DUALITY IN MATHEMATICAL PROGRAMMING INVOLVING CLASS OF E-CONVEX FUNCTIONS

Thesis submitted in partial fulfillment of the requirements for
the award of degree of
Masters of Science
in
Mathematics and Computing

Submitted by

Mehak Goyal
(Reg. No. 331203002)

Under the supervision of
Dr. Navdeep Kailey

School of Mathematics

Thapar University, Patiala - 147001

Punjab (India)
July, 2017

Certificate

I hereby declare that the work which is being presented here in the dissertation entitled "**DUALITY IN MATHEMATICAL PROGRAMMING INVOLVING CLASS OF *E*-CONVEX FUNCTIONS**" in partial fulfillment of the requirement for the award of degree of **Master of Science in Mathematics and Computing** submitted in School of Mathematics, Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Navdeep Kailey**, Assistant Professor, SOM and refer other researcher's work which are duly listed in the reference section.

The matter presented in the thesis has not been submitted to any other University/Institute for the award of my degree.

Mehak
Mehak Goyal
Roll no. 331203002

Dated: 30 July;2017

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Kailey
Dr. Navdeep Kailey
Assistant Professor
SOM, Thapar University
Patiala-147001

Acknowledgement

Apart from my efforts, the success of my dissertation depending mainly on encouragement and guidelines of many others. Every project big or small is successful largely due to the effort of a number of wonderful people who have always given their valuable advice. I sincerely appreciate the inspiration, support and guidance of all those people who have been instrumental in making this dissertation a success. First of all, I would like to thank God for providing me the opportunity to pursue the study under the supervision of **Dr. Navdeep Kailey, Assistant Professor, School of Mathematics, Thapar University, Patiala**, for continuous support in my study and research, for his patience, motivation, enthusiasm, and immense knowledge. His infectious enthusiasm and unlimited zeal have been the driving forces. His guidance helped me in all the time of research and writing of this dissertation.

I take the opportunity to thanks **Dr. A.K. lal, Head SOM, Thapar University, Patiala,** for providing all the necessary facilities in the department. His support and sincere attitude towards department have always been encouraging and helped me pave my way for the dissertation.

I owe and respectfully offer my gratitude to my beloved family for their understanding, endless love, affection and blessings that lead to this achievement. My heartfelt thanks are to Sonali, without whose support this would not have been possible. She supported me unconditionally all through my dissertation. It is difficult to find adequate words to express appreciation for the help given by her. Finally, I am also thankful to my friends who devoted their valuable time and helped me in all possible ways towards successful completetion of their work.

Patiala Mehak Goyal
July 30, 2017

Abstract

The present thesis is organised into four chapters. The additional objective is to clarify the structure underlying generalized convex functions by presenting analogoues to the properties of convex functions. The material on convex functions and their generalizations is intensely large. The overview of the chapters are described below:

The first chapter of the dissertation is the introduction with the brief description of basic concepts, definitions of convex functions and and other definitions to be used in subsequent chapter are given, that are used throughout work which is useful for understanding the general concept of convexity. A brief account of the related studies made by various authors in the field and a summary of the thesis has also presented in this chapter.

Chapter 2 is devoted to a class of E-convex sets and E-convex functions are introduced by relaxing the definitions of convex sets and convex functions. This kind of generalized convexity is based on the effect of an operator E on the sets and domain of definition of the functions. The optimality results for E-convex programming problems are established.

In Chapter 3, we have reviewed a class of functions, F-convex functions and Second order F-convex functions as a generalization of convex functions. Under F-convexity, F-concavity, F-pseudoconvexity, F-pseudoconcavity, duality results for pair of Wolfe and Mond-Weir type symmetric dual nonlinear programming problems and second order symmetric duality are established.

Chapter 4, we discussed a class of functions called F-E-convex functions which are generalizations of F-convex and E-convex function. We proved the weak duality and strong duality results for the second order Mond-Weir type dual problem with cone constraints.

Contents

Chapter 1

1 Introduction

Optimization constitues a very important branch of modern applied mathematics. A variety of problems arising in the areas of engineering design, operation research, management science, computer science, financial engineering and economics. Optimization problems are sometimes difficult to solve. The difficulties may be in the objective function or the constraints. Some of the difficulties can be eliminated if we restrict our problems to convex feasible regions, so there is need for the notion of the convexity. Convexity is one of the most customarily used hypotheses in optimization theory. The convexity and concavity of sets and functions have been the matter of so many studies during the past one hundred years. The importance of convex functions is well known in optimization problems. Convex functions come up in many mathematical models used in economics, engineering, etc. The property of convexity is invariant with respect to certain operations and transformations. But for many mathematical models used in decision sciences, applied mathematics and engineering, the notion of convexity does no longer suffice. Hence, it is mandatory to explore the notion of convexity and to extend the validity of results to bigger class of optimization problems. This field has grown remarkably in different directions. In the recent years, several extensions have been considered for the classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson [13]. Initial work and results of Hanson inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. The study of convex functions begins in the context of real-valued functions of a real variable. Here we find a rich variety of results with significant applications. The optimization problems in which the objective function is a convex function and the constraint set is a convex set. Such problems are called Convex Optimization Problems. They includes linear programs, quadratic objective function with linear constraints. This is a very interesting concept. Numerical algorithms are maturing rapidly, providing accuracy in the solution set of the constrained convex problems. The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex set. Nonetheless it is a theory which touches almost all branches of mathematics.

1.1 Preliminaries

Notations

We will use the following notations throughout the thesis. We denote R^n as the n-dimensional Euclidean space, $R^1 = R$ the set of all real numbers, ${R^n}_+$ be non-negative orthant of R^n.
Let $f(x, y)$ be real-valued twice differentiable function defined on an open set in $R^n \times R^m$. Let $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ denote the partial derivative of f with respect to x, y respectively. Also let ${\nabla^2}_x f(x, y)$ denote the Hessian matrix evaluated at (x, y). $\nabla_{xy} f(x, y)$, $\nabla_{yx} f(x, y)$ and ${\nabla^2}_y f(x, y)$ are defined similarly.

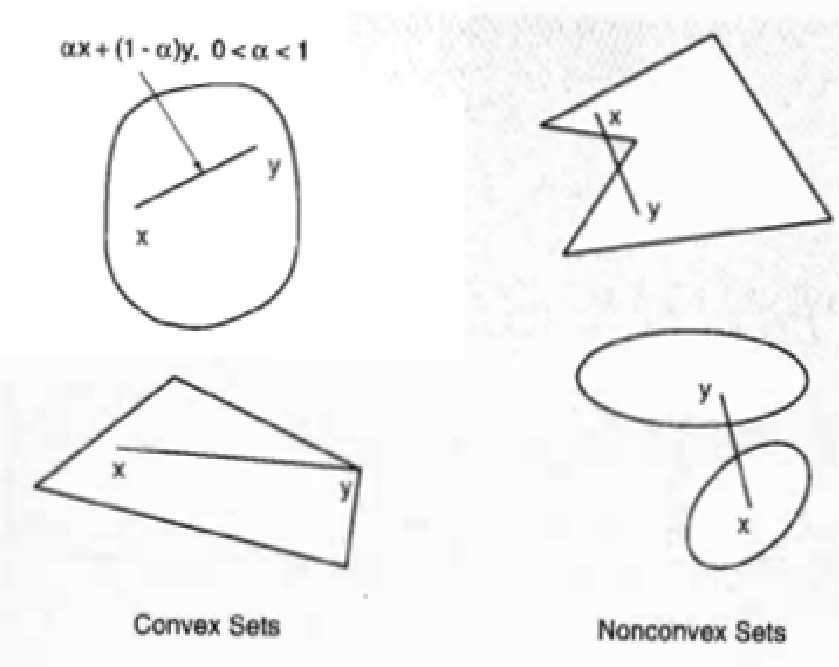
1.2 Convex functions and its extensions

The concept of convexity is of great importance in the study of optimization problems. Convexity is a central concept in non-linear programming. There is a vast literature on convex sets and convex functions. In this section, we shall recall the main properties of convex function and discuss about various concepts of generalized convex functions introduced in the literature in last thirty years for the purpose of weakening the limitations of convexity in the mathematical programming [25]. A discussion of generalized notions of convexity, including convex sets, convex functions, quasi-convexity, Pseudo-convexity.

1.2.1 Convex Sets

Convexity is essentially a one-dimensional concept, as it bases its definition on a line joining two arbitrary points. Convex sets and convex functions play an important role in the study of optimization models. We begin with the definition of a convex set:

A set $X \subseteq R^n$ is convex if $\forall\ x,\ y \in X,\ \alpha x + (1-\alpha)y \in X$, for any $\alpha \in [0,\ 1]$.
In another word, the line segment that connects any two elements lies entirely in the set.



Propositions

Convex sets in E^n satisfy the following relations:

1. If C is a convex set and β is a real number, the set $\beta C = \{x :\ x = \beta c,\ c \in C\}$ is convex.
2. If C and D are convex sets, then the set $C + D = \{x :\ x = c + d,\ c \in C,\ d \in D\}$ is convex.
3. Under linear transformation, the image of a convex set is convex.

4. Inverse image of a convex set under a linear transformation is convex.
5. The intersection of any collection of convex sets is convex.

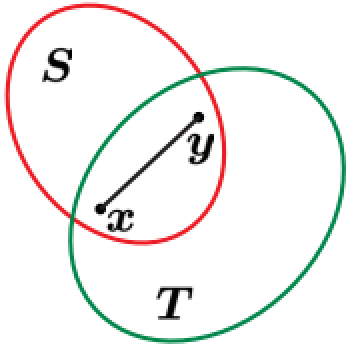


Illustration of intersection of two convex sets.

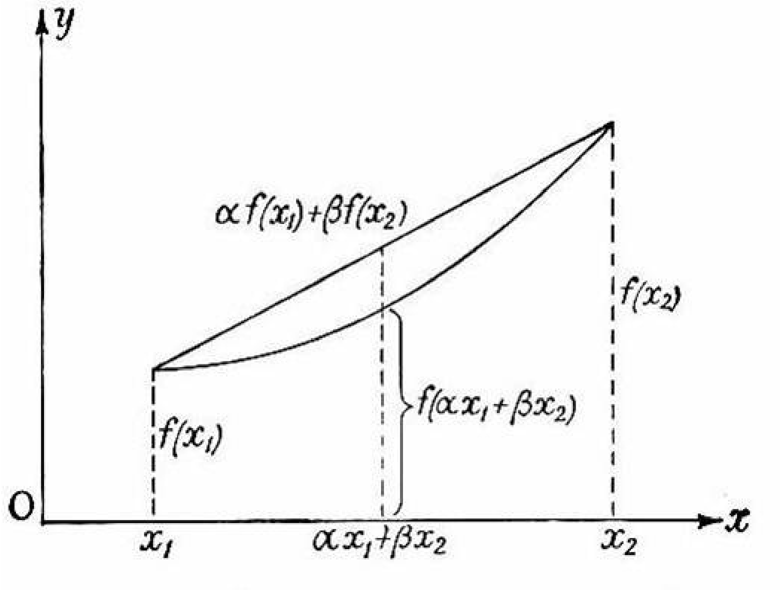
Examples Of Convex Sets

1. A half plane $S = \{x : \; P^t x \leq a\}$ in E^n is a convex set.
2. $S = \{(x_1, x_2) : \; x_1^2 + x_2^2 \; \leq 4\} \subset E_2$
   This set represents points on and inside a circle with center (0, 0) and radius 2.
3. The set $S = \{x : \; Ax \; \leq b\}$ is a convex set.

