METHODS FOR SOLVING DECISION-MAKING PROBLEMS UNDER NEUTROSOPHIC ENVIRONMENT

A Thesis
Submitted in partial fulfillment of the
requirement for the award of the degree
of
Doctor of Philosophy
in
Mathematics

by
NANCY
Roll No. 901511009

Under the guidance of
Dr. Harish Garg
Assistant Professor
School of Mathematics

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

Thapar Institute of Engineering & Technology
(Deemed to be University)
Patiala – 147004 (Punjab)
INDIA
April 2019

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Methods For Solving Decision-Making Problems Under Neutrosophic Environment**" in partial fulfillment of the requirement for the award of degree of Degree of Philosophy and submitted in the School of Mathematics (SoM), Thapar Institute of Engineering & Technology, Patiala is an authentic record of my own carried out during a period from January, 2016 to April, 2019 under the supervision of Dr. Harish Garg, Assistant Professor, SoM, Thapar Institute of Engineering & Technology, Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Nancy

(NANCY)

Reg. No. 901511009

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Harish

Date: April 26, 2019

(Dr. Harish Garg)

Supervisor

Abstract

Multiple-criteria decision-making (MCDM) problems are the imperative part of modern decision theory where a set of alternatives has to be assessed against the multiple influential attributes before the best alternative is selected. In a decision-making(DM) process, an important problem is how to express the preference value. Due to the increasing complexity of the socioeconomic environment and the lack of knowledge or the data about the DM problems, it is difficult for the decision maker to give the exact decision as there is always an imprecise, vague or uncertain information. To deal with this, the theory of the fuzzy sets (FSs) or its extensions such as intuitionistic fuzzy sets (IFSs), interval-valued intuitionistic fuzzy sets (IVIFSs), type-2 fuzzy sets (T2FSs), etc., are widely used by the researchers so as to minimize the uncertainty level. During the last decades, the researchers are paying more attention to these theories and have successfully applied it to the various situations in the DM process. Nevertheless, neither the FS nor IFS theory is able to deal with indeterminate and inconsistent information. For instance, we take a person giving their opinion about an object with 0.5 being the possibility that the statement is true, 0.7 is the possibility that the statement is false and 0.2 being the possibility that he or she is not sure. To resolve this, Smarandache in 1998, introduced a new component called the "indeterminacy-membership function" and added the "truth membership function" and "falsity membership function", all which are independent components lying in $]0^-, 1^+[$, and hence the corresponding set is known as a neutrosophic set (NS), which is the generalization of the IFS and FS. However, without specification, NSs are difficult to apply to real-life problems. Thus, a particular case of the NS called a single-valued NS (SVNS) and the interval neutrosophic set (INS) has been proposed by the researchers.

After this pioneering work, researchers have been engaged in extensions and applications to different disciplines. However, the most important task for the decision-maker is to rank the objects. For this, researchers have made efforts to enrich the concept of information measures as well as aggregation operators in neutrosophic environments. Among these, an aggregation operator is an important part of the DM which usually takes the form of mathematical function to aggregate all the input individual data into a single one. However, an information measure such as the distance and similarity measures, complementary to each other, are defined to differentiate between the two or more objects. Thus in order to handle the information in a more accurate and certain manner, there is a need to plan/adopt suitable methodologies to solve the DM problems.

The objective of this research work is to develop some new methodologies under the environment of the NS by utilizing available information and uncertain data. For it, we define several information measures such as distance measures, similarity measures, divergence measures as well as aggregation operators for solving the DM problems. The various desirable relations between the proposed measures and operators are studied. Later, we developed some new concept of new theory named as Possibility linguistic NSs by embedding the features of the possibility degrees to the linguistic features of the NSs. Afterward, based on all the developed theories, we present an efficient method to solve the DM problems in which information related to each alternative is assessed under the consideration of the experts' features. Several real-life practical examples are taken to demonstrate the approach and compared their performance with some of the existing studies.

The present thesis is organized into eleven chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of DM approaches by using several approaches is presented in the first chapter. In **Chapter 2**, the basics and preliminaries related to the NSs are given.

Chapter 3 presents some new aggregation operators by hybridizing the averaging and geometric aggregation operators with an attitude character parameter. The influence of this parameter will also be analyzed and shows its property. The advantages of adding the parameter are to control the effects of the decision parameters towards the biased ones.

Further, some properties of proposed operators are investigated. To address more features, an algorithm is presented to addressed the problems using SVNS and INS features into the DM problem. Last, a numerical example is presented to show its superiority with respect to the several existing approaches.

In **Chapter 4**, we developed some new aggregation operators based on the Frank t-norm operational law under the SVNS environment. The Frank t-norm operation has an additional parameter which can give a flexible environment to the decision makers to choose their decisions, according to their desired goals. Based on the proposed aggregation operators, an algorithm for solving the MCDM approach is presented and illustrated with a numerical example to show its applicability.

In **Chapter 5**, by taking the features of the Muirhead mean (MM) to consider interrelationships among any number of arguments assigned by a variable vector, we introduce two new prioritized MM aggregation operators, such as the single-valued neutrosophic (SVN) prioritized MM (SVNPMM) and SVN prioritized dual MM (SVNPDMM) under SVN set environment. In addition, some properties of these new aggregation operators are investigated and some special cases are discussed. Furthermore, we propose a new method based on these operators for solving the MCDM problems and illustrate with a numerical example.

In **Chapter 6**, we developed a non-linear programming model based TOPSIS approach for solving MCDM problems with incomplete weight information. The relative closeness coefficient(RCC) degree of the TOPSIS method is formulated based on the distance measures. Further, the importance of the attribute weights is taken in the form of interval numbers rather than a real single number. Several special cases of the proposed approach are discussed in detail. In the developed non-linear programming model of RCC, a Charles-Chooper approach is utilized to solve the problem. The applicability of this presented approach is demonstrated with a numerical example of a power generation project and computed their results with several existing studying results.

Chapter 7 presents an axiomatic definition of divergence measure for SVNSs. The properties of the proposed divergence measure have been studied. Further, we develop a novel technique for order preference by similarity to ideal solution (TOPSIS) method for solving single-valued neutrosophic multi-criteria DM with incomplete weight information. Finally, a numerical example is presented to verify the proposed approach and to present its effectiveness and practicality.

In **Chapter 8**, we developed some bi-parametric generalized distance measure between the pairs of SVNSs. The presented measures utilized the concept of L_p norm and the degree of uncertainties. The various properties and relationships between them are investigated. Later on, the applicability of these measures is explained with the help of pattern recognition and medical diagnosis problems.

Chapter 9 discusses the new concept of the logarithm operational laws for the pairs of SVNNs. Further, based on these proposed laws, some new aggregation operators named as logarithmic weighted averaging and geometric operators for SVNSs are defined and investigated their properties. To further strengthen their applicability, we developed an algorithm based on these operators for solving MCDM problems and demonstrate it with a numerical example. Finally, the influence of the logarithm is examined to show the behavior of the decision makers towards their optimal decision.

Chapter 10 deal with DM problems to address qualitative information rather than quantitative information. For it, a concept of linguistic SVNS is utilized with linguistic variables to represent the data. Further, to add prioritized factor during analyzing the data, we developed some new laws and hence based on it, some new prioritized weighted averaging and geometric aggregation operators are developing to address the problems with linguistic SVN (LSVN) information. The technique corresponding to them is demonstrated with a numerical example and compared with the existing studies.

In **Chapter 11**, we present a new theory named as Possibility LSVNS(PLSVNS) for evaluation of DM problems with the qualitative preference values. The presented idea considered the degree of the possibility value towards each linguistic features of SVNS. Based on the features of PLSVNS, theories of COPRAS method and weighted averaging and geometric aggregation operators have been defined where the weight vector of the attributes are computed with some information measures. The applicability as well feasibility of the developed methods are explained with a numerical example. Finally, the advantages of the presented concepts are explained.

Acknowledgments

At the very outset, acclaims and gratitude to God, the Almighty for His showers of blessings all throughout my research work to complete the research successfully. Without his will, I would never found the right path.

I would like to acknowledge my indebtedness and render my warmest thanks to my enthusiastic supervisor, Dr. Harish Garg, Assistant Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, not only for his tremendous support but also for the patience guidance, encouragement and advice he has provided throughout my time as his student. Because he is very dedicated to his work so he has constantly motivated me to remain focused on achieving my goals. He has given so much of his precious time and exertion to groom and shapes my research ideas. His constructive criticism made me sail through trails and turbulence of the research period smoothly.

I am highly obliged to Dr. Navpreet Singh Noori, Professor, Department of Mathematics, Punjabi University, Patiala, for his constant motivations and for encouraging me to embark on the research path. I am grateful to all members of the thesis committee whose remarks provided me professional guidance.

Special thanks to my friends, Deepika, and Sonali for listening, offering me advice and supporting me through the entire process. To my colleagues Sukhveer Singh, Rishu Arora, Kamal Kumar, Gagandeep Kaur and Dimple Rani for their scientific help during the study.

I owe my deep and special thanks to my family, my Grandmaa, Maa and Mama Ji for their tremendous sacrifice, for selfless encouragement to explore new directions in life and seek my own destiny. I express a profound sense of thanks to my sisters, brother and Harpal for their immense co-operation, inspiration and affection during my research

proposal.

Research fellowship provided by UGC, New Delhi, India during period of my research is thankfully acknowledged.

Nancy

Patiala
(Nancy)

April 26, 2019

List of Publications

Refereed Journals

(J1) Harish Garg and Nancy, Algorithms for possibility linguistic single-valued neutrosophic decision-making based on COPRAS and aggregation operators with new information measures, *Measurement*, 138, 278 - 290, 2019, doi: 10.1016/j.measurement.2019.02.031 (**Impact Factor: 2.791**).

(J2) Nancy and Harish Garg, A novel divergence measure and its based TOPSIS method for multi criteria decision-making under single-valued neutrosophic environment, *Journal of Intelligent & Fuzzy Systems*, 36(1), pp. 101 - 115, 2019, doi: 10.3233/JIFS-18040 (**Impact Factor: 1.637**).

(J3) Harish Garg and Nancy, Some hybrid weighted aggregation operators under neutrosophic set environment and their applications to multicriteria decision-making, *Applied Intelligence*, 48(12), 4871 - 4888, 2018, doi: 10.1007/s10489-018-1244-9 (**Impact Factor: 2.882**)

(J4) Harish Garg and Nancy, New Logarithmic operational laws and their applications to multiattribute decision making for single-valued neutrosophic numbers, *Cognitive Systems Research*, 52, 931 - 946, 2018, doi: 10.1016/j.cogsys.2018.09.001 (**Impact Factor: 1.384**)

(J5) Harish Garg and Nancy, Linguistic single-valued neutrosophic prioritized aggregation operators and their applications to multiple-attribute group decision-making, *Journal of Ambient Intelligence and Humanized Computing*, 9(6), 1975 - 1997, 2018, doi: 10.1007/s12652-018-0723-5 (**Impact Factor: 1.910**)

(J6) Harish Garg and Nancy, Multi-criteria decision-making method based on prioritized Muirhead mean aggregation operator under Neutrosophic set environment, *Symmetry*, 10(7), 280, 2018; doi: 10.3390/sym10070280 (**Impact Factor: 2.143**).

(J7) Harish Garg and Nancy, Non-linear programming method for multi-criteria decision making problems under interval neutrosophic set environment, *Applied Intelligence*, 48(8), 2199 - 2213, 2018, doi: 10.1007/s10489-017-1070-5 (**Impact Factor: 2.882**)

(J8) Harish Garg and Nancy, Some new biparametric distance measures on single-valued neutrosophic sets with applications to pattern recognition and medical diagnosis, *Information*, 8(4), 162, 2017; doi:10.3390/info8040162

(J9) Nancy and Harish Garg, Novel single-valued neutrosophic decision making operators under Frank norm operations and its application, *International Journal for Uncertainty Quantification*, 6(4), 361 - 375, 2016, doi: 10.1615/Int.J.UncertaintyQuantification.2016018603 (**Impact Factor: 3.259**).

(J10) Nancy and Harish Garg, An improved score function for ranking neutrosophic sets and its application to decision - making process, *International Journal for Uncertainty Quantification*, 6(5), 377 - 385, 2016, doi: 10.1615/Int.J.UncertaintyQuantification.2016018441 (**Impact Factor: 3.259**).

Table of Contents

Abstract i

Acknowledgments v

List of Publications vii

Table of Contents ix

List of Tables xvi

List of Figures xvii

1 Introduction 1

- 1.1 Literature review . . . 2
  - 1.1.1 Review of aggregation operators . . . 3
  - 1.1.2 Review of information measures . . . 4
  - 1.1.3 Review of TOPSIS approach . . . 5
  - 1.1.4 Review of linguistic approaches . . . 7
- 1.2 Gaps and motivation towards the work . . . 10
- 1.3 Objective and Structure of the Thesis . . . 12
  - 1.3.1 Objective of the Thesis . . . 12
  - 1.3.2 Structure of the Thesis . . . 13

2 Preliminaries 17

- 2.1 Neutrosophic sets . . . 17
  - 2.1.1 Single-valued Neutrosophic set . . . 18
  - 2.1.2 Interval Neutrosophic set . . . 18
  - 2.1.3 Linguistic single-valued neutrosophic set . . . 20

2.2 Information Measures 21
    2.2.1 Distance measures 21
    2.2.2 Similarity measures 23
2.3 Archimedean t-norms and Archimedean t-conorms 24
2.4 Aggregation operator 27
2.5 Description of the multiattribute decision making 30

3 Hybrid weighted aggregation operators under neutrosophic set environment and their applications to decision-making process 33

3.1 Introduction 33
    3.1.1 Motivation towards the proposed aggregation operators 34
3.2 Aggregation operators with SVN information 35
    3.2.1 Hybrid weighted aggregation operator 35
    3.2.2 Hybrid ordered weighted aggregation operator 42
    3.2.3 Superiority of the proposed aggregation operators 44
3.3 Aggregation operators with INS information 45
3.4 Proposed approaches based on developed operators 47
    3.4.1 Approach I 47
    3.4.2 Approach II 49
3.5 Illustrative example 50
    3.5.1 A case study 50
    3.5.2 Discussion 55
    3.5.3 Sensitivity Analysis 56
3.6 Conclusion 60

4 Novel Frank t-norm based aggregation operators for single-valued neutrosophic numbers with their applications 61

4.1 Introduction 61
4.2 Frank AT and AC based operational laws for SVNNs 62
    4.2.1 Operational laws for SVNNs 63
    4.2.2 Properties of the operational laws 66
4.3 Aggregation operators based on Frank operational laws for SVNSs 69

4.4 Decision making method based on proposed operators 74
4.5 Illustrative example 76
4.5.1 Sensitivity Analysis 77
4.6 Conclusions 78

5 Prioritized Muirhead Mean Aggregation Operator under Neutrosophic Set Environment and its applications 81

5.1 Introduction 81
5.2 Neutrosophic Prioritized Muirhead Mean Operators 82
5.2.1 Single-Valued Neutrosophic Prioritized Muirhead Mean Operator . . 82
5.2.2 Single-Valued Neutrosophic Prioritized Dual Muirhead Mean Operator 91
5.3 Multi-criteria decision-making approach based on proposed operators 94
5.3.1 Proposed decision-making approach 95
5.3.2 Illustrative example 96
5.3.3 Comparison Study 98
5.3.4 Influence of Parameter P on the Decision-Making Process 99
5.3.5 Further Discussion 99
5.4 Conclusions 100

6 Non-linear programming method for multi-criteria decision making problems under interval neutrosophic set environment 105

6.1 Introduction 105
6.2 Proposed Non-linear Programming model for MCDM problem 107
6.2.1 Description of MCDM problem under INSs 107
6.2.2 Determination of ideal solutions 107
6.2.3 Separation measurement values 108
6.2.4 Relative-closeness coefficient and its monotonicity 108
6.2.5 Auxiliary NLP models 110
6.3 Additional features of the proposed NLP models 113
6.4 Illustrative example 116
6.4.1 A case study 116
6.4.2 Comparison Analysis of the obtained results 119

6.4.3 Superiority of the proposed approach 120
6.5 Conclusion 123

7 A novel divergence measure and its based TOPSIS method for single-valued neutrosophic sets 125

7.1 Introduction 125
7.2 Proposed divergence measure 126
7.3 TOPSIS approach based on proposed divergence measure 135
7.3.1 Maximizing divergence method for determine the weights 136
7.3.2 Proposed TOPSIS approach 137
7.4 Illustrative Example 139
7.4.1 A case study 139
7.4.2 Test Criteria for proposed approach 142
7.4.3 Superiority of the proposed approach over the existing approaches . 143
7.5 Conclusion 146

8 Bi-parametric distance measures on single-valued neutrosophic sets with applications to Pattern Recognition and Medical Diagnosis 147

8.1 Introduction 147
8.2 Some New Distance Measures between SVNSs 148
8.2.1 Proposed Distance measures 148
8.2.2 Advantages of the proposed measures 153
8.2.3 Relation between the proposed measures 154
8.3 Generalized Distance Measure 159
8.4 Illustrative examples 162
8.4.1 Application of Distance Measure in Pattern Recognition 162
8.4.2 Comparative analysis of the results with existing measures 163
8.4.3 Application of Distance Measure in Medical Diagnosis 164
8.4.4 Comparative analysis of the results with some existing approaches . 165
8.4.5 Effect of the Parameters p and t on the Ordering 166
8.4.6 Advantages of the Proposed Method 167
8.5 Conclusions 168

9 Logarithmic operational laws for single-valued neutrosophic numbers and their applications to decision making 171

9.1 Introduction 171

9.2 Logarithmic operational laws for SVNSs 172

9.2.1 Logarithmic Operational laws 172

9.2.2 Aggregation operators 179

9.3 Proposed MADM method 186

9.4 Illustrative example 187

9.4.1 Case study 188

9.4.2 Validity test 191

9.4.3 Influence of logarithm operation and λ selection in practice 193

9.4.4 Comparison with other existing methods 196

9.5 Conclusion 198

10 Linguistic single-valued neutrosophic prioritized aggregation operators and their applications to group decision-making 199

10.1 Introduction 199

10.2 Operational laws for Linguistic SVNS 201

10.3 Prioritized AO for linguistic single-valued neutrosophic sets 205

10.3.1 Prioritized weighted averaging operator 206

10.3.2 Prioritized weighted geometric operator 210

10.3.3 Fundamental properties of the developed operators 212

10.4 An approach based on proposed operators for solving DM problems 222

10.4.1 Proposed approach 222

10.4.2 Illustrative example 224

10.4.3 Comparative Analysis 226

10.5 Conclusion 227

11 Possibility linguistic single-valued neutrosophic sets and its application 231

11.1 Introduction 231

11.2 Possibility linguistic SVN sets 234

11.3 Information measures and AOs for the proposed PLSVNSs 239

- 11.3.1 Distance measures 239
- 11.3.2 Aggregation operators for PLSVNSs 240
- 11.4 COPRAS method with PLSVN information 244
- 11.5 Approaches for solving DM problems 247
- 11.5.1 Determination of the weight information 247
- 11.5.2 Approach based on COPRAS method 249
- 11.5.3 Approach based on the AOs 251
- 11.6 Numerical example 254
- 11.6.1 Results based on COPRAS method 255
- 11.6.2 Results based on AOs 256
- 11.6.3 Comparative analysis 257
- 11.6.4 Advantages of proposed approaches 259
- 11.7 Conclusion 260

Bibliography **265**

List of Tables

2.1 Some AT and AC with their relative additive generators 26

3.1 Aggregated number and score value for different value of λ 39

3.2 Neutrosophic decision maker matrix 52

3.3 Neutrosophic aggregated results by using aggregated operators($\lambda = 0.5$) . . 52

3.4 Score values of aggregated SVNNs 53

3.5 Ordering of the alternatives under SVNS environment 54

3.6 Information about each alternative under five characteristics in form of INNs 54

3.7 Neutrosophic aggregated results by using aggregated operators($\lambda = 0.5$) . . 57

3.8 Score values of aggregated INNs 58

3.9 Ordering of the alternatives under INS environment 58

3.10 Effect of parameter λ on score values and ranking 59

4.1 Comparative study with the existing approaches 79

4.2 Effect of the parameter λ on ranking of the alternatives 80

5.1 Single-valued neutrosophic decision making matrix. 97

5.2 Ranking results of alternatives using proposed operators for different values of P. 102

5.3 Comparison of different approaches and aggregation operators. 103

6.1 Interval valued neutrosophic DM matrix 117

6.2 Comparative study with existing approaches 124

8.1 Measurement values and ranking order of Pattern recognition Example . . . 164

8.2 Rating values of diseases in terms of SVNNs 164

8.3 Measurement values and ranking order of Medical diagnosis example 165
8.4 Weighted measurement values and ranking order of Medical diagnosis example 165
8.5 Comparison of diagnosis result using existing measures. 166
8.6 Results of classification of given sample using proposed distance measure. . 167
8.7 Diagnosis result on basis of proposed distance measure. 169

9.1 Neutrosophic aggregated results by averaging operators 197
9.2 Neutrosophic aggregated results by geometric operators 197
9.3 Ordering of the alternatives 198

10.1 Linguistic neutrosophic decision making matrices given by decision makers . 228
10.2 Aggregated Linguistic neutrosophic matrix by using LSVNPOWA operator 229
10.3 Aggregated Linguistic neutrosophic matrix by using LSVNPOWG operator 229
10.4 Aggregated Linguistic neutrosophic decision matrix by existing LNNWAA operator 229

11.1 PLSVN decision matrix given by decision makers 261
11.2 Aggregated decision matrix by PLSVNWA operator used in COPRAS method 262
11.3 Weighted decision matrix 262
11.4 Normalized PLSVN decision matrix for decision makers 263
11.5 Aggregated decision matrix by PLSVNWA operator 264
11.6 Aggregated decision matrix by PLSVNWG operator 264

List of Figures

11.1 Flowchart of the proposed approach based on COPRAS method 250
11.2 Flowchart of the proposed approach based on AOs 253

Chapter 1

Introduction

Decision-making (DM) is a subjective process utilized in both businesses and academic classes for the design of a strategy among the given alternatives scenario by processing the different preferences of the expert(s). To complete the process, the most important problem is to choose the appropriate decision about the task and to find the optimal results. DM is a crucial task for all those professions where experts apply their knowledge in a given zone to take an appropriate decision. However, in many situations, the alternatives among which one must choose or make decisions need to be evaluated in light of multiple criteria. For it, generally, a set of decision makers has taken to evaluate the given alternatives. The focus of such a process is to assign the preferences of decision makers together to solve a specific issue by following some general principle of the DM process.

The general process of multicriteria decision making (MCDM) or multiattribute decision making (MADM) problem consists a set of alternatives $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$ and criteria $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}$ such that it is divided into two mutually disjoint sets namely, the cost and the benefit criteria. The importance factor of these criteria is given in the form of weight vector $(w_1, w_2, \ldots, w_m)$ such that $w_j > 0$ and $\sum\limits_{j=1}^{n} w_j = 1$. The decision matrix corresponding to given alternatives is given as:

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \ldots & \mathcal{G}_n \end{array} \\ \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{m1} & \mathcal{A}_{m2} & \ldots & \mathcal{A}_{mn} \end{pmatrix} \end{array}$$

where $\mathcal{A}_{ij}$ represents the preference of decision maker for alternative $\mathcal{A}_i (i = 1, 2, \ldots, m)$ over the criteria $\mathcal{G}_j (j = 1, 2, \ldots, n)$.

However, due to various constraints in day-to-day life, decision makers may unable to give their judgments in exactly crisp form. Thus, to handle it, they prefer to give their preferences under the uncertain and imprecise in nature by using the theory of fuzzy set(FS)[195]. Using this theory, decision makers use the membership degree(MD) to represent their judgment. But with the growing complexity, there always exists a degree of hesitancy between the preferences of the DM and hence the analysis conducted under such circumstances is not ideal. To address it, an important extension of FSs such as intuitionistic fuzzy set(IFS) [7], type-2 fuzzy set (T2FS) [100], hesitant fuzzy set (HFS) [151], the interval-valued IFS (IVIFS) [6] and so on appears simultaneously. All the extensions are characterized by non-membership degree (NMD) along with MD such that their sum is less than one.

The theories of FS, IFS, and IVIFS are incapable of managing a certain kind of uncertainties such as indeterminate and inconsistent information. To get rid of such uncertainties, a new theory of Neutrosophic sets (NSs) was introduced by Smarandache [141], which is a branch of philosophy, studies the nature and scope of neutralities, and it is a generalization of dialectics. In NSs, the uncertain information is represented as the tuple of membership, non-membership and indeterminacy degrees, all are independent and lies in $]0^-, 1^+[$. However, without specification, NSs are difficult to apply to real-life problems. Thus, an extension of the NS called single-valued neutrosophic set (SVNS) [155] and interval neutrosophic set (INS) [154] theories have been proposed. The aim of this work is to develop some novel techniques to access the best alternative(s) for the decision makers under the NS environment. Brief literature on various issues related to the MCDM process by using existing methods have been reviewed and are given section-wise hereafter.

1.1 Literature review

In this section, we have briefly reviewed the MCDM or MADM approaches under the different scenario summarized as follows.

1.1.1 Review of aggregation operators

The aggregation operators (AOs) is one of the most collective phase and process during ordering the alternatives. The basic principle of AOs is to aggregate the various values into collective ones. Since in MADM process, these always occur more than one attribute values towards a single alternative and hence the process of AOs play a significant role between them. In other words, to evaluate the given information in DM, the important aspect of solving the problem is to design an appropriate mathematical function which aggregates the different preference of the decision makers into the collective ones. Often, AOs are based on the arithmetic, geometric, and integrals means, etc. In that direction, Ye [175] presented the operational laws of SVNSs and SVN weighted average and geometric (WAG) AOs denoted by SVNWA and SVNWG. Liu et al. [79] developed some generalized neutrosophic AOs based on Hamacher operations and named as SVN Hamacher weighted averaging (SVNHWA), SVN Hamacher ordered weighted averaging (SVNHOWA), SVN Hamacher weighted geometric (SVNHWG) and SVN Hamacher ordered weighted geometric (SVNHOWG). Peng et al. [113] defined the improved operations of SVN numbers (SVNNs) and developed their corresponding ordered WAG AOs. Abdel-Basset, Manogaran, Gamal and Smarandache [2] solved the supplier selection problem using the neutrosophic set environment. Peng and Liu [116] presented an algorithm for solving DM problem under neutrosophic soft set environment. Ma, Wang, Wang and Wu [94] presented solved the MCDM problem under an interval neutrosophic linguistic environment. Later on, some different kinds of AOs have been proposed by the authors in [69, 79, 82, 124]. Apart from it, there are many AOs proposed by the researchers such as weighted averaging and weighted geometric operators [113, 175, 182], Hamacher operations based generalized AOs [79], the generalized AOs based on the triangular norms[5, 78, 85, 202], prioritized AOs [87, 166, 186], operators based on exponential or logarithmic operations[92, 133, 144, 184], Einstein AO's[67], AO's based on Dombi and Schweizer-Sklar norms operatos[24, 80, 163] to aggregate different neutrosophic values.

All these above operators, aggregate the given criterion values without considering the precedence relationship among them. To get rid of this flaw, Yang and Li [170] extends the power operator to NS domain which can consider the importance of attributes by considering the support over each other and relieve the influence of unreasonable attribute values of different alternatives given by decision-makers. Wu et al. [166] defined the prioritized WAG operators for SVNNs. Liu and Wang [87] developed the prioritized ordered WAG operators while Ji et al. [57] established the single-valued prioritized Bonferroni mean operator by using the Frank operations. Liu and Wang [86] applied the Bonferroni mean to neutrosophic environment and introduce the SVN normalized weighted Bonferroni mean (SVNNWBM) operator. Wang, Yang and Li [159] proposed the Maclaurin symmetric mean (MSM) aggregation operators to capture the correlation between the aggregated arguments. Liu and Liu [81] developed the prioritized ordered weighted/geometric operators in the neutrosophic environment and Ji et al. [57] established the SVN Frank normalized prioritized Bonferroni mean (SVNFNPBM) operator by using the Frank operations. Apart from this, researchers have proposed many efficient and practical aggregating operators for the problems having correlated criterion, such as the power average operator [60, 76, 84], Heronian mean (HM) [69], Bonferroni mean (BM) operator [57, 86, 164, 171], Choquet integral operators [74, 135, 142], Muirhead mean (MM) operators [88]. A bibliometric analysis of NS is presented in Peng and Dai [115].

1.1.2 Review of information measures

Distance and similarity measures, complementary to each other, are defined to differentiate between the two or more objects. These two measures can be considered as two diverse perspectives of discrimination. The similarity measure utilized to show the proximity whereas the distance measure is utilized to show the contrast between the objects. Since the distance and similarity measures play a significant role in real life, numerous analysts have been defined by various distance and similarity measures in different environments. Under the neutrosophic domain, several researchers have presented different kinds of information measures to solve DM problems. For example, Broumi and Smarandache [17] introduced the Hausdorff distance, while Majumdar [95] presented the Hamming and Euclidean distance for comparing the SVNSs. Ye [173] presented the concept of correlation for single-valued neutrosophic numbers (SVNNs). Additionally, Ye [181] improved the concept of cosine similarity for SVNSs, which was firstly introduced by Kong et al. [62] in a neutrosophic environment. Peng and Dai [114] presented an approach based on similarity measures and TOPSIS ("Technique for order preference by similarity to ideal solution") to solve the DM problem under the SVNS environment. Aside from these, various authors have incorporated the idea of NS theory into the similarity measures [96, 176], distance measures [54, 177], the cosine similarity measure [17, 18, 181]. Thus, it has been concluded that the information measures such as entropy, divergence, distance, similarity, etc., are of key importance in a number of theoretical and applied statistical inference and data processing problems.

1.1.3 Review of TOPSIS approach

The approaches described in Section 1.1.2 are based on the distance/similarity measures to compute the similarities between two sets, by considering only one reference point. However, in our day-to-day life, we all are more curious to select the optimal alternative(s) which is more consistently performed based on the ranking values that are only makes the benefit conceivable but also keep away from as much as a hazard as could be allowed. To deal with this, numerous approaches have developed time to time in the literature such as ELECTRE("ELimination and Choice Expressing REality")[12], COPRAS("Complex proportional assessment")[198], (PROMETHEE)("Preference Ranking Organisation Method for Enrichment Evaluations")[99], VIKOR("VlseKriterijuska Optimizacija I Komoromisno Resenje ")[109], TOPSIS("Technique for order preference by similarity to ideal solution")[56] and so on. Among them, TOPSIS is one of the most successful approaches to deal with it under the underlying principle that it selects the best alternative(s) which has the smallest distance from the PIS("Positive Ideal Solution") and the largest distance from the NIS("Negative Ideal Solution"). In it, PIS is recognized as "speculative elective", which is framed as a composite of the best-characterized rating values, that is, it maximizes the benefit criterion values while NIS represents the worst exhibited rating values, that is, it minimizes the cost criterion values. The prominent characteristic of this approach is that they not only take into account the distance between the sets but also examine the similarity or dissimilarity between the sets so as to avoid the conclusion based on the small distance or large similarity. Further, by considering both the positive and negative ideals in this approach, it will give a suitable choice to the decision makers during ranking the alternatives. Keeping these features in mind, researchers have investigated their problems by enriching the TOPSIS method.

Traditionally, TOPSIS method utilizes crisp strategies to evaluate the objects. However, with the complexities in the data system, it is difficult for the system analyst to record the data and hence uncertainty plays a role. To address it, researchers have utilized IFS and IVIFS to deal the fuzzy information into the system and hence presented the TOPSIS approaches under the evaluations of different problems like evaluation and selection of advanced manufacturing technology [97]; for ranking of the life cycle sustainability performance of different pavement choices [64]; selection process for group DM without aggregating the different decision matrices [194]; applied in portfolio selection problem [59]; selection of knowledge management system by combining the quality function deployment with IFS TOPSIS [68]. Apart from them, many other approaches related to TOPSIS method under IFS/IVIFS environment have been proposed in [21, 22, 26, 27, 32, 33, 35, 36, 42, 43, 45, 50, 65, 66, 104, 108, 130, 153, 162, 201].

As IFS or IVIFS does not handle the complete information about the fuzzy information. However, SVNS or INS is the successful extensions of the existing theories (IFS or IVIFS is a special case of SVNS or INS respectively) to describe the information by using incomplete, indeterminate and inconsistent. Therefore, researchers have presented their theories to extend the TOPSIS approach with SVNS information. For example, Chi and Liu [28] presented a TOPSIS method by utilizing maximizing deviation strategy for solving the DM problems under the INS environment. Sahin and Yiğider [134] presented the TOPSIS approach under SVNS environment for solving the supplier selection problem. Zhang and Wu [202] established a method for solving MCDM where the information about the attribute weights are incomplete. In it, they design an optimization model to compute the criteria weight and have a TOPSIS method to rank alternatives under SVNS and INS environment. Tian, Zhang, Wang, Wang and Chen [150] introduced a comprehensive method for manage the MCDM with INS, involves the two mathematical models for finding optimal weights and form weighted closeness coefficient on the basis of TOPSIS and cross-entropy of IVNSs. Biswas et al. [14] introduced the TOPSIS method for solving group DM problems, where attributes values are represented in terms of SVNNs. Reddy et al. [127] built up the MCDM method with the association of Analytic Hierarchy Process(AHP) and TOPSIS technique to rank the given alternatives under INS environment. Pouresmaeil et al. [117] presented a method to solve the MCDM problem by combining the features of the TOPSIS with the VIKOR method under the SVNS environment. Otay and Kahraman [111] put forward the interval neutrosophic TOPSIS strategy to pick the finest approach in the supply of almost deformity free items. Moreover, Abu-Faty [4] hybridized the game theory with neutrosophic TOPSIS to solve the group DM problems. Peng and Dai [114] presented three algorithms for solving MADM problems based on the modified TOPSIS, Multi-Attribute Border Approximation area Comparison and the similarity measure under the SVNS information. The neutrosophic Analytic Network Process and AHP-TOPSIS strategies which contribute for selection process under SVNS domain have been developed by the authors in [3, 110]. Selvachandran and Peng [137] presented the integrated weighting methods and TOPSIS for a ranking scenario in the neutrosophic domain. Biswas et al. [15] discussed the process of solving the group DM problems by using TOPSIS approach under SVNS and INS domain. However, apart from them some other approaches by using the TOPSIS method to solve the DM problems are investigated by the authors [37, 128, 167] under NS domains.

1.1.4 Review of linguistic approaches

In the previous sections, we discuss the MCDM approaches which can describe the fuzzy essence of the objective world by quantitative measurement. But, as we know, in genuine MCDM process numerous subjective qualities cannot be estimated quantitatively. Like, choosing ventures for distinctive sort of financing arrangement or assessing the 'convenience' or 'luxury' for various types of auto/motors, the specialist typically needs to give his/her inclinations over choices with the data communicated in linguistic characterization similar to, 'very good', 'fair', 'poor',etc. while assessing the vehicle's speed semantics marks like 'quick', 'exceptionally quick', 'moderate' can be utilized. For such scenario, the concept of a linguistic variable (LV) is a more feasible tool for different aspects in the real world which cannot be surveyed in quantitative structures but in qualitative setting [51, 52, 196].

But the use of only LV has a disadvantage as the uncertain and conflicting data cannot be handled by it, that is, the linguistic terms such as 'good' is not associated with acceptance degree and non-acceptance degree in DM process. Consequently, to encourage the specialists to portray both degrees with linguistic assessment, Wang and Li [157] assembled the LV with IFS which consolidated the benefits of both environments. In this manner, IFS with the qualitative linguistic aspects can totally express the uncertain qualitative aspects is a decent prospect in MCDM. Due to this, it has been generally utilized in different fields and in recent years had favorable performances [25, 63, 83, 91, 112, 156]. The above collaboration is truly appropriate to be utilized for portraying the ambiguity of the human mind. Thus, the investigation on the different characteristics of MCDM issues dependent on the association of LVs and IFSs has incredible importance. However, there is a downside of using this concept as it is not able to manage the independent indeterminacies in the different decision processes.

Since, the NSs characterized the indeterminacy factor independently, which can describe the value of a statement between truth and falsehood, therefore, the huge production of work in NS theory has been done. The investigation related to the utilization of SVNSs/INSs to MCDM issues has essentially included the advancement of AOs. NSs have a tendency to manage the independent indeterminacy and the AOs of NSs presents a favorable way to collaborate the NS data. But still, they are unable to represent the qualitative aspects of the given information. In order to eliminate this significant gap in intuitionistic linguistic theory and neutrosophic set theory, Ye [179] initialized the notion of interval neutrosophic linguistic set(INLS) for solving the MCDM problems. Further, Ye [180] proposed the SVN linguistic (SVNL) set (SVNLS), which is represented as $(s_\theta, (\mu, \rho, \nu))$ where (μ, ρ, ν) is SVNN and represents the truth, indeterminacy and falsity degree of some linguistic variable s_θ. Ye [183] investigated the basic AOs in order to aggregate the SVNL numbers (SVNLNs) and solve the MCDM on manufacturing system. Later on, to aggregate the SVNLNs in the decision process in which the criteria values are correlated to each other, Tian, Wang, Zhang, Chen and Wang [149] provided the Bonferroni mean operator in SVNLN domain. Ma, Wang, Wang and Wu [94] applied the concept of INLS in choosing medical treatment preferences in the medical MCDM. The generalization to some SVNL AO's is done by Tian et al. [148]. Wang, Yang and Li [159] founded the Maclurian symmetric mean(MSM) operator for exploring the decision process with SVNLSs. Also, the MSM operator is extended to INLS domain by Geng et al. [47]. Yang et al. [172] introduced the power operator by using the Einstein norms for taking its advantages in INLSs. Afterward, Wang, Tian, Zhang, Zhang and Wang [158] utilized neutrosophic linguistic information in order to introduce the novel neutrosophic cloud model.

However, the assessment in the above theory has only one linguistic variable whose acceptance, indeterminacy and non-acceptance are still the real numbers and is inadequate to express complete qualitative data. Therefore, Li et al. [71] introduced the neutrosophic sets with the linguistic membership(s_θ), linguistic indeterminacy(s_ψ) and linguistic non-membership(s_σ) called linguistic neutrosophic numbers(LNN) with are represented as $(s_\psi, s_\rho, s_\sigma)$. Further, Fang and Ye [39] presented the weighted average and geometric operators for decision process in linguistic neutrosophic set(LNS) domain. Afterward, Fan et al. [38] presented the Bonferroni mean operator for LNNs with its applications to the DM process. Later on, Liang et al. [72] applied the concept of LNNs to solve the human resource management evaluation problem with LNNs as rating values. Cui et al. [31] collaborated the LNNs with uncertain variables and developed weighted average and weighted geometric AOs under the linguistic neutrosophic uncertain number environment. Liu, Mahmood and Khan [82] presented the AO's by combining the power and Heronian mean (HM) AOs by utilizing the LNNs. Liu and You [89] developed the Hamy mean operators for aggregation process and entropy measures for determining the criteria weights to handle the LNN group DM problems.Wang and Liu [161] presented an MCDM technique based on the generalized partitioned BM AO's for neutrosophic linguistic numbers. Ye and Cui [190] hybridized the concept of LNNs and interval linguistic numbers and developed their operational laws for obtaining the AOs.

Apart of AOs, building of information measures like distance measure, similarity measure, entropy measure,etc. and other DM strategies has been broadly examined for their wide applications in DM strategies. Building on SNSs, INSs, SVNLSs, INLSs and LNS, the similarity measure [41, 98, 102, 103, 105, 118, 145, 169, 176, 185, 188, 192, 193], distance measure[30, 58, 75, 129, 136, 152, 191], entropy measure[29, 54, 119, 120, 132, 150, 165, 189] gives the fruitful contribution in investigation DM process. Apart of these measures, several DM methods such as TOPSIS[14, 15, 111, 114, 117, 134], VIKOR[1, 55, 93, 169], ELECTRE[143, 199], GRA[13], MUTIMOORA [10], CODAS[16], DEMATEL[2, 8, 9, 77, 146], WASPAS[107, 197], AHP[123]. Later on, a MULTIMOORA approach is proposed by Tian et al. [147] for ranking the given choices with SVNLNs as preference values. Tian et al. [148] developed an appropriate technique which assimilates the TOPSIS based QUALIFLEX approach to investigate the selection problem under the neutrosophic linguistic(NL) data and for choosing project delivery system NL preference relations are designed by Liu and Luo [76]. For the selection of mining project with LNN preference values, Liang et al. [73] extended the TOPSIS approach for making the best investment. For finding the similarity among the LNNs, Shi and Ye [138] developed the cosine measures for the decision process. In 2018, Shi and Ye [139] introduced the concept of the correlation coefficient between the LNNs and apply it to group DM problems. Li, Wang and Wang [70] developed the EDAS method for the most desirable alternative in the group decision process under the LNN environment. Recently, Liu and You [90] presented the projection measures for LNNs which not only consider the distance but also the included angle in the MCDM problems. Researchers have taken a keen interest in LNN environment to investigate the different kinds of MCDM problems and introduce the innovative findings for various decision problems.

1.2 Gaps and motivation towards the work

As from the above sections, it is noticed that the NSs gave a representative approach whereas the determinations imposed by the IFSs are very narrow towards the evaluation of the data under the imprecise environment. This is due to the fact that in IFSs, the membership degree μ and non-membership degree ν have the property that $\mu + \nu \leq 1$. However, if the condition is ruled out by an expert during an evaluation then under such cases NSs play a dominant role. Further, by taking its advantages, NSs turned into a scientific tool, got consideration from numerous DMs and scholarly analysts for creating and improving the neutrosophic techniques. But sometimes these existing techniques may face some drawbacks during evaluating the objects and hence leads to an inaccurate result. Therefore, new approaches ought to be developed to intensify the DM problem and to bridge the gaps between the numerous existing research works. Some of the gaps, existing in the NS literature are listed as below:

(i) In the investigation of AO's, it has been found that their significance highly depends on its final compiled value. This means that the final value should be independent of some irrelevant factors. But, the obtained values by some existing weighted averaging and geometric operators[113] may tend towards the maximum arguments or the argument with higher importance respectively. Therefore, such operators may lead to an inaccurate result during the ordering.

(ii) The existing AO's [69, 79, 86, 113, 166], and their corresponding extensions have aggregated the information with either the assumption that they are independent or by taking two or more interrelation criteria argument during the process. By doing these, the results computed by them are totally independent or uncorrelated to each other. Also, it is quite understood that during the DM process, the various features are correlated to each other and hence the factor of the proper prioritization should be included into the process to get the desirable results.

(iii) Most of the existing studies under the SVNS/INSs are conducted with the assumption that the attribute weights about the criteria are known as prior and a constant real number. Also, it is quite reasonable that the final ranking order highly depends on the attribute weights. Thus there is a need to include the information about the attribute weights in the analysis and computed by the formulating some optimization model to get the optimal weights.

(iv) In the light of an information measure such as distance or similarity or entropy under the SVNS or INS environment, it has been observed that the measure defined in [17, 95, 181, 192] have failed to classified the objects under some certain cases. This is due to the non-consideration of the proper factors during the formulation of the DM process. Thus, there's ought to develop a more appropriate and adaptable measure to tackle such situations also.

(v) To address some more realistic information during the consideration of the parameter. Thus, to address it, we develop new logarithm operational laws based SVNN operations and explore their studies to solve DM problems.

(vi) From the linguistic neutrosophic environment, it is observed that the existing studies are limited to the access where there is precise possibility degree towards the rating of linguistic numbers to handle the qualitative information. But there is always occur a degree called as possibility degree towards the accessibility of the linguistic rating. Hence, there is a need to extend the concept of linguistic SVNS to the possibility environment to select the appropriate objects.

1.3 Objective and Structure of the Thesis

By motivating from the above gaps and to modify them for solving the DM problems under the SVNS/INS environment, the purpose of the present thesis is to develop some new methods and algorithm under the imprecise and fuzzy environment.

1.3.1 Objective of the Thesis

The complete objectives of the work are summarized as below:

(O1) To develop some information measures under neutrosophic environment for decision making process.

(O2) To develop some aggregation operator in neutrosophic domain.

(O3) To formulate some strategical optimization model to optimize the decision parameters under uncertain environment.

(O4) To develop some probabilistic based operator/measures under the neutrosophic set environment.

1.3.2 Structure of the Thesis

The present thesis is assembled into eleven chapters including the present one that contains mainly the literature review. The rest of the chapters are described below:

In **Chapter 2** the basics and the preliminaries related to the SVNS and/or INS which are to be used in the subsequent chapters are given.

Chapter 3 presents some new AO's by hybridizing the averaging and geometric aggregation operators with an attitude character parameter. The influence of this parameter will also be analyzed and shows its property. The advantages of adding the parameter are to control the effects of the decision parameters towards the biased ones. Further, some properties of proposed operators are investigated. To address more features, an algorithm is presented to address the problem of SVNS/INS into the DM problem. In the end, a numerical example is presented to show its superiority with respect to the several existing approaches.

In **Chapter 4**, we developed some new AO's based on the Frank t-norm operational law under the SVNS environment. The Frank t-norm operation has an additional parameter which can give a flexible environment to the decision makers to choose their decisions, according to their desired goals. Based on the proposed aggregation operators, an algorithm for solving the MCDM approach is presented and illustrated with a numerical example to show its applicability.

Chapter 5 discusses the concept of Muirhead mean(MM) operator and extend it to the SVNS environment by developing some new aggregation operators. Under it, the interrelationship among the different criteria is addressed properly by the property of MM operator. Based on it, we developed MM operator and its dual operator for the different SVNNs. In addition, the involvement of the parameter vector makes the accumulation procedure increasingly adaptable. The examination of the developed operators is tested by presenting an algorithm to solve the MCDM approach and explain it with a numerical example.

Chapter 6 developed a nonlinear programming model based TOPSIS approach for solving MCDM problems with incomplete weight information. The relative closeness coefficient(RCC) degree of the TOPSIS method is formulated based on the distance measures.

Further, the importance of the attribute weights is taken in the form of interval numbers rather than a real single number. Several special cases of the proposed approach are discussed in detail. In the developed non-linear programming model of RCC, a Charles-Chooper approach is utilized to solve the problem. The applicability of this presented approach is demonstrated with a numerical example of a power generation project and computed their results with several existing studying results.

Chapter 7 presents an axiomatic definition of divergence measure for single-valued neutrosophic sets (SVNSs). The properties of the proposed divergence measure have been studied. Further, we develop a novel technique for order preference by similarity to ideal solution (TOPSIS) method for solving single-valued neutrosophic MCDM with incomplete weight information. Finally, a numerical example is presented to verify the proposed approach and to present its effectiveness and practicality.

In **Chapter 8**, we developed some bi-parametric generalized distance measure between the pairs of SVNSs. The presented measures utilized the concept of L_p norm and the degree of uncertainties. The various properties and relationships between them are investigated. Later on, the applicability of these measures is explained with the help of pattern recognition and medical diagnosis problems.

Chapter 9 discusses the new concept of the logarithm operational laws for the pairs of SVNNs. Further, based on these proposed laws, some new aggregation operators named as logarithmic weighted averaging and geometric operators for SVNSs are defined and investigated their properties. To further strengthen their applicability, we developed an algorithm based on these operators for solving MCDM problems and demonstrate it with a numerical example. Finally, the influence of the logarithm is examined to show the behavior of the decision makers towards their optimal decision.

Chapter 10 deal with DM problems to address qualitative information rather than quantitative information. For it, a concept of linguistic SVNS is utilized with linguistic variables to represent the data. Further, to add prioritized factor during analyzing the data, we developed some new laws and hence based on it, some new prioritized weighted averaging and geometric aggregation operators are developing to address the problems with linguistic SVN (LSVN) information. The technique corresponding to them is demonstrated with a numerical example and compared with the existing studies.

In **Chapter 11**, we present a new theory named as Possibility LSVNS(PLSVNS) for evaluation of DM problems with the qualitative preference values. The presented idea considered the degree of the possibility value towards each linguistic features of SVNS. Based on the features of PLSVNS, theories of COPRAS method and weighted averaging and geometric AO's have been defined where the weight vector of the attributes are computed with some information measures. The applicability as well feasibility of the developed methods are explained with a numerical example. Finally, the advantages of the presented concepts are explained.

Chapter 2

Preliminaries

In this chapter, we present the basic concepts and the mathematical structure related to SVNSs, INSs, LSVNS, aggregation operators, etc., over the universal set $\mathcal{X}$.

2.1 Neutrosophic sets

In 1998, Smarandache [140] presented the idea of the neutrosophic set(NS) by considering the features of these independent degrees namely truth degree (TD), indeterminacy degree (ID) and the falsity degree (FD) to represent the preference of an object. The NS is an extension of IFS and has been widely applied in many domains. The formal definition of the NS over the universal set $\mathcal{X}$ is given as follows.

Definition 2.1.1. [140] A NS $\mathcal{A}$ over $\mathcal{X}$ is defined as

$$\mathcal{A} = \{(x, \zeta_{\mathcal{A}}(x), \kappa_{\mathcal{A}}(x), \varphi_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\} \tag{2.1}$$

where $\zeta_{\mathcal{A}}(x)$, $\kappa_{\mathcal{A}}(x)$, $\varphi_{\mathcal{A}}(x)$ represents the TD, ID, FD respectively, and are real standard or non-standard subsets of $]0^-, 1^+[$ such that $0^- \leq \sup \zeta_{\mathcal{A}}(x) + \sup \kappa_{\mathcal{A}}(x) + \sup \varphi_{\mathcal{A}}(x) \leq 3^+$ for all $x \in \mathcal{X}$. Here, sup represents the supremum of the set.

Since, this set is non-standard subset of $]0^-, 1^+[$, so it is difficult to apply in real-life problem. To enhance it, Wang et al. [155] presented the concept of single-valued neutrosophic (SVN) set (SVNS), which is defined as follows.

2.1.1 Single-valued Neutrosophic set

Definition 2.1.2. [155] A SVNS $\mathcal{A}$ in $\mathcal{X}$ is defined as

$$\mathcal{A} = \{(x, \zeta_{\mathcal{A}}(x), \kappa_{\mathcal{A}}(x), \varphi_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\}$$

where $\zeta_{\mathcal{A}}, \kappa_{\mathcal{A}}, \varphi_{\mathcal{A}} : \mathcal{X} \to [0,1]$, be TDs, IDs and FDs such that with $0 \leq \zeta_{\mathcal{A}}(x) + \kappa_{\mathcal{A}}(x) + \varphi_{\mathcal{A}}(x) \leq 3$ for all $x \in \mathcal{X}$. The pairs of these degrees are denoted by $\mathcal{A} = (\zeta, \kappa, \varphi)$ and called as single-valued neutrosophic number (SVNN).

Definition 2.1.3. [155] For two SVNSs $\mathcal{A}_1 = (\zeta_1, \kappa_1, \varphi_1)$ and $\mathcal{A}_2 = (\zeta_2, \kappa_2, \varphi_2)$, the operations are defined as follows:

(i) (Union:) $\mathcal{A}_1 \cup \mathcal{A}_2 = (\max(\zeta_1, \zeta_2), \min(\kappa_1, \kappa_2), \min(\varphi_1, \varphi_2))$.

(ii) (Intersection:) $\mathcal{A}_1 \cap \mathcal{A}_2 = (\min(\zeta_1, \zeta_2), \max(\kappa_1, \kappa_2), \max(\varphi_1, \varphi_2))$.

(iii) (Complement:) $\mathcal{A}_1^c = (\varphi_1, 1 - \kappa_1, \zeta_1)$.

(iv) (Containment:) $\mathcal{A}_1 \subseteq \mathcal{A}_2$ if $\zeta_1 \leq \zeta_2, \kappa_1 \geq \kappa_2$ and $\varphi_1 \geq \varphi_2$.

(v) (Equality:) $\mathcal{A}_1 = \mathcal{A}_2$ if and only if $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_2 \subseteq \mathcal{A}_1$.

Definition 2.1.4. [155] For a SVNN $\mathcal{A} = (\zeta, \kappa, \varphi)$, a score function is defined as

$$\mathcal{S}(\mathcal{A}) = \zeta - \kappa - \varphi \tag{2.2}$$

while an accuracy function is

$$\mathcal{H}(\mathcal{A}) = \zeta + \kappa + \varphi \tag{2.3}$$

Based on these two functions, an order relation between two SVNNs $\mathcal{A}$ and $\mathcal{B}$ denoted by $\mathcal{A} \succ \mathcal{B}$ if either the inequality $\mathcal{S}(\mathcal{A}) > \mathcal{S}(\mathcal{B})$ or $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{B}) \wedge \mathcal{H}(\mathcal{A}) > \mathcal{H}(\mathcal{B})$ holds. Here ‘ $\succ$ means “preferred to”.

