CONVERGENCE OF COMPLEX TRIGONOMETRIC SUMS IN THE METRIC SPACE L

A

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CERTIFICATE

Certified that the dissertation entitled, **"CONVERGENCE OF COMPLEX TRIGONOMETRIC SUMS IN THE METRIC SPACE L"**, which is being submitted by **Mr. Suraj Partap Jindal** (Roll No. 300903019), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of Dr. S.S.Bhatia and Dr. Jatinderdeep Kaur during the period from January 2011 to June 2011. The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University / Institute for the award of any degree.

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ABSTRACT

The present dissertation entitled "Convergence of Complex trigonometric Sums in the metric space L" embodies a brief acocunt of investigations carried out by various authors and by me on L^1-convergence of complex trigonometric sums under the supervision of Dr. S.S. Bhatia, Professor and Dr. Jatinderdeep Kaur, Lecturer, School of Mathematics and Computer Applications, Thapar University, Patiala.

The aim of this work is, study and obtain some results on L^1-convergence of Complex Trigonometric series with special coefficients.

The whole work of my dissertation is divided into four chapters. The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in the sequel, I have given a brief account of some results along with a brief plan of results presented in the subsequent chapters . In second chapter, I have studied the "L^1-Convergence of Modified Complex Trigonometric Sums" with coefficients belonging to class R^*. In chapter third, I have studied the L^1-convergence of r^{th} differential of the complex trigonometric sums introduced in second chapter. The aim of chapter four is to study L^1-convergence of newly modified complex form of Trigonometric sums with semi-convex coefficients.

Towards the end, I have give references of various publications cited in the present dissertation.

DEDICATED TO MY PARENTS AND GOD

CONTENTS

CHAPTER I

INTRODUCTION

1.1 The present dissertation comprises the results associated with the various authors On "Convergence of complex Trigonometric Sums in the metric space L ". It is well known that if a trigonometric series converges in L^1 -metric to a function $f \in L^1(T)$,then it is the Fourier series of the function *f*. Riesz {[2],Vol.II,Ch.VIII§22} gave a counter example to show that the converse of the above said result does not hold good in L^1 metric. This motivated various authors to study the L^1-convergence of trigonometric series with special coefficients.

L^1-convergence of trigonometric series with special coefficients have been studied by number of authors. The work on this topic was initiated by Young W.H.[29] and Kolmogorov A.N.[16] by taking classes of convex sequences ($\Delta^2 a_n \geq 0$) and quasi-convex sequences ($\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty$) respectively. Teljakovskǐi S.A.[27] yet considered another class S introduced by Sidon[23] for L^1-convergence of trigonometric series. The results obtained by these authors have further been generalized and extended by Hardy G.H. and Littlewood J.E.[9], Kano T.[11], Garrett J.W. and Stanojević C.V.([7],[8]), Ram B.([18],[19]), Bala R. and Ram B.[1], Kaur K.,Bhatia S.S. and Ram B.[12], Kaur K.[13], Xhevat Z.Krasniqi[28] by considering various generalizations of classes of sequences mentioned above for one-dimensional trigonometric series.

Recently, L^1-convergence of complex trigonometric series was considered by Stanojević C.V. [24] , Stanojević C.V. and Stanojević V.B [25], Sheng Shu Yun [22] and Bhatia S.S[5, 6] for various classes of complex sequences assuming some additional conditions to control the sine part of complex trigonometric series.

During their investigations, some authors introduced modified trigonometric sums, as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in L^1-metric to the sum of trigonometric series whereas the classical series itself may not. In this concern, various authors like Rees, C.S.and Stanojević C.V.[21],Kumari S. and Ram B.[17],Ram B. and Kumari S.[20], Hooda N.,Ram B. and Bhatia S.S[10], Kaur K., Bhatia S.S and Ram B.[14], Kaur J. and Bhatia S.S.[15] have introduced various new modified trigonometric sums and have studied their L^1- convergence under various classes of coefficient sequences. Bhatia S.S and Ram B.[4] generalized the results of Kumari S. and Ram B.[17] and have obtained their results as corollary.

In the present dissertation , some results of above mentioned authors have been studied and some new results have been obtained by me.

To provide sufficient background for later chapters, a summary of basic concepts, techniques and a brief chapter wise resume of the results contained in the dissertation has been given in this introductory chapter. However, some of the definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

1.2 DEFINITIONS AND NOTATIONS

Let $\{a_n\}$ be a sequence. Then we write

$$\Delta a_n = a_n - a_{n+1}$$

$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Abel's transformation([2],Vol.I,p.1). If $a_0,a_1,a_2,.....,v_0,v_1,.....v_n,....$ are any real numbers, let us assume that

$$V_n = v_0 + v_1 + ... + v_n$$

Then for any values of m and n, we find that

$$\sum_{k=m}^{n} a_k v_k = \sum_{k=m}^{n-1} \Delta a_k V_k + a_n V_n - a_m V_{m-1}$$

(under the condition that if m=0,V_{-1}=0).

Null sequence. A sequence $\{a_n\}$ is said to be null if $a_n \to 0 \quad as \quad n \to \infty$.

Convex sequence. A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$.

Quasi-Convex sequence([2],Vol.II,p.202). A sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty.$$

Semi-convex sequence [11]. A null sequence $\{a_n\}$ is said to be semi-convex if

$$\sum_{n=1}^{\infty} n|\triangle^2 a_{n-1} + \triangle^2 a_n| < \infty, \quad (a_0 = 0)$$

Weakly even sequence [24]. A complex null sequence $\{c_n\}$ satisfying

$$\sum_{n=1}^{\infty} |\triangle(c_n - c_{-n})| \log n < \infty,$$

is called weakly even sequence and is denoted by $\{c_n\} \in \mathbf{W}$. Clearly if $\{c_n\}$ is an even sequence ($c_n = c_{-n}$), then it is weakly even sequence.

If $\{c_n\}$ satisfies

$$\sum_{n=1}^{\infty} |\triangle(c_n - c_{-n})| n^r \log n < \infty, \quad r \in \{0, 1, 2, 3, ..., \}$$

then we say that $\{c_n\} \in \mathbf{W}_r$, obviously $\mathbf{W}_0 = \mathbf{W}$.

O-o Relations[2]. If $v_n \geq 0 \ \ for \ \ n = 0, 1, 2, ...$ and $\dfrac{u_n}{v_n} \to 0$ as $n \to \infty$, then we write

$$u_n = o(v_n)$$

Again if $\dfrac{u_n}{v_n}$ is bounded, then we write

$$u_n = O(v_n)$$

Dirichlet kernel ([2], Vol.I, p.85). Let

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + ... + \cos nx$$

then

$$\begin{aligned} 2\sin\frac{x}{2} D_n(x) &= \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + ... + 2\sin\frac{x}{2}\cos nx \\ &= \sin\left(n + \frac{1}{2}\right)x, \end{aligned}$$

hence
$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

This expression is known as Dirichlet's kernel. Moreover,

$$\begin{aligned} \tilde{D}_n(x) &= \sin x + \sin 2x + ... + \sin nx \\ &= \frac{\cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} \end{aligned}$$

is called the kernel conjugate to the Dirichlet kernel.

If $x \neq 0(\mod 2\pi)$, then

$$|D_n(x)| \leq \frac{\pi}{2x}, \quad for \ \ 0 < |x| \leq \pi$$

and

$$|\tilde{D}_n(x)| \leq \frac{\pi}{x}, \quad for \ \ 0 < |x| \leq \pi$$

Also, we shall use the uniform estimate

$$|D_n(x)| \leq n + \frac{1}{2}, \quad for \ \ any \ \ x$$

Fejér kernel ([2], [30]). The Fejér kernel $K_n(x)$ is defined as

$$\begin{aligned} K_n(x) &= \frac{1}{n+1}\sum_{j=0}^{n} D_j(x) \\ &= \frac{1}{n+1}\sum_{j=0}^{n} \frac{\sin\left(j+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}. \end{aligned}$$

Using $|D_n(x)| \leq n+1$, it follows that $K_n(x) \leq n+1$.

It has the properties

$$\begin{aligned} &(i) \quad K_n(x) \geq 0, \\ &(ii) \quad \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1. \end{aligned}$$

The conjugate Fejér kernel is defined as

$$\tilde{K}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} \tilde{D}_j(x)$$

We have

$$\tilde{K}_n(x) > 0 \quad for \quad 0 < x < \pi, \quad n = 1, 2, 3, ...$$

and

$$|\tilde{K}_n(x)| < \frac{1}{2}n.$$

Fourier series. A trigonometric series

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right),$$

the coefficients a_0, a_n *and* b_n of which are determined by the Fourier formulae

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos nx\ dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\ dx \end{aligned}$$

derived from the function $f(x)$, is called the Fourier series of the function $f(x)$. We will write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and $f(x)$ is a 2π-periodic function on the given interval.

The Class S ([23], [27]). A sequence $\{a_n\}$ belongs to class S, if $a_n \to 0 \ as \ n \to \infty$ and there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum\limits_{n=0}^{\infty} A_n < \infty$, and $|\Delta a_n| \leq A_n$, for all n. The class S is usually called as Sidon-Teljakovskǐi class. Since, firstly, Teljakovskǐi expressed Sidon's conditions [23] in a succinct equivalent form and secondly, he showed that the class S is also a class of L^1-convergence.

The Class R [11]. A null sequence $\{a_n\}$ belongs to the class R, if

$$\sum_{n=1}^{\infty} n^2 \left| \triangle^2 \left(\frac{a_n}{n} \right) \right| < \infty.$$

Complex form of Fourier Series

The Fourier series of function $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

we know that $e^{inx} = \cos nx + i \sin nx$ and $e^{-inx} = \cos nx - i \sin nx$

therefore

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$$

$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$$

putting these values in the fourier series of $f(x)$, we get

$$f(x) \sim a_0 + \tfrac{1}{2} \sum_{n=1}^{\infty} (a_n(e^{inx} + e^{-inx}) - ib_n(e^{inx} - e^{-inx}))$$

after arranging we get

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(\frac{1}{2}(a_n - ib_n)e^{inx} - \frac{1}{2}(a_n + ib_n)e^{-inx}))$$
$$f(x) \sim c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} - c_{-n}e^{-inx})$$

Hence

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_0 = a_0, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$

To determine coefficients

$$c_0 = a_0 {=} \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx,$$

$$c_n = \tfrac{1}{2}(a_n - ib_n) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$c_{-n} = \tfrac{1}{2}(a_n + ib_n) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$$

Hence for Complex form

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

coefficients are determined by the formulae

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \qquad for(n = ..., -2, -1, 0, 1, 2, ...)$$

The Class S_p^* [24]. A weakly even null-sequence $\{c_n\}$ of complex numbers belongs to the class S_p^*, if for some $1 < p \leq 2$ and some monotone sequence $\{A_n\}$ such that $\sum_{n=0}^{\infty} A_n < \infty$, the condition $\frac{1}{n}\sum_{k=1}^{n}\frac{|\triangle c_k|^p}{A_k^p} = O(1), \quad n \to \infty$ holds. This class is further generalization of the class S for complex series.

