proposed the function principle, which could be used as the fuzzy arithmetical operations between generalized fuzzy numbers, where these fuzzy arithmetical operations can deal with the generalized fuzzy numbers (i.e., nonnormal fuzzy numbers). Hsieh and Chen [24] pointed out that fuzzy arithmetical operators presented in [6], do not change the type of membership function of fuzzy number after arithmetical operations, but also can reduce the troublesomeness and tediousness of arithmetical operations. Thus, we use Chen's fuzzy arithmetical operators to deal with the fuzzy arithmetical operations between generalized fuzzy numbers. Assume that there are two generalized trapezoidal fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$, where $\tilde{A}_1 = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2; w_2)$.

The arithmetic operations between the generalized trapezoidal fuzzy numbers are given as follows:

1) Fuzzy Numbers Addition $\oplus$:

$$\tilde{A}_1 \oplus \tilde{A}_2 = \left(a_1, b_1, c_1, d_1; w_1\right) \oplus \left(a_2, b_2, c_2, d_2; w_2\right) = \left(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \text{Minimum}\left(w_1, w_2\right)\right)$$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2$ and d_2 are any real numbers.

2) Fuzzy Numbers Subtraction $\ominus$:

$$\tilde{A}_1 \ominus \tilde{A}_2 = (a_1, b_1, c_1, d_1; w_1) \ominus (a_2, b_2, c_2, d_2; w_2) =$$

$$(a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \text{Minimum}\ (w_1, w_2))$$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2$ and d_2 are any real numbers.

3) Fuzzy Numbers Multiplication $\otimes$:

$$\tilde{A}_1 \otimes \tilde{A}_2 = \left(a, b, c, d; \text{Minimum}\left(w_1, w_2\right)\right);$$

where $a = \text{Minimum}\left(a_1 \times a_2, a_1 \times d_2, d_1 \times a_2, d_1 \times d_2\right)$

$$b = \text{Minimum}\left(b_1 \times b_2, b_1 \times c_2, c_1 \times b_2, c_1 \times c_2\right)$$

$$c = \text{Maximum}\left(b_1 \times b_2, b_1 \times c_2, c_1 \times b_2, c_1 \times c_2\right)$$

$$d = \text{Maximum}\left(a_1 \times a_2, a_1 \times d_2, d_1 \times a_2, d_1 \times d_2\right)$$

It is obvious that if $a_1, b_1, c_1, d_1, a_2, b_2, c_2$ and d_2 are all positive real numbers, then

$$\tilde{A}_1 \otimes \tilde{A}_2 = \left(a_1 \times a_2, b_1 \times b_2, c_1 \times c_2, d_1 \times d_2; \text{Minimum}\left(w_1, w_2\right)\right);$$

4) Fuzzy Numbers Division $\varnothing$:

The inverse of the fuzzy number $\tilde{A}_2$ is $1/\tilde{A}_2 = (1/d_2, 1/c_2, 1/b_2, 1/a_2; w_2)$, where a_1, b_2, c_2 and d_2 all are positive real numbers. If $a_1, b_1, c_1, d_1, a_2, b_2, c_2$ and d_2 all are positive real numbers, then the division of $\tilde{A}_1$ and $\tilde{A}_2$ is

$$\tilde{A}_1 \varnothing \tilde{A}_2 = \left(\frac{a_1}{d_2}, \frac{b_1}{c_2}, \frac{c_1}{b_2}, \frac{d_1}{a_2}; \text{Minimum}\left(w_1, w_2\right)\right);$$

The difference between the generalized fuzzy numbers arithmetic operations and the traditional fuzzy numbers arithmetic operations is that the former can deal with both non-normalized and normalized fuzzy numbers, but the later only can deal with normalized fuzzy numbers.

3.1.3 Traditional center of gravity (COG) method

The traditional COG method is very useful to deal with defuzzification problems and fuzzy ranking problems. The formula for calculating the COG of a fuzzy number is shown as follows:

$$x^*_{\tilde{A}} = \frac{\int x\mu_{\tilde{A}}(x)dx}{\int \mu_{\tilde{A}}(x)dx} \tag{3.1}$$

where $\mu_{\tilde{A}}(x)$ indicates the membership value of the element x in $\tilde{A}$, and $\mu_{\tilde{A}}(x) \in [0, 1]$.

Assume that there is a trapezoidal fuzzy number $\tilde{A}$, where $\tilde{A} = (a,b,c,d)$ and the membership function of the trapezoidal fuzzy number is shown as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu^L_{\tilde{A}}(x), & a \le x < b \\ 1, & b \le x < c \\ \mu^R_{\tilde{A}}(x), & c \le x < d \\ 0, & otherwise \end{cases}$$

where $\mu^L_{\tilde{A}} : [a,b] \to [0,1]$ is continuous and strictly increasing and $\mu^R_{\tilde{A}} : [c,d] \to [0,1]$ is continuous and strictly decreasing, where

$$\mu^L_{\tilde{A}}(x) = \frac{x-a}{b-a} \text{ and } \mu^R_{\tilde{A}}(x) = \frac{x-d}{c-d} \tag{3.2}$$

Then the transformed formula of (3.1) is shown as follows;

$$x^*_{\tilde{A}} = \frac{\int_a^b (x\mu^L_{\tilde{A}}(x))dx + \int_b^c xdx + \int_c^d (x\mu^R_{\tilde{A}}(x))dx}{\int_a^b (\mu^L_{\tilde{A}}(x))dx + \int_b^c 1dx + \int_c^d (\mu^R_{\tilde{A}}(x))dx} \tag{3.3}$$

There are some drawbacks in the traditional COG method. According to (3.3), we can see that it cannot directly calculate the COG of a crisp interval number due to the fact that the denominators of (3.2) will become zero. Furthermore, it is very time consuming to calculate the COG point of a triangular or a trapezoidal fuzzy number. Thus, a new COG method called simple center of gravity method is used to overcome the drawbacks of the traditional COG method, which is explained later.

3.1.4 Similarity measure between fuzzy numbers

Assume that there are two trapezoidal fuzzy numbers, where $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$, then the degree of similarity $S(\tilde{A}, \tilde{B})$ between the trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ can be calculated as follows :

$$S(\tilde{A}, \tilde{B}) = 1 - \frac{\sum_{i=1}^{4} |a_i - b_i|}{4} \tag{3.4}$$

where $S(\tilde{A}, \tilde{B}) \in [0,1]$. If $\tilde{A}$ and $\tilde{B}$ are triangular fuzzy numbers, where $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$, then the degree of similarity $S(\tilde{A}, \tilde{B})$ between $\tilde{A}$ and $\tilde{B}$ can be calculated as follows :

$$S(\tilde{A}, \tilde{B}) = 1 - \frac{\sum_{i=1}^{3} |a_i - b_i|}{3} \tag{3.5}$$

The larger the value of $S(\tilde{A}, \tilde{B})$, the more the similarity between the fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Lee [31] proposed a similarity measure for trapezoidal fuzzy numbers and used the similarity measure to deal with fuzzy opinions for group decision making, where the degree of similarity $S(\tilde{A}, \tilde{B})$ between the trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ can be calculated as follows:

$$S(\tilde{A},\tilde{B}) = 1 - \frac{\left\| \tilde{A} - \tilde{B} \right\|_{l_p}}{\left\| U \right\|} \times 4^{-1/p} \tag{3.6}$$

where U is universe of discourse

$$\left\| \tilde{A} - \tilde{B} \right\|_{l_p} = (\sum_{i=1}^{4} |a_i - b_i|)^{1/p} \tag{3.7}$$

and

$$\left\| U \right\| = \text{Maximum}(U) - \text{Minimum}(U) \tag{3.8}$$

The larger the value of $S(\tilde{A},\tilde{B})$, the more the similarity between the trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$. For example, assume $\tilde{A} = (0.2, 0.3, 0.4, 0.5), \tilde{B} = (0.3, 0.4, 0.5, 0.6)$ and $p = 1$ then using (3.6) we can calculate similarity measure between $\tilde{A}$ and $\tilde{B}$

$$S(\tilde{A},\tilde{B}) = 1 - \frac{|0.2-0.3| + |0.3-0.4| + |0.4-0.5| + |0.5-0.6|}{0.6-0.2} \times 4^{-1}$$

$$= 0.75.$$

3.2 Simple center of gravity method (SCGM)

In this section, a method, called SCGM is presented, to calculate the COG point of a generalized fuzzy number. The presented SCGM method is based on the concept of the medium curve [45]. In the following, we briefly review the concept of the medium curve from [45]. Assume that there is a generalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$, the definition of the medium curve is defined as follows:

Definition 3.1

A medium curve of the fuzzy number $\tilde{A}$ is a function $\eta_{\tilde{A}}(x)$ shown as follows:

$$\eta_{\tilde{A}}(x) = \begin{cases} a, & \text{if } x = \operatorname{med}\left(A_\alpha\right) \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

where $\alpha \in (0, w]$; A_α is called the α-cut of the fuzzy number $\tilde{A}$ and is defined as $A_\alpha = \{x; \mu_{\tilde{A}}(x) > \alpha\}$; Infimum(A_α) denotes the lower bound of A_α, Supremum(A_α) denotes the upper bound of A_α, then median is given by

$$\operatorname{Med}\left(A_\alpha\right) = \frac{[\text{Infimum}(A_\alpha) + \text{Supremum}(A_\alpha)]}{2} \tag{3.10}$$

For example

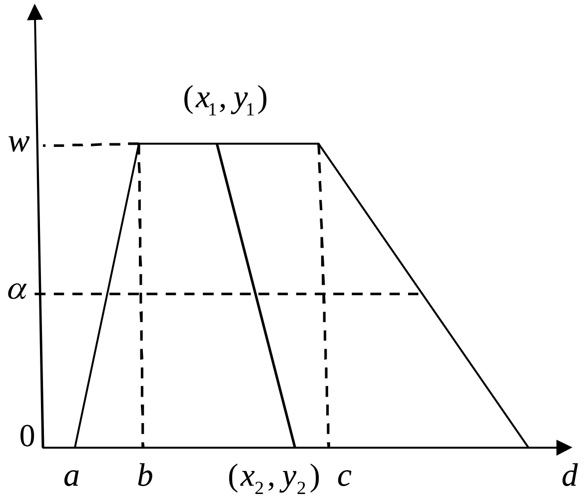

Fig. 3.2 A generalized trapezoidal fuzzy number

Fig. 3.2 shows the generalized trapezoidal fuzzy number $\tilde{A}$ with the function $\eta_{\tilde{A}}(x)$ of medium curve, where the medium curve is a straight line, and there are two points (x_1, y_1) and (x_2, y_2) defined as follows:

$$x_1 = \frac{c+b}{2}, \qquad y_1 = w;\ 0 < w \le 1 \ \text{ and } \ x_2 = \frac{d+a}{2}, \qquad y_2 = 0\,. \tag{3.11}$$

From Fig. 3.2, we can see that the formula of the medium curve is as follows:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1} \tag{3.12}$$

where the linear equation shown in (3.12) is the medium curve of the generalized trapezoidal fuzzy number $\tilde{A}$.

3.2.1 Center of gravity (COG) of a triangle

Fig. 3.3 shows a triangle. From Fig. 3.3, we can see that the center of gravity $G = (x^*, y^*)$ of the triangle is on the medium curve denoted by a dotted line, where

$$x^* = \frac{x_1 + x_2 + x_3}{3} \tag{3.13}$$

$$y^* = \frac{y_1 + y_2 + y_3}{3} \tag{3.14}$$

Because $y_1 = y_3 = 0$ and $y_2 = w$, we can see that $y^* = \frac{w}{3}$, where $0 < w \leq 1$.

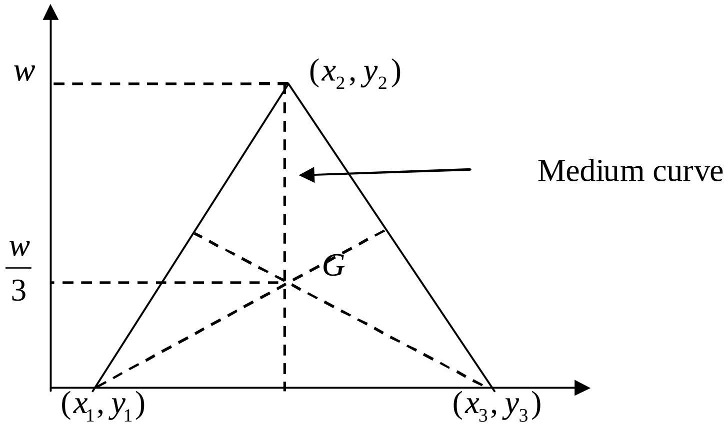

Fig. 3.3 A triangle

3.2.2 Center of gravity (COG) of a rectangle

Fig. 3.4 shows a rectangle. From Fig. 3.4, we can see that the center of gravity of the rectangle is on the medium curve, where

$$x^* = \frac{x_1 + x_2}{2} \tag{3.15}$$

$$y^* = \frac{w}{2} \tag{3.16}$$

where $0 < w \leq 1$.

