Fixed Points of Contraction Mappings on Metric and G–Metric Spaces

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CERTIFICATE

Certified that the dissertation entitled, **"Fixed Points of Contraction Mappings on Metric and G-Metric Spaces"**, which is being submitted by **Miss. Sandeep Kaur Chouhan** (Roll No. 301103018), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of **Dr. Jatinderdeep Kaur** during the period from January 2013 to July 2013.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/ Institute for the award of any degree.

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ABSTRACT

The present dissertation entitled, **“Fixed Points of Contraction Mappings on Metric and *G*-Metric Spaces”,** contains a brief account of investigations carried out by various authors and by me on existence of fixed points of self mappings in metric space under the supervision of **Dr. Jatinderdeep Kaur,** Assistant Professor, School of Mathematices and Computer Applications, Thapar University, Patiala.

The aim of this work is to study and obtain some result on existence and uniqueness of fixed points. Fixed point theory has wide ranging application in many areas of mathematics. For example, in finding the solution of the system of linear equations, in proving the existence of solutions of ordinary and differential equation, integral equations, analysis and many other disciplines.

The whole work is divided into three chapters. Chapter I is introduction which includes brief account of definitions and results which will be required for the later chapters. In Chapter II, we have studied Banach Contraction Theorem which guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and applications of Banach Contraction Theorem to system of linear equations and integral equations. The aim of chapter III is to study fixed point results for mapping satisfying sufficient contractive conditions on a complete G-metric space and studied the existence and uniqueness of fixed points.

Towards the end, references of various publications cited in the present dissertation have been reported.

Contents

CHAPTER I

INTRODUCTION

1.1 Introduction

A fixed point of a function is a point that is mapped to itself by the function. Let X be any non-empty set. Given a function $f : X \to X$, a fixed point of f is a point $x \in X$ such that $f(x) = x$, that is, a point which remains invariant under the mapping f. A set of fixed points is sometimes called a fixed set. This is to say, c is a fixed point of the function $f(x)$ if and only if $f(c) = c$. For example, if a function f defined on the real numbers by $f(x) = x^2 - 3x + 4$, then 2 is a fixed point of f. Not all functions have fixed points. For example if f is a function defined on the real numbers as $f(x) = x + 1$, then it has no fixed points. In graphical terms, a fixed point means the point $(x, f(x))$ is on the line $y = x$ or in other words the graph of f has a point in common with the line. The example $f(x) = x + 1$, is a case where the graph and the line are a pair of parallel lines.

The origin of metric contraction principles rest in the method of successive approximations for proving existence and uniqueness of solutions of differential equations, integral equations, analysis and many other disciplines. The first fixed point theorem was given by L.E.J Brouwer [3]. However, it is the Polish Mathematician Stefan Banach who is credited with placing the ideas underlying the method into an abstract frame work suitable for broad applications well beyond the scope of elementary differential and integral equations. Although the basic idea about the fixed point theory was known to others earlier, but the credit of making it useful and popular goes to Polish mathematician Stefan Banach. In 1922, he proved a common fixed point theorem, which ensured the existence and uniqueness of a fixed point under appropriate conditions. This result of S. Banach [6] is known as Banach Fixed Point Theorem or Contraction Mapping Principle.

Fixed point theory has wide ranging applications in many areas of Mathematics. For example, in proving the existence of solutions of ordinary and partial differential equations, integral equations and many other disciplines.

1.2 Basic Concepts and Definitions

We now present a brief account of basic definitions, which will be used in the subsequent chapters for the sake of convenience.

Metric Space ([4], p. 11).

Let X be non empty set and a mapping $d: X \times X \to R$ (set of reals) is said to be metric if d satisfies the following properties:

(i) $d(x, y) \geq 0 \quad \forall\ x, y \in X$

(ii) $d(x, y) = 0 \text{ iff } x = y \quad \forall\ x, y \in X$

(iii) $d(x, y) = d(y, x) \quad \forall\ x, y \in X$ (symmetry)

(iv) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall\ x, y, z \in X$ (triangle inequality)

If d is a metric for X, then the ordered pair (X, d) is called a metric space and $d(x, y)$ is called the distance between x and y.

Note that if $x \neq y$, then $d(x, y) > 0$.

<u>*Examples of Metric Space.*</u>

(i) Let R be set of real no's and let $d: R \times R \to R$ defined by

$$d(x, y) = |x - y| \quad \forall\ x, y \in R$$

then d is metric on R and (X, d) is known as usual metric space.

(ii) Let X be a non empty set. Consider a mapping $d: X \times X \to X$ defined by

$$d(x, y) = \left.\begin{cases} 0, & when\ x = y \\ 1, & when\ x \neq y \end{cases} \quad \forall x, y \in X \right\}$$

then d is metric on X and (X,d) is called discrete metric space.

Pseudo-metric Space [4].

A mapping $d: X \times X \to R$ is called a pseudo metric (or semi metric) for X if d satisfies the following axioms:

(i) $d(x,y) \geq 0 \quad \forall\ x,y \in X$

(ii) $d(x,y) = 0$ if $x = y \quad \forall\ x,y \in X$

(iii) $d(x,y) = d(y,x) \quad \forall\ x,y \in X$ (symmetry)

(iv) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\ x,y,z \in X$ (triangle inequality)

Example.

Let $f(x) = x^2$ be a real valued function on R and $d: X \times X \to R$ defined by $d\ (x,y) = \left| f(x) - f(y) \right| \quad \forall\ x,y \in X$ is pseudo metric on X.

Solution. $f(3) = 9,\ \ f(-3) = 9$

$$d(-3,3) = \left| f(-3) - f(3) \right|$$

$$= \left| 9 - 9 \right| = \left| 0 \right|$$

$$d(-3,3) = 0 \text{ but } -3 \neq 3$$

Note. For a pseudo-metric space, we may have $d\ (x,y) = 0$ even if $x \neq y$.

Thus for a pseudo metric $x = y \Rightarrow d(x,y) = 0$ but not conversely.

It follows that every metric is pseudo-metric but pseudo-metric is not necessarily metric.

Open Sphere.

Let (X,d) be a metric space and let $x_0 \in X$. If r is positive real number, then the set $\{x \in X : d(x_0, x) < r\}$ is called an open sphere. The point x_0 is called centre and r is the radius of sphere. It is denoted by $S(x_0, r)$ or $S_r(x_0)$.

Closed Sphere.

It is denoted and defined by

$$S[x_0, r] = \{x \in X \ : \ d(x, x_0) \leq r\} .$$

Note. A Sphere is always non-empty since it contains its centre.

Open Set.

Let (X, d) be a metric space. A subset G of X is said to be open set in (X, d) if for each $x \in G$, $\exists$ a real number $r > 0$ such that $S(x, r) \subseteq G$.

Neighbourhood of a point.

Let (X, d) be a metric space and $x \in X$ be any element then a subset N of X is neighbourhood of x if $\exists$ a open set G such that $x \in G \subset N$.

Interior point.

Let (X, d) be a metric space and $A \subseteq X$, then a point $x \in A$ is interior point of A iff $\exists$ open sphere $S(x, r)$ such that $x \in S(x, r) \subseteq A$ i.e x is interior of A iff A is nhd of x.

Interior of a set.

The set of all interior points of a set A is called interior of A and is denoted by A^0 .

Exterior point.

Let (X, d) be a metric space and $A \subseteq X$ be any set ,then a point $x \in X$ is called exterior point of A iff x is interior of A^c.

Exterior of a set.

The set of all exterior points of a set A is called exterior of A and is denoted by ext(A).

Remark. $ext(A) = int(A^c)$.

Frontier point.

Let (X,d) be a metric space and $A \subseteq X$ be any set ,then a point $x \in X$ is called frontier point of A iff x is neither interior point of A nor exterior point of A .

Frontier of a set.

The set of all frontier points of a set A is called frontier of set and is denoted by $\mathrm{Fr}(A)$.

Boundary point.

Let (X,d) be a metric space and $A \subseteq X$ be any set , then a point $x \in X$ is called boundary point of A iff x is frontier point of A and also $x \in A$.

Boundary of a set.

The set of all boundary points of set A is called boundary of A and is denoted by $\mathrm{b}(A)$.

Remark. $b(A) \subseteq Fr(A)$.

Adherent point.

Let (X,d) be a metric space and $A \subseteq X$, then $x \in X$ is called adherent point of a set A if every neighbourhood of x contains at least one point of A . In symbols, a point $x \in X$ is an adherent point of $A \subseteq X$ iff for each nhd N of x , $N \cap A \neq \phi$.

Isolated point.

Let (X,d) be a metric space and $A \subseteq X$ be any set, then a point $x \in X$ is called isolated point of A if each neighbourhood of x contains no point of A other than x i.e $N \cap A = \{x\}$ where N is nhd of x .

Limit point. Let (X,d) be a metric space and $A \subseteq X$ be any set, then a point $x \in X$ is called limit point of A if each neighbourhood of x contains at least one point of A other than x.Thus $x \in X$ is limit point of $A \subseteq X$ iff for each nhd N of x, $N \cap A \neq \phi$ and $\neq \{x\}$.

Closure of a set.

Collection of all adherent points of a set A is called closure of A and is denoted by $\overline{A}$.