The Class $\mathrm{S}^*_{p\alpha r}$ [22]. A null sequence of complex numbers $\{c_n\} \in W_r$ belongs to class $\mathrm{S}^*_{p\alpha r}$, if for some $1 < p \leq 2$ there exists a monotone sequence $\{A_n\}$ such that

$$(i) \ \sum_{n=1}^{\infty} n^{\alpha} A_n < \infty, \quad for \quad some \quad \alpha \geq 0,$$

$$(ii) \ \frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\triangle c_k|^p}{A_k^p} = O(1), \quad r \in \{0, 1, 2, ..., [\alpha]\}, \quad n \to \infty$$

The case $\alpha = r = 0$ of this class reduces to the class S^*_p.

The Class R^* [5]. A null sequence $\{c_n\}$ of complex numbers belongs to the class R^*, if

$$(i) \ \sum_{n=1}^{\infty} \left| \triangle \left(\frac{c_{-n} - c_n}{n} \right) \right| n \log n < \infty,$$

$$(ii) \ \sum_{n=1}^{\infty} n^2 \left| \triangle^2 \left(\frac{c_n}{n} \right) \right| < \infty.$$

The Class R^*_r [3]. A null sequence $\{c_n\}$ of complex numbers belongs to the class R^*_r, if

$$(i) \ \sum_{n=1}^{\infty} \left| \triangle \left(\frac{c_{-n} - c_n}{n} \right) \right| n^{r+1} \log k < \infty,$$

$$(ii) \ \sum_{n=1}^{\infty} n^{r+2} \left| \triangle^2 \left(\frac{c_n}{n} \right) \right| < \infty.$$

If $r = 0$, class R^*_r reduces to R^*.

The Class J^* [15]. A null sequence $\{c_n\}$ of complex numbers belongs to class J^* if there exists a sequence $\{A_n\}$ such that

(i) $\quad A_n \downarrow 0, \quad as \quad n \to \infty,$

(ii) $\quad \displaystyle\sum_{n=1}^{\infty} n A_n < \infty$

(iii) $\quad \left| \triangle \left(\frac{c_n - c_{-n}}{n} \right) \right| \leq \frac{A_n}{n}$, for all $n = 1, 2, 3, ...$

1.3 The following results about the behavior of cosine and sine series are known:

Theorem 1([2], [16], [29]). If $\{a_n\}$ is a quasi-convex null sequence, then

$$f(x)=\sum_{n=1}^{\infty} a_n \cos nx \in L^1[0,\pi] \tag{1.3.1}$$

Theorem 2([2], [26]) . If $\{a_n\}$ is a quasi-convex null sequence, then

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{1.3.2}$$

is a Fourier series if and only if $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$.

In 1968, Kano [11] generalized Theorem 1 and Theorem 2 in the following form:

Theorem 3. If $\{a_n\}$ is a null sequence such that

$$\sum_{n=1}^{\infty} n^2 \left|\triangle^2 \left(\frac{a_n}{n}\right)\right| < \infty, \tag{1.3.3}$$

then (1.3.1) and (1.3.2) are the Fourier series, or equivalently they represent integrable functions.

Further, Ram and Kumari ([17], [20]) introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle \left(\frac{a_j}{j}\right) k \cos kx \tag{1.3.4}$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle \left(\frac{a_j}{j}\right) k \sin kx \tag{1.3.5}$$

and studied their L^1-convergence under the condition that the cosine series and sine series belong to the classes R and S. They also deduced the results about L^1-convergence of cosine and sine series. Their results state as below:

Theorem 4. Let $\{a_n\}$ belong to the class S. If $\lim\limits_{n\to\infty} |a_{n+1}| \log n = 0$, then

$$||f - f_n||_{L^1} = o(1) \quad n \to \infty.$$

Theorem 5. Let $\{a_n\}$ belong to the class R. If $t_n(x)$ represents $f_n(x)$ and $g_n(x)$, then

$$||f - t_n||_{L^1} = o(1) \quad n \to \infty.$$

Later, Hooda and Ram [10] have proved the following theorem regarding L^1-convergence of Ram and kumari sums for the class $S^{'}$ of coefficient sequences.
Theorem 6. Let $\{a_n\}$ belong to the class S′. Then $\quad ||f - f_n||_{L^1} = o(1), \quad n \to \infty.$

1.4 Let

$$S_n(C,t) = \sum_{k=-n}^{n} c_k e^{ikt}, \quad t \in T = \mathbb{R}/2\pi Z,$$

denotes the partial sums of the complex trigonometric series $\sum\limits_{k=-\infty}^{\infty} c_k e^{ikt}$. In case the trigonometric series is a Fourier series of some Lebesgue integrable function f, we write $c_n = \hat{f}(n)$ for all n and $S_n(C,t) = S_n(f)$.

Concerning the integrability and L^1-convergence of complex trigonometric series, Stanojević, Č.V and Stanojević, V.B. [24] proved the following result, which in turn generalizes the Teljakovskǐi [27] result.

Theorem 7. Let $\{c_n\} \in S_p^*$. Then
(i) $\quad \lim\limits_{n\to\infty} S_n(C,t) = f(t)$ exists, $for\ t \neq 0$
(ii)) $\quad f \in L^1(T)$;
(iii) $\quad ||S_n(f) - f||_{L^1} = o(1)$, $n \to \infty$,
is equivalent to

$$\hat{f}(n) \log |n| = o(1), \quad |n| \to \infty.$$

In the proof of Theorem 7, the authors used the complex form

$$g_n(C,t) = S_n(C,t) - (c_n E_n(t) + c_{-n} E_{-n}(t))$$

of modified trigonometric sums of Rees-Stanojević[21]

$$g_n(x) = \tfrac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j)\cos kx. \tag{1.4.1}$$

Sheng [22] extended the results of Stanojević C.V. and Stanojević V.B. [24] by studing the L^1-convergence of r^{th} differential of complex form of Rees-Stanojević sums (1.4.1) with coefficients belonging to the class $S^*_{p\alpha r}$

Further, in 1995, Bhatia and Ram [5] introduced the complex form

$$g_n(C,t) = S_n(C,t) + \frac{i}{n+1}[c_{n+1}E'_n(t) - c_{-(n+1)}E'_{-n}(t)] \tag{1.4.2}$$

of modified trigonometric sums (1.3.4) and (1.3.5) and studied their L^1-convergence under a new class R* of coefficient sequences and obtained the necessary and sufficient condition for the L^1-convergence of complex trigonometric series.

Later, Bhatia and Ram [6] introduced the complex form of r^{th} derivative of complex trigonometric sums (1.4.1) as

$$g_n{}^r(C,t) = S_n{}^r(C,t) + \frac{i}{n+1}[c_{n+1}E_n^{r+1}(t) - c_{-(n+1)}E_{-n}^{r+1}(t)] \tag{1.4.3}$$

and gave the sufficient condition for the integrability of r times differentiated trigonometric series using the complex trigonometric sums (1.4.2) and obtained new necessary and sufficient condition for L^1-convergence of r^{th} derivative of complex trigonometric series.

Recently, Xhevat Z.Krasniqi [28] have introduced new modified sine and cosine sums as:

$$(1.4.4) \qquad H_n(x) = \frac{-1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n} \triangle\left[(a_{j-1} - a_{j+1})\cos jx\right]$$

and

$$(1.4.5) \qquad G_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left[(a_{j-1} - a_{j+1})\sin jx\right]$$

and have studied their L^1-convergence.

Now, I give a brief chapterwise resume of the results contains in this dissertation. In Chapter II, I have studied the L^1-convergence of the complex form (1.4.2) of modified trigonometric sums (1.3.4) and (1.3.5) introduced by Ram and Kumari ([17], [20]) and obtained the necessary and sufficient condition for the L^1-convergence of the complex trigonometric series with coefficients belonging to R^*[5] have been studied.

In Chapter III, I have studied the L^1-convergence of the r-th differential (1.4.3) of complex trigonometric sums (1.4.2) given by Bhatia and Ram [6]

In Chapter IV, I have obtained the complex form

$$g_n(C,t) = S_n(C,t) + \frac{i}{2\sin t}\left[\begin{array}{l} c_n e^{i(n+1)t} - c_{-n}e^{-i(n+1)t} + c_{n+1}e^{int} - c_{-(n+1)}e^{-int} \\ + (n+1)(c_n - c_{n+2})e^{i(n+1)t} + (n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)t} \end{array}\right]$$

of modified trigonometric sums (1.4.4) and (1.4.5) of Xhevat Z.Krasniqi [28] and have obtained its L^1-convergence under the class J^*[15].

CHAPTER II

L^1-Convergence of Modified Complex Trigonometric Sums

2.1. Introduction. Let the partial sums of complex trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{int}$ be denoted by $S_n(C,t)=\sum_{k=-n}^{n} c_k e^{ikt}, t \in \mathrm{T}=\frac{R}{2\pi Z}$. If the trigonometric series is Fourier series of some $f \in L^1$, we shall write $c_n=\hat{f}(n)$ for all n and $S_n(C,t)=S_n(f,t)=S_n(f)$.

In 1987, Stanojević C.V. and Stanojević V.B. [24] considered the integrability and L^1-convergence of complex trigonometric series with coefficients satisfying a specific condition. To control the sine part of the complex trigonometric seires, they needed a technical condition alongwith a class S_p^* of sequences respectively defined as :

Definition. A complex null sequence $\{c_n\}$ satisfying

$$\sum_{n=1}^{\infty} |\triangle(c_n - c_{-n})| \log n < \infty$$

is called weakly even. Obviously if $\{c_n\}$ is an even sequence ($c_n=c_{-n}$), then it is weakly even.