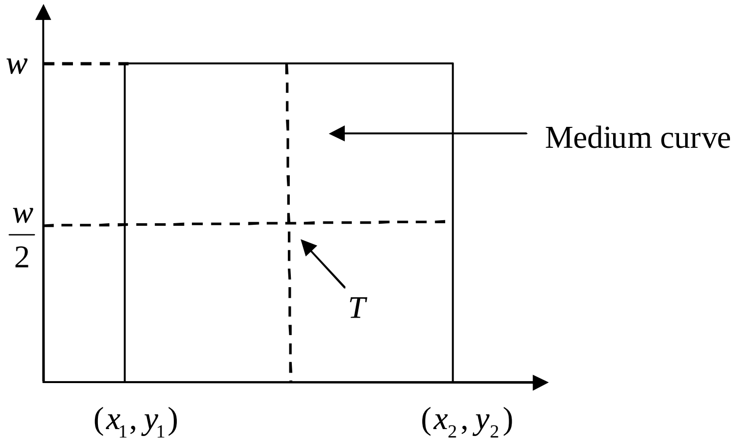

Fig. 3.4 A rectangle

From Figs. 3.2, 3.3 and 3.4, when we assume that a triangle is a triangular fuzzy number and a rectangle is a crisp interval, and a trapezoidal is a trapezoidal fuzzy number, we can see that the COG point of a generalized fuzzy number $\tilde{A}$ is on the medium curve. If we use the value y^* of a COG point and the medium curve, we can obtain the value x^* of the COG point. Based on the previous discussions, we presented a new COG method called the SCGM described as follows. If $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$ is a generalized trapezoidal fuzzy number then we see that the value $y^*_{\tilde{A}}$ of the COG point of $\tilde{A}$ is as follows:

$$y^*_{\tilde{A}} = \begin{cases} \dfrac{w_{\tilde{A}} \times \left(\dfrac{a_3 - a_2}{a_4 - a_1} + 2 \right)}{6}, & \text{if } a_1 \neq a_4 \text{ and } 0 < w_{\tilde{A}} \leq 1, \\ \dfrac{w_{\tilde{A}}}{2}, & \text{if } a_1 = a_4 \text{ and } 0 < w_{\tilde{A}} \leq 1. \end{cases} \tag{3.17}$$

If $\tilde{A}$ is a generalized triangular fuzzy number, where $\tilde{A}=(a_1,a_2,a_2,a_3;w_{\tilde{A}})$, we can see that the value y^* of the COG point of is as follows:

$$\begin{aligned} y^*_{\tilde{A}} &= \frac{w_{\tilde{A}} \times \left(\frac{a_2-a_2}{a_3-a_1} + 2 \right)}{6} \\ &= \frac{w_{\tilde{A}} \times (0+2)}{6} \\ &= \frac{w_{\tilde{A}}}{3} \end{aligned}$$

If $\tilde{A}$ is a crisp interval, where $\tilde{A}=(a_1,a_1,a_4,a_4;w_{\tilde{A}})$, then we can see that the value of the COG point of $\tilde{A}$ is as follows:

$$\begin{aligned} y^*_{\tilde{A}} &= \frac{w_{\tilde{A}} \times \left(\frac{a_4-a_1}{a_4-a_1} + 2 \right)}{6} \\ &= \frac{w_{\tilde{A}} \times (1+2)}{6} \\ &= \frac{w_{\tilde{A}}}{2} \end{aligned}$$

Based on (3.11), and (3.12), we can obtain the value $x^*_{\tilde{A}}$ of the COG point of $\tilde{A}$ as follows:

$$\frac{y_2-y_1}{x_2-x_1} = \frac{y^*_{\tilde{A}}-y_1}{x^*_{\tilde{A}}-x_1}$$

$$\Rightarrow \quad x^*_{\tilde{A}} = \frac{y^*_{\tilde{A}}(x_2-x_1) - x_2y_1 + x_1y_2}{y_2-y_1} \qquad (3.18)$$

where $x_1 = \frac{a_3+a_2}{2}, x_2 = \frac{a_4+a_1}{2}, y_2 = 0, y_1 = w_{\tilde{A}},$ and $0 < w_{\tilde{A}} \le 1$. Thus, (3.18) can be rewritten as

$$x^*_{\tilde{A}} = \frac{y^*_{\tilde{A}}(a_3 + a_2) + (a_4 + a_1)(w_{\tilde{A}} - y^*_{\tilde{A}})}{2w_{\tilde{A}}} \tag{3.19}$$

Based on (3.17) and (3.19), we can obtain the COG point COG($\tilde{A}$) of a generalized trapezoidal fuzzy number , where $\text{COG}\left(\tilde{A}\right) = \left(x^*_{\tilde{A}}, y^*_{\tilde{A}}\right)$.

3.2.3 Similarity measure between generalized fuzzy numbers

In this section, a method is presented to calculate the degree of similarity between generalized fuzzy numbers. Assume that there are two generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$,

Where $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$, $\tilde{B} = (b_1, b_2, b_3, b_4; w_{\tilde{B}})$, $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$.

Step 1. First we use (3.17) and (3.19) to obtain the COG points COG($\tilde{A}$) and COG($\tilde{B}$) of the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively, where $\text{COG}\left(\tilde{A}\right) = \left(x^*_{\tilde{A}}, y^*_{\tilde{A}}\right)$ and $\text{COG}\left(\tilde{B}\right) = \left(x^*_{\tilde{B}}, y^*_{\tilde{B}}\right)$.

Step 2. Now we calculate the degree of similarity $S(\tilde{A}, \tilde{B})$ between the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ as follows:

$$S(\tilde{A}, \tilde{B}) = \left[1 - \frac{\sum_{i=1}^{4} |a_i - b_i|}{4}\right] \times (1 - \left|x^*_{\tilde{A}} - x^*_{\tilde{B}}\right|)^{B(S_{\tilde{A}}, S_{\tilde{B}})} \times \frac{\text{Minimum}(y^*_{\tilde{A}}, y^*_{\tilde{B}})}{\text{Maximum}(y^*_{\tilde{A}}, y^*_{\tilde{B}})} \tag{3.20}$$

where $S(\tilde{A}, \tilde{B}) \in [0,1]$, $B(S_{\tilde{A}}, S_{\tilde{B}})$ is defined as follows:

$$B(S_{\tilde{A}}, S_{\tilde{B}}) = \begin{cases} 1, & if\ S_{\tilde{A}} + S_{\tilde{B}} > 0 \\ 0, & if\ S_{\tilde{A}} + S_{\tilde{B}} = 0 \end{cases} \tag{3.21}$$

where $S_{\tilde{A}}$ and $S_{\tilde{B}}$ are the lengths of the bases of the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively, defined as follows;

$$S_{\tilde{A}} = a_4 - a_1 \tag{3.22}$$

$$S_{\tilde{B}} = b_4 - b_1 \tag{3.23}$$

The presented similarity measure integrates the concepts of the geometric distance which is given by $\left[1 - (\sum_{i=1}^{4} \left| a_i - b_i \right| / 4)\right]$ and the center-of-gravity distance given by $(1 - \left| x^*_{\tilde{A}} - x^*_{\tilde{B}} \right|)^{B(S_{\tilde{A}}, S_{\tilde{B}})}$. The value $B(S_{\tilde{A}}, S_{\tilde{B}})$ is used to determine whether we consider the COG distance or not. If the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are real numbers (i.e., $S_{\tilde{A}} + S_{\tilde{B}} = 0$), then we don't consider the center-of-gravity distance (i.e., $B(S_{\tilde{A}}, S_{\tilde{B}}) = 0$). If either $\tilde{A}$ or $\tilde{B}$ is a generalized fuzzy number (i.e., $S_{\tilde{A}} + S_{\tilde{B}} \neq 0$), then we must consider the COG distance (i.e., $B(S_{\tilde{A}}, S_{\tilde{B}}) = 1$). The larger the value of $S(\tilde{A}, \tilde{B})$, the more the similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Example 3.1

Let us consider the generalized triangular fuzzy numbers $\tilde{A} = (0.1, 0.3, 0.5; 1.0)$, $\tilde{B} = (0.2, 0.4, 0.7; 1.0)$, $\tilde{C} = (0.5, 0.7, 0.9; 1.0)$ as shown in Fig. 3.5. It is obvious that the generalized triangular fuzzy numbers also can be represented by the generalized trapezoidal fuzzy numbers representations shown as follows:

$$\tilde{A} = (0.1, 0.3, 0.3, 0.5; 1.0)$$

$$\tilde{B} = (0.2, 0.4, 0.4, 0.7; 1.0)$$

$$\tilde{C} = (0.5, 0.7, 0.7, 0.9; 1.0)$$

From Fig. 3.5, we can see that generalized fuzzy number $\tilde{B}$ is more similar to generalized fuzzy number $\tilde{A}$ than generalized fuzzy number $\tilde{C}$. According to (3.17) and (3.19), we calculate the values

$$y^*_{\tilde{A}} = y^*_{\tilde{B}} = y^*_{\tilde{C}} = \frac{1}{3}$$

$$x^*_{\tilde{A}} = 0.3,\ x^*_{\tilde{B}} = 0.433,\ x^*_{\tilde{C}} = 0.7,$$

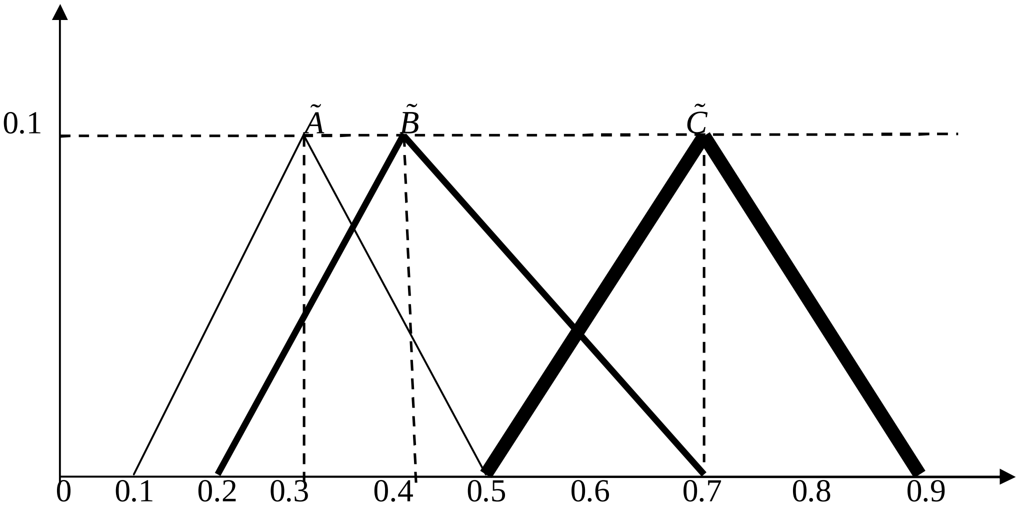

Fig. 3.5 Three generalized triangular fuzzy numbers

So the COG($\tilde{A}$) = $(0.3, 1/3)$, COG($\tilde{B}$) = $(0.433, 1/3)$ and COG($\tilde{C}$) = $(0.7, 1/3)$ of the generalized fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, respectively. From (3.22) and (3.23), we can see that $S_{\tilde{A}} = 0.4$, $S_{\tilde{B}} = 0.5$ and $S_{\tilde{C}} = 0.4$. Thus, according to (3.21), we can see that $B(S_{\tilde{A}}, S_{\tilde{B}}) = 1$, $B(S_{\tilde{A}}, S_{\tilde{C}}) = 1$ and $B(S_{\tilde{B}}, S_{\tilde{C}}) = 1$. Then, we use (3.20) to calculate the degrees of similarity between these generalized fuzzy numbers shown as follows:

$$S(\tilde{A}, \tilde{B}) = [1 - 0.125] \times (1 - 0.13333)^1 \times 1 = 0.75833$$

$$S(\tilde{A}, \tilde{C}) = [1 - 0.4] \times (1 - 0.4)^1 \times 1 = 0.36$$

$S(\tilde{B},\tilde{C}) = [1-0.275] \times (1-0.26667)^1 \times 1 = 0.53167$

Because $S(\tilde{A},\tilde{B}) > S(\tilde{B},\tilde{C})$, we see that the generalized fuzzy number $\tilde{B}$ is more similar to $\tilde{A}$ than the generalized fuzzy number $\tilde{C}$.

3.2.4 Some properties of the presented similarity measure

Property 1: Two generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are identical if and only if $S(\tilde{A},\tilde{B}) = 1$.

Proof*:* From (3.22) and (3.23), we can see that $S_{\tilde{A}} \neq 0$ and $S_{\tilde{B}} \neq 0$. Thus, according to (3.21), we can see that the value $B(S_{\tilde{A}}, S_{\tilde{B}}) = 1$.

1) If $\tilde{A}$ and $\tilde{B}$ are identical, then $a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4$, and $w_{\tilde{A}} = w_{\tilde{B}}$. Then the degree of similarity between $\tilde{A}$ and $\tilde{B}$ can be calculated as follows:

$$S(\tilde{A},\tilde{B}) = \left[1 - \frac{\sum_{i=1}^{4}|a_i - b_i|}{4}\right] \times (1 - |x^*_{\tilde{A}} - x^*_{\tilde{B}}|)^{B(S_{\tilde{A}},S_{\tilde{B}})} \times \frac{\text{Minimum}(y^*_{\tilde{A}}, y^*_{\tilde{B}})}{\text{Maximum}(y^*_{\tilde{A}}, y^*_{\tilde{B}})}$$

$$\begin{aligned} &= [1-0] \times (1-0)^1 \times 1 \\ &= 1 \end{aligned}$$

2) If $S(\tilde{A},\tilde{B}) = 1$,then

$$S(\tilde{A},\tilde{B}) = \left[1 - \frac{\sum_{i=1}^{4}|a_i - b_i|}{4}\right] \times (1 - |x^*_{\tilde{A}} - x^*_{\tilde{B}}|)^{B(S_{\tilde{A}},S_{\tilde{B}})} \times \frac{\text{Minimum}(y^*_{\tilde{A}}, y^*_{\tilde{B}})}{\text{Maximum}(y^*_{\tilde{A}}, y^*_{\tilde{B}})}$$

$$= 1$$

$\Rightarrow$ $a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4, y^*_{\tilde{A}} = y^*_{\tilde{B}}$ and $x^*_{\tilde{A}} = x^*_{B}$. From (3.17) and (3.19) we can see that $w_{\tilde{A}} = w_{\tilde{B}}$. Therefore, the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are identical.

Property 2: $S(\tilde{A},\tilde{B}) = S(\tilde{B},\tilde{A})$

Proof: $S\left(\tilde{A},\tilde{B}\right)$ is given by

$$S(\tilde{A},\tilde{B}) = \left[1 - \frac{\sum_{i=1}^{4}|a_i - b_i|}{4}\right] \times (1 - |x^*_{\tilde{A}} - x^*_{\tilde{B}}|)^{B(S_{\tilde{A}},S_{\tilde{B}})} \times \frac{\text{Minimum}(y^*_{\tilde{A}}, y^*_{\tilde{B}})}{\text{Maximum}(y^*_{\tilde{A}}, y^*_{\tilde{B}})} \qquad \text{and}$$

$$S(\tilde{B},\tilde{A}) = \left[1 - \frac{\sum_{i=1}^{4}|b_i - a_i|}{4}\right] \times (1 - |x^*_{\tilde{B}} - x^*_{\tilde{A}}|)^{B(S_{\tilde{A}},S_{\tilde{B}})} \times \frac{\text{Minimum}(y^*_{\tilde{B}}, y^*_{\tilde{A}})}{\text{Maximum}(y^*_{\tilde{B}}, y^*_{\tilde{A}})}$$

where $\sum_{i=1}^{4}|a_i - b_i| = \sum_{i=1}^{4}|b_i - a_i|$, $|x^*_{\tilde{A}} - x^*_{\tilde{B}}| = |x^*_{\tilde{B}} - x^*_{\tilde{A}}|$, $B(S_{\tilde{A}}, S_{\tilde{B}}) = B(S_{\tilde{B}}, S_{\tilde{A}})$

and $(\text{Minimum}(y^*_{\tilde{A}}, y^*_{\tilde{B}}) / \text{Maximum}\,(y^*_{\tilde{A}}, y^*_{\tilde{B}})) = (\text{Minimum}(y^*_{\tilde{B}}, y^*_{\tilde{A}}) / \text{Maximum}(y^*_{\tilde{B}}, y^*_{\tilde{A}}))$.