2.1.2 Interval Neutrosophic set

In our day-to-day DM problems, it is always preferred by a decision maker to give their decision in the form of interval sets rather than a crisp number to remove the ambiguity during the process. To address it in NS environment, Wang et al. [154] presented the concept of interval neutrosophic set(INS) in which TD, ID, and FD are represented as interval numbers. Mathematically, an INS is defined as follows:

Definition 2.1.5. [154] A INS $\mathcal{A}$ over $\mathcal{X}$ is defined as

$$\mathcal{A} = \left\{ \left(x, \widetilde{\zeta}_{\mathcal{A}}(x), \widetilde{\kappa}_{\mathcal{A}}(x), \widetilde{\varphi}_{\mathcal{A}}(x) \mid x \in \mathcal{X} \right) \right\} \tag{2.4}$$

where $\widetilde{\kappa}_{\mathcal{A}}, \widetilde{\kappa}_{\mathcal{A}}, \widetilde{\varphi}_{\mathcal{A}} : \mathcal{X} \to [0, 1]$ are respectively represents the TD, ID and FD. Also, for each point $x \in \mathcal{X}$, these membership degrees be represented in terms of interval numbers as $\widetilde{\zeta}_{\mathcal{A}}(x) = [\zeta^L_{\mathcal{A}}(x), \zeta^U_{\mathcal{A}}(x)]$, $\widetilde{\kappa}_{\mathcal{A}}(x) = [\kappa^L_{\mathcal{A}}(x), \kappa^U_{\mathcal{A}}(x)]$ and $\widetilde{\varphi}_{\mathcal{A}}(x) = [\varphi^L_{\mathcal{A}}(x), \varphi^U_{\mathcal{A}}(x)]$ with $0 \leq \zeta^U_{\mathcal{A}}(x) + \kappa^U_{\mathcal{A}}(x) + \varphi^U_{\mathcal{A}}(x) \leq 3$. For simplicity, the triplet $\mathcal{A} = \left([\zeta^L_{\mathcal{A}}, \zeta^U_{\mathcal{A}}], [\kappa^L_{\mathcal{A}}, \kappa^U_{\mathcal{A}}], [\varphi^L_{\mathcal{A}}, \varphi^U_{\mathcal{A}}]\right)$ is called interval neutrosophic numbers (INNs) if $0 \leq \mu^L_{\mathcal{A}} \leq \mu^U_{\mathcal{A}} \leq 1$; $0 \leq \rho^L_{\mathcal{A}} \leq \rho^U_{\mathcal{A}} \leq 1$, $0 \leq \nu^L_{\mathcal{A}} \leq \nu^U_{\mathcal{A}} \leq 1$ and $\mu^U_{\mathcal{A}} + \rho^U_{\mathcal{A}} + \nu^U_{\mathcal{A}} \leq 3$ holds.

Definition 2.1.6. [154] Let $\mathcal{A}_j = \left([\zeta^L_j, \zeta^U_j], [\kappa^L_j, \kappa^U_j], [\varphi^L_j, \varphi^U_j]\right)$, $j = 1, 2$ be two INNs, then the set theoretic operations between them are defined as:

(i) $\mathcal{A}_1 \cup \mathcal{A}_2 = \left(\begin{array}{c} [\max\left(\zeta^L_1, \zeta^L_2\right), \max\left(\zeta^U_1, \zeta^U_2\right)], [\min\left(\kappa^L_1, \kappa^L_2\right), \\ \min\left(\kappa^U_1, \kappa^U_2\right)], [\min\left(\varphi^L_1, \varphi^L_2\right), \min\left(\varphi^U_1, \varphi^U_2\right)] \end{array} \right).$

(ii) $\mathcal{A}_1 \cap \mathcal{A}_2 = \left(\begin{array}{c} [\min\left(\zeta^L_1, \zeta^L_2\right), \min\left(\zeta^U_1, \zeta^U_2\right)], [\max\left(\kappa^L_1, \kappa^L_2\right), \\ \max\left(\kappa^U_1, \kappa^U_2\right)], [\max\left(\varphi^L_1, \varphi^L_2\right), \max\left(\varphi^U_1, \varphi^U_2\right)] \end{array} \right).$

(iii) $\mathcal{A}^c_1 = \left([\varphi^L_1, \varphi^U_1], [1 - \kappa^U_1, 1 - \kappa^L_1], [\zeta^L_1, \zeta^U_1]\right)$.

(iv) $\mathcal{A}_1 \subseteq \mathcal{A}_2$, if $\zeta^L_1 \leq \zeta^L_2$, $\zeta^U_1 \leq \zeta^U_2$, $\kappa^L_1 \geq \kappa^L_2$, $\kappa^U_1 \geq \kappa^U_2$, $\varphi^L_1 \geq \varphi^L_2$, $\varphi^U_1 \geq \varphi^U_2$.

(v) $\mathcal{A}_1 = \mathcal{A}_2 \Leftrightarrow \mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_1 \supseteq \mathcal{A}_2$.

Definition 2.1.7. [200] Let $\mathcal{A} = \left([\zeta^L_{\mathcal{A}}, \zeta^U_{\mathcal{A}}], [\kappa^L_{\mathcal{A}}, \kappa^U_{\mathcal{A}}], [\varphi^L_{\mathcal{A}}, \varphi^U_{\mathcal{A}}]\right)$ be INN. Then, the score function of $\mathcal{A}$ is defined as

$$\mathcal{S}(\mathcal{A}) = \frac{\zeta^L_{\mathcal{A}} + \zeta^U_{\mathcal{A}} - \kappa^L_{\mathcal{A}} - \kappa^U_{\mathcal{A}} - \varphi^L_{\mathcal{A}} - \varphi^U_{\mathcal{A}}}{2} \tag{2.5}$$

and the accuracy is:

$$\mathcal{H}(\mathcal{A}) = \frac{\zeta^L_{\mathcal{A}} + \zeta^U_{\mathcal{A}} + \kappa^L_{\mathcal{A}} + \kappa^U_{\mathcal{A}} + \varphi^L_{\mathcal{A}} + \varphi^U_{\mathcal{A}}}{2} \tag{2.6}$$

Then, the comparison law for any two different INNs $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined as:

(i) if $\mathcal{S}(\mathcal{A}_1) > \mathcal{S}(\mathcal{A}_2)$ then $\mathcal{A}_1 \succ \mathcal{A}_2$; where " $\succ$" means "preferred to".

(ii) if $\mathcal{S}(\mathcal{A}_1) = \mathcal{S}(\mathcal{A}_2)$ then,

  (a) $\mathcal{H}(\mathcal{A}_1) > \mathcal{H}(\mathcal{A}_2)$ then $\mathcal{A}_1 \succ \mathcal{A}_2$;

  (b) $\mathcal{H}(\mathcal{A}_1) = \mathcal{H}(\mathcal{A}_2)$ then $\mathcal{A}_1 = \mathcal{A}_2$.

2.1.3 Linguistic single-valued neutrosophic set

In the DM problems, it is sometimes difficult to access the information in terms of quantitative data, perhaps it is better to represent in terms of qualitative information. For example, to rate a certain food item in the restaurant, we generally used the word "good", "tasty", etc. So to deal with qualitative information, the concept of linguistic variables (LVs) [196] is used. The characteristics of LV are stated as below.

Definition 2.1.8. [53]Let $Q = \{s_h \mid h = 0, 1, \ldots, t\}$ be the linguistic term set (LTS) with odd cardinality where s_h represents a possible value for a LV, satisfies the following characteristics:

(i) $s_z \leq s_h \Leftrightarrow z \leq h$.

(ii) Negation operator: $\mathrm{Neg}(s_k) = s_{t-k}$.

Further, Xu [168] extended the LTS Q from discrete to continuous LTS $\bar{Q}$ defined as $\bar{Q} = \{s_h | s_0 \leq s_h \leq s_t, h \in [0, t]\}$ where 't' is even number. Fang and Ye [39] utilize the concept of LVs and SVNNs to define a linguistic single-valued neutrosophic set (LSVNS) as:

Definition 2.1.9. [39] A LSVNS "$\mathcal{A}$" in universe of discourse $\mathcal{X}$ is defined as

$$\mathcal{A} = \{(x, s_\theta(x), s_\psi(x), s_\sigma(x)) \mid x \in \mathcal{X}\}, \tag{2.7}$$

where $s_\theta(x), s_\psi(x), s_\sigma(x) \in \bar{Q}$ represent the linguistic membership, linguistic indeterminacy and linguistic non-membership degrees of x to $\mathcal{A}$, respectively, with condition that $0 \leq \theta + \psi + \sigma \leq 3t \quad \forall x \in \mathcal{X}$. The triple $(s_\theta, s_\psi, s_\sigma)$ is called linguistic single-valued neutrosophic number (LSVNN).

Definition 2.1.10. [39] For LSVNN $\mathcal{A} = (s_\theta, s_\psi, s_\sigma)$, the score value of $\mathcal{A}$ is defined as

$$\mathcal{S}(\mathcal{A}) = s_{(2t+\theta-\psi-\sigma)/3} \tag{2.8}$$

while an accuracy function is

$$\mathcal{H}(\mathcal{A}) = s_{(\theta+\psi+\sigma)/3} \tag{2.9}$$

Definition 2.1.11. [39] For two LSVNNs $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$, we have

(i) $\mathcal{A}_1 \subseteq \mathcal{A}_2$ if $\theta_1 \leq \theta_2$, $\psi_1 \geq \psi_2$, and $\sigma_1 \geq \sigma_2$.

(ii) $\mathcal{A}_1 = \mathcal{A}_2$ iff $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_1 \supseteq \mathcal{A}_2$.

(iii) $\mathcal{A}_1^c = (s_{\sigma_1}, s_{\psi_1}, s_{\theta_1})$.

(iv) $\mathcal{A}_1 \cup \mathcal{A}_2 = (\max(s_{\theta_1}, s_{\theta_2}), \min(s_{\psi_1}, s_{\psi_2}), \min(s_{\sigma_1}, s_{\sigma_2}))$.

(v) $\mathcal{A}_1 \cap \mathcal{A}_2 = (\min(s_{\theta_1}, s_{\theta_2}), \max(s_{\psi_1}, s_{\psi_2}), \max(s_{\sigma_1}, s_{\sigma_2}))$.

2.2 Information Measures

In the literature, information measures such as distance, similarity, entropy, etc., play a dominant role during the DM process. Similarity measure and distance measure are functions that give the degree of similarity and discrimination respectively among the two objects. Entropy measure quantifies the degree of fuzziness or uncertain information and the inclusion measure between two sets gives the extent to which a set is contained in another set. All these measures of information have been extensively explored by many researchers and scholars as vital topics. Here, we have presented some overview of these measures between the different SVNSs and INSs. For it, let $\Phi(\mathcal{X})$ be either collection of SVNSs/INSs over the universal set $\mathcal{X}$.

2.2.1 Distance measures

Definition 2.2.1. [17] A real-valued function $\mathcal{D} : \Phi(\mathcal{X}) \times \Phi(\mathcal{X}) \to [0,1]$ is called as distance measure, if $\mathcal{D}$ satisfies the following properties for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Phi(\mathcal{X})$:

(P1) $0 \leq \mathcal{D}(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{D}(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}(\mathcal{A}, \mathcal{B}) = \mathcal{D}(\mathcal{B}, \mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}(\mathcal{A}, \mathcal{B}) \leq \mathcal{D}(\mathcal{A}, \mathcal{C})$ and $\mathcal{D}(\mathcal{B}, \mathcal{C}) \leq \mathcal{D}(\mathcal{A}, \mathcal{C})$.

For two SVNSs $\mathcal{A}_1 = \{(x_j, \zeta_1(x_j), \kappa_1(x_j), \varphi_1(x_j)) \mid x_j \in \mathcal{X}\}$ and $\mathcal{A}_2 = \{(x_j, \zeta_2(x_j), \kappa_2(x_j), \varphi_2(x_j)) \mid x_j \in \mathcal{X}\}, j = 1, 2, \ldots, n$, some of the existing distance measures are defined as follows:

(i) The extended Hausdorff distance [17]:

$$\mathcal{D}_H(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{n}\sum_{j=1}^{n} \max \left\{ \begin{array}{c} |\zeta_{\mathcal{A}_1}(x_j) - \zeta_{\mathcal{A}_2}(x_j)|, |\kappa_{\mathcal{A}_1}(x_j) - \kappa_{\mathcal{A}_2}(x_j)|, \\ |\varphi_{\mathcal{A}_1}(x_j) - \varphi_{\mathcal{A}_2}(x_j)| \end{array} \right\} \tag{2.10}$$

(ii) The normalized Hamming distance is given by Majumdar [95]:

$$\mathcal{D}_{NH}(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{3n}\sum_{j=1}^{n} \left\{ \begin{array}{c} |\zeta_{\mathcal{A}_1}(x_j) - \zeta_{\mathcal{A}_2}(x_j)| + |\kappa_{\mathcal{A}_1}(x_j) - \kappa_{\mathcal{A}_2}(x_j)| \\ + |\varphi_{\mathcal{A}_1}(x_j) - \varphi_{\mathcal{A}_2}(x_j)| \end{array} \right\} \tag{2.11}$$

(iii) The normalized Euclidean distance is given by Majumdar [95]:

$$\mathcal{D}_{NE}(\mathcal{A}_1, \mathcal{A}_2) = \left(\frac{1}{3n}\sum_{j=1}^{n} \left\{ \begin{array}{c} \mid \zeta_{\mathcal{A}_1}(x_j) - \zeta_{\mathcal{A}_2}(x_j) \mid^2 + \mid \kappa_{\mathcal{A}_1}(x_j) - \kappa_{\mathcal{A}_2}(x_j) \mid^2 \\ + \mid \varphi_{\mathcal{A}_1}(x_j) - \varphi_{\mathcal{A}_2}(x_j) \mid^2 \end{array} \right\} \right)^{1/2} \tag{2.12}$$

(iv) The generalized weighted distance measure [174]:

$$\mathcal{D}_g(\mathcal{A}_1, \mathcal{A}_2) = \left(\frac{1}{3}\sum_{j=1}^{n} w_j \left\{ \begin{array}{c} \mid \zeta_{\mathcal{A}_1}(x_j) - \zeta_{\mathcal{A}_2}(x_j) \mid^g + \mid \kappa_{\mathcal{A}_1}(x_j) - \kappa_{\mathcal{A}_2}(x_j) \mid^g \\ + \mid \varphi_{\mathcal{A}_1}(x_j) - \varphi_{\mathcal{A}_2}(x_j) \mid^g \end{array} \right\} \right)^{1/g} \tag{2.13}$$

where $g > 0$ and $w_j > 0$ is the weight of an element $x_j; (j = 1, 2, \ldots, n)$ with $\sum\limits_{j=1}^{n} w_j = 1$.

Moreover, when $g = 1, 2$, we can obtain weighted Hamming distance and weighted Euclidean distance for SVNSs, respectively, as follows:

$$\mathcal{D}_1(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{3}\sum_{j=1}^{n} w_j \left\{ \begin{array}{c} \mid \zeta_{\mathcal{A}_1}(x_j) - \zeta_{\mathcal{A}_2}(x_j) \mid + \mid \kappa_{\mathcal{A}_1}(x_j) - \kappa_{\mathcal{A}_2}(x_j) \mid \\ + \mid \varphi_{\mathcal{A}_1}(x_j) - \varphi_{\mathcal{A}_2}(x_j) \mid \end{array} \right\} \tag{2.14}$$

$$\mathcal{D}_2(\mathcal{A}_1, \mathcal{A}_2) = \left(\tfrac{1}{3}\sum_{j=1}^{n} w_j \left\{ \begin{array}{c} \mid \zeta_{\mathcal{A}_1}(x_j) - \zeta_{\mathcal{A}_2}(x_j) \mid^2 + \mid \kappa_{\mathcal{A}_1}(x_j) - \kappa_{\mathcal{A}_2}(x_j) \mid^2 \\ + \mid \varphi_{\mathcal{A}_1}(x_j) - \varphi_{\mathcal{A}_2}(x_j) \mid^2 \end{array} \right\} \right)^{1/2} \tag{2.15}$$

On the other hand, for any two LSVNSs, Liang et al. [73] introduced a distance formula of LSVNNs $\mathcal{A}_j = \left(s_{\theta_j}, s_{\psi_j}, s_{\sigma_j}\right), (j = 1, 2)$, as follows:

$$\mathcal{D}_g(\mathcal{A}_1, \mathcal{A}_2) = \left(\frac{1}{3} \left\{ \begin{array}{c} \mid f\left(s_{\theta_1}\right) - f\left(s_{\theta_2}\right) \mid^g + \mid f\left(s_{(t-\psi_1)}\right) - f\left(s_{(t-\psi_2)}\right) \mid^g \\ + \mid f\left(s_{(t-\sigma_1)}\right) - f\left(s_{(t-\sigma_2)}\right) \mid^g \end{array} \right\} \right)^{1/g} \tag{2.16}$$

where $f(s_h) = \frac{h}{t}; h \in [0, t]$ is the linguistic scale function.

2.2.2 Similarity measures

Definition 2.2.2. [181] A real valued function $\mathcal{S}' : \Phi(\mathcal{X}) \times \Phi(\mathcal{X}) \to [0, 1]$ is called the similarity measure, if $\mathcal{S}'$ for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Phi(\mathcal{X})$ satisfies the following properties :

(P1) $0 \leq \mathcal{S}'(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{S}'(\mathcal{A}, \mathcal{B}) = 1 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{S}'(\mathcal{A}, \mathcal{B}) = \mathcal{S}'(\mathcal{B}, \mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{S}'(\mathcal{A}, \mathcal{B}) \geq \mathcal{S}'(\mathcal{A}, \mathcal{C})$ and $\mathcal{S}'(\mathcal{B}, \mathcal{C}) \geq \mathcal{S}'(\mathcal{A}, \mathcal{C})$.

For a collection of SVNSs, some existing similarity measures are defined as follows over the set $\mathcal{X}$.

(i) The cosine similarities [181]:

$$S_{CS1}(\mathcal{A}, \mathcal{B}) = \frac{1}{n}\sum_{j=1}^{n} \cos\left[\frac{\pi}{6} \left(\begin{array}{c} |\zeta_{\mathcal{A}}(x_j) - \zeta_{\mathcal{B}}(x_j)| \vee |\kappa_{\mathcal{A}}(x_j) - \kappa_{\mathcal{B}}(x_j)| \\ \vee |\varphi_{\mathcal{A}}(x_j) - \varphi_{\mathcal{B}}(x_j)| \end{array} \right) \right] \tag{2.17}$$

and

$$S_{CS2}(\mathcal{A},\mathcal{B}) = \frac{1}{n}\sum_{j=1}^{n}\cos\left[\frac{\pi}{6}\begin{pmatrix}|\zeta_{\mathcal{A}}(x_j)-\zeta_{\mathcal{B}}(x_j)|+|\kappa_{\mathcal{A}}(x_j)-\kappa_{\mathcal{B}}(x_j)|\\ +|\varphi_{\mathcal{A}}(x_j)-\varphi_{\mathcal{B}}(x_j)|\end{pmatrix}\right] \tag{2.18}$$

and their corresponding distances denoted by $D_{CS1} = 1-S_{CS1}$ and $D_{CS2} = 1-S_{CS2}$.

(ii) The tangent similarities [192]:

$$S_{T1}(\mathcal{A},\mathcal{B}) = 1-\frac{1}{n}\sum_{j=1}^{n}\tan\left[\frac{\pi}{4}\begin{pmatrix}|\zeta_{\mathcal{A}}(x_j)-\zeta_{\mathcal{B}}(x_j)|\vee|\kappa_{\mathcal{A}}(x_j)-\kappa_{\mathcal{B}}(x_j)|\vee\\ |\varphi_{\mathcal{A}}(x_j)-\varphi_{\mathcal{B}}(x_j)|\end{pmatrix}\right] \tag{2.19}$$

and

$$S_{T2}(\mathcal{A},\mathcal{B}) = 1-\frac{1}{n}\sum_{j=1}^{n}\tan\left[\frac{\pi}{12}\begin{pmatrix}|\zeta_{\mathcal{A}}(x_j)-\zeta_{\mathcal{B}}(x_j)|+|\kappa_{\mathcal{A}}(x_j)-\kappa_{\mathcal{B}}(x_j)|\\ +|\varphi_{\mathcal{A}}(x_j)-\varphi_{\mathcal{B}}(x_j)|\end{pmatrix}\right] \tag{2.20}$$

and their corresponding distances denoted by $D_{T1} = 1-S_{T1}$ and $D_{T2} = 1-S_{T2}$.

2.3 Archimedean t-norms and Archimedean t-conorms

A t-norm (fuzzy intersection)[61, 101] '$\mathcal{T}$' is a binary operation on $[0,1]$ i.e.

$$\mathcal{T} : [0,1]\times[0,1]\to[0,1] \tag{2.21}$$

defined by

$$(\mathcal{B}_1\cap\mathcal{B}_2)(x) = \mathcal{T}(\mathcal{B}_1(x),\mathcal{B}_2(x)) \qquad \forall x\in[0,1] \tag{2.22}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are arbitrary fuzzy sets. Further, the mapping $\mathcal{T}$ satisfies the following axioms for all $a,b,d\in[0,1]$

Axiom 1: $\mathcal{T}(a,1) = a$ (Boundary Condition)

Axiom 2: If $b\le d$ then $\mathcal{T}(a,b)\le\mathcal{T}(a,d)$ (Monotonicity)

Axiom 3: $\mathcal{T}(a,b) = \mathcal{T}(b,a)$ (Commutativity)

Axiom 4: $\mathcal{T}(a,\mathcal{T}(b,d)) = \mathcal{T}(\mathcal{T}(a,b),d)$ (Associativity)

Alternatively, a t-conorm (fuzzy union)[61, 101] 'S' is also a binary operation on $[0,1]$ given by

$$S : [0,1] \times [0,1] \to [0,1] \tag{2.23}$$

defined by

$$(\mathcal{B}_1 \cup \mathcal{B}_2)(x) = S\left(\mathcal{B}_1(x), \mathcal{B}_2(x)\right) \qquad \forall x \in [0,1] \tag{2.24}$$

Also, the mapping 'S' further satisfies the boundary, monotonicity, commutativity and associativity conditions.

The relation between 'S' and '$\mathcal{T}$' norms is given as

$$S(a,b) = 1 - \mathcal{T}(1-a, 1-b) \quad \forall a, b \in [0,1] \tag{2.25}$$

A class of fuzzy intersection (t-norm) is obtained if t-norm also satisfies the additional axioms [61, 101], i.e.,

Axiom 5: $\mathcal{T}$ is continuous function (Continuity)

Axiom 6: $\mathcal{T}(a,a) < a$ (Subidempotency)

Axiom 7: If $a_1 < a_2$ and $b_1 < b_2$ implies $\mathcal{T}(a_1,b_1) < \mathcal{T}(a_2,b_2)$ (Strict monotonicity)

Similarly, for t-conorm, Axiom 6 is replaced by $S(a,a) > a$ and is called superidempotency.

A continuous t-norm that satisfy the subidempotency i.e., $\mathcal{T}(a,a) < a$ is called an Archimedean t-norm(AT)[61, 101]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-norm. On the other hand, a continuous t-conorm that satisfies the superidempotency i.e. $S(a,a) > a$ is called an Archimedean t-conorm (AC) [61, 101]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-conorm.

Furthermore, strict AT and AC can be expressed in the form of continuous function $y : [0,1) \to [0,1]$ and $z : (0,1] \to [0,1]$ respectively for $a, b \in [0,1]$ as

$$\mathcal{T}(a,b) = z^{-1}(z(a) + z(b)) \quad \text{and} \quad S(a,b) = y^{-1}(y(a) + y(b))$$

where z(or y) is a decreasing(or increasing) function with $z(1) = 0$, $y(0) = 0$ and $z(a) = y(1-a)$. However, some standard union and intersection form for $a, b \in [0,1]$ are defined as [61, 101]:

(i) Standard intersection and union

$$\mathcal{T}(a,b) = \min(a,b) \qquad ; \qquad S(a,b) = \max(a,b)$$

(ii) Algebraic product and algebraic sum

$$\mathcal{T}(a,b) = ab \qquad ; \qquad S(a,b) = a+b-ab$$

(iii) Bounded Difference and Sum

$$\mathcal{T}(a,b) = \max(0, a+b-1) \qquad ; \qquad S(a,b) = \min(1, a+b)$$

(iv) Drastic intersection and union

$$\mathcal{T}(a,b) = \begin{cases} a & ;\text{when } b=1 \\ b & ;\text{when } a=1 \\ 0 & ;\text{otherwise} \end{cases} \quad ; \quad S(a,b) = \begin{cases} a & ;\text{when } b=0 \\ b & ;\text{when } a=0 \\ 1 & ;\text{otherwise} \end{cases}$$

(v) Yagar class of t-norm and t-conorm

$$\begin{aligned} &\mathcal{T}(a,b) = 1 - \min\left(1, [(1-a)^p + (1-b)^p]^{1/p}\right); \\ &S(a,b) = \min\left[1, (a^p + b^p)^{1/p}\right] \end{aligned}$$

where $p > 0$.

Apart from them, some other well known ATs and ACs with their generator function are summarized in Table 2.1 [61, 101].

Table 2.1: Some AT and AC with their relative additive generators

Name	t-norm	Additive generator	t-conorm	Additive generator
	$\mathcal{T}(a,b)$	$z(t)$	$S(a,b)$	$y(t)$
Algebraic	ab	$-\log(t)$	$a+b-ab$	$-\log(1-t)$
Einstein	$\frac{ab}{1+(1-a)(1-b)}$	$\log\left(\frac{2-t}{t}\right)$	$\frac{a+b}{1+ab}$	$\log\left(\frac{1+t}{1-t}\right)$
Hamacher ($\gamma > 0$)	$\frac{ab}{\gamma+(1-\gamma)(a+b-ab)}$	$\log\left(\frac{\gamma+(1-\gamma)t}{t}\right)$	$\frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab}$	$\log\left(\frac{\gamma+(1-\gamma)(1-t)}{1-t}\right)$
Frank($\lambda > 1$)	$\log_\lambda\left(1+\frac{(\lambda^a-1)\left(\lambda^b-1\right)}{\lambda-1}\right)$	$-\log\left(\frac{\lambda-1}{\lambda^t-1}\right)$	$\log_\lambda\left(1+\frac{\left(\lambda^{1-a}-1\right)\left(\lambda^{1-b}-1\right)}{\lambda-1}\right)$	$-\log\left(\frac{\lambda-1}{\lambda^{1-t}-1}\right)$

2.4 Aggregation operator

The operations AT and AC are commonly used to combine several fuzzy sets into a single set by using desirable pattern is called as aggregation operators (AOs).

Definition 2.4.1. An aggregation operator on 'n' fuzzy sets ($n \geq 2$) is defined as

$$\mathcal{F} : [0,1]^n \to [0,1] \tag{2.26}$$

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_n$ be first 'n' fuzzy sets defined on $\mathcal{X}$, then function '$\mathcal{F}$' produce an aggregate fuzzy set $\mathcal{B}$ by operating the membership degrees of these sets for each $x \in \mathcal{X}$, i.e.,

$$\mathcal{B}(x) = \mathcal{F}\left(\mathcal{B}_1(x), \mathcal{B}_2(x), \ldots, \mathcal{B}_n(x)\right) \tag{2.27}$$

Further, an aggregation operator $\mathcal{F}$ must satisfies the three axiomatic conditions.

Axiom 1: $\mathcal{F}(0,0,\ldots,0) = 0$ and $\mathcal{F}(1,1,\ldots,1) = 1$ (Boundary condition)

Axiom 2: If $a_i \leq b_i$ for all i, then $\mathcal{F}(a_1, a_2, \ldots, a_n) \leq \mathcal{F}(b_1, b_2, \ldots, b_n)$ (Monotonicity)

Axiom 3: $\mathcal{F}$ is continuous. (Continuity)

Based on the AT and AC operation as defined in Table 2.1, researchers have presented several kinds of AOs under the neutrosophic sets and its extensions, which are defined as follows.

Definition 2.4.2. [113, 155, 184] For two SVNNs $\mathcal{A}_1 = (\zeta_1, \kappa_1, \varphi_1)$ and $\mathcal{A}_2 = (\zeta_2, \kappa_2, \varphi_2)$, and a real number $\lambda > 0$, the basic operational laws by using Algebraic AT and AC operations between them are defined as

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(\zeta_1 + \zeta_2 - \zeta_1\zeta_2, \kappa_1\kappa_2, \varphi_1\varphi_2\right);$

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \left(\zeta_1\zeta_2, \kappa_1 + \kappa_2 - \kappa_1\kappa_2, \varphi_1 + \varphi_2 - \varphi_1\varphi_2\right);$

(iii) $\lambda\mathcal{A}_1 = \left(1 - (1-\zeta_1)^\lambda, \kappa_1^\lambda, \varphi_1^\lambda\right);$

(iv) $\mathcal{A}_1^\lambda = \left(\zeta_1^\lambda, 1 - (1-\kappa_1)^\lambda, 1 - (1-\varphi_1)^\lambda\right);$

$$\text{(v)}\ \lambda^{\mathcal{A}} = \begin{cases} \left(\lambda^{1-\zeta}, 1-\lambda^{\kappa}, 1-\lambda^{\varphi}\right) & \text{if } \lambda \in (0,1) \\ \left((1/\lambda)^{1-\zeta}, 1-(1/\lambda)^{\kappa}, 1-(1/\lambda)^{\varphi}\right) & \text{if } \lambda \geq 1 \end{cases}$$

and are all also SVNNs.

Based on these operations, Peng et al. [113] defined the certain geometric and averaging AOs for the collection of SVNNs $\mathcal{A}_j = (\zeta_j, \kappa_j, \varphi_j); j = 1, 2, \ldots, n$, which are defined as follows:

$$\text{SVNWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\prod_{j=1}^{n} \zeta_j^{\omega_j}, 1 - \prod_{j=1}^{n}(1-\kappa_j)^{\omega_j}, 1 - \prod_{j=1}^{n}(1-\varphi_j)^{\omega_j}\right) \tag{2.28}$$

and

$$\text{SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(1 - \prod_{j=1}^{n}(1-\zeta_j)^{\omega_j}, \prod_{j=1}^{n}(\kappa_j)^{\omega_j}, \prod_{j=1}^{n}(\varphi_j)^{\omega_j}\right) \tag{2.29}$$

where $\omega_j > 0$ be the weight vectors of $\mathcal{A}_j$ with $\sum\limits_{j=1}^{n} \omega_j = 1$. Then, SVNWG and SVNWA represents the SVN weighted geometric and averaging operator respectively.

Further, Peng et al. [113] defined the ordered weighted averaging and geometric AOs, denoted by SVNOWA and SVNOWG, for a collection of SVNNs, which are defined as below

$$\text{SVNOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(1 - \prod_{j=1}^{n}(1-\zeta_{\xi(j)})^{\omega_j}, \prod_{j=1}^{n}\left(\kappa_{\xi(j)}\right)^{\omega_j}, \prod_{j=1}^{n}\left(\varphi_{\xi(j)}\right)^{\omega_j}\right) \tag{2.30}$$

and

$$\text{SVNOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\prod_{j=1}^{n}\left(\zeta_{\xi(j)}\right)^{\omega_j}, 1 - \prod_{j=1}^{n}(1-\kappa_{\xi(j)})^{\omega_j}, 1 - \prod_{j=1}^{n}(1-\varphi_{\xi(j)})^{\omega_j}\right) \tag{2.31}$$

where ξ is a permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\xi(j-1)} \geq \mathcal{A}_{\xi(j)}$ for $j = 2, \ldots, n$.

Later on Wu et al. [166], define some prioritized weighted AO's named as SVN prioritized weighted average (SVNPWA) and SVN prioritized geometric average operators, for a collection of SVNNs $\mathcal{A}_j = (\zeta_j, \kappa_j, \vartheta_j)(j = 1, 2, \ldots, n)$ as

$$\text{SVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(1 - \prod_{j=1}^{n}(1-\zeta_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}, \prod_{j=1}^{n}(\kappa_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}, \prod_{j=1}^{n}(\varphi_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}\right) \tag{2.32}$$

and

$$\text{SVNPGA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\prod_{j=1}^{n} (\zeta_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}, 1 - \prod_{j=1}^{n} (1-\kappa_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}, 1 - \prod_{j=1}^{n} (1-\varphi_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}} \right) \quad (2.33)$$

where $H_1 = 1$ and $H_j = \prod_{k=1}^{j-1} \mathcal{S}(\mathcal{A}_k); (j = 2, \ldots, n)$.

However, on the different collections of the INNs $\mathcal{A}_j = ([\zeta_j^L, \zeta_j^U], [\kappa_j^L, \kappa_j^U], [\varphi_j^L, \varphi_j^U])$, $(j = 1, 2, \ldots, n)$ and by using Algebraic AT and AC operations, [200] defined some weighted average and geometric AOs which are represented as

$$\text{INWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\begin{array}{c} \left[1 - \prod_{j=1}^{n} \left(1 - \zeta_j^L\right)^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \zeta_j^U\right)^{\omega_j} \right], \\ \left[\prod_{j=1}^{n} \left(\kappa_j^L\right)^{\omega_j}, \prod_{j=1}^{n} \left(\kappa_j^U\right)^{\omega_j} \right], \left[\prod_{j=1}^{n} \left(\varphi_j^L\right)^{\omega_j}, \prod_{j=1}^{n} \left(\varphi_j^U\right)^{\omega_j} \right] \end{array} \right) \quad (2.34)$$

and

$$\text{INWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\begin{array}{c} \left[\prod_{j=1}^{n} \left(\zeta_j^L\right)^{\omega_j}, \prod_{j=1}^{n} \left(\zeta_j^U\right)^{\omega_j} \right], \left[1 - \prod_{j=1}^{n} \left(1 - \kappa_j^L\right)^{\omega_j}, \right. \\ \left. 1 - \prod_{j=1}^{n} \left(1 - \kappa_j^U\right)^{\omega_j} \right], \left[1 - \prod_{j=1}^{n} \left(1 - \varphi_j^L\right)^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \varphi_j^U\right)^{\omega_j} \right] \end{array} \right) \quad (2.35)$$

where $\omega_j > 0$ be the weight vector of INNs $\mathcal{A}_j$ such that $\sum_{j=1}^{n} \omega_j = 1$.

As similar to SVNSs/INSs and the features of AT and AC operations as defined in Table 2.1, many scholars have presented the operational laws for the LSVNSs also.

Definition 2.4.3. [39] For LSVNNs $\mathcal{A} = (s_\theta, s_\psi, s_\sigma)$, $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ and for real $\lambda > 0$, the operational laws based on Algebraic AT and AC are defined as:

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(s_{t(\theta_1/t + \theta_2/t - \theta_1\theta_2/t^2)}, s_{t(\psi_1\psi_2/t^2)}, s_{t(\sigma_1\sigma_2/t^2)}\right)$.

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \left(s_{t(\theta_1\theta_2/t^2)}, s_{t(\psi_1/t + \psi_2/t - \psi_1\psi_2/t^2)}, s_{t(\sigma_1/t + \sigma_2/t - \sigma_1\sigma_2/t^2)}\right)$.

(iii) $\lambda\mathcal{A} = \left(s_{t(1-(1-\theta/t)^\lambda)}, s_{t(\psi/t)^\lambda}, s_{t(\sigma/t)^\lambda}\right)$.

(iv) $A^\lambda = \left(s_{t(\theta/t)^\lambda}, s_{t(1-(1-\psi/t)^\lambda)}, s_{t(1-(1-\sigma/t)^\lambda)}\right)$.

Based on these operational laws, Fang and Ye [39] discussed the weighted averaging and geometric AOs, denoted by LNNWAA and LNNWGA respectively, for a collection of LSVNNs $\mathcal{A}_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j}), (j = 1, 2, \ldots, n)$ as follows.

$$\text{LNNWAA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \begin{pmatrix} s_{t\left(1-\prod_{j=1}^{n}(1-\theta_j/t)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}(\psi_j/t)^{\omega_j}\right)}, \\ s_{t\left(\prod_{j=1}^{n}(\sigma_j/t)^{\omega_j}\right)} \end{pmatrix} \tag{2.36}$$

and

$$\text{LNNWGA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \begin{pmatrix} s_{t\left(\prod_{j=1}^{n}(\theta_j/t)^{\omega_j}\right)}, s_{t\left(1-\prod_{j=1}^{n}(1-\psi_j/t)^{\omega_j}\right)}, \\ s_{t\left(1-\prod_{j=1}^{n}(1-\sigma_j/t)^{\omega_j}\right)} \end{pmatrix} \tag{2.37}$$

2.5 Description of the multiattribute decision making

In this thesis, we have presented various algorithms to solve the MADM problem by utilizing the input features of the SVNS or INSs. For it, the brief description of the MADM process is given as below.

Assume that there are m alternatives, denoted by $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$, which are evaluated under the set of n attributes, denoted by $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$, and give their preferences either in the form of the SVNNs $\widetilde{\alpha}_{ij} = (\zeta_{ij}, \kappa_{ij}, \varphi_{ij})$ or in INNs $\widetilde{\alpha}_{ij} = ([\zeta_{ij}^L, \zeta_{ij}^U], [\kappa_{ij}^L, \kappa_{ij}^U], [\varphi_{ij}^L, \varphi_{ij}^U])$; $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ where ζ_{ij} (or $([\zeta_{ij}^L, \zeta_{ij}^U])$) be the degree that the alternative $\mathcal{A}_i$ satisfies the attribute $\mathcal{G}_j$; κ_{ij} (or $([\kappa_{ij}^L, \kappa_{ij}^U])$) be the dissatisfaction degree of the alternative $\mathcal{A}_i$ under the attribute $\mathcal{G}_j$ and φ_{ij} (or $([\varphi_{ij}^L, \varphi_{ij}^U])$) be the degree of indeterminacy such that $\zeta_{ij}, \kappa_{ij}, \varphi_{ij} \in [0, 1]$, $\zeta_{ij} + \kappa_{ij} + \varphi_{ij} \leq 3$ or $\zeta_{ij}^L, \zeta_{ij}^U, \kappa_{ij}^L, \kappa_{ij}^U, \varphi_{ij}^L, \varphi_{ij}^U \in [0, 1]$, $\zeta_{ij}^U + \kappa_{ij}^U + \varphi_{ij}^U \leq 3$. Thus, the rating values corresponding to each alternative are represented in the form of SVNNs over the universal set $\mathcal{X}$ as follows:

$$\mathcal{A}_i = \{(\mathcal{G}_j, \zeta_{ij}(\mathcal{G}_j), \kappa_{ij}(\mathcal{G}_j), \varphi_{ij}(\mathcal{G}_j)) \mid \mathcal{G}_j \in \mathcal{G}\},$$

while under INNs, these rating values are represented as

$$\mathcal{A}_i = \{(\mathcal{G}_j, [\zeta_{ij}^L(\mathcal{G}_j), \zeta_{ij}^U(\mathcal{G}_j)], [\kappa_{ij}^L(\mathcal{G}_j), \kappa_{ij}^U(\mathcal{G}_j)], [\varphi_{ij}^L(\mathcal{G}_j), \varphi_{ij}^U(\mathcal{G}_j)]) \mid \mathcal{G}_j \in \mathcal{G}\}$$

where $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Thus, the complete decision matrix of the given alternative is summarized as $\mathcal{M} = (\alpha_{ij})_{m \times n}$. Let $\omega_j (j = 1, 2, \ldots, n)$ be the weight of the criteria $\mathcal{G}_j$ such that $\omega_j > 0$ and $\sum\limits_{j=1}^{n} \omega_j = 1$. In order to access these given alternatives, a set of 'l' decision makers $\mathcal{DM} = \{\mathcal{DM}^{(1)}, \mathcal{DM}^{(2)}, \ldots, \mathcal{DM}^{(l)}\}$ are invited to evaluate them and find the finest alternative(s). The general procedure to access the best alternative(s) is summarized in the following steps:

Step 1: Discretion the attribute set $\mathcal{G}$ into two disjoint sets namely the benefit $(\mathcal{F}_1)$ and the cost $(\mathcal{F}_2)$ attributes, if required, such that $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{G}$, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ and $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{G}$.

Step 2: Normalize the rating values of the cost type attributes into the benefit type by using the following normalization equation

$$r_{ij} = \begin{cases} \alpha_{ij} & ; \text{ if } j \in \mathcal{F}_1 \\ \alpha_{ij}^c & ; \text{ if } j \in \mathcal{F}_2 \end{cases} \tag{2.38}$$

where α_{ij}^c represent the complement of the number α_{ij} under the considered environment.

Step 3: Aggregate the preference values of the all the alternatives into the overall collective values by using appropriate method/technique.

Step 4: Utilize the appropriate deneutrosophication method to compute the crisp value of the alternatives.

Step 5: Rank the alternatives based on the deneutrosophication values and select the best one(s) according to them.

Chapter 3

Hybrid weighted aggregation operators under neutrosophic set environment and their applications to decision-making process[1]

In this chapter, we present some new AOs by hybridizing the averaging and geometric features with an attitude character parameter. The influence of the parameters will also be analyzed and shows its property. The advantages of adding the parameter are to control the effects of the decision parameters towards the biased ones. Further, some properties of proposed operators are investigated. To address more features, an algorithm is presented to addressed the problems using SVNS and INS features into the decision-making problems (DMPs). Finally, a numerical example is presented to show its superiority.

3.1 Introduction

The extensive literature related to AOs under the SVNSs and INSs environment are reviewed in Section 1.1.1 of Chapter 1. However, from the existing studies, it is concluded that an effective aggregation is one of the most important research areas in the field of decision-making. Aggregation, which more often than not, includes mathematical operators, isn't only an average; rather, it represents a more general notion. The results of the

---

[1] The content of this chapter is published as "Some hybrid weighted aggregation operators under neutrosophic set environment and their applications to multicriteria decision-making", *Applied Intelligence*, 48(12), 4871 - 4888, 2018 (**Impact Factor: 1.983**).

AO are meaningful if the aggregated value given by it is unbiased, that is, it should never tend to one or some number(s)(among to be combined) whose weight is on the higher side and doesn't tend to a maximum or the minimum arguments.

After a deeper study of these AOs, it has been observed that if we utilized the existing SVNWA or SVNOWA operators then under some certain cases their aggregated values may result in a tendency to the maximum value. On the other hand, if we take SVNWG or SVNOWG operators then aggregated values may result in a tendency to the argument with higher importance respectively. Hence, they give unrealistic results in some situations. Thus, it is necessary to handle such issue of removing the biases during the process. Therefore, we focus our attention to extend these operators under the SVNS as well as INS environment. Hence, the main objective of this chapter is to introduce some new AOs which have characteristics of both the averaging and geometric operators. In order to achieve this objective, we propose some hybrid weighted AOs under the neutrosophic set environment. These AOs are called as hybrid SVN weighted averaging and geometric (H-SVNWAG) and hybrid SVN ordered weighted averaging and geometric (H-SVNOWAG). Further, some special cases of these proposed operators are investigating and showed that it reduces to some existing operators. The characteristics of the proposed ones are also studied. Also, in the chapter, the proposed operators are studied under INS environment. Finally, an MCDM approach has been presented and illustrated with a numerical example to validate it.

3.1.1 Motivation towards the proposed aggregation operators

The extensive literature on the AOs under SVNS environment are described in Section 1.1.1 of Chapter 1, while the expression of some of these existing AOs are summarized in Section 2.4 of Chapter 2. After reviewing from these existing operators, it has been observed that the operators such as SVNWA, SVNWG, SVNOWA, SVNOWG defined in Eqs. (2.28), (2.29), (2.30), (2.31) of Chapter 2 respectively have some shortcomings during aggregating the collections of SVNNs. The shortcomings of them are explained below with some numerical examples.

Example 3.1.1. Consider two NSs represented in the form of single-valued as $\mathcal{A}_1 =$

$(0.0001, 0, 0)$ and $\mathcal{A}_2 = (1, 0, 0)$. In order to aggregate these numbers, we utilize Eqs. (2.28), (2.29), (2.30), (2.31) corresponding to the weight vector $\omega = (0.9, 0.1)^T$. The aggregated values by the existing AOs are summarized as follows:

operator	SVNWA	SVNWG	SVNOWA	SVNOWG
Value	(1,0,0)	(0.00025, 0, 0)	(1,0,0)	(0.3981, 0, 0)

On the other hand, if we take $\omega = (0.1, 0.9)$ as weight vector, then the aggregated values corresponding to these existing operators are summarized as

operator	SVNWA	SVNWG	SVNOWA	SVNOWG
Value	(1,0,0)	(0.3981, 0, 0)	(1,0,0)	(0.00025, 0, 0)

Thus, from the above cases, we noticed that the aggregated values of SVNWA and SVNOWA operators tend to the maximum argument while the aggregated values of the SVNWG and SVNOWG operator may tend to the value having maximum weight. So, it is concluded from these results that the existing operators may not give reasonable results to rank the alternatives. Thus, there is a need to modify these AOs so as to get a fair decision during the process. To address them, we define some new AOs under the NSs domain, which is presented in the next section.

3.2 Aggregation operators with SVN information

Let $\Phi(\mathcal{X})$ be the collection of all nonzeros SVNNs over the set $\mathcal{X}$. In this section, we propose some hybrid AO's namely, Hybrid SVN weighted averaging and geometric (H-SVNWAG) and Hybrid SVN ordered weighted averaging and geometric (H-SVNOWAG) AOs to aggregate the collections of SVNNs.

3.2.1 Hybrid weighted aggregation operator

Definition 3.2.1. For a collection of SVNNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$, a H-SVNWAG operator is a mapping H-SVNWAG : $\Phi(\mathcal{X})^n \to \Phi(\mathcal{X})$ defined as

$$\text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\bigoplus_{j=1}^{n} \omega_j \mathcal{A}_j \right)^{\lambda} \bigotimes \left(\bigotimes_{j=1}^{n} \mathcal{A}_j^{\omega_j} \right)^{1-\lambda} \tag{3.1}$$

where $\lambda \in [0,1]$ be a real number which represents the attitudinal characteristic of the decision maker towards an aggregation process, as it gives the weighted average to the averaging and geometric AOs, and $\omega_j > 0$ is the standardized weight vector of $\mathcal{A}_j$ with $\sum\limits_{j=1}^{n} \omega_j = 1$.

Theorem 3.2.1. Let $\mathcal{A}_j = (\zeta_j, \kappa_j, \varphi_j)$ be the collection of "n" SVNNs. Then, the aggregated value of such numbers by H-SVNWAG operator is again a SVNN and is given by

$$\text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \begin{pmatrix} \left(1 - \prod\limits_{j=1}^{n} (1-\zeta_j)^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} \zeta_j^{\omega_j}\right)^{1-\lambda}, \\ 1 - \left(1 - \prod\limits_{j=1}^{n} \kappa_j^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (1-\kappa_j)^{\omega_j}\right)^{1-\lambda}, \\ 1 - \left(1 - \prod\limits_{j=1}^{n} \varphi_j^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (1-\varphi_j)^{\omega_j}\right)^{1-\lambda} \end{pmatrix} \tag{3.2}$$

Proof. By using the operational laws for SVNNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$ and a real number $\lambda \in [0,1]$, we have

$$\begin{aligned} \bigoplus_{j=1}^{n} \omega_j \mathcal{A}_j &= \left(1 - \prod_{j=1}^{n} (1-\zeta_j)^{\omega_j}, \prod_{j=1}^{n} (\kappa_j)^{\omega_j}, \prod_{j=1}^{n} (\varphi_j)^{\omega_j}\right) \\ \text{and} \quad \prod_{j=1}^{n} \mathcal{A}_j^{\omega_j} &= \left(\prod_{j=1}^{n} (\zeta_j)^{\omega_j}, 1 - \prod_{j=1}^{n} (1-\kappa_j)^{\omega_j}, 1 - \prod_{j=1}^{n} (1-\varphi_j)^{\omega_j}\right) \end{aligned}$$

which implies that

$$\left(\bigoplus_{j=1}^{n} \omega_j \mathcal{A}_j\right)^{\lambda} = \begin{pmatrix} \left(1 - \prod\limits_{j=1}^{n} (1-\zeta_j)^{\omega_j}\right)^{\lambda}, 1 - \left(1 - \prod\limits_{j=1}^{n} (\kappa_j)^{\omega_j}\right)^{\lambda}, \\ 1 - \left(1 - \prod\limits_{j=1}^{n} (\varphi_j)^{\omega_j}\right)^{\lambda} \end{pmatrix} \tag{3.3}$$

and

$$\left(\prod_{j=1}^{n}\mathcal{A}_j^{\omega_j}\right)^{1-\lambda} = \left(\begin{array}{c}\left(\prod_{j=1}^{n}(\zeta_j)^{\omega_j}\right)^{1-\lambda}, 1-\left(\prod_{j=1}^{n}(1-\kappa_j)^{\omega_j}\right)^{1-\lambda}, \\ 1-\left(\prod_{j=1}^{n}(1-\varphi_j)^{\omega_j}\right)^{1-\lambda}\end{array}\right) \tag{3.4}$$

Thus, by Definition 3.2.1, we have

$$\begin{aligned}
&\text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
&= \left(\begin{array}{c}\left(1-\prod_{j=1}^{n}(1-\zeta_j)^{\omega_j}\right)^{\lambda}, \\ 1-\left(1-\prod_{j=1}^{n}(\kappa_j)^{\omega_j}\right)^{\lambda}, \\ 1-\left(1-\prod_{j=1}^{n}(\varphi_j)^{\omega_j}\right)^{\lambda}\end{array}\right) \bigotimes \left(\begin{array}{c}\left(\prod_{j=1}^{n}(\zeta_j)^{\omega_j}\right)^{1-\lambda}, \\ 1-\left(\prod_{j=1}^{n}(1-\kappa_j)^{\omega_j}\right)^{1-\lambda}, \\ 1-\left(\prod_{j=1}^{n}(1-\varphi_j)^{\omega_j}\right)^{1-\lambda}\end{array}\right) \\
&= \left(\begin{array}{l}\left(1-\prod_{j=1}^{n}(1-\zeta_j)^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{n}(\zeta_j)^{\omega_j}\right)^{1-\lambda}, \left\{1-\left(1-\prod_{j=1}^{n}(\kappa_j)^{\omega_j}\right)^{\lambda}\right\} + \left\{1-\left(\prod_{j=1}^{n}(1-\kappa_j)^{\omega_j}\right)^{1-\lambda}\right\} \\ -\left\{1-\left(1-\prod_{j=1}^{n}(\kappa_j)^{\omega_j}\right)^{\lambda}\right\}\left\{1-\left(\prod_{j=1}^{n}(1-\kappa_j)^{\omega_j}\right)^{1-\lambda}\right\}, \left\{1-\left(1-\prod_{j=1}^{n}(\varphi_j)^{\omega_j}\right)^{\lambda}\right\} \\ +\left\{1-\left(\prod_{j=1}^{n}(1-\varphi_j)^{\omega_j}\right)^{1-\lambda}\right\} - \left\{1-\left(1-\prod_{j=1}^{n}(\varphi_j)^{\omega_j}\right)^{\lambda}\right\}\left\{1-\left(\prod_{j=1}^{n}(1-\varphi_j)^{\omega_j}\right)^{1-\lambda}\right\}\end{array}\right) \\
&= \left(\begin{array}{l}\left(1-\prod_{j=1}^{n}(1-\zeta_j)^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{n}(\zeta_j)^{\omega_j}\right)^{1-\lambda}, 1-\left(1-\prod_{j=1}^{n}(\kappa_j)^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{n}(1-\kappa_j)^{\omega_j}\right)^{1-\lambda} \\ 1-\left(1-\prod_{j=1}^{n}(\varphi_j)^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{n}(1-\varphi_j)^{\omega_j}\right)^{1-\lambda}\end{array}\right)
\end{aligned}$$

Hence, the result. □

To demonstrate the working of the H-SVNWAG operator, we explain it with a numerical example as below.

Example 3.2.1. Let $\mathcal{A}_1 = (0.9, 0.1, 0.1)$, $\mathcal{A}_2 = (0.92, 0.1, 0.05)$ and $\mathcal{A}_3 = (0.7, 0.1, 0.2)$ be three SVNNs and $\omega = (0.3, 0.5, 0.2)^T$ be its weight vector. Without loss of generality, we assume that $\lambda = 0.5$ be a real number. Then, by utilizing the given information, we have

$$\begin{aligned}
& \left(1-\prod_{j=1}^{3}(1-\zeta_j)^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{3}\zeta_j{}^{\omega_j}\right)^{1-\lambda} \\
= \ & \left(1-(1-0.9)^{0.3}\times(1-0.92)^{0.5}\times(1-0.7)^{0.2}\right)^{0.5}\times\left(0.9^{0.3}\times0.92^{0.5}\times0.7^{0.2}\right)^{1-0.5} \\
= \ & 0.8769 \\
& 1-\left(1-\prod_{j=1}^{3}\kappa_j{}^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{3}(1-\kappa_j)^{\omega_j}\right)^{1-\lambda} \\
= \ & 1-\left(1-(0.1)^{0.3}\times(0.1)^{0.5}\times(0.1)^{0.2}\right)^{0.5}\times\left((1-0.1)^{0.3}\times(1-0.1)^{0.5}\times(1-0.1)^{0.2}\right)^{0.5} \\
= \ & 0.100
\end{aligned}$$

and

$$\begin{aligned}
& 1-\left(1-\prod_{j=1}^{3}\varphi_j{}^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{3}(1-\varphi_j)^{\omega_j}\right)^{1-\lambda} \\
= \ & 1-\left(1-(0.1)^{0.3}\times(0.05)^{0.5}\times(0.2)^{0.2}\right)^{0.5}\left((1-0.1)^{0.3}\times(1-0.05)^{0.5}\times(1-0.2)^{0.2}\right)^{0.5} \\
= \ & 0.0891
\end{aligned}$$

Hence, by Eq. (3.2), we get H-SVNWAG$(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = (0.8769, 0.100, 0.0891)$.