Definition. A weakly even null sequence $\{c_n\}$ of complex numbers belongs to class S_p^* if for some $1 < p \leq 2$ and some monotone sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$, the condition

(2.1.1) $$\frac{1}{n}\sum_{k=1}^{n} \frac{|\triangle c_k|^p}{A_k^p} = O(1), \quad n \to \infty$$

holds.

Stanojević C.V. and Stanojević V.B. [24] proved the following theorem that generalizes the Teljakovskǐi result.

Theorem A. Let $\{c_n\} \in S_p^*$. Then

(2.1.2) $\lim\limits_{n\to\infty} S_n(C,t) = f(t)$ exists, $for\ t \neq 0$

(2.1.3) $f(t) \in L^1(T)$;

(2.1.4) $||S_n(f,t) - f(t)||_{L^1} = o(1),\ n \to \infty$, is equivalent to

$\hat{f}(n)\log|r| = o(1), \quad |n| \to \infty.$

In the proof of Theorem A, the authors used the complex form

$$g_n(C,t) = S_n(C,t) - (c_n E_n(t) + c_{-n}E_{-n}(t))$$

of the modified trigonometric sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j)\cos kx.$$

introduced by Rees-Stanojević[21].

Later, Ram and Kumari ([17], [20]) introduced new modified cosine and sine sums

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k\cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k\sin kx$$

and studied their L^1-convergence for the sequences belonging to classes R and S.

The Class S ([23], [27]). A sequence $\{a_n\}$ belongs to class S, if $a_n \to 0\ \ as\ \ n \to \infty$ and there exists a sequence $\{A_n\}$ such that

$(i)\ \ A_n \downarrow 0,\ \ as\ n \to \infty,$

$(ii)\ \ \sum\limits_{n=0}^{\infty} A_n < \infty,$

$(iii)\ \ |\Delta a_n| \leq A_n, \quad for\ all\ n.$

The aim of this chapter is to study the L^1-convergence of complex form of the Ram and Kumari ([17], [20]) sums with coefficients belonging to class R^*.

The Class R* [5]. A null sequence $\{c_n\}$ of complex numbers belongs to the class R*, if

$$(i)\ \sum_{n=1}^{\infty}\left|\triangle\left(\frac{c_{-n}-c_n}{n}\right)\right| n\log n<\infty,$$

$$(ii)\ \sum_{n=1}^{\infty} n^2\left|\triangle^2\left(\frac{c_n}{n}\right)\right|<\infty.$$

Let $D_n(t), \tilde{D}_n(t), \tilde{K}_n(t)$ denote the Dirichlet kernel,conjugate Dirichlet kernel,cojugate Fejé́r kernel respectively. Then the first differentials, $D_n'(t)$ and $\tilde{D}_n'(t)$ of $D_n(t)$ and $\tilde{D}_n(t)$ respectively can be written as

(i) $\qquad 2D_n'(t) = E_n'(t) + E_{-n}'(t),$

(ii) $\qquad 2i\tilde{D}_n'(t) = E_n'(t)$ - $E_{-n}'(t),$

where $E_n'(t)$ and $E_{-n}'(t)$ denotes the first differentials of $E_n(t)=\sum_{k=0}^{n} e^{ikt}$

and $\quad E_{-n}(t)=\sum_{k=0}^{n} e^{-ikt}$ respectively

we therefore have

$$E_n'(t)=i\sum_{k=0}^{n} ke^{ikt}\quad and\quad E_{-n}'(t)=-i\sum_{k=0}^{n} ke^{-ikt}$$

We know that

$$D_n(t)=\frac{1}{2}+\cos t+\cos 2t+...+\cos nt=\frac{1}{2}+\sum_{k=0}^{n}\cos kt$$

$$D_n'(t)=-\sin t-2\sin 2t-...-n\sin nt=-\sum_{k=1}^{n} k\sin kt$$

therefore

$$E_n^{'}(t)+E_{-n}^{'}(t)=i\sum_{k=0}^{n}k(2i\sin kt)\ \ =-2\sum_{k=0}^{n}k\sin kt$$

Hence $\qquad 2D_n^{'}(t)=E_n^{'}(t)+E_{-n}^{'}(t)$, which proves (i).

Again

$$\tilde{D}_n(t)=\sin t+\sin 2t+...+\sin nt$$

$$\tilde{D}_n^{'}(t)=\cos t+2\cos 2t+...+n\cos nt$$

therefore, $\quad E_n^{'}(t)-E_{-n}^{'}(t)=2i\sum_{k=0}^{n}k\cos kt\ \ =2i\tilde{D}_n^{'}(t)$, which proves (ii).

2.2 We now obtain the complex form of the modified sums of Ram and Kumari ([17], [20])

Let

$$g_n(C,t)=\frac{a_0}{2}+\sum_{k=1}^{n}\sum_{j=k}^{n}\triangle\left(\frac{a_j}{j}\right)k\cos kt+\sum_{k=1}^{n}\sum_{j=k}^{n}\triangle\left(\frac{b_j}{j}\right)k\sin kt$$

$$=\frac{a_0}{2}+\sum_{k=1}^{n}\left(\frac{a_k}{k}-\frac{a_{n+1}}{n+1}\right)k\cos kt+\sum_{k=1}^{n}\left(\frac{b_k}{k}-\frac{b_{n+1}}{n+1}\right)k\sin kt$$

we know that $\qquad \cos kt=\dfrac{e^{ikt}+e^{-ikt}}{2}\quad$ *and* $\quad \sin kt=\dfrac{e^{ikt}-e^{-ikt}}{2i}$, then we get

$$g_n(C,t)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(\frac{a_k}{k}-\frac{a_{n+1}}{n+1}\right)k\left(\frac{e^{ikt}+e^{-ikt}}{2}\right)+\sum_{k=1}^{n}\left(\frac{b_k}{k}-\frac{b_{n+1}}{n+1}\right)k\left(\frac{e^{ikt}-e^{-ikt}}{2i}\right)$$

after arranging the terms, we get

$$g_n(C,t)=c_0+\sum_{k=1}^{n}c_ke^{ikt}+\sum_{k=1}^{n}c_{-k}e^{-ikt}-\frac{1}{n+1}\left[c_{n+1}\sum_{k=1}^{n}ke^{ikt}+c_{-(n+1)}\sum_{k=1}^{n}ke^{-ikt}\right]$$

where $\quad c_0=\dfrac{a_0}{2},\quad c_k=\dfrac{a_k-ib_k}{2},\quad c_{-k}=\dfrac{a_k+ib_k}{2}$

Thus,

$$g_n(C,t)=S_n(C,t)-\frac{1}{n+1}\left[c_{n+1}\sum_{k=1}^{n}ke^{ikt}+c_{-(n+1)}\sum_{k=1}^{n}ke^{-ikt}\right]$$

On using $E_n^{'}(t) = \sum_{k=0}^{n} ike^{ikt}$, we get the complex form as:

$$g_n(C,t) = S_n(C,t) + \frac{i}{n+1}\left[c_{n+1}E_n^{'}(t) - c_{-(n+1)}E_{-n}^{'}(t)\right], \qquad (c_0 = 0)$$

2.3. Lemmas. The proof of our result is based upon the following lemmas of which first two are due to **Sheng [22]**:

Lemma 1. For the r^{th} derivatives of the Dirichlet's kernels $D_n(t)$ and $\tilde{D}_n(t)$ the following estimates hold

$$(i) \quad ||D_n^{(r)}(t)||_{L^1} = \frac{4}{\pi}(n^r \log n) + O(n^r),\ r = 0, 1, 2, ...$$
$$(ii) \quad ||\tilde{D}_n^{(r)}(t)||_{L^1} = O(n^r \log n),\ r = 0, 1, 2, ...$$

Proof. Let r be any non-negative integer.

Then the r^{th} derivative of the Dirichlet's kernel is given as

$$D_n^{(r)}(t) = \sum_{k=0}^{r-1} \frac{(n+\frac{1}{2})^k \sin\left[(n+\frac{1}{2})t + \frac{k\pi}{2}\right]}{(\sin(\frac{t}{2}))^{r+1-k}}\phi + \frac{(n+\frac{1}{2})^r \sin\left[(n+\frac{1}{2})t + \frac{r\pi}{2}\right]}{2\sin\frac{t}{2}}$$

where the same function ϕ denotes various analytical functions of t independent of n. it follows that $D_n^{(r)}(t) = O(n^{r+1})$.

Moreover,

$$||D_n^{(r)}(t)||_{L^1} = 2\left[\int_0^{\pi} |D_n^{(r)}(t)|dt\right]$$

$$= 2\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right] |D_n^{(r)}(t)|dt$$

$$= 2n^r \int_{\pi/n}^{\pi} \left|\frac{\sin(nt + \frac{r\pi}{2})}{t}\right| dt + \sum_{k=0}^{r-1}\int_{\pi/n}^{\pi} n^k t^{k-1-r} dt + O(n^r)$$

therefore,

$$||D_n^{(r)}(t)||_{L^1} = \frac{4}{\pi}(n^r \log n) + O(n^r), \quad \text{ist part proved.}$$

Now, with the use of Bernstein inequality in L^p-space, we have

$$\int_0^\pi |\tilde{D}_n^{(r)}(t)|dt \leq n^r \int_0^\pi |\tilde{D}_n(t)|dt.$$

On the other hand,

$$\int_0^\pi |\tilde{D}_n(t)|dt \leq \int_0^\pi \frac{\sin^2\left(\frac{nt}{2}\right)}{t}dt + O(1)$$

$$= \log(1+n\pi) + O(1) = O(\log n).$$

Therefore

$$||\tilde{D}_n^{(r)}(t)||_{L^1} = O(n^r \log n)$$

Lemma 2. For each non-negative integer n ,there holds

$$||c_n E_n^{(r)}(t) + c_{-n}E_{-n}^{(r)}(t)||_{L^1} = o(1), \qquad |n| \to \infty$$

if and only if $|n|^r c_n \log|n| = o(1), \quad as\ n \to \infty$,where $< c_n >$ is a complex sequence and $t \in \frac{R}{2\pi Z}$.