Derived set of a set.

Collection of all limit points of a set A is called derived set of a set and is denoted by A'.

Dense set.

Let (X,d) be a metric space and $A \subseteq X$ be any set , then A is said to be dense in X if $\overline{A} = X$.

Closed set.

Let (X,d) be a metric space and $A \subseteq X$,then A is said to be closed if it contains all its limit point, i.e $A' \subseteq A$.

Example.

Let (X,d) be a metric space where $X = R$ and $d(x,y) = |x - y|$. If we take $A = [0,1]$ then every limit point of A is contained in A. Thus, A is a closed set.

Sequence in metric space ([4], p.5).

Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is a function from N into X.

Convergence of a sequence in metric space ([4], p.47).

Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for each $\in > 0$, however small there exist a positive integer $m \in N$ such that

$$d(x_n, x) < \in \qquad \forall \ n \geq m$$

$$\text{i.e } x_n \in B(x,\in) \quad \forall \ n \geq m$$

if x_n converges to x then we write as $\lim_{n\to\infty} d(x_n, x) = 0$.

<u>*Example.*</u>

Let (X,d) be a usual metric space and let sequence $\{x_n\} = \frac{1}{n}$. Given $\in > 0$, however small, we choose a natural number m such that

$$m > \frac{1}{\in} \;\Rightarrow\; \frac{1}{m} < \in$$

$$\therefore \ \forall \ n \geq m, \ \frac{1}{n} \leq \frac{1}{m} < \in$$

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \in, \ \forall \ n \geq m.$$

$\Rightarrow$ $\{x_n\}$ converges to 0 in X.

Cauchy sequence ([4], p.52).

Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for every $\in > 0$, there exist $m_0 \in N$ such that

$$d(x_n, x_m) < \in \ \forall \ n, m \geq m_0.$$

Remark. *Every convergent sequence in a metric space is Cauchy sequence but converse may not be true.*

Example.

Let $X =]0,1]$ with usual metric and $x_n = \frac{1}{n}$ be a sequence in X. Then $\{x_n\}$ is Cauchy sequence, since for each $\in > 0$, we have

$$d\,(x_m, x_n) = |\,x_m - x_n\,| \le |\,x_m\,| + |\,x_n\,| = \left|\frac{1}{m}\right| + \left|\frac{1}{n}\right|$$

$$= \frac{1}{m} + \frac{1}{n} < \frac{\in}{2} + \frac{\in}{2} = \in, \ \forall \ m, n > \frac{2}{\in}.$$

i.e $d\,(x_m, x_n) < \in \quad \Rightarrow \quad \{x_n\}$ is a Cauchy sequence.

But $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0,$ which is not a point of X.

Thus $\{x_n\}$ is a Cauchy sequence but it is not convergent.

Complete Metric Space ([4], p.55).

A metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point of X.

In other words, (X,d) is a complete metric space if, the sequence $\{x_n\}$ in X is such that

$$d(x_m, x_n) \to 0 \text{ as } m,n \to \infty \text{ and } x \in X \text{ with } d(x_n, x) \to 0 \text{ as } n \to \infty.$$

Example.

Let X be an arbitrary non empty set. For $x, y \in X,$ define $d : X \times X \to R$ by

$$d(x,y) = \left\{ \begin{matrix} 0, & when\ x = y \\ 1, & when\ x \ne y \end{matrix} \ \forall\ x, y \in X \right\}$$

Then (X,d) is a metric space and called as the discrete metric space.

As every Cauchy sequence in X is convergent. Therefore (X,d) is a complete metric space.

Fixed point ([4], p.127).

Let (X,d) be a metric space and $T:X\to X$ be a mapping. Then a point $x\in X$ is called a fixed point of T if x is mapped into itself i.e $T(x)=x$.

<u>*Examples.*</u>

(i) The mapping $T:R\to R$ defined as $T(x)=\frac{x}{2}$ has unique fixed point. Clearly, 0 is the only fixed point.

(ii) The mapping $T:R\to R$ defined as $T(x)=x^2$ has two only fixed points. Here 0 and 1 are fixed points.

(iii) The mapping $T:R^2\to R^2$ defined as $T(x,y)=x$ has infinitely many fixed points. In fact all points of $x-axis$ are fixed points.

Mapping may have unique fixed points, more than one and even infinitely many fixed points.

Remark. *There may exist mappings which have no fixed point*

<u>*Example.*</u>

Let X be a non empty set. We can define a mapping $T:X\to X$ as $T(x)=x+a$ where 'a' is any constant. Clearly, it has no fixed point.

Contraction Mapping ([4], p.128).

Let (X,d) be a metric space. A mapping $T:X\to X$ is called contraction on X if there exist a real number α with $0\le\alpha<1$ such that

$$d\,(Tx,Ty)\le\alpha\,d\,(x,y)\quad\forall\quad x,y\in X.$$

<u>*Example.*</u>

If $T(x)=x^2$, $0\le x\le\frac{1}{3}$, then T is contraction mapping on $\left[0,\frac{1}{3}\right]$ with usual metric d.

Consider $d\,(Tx,Ty)=d\,(x^2,y^2)$

$$= \left| x^2 - y^2 \right|$$

$$= \left| x - y \right| \left| x + y \right|$$

$$\le \frac{2}{3} \left| x - y \right| \qquad \left[\begin{array}{l} \because \; 0 \le x \le \frac{1}{3}, 0 \le y \le \frac{1}{3}, \\ \Rightarrow 0 \le x + y \le \frac{2}{3} \Rightarrow \left| x + y \right| \le \frac{2}{3} \end{array} \right]$$

$$= \frac{2}{3} d(x, y)$$

i.e $d(Tx, Ty) \le \frac{2}{3} d(x, y)$

Thus, T is contraction mapping.

Strict Contraction Mapping [1].

Let (X, d) be a metric space. A mapping $T : X \to X$ is called strict contraction of X if there exist a real number α with $0 \le \alpha < 1$ such that

$$d(Tx, Ty) < \alpha \, d(x, y) \quad \forall \; x, y \in X .$$

2-Metric Space [7],[8].

In 1960, Gahler [7],[8] introduced 2-metric space as follows:

Let X be a non empty set and let R denote the real numbers. A function $d : X \times X \times X \to R^+$ satisfying the following axioms:

(i) For distinct points $x, y \in X$, there is $z \in X$, such that $d(x, y, z) \ne 0$.

(ii) $d(x, y, z) = 0$ if two of the three variables $x, y, z \in X$ are equal.

(iii) $d(x, y, z) = d(x, z, y) = \; . \; . \; .$ (symmetry in all three variables),

(iv) $d(x, y, z) \le d(x, y, a) + d(x, a, z) + d(a, y, z)$, for all $x, y, z \in X$.

is called a 2-metric, on X. The set X equipped with such a 2-metric is called a 2-metric space.

D-Metric space [2].

Bapure Dhage [2] introduced a new class of generalized metrics called in 1992 as follows :

A function $D: X \times X \times X \to R^{+}$ is D-metric if it satisfy the following axioms:

(i) $D(x,y,z)=0$ if and only if $x=y=z$.

(ii) $D(x,y,z)=D(x,z,y)= \ . \ . \ .$ (symmetry in all three variables),

(iii) $D(x,y,z) \le D(x,y,a)+D(x,a,z)+D(a,y,z)$, for all $x,y,z,a \in X$.

(iv) $D(x,y,y) \le D(x,z,z)+D(z,y,y)$, for all $x,y,z \in X$.

In 2003, we demonstrated in [10], [11] that most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect. For a instance D-metric need not be a continuous function of its variables. These considerations lead us to seek a more appropriate notion of generalized metric space.

G -Metric space [12], [13].

Let X be a non empty set, and $G: X \times X \times X \to R^{+}$ be a function satisfies the following axioms:

(i) $G(x,y,z)=0 \ \ if \ \ x=y=z \quad \forall\ x,y,z \in X$

(ii) $G(x,x,y)>0$; $\forall\ x,y \in X$ with $x \neq y$,

(iii) $G(x,x,y) \le G(x,y,z), \quad \forall\ x,y,z \in X$ with $z \neq y$

(iv) $G(x,y,z)=G(x,z,y)=G(y,z,x)= \ . \ . \ .$ (symmetry in all three variables)

(v) $G(x,y,z) \le G(x,a,a)+G(a,y,z), \quad \forall\ x,y,z,a \in X$, (quadrilateral inequality),

Then the mapping G is called a generalized metric or G-metric on X and the pair (X,G) is called a G-metric space.

Here $G(x,y,z)$ denote the perimeter of the triangle with vertices at x, y and z in R^{2}.

Example [12].

For any metric space (X,d), define G_s and G_m on X as

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z) \text{ and}$$

$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$$

for all $x,y,z \in X$. Then (X,G_s) and (X,G_m) are G-metric spaces.

Symmetric G -metric space [13].

G-metric space (X,G) is symmetric if

$$G(x,y,y) = G(x,x,y), \quad \forall\ x,y \in X$$

But if **,** $G(x,y,y) \neq G(x,x,y), \quad \forall\ x,y \in X$, then it is not symmetric.

Example[13].