Remark 3.2.1. It is evident from the proposed operator that if we assign some particular value to λ, then it reduces to some existing AO's. For instance,

(i) if $\lambda = 1$, then H-SVNWAG is reduced to SVNWA [113] operator, and

(ii) if $\lambda = 0$ then it is reduced to SVNWG [113] operator.

Example 3.2.2. Consider two NSs represented in the form of single-valued as $\mathcal{A}_1 = (0.3, 0.4, 0.2)$ and $\mathcal{A}_2 = (0.5, 0.4, 0.4)$. In order to aggregate these two numbers, we take the importance of these numbers as $\omega_1 = 0.3$ and $\omega_2 = 0.7$. Now, in order to analyze the influence of the parameter λ, the aggregated value corresponding to $\mathcal{A}_1$ and $\mathcal{A}_2$ are computed for different values of λ ranging from 0 to 1 and their results along with their score values are summarized in Table 3.1. From their corresponding results, it has been observed that with the increase of the parameter λ, the score value corresponding to the

Table 3.1: Aggregated number and score value for different value of λ

$\lambda \rightarrow$	0	0.2	0.4	0.5
Aggregated value	$(0.4290, 0.4000, 0.3459)$	$(0.4325, 0.4000, 0.3418)$	$(0.4360, 0.4000, 0.3376)$	$(0.4378, 0.4000, 0.3355)$
Score value	-0.3170	-0.3093	-0.3015	-0.2977
$\lambda \rightarrow$	0.6	0.8	1.0	
	$(0.4396, 0.4000, 0.3334)$	$(0.4432, 0.4000, 0.3292)$	$(0.4469, 0.4000, 0.3249)$	
score value	-0.2938	-0.2859	-0.2780	

aggregated value also increases. Also, it is observed from it that when $\lambda = 0.5$ then it shows a neutral effect on the operator, while when $\lambda = 1$, then proposed operator reduces to SVNWA [113] operator, and $\lambda = 0$ then it reduces to SVNWG [113] operator. Further, from the table, it has been concluded that with the increase of λ from 0 to 1, the relative score of the alternatives increases. Thus, based on these different parametric values, system analyst/decision maker may possess the power to alter the ranks of the alternatives. For instance, if the decision makers want to pay more attention towards the optimistic approach towards the aggregation, then he or she may choose the largest value of λ so that the score value increases and consequently the membership degree increases. Contrary to the optimism decision makers, the pessimistic decision makers tend to choose the smaller value of λ. Therefore, based on the attitude, behavioral characteristic of the decision maker toward the ranking order related to optimistic and pessimistic behavior, they can choose the desired one accordingly. Therefore, the proposed hybrid AO is more generic to aggregate the different preferences of the decision makers.

For a collection of SVNNs $\mathcal{A}_j = (\zeta_j, \kappa_j, \varphi_j); (j = 1, 2, \ldots, n)$, the H-SVNWAG operator satisfy the certain properties namely, idempotency, boundedness and monotonicity which are stated as follows.

(P1) (Idempotency) Let $\mathcal{A}$ be another SVNN and if $\mathcal{A}_j = \mathcal{A}$ for all j, then we have

$$\text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$$

(P2) (Boundedness) For SVNNs $\mathcal{A}_j; (j = 1, 2, \ldots, n)$, we have

$$\mathcal{A}^- \leq \text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

where $\mathcal{A}^- = (\min_j\{\zeta_j\}, \max_j\{\kappa_j\}, \max_j\{\varphi_j\})$ and $\mathcal{A}^+ = (\max_j\{\zeta_j\}, \min_j\{\kappa_j\}, \min_j\{\varphi_j\})$.

(P3) (Monotonicity) If $\mathcal{A}_j$ and $\mathcal{B}_j$ be two collections of SVNNs such that $\mathcal{A}_j \leq \mathcal{B}_j$ for all j, then

$$\text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{H-SVNWAG}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$$

Since SVNWA and SVNWG operators satisfy these properties so by the definition of H-SVNWAG operator, it follows directly and hence we omit their proofs.

Theorem 3.2.2. For a collection of SVNNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$, the existing operators SVNWA, SVNWG and the proposed operator H-SVNWAG satisfy the following inequality

$$\text{SVNWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

Proof. For SVNNs $\mathcal{A}_j = (\zeta_j, \kappa_j, \varphi_j)(j = 1, 2, \ldots, n)$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be their normalized weight vector, then we have $\prod_{j=1}^{n} \zeta_j^{\omega_j} \leq 1 - \prod_{j=1}^{n} (1 - \zeta_j)^{\omega_j}$ and $0 \leq \prod_{j=1}^{n} \zeta_j^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \zeta_j)^{\omega_j} \leq 1$. Therefore, for a real number $\lambda \in [0, 1]$ we have

$$\left(\prod_{j=1}^{n} \zeta_j^{\omega_j} \right)^{1-\lambda} \leq \left(1 - \prod_{j=1}^{n} (1 - \zeta_j)^{\omega_j} \right)^{1-\lambda}$$

which implies that

$$\left(1 - \prod_{j=1}^{n} (1 - \zeta_j)^{\omega_j} \right)^{\lambda} \left(\prod_{j=1}^{n} \zeta_j^{\omega_j} \right)^{1-\lambda} \leq 1 - \prod_{j=1}^{n} (1 - \zeta_j)^{\omega_j} \tag{3.5}$$

Similarly, for $\prod_{j=1}^{n} \kappa_j^{\omega_j} \geq 1 - \prod_{j=1}^{n} (1 - \kappa_j)^{\omega_j}$ and $\prod_{j=1}^{n} \varphi_j^{\omega_j} \geq 1 - \prod_{j=1}^{n} (1 - \varphi_j)^{\omega_j}$, we have

$$\prod_{j=1}^{n} \kappa_j^{\omega_j} \leq 1 - \left(1 - \prod_{j=1}^{n} \kappa_j^{\omega_j} \right)^{\lambda} \left(\prod_{j=1}^{n} (1 - \kappa_j)^{\omega_j} \right)^{1-\lambda} \tag{3.6}$$

and

$$\prod_{j=1}^{n} \varphi_j^{\omega_j} \leq 1 - \left(1 - \prod_{j=1}^{n} \varphi_j^{\omega_j} \right)^{\lambda} \left(\prod_{j=1}^{n} (1 - \varphi_j)^{\omega_j} \right)^{1-\lambda} \tag{3.7}$$

Thus, by using Eqs. (3.5), (3.6), (3.7) and the definition of score function, we get

$$\begin{aligned}
&\mathcal{S}(\text{SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n)) \\
= \;& 1 - \prod_{j=1}^{n}(1-\zeta_j)^{\omega_j} - \prod_{j=1}^{n}\kappa_j^{\omega_j} - \prod_{j=1}^{n}\varphi_j^{\omega_j} \\
\geq \;& \left(1 - \prod_{j=1}^{n}(1-\zeta_j)^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{n}\zeta_j^{\omega_j}\right)^{1-\lambda} - 1 + \left(1 - \prod_{j=1}^{n}\kappa_j^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{n}(1-\kappa_j)^{\omega_j}\right)^{1-\lambda} \\
& -1 + \left(1 - \prod_{j=1}^{n}\varphi_j^{\omega_j}\right)^{\lambda}\left(\prod_{j=1}^{n}(1-\varphi_j)^{\omega_j}\right)^{1-\lambda} \\
= \;& \mathcal{S}(\text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n))
\end{aligned}$$

Hence, H-SVNWAG$(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n) \leq$ SVNWA$(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n)$. Similarly, we can obtain that H-SVNWAG$(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n) \geq$ SVNWG$(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n)$. Thus, we get the required proof. □

Example 3.2.3. Consider the collection of three SVNNs $\mathcal{A}_1 = (0.9, 0.1, 0.1)$, $\mathcal{A}_2 = (0.92, 0.1, 0.05)$ and $\mathcal{A}_3 = (0.7, 0.1, 0.2)$ and $\omega = (0.3, 0.5, 0.2)^T$ be the weight vector of $\mathcal{A}_j (j = 1, 2, 3)$. Without loss of generality, we assume that $\lambda = 0.5$. Now, by using the existing AOs SVNWA and SVNWG as well as the proposed operator H-SVNWAG defined in Eqs. (2.29), (2.28) and (3.2) respectively, we get

$$\begin{aligned}
\text{SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) &= (0.8886, 0.1, 0.0812) \\
\text{SVNWG}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) &= (0.8653, 0.1, 0.0969) \\
\text{and H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) &= (0.8769, 0.1, 0.0891)
\end{aligned}$$

Thus, the score value of these aggregated numbers are computed by using Eq. (2.2) and we get $\mathcal{S}$(SVNWA) $= 0.7074$, $\mathcal{S}$(SVNWG) $= 0.6685$ and $\mathcal{S}$(H-SVNWAG) $= 0.6878$. Since $\mathcal{S}$(SVNWG) $<$ $\mathcal{S}$(H-SVNWAG) $<$ $\mathcal{S}$(SVNWA) and hence by Definition 2.1.4 of Chapter 2, we get

$$\text{SVNWG}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \leq \text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \leq \text{SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$$

which validates the Theorem 3.2.2.

3.2.2 Hybrid ordered weighted aggregation operator

In this section, we define an ordered weighted hybrid aggregation operator for a collection of SVNNs.

Definition 3.2.2. A H-SVNOWAG operator is a mapping H-SVNOWAG : $\mathcal{A}(\mathcal{X})^n \rightarrow \mathcal{A}(\mathcal{X})$ defined on $\mathcal{A}(\mathcal{X})$ as

$$\text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\bigoplus_{j=1}^{n} w_j \mathcal{A}_{\xi(j)} \right)^{\lambda} \bigotimes \left(\bigotimes_{j=1}^{n} \mathcal{A}_{\xi(j)}^{w_j} \right)^{1-\lambda} \tag{3.8}$$

where $w_j > 0$ be an associated positional weight vector with $\sum\limits_{j=1}^{n} w_j = 1$ and $\xi : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ is a permutation function such that $\mathcal{A}_{\xi(j-1)} \geq \mathcal{A}_{\xi(j)}$ for $j = 2, 3, \ldots, n$ i.e., $\mathcal{A}_{\xi(j-1)}$ is the jth largest element in the set $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ and λ be any real number in $[0, 1]$.

Theorem 3.2.3. The aggregated value for a collection of SVNNs $\mathcal{A}_j = (\zeta_j, \kappa_j, \varphi_j)$, $(j = 1, 2, \ldots, n)$ by using H-SVNOWAG is still a SVNN and is given by

$$\text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \begin{pmatrix} \left(1 - \prod\limits_{j=1}^{n} (1 - \zeta_{\xi(j)})^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} \zeta_{\xi(j)}^{w_j}\right)^{1-\lambda}, \\ 1 - \left(1 - \prod\limits_{j=1}^{n} \kappa_{\xi(j)}^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (1 - \kappa_{\xi(j)})^{w_j}\right)^{1-\lambda}, \\ 1 - \left(1 - \prod\limits_{j=1}^{n} \varphi_{\xi(j)}^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (1 - \varphi_{\xi(j)})^{w_j}\right)^{1-\lambda} \end{pmatrix} \tag{3.9}$$

Proof. Based on the SVNOWA and SVNOWG operators and the operational laws of SVNNs, we have

$$\text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\bigoplus_{j=1}^{n} w_j \mathcal{A}_{\xi(j)} \right)^{\lambda} \bigotimes \left(\bigotimes_{j=1}^{n} \mathcal{A}_{\xi(j)}^{w_j} \right)^{1-\lambda}$$

$$= \begin{pmatrix} 1 - \prod\limits_{j=1}^{n} (1 - \zeta_{\xi(j)})^{w_j}, \prod\limits_{j=1}^{n} \kappa_{\xi(j)}^{w_j}, \\ \prod\limits_{j=1}^{n} \varphi_{\xi(j)}^{w_j} \end{pmatrix}^{\lambda} \bigotimes \begin{pmatrix} \prod\limits_{j=1}^{n} \zeta_{\xi(j)}^{w_j}, 1 - \prod\limits_{j=1}^{n} (1 - \kappa_{\xi(j)})^{w_j}, \\ 1 - \prod\limits_{j=1}^{n} (1 - \varphi_{\xi(j)})^{w_j} \end{pmatrix}^{1-\lambda}$$

$$= \begin{pmatrix} \left(1-\prod\limits_{j=1}^{n}\left(1-\zeta_{\xi(j)}\right)^{w_j}\right)^{\lambda}, \\ 1-\left(1-\prod\limits_{j=1}^{n}\kappa_{\xi(j)}^{w_j}\right)^{\lambda}, \\ 1-\left(1-\prod\limits_{j=1}^{n}\varphi_{\xi(j)}^{w_j}\right)^{\lambda} \end{pmatrix} \bigotimes \begin{pmatrix} \left(\prod\limits_{j=1}^{n}\zeta_{\xi(j)}^{w_j}\right)^{1-\lambda}, \\ 1-\left(\prod\limits_{j=1}^{n}\left(1-\kappa_{\xi(j)}\right)^{w_j}\right)^{1-\lambda}, \\ 1-\left(\prod\limits_{j=1}^{n}\left(1-\varphi_{\xi(j)}\right)^{w_j}\right)^{1-\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-\prod\limits_{j=1}^{n}\left(1-\zeta_{\xi(j)}\right)^{w_j}\right)^{\lambda}\left(\prod\limits_{j=1}^{n}\zeta_{\xi(j)}^{w_j}\right)^{1-\lambda}, 1-\left(1-\prod\limits_{j=1}^{n}\kappa_{\xi(j)}^{w_j}\right)^{\lambda}\left(\prod\limits_{j=1}^{n}\left(1-\kappa_{\xi(j)}\right)^{w_j}\right)^{1-\lambda}, \\ 1-\left(1-\prod\limits_{j=1}^{n}\varphi_{\xi(j)}^{w_j}\right)^{\lambda}\left(\prod\limits_{j=1}^{n}\left(1-\varphi_{\xi(j)}\right)^{w_j}\right)^{1-\lambda} \end{pmatrix}$$

Hence, the result. □

The working of the proposed H-SVNOWAG operator is demonstrated with a numerical example as follows.

Example 3.2.4. Consider three SVNNs $\mathcal{A}_1 = (0.4, 0.2, 0.6)$, $\mathcal{A}_2 = (0.3, 0.1, 0.4)$ and $\mathcal{A}_3 = (0.7, 0.2, 0.1)$ with their importance vector $w = (0.6, 0.3, 0.1)^T$. In order to give the permutation to the SVNNs, we compute the score values of them by using Eq. (2.2) and hence get $\mathcal{S}(\mathcal{A}_1) = -0.4$, $\mathcal{S}(\mathcal{A}_2) = -0.2$ and $\mathcal{S}(\mathcal{A}_3) = 0.4$. Since, $\mathcal{S}(\mathcal{A}_3) > \mathcal{S}(\mathcal{A}_2) > \mathcal{S}(\mathcal{A}_1)$ so therefore, $\mathcal{A}_{\xi(1)} = \mathcal{A}_3, \mathcal{A}_{\xi(2)} = \mathcal{A}_2$ and $\mathcal{A}_{\xi(3)} = \mathcal{A}_1$. Without loss of generality, we take $\lambda = 0.5$ and by utilizing Eq. (3.9), we have

$$\begin{aligned} &\text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \\ = &\begin{pmatrix} \left(1-\prod\limits_{j=1}^{3}(1-\zeta_{\xi(j)})^{w_j}\right)^{\lambda}\left(\prod\limits_{j=1}^{3}\zeta_{\xi(j)}^{w_j}\right)^{1-\lambda}, 1-\left(1-\prod\limits_{j=1}^{3}\kappa_{\xi(j)}^{w_j}\right)^{\lambda}\left(\prod\limits_{j=1}^{3}(1-\kappa_{\xi(j)})^{w_j}\right)^{1-\lambda}, \\ 1-\left(1-\prod\limits_{j=1}^{3}\varphi_{\xi(j)}^{w_j}\right)^{\lambda}\left(\prod\limits_{j=1}^{3}(1-\varphi_{\xi(j)})^{w_j}\right)^{1-\lambda} \end{pmatrix} \\ = &\begin{pmatrix} \left(1-(1-0.7)^{0.6}\times(1-0.3)^{0.3}\times(1-0.4)^{0.1}\right)^{0.5}\times\left(0.7^{0.6}\times0.3^{0.3}\times0.4^{0.5}\right)^{0.5}, \\ 1-\left(1-(0.2)^{0.6}\times(0.1)^{0.3}\times(0.2)^{0.1}\right)^{0.5}\times\left((1-0.2)^{0.6}\times(1-0.1)^{0.3}\times(1-0.2)^{0.1}\right), \\ 1-\left(1-(0.1)^{0.6}\times(0.4)^{0.3}\times(0.6)^{0.1}\right)^{0.5}\times\left((1-0.1)^{0.6}\times(1-0.4)^{0.3}\times(1-0.6)^{0.1}\right) \end{pmatrix} \\ = &(0.5482, 0.1669, 0.2244) \end{aligned}$$

According to the properties of the SVNOWA and SVNOWG operators , it is clear that the H-SVNOWAG operator also satisfies the following properties

(P1) (Idempotency) Let $\mathcal{A}$ is another SVNN and if $\mathcal{A}_j = \mathcal{A}$, for all j, then

$$\text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$$

(P2) (Boundedness) For a collection of SVNNs $\mathcal{A}_j$, we have

$$\mathcal{A}^- \leq \text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

where $\mathcal{A}^- = (\min_j\{\zeta_j\}, \max_j\{\kappa_j\}, \max_j\{\varphi_j\})$ and $\mathcal{A}^+ = (\max_j\{\zeta_j\}, \min_j\{\kappa_j\}, \min_j\{\varphi_j\})$.

(P3) (Monotonicity) For two different SVNNs $\mathcal{A}_j$ and $\mathcal{B}_j$, $j = 1, 2, \ldots, n$ such that $\mathcal{A}_j \leq \mathcal{B}_j$ for all j, then

$$\text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{H-SVNOWAG}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$$

(P4) (Commutativity) If $(\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_n)$ be any permutation of $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$, then

$$\text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \text{H-SVNOWAG}(\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_n)$$

Theorem 3.2.4. For a collection of SVNNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$, the operators SVNOWA, SVNOWG and H-SVNOWAG satisfies the following inequality

$$\text{SVNOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{SVNOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

Proof. It can be obtained as similar to Theorem 3.2.2, so we omit here. □

3.2.3 Superiority of the proposed aggregation operators

In order to show the superiority of the aggregation operator H-SVNWAG and H-SVNOWAG with respect to the existing operators, we revisit the Example 3.1.1 as considered in Section 3.1.1.

Example 3.2.5. Consider two SVNNs $\mathcal{A}_1 = (0.0001, 0, 0)$ and $\mathcal{A}_2 = (1, 0, 0)$ as taken in the Example 3.1.1 and $\omega = w = (0.9, 0.1)^T$ be the weight vector corresponding to them.

Without loss of generality, we take $\lambda = 0.5$ and utilized the proposed H-SVNWAG and H-SVNOWAG AOs given in Eqs. (3.2) and (3.9) respectively to aggregate the numbers. The results corresponding to these operators are

$$\begin{aligned} & \text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2) = (0.0158, 0, 0), \\ \text{and} \quad & \text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2) = (0.6310, 0, 0). \end{aligned}$$

Clearly, it has been seen that H-SVNWAG lie between SVNWA and SVNWG, while H-SVNOWAG operator value lie between SVNOWA and SVNOWG.

On the other hand, if we utilize the weight vector $\omega = w = (0.1, 0.9)^T$ to the SVNNs $\mathcal{A}_1$ and $\mathcal{A}_2$, then the aggregated values by utilizing the proposed AOs H-SVNWAG and H-SVNOWAG defined in Eqs. (3.2) and (3.9) corresponding to parameter $\lambda = 0.5$ are

$$\begin{aligned} & \text{H-SVNWAG}(\mathcal{A}_1, \mathcal{A}_2) = (0.6310, 0, 0), \\ \text{and} \quad & \text{H-SVNOWAG}(\mathcal{A}_1, \mathcal{A}_2) = (0.0158, 0, 0) \end{aligned}$$

Again, from it, we observed that the aggregated value obtained by using H-SVNWAG operator lies between the existing SVNWA and SVNWG operators, while value obtained through H-SVNOWAG operator lies between SVNOWA and SVNOWG operators.

Hence, from the above, we conclude that the proposed hybrid operators are unbiased as they are not tending towards maximum argument as shown by SVNWA and SVOWA operators and also not tending towards the maximum weight value as shown by SVNWG and SVNOWG operators. Therefore, the proposed operator gives the aggregated values which are much more informative.

3.3 Aggregation operators with INS information

In this section, the above proposed hybrid operators are extended to INS environment where the data towards accessing the information is represented in the form of intervals. For it, let $G(\mathcal{X})$ be the collection of the INNs $\mathcal{A}_j = ([\zeta_j^L, \zeta_j^U], [\kappa_j^L, \kappa_j^U], [\varphi_j^L, \varphi_j^U])$, $(j = 1, 2, \ldots, n)$ over the universal set $\mathcal{X}$ such that $[\zeta_j^L, \zeta_j^U], [\kappa_j^L, \kappa_j^U], [\varphi_j^L, \varphi_j^U]) \subseteq [0, 1]$ with $\zeta_j^U + \kappa_j^U + \varphi_j^U \leq 3$ for all j. Then, in the following, we present some new hybrid

AOs namely, hybrid interval neutrosophic weighted and ordered weighted averaging and geometric operators denoted by H-INWAG and H-INOWAG for a collection of the INNs.

Definition 3.3.1. For any real number $\lambda \in [0,1]$ and for a collection of INNs $\mathcal{A}_j = ([\zeta_j^L, \zeta_j^U], [\kappa_j^L, \kappa_j^U], [\varphi_j^L, \varphi_j^U])$, $(j = 1, 2, \ldots, n)$, a H-INWAG operator is a mapping H-INWAG : $G(\mathcal{X})^n \to G(\mathcal{X})$ defined as

$$\text{H-INWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\bigoplus_{j=1}^{n} \omega_j \mathcal{A}_j\right)^{\lambda} \bigotimes \left(\bigotimes_{j=1}^{n} \mathcal{A}_j^{\omega_j}\right)^{1-\lambda} \tag{3.10}$$

$$= \left(\begin{array}{c} \left[\left(1 - \prod_{j=1}^{n} (1-\zeta_j^L)^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (\zeta_j^L)^{\omega_j}\right)^{1-\lambda}, \left(1 - \prod_{j=1}^{n} (1-\zeta_j^U)^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (\zeta_j^U)^{\omega_j}\right)^{1-\lambda}\right] \\ \left[1 - \left(1 - \prod_{j=1}^{n} (\kappa_j^L)^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (1-\kappa_j^L)^{\omega_j}\right)^{1-\lambda}, 1 - \left(1 - \prod_{j=1}^{n} (\kappa_j^U)^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (1-\kappa_j^U)^{\omega_j}\right)^{1-\lambda}\right], \\ \left[1 - \left(1 - \prod_{j=1}^{n} (\varphi_j^L)^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (1-\varphi_j^L)^{\omega_j}\right)^{1-\lambda}, 1 - \left(1 - \prod_{j=1}^{n} (\varphi_j^U)^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (1-\varphi_j^U)^{\omega_j}\right)^{1-\lambda}\right] \end{array} \right)$$

where $\omega_j > 0$ is the normalized weight of $\mathcal{A}_j$; $(j = 1, 2, \ldots, n)$ and λ represents the attitudinal characters of the decision makers towards the AO.

Theorem 3.3.1. For a collection of INNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$, the existing AOs INWA, INWG and the proposed operator H-INWAG satisfies the following inequality

$$\text{INWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{H-INWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{INWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

Proof. It can be easily proved as similar Theorem 3.2.2, so we omit here. □

Definition 3.3.2. Let $\mathcal{A}_j = ([\zeta_j^L, \zeta_j^U], [\kappa_j^L, \kappa_j^U], [\varphi_j^L, \varphi_j^U])$, $j = 1, 2, \ldots, n$ be a collection of INNs. A H-INOWAG operator of dimension n is a mapping H-INOWAG : $G(\mathcal{X})^n \to G(\mathcal{X})$, that has an associated positional weight vector $w_j > 0$ such that $\sum_{j=1}^{n} w_j = 1$, defined as

$$\begin{aligned} &\text{H-INOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &= \left(\bigoplus_{j=1}^{n} w_j \mathcal{A}_{\xi(j)}\right)^{\lambda} \bigotimes \left(\bigotimes_{j=1}^{n} \mathcal{A}_{\xi(j)}^{w_j}\right)^{1-\lambda} \end{aligned} \tag{3.11}$$

$$= \left(\begin{array}{c} \left[\left(1-\prod_{j=1}^{n}\left(1-\zeta_{\xi(j)}^{L}\right)^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n}\left(\zeta_{\xi(j)}^{L}\right)^{w_j}\right)^{1-\lambda}, \left(1-\prod_{j=1}^{n}\left(1-\zeta_{\xi(j)}^{U}\right)^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n}\left(\zeta_{\xi(j)}^{U}\right)^{w_j}\right)^{1-\lambda} \right], \\ \left[1-\left(1-\prod_{j=1}^{n}\left(\kappa_{\xi(j)}^{L}\right)^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n}\left(1-\kappa_{\xi(j)}^{L}\right)^{w_j}\right)^{1-\lambda}, 1-\left(1-\prod_{j=1}^{n}\left(\kappa_{\xi(j)}^{U}\right)^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n}\left(1-\kappa_{\xi(j)}^{U}\right)^{w_j}\right)^{1-\lambda} \right], \\ \left[1-\left(1-\prod_{j=1}^{n}\left(\varphi_{\xi(j)}^{L}\right)^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n}\left(1-\varphi_{\xi(j)}^{L}\right)^{w_j}\right)^{1-\lambda}, 1-\left(1-\prod_{j=1}^{n}\left(\varphi_{\xi(j)}^{U}\right)^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n}\left(1-\varphi_{\xi(j)}^{U}\right)^{w_j}\right)^{1-\lambda} \right] \end{array} \right)$$

where $(\xi(1), \xi(2), \ldots, \xi(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\xi(j-1)} \geq \mathcal{A}_{\xi(j)}$ for $j = 2, 3, \ldots, n$ and $\lambda \in [0, 1]$ be a real number.

Theorem 3.3.2. For a collection of INNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$, the existing AOs INOWA, INOWG and the proposed operator H-INOWAG satisfies the following inequality

$$\text{INOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{H-INOWAG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{INOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

Proof. Follows from Theorem 3.2.2. □

3.4 Proposed approaches based on developed operators

In this section, we develop two approaches based on the developed operators to solve the DMPs under the NSs environment.

The general brief description of the problem is given in Section 2.5 of Chapter 2 in which rating information of the "m" alternatives $\mathcal{A}_i$ under the "n" different attributes $\mathcal{G}_j$ are given by an experts in terms of either SVNNs or INNs. The collection information obtained from them is summarized in the decision matrix $\mathcal{M} = (\alpha_{ij})_{m \times n}$. Then, based on the considered environment and the developed operators, we define two new approaches under the SVNSs and INSs, to solve the DMPs and hence to find the most desirable alternative(s) as below.

3.4.1 Approach I

In this approach, assume that rating values α_{ij} of the alternatives are assessed in terms of SVNNs as $\alpha_{ij} = (\zeta_{ij}, \kappa_{ij}, \varphi_{ij})$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ where ζ_{ij}, κ_{ij}, φ_{ij} represents the degree of satisfaction, indeterminacy, and dissatisfaction, respectively, of $\mathcal{A}_i$

corresponding to attribute $\mathcal{G}_j$ such that $0 \leq \zeta_{ij}, \kappa_{ij}, \varphi_{ij} \leq 1$ and $\zeta_{ij} + \kappa_{ij} + \varphi_{ij} \leq 3$. Then, the following steps of the proposed steps, based on either H-SVNWAG or H-SVNOWAG operator, are summarized to solve the problems.

Step 1. Arrange the information about the alternatives in the matrix $\mathcal{M} = (\alpha_{ij})_{m \times n}$.

Step 2. Aggregate the individual values of each alternative under different attributes either by using H-SVNWAG or H-SVNOWAG operator. For example, if a person wants to utilize the H-SVNWAG operator, then the aggregated values $r_i = (\zeta_i, \kappa_i, \varphi_i)$ for each alternative are computed as

$$\begin{aligned} r_i &= \text{H-SVNWAG}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \\ &= \left(\begin{array}{c} \left(1 - \prod_{j=1}^{n} (1 - \zeta_{ij})^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} \zeta_{ij}^{\omega_j}\right)^{1-\lambda}, 1 - \left(1 - \prod_{j=1}^{n} \kappa_{ij}^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (1 - \kappa_{ij})^{\omega_j}\right)^{1-\lambda}, \\ 1 - \left(1 - \prod_{j=1}^{n} \varphi_{ij}^{\omega_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (1 - \varphi_{ij})^{\omega_j}\right)^{1-\lambda} \end{array} \right) \end{aligned}$$

while by using H-SVNOWAG operator, we get the aggregated values r_i as

$$\begin{aligned} r_i &= \text{H-SVNOWAG}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \\ &= \left(\begin{array}{c} \left(1 - \prod_{j=1}^{n} (1 - \zeta_{i\xi(j)})^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n} \zeta_{i\xi(j)}^{w_j}\right)^{1-\lambda}, 1 - \left(1 - \prod_{j=1}^{n} \kappa_{i\xi(j)}^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (1 - \kappa_{i\xi(j)})^{w_j}\right)^{1-\lambda}, \\ 1 - \left(1 - \prod_{j=1}^{n} \varphi_{i\xi(j)}^{w_j}\right)^{\lambda} \left(\prod_{j=1}^{n} (1 - \varphi_{i\xi(j)})^{w_j}\right)^{1-\lambda} \end{array} \right) \end{aligned}$$

to aggregate all the individual SVNNs $\alpha_{ij} (j = 1, 2, \ldots, n)$ into the collective SVNN $r_i (i = 1, 2, \ldots, m)$.

Step 3. Compute the score values $\mathcal{S}(r_i)$ of each of the aggregated number $r_i (i = 1, 2, \ldots, m)$ by using Eq. (2.2). If there is no difference between two score values $\mathcal{S}(r_{b_1})$ and $\mathcal{S}(r_{b_2})$ for any two positive b_1, b_2, then we need to calculate the accuracy values of the alternatives as $\mathcal{H}(r_{b_1})$ and $\mathcal{H}(r_{b_2})$ by using Eq. (2.3).

Step 4. Rank all the feasible alternatives $\mathcal{A}_i (i = 1, 2, \ldots, m)$ according to the Definition 2.1.4 and hence select the most desirable alternative(s).

3.4.2 Approach II

In this approach, we assume that an expert give their preferences in terms of interval numbers as

$$\mathcal{A}_i = \left\{ \begin{pmatrix} \mathcal{G}_j, \left[\inf \zeta_{\mathcal{A}}(\mathcal{G}_j), \sup \zeta_{\mathcal{A}}(\mathcal{G}_j)\right], \left[\inf \kappa_{\mathcal{A}}(\mathcal{G}_j), \right. \\ \left. \sup \kappa_{\mathcal{A}}(\mathcal{G}_j)\right], \left[\inf \varphi_{\mathcal{A}}(\mathcal{G}_j), \sup \varphi_{\mathcal{A}}(\mathcal{G}_j)\right] \end{pmatrix} \middle| \mathcal{G}_j \in \mathcal{G} \right\}$$

where $[\inf \zeta_{\mathcal{A}}(\mathcal{G}_j), \sup \zeta_{\mathcal{A}}(\mathcal{G}_j)] \subseteq [0,1]$, $[\inf \kappa_{\mathcal{A}}(\mathcal{G}_j), \sup \kappa_{\mathcal{A}}(\mathcal{G}_j)] \subseteq [0,1]$, $[\inf \varphi_{\mathcal{A}}(\mathcal{G}_j), \sup \varphi_{\mathcal{A}}(\mathcal{G}_j)] \subseteq [0,1]$ represents the interval degrees of “satisfaction”, “indeterminacy”, and “dissatisfaction”, respectively of $\mathcal{A}_i$ corresponding to attribute $\mathcal{G}_j$ such that $\sup \zeta_{\mathcal{A}}(\mathcal{G}_j) + \sup \kappa_{\mathcal{A}}(\mathcal{G}_j) + \sup \varphi_{\mathcal{A}}(\mathcal{G}_j) \leq 3$. For simplicity, this attribute value is denoted by $\alpha_{ij} = ([\zeta_{ij}^L, \zeta_{ij}^U], [\kappa_{ij}^L, \kappa_{ij}^U], [\varphi_{ij}^L, \varphi_{ij}^U])$, where $[\zeta_{ij}^L, \zeta_{ij}^U], [\kappa_{ij}^L, \kappa_{ij}^U], [\varphi_{ij}^L, \varphi_{ij}^U] \subseteq [0,1]$ and $\zeta_{ij}^U + \kappa_{ij}^U + \varphi_{ij}^U \leq 3$. Under this environment, the following steps are proposed to solve the MCDM problem based on either H-INWAG or H-INOWAG operator.

Step 1. Arrange the information in terms of the interval-valued decision matrix $\mathcal{M} = (\alpha_{ij})_{m\times n}$.

Step 2. Utilize either appropriately H-INWAG operator

$$\begin{aligned} r_i &= \text{H-INWAG}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \\ &= \begin{pmatrix} \left[\left(1 - \prod\limits_{j=1}^{n} (1-\zeta_{ij}^L)^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (\zeta_{ij}^L)^{\omega_j}\right)^{1-\lambda}, \left(1 - \prod\limits_{j=1}^{n} (1-\zeta_{ij}^U)^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (\zeta_{ij}^U)^{\omega_j}\right)^{1-\lambda}\right] \\ \left[1 - \left(1 - \prod\limits_{j=1}^{n} (\kappa_{ij}^L)^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (1-\kappa_{ij}^L)^{\omega_j}\right)^{1-\lambda}, 1 - \left(1 - \prod\limits_{j=1}^{n} (\kappa_{ij}^U)^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (1-\kappa_{ij}^U)^{\omega_j}\right)^{1-\lambda}\right], \\ \left[1 - \left(1 - \prod\limits_{j=1}^{n} (\varphi_{ij}^L)^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (1-\varphi_{ij}^L)^{\omega_j}\right)^{1-\lambda}, 1 - \left(1 - \prod\limits_{j=1}^{n} (\varphi_{ij}^U)^{\omega_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} (1-\varphi_{ij}^U)^{\omega_j}\right)^{1-\lambda}\right] \end{pmatrix} \end{aligned}$$

or the H-INOWAG operator

$$\begin{aligned} r_i &= \text{H-INOWAG}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \\ &= \begin{pmatrix} \left[\left(1 - \prod\limits_{j=1}^{n} \left(1-\zeta_{i\xi(j)}^L\right)^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} \left(\zeta_{i\xi(j)}^L\right)^{w_j}\right)^{1-\lambda}, \left(1 - \prod\limits_{j=1}^{n} \left(1-\zeta_{i\xi(j)}^U\right)^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} \left(\zeta_{i\xi(j)}^U\right)^{w_j}\right)^{1-\lambda}\right], \\ \left[1 - \left(1 - \prod\limits_{j=1}^{n} \left(\kappa_{i\xi(j)}^L\right)^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} \left(1-\kappa_{i\xi(j)}^L\right)^{w_j}\right)^{1-\lambda}, 1 - \left(1 - \prod\limits_{j=1}^{n} \left(\kappa_{i\xi(j)}^U\right)^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} \left(1-\kappa_{i\xi(j)}^U\right)^{w_j}\right)^{1-\lambda}\right], \\ \left[1 - \left(1 - \prod\limits_{j=1}^{n} \left(\varphi_{i\xi(j)}^L\right)^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} \left(1-\varphi_{i\xi(j)}^L\right)^{w_j}\right)^{1-\lambda}, 1 - \left(1 - \prod\limits_{j=1}^{n} \left(\varphi_{i\xi(j)}^U\right)^{w_j}\right)^{\lambda} \left(\prod\limits_{j=1}^{n} \left(1-\varphi_{i\xi(j)}^U\right)^{w_j}\right)^{1-\lambda}\right] \end{pmatrix} \end{aligned}$$

to aggregate the preference values into the collected INN $r_i (i = 1, 2, \dots, m)$.

Step 3. Compute the score values $\mathcal{S}(r_i)$ of each of the aggregated INNs $r_i (i = 1, 2, \dots, m)$ by using Eq. (2.5). If there is no difference between two score values $\mathcal{S}(r_{b_1})$ and $\mathcal{S}(r_{b_2})$ for any two positive b_1, b_2, then we need to calculate the accuracy values of the alternatives as $\mathcal{H}(r_{b_1})$ and $\mathcal{H}(r_{b_2})$ by using Eq. (2.6).

Step 4. Rank all the feasible alternatives $\mathcal{A}_i (i = 1, 2, \dots, m)$ according to the Definition 2.1.7 and hence select the most desirable alternative(s).

3.5 Illustrative example

The above developed approaches are illustrated with a numerical example, which can be read as.

3.5.1 A case study

Demonetization is the withdrawal of a particular form of currency from circulation. On 8 November 2016, Government of India announced in a broadcast to the nation that Rs. 500 and Rs. 1,000 currency notes would no longer be recognized legally as currency. This step was taken to crack down the use of illicit and counterfeit cash to fund illegal activity and terrorism. The government knew that demonetization will affect the Indian economy and may also drop the country's Gross domestic product (GDP) growth. So before the announcement of the demonetization, the Government wanted to conduct the survey of the effect of this bold move on various sectors of the Indian economy to take further decisions. For doing this, Indian government hired an economist or decision maker who is able to handle this kind of situation and able to crack that which sector (alternative) of the Indian economy will be affected by the demonetization. For this, decision maker assumes the five important sectors on which our Indian economy depends and were given as $\mathcal{A}_1$ (Agricultural Sector), $\mathcal{A}_2$ (Real-Estate Sector), $\mathcal{A}_3$ (Information Technology Sector), $\mathcal{A}_4$ (Education Sector), $\mathcal{A}_5$ (Industrial Sector). For evaluation, decision maker considered criterion in the terms how much the effect of the demonetization on particular sector which are summarized as: $\mathcal{G}_1$(Very low effect), $\mathcal{G}_2$(Low effect), $\mathcal{G}_3$(Regular effect), $\mathcal{G}_4$(High effect) and $\mathcal{G}_5$(Very high effect). The importance of each attribute $\mathcal{G}_j; (j = 1, 2, 3, 4, 5)$ is taken in the form of weight vector as $\omega = (0.2, 0.25, 0.15, 0.3, 0.1)^T$ in the DMP and in order to make the decision more pessimistic for future goals then manipulate the aggregation by using the OWA weighted vector $w = (0.1, 0.2, 0.2, 0.2, 0.3)^T$.

By following the steps of the **Approach I**, the best alternative(s) is obtained as follows:

Step 1. The characteristics of each alternative is accessed in terms of SVNNs and hence the decision matrix is summarized in Table 3.2.

Step 2. By utilizing the proposed operators along with some existing operators to aggregate the given information, their corresponding results are summarized in Table 3.3 which includes the SVNWA, SVNWG, SVNOWA, SVNOWG, SVNHWA, SVNHWG, H-SVNWAG and H-SVNOWAG operators.

Step 3. By using Eq. (2.2), the score values of aggregated numbers shown in Table 3.3, are summarized in Table 3.4.

Step 4. According to score values, the ordering of the alternatives is shown in Table 3.5 in which $\succ$ means "preferred to". From these results, it has been seen that the best alternative is $\mathcal{A}_2$ by all the operators while the different AOs have different ranking strategies which are slightly different. Thus, based on the decision makers preference in terms of their AOs used, the results may lead to different decisions.

Table 3.2: Neutrosophic decision maker matrix

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{A}_1$	$(0.5, 0.3, 0.4)$	$(0.5, 0.2, 0.3)$	$(0.2, 0.2, 0.6)$	$(0.3, 0.2, 0.4)$	$(0.3, 0.3, 0.4)$
$\mathcal{A}_2$	$(0.7, 0.1, 0.3)$	$(0.7, 0.2, 0.3)$	$(0.6, 0.3, 0.2)$	$(0.6, 0.4, 0.2)$	$(0.7, 0.1, 0.2)$
$\mathcal{A}_3$	$(0.5, 0.3, 0.4)$	$(0.6, 0.2, 0.4)$	$(0.6, 0.1, 0.2)$	$(0.5, 0.1, 0.3)$	$(0.6, 0.4, 0.3)$
$\mathcal{A}_4$	$(0.7, 0.3, 0.2)$	$(0.7, 0.2, 0.2)$	$(0.4, 0.5, 0.2)$	$(0.5, 0.2, 0.2)$	$(0.4, 0.5, 0.4)$
$\mathcal{A}_5$	$(0.4, 0.1, 0.3)$	$(0.5, 0.1, 0.2)$	$(0.4, 0.1, 0.5)$	$(0.4, 0.3, 0.6)$	$(0.3, 0.2, 0.4)$

Table 3.3: Neutrosophic aggregated results by using aggregated operators($\lambda = 0.5$)

	SVNWA [113]	SVNWG [113]	SVNHWA [79]	
			$\gamma = 2$	$\gamma = 3$
$\mathcal{A}_1$	$(0.3862, 0.2259, 0.3956)$	$(0.3553, 0.2314, 0.4132)$	$(0.3812, 0.2264, 0.3980)$	$(0.3782, 0.2266, 0.3992)$
$\mathcal{A}_2$	$(0.6585, 0.2125, 0.2400)$	$(0.6531, 0.2548, 0.2467)$	$(0.6578, 0.2165, 0.2407)$	$(0.6575, 0.2181, 0.2410)$
$\mathcal{A}_3$	$(0.5528, 0.1702, 0.3213)$	$(0.5477, 0.2020, 0.3337)$	$(0.5520, 0.1726, 0.3230)$	$(0.5516, 0.1735, 0.3237)$
$\mathcal{A}_4$	$(0.5842, 0.2727, 0.2144)$	$(0.5502, 0.3074, 0.2227)$	$(0.5790, 0.2765, 0.2151)$	$(0.5766, 0.2781, 0.2154)$
$\mathcal{A}_5$	$(0.4178, 0.1490, 0.3708)$	$(0.4110, 0.1751, 0.4271)$	$(0.4167, 0.1508, 0.3788)$	$(0.4160, 0.1515, 0.3825)$
	SVNOWA [113]	SVNOWG [113]	H-SVNWAG	H-SVNOWAG
$\mathcal{A}_1$	$(0.3413, 0.2352, 0.4389)$	$(0.3096, 0.2416, 0.4605)$	$(0.3704, 0.2286, 0.4045)$	$(0.3251, 0.2384, 0.4498)$
$\mathcal{A}_2$	$(0.6536, 0.2169, 0.2352)$	$(0.6481, 0.2598, 0.2416)$	$(0.6558, 0.2340, 0.2434)$	$(0.6508, 0.2386, 0.2384)$
$\mathcal{A}_3$	$(0.5528, 0.2107, 0.3326)$	$(0.5477, 0.2483, 0.3432)$	$(0.5502, 0.1862, 0.3275)$	$(0.5502, 0.2297, 0.3380)$
$\mathcal{A}_4$	$(0.5301, 0.3429, 0.2462)$	$(0.4947, 0.3842, 0.2661)$	$(0.5669, 0.2903, 0.2185)$	$(0.5121, 0.3639, 0.2563)$
$\mathcal{A}_5$	$(0.4193, 0.1261, 0.3455)$	$(0.3862, 0.1848, 0.4563)$	$(0.4144, 0.1622, 0.3996)$	$(0.3893, 0.1723, 0.4365)$

Table 3.4: Score values of aggregated SVNNs

	SVNWA [113]	SVNWG [113]	SVNOWA[113]	SVNOWG [113]
$\mathcal{A}_1$	-0.2353	-0.2894	-0.3328	-0.3924
$\mathcal{A}_2$	0.2060	0.1516	0.2015	0.1467
$\mathcal{A}_3$	0.0613	0.0120	0.0094	-0.0438
$\mathcal{A}_4$	0.0971	0.0201	-0.0591	-0.1556
$\mathcal{A}_5$	-0.1020	-0.1913	-0.1834	-0.2549

	SVNHWA[79]		Proposed Operators	
	$\gamma = 2$	$\gamma = 3$	H-SVNWAG	H-SVNOWAG
$\mathcal{A}_1$	-0.2432	-0.2476	-0.2627	-0.3631
$\mathcal{A}_2$	0.2006	0.1984	0.1785	0.1738
$\mathcal{A}_3$	0.0564	0.0544	0.0365	-0.0174
$\mathcal{A}_4$	0.0874	0.0831	0.0581	-0.1081
$\mathcal{A}_5$	-0.1129	-0.1180	-0.1474	-0.2196

Table 3.5: Ordering of the alternatives under SVNS environment

Existing operators	Ordering	Proposed operators	Ordering
SVNWA [113]	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$	H-SVNWAG	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
SVNOWA [113]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$	H-SVNOWAG	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
SVNWG [113]	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SVNOWG [113]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SVNHWA [79]	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		

Table 3.6: Information about each alternative under five characteristics in form of INNs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{A}_1$	$([0.4,0.6],[0.2,0.3],[0.3,0.5])$	$([0.45,0.55],[0.1,0.2],[0.3,0.4])$	$([0.2,0.3],[0.1,0.2],[0.6,0.7])$	$([0.2,0.4],[0.2,0.3],[0.3,0.4])$	$([0.2,0.4],[0.3,0.4],[0.4,0.6])$
$\mathcal{A}_2$	$([0.7,0.8],[0.1,0.2],[0.3,0.5])$	$([0.65,0.80],[0.2,0.3],[0.3,0.5])$	$([0.5,0.7],[0.2,0.3],[0.2,0.3])$	$([0.6,0.7],[0.4,0.5],[0.2,0.3])$	$([0.6,0.7],[0.1,0.3],[0.2,0.3])$
$\mathcal{A}_3$	$([0.5,0.6],[0.3,0.4],[0.4,0.5])$	$([0.60,0.65],[0.2,0.3],[0.3,0.4])$	$([0.4,0.6],[0.1,0.2],[0.2,0.3])$	$([0.4,0.5],[0.1,0.3],[0.3,0.4])$	$([0.6,0.7],[0.4,0.5],[0.3,0.4])$
$\mathcal{A}_4$	$([0.6,0.7],[0.3,0.5],[0.2,0.4])$	$([0.70,0.80],[0.1,0.2],[0.1,0.2])$	$([0.4,0.5],[0.5,0.6],[0.1,0.2])$	$([0.3,0.5],[0.1,0.2],[0.2,0.3])$	$([0.3,0.4],[0.5,0.6],[0.4,0.5])$
$\mathcal{A}_5$	$([0.4,0.5],[0.1,0.2],[0.3,0.4])$	$([0.50,0.60],[0.1,0.2],[0.2,0.3])$	$([0.3,0.4],[0.1,0.2],[0.5,0.6])$	$([0.3,0.4],[0.2,0.3],[0.5,0.6])$	$([0.2,0.3],[0.2,0.4],[0.3,0.4])$

On the other hand, if we follow the steps of the **Approach II** then the best alternative(s) is obtained as follows:

Step 1: The rating values of each alternative is measured in the form of INNs and are summarized in Table 3.6.

Step 2: Aggregate these values into the collective one by using proposed operators as well as some existing operators such as INWA, INWG, INOWA, INOWG, INFWA, INFWG, INHWA and INHWG. Their results are presented in Table 3.7.

Step 3: Utilize Eq. (2.5) and the score values are summarized in Table 3.8.

Step 4: The ordering of the alternatives are summarized in Table 3.9 which indicates that the best alternative is $\mathcal{A}_2$ by all the approaches.

3.5.2 Discussion

It is observed from these computed values that the score values of the alternatives obtained by using proposed H-SVNWAG (or H-SVNOWAG) operator lie between the score values obtained by using SVNWA and SVNWG (or SVNOWA and SVNOWG) operators. Thus, the aggregated values of the proposed operators are more or less closed to moderate values between the aggregated values of the existing operators. Also, from Table 3.5 and Table 3.9, it is clearly seen that the final ranking order of the alternatives based on SVNWA, SVNWG, and H-SVNWAG (or INWA, INWA, H-INWAG) operators are different from the ones based on SVNOWA, SVNOWG and H-SVNOWAG (or INOWA, INOWA, H-INOWAG) operators. For instance, in the former one, the alternative $\mathcal{A}_3$ is preferable over $\mathcal{A}_4$ while in the order position weight values $\mathcal{A}_4$ is preferable over $\mathcal{A}_3$. Since the decision maker in the proposed operators can choose the desired value of the parameter λ according to their preferences and hence the developed approach is more flexible to choose the alternatives according to his or her desired goals than the existing ones.

Further, it is revealed from Tables 3.5 and 3.9 that the best and worst alternatives are same for the considered problem, but it is observed from the aforementioned mentioned examples (Examples 3.1.1 and 3.2.5) that the existing studies produce some unreliable results while the proposed ones can overcome these shortcomings. Also, it is noted that the existing averaging operators SVNWA and SVNOWA emphasize more on the group points while geometric operators SVNWG and SVNOWG emphasize personal major points. On the other hand, the proposed operator aggregates their preferences with the parameter λ and show their rationality in the information aggregations. By assigning a particular value to λ, we can observe that the proposed operators reduce to the existing operator. For instance, if $\lambda = 1$ then it reduces to SVNWA while for $\lambda = 0$ it becomes SVNWG operator. Therefore, the proposed approach is more generic to solve the stated problem under SVN decision-making environment.

3.5.3 Sensitivity Analysis

Keeping in mind the end goal is to get the effect of the parameter λ on the score values and positioning of the alternatives, an examination has been finished by increasing the values of the parameter λ from 0 to 1. The overall score values computed by using proposed operators are summarized in Table 3.10. From this, it validates that with the increase of the parameter λ, the score values of the proposed operator increases. This impact of λ values on the decision makes our proposed approach more flexible as the decision-maker(s) can choose the parameters according to their preferences and practical situations. For instance, if the decision makers can take a decision as per the pessimistic nature, then at that point a smaller value of λ can be chosen to aggregate so as to decrease the scores values. On to contrary, if a higher value is allocated to parameter λ then score value will increase and it makes an optimism decision to the decision makers. Nonetheless, with the variation of these values, the best option is the same which affected that the outcomes are objective and can't be changed by decision makes' inclination of cynicism and hopefulness. Thus, in this manner, the ranking results are reliable. Further, it is seen that when $\lambda = 0$, then the proposed H-SVNWAG/H-INWAG reduces to SVNWG/INWG and H-SVNOWAG/H-INOWAG reduces to SVNOWG/INOWG. Also, it has been highlighted that if $\lambda = 1$, then the H-SVNWAG/H-INWAG reduces to SVNWA/INWA and H-SVNOWAG/ H-INOWAG reduces to SVNOWA/INOWA.