Thus we can see that $S(\tilde{A},\tilde{B}) = S(\tilde{B},\tilde{A})$.

Property 3: If $\tilde{A} = (a,a,a,a;1.0)$ and $\tilde{B} = (b,b,b,b;1.0)$ are two real numbers, then $S(\tilde{A},\tilde{B}) = 1 - |a-b|$.

Proof*:* Based on (3.22) and (3.23), we can see that $S_{\tilde{A}} = 0$ and $S_{\tilde{B}} = 0$. From (3.21), we can see that $B(S_{\tilde{A}}, S_{\tilde{B}}) = 0$. From (3.17), we can see that $y^*_{\tilde{A}} = y^*_{\tilde{B}} = 1/2$. Based on (3.20), we get

$$S(\tilde{A},\tilde{B}) = \left[1 - \frac{\sum_{i=1}^{4} |a-b|}{4} \right] \times (1 - |x_{\tilde{A}}^{*} - x_{\tilde{B}}^{*}|)^{0} \times \frac{\text{Minimum}(y_{\tilde{A}}^{*}, y_{\tilde{B}}^{*})}{\text{Maximum}(y_{\tilde{A}}^{*}, y_{\tilde{B}}^{*})}$$

$$= \left[1 - \frac{4|a-b|}{4} \right] \times 1 \times 1$$

$$= 1 - |a-b|$$

3.3 Conclusion

In this chapter, new method is presented to calculate similarity measure of generalized fuzzy numbers. First, we present a method to calculate the COG point of generalized fuzzy numbers called the SCGM. Then, we present a new similarity measure to calculate the degree of similarity between generalized fuzzy numbers based on SCGM. We also prove some properties of the proposed similarity measure. The proposed similarity measure can overcome the drawbacks of the existing similarity measures.

Chapter 4

SIMILARITY MEASURE OF INTERVAL–VALUED FUZZY NUMBERS

Interval-valued fuzzy numbers are very useful to represent evaluating values in real-world problems. Guijun and Xiaoping [21] defined the interval-valued fuzzy numbers and their extended operations. In recent years, some methods have been presented to calculate the degree of similarity between fuzzy numbers [8,11] also in last chapter we present a method to calculate similarity measure between generalized fuzzy numbers However, these similarity measures cannot calculate the degree of similarity between interval-valued fuzzy numbers.

In this chapter, we calculate the similarity measure between interval-valued fuzzy numbers by applying the method which is already discussed in last chapter. First, we propose a similarity measure to calculate the degree of similarity between interval-valued fuzzy numbers. The proposed similarity measure uses the concept of geometry to calculate the (COG) points of the lower fuzzy numbers and the upper fuzzy numbers of interval-valued fuzzy numbers, respectively, to calculate the degree of similarity between interval-valued fuzzy numbers. We also prove some properties of the proposed similarity measure.

4.1 Interval valued fuzzy numbers

Definition 4.1

An interval-valued fuzzy set $\tilde{\tilde{A}}$ defined in the universe of discourse X is given by

$$\tilde{\tilde{A}} = \{(x, [\mu^L_{\tilde{\tilde{A}}}(x), \mu^U_{\tilde{\tilde{A}}}(x)]; x \in X)\}$$

where $0 \le \mu^L_{\tilde{\tilde{A}}}(x) \le \mu^U_{\tilde{\tilde{A}}}(x) \le 1$ and the membership grade $\mu_{\tilde{\tilde{A}}}(x)$ of the element x belongs to the interval-valued fuzzy set $\tilde{\tilde{A}}$, which can be represented by an interval $\mu_{\tilde{\tilde{A}}}(x) = [\mu^L_{\tilde{\tilde{A}}}(x), \mu^U_{\tilde{\tilde{A}}}(x)]$.

Definition 4.2 [9]

If an interval-valued fuzzy set $\tilde{\tilde{A}}$ satisfies the following properties:

(1) $\tilde{\tilde{A}}$ is defined in a closed bounded interval,

(2) $\tilde{\tilde{A}}$ is a convex set,

then $\tilde{\tilde{A}}$ is called an interval-valued fuzzy number in the universe of discourse X.

Assume that there is an interval-valued fuzzy number $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U]$ as shown in Fig. 4.1. From Fig. 4.1, we can see that the interval-valued fuzzy number $\tilde{\tilde{A}}$ has two elements, where the one is the lower fuzzy number $\tilde{\tilde{A}}^L$, and the other one is the upper fuzzy number $\tilde{\tilde{A}}^U$. Furthermore, the interval-valued fuzzy number $\tilde{\tilde{A}}$ can be represented as :

$$\tilde{\tilde{A}} = \left[(a_1^L, a_2^L, a_3^L, a_4^L; w^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; w^U_{\tilde{\tilde{A}}}) \right]$$

where $a_1^L \le a_2^L \le a_3^L \le a_4^L$, $a_1^U \le a_2^U \le a_3^U \le a_4^L$, $0 < w^L_{\tilde{\tilde{A}}} \le w^U_{\tilde{\tilde{A}}} \le 1$ and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$.

If $a_1^L = a_1^U, a_2^L = a_2^U, a_3^L = a_3^U, a_4^L = a_4^U$ and $w^L_{\tilde{\tilde{A}}} = w^U_{\tilde{\tilde{A}}} = w_{\tilde{A}}$, then the interval-valued fuzzy number $\tilde{\tilde{A}}$ can be regarded as a generalized fuzzy number, denoted as $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$.

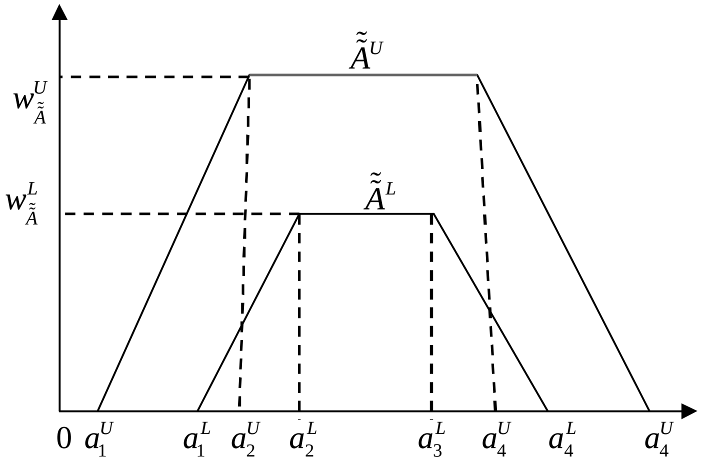

Fig. 4.1 An interval-valued fuzzy number

4.2 Arithmetic operations

Assume that there are two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, where

$\tilde{\tilde{A}} = \left[(a_1^L, a_2^L, a_3^L, a_4^L; w^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; w^U_{\tilde{\tilde{A}}})\right]$ and

$\tilde{\tilde{B}} = \left[(b_1^L, b_2^L, b_3^L, b_4^L; w^L_{\tilde{\tilde{A}}}), (b_1^U, b_2^U, b_3^U, b_4^U; w^U_{\tilde{\tilde{A}}})\right]$.

The arithmetic operations between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are as follows:

1) Interval-Valued Fuzzy Number Addition $\oplus$:

$$\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} = \left[(a_1^L, a_2^L, a_3^L, a_4^L; w^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; w^U_{\tilde{\tilde{A}}})\right] \oplus \left[(b_1^L, b_2^L, b_3^L, b_4^L; w^L_{\tilde{\tilde{A}}}), (b_1^U, b_2^U, b_3^U, b_4^U; w^U_{\tilde{\tilde{A}}})\right]$$

$$\begin{aligned} &= [(a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; \text{Minimum}(w^L_{\tilde{\tilde{A}}}, w^L_{\tilde{\tilde{B}}})), \\ &\quad (a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U; \text{Minimum}(w^U_{\tilde{\tilde{A}}}, w^U_{\tilde{\tilde{B}}}))] \end{aligned}$$

where $a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U$ and b_4^U are any real values,

$0 < w^L_{\tilde{\tilde{A}}} \le w^U_{\tilde{\tilde{A}}} \le 1$ and $0 < w^L_{\tilde{\tilde{B}}} \le w^U_{\tilde{\tilde{B}}} \le 1$.

2) Interval-Valued Fuzzy Number Subtraction $\ominus$

$$\tilde{\tilde{A}} \ominus \tilde{\tilde{B}} = \left[\left(a^L_1, a^L_2, a^L_3, a^L_4; w^L_{\tilde{\tilde{A}}}\right), \left(a^U_{\tilde{\tilde{A}}}, a^U_{\tilde{\tilde{A}}}, a^U_{\tilde{\tilde{A}}}, a^{\mathsf{U}}_{\tilde{\tilde{A}}}; w^U_{\tilde{\tilde{A}}}\right)\right] \ominus$$

$$\left[\left(b^L_1, b^L_2, b^L_3, b^L_4; w^L_{\tilde{\tilde{B}}}\right), \left(b^U_1, b^U_2, b^U_3, b^U_4; w^U_{\tilde{\tilde{B}}}\right)\right]$$

$$\begin{aligned} &= [(a^L_1 - b^L_4, a^L_2 - b^L_3, a^L_3 - b^L_2, a^L_4 - b^L_1; \text{Minimum}(w^L_{\tilde{\tilde{A}}}, w^L_{\tilde{\tilde{B}}})), \\ &\quad (a^U_1 - b^U_4, a^U_2 - b^U_3, a^U_3 - b^U_2, a^U_4 - b^U_1; \text{Minimum}(w^U_{\tilde{\tilde{A}}}, w^U_{\tilde{\tilde{B}}}))] \end{aligned}$$

where $a^L_1, a^L_2, a^L_3, a^L_4, a^U_1, a^U_2, a^U_3, a^U_4, b^L_1, b^L_2, b^L_3, b^L_4, b^U_1, b^U_2, b^U_3$ and b^U_4 are any real values, $0 < w^L_{\tilde{\tilde{A}}} \le w^U_{\tilde{\tilde{A}}} \le 1$ and $0 < w^L_{\tilde{\tilde{B}}} \le w^U_{\tilde{\tilde{B}}} \le 1$.

3) Interval-Valued Fuzzy Number Multiplication $\otimes$:

$$\tilde{\tilde{A}} \otimes \tilde{\tilde{B}} = \left[(a^L_1, a^L_2, a^L_3, a^L_4; w^L_{\tilde{\tilde{A}}}), (a^U_1, a^U_2, a^U_3, a^U_4; w^U_{\tilde{\tilde{A}}})\right] \otimes \left[(b^L_1, b^L_2, b^L_3, b^L_4; w^L_{\tilde{\tilde{A}}}), (b^U_1, b^U_2, b^U_3, b^U_4; w^U_{\tilde{\tilde{A}}})\right]$$

$$\begin{aligned} &= [(a^L_1 \times b^L_1, a^L_2 \times b^L_2, a^L_3 \times b^L_3, a^L_4 \times b^L_4; \text{Minimum}(w^L_{\tilde{\tilde{A}}}, w^L_{\tilde{\tilde{B}}})), \\ &\quad (a^U_1 \times b^U_1, a^U_2 \times b^U_2, a^U_3 \times b^U_3, a^U_4 \times b^U_4; \text{Minimum}(w^U_{\tilde{\tilde{A}}}, w^U_{\tilde{\tilde{B}}}))] \end{aligned}$$

where $a^L_1, a^L_2, a^L_3, a^L_4, a^U_1, a^U_2, a^U_3, a^U_4, b^L_1, b^L_2, b^L_3, b^L_4, b^U_1, b^U_2, b^U_3$ and b^U_4 are any real values, $0 < w^L_{\tilde{\tilde{A}}} \le w^U_{\tilde{\tilde{A}}} \le 1$ and $0 < w^L_{\tilde{\tilde{B}}} \le w^U_{\tilde{\tilde{B}}} \le 1$.

4) Interval-Valued Fuzzy Number Division $\varnothing$:

$$\begin{aligned} \tilde{\tilde{A}} \varnothing \tilde{\tilde{B}} &= \left[(a^L_1, a^L_2, a^L_3, a^L_4; w^L_{\tilde{\tilde{A}}}), (a^U_1, a^U_2, a^U_3, a^U_4; w^U_{\tilde{\tilde{A}}})\right] \varnothing \left[(b^L_1, b^L_2, b^L_3, b^L_4; w^L_{\tilde{\tilde{A}}}), (b^U_1, b^U_2, b^U_3, b^U_4; w^U_{\tilde{\tilde{A}}})\right] \\ &= [(a^L_1 / b^L_4, a^L_2 / b^L_3, a^L_3 / b^L_2, a^L_4 / b^L_1; \text{Minimum}(w^L_{\tilde{\tilde{A}}}, w^L_{\tilde{\tilde{B}}})), \\ &\quad (a^U_1 / b^U_4, a^U_2 / b^U_3, a^U_3 / b^U_2, a^U_4 / b^U_1; \text{Minimum}(w^U_{\tilde{\tilde{A}}}, w^U_{\tilde{\tilde{B}}}))] \end{aligned}$$

where $a^L_1, a^L_2, a^L_3, a^L_4, a^U_1, a^U_2, a^U_3, a^U_4, b^L_1, b^L_2, b^L_3, b^L_4, b^U_1, b^U_2, b^U_3$ and b^U_4 are all non-zero positive real numbers or all non-zero negative real numbers, $0 < w^L_{\tilde{\tilde{A}}} \le w^U_{\tilde{\tilde{A}}} \le 1$ and

$0 < w^L_{\tilde{\tilde{B}}} \leq w^U_{\tilde{\tilde{B}}} \leq 1$.

4.3 Similarity measure between interval-valued fuzzy numbers

In this section, we propose a similarity measure to calculate the degree of similarity between interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$. Let U be the universe of discourse, $U = [0,1]$. Assume that there are two interval-valued fuzzy numbers

$$\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U] = \left[(a_1^L, a_2^L, a_3^L, a_4^L; w^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; w^U_{\tilde{\tilde{A}}}) \right]$$

And $$\tilde{\tilde{B}} = [\tilde{\tilde{B}}^L, \tilde{\tilde{B}}^U] = \left[(b_1^L, b_2^L, b_3^L, b_4^L; w^L_{\tilde{\tilde{A}}}), (b_1^U, b_2^U, b_3^U, b_4^U; w^U_{\tilde{\tilde{A}}}) \right]$$

Where

$0 \leq a_1^L \leq a_2^L \leq a_3^L \leq a_4^L \leq 1,\ 0 \leq a_1^U \leq a_2^U \leq a_3^U \leq a_4^L \leq 1,\ 0 < w^L_{\tilde{\tilde{A}}} \leq w^U_{\tilde{\tilde{A}}} \leq 1$ and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{B}}^U$

$0 \leq b_1^L \leq b_2^L \leq b_3^L \leq b_4^L \leq 1,\ 0 \leq b_1^U \leq b_2^U \leq b_3^U \leq b_4^L \leq 1,\ 0 < w^L_{\tilde{\tilde{B}}} \leq w^U_{\tilde{\tilde{B}}} \leq 1$ and $\tilde{\tilde{B}}^L \subset \tilde{\tilde{B}}^U$.