Let $X = \{a,b\}$, let,

$$G(a,a,a) = G(b,b,b) = 0$$
$$G(a,a,b) = 1,\ G(a,b,b) = 2$$

and extend G to all of $X \times X \times X$ by symmetry in the variables.

We note that it follows all the five axioms of the G-metric space

(i) $G(x,y,z) = 0$ if $x = y = z \quad \forall\ x,y,z \in X$

i.e $G(a,a,a) = 0$

(ii) $G(x,x,y) > 0$; $\forall\ x,y \in X$ with $x \neq y$

$G(a,a,b) > 0 \quad \forall\ a,b \in X$

(iii) $G(x,x,y) \leq G(x,y,z), \quad \forall\ x,y,z \in X$ with $z \neq y$

$G(a,a,b) \leq G(a,b,a) \quad \forall\ a,b \in X$

(iv) Symmetry in variables $G(a,a,b) = G(a,b,a)$.

(v) $G(x,y,z) \le G(x,a,a) + G(a,y,z), \quad \forall\ x,y,z,a \in X,$

$$G(a,a,b) \le G(a,a,a) + G(a,a,b) \qquad \forall\ a,b \in X$$

Therefore, it is G-metric space. But $G(a,a,b) \ne G(a,b,b)$,

Hence it is not symmetric.

G-metric topology $\tau(G)$ [13].

Every G-metric space (X,G) induces a (X,d_G) defined as

$$d_G(x,y) = G(x,y,y) + G(x,x,y), \ \ \forall\ x,y \in X.$$

G-ball [13].

Let (X,G) be a G-metric space then for $x_0 \in X,\ r > 0,$ the G-ball with centre x_0 and radius r is

$$B_G(x_0,r) = \{y \in X : G(x_0,y,y) < r\}$$

G-convergent metric space [13].

Let (X,G) be G-metric space. The sequence $\{x_n\} \subseteq X$ is G-convergent to x if it converges to x in the G-metric topology, $\tau(G)$.

G-continuous function [13].

Let $(X,G),(X',G')$ be two G-metric spaces, a function $f : X \to X'$ is G-continuous at a point $x_0 \in X$ if and only if, given $\in > 0$, there exist $\delta > 0$ such that $x,y \in X$ and $G\,(x_0,x,y) < \delta$ implies $G'(f(x_0),f(x),f(y)) < \in$. In other words f is G-continuous if it is G-continuous at all points of X; i.e continuous as a function from X with the $\tau(G)$- topology to X' with the $\tau(G')$- topology.

G-Cauchy sequence [13].

Let (X,G) be a G-metric space, then a sequence $(x_n) \subseteq X$ is said to be G-Cauchy if for $\in > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \in$ for all $n,m,l \geq N$.

G-complete metric space [13]. A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

Now, I give a brief chapter wise resume of the results contains in this dissertation.

In chapter II, we study the existence and uniqueness of Banach fixed points and also study the applications of Banach contraction theorem to the system of linear equations and integral equations.

The aim of chapter III is to study some Fixed point results using complete metric space and study the existence and uniqueness of fixed points.

CHAPTER II

BANACH CONTRACTION THEOREM AND ITS APPLICATIONS

2.1 Introduction

In mathematics, Banach Fixed Point Theorem [6] (also known as the contraction mapping theorem or contraction mapping principle) is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The theorem is named after Stefan Banach $(1892-1945)$, and was first stated by him in 1922.

Fixed point ([4], p.127).

Let (X,d) be a metric space and $T:X\to X$ be a mapping. Then a point $x\in X$ is called a fixed point of T if x is mapped into itself, i.e $T(x)=x$

Example.

A mapping $T:R\to R$ defined by $T(x)=x^3$ have three fixed points namely $0, 1, -1$.

Contraction mapping ([4], p.128).

Let (X,d) be a metric space. A mapping $T:X\to X$ is called contraction on X if there exist a real number α with $0\le\alpha<1$ such that

$$d(Tx,Ty)\le\alpha\, d(x,y)\quad\forall\quad x,y\in X.$$

Example.

Consider the usual metric d for R^2 and the mapping $T:R^2\to R^2$ such that $T(x)=\dfrac{x}{3},\ \forall\ x\in R^2$, where $x=(x_1,x_2)$, we have

$$d\,(T(x),T(y))=d\left(\frac{x}{3},\frac{y}{3}\right)$$

$$= d\left(\frac{1}{3}(x_1, x_2), \frac{1}{3}(y_1, y_2)\right)$$

$$= d\left(\left(\frac{x_1}{3}, \frac{x_2}{3}\right), \left(\frac{y_1}{3}, \frac{y_2}{3}\right)\right)$$

$$= \sqrt{\left(\frac{x_1}{3} - \frac{y_1}{3}\right)^2 + \left(\frac{x_2}{3} - \frac{y_2}{3}\right)^2}$$

$$= \frac{1}{3}\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$= \frac{1}{3}\, d(x, y)$$

Thus $d(T(x), T(y)) = \alpha\, d(x, y)$, where $\alpha = \frac{1}{3} < 1$

Therefore, T is contraction on R^2.

The aim of this chapter is to study the Existence and Uniqueness of Banach fixed points and also to study the applications of Banach contraction theorem to the system of linear equations and integral equations.

2.2 Main results

Theorem 2.2.1 (Banach Contraction Theorem)([4],p.128). *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a contraction on* X. *Then* T *has a unique fixed point in* X *.*

Proof. We will prove this theorem in two parts. In the first part we prove the existence of fixed point and uniqueness of fixed point in the second part.

<u>*Existence of fixed point.*</u> Let $T : X \to X$ be a contraction mapping, then there exist a real number α with $0 \le \alpha < 1$ such that

(2.2.1) $\quad d\,(T(x), T(y)) \le \alpha\; d(x, y) \quad \forall\; x, y \in X$

Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by

$$\begin{aligned}
x_1 &= T(x_0) \\
x_2 &= T(x_1) = T(T(x_0)) = T^2(x_0) \\
x_3 &= T(x_2) = T(T^2(x_0)) = T^3(x_0) \\
&\quad . \qquad . \qquad . \\
&\quad . \qquad . \qquad . \\
&\quad . \qquad . \qquad . \\
x_n &= T(x_n) = T^n(x_0)
\end{aligned}$$

we prove this part in three steps :

Step (i) firstly we prove that for all $n \in N$, $d(x_n, x_{n+1}) \le \alpha^n \ d(x_0, x_1)$

we will proceed using induction, the base of induction i.e $n = 1$ holds using (2.2.1).

$$d(x_1, x_2) = d(T(x_0, x_1) \le \alpha \ d(x_0, x_1)$$

Suppose it holds for some $k \in N$ i.e for $n = k$

(2.2.2) $\quad d(x_k, x_{k+1}) \le \alpha^k \, d(x_0, x_1)$

Let $n = k+1$, then we consider,

$$d(x_{k+1}, x_{k+2}) = d(T(x_k), T(x_{k+1}))$$

By using (2.2.1), we get

$$d(x_{k+1}, x_{k+2}) \le \alpha \, d(x_k, x_{k+1})$$

Further, by using (2.2.2), we have

$$d(x_{k+1}, x_{k+2}) \le \alpha . \alpha^k \, d(x_0, x_1)$$

$$d(x_{k+1}, x_{k+2}) \le \alpha^{k+1} \, d(x_0, x_1)$$

Thus by the principle of mathematical induction *step (i)* holds.

Step (ii) In this step, we will prove that $\{x_n\}$ is a Cauchy sequence in X .

For this we consider $m,n \in N$ s.t $n > m$,

$$d(x_m,x_n) \le d(x_m,x_{m+1}) + d(x_{m+1},x_{m+2}) + \dots\dots\dots\dots + d(x_{n-1},x_n)$$

(by triangle inequality)

By use of *Step (i)*, we have

$$d(x_m,x_n) \le \alpha^m\ d(x_0,x_1) + \alpha^{m+1}\ d(x_0,x_1) + \alpha^{m+2}\ d(x_0,x_1) + \dots\dots\dots + \alpha^{n-1}\ d(x_0,x_1)$$

$$= (\alpha^m + \alpha^{m+1} + \alpha^{m+2} + \dots + \alpha^{n-1})\, d(x_0,x_1)$$

$$= \alpha^m (1 + \alpha + \alpha^2 + \dots + \alpha^{n-m-1})\, d(x_0,x_1) = \alpha^m \left(\sum_{k=0}^{n-m-1} \alpha^k \right) d(x_0,x_1)$$

$$< \alpha^m \left(\sum_{k=0}^{\infty} \alpha^k \right) d(x_0,x_1) = \alpha^m \left(\frac{1}{1-\alpha} \right) d(x_0,x_1)$$

Since $0 \le \alpha < 1$, therefore $\alpha^m \to 0$ *as* $m \to \infty$, thus $d(x_m,x_n) < \in \quad \forall\, n > m$.

Hence, $\{x_n\}$ is a Cauchy sequence in X .

Step (iii) In this step, we will show that x is a fixed point of T .

By given hypothesis, X is complete , therefore the cauchy sequence $\{x_n\}$ is convergent in X , let $\{x_n\}$ converges to x (where $x \in X$) i.e $\lim_{n\to\infty} x_n = x$.