1.2.2 [7]Convex Function

Let $S \subseteq R^n$ be a convex set and $f : S \to R$, then f is called a convex function, if for all $x_1, \; x_2 \in S$ and for all $0 \leq \alpha, \beta \leq 1$ where $\alpha + \beta = 1$, we have

$$f(\alpha x_1 + \beta x_2) \leq \alpha f(x_1) + \beta f(x_2).$$



The Geometrical representation of Convex Function.

Properties

1. If the inequality is strict then the function f is called a *Strictly Convex Function.*
2. Every Linear function is both convex and concave function.
3. The Affine functions are those which are both convex and concave functions.
4. A function may be neither convex nor concave, e.g. $f(x) = \sin x,\ \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ or $f(x) = x^3,\ x \in R$.
5. The domain of a convex function has to be a convex set.
6. A convex function need not be differentiable; e.g. $f(x) = |x|,\ x \in R$ is a convex function but is not differentiable at $x = 0$.

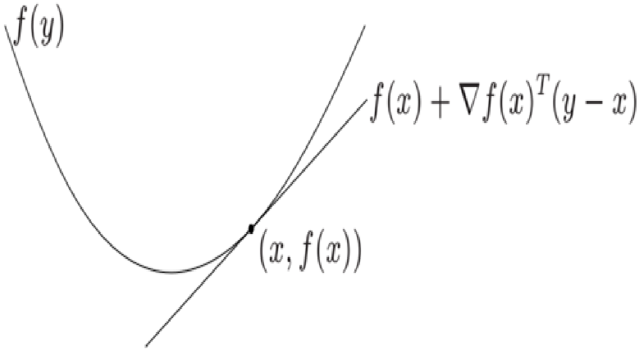
Examples

1. $f(x) = x^2,\ x \in R.$
2. $f(x) = |x|,\ x \in R.$
3. $f(x) = e^x,\ x \in R.$
4. $f(x) = exp(-x),\ x \in R.$
5. $f(x) = -\sqrt{1-x^2},\ -1 \leq x \leq 1.$

Differentiable Convex Functions

Assume that the function $f : R^n \to R$ is differentiable. Then f is convex, iff for every $x,\ y \in R^n$, the inequality

$$f(y) \geq f(x) + \nabla f(x)^T (y - x).$$



1.2.3 Quasiconvex function

Let $X \subset R^n$ a convex set. Then, the function $f : X \to R$ is said to be on X if for all $x, y \in X$ and for $\lambda \in [0, 1]$, we have

$$f(x) \leq f(y) \implies f(\lambda x + (1 - \lambda)y) \leq f(y),$$

or, equivalently in non-Euclidean form

$$f(\lambda x + (1 - \lambda)y) \leq max\{f(x), f(y)\},\ \forall x,\ y \in X,\ \forall\ \lambda \in [0, 1].$$

Every convex function is quasiconvex, but the converse is not true. For example $f(x) = x^3$ is quasiconvex but not convex.

1.2.4 Pseudoconvex function

Let $X \subset R^n$ a convex set. Then, the function $f : X \to R$ is said to be on X if f is differentiable at u and for all $x \in X$, we have

$$\nabla f(u)^T (x - u) \geq 0 \implies f(x) \geq f(u),$$

or equivalently, if

$$f(x) < f(u) \implies \nabla f(u)^T (x - u) < 0.$$

Every pseudoconvex function is quasiconvex function but converse is not true eg. $f(x) = x^3$ is quasiconvex but not pseudoconvex. Also every convex function is pseudoconvex function under differentiability but converse is not true For example, $f(x) = x^3 + x$ is pseudoconvex.

Youness [31] introduced the concepts of E-convex sets and E-convex functions. This kind of generalized convexity is based on the effect of an operator E on the sets and domain of definition of the functions. We study the extended notion of E-convexity to generalized E-convex functions and studied their properties. Yu-Ru Syau, E. Stanley Lee [28] introduced the concept of E-quasiconvex functions, strictly E-quasiconvex functions.

1.2.5 E-convex set

A set $M \subset R^n$ is said to be E-convex iff there is a map $E: R^n \to R^n$ such that $(1-\lambda)E(x)+\lambda E(y) \in M$, for each $x, y \in M$ and $0 \leq \lambda \leq 1$.

1.2.6 E-convex functions

A function $f : R^n \to R$ is said to be E-convex on a set $M \subset R^n$ iff there is a map $E : R^n \to R^n$ such that M is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(E(x)) + (1-\lambda)f(E(y)),$$

for each x, $y \in M$ and $0 \leq \lambda \leq 1$.

Hanson and Mond [14] introduced F-convex functions as a generalization of convex functions. Like convex functions, the set of all F-convex functions is closed under addition and non-negative scalar multiplication.

Let $f(x, y)$ be real-valued twice differentiable function defined on an open set in $R^n \times R^m$. Let $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ denote the partial derivative of f with respect to x, y respectively. Also let ${\nabla^2}_x f(x, y)$ denote the Hessian matrix evaluated at (x, y). $\nabla_{xy} f(x, y)$, $\nabla_{yx} f(x, y)$ and ${\nabla^2}_y f(x, y)$ are defined similarly.

1.2.7 Sublinear functional

A functional $F : X \times X \times R^n \to R$ where $X \subseteq R^n$ is said to be **sublinear** if for any x, $x^o \in X$

1. $F(x, x^o, a_1 + a_2) \leq F(x, x^o, a_1) + F(x, x^o, a_2)$, for any $a_1,\ a_2 \in R^n$,
2. $F(x, x^o, \alpha a) = \alpha F(x, x^o, a)$, for any $\alpha \in R_+$ and a $\in R^n$.

The value $F(x, x^o, a)$ will be denoted by $F_{x,x^o}(a)$. Thus from (2), it is clear that $F_{x,x^o}(0) = 0$.

1.2.8 F-convex function

A function $f(\cdot, y)$ is said to be F-convex at $\bar{x}$, for fixed $y \in Y$, if

$$f(x, y) - f(\bar{x}, y) \geq F_{x,\bar{x}}(\nabla_x f(\bar{x}, y)),$$

for all $x \in X$ and for some arbitrary sublinear functional F.

1.3 Review of literature work

1.3.1 Convex Optimization Problem

A convex optimization problem is a problem consisting of minimizing a convex function over a closed and convex set [4]. The purpose is to discuss the problem of minimizing a convex function over a convex set defined by a system of convex inequalities. The main result is the equivalence of this problem to the saddle-point problem. Assuming the differentiability of the functions concerned, the solution of the saddle-point problem is characterized by the Karush-Kuhn-Tucker conditions. Specifically, a convex problem is of the form

$$\begin{aligned} &\text{Minimize } f_0(x) \\ &\text{subject to } f_1(x) \le 0, \\ &\vdots \\ &f_m(x) \le 0. \end{aligned}$$

Objective and Constraint functions are convex: for $0 \le \theta \le 1$,

$$f_i(\theta x + (1-\theta)y) \le \theta f_i(x) + (1-\theta) f_i(y).$$

Properties: Convex programs have many useful properties:

1. The set of all feasible solutions is convex.
2. Any local minimum is a global minimum.
3. The Karush-Kuhn-Tucker optimality condition are sufficent for a minimum.
4. A minimum is unique if the objective function is strictly convex.

Example of a convex problem

$$\begin{aligned} &\text{Min} \quad x^2 + e^y \\ &\text{subject to } x^2 + y^2 \le 64, \\ &x + y \le 9, \\ &(x-10)^2 + y^2 \le 25, \\ &x \ge 0,\ y \ge 0. \end{aligned}$$

1.3.2 Duality in Mathematical Programming

Neumann [24] introduced the concept of duality in linear programming. He formulated the following dual pair and proved duality relations:

Primal Minimize $z(x) = c^T x$ subject to $Ax \ge b;\ x \ge 0.$

Dual Maximize $w(y) = b^T y$ subject to $A^T y \le c$, $y \ge 0.$

The above pair shows that if the primal problem is a minimization of a linear function over a set of linear constraints, then the dual is a maximization of another linear function over a set of linear constraints. Moreover, dual of the dual is again the primal problem. In mathematical programming, a pair of primal and dual problems is called **symmetric** if the dual of the dual is the primal problem.

Duality in nonlinear programming has also been developed extensively [15],[2],[12],[1],[29]. It originated with the duality results of quadratic programming given by Dennis [10]. Wolfe [29] and Mangasarian [16] gave duality results for convex primal and its dual program. Wolfe [29] formulated the following dual to Primal:

$$\text{Maximize} \quad \theta(y) + \mu^T g(y)$$
$$\text{subject to}$$

$$\nabla\theta(y) + \nabla\mu^T g(y) = 0,$$
$$y \in X,$$
$$\mu \geq 0.$$

and proved weak and strong duality theorems assuming θ and g to be convex. Mangasarian [16] pointed out that these duality relations do not hold under weaker convexity assumptions. Mond and Weir [22] introduced the following dual to Primal:

$$\text{Maximize} \quad \theta(y)$$
$$\text{subject to}$$

$$\nabla\theta(y) + \nabla\mu^T g(y) = 0,$$
$$\mu^T g(y) \geq 0,$$
$$y \in X,$$
$$\mu \geq 0.$$

and proved duality theorems by weakening the convexity assumptions θ and g to pseudoconvexity of θ and quasiconvexity of $\mu^T g$. They also discussed duality results for the problems involving both equality and inequality constraints.

1.3.3 Symmetric duality over cones

Bazaraa and Goode [3] generalized the symmetric dual programs, introduced by Dantzig et al. [9] to include the case where the inequality constraints are defined through closed convex cones and their polars. Such a formulation enables one to consider infinitely many constraints of the inequality type. The new formulation retains the symmetric duality of the original programs. They studied the following Wolfe's type symmetric dual pair with arbitrary cone constraints and established the duality results under convexity/concavity assumptions:

Primal

$$\text{Minimize} \quad G(x,y) - y^T \nabla_y G(x,y)$$
$$\text{subject to}$$

$$\nabla_y G(x,y) \in C_2^*,$$
$$(x,y) \in C_1 \times C_2,$$

Dual

$$\text{Maximize} \quad G(u,v) - u^T \nabla_x G(u,v)$$
$$\text{subject to}$$

$$-\nabla_x G(u,v) \in C_1^*,$$
$$(u,v) \in C_1 \times C_2,$$

where

1. C_1 and C_2 are closed convex cones in R^n and R^m respectively.
2. C_i^* is the polar of C_i for $i = 1, 2$.
3. $S_1 \subseteq R^n$, $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subseteq S_1 \times S_2$ and $G : S_1 \times S_2 \to R$ is a twice differentiable function.