The proposed method is now presented as follows:

Step 1. Based on formulas (3.17) and (3.19), find the COG points of the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, respectively. For example, the interval-valued fuzzy number $\tilde{\tilde{A}}$ has two different COG points $(x^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{A}}^L})$, of the lower fuzzy number $\tilde{\tilde{A}}^L$, and the other one is the COG point $(x^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{A}}^U})$ of the upper fuzzy number $\tilde{\tilde{A}}^U$, shown as follows

$$y^*_{\tilde{\tilde{A}}^L} = \begin{cases} \dfrac{w^L_{\tilde{\tilde{A}}} \times \left(\dfrac{a_3^L - a_2^L}{a_4^L - a_1^L} + 2\right)}{6}, & \text{if } a_1^L \neq a_4^L \text{ and } 0 < w^L_{\tilde{\tilde{A}}} \leq 1, \\ \dfrac{w^L_{\tilde{\tilde{A}}}}{2}, & \text{if } a_1^L = a_4^L \text{ and } 0 < w^L_{\tilde{\tilde{A}}} \leq 1, \end{cases} \tag{4.1}$$

$$x^*_{\tilde{\tilde{A}}^L} = \frac{y^*_{\tilde{\tilde{A}}^L}(a_3^L + a_2^L) + (a_4^L + a_1^L)(w^L_{\tilde{\tilde{A}}} - y^*_{\tilde{\tilde{A}}^L})}{2w^L_{\tilde{\tilde{A}}}}, \tag{4.2}$$

$$y^*_{\tilde{\tilde{A}}^U} = \begin{cases} \dfrac{w^U_{\tilde{\tilde{A}}} \times (\dfrac{a_3^U - a_2^U}{a_4^L - a_1^L} + 2)}{6}, & \text{if } a_1^U \neq a_4^U \text{ and } 0 < w^U_{\tilde{\tilde{A}}} \leq 1, \\ \dfrac{w^U_{\tilde{\tilde{A}}}}{2}, & \text{if } a_1^U = a_4^U \text{ and } 0 < w^U_{\tilde{\tilde{A}}} \leq 1, \end{cases} \tag{4.3}$$

$$x^*_{\tilde{\tilde{A}}^U} = \frac{y^*_{\tilde{\tilde{A}}^U}(a_3^U + a_2^U) + (a_4^U + a_1^U)(w^U_{\tilde{\tilde{A}}} - y^*_{\tilde{\tilde{A}}^U})}{2w^U_{\tilde{\tilde{A}}}} \tag{4.4}$$

In the same way, we can find the COG points $(x^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{B}}^L})$ of the lower fuzzy number $\tilde{\tilde{B}}^L$ and $(x^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{B}}^U})$ of the upper fuzzy number $\tilde{\tilde{B}}^U$ of the interval-valued fuzzy number $\tilde{\tilde{B}}$, respectively.

Step 2. Based on formula (3.20), calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$ and calculate the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$, respectively, as follows:

$$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = \left[1 - \frac{\sum_{i=1}^{4} \left| a_i^L - b_i^L \right|}{4}\right] \times (1 - \left| x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L} \right|)^{B(S_{\tilde{\tilde{A}}^L}, S_{\tilde{\tilde{B}}^L})} \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} \tag{4.5}$$

where $B(S^L_{\tilde{\tilde{A}}}, S^L_{\tilde{\tilde{B}}})$ is defined as:

$$B(S^L_{\tilde{\tilde{A}}}, S^L_{\tilde{\tilde{B}}}) = \begin{cases} 1, & \text{if } S^L_{\tilde{\tilde{A}}} + S^L_{\tilde{\tilde{B}}} > 1 \\ 0, & \text{if } S^L_{\tilde{\tilde{A}}} + S^L_{\tilde{\tilde{B}}} = 0. \end{cases} \tag{4.6}$$

where $S_{\tilde{\tilde{A}}}$ and $S_{\tilde{\tilde{B}}}$ are the lengths of the bases of the lower interval-valued trapezoidal fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, respectively, defined as follows:

$$S^L_{\tilde{\tilde{A}}} = a_4^L - a_1^L \tag{4.7}$$

$$S^L_{\tilde{\tilde{B}}} = b_4^L - b_1^L \tag{4.8}$$

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[1 - \frac{\sum_{i=1}^{4} \left| a_i^U - b_i^U \right|}{4} \right] \times (1 - \left| x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U} \right|)^{B(S_{\tilde{\tilde{A}}^U}, S_{\tilde{\tilde{B}}^U})} \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})} \tag{4.9}$$

where $B(S_{\tilde{\tilde{A}}^U}, S_{\tilde{\tilde{B}}^U})$ is defined as follows:

$$B(S_{\tilde{\tilde{A}}^U}, S_{\tilde{\tilde{B}}^U}) = \begin{cases} 1, & \text{if } S_{\tilde{\tilde{A}}^U} + S_{\tilde{\tilde{B}}^U} > 0 \\ 0, & \text{if } S_{\tilde{\tilde{A}}^U} + S_{\tilde{\tilde{B}}^U} = 0 \end{cases} \tag{4.10}$$

$$S^U_{\tilde{\tilde{A}}} = a_4^U - a_1^U \tag{4.11}$$

$$S^U_{\tilde{\tilde{B}}} = b_4^U - b_1^U \tag{4.12}$$

The value $B(S^L_{\tilde{\tilde{A}}}, S^L_{\tilde{\tilde{B}}})$ and $B(S^U_{\tilde{\tilde{A}}}, S^U_{\tilde{\tilde{B}}})$ is used to determine whether we consider the COG distance or not. If the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are real numbers i.e. $S^L_{\tilde{\tilde{A}}} + S^L_{\tilde{\tilde{B}}} = 0, \text{and}\, S^U_{\tilde{\tilde{A}}} + S^U_{\tilde{\tilde{B}}} = 0$ then we don't consider the center-of-gravity distance i.e. $B(S^L_{\tilde{\tilde{A}}}, S^L_{\tilde{\tilde{B}}}) = 0$ and $B(S^U_{\tilde{\tilde{A}}}, S^U_{\tilde{\tilde{B}}}) = 0$. If either $\tilde{\tilde{A}}$ or $\tilde{\tilde{B}}$ is a generalized fuzzy number i.e. $S^L_{\tilde{\tilde{A}}} + S^L_{\tilde{\tilde{B}}} \neq 0$ and $S^U_{\tilde{\tilde{A}}} + S^U_{\tilde{\tilde{B}}} \neq 0$ then we must consider the COG distance i.e. $B(S^L_{\tilde{\tilde{A}}}, S^L_{\tilde{\tilde{B}}}) = 1$ and $B(S^U_{\tilde{\tilde{A}}}, S^U_{\tilde{\tilde{B}}}) = 1$.

where $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \in [0,1]$ and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) \in [0,1]$. The larger the value of $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$, the higher the similarity between the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$; the larger the value of $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ the higher the similarity between the upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$.

Step 3. Calculate the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ as follows:

$$S(\tilde{\tilde{A}},\tilde{\tilde{B}})=\sqrt{S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)\times S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)} \tag{4.13}$$

where $S(\tilde{\tilde{A}},\tilde{\tilde{B}})\in[0,1]$. The larger the value of $S(\tilde{\tilde{A}},\tilde{\tilde{B}})$, the higher the similarity between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

4.4 Some properties of presented method

Property 1. Two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical if and only if $S(\tilde{\tilde{A}},\tilde{\tilde{B}})=1$

Proof: 1) If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical then we have to prove that

$$S(\tilde{\tilde{A}},\tilde{\tilde{B}})=\sqrt{S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)\times S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)}=1$$

If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical then, $\tilde{\tilde{A}}^L$ is identical with $\tilde{\tilde{B}}^L$ iff $S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)=1$ and $\tilde{\tilde{A}}^U$ is identical with $\tilde{\tilde{B}}^U$ iff $S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)=1$. where

$$S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)=\left[1-\frac{\sum_{i=1}^{4}\left|a_i^L-b_i^L\right|}{4}\right]\times\left(1-\left|x^*_{\tilde{\tilde{A}}^L}-x^*_{\tilde{\tilde{B}}^L}\right|\right)^{B(S_{\tilde{\tilde{A}}^L},S_{\tilde{\tilde{B}}^L})}\times\frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^L},y^*_{\tilde{\tilde{B}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^L},y^*_{\tilde{\tilde{B}}^L})}$$

If $\tilde{\tilde{A}}^L$ is identical with $\tilde{\tilde{B}}^L$ then $a_1^L=b_1^L, a_2^L=b_2^L, a_3^L=b_3^L, a_4^L=b_4^L$, $w^L_{\tilde{\tilde{A}}}=w^L_{\tilde{\tilde{B}}}$, therefore $y^*_{\tilde{\tilde{A}}^L}=y^*_{\tilde{\tilde{B}}^L}$ and $x^*_{\tilde{\tilde{A}}^L}=x^*_{\tilde{\tilde{B}}^L}$, also from (4.7) and (4.8), $S^L_{\tilde{\tilde{A}}}\neq 0$, $S^L_{\tilde{\tilde{B}}}\neq 0$. So according to (4.6), $B(S^L_{\tilde{\tilde{A}}},S^L_{\tilde{\tilde{B}}})=1$.

So $$S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)=\left[(1-0)\times(1-0)\right]^{1/2}\times 1$$

$$=1$$

Now $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ is given as

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[1 - \frac{\sum_{i=1}^{4} \left| a_i^U - b_i^U \right|}{4} \right] \times (1 - \left| x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U} \right|)^{B(S_{\tilde{\tilde{A}}^U}, S_{\tilde{\tilde{B}}^U})} \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}$$

Similarly if $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ are identical then $a_1^U = b_1^U, a_2^U = b_2^U, a_3^U = b_3^U, a_4^U = b_4^U$, $w^U_{\tilde{\tilde{A}}} = w^U_{\tilde{\tilde{B}}}$, therefore $y^*_{\tilde{\tilde{A}}^U} = y^*_{\tilde{\tilde{B}}^U}$ and $x^*_{\tilde{\tilde{A}}^U} = x^*_{\tilde{\tilde{B}}^U}$, also from (4.11) and (4.12), $S^U_{\tilde{\tilde{A}}} \neq 0$, $S^U_{\tilde{\tilde{B}}} \neq 0$ so according to (4.10), $B(S^U_{\tilde{\tilde{A}}}, S^U_{\tilde{\tilde{B}}}) = 1$

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[(1-0) \times (1-0) \right]^{1/2} \times 1$$

$$= 1$$

So $$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{1 \times 1}$$

$$=1$$

2) If $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ = 1, then

$$\sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S\left(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U\right)} = 1$$

i.e. $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ = 1 and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ = 1

$$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = \left[1 - \frac{\sum_{i=1}^{4} \left| a_i^L - b_i^L \right|}{4} \right] \times (1 - \left| x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L} \right|)^{B(S_{\tilde{\tilde{A}}^L}, S_{\tilde{\tilde{B}}^L})} \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} = 1$$

It implies that $a_1^L = b_1^L, a_2^L = b_2^L, a_3^L = b_3^L, a_4^L = b_4^L$, $w^L_{\tilde{\tilde{A}}} = w^L_{\tilde{\tilde{B}}}$, $x^*_{\tilde{\tilde{A}}^L} = x^*_{\tilde{\tilde{B}}^L}$ and $y^*_{\tilde{\tilde{A}}^L} = y^*_{\tilde{\tilde{B}}^L}$.

Therefore the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^U$ are identical.

If $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ =1,then

$$S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)=\left[1-\frac{\sum_{i=1}^{4}\left|a_i^U-b_i^U\right|}{4}\right]\times(1-\left|x^*_{\tilde{\tilde{A}}^U}-x^*_{\tilde{\tilde{B}}^U}\right|)^{B(S_{\tilde{\tilde{A}}^U},S_{\tilde{\tilde{B}}^U})}\times\frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^U},y^*_{\tilde{\tilde{B}}^U})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^U},y^*_{\tilde{\tilde{B}}^U})}=1$$

It implies that $a_1^U=b_1^U, a_2^U=b_2^U, a_3^U=b_3^U, a_4^U=b_4^U$, $w^U_{\tilde{\tilde{A}}}=w^U_{\tilde{\tilde{B}}}$, $x^*_{\tilde{\tilde{A}}^U}=x^*_{\tilde{\tilde{B}}^U}$ and $y^*_{\tilde{\tilde{A}}^U}=y^*_{\tilde{\tilde{B}}^U}$.

Therefore, the upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ are identical.