Also, T is contraction on X ,therefore,

$$d(T(x),T(y)) \le \alpha\ d(x,y) \qquad \forall\ x,y \in X$$

$$< d(x,y) \qquad (\because \alpha < 1)$$

Taking $d(x,y) < \in$, we get $d(T(x),T(y)) < \in$

Thus, for given $\in > 0$, there exist $\delta\ (= \in)$ such that,

$$d(T(x),T(y)) < \in \ \text{ for } d(x,y) < \delta$$

$\Rightarrow T$ is continuous mapping.

Therefore, $T(x) = T(\lim_{n\to\infty} x_n)$

$$= \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

i.e $T(x) = x$

$\Rightarrow$ x is a fixed point of T.

Thus the existence of a fixed point is established.

Uniqueness. Let if possible, x and y be two fixed points of T in X.

Then $T(x) = x$ and $T(y) = y$ for some $x, y \in X$

Now consider,

$$d(x, y) = d(T(x), T(y))$$

Using the equation (2.2.1), we get

(2.2.3) $d(x, y) \le \alpha\, d(x, y)$

But as we know that $d(x, y) \ge 0$ this implies that either $d(x, y) > 0$ or $d(x, y) = 0$

Therefore, first we suppose that $d(x, y) > 0$, then by the use of (2.2.3), we have

$$\frac{d(x, y)}{d(x, y)} \le \alpha \Rightarrow 1 \le \alpha,$$

which is a contradiction to the given hypothesis that $0 \le \alpha < 1$

Therefore, our supposition is wrong. Hence, $d(x, y)$ must be equal to zero which further implies that $x = y$.

This proves the uniqueness of fixed point in X.

Remarks:

(i) If X is not complete in above theorem, then T may not have a fixed point.

For example:

Consider $X =]0,1[$ and the mapping $T : X \to X$ defined by $T(x) = \frac{x}{2}$. Clearly X is not complete with the usual metric and also we see that T does not have any fixed point. In fact , $T(0)=0$ but 0 does not belong to X.

(ii) *If T is not contraction in above theorem, then it may not have a fixed point.*

<u>*For example:*</u>

Consider the metric space $X=[1,\infty[$ with the usual metric and the mapping $T:X\to X$ given by $T(x)=x+\dfrac{1}{x}$.

Here, X is a complete metric space but T is not a Contraction mapping.

$$\because\ d(Tx,Ty)=|T(x)-T(y)|=\left|x+\frac{1}{x}-y-\frac{1}{y}\right|=\left|(x-y)\left(1-\frac{1}{xy}\right)\right|$$

$$d(Tx,Ty)\le|x-y|\left|1-\frac{1}{xy}\right|$$

$$d(Tx,Ty)<|x-y| \qquad \forall\ x,y\in X \qquad \left(\because\ 0\le\left|1-\frac{1}{xy}\right|<1\right)$$

2.3 Applications of Banach contraction theorem:

Here we discuss the applications of Banach contraction Theorem

<u>2.3.1 Applications to system of linear equations</u> **:**

Firstly we study the application of Banach contraction Theorem to find the solution of the following system of n linear equations with n unknowns.

$$(2.3.1)\qquad \left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1\\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2\\ \quad .\qquad\qquad .\qquad\qquad .\\ \quad .\qquad\qquad .\qquad\qquad .\\ \quad .\qquad\qquad .\qquad\qquad .\\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n=b_n \end{array}\right\}$$

This system can be written

$$
(2.3.2)\qquad \left.\begin{array}{l}
x_1 = (1-a_{11})x_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n + b_1 \\
x_2 = -a_{21}x_1 + (1-a_{22})x_2 - a_{13}x_3 - \ldots - a_{2n}x_n + b_2 \\
x_3 = -a_{31}x_1 - a_{32}x_2 + (1-a_{33})x_3 - \ldots - a_{3n}x_n + b_3 \\
\quad . \qquad\qquad\qquad . \qquad\qquad\qquad . \\
\quad . \qquad\qquad\qquad . \qquad\qquad\qquad . \\
\quad . \qquad\qquad\qquad . \qquad\qquad\qquad . \\
x_n = -a_{n1}x_1 - a_{n2}x_2 - a_{n3}x_3 - \ldots + (1-a_{nn})x_n + b_n
\end{array}\right\}
$$

By letting $\alpha_{ij} = -a_{ij} + \delta_{ij}$, where

$$
\delta_{ij} = \begin{cases} 1 \ for \ i = j \\ 0 \ \ for \ i \neq j \end{cases}
$$

System (2.3.2) is equivalent to the following system:

(2.3.3) $$x_i = \sum_{j=1}^{n} \alpha_{ij}\, x_j + b_i \qquad\qquad i = 1,2,3,\ldots,n$$

If $x = (\ x_1, x_2, x_3\ \ldots, x_n\ ) \in R^n$ and

$$
A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & & . & . \\ . & & . & . \\ . & & . & . \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}
$$

and $(\ b_1, b_2, b_3, \ldots, b_n) \in R^n$,then the system (2.3.3) is equivalent to

(2.3.4) $$x = Ax + b$$

In other words, the problem is to find the fixed point of the transformation $T : R^n \to R^n$ defined by

(2.3.5) $$T(x) = Ax + b$$

If T is a contraction mapping, then we can use Banach Contraction Theorem and obtain the unique solution of $T(x) = x$ by the method of successive approximation.

The condition under which T is a contraction mapping depend on the choice of the metric on $X = R^n$.

Theorem A ([5], p.128). *Let* $X = R^n$ *be a metric space with the metric*

$$d_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|$$

(2.3.6) $$\text{if } \sum_{j=1}^{n} |\alpha_{ij}| \le \alpha < 1 \qquad \text{for all } i = 1, 2, 3, \ldots, n$$

then the linear system (2.3.1) *of* n *linear equations in* n *unknowns has a unique solution.*

Proof. Since $X = R^n$ with respect to the metric d_∞ is complete it is sufficient to prove that the mapping T defined by (2.3.5) is a contraction

$$d_\infty(T(x), T(y)) = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} \alpha_{ij}(x_i - y_i) \right|$$

$$\le \max_{1 \le i \le n} \sum_{j=1}^{n} |\alpha_{ij}| |x_i - y_i|$$

$$\le \max_{1 \le i \le n} \left(\max_{1 \le j \le n} |x_j - y_j| \right) \sum_{j=1}^{n} |\alpha_{ij}|$$

$$= \max_{1 \le i \le n} \sum_{j=1}^{n} |\alpha_{ij}| \, d_\infty(x, y)$$

By the use of (2.3.6), we get

$$d_\infty(T(x), T(y)) \le \alpha \, d_\infty(x, y)$$

Thus T is a contraction mapping, therefore by using Banach contraction theorem, the linear system has a unique solution.

Theorem B ([5], p.128). *Let* $X = R^n$ *be a metric space with the metric*

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

(2.3.7) $$\text{if } \sum_{i=1}^{n} |\alpha_{ij}| \le \alpha \qquad \forall \ j = 1, 2, 3, \ldots, n$$

Then prove that the linear system (2.3.1) *of* n *linear equations in* n *unknowns has a unique solution.*

Proof. Since $X = R^n$ with respect to the metric d_1 is complete, it is sufficient to prove that the mapping T defined by (2.3.5) is a contraction.

$$d_1(T(x), T(y)) = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \alpha_{ij}(x_j - y_j) \right|$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_{ij}||x_j - y_j|$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} |\alpha_i||x_j - y_j|$$

$$\leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} |\alpha_{ij}| \, d_1(x, y)$$

By using (2.3.7), we get

$$d_1(T(x), T(y)) \leq \alpha \; d_1(x, y).$$

Therefore, T is a contraction mapping .Thus the linear system (2.3.1) has unique solution by Banach contraction theorem

Theorem C ([5], p.129). *Let* $X = R^n$ *be a metric space with the metric*

$$d_2(x, y) = \left(\sum_{i=1}^{n} | x_i - y_i |^2 \right)^{\frac{1}{2}}$$

(2.3.8) $$\text{if} \;\; \sum_{i=1}^{n} \sum_{j=1}^{n} | \alpha_{ij} |^2 \leq \alpha^2 < 1$$

Then prove that the linear system (2.3.1) *of* n *linear equations in* n *unknowns has a unique solution.*

Proof. Since $X = R^n$ with respect to the metric d_2 is complete, it is sufficient to prove that the mapping T defined by (2.3.8) is a contraction.

$$[d_2(T(x), T(y))]^2 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \alpha_{ij}(x_j - y_j) \right|^2$$

$$\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} | \alpha_{ij} | \sum_{j=1}^{n} |x_j - y_j| \right)^2$$

$$\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} | \alpha_{ij} |^2 \sum_{j=1}^{n} |x_j - y_j|^2 \right)$$

So, $$[d_2(T(x), T(y))]^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} | \alpha_{ij} |^2 \, d_2(x, y)$$

By using (2.3.8), we get

$$[d_2(T(x), T(y))]^2 \leq \alpha^2 d_2(x, y)$$

Therefore, T is a contraction mapping and by Banach contraction theorem, the linear system (2.3.1) has unique solution.