Table 3.7: Neutrosophic aggregated results by using aggregated operators($\lambda = 0.5$)

	INWA [200]	INWG [200]
$\mathcal{A}_1$	([0.3123, 0.4731], [0.1578, 0.2625], [0.3426, 0.4737])	([0.2814, 0.4499], [0.1725, 0.2729], [0.3662, 0.4993])
$\mathcal{A}_2$	([0.6223, 0.7500], [0.2000, 0.3224], [0.2400, 0.3775])	([0.6143, 0.7434], [0.2398, 0.3501], [0.2467, 0.3984])
$\mathcal{A}_3$	([0.4980, 0.5981], [0.1702, 0.3147], [0.2990, 0.4006])	([0.4820, 0.5885], [0.2020, 0.3304], [0.3075, 0.4080])
$\mathcal{A}_4$	([0.5052, 0.6344], [0.1863, 0.3162], [0.1625, 0.2844])	([0.4447, 0.5882], [0.2611, 0.3876], [0.1852, 0.3077])
$\mathcal{A}_5$	([0.3676, 0.4691], [0.1320, 0.2421], [0.3590, 0.4468])	([0.3467, 0.4497], [0.1414, 0.2532], [0.3985, 0.4804])
	INOWA [200]	INOWG[200]
$\mathcal{A}_1$	([0.2725, 0.4456], [0.1835, 0.2896], [0.3757, 0.5283])	([0.2491, 0.4228], [0.2038, 0.3043], [0.4024, 0.5541])
$\mathcal{A}_2$	([0.6043, 0.7344], [0.2000, 0.3358], [0.2259, 0.3497])	([0.5970, 0.7286], [0.2398, 0.3587], [0.2314, 0.3672])
$\mathcal{A}_3$	([0.5170, 0.6155], [0.2107, 0.3478], [0.3140, 0.4156])	([0.5030, 0.6063], [0.2483, 0.3667], [0.3226, 0.4231])
$\mathcal{A}_4$	([0.4424, 0.5649], [0.2786, 0.4161], [0.2000, 0.3280])	([0.3973, 0.5242], [0.3621, 0.4851], [0.2398, 0.3613])
$\mathcal{A}_5$	([0.3169, 0.4182], [0.1414, 0.2670], [0.4119, 0.4571])	([0.2961, 0.3995], [0.1515, 0.2855], [0.4395, 0.4819])
	INHWA[79]	
	$\gamma = 2$	$\gamma = 3$
$\mathcal{A}_1$	([0.3072, 0.4693], [0.1589, 0.2637], [0.3454, 0.4774])	([0.3040, 0.4672], [0.1593, 0.2641], [0.3468, 0.4794])
$\mathcal{A}_2$	([0.6212, 0.7493], [0.2035, 0.3259], [0.2407, 0.3805])	([0.6207, 0.7490], [0.2049, 0.3275], [0.2410, 0.3819])
$\mathcal{A}_3$	([0.4953, 0.5967], [0.1726, 0.3167], [0.3001, 0.4017])	([0.4940, 0.5960], [0.1735, 0.3175], [0.3005, 0.4022])
$\mathcal{A}_4$	([0.4951, 0.6277], [0.1923, 0.3249], [0.1641, 0.2870])	([0.4899, 0.6247], [0.1948, 0.3290], [0.1647, 0.2882])
$\mathcal{A}_5$	([0.3641, 0.4658], [0.1325, 0.2432], [0.3647, 0.4521])	([0.3621, 0.4641], [0.1327, 0.2437], [0.3672, 0.4547])
	H-INWAG	H-INOWAG
$\mathcal{A}_1$	([0.2964, 0.4613], [0.1652, 0.2677], [0.3545, 0.4867])	([0.2606, 0.4340], [0.1937, 0.2970], [0.3892, 0.5414])
$\mathcal{A}_2$	([0.6183, 0.7467], [0.2201, 0.3076], [0.2434, 0.3880])	([0.6006, 0.7315], [0.2201, 0.3184], [0.2286, 0.3585])
$\mathcal{A}_3$	([0.4900, 0.5933], [0.1862, 0.3226], [0.3033, 0.4043])	([0.5099, 0.6109], [0.2297, 0.3573], [0.3184, 0.4194])
$\mathcal{A}_4$	([0.4740, 0.6109], [0.2246, 0.3529], [0.1739, 0.2961])	([0.4193, 0.5442], [0.3216, 0.4517], [0.2201, 0.3449])
$\mathcal{A}_5$	([0.3570, 0.4593], [0.1367, 0.2477], [0.3598, 0.4639])	([0.3170, 0.4191], [0.1465, 0.2670], [0.3863, 0.4894])

Table 3.8: Score values of aggregated INNs

	INWG [200]	INWA [200]	INOWA[200]	INOWG [200]
$\mathcal{A}_1$	-0.2898	-0.2256	-0.3295	-0.3963
$\mathcal{A}_2$	0.0614	0.1162	0.1137	0.0643
$\mathcal{A}_3$	-0.0887	-0.0442	0.0778	-0.1258
$\mathcal{A}_4$	-0.0544	0.0952	-0.1077	-0.2633
$\mathcal{A}_5$	-0.2283	-0.1626	-0.2712	-0.3046
	INHWA[79]		Proposed operators	
	$\gamma = 2$	$\gamma = 3$	H-INWAG	H-INOWAG
$\mathcal{A}_1$	-0.2345	-0.2392	-0.2582	-0.3633
$\mathcal{A}_2$	0.1099	0.1072	0.1029	0.1032
$\mathcal{A}_3$	-0.0495	-0.0519	-0.0666	-0.1020
$\mathcal{A}_4$	0.0772	0.0690	0.0187	-0.1874
$\mathcal{A}_5$	-0.1813	-0.1861	-0.1959	-0.2765

Table 3.9: Ordering of the alternatives under INS environment

Existing operators	Ordering	Proposed operators	Ordering
INWA [200]	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
INOWA [200]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$	H-INWAG	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
INWG [200]	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$	H-INOWAG	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
INOWG [200]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
INHWA [79]	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		

Table 3.10: Effect of parameter λ on score values and ranking

	Score Values ($\lambda = 0$)				Score Values ($\lambda = 0.2$)				Score Values ($\lambda = 0.5$)			
	H-SVNWAG	H-SVNOWAG	H-INWAG	H-INOWAG	H-SVNWAG	H-SVNOWAG	H-INWAG	H-INOWAG	H-SVNWAG	H-SVNOWAG	H-INWAG	H-INOWAG
$\mathcal{A}_1$	-0.2894	-0.394	-0.2898	-0.3963	-0.2788	-0.3808	-0.2773	-0.3832	-0.2627	-0.3631	-0.2582	-0.3633
$\mathcal{A}_2$	0.1516	0.1467	0.0797	0.0823	0.1623	0.1574	0.0889	0.0906	0.1785	0.1738	0.1029	0.1032
$\mathcal{A}_3$	0.0120	-0.0438	-0.0887	-0.1258	0.0218	-0.0333	-0.0799	-0.1163	0.0350	-0.0174	-0.0666	-0.1020
$\mathcal{A}_4$	0.0201	-0.01556	-0.0544	-0.2633	0.0352	-0.1368	-0.0255	-0.2334	0.0581	-0.1081	0.0187	-0.1874
$\mathcal{A}_5$	-0.1913	-0.2549	-0.2283	-0.3046	-0.1739	-0.2409	-0.2154	-0.2934	-0.1474	-0.2196	-0.1958	-0.2765
Ranking	(24351)	(24351)	(24351)	(23451)	(24351)	(24351)	(24351)	(23451)	(24351)	(24351)	(24351)	(23451)
	Score Values ($\lambda = 0.7$)				Score Values ($\lambda = 0.9$)				Score Values ($\lambda = 1$)			
$\mathcal{A}_1$	-0.2518	-0.3511	-0.2453	-0.3499	-0.2408	-0.3389	-0.2322	-0.3363	-0.2353	-0.3328	-0.2257	-0.3295
$\mathcal{A}_2$	0.1894	0.1848	0.1124	0.1117	0.2004	0.1959	0.1218	0.1202	0.2060	0.2015	0.1266	0.1245
$\mathcal{A}_3$	0.0464	-0.0068	-0.0577	-0.0924	0.0563	0.0040	-0.0487	-0.0827	0.0613	0.0094	-0.0442	-0.0778
$\mathcal{A}_4$	0.0736	-0.0886	0.0488	-0.1560	0.0892	-0.0690	0.0796	-0.1239	0.0971	-0.0591	0.0952	-0.1076
$\mathcal{A}_5$	-0.1295	-0.2053	-0.1826	-0.2651	-0.1112	-0.1907	-0.1693	-0.2537	-0.1020	-0.1834	-0.1626	-0.2479
Ranking	(24351)	(24351)	(24351)	(23451)	(23451)	(23451)	(23451)	(23451)	(24351)	(24351)	(24351)	(23451)

3.6 Conclusion

The fundamental goal of this work is to introduce some new hybrid AOs which are proficient to aggregate the behavior of the averaging and geometric operators. However, it has been considered from the existing studies that the aggregated values obtained by averaging or geometric operators are either tending towards the most extreme arguments or towards the greatest weight value. Thus, they don't give the unbiased aggregated values. By taking care of their restrictions, some new hybrid neutrosophic weighted average and geometric AOs have been developed under the SVNS as well as INS environment which can give the moderate values in the aggregation process. The desirable properties of these are also investigated. The major advantages of these proposed operators are that the various existing AOs such as averaging and geometric are the special case of the proposed one. Further, depending on the standardize decision-matrix and the proposed operators, two methods for solving DMPs have been developed under the SVNS as well as INS environment. The approach has been illustrated with a numerical example and their results are compared with some of the existing approaches to show the validity of it. From the study, it is observed that the existing operators under the SVNS/INS environment can be taken as a special case of the proposed measure. Finally, a parameter λ will give various choices to the decision makers to make a decision according to his or her desired goals.

Chapter 4

Novel Frank t-norm based aggregation operators for single-valued neutrosophic numbers with their applications[1]

In this chapter, we developed some new aggregation operators (AOs) based on the Frank t-norm operational law under the SVNS environment. The Frank t-norm operation has an additional parameter which can give a flexible environment to the decision makers to chose their decisions, according to their desired goals. Based on the proposed AOs , an algorithm for solving the MCDM approach is presented and illustrated with a numerical example to show its applicability.

4.1 Introduction

The extensive literature related to the AOs for solving the decision-making problems under the SVNSs environment is described in Section 1.1.1 of Chapter 1. However, all aforementioned AOs are mainly based on the algebraic operational laws of general t-norms and their dual t-conorms. The most widely used norm operations are a product and probabilistic sum [34], because these pairs of the triangular norm are useful to carry out computing. However, the main drawbacks of the probability triangular norms are that

---

[1] The content of this chapter is published as "Novel single-valued neutrosophic decision making operators under Frank norm operations and its application", *International Journal for Uncertainty Quantification*, 6(4), 361 - 375, 2016. (**Impact Factor: 0.967**).

lack of flexibility and robustness. Apart from them, some of the existing AOs are derived by either the fuzzy extension principle or based on the triangular norms. For more details about them under the SVNSs, we refer to read [17, 79, 96, 131, 173, 177, 178].

Frank triangular norms [40], is one of the most important norm operations and the generalizations of probabilistic and product t-norm and t-conorm. Also, it satisfying the compatibility property. Since the Frank triangular norms involve the parameter (shown in Table 2.1 of Chapter 2) which can provide more flexibility and robustness in the process of information fusion and make it more adequate to model practical DM problems than other triangular norms. To the best of our knowledge, very fewer investigations on AOs based on Frank t-norms for the DM problems have been done by the authors. For instance, Qin and Liu [121] have presented an AO for triangular interval type-2 fuzzy set during the DM process. Qin et al. [122] presented a hesitant fuzzy AO based on the Frank t-norm operations. Garg and Singh [46] presented some series of AOs for the triangular interval type-2 IFSs based on the Frank t-norms for DM problems. Thus, to enrich the study of the SVNSs, an AO is the crucial factor so it is meaningful to study AOs based on the triangular norm operations.

Therefore, keeping inspiration from the fact that SVNS have a great powerful ability to model the imprecise and ambiguous information, the present chapter discusses the various AOs for the SVNNs based on Frank's t-norms for DM problems. For it, an operational law on different SVNSs and their corresponding averaging and geometric AOs have been proposed. Further, a method within the multi-criteria decision analysis based on these operators of SVNSs has been proposed for handling the uncertainties in the collective information. Finally, a numerical example is given to illustrate the approach and compare with the several existing algorithms.

4.2 Frank AT and AC based operational laws for SVNNs

Based on the definition of AT and AC, as described in Section 2.3 of Chapter 2, we have taken the Frank AT and AC operational laws whose additive generators are defined in Table 2.1 of Chapter 2. In brief, Frank operations include the Frank product($\otimes_F$) and

Frank sum ($\oplus_F$) which are defined in the following ways:

$$\begin{aligned}
x \oplus_F y &= 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right), \lambda > 1 \quad \forall (x,y) \in [0,1]^2 \\
x \otimes_F y &= \log_\lambda\left(1 + \frac{(\lambda^{x}-1)(\lambda^{y}-1)}{\lambda-1}\right), \lambda > 1 \quad \forall (x,y) \in [0,1]^2
\end{aligned}$$

Further, it can be easily verified that the Frank sum and product satisfy the following properties, i.e.,

- $\big(x \oplus_F y\big) + \big(x \otimes_F y\big) = x + y$
- $\frac{\partial(x \oplus_F y)}{\partial x} + \frac{\partial(x \otimes_F y)}{\partial x} = \frac{\partial(x \oplus_F y)}{\partial y} + \frac{\partial(x \otimes_F y)}{\partial y} = 1$

In addition, Frank t-norm and t-conorm have also two special cases given as follows.

(i) When $\lambda \to 1$, we have $x \oplus_F y = x + y - xy$ and $x \otimes_F y = xy$ which are the algebraic t-norm and the algebraic t-conorm, respectively.

(ii) When $\lambda \to +\infty$, we have $x \oplus_F y = \min(x + y, 1)$, $x \otimes_F y = \max(0, x + y - 1)$, which are the Lukasiewicz t-norm and Lukasiewicz t-conorm, respectively.

4.2.1 Operational laws for SVNNs

By keeping the features of Frank AT and AC, we define the basic operational laws for the different SVNNs as follows:

Definition 4.2.1. Let $\mathcal{A}_1 = (\zeta_1, \kappa_1, \varphi_1)$ and $\mathcal{A}_2 = (\zeta_2, \kappa_2, \varphi_2)$ be two SVNNs and $\lambda > 1$ be a real number, then the operational rules based on Frank norms are defined as follows:

$$\begin{aligned}
\mathcal{A}_1 \oplus_F \mathcal{A}_2 &= \Bigg(1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta_1}-1)(\lambda^{1-\zeta_2}-1)}{\lambda-1}\right), \log_\lambda\left(1 + \frac{(\lambda^{\kappa_1}-1)(\lambda^{\kappa_2}-1)}{\lambda-1}\right), \\
&\quad \log_\lambda\left(1 + \frac{(\lambda^{\varphi_1}-1)(\lambda^{\varphi_2}-1)}{\lambda-1}\right)\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_1 \otimes_F \mathcal{A}_2 &= \Bigg(\log_\lambda\left(1 + \frac{(\lambda^{\zeta_1}-1)(\lambda^{\zeta_2}-1)}{\lambda-1}\right), 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\kappa_1}-1)(\lambda^{1-\kappa_2}-1)}{\lambda-1}\right), \\
&\quad 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\varphi_1}-1)(\lambda^{1-\varphi_2}-1)}{\lambda-1}\right)\Bigg)
\end{aligned}$$

Theorem 4.2.1. The operations defined in Definition 4.2.1 for two SVNNs $\mathcal{A}_1$ and $\mathcal{A}_2$ are also SVNNs.

Proof. Let $\mathcal{A}_1 = (\zeta_1, \kappa_1, \varphi_1)$ and $\mathcal{A}_2 = (\zeta_2, \kappa_2, \varphi_2)$ be two SVNNs such that $0 \leq \zeta_j, \kappa_j, \varphi_j \leq 1$ and $\zeta_j + \kappa_j + \varphi_j \leq 3$ for $j = 1, 2$. Now for real number $\lambda > 1$, we have

$$\begin{aligned} \log_\lambda\left(1 + \frac{(\lambda^{1-1}-1)(\lambda^{1-1}-1)}{\lambda-1}\right) &\leq \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta_1}-1)(\lambda^{1-\zeta_2}-1)}{\lambda-1}\right) \\ &\leq \log_\lambda\left(1 + \frac{(\lambda^{1-0}-1)(\lambda^{1-0}-1)}{\lambda-1}\right) \\ \Rightarrow \quad 0 \leq \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta_1}-1)(\lambda^{1-\zeta_2}-1)}{\lambda-1}\right) &\leq 1 \end{aligned}$$

Hence, $0 \leq 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta_1}-1)(\lambda^{1-\zeta_2}-1)}{\lambda-1}\right) \leq 1$. Similarly, for $0 \leq \kappa_1, \kappa_2 \leq 1$ and $0 \leq \varphi_1, \varphi_2 \leq 1$, we get

$$\begin{aligned} & 0 \leq \log_\lambda\left(1 + \frac{(\lambda^{\kappa_1}-1)(\lambda^{\kappa_2}-1)}{\lambda-1}\right) \leq 1 \\ \text{and} \qquad & 0 \leq \log_\lambda\left(1 + \frac{(\lambda^{\varphi_1}-1)(\lambda^{\varphi_2}-1)}{\lambda-1}\right) \leq 1 \end{aligned}$$

Further, since $\zeta_j, \kappa_j, \varphi_j$ are independent to each other and hence we get

$$\begin{aligned} 0 \quad \leq \quad & 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta_1}-1)(\lambda^{1-\zeta_2}-1)}{\lambda-1}\right) \\ & + \log_\lambda\left(1 + \frac{(\lambda^{\kappa_1}-1)(\lambda^{\kappa_2}-1)}{\lambda-1}\right) \\ & + \log_\lambda\left(1 + \frac{(\lambda^{\varphi_1}-1)(\lambda^{\varphi_2}-1)}{\lambda-1}\right) \leq 3 \end{aligned}$$

which indicates $\mathcal{A}_1 \oplus_F \mathcal{A}_2$ is SVNN. Similarly, we can prove that $\mathcal{A}_1 \otimes_F \mathcal{A}_2$ is also SVNN. □

Theorem 4.2.2. Let $\mathcal{A} = (\zeta, \kappa, \varphi)$ is a SVNN and $\eta > 0$ be a real number. Then, we have

$$\begin{aligned} \eta\mathcal{A} \quad = \quad & \left(1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta}-1)^\eta}{(\lambda-1)^{\eta-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{\kappa}-1)^\eta}{(\lambda-1)^{\eta-1}}\right),\right. \\ & \left.\log_\lambda\left(1 + \frac{(\lambda^{\varphi}-1)^\eta}{(\lambda-1)^{\eta-1}}\right)\right) \end{aligned} \tag{4.1}$$

is also SVNN.

Proof. We prove the results by induction on η and then by analogy, we can obtain the same for any real positive number η. For $\eta = 2$ and by Definition 4.2.1, we have

$$\begin{aligned}
2\mathcal{A} &= \mathcal{A} \oplus_F \mathcal{A} \\
&= \left(1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta} - 1)^1(\lambda^{1-\zeta} - 1)^1}{(\lambda - 1)}\right), \log_\lambda\left(1 + \frac{(\lambda^{\kappa} - 1)^1(\lambda^{\kappa} - 1)^1}{(\lambda - 1)}\right),\right. \\
&\qquad \left.\log_\lambda\left(1 + \frac{(\lambda^{\varphi} - 1)^1(\lambda^{\varphi} - 1)^1)}{(\lambda - 1)}\right)\right) \\
&= \left(1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta} - 1)^2}{(\lambda - 1)^{2-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{\kappa} - 1)^2}{(\lambda - 1)^{2-1}}\right),\right. \\
&\qquad \left.\log_\lambda\left(1 + \frac{(\lambda^{\varphi} - 1)^2}{(\lambda - 1)^{2-1}}\right)\right)
\end{aligned}$$

Thus, result holds for $\eta = 2$. Assume it holds for $\eta = k$. Now, for $\eta = k + 1$, we have to prove

$$\begin{aligned}
(k+1)\mathcal{A} &= \left(1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta} - 1)^{k+1}}{(\lambda - 1)^{k}}\right), \log_\lambda\left(1 + \frac{(\lambda^{\kappa} - 1)^{k+1}}{(\lambda - 1)^{k}}\right),\right. \\
&\qquad \left.\log_\lambda\left(1 + \frac{(\lambda^{\varphi} - 1)^{k+1}}{(\lambda - 1)^{k}}\right)\right)
\end{aligned}$$

The left hand side can be rewritten as $(k+1)\mathcal{A} = k\mathcal{A} \oplus_F \mathcal{A}$, and based on operations defined in Definition 4.2.1, we have

$$\begin{aligned}
(k\mathcal{A}) \oplus_F \mathcal{A} &= \left(1 - \log_\lambda\left(1 + \frac{(\lambda^{1-\zeta} - 1)^{k}}{(\lambda - 1)^{k-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{\kappa} - 1)^{k}}{(\lambda - 1)^{k-1}}\right),\right. \\
&\qquad \left.\log_\lambda\left(1 + \frac{(\lambda^{\varphi} - 1)^{k}}{(\lambda - 1)^{k-1}}\right)\right) \oplus_F (\zeta, \kappa, \varphi) \\
&= \left(1 - \log_\lambda\left(1 + \frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{1-\zeta}-1)^k}{(\lambda-1)^{k-1}}\right)\right)} - 1\right)(\lambda^{1-\zeta} - 1)}{(\lambda - 1)}\right),\right. \\
&\qquad \left.\log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\kappa}-1)^k}{(\lambda-1)^{k-1}}\right)} - 1\right)(\lambda^{\kappa} - 1)}{(\lambda-1)}\right), \log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\varphi}-1)^k}{(\lambda-1)^{k-1}}\right)} - 1\right)(\lambda^{\varphi} - 1)}{(\lambda-1)}\right)\right)
\end{aligned}$$

$$= \left(1-\log_\lambda\left(1+\frac{(\lambda^{1-\zeta}-1)^{k+1}}{(\lambda-1)^k}\right), \log_\lambda\left(1+\frac{(\lambda^{\kappa}-1)^{k+1}}{(\lambda-1)^k}\right), \log_\lambda\left(1+\frac{(\lambda^{\varphi}-1)^{k+1}}{(\lambda-1)^k}\right)\right)$$

Therefore, result holds for $\eta = k+1$. Further, it can be easily verified that

$$\begin{aligned}
0 = 1-\log_\lambda\left(1+\frac{(\lambda^{1-0}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) &\leq 1-\log_\lambda\left(1+\frac{(\lambda^{1-\zeta}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) \\
&\leq 1-\log_\lambda\left(1+\frac{(\lambda^{1-1}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) = 1 \\
0 = \log_\lambda\left(1+\frac{(\lambda^{0}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) &\leq \log_\lambda\left(1+\frac{(\lambda^{\kappa}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) \\
&\leq \log_\lambda\left(1+\frac{(\lambda^{1}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) = 1 \\
\text{and} \qquad 0 = \log_\lambda\left(1+\frac{(\lambda^{0}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) &\leq \log_\lambda\left(1+\frac{(\lambda^{\varphi}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) \\
&\leq \log_\lambda\left(1+\frac{(\lambda^{1}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) = 1
\end{aligned}$$

Also,

$$\begin{aligned}
0 \leq 1-\log_\lambda\left(1+\frac{(\lambda^{1-\zeta}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) &+ \log_\lambda\left(1+\frac{(\lambda^{\kappa}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) \\
&+ \log_\lambda\left(1+\frac{(\lambda^{\varphi}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right) \leq 3
\end{aligned}$$

and hence $\eta\mathcal{A}$ is a SVNN. □

Theorem 4.2.3. For a real number $\eta > 0$ and a SVNN $\mathcal{A} = (\zeta, \kappa, \varphi)$, the operation $\mathcal{A}^{\eta}$ defined by

$$\begin{aligned}
\mathcal{A}^{\eta} \;=\; &\left(\log_\lambda\left(1+\frac{(\lambda^{\zeta}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-\kappa}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right),\right. \\
&\left. 1-\log_\lambda\left(1+\frac{(\lambda^{1-\varphi}-1)^{\eta}}{(\lambda-1)^{\eta-1}}\right)\right)
\end{aligned}$$

is again a SVNN.

Proof. It can be easily follows from the Theorem 4.2.2, so it is omit them here. □

4.2.2 Properties of the operational laws

In this section, we investigate some properties of the proposed operations laws as follows.

Theorem 4.2.4. (Commutative law) Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2)$ be two SVNNs, then

(i) $\mathcal{A}_1 \oplus_F \mathcal{A}_2 = \mathcal{A}_2 \oplus_F \mathcal{A}_1$

(ii) $\mathcal{A}_1 \otimes_F \mathcal{A}_2 = \mathcal{A}_2 \otimes_F \mathcal{A}_1$

Theorem 4.2.5. (Associative law) Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2, 3)$ be three SVNNs, then

(i) $(\mathcal{A}_1 \oplus_F \mathcal{A}_2) \oplus_F \mathcal{A}_3 = \mathcal{A}_1 \oplus_F (\mathcal{A}_2 \oplus_F \mathcal{A}_3)$.

(ii) $(\mathcal{A}_1 \otimes_F \mathcal{A}_2) \otimes_F \mathcal{A}_3 = \mathcal{A}_1 \otimes_F (\mathcal{A}_2 \otimes_F \mathcal{A}_3)$.

Theorems 4.2.4 and 4.2.5 are straightforward and we omit their proofs.

Theorem 4.2.6. If $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2)$ be two SVNNs, $\eta > 0$ be a real number, then

(i) $\eta(\mathcal{A}_1 \oplus_F \mathcal{A}_2) = \eta\mathcal{A}_1 \oplus_F \eta\mathcal{A}_2$.

(ii) $(\mathcal{A}_1 \otimes_F \mathcal{A}_2)^\eta = (\mathcal{A}_1)^\eta \otimes_F (\mathcal{A}_2)^\eta$.

(iii) $\eta_1\mathcal{A}_1 \oplus_F \eta_2\mathcal{A}_1 = (\eta_1 + \eta_2)\mathcal{A}_1$.

(iv) $(\mathcal{A}_1)^{\eta_1} \otimes_F (\mathcal{A}_1)^{\eta_2} = (\mathcal{A}_1)^{\eta_1+\eta_2}$.

Proof. We prove the parts (i) and (iii) and hence similar for other.

(i) For SVNNs $\mathcal{A}_1, \mathcal{A}_2$ and real number $\eta > 0$, we have

$$\begin{aligned} & \eta(\mathcal{A}_1 \oplus_F \mathcal{A}_2) \\ = & \left(1 - \log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\left(\lambda^{1-\zeta_1}-1\right)\left(\lambda^{1-\zeta_2}-1\right)/\lambda-1\right)} - 1\right)^\eta}{(\lambda-1)^{\eta-1}}\right),\right. \\ & \left.\log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\kappa_1}-1)(\lambda^{\kappa_2}-1)}{\lambda-1}\right)}-1\right)^\eta}{(\lambda-1)^{\eta-1}}\right), \log_\lambda\left(1 + \frac{\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\varphi_1}-1)(\lambda^{\varphi_2}-1)}{\lambda-1}\right)}-1\right)^\eta}{(\lambda-1)^{\eta-1}}\right)\right) \end{aligned}$$

$$
\begin{aligned}
= & \left(1-\log_\lambda\left(1+\frac{\frac{\left(\lambda^{(1-\zeta_1)}-1\right)^\eta\left(\lambda^{(1-\zeta_2)}-1\right)^\eta}{(\lambda-1)^\eta}}{(\lambda-1)^{\eta-1}}\right), \log_\lambda\left(1+\frac{\frac{(\lambda^{\kappa_1}-1)^\eta(\lambda^{\kappa_2}-1)^\eta}{(\lambda-1)^\eta}}{(\lambda-1)^{\eta-1}}\right),\right. \\
& \left.\log_\lambda\left(1+\frac{\frac{(\lambda^{\varphi_1}-1)^\eta(\lambda^{\varphi_2}-1)^\eta}{(\lambda-1)^\eta}}{(\lambda-1)^{\eta-1}}\right)\right) \\
= & \left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\zeta_1}-1\right)^\eta\left(\lambda^{1-\zeta_2}-1\right)^\eta}{(\lambda-1)^{2\eta-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{\kappa_1}-1)^\eta(\lambda^{\kappa_2}-1)^\eta}{(\lambda-1)^{2\eta-1}}\right),\right. \\
& \left.\log_\lambda\left(1+\frac{(\lambda^{\varphi_1}-1)^\eta(\lambda^{\varphi_2}-1)^\eta}{(\lambda-1)^{2\eta-1}}\right)\right) \\
= & \left(1-\log_\lambda\left(1+\frac{\left(\frac{\left(\lambda^{1-\zeta_1}-1\right)^\eta}{(\lambda-1)^{\eta-1}}\right)\left(\frac{\left(\lambda^{1-\zeta_2}-1\right)^\eta}{(\lambda-1)^{\eta-1}}\right)}{\lambda-1}\right), \log_\lambda\left(1+\frac{\left(\frac{(\lambda^{\kappa_1}-1)^\eta}{(\lambda-1)^{\eta-1}}\right)\left(\frac{(\lambda^{\kappa_2}-1)^\eta}{(\lambda-1)^{\eta-1}}\right)}{\lambda-1}\right),\right. \\
& \left.\log_\lambda\left(1+\frac{\left(\frac{(\lambda^{\varphi_1}-1)^\eta}{(\lambda-1)^{\eta-1}}\right)\left(\frac{(\lambda^{\varphi_2}-1)^\eta}{(\lambda-1)^{\eta-1}}\right)}{\lambda-1}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \left(1-\log_\lambda\left(\frac{\left(\lambda^{\log_\lambda\left(1+\left(\lambda^{(1-\zeta_1)^\eta}-1\right)/(\lambda-1)^{\eta-1}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\left(\lambda^{(1-\zeta_2)^\eta}-1\right)/(\lambda-1)^{\eta-1}\right)}-1\right)}{\lambda-1}\right),\right. \\
& \log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+(\lambda^{\kappa_1}-1)^\eta/(\lambda-1)^{\eta-1}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\left(\lambda^{\zeta_2}-1\right)^\eta/(\lambda-1)^{\eta-1}\right)}-1\right)}{\lambda-1}\right), \\
& \left.\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+(\lambda^{\varphi_1}-1)^\eta/(\lambda-1)^{\eta-1}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+(\lambda^{\varphi_2}-1)^\eta/(\lambda-1)^{\eta-1}\right)}-1\right)}{\lambda-1}\right)\right) \\
= & \ \eta\mathcal{A}_1 \oplus_F \eta\mathcal{A}_2
\end{aligned}
$$

(iii) For real numbers $\eta_1, \eta_2 > 0$, we have

$$
\begin{aligned}
& \eta_1\mathcal{A}_1 \oplus_F \eta_2\mathcal{A}_1 \\
= & \left(1-\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+\frac{\left(\lambda^{1-\zeta_1}-1\right)^{\eta_1}}{(\lambda-1)^{\eta_1-1}}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\frac{\left(\lambda^{1-\zeta_1}-1\right)^{\eta_2}}{(\lambda-1)^{\eta_2-1}}\right)}-1\right)}{\lambda-1}\right),\right.
\end{aligned}
$$

$$
\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\kappa_1}-1)^{\eta_1}}{(\lambda-1)^{\eta_1-1}}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\kappa_1}-1)^{\eta_2}}{(\lambda-1)^{\eta_2-1}}\right)}-1\right)}{\lambda-1}\right),
$$

$$
\left.\left.\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+\frac{\left(\lambda^{(\varphi_1)}-1\right)^{\eta_1}}{(\lambda-1)^{\eta_1-1}}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\frac{\left(\lambda^{(\varphi_1)}-1\right)^{\eta_2}}{(\lambda-1)^{\eta_2-1}}\right)}-1\right)}{\lambda-1}\right)\right)\right)
$$

$$
\begin{aligned}
&= \left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\zeta_1}-1\right)^{\eta_1+\eta_2}}{(\lambda-1)^{\eta_1+\eta_2-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{\kappa_1}-1\right)^{\eta_1+\eta_2}}{(\lambda-1)^{\eta_1+\eta_2-1}}\right),\right.\\
&\qquad \left.\log_\lambda\left(1+\frac{\left(\lambda^{\varphi_1}-1\right)^{\eta_1+\eta_2}}{(\lambda-1)^{\eta_1+\eta_2-1}}\right)\right)\\
&= (\eta_1+\eta_2)\mathcal{A}_1
\end{aligned}
$$

□

4.3 Aggregation operators based on Frank operational laws for SVNSs

Based on the Definition 4.2.1, we will discuss some averaging and geometric AOs for set of all SVNNs. For it, let $\Phi(\mathcal{X})$ be the collection of SVNNs defined over $\mathcal{X}$.

Definition 4.3.1. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)$ be the collections of "n" SVNNs. A mapping SVNFWA : $\Phi(\mathcal{X})^n \to \Phi(\mathcal{X})$, given by

$$
\mathrm{SVNFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = (\omega_1\mathcal{A}_1) \oplus_F (\omega_2\mathcal{A}_2) \oplus_F \ldots \oplus_F (\omega_n\mathcal{A}_n) \tag{4.2}
$$

where $\omega_i > 0$ be the weight vector of the $\mathcal{A}_i$'s with $\sum\limits_{i=1}^{n}\omega_i = 1$. Then SVNFWA is called SVN Frank weighted averaging operator.

Definition 4.3.2. Let $\mathcal{A}_i$ be the collection of "n" SVNNs. A mapping SVNFWG : $\Phi(\mathcal{X})^n \to \Phi(\mathcal{X})$, defined by

$$
\mathrm{SVNFWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}_1^{\omega_1} \otimes_F \mathcal{A}_2^{\omega_2} \otimes_F \ldots \otimes_F \mathcal{A}_n^{\omega_n} \tag{4.3}
$$

is called SVN weighted geometric AO, where $\omega_i > 0$ be the weight vector of SVNN $\mathcal{A}_i$'s with $\sum\limits_{i=1}^{n} \omega_i = 1$.

Theorem 4.3.1. For a collection of "n" SVNNs $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)$, the aggregated value by using SVNFWA operator is also a SVNN and is expressed as

$$\begin{aligned} &\text{SVNFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = &\left(\begin{array}{c} 1 - \log_\lambda \left(1 + \prod\limits_{i=1}^{n} \left(\lambda^{1-\zeta_i} - 1 \right)^{\omega_i} \right), \log_\lambda \left(1 + \prod\limits_{i=1}^{n} \left(\lambda^{\kappa_i} - 1 \right)^{\omega_i} \right), \\ \log_\lambda \left(1 + \prod\limits_{i=1}^{n} \left(\lambda^{\varphi_i} - 1 \right)^{\omega_i} \right) \end{array} \right) \end{aligned} \tag{4.4}$$

Proof. In order to prove the above result, it is sufficient to prove that Eq. (4.5) holds for any vector ω.

$$\begin{aligned} &\text{SVNFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = &\left(1 - \log_\lambda \left(1 + \frac{\prod\limits_{i=1}^{n} \left(\lambda^{1-\zeta_i} - 1 \right)^{\omega_i}}{(\lambda - 1)^{\sum\limits_{i=1}^{n} \omega_i - 1}} \right), \log_\lambda \left(1 + \frac{\prod\limits_{i=1}^{n} \left(\lambda^{\kappa_i} - 1 \right)^{\omega_i}}{(\lambda - 1)^{\sum\limits_{i=1}^{n} \omega_i - 1}} \right), \right. \\ &\left. \log_\lambda \left(1 + \frac{\prod\limits_{i=1}^{n} \left(\lambda^{\varphi_i} - 1 \right)^{\omega_i}}{(\lambda - 1)^{\sum\limits_{i=1}^{n} \omega_i - 1}} \right) \right) \end{aligned} \tag{4.5}$$

We prove this by induction on n. Now, for $n = 2$, we have

$$\begin{aligned} \omega_1 \mathcal{A}_1 &= \left(1 - \log_\lambda \left(1 + \frac{(\lambda^{1-\zeta_1} - 1)^{\omega_1}}{(\lambda - 1)^{\omega_1 - 1}} \right), \log_\lambda \left(1 + \frac{(\lambda^{\kappa_1} - 1)^{\omega_1}}{(\lambda - 1)^{\omega_1 - 1}} \right), \right. \\ &\qquad \left. \log_\lambda \left(1 + \frac{(\lambda^{\varphi_1} - 1)^{\omega_1}}{(\lambda - 1)^{\omega_1 - 1}} \right) \right) \\ \omega_2 \mathcal{A}_2 &= \left(1 - \log_\lambda \left(1 + \frac{(\lambda^{1-\zeta_2} - 1)^{\omega_2}}{(\lambda - 1)^{\omega_2 - 1}} \right), \log_\lambda \left(1 + \frac{(\lambda^{\kappa_2} - 1)^{\omega_1}}{(\lambda - 1)^{\omega_2 - 1}} \right), \right. \\ &\qquad \left. \log_\lambda \left(1 + \frac{(\lambda^{\varphi_2} - 1)^{\omega_2}}{(\lambda - 1)^{\omega_2 - 1}} \right) \right) \end{aligned}$$

and by using Definition 4.2.1, we get

$$\text{SVNFWA}(\mathcal{A}_1, \mathcal{A}_2) = (\omega_1 \mathcal{A}_1) \oplus_F (\omega_2 \mathcal{A}_2)$$

$$= \left(1-\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+\left(\lambda^{1-\zeta_1}-1\right)^{\omega_1}/(\lambda-1)^{\omega_1-1}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\left(\lambda^{1-\zeta_2}-1\right)^{\omega_2}/(\lambda-1)^{\omega_2-1}\right)}-1\right)}{\lambda-1}\right),\right.$$

$$\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda(1+\frac{(\lambda^{\kappa_1}-1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}})}-1\right)\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\kappa_2}-1)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)}-1\right)}{\lambda-1}\right),$$

$$\left.\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\varphi_1}-1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\frac{(\lambda^{\varphi_2}-1)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)}-1\right)}{\lambda-1}\right)\right)$$

$$= \left(1-\log_\lambda\left(1+\frac{\prod_{i=1}^{2}\left(\lambda^{1-\zeta_i}-1\right)^{\omega_i}}{(\lambda-1)^{\omega_1+\omega_2-1}}\right), \log_\lambda\left(1+\frac{\prod_{i=1}^{2}(\lambda^{\kappa_i}-1)^{\omega_i}}{(\lambda-1)^{\omega_1+\omega_2-1}}\right),\right.$$

$$\left.\log_\lambda\left(1+\frac{\prod_{i=1}^{2}(\lambda^{\varphi_i}-1)^{\omega_i}}{(\lambda-1)^{\omega_1+\omega_2-1}}\right)\right)$$

Thus, result holds for $n=2$. Assume result holds for $n=k$, then for $n=k+1$, we have

$$\text{SVNFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k,\mathcal{A}_{k+1}) = \text{SVNFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k)\oplus_F(\omega_{k+1}\mathcal{A}_{k+1})$$

$$= \left(1-\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+\prod_{i=1}^{k}\left(\lambda^{1-\zeta_i}-1\right)^{\omega_i}\Big/(\lambda-1)^{\sum_{i=1}^{k}\omega_i-1}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\left(\lambda^{1-\zeta_{k+1}}-1\right)^{\omega_{k+1}}\Big/(\lambda-1)^{\omega_{k+1}-1}\right)}-1\right)}{\lambda-1}\right),\right.$$

$$\log_\lambda\left(1+\frac{\left(\lambda^{\log_\lambda\left(1+\prod_{i=1}^{k}(\lambda^{\kappa_i}-1)^{\omega_i}\Big/(\lambda-1)^{\sum_{i=1}^{k}\omega_i-1}\right)}-1\right)\left(\lambda^{\log_\lambda\left(1+\left(\lambda^{\kappa_{k+1}}-1\right)^{\omega_{k+1}}\Big/(\lambda-1)^{\omega_{k+1}-1}\right)}-1\right)}{\lambda-1}\right),$$

$$\log_{\lambda}\left(1+\frac{\left(\lambda^{\log_{\lambda}\left(1+\prod\limits_{i=1}^{k}(\lambda^{\varphi_i}-1)^{\omega_i}\Big/(\lambda-1)^{\sum\limits_{i=1}^{k}\omega_i-1}\right)}-1\right)\left(\lambda^{\log_{\lambda}\left(1+\left(\lambda^{\varphi_{k+1}}-1\right)^{\omega_{k+1}}\Big/(\lambda-1)^{\omega_{k+1}-1}\right)}-1\right)}{\lambda-1}\right)\Bigg)\Bigg)$$

$$= \left(1-\log_{\lambda}\left(1+\frac{\prod\limits_{i=1}^{k+1}(\lambda^{1-\zeta_i}-1)^{\omega_i}}{(\lambda-1)^{\sum\limits_{i=1}^{k+1}\omega_i-1}}\right), \log_{\lambda}\left(1+\frac{\prod\limits_{i=1}^{k+1}(\lambda^{\kappa_i}-1)^{\omega_i}}{(\lambda-1)^{\sum\limits_{i=1}^{k+1}\omega_i-1}}\right)\right.$$
$$\left.\log_{\lambda}\left(1+\frac{\prod\limits_{i=1}^{k+1}(\lambda^{\varphi_i}-1)^{\omega_i}}{(\lambda-1)^{\sum\limits_{i=1}^{k+1}\omega_i-1}}\right)\right)$$

Therefore, result holds for $n = k + 1$. □

Theorem 4.3.2. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)$ be "n" SVNNs. The aggregated value by using Definition 4.3.2 is again a SVNN and is given by

$$\begin{aligned}&\text{SVNFWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\\ &= \begin{pmatrix} \log_{\lambda}\left(1+\prod\limits_{i=1}^{n}\left(\lambda^{\zeta_i}-1\right)^{\omega_i}\right), 1-\log_{\lambda}\left(1+\prod\limits_{i=1}^{n}\left(\lambda^{1-\kappa_i}-1\right)^{\omega_i}\right), \\ 1-\log_{\lambda}\left(1+\prod\limits_{i=1}^{n}\left(\lambda^{1-\varphi_i}-1\right)^{\omega_i}\right)\end{pmatrix}\end{aligned} \tag{4.6}$$

Proof. As similar to above theorem, we can obtained their proof. □

Further, it is observed that both the developed operators namely SVNFWA and SVNFWG have certain properties such as idempotency, monotonicity, and boundedness. As an example, we have explained it by taking SVNFWA operator, while it is similar to other ones.

Property 4.3.1. If all SVNNs $\mathcal{A}_i$'s are equal to $\mathcal{A}$, then

$$\text{SVNFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$$

Proof. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)$ be SVNNs such that $\mathcal{A}_i = \mathcal{A} = (\zeta, \kappa, \varphi)$ then by definition of the equality of two SVNNs we have $\zeta_i = \zeta$, $\kappa_i = \kappa$ and $\varphi_i = \varphi$. Thus, by Eq. (4.4) and $\sum\limits_{i=1}^{n} \omega_i = 1$, we have

$$\begin{aligned}
\mathrm{SVNFWA}(\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A}) &= \Bigg(1 - \log_\lambda \Bigg(1 + \prod_{i=1}^{n} \left(\lambda^{1-\zeta} - 1\right)^{\omega_i}\Bigg), \log_\lambda \Bigg(1 + \prod_{i=1}^{n} (\lambda^{\kappa} - 1)^{\omega_i}\Bigg), \\
&\qquad \log_\lambda \Bigg(1 + \prod_{i=1}^{n} (\lambda^{\varphi} - 1)^{\omega_i}\Bigg)\Bigg) \\
&= \Bigg(1 - \log_\lambda \Bigg(1 + \left(\lambda^{1-\zeta} - 1\right)^{\sum\limits_{i=1}^{n}\omega_i}\Bigg), \log_\lambda \Bigg(1 + (\lambda^{\kappa} - 1)^{\sum\limits_{i=1}^{n}\omega_i}\Bigg), \\
&\qquad \log_\lambda \Bigg(1 + (\lambda^{\varphi} - 1)^{\sum\limits_{i=1}^{n}\omega_i}\Bigg)\Bigg) \\
&= \Bigg(1 - \log_\lambda \left(1 + \left(\lambda^{1-\zeta} - 1\right)\right), \log_\lambda \left(1 + (\lambda^{\kappa} - 1)\right), \\
&\qquad \log_\lambda \left(1 + (\lambda^{\varphi} - 1)\right)\Bigg) \\
&= (\zeta, \kappa, \varphi) \\
&= \mathcal{A}
\end{aligned}$$

□

Property 4.3.2. (Monotonicity) Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)$ and $\mathcal{A}'_i = (\zeta'_i, \kappa'_i, \varphi'_i)$, $(i = 1, 2, \ldots, n)$ be two collections of SVNNs with $\zeta_i \leq \zeta'_i$, $\kappa_i \geq \kappa'_i$ and $\varphi_i \geq \varphi'_i$, for all i, then

$$\mathrm{SVNFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathrm{SVNFWA}(\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_n).$$

Proof. Since $\mathcal{A}_i$ and $\mathcal{A}'_i$ be two SVNNs such that for all i, $\zeta_i \leq \zeta'_i$, $\kappa_i \geq \kappa'_i$ and $\varphi_i \geq \varphi'_i$ and $\lambda > 1$ be a real number. Therefore, $\lambda^{1-\zeta_i} \geq \lambda^{1-\zeta'_i}$ implies that

$$\begin{aligned}
& 1 + \prod_{i=1}^{n} (\lambda^{1-\zeta_i} - 1)^{\omega_i} \geq 1 + \prod_{i=1}^{n} (\lambda^{1-\zeta'_i} - 1)^{\omega_i} \\
\Leftrightarrow \quad & 0 \leq \frac{1 + \prod\limits_{i=1}^{n} (\lambda^{1-\zeta'_i} - 1)^{\omega_i}}{1 + \prod\limits_{i=1}^{n} (\lambda^{1-\zeta_i} - 1)^{\omega_i}} \leq 1
\end{aligned}$$

$$\Leftrightarrow \ \log_\lambda \left(\frac{1+\prod\limits_{i=1}^{n}(\lambda^{1-\zeta_i'}-1)^{\omega_i}}{1+\prod\limits_{i=1}^{n}(\lambda^{1-\zeta_i}-1)^{\omega_i}} \right) \leq 0$$

$$\Leftrightarrow \ 1-\log_\lambda \left(1+\prod_{i=1}^{n}(\lambda^{1-\zeta_i}-1)^{\omega_i}\right) \leq 1-\log_\lambda \left(1+\prod_{i=1}^{n}(\lambda^{1-\zeta_i'}-1)^{\omega_i}\right)$$

Further, $\kappa_i \geq \kappa_i'$ implies that

$$1+\prod_{i=1}^{n}(\lambda^{\kappa_i}-1)^{\omega_i} \geq 1+\prod_{i=1}^{n}(\lambda^{\kappa_i'}-1)^{\omega_i}$$

$$\Leftrightarrow \ \log_\lambda \left(\frac{1+\prod\limits_{i=1}^{n}(\lambda^{\kappa_i}-1)^{\omega_i}}{1+\prod\limits_{i=1}^{n}(\lambda^{\kappa_i'}-1)^{\omega_i}} \right) \geq 0$$

$$\Leftrightarrow \ \log_\lambda \left(1+\prod_{i=1}^{n}(\lambda^{\kappa_i}-1)^{\omega_i}\right) \geq \log_\lambda \left(1+\prod_{i=1}^{n}(\lambda^{\kappa_i'}-1)^{\omega_i}\right)$$

Similarly, for $\varphi_i \geq \varphi_i'$, we get

$$\log_\lambda \left(1+\prod_{i=1}^{n}(\lambda^{\varphi_i}-1)^{\omega_i}\right) \geq \log_\lambda \left(1+\prod_{i=1}^{n}(\lambda^{\varphi_i'}-1)^{\omega_i}\right).$$

Therefore, by score function of SVNN, we get

$$\text{SVNFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{SVNFWA}(\mathcal{A}_1', \mathcal{A}_2', \ldots, \mathcal{A}_n')$$

□

Property 4.3.3. For a collection of SVNNs $\mathcal{A}_i$'s, take $\mathcal{A}^- = (\min\limits_i \zeta_i, \max\limits_i \kappa_i, \max\limits_i \varphi_i)$ and $\mathcal{A}^+ = (\max\limits_i \zeta_i, \min\limits_i \kappa_i, \min\limits_i \varphi_i)$ then $\mathcal{A}^- \leq \text{SVNFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$

Proof. By using idempotency and monotonicity properties of SVNFWA operator, this result can be followed directly. Hence, we omit their proof at here. □

4.4 Decision making method based on proposed operators

The general description of the decision making problem is given in Section 2.5 of Chapter 2 under the SVNS environment. Under it, the given alternatives $\mathcal{A}_i (i = 1, 2, \ldots, m)$ are assessed under the different attributes $\mathcal{G}_j (j = 1, 2, \ldots, n)$ by an expert. The rating information provided by an expert are summarized in the decision matrix $\mathcal{M} = (\alpha_{ij})_{m\times n}$ where $\alpha_{ij} = (\zeta_{ij}, \kappa_{ij}, \varphi_{ij})$ such that $0 \le \zeta_{ij}, \kappa_{ij}, \varphi_{ij} \le 1$ and $\zeta_{ij} + \kappa_{ij} + \varphi_{ij} \le 3$. Here, ζ_{ij}, κ_{ij} and φ_{ij} represents the degrees of "truth membership function", "indeterminacy-membership function" and a "falsity membership function" of the alternative $\mathcal{A}_i$ under attribute $\mathcal{G}_j$ respectively.

Then, based on the developed operators, we present an approach to solve the MADM problems, whose steps are summarized as follows:

Step 1: Normalize the matrix $\mathcal{M}$, if required, by using Eq. (2.38) of Chapter 2 and hence get the normalized matrix $\mathcal{R} = (r_{ij})_{m\times n}$.

Step 2: Aggregate the different preference values of the alternative r_{ij} by using either SVNFWA or SVNFWG operator to get the collective information $r_i = (\zeta_i, \kappa_i, \varphi_i)$ towards the each alternative $\mathcal{A}_i$, $(i = 1, 2, \ldots, m)$. For example, by utilizing SVNFWA operator, we get

$$\begin{aligned} r_i &= \text{SVNFWA}(r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \left(1 - \log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{1-\zeta_{ij}} - 1\right)^{\omega_j}\right), \log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{\kappa_{ij}} - 1\right)^{\omega_j}\right),\right. \\ &\qquad \left.\log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{\varphi_{ij}} - 1\right)^{\omega_j}\right)\right) \end{aligned} \tag{4.7}$$

and by SVNFWG operator, we get

$$\begin{aligned} r_i &= \text{SVNFWG}(r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \left(\log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{\zeta_{ij}} - 1\right)^{\omega_j}\right), 1 - \log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{1-\kappa_{ij}} - 1\right)^{\omega_j}\right),\right. \\ &\qquad \left.1 - \log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{1-\varphi_{ij}} - 1\right)^{\omega_j}\right)\right) \end{aligned} \tag{4.8}$$

Step 3: Compute the score values of r_i, $(i = 1, 2, \ldots, m)$ as

$$\mathcal{S}(r_i) = \zeta_i - \kappa_i - \varphi_i \tag{4.9}$$

Step 4: Rank the alternatives based on the decreasing values of $\mathcal{S}(\cdot)$ and select the best one(s).

4.5 Illustrative example

The above proposed approach is illustrated with a numerical example, which can be read as

Consider a certain university, in which a computer center wants to improve the work productivity. For it they wants to select a new information system from the set of four different alternatives $\mathcal{A}_i, i = 1, 2, 3, 4$ which are evaluated by the decision maker under the different criteria, namely the "cost of hardware/software" ($\mathcal{G}_1$), "contribution to organizational performance" ($\mathcal{G}_2$) and "effort to transform from current system" ($\mathcal{G}_3$) whose weight vector is $\omega = (0.4, 0.2, 0.4)^T$. After evaluation, the rating values of these alternatives are summarized in the form of SVNNs as below.

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \mathcal{A}_4 \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1\quad & \quad\mathcal{G}_2\quad & \quad\mathcal{G}_3\quad \end{array} \\ \begin{bmatrix} (0.265, 0.350, 0.385) & (0.280, 0.610, 0.330) & (0.245, 0.275, 0.480) \\ (0.345, 0.245, 0.410) & (0.280, 0.710, 0.430) & (0.245, 0.375, 0.380) \\ (0.365, 0.300, 0.335) & (0.205, 0.685, 0.480) & (0.340, 0.370, 0.290) \\ (0.430, 0.300, 0.270) & (0.295, 0.755, 0.460) & (0.310, 0.520, 0.170) \end{bmatrix} \end{array}$$

Then, the steps of the Section 4.4 corresponding to SVNFWA operator are executed here for the considered problem.

Step 1: Since $\mathcal{G}_1$ & $\mathcal{G}_3$ are the cost criteria and hence the normalized matrix $\mathcal{R}$ is obtained, by using Eq. (2.38) of Chapter 2, as

$$\mathcal{R} = \begin{bmatrix} (0.265, 0.350, 0.385) & (0.330, 0.390, 0.280) & (0.245, 0.275, 0.480) \\ (0.345, 0.245, 0.410) & (0.430, 0.290, 0.280) & (0.245, 0.375, 0.380) \\ (0.365, 0.300, 0.335) & (0.480, 0.315, 0.205) & (0.340, 0.370, 0.290) \\ (0.430, 0.300, 0.270) & (0.460, 0.245, 0.295) & (0.310, 0.520, 0.170) \end{bmatrix}$$

Step 2: Without loss of generality, we use $\lambda = 2$ and hence by Eq. (4.7), the aggregated values of each alternative are obtained as $r_1 = (0.2705, 0.3955, 0.3251)$, $r_2 = (0.3249, 0.3689, 0.3010)$, $r_3 = (0.3799, 0.2871, 0.3296)$, and $r_4 = (0.3907, 0.2289, 0.3612)$.

Step 3: By Eq. (4.9), we get $\mathcal{S}(r_1) = -0.4501$; $\mathcal{S}(r_2) = -0.3450$; $\mathcal{S}(r_3) = -0.2368$ and $\mathcal{S}(r_4) = -0.1994$.

Step 4: As $\mathcal{S}(r_4) > \mathcal{S}(r_3) > \mathcal{S}(r_2) > \mathcal{S}(r_1)$ and thus the ranking order of the alternatives is $\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$. Therefore, the best alternative is $\mathcal{A}_4$.

Further, if we utilize the SVNFWG operator instead of SVNFWA to aggregate the given information during Step 2 of the developed approach, then their corresponding results are summarized as follows.

Step 1: As similar as that of above.