Proof. Assuming $r \geq 1$ and denoting $J_n = ||c_n E_n^{(r)}(t) + c_{-n}E_{-n}^{(r)}(t)||_{L^1}$, from Lemma 1, we have

$$\begin{aligned} J_n &= \int_0^\pi [|c_n E_n^{(r)}(t) + c_{-n}E_{-n}^{(r)}(t)| + |c_n E_{-n}^{(r)}(t) + c_{-n}E_n^{(r)}(t)|]dt \\ &\geq |c_n + c_{-n}| \int_0^\pi |E_n^{(r)}(t) + E_{-n}^{(r)}(t)|dt \qquad (2.3.1) \\ &= |c_n + c_{-n}| \int_0^\pi 2|D_n^{(r)}(t)|dt \geq \frac{4}{\pi}|c_n + c_{-n}|n^r \log n + O(1). \end{aligned}$$

On the other hand,

$$J_n = \int_{-\pi}^{\pi} \left|(c_n + c_{-n})E_n^{(r)}(t) + c_{-n}(E_{-n}^{(r)}(t) - E_n^{(r)}(t)\right| dt \qquad (2.3.2)$$

$$\leq |c_n + c_{-n}| \int_{-\pi}^{\pi} |E_n^{(r)}(t)|dt + |c_{-n}| \int_{-\pi}^{\pi} |E_{-n}^{(r)}(t) - E_n^{(r)}(t)|dt$$

by the use of Lemma 1, the right-hand side of this equation can be written as

$$O\left[|c_n + c_{-n}|n^r \log n\right] + O\left[|c_{-n}|n^r \log n\right] = O\left[|c_n + c_{-n}|n^r \log n\right] \tag{2.3.3}$$

Combining (2.3.1),(2.3.2),and (2.3.3), completes the proof.

Lemma 3. (i) There exist positive constants α and β such that

$$\alpha(\log n) \leq ||\tilde{K}_n(t)||_{L^1} \leq \beta(\log n)$$

(ii) $$||\tilde{K}_n'(t)||_{L^1} = O(n).$$

Proof. The existence of β follows from the fact that

$$||\tilde{D}_n(t)||_1 = O(\log n).$$

Further we have,

$$\begin{aligned} 2\pi||\tilde{K}_n(t)||_1 &\geq \int_0^{\pi} \tilde{K}_n(t)dt \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \int_0^{\pi} \sin kt \ dt \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \frac{(1-\cos k\pi)}{k} \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \frac{(1-\cos j\pi)}{j}\right] \geq \frac{M(\log n!)}{n+1} \end{aligned}$$

For some constant M,the last step being the consequence of the relation $\sum_{v=1}^{n} \log v = \log n!$. Using Sterling asymptotic formula $n! \sim \sqrt{2\pi n} n^n e^{-n}$,we then have

$$||\tilde{K}_n(t)||_1 \geq \alpha \log n.$$

This completes the proof of (i).

To prove (ii) we have,

$$|\tilde{D}_n'(t)| = \left|\sum_{k=0}^{n} k\cos kt\right| \le \left(\frac{n(n+1)}{2}\right)$$

and so

$$|\tilde{K}_n'(t)| \le (n+1)^{-1}\sum_{k=0}^{n}|\tilde{D}_k'(t)| = O(n^2).$$

This implies that

$$\int_{|t|\le\pi/n}|\tilde{K}_n'(t)|dt = O(n),$$

We know that

$$\tilde{K}_n(t) = \frac{1}{n+1}\sum_{m=0}^{n}\tilde{D}_m(t) = \frac{1}{4\sin^2\frac{t}{2}}[\sin t - \frac{\sin(n+1)t}{n+1}]$$

Differentiating $\tilde{K}_n(t)$ we get

$$\tilde{K}_n'(t) = \sum_{1n}(t) - \sum_{2n}(t) + \sum_{3n}(t),$$

Where

$$\sum_{1n}(t) = \left[\frac{\cos t - \cos{(n+1)t}}{4\sin^2\frac{t}{2}}\right].$$

$$\sum_{2n}(t) = \left[\frac{2\sin^2 t}{\left(2\sin\frac{t}{2}\right)^4}\right],$$

$$\sum_{3n}(t) = \left[\frac{2\sin t\sin{(n+1)t}}{(n+1)\left(2\sin\frac{t}{2}\right)^4}\right],$$

Obviously, $$\left|\sum_{jn}(t)\right| = O(|t|^{-2}) \quad for \;\; j=1,2$$

and $$(n+1)\left|\sum_{3n}(t)\right| = O(|t|^{-3}).$$

Using these estimates, we get

$$\int_{\pi/n\le|t|\le\pi} |\tilde{K}_n^{'}(t)|dt = O\left(\int_{\pi/n\le|t|\le\pi} t^{-2}dt\right) + O\left(\frac{1}{n+1}\int_{\pi/n\le|t|\le\pi} t^{-3}dt\right)$$

$$= O(n).$$

Combining above estimates, we get

$$||\tilde{K}_n^{'}(t)||_1 = O(n).$$

Lemma 4. Let $n \ge 1$ and $0 < \epsilon < \pi$.Then there exists $A_\epsilon > 0$ such that for all $\epsilon \le |t| \le \pi$

(2.3.4) $\quad |E_n^{'}(t)| \le \dfrac{A_\epsilon n}{|t|}$

(2.3.5) $\quad |E_{-n}^{'}(t)| \le \dfrac{A_\epsilon n}{|t|}$

(2.3.6) $\quad |D_n^{'}(t)| \le \dfrac{A_\epsilon n}{|t|}$

(2.3.7) $\quad |\tilde{D}_n^{'}(t)| \le \dfrac{A_\epsilon n}{|t|}$

Proof. We have

$$E_n^{'}(t) = \sum_{k=0}^{n} ike^{ikt}$$

.

$$i^{-1}E_n^{'}(t) = \sum_{k=0}^{n} ke^{ikt}.$$

$$= \sum_{k=0}^{n} k(E_k(t) - E_{k-1}(t))$$

$$= \sum_{k=0}^{n} (\triangle k)E_k(t) + (n+1)E_n(t).$$

Since

$$|E_n(t)| \le \frac{A_\epsilon}{|t|} \quad for\ some\ constant\ A_\epsilon,$$

we have

$$|E_n^{'}(t)| \le \frac{A_\epsilon}{|t|}\left[\sum_{k=0}^{n} -1 + (n+1)\right] \le \frac{A_\epsilon n}{|t|}$$

which proves (2.3.4).

Further, since $|E'_{-n}(t)| = (-1)|E'_n(-t)|$

we have

$$|E'_{-n}(t)| \leq \frac{A_\epsilon n}{|t|}.$$

which proves (2.3.5).

Moreover, we know that $2D'_n(t) = E'_n(t) + E'_{-n}(t)$

therefore,

$$|D'_n(t)| \leq \frac{A_\epsilon n}{|t|}$$

and

$$2i\tilde{D}'_n(t) = E'_n(t) - E'_{-n}(t),$$

therefore

$$|\tilde{D}'_n(t)| \leq \frac{A_\epsilon n}{|t|}.$$

2.4. **Main Theorem:** The main result of this chapter is the following theorem:

Theorem: Let $c_n \in R^*$. Then there exists $f(t)$ such that

(2.4.1) $\lim_{n\to\infty} g_n(C,t) = f(t)\ for\ all\ 0 < |t| \leq \pi$,

(2.4.2) $f(t) \in L^1(T)$ and $||g_n(C,t) - f(t)||_1$=o(1) as $n \to \infty$,

(2.4.3) $||S_n(f,t) - f(t)||_{L^1}$=o(1) as $n \to \infty$ if and only if $\hat{f}(n)\log|n|$=o(1) as $|n| \to \infty$.

Proof . Complex form of Modified Trigonometric sums is

$$g_n(C,t) = S_n(C,t) + \frac{i}{n+1}\left[c_{n+1}E'_n(t) - c_{-(n+1)}E'_{-n}(t)\right]$$

Since

$$S_n(C,t) = \sum_{k=-n}^{n} c_k e^{ikt} = \sum_{k=-n}^{-1} c_k e^{ikt} + \sum_{k=1}^{n} c_k e^{ikt}$$

we can write it as

$$S_n(C,t) = \sum_{k=1}^{n} c_{-k} e^{-ikt} + \sum_{k=1}^{n} c_k e^{ikt}$$

$$S_n(C,t) = \frac{-1}{i}\sum_{k=1}^{n}\frac{c_{-k}}{k}e^{-ikt}(-ik) + \frac{1}{i}\sum_{k=1}^{n}\frac{c_k}{k}e^{ikt}(ik)$$

by using Abel's transformation, we get

$$S_n(C,t) = i\left[\sum_{k=1}^{n-1}\triangle\left(\frac{c_{-k}}{k}\right)E'_{-k}(t) + \frac{c_{-n}}{n}E'_{-n}(t)\right] - i\left[\sum_{k=1}^{n-1}\triangle\left(\frac{c_k}{k}\right)E'_k(t) + \frac{c_n}{n}E'_n(t)\right]$$

Putting this value in general form, we get

$$g_n(C,t) = i\sum_{k=1}^{n}\triangle\left(\frac{c_{-k}}{k}\right)E'_{-k}(t) - i\sum_{k=1}^{n}\triangle\left(\frac{c_k}{k}\right)E'_k(t)$$

Now, we know that $2i\tilde{D}'_k(t)$=$E'_k(t)$-$E'_{-k}(t)$,

therefore $E'_k(t) = 2i\tilde{D}'_k(t) + E'_{-k}(t)$ and thus we have

$$g_n(C,t) = i\sum_{k=1}^{n}\triangle\left(\frac{c_{-k}}{k}\right)E'_{-k}(t) - i\sum_{k=1}^{n}\triangle\left(\frac{c_k}{k}\right)(2i\tilde{D}'_k(t) + E'_{-k}(t))$$

$$= 2\sum_{k=1}^{n}\triangle\left(\frac{c_k}{k}\right)\tilde{D}'_k(t) - i\sum_{k=1}^{n}\triangle\left(\frac{c_k}{k}\right)E'_{-k}(t) + i\sum_{k=1}^{n}\triangle\left(\frac{c_{-k}}{k}\right)E'_{-k}(t)$$

Therefore, we get

$$g_n(C,t) = 2\sum_{k=1}^{n}\triangle\left(\frac{c_k}{k}\right)\tilde{D}'_k(t) + \sum_{k=1}^{n}\triangle\left(\frac{c_{-k}-c_k}{k}\right)iE'_{-k}(t)$$

by Lemma 4, we get

$$\sum_{k=1}^{\infty}\left|\triangle\left(\frac{c_k}{k}\right)\tilde{D}'_k(t)\right| \leq \left(\frac{A_1}{|t|}\right)\sum_{k=1}^{\infty}\left[k\left|\triangle\left(\frac{c_k}{k}\right)\right|\right]$$

$$\leq \left(\frac{A_1}{|t|}\right)\left[\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}k\left|\triangle^2\left(\frac{c_j}{j}\right)\right|\right]$$

$$= \left(\frac{A_1}{|t|}\right)\left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j} k\right)\left|\triangle^2\left(\frac{c_j}{j}\right)\right|\right]$$

$$= O\left[\left(\frac{A_1}{|t|}\right)\left(\sum_{j=1}^{\infty} j^2\right)\left|\triangle^2\left(\frac{c_j}{j}\right)\right|\right] < \infty$$

and

$$\sum_{k=3}^{\infty}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)\right| \leq \left(\frac{A_1}{|t|}\right)\left[\sum_{k=3}^{\infty} k\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|\right]$$

$$\leq \left(\frac{A_1}{|t|}\right)\left[\sum_{k=3}^{\infty} k(\log k)\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|\right]$$

$$< \infty$$

where A_1 is a suitable constant. This implies that

$$f(t) = 2\left[\sum_{k=1}^{\infty}\triangle\left(\frac{c_k}{k}\right)\tilde{D}'_k(t)\right] + i\left[\sum_{k=1}^{\infty}\triangle\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)\right]$$

exists and thus (2.4.1) follows.