Theorem D [5]. *Let* (X,d) *be a complete metric space and* $T: X \to X$ *be a mapping such that for some integer* m, $T^m = T \circ T \circ \ldots \circ T$ $(m\ times)$ *.Then* T *has unique fixed point.*

Proof. By the Banach Contraction Theorem, T^m has unique fixed point $x \in X$, i.e $T^m(x) = x$. Then

$$T(x) = T(T^m(x)) = T^m(T(x))$$

So, $T(x)$ is fixed point of T^m. Since the fixed point of T^m is unique, $T(x) = x$. To prove the uniqueness , we assume that y is another fixed point of T. Then $T(y) = y$ and so $T^m(y) = y$. Again by uniqueness of the fixed point of T^m we have $x = y$. Hence $x \in X$ is unique fixed point of T.

2.3.2 Application of fixed point theorems for system of integral equations

The most interesting applications of fixed point theorems arise when the underlying metric space is a function space. Here we study the existence and uniqueness of the Volterra integral equation by using the Theorem 2.2.1

Volterra integral equation [5].

Let K be a continuous function on $[a,b]\times[a,b]$ and let φ be a continuous function on $[a,b]$. Consider the equation

(2.3.9) $$f(x) = \varphi(x) + \lambda \int_a^x K(x,y) f(y)\,dy \qquad \textit{for all } x \in [a,b]$$

where λ is a parameter. It is called volterra equation.

Theorem E [5]. *For each* $\lambda \in R$*, the volterra equation (2.3.9) has a unique solution* f *that is continuous on* $[a,b]$.

Proof. Let $X = C[a,b]$, the set of all continuous function defined on $[a,b]$ with the uniform metric. Since K is continuous, there exist a constant $k > 0$ such that $|K(x,y)| \le k$ for all $x, y \in [a,b]$.

Define the transformation $T: f \to T(f)$ on X by

$$T(f(x)) = \varphi(x) + \lambda \int_a^x K(x,y)\, f(y)\, dy$$

For all $f, g \in X$ we have,

$$\left| T(f(x)) - T(g(x)) \right| = \left| \lambda \int_a^x K(x,y) \left| f(y) - g(y) \right| dy \right|$$

$$\left| T(f(x)) - T(g(x)) \right| = \left| \lambda \right| k\,(x-a)\, d\,(f,g) \qquad \textit{for all } \; x \in [a,b]$$

Since $T^2(f) - T^2(g) = T(T(f)) - T(T(g))$, we have

$$\left| T^2(f(x) - T^2(g(x)) \right| = \left| \lambda \int_a^x K(x,y) \left| T(f(y) - T(g(y)) \right| dy \right|$$

$$\leq \left| \lambda \right| \int_a^x \left| K(x,y) \right| \left| \lambda \right| k(y-a)\; d\,(f,g)\, dy$$

$$\leq \left| \lambda \right|^2 k^2 \int_a^x (y-a)\; dy\; d\,(f,g)$$

$$\leq \frac{\left| \lambda \right|^2 k^2\, (x-a)^2}{2}\; d\,(f,g)$$

Repeating the above steps successively, we get

$$\left| T^n(f(x)) - T^n(g(x)) \right| \leq \frac{\left[\, |\lambda|\, k\,(b-a) \right]^n}{n!}\; d\,(f,g)$$

Recalling that $\frac{r^n}{n!} \to 0$ as $n \to \infty$ for any $r \in R$, we conclude that there exist n such that T^n is contraction mapping. Take n sufficiently large, we have $\frac{\left[|\lambda|\, k\,(b-a) \right]^n}{n!} < 1$.

Hence, by using Theorem D there exist a unique solution $f \in X$ satisfying $T(f) = f$. Obviously if $T(f) = f$, then f solves volterra equation.

CHAPTER III

GENERALIZED METRIC SPACE

3.1 INTRODUCTION

The study of fixed points of a function satisfying certain contractive conditions has been at the center of vigorous activity, because it has a wide range of applications in different areas such as variational, linear inequalities, optimization and parameterize estimation problems.

In 2005, Zead Mustafa and B. Sims [12], [13] introduced a new class of generalized metric spaces, which are called G-metric spaces as generalization of metric spaces and thereby introducing a new fixed point theory for a variety of mapping in this new setting, also to extend known metric space theorems to a more general setting.

In this chapter, we have studied some fixed points results for mapping satisfying sufficient contractive conditions on a complete G-metric space, and also we showed that if the G-metric space (X,G) is symmetric, then the existence and uniqueness of these fixed point results follows from Reich theorems in usual metric space (X,d_G), where (X,d_G) the metric induced by the G-metric space (X,G).

G-metric space [12], [13].

Let X be a non empty set, and $G : X \times X \times X \to R^+$ be a function satisfies the following axioms:

(3.1.1) $G(x,y,z)=0$ *if* $x=y=z \quad \forall\, x,y,z \in X$

(3.1.2) $G(x,x,y)>0$; $\forall\, x,y \in X$ with $x \neq y$,

(3.1.3) $G(x,x,y) \le G(x,y,z)$, $\forall\, x,y,z \in X$ with $z \neq y$

(3.1.4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=\ldots$(symmetry in all three variables)

(3.1.5) $G(x,y,z) \le G(x,a,a) + G(a,y,z), \quad \forall\ x,y,z,a \in X,$

(quadrilateral inequality),

Then the mapping G is called a generalized metric or G-metric on X and the pair (X,G) is called a G-metric space.

Example [12].

Let (X,d) be a usual metric space, and define G_s and G_m on $X \times X \times X \to R^+$ by

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z),$$
$$G_m(x,y,z) = \max\{\ d(x,y), d(y,z), d(x,z)\ \}$$

for all $x,y,z \in X$. Then (X,G_s) and (X,G_m) are G-metric spaces.

Symmetric G-metric space [13].

G-metric space (X,G) is symmetric if

$$G(x,y,y) = G(x,x,y), \quad \forall\ x,y \in X$$

Proposition 3.1.1 [13]. *Let (X,G) be a G-metric space, then for any $x,y,z,a \in X$ then the following holds:*

(i) *if* $G(x,y,z) = 0$, *then* $x = y = z$

(ii) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$

(iii) $G(x,y,y) \le 2G(y,x,x)$

(iv) $G(x,y,z) \le G(x,a,z) + G(a,y,z)$

Proof. we can prove each part as

(i) It follows directly from (3.1.1)

(ii) From (3.1.5) we have $G(x,y,z) \le G(x,a,a) + G(a,y,z), \ \forall\ x,y,z,a \in X,$

Put $a = y$, then we have $G(x,y,z) \le G(x,y,y) + G(y,y,z)$

Interchange x and y, $G(y,x,z) \le G(y,x,x) + G(x,x,z)$

Using (3.1.4), we get $G(x,y,z) \le G(x,x,y) + G(x,x,z)$

(iii) From (3.1.5) we have $G(x,y,z) \le G(x,a,a)+G(a,y,z), \ \forall\ x,y,z,a \in X,$

Put $a=y$ and $z=x$, we have $G(x,y,x) \le G(x,y,y) + G(y,y,x)$

Interchange x and y, we have

$$G(y,x,y) \le G(y,x,x) + G(x,x,y)$$

Using (3.1.4), we get $G(x,y,y) \le 2G(y,x,x)$

(iv) From (3.1.5), we have $G(x,y,z) \le G(x,a,a) + G(a,y,z), \ \forall\, x,y,z,a \in X$

Using (3.1.3), we get $G(x,y,z) \le G(x,a,z) + G(a,y,z)$

Proposition 3.1.2 [13]. *Let* (X,G) *be a* G*-metric space and let* $k>0$*, then* $G_1(x,y,z) = \min\{k, G(x,y,z)\}$ is also G*-metric on* X *.*

Proof. G_1 is G-metric on X as it satisfy all the axioms of G-metric as following:

(i) Since (X,G) is G-metric space , therefore $G(x,y,z)=0$ if $x=y=z$

Given that $G_1(x,y,z) = \min\{k, G(x,y,z)\}$ and $k>0$, we have

$$G_1(x,y,z) = G(x,y,z)$$

Hence, $G_1(x,y,z)=0$ if $x=y=z$

(ii) Since (X,G) is G-metric space , therefore $G(x,x,y) > 0$

Given that $G_1(x,y,z) = \min\{k, G(x,y,z)\}$ and $k>0$

Similarly, $G_1(x,x,y) = \min\{k, G(x,x,y)\}$

$$G_1(x,x,y) = \min\{k, G(x,x,y)\} > 0$$

Hence , $G_1(x,x,y) > 0$

(iii) Since (X,G) is G-metric space , therefore $G(x,x,y) \le G(x,y,z)$

Given that $G_1(x,x,y) = \min\{k, G(x,x,y)\} \le \min\{k, G(x,y,z)\}$

$$G_1(x,y,y) \leq G_1(x,y,z)$$

(iv) Given that $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$

$$G_1(x,y,z) = \min\{k, G(x,y,z)\} = \min\{k, G(x,z,y)\} = \min\{k, G(y,z,x)\} = \ldots$$

$$G_1(x,y,z) = G_1(x,z,y) = G_1(y,z,x) = \ldots$$

(v) Since (X,G) is a G-metric space, therefore

$$G(x,y,z) \leq G(x,a,a) + G(a,y,z)$$

Given that $G_1(x,y,z) = \min\{k, G(x,y,z)\} \leq \min\{k, G(x,a,a) + G(a,y,z)\}$

$$G_1(x,y,z) \leq \min\{k, G(x,a,a)\} + \min\{k, G(a,y,z)\}$$

$$G_1(x,y,z) \leq G_1(x,a,a) + G_1(a,y,z)$$

Hence G_1 satisfy all the five axioms of G-metric space.