Chandra and Kumar [8] studied the following Mond-Weir type symmetric dual programs with arbitrary cone constraints :

Primal

$$\text{Minimize} \quad G(x,y)$$
subject to

$$\nabla_y G(x,y) \in C_2^*,$$
$$y^T \nabla_y G(x,y) \geq 0,$$
$$x \in C_1,$$

Dual

$$\text{Maximize} \quad G(u,v)$$
subject to

$$-\nabla_x G(u,v) \in C_1^*,$$
$$u^T \nabla_x G(x,v) \geq 0,$$
$$v \in C_2.$$

and proved usual duality theorems under pseudoinvexity type assumptions.

1.3.4 Second order Symmetric duality

The study of second-order duality is significant due to the computational advantage over the first-order duality as it provides tighter bounds for the value of the objective function, when approximations are used.
Mangasarian [17] first formulated the second-order dual for a nonlinear programming problem and established duality results under some involved assumptions. Mond [23] considered the following second-order symmetric dual programs:

Primal

$$\text{Minimize} \quad G(x,y) - y^T \nabla_y G(x,y) - y^T \nabla_{yy} G(x,y) p - \tfrac{1}{2} p^T \nabla_{yy} G(x,y) p$$
subject to

$$\nabla_y G(x,y) + \nabla_{yy} G(x,y)p \leq 0,$$
$$x \geq 0.$$

Dual

$$\text{Maximize} \quad G(u,v) - u^T \nabla_x G(u,v) - u^T \nabla_{xx} G(u,v)q - \tfrac{1}{2} q^T \nabla_{xx} G(u,v)q$$
$$\text{subject to}$$

$$\nabla_x G(u,v) + \nabla_{xx} G(u,v)q \geq 0,$$
$$v \geq 0.$$

and reproved second-order duality theorems [17] under simpler assumptions. Bector and Chandra [5] studied the following pair of second-order Mond-Weir type symmetric dual programs and established duality theorems involving pseudoconvex functions:

Primal

$$\text{Minimize} \quad G(x,y) - \tfrac{1}{2} p^T \nabla_{yy} G(x,y)p$$
$$\text{subject to}$$

$$\nabla_y G(x,y) + \nabla_{yy} G(x,y)p \leq 0,$$
$$y^T \nabla_y G(x,y) + y^T \nabla_{yy} G(x,y)p \geq 0,$$
$$x \geq 0.$$

Dual

$$\text{Maximize} \quad G(u,v) - \tfrac{1}{2} q^T \nabla_{xx} G(u,v)q$$
$$\text{subject to}$$

$$\nabla_x G(u,v) + \nabla_{xx} G(u,v)q \geq 0,$$
$$u^T \nabla_x G(u,v) + u^T \nabla_{xx} G(u,v)q \geq 0,$$
$$v \geq 0.$$

1.4 Summary of the thesis

The aim of the present thesis is to study the duality results for pair of Wolfe and Mond-Weir type symmetric dual nonlinear programming problems under Generalized convexity assumptions.

In Chapter 2, we have reviewed the following Mond-Weir type duality in non-linear programming problem considered by E.A. Youness et al. [18] and X.M. Yang [30].

Let $E : R^n \to R^n$ be a mapping, $f : R^n \to R^n$ and $g_i : R^n \to R^m$, $i = 1, 2, ..., m$ are E-convex functions on R^n and M is an E-convex set, an E-convex programming problem is formulated as

Primal (P_E)

$$\begin{gathered}\text{Min } f(x)\\ \text{subject to}\\ x \in M = \{x \in R^n /\ g_i(x) \leq 0,\ i = 1, 2, ..., m\}.\end{gathered}$$

The **Mond-Weir type dual** for problem P_E can be consider as follows:
Dual (D_E)

$$\begin{gathered}\text{Max } Z_E(x, u)\\ \text{subject to}\\ \nabla Z_E(x, u) = \nabla(f \circ E)(x) + u\nabla(g_i \circ E)(x) = 0,\\ u\nabla(g_i \circ E)(x) \geq 0,\ u \geq 0,\ x \in M.\end{gathered}$$

where $Z_E(x, u) = (f \circ E)(x) + u(g_i \circ E)(x)$, f and g_i are E-convex functions on R^n.
By proving the duality theorem, we will clear the relationship between the P_E and D_E.

In Chapter 3, we have studied the duality results for pair of Wolfe and Mond-Weir type symmetric dual nonlinear programming problems considered by Chandra et al.[6] and S.K. Mishra [19].

Wolfe Type Symmetric Duality
Primal (WP)

$$\begin{gathered}\text{Minimize } \phi(x, y) = f(x, y) - y^T \nabla_y f(x, y)\\ \text{subject to}\end{gathered}$$

$$\begin{aligned}\nabla_y f(x, y) &\leq 0,\\ x &\geq 0.\end{aligned}$$

Dual(WD)

$$\begin{gathered}\text{Maximize } \psi(u, v) = f(u, v) - u^T \nabla_x f(u, v)\\ \text{subject to}\end{gathered}$$

$$\begin{aligned}\nabla_x f(u, v) &\leq 0,\\ u &\geq 0.\end{aligned}$$

Mond-Weir Type Symmetric Duality
Primal (MP)

$$\begin{gathered}\text{Minimize } f(x, y)\\ \text{subject to}\end{gathered}$$

$$\begin{aligned}\nabla_y f(x, y) &\leq 0,\\ y^T \nabla_y f(x, y) &\geq 0,\\ x &\geq 0.\end{aligned}$$

Dual(MD)

Maximize $f(u,v)$
subject to

$$\nabla_x f(u,v) \geq 0,$$
$$u^T \nabla_x f(u,v) \leq 0,$$
$$v \geq 0.$$

Second order Wolfe Type Symmetric Duality
Primal (WP)

$$\text{Minimize } \phi(x,y,p) = f(x,y) - y^T \nabla_y f(x,y) - y^T \nabla_{yy} f(x,y)p - \tfrac{1}{2} p^T \nabla_{yy} f(x,y)p$$
subject to

$$\nabla_y f(x,y) + \nabla_{yy} f(x,y)p \leq 0,$$
$$x \geq 0.$$

Dual(WD)

$$\text{Maximize } \psi(u,v) = f(u,v) - u^T \nabla_{xx} f(u,v)p_1 - \tfrac{1}{2} p^T{}_1 \nabla xxf(u,v)p_1$$
subject to

$$\nabla_x f(u,v) + \nabla_{xx} f(u,v)p_1 \geq 0,$$
$$v \geq 0.$$

Second order Mond-Weir Type Symmetric Duality
Primal (WP)

$$\text{Minimize } f(x,y) - \tfrac{1}{2} p^T \nabla_{yy} f(x,y)p$$
subject to

$$\nabla_y f(x,y) + \nabla_{yy} f(x,y)p \leq 0,$$
$$y^T \nabla_y f(x,y) + y^T \nabla_{yy} f(x,y)p \geq 0,$$
$$x \geq 0.$$

Dual(WD)

$$\text{Maximize } f(u,v) - \tfrac{1}{2} p^T{}_1 \nabla_{xx} f(u,v)p_1$$
subject to

$$\nabla_x f(u,v) + \nabla_{xx} f(u,v)p_1 \leq 0,$$
$$u^T \nabla_x f(u,v) + u^T \nabla_{xx} f(u,v)p_1 \leq 0,$$
$$v \geq 0.$$

In Chapter 4, we discussed a class of functions called F-E-convex functions which are generalizations of F-convex and E-convex function. We have formulated the following pair of second order Mond-Weir type dual problems with cone constraints and proved the usual duality results under second order F-E-convexity assumptions.

Mond-Weir type second order duality
(**PM**)

$$\text{Minimize } f(E(x))$$
$$\text{subject to}$$

$$M = \{x \in C_1 : -g(E(x)) \in C_2^*\}$$

(**DP**)

$$\text{Maximize } f(E(u)) - \tfrac{1}{2}p^T \nabla^2 f(E(u))p$$
$$\text{subject to}$$

$$\begin{aligned} \nabla \lambda f(E(u)) + \nabla^2 \lambda f(E(u))p + \nabla \gamma^T g(E(u)) + \nabla^2 \gamma^T g(E(u))p = 0, \\ g(E(u)) - \frac{1}{2}p^T \nabla^2 g(E(u))p \in C_2^*, \\ \gamma \in C_2,\ \lambda \geq 0, \end{aligned}$$

where

1. The function f and set M are E-convex with respect to the map $E : R^n \to R^n$, and f and g are twice differentiable functions.

2. C_1 and C_2 are closed convex cones in R^n and R^m with non-empty interiors respectively. C_1^* and C_2^* are polar cones of C_1 and C_2, respectively.

Chapter 2

2 E-convex sets, E-convex function and E-convex programming

2.1 Introduction

The concept of convexity and its various generalizations is important for quantitative and qualitative studies in operations research or applied mathematics. Youness [31] introduced the concepts of E-convex sets and E-convex functions which have applications in various branches of Mathematical Sciences. This kind of generalized convexity is based on the effect of an operator E on the sets and domain of definition of the functions. In this chapter, we discuss some basic properties of E-convex functions and the E-convex programming. A nonlinear programming problem, in which the objective function is E-convex and the constraint set is E-convex is called an E-convex programming problem. An E-convex programming problem is an extension of a convex programming problem.

2.2 Preliminaries

Notations and definitions

We will use the following notations throughout the thesis. We denote R^n as the n-dimensional Euclidean space, $R^1 = R$ the set of all real numbers, ${R^n}_+$ be non-negative orthant of R^n.
Let $f(x, y)$ be real-valued twice differentiable function defined on an open set in $R^n \times R^m$. Let $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ denote the partial derivative of f with respect to x, y respectively. Also let ${\nabla^2}_x f(x, y)$ denote the Hessian matrix evaluated at (x, y). $\nabla_{xy} f(x, y)$, $\nabla_{yx} f(x, y)$ and ${\nabla^2}_y f(x, y)$ are defined similarly.

Definition 2.2.1. A set $M \subset R^n$ is said to be E-convex iff there is a map $E : R^n \to R^n$ such that $(1 - \lambda)E(x) + \lambda E(y) \in M$, for each $x, y \in M$ and $0 \leq \lambda \leq 1$.

Proposition 2.2.1. Every convex set $M \subset R^n$ is E-convex.
The proof is clear by taking a map $E : R^n \to R^n$ as the identity map.

Proposition 2.2.2. If a set $M \subset R^n$ is E-convex, then $E(M) \subseteq M$.
Proof. Since M is E-convex, then for any x, $y \in M$ and $0 \leq \lambda \leq 1$, we have

$$(1 - \lambda)E(x) + \lambda E(y) \in M.$$

Thus for $\lambda = 1$, $E(y) \in M$. Hence, $E(M) \subseteq M$.