Further, for $t \neq 0$,we have

$$f(t) - g_n(C,t) = 2\sum_{k=n+1}^{\infty}\triangle\left(\frac{c_k}{k}\right)\tilde{D}'_k(t) + i\sum_{k=n+1}^{\infty}\triangle\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)$$

Again on applying Abel's Transformation on 1st term of right side, we get

$$f(t) - g_n(C,t) = 2\sum_{k=n+1}^{\infty}(k+1)\triangle^2\left(\frac{c_k}{k}\right)\tilde{K}'_k(t) - 2(n+1)\triangle\left(\frac{c_{n+1}}{n+1}\right)\tilde{K}'_n(t)$$
$$+ i\sum_{k=n+1}^{\infty}\triangle\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)$$

Thus

$$\begin{aligned}||f(t)-g_n(C,t)||_{L^1} \leq\ & 2\sum_{k=n+1}^{\infty}(k+1)\left|\triangle^2\left(\frac{c_k}{k}\right)\right|\int_{-\pi}^{\pi}|\tilde{K}_k'(t)|dt \\ & +2(n+1)\left|\triangle\left(\frac{c_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}|\tilde{K}_{n+1}'(t)|dt \\ & +\sum_{k=n+1}^{\infty}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|\int_{-\pi}^{\pi}|E_{-k}'(t)|dt\end{aligned}$$

But, by Lemma 3,

$$\int_{-\pi}^{\pi}|\tilde{K}_k'(t)|dt = O(n)$$

and also

$$\begin{aligned}\left|\triangle\left(\frac{c_{n+1}}{n+1}\right)\right| &= \left|\sum_{k=n+1}^{\infty}\triangle^2\left(\frac{c_k}{k}\right)\right| \\ &\leq \sum_{k=n+1}^{\infty}\frac{k^2}{k^2}\left|\triangle^2\left(\frac{c_k}{k}\right)\right| \\ &\leq (n+1)^{-2}\sum_{k=n+1}^{\infty}k^2\left|\triangle^2\left(\frac{c_k}{k}\right)\right| \\ &= o((n+1)^{-2}), \quad \text{by the hypothesis of the theorem.}\end{aligned}$$

Lemma 1 implies that

$$\int_{-\pi}^{\pi}|E_{-k}'(t)|dt = o(k\log k).$$

Therefore, we have

$$\begin{aligned}||f(t)-g_n(C,t)||_{L^1} &= O\left(\sum_{k=n+1}^{\infty}(k+1)^2\left|\triangle^2\left(\frac{c_k}{k}\right)\right|\right)+o(1) \\ &\quad +o\left(\sum_{k=n+1}^{\infty}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|k\log k\right) \\ &= o(1),\text{by the hypothesis of the theorem.}\end{aligned}$$

Since $g_n(C,t)$ is a polynomial, it follows that $f \in L^1(T)$, which proves (2.4.2).

We notice further that

$$||f - S_n(f)||_{L^1} = ||f - g_n(C,t) + g_n(C,t) - S_n(f)||_{L^1}$$

$$\leq ||f - g_n(C,t)||_{L^1} + ||g_n(C,t) - S_n(f)||_{L^1}$$

$$= ||f - g_n(C,t)||_{L^1} + \left|\left|\tfrac{i}{n+1}[\hat{f}(n+1)E_n^{'}(t) - \hat{f}(-(n+1))E_{-n}^{'}(t)]\right|\right|_{L^1}$$

and

$$\left|\left|\frac{i}{n+1}[\hat{f}(n+1)E_n^{'}(t) - \hat{f}(-(n+1))E_{-n}^{'}(t)]\right|\right|_{L^1} = ||g_n(C,t) - S_n(f)||_{L^1}$$

$$\leq ||f - S_n(f)||_{L^1} + ||f - g_n(C,t)||_{L^1}.$$

Since $||f - g_n(C,t)||_{L^1} = o(1), \quad n \to \infty$ by (2.4.2), and Lemma 2,

$$||\hat{f}(n)E_n^{'}(t) - \hat{f}(-n)E_{-n}^{'}(t)||_{L^1} = o(n), \ n \to \infty,$$

if and only if $\hat{f}(n)\log|n| = o(1), \ |n| \to \infty$, the assertion (2.4.3) follows.

CHAPTER III

L^1-Convergence of r^{th} Differential of Modified Complex Trigonometric Sums

3.1. Introduction. In Chapter II, I have studied L^1-convergence of the complex form of the trigonometric sums introduced by Ram and Kumari ([17], [20]) and obtained necessary and sufficient condition for the L^1-convergence of the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$.

The aim of this chapter is to study the L^1-convergence of r^{th} derivative of the complex form

$$g_n(C,t) = S_n(C,t) + \frac{i}{n+1}\left[c_{n+1}E_n^{'}(t) - c_{-(n+1)}E_{-n}^{'}(t)\right] \tag{3.1.1}$$

of trigonometric sums.

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k\cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k\sin kx$$

under the class R_r^*.

The Class R_r^* [3]. A null sequence $\{c_n\}$ of complex numbers belongs to the class R_r^*, if

(i) $\sum_{k=1}^{\infty}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right| k^{r+1}\log k < \infty,$

(ii) $\sum_{k=1}^{\infty} k^{r+2}\left|\triangle^2\left(\frac{c_k}{k}\right)\right| < \infty.$

if $r = 0$, class R_r^* reduces to R^*.

The r^{th} differentials $D_n^{(r)}(t)$ and $\tilde{D}_n^{(r)}(t)$ of the Dirichlet kernel $D_n(t)$ and conjugate Dirichlet kernel $\tilde{D}_n(t)$ respectively can be written as

$$2D_n^{(r)}(t) = E_n^{(r)}(t) + E_{-n}^{(r)}(t),$$

$$2i\tilde{D}_n^{(r)}(t) = E_n^{(r)}(t) - E_{-n}^{(r)}(t)$$

where $E_n^{(r)}(t)$ denotes the r^{th} derivative of $E_n(t) = \sum_{k=0}^{n} e^{ikt}$.

The complex form of the r^{th} derivative of (3.1.1) is

$$G_n^{(r)}(C,t) = S_n^r(C,t) + \frac{i}{n+1}\left[c_{n+1}E_n^{(r+1)}(t) - c_{-(n+1)}E_{-n}^{(r+1)}(t)\right]$$

3.2. Lemmas. The proof of our result is based upon the following lemmas of which first two are due to Sheng [22] :

Lemma 1. For the r^{th} derivatives of the Dirichlet's kernels $D_n(t)$ and $\tilde{D}_n(t)$ the following estimates hold

$$(i) \quad ||D_n^{(r)}(t)||_{L^1} = \frac{4}{\pi}(n^r \log n) + O(n^r),\ r = 0,1,2,....$$
$$(ii) \quad ||\tilde{D}_n^{(r)}(t)||_{L^1} = O(n^r \log n),\ r = 0,1,2,...$$

Lemma 2. For each non-negative integer n ,there holds

$$||c_n E_n^{(r)}(t) + c_{-n}E_{-n}^{(r)}(t)||_{L^1} = o(1), \qquad |n| \to \infty$$

if and only if $|n|^r c_n \log|n| = o(1),\ \ as\ |n| \to \infty$,where $< c_n >$ is a complex sequence and $t \in \frac{R}{2\pi Z}$.

Proofs of these two lemmas have been reported in Chapter II.

Lemma 3. (i) There exist positive constants α and β such that

$$\alpha(\log n) \leq ||\tilde{K}_n(t)||_{L^1} \leq \beta(\log n)$$

(ii) $$||\tilde{K}_n^{(r)}(t)||_{L^1} = O(n^r) \ \ for\ all \ \ r \geq 1\ .$$

Proof. The proof of 1st part has already been proved in Chapter II. To prove second part we notice that (Chapter II, Lemma 3) $||\tilde{K}_n'(t)|| = O(n)$. By Induction on r and Zygmund's Theorem[2,Vol.II.458], the desired result follows.

Lemma 4. Let r be a non-negative integer and $0 < \epsilon < \pi$. Then there exists $A_{r\epsilon} > 0$ such that for all $\epsilon \leq |t| \leq \pi$ and all $n \geq 1$

(3.2.1) $$|E_n^{(r)}(t)| \leq \frac{A_{r\epsilon} n^r}{|t|}$$

(3.2.2) $$|E_{-n}^r(t)| \leq \frac{A_{r\epsilon} n^r}{|t|}$$

(3.2.3) $$|D_n^r(t)| \leq \frac{A_{r\epsilon} n^r}{|t|}$$

(3.2.4) $$|\tilde{D}_n^r(t)| \leq \frac{A_{r\epsilon} n^r}{|t|}$$

Proof. The case $r = 0$ is trivial. For $r \geq 1$, We have

$$E_n^r(t) = \sum_{k=0}^{n} i^r k^r e^{ikt}$$

$$\begin{aligned} i^{-r} E_n^r(t) &= \sum_{k=0}^{n} k^r e^{ikt} \\ &= \sum_{k=0}^{n} k^r (E_k(t) - E_{k-1}(t)) \\ &= \sum_{k=0}^{n} (\triangle k^r) E_k(t) + (n+1)^r E_n(t). \end{aligned}$$

since

$$|E_n(t)| \leq \frac{A_{r\epsilon}}{|t|} \ \ for\ some\ constant\ A_{r\epsilon}$$

we have

$$|E_n^r(t)| \leq \frac{A_{r\epsilon}}{|t|}\left[\sum_{k=0}^{n} |\triangle k^r| + (n+1)^r\right] \leq \frac{A_{r\epsilon}n^r}{|t|}$$

which proves (3.2.1) part.