Therefore, G_1 is G-metric on X.

Proposition 3.1.3 [13]. *Let (X,G) be a G-metric space, then the following are equivalent.*

(i) (X,G) *is symmetric.*

(ii) $G(x,y,y) \leq G(x,y,a)$, *for all* $x,y,a \in X$

(iii) $G(x,y,z) \leq G(x,y,a) + G(z,y,b)$, *for all* $x,y,z,a,b \in X$

Proof. (i) $\Rightarrow$ (ii) follows from (3.1.3),

$$G(x,x,y) \leq G(x,y,z), \text{ for all } x,y,z \in X \text{ with } y \neq z.$$

Since (X,G) being symmetric.

$$G(x,y,y) \leq G(x,y,z), \text{ for all } x,y,z \in X \text{ with } x \neq z$$

Put $z = a$, $G(x,y,y) \leq G(x,y,a)$, for all $x,y,a \in X$ with $x \neq a$

(ii) $\Rightarrow$ (i) (X,G) being symmetric when $a = x$ in (ii)

(ii) $\Rightarrow$ (iii) From (ii) of Proposition 3.1.1,

$$G(x,y,z) \le G(x,x,y) + G(x,x,z)$$

Similarly, $G(x,y,z) \le G(x,y,y) + G(z,y,y)$

From (ii), $G(x,y,z) \le G(x,y,a) + G(z,y,b)$

(iii) $\Rightarrow$ (i) follows by taking $a = x$ and $b = y$ in (ii)

$$G(x,y,z) \le G(x,y,x) + G(z,y,y)$$

***G*-ball [13].**

Let (X,G) be a G-metric space then for $x_0 \in X$, $r > 0$, the G-ball with centre x_0 and radius r is

$$B_G(x_0,r) = \{y \in X : G(x_0,y,y) < r\}$$

Proposition 3.1.4 [13]. *Let (X,G) be a G-metric space, then for any $x_0 \in X$ and $r > 0$, we have*

(i) *if* $G(x_0,x,y) < r$ *then* $x, y \in B_G(x_0,r)$,

(ii) *if* $y \in B_G(x_0,r)$ *then there exist a* $\delta > 0$ *such that* $B_G(y,\delta) \subseteq B_G(x_0,r)$.

Proof. (i) follows from (3.1.3), i.e $G(x,x,y) \le G(x,y,z)$ with $y \ne z$

Similarly, $G(x_0,x,x) \le G(x_0,x,y)$ with $x_0 \ne y$

$$G(x_0,x,x) \le G(x_0,x,y) < r$$

$$\Rightarrow x \in B_G(x_0,r)$$

Also, $G(x_0,y,y) \le G(x_0,y,x)$ with $x_0 \ne x$

$$G(x_0,y,y) \le G(x_0,x,y) < r$$

$$\Rightarrow y \in B_G(x_0,r)$$

Hence, if $G(x_0,x,y) < r$, then $x, y \in B_G(x_0,r)$

(ii) follows from the definition of G-metric space, we have

$$G(x,y,z) \leq G(x,a,a) + G(a,y,z) \text{ with, } \delta = r - G(x_0,y,y)$$

Thus, $B_G(y,\delta) \subseteq B_G(x_0,r)$.

Proposition 3.1.5 [13]. *Let* (X,G) *be a* G*-metric space, then for all* $x_0 \in X$ *and* $r > 0$, *we have*

$$B_G(x_0,\frac{1}{3}r) \subseteq B_{d_G}(x_0,r) \subseteq B_G(x_0,r)$$

G-convergent metric space [13].

Let (X,G) be a G-metric space and the sequence $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$.

Proposition 3.1.6 [13]. *Every* G*-metric space* (X,G) *induces a metric space* (X,d_G) *defined by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \quad \forall\ x,y \in X.$$

(i) *if* (X,G) *is symmetric, then* $d_G(x,y) = 2G(x,y,y), \ \forall\ x,y \in X.$

(ii) *if* (X,G) *is not symmetric then, it holds by the* G*-metric properties that*

$$\frac{3}{2}G(x,y,y) \leq d_G(x,y) \leq 3G(x,y,y), \quad \forall\ x,y \in X.$$

Proposition 3.1.7 [13]. *Let* (X,G) *be a* G*-metric space, then for a sequence* $\{x_n\} \subseteq X$ *and point* $x \in X$ *the following are equivalent.*

(i) $\{x_n\}$ *is* G*-convergent to* x.

(ii) $d_G(x_n,x) \to 0$ as $n \to \infty$

(iii) $G(x_n,x_n,x) \to 0$ as $n \to \infty$

(iv) $G(x_n,x,x) \to 0$ as $n \to \infty$

(v) $G(x_m,x_n,x) \to 0$ as $m,n \to \infty$

Proof. The equivalence (i) and (ii) follows from Proposition 3.1.5

i.e $B_G(x_0,\frac{1}{3}r)\subseteq B_{d_G}(x_0,r)\subseteq B_G(x_0,r)$

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) follows from Proposition 3.1.6

$$d_G(x,y)=G(x,y,y)+G(x,x,y)$$

Hence, $d_G(x_n,x)\to 0$ *as* $n\to\infty$

$\Rightarrow$ $G(x_n,x_n,x)\to 0$ as $n\to\infty$ and $G(x_n,x,x)\to 0$ as $n\to\infty$

(iii) $\Rightarrow$ (iv) follows from (iii) of Proposition 3.1.1

$$G(x,y,y)\le 2G(y,x,x)$$

Hence, $G(x_n,x_n,x)\to 0$ as $n\to\infty$ $\Rightarrow$ $G(x_n,x,x)\to 0$ *as* $n\to\infty$

(iv) $\Rightarrow$ (v) follows from (ii) of Proposition 3.1.1,

$$G(x,y,z)\le G(x,x,y)+G(x,x,z)$$

Hence, $G(x_n,x_n,x)\to 0$ *as* $n\to\infty$ and $G(x_n,x,x)\to 0$ *as* $n\to\infty$

$\Rightarrow$ $G(x_m,x_n,x)\to 0$ *as* $m,n\to\infty$

(v) $\Rightarrow$ (ii) follows from Proposition 3.1.6 and (iii) of Proposition 3.1.1.

G-continuous function [13].

Let $(X,G),(X',G')$ be G-metric spaces, Then a function $f:(X,G)\to(X',G')$ be a function, then f is said to be G-continuous at a point $a\in X$ if and only if, given $\in>0$, there exists $\delta>0$ such that $x,y\in X$; and $G(a,x,y)<\delta$ implies $G'(f(a),f(x),f(y))<\in$. A function f is G-continuous at X if and only if it is G-continuous at all $a\in X$.

Proposition 3.1.8 [13]. *Let $(X,G),(X',G')$ be G-metric spaces, then a function $f:X\to X'$ is G-continuous at a point $x\in X$ if and only if it is G-sequentially continuous at x; i.e whenever $\{x_n\}$ is G-convergent to x we have $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 3.1.9 [13]. *Let* (X,G) *be a* G*-metric space, then the function* $G(x,y,z)$ *is jointly continuous in all three of its variables.*

Proof. Suppose $\{x_k\}, \{y_m\}, \{z_n\}$ are G-convergent to x,y,z respectively. Then, by (3.1.5) we have,

$$G(x,y,z) \le G(y,y_m,y_m) + G(y_m,x,z)$$

$$G(z,x,y_m) \le G(x,x_k,x_k) + G(x_k,y_m,z)$$

$$G(z,x_k,y_m) \le G(z,z_n,z_n) + G(z_n,y_m,x_k)$$

So, by adding these three conditions, we get

(3.1.6) $$G(x,y,z) - G(x_k,y_m,z_n) \le G(y,y_m,y_m) + G(x,x_k,x_k) + G(z,z_n,z_n)$$

Similarly,

$$G(x_k,y_m,z_n) \le G(x_k,x,x) + G(x,z_n,y_m)$$

$$G(z,y_m,x) \le G(y_m,y,y) + G(y,x,z)$$

$$G(x,z_n,y_m) \le G(z_n,z,z) + G(z,y_m,x)$$

By adding these,

(3.1.7) $$G(x_k,y_m,z_n) - G(x,y,z) \le G(x_k,x,x) + G(y_m,y,y) + G(z_n,z,z)$$

Combining (3.1.6) and (3.1.7) using (iii) of Proposition 3.1.1

$$\left| G(x_k,y_m,z_n) - G(x,y,z) \right| \le 2\left(G(x,x_k,x_k) + G(y,y_m,y_m) + G(z,z_n,z_n)\right)$$

So, $$G(x_k,y_m,z_n) \to G(x,y,z), \text{ as } k,m,n \to \infty$$

G-Cauchy sequence [13].