Step 2: Without loss of generality, we use $\lambda = 2$ and hence by Eq. (4.8), we get the collective values of the alternatives are r_1=(0.2685, 0.3292,0.4056), r_2=(0.3152, 0.3080, 0.3734), r_3=(0.3752, 0.3316, 0.2922) and r_4=(0.3831, 0.3865, 0.2364).

Step 3: By Eq. (4.9), we get $\mathcal{S}(r_1) = -0.4664$, $\mathcal{S}(r_2) = -0.3661$, $\mathcal{S}(r_3) = -0.2486$ and $\mathcal{S}(r_4) = -0.2399$.

Step 4: As $\mathcal{S}(r_4) > \mathcal{S}(r_3) > \mathcal{S}(r_2) > \mathcal{S}(r_1)$ and thus the ranking order of the alternatives is $\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$. Therefore, the best alternative is again $\mathcal{A}_4$.

On the other hand, if we apply the various existing approaches [17, 79, 96, 131, 173, 177, 178] from the field of DM to the considered problem, then their corresponding rating values as well as the ranking of the alternatives are summarized in Table 4.1. From these results, it has been analyzed the best alternatives coincides with the proposed ones and hence the proposed methods have a suitable tool for solving the DM problems under the uncertain environment.

4.5.1 Sensitivity Analysis

In order to see the influence of the parameter λ on the decision making, an analysis has been conducted in which different values of $\lambda (= 1, 1.5, 2, 2.5, 3, 5, 10, 15)$ have been taken for the considered problem. Based on these parameters, the proposed approach has been applied and their corresponding score values, as well as the ranking of the alternatives, are summarized in Table 4.2. From it, it has been observed that with the increase of λ, score value by SVNFWA operators is decreasing while increases for SVNFWG operators.

Further, it has been concluded that the ranking of the given alternative is symmetric and found that the most suitable alternative is $\mathcal{A}_4$ and $\mathcal{A}_1$ is the least one.

4.6 Conclusions

Aggregation operators play a crucial role during the DM process as most of the data related to system identification are uncertain in nature. For this, the neutrosophic set theory has been utilized in the present manuscript and hence the performance of each object has been measured in terms of SVNNs. In order to aggregate all these preferences, a Frank-operator based an AO such as SVNFWA and SVNFWG have been proposed in this chapter. Some of its desirable properties have also been investigated. Further, a DM approach has been presented based on these operators and hence illustrated with a numerical example in which each alternative is assessed in terms of SVNNs. By comparison with the existing approaches, it has been concluded that the proposed operators show a more stable, practical and optimistic nature to the decision makers during the aggregation process. Measure values corresponding to different values of λ will offer the various choices to the decision-makers in assessing the alternatives. Therefore, the present approach becomes more consistent and reliable to present the degree of fuzziness.

Table 4.1: Comparative study with the existing approaches

	Method	Calculated Values of				Ranking
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	
Liu et al. [79]	Hamacher Operator					
	$\gamma = 1$	0.2707	0.3257	0.3804	0.3913	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\gamma = 2$	-0.0445	0.0555	0.1544	0.1628	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\gamma = 2.5$	-0.0804	0.0332	0.1429	0.1435	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\gamma = 3$	-0.0922	0.0303	0.1462	0.1398	$\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	$\gamma = 5$	-0.0661	0.0694	0.1900	0.1656	$\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Ye [178]	Cross entropy	1.9099	1.7331	1.5431	1.5296	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Ye [173]	Correlation Coefficient	0.4559	0.5471	0.6453	0.6387	$\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Majumdar and Samant [96]	Similarity measure	0.5200	0.5600	0.5967	0.6000	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Broumi and Smarandache [17]	Distance measure	0.7300	0.6780	0.6220	0.6120	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Sahin [131]	Score function	0.1133	0.1782	0.2174	0.2225	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Ye [177]	Hamming distance	0.4867	0.4520	0.4147	0.4080	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	Euclidean distance	0.5196	0.4839	0.4424	0.4446	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Ye [181]	Cosine Similarity measure	0.4110	0.4815	0.5575	0.5695	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	Cosine Similarity measure	0.7214	0.7563	0.7941	0.7997	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$

Table 4.2: Effect of the parameter λ on ranking of the alternatives

λ	Operator	Score value of alternative				Ranking
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	
$\to 1$	SVNFWA	-0.4486	-0.3433	-0.2357	-0.1960	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	SVNFWG	-0.4677	-0.3678	-0.2496	-0.2434	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
1.5	SVNFWA	-0.4495	-0.3443	-0.2363	-0.1980	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	SVNFWG	-0.4669	-0.3668	-0.2490	-0.2413	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
2	SVNFWA	-0.4501	-0.3450	-0.2368	-0.1994	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	SVNFWG	-0.4664	-0.3661	-0.2486	-0.2399	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
2.5	SVNFWA	-0.4506	-0.3456	-0.2371	-0.2004	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	SVNFWG	-0.4660	-0.3656	-0.2484	-0.2389	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
3	SVNFWA	-0.4509	-0.3460	-0.2373	-0.2012	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	SVNFWG	-0.4657	-0.3652	-0.2482	-0.2381	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
5	SVNFWA	-0.4518	-0.3471	-0.2380	-0.2033	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	SVNFWG	-0.4649	-0.3642	-0.2476	-0.2360	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
10	SVNFWA	-0.4530	-0.3484	-0.2388	-0.2074	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	SVNFWG	-0.4639	-0.3629	-0.2470	-0.2336	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
15	SVNFWA	-0.4536	-0.3484	-0.2388	-0.2060	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
	SVNFWG	-0.4635	-0.3623	-0.2466	-0.2324	$\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$

Chapter 5

Prioritized Muirhead Mean Aggregation Operator under Neutrosophic Set Environment and its applications[1]

SVNS, as an extension and generalization of an IFS, is a powerful tool to describe the fuzziness and uncertainty, and Muirhead mean (MM) is a well-known AO which can consider interrelationships among any number of arguments assigned by a variable vector. In order to take advantages of both, in this chapter, we introduce two new prioritized MM AOs, such as the SVN prioritized MM (SVNPMM) and SVN prioritized dual MM (SVNPDMM) under SVN set environment. In addition, some properties of these new AOs are investigated and some special cases are discussed. Furthermore, we propose a new method based on these operators for solving the MCDM problems and illustrate with a numerical example.

5.1 Introduction

From the literature of AOs in Chapter 1, it is analyzed that the input arguments used during the aggregation may not be independent of each other and in complex problems, the involvement of interrelationship between the argument values always occurs in DM

---

[1] The content of this chapter is published as "Multi-criteria decision-making method based on prioritized Muirhead mean aggregation operator under Neutrosophic set environment", *Symmetry*, 10(7), 280, 2018; doi: 10.3390/sym10070280 (**Impact Factor: 1.256**).

process. Moreover, the DM process always encounters a problem of some criterion values, provided by the decision makers, whose impact on the DM process are unduly high or unduly low; this consequently affects the final output of the process. To handle both the scenarios in the DM process, we develop two new AOs named as SVN prioritized MM (SVNPMM) and SVN prioritized dual MM (SVNPDMM) operators, by extending the operations of SVNNs by using MM and prioritized operators. MM operator is a powerful and useful aggregation technique with the feature that it considers the interrelationships among all arguments which makes it more powerful and comprehensive than BM, MSM and HM. Moreover, the MM has a parameter vector which can make the aggregation process more flexible. Further, an MCDM method based on these proposed operators under the SVNS environment where preferences related to each alternative is expressed in terms of SVNNs has been established. An illustrative example is presented to testify the efficiency and superiority of the proposed method by comparative analysis with the other existing methods. Apart from these, we verify that the proposed methods have advantages with respect to existing operators as follows: (1) some of the existing AOs can be taken as a special case of the proposed operators under NSs environment, (2) they consider the interrelationship among all arguments, (3) they are more adaptable and feasible than the existing AOs based on the parameter vector, (4) the presented approach considers the preferences of the decision maker in terms of risk preference as well as risk aversion.

5.2 Neutrosophic Prioritized Muirhead Mean Operators

Let $\Phi(\mathcal{X})$ be a collection of "n" SVNNs $\mathcal{A}_j$ and σ is the permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\sigma(j-1)} \leq \mathcal{A}_{\sigma(j)}$ for $j = 2, 3, \ldots, n$. Then, in this section, we develop some new prioritized based MM AOs on $\Phi(\mathcal{X})$.

5.2.1 Single-Valued Neutrosophic Prioritized Muirhead Mean Operator

Definition 5.2.1. For a collection of SVNNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$, a SVNPMM operator is a mapping SVNPMM : $\Phi(\mathcal{X})^n \to \Phi(\mathcal{X})$ defined as

$$\mathrm{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \quad = \quad \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \prod_{j=1}^{n} \left(n\xi_j \mathcal{A}_{\sigma(j)} \right)^{p_j} \right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}}, \tag{5.1}$$

where $\xi_j = \frac{H_{\sigma(j)}}{\sum\limits_{j=1}^{n} H_j}$, $H_1 = 1$, $H_j = \prod\limits_{k=1}^{j-1} \mathcal{S}(\mathcal{A}_k); (j = 2, \ldots, n)$, S_n is collection of all permutations of $(1, 2, \ldots, n)$ and $P = (p_1, p_1, \ldots, p_n) \in R^n$ be a vector of parameters.

Theorem 5.2.1. For a collection of SVNNs $\mathcal{A}_j = (\zeta_j, \kappa_j, \varphi_j), (j = 1, 2, \ldots, n)$, the aggregated value by Eq. (5.1) is again a SVNN and given by

$$\begin{aligned} &\mathrm{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = &\left(\begin{array}{c} \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - (1 - \zeta_{\sigma(j)})^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - {\kappa_{\sigma(j)}}^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - {\varphi_{\sigma(j)}}^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}} \end{array} \right). \end{aligned} \tag{5.2}$$

Proof. For SVNN $\mathcal{A}_j (j = 1, 2, \ldots, n)$ and by Definition 2.4.2 of Chapter 2, we have

$$\begin{aligned} n\xi_j \mathcal{A}_\sigma(j) \quad &= \quad \left(1 - (1 - \zeta_{\sigma(j)})^{n\xi_j}, {\kappa_{\sigma(j)}}^{n\xi_j}, {\varphi_{\sigma(j)}}^{n\xi_j} \right) \\ \text{and} \qquad \left(n\xi_j \mathcal{A}_{\sigma(j)} \right)^{p_j} \quad &= \quad \left(\begin{array}{c} \left(1 - (1 - \zeta_{\sigma(j)})^{n\xi_j} \right)^{p_j}, 1 - \left(1 - {\kappa_{\sigma(j)}}^{n\xi_j} \right)^{p_j}, \\ 1 - \left(1 - {\varphi_{\sigma(j)}}^{n\xi_j} \right)^{p_j} \end{array} \right). \end{aligned}$$

Thus,

$$\bigoplus_{\sigma \in S_n} \prod_{j=1}^{n} \left(n\xi_j \mathcal{A}_{\sigma(j)} \right)^{p_j} \quad = \quad \left(\begin{array}{c} 1 - \prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - (1 - \zeta_{\sigma(j)})^{n\xi_j} \right)^{p_j} \right), \\ \prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - {\kappa_{\sigma(j)}}^{n\xi_j} \right)^{p_j} \right), \\ \prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - {\varphi_{\sigma(j)}}^{n\xi_j} \right)^{p_j} \right) \end{array} \right).$$

Now,

$$\mathrm{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \prod_{j=1}^{n} \left(n\xi_j \mathcal{A}_{\sigma(j)} \right)^{p_j} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}$$

$$= \left(\begin{array}{c} \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \zeta_{\sigma(j)})^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \kappa_{\sigma(j)}{}^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \varphi_{\sigma(j)}{}^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \end{array} \right).$$

Thus Eq. (5.2) holds. Furthermore, $0 \le \zeta_{\sigma(j)}, \kappa_{\sigma(j)}, \varphi_{\sigma(j)} \le 1$ so we have $1 - \left(1 - \zeta_{\sigma(j)}\right)^{n\xi_j} \in [0,1]$ and $\prod_{j=1}^{n} \left(1 - (1 - \zeta_{\sigma(j)})^{n\xi_j}\right)^{p_j} \in [0,1]$, which implies that

$$1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \zeta_{\sigma(j)})^{n\xi_j} \right)^{p_j} \right) \right) \in [0,1].$$

Hence,

$$0 \le \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \zeta_{\sigma(j)})^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \le 1.$$

Similarly, we have

$$0 \le 1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \kappa_{\sigma(j)}{}^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \le 1$$

and

$$0 \le 1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \varphi_{\sigma(j)}{}^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \le 1.$$

which complete the proof. □

The working of the proposed operator is demonstrated through a numerical example, which is illustrated as follow.

Example 5.2.1. Let $\mathcal{A}_1 = (0.5, 0.2, 0.3)$, $\mathcal{A}_2 = (0.3, 0.5, 0.4)$ and $\mathcal{A}_3 = (0.6, 0.5, 0.2)$ be three SVNNs and $P = (1, 0.5, 0.3)$ be the given parameter vector. By utilizing the given information and $H_j = \prod\limits_{k=1}^{j-1} \mathcal{S}(\mathcal{A}_k); (j = 2, 3)$, we get $H_1 = 1$, $H_2 = 0.74$ and $H_3 = 0.2257$. Hence, $\xi_j = \frac{H_{\sigma(j)}}{\sum\limits_{j=1}^{n} H_j}$ becomes $\xi_1 = 0.5087$, $\xi_2 = 0.3765$, $\xi_3 = 0.1148$. Therefore,

$$
\begin{aligned}
&\prod_{\sigma\in S_3}\left(1-\prod_{j=1}^{3}\left(1-(1-\zeta_{\sigma(j)})^{3\xi_j}\right)^{p_j}\right)\\
=\;&\left\{1-\left(1-(1-0.5)^{3\times0.5087}\right)^{1}\times\left(1-(1-0.3)^{3\times0.3765}\right)^{0.5}\times\left(1-(1-0.6)^{3\times0.1148}\right)^{0.3}\right\}\\
&\times\left\{1-\left(1-(1-0.3)^{3\times0.3765}\right)^{1}\times\left(1-(1-0.5)^{3\times0.5087}\right)^{0.5}\times\left(1-(1-0.6)^{3\times0.1148}\right)^{0.3}\right\}\\
&\times\left\{1-\left(1-(1-0.6)^{3\times0.1148}\right)^{1}\times\left(1-(1-0.3)^{3\times0.3765}\right)^{0.5}\times\left(1-(1-0.5)^{3\times0.5087}\right)^{0.3}\right\}\\
&\times\left\{1-\left(1-(1-0.3)^{3\times0.3765}\right)^{1}\times\left(1-(1-0.6)^{3\times0.1148}\right)^{0.5}\times\left(1-(1-0.5)^{3\times0.5087}\right)^{0.3}\right\}\\
&\times\left\{1-\left(1-(1-0.5)^{3\times0.5087}\right)^{1}\times\left(1-(1-0.6)^{3\times0.1148}\right)^{0.5}\times\left(1-(1-0.3)^{3\times0.3765}\right)^{0.3}\right\}\\
&\times\left\{1-\left(1-(1-0.6)^{3\times0.1148}\right)^{1}\times\left(1-(1-0.5)^{3\times0.5087}\right)^{0.5}\times\left(1-(1-0.3)^{3\times0.3765}\right)^{0.3}\right\}\\
=\;&0.0052.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\prod_{\sigma\in S_3}\left(1-\prod_{j=1}^{3}\left(1-\kappa_{\sigma(j)}^{3\xi_j}\right)^{p_j}\right)\\
=\;&\left\{1-\left(1-(0.2)^{3\times0.5087}\right)^{1}\times\left(1-(0.5)^{3\times0.3765}\right)^{0.5}\times\left(1-(0.5)^{3\times0.1148}\right)^{0.3}\right\}\\
\times\;&\left\{1-\left(1-(0.5)^{3\times0.3765}\right)^{1}\times\left(1-(0.2)^{3\times0.5087}\right)^{0.5}\times\left(1-(0.5)^{3\times0.1148}\right)^{0.3}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\times &\quad \left\{1-\left(1-(0.5)^{3\times 0.1148}\right)^{1}\times\left(1-(0.5)^{3\times 0.3765}\right)^{0.5}\times\left(1-(0.2)^{3\times 0.5087}\right)^{0.3}\right\}\\
\times &\quad \left\{1-\left(1-(0.5)^{3\times 0.3765}\right)^{1}\times\left(1-(0.5)^{3\times 0.1148}\right)^{0.5}\times\left(1-(0.2)^{3\times 0.5087}\right)^{0.3}\right\}\\
\times &\quad \left\{1-\left(1-(0.2)^{3\times 0.5087}\right)^{1}\times\left(1-(0.5)^{3\times 0.1148}\right)^{0.5}\times\left(1-(0.5)^{3\times 0.3765}\right)^{0.3}\right\}\\
\times &\quad \left\{1-\left(1-(0.5)^{3\times 0.1148}\right)^{1}\times\left(1-(0.2)^{3\times 0.5087}\right)^{0.5}\times\left(1-(0.5)^{3\times 0.3765}\right)^{0.3}\right\}\\
= &\quad 0.000093196
\end{aligned}
$$

and

$$
\begin{aligned}
&\prod_{\sigma\in S_3}\left(1-\prod_{j=1}^{3}\left(1-\varphi_{\sigma(j)}^{3\xi_j}\right)^{p_j}\right)\\
= &\quad \left\{1-\left(1-(0.3)^{3\times 0.5087}\right)^{1}\times\left(1-(0.4)^{3\times 0.3765}\right)^{0.5}\times\left(1-(0.2)^{3\times 0.1148}\right)^{0.3}\right\}\\
\times &\quad \left\{1-\left(1-(0.4)^{3\times 0.3765}\right)^{1}\times\left(1-(0.3)^{3\times 0.5087}\right)^{0.5}\times\left(1-(0.2)^{3\times 0.1148}\right)^{0.3}\right\}\\
\times &\quad \left\{1-\left(1-(0.2)^{3\times 0.1148}\right)^{1}\times\left(1-(0.4)^{3\times 0.3765}\right)^{0.5}\times\left(1-(0.3)^{3\times 0.5087}\right)^{0.3}\right\}\\
\times &\quad \left\{1-\left(1-(0.4)^{3\times 0.3765}\right)^{1}\times\left(1-(0.2)^{3\times 0.1148}\right)^{0.5}\times\left(1-(0.3)^{3\times 0.5087}\right)^{0.3}\right\}\\
\times &\quad \left\{1-\left(1-(0.3)^{3\times 0.5087}\right)^{1}\times\left(1-(0.2)^{3\times 0.1148}\right)^{0.5}\times\left(1-(0.4)^{3\times 0.3765}\right)^{0.3}\right\}\\
\times &\quad \left\{1-\left(1-(0.2)^{3\times 0.1148}\right)^{1}\times\left(1-(0.3)^{3\times 0.5087}\right)^{0.5}\times\left(1-(0.4)^{3\times 0.3765}\right)^{0.3}\right\}\\
= &\quad 0.00000093195.
\end{aligned}
$$

Hence, by using Eq. (5.2), we get the aggregated value by SVNPMM is

$$
\begin{aligned}
&\mathrm{SVNPMM}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)\\
= &\left(\begin{array}{c}\left(1-(0.0052)^{1/6}\right)^{1/1.8},1-\left(1-(0.000093196)^{1/6}\right)^{1/1.8},\\ 1-\left(1-(0.00000093195)^{1/6}\right)^{1/1.8}\end{array}\right)\\
= &\ (0.7415, 0.1246, 0.0562).
\end{aligned}
$$

It is observed from the proposed operator that it satisfies the certain properties which are stated as follows.

Theorem 5.2.2. If $\mathcal{A}_j = (\zeta_j, \kappa_j, \varphi_j)$ and $\mathcal{A}'_j = (\zeta'_j, \kappa'_j, \varphi'_j)$ are two SVNNs such that $\zeta_j \leq \zeta'_j$, $\kappa_j \geq \kappa'_j$ and $\varphi_j \geq \varphi'_j$ for all j, then

$$\text{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{SVNPMM}(\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_n).$$

This property is called monotonicity.

Proof. For two SVNNs $\mathcal{A}_j$ and $\mathcal{A}'_j$. Let $\xi_j = \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}$ and $\xi'_j = \frac{H'_{\sigma(j)}}{\sum_{j=1}^{n} H'_j}$ where $H_1 = 1$, $H_j = \prod_{k=1}^{j-1} \mathcal{S}(\mathcal{A}_k)$ and $H'_1 = 1$, $H'_j = \prod_{k=1}^{j-1} \mathcal{S}(\mathcal{A}'_k)$ for $(j = 2, 3, \ldots, n)$, such that $\xi_j, \xi'_j \in [0, 1]$. As for all j we have $\zeta_{\sigma(j)} \leq \zeta'_{\sigma(j)}$ which implies that $(1 - \zeta_{\sigma(j)})^{n\xi_j} \geq (1 - \zeta'_{\sigma(j)})^{n\xi'_j}$, Thus,

$$\left(1 - \left(1 - \zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j} \leq \left(1 - \left(1 - \zeta'_{\sigma(j)}\right)^{n\xi'_j}\right)^{p_j}$$

$$\text{and} \qquad \prod_{j=1}^{n} \left(1 - \left(1 - \zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j} \leq \prod_{j=1}^{n} \left(1 - \left(1 - \zeta'_{\sigma(j)}\right)^{n\xi'_j}\right)^{p_j}.$$

Further, we have

$$\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j}\right) \geq \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \zeta'_{\sigma(j)}\right)^{n\xi'_j}\right)^{p_j}\right)$$

and

$$\left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}} \geq \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \zeta'_{\sigma(j)}\right)^{n\xi'_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}.$$

Hence, we get

$$\left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \leq \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \zeta'_{\sigma(j)}\right)^{n\xi'_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}.$$

Similarly, we have

$$1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-{\kappa_{\sigma(j)}}^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

$$\geq 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-{\kappa'_{\sigma(j)}}^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

and

$$1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-{\varphi_{\sigma(j)}}^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

$$\geq 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-{\varphi'_{\sigma(j)}}^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

Therefore, by Definition 2.1.4 of Chapter 2, we have

$$\mathrm{SVNPMM}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\leq \mathrm{SVNPMM}(\mathcal{A}'_1,\mathcal{A}'_2,\ldots,\mathcal{A}'_n).$$

□

Theorem 5.2.3. For a collection of SVNNs $\mathcal{A}_j=(\zeta_j,\kappa_j,\varphi_j)(j=1,2,\ldots,n)$. Let $\mathcal{A}^-=(\zeta^-,\kappa^-,\varphi^-)$ and $\mathcal{A}^+=(\zeta^+,\kappa^+,\varphi^+)$ be the lower and upper bound, respectively, of the SVNNs where $\zeta^-=\min\limits_j\{\zeta_j\}$, $\kappa^-=\max\limits_j\{\kappa_j\}$, $\varphi^-=\max\limits_j\{\varphi_j\}$, $\zeta^+=\max\limits_j\{\zeta_j\}$, $\kappa^+=\min\limits_j\{\kappa_j\}$ and $\varphi^+=\min\limits_j\{\varphi_j\}$, then

$$\mathcal{A}^-\leq \mathrm{SVNPMM}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\leq \mathcal{A}^+.$$

This property is called boundedness.

Proof. Since $\min\limits_j\{\zeta_j\}\leq\zeta_j$, therefore $\min\limits_j\{\zeta_j\}\leq\zeta_{\sigma(j)}$, which implies

$$\left(1-\min_j\zeta_j\right)^{n\xi_j}\geq\left(1-\zeta_{\sigma(j)}\right)^{n\xi_j}$$

and

$$\left(1-\left(1-\min_j\zeta_j\right)^{n\xi_j}\right)^{p_j}\leq\left(1-\left(1-\zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j}.$$

Then,

$$\prod_{j=1}^{n}\left(1-\left(1-\min_{j}\zeta_j\right)^{n\xi_j}\right)^{p_j} \leq \prod_{j=1}^{n}\left(1-\left(1-\zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j}.$$

Further,

$$\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\min_{j}\zeta_j\right)^{n\xi_j}\right)^{p_j}\right) \geq \prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j}\right),$$

which implies that

$$\begin{aligned}&\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\min_{j}\zeta_j\right)^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}} \\ &\quad \leq \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},\end{aligned}$$

i.e.,

$$\zeta^{-} \leq \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\zeta_{\sigma(j)}\right)^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

In the same manner, we get

$$\kappa^{-} \geq 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-{\kappa_{\sigma(j)}}^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

and

$$\varphi^{-} \geq 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-{\varphi_{\sigma(j)}}^{n\xi_j}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

Hence, $(\zeta^{-},\kappa^{-},\varphi^{-}) \leq \mathrm{SVNPMM}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)$. Similarly, we have

$$\mathrm{SVNPMM}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \leq (\zeta^{+},\kappa^{+},\varphi^{+}),$$

which completes the proof. □

Theorem 5.2.4. Let $\tilde{\mathcal{A}}_j$ be any permutation of $\mathcal{A}_j$ then we have

$$\text{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \text{SVNPMM}(\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2, \ldots, \tilde{\mathcal{A}}_n).$$

This property is called commutativity.

Proof. The proof of this theorem can be easily followed from Eq. (5.2), so we omit it here. □

Theorem 5.2.5. If the priority level of all the SVNNs is taken to be the same then SVNPMM operator reduces to SVN Muirhead mean (SVNMM) operator. This property is called reducibility.

Proof. Take $\xi_j = \frac{H_j}{\sum\limits_{j=1}^{n} H_j} = \frac{1}{n}$ for all j denotes the prioritized level. As ξ_j is same for all j, so, we have $(n\xi_j)\mathcal{A}_{\sigma(j)} = \mathcal{A}_{\sigma(j)}$, which implies

$$\begin{aligned} \text{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \prod_{j=1}^{n} \mathcal{A}_{\sigma(j)}^{p_j} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \\ &= \text{SVNMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n). \end{aligned}$$

□

However, apart from these, the following particular cases are observed from the proposed SVNPMM operator by assigning different values to $P = (p_1, p_2, \ldots, p_n)$.

1) If $P = (1, 0, \ldots, 0)$, then SVNPMM operator becomes the SVN prioritized weighted average (SVNPWA) operator which is given as

$$\begin{aligned} \text{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \left(n \frac{H_{\sigma(1)}}{\sum\limits_{j=1}^{n} H_j} \mathcal{A}_{\sigma(1)} \right) \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \\ &= \bigoplus_{j=1}^{n} \xi_j \mathcal{A}_j \\ &= \text{SVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n). \end{aligned}$$

2) When $P = (\lambda, 0, \ldots, 0)$, then SVNPMM operator yields to SVN generalized hybrid prioritized weighted average (SVNGHPWA) operator as shown below

$$\begin{aligned}
\mathrm{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \left(n \frac{H_{\sigma(1)}}{\sum\limits_{j=1}^{n} H_j} \mathcal{A}_{\sigma(1)} \right)^{\lambda} \right)^{\frac{1}{\lambda}} \\
&= \left(\frac{1}{n} \bigoplus_{j=1}^{n} (n \xi_j \mathcal{A}_j)^{\lambda} \right)^{\frac{1}{\lambda}} \\
&= \mathrm{SVNGHPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n).
\end{aligned}$$

3) If $P = (1, 1, 0, \ldots, 0)$, then Eq. (5.1) reduces to SVN prioritized Bonferroni mean (SVNPBM) operator as below

$$\begin{aligned}
\mathrm{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \left(n \frac{H_{\sigma(1)}}{\sum\limits_{j=1}^{n} H_j} \mathcal{A}_{\sigma(1)} \right) \left(n \frac{H_{\sigma(2)}}{\sum\limits_{j=1}^{n} H_j} \mathcal{A}_{\sigma(2)} \right) \right)^{\frac{1}{2}} \\
&= \left(\frac{n^2}{n!} \bigoplus_{\substack{r,s=1 \\ r \neq s}}^{n} \left(\frac{H_r}{\sum\limits_{r=1}^{n} H_r} \mathcal{A}_r \right) \left(\frac{H_s}{\sum\limits_{s=1}^{n} H_s} \mathcal{A}_s \right) \right)^{\frac{1}{2}} \\
&= \mathrm{SVNPBM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n).
\end{aligned}$$

4) If $P = (\overbrace{1, 1 \ldots, 1}^{t \text{ terms}}, \overbrace{0, 0 \ldots, 0}^{n-t \text{ terms}})$, then SVNPMM operator yields to SVN prioritized Maclaurin symmetric mean (SVNPMSM) operator as follows

$$\mathrm{SVNPMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\frac{2n^t t}{n!} \bigoplus_{\substack{1 < j_1 < \cdots \\ < j_t < n}} \bigotimes_{q=1}^{t} \left(\frac{H_{j_q}}{\sum\limits_{r=1}^{n} H_r} \mathcal{A}_{j_q} \right) \right)^{\frac{1}{t}}.$$

5.2.2 Single-Valued Neutrosophic Prioritized Dual Muirhead Mean Operator

In this section, we propose prioritized dual AO based on the MM under the SVNS environment.

Definition 5.2.2. A SVNPDMM operator is a mapping SVNPDMM : $\Phi(\mathcal{X})^n \to \Phi(\mathcal{X})$ given by

$$\text{SVNPDMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \frac{1}{\sum\limits_{j=1}^{n} p_j} \left(\prod_{\sigma \in S_n} \bigoplus_{j=1}^{n} \left(p_j \mathcal{A}_{\sigma(j)} \right)^{n\xi_j} \right)^{\frac{1}{n!}}. \tag{5.3}$$

Theorem 5.2.6. The collective value by using Eq. (5.3) is still a SVNN and is given as

$$\begin{aligned} &\text{SVNPDMM}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = &\left(\begin{array}{l} 1 - \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \zeta_{\sigma(j)}{}^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - (1 - \kappa_{\sigma(j)})^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - (1 - \varphi_{\sigma(j)})^{n\xi_j} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \end{array} \right). \end{aligned} \tag{5.4}$$

Proof. The proof follows from Theorem 5.2.1. □

In order to illustrate the working of this operator, we demonstrate it through an illustrative example as follows.

Example 5.2.2. If we have taken the data as considered in Example 5.2.1 to illustrate the AO as defined in Theorem 5.2.6 then, we have

$$\begin{aligned} &\prod_{\sigma \in S_3} \left(1 - \prod_{j=1}^{3} \left(1 - \zeta_{\sigma(j)}^{3\xi_j} \right)^{p_j} \right) \\ = &\left\{ 1 - \left(1 - (0.5)^{3 \times 0.5087} \right)^{1} \times \left(1 - (0.3)^{3 \times 0.3765} \right)^{0.5} \times \left(1 - (0.6)^{3 \times 0.1148} \right)^{0.3} \right\} \\ \times &\left\{ 1 - \left(1 - (0.3)^{3 \times 0.3765} \right)^{1} \times \left(1 - (0.5)^{3 \times 0.5087} \right)^{0.5} \times \left(1 - (0.6)^{3 \times 0.1148} \right)^{0.3} \right\} \\ \times &\left\{ 1 - \left(1 - (0.6)^{3 \times 0.1148} \right)^{1} \times \left(1 - (0.3)^{3 \times 0.3765} \right)^{0.5} \times \left(1 - (0.5)^{3 \times 0.5087} \right)^{0.3} \right\} \end{aligned}$$

$$
\begin{aligned}
&\times \left\{1-\left(1-(0.3)^{3\times 0.3765}\right)^{1}\times\left(1-(0.6)^{3\times 0.1148}\right)^{0.5}\times\left(1-(0.5)^{3\times 0.5087}\right)^{0.3}\right\}\\
&\times \left\{1-\left(1-(0.5)^{3\times 0.5087}\right)^{1}\times\left(1-(0.6)^{3\times 0.1148}\right)^{0.5}\times\left(1-(0.3)^{3\times 0.3765}\right)^{0.3}\right\}\\
&\times \left\{1-\left(1-(0.6)^{3\times 0.1148}\right)^{1}\times\left(1-(0.5)^{3\times 0.5087}\right)^{0.5}\times\left(1-(0.3)^{3\times 0.3765}\right)^{0.3}\right\}\\
&= 0.00042495.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\prod_{\sigma\in S_3}\left(1-\prod_{j=1}^{3}\left(1-(1-\kappa_{\sigma(j)})^{3\xi_j}\right)^{p_j}\right)\\
=\;&\left\{1-\left(1-(1-0.2)^{3\times 0.5087}\right)^{1}\times\left(1-(1-0.5)^{3\times 0.3765}\right)^{0.5}\times\left(1-(1-0.5)^{3\times 0.1148}\right)^{0.3}\right\}\\
\times\;&\left\{1-\left(1-(1-0.5)^{3\times 0.3765}\right)^{1}\times\left(1-(1-0.2)^{3\times 0.5087}\right)^{0.5}\times\left(1-(1-0.5)^{3\times 0.1148}\right)^{0.3}\right\}\\
\times\;&\left\{1-\left(1-(1-0.5)^{3\times 0.1148}\right)^{1}\times\left(1-(1-0.5)^{3\times 0.3765}\right)^{0.5}\times\left(1-(1-0.2)^{3\times 0.5087}\right)^{0.3}\right\}\\
\times\;&\left\{1-\left(1-(1-0.5)^{3\times 0.3765}\right)^{1}\times\left(1-(1-0.5)^{3\times 0.1148}\right)^{0.5}\times\left(1-(1-0.2)^{3\times 0.5087}\right)^{0.3}\right\}\\
\times\;&\left\{1-\left(1-(1-0.2)^{3\times 0.5087}\right)^{1}\times\left(1-(1-0.5)^{3\times 0.1148}\right)^{0.5}\times\left(1-(1-0.5)^{3\times 0.3765}\right)^{0.3}\right\}\\
\times\;&\left\{1-\left(1-(1-0.5)^{3\times 0.1148}\right)^{1}\times\left(1-(1-0.2)^{3\times 0.5087}\right)^{0.5}\times\left(1-(1-0.5)^{3\times 0.3765}\right)^{0.3}\right\}\\
=\;&0.0268
\end{aligned}
$$

and

$$
\begin{aligned}
&\prod_{\sigma\in S_3}\left(1-\prod_{j=1}^{3}\left(1-(1-\varphi_{\sigma(j)})^{3\xi_j}\right)^{p_j}\right)\\
=\;&\left\{1-\left(1-(1-0.3)^{3\times 0.5087}\right)^{1}\times\left(1-(1-0.4)^{3\times 0.3765}\right)^{0.5}\times\left(1-(1-0.2)^{3\times 0.1148}\right)^{0.3}\right\}\\
\times\;&\left\{1-\left(1-(1-0.4)^{3\times 0.3765}\right)^{1}\times\left(1-(1-0.3)^{3\times 0.5087}\right)^{0.5}\times\left(1-(1-0.2)^{3\times 0.1148}\right)^{0.3}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \times \left\{1-\left(1-(1-0.2)^{3\times 0.1148}\right)^{1}\times\left(1-(1-0.4)^{3\times 0.3765}\right)^{0.5}\times\left(1-(1-0.3)^{3\times 0.5087}\right)^{0.3}\right\} \\
& \times \left\{1-\left(1-(1-0.4)^{3\times 0.3765}\right)^{1}\times\left(1-(1-0.2)^{3\times 0.1148}\right)^{0.5}\times\left(1-(1-0.3)^{3\times 0.5087}\right)^{0.3}\right\} \\
& \times \left\{1-\left(1-(1-0.3)^{3\times 0.5087}\right)^{1}\times\left(1-(1-0.2)^{3\times 0.1148}\right)^{0.5}\times\left(1-(1-0.4)^{3\times 0.3765}\right)^{0.3}\right\} \\
& \times \left\{1-\left(1-(1-0.2)^{3\times 0.1148}\right)^{1}\times\left(1-(1-0.3)^{3\times 0.5087}\right)^{0.5}\times\left(1-(1-0.4)^{3\times 0.3765}\right)^{0.3}\right\} \\
= {} & 0.0791.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\text{SVNPDMM}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3) &= \begin{pmatrix} 1-\left(1-(0.00042495)^{\frac{1}{6}}\right)^{\frac{1}{1.8}}, \left(1-(0.0268)^{\frac{1}{6}}\right)^{\frac{1}{1.8}}, \\ \left(1-(0.0791)^{\frac{1}{6}}\right)^{\frac{1}{1.8}} \end{pmatrix} \\
&= (0.1631, 0.6441, 0.5535).
\end{aligned}
$$

Similar to SVNPMM operator, it is observed that this SVNPDMM operator also satisfies same properties for a collection of SVNNs $\mathcal{A}_j (j = 1, 2, \ldots, n)$ which are stated without proof as below.

(P1) Monotonicity: If $\mathcal{A}_j \leq \mathcal{A}'_j$ for all j, then

$$\text{SVNPDMM}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \leq \text{SVNPDMM}(\mathcal{A}'_1,\mathcal{A}'_2,\ldots,\mathcal{A}'_n).$$

(P2) Boundedness: If $\mathcal{A}^-$, and $\mathcal{A}^+$ are lower and upper bound of SVNNs then

$$\mathcal{A}^- \leq \text{SVNPDMM}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \leq \mathcal{A}^+.$$

(P3) Commutativity: For any permutation $(\tilde{\mathcal{A}}_1,\tilde{\mathcal{A}}_2,\ldots,\tilde{\mathcal{A}}_n)$ of the $(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)$, we have

$$\text{SVNPDMM}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) = \text{SVNPDMM}(\tilde{\mathcal{A}}_1,\tilde{\mathcal{A}}_2,\ldots,\tilde{\mathcal{A}}_n).$$

5.3 Multi-criteria decision-making approach based on proposed operators

In this section, we present an MCDM approach for solving the problems under the SVNS environment by using the proposed operators.

5.3.1 Proposed decision-making approach

The description of the MCDM process is given in Section 2.5 of Chapter 2. The evaluation of each alternative is done under the SVNS environment and represent their rating values in a SVN decision matrix $\mathcal{M} = (\alpha_{ij})_{m\times n}$ where $\alpha_{ij} = (\zeta_{ij}, \kappa_{ij}, \varphi_{ij})$ such that $0 \leq \zeta_{ij}, \kappa_{ij}, \varphi_{ij} \leq 1$ and $\zeta_{ij} + \kappa_{ij} + \varphi_{ij} \leq 3$.

Then, the following steps have been proposed to solve the decision making problems by using proposed operators.

Step 1: Normalize the data of the matrix $\mathcal{M}$, if required, by using Eq. (2.38) of Chapter 2 and hence obtained the normalized decision matrix $\mathcal{R} = (r_{ij})$.

Step 2: Compute $H_{ij} (i = 1, 2, \ldots, m)$ as

$$H_{ij} = \begin{cases} 1 & ; \quad j = 1, \\ \prod\limits_{k=1}^{j-1} \mathcal{S}(r_{ik}) & ; \quad j = 2, \ldots, n. \end{cases} \tag{5.5}$$

Step 3: For a given parameter $P = (p_1, p_2, \ldots, p_n)$, utilize either SVNPMM or SVNPDMM operator to get the collective values $r_i = (\zeta_i, \kappa_i, \varphi_i) (i = 1, 2, \ldots, m)$ for each alternative as

$$\begin{aligned} r_i &= \mathrm{SVNPMM}(r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \left(\begin{array}{l} \left(1 - \left(\prod\limits_{\sigma\in S_n}\left(1 - \prod\limits_{j=1}^{n}\left(1 - (1-\zeta_{i\sigma(j)})^{n\xi_{ij}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod\limits_{\sigma\in S_n}\left(1 - \prod\limits_{j=1}^{n}\left(1 - \kappa_{i\sigma(j)}^{n\xi_{ij}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod\limits_{\sigma\in S_n}\left(1 - \prod\limits_{j=1}^{n}\left(1 - \varphi_{i\sigma(j)}^{n\xi_{ij}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \end{array} \right) \end{aligned} \tag{5.6}$$

or

$$\begin{aligned} r_i &= \text{SVNPDMM}(r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \left(\begin{array}{c} 1 - \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \zeta_{i\sigma(j)}^{n\xi_{ij}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}}, \\ \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - (1 - \kappa_{i\sigma(j)})^{n\xi_{ij}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}}, \\ \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - (1 - \varphi_{i\sigma(j)})^{n\xi_{ij}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}} \end{array} \right) \end{aligned} \tag{5.7}$$

where $\xi_{ij} = \frac{H_{i\sigma(j)}}{\sum\limits_{j=1}^{n} H_{ij}} \geq 0$.

Step 4: Calculate score values of the overall aggregated values $r_i = (\zeta_i, \kappa_i, \varphi_i)$ $(i = 1, 2, \ldots, m)$ by using equation

$$\mathcal{S}(r_i) = \frac{1 + (\zeta_i - 2\kappa_i - \varphi_i)(2 - \zeta_i - \varphi_i)}{2}. \tag{5.8}$$

Step 5: Rank all the feasible alternatives $\mathcal{A}_i (i = 1, 2, \ldots, m)$ according to Definition 2.1.4 of Chapter 2 and hence select the most desirable alternative(s).

The above mentioned approach has been illustrated with a numerical example discussed in Section 5.3.2.

5.3.2 Illustrative example

A travel agency named Marricot Tripmate has excelled in providing travel related services to domestic and inbound tourists. The agency wants to provide more facilities like detailed information, online booking capabilities, the ability to book and sell airline tickets, and other travel related services to their customers. For this purpose, the agency intends to find an appropriate information technology (IT) software company that delivers affordable solutions through software development. To complete this motive, the agency forms a set of five companies (alternatives), namely, Zensar Tech ($\mathcal{A}_1$), NIIT Tech ($\mathcal{A}_2$), HCL

Tech($\mathcal{A}_3$), Hexaware Tech($\mathcal{A}_4$), and Tech Mahindra ($\mathcal{A}_5$) and the selection is held on the basis of the different criteria, namely, technology expertise ($\mathcal{G}_1$), service quality ($\mathcal{G}_2$), project management ($\mathcal{G}_3$) and industry experience ($\mathcal{G}_4$). The prioritization relationship for the criterion is $\mathcal{G}_1 \succ \mathcal{G}_2 \succ \mathcal{G}_3 \succ \mathcal{G}_4$. In order to access these alternatives, an expert was invited and he gives their preferences toward each alternative in the form of SVNN. Their complete preferences of the expert are summarized in Table 5.1.

Table 5.1: Single-valued neutrosophic decision making matrix.

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$(0.5, 0.3, 0.4)$	$(0.5, 0.2, 0.3)$	$(0.2, 0.2, 0.6)$	$(0.3, 0.2, 0.4)$
$\mathcal{A}_2$	$(0.7, 0.1, 0.3)$	$(0.7, 0.2, 0.3)$	$(0.6, 0.3, 0.2)$	$(0.6, 0.4, 0.2)$
$\mathcal{A}_3$	$(0.5, 0.3, 0.4)$	$(0.6, 0.2, 0.4)$	$(0.6, 0.1, 0.2)$	$(0.5, 0.1, 0.3)$
$\mathcal{A}_4$	$(0.7, 0.3, 0.2)$	$(0.7, 0.2, 0.2)$	$(0.4, 0.5, 0.2)$	$(0.5, 0.2, 0.2)$
$\mathcal{A}_5$	$(0.4, 0.1, 0.3)$	$(0.5, 0.1, 0.2)$	$(0.4, 0.1, 0.5)$	$(0.4, 0.3, 0.6)$

Then, the following steps of the proposed approach have been executed as below

Step 1: As all the criteria values are of the same types, the original decision matrix need not be normalized.

Step 2: Compute $H_{ij}(j = 1, 2, 3, 4)$ by using Eq. (5.5), we get

$$H = \begin{bmatrix} 1 & 0.6650 & 0.4921 & 0.3642 \\ 1 & 0.9000 & 0.7200 & 0.4464 \\ 1 & 0.6650 & 0.5320 & 0.4575 \\ 1 & 0.6650 & 0.5154 & 0.1134 \\ 1 & 0.8250 & 0.6806 & 0.6024 \end{bmatrix}.$$

Step 3: Without loss of generality, we take $P = (0.25, 0.25, 0.25, 0.25)$ and use SVNPMM operator given in Eq. (5.6) to aggregate $r_{ij}(j = 1, 2, 3, 4)$ and hence we get $r_1 = (0.9026,\ 0.0004,\ 0.0118)$; $r_2 = (0.9963,\ 0.0008,\ 0.0007)$; $r_3 = (0.9858,\ 0.0001,\ 0.0029)$; $r_4 = (0.9877,\ 0.0021,\ 0.0002)$ and $r_5 = (0.9474,\ 0.0000,\ 0.0093)$.

Step 4: By Eq. (5.8), we get $\mathcal{S}(r_1) = 0.9959$, $\mathcal{S}(r_2) = 0.9992$, $\mathcal{S}(r_3) = 0.9998$, $\mathcal{S}(r_4) = 0.9978$ and $\mathcal{S}(r_5) = 0.9990$.

Step 5: Since $\mathcal{S}(r_3) > \mathcal{S}(r_2) > \mathcal{S}(r_5) > \mathcal{S}(r_4) > \mathcal{S}(r_1)$ and thus ranking order of their corresponding alternatives is $\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1$. Here $\succ$ refers "preferred to". Therefore, $\mathcal{A}_3$ is the best one according to the requirement of the travel agency.

Contrary to this, if we utilize SVNPDMM operator then the following steps are executed as:

Step 1: Similar to above Step 1.

Step 2: Similar to above Step 2.

Step 3: For a parameter $P = (0.25, 0.25, 0.25, 0.25)$, use SVNPDMM operator given in Eq. (5.7) we get $r_1 = (0.0069,\ 0.7379,\ 0.9413)$; $r_2 = (0.1034,\ 0.7423,\ 0.7782)$; $r_3 = (0.0428,\ 0.6021,\ 0.8672)$; $r_4 = (0.0625,\ 0.8271,\ 0.6966)$ and $r_5 = (0.0109,\ 0.5340,\ 0.9125)$.

Step 4: The evaluated score values by using Eq. (5.8) are $\mathcal{S}(r_1) = 0.2226$, $\mathcal{S}(r_2) = 0.1628$, $\mathcal{S}(r_3) = 0.3396$, $\mathcal{S}(r_4) = -0.0554$ and $\mathcal{S}(r_5) = 0.4222$.

Step 5: The ranking order of the alternatives, based on the score values, is $\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$ and hence $\mathcal{A}_5$ as the best alternative among the others.

5.3.3 Comparison Study

If we apply the existing prioritized AO named as SVN prioritized operator [166] on the considered problem, then the following steps of the Wu et al. [166] approach have been executed as follows:

Step 1: Use SVNPWA operator as given in Eq. (2.32) of Chapter 2 to calculate the aggregated values $\beta_i (i = 1, 2, 3, 4, 5)$ of each alternative $\mathcal{A}_i$ are $\beta_1 = (0.4392, 0.2407, 0.3981)$, $\beta_2 = (0.6681, 0.1864, 0.2602)$, $\beta_3 = (0.5461, 0.1929, 0.3414)$, $\beta_4 = (0.6294, 0.2844, 0.2000)$ and $\beta_5 = (0.4291, 0.1141, 0.3232)$.

Step 2: Compute the cross entropy E for each β_i from $\mathcal{A}^+ = (1, 0, 0)$ and $\mathcal{A}^- = (0, 0, 1)$ based on the equation $E(\beta_1, \beta_2) = (\sin\zeta_1 - \sin\zeta_2) \times (\sin(\zeta_1 - \zeta_2)) + (\sin\kappa_1 - \sin\kappa_2) \times (\sin(\kappa_1 - \kappa_2)) + (\sin\varphi_1 - \sin\varphi_2) \times (\sin(\varphi_1 - \varphi_2))$ and then evaluate S_{β_i} by using equation $S_{\beta_i} = \frac{E(\beta_i, \mathcal{A}^+)}{E(\beta_i, \mathcal{A}^+) + E(\beta_i, \mathcal{A}^-)}$. The values corresponding to it are: $S_{\beta_1} = 0.4642$, $S_{\beta_2} = 0.1755$, $S_{\beta_3} = 0.3199$, $S_{\beta_4} = 0.1914$ and $S_{\beta_5} = 0.4007$.

Step 3: The final ranking of alternative, according to the values of S_{β_i}, is $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$.

From above, we have concluded that the $\mathcal{A}_2$ is the best alternative and $\mathcal{A}_1$ is the worst one. However, from their approach [166], it has been concluded that they have completely ignored the interrelationships among the multi-input arguments and hence the ranking order are quite different.

5.3.4 Influence of Parameter P on the Decision-Making Process

The proposed AOs have two prominent advantages. First, it can reduce the bad effects of the unduly high and low assessments on the final results. Second, it can capture the interrelationship between SVN attributes values. Moreover, both of the two AOs have a parameter vector P, which leads to more flexibility during the aggregation process. Further, the parameter vector P plays a significant role in the final ranking results. In order to illustrate the influence of the parameter vector $P = (p_1, p_2, \ldots, p_n)$ on the score functions and the ranking results, we set different values to P in the SVNPMM and SVNPDMM operators and their corresponding results are summarized in Table 5.2. From this table, it is concluded that the score value of each alternative decreases by SVNPMM operator while it increases by SVNPDMM operator. Therefore, based on the decision maker behavior, either $\mathcal{A}_3$ or $\mathcal{A}_5$ are the best alternatives to be chosen for their desired goals. Thus, the parameter vector P can be viewed as decision makers' risk preference.

5.3.5 Further Discussion

The prominent advantage of the proposed AOs is that the interrelationship among all SVNNs can be taken into consideration. Moreover, it has a parameter vector that leads to flexible AOs. To show the validity and superiorities of the proposed operators, we conduct a comparative analysis whose characteristics are presented in Table 5.3. From this table, it is concluded that the approaches presented in [79, 113, 175] are based on a simple weighted averaging operator. However, in these approaches, some of the weakness is (1) they assume that all the input arguments are independent, which is somewhat inconsistent with reality; (2) they cannot consider the interrelationship among input arguments. However, on the contrary, the proposed method can capture the interrelationship among input arguments. In addition to that, the proposed operator has an additional parameter of P which provide a feasible aggregation process. In addition, some of the existing operators are deduced from the proposed operators. Thus, the proposed method is more powerful and flexible than the methods in [79, 113, 175].

In [57, 86], authors presented an approach based on the BM AO where they considered the interrelationship between the arguments. However, the main flaws of these approaches are that they consider only two arguments during the interrelationship. On the other hand, in [159] authors have presented an AO based on MSM by considering two or more arguments during the interrelationship; however, these methods [57, 86, 159] fail to reflect the interrelationship among all input arguments. Finally, in [69] authors used the Heronian mean AOs without considering any interrelationship between the arguments.

As compared with these existing approaches, the merits of the proposed approach are that it can reflect the interrelationships among all the input arguments. In addition, the proposed operators have an additional parameter of P which makes the proposed approach more flexible and feasible.

5.4 Conclusions

Muirhead means AO is more flexible by using a variable and considering the multiple interrelationships between the pairs of the input arguments. On the other hand, SVNS is more of a generalization of the FS, IFS to describe the uncertainties in the data. In order to combine their advantages, in the present chapter, we develop some new MM AOs for the SVNSs including the SVNPMM and the SVNPDMM. The desirable properties of these proposed operators and some special cases are discussed in detail. Moreover, we presented two new methods to solve the MCDM problem based on the proposed operators. The proposed method is more general and flexible, not only by considering the parametric vector P but also by taking into account the multiple interrelationships between the input argument. Apart from this, the remarkable characteristic of the proposed operator is to reflect the correlations of the aggregated arguments by considering the fact that those different criteria having different priority levels. The mentioned approach has been demonstrated through a numerical example and compares their corresponding proposed results with some of the results of existing approaches. Furthermore, by changing the values of the parameter P, an analysis has been done which concludes that the proposed operators provide more choices to the decision makers according to their preferences. In addition, it is also regarded as considering the risk preference of decision makers by the parameter P. So, the proposed approach is more suitable and flexible to solve practical and complex MCDM problems.

Table 5.2: Ranking results of alternatives using proposed operators for different values of P.

Parameter Vector P	**Operator**	Score Values of Alternatives					Ranking Results
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	
(1,0,0,0)	SVNPMM	0.9975	0.9997	0.9999	0.9989	0.9990	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
	SVNPDMM	0.2184	0.0876	0.2942	-0.1233	0.3632	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
(1,1,0,0)	SVNPMM	0.9844	0.9969	0.9988	0.9920	0.9940	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
	SVNPDMM	0.3638	0.2891	0.4851	0.0162	0.5597	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
(1,1,1,0)	SVNPMM	0.9723	0.9926	0.9968	0.9809	0.9887	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
	SVNPDMM	0.4268	0.3846	0.5529	0.1219	0.6053	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
(1,1,1,1)	SVNPMM	0.9624	0.9868	0.9942	0.9659	0.9851	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
	SVNPDMM	0.4617	0.4507	0.5955	0.2079	0.6341	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
(2,2,2,2)	SVNPMM	0.9443	0.9633	0.9836	0.9189	0.9767	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	SVNPDMM	0.5165	0.5024	0.640	0.3016	0.6698	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
(3,3,3,3)	SVNPMM	0.9322	0.9440	0.9744	0.8896	0.9715	$\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
	SVNPDMM	0.5369	0.5018	0.6490	0.3142	0.6853	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	SVNPMM	0.9824	0.9965	0.9987	0.9903	0.9943	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
	SVNPDMM	0.3652	0.3217	0.4982	0.0490	0.5661	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	SVNPMM	0.9959	0.9992	0.9998	0.9978	0.9990	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
	SVNPDMM	0.2226	0.1628	0.3396	-0.0554	0.4222	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
(2,0,0,0)	SVNPMM	0.9890	0.9984	0.9990	0.9953	0.9931	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
	SVNPDMM	0.3571	0.1886	0.4228	-0.1009	0.4781	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
(3,0,0,0)	SVNPMM	0.9814	0.9964	0.9974	0.9898	0.9860	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
	SVNPDMM	0.4139	0.2426	0.4645	-0.0595	0.5008	$\mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$

SVNPMM: single-valued neutrosophic prioritized Muirhead mean, SVNPDMM: single-valued neutrosophic prioritized dual Muirhead mean.