Further, since

$$|E_{-n}^r(t)| = (-1)^r|E_n^r(-t)|,$$

we have

$$|E_{-n}^r(t)| \leq \frac{A_{r\epsilon}n^r}{|t|}.$$

which proves (3.2.2) part.

Moreover, we know that

$$2D_n^r(t) = E_n^r(t) + E_{-n}^r(t),$$

therefore

$$|D_n^r(t)| \leq \frac{A_{r\epsilon}n^r}{|t|}$$

and

$$2i\tilde{D}_n^r(t) = E_n^r(t) - E_{-n}^r(t),$$

Therefore

$$|\tilde{D}_n^r(t)| \leq \frac{A_{r\epsilon}n^r}{|t|}.$$

3.3 Main result: The main result of this chapter is the following theorem:

Theorem . Let $c_n \in R^*(r)$.Then there exists $f^{(r)}(t)$ such that

(3.3.1) $\lim_{n\to\infty} G_n^{(r)}(C,t) = f^{(r)}(t)\ for\ all\ 0 < |t| \leq \pi,$

(3.3.2) $f^{(r)}(t) \in L^1(T)$ and $||G_n^{(r)}(C,t) - f^{(r)}(t)||_{L^1}$=o(1) as $n \to \infty$,

(3.3.3) $||S_n^{(r)}(f,t) - f^{(r)}(t)||_{L^1}$=o(1) as $n \to \infty$ if and only if $|n|^r\hat{f}(n)\log|n|$=o(1) as $|n| \to \infty$.

Proof. The r^{th} derivative of complex trigonometric sums is

$$G_n^{(r)}(C,t) = S_n^{(r)}(C,t) + \frac{i}{n+1}\left[c_{n+1}E_n^{(r+1)}(t) - c_{-(n+1)}E_{-n}^{(r+1)}(t)\right]$$

by the use of Abel's transformation, we get

$$G_n^{(r)}(C,t) = 2\sum_{k=1}^{n} \triangle\left(\frac{c_k}{k}\right)\tilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{n}\triangle\left(\frac{c_{-k}-c_k}{k}\right) iE_{-k}^{(r+1)}(t).$$

By Lemma 4, we get

$$\sum_{k=1}^{\infty}\left|\triangle\left(\frac{c_k}{k}\right)\tilde{D}_k^{(r+1)}(t)\right| \le \left(\frac{A_{(r+1)\epsilon}}{|t|}\right)\sum_{k=1}^{\infty}\left[k^{r+1}\left|\triangle\left(\frac{c_k}{k}\right)\right|\right]$$

$$\le \left(\frac{A_{(r+1)\epsilon}}{|t|}\right)\left[\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}k^{r+1}\left|\triangle^2\left(\frac{c_j}{j}\right)\right|\right]$$

$$= \left(\frac{A_{(r+1)\epsilon}}{|t|}\right)\left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j}k^{r+1}\right)\left|\triangle^2\left(\frac{c_j}{j}\right)\right|\right]$$

$$= O\left[\left(\frac{A_{(r+1)\epsilon}}{|t|}\right)\left(\sum_{j=1}^{\infty}j^{r+2}\right)\left|\triangle^2\left(\tfrac{c_j}{j}\right)\right|\right] < \infty$$

and

$$\sum_{k=3}^{\infty}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)E_{-k}^{(r+1)}(t)\right| \le \left(\frac{A_{(r+1)\epsilon}}{|t|}\right)\left[\sum_{k=3}^{\infty}k^{r+1}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|\right]$$

$$\le \frac{A_{(r+1)\epsilon}}{|t|}\left(\sum_{k=3}^{\infty}k^{r+1}(\log k)\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|\right) < \infty$$

where $A_{(r+1)\epsilon}$ is suitable constant. This implies that

$$f^{(r)}(t) = 2\left[\sum_{k=1}^{\infty}\triangle\left(\frac{c_k}{k}\right)\tilde{D}_k^{(r+1)}(t)\right] + i\left[\sum_{k=1}^{\infty}\triangle\left(\frac{c_{-k}-c_k}{k}\right)E_{-k}^{(r+1)}(t)\right]$$

exists and thus (3.3.1) follows.

Further, for $t \neq 0$, we have

$$f^{(r)}(t) - G_n^{(r)}(C,t) = 2\sum_{k=n+1}^{\infty} \triangle\left(\frac{c_k}{k}\right)\tilde{D}_k^{(r+1)}(t) + i\sum_{k=n+1}^{\infty}\triangle\left(\frac{c_{-k}-c_k}{k}\right)E_{-k}^{(r+1)}(t)$$

Again apply Abel's Transformation on 1st term of right side,we get

$$\begin{aligned} f^{(r)}(t)-G_n^{(r)}(C,t) &= 2\sum_{k=n+1}^{\infty}(k+1)\triangle^2\left(\frac{c_k}{k}\right)\tilde{K}_k^{(r+1)}(t)-2(n+1)\triangle\left(\frac{c_{n+1}}{n+1}\right)\tilde{K}_n^{(r+1)}(t) \\ &\quad +i\sum_{k=n+1}^{\infty}\triangle\left(\frac{c_{-k}-c_k}{k}\right)E_{-k}^{(r+1)}(t) \end{aligned}$$

Thus

$$\begin{aligned} ||f^{(r)}(t)-g_n^{(r)}(C,t)||_{L^1} &\leq 2\sum_{k=n+1}^{\infty}(k+1)\left|\triangle^2\left(\frac{c_k}{k}\right)\right|\int_{-\pi}^{\pi}|\tilde{K}_k^{(r+1)}(t)|dt \\ &\quad +2(n+1)\left|\triangle\left(\frac{c_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}|\tilde{K}_{n+1}^{(r+1)}(t)|dt \\ &\quad +\sum_{k=n+1}^{\infty}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|\int_{-\pi}^{\pi}|E_{-k}^{(r+1)}(t)|dt \end{aligned}$$

but, by Lemma 3,

$$\int_{-\pi}^{\pi}|\tilde{K}_k^{(r+1)}(t)|dt = O(n^{r+1})$$

and also

$$\begin{aligned} \left|\triangle\left(\frac{c_{n+1}}{n+1}\right)\right| &= \left|\sum_{k=n+1}^{\infty}\triangle^2\left(\frac{c_k}{k}\right)\right| \\ &\leq \sum_{k=n+1}^{\infty}\frac{k^{r+2}}{k^{r+2}}\left|\triangle^2\left(\frac{c_k}{k}\right)\right| \\ &\leq (n+1)^{-r-2}\sum_{k=n+1}^{\infty}k^{r+2}\left|\triangle^{r+2}\left(\frac{c_k}{k}\right)\right| \\ &= o((n+1)^{-r-2}), \quad \text{by the hypothesis of the theorem.} \end{aligned}$$

Lemma 1 implies that

$$\int_{-\pi}^{\pi} |E_{-k}^{(r+1)}(t)|dt = O(k^{r+1}\log k).$$

Therefore, we have

$$\begin{aligned}||f^{(r)}(t) - G_n^{(r)}(C,t)||_{L^1} &= O\left(\sum_{k=n+1}^{\infty}(k+1)^{r+2}\left|\triangle^2\left(\frac{c_k}{k}\right)\right|\right) + o(1)\\ &+O\left(\sum_{k=n+1}^{\infty}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|k^{r+1}\log k\right)\\ &= o(1),\text{by the hypothesis of the theorem.}\end{aligned}$$

Since $G_n^{(r)}(C,t)$ is a polynomial, it follows that $f^{(r)}(t) \in L^1(T)$, which proves (3.3.2).

We notice further that

$$||f^{(r)} - S_n^{(r)}(f)||_{L^1} = ||f^{(r)} - g_n^{(r)}(C,t) + g_n^{(r)}(C,t) - S_n^{(r)}(f)||_{L^1}$$

$$\leq ||f^{(r)} - g_n^{(r)}(C,t)||_{L^1} + ||g_n^{(r)}(C,t) - S_n^{(r)}(f)||_{L^1}$$

$$= ||f^{(r)} - g_n^{(r)}(C,t)||_{L^1} + \left|\left|\frac{i}{n+1}[\hat{f}(n+1)E_n^{(r+1)}(t) - \hat{f}(-(n+1))E_{-n}^{(r+1)}(t)]\right|\right|_{L^1}$$

and

$$\left|\left|\frac{i}{n+1}[\hat{f}(n+1)E_n^{(r+1)}(t) - \hat{f}(-(n+1))E_{-n}^{(r+1)}(t)]\right|\right|_{L^1} = ||g_n^{(r)}(C,t) - S_n^{(r)}(f)||_{L^1}$$

$$\leq ||f^{(r)} - S_n^{(r)}(f)||_{L^1} + ||f^{(r)} - g_n^{(r)}(C,t)||_{L^1}.$$

Since $||f^{(r)} - g_n^{(r)}(C,t)||_{L^1} = o(1), \quad n \to \infty$ by (3.3.2), and Lemma 2,

$$||\hat{f}(n)E_n^{(r+1)}(t) - \hat{f}(-n)E_{-n}^{(r+1)}(t)||_{L^1} = o(n), \ n \to \infty,$$

if and only if $|n|^r\hat{f}(n)\log|n| = o(1), \ |n| \to \infty$, the assertion (3.3.3) follows.

CHAPTER IV

Convergence of New Modified Complex Trigonometric Sums in the Metric Space L

4.1 Introduction

Let the partial sums of the complex trigonometric series

$$\sum_{n=-\infty}^{\infty} c_n e^{int}$$

be denoted by

$$S_n(C,t) = \sum_{k=-n}^{n} c_k e^{ikt}, t \in T = R/2\pi Z.$$

If a trigonometric series is the Fourier series of some function $f \in L^1$, we shall write $c_n = \hat{f}(n) \ for \ all \ n$ and $S_n(C,t)$=$S_n(f,t)$=$S_n(f)$.