Property 2: $S(\tilde{\tilde{A}},\tilde{\tilde{B}})=S(\tilde{\tilde{B}},\tilde{\tilde{A}})$

Proof : Based on formula (4.13)

$$S(\tilde{\tilde{A}},\tilde{\tilde{B}})=\sqrt{S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)\times S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)}$$

$$S(\tilde{\tilde{B}},\tilde{\tilde{A}})=\sqrt{S(\tilde{\tilde{B}}^L,\tilde{\tilde{A}}^L)\times S(\tilde{\tilde{B}}^U,\tilde{\tilde{A}}^U)}$$

where

$$S(\tilde{\tilde{A}}^L,\tilde{\tilde{B}}^L)=\left[1-\frac{\sum_{i=1}^{4}\left|a_i^L-b_i^L\right|}{4}\right]\times(1-\left|x^*_{\tilde{\tilde{A}}^L}-x^*_{\tilde{\tilde{B}}^L}\right|)^{B(S_{\tilde{\tilde{A}}^L},S_{\tilde{\tilde{B}}^L})}\times\frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^L},y^*_{\tilde{\tilde{B}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^L},y^*_{\tilde{\tilde{B}}^L})}$$

$$S(\tilde{\tilde{B}}^L,\tilde{\tilde{A}}^L)=\left[1-\frac{\sum_{i=1}^{4}\left|b_i^L-a_i^L\right|}{4}\right]\times(1-\left|x^*_{\tilde{\tilde{B}}^L}-x^*_{\tilde{\tilde{A}}^L}\right|)^{B(S_{\tilde{\tilde{B}}^L},S_{\tilde{\tilde{A}}^L})}\times\frac{\text{Minimum}(y^*_{\tilde{\tilde{B}}^L},y^*_{\tilde{\tilde{A}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{B}}^L},y^*_{\tilde{\tilde{A}}^L})}$$

$$S(\tilde{\tilde{A}}^U,\tilde{\tilde{B}}^U)=\left[1-\frac{\sum_{i=1}^{4}\left|a_i^U-b_i^U\right|}{4}\right]\times(1-\left|x^*_{\tilde{\tilde{A}}^U}-x^*_{\tilde{\tilde{B}}^U}\right|)^{B(S_{\tilde{\tilde{A}}^U},S_{\tilde{\tilde{B}}^U})}\times\frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^U},y^*_{\tilde{\tilde{B}}^U})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^U},y^*_{\tilde{\tilde{B}}^U})}$$

$$S(\tilde{\tilde{B}}^U, \tilde{\tilde{A}}^U) = \left[1 - \frac{\sum_{i=1}^{4} \left| b_i^U - a_i^U \right|}{4}\right] \times \left(1 - \left| x^*_{\tilde{\tilde{B}}^U} - x^*_{\tilde{\tilde{A}}^U} \right|\right)^{B(S_{\tilde{\tilde{B}}^U}, S_{\tilde{\tilde{A}}^U})} \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U})}{\text{Maximum}(y^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U})}$$

where

$$\sum_{i=1}^{4} \left| a_i^L - b_i^L \right| = \sum_{i=1}^{4} \left| b_i^L - a_i^L \right|, \sum_{i=1}^{4} \left| a_i^U - b_i^U \right| = \sum_{i=1}^{4} \left| b_i^U - a_i^U \right|, \left| x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L} \right| = \left| x^*_{\tilde{\tilde{B}}^L} - x^*_{\tilde{\tilde{A}}^L} \right|, \left| x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U} \right| = \left| x^*_{\tilde{\tilde{B}}^U} - x^*_{\tilde{\tilde{A}}^U} \right|,$$

$$\frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} = \frac{\text{Minimum}(y^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{A}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{A}}^L})}, \quad \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})} = \frac{\text{Minimum}(y^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U})}{\text{Maximum}(y^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U})},$$

$B(S^L_{\tilde{\tilde{A}}}, S^L_{\tilde{\tilde{B}}}) = B(S^L_{\tilde{\tilde{B}}}, S^L_{\tilde{\tilde{A}}})$ and $B(S^U_{\tilde{\tilde{A}}}, S^U_{\tilde{\tilde{B}}}) = B\left(S^U_{\tilde{\tilde{B}}}, S^U_{\tilde{\tilde{A}}}\right)$. Thus we can see that

$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = S(\tilde{\tilde{B}}^L, \tilde{\tilde{A}}^L)$ and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = S(\tilde{\tilde{B}}^U, \tilde{\tilde{A}}^U)$.

Therefore, we can see that $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$.

Property 3. If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are two real numbers, then $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1 - |a - b|$.

Proof : If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are real two numbers, then we can see that

$$\tilde{\tilde{A}} = \left[\left(a_1^L, a_2^L, a_3^L, a_4^L; w^L_{\tilde{\tilde{A}}}\right), \left(a_1^U, a_2^U, a_3^U, a_4^U; w^U_{\tilde{\tilde{A}}}\right)\right]$$

$$= \left[(a, a, a, a; 1), (a, a, a, a; 1)\right]$$

$$= (a, a, a, a; 1)$$

$$= a,$$

$$\tilde{\tilde{B}} = \left[\left(b_1^L, b_2^L, b_3^L, b_4^L; w^L_{\tilde{\tilde{B}}}\right), \left(b_1^U, b_2^U, b_3^U, b_4^U; w^U_{\tilde{\tilde{B}}}\right)\right]$$

$$= \left[(b, b, b, b; 1), (b, b, b, b; 1)\right]$$

$$= (b, b, b, b; 1)$$

$$= b$$

Based on (4.7) and (4.8), we can see that $S^L_{\tilde{\tilde{B}}} = 0$ and $S^L_{\tilde{\tilde{B}}} = 0$. From (4.6), we can see that $B(S^L_{\tilde{\tilde{A}}}, S^L_{\tilde{\tilde{B}}}) = 0$. Similarly from (4.11) and (4.12) , $S^U_{\tilde{\tilde{B}}} = 0$ and $S^U_{\tilde{\tilde{B}}} = 0$. From (4.10) we can see that $B(S^U_{\tilde{\tilde{A}}}, S^U_{\tilde{\tilde{B}}}) = 0$. From the formulas (3.20) and (4.2), we can see that $y^*_{\tilde{\tilde{A}}^U} = y^*_{\tilde{\tilde{B}}^U} = y^*_{\tilde{\tilde{B}}^U} = y^*_{\tilde{\tilde{B}}^U} = \frac{1}{2}$.

So from (4.5)

$$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = \left[1 - \frac{\sum_{i=1}^{4} \left| a_i^L - b_i^L \right|}{4} \right] \times (1 - \left| x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L} \right|)^0 \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}$$

$$= \left[1 - \frac{4\left| a - b \right|}{4} \right] \times 1 \times 1$$

$$= 1 - \left| a - b \right|$$

and from (4.9)

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[1 - \frac{\sum_{i=1}^{4} \left| a_i^U - b_i^U \right|}{4} \right] \times (1 - \left| x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U} \right|)^0 \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}$$

$$= \left[1 - \frac{4\left| a - b \right|}{4} \right] \times 1 \times 1$$

$$= 1 - \left| a - b \right| .$$

Therefore, we can see that

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)}$$

$$=\sqrt{(1-|a-b|)\times(1-|a-b|)}$$

$$=1-|a-b|.$$

4.5 Numerical examples

Example 4.1. Assume that there are two interval-value fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$,where

$$\tilde{\tilde{A}}=\left[\left(0.1,0.2,0.3,0.4;0.5\right),\left(0.1,0.2,0.3,0.4;1\right)\right]$$

$$\tilde{\tilde{B}}=\left[\left(0.4,0.5,0.6,0.7;0.5\right),\left(0.4,0.5,0.6,0.7;1\right)\right]$$

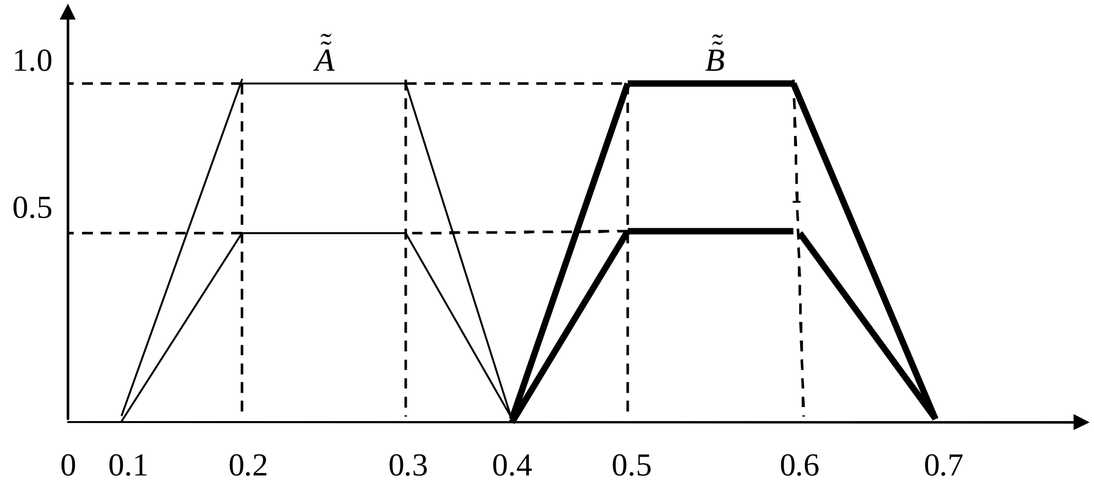

Fig.4. 2 Two interval-valued fuzzy numbers

Solution:

Step 1 From (4.1) and (4.2), we get COG point COG($\tilde{\tilde{A}}^L$) = $(x^*_{\tilde{\tilde{A}}^L}, x^*_{\tilde{\tilde{B}}^L})$ of the lower fuzzy number $\tilde{\tilde{A}}^L$, shown as follows:

$$y^*_{\tilde{\tilde{A}}^L}=\frac{w^L_{\tilde{\tilde{A}}}\times\left(\frac{a^L_3-a^L_2}{a^L_4-a^L_1}+2\right)}{6}$$

$$=\frac{0.5\times\left(\frac{0.3-0.2}{0.4-0.1}\right)}{6}$$

$$= 0.1944,$$

$$x^*_{\tilde{\tilde{A}}^L} = \frac{y^*_{\tilde{\tilde{A}}^L}(a_3^L + a_2^L) + (a_4^L + a_1^L)(w^L_{\tilde{\tilde{A}}} - y^*_{\tilde{\tilde{A}}^L})}{2w^L_{\tilde{\tilde{A}}}}$$

$$= \frac{0.1944 \times (0.3 + 0.2) + (0.4 + 0.1) + (0.5 - 0.1994)}{2 \times 0.5}$$

$$= 0.25$$

Similarly COG($\tilde{\tilde{B}}^L$) = (0.55, 0.1944), COG($\tilde{\tilde{A}}^U$) = (0.25, 0.3889), and COG($\tilde{\tilde{B}}^U$) = (0.55, 0.3889). From (4.7) and (4.8)

$$S^L_{\tilde{\tilde{A}}} = a_4^L - a_1^L = 0.4 - 0.1 = 0.3$$

$$S^L_{\tilde{\tilde{B}}} = b_4^L - b_1^L = 0.7 - 0.4 = 0.3$$

$S^L_{\tilde{\tilde{A}}} + S^L_{\tilde{\tilde{B}}} > 0$ so its clear from (4.5) $B(S^L_{\tilde{\tilde{A}}}, S^L_{\tilde{\tilde{B}}}) = 1$, similarly $B(S^U_{\tilde{\tilde{A}}}, S^U_{\tilde{\tilde{B}}}) = 1$.

Step 2 From (4.5), we calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$, as shown follows:

$$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = \left[1 - \frac{\sum_{i=1}^{4} \left|a_i^L - b_i^L\right|}{4}\right] \times (1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|)^{B(S_{\tilde{\tilde{A}}^L}, S_{\tilde{\tilde{B}}^L})} \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}$$

$$= \left[1 - \frac{|0.1 - 0.4| + |0.2 - 0.5| + |0.3 - 0.6| + |0.4 - 0.7|}{4}\right] \times (1 - |0.25 - 0.55|)^1 \times \frac{\text{Minimun }(0.1944, 0.1944)}{\text{Maximum}(0.1944, 0.1944)}$$

$$= \left[1 - \frac{4 \times 0.3}{4}\right] \times (1 - 0.30) \times 1$$

$$= 0.7 \times 0.7 = 0.49$$

Similarly by using (4.9), we can calculate the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$, where $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = 0.49$.

step 3. Now we calculate degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ as follows:

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)}$$

$$= \sqrt{0.49 \times 0.49} = 0.49$$

i.e., degree of similarity between interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is about 0.49.

Example 4.2. Assume that $\tilde{\tilde{A}}$ is an interval-valued fuzzy number and $\tilde{B}$ is a generalized fuzzy number, where

$$\tilde{\tilde{A}} = \left[(0.1, 0.2, 0.3, 0.4; 0.5), (0.1, 0.2, 0.3, 0.4; 1)\right] \text{ and } \tilde{B} = \left[(0.4, 0.5, 0.6, 0.7; 0.5)\right]$$

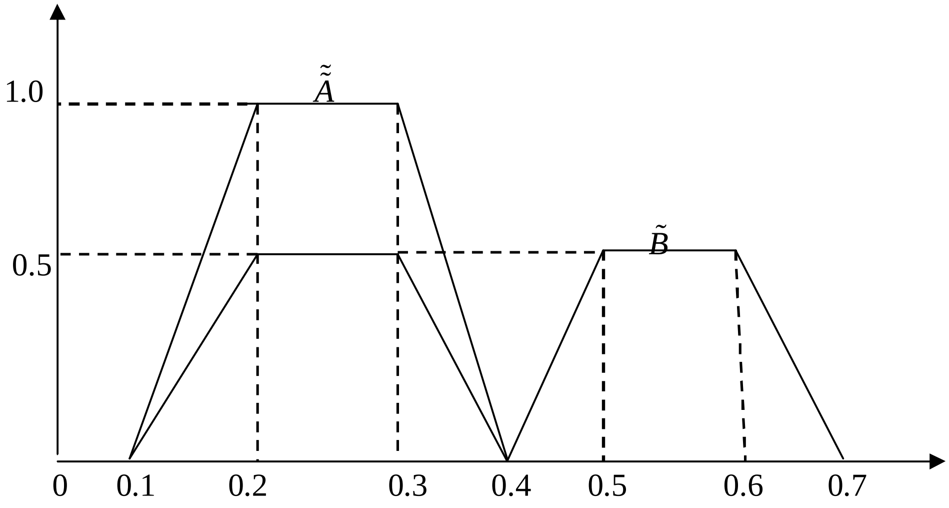

Fig. 4.3. An interval-valued fuzzy number $\tilde{\tilde{A}}$ and a generalized fuzzy number $\tilde{B}$

Solution:

As shown in Fig. 4.3 we can see that the generalized fuzzy number $\tilde{B}$ can be represented as follows:

$$\tilde{B} = \left[\left(0.4, 0.5, 0.6, 0.7; 0.5\right), \left(0.4, 0.5, 0.6, 0.7; 0.5\right) \right].$$

Step 1 From (4.1) and (4.2) , we get COG point COG($\tilde{\tilde{A}}^L$) = ($x^*_{\tilde{\tilde{A}}^L}, x^*_{\tilde{\tilde{B}}^L}$) of the lower fuzzy number $\tilde{\tilde{A}}^L$, shown as follows:

$$y^*_{\tilde{\tilde{A}}^L} = \frac{w^L_{\tilde{\tilde{A}}} \times \left(\dfrac{a_3^L - a_2^L}{a_4^L - a_1^L} + 2 \right)}{6}$$

$$= \frac{0.5 \times \left(\dfrac{0.3 - 0.2}{0.4 - 0.1} \right)}{6}$$

$$= 0.1944$$

$$x^*_{\tilde{\tilde{A}}^L} = \frac{y^*_{\tilde{\tilde{A}}^L} (a_3^L + a_2^L) + (a_4^L + a_1^L)(w^L_{\tilde{\tilde{A}}} - y^*_{\tilde{\tilde{A}}^L})}{2 w^L_{\tilde{\tilde{A}}}}$$

$$= \frac{0.1944 \times (0.3 + 0.2) + (0.4 + 0.1) + (0.5 - 0.1994)}{2 \times 0.5}$$

$$= 0.25$$

Similarly COG($\tilde{\tilde{A}}^U$) = (0.25, 0.3889), and from the formula (3.17) and (3.19) COG($\tilde{B}$) = (0.55, 0.1944). From (4.7) and (4.8)

$$S^L_{\tilde{\tilde{A}}} = a_4^L - a_1^L = 0.4 - 0.1 = 0.3$$

$$S_{\tilde{B}} = b_4 - b_1 = 0.7 - 0.4 = 0.3$$

$S^L_{\tilde{\tilde{A}}} + S_{\tilde{B}} > 0$ so its clear from (4.5) $B(S^L_{\tilde{\tilde{A}}}, S_{\tilde{B}})$ = 1, similarly $B(S_{\tilde{\tilde{A}}}, S_{\tilde{B}})$ = 1.