Table 5.3: Comparison of different approaches and aggregation operators.

Approaches	Whether the Interrelationship of Two Attributes Is Captured	Whether the Interrelationship of Three Attributes Is Captured	Whether the Relationship of Multiple Attributes Is Captured	Whether the Bad Effects of the Unduly High Unduly Low Arguments Can Be Reduced	Whether It Makes the Method Flexible by the Parameter Vector
NWA [175]	×	×	×	×	×
SVNWA [113]	×	×	×	×	×
SVNOWA [113]	×	×	×	×	×
SVNWG [113]	×	×	×	×	×
SVNOWG [113]	×	×	×	×	×
SVNHWA [79]	×	×	×	×	×
SVNHWG [79]	×	×	×	×	×
NWG [175]	×	×	×	×	×
SVNFNPBM [57]	✓	×	×	×	✓
WSVNLMSM [159]	✓	✓	✓	×	✓
SVNNWBM [86]	✓	×	×	×	✓
SVNIGWHM [69]	✓	✓	✓	×	✓
GNNHWA [79]	×	×	×	×	✓
The proposed method	✓	✓	✓	✓	✓

NWA: neutrosophic weighted averaging , SVNWA: single-valued neutrosophic weighted averaging, SVNOWA: single-valued neutrosophic ordered weighted averaging, SVNWG: single-valued neutrosophic weighted geometric , SVNOWG: single-valued neutrosophic ordered weighted geometric, SVNHWA: single-valued neutrosophic hybrid weighted averaging, SVNHWG: single-valued neutrosophic hybrid weighted geometric, NWG: neutrosophic weighted geometric, SVNFNPBM: single-valued neutrosophic Frank normalized prioritized Bonferroni mean, WSVNLMSM: weighted single-valued neutrosophic linguistic Maclaurin symmetric mean, SVNNWBM: single-valued neutrosophic normalized weighted Bonferroni mean, IGWHM: single-valued neutrosophic improved generalized weighted Heronian mean, GNNHWA: generalized neutrosophic number Hamacher weighted averaging.

Chapter 6

Non-linear programming method for multi-criteria decision making problems under interval neutrosophic set environment[1]

In this chapter, we developed a nonlinear programming model based TOPSIS approach for solving MCDM problems with incomplete weight information. The relative closeness coefficient(RCC) degree of the TOPSIS method is formulated based on the distance measures. Further, the importance of the attribute weights is taken in the form of interval numbers rather than a real single number. Several special cases of the proposed approach are discussed in detail. In the developed non-linear programming model of RCC, a Charles-Chooper approach is utilized to solve the problem. The applicability of this presented approach is demonstrated with a numerical example of a power generation project and computed their results with several existing studying results.

6.1 Introduction

The extensive literature related to approaches based on the TOPSIS and the distance measures to solve the decision making (DM) problems under the SVNSs and INSs is presented in Chapter 1. From these studies, it is reviewed that the aforementioned theories

---

[1] The content of this chapter is published as “Non-linear programming method for multi-criteria decision making problems under interval neutrosophic set environment”, *Applied Intelligence*, 48(8), 2199 - 2213, 2018 (**Impact Factor: 1.983**).

have a great contribution in various fields; however, some limitations have been observed which is outlined below:

(i) The evaluation of given data on MCDM problem, by using existing approaches needs complete information about weights. However, due to increasing complexity, time pressure or lack of data in practical situations, it is difficult to provide exact weights to the criteria under consideration.

(ii) For ranking purposes in MCDM problems, various aggregating operators, distance, correlation coefficients [19, 20, 132, 177, 181, 200] utilize the interval neutrosophic information and processing that information into a crisp number. The procedure of aggregating the interval-valued information into the real number leads to the loss of a lot of information and final decision obtained from them is not always trustworthy.

(iii) Some methods for ranking the alternatives are highly tedious and not concentrate on reducing the computational complexities. So processing the information by such methods is quite time-consuming and will not help in the situations when the expert requires immediate result.

So, an interesting and important issue is how to utilize the collective interval neutrosophic decision matrix and the unknown preferences information to find the most desirable alternative(s) during the DM process. In the light of this, in this chapter, we have presented a concept of new nonlinear programming (NLP) method for solving the DM problems with the concept of the closeness coefficient of the TOPSIS method. During the formulation, the information related to each alternative is measured in the forms of interval neutrosophic numbers (INNs). A weighted distance between an alternative and the interval neutrosophic positive ideal solution (INPIS), as well as the interval neutrosophic negative ideal solution (INNIS) is used to define the concept of closeness coefficient, which is continuous and monotonic functions with respect to the degrees of the truth, the indeterminacy, and the falsity degrees of the alternatives and preferences. Then, a pair of two auxiliary nonlinear fractional programming models is constructed to calculate the closeness coefficients' interval of the alternatives with respect to the INPIS, which can be used to generate ranking order of the alternatives.

6.2 Proposed Non-linear Programming model for MCDM problem

MCDM problem plays a very important role to find out the most desirable alternative among the set of feasible alternatives having criterion with unequal importance. These problems which incorporate uncertainty, imprecision, inconsistency can also be represented by INSs. An MCDM problem under this environment can be outlined as:

6.2.1 Description of MCDM problem under INSs

The description of the MCDM problem is given in Section 2.5 of Chapter 2 under the INS environment, i.e., the overall representation of these rating values can be framed into the DM matrix as $\mathcal{M} = (\alpha_{ij})_{m\times n} = (([\zeta_{ij}^L, \zeta_{ij}^U], [\kappa_{ij}^L, \kappa_{ij}^U], [\varphi_{ij}^L, \varphi_{ij}^U]))_{m\times n}$. Further, it is noted that the different criteria plays an important role during the evaluations of the alternatives thus it is necessary to consider their different priority levels, instead of giving equal priorities, in real-life problems. For it, assume that the weight vector for each criterion $\mathcal{G}_j (j = 1, 2, \ldots, n)$ is expressed as an interval set $W_j = \{(\mathcal{G}_j, [\omega_j^L, \omega_j^U], [\xi_j^L, \xi_j^U], [\eta_j^L, \eta_j^U] \mid \mathcal{G}_j \in \mathcal{G}\}$, which is usually denoted by $W_j = (([\omega_j^L, \omega_j^U], [\xi_j^L, \xi_j^U], [\eta_j^L, \eta_j^U]))_{1\times n}$ for short such that $0 \leq \omega_j^L \leq \omega_j^U \leq 1$; $0 \leq \xi_j^L \leq \xi_j^U \leq 1$; $0 \leq \eta_j^L \leq \eta_j^U \leq 1$ and $\omega_j^U + \xi_j^U + \eta_j^U \leq 1$. Here, the interval $[\omega_j^L, \omega_j^U]$ represents that the decision maker thinks the membership degree of the importance factor of criteria $\mathcal{G}_j (j = 1, 2, \ldots, n)$ must be at least ω_j^L, whereas at most ω_j^U. Similarly, for other terms.

6.2.2 Determination of ideal solutions

As the matrix $\mathcal{M}$ is an INS, so the two ideals namely, interval neutrosophic positive and negative ideal solutions, denoted by INPIS ($\mathcal{A}^+$) and INNIS ($\mathcal{A}^-$) respectively may be taken as $\mathcal{A}^+ = ([1, 1], [0, 0], [0, 0])_{1\times n}$ and $\mathcal{A}^- = ([0, 0], [1, 1], [1, 1])_{1\times n}$ respectively. From these, it has been seen that $\mathcal{A}^+$ and $\mathcal{A}^-$ are complement to each other. However, apart from fixed values, INPIS $\mathcal{A}^+$ and INNIS $\mathcal{A}^-$ can also be defined as $\mathcal{A}^+ = (([g_j^+, g_j^+], [h_j^+, h_j^+], [k_j^+, k_j^+]))_{1\times n}$, and $\mathcal{A}^- = (([g_j^-, g_j^-], [h_j^-, h_j^-], [k_j^-, k_j^-]))_{1\times n}$ where $g_j^+ = \max\limits_j\{\zeta_{ij}^U \mid i = 1, 2, \ldots, m\}$, $h_j^+ = \min\limits_j\{\kappa_{ij}^L \mid i = 1, 2, \ldots, m\}$, $k_j^+ = \min\limits_j\{\varphi_{ij}^L \mid i = 1, 2, \ldots, m\}$, $g_j^- =$

$\min_j\{\zeta^L_{ij} \mid i = 1,2,\ldots,m\}$, $h^-_j = \max_j\{\kappa^U_{ij} \mid i = 1,2,\ldots,m\}$, $k^-_j = \max_j\{\varphi^U_{ij} \mid i = 1,2,\ldots,m\}$. Since $g^-_j \leq \zeta^L_{ij} \leq \zeta^U_{ij} \leq g^+_j$, $h^+_j \leq \kappa^L_{ij} \leq \kappa^U_{ij} \leq h^-_j$ and $k^+_j \leq \varphi^L_{ij} \leq \varphi^U_{ij} \leq k^-_j$, therefore, for all i, j, $\Big([g^-_j, g^-_j], [h^-_j, h^-_j], [k^-_j, k^-_j]\Big) \subseteq \Big([\zeta^L_{ij}, \zeta^U_{ij}], [\kappa^L_{ij}, \kappa^U_{ij}], [\eta^L_{ij}, \eta^U_{ij}]\Big) \subseteq \Big([g^+_j, g^+_j], [h^+_j, h^+_j], [k^+_j, k^+_j]\Big)$.

6.2.3 Separation measurement values

The weighted Hamming distance measure for INS are taken to compute measurement values between the alternatives $\mathcal{A}_i$ from its reference values $\mathcal{A}^+$ and $\mathcal{A}^-$. For it, assume that for any $\zeta_{ij} \in [\zeta^L_{ij}, \zeta^U_{ij}]$, $\kappa_{ij} \in [\kappa^L_{ij}, \kappa^U_{ij}]$, $\varphi_{ij} \in [\varphi^L_{ij}, \varphi^U_{ij}]$, $\omega_j \in [\omega^L_j, \omega^U_j], \xi_j \in [\xi^L_j, \xi^U_j]$ and $\eta_j \in [\eta^L_j, \eta^U_j]$, the separation measure of $\mathcal{A}_i$ from $\mathcal{A}^+ = \big(([1,1],[0,0]),(0,1)\big)$ and $\mathcal{A}^- = (([0,0],[1,1]),(1,0))$ are defined as

$$\mathcal{D}(\mathcal{A}_i, \mathcal{A}^+) = \sum_{j=1}^{n} \Big\{\omega_j(1-\zeta_{ij}) + \xi_j\kappa_{ij} + \eta_j\varphi_{ij}\Big\} \tag{6.1}$$

and

$$\mathcal{D}(\mathcal{A}_i, \mathcal{A}^-) = \sum_{j=1}^{n} \Big\{\omega_j\zeta_{ij} + \xi_j(1-\kappa_{ij}) + \eta_j(1-\varphi_{ij})\Big\} \tag{6.2}$$

However, if an expert utilize $\mathcal{A}^+ = ([g^+_j, g^+_j], [h^+_j, h^+_j], [k^+_j, k^+_j])_{1\times n}$, and $\mathcal{A}^- = ([g^-_j, g^-_j], [h^-_j, h^-_j], [k^-_j, k^-_j])_{1\times n}$ then the separation measures between them are defined as

$$\mathcal{D}(\mathcal{A}_i, \mathcal{A}^+) = \sum_{j=1}^{n} \Big\{\omega_j(g^+_j - \zeta_{ij}) + \xi_j(\kappa_{ij} - h^+_j) + \eta_j(\varphi_{ij} - k^+_j)\Big\} \tag{6.3}$$

and

$$\mathcal{D}(\mathcal{A}_i, \mathcal{A}^-) = \sum_{j=1}^{n} \Big\{\omega_j(\zeta_{ij} - g^-_j) + \xi_j(h^-_j - \kappa_{ij}) + \eta_j(k^-_j - \varphi_{ij})\Big\} \tag{6.4}$$

6.2.4 Relative-closeness coefficient and its monotonicity

To measure the relative strength of the alternatives $\mathcal{A}_i, (i = 1,2,\ldots,m)$ with respect to $\mathcal{A}^+$ and $\mathcal{A}^-$, we define the closeness-coefficients of the alternatives by Eq. (6.5) as

$$\Re_i(\zeta_{ij}, \kappa_{ij}, \varphi_{ij}, \omega_j, \xi_j, \eta_j) = \frac{\mathcal{D}(\mathcal{A}_i, \mathcal{A}^-)}{\mathcal{D}(\mathcal{A}_i, \mathcal{A}^-) + \mathcal{D}(\mathcal{A}_i, \mathcal{A}^+)}, \tag{6.5}$$

provided $\mathcal{D}(\mathcal{A}_i, \mathcal{A}^+) \neq 0$. Here, $\zeta_{ij}, \kappa_{ij}, \varphi_{ij}$ are $m \times n$ matrices, (ω_j), (ξ_j), (η_j) are the $n-$ dimensional column weights corresponding to the truth, indeterminacy, and falsity degrees of criteria $\mathcal{G}_j$. Further, it is noticed that $0 \leq \mathcal{D}(\mathcal{A}_i, \mathcal{A}^-) \leq \mathcal{D}(\mathcal{A}_i, \mathcal{A}^-) + \mathcal{D}(\mathcal{A}_i, \mathcal{A}^+)$ and hence $0 \leq \mathfrak{R}_i((\zeta_{ij})_{m\times n}, (\kappa_{ij})_{m\times n}, (\varphi_{ij})_{m\times n}, (\omega_j)_{1\times n}, (\xi_j)_{1\times n}, (\eta_j)_{1\times n}) \leq 1$. Also, from Eq. (6.5), it has been seen that $\mathfrak{R}_i$ is a continuous function of $3n(m+1)$ variables, which rises sharply even for a small value of m and n. For example, if $m = 2$ and $n = 3$ then there are 27 unknown variables, whereas if $m = 4$ and $n = 5$ then there are 75 unknown variables. Therefore, it is a tedious task to invest effort leading to large computational time and excess cost. So, there is a need for construction of a time-efficient algorithm to resolve this problem so that the number of unknowns gets reduced.

To do so, firstly we check the monotonicity, boundedness and continuity of the function $\mathfrak{R}_i$ with respect to the unknown variables $\zeta_{ij} \in [\zeta_{ij}^L, \zeta_{ij}^U]$, $\kappa_{ij} \in [\kappa_{ij}^L, \kappa_{ij}^U]$ and $\varphi_{ij} \in [\varphi_{ij}^L, \varphi_{ij}^U]$, respectively. For it, by taking the measures given in Eqs. (6.1) and (6.2), we explicit the expression of $\mathfrak{R}_i$ given in Eq. (6.5) as

$$\begin{aligned}\mathfrak{R}_i &= \frac{\sum\limits_{j=1}^{n}\left\{\omega_j\zeta_{ij} + \xi_j(1-\kappa_{ij}) + \eta_j(1-\varphi_{ij})\right\}}{\sum\limits_{j=1}^{n}\left\{\omega_j\zeta_{ij} + \xi_j(1-\kappa_{ij}) + \eta_j(1-\varphi_{ij})\right\} + \sum\limits_{j=1}^{n}\left\{\omega_j(1-\zeta_{ij}) + \xi_j\kappa_{ij} + \eta_j\varphi_{ij}\right\}} \\ &= \frac{\sum\limits_{j=1}^{n}\left\{\omega_j\zeta_{ij} + \xi_j(1-\kappa_{ij}) + \eta_j(1-\varphi_{ij})\right\}}{\sum\limits_{j=1}^{n}(\omega_j + \xi_j + \eta_j)} \end{aligned} \tag{6.6}$$

To address the monotonic behavior of $\mathfrak{R}_i (i = 1, 2, \ldots, m)$, differentiate it partially with respect to ζ_{ij}; $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, we get the following set of equations

$$\frac{\partial \mathfrak{R}_i}{\partial \zeta_{ij}} = \frac{\omega_j}{\sum_{j=1}^{n}(\omega_j + \xi_j + \eta_j)} \tag{6.7}$$

Since, $\omega_j, \xi_j, \eta_j \geq 0$ therefore it follows that $\frac{\partial \mathfrak{R}_i}{\partial \zeta_{ij}} \geq 0$. Especially, for $\omega_j \neq 0$, $\frac{\partial \mathfrak{R}_i}{\partial \zeta_{ij}} > 0$. Therefore, $\mathfrak{R}_i$ are monotonic and non-decreasing functions of the variables ζ_{ij}.

In same manner, partial derivatives of $\mathfrak{R}_i$ with respect to κ_{ij} and φ_{ij} are computed as

$$\frac{\partial \mathfrak{R}_i}{\partial \kappa_{ij}} = \frac{-\xi_j}{\sum_{j=1}^{n}(\omega_j + \xi_j + \eta_j)} \tag{6.8}$$

and

$$\frac{\partial \mathfrak{R}_i}{\partial \varphi_{ij}} = \frac{-\eta_j}{\sum_{j=1}^{n}(\omega_j + \xi_j + \eta_j)} \tag{6.9}$$

respectively. Since $\xi_j, \eta_j \geq 0$, therefore $\frac{\partial \mathfrak{R}_i}{\partial \xi_{ij}} \leq 0$ and $\frac{\partial \mathfrak{R}_i}{\partial \eta_{ij}} \leq 0$. Especially, for $\xi_j \neq 0, \eta_j \neq 0$, $\frac{\partial \mathfrak{R}_i}{\partial \kappa_{ij}}, \frac{\partial \mathfrak{R}_i}{\partial \varphi_{ij}} < 0$, therefore, $\mathfrak{R}_i$ are monotonic and non-increasing functions of the variables κ_{ij} and φ_{ij}.

Further, $\zeta_{ij}, \kappa_{ij}, \varphi_{ij}, \omega_j, \xi_j, \eta_j$ are all closed and bounded elements of the subintervals of the interval $[0,1]$ then the continuous functions $\mathfrak{R}_i$ are bounded, that is, the value of $\mathfrak{R}_i$ lies in the range $[0,1]$ which is denoted by $[\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ respectively. Therefore, by the basic definition as well as the continuous nature of the function $\mathfrak{R}_i$, we have $0 \leq \mathfrak{R}_i^L \leq \mathfrak{R}_i^U \leq 1$ for $\zeta_{ij} \in [\zeta_{ij}^L, \zeta_{ij}^U]$ and so on. Therefore, the RCC of an alternative $\mathcal{A}_i$ is an interval-valued set $[\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ and lies in the interval $[0,1]$. Further, it can be seen that $\mathfrak{R}_i^L + (1 - \mathfrak{R}_i^U) = 1 - (\mathfrak{R}_i^U - \mathfrak{R}_i^L) \leq 1$ and thus the set $[\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ can be equivalently expressed as an IFS $\mathfrak{R}_i = (\mathfrak{R}_i^L, 1 - \mathfrak{R}_i^U)$ which shows that closeness degree of alternatives A_i to $\mathcal{A}^+$ is $\mathfrak{R}_i^L$ whereas their non-closeness is $1 - \mathfrak{R}_i^U$. Hence, the set $\mathfrak{R}_i = [\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ can be used to determine the rankings.

6.2.5 Auxiliary NLP models

Since $\mathfrak{R}_i^L$ and $\mathfrak{R}_i^U$ are the lower and upper bounds of $\mathfrak{R}_i$, respectively, which is bounded and continuous functions of the variables $\zeta_{ij}, \kappa_{ij}, \varphi_{ij}, \omega_j, \xi_j$ and η_j $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Now by combining with Eq. (6.6), $\mathfrak{R}_i^L$ and $\mathfrak{R}_i^U$ can be captured by solving the NLP models, constructed as follows:

$$\mathfrak{R}_i^L = \min\left\{\mathfrak{R}_i\left((\zeta_{ij})_{m\times n}, (\kappa_{ij})_{m\times n}, (\varphi_{ij})_{m\times n}, (\omega_j)_{1\times n}, (\xi_j)_{1\times n}, (\eta_j)_{1\times n}\right)\right\} \tag{6.10}$$

and

$$\mathfrak{R}_i^U = \max\left\{\mathfrak{R}_i\left((\zeta_{ij})_{m\times n}, (\kappa_{ij})_{m\times n}, (\varphi_{ij})_{m\times n}, (\omega_j)_{1\times n}, (\xi_j)_{1\times n}, (\eta_j)_{1\times n}\right)\right\} \tag{6.11}$$

subject to

$$\begin{aligned} &((\zeta_{ij})_{m\times n}, (\kappa_{ij})_{m\times n}, (\varphi_{ij})_{m\times n}, (\omega_j)_{1\times n}, (\xi_j)_{1\times n}, (\eta_j)_{1\times n}) \\ &\qquad \in \Omega_\zeta \times \Omega_\kappa \times \Omega_\varphi \times \Omega_\omega \times \Omega_\xi \times \Omega_\eta \end{aligned}$$

Here, $\Omega_\zeta = \{(\zeta_{ij})_{m\times n} \mid \zeta_{ij}^L \leq \zeta_{ij} \leq \zeta_{ij}^U\}$, $\Omega_\kappa = \{(\kappa_{ij})_{m\times n} \mid \kappa_{ij}^L \leq \kappa_{ij} \leq \kappa_{ij}^U\}$, $\Omega_\varphi = \{(\varphi_{ij})_{m\times n} \mid \varphi_{ij}^L \leq \varphi_{ij} \leq \varphi_{ij}^U\}$, $\Omega_\omega = \{(\omega_j)_{1\times n} \mid \omega_j^L \leq \omega_j \leq \omega_j^U\}$, $\Omega_\xi = \{(\xi_j)_{1\times n} \mid \xi_j^L \leq \xi_j \leq \xi_j^U\}$ and $\Omega_\eta = \{(\eta_j)_{1\times n} \mid \eta_j^L \leq \eta_j \leq \eta_j^U\}$ and $\Omega_\zeta \times \Omega_\kappa \times \Omega_\varphi \times \Omega_\omega \times \Omega_\xi \times \Omega_\eta$ is the cartesian product of all the sets. As stated above, $\mathfrak{R}_i$ are the monotonic and continuous functions and hence it reaches its minimum at the lower bounds ζ_{ij}^L of the intervals $[\zeta_{ij}^L, \zeta_{ij}^U]$ and the upper bounds of κ_{ij}^U and φ_{ij}^U of the intervals $[\kappa_{ij}^L, \kappa_{ij}^U]$ and $[\varphi_{ij}^L, \varphi_{ij}^U]$ respectively. Therefore, model (6.10) can be further reduce to

$$\begin{aligned} \mathfrak{R}_i^L &= \min\left\{\mathfrak{R}_i\left((\zeta_{ij}^L)_{m\times n}, (\kappa_{ij}^U)_{m\times n}, (\varphi_{ij}^U)_{m\times n}, (\omega_j)_{1\times n}, (\xi_j)_{1\times n}, (\eta_j)_{1\times n}\right)\right\} \\ s.t. \quad & ((\omega_j)_{1\times n}, (\xi_j)_{1\times n}, (\eta_j)_{1\times n}) \in \Omega_\omega \times \Omega_\xi \times \Omega_\eta \end{aligned} \tag{6.12}$$

which may equivalently be written as

$$\mathfrak{R}_i^L = \min\left\{\frac{\sum\limits_{j=1}^{n}\left\{\omega_j\zeta_{ij}^L + \xi_j(1-\kappa_{ij}^U) + \eta_j(1-\varphi_{ij}^U)\right\}}{\sum\limits_{j=1}^{n}(\omega_j + \xi_j + \eta_j)}\right\} \tag{6.13}$$

$$\text{s.t.} \quad \begin{cases} \omega_j^L \leq \omega_j \leq \omega_j^U & ; \quad j = 1, 2, \ldots, n \\ \xi_j^L \leq \xi_j \leq \xi_j^U & ; \quad j = 1, 2, \ldots, n \\ \eta_j^L \leq \eta_j \leq \eta_j^U & ; \quad j = 1, 2, \ldots, n \end{cases} \tag{6.14}$$

Obviously, the model (6.13) is a nonlinear fractional programming model (NLFPM) and the unknown variables are reduced to $3n$, which is less than that of unknown variables in the model (6.10), and hence it can be easily solved as compared to the model (6.10).

Similarly, $\mathfrak{R}_i^U$ given in model (6.11) attains its maximum at the upper bounds of the interval $[\zeta_{ij}^L, \zeta_{ij}^U]$ and the lower bounds of the intervals $[\kappa_{ij}^L, \kappa_{ij}^U]$ and $[\varphi_{ij}^L, \varphi_{ij}^U]$ respectively. Thus, model (6.11) reduces to

$$\mathfrak{R}_i^U = \max\left\{\frac{\sum\limits_{j=1}^{n}\left\{\omega_j\zeta_{ij}^U + \xi_j(1-\kappa_{ij}^L) + \eta_j(1-\varphi_{ij}^L)\right\}}{\sum\limits_{j=1}^{n}(\omega_j + \xi_j + \eta_j)}\right\} \tag{6.15}$$

which consists of $3n$ unknown variables subject to constraints given in Eq. (6.14). Also, this model has lesser number of unknown variables than the model (6.11).

Now, these models (6.13) and (6.15) can be simplified as linear programming models based on Charnes and Cooper's transformations [49]. For it, let $z = \frac{1}{\sum\limits_{j=1}^{n}(\omega_j+\xi_j+\eta_j)}$, $t_j = z\omega_j$, $r_j = z\xi_j$, $y_j = z\eta_j$ and hence based on these transformations, the above NLFPMs are transformed into the equivalent linear programming models as follows:

$$\begin{aligned} \mathfrak{R}_i^L &= \min \sum_{j=1}^{n} \left\{ t_j \zeta_{ij}^L + r_j(1-\kappa_{ij}^U) + y_j(1-\varphi_{ij}^U) \right\} \\ s.t. & \quad \begin{cases} z\omega_j^L \leq t_j \leq z\omega_j^U; & z\xi_j^L \leq r_j \leq z\xi_j^U; \\ z\eta_j^L \leq y_j \leq z\eta_j^U; & \sum\limits_{j=1}^{n}(t_j + r_j + y_j) = 1 \\ z \geq 0 \end{cases} \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} \mathfrak{R}_i^U &= \max \sum_{j=1}^{n} \left\{ t_j \zeta_{ij}^U + r_j(1-\kappa_{ij}^L) + y_j(1-\varphi_{ij}^L) \right\} \\ s.t. & \quad \begin{cases} z\omega_j^L \leq t_j \leq z\omega_j^U; & z\xi_j^L \leq r_j \leq z\xi_j^U; \\ z\eta_j^L \leq y_j \leq z\eta_j^U; & \sum\limits_{j=1}^{n}(t_j + r_j + y_j) = 1 \\ z \geq 0 \end{cases} \end{aligned} \tag{6.17}$$

which can be solved easily by using any existing method and thus RCC of each alternative $\mathcal{A}_i (i = 1, 2, \ldots, m)$ can be obtained in the form of the interval set $\mathfrak{R}_i = [\mathfrak{R}_i^L, \mathfrak{R}_i^U]$.

In order to rank the different alternative, the inclusion-comparison probability of $\mathcal{A}_i \succeq \mathcal{A}_k$; $i, k \in \{1, 2, \ldots, m\}$ is denoted by $p(\mathcal{A}_i \succeq \mathcal{A}_k)$ and hence the likelihood of the alternatives $p(\mathcal{A}_i \succeq \mathcal{A}_k) = p(\mathfrak{R}_i \supseteq \mathfrak{R}_k)$ is defined by using Eq. (6.18) as

$$p(\mathcal{A}_i \succeq \mathcal{A}_k) = p(\mathfrak{R}_i \supseteq \mathfrak{R}_k) = \max\left\{1 - \max\left\{\frac{\mathfrak{R}_k^U - \mathfrak{R}_i^L}{L(\mathfrak{R}_i) + L(\mathfrak{R}_k)}, 0\right\}, 0\right\} \tag{6.18}$$

where $\mathfrak{R}_i$ and $\mathfrak{R}_k$ are corresponding closeness coefficient interval numbers of alternatives $\mathcal{A}_i$ and $\mathcal{A}_k (i, k = 1, 2, \ldots, m)$, $L(\mathfrak{R}_k) = \mathfrak{R}_k^U - \mathfrak{R}_k^L$ and $L(\mathfrak{R}_i) = \mathfrak{R}_i^U - \mathfrak{R}_i^L$ are the hesitation degrees of $\mathcal{A}_k$ and $\mathcal{A}_i$ respectively and their corresponding likelihood matrix is denoted by $P = (p_{ik})_{m \times m}$ where $p_{ik} = p(\mathcal{A}_i \succeq \mathcal{A}_k); (i, k = 1, 2, \ldots, m)$. Further, it has been clearly observed that $0 \leq p_{ik} \leq 1$ and $p_{ik} + p_{ki} = 1; (i, k = 1, 2, \ldots, m)$ which implies that P is the fuzzy-complementary-judgement matrix. Now, based on the values of p_{ik}, the optimal degrees of the alternative $\mathcal{A}_i(i = 1, 2, \ldots, m)$ are computed as:

$$\theta_i = \frac{1}{m(m-1)} \left(\sum_{k=1}^{m} p_{ik} + \frac{m}{2} - 1 \right) \tag{6.19}$$

and select the best alternative(s), according to the decreasing order of values of θ_i's.

Based on the above analysis, the following steps are summarized to proposed a TOPSIS method based on NLP model to solve MCDM problem.

Step 1: Arrange the rating values of each alternative $\mathcal{A}_i(i = 1, 2, \ldots, m)$ given by an expert in terms of matrix $\mathcal{M} = (([\zeta_{ij}^L, \zeta_{ij}^U], [\kappa_{ij}^L, \kappa_{ij}^U], [\varphi_{ij}^L, \varphi_{ij}^U]))_{m\times n}$.

Step 2: Summarize the subjective weight vector $W_j, (j = 1, 2, \ldots, n)$ to each criterion $\mathcal{G}_j$.

Step 3: Construct the models (6.16) and (6.17) for each alternative $\mathcal{A}_i$ and hence obtain their RCCs $\mathfrak{R}_i = [\mathfrak{R}_i^L, \mathfrak{R}_i^U]$ for $i = 1, 2, \ldots, m$.

Step 4: Construct the likelihood matrix and obtain the values of θ_i's by using Eq. (6.19).

Step 5: Based on the descending values of θ_i's, rank the given alternatives.

6.3 Additional features of the proposed NLP models

The proposed designed models (6.13) and (6.15) also has some additional features as compared to the other approaches which are enlisted below:

(i) It is noticed that the discussion given in Section 6.2.2 is based on fixed $\mathcal{A}^+$ and $\mathcal{A}^-$ which results that RCCs could not show any variation even if the alternatives are changed. However, if a decision-maker can define these references points as $\mathcal{A}^+ = ([g_j^+, g_j^+], [h_j^+, h_j^+], [k_j^+, k_j^+])_{1\times n}$, and $\mathcal{A}^- = ([g_j^-, g_j^-], [h_j^-, h_j^-], [k_j^-, k_j^-])_{1\times n}$ as given in Section 6.2.2, then by using Eqs. (6.3) and (6.4), we explicitly defined the RCC of the alternatives $\mathcal{A}_i(i = 1, 2, \ldots, m)$ as follows:

$$\begin{aligned}\mathfrak{R}_i &= \frac{\sum\limits_{j=1}^{n}\big(\omega_j(\zeta_{ij}-g_j^-)+\xi_j(h_j^- -\kappa_{ij})+\eta_j(k_j^- -\varphi_{ij})\big)}{\sum\limits_{j=1}^{n}\big(\omega_j(g_j^+-\zeta_{ij})+\xi_j(\kappa_{ij}-h_j^+)+\eta_j(\varphi_{ij}-k_j^+)\big)+\sum\limits_{j=1}^{n}\big(\omega_j(\zeta_{ij}-g_j^-)+\xi_j(h_j^- -\kappa_{ij})+\eta_j(k_j^- -\varphi_{ij})\big)}\\ &= \frac{\sum\limits_{j=1}^{n}\Big(\omega_j(\zeta_{ij}-g_j^-)+\xi_j(h_j^- -\kappa_{ij})+\eta_j(k_j^- -\varphi_{ij})\Big)}{\sum\limits_{j=1}^{n}\Big(\omega_j(g_j^+-g_j^-)+\xi_j(h_j^- -h_j^+)+\eta_j(k_j^- -k_j^+)\Big)}\end{aligned}$$

Thus, in view of (6.10) and (6.11), $\mathfrak{R}_i^L$ and $\mathfrak{R}_i^U$ can similarly be captured by solving the NLP models as follows:

$$\begin{aligned}\mathfrak{R}_i^L &= \min\big\{\mathfrak{R}_i\big((\zeta_{ij}^L)_{m\times n},(\kappa_{ij}^U)_{m\times n},(\varphi_{ij}^U)_{m\times n},(\omega_j)_{1\times n},(\xi_j)_{1\times n},(\eta_j)_{1\times n}\big)\big\}\\ s.t. &\quad ((\omega_j)_{1\times n},(\xi_j)_{1\times n},(\eta_j)_{1\times n})\in\Omega_\omega\times\Omega_\xi\times\Omega_\eta \end{aligned}\tag{6.20}$$

and

$$\begin{aligned}\mathfrak{R}_i^U &= \max\big\{\mathfrak{R}_i\big((\zeta_{ij}^U)_{m\times n},(\kappa_{ij}^L)_{m\times n},(\varphi_{ij}^L)_{m\times n},(\omega_j)_{1\times n},(\xi_j)_{1\times n},(\eta_j)_{1\times n}\big)\big\}\\ s.t. &\quad ((\omega_j)_{1\times n},(\xi_j)_{1\times n},(\eta_j)_{1\times n})\in\Omega_\omega\times\Omega_\xi\times\Omega_\eta \end{aligned}\tag{6.21}$$

respectively. Thus, RCCs and ranking of the alternatives can be found for the given set of alternatives.

(ii) If the weights of the criterion are known *a priori*, then we can still utilize the Eqs. (6.20) and (6.21) provided all the constraints $\big((\omega_j)_{1\times n},\ (\xi_j)_{1\times n},\ (\eta_j)_{1\times n}\big)\in\Omega_\omega\times\Omega_\xi\times\Omega_\eta$ and regard all variables weights as known constants. Then, $\mathfrak{R}_i^L$ and $\mathfrak{R}_i^U$ of alternative $\mathcal{A}_i$ $(i=1,2,\ldots,m)$ to $\mathcal{A}^+$ are obtained directly as

$$\mathfrak{R}_i^L = \frac{\sum\limits_{j=1}^{n}\Big(\omega_j(\zeta_{ij}^L-g_j^-)+\xi_j(h_j^- -\kappa_{ij}^U)+\eta_j(k_j^- -\varphi_{ij}^U)\Big)}{\sum\limits_{j=1}^{n}\Big(\omega_j(g_j^+-g_j^-)+\xi_j(h_j^- -h_j^+)+\eta_j(k_j^- -k_j^+)\Big)}\tag{6.22}$$

and

$$\mathfrak{R}_i^U = \frac{\sum\limits_{j=1}^{n}\Big(\omega_j(\zeta_{ij}^U-g_j^-)+\xi_j(h_j^- -\kappa_{ij}^L)+\eta_j(k_j^- -\varphi_{ij}^L)\Big)}{\sum\limits_{j=1}^{n}\Big(\omega_j(g_j^+-g_j^-)+\xi_j(h_j^- -h_j^+)+\eta_j(k_j^- -k_j^+)\Big)}\tag{6.23}$$

respectively.

(iii) If in the proposed approach, we utilize the weighted Euclidean distances

$$\mathcal{D}(\mathcal{A}_i,\mathcal{A}^+) = \sqrt{\sum_{j=1}^{n}\left\{\omega_j(1-\zeta_{ij})^2+\xi_j(\kappa_{ij})^2+\eta_j(\varphi_{ij})^2\right\}} \tag{6.24}$$

and

$$\mathcal{D}(\mathcal{A}_i,\mathcal{A}^-) = \sqrt{\sum_{j=1}^{n}\left\{\omega_j(\zeta_{ij})^2+\xi_j(1-\kappa_{ij})^2+\eta_j(1-\varphi_{ij})^2\right\}} \tag{6.25}$$

in place of the weighted Hamming distances between the alternatives $\mathcal{A}_i$ and $\mathcal{A}^+$, as well as $\mathcal{A}^-$, then the RCC of the alternatives $\mathcal{A}_i$ to $\mathcal{A}^+$ is written as follows:

$$\mathfrak{R}_i = \frac{\sqrt{\sum\limits_{j=1}^{n}\left\{\omega_j(\zeta_{ij})^2+\xi_j(1-\kappa_{ij})^2+\eta_j(1-\varphi_{ij})^2\right\}}}{\sqrt{\sum\limits_{j=1}^{n}\left\{\omega_j(1-\zeta_{ij})^2+\xi_j(\kappa_{ij})^2+\eta_j(\varphi_{ij})^2\right\}}+\sqrt{\sum\limits_{j=1}^{n}\left\{\omega_j(\zeta_{ij})^2+\xi_j(1-\kappa_{ij})^2+\eta_j(1-\varphi_{ij})^2\right\}}} \tag{6.26}$$

Now, according to the monotonicity of $\mathfrak{R}_i$ as stated earlier, the bounds of RCC i.e., $\mathfrak{R}_i^L$ and $\mathfrak{R}_i^U$ are obtained by solving the models

$$\mathfrak{R}_i^L = \min\left\{\mathfrak{R}_i\big((\zeta_{ij}^L)_{m\times n},(\kappa_{ij}^U)_{m\times n},(\varphi_{ij}^U)_{m\times n},(\omega_j)_{1\times n},(\xi_j)_{1\times n},(\eta_j)_{1\times n}\big)\right\} \tag{6.27}$$

and

$$\mathfrak{R}_i^U = \max\left\{\mathfrak{R}_i\big((\zeta_{ij}^U)_{m\times n},(\kappa_{ij}^L)_{m\times n},(\varphi_{ij}^L)_{m\times n},(\omega_j)_{1\times n},(\xi_j)_{1\times n},(\eta_j)_{1\times n}\big)\right\} \tag{6.28}$$

respectively, subject to $((\omega_j)_{1\times n},(\xi_j)_{1\times n},(\eta_j)_{1\times n}) \in \Omega_\omega \times \Omega_\xi \times \Omega_\eta$.

(iv) If the weights of the each criterion are known *a priori* and a real numbers, say ω_j such that $\omega_j > 0, \sum_{j=1}^{n}\omega_j = 1$ instead of interval numbers, then we can still utilize the Eqs. (6.13) and (6.15) provided all the constraints and variables weights as known as constants. Thus, $\mathfrak{R}_i^L$ and $\mathfrak{R}_i^U$ of alternatives $\mathcal{A}_i$ to $\mathcal{A}^+$ are obtained directly as

$$\mathfrak{R}_i^L = \frac{1}{3}\sum_{j=1}^{n}\omega_j\left\{2+\zeta_{ij}^L-\kappa_{ij}^U-\varphi_{ij}^U\right\} \tag{6.29}$$

$$\text{and}\quad \mathfrak{R}_i^U = \frac{1}{3}\sum_{j=1}^{n}\omega_j\left\{2+\zeta_{ij}^U-\kappa_{ij}^L-\varphi_{ij}^L\right\} \tag{6.30}$$

respectively.

6.4 Illustrative example

In this section, a real example related to the selection of a suitable company to set up power plants in a country is used to demonstrate the proposed method under INS environment.

6.4.1 A case study

Electric power generation is a state-controlled subject. The government has liberalized power generation of the user industry to bridge the demand and supply gap. Punjab's, one of the states of India, installed capacity at the end of the 10th plan is 6088 MW, including own generation, share from Bhakra Beas Management Board (BBMB) and various central sector projects. During the 11th plan, two thermal generation projects, namely 500 MW Lehra Mohabat Stage-II and 540 MW Goindwal Sahib Thermal Power Project are expected to be commissioned. In addition to this, 263 MW Hydel Generation projects are also expected to be commissioned during the 11th Plan. Thus, the total generation is expected to be about 7451 MW at the end of the 11th plan against the expected peak demand of 11000 MW during the year 2011-12. Therefore, the Punjab Government has planned to develop three Thermal Power Plants at Rajpura, Talwandi Sabo and Gidderbaha having a capacity of 1320 MW, 1980 MW, and 2640 MW respectively on Build-Own-Operate (BOO) basis. For this purpose, Government of Punjab wants to select a developer(company) of the project based on the Tariff Based Competitive Bidding Process (Case-2) on BOO basis for the supply of 100 percent power to the Punjab State Electricity Board (PSEB) for 25 years as per the guidelines of Government of India. In order to do this, PSEB makes a list of integrated power companies (alternatives) as follows: $\mathcal{A}_1$ (KSK Energy Ventures limited), $\mathcal{A}_2$ (Hartek Power Private Limited), $\mathcal{A}_3$ (JSW Energy Limited), $\mathcal{A}_4$ (CESC Limited), $\mathcal{A}_5$ (NTPC Limited) which can handle the development of considering power project. A decision maker is hired by Government wants to select the best Project company based on the following four criteria: $\mathcal{G}_1$ (Reliability), $\mathcal{G}_2$ (Adapted strategy), $\mathcal{G}_3$ (Financial model for a project) and $\mathcal{G}_4$ (Availability of fuel supplier). Then the steps of the mentioned approach are executed as follows:

Step 1: The rating values of the alternatives given by an expert in terms of INNs are summarized in Table 6.1.

Table 6.1: Interval valued neutrosophic DM matrix

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$([0.7,0.8],[0.5,0.7],[0.1,0.2])$	$([0.6,0.8],[0.4,0.5],[0.3,0.3])$	$([0.8,0.8],[0.4,0.6],[0.1,0.2])$	$([0.7,0.9],[0.3,0.4],[0.2,0.2])$
$\mathcal{A}_2$	$([0.6,0.8],[0.4,0.6],[0.1,0.3])$	$([0.5,0.7],[0.3,0.5],[0.1,0.3])$	$([0.6,0.6],[0.2,0.3],[0.4,0.5])$	$([0.6,0.8],[0.4,0.4],[0.2,0.4])$
$\mathcal{A}_3$	$([0.4,0.6],[0.2,0.2],[0.2,0.4])$	$([0.6,0.7],[0.4,0.6],[0.3,0.4])$	$([0.7,0.8],[0.6,0.7],[0.1,0.2])$	$([0.5,0.6],[0.5,0.6],[0.2,0.3])$
$\mathcal{A}_4$	$([0.4,0.5],[0.5,0.6],[0.4,0.4])$	$([0.5,0.6],[0.3,0.4],[0.4,0.5])$	$([0.6,0.7],[0.7,0.8],[0.2,0.3])$	$([0.8,0.9],[0.3,0.4],[0.1,0.2])$
$\mathcal{A}_5$	$([0.6,0.7],[0.4,0.5],[0.4,0.5])$	$([0.8,0.9],[0.3,0.4],[0.1,0.2])$	$([0.7,0.8],[0.5,0.6],[0.1,0.2])$	$([0.5,0.7],[0.5,0.5],[0.2,0.3])$

Step 2: Assume the interval weights of the attribute $\mathcal{G}_j, (j = 1, 2, 3, 4)$ as $W = \{([0.1, 0.3], [0.1, 0.2], [0.2, 0.4]), ([0.2, 0.5], [0.1, 0.2], [0.15, 0.25]), ([0.25, 0.4], [0.2, 0.3], [0.15, 0.3]), ([0.15, 0.3], [0.1, 0.3], [0.3, 0.4])\}$.

Step 3: Based on the gathered data and on the basis of models (6.16) and (6.17), the optimization models for an alternative $\mathcal{A}_1$ are formulated as:

$$\begin{aligned}
\mathfrak{R}_1^L &= \min(0.7t_1 + 0.6t_2 + 0.8t_3 + 0.7t_4 + 0.3r_1 + 0.5r_2 + 0.4r_3 + 0.6r_4 \\
&\qquad + 0.8y_1 + 0.7y_2 + 0.8y_3 + 0.8y_4) \\
\text{s.t.} &\quad \begin{cases}
0.10z \le t_1 \le 0.3z; \quad 0.10z \le r_1 \le 0.20z; \quad 0.20z \le y_1 \le 0.40z \\
0.20z \le t_2 \le 0.5z; \quad 0.10z \le r_2 \le 0.20z; \quad 0.15z \le y_2 \le 0.25z \\
0.25z \le t_3 \le 0.4z; \quad 0.20z \le r_3 \le 0.30z; \quad 0.15z \le y_3 \le 0.30z \\
0.15z \le t_4 \le 0.3z; \quad 0.10z \le r_4 \le 0.30z; \quad 0.30z \le y_4 \le 0.40z \\
\sum_{j=1}^{4}(t_j + r_j + y_j) = 1; \quad z \ge 0
\end{cases}
\end{aligned} \tag{6.31}$$

and

$$\begin{aligned}
\mathfrak{R}_1^U &= \max(0.8t_1 + 0.8t_2 + 0.8t_3 + 0.9t_4 + 0.5r_1 + 0.6r_2 + 0.6r_3 + 0.7r_4 \\
&\qquad + 0.9y_1 + 0.7y_2 + 0.9y_3 + 0.8y_4) \\
\text{s.t.} &\quad \begin{cases}
0.10z \le t_1 \le 0.3z; \quad 0.10z \le r_1 \le 0.20z; \quad 0.20z \le y_1 \le 0.40z \\
0.20z \le t_2 \le 0.5z; \quad 0.10z \le r_2 \le 0.20z; \quad 0.15z \le y_2 \le 0.25z \\
0.25z \le t_3 \le 0.4z; \quad 0.20z \le r_3 \le 0.30z; \quad 0.15z \le y_3 \le 0.30z \\
0.15z \le t_4 \le 0.3z; \quad 0.10z \le r_4 \le 0.30z; \quad 0.30z \le y_4 \le 0.40z \\
\sum_{j=1}^{4}(t_j + r_j + y_j) = 1; \quad z \ge 0
\end{cases}
\end{aligned} \tag{6.32}$$

The optimal values of these models are obtained as $\mathfrak{R}_1^L = 0.62857$ and $\mathfrak{R}_1^U = 0.79538$, i.e., the RCC interval of $\mathcal{A}_1$ is $\mathfrak{R}_1 = [0.62857,\ 0.79538]$. Similarly, the RCCs of other alternatives are obtained as $\mathfrak{R}_2 = [0.56792,\ 0.77017]$; $\mathfrak{R}_3$=[0.53636, 0.73148]; $\mathfrak{R}_4$=[0.53214, 0.69824] and $\mathfrak{R}_5$=[0.5990, 0.77142] respectively.

Step 4: By Eq. (6.18), the likelihood probability matrix is given as

$$P = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \mathcal{A}_4 \\ \mathcal{A}_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 & \mathcal{A}_4 & \mathcal{A}_5 \end{array} \\ \begin{bmatrix} 0.5 & 0.6163 & 0.7157 & 0.7907 & 0.5778 \\ 0.3837 & 0.5 & 0.5884 & 0.6462 & 0.4555 \\ 0.2843 & 0.4116 & 0.5 & 0.5519 & 0.3589 \\ 0.2093 & 0.3538 & 0.4481 & 0.5 & 0.2913 \\ 0.4222 & 0.5445 & 0.6411 & 0.7087 & 0.5 \end{bmatrix} \end{array}$$

and hence by Eq. (6.19) we get $\theta_1 = 0.2350$; $\theta_2 = 0.2037$; $\theta_3 = 0.1803$; $\theta_4 = 0.1651$; and $\theta_5 = 0.2158$.

Step 5: Based on values of θ_i's, we get the ranking order of the alternatives as $\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$ and conclude that $\mathcal{A}_1$ i.e., the KSK Energy ventures Limited, company is the best for the required task.

On the other hand, by taking a real weight vector $\omega = \rho = \eta = (0.15, 0.25, 0.375, 0.225)^T$ of the attribute instead of interval one, then the develop method is still applicable to to rank the alternative. The necessary steps corresponding to them are performed as below:

Step 1: The information related to each alternative is summarized in Table 6.1.

Step 2: The weight information is taken as $\omega = \rho = \eta = (0.15, 0.25, 0.375, 0.225)^T$, the INPIS and INNIS are taken as $\mathcal{A}^+ = (([1,1],\ [0,0],\ [0,0]))_{2\times3}$ and $\mathcal{A}^- = (([0,0],\ [1,1],\ [1,1]))_{2\times3}$, respectively.

Step 3: By Eqs. (6.29) and (6.30), the bounds of RCCs of each alternative $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$ are computed as $\mathfrak{R}_1 = [0.6475, 0.7525]$, $\mathfrak{R}_2 = [0.5867, 0.6800]$, $\mathfrak{R}_3 = [0.5683, 0.6817]$, $\mathfrak{R}_4 = [0.5558, 0.6508]$ and $\mathfrak{R}_5 = [0.6283, 0.7283]$.

Step 4: The likelihood probability matrix is represented as

$$P = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \mathcal{A}_4 \\ \mathcal{A}_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 & \mathcal{A}_4 & \mathcal{A}_5 \end{array} \\ \begin{bmatrix} 0.5 & 0.8361 & 0.8434 & 0.9833 & 0.6057 \\ 0.1639 & 0.5 & 0.5403 & 0.6593 & 0.2672 \\ 0.1565 & 0.4597 & 0.5 & 0.6040 & 0.2500 \\ 0.0167 & 0.3407 & 0.3960 & 0.5 & 0.1154 \\ 0.3943 & 0.7328 & 0.7500 & 0.8846 & 0.5 \end{bmatrix} \end{array}$$

and hence by Eq. (6.19) we get $\theta_1 = 0.2634$, $\theta_2 = 0.1815$, $\theta_3 = 0.1735$, $\theta_4 = 0.1434$ and $\theta_5 = 0.2381$.

Step 5: Thus, the ranking order of the alternatives is $\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$ and the best one is $\mathcal{A}_1$.

6.4.2 Comparison Analysis of the obtained results

In order to conduct a comparative analysis of the proposed method with some existing approaches under the INS environment, an analysis has been conducted by taking the real weight vectors of the criteria as a constant rather than the interval numbers. For it, we have taken the weight of the criteria be either $\omega = (0.15, 0.25, 0.375, 0.225)^T$ or giving an equal importance to each criteria, i.e., by taking a weight vector to be $\omega = (0.25, 0.25, 0.25, 0.25)^T$. Based on these vectors and the considered data, given in Table 6.1, corresponding to the alternatives, the comparative analysis of the existing approaches as proposed by the authors in [19, 20, 132, 177, 181, 200] has been summarized in Table 6.2. From this table, it has been concluded that the best alternative is $\mathcal{A}_1$ and the ranking order of the five alternatives is $\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2$, which is same as that obtained by using the existing measures [19, 132, 177, 181, 200] except the cosine similarity measure as proposed in [20], which validate the existence of the approach.

From this analysis, it has been concluded that results obtained by the existing methods and our methodology are the same. However, the principles, as well as the procedure of the approaches, are remarkably different. The former methods simply summed the values of the INSs on the alternatives $\mathcal{A}_i (i = 1, 2, \ldots, m)$ using the set operations, distance measures, etc., and ranked the alternatives based on their overall collective values. The latter takes into consideration not only both the INPIS $\mathcal{A}^+$ and the INNIS $\mathcal{A}^-$ but the relative importance of the weighted-Hamming as well as weighted Euclidean distances between $\mathcal{A}_i$ and $\mathcal{A}^-$, as well as $\mathcal{A}^-$. Also, we can easily see that the former approaches are unable to solve the DM problems with weights of the interval-number sets. Thus, the proposed methodology can suitably solve the problem in a real-life situation and can be found as an alternative place than of the existing measures in the INS environment.

6.4.3 Superiority of the proposed approach

In this section, we have presented some counterexamples which show that the existing approaches under the INS environment fail to rank the given alternatives while the proposed approach can overcome their shortcoming.