Definition. A sequence $\{a_n\}$ is said to semi-convex if $a_n \to 0 \ \ as \ \ n \to \infty$ and

$$\sum_{n=1}^{\infty} n|\triangle^2 a_{n-1} + \triangle^2 a_n| \ < \infty$$

Xhevat Z. Krasniqi [28] introduced new modified cosine and sine sums

$$(4.1.1) \qquad H_n(x) = \frac{-1}{2\sin x} \sum_{k=0}^{n} \sum_{j=k}^{n} \triangle \left[(a_{j-1} - a_{j+1}) \cos jx\right]$$

and

$$(4.1.2) \qquad G_n(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left[(a_{j-1} - a_{j+1}) \sin jx\right]$$

and studied their L^1-convergence with semi-convex coefficients.

The aim of this chapter is to obtain the complex form of above modified sums (4.1.1) and (4.1.2) of Xhevat Z. Krasniqi and to study its L^1-convergence.

Let

$$D_n(t) = \frac{1}{2} + \cos t + \cos 2t + ... + \cos nt \ = \ \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\tilde{D}_n(t) = \sin t + \sin 2t + ... + \sin nt \ = \ \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

and

$$\tilde{K}_n(t) = \frac{1}{n+1}\sum_{j=0}^{n} \tilde{D}_j(t) = \frac{1}{4\sin^2\frac{t}{2}}\left[\sin t - \frac{\sin(n+1)t}{n+1}\right]$$

denote the Dirichlet's kernel, the conjugate Dirichlet's kernel, and the conjugate Fejér's kernel respectively. Let $E_n(t) = \sum_{k=0}^{n} e^{ikt}$, Then the first differentials $D_n^{'}(t)$ and $\tilde{D}_n^{'}(t)$ of $D_n(t)$ and $\tilde{D}_n(t)$ can be written as

$$2D_n^{'}(t) = E_n^{'}(t) + E_{-n}^{'}(t)$$

$$2i\tilde{D}_n^{'}(t) = E_n^{'}(t) - E_{-n}^{'}(t)$$

where $E_n^{'}(t)$ denotes the first differential of $E_n(t)$.

The Complex form of modified sums (4.1.1) and (4.1.2) is obtained as:

Since

$$H_n(t) = \frac{-1}{2\sin t}\sum_{k=0}^{n}\sum_{j=k}^{n} \triangle\left[(b_{j-1} - b_{j+1})\cos jt\right]$$

$$H_n(t) = \frac{-1}{2\sin t}\left[\sum_{k=0}^{n}\left((b_{k-1} - b_{k+1})\cos kt - (b_n - b_{n+2})\cos(n+1)t\right)\right]$$

$$= \frac{-1}{2\sin t}[-b_1 + (b_0 - b_2)\cos t + (b_1 - b_3)\cos 2t + ...$$

$$+(b_{n-2} - b_n)\cos(n-1)t + (b_{n-1} - b_{n+1})\cos nt]$$

$$+ \frac{1}{2\sin t}\sum_{k=0}^{n}(b_n - b_{n+2})\cos(n+1)t$$

$$= \frac{-1}{2\sin t}[-b_1(1-\cos 2t) - b_2(\cos t - \cos 3t) - ....$$

$$-b_{n-1}(\cos(n-2)t - \cos nt) - b_n \cos(n-1)t - b_{n+1}\cos nt]$$
$$+ \frac{(n+1)}{2\sin t}(b_n - b_{n+2})\cos(n+1)t$$

therefore,

$$H_n(t) = \frac{-1}{2\sin t}[-2b_1\sin^2 t - 2b_2\sin 2t\sin t - .... - 2b_{n-1}\sin(n-1)t\sin t$$

$$-b_n(\cos(n-1)t - \cos(n+1)t) - b_n\cos(n+1)t - b_{n+1}\cos nt]$$

$$+ \frac{n+1}{2\sin t}(b_n - b_{n+2})\cos(n+1)t$$

$$= b_1\sin t + b_2\sin 2t + ... + b_n \sin nt$$

$$+\frac{1}{2\sin t}\left[b_n\cos(n+1)t + b_{n+1}\cos nt + (n+1)(b_n - b_{n+2})\cos(n+1)t\right]$$

hence,

$$(4.1.3)\quad H_n(t) = \sum_{k=1}^{n} b_k \sin kt + \frac{1}{2\sin t}\left[b_n\cos(n+1)t + b_{n+1}\cos nt + (n+1)(b_n - b_{n+2})\cos(n+1)t\right]$$

Again

$$G_n(t) = \frac{1}{2\sin t}\sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left[(a_{j-1} - a_{j+1})\sin jt\right]$$

$$G_n(t) = \frac{1}{2\sin t}\left[\sum_{k=1}^{n}((a_{k-1} - a_{k+1})\sin kt - (a_n - a_{n+2})\sin(n+1)t)\right]$$

$$= \frac{1}{2\sin t}\big[(a_0\sin t + a_1\sin 2t - a_2(\sin t - \sin 3t) - a_3(\sin 2t - \sin 4t) - ....$$

$$-a_{n-1}(\sin(n-2)t - \sin nt) - a_n(\sin(n-1)t - \sin(n+1)t)$$

$$-\, a_n\sin(n+1)t - a_{n+1}\sin nt\big] - \frac{(n+1)}{2\sin t}(a_n - a_{n+2})\sin(n+1)t$$

therefore,

$$G_n(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + ... + a_n \cos nt$$

$$-\frac{1}{2\sin t}\left[a_n \sin{(n+1)t} + a_{n+1} \sin nt + (n+1)(a_n - a_{n+2}) \sin{(n+1)t}\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kt - \frac{1}{2\sin t}\left[a_n \sin{(n+1)t} + a_{n+1} \sin nt + (n+1)(a_n - a_{n+2}) \sin{(n+1)t}\right]$$

.... (4.1.4)

Adding (4.1.3) and (4.1.4), we get

$$G_n(t) + H_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kt + \sum_{k=1}^{n} b_k \sin kt$$

$$+ \frac{1}{2\sin t}\big[b_n \cos{(n+1)t} + b_{n+1} \cos nt - a_n \sin{(n+1)t} - a_{n+1} \sin nt\big]$$

$$+ \frac{(n+1)}{2\sin t}\big[(b_n - b_{n+2}) \cos{(n+1)t} - (a_n - a_{n+2}) \sin{(n+1)t}\big]$$

$$= S_n(C,t) + \frac{i}{2\sin t}\begin{bmatrix} c_n e^{i(n+1)t} - c_{-n} e^{-i(n+1)t} + c_{n+1} e^{int} - c_{-(n+1)} e^{-int} \\ + (n+1)(c_n - c_{n+2}) e^{i(n+1)t} + (n+1)(c_{-(n+2)} - c_{-n}) e^{-i(n+1)t} \end{bmatrix}$$

Therefore, new modified complex form of modified sums (4.1.1) and (4.1.2) is

$$g_n(C,t) = S_n(C,t) + \frac{i}{2\sin t}\begin{bmatrix} c_n e^{i(n+1)t} - c_{-n} e^{-i(n+1)t} + c_{n+1} e^{int} - c_{-(n+1)} e^{-int} \\ + (n+1)(c_n - c_{n+2}) e^{i(n+1)t} + (n+1)(c_{-(n+2)} - c_{-n}) e^{-i(n+1)t} \end{bmatrix}$$

Kaur J. and Bhatia S.S. [15] have introduced a new class J^* of coefficient sequences as :

Definition. A null sequence $\{c_n\}$ of complex numbers belongs to class J^* if there exists a sequence $\{A_n\}$ such that

(4.1.5) $\quad A_n \downarrow 0, \quad as \quad n \to \infty,$

(4.1.6) $\quad \sum_{n=1}^{\infty} nA_n < \infty$

(4.1.7) $\quad \left| \triangle \left(\frac{c_n - c_{-n}}{n} \right) \right| \le \frac{A_n}{n}$, for all $n = 1, 2, 3, ...$

4.2 Lemmas. The proof of our result is based upon the following lemmas.

Lemma 1. Let r be a non-negative integer and $0 < \epsilon < \pi$. Then there exists $M_{r\epsilon} > 0$ such that for all $\epsilon \le |t| \le \pi$ and all $n \ge 1$,

(i) $\quad |E_n^r(t)| \le A_{r\epsilon} n^r/|t|,$

(ii) $\quad |E_{-n}^r(t)| \le A_{r\epsilon} n^r/|t|,$

(iii) $\quad |D_n^r(t)| \le A_{r\epsilon} n^r/|t|,$

(iv) $\quad |\tilde{D}_n^r(t)| \le A_{r\epsilon} n^r/|t|.$

The proof of this Lemma has been given in Chapter III.

Lemma 2. For $n \ge 1$, we have

(i) $\quad \left|\left| \frac{E_n(t)}{2\sin t} \right|\right| = o(n), \ n \to \infty$

(ii) $\quad \left|\left| \frac{E_{-n}(t)}{2\sin t} \right|\right| = o(n), \ n \to \infty$

(iii) $\quad \left|\left| \frac{e^{int}}{2\sin t} \right|\right| = o(\log n), \ n \to \infty$

Proof. For $t \neq 0$, we note that $\frac{\sin t}{t} \ge \frac{2}{\pi}$ in $(0,\pi/2)$ and using Lemma 1, we get

$$\left|\left| \frac{E_n(t)}{2\sin t} \right|\right| = \int_0^{\pi} \left| \frac{E_n(t)}{2\sin t} \right| dt \le \int_0^{\pi} \frac{M_\epsilon}{2|t\sin t|} dt \le 2\int_0^{\pi/2} \frac{M_\epsilon}{2|t||\sin t|} dt$$

Therefore

$$\left|\left|\frac{E_n(t)}{2\sin t}\right|\right| \leq \int_0^{\pi/2} \frac{M_\epsilon}{t^2}dt = \lim_{n\to\infty}\left[\frac{-M_\epsilon}{t}\right]_{\pi/n}^{\pi/2} = o(n),\ n\to\infty.$$

Similarly, $\left|\left|\dfrac{E_{-n}(t)}{2\sin t}\right|\right| = o(n)$, and to prove (iii), we consider,

$$\left|\left|\frac{e^{int}}{2\sin t}\right|\right| = \int_0^{\pi}\left|\frac{e^{int}}{2\sin t}\right|dt = 2\int_0^{\pi/2}\left|\frac{1}{2\sin t}\right|dt = \int_0^{\pi/2}\left|\frac{1}{t}\right|dt$$

Therefore

$$\left|\left|\frac{e^{int}}{2\sin t}\right|\right| = \lim_{n\to\infty}[\log t]_{\pi/n}^{\pi/2} = o(\log n),\ n\to\infty.$$

4.3 Result. The main result of this chapter is the following theorem:

Theorem Let $c_n \in J^*$. Then there exists $f(t)$ such that

(4.3.1) $\lim_{n\to\infty} g_n(C,t) = f(t)\ for\ |t| \in (0,\pi]$,

(4.3.2) $f(t) \in L^1(0,\pi]$ and $||g_n(C,t) - f(t)||_{L^1}$=o(1) as $n\to\infty$,

(4.3.3) $||S_n(f,t) - f(t)||_{L^1}$=o(1) as $|n|\to\infty$.