Let (X,G) be a G-metric space, then a sequence $\{x_n\} \subseteq X$ is said to be G-Cauchy if for $\in > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n,x_m,x_l) < \in$ for all $n,m,l \ge N$. That is $G(x_n,x_m,x_l) \to 0$ *as* $n,m,l \to \infty$.

Proposition 3.1.10 [13]. In *a G-metric space,* (X,G) *the following are equivalent.*

(i) *The sequence* $\{x_n\}$ *is* G*-Cauchy.*

(ii) *For every* $\in > 0$, *there exist* $N \in N$ *such that* $G(x_n, x_m, x_m) < \in$ *for all* $n, m, l \geq N$.

(iii) $\{x_n\}$ *is a Cauchy sequence in the metric space* (X, d_G).

Remarks.

1. *Every* G *- convergent sequence in a* G*-metric space is* G*-Cauchy.*
2. *If a* G*-Cauchy sequence in a* G*-metric space* (X,G) *contains a* G*-convergent subsequence, then the sequence itself is* G*-convergent.*

G-complete metric [13].

A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

Proposition 3.1.11 [13]. *A* G*-metric space* (X,G) *is* G*-complete if and only if* (X, d_G) *is a complete metric space.*

In 1971, S., Reich gaves some remarks concerning contraction mapping.

Theorem 3.1.1 [9]. *Let* (X,d) *be a complete metric space, and T be a function mapping* X *into itself, satisfy the following condition,*

(3.1.8) $$d(T(x),T(y)) \leq a\,d(x,T(x)) + b\,d(y,T(y)) + c\,d(x,y), \ \ \forall\ x,y \in X$$

where a,b,c *are nonnegative numbers satisfying* $a+b+c<1$.

Then, T *has unique fixed point.*

3.2 Main Results

In this section, we will present fixed point results on a complete G- metric space.

Theorem 3.2.1[13]. *Let* (X,G) *be a complete* G- *metric space, and let* $T:X\to X$ *be a mapping satisfies the following condition*

(3.2.1) $$G(T(x),T(y),T(z))\le k\{G(x,T(x),T(x))+G(y,T(y),T(y))+G(z,T(z),T(z))\}$$

for all $x,y,z\in X$, *where* $k\in\left[0,\frac{1}{3}\right)$. *Then* T *has a unique fixed point (say* u*), and* T *is* G*-continuous at* u.

Proof. Let T satisfy the condition (3.2.1), then for all $x,y\in X$, we get

(3.2.2) $$G(Tx,Ty,Ty)\le k[G(x,Tx,Tx)+2G(y,Ty,Ty)],$$ and

(3.2.3) $$G(Ty,Tx,Tx)\le k[G(y,Ty,Ty)+2G(x,Tx,Tx)].$$

Let (X,G) be symmetric, then from the eq. (i) of Proposition 3.1.6, we have

$$d_G(Tx,Ty)=2G(Tx,Ty,Ty)$$

and by using (3.2.2),

$$d_G(Tx,Ty)\le 2\{k[G(x,Tx,Tx)+2G(y,Ty,Ty)]\}$$

(3.2.4) $$d_G(Tx,Ty)\le k\,d_G(x,Tx)+2k\,d_G(y,Ty),\quad \forall\, x,y\in X$$

Since $0<k+2k<1$, then the condition (3.2.4) become the special case of condition (3.1.8), so the existence and uniqueness of the fixed point follows from Theorem 3.1.1

However, If (X,G) is not symmetric then we note that for all $x,y\in X$

$$d_G(Tx,Ty)=G(Tx,Ty,Ty)+G(Ty,Tx,Tx)\le 3k\,G(x,Tx,Tx)+3k\,G(y,Ty,Ty)$$

Again by the use of definition of metric (X, d_G) and (ii) of Proposition 3.1.6, we get $$d_G(Tx,Ty) \le 2k\, d_G(x,Tx) + 2k\, d_G(y,Ty), \qquad \forall\ x, y \in X$$

Since $0 < 2k + 2k$ need not be less than one. Therefore the above condition gives us no information about this map.

But the existence of a fixed point can be proved using properties of a G-metric.

Let $x_0 \in X$, be an arbitrary point, and define the sequence $\{x_n\}$ by $x_n = T^n(x_0)$ then by the use of condition (3.2.1) we get

$$G(x_n, x_{n+1}, x_{n+1}) \le k\, G(x_{n-1}, x_n, x_n) + 2k\, G(x_n, x_{n+1}, x_{n+1}),$$

Hence, $$G(x_n, x_{n+1}, x_{n+1}) \le \frac{k}{1-2k} G(x_{n-1}, x_n, x_n)$$

Suppose $q = \frac{k}{1-2k}$, as $0 \le k < \frac{1}{3} \Rightarrow 0 \le q < 1$

therefore, $$G(x_n, x_{n+1}, x_{n+1}) \le q\, G(x_{n-1}, x_n, x_n)$$

Similarly, $$G(x_{n-1}, x_n, x_n) \le k\, G(x_{n-2}, x_{n-1}, x_{n-1}) + 2k\, G(x_{n-1}, x_n, x_n)$$

$$G(x_{n-1}, x_n, x_n) \le \frac{k}{1-2k} G(x_{n-2}, x_{n-1}, x_{n-1})$$

$$G(x_{n-1}, x_n, x_n) \le q\, G(x_{n-2}, x_{n-1}, x_{n-1})$$

Hence, $$G(x_n, x_{n+1}, x_{n+1}) \le q^2\, G(x_{n-2}, x_{n-1}, x_{n-1})$$

Continuing in the same manner, we get

(3.2.5) $$G(x_n, x_{n+1}, x_{n+1}) \le q^n\, G(x_0, x_1, x_1)$$

Also, for all $n, m \in \mathbf{N}$, $n < m$ by the use of rectangle inequality and equation (3.2.5) we obtain,

$$G(x_n, x_m, x_m) \le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots + G(x_{m-1}, x_m, x_m)$$

$$G(x_n, x_m, x_m) \le (q^n + q^{n+1} + q^{n+2} + \ldots + q^{m-1}) G(x_0, x_1, x_1)$$

$$\le \frac{q^n}{1-q} G(x_0, x_1, x_1)$$

Since $0 \le q < 1$, $\therefore \lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. Thus $\{x_n\}$ is G-Cauchy sequence, and also (X,G) is complete, therefore there exists $u \in X$ such that $\{x_n\}$ is G-convergent to u.

Next, we will show that u is a fixed point.

Let if possible, $T(u) \ne u$

Then, $$G(x_{n+1}, T(u), T(u)) \le k\{G(x_n, x_{n+1}, x_{n+1}) + 2G(u, T(u), T(u))\}$$

Since the function G is continuous on its variable, therefore by letting $n \to \infty$, we get

$$G(u, T(u), T(u)) \le 2k\, G(u, T(u), T(u)).$$

which is a contradiction to our supposition. Therefore $T(u)$ must be equal to u. This implies u is a fixed point.

Now, we will prove uniqueness of fixed point. For this, we consider two fixed points for T say u and v.

$$G(T(u), T(v), T(v)) \le k\, G(u, T(u), T(u)) + 2k\, G(v, T(v), T(v))$$

As $T(u) = u$ and $T(v) = v$, this implies that

$$G(u, v, v) \le k\, G(u, T(u), T(u)) + 2k\, G(v, T(v), T(v)) = 0,$$

This further implies that $u = v$.

Next we show that T is G-continuous at u, let $\{y_n\} \subseteq X$ be a sequence converges to u in (X,G), then we see that

$$G(u,T(y_n),T(y_n) \le k\{G(u,T(u),T(u)) + 2G(y_n,T(y_n),T(y_n))\}$$

(3.2.6) $$G(u,T(y_n),T(y_n)) \le 2k\,G(y_n,T(y_n),T(y_n))$$

Also by the use of definition of G-metric , we note that

$$G(y_n,T(y_n),T(y_n)) \le G(y_n,u,u) + G(u,T(y_n),T(y_n)),$$

But from equation (3.2.6), we have

$$G(u,T(y_n),T(y_n)) \le 2k\,G(y_n,u,u) + 2k\,G(u,T(y_n),T(y_n))$$

$$G(u,T(y_n),T(y_n)) \le \frac{2k}{1-2k}G(y_n,u,u).$$

As $n \to \infty$, $G(y_n,T(y_n),T(y_n)) \to 0$ and thus by Proposition 3.1.8 $T(y_n) \to u = Tu$, therefore T is G- continuous at u.

Thus the conclusion of theorem holds.

Corollary 3.1[13]. *Let* (X,G) *be a complete* G*- metric spaces, and let* $T: X \to X$ *be a mapping satisfying the following condition for some* $m \in \boldsymbol{N}$

(3.2.7) $$G(T^m(x),T^m(y),T^m(z)) \le k\begin{Bmatrix} G(x,T^m(x),T^m(x)) + G(y,T^m(y),T^m(y) + \\ G(z,T^m(z),T^m(z)) \end{Bmatrix}$$

for all $x,y,z \in X$*, where* $k \in \left[0,\frac{1}{3}\right)$. *Then* T *has unique fixed point (say* u*), and* T^m *is* G*-continuous at* u.