Proposition 2.2.3. Let $E(M)$ be convex and $E(M) \subseteq M$. Then, M is E-convex.
Proof. Assume that x, $y \in M$. Then, $E(x)$ and $E(y) \in E(M)$. Since $E(M)$ is convex, then for each $0 \leq \lambda \leq 1$, we have

$$(1 - \lambda)E(x) + \lambda E(y) \in E(M) \subseteq M.$$

Hence, M is E-convex.

Example 2.2.1. Let $E : R^2 \to R^2$ be defined as $E(x,\ y)$=(0, y). Then, the set
$M = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(2,1) + \lambda_3(0,3)\}$
$\cup \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(0,-3) + \lambda_3(-2,-1)\}$,
with $\lambda_i \geq 0$, $\sum\limits_{i=1}^{3} \lambda_i = 1$, is E-convex but is not convex ; see Fig.1.

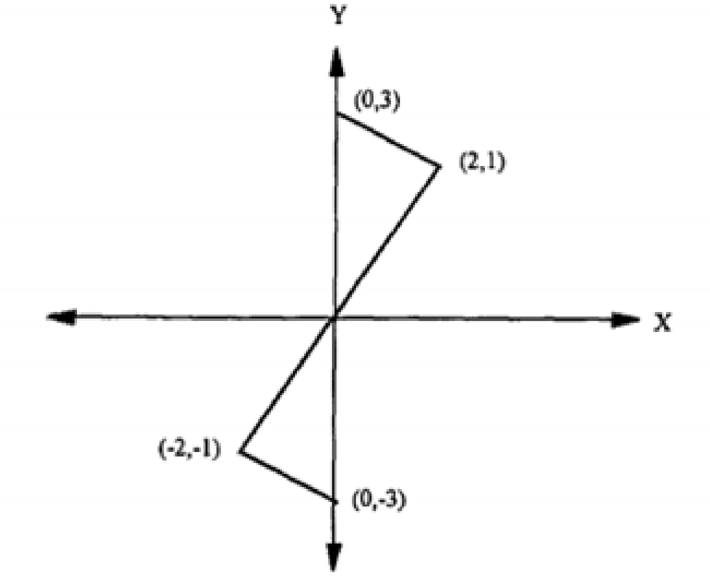


Fig.1. The set M is E-convex but not convex.

Example 2.2.2. Let $E : R^2 \to R^2$ be defined as

$$E(x,y) = (2y/3 - x/3, y/3 + 4x/3).$$

Consider the set M given in Example 2.2.1. It is clear that $E(M) = M$, which is not convex and is not E-convex, because

$$\lambda E(0,3) + (1-\lambda)E(-2,-1) \notin M,$$

for some $0 < \lambda < 1$.

Proposition 2.2.4. Let M_1 and M_2 be two E-convex sets, then $M_1 \cap M_2$ is an E-convex set.

Remark 2.2.1. If Let M_1 and M_2 be two E-convex sets, then $M_1 \cup M_2$ is not necessarily E-convex set; We can show this by the following example.

Example 2.2.3. Consider the map $E : R^2 \to R^2$ given in Example 2.2.2., and consider the two sets
$M_1 = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(2,1) + \lambda_3(0,3)\}$,

$$M_2 = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(0,-3) + \lambda_3(-2,-1)\}$$

with $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$, $\sum\limits_{i=1}^{3} \lambda_i = 1$. The two sets M_1 and M_2 are E-convex, but $M_1 \cup M_2$ is not E-convex; see Figs. 2 and 3.

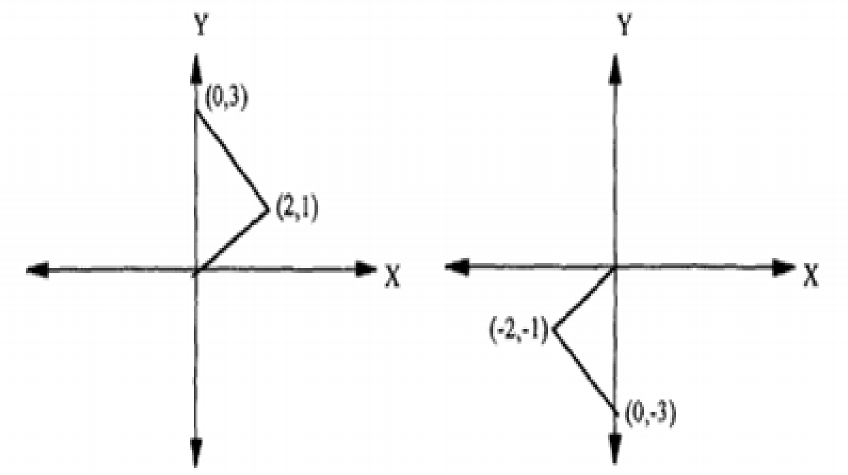


Fig.2. Example 2.2.3. The sets M_1 and M_2 are E-convex .

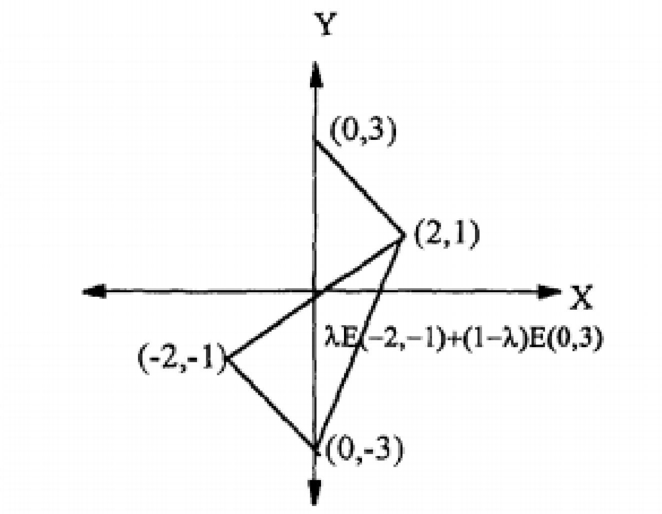


Fig.3. Example 2.2.3. The set $M_1 \cup M_2$ is not E-convex.

Lemma 2.2.1. Let $M \subset R^n$ be an E_1- and E_2-convex set. Then, M is an $(E_1 \circ E_2)$ and $(E_2 \circ E_1)$-convex set.

Proof. Assume that x, $y \in M$, and let

$$\lambda(E_1 \circ E_2)x + (1-\lambda)(E_1 \circ E_2)y \notin M, \qquad \mathit{for} \quad 0 \leq \lambda \leq 1,$$

that is,

$$\lambda E_1(E_2(x)) + (1-\lambda)E_1(E_2(y)) \notin M, \qquad \mathit{for} \quad 0 \leq \lambda \leq 1.$$

Since, from Proposition 2.2.2., $E_2(x)$ and $E_2(y) \in M$, then

$$\lambda(E_1(E_2(x))) + (1-\lambda)(E_1(E_2(y))) \notin M,$$

contradicting the E_1-convexity of M. Hence, M is an $(E_1 \circ E_2)$-convex set. Similarly, M is an $(E_2 \circ E_1)$-convex set.

Lemma 2.2.2. Let $E : R^n \to R^n$ be linear map and M_1 and $M_2 \subset R^n$ be E-convex sets, then $M_1 + M_2$ is an E-convex set.
Proof. Let $(p+q)$, $(x+y) \in M_1 + M_2$, where p, $x \in M_1$ and q, $y \in M_2$ Then, for $0 \leq \lambda \leq 1$, we have,

$$\lambda E(p+q) + (1-\lambda)E(x+y) = (\lambda E(p) + (1-\lambda)E(x)) + (\lambda E(q) + (1-\lambda)E(y)) \in M_1 + M_2.$$

Thus, $M_1 + M_2$ is an E-convex set.

Definition 2.2.2. A function $f : R^n \to R$ is said to be E-convex on a set $M \subset R^n$ iff there is a map $E : R^n \to R^n$ such that M is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(E(x)) + (1-\lambda) f(E(y)),$$

for each x, $y \in M$ and $0 \leq \lambda \leq 1$.

If $f(\lambda E(x) + (1-\lambda)E(y)) \geq \lambda f(E(x)) + (1-\lambda) f(E(y))$, then f is called E-concave on M. When the inequality is stricted, then f is called strictly E-convex function. Every strictly E-convex function is E-convex.

Remark 2.2.2. Every convex function f on a convex set M is an E-convex function, where E is the Identity map.
The following two examples give E-convex functions which are not convex.

Example 2.2.4. Consider $E : R^2 \to R^2$ be defined as $E(x, y)$=(0, y). $M \subset R^n$ is given as $M = \{(x, y) \in R^2 : (x, y) = \lambda_1(0,0) + \lambda_2(2,1) + \lambda_3(0,3)\}$ with $\lambda_i \geq 0$, $\sum\limits_{i=1}^{3} \lambda_i = 1$. The function $f : R^2 \to R$ defined by

$$f(x, y) = \begin{cases} x^3, & \text{if } y < 1, \\ xy^3, & \text{if } y \geq 1, \end{cases}$$

is E-convex on M but is not convex.

Example 2.2.5. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -x, & \text{if } x \leq 0, \end{cases}$$

and let $E : R \to R$ be defined as $E(x) = -x^2$. Then, R is an E-convex set and f is E-convex but is not convex.

Definition 2.2.3. Let S $\subset R^n \times R$ and $E : R^n \to R^n$. A set S is said to be E-convex if (x, α), $(y, \beta) \in S$ imply

$$(\lambda E(x) + (1-\lambda)E(y), \lambda\alpha + (1-\lambda)\beta) \in S, \qquad 0 \leq \lambda \leq 1.$$

Now, we define an E-**epigraph**, $E - e(f)$ of f as follow :

$$E - e(f) = \{(x, \alpha) : x \in M, \quad \alpha \in R, \quad f(E(x)) \leq \infty\}.$$

The following Theorem gives a characterization of an E-convex function f in terms of its E- epigraph $E - e(f)$.

Theorem 2.2.1. (E.A. Youness) A numerical function f defined on an E-convex set $M \subset R^n$ is E-convex on M iff $E - e(f)$ is E-convex in $R^n \times R$.

Remark. Sheiba Grace,J. and Thangavelu,P [26] shown that theorem 2.2.1. of Youness is incorrect by giving some counter examples. In the above theorem, the sufficent part namely " If the E-epigraph of a numerical function f defined on an E-convex set $M \subseteq R^n$ is E-convex in $R^n \times R$, then f is E-convex on M " is true. However the necessary part namely " If a numerical function f defined on M , then the E-epigraph of f is E-convex in $R^n \times R$ is not true "are shown in the following example.