Step 2 From (4.5), we calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$, as shown follows:

$$S(\tilde{\tilde{A}}^U, \tilde{B}) = \left[1 - \frac{\sum_{i=1}^{4} \left| a_i^U - b_i \right|}{4} \right] \times (1 - \left| x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}} \right|)^{B(S_{\tilde{\tilde{A}}^U}, S_{\tilde{\tilde{B}}})} \times \frac{\text{Minimum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}})}{\text{Maximum}(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}})}$$

$$= \left[1 - \frac{|0.1 - 0.4| + |0.2 - 0.5| + |0.3 - 0.6| + |0.4 - 0.7|}{4} \right] \times (1 - |0.25 - 0.55|)^1$$

$$\times \frac{\text{Minimum}\ (0.3889, 0.1944)}{\text{Maximum}(0.3889, 0.1944)}$$

$$= \left[1 - \frac{4 \times 0.3}{4} \right] \times (1 - 0.30) \times \frac{0.1944}{0.3889}$$

$$= 0.7 \times 0.7 \times 0.4998714$$

$$= 0.2449369\,.$$

Similarly, we can calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{B})$ between interval-valued fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{B}$, where $S(\tilde{\tilde{A}}^L, \tilde{B}) = 0.49$.

step 3 Now we calculate degree of similarity $S(\tilde{\tilde{A}}, \tilde{B})$ between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ as follows:

$$S(\tilde{\tilde{A}}, \tilde{B}) = \sqrt{S(\tilde{\tilde{A}}^L, \tilde{B}) \times S(\tilde{\tilde{A}}^U, \tilde{B})}$$

$$= \sqrt{0.49 \times 0.2449369}$$

$$= 0.346437.$$

That is, degree of similarity between interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is about 0.346437.

4.6 Conclusion

In this chapter, we have presented a new method to calculate measures of similarity between interval-valued fuzzy numbers. The presented similarity measure uses the concept of geometry to calculate the center-of-gravity points of the lower fuzzy numbers and upper fuzzy numbers of interval valued fuzzy numbers, respectively, to calculate the degree of similarity between interval-valued fuzzy numbers. We also proved some properties of the proposed similarity measure. The proposed method is more flexible and more intelligent due to the fact that it uses interval-valued fuzzy numbers rather than fuzzy numbers or generalized fuzzy numbers.

Chapter 5

SIMILARITY MEASURE BETWEEN INTUITIONISTIC FUZZY SETS

Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFSs) as a generalization of fuzzy sets introduced by Zadeh [54]. The concept of vague sets introduced by Gau and Buehrer [20] is another generalization of fuzzy sets. Bustince and Burfllio [5] pointed out that the notion of vague set is the same as that of intuitionistic fuzzy set. Chen [7, 9] proposed a set of methods for measuring the degree of similarity between vague sets. Hong and Kim [23] showed by examples that the similarity measures proposed by Chen [7, 9] do not fit well in some cases, and proposed a set of modified measures. Recently, Li and Cheng [32] proposed several new similarity measures between IFSs.

In this chapter, it is shown that Li and Cheng's methods have the same disadvantages as Chen's methods. At the same time, by example it is shown that Hong and Kim's modified similarity measures also do not fit well in some cases. Therefore, by means of the geometrical representation of an IFSs and the distances between IFSs proposed by Szmidt and Kacprzyk [46], a set of methods for measuring the degree of similarity between IFSs and between elements. Comparing the similarity degree of each measure, it is shown that the similarity measures presented in this chapter are more reasonable than the previous.

5.1 Intuitionistic fuzzy sets (IFSs) and vague sets

In this section, we review the definitions of intuitionistic fuzzy sets and vague sets.

Definition 5.1 [2]

Let X be a fixed set. IFS $\tilde{A}$ in X , is an object having the form,

$$\tilde{A}=\left\{\left\langle x,\mu_{\tilde{A}}(x),\nu_{\tilde{A}}(x)\right\rangle \mid x\in X\right\}$$

where the functions $\mu_{\tilde{A}},\nu_{\tilde{A}}:X\rightarrow[0,1]$ are the degree of membership and the degree of non-membership of the element $x\in X$ to $\tilde{A}$, respectively, moreover, $0\le\mu_{\tilde{A}}(x)+\nu_{\tilde{A}}(x)\le 1$ must hold.

For each IFS $\tilde{A}$ in X , we will call $\pi_{\tilde{A}}(x)=1-\mu_{\tilde{A}}(x)-\nu_{\tilde{A}}(x)$, the intuitionistic index of X in $\tilde{A}$. It is a hesitancy degree of X to $\tilde{A}$. It is obvious that $0\le\pi_{\tilde{A}}(x)\le 1$, for each $x\in X$.

Definition 5.2 [20]

Let X be the universe of discourse. A vague set $\tilde{A}$ in X is characterized by a truth-membership function $t_{\tilde{A}}$ and false-membership function $f_{\tilde{A}}$,

$$t_{\tilde{A}}:X\rightarrow[0,1],\qquad\qquad f_{\tilde{A}}:X\rightarrow[0,1],$$

where $t_{\tilde{A}}(x)$ is a lower bound on the grade of membership of x derived from the evidence for $x\in X$, $f_{\tilde{A}}(x)$ is a lower bound on the negation of x derived from the evidence against x, and $t_{\tilde{A}}(x)+f_{\tilde{A}}(x)\le 1$.

5.2 Similarity measure between IFSs

In this section, we will first review the similarity measures proposed by Chen [7, 9], by Hong and Kim [23], and by Li and Cheng [32], respectively. Then, we will show by examples that these similarity measures are not reasonable in some cases. Finally, a similarity measure between IFSs will be presented and its properties will be discussed.

Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. $\tilde{A}, \tilde{B}$ be two IFSs in X , where

$$\tilde{A} = \left\{ \left\langle x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i) \right\rangle \mid x_i \in X \right\}, \text{ and } \tilde{B} = \left\{ \left\langle x_{i,} \mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i) \right\rangle \right\}$$

Chen [7, 9] proposed the following similarity measure between $\tilde{A}$ and $\tilde{B}$,

$$T_C = 1 - \frac{1}{n} \sum_{i=1}^{n} \frac{\left| \left(\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i) \right) - \left(\nu_{\tilde{A}}(x_i) - \nu_{\tilde{B}}(x_i) \right) \right|}{2} \tag{5.1}$$

where n denotes the number of terms.

Hong and Kim [23] used the following example to show that the similarity measure proposed by Chen is not reasonable.

Example 5.1: Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse, $\tilde{A}, \tilde{B}$ and $\tilde{C}$ be IFSs in X , where

$$\begin{aligned} \tilde{A} &= \left\{ \left\langle x_i, 0, 0 \right\rangle \mid x_i \in X \right\}, \\ \tilde{B} &= \left\{ \left\langle x_i, 0.5, 0.5 \right\rangle \mid x_i \in X \right\}, \\ \tilde{C} &= \left\{ \left\langle x_i, 0.499, 0.501 \right\rangle \mid x_i \in X \right\}. \end{aligned}$$

Intuitively, we can see that the IFS $\tilde{B}$ is much more similar to the IFS $\tilde{C}$ than the IFS $\tilde{A}$, and that the IFS $\tilde{A}$ is not so similar to the IFS $\tilde{B}$. Based on (5.1), however, we get the degrees of similarity between $\tilde{A}$ and $\tilde{B}$ and between $\tilde{B}$ and $\tilde{C}$ as follows,

In this case $n = 1$

$$T_C\left(\tilde{A}, \tilde{B}\right) = 1 - \left(\frac{\left| (0 - 0.5) - (0 - 0.5) \right|}{2} \right)$$

$$T_C\left(\tilde{A}, \tilde{B}\right) = 1,$$

Similarly $$T_C\left(\tilde{B}, \tilde{C}\right) = 1 - \left(\frac{\left| (0.5 - 0.499) - (0.5 - 0.501) \right|}{2} \right)$$

$$T_C\left(\tilde{B},\tilde{C}\right)=0.999$$

i.e., $T_C\left(\tilde{A},\tilde{B}\right)>T_C\left(\tilde{B},\tilde{C}\right)$. It can be seen that the similarity measure T_C is not so reasonable as we expect.

Li and Cheng [32] defined the degree of similarity between IFSs $\tilde{A}$ and $\tilde{B}$ as follows

$$T_L\left(\tilde{A},\tilde{B}\right)=1-\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left|\left(\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right)-\left(\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right)\right|}{2}\right]^p} \tag{5.2}$$

where $1\le p<+\infty$. We clearly see that Li and Cheng's similarity measure T_L, has the same disadvantages as Chen's T_C. So, the similarity measure T_L is also not reasonable.

To overcome above problem, Hong and Kim [23] suggested a modified similarity measure between IFSs $\tilde{A}$ and $\tilde{B}$ as follows,

$$T_H=1-\frac{1}{n}\sum_{i=1}^{n}\frac{\left|\left(\mu_{\tilde{A}}(x_i)-\mu_{\tilde{B}}(x_i)\right)+\left(\nu_{\tilde{A}}(x_i)-\nu_{\tilde{B}}(x_i)\right)\right|}{2} \tag{5.3}$$

In the following, we give an example to show that Hong and Kim's modified similarity measure is also not reasonable in some cases.

Example 5.2: Let $X=\{x_1,x_2,...,x_n\}$ be the universe of discourse, $\tilde{A}$, $\tilde{B}_1$, $\tilde{B}_2$ and $\tilde{B}_3$, be IFSs in X, where

$$\begin{aligned}
\tilde{A}&=\left\{\left\langle x_i,0,0\right\rangle \mid x_i\in X\right\},\\
\tilde{B}_1&=\left\{\left\langle x_i,0.2,0.8\right\rangle \mid x_i\in X\right\},\\
\tilde{B}_2&=\left\{\left\langle x_i,0.4,0.6\right\rangle \mid x_i\in X\right\},\\
\tilde{B}_3&=\left\{\left\langle x_i,0.5,0.5\right\rangle \mid x_i\in X\right\}.
\end{aligned}$$

In this case $n=1$

using (5.3), we get

$$T_H\left(\tilde{A},\tilde{B}_1\right)=1-\left(\frac{\left|(0-0.2)+(0-0.8)\right|}{2}\right)=0.5$$

Similarly $T_H\left(\tilde{A},\tilde{B}_3\right)=0.5$, and $T_H\left(\tilde{A},\tilde{B}_2\right)=0.5$

$$T_H\left(\tilde{A},\tilde{B}_1\right)=T_H\left(\tilde{A},\tilde{B}_2\right)=T_H\left(\tilde{A},\tilde{B}_3\right)=0.5$$

In fact, for $\tilde{A}=\left\{\left\langle x_i,0,0\right\rangle \mid x_i\in X\right\}$ and $\tilde{B}_k=\left\{\left\langle x_i,\mu_{\tilde{B}_k}\left(x_i\right),\nu_{\tilde{B}_k}\left(x_i\right)\right\rangle \mid x_i\in X\right\}$ satisfying $\mu_{\tilde{B}_k}\left(x_i\right)+\nu_{\tilde{B}_k}\left(x_i\right)=1\left(\forall x_i\in X,k=1,2,...\right)$, we always get $T_H\left(\tilde{A},\tilde{B}_k\right)=0.5$ i.e., $T_H\left(\tilde{A},\tilde{B}_1\right)=T_H\left(\tilde{A},\tilde{B}_2\right)=...=0.5$

Obviously, we cannot carry out our comparison and recognition in such case. This will get decision maker into trouble in practical application. To overcome this problem, we presented a new method for measuring the degree of similarity between IFSs as follows according to the geometrical representation of an intuitionistic fuzzy set and the distance between IFSs proposed by Szmidt and Kacprzyk [46],

$$T\left(\tilde{A},\tilde{B}\right)=1-\sqrt[p]{\frac{1}{2n}\sum_{i=1}^{n}\left|\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right|^p} \qquad (5.4)$$

where $\pi_{\tilde{A}}\left(x_i\right)=1-\mu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{A}}\left(x_i\right)$, $\pi_{\tilde{B}}\left(x_i\right)=1-\mu_{\tilde{B}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)$ and $1<p<+\infty$. Now, we use this new similarity measure T to calculate the similarity measures of Example 5.1 and 5.2. By taking $p=2$ in (5.4), we have for Example 5.1,

$$T\left(\tilde{A},\tilde{B}\right)=1-\sqrt{\frac{1}{2}\sum_{i=1}^{n}(0-0.5)^2+(0-0.5)^2+(1-0)^2}\quad=0.134$$

$$T\left(\tilde{B},\tilde{C}\right)=1-\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(0.5-0.499\right)^2+\left(0.5-0.501\right)^2+\left(0-0\right)^2}\quad=0.999$$

and for Example 5.2,

$$T\left(\tilde{A},\tilde{B}_1\right)=1-\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(0-0.2\right)^2+\left(0-0.8\right)^2+\left(1-0\right)^2}\quad=0.083$$

$$T\left(\tilde{A},\tilde{B}_2\right)=1-\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(0-0.4\right)^2+\left(0-0.6\right)^2+\left(1-0\right)^2}\quad=0.128$$

$$T\left(\tilde{A},\tilde{B}_3\right)=1-\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(0-0.5\right)^2+\left(0-0.5\right)^2+\left(1-0\right)^2}\quad=0.134$$

Therefore, the similarity measure T is illustrated to be more reasonable than T_C, T_L and T_H.