Example 6.4.1. Consider a DM problem which consists of two alternatives, namely $\mathcal{A}_1$ and $\mathcal{A}_2$ which are going to be evaluated by the decision maker under the set of three criteria, $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$. Assume that the importance of these criteria are given in the form of the weight vector $\omega = (0.5, 0.3, 0.2)^T$. The main objective of the problem is to find the best alternative. For this, a decision maker evaluates each alternative and give their preferences in form of INN which are summarized as follows:

$$\mathcal{M}_1 = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \qquad\mathcal{G}_1\qquad & \qquad\mathcal{G}_2\qquad & \qquad\mathcal{G}_3\qquad \end{array} \\ \begin{bmatrix} ([0.4,0.5],[0.1,0.2],[0.2,0.3]) & ([0.7,0.8],[0.1,0.2],[0.1,0.2]) & ([0.3,0.4],[0.1,0.2],[0.4,0.5]) \\ ([0.5,0.6],[0.1,0.2],[0.3,0.4]) & ([0.6,0.7],[0.1,0.2],[0.1,0.2]) & ([0.3,0.5],[0.1,0.2],[0.3,0.4]) \end{bmatrix} \end{array}$$

Based on this decision-matrix, if we utilize the existing TOPSIS approach [28] to find the best alternative, then the following steps have been executed as:

Step 1: The information of each alternative in INS environment is represented in the form of decision matrix $\mathcal{M}_1$.

Step 2: The positive and negative ideal solutions considered by decision maker are $\mathcal{A}^+ = (([1,1],[0,0],[0,0]))_{2\times 3}$ and $\mathcal{A}^- = (([0,0],[1,1],[1,1]))_{2\times 3}$, respectively.

Step 3: Based on these values the normalized Hamming distance, $(\mathcal{D})$, between the alternatives $\mathcal{A}_i (i = 1, 2)$ and its ideals values are evaluated as $\mathcal{D}(\mathcal{A}_1, \mathcal{A}^+) = \mathcal{D}(\mathcal{A}_2, \mathcal{A}^+) = 0.2800$ and $\mathcal{D}(\mathcal{A}_1, \mathcal{A}^-) = \mathcal{D}(\mathcal{A}_2, \mathcal{A}^-) = 0.7200$.

Step 4: The RCC $C(\cdot)$ of each alternative is $C(\mathcal{A}_1) = \frac{\mathcal{D}(\mathcal{A}_1,\mathcal{A}^-)}{\mathcal{D}(\mathcal{A}_1,\mathcal{A}^-)+\mathcal{D}(\mathcal{A}_1,\mathcal{A}^+)} = 0.5$ and $C(\mathcal{A}_2) = \frac{\mathcal{D}(\mathcal{A}_2,\mathcal{A}^-)}{\mathcal{D}(\mathcal{A}_2,\mathcal{A}^-)+\mathcal{D}(\mathcal{A}_2,\mathcal{A}^+)} = 0.5$. Since, $C(\mathcal{A}_1) = C(\mathcal{A}_2)$ and hence we conclude that the existing TOPSIS approach is unable to rank the given alternatives.

On the other hand, if we utilize the proposed approach for above considered data, then the following steps have been executed as:

Step 1: As same as that of above.

Step 2: As same as that of above.

Step 3: Using the Eqs. (6.22) and (6.23) for given information, we obtain the RCC $\mathfrak{R}_i (i = 1, 2)$ for alternative $\mathcal{A}_i$ as $\mathfrak{R}_1 = [0.6533, 0.7533]$ and $\mathfrak{R}_2 = [0.6500,\ 0.7567]$.

Step 4: The likelihood probabilities matrix is expressed as

$$P = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{A}_1 & \mathcal{A}_2 \end{array} \\ \begin{bmatrix} 0.5000 & 0.49975 \\ 0.50024 & 0.5000 \end{bmatrix} \end{array}$$

and values of θ_i's are $\theta_1 = 0.4998$ and $\theta_2 = 0.5001$.

Step 5: Since $\theta_2 > \theta_1$ and hence $\mathcal{A}_2$ is the best one.

Thus, from the example, we conclude that the proposed approach can easily distinguish between the alternatives $\mathcal{A}_1$ and $\mathcal{A}_2$ and found that alternative $\mathcal{A}_2$ is superior than the alternative $\mathcal{A}_1$ while the existing TOPSIS approach as proposed in [28] is unable to identify the best one.

Example 6.4.2. Consider a DM problem which consists of two alternatives $\mathcal{A}_1$ and $\mathcal{A}_2$ and three criteria $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ whose weight vector is 0.5, 0.25 and 0.25. In order to get the best alternative under the given set, decision maker evaluates all the alternatives and give their preference values in terms of INNs, which are summarized as follows:

$$\mathcal{M}_2 = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \end{array} \\ \begin{bmatrix} ([0.2,0.6],[0.1,0.2],[0.10,0.30]) & ([0.3,0.5],[0.1,0.2],[0.10,0.30]) & ([0.1,0.7],[0.1,0.2],[0.10,0.30]) \\ ([0.2,0.6],[0.1,0.2],[0.15,0.25]) & ([0.3,0.5],[0.1,0.2],[0.15,0.25]) & ([0.1,0.7],[0.1,0.2],[0.15,0.25]) \end{bmatrix} \end{array}$$

If we utilize the existing approach [58] to compute the distance measures of the alternative from its fixed ideal alternatives, then we get their respective measure values are $\mathcal{D}(\mathcal{A}_1, A^+) = \mathcal{D}(\mathcal{A}_2, A^+) = 0.2$ and $\mathcal{D}(\mathcal{A}_1, A^-) = \mathcal{D}(\mathcal{A}_2, A^-) = 0.2667$. Thus, RCCs for these two alternatives are equal and hence, it is unable to choose the best one.

On the other hand, if we apply the Eqs. (6.22) and (6.23), then we obtain the RCC intervals of the alternatives $\mathcal{A}_1$ and $\mathcal{A}_2$ as $\mathfrak{R}_1 = [0.5667, 0.8]$ and $\mathfrak{R}_2 = [0.583, 0.7833]$. The likelihood probabilities values are summarized as

$$P \;=\; \begin{array}{c|cc} & \mathcal{A}_1 & \mathcal{A}_2 \\ \hline \mathcal{A}_1 & 0.5000 & 0.5001 \\ \mathcal{A}_2 & 0.4998 & 0.5000 \end{array}$$

and hence by Eq. (6.19), we get $\theta_1 = 0.5$ and $\theta_2 = 0.4995$. Since $\theta_1 > \theta_2$ and hence we conclude that the alternative $\mathcal{A}_1$ is better alternative than the alternative $\mathcal{A}_2$.

From the results of two above examples, it is clear that the proposed TOPSIS approach is more suitable and effective comparing with the other existing TOPSIS approaches. According to the above comparative and superiority analysis to address the DM problems, the proposed approach has the following advantages:

(i) The existing approaches [19, 20, 132, 177, 181, 200] under the NS environment have considered only the PIS for the ranking purposes. On the other hand, the proposed method has considered both the PIS and NIS during the analysis. It means that all the given data is utilized and chances of loss of information are reduced, which implies that the proposed approach may enhance the profits without ignoring the risk factors while taking the decisions.

(ii) The approach in the IFS is a special case of the NSs. Also, the weight vector information to the criteria is taken in the form of the intervals rather than point values which reduce the discrimination during the analysis. Hence the proposed approach is more generalized and suitable for solving the DM problems than the existing ones.

6.5 Conclusion

This chapter addressed the methodology to solve the DM problems under the INS environment where rating values of all alternatives, as well as the preference information on different criteria, are expressed with the interval numbers rather than real numbers. To solve the model, two fractional programming models based on the TOPSIS method are established to obtain an RCC interval where preference information is independently determined for each alternative. Later, with the Charnes and Cooper's transformation technique, the proposed models are transformed into two simpler linear programming models to compute the RCC of the alternatives. Finally, using the likelihood probabilities values of the interval numbers, the given alternatives are ranked. The relevant properties of the proposed models are also examined. Further, an algorithm to solve the DM problems is presented and a practical example is considered to verify its feasibility with the several existing approaches. Also, it is examined that some of the existing studies such as IFSs and IVIFSs are the special cases of the proposed one. Finally, the superiority of the proposed approach has been presented and showed that some of the existing approaches under INS environment fail to rank the alternatives while the proposed approach can suitably work for it.

Table 6.2: Comparative study with existing approaches

Methods	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	Ranking order
	When weight vector $\omega = (0.25, 0.25, 0.25, 0.25)^T$ has considered					
Cosine similarity [20]	0.2085	0.2105	0.2028	0.1971	0.2077	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Similarity [19]	0.4242	0.3836	0.3735	0.3743	0.4182	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Similarity [19]	0.5833	0.5083	0.5167	0.5333	0.5750	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2$
Normalized Hamming distance [177]	0.3958	0.4083	0.4333	0.4458	0.4000	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Weighted arithmetic operator [200]	0.3247	0.2806	0.2544	0.2160	0.2986	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Weighted geometric operator [200]	0.2944	0.2431	0.1685	0.1087	0.2542	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Improved cosine similarity [181]	0.2030	0.2001	0.1938	0.1894	0.2016	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Cross entropy [132]	0.4667	0.4750	0.5333	0.5667	0.4917	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Proposed approach	0.2607	0.1862	0.1777	0.1482	0.2273	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
	When weight vector $\omega = (0.15, 0.25, 0.375, 0.225)^T$ has considered					
Cosine similarity[20]	0.2095	0.2149	0.1976	0.1949	0.2057	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Similarity[19]	0.4282	0.3915	0.3751	0.3758	0.4180	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Similarity[19]	0.5867	0.5158	0.5292	0.5408	0.5733	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2$
Normalized Hamming distance[177]	0.3917	0.4008	0.4408	0.4492	0.3992	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Weighted arithmetic operator [200]	0.3326	0.2972	0.2227	0.2000	0.3069	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Weighted geometric operator[200]	0.3057	0.2569	0.1498	0.0870	0.2669	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Improved cosine similarity[181]	0.2040	0.2018	0.1921	0.1886	0.2018	$\mathcal{A}_1 \succ \mathcal{A}_5 = \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Cross entropy[132]	0.4592	0.4658	0.5492	0.5758	0.4833	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Proposed approach	0.2634	0.1815	0.1735	0.1434	0.2381	$\mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$

Chapter 7

A novel divergence measure and its based TOPSIS method for single-valued neutrosophic sets[1]

In this chapter, we present an axiomatic definition of divergence measure for single-valued neutrosophic sets (SVNSs). The properties of the proposed divergence measure have been studied. Further, we develop a novel technique for order preference by similarity to ideal solution (TOPSIS) method for solving single-valued neutrosophic MCDM with incomplete weight information. Finally, a numerical example is presented to verify the proposed approach and to present its effectiveness and practicality.

7.1 Introduction

The literature related to the information measure approaches for SVNSs are presented in Chapter 1. However, apart from them, TOPSIS developed by Hwang and Yoon [56] is well-known DM approach to find the best alternative(s) based on its ideal values. The chief advantages of the TOPSIS are to consider positive and negative ideal solutions as anchor points to reflect the contrast of the currently achievable criterion performances. In TOPSIS, the preferred alternative should have the shortest distance from the positive-ideal solution and the farthest distance from the negative-ideal solution. The difference between

---

[1] The content of this chapter is published as “A novel divergence measure and its based TOPSIS method for multi criteria decision-making under single-valued neutrosophic environment”, *Journal of Intelligent & Fuzzy Systems*, 36(1), pp. 101 - 115, 2019 (**Impact Factor: 1.426**).

the TOPSIS and the neutrosophic TOPSIS is in their rating approaches. The merit of neutrosophic TOPSIS is to designate the importance of attributes and the performance of alternatives with respect to various attributes by using SVNNs instead of precise numbers. Under this environment, Biswas et al. [14], Ye [180] presented the TOPSIS method for DM problems based on the distance measure to rank the alternatives.

Consequently, keeping the flexibility and efficiency of SVNS, the theme of this work is to present a novel divergence measure for SVNSs to find discrimination between them. Further, based on it, a TOPSIS method for solving the neutrosophic DM problem has been presented. The proposed measure has elegant properties, which are expressed and tested in the work to enhance the employability of this measure. In contrast to the classical TOPSIS method, which is based on a distance measure, this chapter applies the proposed divergence measure to establish the comparative index of closeness coefficient. The strength of this extension has been demonstrated by an example of the DM process. Finally, through an example, the superiority of divergence based TOPSIS method over classical TOPSIS has been shown.

7.2 Proposed divergence measure

Let $\Phi(\mathcal{X})$ be the class of SVNSs over the universal set $\mathcal{X}$. Then for any $\mathcal{A}, \mathcal{B} \in \Phi(\mathcal{X})$, a real function $\mathcal{D}v : \Phi(\mathcal{X}) \times \Phi(\mathcal{X}) \to \mathbb{R}^+$ is called a divergence measure, denoted by $\mathcal{D}v(\mathcal{A}|\mathcal{B})$, if it satisfies the following axioms:

(P1) $\mathcal{D}v(\mathcal{A}|\mathcal{B}) \geq 0$.

(P2) $\mathcal{D}v(\mathcal{A}|\mathcal{B}) = \mathcal{D}v(\mathcal{B}|\mathcal{A})$.

(P3) $\mathcal{D}v(\mathcal{A}|\mathcal{B}) = 0$ if $\mathcal{A} = \mathcal{B}$.

(P4) $\mathcal{D}v(\mathcal{A}|\mathcal{B}) = \mathcal{D}v(\mathcal{A}^c|\mathcal{B}^c)$.

Definition 7.2.1. For two SVNSs $\mathcal{A} = (\zeta_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j),\ \varphi_{\mathcal{A}}(x_j) \mid x_j \in \mathcal{X})$ and $\mathcal{B} = (\zeta_{\mathcal{B}}(x_j),\ \kappa_{\mathcal{B}}(x_j),\ \varphi_{\mathcal{B}}(x_j) \mid x_j \in \mathcal{X})$, the divergence measure of $\mathcal{A}$ against $\mathcal{B}$ is to measure the degree of discrimination between the pair is defined as:

$$\mathcal{D}v(\mathcal{A}|\mathcal{B}) = \frac{1}{n(\sqrt{2}-1)} \sum_{j=1}^{n} \left[\begin{array}{r} \sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2 + (\zeta_{\mathcal{B}}(x_j))^2}{2}} - \frac{\zeta_{\mathcal{A}}(x_j) + \zeta_{\mathcal{B}}(x_j)}{2} \\ + \sqrt{\frac{(\kappa_{\mathcal{A}}(x_j))^2 + (\kappa_{\mathcal{B}}(x_j))^2}{2}} - \frac{\kappa_{\mathcal{A}}(x_j) + \kappa_{\mathcal{B}}(x_j)}{2} \\ + \sqrt{\frac{(\varphi_{\mathcal{A}}(x_j))^2 + (\varphi_{\mathcal{B}}(x_j))^2}{2}} - \frac{\varphi_{\mathcal{A}}(x_j) + \varphi_{\mathcal{B}}(x_j)}{2} \end{array} \right] \tag{7.1}$$

Theorem 7.2.1. The divergence measure $\mathcal{D}v(\mathcal{A}|\mathcal{B})$, as defined in Definition 7.2.1, for two SVNSs $\mathcal{A}$ and $\mathcal{B}$ satisfies the following four axioms.

(P1) $\mathcal{D}v(\mathcal{A}|\mathcal{B}) \geq 0$.

(P2) $\mathcal{D}v(\mathcal{A}|\mathcal{B}) = \mathcal{D}v(\mathcal{B}|\mathcal{A})$.

(P3) $\mathcal{D}v(\mathcal{A}|\mathcal{B}) = 0$ if $\mathcal{A} = \mathcal{B}$.

(P4) $\mathcal{D}v(\mathcal{A}|\mathcal{B}) = \mathcal{D}v(\mathcal{A}^c|\mathcal{B}^c)$.

Proof. For two SVNSs $\mathcal{A} = (\zeta_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j), \varphi_{\mathcal{A}}(x_j) | x_j \in \mathcal{X})$ and $\mathcal{B} = (\zeta_{\mathcal{B}}(x_j), \kappa_{\mathcal{B}}(x_j), \varphi_{\mathcal{B}}(x_j) \ | x_j \in \mathcal{X})$, we have

(P1) Since $0 \leq \zeta_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j), \varphi_{\mathcal{A}}(x_j) \leq 1$ and $0 \leq \zeta_{\mathcal{B}}(x_j), \kappa_{\mathcal{B}}(x_j), \varphi_{\mathcal{B}}(x_j) \leq 1$ which implies that $\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{B}}(x_j))^2}{2}} \geq \frac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{B}}(x_j)}{2}$, $\sqrt{\frac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{B}}(x_j))^2}{2}} \geq \frac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{B}}(x_j)}{2}$ and $\sqrt{\frac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{B}}(x_j))^2}{2}} \geq \frac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{B}}(x_j)}{2}$ for each j. Therefore, from Eq. (7.1), we get $\mathcal{D}v(\mathcal{A}|\mathcal{B}) \geq 0$.

(P2) It is trivial from the Eq. (7.1).

(P3) Assume that $\mathcal{A} = \mathcal{B}$ which implies $\zeta_{\mathcal{A}}(x_j) = \zeta_{\mathcal{B}}(x_j)$, $\kappa_{\mathcal{A}}(x_j) = \kappa_{\mathcal{B}}(x_j)$ and $\varphi_{\mathcal{A}}(x_j) = \varphi_{\mathcal{B}}(x_j)$ for each j, which implies that

$\mathcal{D}v(\mathcal{A}|\mathcal{B}) = \frac{1}{n(\sqrt{2}-1)} \sum\limits_{j=1}^{n} \Big[\zeta_{\mathcal{A}}(x_j) - \zeta_{\mathcal{A}}(x_j) + \kappa_{\mathcal{A}}(x_j) - \kappa_{\mathcal{A}}(x_j) + \varphi_{\mathcal{A}}(x_j) - \varphi_{\mathcal{A}}(x_j)\Big] = 0.$

(P4) For SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{A}^c = (\varphi_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j), \zeta_{\mathcal{A}}(x_j))$ and $\mathcal{B}^c = (\varphi_{\mathcal{B}}(x_j), \kappa_{\mathcal{B}}(x_j), \zeta_{\mathcal{B}}(x_j))$, so by Eq. (7.1), we have

$$\begin{aligned}\mathcal{D}v(\mathcal{A}^c|\mathcal{B}^c) &= \frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}\left[\begin{array}{l}\sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{B}}(x_j)}{2}\\ +\sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{B}}(x_j)}{2}\\ +\sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{B}}(x_j)}{2}\end{array}\right]\\ &= \mathcal{D}v(\mathcal{A}|\mathcal{B})\end{aligned}$$

Hence, $\mathcal{D}v(\mathcal{A}|\mathcal{B})$ is a valid divergence measure. □

Also, it is observed that $\mathcal{D}v(\mathcal{A}|\mathcal{B})$ satisfies certain properties which are stated as below:

Theorem 7.2.2. For SVNS $\mathcal{A} = (\zeta_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j), \varphi_{\mathcal{A}}(x_j)|\ x_j \in \mathcal{X})$, $\mathcal{D}v(\mathcal{A}|\mathcal{A}^c) = 0$ if and only if $\zeta_{\mathcal{A}}(x_j) = \varphi_{\mathcal{A}}(x_j)$ for each $x_j \in \mathcal{X}$.

Proof. For SVNS $\mathcal{A} = (\zeta_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j), \varphi_{\mathcal{A}}(x_j)|\ x_j \in \mathcal{X})$, we have $\mathcal{A}^c = (\varphi_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j), \zeta_{\mathcal{A}}(x_j)\ |x_j \in \mathcal{X})$. Thus, from Eq. (7.1), we get

$$\begin{aligned}&\mathcal{D}v(\mathcal{A}|\mathcal{A}^c) = 0\\ \Leftrightarrow\quad &\frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}2\left[\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{A}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\varphi_{\mathcal{A}}(x_j)}{2}\right]=0\\ \Leftrightarrow\quad &\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{A}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\varphi_{\mathcal{A}}(x_j)}{2}=0\quad \text{for each } x_j\in\mathcal{X}\\ \Leftrightarrow\quad &\left(\frac{\zeta_{\mathcal{A}}(x_j)}{\sqrt{2}}-\frac{\varphi_{\mathcal{A}}(x_j)}{\sqrt{2}}\right)^2=0\quad \text{for each } x_j\in\mathcal{X}\\ \Leftrightarrow\quad &\zeta_{\mathcal{A}}(x_j)=\varphi_{\mathcal{A}}(x_j)\quad \text{for each } x_j\in\mathcal{X}\end{aligned}$$

Hence, $\mathcal{D}v(\mathcal{A}|\mathcal{A}^c) = 0$ if and only if $\zeta_{\mathcal{A}}(x_j) = \varphi_{\mathcal{A}}(x_j)$ for each $x_j \in \mathcal{X}$. □

Theorem 7.2.3. For SVNS $\mathcal{A} = (\zeta_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j), \varphi_{\mathcal{A}}(x_j)|\ x_j \in \mathcal{X})$, $\mathcal{D}v(\mathcal{A}|\mathcal{A}^c) = 1$ if and only if either $\zeta_{\mathcal{A}}(x_j) = 0$, $\varphi_{\mathcal{A}}(x_j) = 1$ or $\zeta_{\mathcal{A}}(x_j) = 1$, $\varphi_{\mathcal{A}}(x_j) = 0$.

Proof. For SVNS $\mathcal{A}$, we have $\mathcal{D}v(\mathcal{A}|\mathcal{A}^c)=1$ which implies that

$$\begin{aligned}
&\frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}2\Big[\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{A}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\varphi_{\mathcal{A}}(x_j)}{2}\Big]=1\\
\Leftrightarrow\ &\sum_{j=1}^{n}\Big[\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{A}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\varphi_{\mathcal{A}}(x_j)}{2}\Big]=\frac{n(\sqrt{2}-1)}{2}\\
\Leftrightarrow\ &\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{A}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\varphi_{\mathcal{A}}(x_j)}{2}=\frac{(\sqrt{2}-1)}{2},\ \text{for each } x_j\\
\Leftrightarrow\ &(\zeta_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{A}}(x_j))^2=1 \text{ and } \zeta_{\mathcal{A}}(x_j)+\varphi_{\mathcal{A}}(x_j)=1
\end{aligned}$$

which implies that $\zeta_{\mathcal{A}}(x_j)\varphi_{\mathcal{A}}(x_j)=0$ and hence $\mathcal{D}v(\mathcal{A}|\mathcal{A}^c)=1$ if and only if either $\zeta_{\mathcal{A}}(x_j)=0$, $\varphi_{\mathcal{A}}(x_j)=1$ or $\zeta_{\mathcal{A}}(x_j)=1$, $\varphi_{\mathcal{A}}(x_j)=0$. □

Theorem 7.2.4. For SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{D}v(\mathcal{A}|\mathcal{B}^c)=\mathcal{D}v(\mathcal{A}^c|\mathcal{B})$.

Proof. As $\mathcal{A}^c=(\varphi_{\mathcal{A}}(x_j),\kappa_{\mathcal{A}}(x_j),\zeta_{\mathcal{A}}(x_j)|x_j\in\mathcal{X})$ and $\mathcal{B}^c=(\varphi_{\mathcal{B}}(x_j),\kappa_{\mathcal{B}}(x_j),\zeta_{\mathcal{B}}(x_j)|x_j\in\mathcal{X})$ be the complement of the SVNSs $\mathcal{A}$ and $\mathcal{B}$, then from Eq. (7.1) we have

$$\begin{aligned}
\mathcal{D}v(\mathcal{A}^c|\mathcal{B}) &= \frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}\left[\begin{array}{l}\sqrt{\frac{(\varphi_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{B}}(x_j))^2}{2}}-\frac{\varphi_{\mathcal{A}}(x_j)+\zeta_{\mathcal{B}}(x_j)}{2}\\ +\sqrt{\frac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{B}}(x_j))^2}{2}}-\frac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{B}}(x_j)}{2}\\ +\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{B}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\varphi_{\mathcal{B}}(x_j)}{2}\end{array}\right]\\
&= \frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}\left[\begin{array}{l}\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{B}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\varphi_{\mathcal{B}}(x_j)}{2}\\ +\sqrt{\frac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{B}}(x_j))^2}{2}}-\frac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{B}}(x_j)}{2}\\ +\sqrt{\frac{(\varphi_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{B}}(x_j))^2}{2}}-\frac{\varphi_{\mathcal{A}}(x_j)+\zeta_{\mathcal{B}}(x_j)}{2}\end{array}\right]\\
&= \mathcal{D}v(\mathcal{A}|\mathcal{B}^c)
\end{aligned}$$

□

Divide the universe $\mathcal{X}$ into two disjoint parts $\mathcal{X}_1$ and $\mathcal{X}_2$, where $\mathcal{X}_1=\{x_j : x_j\in\mathcal{X},\mathcal{A}(x_j)\subseteq\mathcal{B}(x_j)\}$ and $\mathcal{X}_2=\{x_j : x_j\in\mathcal{X},\mathcal{B}(x_j)\subseteq\mathcal{A}(x_j)\}$. Based on these considerations, we further propose some properties of the divergence measure which are explained as follows:

Theorem 7.2.5. If $\mathcal{A}$ and $\mathcal{B}$ be two SVNSs defined on universal set $\mathcal{X}$, then

$$\mathcal{D}v(\mathcal{A}\cap\mathcal{B}|\mathcal{A}\cup\mathcal{B})=\mathcal{D}v(\mathcal{A}|\mathcal{B}).$$

Proof. Let $A=(\zeta_{\mathcal{A}}(x_j),\kappa_{\mathcal{A}}(x_j),\varphi_{\mathcal{A}}(x_j)|x_j\in\mathcal{X})$ and $B=(\zeta_{\mathcal{B}}(x_j),\kappa_{\mathcal{B}}(x_j),\varphi_{\mathcal{B}}(x_j)|x_j\in\mathcal{X})$ be two SVNSs defined on the universal set $\mathcal{X}$, then by Eq. (7.1), we have

$$\begin{aligned}
&\mathcal{D}v(\mathcal{A}\cap\mathcal{B}|\mathcal{A}\cup\mathcal{B})\\
=&\ \frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}\left[\begin{array}{l}
\sqrt{\dfrac{(\zeta_{\mathcal{A}\cap\mathcal{B}}(x_j))^2+(\zeta_{\mathcal{A}\cup\mathcal{B}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}\cap\mathcal{B}}(x_j)+\zeta_{\mathcal{A}\cup\mathcal{B}}(x_j)}{2}\\
+\sqrt{\dfrac{(\kappa_{\mathcal{A}\cap\mathcal{B}}(x_j))^2+(\kappa_{\mathcal{A}\cup\mathcal{B}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}\cap\mathcal{B}}(x_j)+\kappa_{\mathcal{A}\cup\mathcal{B}}(x_j)}{2}\\
+\sqrt{\dfrac{(\varphi_{\mathcal{A}\cap\mathcal{B}}(x_j))^2+(\varphi_{\mathcal{A}\cup\mathcal{B}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}\cap\mathcal{B}}(x_j)+\varphi_{\mathcal{A}\cup\mathcal{B}}(x_j)}{2}
\end{array}\right]\\
=&\ \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_1}\left[\begin{array}{l}
\sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{B}}(x_j)}{2}\\
+\sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{B}}(x_j)}{2}\\
+\sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{B}}(x_j)}{2}
\end{array}\right]\\
+&\ \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_2}\left[\begin{array}{l}
\sqrt{\dfrac{(\zeta_{\mathcal{B}}(x_j))^2+(\zeta_{\mathcal{A}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{B}}(x_j)+\zeta_{\mathcal{A}}(x_j)}{2}\\
+\sqrt{\dfrac{(\kappa_{\mathcal{B}}(x_j))^2+(\kappa_{\mathcal{A}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{B}}(x_j)+\kappa_{\mathcal{A}}(x_j)}{2}\\
+\sqrt{\dfrac{(\varphi_{\mathcal{B}}(x_j))^2+(\varphi_{\mathcal{A}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{B}}(x_j)+\varphi_{\mathcal{A}}(x_j)}{2}
\end{array}\right]\\
=&\ \mathcal{D}v(\mathcal{A}|\mathcal{B})
\end{aligned}$$

Thus, the result holds. □

Theorem 7.2.6. For two SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have

(i) $\mathcal{D}v(\mathcal{A}|\mathcal{A}\cup\mathcal{B})=\mathcal{D}v(\mathcal{B}|\mathcal{A}\cap\mathcal{B})$

(ii) $\mathcal{D}v(\mathcal{A}|\mathcal{A}\cap\mathcal{B})=\mathcal{D}v(\mathcal{B}|\mathcal{A}\cup\mathcal{B})$

(iii) $\mathcal{D}v(\mathcal{A}|\mathcal{A}\cup\mathcal{B})+\mathcal{D}v(\mathcal{A}|\mathcal{A}\cap\mathcal{B})=\mathcal{D}v(\mathcal{A}|\mathcal{B})$

(iv) $\mathcal{D}v(\mathcal{B}|\mathcal{A}\cup\mathcal{B})+\mathcal{D}v(\mathcal{B}|\mathcal{A}\cap\mathcal{B})=\mathcal{D}v(\mathcal{A}|\mathcal{B})$

Proof. We will prove the first part and rest can be proved in the same manner. By the definition of divergence, we have

$$
\begin{aligned}
&\mathcal{D}v(\mathcal{A}|\mathcal{A}\cup\mathcal{B}) \\
= \;& \frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}\left[\begin{array}{l}
\sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{A}\cup\mathcal{B}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{A}\cup\mathcal{B}}(x_j)}{2} \\
+\sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{A}\cup\mathcal{B}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{A}\cup\mathcal{B}}(x_j)}{2} \\
+\sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{A}\cup\mathcal{B}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{A}\cup\mathcal{B}}(x_j)}{2}
\end{array}\right] \\
= \;& \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_1}\left[\begin{array}{l}
\sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{B}}(x_j)}{2} \\
+\sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{B}}(x_j)}{2} \\
+\sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{B}}(x_j)}{2}
\end{array}\right] \\
+ \;& \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_2}\left[\begin{array}{l}
\sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{A}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{A}}(x_j)}{2} \\
+\sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{A}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{A}}(x_j)}{2} \\
+\sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{A}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{A}}(x_j)}{2}
\end{array}\right] \\
= \;& \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_1}\left[\begin{array}{l}
\sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{B}}(x_j)}{2} \\
+\sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{B}}(x_j)}{2} \\
+\sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{B}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{B}}(x_j)}{2}
\end{array}\right] \qquad (7.2)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\mathcal{D}v(\mathcal{B}|\mathcal{A}\cap\mathcal{B}) \\
= \;& \frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}\left[\begin{array}{l}
\sqrt{\dfrac{(\zeta_{\mathcal{B}}(x_j))^2+(\zeta_{\mathcal{A}\cap\mathcal{B}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{B}}(x_j)+\zeta_{\mathcal{A}\cap\mathcal{B}}(x_j)}{2} \\
+\sqrt{\dfrac{(\kappa_{\mathcal{B}}(x_j))^2+(\kappa_{\mathcal{A}\cap\mathcal{B}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{B}}(x_j)+\kappa_{\mathcal{A}\cap\mathcal{B}}(x_j)}{2} \\
+\sqrt{\dfrac{(\varphi_{\mathcal{B}}(x_j))^2+(\varphi_{\mathcal{A}\cap\mathcal{B}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{B}}(x_j)+\varphi_{\mathcal{A}\cap\mathcal{B}}(x_j)}{2}
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n(\sqrt{2}-1)} \sum_{x_j \in \mathcal{X}_1} \left[\begin{array}{l} \sqrt{\dfrac{(\zeta_{\mathcal{B}}(x_j))^2 + (\zeta_{\mathcal{A}}(x_j))^2}{2}} - \dfrac{\zeta_{\mathcal{B}}(x_j) + \zeta_{\mathcal{A}}(x_j)}{2} \\ + \sqrt{\dfrac{(\kappa_{\mathcal{B}}(x_j))^2 + (\kappa_{\mathcal{A}}(x_j))^2}{2}} - \dfrac{\kappa_{\mathcal{B}}(x_j) + \kappa_{\mathcal{A}}(x_j)}{2} \\ + \sqrt{\dfrac{(\varphi_{\mathcal{B}}(x_j))^2 + (\varphi_{\mathcal{A}}(x_j))^2}{2}} - \dfrac{\varphi_{\mathcal{B}}(x_j) + \varphi_{\mathcal{A}}(x_j)}{2} \end{array} \right] \\
&+ \frac{1}{n(\sqrt{2}-1)} \sum_{x_j \in \mathcal{X}_2} \left[\begin{array}{l} \sqrt{\dfrac{(\zeta_{\mathcal{B}}(x_j))^2 + (\zeta_{\mathcal{B}}(x_j))^2}{2}} - \dfrac{\zeta_{\mathcal{B}}(x_j) + \zeta_{\mathcal{B}}(x_j)}{2} \\ + \sqrt{\dfrac{(\kappa_{\mathcal{B}}(x_j))^2 + (\kappa_{\mathcal{B}}(x_j))^2}{2}} - \dfrac{\kappa_{\mathcal{B}}(x_j) + \kappa_{\mathcal{B}}(x_j)}{2} \\ + \sqrt{\dfrac{(\varphi_{\mathcal{B}}(x_j))^2 + (\varphi_{\mathcal{B}}(x_j))^2}{2}} - \dfrac{\varphi_{\mathcal{B}}(x_j) + \varphi_{\mathcal{B}}(x_j)}{2} \end{array} \right] \\
&= \frac{1}{n(\sqrt{2}-1)} \sum_{x_j \in \mathcal{X}_1} \left[\begin{array}{l} \sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2 + (\zeta_{\mathcal{B}}(x_j))^2}{2}} - \dfrac{\zeta_{\mathcal{A}}(x_j) + \zeta_{\mathcal{B}}(x_j)}{2} \\ + \sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2 + (\kappa_{\mathcal{B}}(x_j))^2}{2}} - \dfrac{\kappa_{\mathcal{A}}(x_j) + \kappa_{\mathcal{B}}(x_j)}{2} \\ + \sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2 + (\varphi_{\mathcal{B}}(x_j))^2}{2}} - \dfrac{\varphi_{\mathcal{A}}(x_j) + \varphi_{\mathcal{B}}(x_j)}{2} \end{array} \right] \qquad (7.3)
\end{aligned}
$$

Then, from Eqs. (7.2) and (7.3), we get $\mathcal{D}v(\mathcal{A}|\mathcal{A} \cup \mathcal{B}) = \mathcal{D}v(\mathcal{B}|\mathcal{A} \cap \mathcal{B})$. □

Theorem 7.2.7. If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three SVNSs defined on the universal sets $\mathcal{X}$, then

(i) $\mathcal{D}v(\mathcal{A}|\mathcal{C}) + \mathcal{D}v(\mathcal{B}|\mathcal{C}) - \mathcal{D}v(\mathcal{A} \cup \mathcal{B}|\mathcal{C}) \geq 0$,

(ii) $\mathcal{D}v(\mathcal{A}|\mathcal{C}) + \mathcal{D}v(\mathcal{B}|\mathcal{C}) - \mathcal{D}v(\mathcal{A} \cap \mathcal{B}|\mathcal{C}) \geq 0$

Proof. In this property, we prove only the first part because of having analogously similar proofs.

$$
\mathcal{D}v(\mathcal{A}|\mathcal{C}) = \frac{1}{n(\sqrt{2}-1)} \sum_{j=1}^{n} \left[\begin{array}{l} \sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2 + (\zeta_{\mathcal{C}}(x_j))^2}{2}} - \dfrac{\zeta_{\mathcal{A}}(x_j) + \zeta_{\mathcal{C}}(x_j)}{2} \\ + \sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2 + (\kappa_{\mathcal{C}}(x_j))^2}{2}} - \dfrac{\kappa_{\mathcal{A}}(x_j) + \kappa_{\mathcal{C}}(x_j)}{2} \\ + \sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2 + (\varphi_{\mathcal{C}}(x_j))^2}{2}} - \dfrac{\varphi_{\mathcal{A}}(x_j) + \varphi_{\mathcal{C}}(x_j)}{2} \end{array} \right]
$$

$$
\mathcal{D}v(\mathcal{B}|\mathcal{C}) = \frac{1}{n(\sqrt{2}-1)} \sum_{j=1}^{n} \left[\begin{array}{l} \sqrt{\dfrac{(\zeta_{\mathcal{B}}(x_j))^2 + (\zeta_{\mathcal{C}}(x_j))^2}{2}} - \dfrac{\zeta_{\mathcal{B}}(x_j) + \zeta_{\mathcal{C}}(x_j)}{2} \\ + \sqrt{\dfrac{(\kappa_{\mathcal{B}}(x_j))^2 + (\kappa_{\mathcal{C}}(x_j))^2}{2}} - \dfrac{\kappa_{\mathcal{B}}(x_j) + \kappa_{\mathcal{C}}(x_j)}{2} \\ + \sqrt{\dfrac{(\varphi_{\mathcal{B}}(x_j))^2 + (\varphi_{\mathcal{C}}(x_j))^2}{2}} - \dfrac{\varphi_{\mathcal{B}}(x_j) + \varphi_{\mathcal{C}}(x_j)}{2} \end{array} \right]
$$

and

$$\begin{aligned}
\mathcal{D}v(\mathcal{A}\cup\mathcal{B}|\mathcal{C}) &= \frac{1}{n(\sqrt{2}-1)}\sum_{j=1}^{n}\left[\begin{array}{l}\sqrt{\dfrac{(\zeta_{\mathcal{A}\cup\mathcal{B}}(x_j))^2+(\zeta_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}\cup\mathcal{B}}(x_j)+\zeta_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\kappa_{\mathcal{A}\cup\mathcal{B}}(x_j))^2+(\kappa_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}\cup B}(x_j)+\kappa_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\varphi_{\mathcal{A}\cup\mathcal{B}}(x_j))^2+(\varphi_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}\cup\mathcal{B}}(x_j)+\varphi_{\mathcal{C}}(x_j)}{2}\end{array}\right]\\
&= \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_1}\left[\begin{array}{l}\sqrt{\dfrac{(\zeta_{\mathcal{B}}(x_j))^2+(\zeta_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{B}}(x_j)+\zeta_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\kappa_{\mathcal{B}}(x_j))^2+(\kappa_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{B}}(x_j)+\kappa_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\varphi_{\mathcal{B}}(x_j))^2+(\varphi_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{B}}(x_j)+\varphi_{\mathcal{C}}(x_j)}{2}\end{array}\right]\\
&+ \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_2}\left[\begin{array}{l}\sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{C}}(x_j)}{2}\end{array}\right]
\end{aligned}$$

Then,

$$\begin{aligned}
&\mathcal{D}v(\mathcal{A}|\mathcal{C})+\mathcal{D}v(\mathcal{B}|\mathcal{C})-\mathcal{D}v(\mathcal{A}\cup\mathcal{B}|\mathcal{C})\\
&= \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_1}\left[\begin{array}{l}\sqrt{\dfrac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{C}}(x_j)}{2}\end{array}\right]\\
&+ \frac{1}{n(\sqrt{2}-1)}\sum_{x_j\in\mathcal{X}_2}\left[\begin{array}{l}\sqrt{\dfrac{(\zeta_{\mathcal{B}}(x_j))^2+(\zeta_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\zeta_{\mathcal{B}}(x_j)+\zeta_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\kappa_{\mathcal{B}}(x_j))^2+(\kappa_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\kappa_{\mathcal{B}}(x_j)+\kappa_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\dfrac{(\varphi_{\mathcal{B}}(x_j))^2+(\varphi_{\mathcal{C}}(x_j))^2}{2}}-\dfrac{\varphi_{\mathcal{B}}(x_j)+\varphi_{\mathcal{C}}(x_j)}{2}\end{array}\right]
\end{aligned}$$

Since, $\zeta(x_j),\kappa(x_j),\varphi(x_j)\in[0,1]$ for all $x_j\in\mathcal{X}$. Thus, we get $\mathcal{D}v(\mathcal{A}|\mathcal{C})+\mathcal{D}v(\mathcal{B}|\mathcal{C})-\mathcal{D}v(\mathcal{A}\cup\mathcal{B}|\mathcal{C})\geq 0$. □

Theorem 7.2.8. For three SVNSs $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we have

$$\mathcal{D}v(\mathcal{A}\cap\mathcal{B}|\mathcal{C})+\mathcal{D}v(\mathcal{A}\cup\mathcal{B}|\mathcal{C})=\mathcal{D}v(\mathcal{A}|\mathcal{C})+\mathcal{D}v(\mathcal{B}|\mathcal{C})$$

Proof. For three SVNSs $\mathcal{A} = (\zeta_{\mathcal{A}}(x_j), \kappa_{\mathcal{A}}(x_j),\ \varphi_{\mathcal{A}}(x_j)|x_j \in \mathcal{X})$, $\mathcal{B} = (\zeta_{\mathcal{B}}(x_j), \kappa_{\mathcal{B}}(x_j),$ $\varphi_{\mathcal{B}}(x_j)|x_j \in \mathcal{X})$ and $\mathcal{C} = (\zeta_{\mathcal{C}}(x_j), \kappa_{\mathcal{C}}(x_j),\ \varphi_{\mathcal{C}}(x_j)|x_j \in \mathcal{X})$ and by definition of the divergence measure, we have

$$
\begin{aligned}
&\mathcal{D}v(\mathcal{A} \cap \mathcal{B}|\mathcal{C})\\
= \quad & \frac{1}{n(\sqrt{2}-1)} \sum_{j=1}^{n} \left[\begin{array}{l}
\sqrt{\frac{(\zeta_{A\cap B}(x_j))^2 + (\zeta_{\mathcal{C}}(x_j))^2}{2}} - \frac{\zeta_{A\cap B}(x_j) + \zeta_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\kappa_{A\cap B}(x_j))^2 + (\kappa_{\mathcal{C}}(x_j))^2}{2}} - \frac{\kappa_{A\cap B}(x_j) + \kappa_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\varphi_{A\cap B}(x_j))^2 + (\varphi_{\mathcal{C}}(x_j))^2}{2}} - \frac{\varphi_{A\cap B}(x_j) + \varphi_{\mathcal{C}}(x_j)}{2}
\end{array}\right]\\
= \quad & \frac{1}{n(\sqrt{2}-1)} \sum_{j\in\mathcal{X}_1} \left[\begin{array}{l}
\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2 + (\zeta_{\mathcal{C}}(x_j))^2}{2}} - \frac{\zeta_{\mathcal{A}}(x_j) + \zeta_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\kappa_{\mathcal{A}}(x_j))^2 + (\kappa_{\mathcal{C}}(x_j))^2}{2}} - \frac{\kappa_{\mathcal{A}}(x_j) + \kappa_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\varphi_{\mathcal{A}}(x_j))^2 + (\varphi_{\mathcal{C}}(x_j))^2}{2}} - \frac{\varphi_{\mathcal{A}}(x_j) + \varphi_{\mathcal{C}}(x_j)}{2}
\end{array}\right]\\
+ \quad & \frac{1}{n(\sqrt{2}-1)} \sum_{j\in\mathcal{X}_2} \left[\begin{array}{l}
\sqrt{\frac{(\zeta_{\mathcal{B}}(x_j))^2 + (\zeta_{\mathcal{C}}(x_j))^2}{2}} - \frac{\zeta_{\mathcal{B}}(x_j) + \zeta_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\kappa_{\mathcal{B}}(x_j))^2 + (\kappa_{\mathcal{C}}(x_j))^2}{2}} - \frac{\kappa_{\mathcal{B}}(x_j) + \kappa_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\varphi_{\mathcal{B}}(x_j))^2 + (\varphi_{\mathcal{C}}(x_j))^2}{2}} - \frac{\varphi_{\mathcal{B}}(x_j) + \varphi_{\mathcal{C}}(x_j)}{2}
\end{array}\right]
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\mathcal{D}v(\mathcal{A} \cup \mathcal{B}|\mathcal{C})\\
= \quad & \frac{1}{n(\sqrt{2}-1)} \sum_{j=1}^{n} \left[\begin{array}{l}
\sqrt{\frac{(\zeta_{A\cup B}(x_j))^2 + (\zeta_{\mathcal{C}}(x_j))^2}{2}} - \frac{\zeta_{A\cup B}(x_j) + \zeta_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\kappa_{A\cup B}(x_j))^2 + (\kappa_{\mathcal{C}}(x_j))^2}{2}} - \frac{\kappa_{A\cup B}(x_j) + \kappa_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\varphi_{A\cup B}(x_j))^2 + (\varphi_{\mathcal{C}}(x_j))^2}{2}} - \frac{\varphi_{A\cup B}(x_j) + \varphi_{\mathcal{C}}(x_j)}{2}
\end{array}\right]\\
= \quad & \frac{1}{n(\sqrt{2}-1)} \sum_{j\in\mathcal{X}_1} \left[\begin{array}{l}
\sqrt{\frac{(\zeta_{\mathcal{B}}(x_j))^2 + (\zeta_{\mathcal{C}}(x_j))^2}{2}} - \frac{\zeta_{\mathcal{B}}(x_j) + \zeta_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\kappa_{\mathcal{B}}(x_j))^2 + (\kappa_{\mathcal{C}}(x_j))^2}{2}} - \frac{\kappa_{\mathcal{B}}(x_j) + \kappa_{\mathcal{C}}(x_j)}{2}\\
+\sqrt{\frac{(\varphi_{\mathcal{B}}(x_j))^2 + (\varphi_{\mathcal{C}}(x_j))^2}{2}} - \frac{\varphi_{\mathcal{B}}(x_j) + \varphi_{\mathcal{C}}(x_j)}{2}
\end{array}\right]
\end{aligned}
$$

$$+\quad \frac{1}{n(\sqrt{2}-1)}\sum_{j\in\mathcal{X}_2}\left[\begin{array}{l}\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{C}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\frac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{C}}(x_j))^2}{2}}-\frac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{C}}(x_j)}{2}\\ +\sqrt{\frac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{C}}(x_j))^2}{2}}-\frac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{C}}(x_j)}{2}\end{array}\right]$$

Thus, by adding these equations, we get $\mathcal{D}v(\mathcal{A}\cap\mathcal{B}|\mathcal{C})+\mathcal{D}v(\mathcal{A}\cup\mathcal{B}|\mathcal{C})=\mathcal{D}v(\mathcal{A}|\mathcal{C})+\mathcal{D}v(\mathcal{B}|\mathcal{C})$.

□

Theorem 7.2.9. For two SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have

(i) $\mathcal{D}v(\mathcal{A}|\mathcal{B})=\mathcal{D}v(\mathcal{A}^c|\mathcal{B}^c)$

(ii) $\mathcal{D}v(\mathcal{A}|\mathcal{B}^c)=\mathcal{D}v(\mathcal{A}^c|\mathcal{B})$

(iii) $\mathcal{D}v(\mathcal{A}|\mathcal{B})+\mathcal{D}v(\mathcal{A}^c|\mathcal{B})=\mathcal{D}v(\mathcal{A}^c|\mathcal{B}^c)+\mathcal{D}v(\mathcal{A}|\mathcal{B}^c)$

Proof. First and second parts are similar and third part can be proved by adding first and second one. Proof of the part (ii) follows from Theorem 7.2.4. □

Definition 7.2.2. For two SVNSs $\mathcal{A}=(\zeta_{\mathcal{A}}(x_j),\kappa_{\mathcal{A}}(x_j),\varphi_{\mathcal{A}}(x_j)|\ x_j\in\mathcal{X})$ and $\mathcal{B}=(\zeta_{\mathcal{B}}(x_j),$ $\kappa_{\mathcal{B}}(x_j),\ \varphi_{\mathcal{B}}(x_j)|\ x_j\in\mathcal{X})$, the weighted divergence measure of $\mathcal{A}$ against $\mathcal{B}$ is to measure the degree of discrimination between the pair is defined as:

$$\mathcal{D}v_w(\mathcal{A}|\mathcal{B})=\frac{1}{(\sqrt{2}-1)}\sum_{j=1}^{n}w_j\left[\begin{array}{l}\sqrt{\frac{(\zeta_{\mathcal{A}}(x_j))^2+(\zeta_{\mathcal{B}}(x_j))^2}{2}}-\frac{\zeta_{\mathcal{A}}(x_j)+\zeta_{\mathcal{B}}(x_j)}{2}\\ +\sqrt{\frac{(\kappa_{\mathcal{A}}(x_j))^2+(\kappa_{\mathcal{B}}(x_j))^2}{2}}-\frac{\kappa_{\mathcal{A}}(x_j)+\kappa_{\mathcal{B}}(x_j)}{2}\\ +\sqrt{\frac{(\varphi_{\mathcal{A}}(x_j))^2+(\varphi_{\mathcal{B}}(x_j))^2}{2}}-\frac{\varphi_{\mathcal{A}}(x_j)+\varphi_{\mathcal{B}}(x_j)}{2}\end{array}\right]\qquad(7.4)$$

7.3 TOPSIS approach based on proposed divergence measure

In this section, a TOPSIS method based on the proposed divergence measure for SVNSs has been presented.

7.3.1 Maximizing divergence method for determine the weights

In this subsection, we construct a nonlinear programming model by maximizing the divergence value $\mathcal{D}v(w)$ to find the optimal weights of the criteria. For this, let Δ be the set of the partially known weight information. For j^{th} criteria, the divergence of the i^{th} alternative to all the other alternatives can be defined as follows:

$$D_{ij} = \sum_{q=1}^{m} \mathcal{D}v(\mathcal{A}_{ij}|\mathcal{A}_{qj})$$

where $\mathcal{D}v(\mathcal{A}_{ij}|\mathcal{A}_{qj})$ denotes the divergence measure between the two alternatives $\mathcal{A}_{ij}$ and $\mathcal{A}_{qj}$ defined in Definition 7.2.1.

Let $D_j = \sum\limits_{i=1}^{m} D_{ij} = \sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m} \mathcal{D}v(\mathcal{A}_{ij}|\mathcal{A}_{qj})$ represents the total divergence values of all the alternatives to other alternatives for j^{th} attributes and hence

$$\mathcal{D}v(w) = \sum_{j=1}^{n} w_j D_j = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q=1}^{m} w_j \mathcal{D}v(\mathcal{A}_{ij}|\mathcal{A}_{qj})$$

represents the total divergence values of all the alternatives with respect to all criteria.

Under these, a nonlinear optimization model has been constructed to determine the optimal weight vector as follow:

$$\left.\begin{aligned} \max \mathcal{D}v(w) &= \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q=1}^{m} w_j \mathcal{D}v(\mathcal{A}_{ij}|\mathcal{A}_{qj}) \\ \text{subject to } & w_j \in \Delta \quad , w_j \geq 0, \\ & \sum_{j=1}^{n} w_j = 1 \end{aligned}\right\} \tag{7.5}$$

By solving this model, we get the optimal weight $w = (w_1, w_2, \ldots, w_n)^T$ of the criteria.

On the other hand, if information about the criteria weights are completely unknown, then we establish another nonlinear optimization model as

$$\begin{aligned} \max \mathcal{D}v(w) \quad &= \quad \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q=1}^{m} w_j \mathcal{D}v(\mathcal{A}_{ij}|\mathcal{A}_{qj}) \\ \text{subject to} \quad & \quad w_j \geq 0, \\ & \quad \sum_{j=1}^{n} w_j^2 = 1 \end{aligned} \tag{7.6}$$

Solve the above model by using the Lagrangian multiplier method and hence get the normalized weight vector as

$$w_j = \frac{\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m} \mathcal{D}v(\mathcal{A}_{ij}|\mathcal{A}_{qj})}{\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m} \mathcal{D}v(\mathcal{A}_{ij}|\mathcal{A}_{qj})} \tag{7.7}$$

7.3.2 Proposed TOPSIS approach

TOPSIS method [56] is a simple and effective tool to solve the DM problems which aims to pick out the best alternative(s) with the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). In this section, instead of using the distance measures to figure the relative closeness coefficient, the concept of divergence measure is appropriately applied to the main structure of the TOPSIS method.

The brief description of the decision making process is given in Section 2.5 of Chapter 2, where rating values of each alternative is assessed in terms of the SVNNs and is denoted by $\alpha_{ij} = (\zeta_{ij}, \kappa_{ij}, \varphi_{ij})$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$, where ζ_{ij}, κ_{ij}, φ_{ij} represents the degree of satisfaction, indeterminacy, and dissatisfaction, respectively, of $\mathcal{A}_i$ corresponding to criteria $\mathcal{G}_j$ such that $0 \leq \zeta_{ij}, \kappa_{ij}, \varphi_{ij} \leq 1$ and $\zeta_{ij} + \kappa_{ij} + \varphi_{ij} \leq 3$. Then, the following steps of the TOPSIS method have been summarized for solving the DM problems under SVNN information by using the proposed measures as:

Step 1: *Arrange the collective information:* Collect the information related to each alternative $\mathcal{A}_i (i = 1, 2, \ldots, m)$ and arranged them in the form of the neutrosophic decision-matrix $\mathcal{M} = (\alpha_{ij})_{m \times n}$.

Step 2: *Normalize the decision matrix:* By using Eq. (2.38) of Chapter 2, we normalize the matrix rating values of $\mathcal{M}$ into the normalized decision matrix $\mathcal{R} = (r_{ij})$.

Step 3: *Determine the attribute weights:* The weights of the different criteria $w = (w_1, w_2, \ldots, w_n)^T$ are determined by solving the optimization model (7.5) or (7.6) accordingly whether the information about the criteria weights are partially known and completely unknown.