Example 2.2.5. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

and let $E : R \to R$ be defined as $E(x) = -x^2$.
then R is an E-convex set and f is E-convex on R. The E-epigraph,

$$\begin{aligned} E - e(f) &= \{(x, \alpha) : x \in M, \quad \alpha \in R, \quad f(E(x)) \leq \alpha\} \\ &= \{(x, \alpha) : x \in M, \quad \alpha \in R, \quad f(-x^2) \leq \alpha\} \\ &= \{(x, \alpha) : x \in M, \quad \alpha \in R, \quad x^2 \leq \alpha\} \end{aligned}$$

Clearly (1 , 2), (-3 , 9) $\in$ E-epigraph of f. For $\lambda = 1/2$, Now $\lambda E(1) + (1-\lambda)E(-3) = -5$ and $2\lambda + 9(1-\lambda) = 5.5$. As (-5)2 = 25 > 5.5, (-5 , 5.5) $\notin$ E-epigraph of f.
we see that (λ E(1)+(1 - λ)E(-3), 2λ +9(1 - λ)) $\notin$ E-epigraph of f. Therefore the E-epigraph of f is not E-convex in $R \times R$.

Rectification in the above theorem
In this theorem, the necessary part of theorem is modified and its validity is established.

Theorem 2.2.2. Let $M \subseteq R^n$ be an E-convex set. Suppose $E : R^n \to R^n$ is linear and idempotent. Let a numerical function f defined on M be E-convex on M. Then the E-epigraph of f is E-convex in $R^n \times R$.

Proof. Let (x, α) , $(y, \beta) \in$ E-epigraph of f. Then $f(E(x)) \leq \alpha$; $f(E(y)) \leq \beta$.

$$f(E(\lambda E(x) + (1-\lambda)E(y))) = f(\lambda E^2(x) + (1-\lambda)E^2(y))$$

$$= f(\lambda E(x) + (1-\lambda)E(y))$$
$$\leq \lambda f(E(x)) + (1-\lambda) f(E(y))$$
$$\leq \lambda\alpha + (1-\lambda)\beta.$$

Then using the definition of E-epigraph, $(\lambda E(x) + (1-\lambda)E(y)\ ,\ \lambda\alpha + (1-\lambda)\beta) \in E$-epigraph of f. This proves that the E-epigraph of f is E-convex. This completes the proof.

2.3 Generalization of E-convex Functions

Recently, it was shown by Youness [31] that many results for convex sets and convex functions actually hold for a wider class of sets and functions, called E-convex sets and E-convex functions. The initial results of Youness inspired a great deal of subsequent work which has greatly expanded the role of E-convexity in optimization theory. Especially, much attention has been paid on extending the notion of E-convexity to the new classes of generalized E-convex functions and studied their properties [32]. Motivated both by earlier research works and by the importance of the concept of convexity, Yu-Ru Syau, E. Stanley Lee introducd the concept of E-quasiconvex functions, strictly E-quasiconvex functions [28],[27], E-pseudoconvex functions and strictly E-pseudoconvex functions.

Definition 2.3.1. A real-valued function $f : M \to R^1$, $E : R^n \to R^n$ are differentiable in an open E-convex set $X \subset R^n$, is said to be E-quasiconvex if

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq max\{f(E(x)), f(E(y))\}$$

for all x , $y \in M$ and $\lambda \in [0,1]$; and strictly E-quasiconvex if strict inequality holds for
all x , $y \in M$, $E(x) \neq E(y)$ and $\lambda \in (0,1)$.

Proposition 2.3.1. Every strictly E-quasiconvex function is E-quasiconvex.

Definition 2.3.2. A real-valued function $f : M \to R^1$, $E : R^n \to R^n$ are differentiable in an open E-convex set $X \subset R^n$, is said to be E-pseudoconvex function if

$$\nabla f(E(y))^T(E(x) - E(y)) \geq 0 \Rightarrow f(E(x)) \geq f(E(y)), \quad \forall\ x,\ y \in X.$$

Example 2.3.1. Let $E(x) = |x|$ and

$$f(x) = \begin{cases} 1 - \sqrt{1-(x-1)^2}, & \text{if } 1 \leq x < 2, \\ 0, & \text{if } -1 < x < 1, \\ 1 - \sqrt{1-(x+1)^2}, & \text{if } -2 < x \leq -1, \end{cases}$$

Then f is a E-pseudoconvex function on the interval (-2 , 2). In fact, f is also a E-quasiconvex function.
It is obviously that E-convex functions are E-pseudoconvex. But, the reverse is not correct.

Example 2.3.2. Assume that $f(x) = \cos x,\ x \in [0, 2\pi]$, and

$$E(x) = \begin{cases} x, & \text{if } 0 \le x \le \pi, \\ x - \pi, & \text{if } \pi < x \le 2\pi, \end{cases}$$

Then f is E-pseudoconvex on the interval $[0, 2\pi]$. However, f is not E-convex on the interval $[0, \frac{\pi}{2}]$.

Lemma 2.3.1. Let $f : R^n \to R^1$, $E : R^n \to R^n$ are differentiable in an open E-convex set $X \subset R^n$. If f is E-pseudoconvex then f is strictly E-quasiconvex and E-quasiconvex.
Proof. Assuming that f is not strictly E-quasiconvex on X, then there exist $x, y \in X$, $f(E(x)) < f(E(y))$ and $\lambda \in (0, 1)$ such that

$$f(\lambda E(x) + (1 - \lambda)E(y)) \ge f(E(y)) > f(E(x)),$$

Let $E(u) = \lambda E(x) + (1 - \lambda)E(y)$. It follows from the E-pseudoconvexity of f that

$$\nabla f(E(u))^T (E(x) - E(u)) < 0,$$

Since $E(x) - E(u) = -\frac{1-\lambda}{\lambda}(E(y) - E(u))$, we get

$$\nabla f(E(u))^T (E(y) - E(u)) > 0,$$

By the E-pseudoconvexity of f again, we have

$$f(E(y)) \ge f(E(u)),$$

Hence $f(E(y)) = f(E(u))$.
Noticing that

$$0 < \nabla f(E(u))^T (E(y) - E(u)) = \lim_{t \to 0^+} \frac{f(E(x) + t(E(y) - E(u))) - f(E(u))}{t}.$$

For small enough $1 > t > 0$, we get

$$f(E(u) + t(E(y) - E(u))) > f(E(u)) = f(E(y)).$$

Let $E(v) = tE(y) + (1 - t)E(u)$, by the E-pseudoconvexity of f again, it follows that

$$\nabla f(E(v))^T (E(y) - E(v)) < 0,$$

$$\nabla f(E(v))^T (E(u) - E(v)) < 0.$$

Since $E(u) - E(v) = \frac{t}{1-t}(E(v) - E(y))$, the above two inequalities can not hold simultaneously. Thus, we get a contradiction. So, f is strictly E-quasiconvex. Finally, we get f is also E-quasiconvex. This completes the proof.

2.4 E-convex Programming

A nonlinear programming problem, in which the objective function is E-convex and the constraint set is E-convex is called an E-convex programming problem. An E-convex programming problem is an extension of a convex programming problem [18]. Let $E : R^n \to R^n$ be a mapping, $f : R^n \to R^n$ and $g_i : R^n \to R^m,\ i = 1, 2, ..., m$ are E-convex functions on R^n and M is an E-convex set, an E-convex programming problem is formulated as

Primal (P_E)

$$\begin{gathered}\text{Min } f(x)\\ \text{subject to}\\ x \in M = \{x \in R^n /\ g_i(x) \leq 0,\ i = 1, 2, ..., m\}.\end{gathered}$$

2.5 Duality of E-convex Programming

Given a nonlinear programming problem, there is another nonlinear problem closely associated with it. The former is called primal problem, and the latter is called the Lagrangian dual problem. Under certain convexity assumptions, the primal and dual problems have equal optimal objective values, and hence it is possible to solve the primal problem indirectly by solving the dual problem. The Mond-Weir type dual for problem P_E can be consider as follows:

Dual (D_E)

$$\begin{gathered}\max\ Z_E(x, u)\\ \text{subject to}\\ \nabla Z_E(x, u) = \nabla(f \circ E)(x) + u\nabla(g_i \circ E)(x) = 0,\\ u\nabla(g_i \circ E)(x) \geq 0,\ u \geq 0,\ x \in M.\end{gathered}$$

where $Z_E(x, u) = (f \circ E)(x) + u(g_i \circ E)(x)$, f and g_i are E-convex functions on R^n.
By proving the duality theorem, we will clear the relationship between the P_E and D_E.

Theorem 2.5.1.(Weak Duality) Let x be a feasible solution to the problem P_E , $E : R^n \to R^n$ is a mapping, let f and g are E-convex functions and differentiable, also let (y, u) be a feasible solution to the problem D_E, then $f(E(x)) \geq Z_E(y, u)$.

Proof. Since f is E-convex and differentiable at x, then

$$f(E(x)) = f(E(y)) + (E(x) - E(y))\nabla f(E(y)),$$

$$f(E(x)) = f(E(y)) - (E(x) - E(y))u\nabla g_i(E(y)).$$

where $\nabla(f \circ E)(y) = -u\nabla(g_i \circ E)(y)$.
Since $g_i,\ i = 1, 2, 3, ..., m$ is E-convex and differentiable at x, then

$$f(E(x)) = f(E(y)) + u[g_i(E(y)) - g_i(E(x))],$$

and $f(E(x)) \geq f(E(y)) + ug_i(E(y))$, where $u \geq 0$, $(g_i \circ E)(x) \leq 0$.
Thus $(f \circ E)(x) \geq Z_E(y, u)$.

Theorem 2.5.2. (Strong Duality) Let $\bar{x}$ be an optimal solution of the problem P_E, and assume

that the Kuhn-Tucker constraint qualification is satisfied. Then there exists $\bar{u} \in R^m$, such that $(\bar{x}, \bar{u})$ be an optimal solution for D_E and the objective values of P_E and D_E are equal.
Proof. Since the Kuhn-Tucker constraint qualification is satisfied, then there exists $\bar{u} \in R^m$ such that

$$\nabla(f \circ E)(\bar{x}) + \bar{u}\nabla(g_i \circ E)(\bar{x}) = 0,$$

$$\bar{u}(g_i \circ E)(\bar{x}) = 0, \quad \bar{u} \geq 0.$$

which it yields that $(\bar{x}, \bar{u})$ is feasible solution for the problem D_E.
Since $\bar{u}\nabla(g_i \circ E)(\bar{x}) = 0$, then the objective values of P_E and D_E are equal.
Now, let $(\bar{x}, \bar{u})$ is not optimal solution for D_E, then there exists (x', u') be a feasible for D_E such that

$$(f \circ E)(x') + u'(g_i \circ E)(x') > (f \circ E)(\bar{x}) + \bar{u}(g_i \circ E)(\bar{x}),$$

Since $u'(g_i \circ E)(\bar{x}) = 0$, then $(f \circ E)(x') + u'(g_i \circ E)(x') > (f \circ E)(\bar{x})$. This contradicts the weak duality theorem. Hence $(\bar{x}, \bar{u})$ is an optimal solution for problem D_E.