Proof. By above theorem, we can easily note that that T^m has a unique fixed point (say u), that is, $T^m(u) = u$, and $T^m(u)$ is G-continuous at u.

But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u)$,

So, $T(u)$ is another fixed point for T^m and by uniqueness of fixed point $Tu = u$.

Theorem 3.2.2[13]. *Let* (X,G) *be a complete* G*-metric space, and let* $T: X \to X$*, be a mapping satisfying the following condition*

(3.2.8) $$G(T(x),T(y),T(z)) \le \alpha\, G(x,y,z) + \beta \begin{Bmatrix} G(y,T(y),T(y)) + G(z,T(z),T(z)) + \\ G(x,T(x),T(x)) \end{Bmatrix}$$

for all $x,y,z \in X$*, where* $0 \le \alpha + 3\beta < 1$*.Then* T *has unique fixed points (say* u*), and* T *is* G*-continuous at* u*.*

Proof. Suppose that T satisfies condition (3.2.8). Then for all $x, y \in X$

(3.2.9) $$G(Tx,Ty,Ty) \le \alpha\, G(x,y,y) + \beta[G(x,Tx,Tx) + 2G(y,Ty,Ty)],$$

(3.2.10) $$G(Ty,Tx,Tx) \le \alpha\, G(y,x,x) + \beta[G(y,Ty,Ty) + 2G(x,Tx,Tx)]$$

Let (X,G) is symmetric, then from the equation (i) of Proposition 3.1.6,

we get $$d_G(Tx,Ty) = 2G(Tx,Ty,Ty)$$

and by using (3.2.8)

$$d_G(Tx,Ty) \le 2\alpha\, G(x,y,y) + 2\beta\, G(x,Tx,Tx) + 4\beta\, G(y,Ty,Ty)\}$$

(3.2.11) $$d_G(Tx,Ty) \le \alpha\, d_G(x,y) + \beta\, d_G(x,Tx) + 2\beta\, d_G(y,Ty)$$

Since $0 < \alpha + 3\beta < 1$, then the condition (3.2.11) satisfy the condition (3.1.8), so the existence and uniqueness of the fixed point follows from Theorem 3.1.1.

If (X,G) is not symmetric then from Proposition 3.1.6, we find that

$$d_G(Tx,Ty) = G(Tx,Ty,Ty) + G(Ty,Tx,Tx)$$

Using (3.2.9) and (3.2.10), we get

$$d_G(Tx,Ty) \le \alpha\, \left[G(x,y,y) + G(y,x,x)\right] + 3\beta\, G(x,Tx,Tx) + 3\beta\, G(y,Ty,Ty), \ \forall\, x,y \in X$$

Again by the use of definition of metric (X,d_G) and (ii) of Proposition 3.1.6, we get, $d_G(Tx,Ty) \le \alpha\, d_G(x,y) + 2\beta\, d_G(x,Tx) + 2\beta\, d_G(y,Ty), \quad \forall\, x, y \in X.$

Since $0 < \alpha + 2\beta + 2\beta$ need not be less than one, therefore the above condition gives no information about this map.

But existence of fixed point can be proved using property of G-metric.

Let $x_0 \in X$, be an arbitrary point, and define the sequence $\{x_n\}$ by $x_n = T^n(x_0)$,

then by the use of condition (3.2.8), we get

$$G(x_n, x_{n+1}, x_{n+1}) \le \alpha\, G(x_{n-1}, x_n, x_n) + \beta\{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\}$$

$$(1-2\beta)G(x_n, x_{n+1}, x_{n+1}) \le (\alpha + \beta)G(x_{n-1}, x_n, x_n)$$

Hence, $$G(x_n, x_{n+1}, x_{n+1}) \le \frac{\alpha+\beta}{1-2\beta}\, G(x_{n-1}, x_n, x_n)$$

Suppose $q = \dfrac{\alpha+\beta}{1-2\beta}$, as $0 \le \alpha + 3\beta < 1 \;\Rightarrow\; 0 \le q < 1$

therefore, $G(x_n, x_{n+1}, x_{n+1}) \le q\, G(x_{n-1}, x_n, x_n)$

Similarly, $$G(x_{n-1}, x_n, x_n) \le \alpha\, G(x_{n-2}, x_{n-1}, x_{n-1}) + \beta\{G(x_{n-2}, x_{n-1}, x_{n-1}) + 2G(x_{n-1}, x_n, x_n)\}$$

$$G(x_{n-1}, x_n, x_n) \le \frac{\alpha+\beta}{1-2\beta} G(x_{n-2}, x_{n-1}, x_{n-1})$$

$$G(x_{n-1}, x_n, x_n) \le q\, G(x_{n-2}, x_{n-1}, x_{n-1})$$

Therefore, $G(x_n, x_{n+1}, x_{n+1}) \le q^2\, G(x_{n-2}, x_{n-1}, x_{n-1})$

Continuing in the same manner, we get

(3.2.12) $G(x_n, x_{n+1}, x_{n+1}) \le q^n\, G(x_0, x_1, x_1).$

Also for all $n, m \in \mathbf{N}$, $n < m$, by the use of rectangle inequality, we obtain

$$G(x_n, x_m, x_m) \le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots\ldots + G(x_{m-1}, x_m, x_m)$$

and by using (3.2.12), we get

$$G(x_n, x_m, x_m) \le (q^n + q^{n+1} + \dots\dots + q^{m-1}) G(x_0, x_1, x_1)$$

$$\le \frac{q^n}{1-q} G(x_0, x_1, x_1)$$

Since $0 \le q < 1$ $\therefore \lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. Thus $\{x_n\}$ is G-Cauchy sequence and (X,G) is complete therefore there exists $u \in X$ such that $\{x_n\}$ is G-convergent to u in (X,G).

Next we will show that u is a fixed point.

Let if possible, $T(u) \neq u$.

Then, $$G(x_n, T(u), T(u)) \le \alpha G(x_{n-1}, u, u) + \beta\{G(x_{n-1}, x_n, x_n) + 2G(u, T(u), T(u))\},$$

Since the function G is continuous on its variables, therefore by letting $n \to \infty$, we get

$$G(u, T(u), T(u)) \le 2\beta G(u, T(u), T(u)).$$

which is a contradiction to our supposition. Therefore $T(u)$ must be equal to u. This implies u is a fixed point.

Now we prove the uniqueness of fixed point. For this we consider u and v are two fixed points for T. Then

$$G(T(u), T(v), T(v)) \le \alpha G(u, v, v) + \beta\{G(u, T(u), T(u)) + 2G(v, T(v), T(v))\}$$

As $u = T(u)$ and $v = T(v)$, this implies that

$$G(u, v, v) \le \alpha G(u, v, v) + \beta\{G(u, T(u), T(u)) + 2\beta G(v, T(v), T(v))\}$$
$$= 0 + \alpha G(u, v, v)$$

This further implies that $u = v$, since $0 < \alpha < 1$.

Next we show that T is G-continuous at u, let $\{y_n\} \subseteq X$ be a sequence converges to u in (X,G), then we see that

$$G(u,T(y_n),T(y_n)) \le \alpha G(u,y_n,y_n) + \beta\{G(u,T(u),T(u) + 2G(y_n,T(y_n),T(y_n))\}$$

(3.2.12) $G(u,T(y_n),T(y_n)) \le \alpha G(u,y_n,y_n) + 2G(y_n,T(y_n),T(y_n))$

Also, by the use of definition of G-metric, we note that

$$G(y_n,T(y_n),T(y_n) \le G(y_n,u,u) + G(u,T(y_n),T(y_n)),$$

But from equation (3.2.12) ,we have

$$G(u,T(y_n),T(y_n)) \le \alpha G(u,y_n,y_n) + 2\beta G(y_n,u,u) + 2\beta G(u,T(y_n),T(y_n))$$

$$G(u,T(y_n),T(y_n)) \le \frac{\alpha}{1-2\beta} G(u,y_n,y_n) + \frac{2\beta}{1-2\beta} G(y_n,u,u) .$$

As $n \to \infty$, $G(y_n,T(y_n),T(y_n)) \to 0$, and thus by Proposition 3.1.8, $T(y_n) \to u = Tu$. Therefore T is G-continuous at u.

Thus the conclusion of theorem holds.

Corollary 3.2 [13]. *Let (X,G) be a complete G- metric spaces, and let $T: X \to X$ be a mapping satisfying the following condition for some $m \in$ **N***

(3.2.13) $G(T^m(x),T^m(y),T^m(z)) \le \alpha G(x,y,z) + \beta \begin{Bmatrix} G(x,T^m(x),T^m(x) + \\ G(y,T^m(y),T^m(y) + \\ G(z,T^m(z),T^m(z)) \end{Bmatrix}$

for all $x,y,z \in X$, where $0 \le \alpha + 3\beta < 1$.Then T has a unique fixed point (say u), and T^m is G-continuous at u.

Proof. By above theorem, we can easily seen that T^m has a unique fixed point (say u), that is, $T^m(u) = u$, and $T^m(u)$ is G-continuous at u. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u)$

So, $T(u)$ is another fixed point for T^m and by uniqueness of fixed point $Tu = u$.

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