5.3 Some properties of similarity measure *T*

Property 1. $0\le T\left(\tilde{A},\tilde{B}\right)\le 1$,

Proof: From (5.4) its clear that $0\le T\left(\tilde{A},\tilde{B}\right)\le 1$

Property 2. $T\left(\tilde{A},\tilde{B}\right)=T\left(\tilde{B},\tilde{A}\right)$

Proof : $T\left(\tilde{A},\tilde{B}\right)$ is given by

$$T\left(\tilde{A},\tilde{B}\right)=1-\sqrt[p]{\frac{1}{2n}\sum_{i=1}^{n}\left|\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right|^p}$$

$$T\left(\tilde{B},\tilde{A}\right)=1-\sqrt[p]{\frac{1}{2n}\sum_{i=1}^{n}\left|\mu_{\tilde{B}}\left(x_i\right)-\mu_{\tilde{A}}\left(x_i\right)\right|^p+\left|\nu_{\tilde{B}}\left(x_i\right)-\nu_{\tilde{A}}\left(x_i\right)\right|^p+\left|\pi_{\tilde{B}}\left(x_i\right)-\pi_{\tilde{A}}\left(x_i\right)\right|^p}$$

where $\left|\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right|=\left|\mu_{\tilde{B}}\left(x_i\right)-\mu_{\tilde{A}}\left(x_i\right)\right|,\left|\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right|=\left|\nu_{\tilde{B}}\left(x_i\right)-\nu_{\tilde{A}}\left(x_i\right)\right|$ and

$\left|\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right|=\left|\pi_{\tilde{B}}\left(x_i\right)-\pi_{\tilde{A}}\left(x_i\right)\right|$

It implies that $T\left(\tilde{A},\tilde{B}\right)=T\left(\tilde{B},\tilde{A}\right)$

Property 3. $T\left(\tilde{A},\tilde{B}\right)=1 \Leftrightarrow \tilde{A}=\tilde{B}$

Proof : $T\left(\tilde{A},\tilde{B}\right)$ is given by

$$T\left(\tilde{A},\tilde{B}\right)=1-\sqrt[p]{\frac{1}{2n}\sum_{i=1}^{n}\left|\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right|^p}$$

if $\tilde{A}=\tilde{B}$ then

$\mu_{\tilde{A}}\left(x_i\right)=\mu_{\tilde{B}}\left(x_i\right),\nu_{\tilde{A}}\left(x_i\right)=\nu_{\tilde{B}}\left(x_i\right)$ and $\pi_{\tilde{A}}\left(x_i\right)=\pi_{\tilde{B}}\left(x_i\right)$ so clearly the term

$$\sqrt[p]{\frac{1}{2n}\sum_{i=1}^{n}\left|\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right|^p}=0$$

which implies that $T\left(\tilde{A},\tilde{B}\right)=1$,

now if $T\left(\tilde{A},\tilde{B}\right)=1$, then

$$1-\sqrt[p]{\frac{1}{2n}\sum_{i=1}^{n}\left|\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right|^p+\left|\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right|^p}=1$$

it implies that $\mu_{\tilde{A}}\left(x_i\right)=\mu_{\tilde{B}}\left(x_i\right),\nu_{\tilde{A}}\left(x_i\right)=\nu_{\tilde{B}}\left(x_i\right)$ and $\pi_{\tilde{A}}\left(x_i\right)=\pi_{\tilde{B}}\left(x_i\right)$

i.e., $\tilde{A}=\tilde{B}$

For IFSs $A=\left\{\left\langle x,\mu_{\tilde{A}}\left(x\right),\nu_{\tilde{A}}\left(x\right)\right\rangle \mid x\in\left[a,b\right]\right\}$, $\tilde{B}=\left\{\left\langle x,\mu_{\tilde{B}}\left(x\right)\nu_{\tilde{B}}\left(x\right)\right\rangle \mid x\in\left[a,b\right]\right\}$, Li and Cheng[32] defined the degree of similarity between $\tilde{A}$ and $\tilde{B}$ as follows:

$$T_L^c\left(\tilde{A},\tilde{B}\right)=1-\sqrt[p]{\frac{1}{b-a}\int_a^b\left[\frac{\left|\left(\mu_{\tilde{A}}\left(x\right)-\mu_{\tilde{B}}\left(x\right)\right)-\left(\nu_{\tilde{A}}\left(x\right)-\nu_{\tilde{B}}\left(x\right)\right)\right|}{2}\right]^p dx}\,, \tag{5.5}$$

where $1\leq p<+\infty$.

By the same reasons as before, we modify the similarity measure T_L^c as follows:

$$T^c\left(\tilde{A},\tilde{B}\right)=1-\sqrt[p]{\frac{1}{2\left(b-a\right)}\int_a^b\left[\left|\mu_{\tilde{A}}\left(x\right)-\mu_{\tilde{B}}\left(x\right)\right|^p+\left|\nu_{\tilde{A}}\left(x\right)-\nu_{\tilde{B}}\left(x\right)\right|^p+\left|\pi_{\tilde{A}}\left(x\right)-\pi_{\tilde{B}}\left(x\right)\right|^p\right]dx} \tag{5.6}$$

where $1<p<+\infty$

similarly T^c has the following properties,

i. $0\le T^c\left(\tilde{A},\tilde{B}\right)\le 1$,

ii. $T^c\left(\tilde{A},\tilde{B}\right)=T^c\left(\tilde{B},\tilde{A}\right)$,

iii. $T^c\left(\tilde{A},\tilde{B}\right)=1\Leftrightarrow \tilde{A}=\tilde{B}$

5.4 Modified weighted similarity measure between IFSs

In this section, the weighted similarity measures between IFSs proposed by Chen [9], by Li and Cheng [32] and by Hong and Kim [23] are recalled, and their disadvantages are shown. The modified weighted similarity measures are presented.

Let $X=\left(x_1,x_2,...,x_n\right)$ be the universe of discourse, $\tilde{A}$ and $\tilde{B}$ be two IFSs in X, where

$$\tilde{A}=\left\{\left\langle x_i,\mu_{\tilde{A}}\left(x_i\right),\nu_{\tilde{A}}\left(x_i\right)\right|x\in X\right\rangle\Big\},\ \tilde{B}=\left\{\left\langle x_i,\mu_{\tilde{B}}\left(x_i\right),\nu_{\tilde{B}}\left(x_i\right)\right\rangle|\,x\in X\right\}$$

Chen [9] and, Li and Cheng [32] defined, respectively, the following weighted similarity measures between IFSs $\tilde{A}$ and $\tilde{B}$,

$$W_C\left(\tilde{A},\tilde{B}\right)=1-\frac{1}{a-b}\sum_{i=1}^n w_i\times\left|a\times\left(\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right)+b\times\left(\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right)+c\times\left(\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right)\right|$$

where $w_i\in[0,1]$, $i=1,2,3,...,n$, $\sum_{i=1}^n w_i=1$, and $a\ge c\ge 0\ge b$.

$$W_L\left(\tilde{A},\tilde{B}\right)=1-\sqrt[p]{\sum_{i=1}^{n}\left[\frac{\left|\left(\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right)-\left(\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right)\right|}{2}\right]^p} \tag{5.7}$$

where $1\leq p<+\infty$ and $w_i\in\left[0,1\right],\ i=1,2,\ldots,n,\ \sum_{i=1}^{n}w_i=1.$

if we take $a=1,b=-1,c=0,$ and $w_i=1/n,i=1,2,\ldots,n$, then

$$W_C\left(\tilde{A},\tilde{B}\right)=T_C\left(\tilde{A},\tilde{B}\right),\qquad W_L\left(\tilde{A},\tilde{B}\right)=T_L\left(\tilde{A},\tilde{B}\right).$$

So, W_C and W_L are not reasonable by the same reason as before. So Hong and Kim [23] gave the modified similarity measure as follows:

$$W_H\left(\tilde{A},\tilde{B}\right)=1-\frac{1}{a+b+c}\sum_{i=1}^{n}w_i\times\left[a\times\left|\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right|+b\times\left|\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right|+c\times\left|\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right|\right] \tag{5.8}$$

where $w_i\in\left[0,1\right],\ i=1,2,\ldots,n,\ \sum_{i=1}^{n}w_i=1$, and $a,b,c\geq 0.$

In the following, we will use Example 5.2 to show that Hong and Kim's modified similarity measure does not fit well in some case.

IFSs $\tilde{A},\ \tilde{B}_1,\ \tilde{B}_2$ and $\tilde{B}_3$, are shown in Example 5.2. If $a=b=c\left(\text{or } a=b,c=0\right)$ in (5.8), then we obtain

$$W_H\left(\tilde{A},\tilde{B}_1\right)=W_H\left(\tilde{A},\tilde{B}_2\right)=W_H\left(\tilde{A},\tilde{B}_3\right)=\frac{1}{3},\ \left(\text{or } W_H\left(\tilde{A},\tilde{B}_1\right)=W_H\left(\tilde{A},\tilde{B}_2\right)=W_H\left(\tilde{A},\tilde{B}_3\right)=\frac{1}{2}\right)$$

In fact for $\tilde{A}=\left\{\left\langle x_i,0,0\right\rangle \mid x_i\in X\right\}$, and $\tilde{B}_k=\left\{\left\langle x_i,\mu_{\tilde{B}_k}\left(x_i\right),\nu_{\tilde{B}_k}\left(x_i\right)\right\rangle \mid x_i\in X\right\}$ satisfying $\mu_{\tilde{B}_k}\left(x_i\right)+\nu_{\tilde{B}_k}\left(x_i\right)=1\left(\forall x_i\in X,k=1,2,\ldots\right)$, and $a=b=c\left(\text{or } a=b,c=0\right)$ in (5.8), we always have $W_H\left(\tilde{A},\tilde{B}_k\right)=1/3\left(W_H\left(\tilde{A},\tilde{B}_k\right)=1/2\right),\ k=1,2,\ldots$ hence T_H,W_H is not reasonable.

Because, we cannot carry out our comparison and recognition in such case. This will get decision maker into trouble in practical application. To overcome this problem, we present a modified method for measuring the degree of similarity between IFSs as follows:

$$W\left(\tilde{A},\tilde{B}\right)=1-\left[\sum_{i=1}^{n} w_i\times\left[\alpha\times\left|\mu_{\tilde{A}}\left(x_i\right)-\mu_{\tilde{B}}\left(x_i\right)\right|^p+\beta\times\left|\nu_{\tilde{A}}\left(x_i\right)-\nu_{\tilde{B}}\left(x_i\right)\right|^p+\gamma\times\left|\pi_{\tilde{A}}\left(x_i\right)-\pi_{\tilde{B}}\left(x_i\right)\right|^p\right]\right]^{1/p} \tag{5.9}$$

where $1<p<+\infty,\ w_i\in\left[0,1\right],\ i=1,2,...,n,\ \sum_{i=1}^{n} w_i=1,\ \alpha,\beta,\gamma\in\left[0,1\right]$, and $\alpha+\beta+\gamma=1$.

Now, let us calculate the similarity measures of Example 5.1 and 5.2 using (5.9).

By choosing $w_i=1/n, i=1,2,...,n,\alpha=\beta=\gamma=1/3$, and $p=2$ we get for Example 5.1

$$W\left(\tilde{A},\tilde{B}\right)=1-\left[\sum_{i=1}^{n}\left[\frac{1}{3}\times\left|0-0.5\right|^2+\frac{1}{3}\times\left|0-0.5\right|^2+\frac{1}{3}\times\left|1-0\right|^2\right]\right]^{1/2}=0.293$$

$$W\left(\tilde{B},\tilde{C}\right)=1-\left[\sum_{i=1}^{n}\left[\frac{1}{3}\times\left|0.5-0.499\right|^2+\frac{1}{3}\times\left|0.5-0.501\right|^2+\frac{1}{3}\times\left|0-0\right|^2\right]\right]^{1/2}=0.999$$

and for Example 5.2

$$W\left(\tilde{A},\tilde{B}_1\right)=1-\left[\sum_{i=1}^{n}\left[\frac{1}{3}\times\left|0-0.2\right|^2+\frac{1}{3}\times\left|0-0.8\right|^2+\frac{1}{3}\times\left|1-0\right|^2\right]\right]^{1/2}=0.252$$

$$W\left(\tilde{A},\tilde{B}_2\right)=1-\left[\sum_{i=1}^{n}\left[\frac{1}{3}\times\left|0-0.4\right|^2+\frac{1}{3}\times\left|0-0.6\right|^2+\frac{1}{3}\times\left|1-0\right|^2\right]\right]^{1/2}=0.288$$

$$W\left(\tilde{A},\tilde{B}_3\right)=1-\left[\sum_{i=1}^{n}\left[\frac{1}{3}\times\left|0-0.5\right|^2+\frac{1}{3}\times\left|0-0.5\right|^2+\frac{1}{3}\times\left|1-0\right|^2\right]\right]^{1/2}=0.293$$

From above results, we see that the modified weighted similarity measure W is more reasonable than the previous. Similarly, the function W has following properties,

i. $0 \le W\left(\tilde{A}, \tilde{B}\right) \le 1$,

ii. $W\left(\tilde{A}, \tilde{B}\right) = W\left(\tilde{B}, \tilde{A}\right)$,

iii. if $\alpha, \beta > 0$, then $W\left(\tilde{A}, \tilde{B}\right) = 1 \Leftrightarrow \tilde{A} = \tilde{B}$.

For $X = [a,b]$, assume that the weights of $x \in X$ is $w(x)$, where $0 \le w(x) \le 1$ and $\int_a^b w(x)\,dx = 1$, IFSs $\tilde{A}$ and $\tilde{B}$ on X as follows:

$$\tilde{A} = \left\{ \left\langle x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i) \mid x \in X \right\rangle \right\}, \tilde{B} = \left\{ \left\langle x_i, \mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i) \right\rangle \mid x \in X \right\}$$

Li and Cheng [32] gave the following weighted similarity measure,

$$W_L^c\left(\tilde{A}, \tilde{B}\right) = 1 - \sqrt[p]{\int_a^b w(x) \times \left[\frac{\left| \left(\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x) \right) - \left(\nu_{\tilde{A}}(x) - \nu_{\tilde{B}}(x) \right) \right|}{2} \right]^p dx} \tag{5.10}$$

where $1 \le p < +\infty$.

If we take $w(x) = 1/(b-a)$ for any $x \in [a,b]$, then we have $W_L^c\left(\tilde{A}, \tilde{B}\right) = T_L^c\left(\tilde{A}, \tilde{B}\right)$.

Therefore by the reason as above, W_L^c is not reasonable. So we modify this similarity measure as follows:

$$W^c\left(\tilde{A}, \tilde{B}\right) = 1 - \left[\int_a^b w(x) \times \left[\alpha \times \left| \mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i) \right|^p + \beta \times \left| \nu_{\tilde{A}}(x_i) - \nu_{\tilde{B}}(x_i) \right|^p + \gamma \times \left| \pi_{\tilde{A}}(x_i) - \pi_{\tilde{B}}(x_i) \right|^p \right] dx \right]^{1/p} \tag{5.11}$$

where $1 \le p < +\infty$, $0 \le w(x) \le 1$, $\int_a^b w(x)\,dx = 1$, $\alpha, \beta \in [0,1]$, and $\alpha + \beta + \gamma = 1$.