CHAPTER 3

3 Duality in Mathematical Programming with F-convex function

3.1 Introduction

Hanson and Mond [14] introduced F-convex functions as a generalization of convex functions. Like convex functions, the set of all F-convex functions is closed under addition and non-negative scalar multiplication. Symmetric duality in mathematical programming was introduced by Dorn [11] who also defined a program and its dual to be symmetric if the dual of the dual is the original problem. Dantzig et al. [9], Mond [20] and Bazaraa and Goode [3] formulated a pair of symmetric dual programs involving a scalar function $f(x, y)$, $x \in R^n$, $y \in R^m$ under the condition that $f(\cdot, y)$ is convex for each y and $f(x, \cdot)$ is concave for each x. Mond and Weir [21] presented a different pair of symmetric dual nonlinear programs which allows the weakening of convexity-concavity condition for $f(x, y)$ to pseudoconvexity-pseudoconcavity.

In this chapter, we study second order symmetric duality under second order F-convexity, F-concavity/second order F-pseudoconvexity F-pseudoconcavity for second order Wolfe and Mond-Weir type models, respectively.

3.2 Preliminaries and Notations

Let R^n denote the n-dimensional Euclidean space and ${R^n}_+$ be its non-negative orthant. We now introduce the following definitions on the lines of Hanson, Mond and Preda.

Definition 3.2.1. A functional $F : X \times X \times R^n \to R$ where $X \subseteq R^n$ is said to be sublinear if for any x, $x^o \in X$

1. $F(x, x^o, a_1 + a_2) \leq F(x, x^o, a_1) + F(x, x^o, a_2)$, for any $a_1,\ a_2 \in R^n$,
2. $F(x, x^o, \alpha a) = \alpha F(x, x^o, a)$, for any $\alpha \in R_+$ and a $\in R^n$.

The value $F(x, x^o, a)$ will be denoted by $F_{x,x^o}(a)$. Thus from (b), it is clear that $F_{x,x^o}(0) = 0$. Let $f(x, y)$ be real-valued twice differentiable function defined on an open set in $R^n \times R^m$. Let $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ denote the partial derivative of f with respect to x, y respectively. Also let ${\nabla^2}_x f(x, y)$ denote the Hessian matrix evaluated at (x, y). $\nabla_{xy} f(x, y)$, $\nabla_{yx} f(x, y)$ and ${\nabla^2}_y f(x, y)$ are defined similarly.

Definition 3.2.2. A function $f(\cdot, y)$ is said to be F-convex at $\bar{x}$, for fixed $y \in Y$, if

$$f(x, y) - f(\bar{x}, y) \geq F_{x,\bar{x}}(\nabla_x f(\bar{x}, y)),$$

for all $x \in X$ and for some arbitrary sublinear functional F.

Definition 3.2.3. A function $f(x,\cdot)$ is said to be F-concave at $\bar{y}$, for fixed $x \in X$, if

$$f(x,\bar{y}) - f(x,y) \geq F_{y,\bar{y}}(-\nabla_y f(x,\bar{y})),$$

for all $y \in Y$ and for some arbitrary sublinear functional F.

Definition 3.2.4. A function $f(\cdot,y)$ is said to be F-pseudoconvex at $\bar{x}$, for fixed $y \in Y$, if

$$F_{x,\bar{x}}(\nabla_x f(\bar{x},y)) \geq 0 \Rightarrow f(x,y) \geq f(\bar{x},y),$$

for all $x \in X$ and for some arbitrary sublinear functional F.

Definition 3.2.5. A function $f(x,\cdot)$ is said to be F-pseudoconcave at $\bar{y}$, for fixed $x \in X$, if

$$F_{y,\bar{y}}(-\nabla_y f(x,\bar{y})) \geq 0 \Rightarrow f(x,y) \leq f(x,\bar{y}),$$

for all $y \in Y$ and for some arbitrary sublinear functional F.

Zhang and Mond [33] introduced second order F-convex functions as a generalization of F-convex functions [14] and obtained various second order duality results for multiobjective nonlinear programming problems under the assumption of second order F-convexity.

Definition 3.2.6. A function $f(\cdot,y)$ is said to be second order F-convex at $\bar{x}$, for fixed $y \in Y$, if

$$f(x,y) - f(\bar{x},y) + \frac{1}{2}p(x,\bar{x})\nabla_{xx}f(\bar{x},y)p(x,\bar{x}) \geq F(x,\bar{x};\nabla_x f(\bar{x},y) + \nabla_{xx}f(\bar{x},y)p_1(x,\bar{x})),$$

for all $x \in X$ and some arbitrary sublinear functional F.

Definition 3.2.7. A function $f(x,\cdot)$ is said to be second order F-concave at $\bar{y}$, for fixed $x \in X$, if

$$f(x,\bar{y}) - f(x,y) - \frac{1}{2}p(y,\bar{y})\nabla_{yy}f(x,\bar{y})p(y,\bar{y}) \geq F(y,\bar{y};-\nabla_y f(x,\bar{y}) - \nabla_{yy}f(x,\bar{y})p(y,\bar{y})),$$

for all $y \in Y$ and some arbitrary sublinear functional F.

Definition 3.2.8. A function $f(\cdot,y)$ is said to be second order F-pseudoconvex at $\bar{x}$, for fixed $y \in Y$, if

$$F(x,\bar{x};\nabla_x f(\bar{x},y) + \nabla_{xx}f(\bar{x},y)p(x,\bar{x})) \geq 0 \Rightarrow f(x,y) \geq f(\bar{x},y) - \frac{1}{2}p(x,\bar{x})\nabla_{xx}f(\bar{x},y)p(x,\bar{x}),$$

for all $x \in X$ and some arbitrary sublinear functional F.

Defintion 3.2.9. A function $f(x,\cdot)$ is said to be second order F-pseudoconcave at $\bar{y}$, for fixed $x \in X$, if

$$F(y,\bar{y};-\nabla_y f(x,\bar{y}) - \nabla_{yy}f(x,\bar{y})p(y,\bar{y})) \geq 0 \Rightarrow f(x,y) \leq f(x,\bar{y}) - \frac{1}{2}p(y,\bar{y})\nabla_{yy}f(x,\bar{y})p(y,\bar{y}),$$

for all $y \in Y$ and some arbitrary sublinear functional F.

3.3 Wolfe Type Symmetric Duality

We consider the following pair of Wolfe type problems, establish weak and strong duality theorems. Primal (WP)

$$\text{Minimize } \phi(x,y) = f(x,y) - y^T\nabla_y f(x,y)$$
$$\text{subject to}$$

$$\nabla_y f(x,y) \leq 0,$$
$$x \geq 0.$$

Dual(WD)

$$\text{Maximize } \psi(u,v) = f(u,v) - u^T\nabla_x f(u,v)$$
$$\text{subject to}$$

$$\nabla_x f(u,v) \leq 0,$$
$$u \geq 0.$$

Theorem 3.3.1. Let *f(x,y)* be F-convex in x and F-concave in y and for all feasible (x,y,u,v) to (WP) and (WD):

- $F_{x,u}(\zeta) + u^T\zeta \geq 0$, for $\zeta \in {R_+}^n$, and
- $F_{v,y}(\eta) + y^T\eta \geq 0$, for $\eta \in {R_+}^m$,

then,

$$Inf(WP) \geq Sup(WD).$$

Proof. Because of the F-Convexity and F-Concavity of the function *f*, the following inequalities hold

$$f(x,v) - f(u,v) \geq F_{x,u}(\nabla_x f(u,v)),$$

and

$$f(x,y) - f(x,v) \geq F_{v,y}(-\nabla_y f(x,y)).$$

On adding these, we get

$$f(x,y) - f(u,v) \geq F_{x,u}(\nabla_x f(u,v)) + F_{v,y}(-\nabla_y f(x,y)).$$

Now taking $\zeta = \nabla_x f(u,v) \in {R^n}_+$ and $\eta = -\nabla_y f(x,y) \in {R^m}_+$ and using the hypothesis of the theorem, we have

$$f_{x,u}(\nabla_x f(u,v)) \geq -u^T\nabla_x f(u,v),$$

and

$$f_{v,y}(-\nabla_y f(x,y)) \geq y^T\nabla_y f(x,y).$$

These inequalities yield

$$f(x,y) - f(u,v) \geq -u^T\nabla_x f(u,v) + y^T\nabla_y f(x,y),$$

and hence

$$Inf(WP) \geq Sup(WD).$$

3.4 Mond-Weir Type Symmetric Duality

We consider the following pair of problems, formuated by Mond and Weir.

Primal (MP)

$$\begin{gathered}\text{Minimize } f(x,y)\\ \text{subject to}\end{gathered}$$

$$\begin{aligned}\nabla_y f(x,y) &\leq 0,\\ y^T\nabla_y f(x,y) &\geq 0,\\ x &\geq 0.\end{aligned}$$

Dual(MD)

$$\begin{gathered}\text{Maximize } f(u,v)\\ \text{subject to}\end{gathered}$$

$$\begin{aligned}\nabla_x f(u,v) &\geq 0,\\ u^T\nabla_x f(u,v) &\leq 0,\\ v &\geq 0.\end{aligned}$$

Theorem 3.4.1. Let $f(\cdot,y)$ be F-Pseudoconvex and $f(x,\cdot)$ be F-Pseudoconcave and for all feasible (x,y,u,v) to (MP) and (MD):

- $F_{x,u}(\zeta)+u^T\zeta \geq 0$, for $\zeta \in {R^n}_+$,
- $F_{v,y}(\eta)+y^T\eta \geq 0$, for $\eta \in {R^m}_+$.

  then,

$$Inf(MP) \geq Sup(MD).$$

Proof. On taking $\zeta = \nabla_x f(u,v) \in {R^n}_+$, we have

$$F_{x,u}(\nabla_x f(u,v)) \geq -u^T\nabla_x f(u,v) \geq 0,$$

which by pseudoconvexity of $f(\cdot,y)$ yields

$$f(x,v) \geq f(u,v).$$

On taking $\eta = -\nabla_y f(x,y) \in {R^m}_+$, we have

$$F_{v,y}(-\nabla_y f(x,y)) + y^T(-\nabla_y f(x,y)) \geq 0,$$

which by pseudoconcavity of $f(x,\cdot)$ gives

$$f(x,y) \geq f(x,v).$$

therefore

$$f(x,y) \geq f(u,v),$$

Hence,

$$Inf(MP) \geq Sup(MD).$$

Now we will study second order symmetric duality under second order F-convexity F-concavity/ second order F-pseudoconvexity F-pseudoconcavity for second order Wolfe and Mond-Weir type models, respectively.