Proof. Consider,

$$g_n(C,t) = S_n(C,t) + \frac{i}{2\sin t}\left[\begin{array}{l} c_n e^{i(n+1)t} - c_{-n}e^{-i(n+1)t} + c_{n+1}e^{int} - c_{-(n+1)}e^{-int} \\ + (n+1)(c_n - c_{n+2})e^{i(n+1)t} + (n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)t} \end{array}\right]$$

since $\dfrac{e^{int}}{2\sin t}$ is bounded in $(o,\pi]$. Also, $\{c_n\}$ is a null sequence.

Therefore, we can write

$$\lim_{n\to\infty} g_n(C,t) = \lim_{n\to\infty} S_n(C,t) + \lim_{n\to\infty}\frac{iA_\epsilon}{2}\left[(n+1)(c_n - c_{-n}) + (n+1)(c_{-(n+2)} - c_{n+2})\right]$$

But, for all $n \geq 1$, we note that

$$\frac{n(n+1)(c_n - c_{-n})}{n} = n(n+1)\sum_{k=n}^{\infty} \triangle\left(\frac{c_k - c_{-k}}{k}\right) \leq \sum_{k=n}^{\infty} k(k+1)\frac{A_k}{k}$$

$$= o(1), \text{ by hypothesis of the theorem.}$$

Therefore

$$\lim_{n\to\infty} g_n(C,t) = \lim_{n\to\infty} S_n(C,t) = f(t)$$

Now, we show that $f(t)$ exists in $(0,\pi]$. Consider,

$$S_n(C,t) = \sum_{k=-n}^{n} c_k e^{ikt} = c_0 + \sum_{k=1}^{n}\left[\frac{c_k}{k}ke^{ikt} + \frac{c_{-k}}{k}ke^{-ikt}\right]$$

by the use of Abel's transformation, we get

$$S_n(C,t) = c_0 + \sum_{k=1}^{n-1}\triangle\left(\frac{c_k}{k}\right)(-iE_k'(t)) + \frac{c_n(-iE_n'(t))}{n} + \sum_{k=1}^{n-1}\triangle\left(\frac{c_{-k}}{k}\right)(iE_{-k}'(t))$$
$$+ \frac{c_{-n}(iE_{-n}'(t))}{n}$$

$$S_n(C,t) = c_0 - \sum_{k=1}^{n-1}\left[\triangle\left(\frac{c_k}{k}\right)(iE_k'(t)) - \triangle\left(\frac{c_{-k}}{k}\right)(iE_{-k}'(t))\right]$$
$$- \frac{c_n(iE_n'(t)) - c_{-n}(iE_{-n}'(t))}{n}$$

by using Lemma 1 and (4.1.7), we get

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\triangle\left(\frac{c_k}{k}\right)(iE_k'(t)) - \triangle\left(\frac{c_{-k}}{k}\right)(iE_{-k}'(t))\right| &\leq \left(\frac{A_\epsilon}{|t|}\right)\sum_{k=1}^{\infty} k\left|\triangle\left(\frac{c_k - c_{-k}}{k}\right)\right| \\ &\leq \left(\frac{A_\epsilon}{|t|}\right)\sum_{k=1}^{\infty}\frac{kA_k}{k} \\ &= \left(\frac{A_\epsilon}{|t|}\right)\sum_{k=1}^{\infty} A_k \ < \ \infty,\end{aligned}$$

since $A_n \le nA_n$ *for all* $n \ge 1$ *and* $\sum nA_n$ converges, by hypothesis of the theorem.

Therefore $\sum A_n$ is also convergent.

and

$$\left|\frac{c_n(iE_n^{'}(t)) - c_{-n}(iE_{-n}^{'}(t))}{n}\right| \le \frac{A_\epsilon n}{|t|}\left|\frac{c_n - c_{-n}}{n}\right| = o(1), \quad as\ n \to \infty.$$

Hence $f(t) = \lim\limits_{n\to\infty} S_n(C,t)$ exists and thus (4.3.1) follows.

Further, for $t \neq 0$, we have

$$f(t)-g_n(C,t) = \sum_{|k|>n}^{\infty} c_k e^{ikt} - \frac{i}{2\sin t}\left[\begin{array}{l} c_n e^{i(n+1)t} - c_{-n}e^{-i(n+1)t} + c_{n+1}e^{int} - c_{-(n+1)}e^{-int} \\ + (n+1)(c_n - c_{n+2})e^{i(n+1)t} + (n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)t} \end{array}\right]$$

Using Abel's transformation, we get

$$\begin{aligned} f(t) - g_n(C,t) = &\sum_{k=n+1}^{\infty}\left[\triangle\left(\frac{c_k}{k}\right)(-iE_k^{'}(t)) + \triangle\left(\frac{c_{-k}}{k}\right)(iE_{-k}^{'}(t))\right] \\ &- \frac{c_{n+1}(-iE_n^{'}(t))}{n+1} - \frac{c_{-(n+1)}(iE_{-n}^{'}(t))}{n+1} \\ &- \frac{i}{2\sin t}\left[\begin{array}{l} c_n e^{i(n+1)t} - c_{-n}e^{-i(n+1)t} + c_{n+1}e^{int} - c_{-(n+1)}e^{-int} \\ + (n+1)(c_n - c_{n+2})e^{i(n+1)t} + (n+1)(c_{-(n+2)} - c_{-n})e^{-i(n+1)t} \end{array}\right] \end{aligned}$$

$$\begin{aligned} ||f(t)-g_n(C,t)||_{L^1} \le &\int_0^\pi \sum_{k=n+1}^{\infty} k\left|\triangle\left(\frac{c_k - c_{-k}}{k}\right)\right|\frac{A_\epsilon}{|t|}dt + \int_0^\pi \frac{A_\epsilon n}{|t|}\left|\frac{c_{n+1} - c_{-(n+1)}}{n}\right| dt \\ &+ \int_0^\pi \left|\frac{(c_n - c_{-n}) + (c_{n+1} - c_{-(n+1)}) + (n+1)(c_n - c_{-n}) - (n+1)(c_{n+2} - c_{-(n+2)})}{\sin t}\right| dt \end{aligned}$$

Therefore,

$$
\begin{aligned}
||f(t)-g_n(C,t)||_{L^1} \leq & \int_0^{\pi}\left|\sum_{k=n+1}^{\infty} k\frac{A_k}{k}\frac{A_\epsilon}{|t|}\right| dt + \int_0^{\pi}\left|\frac{A_\epsilon}{|t|}\left((c_{n+1}-c_{-(n+1)})\right)\right| dt \\
& + \int_0^{\pi}\left|\frac{(c_n-c_{-n})+(c_{n+1}-c_{-(n+1)})}{|\sin t|}\right| dt \\
& + \int_0^{\pi}\left|\frac{(n+1)(c_n-c_{-n})-(n+1)(c_{n+2}-c_{-(n+2)})}{|\sin t|}\right| dt \\
& = O\left(\sum_{k=n+1}^{\infty} A_k \log k\right) + o\left(\log n(c_n-c_{-n})\right) \\
& + o\left(\log n(c_n-c_{-n})\right) + o\left(\log n(c_n-c_{-n})\right)
\end{aligned}
$$

But, for all $n \geq 1$, we note that

$$
\log n(c_n-c_{-n}) \leq n^2\frac{(c_n-c_{-n})}{n} \leq \sum_{k=n}^{\infty} k^2 \triangle\left(\frac{c_k-c_{-k}}{k}\right) \leq \sum_{k=n}^{\infty} k^2\left(\frac{A_k}{k}\right)
$$

$= o(1)$, by hypothesis of the theorem.

Therefore,

$||f(t)-g_n(C,t)||_{L^1} = o(1),\ n \to \infty$ and since $g_n(C,t)$ is a polynomial, it follows that $f \in L^1(0,\pi]$, which proves the assertion (4.3.2).

Further, we notice that

$$
\begin{aligned}
||f-S_n||_{L^1} &= ||f-g_n+g_n-S_n||_{L^1} \\
&\leq ||f-g_n||_{L^1} + ||g_n-S_n||_{L^1}
\end{aligned}
$$

$$
||f-S_n||_{L^1} \leq ||f-g_n||_{L^1} + \left|\left|\frac{i}{2\sin t}\begin{bmatrix} c_ne^{i(n+1)t}-c_{-n}e^{-i(n+1)t}+c_{n+1}e^{int}-c_{-(n+1)}e^{-int} \\ +(n+1)(c_n-c_{n+2})e^{i(n+1)t}+(n+1)(c_{-(n+2)}-c_{-n})e^{-i(n+1)t}\end{bmatrix}\right|\right|
$$

and

$$
\left|\left|\frac{i}{2\sin t}\begin{bmatrix} c_ne^{i(n+1)t}-c_{-n}e^{-i(n+1)t}+c_{n+1}e^{int}-c_{-(n+1)}e^{-int} \\ +(n+1)(c_n-c_{n+2})e^{i(n+1)t}+(n+1)(c_{-(n+2)}-c_{-n})e^{-i(n+1)t}\end{bmatrix}\right|\right|_{L^1} = ||g_n(C,t)-S_n(f)||_{L^1}
$$

$$
\leq ||f-S_n(f)||_{L^1} + ||f-g_n(C,t)||_{L^1}
$$

since $\quad ||f-g_n(C,t)||_{L^1} = o(1),\ n \to \infty \quad$ Hence the assertion (4.3.3) follows.

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