W^c has following properties,

i. $0 \le W^c\left(\tilde{A}, \tilde{B}\right) \le 1$,

ii. $W^c\left(\tilde{A},\tilde{B}\right)=W^c\left(\tilde{B},\tilde{A}\right)$,

iii. if $\alpha,\beta>0$, then $W^c\left(\tilde{A},\tilde{B}\right)=1 \Leftrightarrow \tilde{A}=\tilde{B}$.

(proof is similar as T).

5.5 An application of the modified similarity measures between IFSs

In this section, we will use the example to apply the modified similarity measures between IFSs to pattern recognition.

Recognition Principle. Assume that there exist m patterns, which are represented by IFSs $\tilde{A}_k\ (k=1,2,...,m)$ in the universe of discourse X, and there is a sample to be recognized which is represented by IFS $\tilde{B}$ in X.

$$T\left(\tilde{A}_{k_0},\tilde{B}\right)=\underset{1\le k\le m}{\text{Maximum}}\left\{T\left(\tilde{A}_k,\tilde{B}\right)\right\}$$

where $T\left(\tilde{A}_k,\tilde{B}\right)$ expresses the degree of similarity between IFSs $\tilde{A}_k$ and $\tilde{B}$. Then we decide that the sample $\tilde{B}$ should belong to the pattern $\tilde{A}_{k_0}$.

Example 5.3. Let three patterns be represented by IFSs in $X=\{x_1,x_2,x_3\}$.

$$\tilde{A}_1=\{\langle 1,0\rangle,\langle 0.8,0\rangle,\langle 0.7,0.1\rangle\},$$

$$\tilde{A}_2=\{\langle 0.8,0.1\rangle,\langle 1,0\rangle,\langle 0.9,0\rangle\},$$

$$\tilde{A}_3=\{\langle 0.6,0.2\rangle,\langle 0.8,0\rangle,\langle 1,0\rangle\}.$$

Consider a sample $\tilde{B}$ which will be recognized, where

$$\tilde{B}=\{\langle 0.5,0.3\rangle,\langle 0.6,0.2\rangle,\langle 0.8,0.1\rangle\}.$$

For this problem we obtain the following results by choosing $p=2$ in (5.2),

$$T_L\left(\tilde{A}_1,\tilde{B}\right)=0.72,$$

$$T_L\left(\tilde{A}_2,\tilde{B}\right)=0.78,$$

$$T_L\left(\tilde{A}_3,\tilde{B}\right)=0.84,$$

So, according to the recognition principle, sample $\tilde{B}$ belongs to pattern $\tilde{A}_3$.

Now , we take $p=2$ in (5.4), we get

$$T\left(\tilde{A}_1,\tilde{B}\right)=0.72,$$

$$T\left(\tilde{A}_2,\tilde{B}\right)=0.74,$$

$$T\left(\tilde{A}_3,\tilde{B}\right)=0.84.$$

We also draw the conclusion that $\tilde{B}$ belongs to $\tilde{A}_3$ according to the recognition principle.

Assume that the weights of x_1, x_2 and x_3 are 0.5, 0.3, and 0.2 respectively. If we take $p=2$ in (5.7) we get,

$$W_L\left(\tilde{A}_1,\tilde{B}\right)=0.70,$$

$$W_L\left(\tilde{A}_2,\tilde{B}\right)=0.78,$$

$$W_L\left(\tilde{A}_3,\tilde{B}\right)=0.85.$$

and we conclude that $\tilde{B}$ belongs to pattern $\tilde{A}_3$ according to the recognition pattern.

Now , choose $\alpha=\beta=\gamma=1/3$ and $p=2$ in (5.9), we obtain

$$W\left(\tilde{A}_1,\tilde{B}\right)=0.64,$$

$$W\left(\tilde{A}_2,\tilde{B}\right)=0.78,$$

$$W\left(\tilde{A}_3, \tilde{B}\right) = 0.88 .$$

From these results, our conclusion is also that $\tilde{B}$ belongs to pattern $\tilde{A}_3$.

5.6 Conclusion

In this chapter, we showed by some examples that the similarity measure IFSs presented by Li and Cheng [32] and the modified similarity between vague sets proposed by Hong and Kim [23] are not reasonable in some cases. Several modified similarity measures between IFSs were presented. Using the same example given by Li and Cheng [32], we illustrated the application of the new similarity measure turned out to be more reasonable and more effective than the measure introduced by Li and Cheng [32] and by Hong and Kim [23].

BIBLIOGRAPHY

[1] Ashby, F. G., and Perrin, N. A., *Toward a unified theory of similarity and recognition*, Psychological Review, Vol. 95, 1988, pp. 124-150

[2] Atanassov, K., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, Vol. 20, 1986, pp. 87-96.

[3] Ban, X., Lv, X. and Chen, J., *Color image retrieval and classification using fuzzy similarity measure and fuzzy clustering method*, Proceedings 28^{th} Chinese Control Conference Shanghai, P.R. China, 2009, pp. 7777-7782.

[4] Bilgic, T. and Turksen, I. B., *Effective search methods for pattern matching inferencing using specific similarity measures*, IEEE Transactions, Cybernetics and Intelligent Systems, 1992, pp. 161-168.

[5] Bustince, H. and Burillo, P., *Vague sets are intuitionistic fuzzy sets*, Fuzzy Sets and Systems, Vol. 79, 1996, pp. 403-405.

[6] Chen, S. H., *Operations on fuzzy numbers with function principal*, Tamkang Journal of Management Sciences, Vol. 6, 1985, pp. 13-25.

[7] Chen, S. M., *Measures of similarity between vague sets*, Fuzzy Sets and Systems, Vol. 74, 1995, pp. 217-223.

[8] Chen, S. M., *New methods for subjective mental workload assessment and fuzzy risk analysis*, Cybernetics and Systems, Vol. 27, 1996, pp. 449-472.

[9] Chen, S. M., *Similarity measure between vague sets and between elements*, IEEE Transactions on Systems, Man and Cybernetics-Part B: Cybernetics, Vol. 27, 1997, pp. 153-158.

[10] hen, S.J. and Chen, S.M., *A new simple center-of-gravity method for handling the fuzzy ranking and the defuzzification problems*, 8th National Conference Fuzzy Theory Applications, Taipei, Taiwan, Republic of China, 2000.

[11] Chen, S. J. and Chen, S. M., *Fuzzy risk analysis based on similarity measure of generalized fuzzy numbers*, IEEE Transactions on Fuzzy Systems, Vol. 11, 2003, pp. 45-56.

[12] Chen, J. H. and Chen, S. M., *A new method to measure the similarity between interval-valued fuzzy numbers*, Proceedings of the Sixth International Conference on Machine Learning and Cybernetics, Hong Kong, 2007, pp. 1403-1408.

[13] Chen, S. J. and Chen, S. M., *Fuzzy risk analysis based on measures of similarity between interval-valued fuzzy numbers*, Computers and Mathematics with Applications, Vol. 55, 2008, pp. 1670-1685.

[14] Chen, S. M. and Chen, J. H., *Fuzzy risk analysis based on similarity measures between interval-valued fuzzy numbers and arithmetic operators*, Expert Systems with Applications, Vol. 36, 2009, pp. 6309-6317.

[15] Chen, X. and Tian, J., *A new algorithm for distorted fingerprints matching based on normalized fuzzy similarity measure*, IEEE Transaction on Image Processing, Vol. 15, 2006, pp. 767-776.

[16] Chen, S. M., Yeh. S. M. and Hsiao, P. Y., *A comparison of similarity measure of fuzzy values*, Fuzzy Sets and Systems, Vol. 72, 1995, pp. 79-89.

[17] Coombs, C. H., *A theory of data*, New York: John Wiley and sons, Inc., 1964.

[18] Ennis, D. M., *Confusable and discriminable stimuli*, Journal of Experimental Psychology, Vol. 117, 1988, pp. 408-411.

[19] Ennis, D. M. and Johnson, N. L., *Thurstone-Shepard similarity models as special cases of moment generating functions*, Journal of Mathematical Psychology, Vol. 37, 1993, pp. 104-110.

[20] Gau, W. L and Buehrer, D. J., *Vague Sets*, IEEE Transactions, Systems Man, Cybernetics, Vol. 23. 1993, pp. 610-614.

[21] Guijun, W. and Xiaoping, L., *The applications of interval-valued fuzzy numbers and interval-distribution numbers*, Fuzzy Sets and Systems, Vol. 98, 1998, pp. 331-335.

[22] Guo, Z. L. and Tian, J., *Discussion about similarity measures in pattern recognition of fuzzy information,* Proceedings International Conference on Computational Intelligence and Security Workshops, 2007, pp. 299-303.

[23] Hong, D. H. and Kim, C., *A note on similarity measures between vague sets and between elements,* Information Sciences, Vol. 115, 1999, pp. 83-96.

[24] Hsieh, C. H., and Chen, S. H., *A model and algorithm of fuzzy product positioning,* Information Sciences, Vol.121, 1999, pp. 61-82.

[25] Hyung, L. K., Song, Y. S. and Lee, K. M., *Similarity measure between fuzzy sets and between elements,* Fuzzy Sets Systems, Vol. 62, 1994, pp. 291-293.

[26] Kaufmann, A. and Gupta, M. M., *Fuzzy mathematical models in engineering and management science,* Amsterdam, The Netherland Elsevier,1988.

[27] Kim, D. J. and Bien, Z., *A personalized facial expression recognition with fuzzy similarity measure and novel feature selection method,* FUZZ-IEEE, 2004, pp. 33-38.

[28] Kim, K. H., Lim, M. and Kim, H. J., *Music copyright protection system using*

fuzzy similarity measure for music phoneme segmentation, International Conference on Fuzzy Systems (FUZZ-IEEE), 20-24 August, 2009, pp. 159-164.

[29] Klir, G. J. and Yuan, B., *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice-Hall of India Pvt. Ltd.

[30] Lee, C. C., *Fuzzy logic in control system: fuzzy logic controller*, IEEE Transaction Systems, Man, Cybernetics, Vol. 20, 1990, pp. 419-435.

[31] Lee, S. H., *An optimal aggregation method for fuzzy opinions of group decision*, Proceedings IEEE International Conference of Systems, Man, Cybernetics, Vol. 3, 1999, pp. 314-319.

[32] Li, D. and Cheng, C., *New similarity measure of intuitionistic fuzzy sets and application to pattern recognition,* Pattern Recognition Letters, Vol. 23, 2002, pp. 221-225.

[33] Li, Y., Liu, J. M., Ye, W. D. C. X., and Wu, Z. F., *The fuzzy similarity measure for content-based image retrieval*, Proceedings of the Second International Conference on Machine Learning and Cybernetics, Xian, 2-5 November, 2003, pp. 3224-3228.

[34] Li, Y., Olson, D. L. and Qin, Z., *Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis*, Pattern Recognition Letters Vol. 28, 2007, pp. 278-285.

[35] Liang, Z. and Shi, P., *Similarity measure on intuitionistic fuzzy sets*, Pattern Recognition Letters, Vol. 24, 2003, pp. 2687-2693.

[36] Liao, Y., Ma, C. and Zhang, C., *A new fuzzy risk assessment method for the network security based on fuzzy similarity measure*, Proceedings of the 6th World Congress on Intelligent Control and Automation, Dalian, China, June 21-23, 2006,

pp. 8486-8490.

[37] Liu, X., *Entropy distance measure and similarity measure of fuzzy sets and their relations*, Fuzzy Sets and Systems, Vol. 52, 1992, pp. 305-318.

[38] Liu, H. W., *New similarity measure between intuitionistic fuzzy sets and between elements,* Mathematical and Computer Modeling, Vol. 42, 2005, pp. 61-70.

[39] Lu, B. and Chen, J., *Grid resource scheduling based on fuzzy similarity measures*, Proceedings of the 7^{th} Conference on Cybernetics and Intelligent Systems, London, U.K., 2008, pp. 940-944.

[40] Mazhar, R., Gadar, P. D. and Wilson, J. N., *A matching pursuit based similarity measure for fuzzy clustering and classification of signals*, IEEE International Conference on Fuzzy Systems, 2008, pp. 1950-1955.

[41] Pappis, C. P., and Karacatilides, N. I., *A comparative assessment of measure of similarity of fuzzy values*, Fuzzy Sets Systems, Vol. 56, 1993, pp. 171-174.

[42] Sanguansat, K. and Chen, S. M., *Analyzing fuzzy risk based on a new similarity measure between interval-valued fuzzy numbers*, Proceedings 8^{th} Conference on Machine Learning and Cybernetics, Baoding, 2009, pp. 2869-2874.

[43] Santini, S. and Jain, R., *Similarity measure,* IEEE Transactions, Vol. 21, 1999, pp. 871-883.

[44] Shepard, R. N., *Toward a universal law of generalization for psychological science*, Science, Vol. *237*, 1987, pp. 1317-1323.

[45] Subasic, P. and Hirota, K., *Similarity rules and gradual rules for analogical and interpolative reasoning with imprecise data*, Fuzzy Sets and Systems, Vol. 96, 1998, pp. 53-75.

[46] Szmidt, E. and Kacprzyk, J., *Distances between intuitionistic fuzzy sets*, Fuzzy Sets and Systems, Vol. 114, 2000, pp. 505-518.

[47] Tanner, W. P. and Swets, J. A., *A decision-making theory of visual detection, Psychological Review*, Vol. 61, 1954, pp. 401-409.

[48] Thurstone, L. L., *A law of comparative judgment*, Psychological Review, Vol. 34, 1927, pp. 273-286.

[49] Turksen, I. B and Zhong, Z., *An approximate analogical reasoning schema based on similarity measure and interval-valued fuzzy sets*, Fuzzy Sets and Systems, Vol. 34, 1990, pp. 323-346.

[50] Tversky, A., *Features of similarity*, Psychological Review, 1977, Vol. 84, pp. 327-352.

[51] Wang, W. J., *New similarity measure on fuzzy sets and on elements*, Fuzzy Sets and Systems, Vol. 85, 1997, pp. 305-309.

[52] Wei, S. H. and Chen, S. M., *A new approach for fuzzy risk analysis based on similarity measures of generalized fuzzy* numbers, Expert Systems with Applications, Vol. 36, 2009, pp. 589-598.

[53] Xu, Z., Shang, S., Qian, W. and Shu, W., *A method for fuzzy risk analysis based on the new similarity of trapezoidal fuzzy numbers*, Expert Systems with Applications, Vol. 37, 2010, pp. 1920-1927.

[54] Zadeh, L. A., *Fuzzy sets*, Inform. and Control, Vol. 8, 1965, pp. 338-353.

[55] Zwick, R., Carlstein, E. and Budescu, D. V., *Measures of similarity among fuzzy concepts: A comparative analysis*, International Journal of Approximate Reasoning, Vol. 1, 1987, pp. 221-242.