3.5 Second order Wolfe Type Symmetric Program

We consider the following pair of second order Wolfe type problems and establish weak and strong duality theorems.
Primal (WP)

$$\text{Minimize } \phi(x,y,p) = f(x,y) - y^T\nabla_y f(x,y) - y^T\nabla_{yy} f(x,y)p - \tfrac{1}{2}p^T\nabla_{yy} f(x,y)p$$
$$\text{subject to}$$

$$\nabla_y f(x,y) + \nabla_{yy} f(x,y)p \leq 0,$$
$$x \geq 0.$$

Dual(WD)

$$\text{Maximize } \psi(u,v) = f(u,v) - u^T\nabla_{xx} f(u,v)p_1 - \tfrac{1}{2}p^T{}_1\nabla xxf(u,v)p_1$$
$$\text{subject to}$$

$$\nabla_x f(u,v) + \nabla_{xx} f(u,v)p_1 \geq 0,$$
$$v \geq 0.$$

Theorem 3.5.1. (Weak duality) Let $f(x,y)$ be second order F-convex in x and second order F-concave in y and for all feasible (x,y,u,v) to (WP) and (WD):

- $F(x,u;\zeta_1+\zeta_2) + u^T\zeta_1 + u^T\zeta_2 \geq 0$, for $\zeta_1 \in R^n{}_+$, $\zeta_2 \in R^n{}_+$,
- $F(v,y;\eta_1+\eta_2) + y^T\eta_1 + y^T\eta_2 \geq 0$, for $\eta_1 \in R^m{}_+$, $\eta_2 \in R^m{}_+$, then

$$Inf(WP) \geq Sup(WD).$$

Proof. Because of the second order F-convexity and second order F-concavity of the function f, the following inequalities hold:

$$f(x,v) - f(u,v) + \frac{1}{2}p_1(x,u)\nabla_{xx} f(u,v)p_1(x,u) \geq F(x,u;\nabla_x f(u,v) + \nabla_{xx} f(u,v)p_1(x,u)),$$

and

$$f(x,y) - f(x,v) - \frac{1}{2}p(v,y)\nabla_{yy} f(x,y)p(v,y) \geq F(v,y;\nabla_y f(x,y) - \nabla_{yy} f(x,y)p(v,y)).$$

on adding these, we get

$$f(x,y) - f(u,v) \geq F(x,u;\nabla_x f(u,v) + \nabla_{xx} f(u,v)p_1(x,u)) + F(v,y;-\nabla_y f(x,y) - \nabla_{yy} f(x,y)p(x,y))$$
$$-\frac{1}{2}p_1(x,u)\nabla_{xx} f(u,v)p_1(x,u) + \frac{1}{2}p(v,y)\nabla_{yy} f(x,y)p(v,y).$$

Now taking $\zeta_1 = \nabla_x f(u,v) \in R^n{}_+$ and $\zeta_2 = \nabla_{xx} f(u,v) \in R^n{}_+$ and $\eta_1 = -\nabla_y f(u,v) \in R^m{}_+$ and $\eta_2 = -\nabla_{yy} f(x,y) \in R^m{}_+$ and using the hypothesis of the theorem, we have

$$f(x,u;\nabla_x f(u,v) + \nabla_{xx} f(u,v)p_1(x,u)) \geq -u^T \nabla_x f(u,v) - u^T \nabla_{xx} f(u,v)p_1,$$

and

$$f(v,y;-\nabla_y f(x,y) - \nabla_{yy} f(x,y)p(v,y)) \geq y^T \nabla_y f(x,y) + y^T \nabla_{yy} f(x,y)p(x,y)).$$

These inequalities together yield

$$f(x,y) - f(u,v) \geq -u^T \nabla_x f(u,v) - u^T \nabla_{xx} f(u,v)p_1 + y^T \nabla_y f(x,y) + y^T \nabla_{yy} f(x,y)p$$

$$-\frac{1}{2}p_1(x,u)\nabla_{xx} f(u,v)p_1(x,u) + \frac{1}{2}p(v,y)\nabla_{yy} f(x,y)p(v,y),$$

i.e.,

$$f(x,y) - y^T f(x,y) - y^T \nabla_{yy} f(x,y)p(v,y) - \frac{1}{2}p(v,y)\nabla yyf(x,y)p(v,y)$$

$$\geq f(u,v) - u^T \nabla_x f(u,v) - u^T \nabla_{xx} f(u,v)p_1(x,u) - \frac{1}{2}p_1(x,u)\nabla_{xx} f(u,v)p_1(x,u),$$

and hence

$$Inf(WP) \geq Sup(WD).$$

3.6 Second order Mond-Weir Type Symmetric Program

We consider the following pair of second order problems.
Primal (WP)

$$\text{Minimize } f(x,y) - \tfrac{1}{2}p^T \nabla_{yy} f(x,y)p$$
$$\text{subject to}$$

$$\begin{aligned} \nabla_y f(x,y) + \nabla_{yy} f(x,y)p &\leq 0, \\ y^T \nabla_y f(x,y) + y^T \nabla_{yy} f(x,y)p &\geq 0, \\ x &\geq 0. \end{aligned}$$

Dual(WD)

$$\text{Maximize } f(u,v) - \tfrac{1}{2}p^T{}_1 \nabla_{xx} f(u,v)p_1$$
$$\text{subject to}$$

$$\begin{aligned} \nabla_x f(u,v) + \nabla_{xx} f(u,v)p_1 &\leq 0, \\ u^T \nabla_x f(u,v) + u^T \nabla_{xx} f(u,v)p_1 &\leq 0, \\ v &\geq 0. \end{aligned}$$

Theorem 3.6.1. (Weak Duality) Let $f(\cdot,y)$ be second order F-pseudoconvex and $f(x,\cdot)$ be second order F-pseudoconcave and for all feasible (x,y,u,v) to (MP) and (MD):

- $F(x, u; \zeta_1 + \zeta_2) + u^T\zeta_1 + u^T\zeta_2 \geq 0$, for $\zeta_1 \in R^n{}_+$, $\zeta_2 \in R^n{}_+, and$
- $F(v, y; \eta_1 + \eta_2) + y^T\eta_1 + y^T\eta_2 \geq 0$, for $\eta_1 \in R^m{}_+$, $\eta_2 \in R^m{}_+$, then

$$inf(MP) \geq sup(MD).$$

Proof. On taking $\zeta_1 + \zeta_2 = \nabla_x f(u, v) + \nabla_{xx} f(u, v)p_1 \in R^n{}_+$, we have

$$F(x, u; \nabla_x f(u, v) + \nabla_{xx} f(u, v)p_1(x, u)) \geq -u^T\nabla_x f(u, v) - u^T\nabla_{xx} f(u, v)p_1 \geq 0,$$

which by second order F-pseudoconvexity of $f(\cdot, y)$ yields

$$f(x, v) \geq f(u, v) - \frac{1}{2}{p_1}^T\nabla_{xx} f(u, v)p_1,$$

On taking $\eta_1 + \eta_2 = -\nabla_y f(x, y) - \nabla_{yy} f(x, y)p(v, y) \in R^m{}_+$, we have

$$F(v, y; -\nabla_y f(x, y) - \nabla_{yy} f(x, y)p(v, y)) \geq y^T\nabla_y f(x, y) + y^T\nabla_{yy} f(x, y)p \geq 0,$$

which by second order F-pseudoconcavity of $f(x, \cdot)$ gives

$$f(x, y) \geq f(x, v) + \frac{1}{2}p^T(v, y)\nabla_{yy} f(x, y)p(v, y).$$

Combining we have

$$f(x, y) - \frac{1}{2}p^T(v, y)\nabla_{yy} f(x, y)p(v, y) \geq f(u, v) - \frac{1}{2}{p_1}^T(x, u)\nabla_{xx} f(u, v)p_1(x, u),$$

i.e.,

$$inf(MP) \geq sup(MD).$$

CHAPTER 4

4 Second Order Mond-Weir type dual Programs over cones with F-E-convexity

4.1 Introduction

Convexity and generalized convexity play important roles in optimization theory. Various generalizations of convexity have appeared in the literature. Youness[31] introduced a class of sets and a class of functions, called E-convex sets and E-convex functions, which generalize the definitions of convex sets and convex functions based on the effect of an operator E on the sets and domain of definition of the functions. Hanson and Mond [14] introduced F-convex functions as a generalization of convex functions. Motivated both by earlier research works and by the importance of convexity and generalized convexity, we introduce a class of functions called F-E-convex functions which are generalizations of F-convex and E-convex functions.

4.2 Preliminaries

Let R^n denote the n-dimensional Euclidean space. M be the subset of R^n.

Definition 4.2.1. A functional $F : M \times M \times R^n \to R$ is said to be sublinear in its third component, if for all x, $\bar{x} \in M$,

1. $F(x, \bar{x}, a + b) \leq F(x, \bar{x}, a) + F(x, \bar{x}, b)$, $\forall a, b \in R^n$,
2. $F(x, \bar{x}, \beta a) = \beta F(x, \bar{x}, a)$, $\forall \beta \in R$, $\beta \geq 0$, and $a \in R^n$.

Definition 4.2.2. Let $E : R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is differentiable function on M, then f is said to be F-E convex function at $\bar{x}$ on M if for all $x \in M$,

$$f(E(x)) - f(E(\bar{x})) \geq F_{x,\bar{x}}(\nabla f(E(\bar{x}))),$$

for some arbitrary functional F.

Definition 4.2.3. Let $E : R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is differentiable function on M, then f is said to be F-E pseudoconvex function on $\bar{x}$ on M if for all $x \in M$,

$$f(E(x)) < f(E(\bar{x})) \implies F_{x,\bar{x}}[\nabla f(E(\bar{x}))] < 0,$$

for some arbitrary functional F.

Definition 4.2.4. Let $E : R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is differentiable function on M, then f is said to be stricly F-E pseudoconvex function at $\bar{x}$ on M if for all $x \in M$,

$$F_{x,\bar{x}}[\nabla f(E(\bar{x}))] \geq 0 \implies f(E(x)) > f(E(\bar{x})),$$

for some arbitrary functional F.

Definition 4.2.5. Let $E : R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is differentiable function on M, then f is said to be F-E quasiconvex function at $\bar{x}$ on M if for all $x \in M$,

$$f(E(x)) \leq f(E(\bar{x})) \implies F_{x,\bar{x}}[\nabla f(E(\bar{x}))] \leq 0,$$

for some arbitrary functional F.

Now we discuss the connection between the concept of E-convex function and second order F-convex functions by introducing the concept second order F-E convex functions and their generalization.

Definition 4.2.6. Let $E : R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be second order F-E convex function at $\bar{x}$ on M, then there exists a vector $p \in R^n$ such that, if for all $x \in M$,

$$f(E(x)) - f(E(\bar{x})) + \frac{1}{2} p^T \nabla^2 f(E(\bar{x}))p \geq F_{x,\bar{x}}(\nabla f(E(\bar{x})) + \nabla^2 f(E(\bar{x}))p),$$

for some arbitrary functional F.

Definition 4.2.7. Let $E : R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be second order F-E pseudoconvex function on $\bar{x}$ on M, then there exists a vector $p \in R^n$ such that, if for all $x \in M$,

$$f(E(x)) < f(E(\bar{x})) - \frac{1}{2} p^T \nabla^2 f(E(\bar{x}))p \implies F_{x,\bar{x}}[\nabla f(E(\bar{x})) + \nabla^2 f(E(\bar{x}))p] < 0,$$

for some arbitrary functional F.

Definition 4.2.8. Let $E : R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be stricly second order F-E pseudoconvex function at $\bar{x}$ on M if for all $x \in M$, then there exists a vector $p \in R^n$ such that,

$$F_{x,\bar{x}}[\nabla f(E(\bar{x})) + \nabla^2 f(E(\bar{x}))p] \geq 0 \implies f(E(x)) > f(E(\bar{x})) - \frac{1}{2} p^T \nabla^2 f(E(\bar{x}))p,$$

for some arbitrary functional F.

Definition 4.2.9. Let $E : R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be F-E quasiconvex function at $\bar{x}$ on M if for all $x \in M$,

$$f(E(x)) \leq f(E(\bar{x})) - \frac{1}{2} p^T \nabla^2 f(E(\bar{x}))p \implies F_{x,\bar{x}}[\nabla f(E(\bar{x})) + \nabla^2 f(E(\bar{x}))p] \leq 0,$$

for some arbitrary functional F.

Definition 4.2.10. A convex set C of R^n is called a convex cone if for each $x \in C$ and $\lambda \geq 0$, $\lambda x \in C$.

Definition 4.2.11. The positive polar cone C^* of C is defined as $C^* = \{x : \ \zeta^T x \ \geq \ 0; \ \text{for all } \zeta \in C\}$.

4.3 Mond-Weir type second order duality

In this section, we propose the following second order Mond-Weir dual problem:
(PM)

$$\text{Minimize } f(E(x))$$
$$\text{subject to}$$

$$M = \{x \in C_1 : -g(E(x)) \in C_2^*\}$$

(DP)

$$\text{Maximize } f(E(u)) - \tfrac{1}{2}p^T \nabla^2 f(E(u))p$$
$$\text{subject to}$$

$$\begin{aligned}\nabla \lambda f(E(u)) + \nabla^2 \lambda f(E(u))p + \nabla \gamma^T g(E(u)) + \nabla^2 \gamma^T g(E(u))p = 0,\\ g(E(u)) - \frac{1}{2}p^T \nabla^2 g(E(u))p \in C_2^*,\\ \gamma \in C_2,\ \lambda \geq 0,\end{aligned}$$

where

1. The function f and set M are E-convex with respect to the map $E : R^n \to R^n$, and f and g are twice differentiable functions.
2. C_1 and C_2 are closed convex cones in R^n and R^m with non-empty interiors respectively. C_1^* and C_2^* are polar cones of C_1 and C_2, respectively.

Theorem 4.3.1.(Weak duality). Suppose that for all feasible x in the problem PM and all feasible (u, γ, λ, p) in DP,

1. $\gamma^T g(.)$ is second order F-E-quasiconvex at u,
2. $\lambda^T f(.)$ is strictly second order F-E-pseudoconvex at u.

Then the following cannot holds

$$f(E(x)) \leq f(E(u)) - \frac{1}{2}p^T \nabla^2 f(E(u))p. \tag{1}$$

Proof. Let x be any feasible solution in PM and (u, γ, λ, p) be any feasible solution in DP problem.

Since $\gamma \in C_2$ and $-g(E(x)) \in\ C_2^*$,
Therefore we have,

$$-\gamma^T g(E(x))\ \geq\ 0 \Rightarrow\ \gamma^T g(E(x))\ \leq\ 0,$$

Also $\gamma \in C_2$ and $g(E(u)) - \frac{1}{2}p^T \nabla^2 g(E(u))p\ \in\ C_2^*$

Therefore, $\gamma^T g(E(u)) - \frac{1}{2}p^T \nabla^2 \gamma^T g(E(u))p\ \geq\ 0.$

Now,

$$\gamma^T g(E(u)) - \frac{1}{2}p^T\nabla^2\gamma^T g(E(u))p \;\geq\; 0 \geq \gamma^T g(E(x)).$$

Using second order F-E- quasiconvexity of $\gamma^T g(.)$ at u, we get

$$F(x, u, \nabla\gamma^T g(E(u) + \nabla^2\gamma^T g(E(u))p) \leq 0, \tag{2}$$

The first dual constraint and the sublinearity of F gives

$$F(x, u, \nabla\lambda f(E(u))_+\nabla^2\lambda f(E(u))p) \geq -F(x, u, \nabla\gamma^T g(E(u) + \nabla^2\gamma^T g(E(u))p), \tag{3}$$

From (2) and (3), we have

$$F(x, u, \nabla\lambda f(E(u))_+\nabla^2\lambda f(E(u))p) \geq 0, \tag{4}$$

Now suppose on contrary to the result that (1) holds, i.e.

$$f(E(x)) \leq f(E(u)) - \frac{1}{2}p^T\nabla^2 f(E(u))p. \tag{5}$$

By strictly second order F-E-pseudoconvexity of $\lambda^T f(.)$ at u and (4), we get

$$\lambda f(E(x)) > \lambda f(E(u)) - \frac{1}{2}p^T\nabla^2\lambda f(E(u))p. \tag{6}$$

On the other hand, multiply the inequality (5) by λ, we have

$$\lambda f(E(x)) \leq \lambda f(E(u)) - \frac{1}{2}p^T\nabla^2\lambda f(E(u))p. \tag{7}$$

which contradicts (6), Hence proved.

Theorem 4.3.2.(Strong duality theorem). Let $\bar{x}$ is an optimal solution of PM at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{\lambda} \geq 0$ and $\bar{\gamma} \in C_2$, such that $(\bar{x}, \bar{\gamma}, \bar{\lambda}, \bar{p} = 0)$ is feasible for DP and the corresponding values of PM and DP are equal.

Proof. Since $\bar{x}$ is an optimal solution of PM at which the Kuhn-Tucker constraint qualification is satisfied, then by Fritz John Necessary Condition for Optimality Theorem, there exist $\bar{\lambda} \geq 0$ and $\bar{\gamma} \in C_2$, such that

$$\bar{\lambda}\nabla f(E(\bar{x})) + \bar{\gamma}\nabla g(E(\bar{x})) = 0,$$
$$\bar{\gamma} g(E(\bar{x})) = 0,$$
$$\bar{\lambda} \geq 0.$$

Therefore $(\bar{x}, \bar{\gamma}, \bar{\lambda}, \bar{p} = 0)$ is feasible for DP and the corresponding values of PM and DP are equal from weak duality theorem $(\bar{x}, \bar{\gamma}, \bar{\lambda}, \bar{p} = 0)$ is optimal solution of DP.

Bibliography

1. Ahmad I., Second order symmetric duality in nondifferentiable multiobjective programming, Information Sciences 173 (2005) 23-34.

2. Balas E., Minimax and duality for linear and nonlinear mixed integer programming, in: J. Abadie (Ed.), Integer and Nonlinear Programming, North-Holland, Amsterdam, 1970.

3. Bazaraa M.S. and Goode J.J., On symmetric duality in nonlinear programming, Oper. Res., 21 (1973) 1-9.

4. Bazaarra M.S., John Jarvis J., and Shirali Hanif D., Nonlinear Programming, John Wiley and Sons (1990).

5. Bector C.R. and Chandra S. , Second-order symmetric and self dual programs, Opsearch, 23 (1986) 89-95.

6. Chandra S., Goyal A., Husain I., On symmetric duality in mathematical programming with F-convexity, Optimization 43 (1998) 1-18.

7. Chandra S., Jayadeva, Mehra A., Numerical Optimization and Applications, Narosa Publishing House, (2009).

8. Chandra S. and Kumar V., A note on pseudo-invexity and symmetric duality, Euro. J. Oper. Res., 105 (1998) 626-629.

9. Dantzig G.B., Eisenberg E. and Cottle R.W., Symmetric dual nonlinear programs, Pacific J. Math., 15 (1965) 809-812.

10. Dennis J.B., Mathematical Programming and Electrical Network, John Wiley and Sons, New York (1969).

11. Dorn W.S., Self-dual quadratic programs, SIAM Journal on Applied Mathematics 9 (1961) 51-54.

12. Gulati T.R., Islam M.A., Sufficiency and duality in multiobjective programming involving generalized F-convex functions, Journal of Mathematical Analysis and Applications 183 (1994) 181-195.

13. Hanson M.A., On sufficiency of the Kuhn-Tucker conditions. J. Math. Anal. Appl. 80 (1981), 545-550.

14. Hanson M.A., Mond B., Further generalization of convexity in mathematical programming, Journal of Information and Optimation Science 3 (1) (1982) 25-32.

15. Kaul R.N., Kaur S.: Optimality criteria in nonlinear programming involving nonconvex functions. J. Math. Anal. Appl. 105, 104-112 (1985).

16. Mangasarian O.L., Nonlinear Programming, McGraw-Hill, New York (1969).

17. Mangasarian O.L., Second- and higher-order duality in nonlinear programming, J. Math. Anal. Appl., 51 (1975) 607-620 (Computer Science, Technical Report 159, University of Wisconsin, 1972).

18. Megahed A.A., Gomma H.G., Youness E.A., El-Banna A.H., A study on the duality of E-convex non-linear programming problem,Int. Journal of Maths. Analysis, Vol.7, 2013, no.4, 175-185.

19. Mishra S.K., Second order symmetric duality in mathematical programming with F-convexity, European Journal of Operational Research 127 (2000) 507-518.

20. Mond B., A symmetric dual theorem for nonlinear programs, Quarterly Journal of Applied Mathematics 23 (1965) 265-269.

21. Mond B., Weir T., Generalized concavity and duality, in: S. Schaible, W.T. Ziemba (Eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, 1981.

22. Mond B. and Weir T., Generalized convexity and higher order duality, J. Math. Anal. Appl., 16-18 (1981-83) 74-94.

23. Mond B., Second order duality for nonlinear programs, Opsearch, 11 (1974) 90-99.

24. Neumann J.V., On a maximization problem, Institute for advanced study, Princeton, New Jersey (1947).

25. Rockafellar R.T., Convex Analysis, Princeton University Press, New Jersey, 1970.

26. Sheiba Grace J., and Thangavelu P.,: Properties of E-convex sets,Tamsui Oxford Journal of Mathematical Sciences, 25(1) (2009) 1-7.

27. Suneja S.K., Lalitha C.S. and Govil M.G., E–convex and Related functions, International Journal of Management and Systems, 18(2)(2002), 193-206.

28. Syau Y.R., Lee E.S.: Some properties of E-convex functions. Appl. Math. Lett. 18, 1074-1080 (2005).

29. Wolfe P., A duality theorem for nonlinear programming, Quart. Appl. Math. 19 (1961), 239-244.

30. Yang X.M., On E-convex set, E-convex function and E-convex programming, J. Optim. Theory Appl. 109 (2001) 699-703.

31. Youness E.A., On E-convex sets, E-convex functions and E-convex programming, Journal of Optimization Theory and Applications, 102 (1999) 439-450.

32. Youness E.A.: Quasi and strictly quasi E-convex functions. Stat. Manag. Syst. 4, 201-210 (2001).

33. Zhang J. and Mond B., Second order duality for multiobjective nonlinear programming involving generalized convexity in: B.M.Glover, B.D.Craven, D.Ralph (Eds.), Proceedings of the Optimization Miniconference III, University of Ballarat, 1997, 79-95.