Step 4: *Determine the discrimination of each alternative from ideal and anti-ideal alternatives:* As $0 \leq \zeta_{ij}, \kappa_{ij}, \varphi_{ij} \leq 1$ and hence all rating values of the alternative as given in decision matrix $\mathcal{R} = (r_{ij})_{m\times n}$ are SVNNs. Therefore, accordingly the membership degrees of relative positive ideal ideal solution (RPIS) may be expressed as $\mathcal{A}^+ = (1,0,0)_{1\times n}$. Similarly, the membership degrees of the relative negative ideal solution (RNIS) may be summarized as $\mathcal{A}^- = (0,1,1)_{1\times n}$. From these, it has been seen that $\mathcal{A}^+$ and $\mathcal{A}^-$ are complement to each other. Furthermore, instead of fixing the degree of $\mathcal{A}^+$ to be 1, 0 and 0, the decision maker may varying it by defining $\mathcal{A}^+$ and $\mathcal{A}^-$ respectively as $(\zeta_j^+, \kappa_j^+, \varphi_j^+))_{1\times n}$, and $(\zeta_j^-, \kappa_j^-, \varphi_j^-))_{1\times n}$ where $\zeta_j^+ = \max\limits_i\{\zeta_{ij}|i = 1,2,\ldots,m\}$, $\kappa_j^+ = \min\limits_i\{\kappa_{ij}|i = 1,2,\ldots,m\}$, $\varphi_j^+ = \min\limits_i\{\varphi_{ij}|i = 1,2,\ldots,m\}$, $\zeta_j^- = \min\limits_i\{\zeta_{ij}|i = 1,2,\ldots,m\}$, $\kappa_j^- = \max\limits_i\{\kappa_{ij}|i = 1,2,\ldots,m\}$, $\varphi_j^- = \max\limits_i\{\varphi_{ij}|i = 1,2,\ldots,m\}$. From this, it has been clearly seen that $((\zeta_j^-, \kappa_j^-, \varphi_j^-)) \subseteq ((\zeta_{ij}, \kappa_{ij}, \varphi_{ij})) \subseteq ((\zeta_j^+, \kappa_j^+, \varphi_j^+))$.

In order to compare the different alternatives, the divergence measure defined in Eq. (7.4) is used to measures the degree of discrimination between an alternative $\mathcal{A}_i$ and the RPIS $\mathcal{A}^+$ as well as the RNIS $\mathcal{A}^-$ as follows:

$$\mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^+) = \frac{1}{(\sqrt{2}-1)}\sum_{j=1}^{n} w_j \left[\begin{array}{l} \sqrt{\dfrac{(\zeta_{ij}(r_{ij}))^2 + (\zeta_j^+(r_{ij}))^2}{2}} - \dfrac{\zeta_{ij}(r_{ij}) + \zeta_j^+(r_{ij})}{2} \\ + \sqrt{\dfrac{(\kappa_{ij}(r_{ij}))^2 + (\kappa_j^+(r_{ij}))^2}{2}} - \dfrac{\kappa_{ij}(r_{ij}) + \kappa_j^+(r_{ij})}{2} \\ + \sqrt{\dfrac{(\varphi_{ij}(r_{ij}))^2 + (\varphi_j^+(r_{ij}))^2}{2}} - \dfrac{\varphi_{ij}(r_{ij}) + \varphi_j^+(r_{ij})}{2} \end{array}\right] \tag{7.8}$$

and

$$\mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^-) = \frac{1}{(\sqrt{2}-1)}\sum_{j=1}^{n} w_j \left[\begin{array}{l} \sqrt{\dfrac{(\zeta_{ij}(r_{ij}))^2 + (\zeta_j^-(r_{ij}))^2}{2}} - \dfrac{\zeta_{ij}(r_{ij}) + \zeta_j^-(r_{ij})}{2} \\ + \sqrt{\dfrac{(\kappa_{ij}(r_{ij}))^2 + (\kappa_j^-(r_{ij}))^2}{2}} - \dfrac{\kappa_{ij}(r_{ij}) + \kappa_j^-(r_{ij})}{2} \\ + \sqrt{\dfrac{(\varphi_{ij}(r_{ij}))^2 + (\varphi_j^-(r_{ij}))^2}{2}} - \dfrac{\varphi_{ij}(r_{ij}) + \varphi_j^-(r_{ij})}{2} \end{array}\right] \tag{7.9}$$

Step 5: *Compute the relative-closeness coefficient:* Based on Eqs. (7.8) and (7.9), the relative-closeness coefficient of the alternative $\mathcal{A}_i (i = 1,2,\ldots,m)$ with respect to

$\mathcal{A}^+$ and $\mathcal{A}^-$ is defined as follows:

$$\mathfrak{R}_i = \frac{\mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^-)}{\mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^-) + \mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^+)} \tag{7.10}$$

provided $\mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^+) \neq 0$. It has been seen that $0 \leq \mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^-) \leq \mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^-)$ $+\mathcal{D}v_w(\mathcal{A}_i|\mathcal{A}^+)$ and hence $0 \leq \mathfrak{R}_i \leq 1$.

Step 6: *Rank the alternative:* Based on the descending order of the values of $\mathfrak{R}_i$, we rank the alternatives $\mathcal{A}_i (i = 1, 2, \ldots, m)$ and select the best alternative(s).

7.4 Illustrative Example

The above developed approach is illustrated with a numerical example as follows.

7.4.1 A case study

A travel agency naming, Marricot Trip mate , has excelled in providing travel related services to domestic and Inbound tourists . Agency wants to provide more facilities like detailed information, online booking capabilities, allow to book and sell airline tickets, car rentals, hotels, and other travel related services etc. to their customers. For this purpose, agency intends to find an appropriate information technology (IT) software development company that delivers affordable solutions through software development. To complete this motive, agency forms a set of five companies (alternatives), namely, Zensar Tech ($\mathcal{A}_1$), NIIT Tech ($\mathcal{A}_2$), HCL Tech($\mathcal{A}_3$), Hexaware Tech($\mathcal{A}_4$), and Tech Mahindra ($\mathcal{A}_5$) and the selection is held on the basis of the different criteria, namely, Technology Expertise ($\mathcal{G}_1$), Service quality ($\mathcal{G}_2$), Project Management ($\mathcal{G}_3$), Industry Experience ($\mathcal{G}_4$). Now, we can obtain the evaluation of an alternative $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$ with respect to the criterion $\mathcal{G}_j (j = 1, 2, 3, 4)$ from the questionnaire of a domain expert. For instance, corresponding to alternative $\mathcal{A}_1$ under criterion $\mathcal{G}_1$, when we ask the opinion of an expert about the alternative $\mathcal{A}_1$ with respect to the criterion $\mathcal{G}_1$, he or she may that the "possibility degree in which the statement is good" is 0.5, "the statement is false" is 0.4 and "the degree in which he or she is unsure" is 0.3. In this case, the evaluation of this alternative is represented as SVNN $\alpha_{11} = (0.5, 0.3, 0.4)$. In the similar manner, we can obtain all the evaluations of the alternatives $\mathcal{A}_i$ with respect to criterion $\mathcal{G}_j$. Then, the following steps of the proposed approach have been executed to find the most desirable alternative(s).

Step 1: The complete rating values of all the alternatives are given in the decision matrix $\mathcal{M} = (\alpha_{ij})$ as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \mathcal{A}_4 \\ \mathcal{A}_5 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{G}_1 \quad & \quad\mathcal{G}_2\quad & \quad\mathcal{G}_3\quad & \quad\mathcal{G}_4\quad \end{array} \\ \begin{pmatrix} (0.5, 0.3, 0.4) & (0.5, 0.2, 0.5) & (0.2, 0.2, 0.6) & (0.3, 0.2, 0.4) \\ (0.7, 0.1, 0.3) & (0.7, 0.2, 0.3) & (0.6, 0.3, 0.2) & (0.6, 0.4, 0.2) \\ (0.5, 0.3, 0.4) & (0.6, 0.2, 0.4) & (0.6, 0.1, 0.2) & (0.5, 0.1, 0.3) \\ (0.7, 0.3, 0.2) & (0.7, 0.2, 0.2) & (0.4, 0.5, 0.2) & (0.5, 0.2, 0.2) \\ (0.4, 0.1, 0.3) & (0.5, 0.1, 0.2) & (0.4, 0.1, 0.5) & (0.4, 0.3, 0.6) \end{pmatrix} \end{array}$$

Step 2: Since all the criteria are of the benefit types, so there is no need of normalizing process.

Step 3: Consider the partial information about the criterion weights given by $\Delta = \{0.10 \leq w_1 \leq 0.2, 0.2 \leq w_2 \leq 0.3, 0.2 \leq w_3 \leq 0.25, 0.15 \leq w_4 \leq 0.25, w_j \geq 0, \sum_{j=1}^{4} w_j = 1\}$ and hence the optimization model (7.5) is constructed as follows:

$$\begin{aligned} \text{maximize } \mathcal{D}v(w) &= 0.3344w_1 + 0.2372w_2 + 0.7670w_3 + 0.4519w_4 \\ \text{subject to} \quad & \quad w \in \Delta \end{aligned}$$

By solving this model, we get the optimal weight vector of criteria $w = (0.2, 0.3, 0.25, 0.25)^T$.

Step 4: The RPIS and RNIS is calculated from the given information as $\mathcal{A}^+ = \{(\mathcal{G}_1, (0.7, 0.1, 0.2)), (\mathcal{G}_2, (0.7, 0.1, 0.2)), (\mathcal{G}_3, (0.6, 0.1, 0.2)), (\mathcal{G}_4, (0.6, 0.1, 0.2))\}$ and $\mathcal{A}^- = \{(\mathcal{G}_1, (0.4, 0.3, 0.4)), (\mathcal{G}_2, (0.5, 0.2, 0.5)), (\mathcal{G}_3, (0.2, 0.5, 0.6)), (\mathcal{G}_4, (0.3, 0.4, 0.6))\}$. Thus, the degree of discrimination between the alternative $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$ to $\mathcal{A}^+$ and $\mathcal{A}^-$ are computed by using Eqs. (7.8) and (7.9) and get as

$$
\begin{aligned}
&\mathcal{D}v_w(\mathcal{A}_1|\mathcal{A}^+)\\
= \; & \frac{1}{(\sqrt{2}-1)}\left[\begin{array}{l}
0.2\left\{\sqrt{\frac{0.5^2+0.7^2}{2}}-\frac{0.5+0.7}{2}\right\}+0.3\left\{\sqrt{\frac{0.5^2+0.7^2}{2}}-\frac{0.5+0.7}{2}\right\}\\
+0.25\left\{\sqrt{\frac{0.2^2+0.6^2}{2}}-\frac{0.2+0.6}{2}\right\}+0.25\left\{\sqrt{\frac{0.3^2+0.6^2}{2}}-\frac{0.3+0.6}{2}\right\}\\
+0.2\left\{\sqrt{\frac{0.3^2+0.1^2}{2}}-\frac{0.3+0.1}{2}\right\}+0.3\left\{\sqrt{\frac{0.2^2+0.1^2}{2}}-\frac{0.2+0.1}{2}\right\}\\
+0.25\left\{\sqrt{\frac{0.2^2+0.1^2}{2}}-\frac{0.2+0.1}{2}\right\}+0.25\left\{\sqrt{\frac{0.2^2+0.1^2}{2}}-\frac{0.2+0.1}{2}\right\}\\
+0.2\left\{\sqrt{\frac{0.4^2+0.2^2}{2}}-\frac{0.4+0.2}{2}\right\}+0.3\left\{\sqrt{\frac{0.5^2+0.2^2}{2}}-\frac{0.5+0.2}{2}\right\}\\
+0.25\left\{\sqrt{\frac{0.6^2+0.2^2}{2}}-\frac{0.6+0.2}{2}\right\}+0.25\left\{\sqrt{\frac{0.4^2+0.2^2}{2}}-\frac{0.4+0.2}{2}\right\}
\end{array}\right]\\
= \; & 0.1487
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{D}v_w(\mathcal{A}_1|\mathcal{A}^-)\\
= \; & \frac{1}{(\sqrt{2}-1)}\left[\begin{array}{l}
0.2\left\{\sqrt{\frac{0.5^2+0.4^2}{2}}-\frac{0.5+0.4}{2}\right\}+0.3\left\{\sqrt{\frac{0.5^2+0.5^2}{2}}-\frac{0.5+0.5}{2}\right\}\\
+0.25\left\{\sqrt{\frac{0.2^2+0.2^2}{2}}-\frac{0.2+0.2}{2}\right\}+0.25\left\{\sqrt{\frac{0.3^2+0.3^2}{2}}-\frac{0.3+0.3}{2}\right\}\\
+0.2\left\{\sqrt{\frac{0.3^2+0.3^2}{2}}-\frac{0.3+0.3}{2}\right\}+0.3\left\{\sqrt{\frac{0.2^2+0.2^2}{2}}-\frac{0.2+0.2}{2}\right\}\\
+0.25\left\{\sqrt{\frac{0.2^2+0.5^2}{2}}-\frac{0.2+0.5}{2}\right\}+0.25\left\{\sqrt{\frac{0.2^2+0.4^2}{2}}-\frac{0.2+0.4}{2}\right\}\\
+0.2\left\{\sqrt{\frac{0.4^2+0.4^2}{2}}-\frac{0.4+0.4}{2}\right\}+0.3\left\{\sqrt{\frac{0.5^2+0.5^2}{2}}-\frac{0.5+0.5}{2}\right\}\\
+0.25\left\{\sqrt{\frac{0.6^2+0.6^2}{2}}-\frac{0.6+0.6}{2}\right\}+0.25\left\{\sqrt{\frac{0.4^2+0.6^2}{2}}-\frac{0.4+0.6}{2}\right\}
\end{array}\right]\\
= \; & 0.0357
\end{aligned}
$$

Similarly, we can calculate the others and get $\mathcal{D}v_w(\mathcal{A}_2|\mathcal{A}^+) = 0.0512$; $\mathcal{D}v_w(\mathcal{A}_2|\mathcal{A}^-) = 0.1453$; $\mathcal{D}v_w(\mathcal{A}_3|\mathcal{A}^+) = 0.0466$; $\mathcal{D}v_w(\mathcal{A}_3|\mathcal{A}^-) = 0.1457$; $\mathcal{D}v_w(\mathcal{A}_4|\mathcal{A}^+) = 0.0661$; $\mathcal{D}v_w(\mathcal{A}_4|\mathcal{A}^-) = 0.1298$; $\mathcal{D}v_w(\mathcal{A}_5|\mathcal{A}^+) = 0.0914$ and $\mathcal{D}v_w(\mathcal{A}_5|\mathcal{A}^-) = 0.0933$.

Step 5: The closeness coefficients of i^{th} alternative $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$ is calculated by using Eq. (7.10) and are given as $\mathfrak{R}_1 = 0.1936$; $\mathfrak{R}_2 = 0.7396$; $\mathfrak{R}_3 = 0.7577$;

$\mathfrak{R}_4 = 0.6628$; and $\mathfrak{R}_5 = 0.5052$.

Step 6: Therefore, the optimal ranking of these five alternatives are $\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$, and thus, the best alternative is $\mathcal{A}_3$ namely HCL Tech.

7.4.2 Test Criteria for proposed approach

Wang and Triantaphyllou [160] established the following testing criteria to evaluate the validity of such types of methods.

Test criterion 1: "An effective DM method should not change the indication of the best alternative on replacing a non-optimal alternative by another worse alternative without changing the relative importance of each decision criteria."

Test criterion 2: "An effective DM method should follow transitive property."

Test criterion 3: "When a DM problem is decomposed into smaller problems and same method is applied on smaller problems to rank the alternatives, combined ranking of the alternatives should be identical to the original ranking of un-decomposed problem."

The validity of the proposed TOPSIS method is tested using these test criteria.

Validity test under criterion 1

Under this test criterion, the rating values of the non-optimal alternative $\mathcal{A}_1$ and the worse alternative $\mathcal{A}_5$ is replaced with the another ones and hence their updated decision matrix is summarized as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \mathcal{A}_4 \\ \mathcal{A}_5 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{G}_1 \quad & \quad \mathcal{G}_2 \quad & \quad \mathcal{G}_3 \quad & \quad \mathcal{G}_4 \quad \end{array} \\ \begin{pmatrix} (0.4,0.3,0.5) & (0.5,0.2,0.5) & (0.6,0.2,0.2) & (0.4,0.2,0.3) \\ (0.7,0.1,0.3) & (0.7,0.2,0.3) & (0.6,0.3,0.2) & (0.6,0.4,0.2) \\ (0.5,0.3,0.4) & (0.6,0.2,0.4) & (0.6,0.1,0.2) & (0.5,0.1,0.3) \\ (0.7,0.3,0.2) & (0.7,0.2,0.2) & (0.4,0.5,0.2) & (0.5,0.2,0.2) \\ (0.3,0.1,0.4) & (0.2,0.1,0.5) & (0.5,0.1,0.4) & (0.6,0.3,0.4) \end{pmatrix} \end{array}$$

Based on this information, by applying the proposed approach, we compute the divergence measures of each alternative $\mathcal{A}_i (i = 1, 2, \ldots, 5)$ to $(\mathcal{A}^+)$ and $(\mathcal{A}^-)$ are $\mathcal{D}v_w(\mathcal{A}_1|\mathcal{A}^+) =$

0.0889, $\mathcal{D}v_w(\mathcal{A}_1|\mathcal{A}^-) = 0.0703$, $\mathcal{D}v_w(\mathcal{A}_2|\mathcal{A}^+) = 0.0512$, $\mathcal{D}v_w(\mathcal{A}_2|\mathcal{A}^-) = 0.1307$, $\mathcal{D}v_w(\mathcal{A}_3|\mathcal{A}^+) = 0.0466$, $\mathcal{D}v_w(\mathcal{A}_3|\mathcal{A}^-) = 0.1247$, $\mathcal{D}v_w(\mathcal{A}_4\ |\mathcal{A}^+) = 0.0661$, $\mathcal{D}v_w(\mathcal{A}_4|\mathcal{A}^-) = 0.1337$, $\mathcal{D}v_w(\mathcal{A}_5|\mathcal{A}^+) = 0.1309$ and $\mathcal{D}v_w(\mathcal{A}_5|\mathcal{A}^-) = 0.0650$. Thus, the optimal values of the relative closeness coefficient of each alternative are computed by using Eq. (7.10) as $\mathfrak{R}_1 = 0.4416$; $\mathfrak{R}_2 = 0.7187$; $\mathfrak{R}_3 = 0.7279$ $\mathfrak{R}_4 = 0.6693$ and $\mathfrak{R}_5 = 0.3317$. According to the descending order of these values, the alternatives are ranked as $\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_5$. Thus, the best alternative is remain unchanged i.e., $\mathcal{A}_3$. Hence the proposed method is valid under *test criterion 1.*

Validity test using criterion 2 and 3

Under it, if we decomposed the original problem into a set of $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4\}$, $\{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_5\}$, $\{\mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_5\}$, $\{\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5\}$ and $\{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_4\}$, then the proposed approach has been applied to these subproblems. The ranking orders of the alternatives is $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1$, $\mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$, $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5$, $\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5$ and $\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_1$ respectively and hence the combined ranking order is $\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$, which is identical to original problem. Therefore, it displays transitive property and hence the proposed method is valid under the *test criterion 2 and 3.*

7.4.3 Superiority of the proposed approach over the existing approaches

In this section, we have presented some counterexamples to show that the existing approaches [14, 95, 181] under SVNS environment fail to rank the given alternatives while the proposed divergence measure can overcome their shortcoming.

Example 7.4.1. Consider a DM problem in which there are two alternatives, namely, $\mathcal{A}_1$ and $\mathcal{A}_2$ which are evaluated by a decision maker under the three different criteria denoted by $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$. The preference values of each alternative given by the decision maker are summarized under SVNS environment as follows:

$$\mathcal{K} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1 \quad & \quad\mathcal{G}_2\quad & \quad\mathcal{G}_3\quad \end{array} \\ \begin{bmatrix} (0.5, 0.2, 0.3) & (0.6, 0.3, 0.1) & (0.3, 0.2, 0.4) \\ (0.4, 0.3, 0.2) & (0.6, 0.2, 0.2) & (0.4, 0.3, 0.3) \end{bmatrix} \end{array} \tag{7.11}$$

In order to find the best alternative, we utilize the existing TOPSIS approach as proposed by Biswas et al. [14]. For it, the following steps of their approach are followed as

Step 1: The information of each alternative in SVNS environment is represented in the form of decision matrix as given in Eq. (7.11).

Step 2: The positive and negative ideal solutions are obtained as $\mathcal{A}^+ = \{($ 0.5, 0.2,0.2), (0.6, 0.2,0.1), (0.4, 0.2,0.3)$\}$ and $\mathcal{A}^-$ =$\{$(0.4,0.3, 0.3), (0.6,0.3,0.2), (0.3,0.3,0.4)$\}$, respectively.

Step 3: Based on these values, the hamming distance measures between the alternatives $\mathcal{A}_i (i = 1, 2)$ and ideals solutions are evaluated as $d(\mathcal{A}_1, \mathcal{A}^+) = 0.0444$, $d(\mathcal{A}_1, \mathcal{A}^-) = 0.0444$, $d(\mathcal{A}_2, \mathcal{A}^+) = 0.0444$ and $d(\mathcal{A}_2, \mathcal{A}^-) = 0.0444$.

Step 4: Thus, the optimal values of the relative closeness coefficient $R_i = \frac{d(\mathcal{A}_i,\mathcal{A}^-)}{d(\mathcal{A}_i,\mathcal{A}^+)+d(\mathcal{A}_i,\mathcal{A}^-)}$ of each alternative $\mathcal{A}_i (i = 1, 2)$ are computed as $R_1 = 0.5$ and $R_2 = 0.5$. Hence, this method is unable to rank the given alternatives.

On the other hand, if we utilize the proposed divergence measure to compute the discrimination between the alternatives and the ideal solutions then their measurement values are $\mathcal{D}v(\mathcal{A}_1|\mathcal{A}^+) = 0.0137$, $\mathcal{D}v(\mathcal{A}_2|\mathcal{A}^+) = 0.0167$, $\mathcal{D}v(\mathcal{A}_1|\mathcal{A}^-) = 0.0167$ and $\mathcal{D}v(\mathcal{A}_2|\mathcal{A}^-) = 0.0137$. Thus, the relative closeness coefficient $\mathfrak{R}_i (i = 1, 2)$ of the alternative $\mathcal{A}_i$ $(i = 1, 2)$ is computed by using Eq.(7.10) as $\mathfrak{R}_1 = 0.5500$ and $\mathfrak{R}_2 = 0.4500$. Since $\mathfrak{R}_1 > \mathfrak{R}_2$ which gives us that $\mathcal{A}_1$ is better alternative than $\mathcal{A}_2$. Therefore, proposed divergence based TOPSIS approach is able to rank the alternatives in the situation when existing approach fails.

Example 7.4.2. Consider two SVNSs $\mathcal{A}_1$ and $\mathcal{A}_2$ defined over the universal set $\mathcal{X} = \{x_1, x_2, \ldots, x_5\}$ as

$$\mathcal{A}_1 = \Big\{ (x_1, 0.5, 0.3, 0.2), (x_2, 0.5, 0.2, 0.3), (x_3, 0.9, 0.0, 0.1), (x_4, 0.5, 0.4, 0.1), (x_5, 0.7, 0.1, 0.2) \Big\}$$

and

$$\mathcal{A}_2 = \Big\{ (x_1, 0.7, 0.2, 0.1), (x_2, 0.5, 0.4, 0.1), (x_3, 0.9, 0.1, 0.0), (x_4, 0.6, 0.3, 0.1), (x_5, 0.8, 0.0, 0.2) \Big\}$$

Then the aim of this problem is to classify the unknown pattern $\mathcal{B} \in \text{SVNS}(\mathcal{X})$ which is defined as

$$\mathcal{B} = \left\{ \begin{array}{c} (x_1, 0.7, 0.1, 0.2), (x_2, 0.6, 0.3, 0.1), (x_3, 0.7, \\ 0.1, 0.2), (x_4, 0.5, 0.4, 0.1), (x_5, 0.4, 0.5, 0.1) \end{array} \right\}$$

in one of the class of $\mathcal{A}_1$ and $\mathcal{A}_2$. For it, if we apply the existing normalized Hamming (d_N) and Euclidean (q_N) distances [95] as defined in Eqs. (7.12) and (7.13) respectively, between the alternatives $\mathcal{A}_i (i = 1, 2)$ and the unknown pattern $\mathcal{B}$

$$d_N(\mathcal{A}_i, \mathcal{B}) = \frac{1}{3n} \sum_{j=1}^{n} \left\{ \begin{array}{c} |\zeta_{\mathcal{A}_i}(x_j) - \zeta_{\mathcal{B}}(x_j)| + |\kappa_{\mathcal{A}_i}(x_j) - \kappa_{\mathcal{B}}(x_j)| \\ + |\varphi_{\mathcal{A}_i}(x_j) - \varphi_{\mathcal{B}}(x_j)| \end{array} \right\} \tag{7.12}$$

and

$$q_N(\mathcal{A}_i, \mathcal{B}) = \sqrt{\frac{1}{3n} \sum_{j=1}^{n} \left\{ \begin{array}{c} (\zeta_{\mathcal{A}_i}(x_j) - \zeta_{\mathcal{B}}(x_j))^2 + (\kappa_{\mathcal{A}_i}(x_j) - \kappa_{\mathcal{B}}(x_j))^2 \\ + (\varphi_{\mathcal{A}_i}(x_j) - \varphi_{\mathcal{B}}(x_j))^2 \end{array} \right\}} \tag{7.13}$$

then, we get the measurement values corresponding to the set $\mathcal{A}_1$ and $\mathcal{A}_2$ as $d_N(\mathcal{A}_1, \mathcal{B}) = d_N(\mathcal{A}_2, \mathcal{B}) = 0.1333$ and $q_N(\mathcal{A}_1, \mathcal{B}) = q_N(\mathcal{A}_2, \mathcal{B}) = 0.1751$. Thus, this existing approach is unable to classify the pattern $\mathcal{B}$ with any one of the class of $\mathcal{A}_1$ and $\mathcal{A}_2$. On the other hand, if we apply the proposed divergence measure between the alternative $\mathcal{A}_i (i = 1, 2)$ and $\mathcal{B}$, then we get their values as $\mathcal{D}v(\mathcal{A}_1|\mathcal{B}) = 0.0901$ and $\mathcal{D}v(\mathcal{A}_2|\mathcal{B}) = 0.1061$. Hence, we conclude that unknown pattern $\mathcal{B}$ is most likely to belong to the class of $\mathcal{A}_2$. Therefore, the proposed divergence measure is successfully work in those cases where the existing measure fails.

Example 7.4.3. Consider two SVNSs $\mathcal{A}_1 = \{(x_1, 0.6, 0.1, 0.2), (x_2, 0.5, 0.1, 0.3), (x_3, 0.4, 0.1, 0.1)\}$ and $\mathcal{A}_2 = \{(x_1, 0.5, 0.1, 0.2), (x_2, 0.4, 0.1, 0.3), (x_3, 0.4, 0.1, 0.1)\}$ defined over the universal set $\mathcal{X} = \{x_1, x_2, x_3\}$. Let the ideal solution considered by decision maker is $\mathcal{B} = \{(x_1, 0.6, 0.1, 0.2), (x_2, 0.4, 0.1, 0.3), (x_3, 0.5, 0.1, 0.1)\}$. In order to rank these alternatives, we apply the existing improved cosine similarity measures [181], defined in

Eqs. (7.14) and (7.15), as

$$\mathcal{SG}_1(\mathcal{A}_i,\mathcal{B}) = \frac{1}{n}\sum_{j=1}^{n}\cos\left[\frac{\pi}{2}\max\begin{pmatrix}|\zeta_{\mathcal{A}_i}(x_j)-\zeta_{\mathcal{B}}(x_j)|, |\kappa_{\mathcal{A}_i}(x_j)-\kappa_{\mathcal{B}}(x_j)|,\\ |\varphi_{\mathcal{A}_i}(x_j)-\varphi_{\mathcal{B}}(x_j)|\end{pmatrix}\right] \tag{7.14}$$

and

$$\mathcal{SG}_2(\mathcal{A}_i,\mathcal{B}) = \frac{1}{n}\sum_{j=1}^{n}\cos\left[\frac{\pi}{6}\begin{pmatrix}|\zeta_{\mathcal{A}_i}(x_j)-\zeta_{\mathcal{B}}(x_j)| + |\kappa_{\mathcal{A}_i}(x_j)-\kappa_{\mathcal{B}}(x_j)|\\ + |\varphi_{\mathcal{A}_i}(x_j)-\varphi_{\mathcal{B}}(x_j)|\end{pmatrix}\right] \tag{7.15}$$

and hence obtain their corresponding measurement values are $\mathcal{SG}_1(\mathcal{A}_1,\mathcal{B}) = \mathcal{SG}_1(\mathcal{A}_2,\mathcal{B}) = 0.3292$ and $\mathcal{SG}_2(\mathcal{A}_1,\mathcal{B}) = \mathcal{SG}_2(\mathcal{A}_2,\mathcal{B}) = 0.3315$. Thus, their approach is unable to rank the alternatives. On the other hand, if we compute the proposed divergence measurement values for these two alternatives, then we get their respective values are $\mathcal{D}v(\mathcal{A}_1,\mathcal{B}) = 0.0045$ and $\mathcal{D}v(\mathcal{A}_2,\mathcal{B}) = 0.0041$ and hence conclude that $\mathcal{A}_1$ is the best alternative than $\mathcal{A}_2$.

Hence, we can say that the proposed divergence measures, as well as their corresponding TOPSIS approach, are able to solve the DM problem in a better way under SVNS environment where the existing studies fail to rank the alternatives.

7.5 Conclusion

In this chapter, the theory of SVNS has been enriched by proposing a new information measure, called a divergence measure, to evaluate the discrimination between the two SVNSs. Some properties and the correlations of the measure have been investigated in detail. A maximizing divergence method has been presented to determine the optimal criterion weights under SVN environment. Then, a single-valued neutrosophic TOPSIS is proposed to solve the DM problems. A practical example is given to validate its effectiveness and practicality. From the study, we resolve that the proposed measure shows its superiority in those cases also where the existing measures fail to classify the objects.

Chapter 8

Bi-parametric distance measures on single-valued neutrosophic sets with applications to Pattern Recognition and Medical Diagnosis[1]

In this chapter, we developed some bi-parametric generalized distance measure between the pairs of SVNSs. The presented measures utilized the concept of L_p norm and the degree of uncertainties. The various properties and relationships between them are investigated. Later on, the applicability of these measures is explained with the help of pattern recognition and medical diagnosis problems.

8.1 Introduction

The classical measure theory has been widely used to represent uncertainties in data. However, these measures are valid only for precise data, and hence they may be unable to give accurate judgments for data uncertain and imprecise in nature. To handle it, Smarandache [141] introduced a neutrosophic set (NS), which is the generalization of the IFS and FS. However, without specification, NSs are difficult to apply to real-life problems.

---

[1] The content of this chapter is published as "Some new biparametric distance measures on single-valued neutrosophic sets with applications to pattern recognition and medical diagnosis", *Information*, 8(4), 162, 2017; doi:10.3390/info8040162

Thus, a particular case of the NS called a single-valued NS (SVNS) has been proposed by Smarandache [141], Wang et al. [155].

Since the literature presented in Section 1.1.2 of Chapter 1, it is observed that distance measure plays a significant role during the DM process. Further, it has been deduced from studies that similarity, entropy and divergence measures could be induced by the normalized distance measure on the basis of their axiomatic definitions. On the other hand, SVNSs is one of the most successful theories to handle the uncertainties and certainties in the system, but little systematic research has explored these problems. The gap in the research motivates us to develop some families of the distance measures of the SVNS to solve the DM problem.

The main contributions of this work are summarized as follows: (i) to highlight the shortcomings of the various existing distance measures under the single-valued neutrosophic information through illustrative examples; (ii) to overcome the shortcomings of the existing measures, this paper defines some new series of parametric distance measures between SVNSs, which depend on two parameters, namely, p and t, where p is the L_p norm and t identifies the level of uncertainty. The various desirable relations between these have been investigated in detail. Then, we utilized these measures to solve the problem of pattern recognition as well as a medical diagnosis and compared their performance with that of some of the existing approaches.

8.2 Some New Distance Measures between SVNSs

In this section, we present the Hamming and the Euclidean distances between SVNSs, which can be used in real scientific and engineering applications.

8.2.1 Proposed Distance measures

Letting $\Phi(\mathcal{X})$ be the class of SVNSs over the universal set $\mathcal{X}$, then we define the distances for SVNSs, $\mathcal{A} = (\zeta_{\mathcal{A}}(x_i), \kappa_{\mathcal{A}}(x_i), \varphi_{\mathcal{A}}(x_i) \mid x_i \in \mathcal{X})$ and $\mathcal{B} = (\zeta_{\mathcal{B}}(x_i), \kappa_{\mathcal{B}}(x_i), \varphi_{\mathcal{B}}(x_i) \mid x_i \in \mathcal{X})$, by considering the uncertainty parameter t, as follows:

(i) Hamming distance:

$$\mathcal{D}_1(\mathcal{A},\mathcal{B}) = \frac{1}{3(2+t)}\sum_{i=1}^{n}\left(\begin{array}{l} \big| -t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))\big| \\ +\big| -t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big| \\ +\big| -t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big| \end{array}\right) \tag{8.1}$$

(ii) Normalized Hamming distance:

$$\mathcal{D}_2(\mathcal{A},\mathcal{B}) = \frac{1}{3n(2+t)}\sum_{i=1}^{n}\left(\begin{array}{l} \big| -t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))\big| \\ +\big| -t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big| \\ +\big| -t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big| \end{array}\right) \tag{8.2}$$

(iii) Euclidean distance:

$$\mathcal{D}_3(\mathcal{A},\mathcal{B}) = \left(\frac{1}{3(2+t)^2}\sum_{i=1}^{n}\left(\begin{array}{l} \big| -t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))\big|^2 \\ +\big| -t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big|^2 \\ +\big| -t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big|^2 \end{array}\right)\right)^{1/2} \tag{8.3}$$

(iv) Normalized Euclidean distance:

$$\mathcal{D}_4(\mathcal{A},\mathcal{B}) = \left(\frac{1}{3n(2+t)^2}\sum_{i=1}^{n}\left(\begin{array}{l} \big| -t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))\big|^2 \\ +\big| -t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big|^2 \\ +\big| -t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big|^2 \end{array}\right)\right)^{1/2} \tag{8.4}$$

where $t \geq 3$ is a parameter.

Proposition 8.2.1. The above-defined distance $\mathcal{D}_2(\mathcal{A},\mathcal{B})$, between two SVNSs $\mathcal{A}$ and $\mathcal{B}$, satisfies the following properties:

(P1) $0 \leq \mathcal{D}_2(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{D}_2(\mathcal{A},\mathcal{B}) = 0$ iff $\mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}_2(\mathcal{A},\mathcal{B}) = \mathcal{D}_2(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{D}_2(\mathcal{A},\mathcal{C}) \geq \mathcal{D}_2(\mathcal{A},\mathcal{B})$ and $\mathcal{D}_2(\mathcal{A},\mathcal{C}) \geq \mathcal{D}_2(\mathcal{B},\mathcal{C})$, for $\mathcal{C} \in \Phi(\mathcal{X})$.

Proof. For two SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have

(P1) $0 \leq \zeta_{\mathcal{A}}(x_i), \zeta_{\mathcal{B}}(x_i) \leq 1$, $0 \leq \kappa_{\mathcal{A}}(x_i), \kappa_{\mathcal{B}}(x_i) \leq 1$ and $0 \leq \varphi_{\mathcal{A}}(x_i), \varphi_{\mathcal{B}}(x_i) \leq 1$. Thus, $\mid \zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i) \mid \leq 1$, $\mid \kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i) \mid \leq 1$, $\mid \varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i) \mid \leq 1$ and $\mid t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) \mid \leq t$. Therefore,

$$\begin{aligned}
&\mid (t\zeta_{\mathcal{A}}(x_i) - \varphi_{\mathcal{A}}(x_i) - \kappa_{\mathcal{A}}(x_i)) - (t\zeta_{\mathcal{B}}(x_i) - \varphi_{\mathcal{B}}(x_i) - \kappa_{\mathcal{B}}(x_i)) \mid \leq (2+t) \\
&\mid (t\kappa_{\mathcal{A}}(x_i) + \varphi_{\mathcal{A}}(x_i) - \zeta_{\mathcal{A}}(x_i)) - (t\kappa_{\mathcal{B}}(x_i) + \varphi_{\mathcal{B}}(x_i) - \zeta_{\mathcal{B}}(x_i)) \mid \leq (2+t) \\
&\mid (t\varphi_{\mathcal{A}}(x_i) + \kappa_{\mathcal{A}}(x_i) - \zeta_{\mathcal{A}}(x_i)) - (t\varphi_{\mathcal{B}}(x_i) + \kappa_{\mathcal{B}}(x_i) - \zeta_{\mathcal{B}}(x_i)) \mid \leq (2+t)
\end{aligned}$$

Hence, by the definition of $\mathcal{D}_2$, we obtain $0 \leq \mathcal{D}_2(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) Firstly, we assume that $\mathcal{A} = \mathcal{B}$, which implies that $\zeta_{\mathcal{A}}(x_i) = \zeta_{\mathcal{B}}(x_i)$, $\kappa_{\mathcal{A}}(x_i) = \kappa_{\mathcal{B}}(x_i)$, and $\varphi_{\mathcal{A}}(x_i) = \varphi_{\mathcal{B}}(x_i)$ for $i = 1, 2, .., n$. Thus, by the definition of $\mathcal{D}_2$, we obtain $\mathcal{D}_2(\mathcal{A}, \mathcal{B}) = 0$. Conversely, assuming that $\mathcal{D}_2(\mathcal{A}, \mathcal{B}) = 0$ for two SVNSs $\mathcal{A}$ and $\mathcal{B}$, this implies that

$$\left.\begin{array}{rl}
& |-t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))| \\
+ & |-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| \\
+ & |-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|
\end{array}\right\} = 0$$

or

$$\begin{aligned}
|-t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))| &= 0 \\
|-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| &= 0 \\
|-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| &= 0
\end{aligned}$$

After solving, we obtain $\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i) = 0$, $\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i) = 0$ and $\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i) = 0$, which implies $\zeta_{\mathcal{A}}(x_i) = \zeta_{\mathcal{B}}(x_i)$, $\kappa_{\mathcal{A}}(x_i) = \kappa_{\mathcal{B}}(x_i)$ and $\varphi_{\mathcal{A}}(x_i) = \varphi_{\mathcal{B}}(x_i)$. Therefore, $\mathcal{A} = \mathcal{B}$. Hence $\mathcal{D}_2(\mathcal{A}, \mathcal{B}) = 0$ iff $\mathcal{A} = \mathcal{B}$.

(P3) This is straightforward from the definition of $\mathcal{D}_2$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $\zeta_{\mathcal{A}}(x_i) \leq \zeta_{\mathcal{B}}(x_i) \leq \zeta_{\mathcal{C}}(x_i)$, $\kappa_{\mathcal{A}}(x_i) \geq \kappa_{\mathcal{B}}(x_i) \geq \kappa_{\mathcal{C}}(x_i)$ and $\varphi_{\mathcal{A}}(x_i) \geq \varphi_{\mathcal{B}}(x_i) \geq \varphi_{\mathcal{C}}(x_i)$, which implies that $\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i) \geq \zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{C}}(x_i)$,

$\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i) \leq \varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i)$, and $\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i) \leq \kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i)$. Therefore,

$$\begin{aligned}
&|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))| \\
&\qquad \leq |-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{c}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))| \\
&|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))| \\
&\qquad \leq |-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{C}}(x_i))| \\
\text{and}\quad &|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))| \\
&\qquad \leq |-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{C}}(x_i))|
\end{aligned}$$

By adding, we obtain $\mathcal{D}_2(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_2(\mathcal{A},\mathcal{C})$. Similarly, we obtain $\mathcal{D}_2(\mathcal{B},\mathcal{C}) \leq \mathcal{D}_2(\mathcal{A},\mathcal{C})$.

□

Proposition 8.2.2. Distance $\mathcal{D}_4$ as defined in Eq. (8.4) is also a valid measure.

Proof. For two SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have

(P1) $0 \leq \zeta_{\mathcal{A}}(x_i), \zeta_{\mathcal{B}}(x_i) \leq 1$, $0 \leq \kappa_{\mathcal{A}}(x_i), \kappa_{\mathcal{B}}(x_i) \leq 1$ and $0 \leq \varphi_{\mathcal{A}}(x_i), \varphi_{\mathcal{B}}(x_i) \leq 1$. Thus, $|\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i)| \leq 1$, $|\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)| \leq 1$, $|\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)| \leq 1$ and $|t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))| \leq t$. Therefore,

$$\begin{aligned}
&|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|^2 \leq (2+t)^2 \\
&|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2 \leq (2+t)^2 \\
&|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2 \leq (2+t)^2
\end{aligned}$$

Hence, by the definition of $\mathcal{D}_4$, we obtain $0 \leq \mathcal{D}_4(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) Assuming that $\mathcal{A}=\mathcal{B}$ implies that $\zeta_{\mathcal{A}}(x_i)=\zeta_{\mathcal{B}}(x_i)$, $\kappa_{\mathcal{A}}(x_i)=\kappa_{\mathcal{B}}(x_i)$ and $\varphi_{\mathcal{A}}(x_i)=\varphi_{\mathcal{B}}(x_i)$ for $i=1,2,\ldots,n$, and hence using Eq. (8.4), we obtain $\mathcal{D}_4(\mathcal{A},\mathcal{B})=0$. Conversely, assuming that $\mathcal{D}_4(\mathcal{A},\mathcal{B})=0$ implies

$$\begin{aligned}
&|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|^2=0 \\
&|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2=0 \\
\text{and } &|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2=0
\end{aligned}$$

After solving these, we obtain $\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i)=0$, $\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i)=0$ and $\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i)=0$; that is, $\zeta_{\mathcal{A}}(x_i)=\zeta_{\mathcal{B}}(x_i)$, $\kappa_{\mathcal{A}}(x_i)=\kappa_{\mathcal{B}}(x_i)$ and $\varphi_{\mathcal{A}}(x_i)=\varphi_{\mathcal{B}}(x_i)$ for $t\geq 3$. Hence $\mathcal{A}=\mathcal{B}$. Therefore, $\mathcal{D}_4(\mathcal{A},\mathcal{B})=0$ iff $\mathcal{A}=\mathcal{B}$.

(P3) This is straightforward from the definition of $\mathcal{D}_4$.

(P4) If $\mathcal{A}\subseteq\mathcal{B}\subseteq\mathcal{C}$, then $\zeta_{\mathcal{A}}(x_i)\leq\zeta_{\mathcal{B}}(x_i)\leq\zeta_{\mathcal{C}}(x_i)$, $\kappa_{\mathcal{A}}(x_i)\geq\kappa_{\mathcal{B}}(x_i)\geq\kappa_{\mathcal{C}}(x_i)$, and $\varphi_{\mathcal{A}}(x_i)\geq\varphi_{\mathcal{B}}(x_i)\geq\varphi_{\mathcal{C}}(x_i)$. Therefore

$$\begin{aligned}
&|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|^2\\
&\qquad\leq|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{c}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))|^2\\
&|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2\\
&\qquad\leq|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{C}}(x_i))|^2\\
\text{and}\quad&|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2\\
&\qquad\leq|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{C}}(x_i))|^2
\end{aligned}$$

Hence by the definition of $\mathcal{D}_4$, we obtain $\mathcal{D}_4(\mathcal{A},\mathcal{B})\leq\mathcal{D}_4(\mathcal{A},\mathcal{C})$. Similarly, we obtain $\mathcal{D}_4(\mathcal{B},\mathcal{C})\leq\mathcal{D}_4(\mathcal{A},\mathcal{C})$.

□

However, in many practical situations, the different sets may have taken different weights, and thus weight $\omega_i(i=1,2,\ldots,n)$ of the element $x_i\in\mathcal{X}$ should be taken into account. In the following, we develop a weighted Hamming distance and the normalized weighted Euclidean distance between SVNSs.

(i) The normalized weighted Hamming distance:

$$\mathcal{D}_5(\mathcal{A},\mathcal{B})=\frac{1}{3n(2+t)}\sum_{i=1}^{n}\omega_i\left(\begin{array}{l}
|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|\\
+|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|\\
+|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|
\end{array}\right) \tag{8.5}$$

(ii) The normalized weighted Euclidean distance:

$$\mathcal{D}_6(\mathcal{A},\mathcal{B})=\left\{\frac{1}{3n(2+t)^2}\sum_{i=1}^{n}\omega_i\left(\begin{array}{l}
|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|^2\\
+|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2\\
+|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2
\end{array}\right)\right\}^{1/2} \tag{8.6}$$

where $t \geq 3$ is a parameter.

It is straightforward to check that the normalized weighted distance $\mathcal{D}_k(\mathcal{A}, \mathcal{B})(k = 5, 6)$ between SVNSs $\mathcal{A}$ and $\mathcal{B}$ also satisfies the above properties (P1)–(P4).

8.2.2 Advantages of the proposed measures

This section addressed that the existing measures defined in Section 2.2 of Chapter 2 have some shortcoming which is explained with some numerical examples as below:

Example 8.2.1. Consider two known patterns $\mathcal{A}$ and $\mathcal{B}$, which are represented by SVNSs in a universe $\mathcal{X}$ given by $\mathcal{A} = (x, 0.5, 0.0, 0.0 \mid x \in \mathcal{X})$, $\mathcal{B} = (x, 0.0, 0.5, 0.0 \mid x \in \mathcal{X})$. Consider an unknown pattern $\mathcal{C} \in \Phi(\mathcal{X})$, which is recognized where $\mathcal{C} = (x, 0.0, 0.0, 0.5 \mid x \in \mathcal{X})$; then the target of this problem is to classify the pattern $\mathcal{C}$ in one of the classes $\mathcal{A}$ or $\mathcal{B}$. If we apply the existing measures [17, 95, 181, 187] defined in Eqs. (2.10), (2.11), (2.12), (2.17), (2.18), (2.19) and (2.20) of Chapter 2, then we obtain

Pair	$\mathcal{D}_H$	$\mathcal{D}_{NH}$	$\mathcal{D}_{NE}$	$\mathcal{D}_{CS1}$	$\mathcal{D}_{CS2}$	$\mathcal{D}_{T1}$	$\mathcal{D}_{T1}$
$(\mathcal{A}, \mathcal{C})$	0.5	0.3333	0.4048	0.2929	0.1340	0.4142	0.2679
$(\mathcal{B}, \mathcal{C})$	0.5	0.3333	0.4048	0.2929	0.1340	0.4142	0.2679

Thus, from this, we conclude that these existing measures are unable to classify the pattern $\mathcal{C}$ with $\mathcal{A}$ and $\mathcal{B}$. Hence these measures are inconsistent and unable to perform ranking.

On the other hand, if we apply the proposed distance measures $\mathcal{D}_2$ and $\mathcal{D}_4$ on the data considered in example 8.2.1 to classify the pattern $\mathcal{C}$, then corresponding to the parameter $t = 3$, we obtain $\mathcal{D}_2(\mathcal{A}, \mathcal{C}) = 0.3333$, $\mathcal{D}_2(\mathcal{B}, \mathcal{C}) = 0.1333$, $\mathcal{D}_4(\mathcal{A}, \mathcal{C}) = 0.3464$ and $\mathcal{D}_4(\mathcal{B}, \mathcal{C}) = 0.1633$. Thus, the pattern $\mathcal{C}$ is classified with the pattern $\mathcal{B}$ and hence is able to identify the best pattern.

Example 8.2.2. Consider two SVNSs defined on the universal set $\mathcal{X}$ given by $\mathcal{A} = (x, 0.3, 0.2, 0.3 \mid x \in \mathcal{X})$ and $\mathcal{B} = (x, 0.4, 0.2, 0.4 \mid x \in \mathcal{X})$. If we replace the degree of falsity membership of $\mathcal{A}$ (0.3) with 0.4, and that of $\mathcal{B}$ (0.4) with 0.3, then we obtain new SVNSs as $\mathcal{P} = (x, 0.3, 0.2, 0.4 \mid x \in \mathcal{X})$ and $\mathcal{Q} = (x, 0.4, 0.2, 0.3 \mid x \in \mathcal{X})$. Now, by using the distance measures defined in Eqs. (2.10), (2.11), (2.12), (2.17), (2.18), (2.19) and (2.20) of Chapter 2, we obtain their corresponding values as follows:

Pair	$\mathcal{D}_H$	$\mathcal{D}_{NH}$	$\mathcal{D}_{NE}$	$\mathcal{D}_{CS1}$	$\mathcal{D}_{CS2}$	$\mathcal{D}_{T1}$	$\mathcal{D}_{T1}$
$(\mathcal{A},\mathcal{B})$	0.1	0.066	0.077	0.013	0.006	0.078	0.052
$(\mathcal{P},\mathcal{Q})$	0.1	0.066	0.077	0.013	0.006	0.078	0.052

Thus, it has been concluded that by changing the falsity degree of SVNSs and keeping the other degrees unchanged, the values of their corresponding measures remain the same. Thus, there is no effect of the degree of falsity membership on the distance measures. Similarly, we can observe the same for the degree of truth membership functions.

On the other hand, if we utilize the proposed distances $\mathcal{D}_2$ and $\mathcal{D}_4$ for the above-considered example 8.2.2, then their corresponding values are $\mathcal{D}_2(\mathcal{A},\mathcal{B}) = 0.0267$, $\mathcal{D}_2(\mathcal{P},\mathcal{Q}) = 0.0667$, $\mathcal{D}_4(\mathcal{A},\mathcal{B}) = 0.0327$ and $\mathcal{D}_4(\mathcal{P},\mathcal{Q}) = 0.6930$. Therefore, there is a significant effect of the change in the falsity membership on the measure values and hence consequently on the ranking values.

8.2.3 Relation between the proposed measures

In this section, we derive the various inequalities between the developed measures.

Proposition 8.2.3. Measures $\mathcal{D}_1$ and $\mathcal{D}_3$ satisfy the following properties:

(i) $0 \leq \mathcal{D}_1 \leq n$;

(ii) $0 \leq \mathcal{D}_3 \leq n^{1/2}$.

Proof. We can easily obtain that $\mathcal{D}_1(\mathcal{A},\mathcal{B}) = n\mathcal{D}_2(\mathcal{A},\mathcal{B})$, and thus by Proposition 8.2.1, we obtain $0 \leq \mathcal{D}_1(\mathcal{A},\mathcal{B}) \leq n$. Similarly, we can obtain $0 \leq \mathcal{D}_3(\mathcal{A},\mathcal{B}) \leq n^{1/2}$. □

Proposition 8.2.4. Distance measures $\mathcal{D}_2$ and $\mathcal{D}_5$ satisfy the relation $\mathcal{D}_5 \leq \mathcal{D}_2$.

Proof. Because $\omega_i \geq 0$, $\sum\limits_{i=1}^{n}\omega_i = 1$, then for any two SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{D}_5(\mathcal{A},\mathcal{B}) = \frac{1}{3n(2+t)}\sum\limits_{i=1}^{n}\omega_i\Big\{\Big(|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|+|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|+|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|\Big)\Big\} \leq \frac{1}{3n(2+t)}\sum\limits_{i=1}^{n}\Big(|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|+|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|+|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|\Big)$; that is, $\mathcal{D}_5(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_2(\mathcal{A},\mathcal{B})$. □

Proposition 8.2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two SVNSs in $\mathcal{X}$; then $\mathcal{D}_5$ and $\mathcal{D}_6$ are the distance measures.

Proof. Because $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$ then we can easily obtain $0 \leq \mathcal{D}_5(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_2(\mathcal{A},\mathcal{B})$. Thus, $\mathcal{D}_5(\mathcal{A},\mathcal{B})$ satisfies (P1). The proofs of (P2)–(P4) are similar to those of Proposition 8.2.1. Similar is true for $\mathcal{D}_6$. ☐

Proposition 8.2.6. The distance measures $\mathcal{D}_4$ and $\mathcal{D}_6$ satisfy the relation $\mathcal{D}_6 \leq \mathcal{D}_4$.

Proof. The proof follows from Proposition 8.2.4. ☐

Proposition 8.2.7. The distance measures $\mathcal{D}_2$ and $\mathcal{D}_4$ satisfy the inequality $\mathcal{D}_4 \leq \sqrt{\mathcal{D}_2}$.

Proof. For two SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have

$$|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|^2 \leq (2+t)^2$$
$$|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2 \leq (2+t)^2$$
$$|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2 \leq (2+t)^2$$

which implies that

$$\left|\frac{-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))}{2+t}\right|^2 \leq 1$$
$$\left|\frac{-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))}{2+t}\right|^2 \leq 1$$
$$\left|\frac{-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))}{2+t}\right|^2 \leq 1$$

For any $a \in [0,1]$, we have $a^2 \leq a$. Therefore,

$$\left|\frac{-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))}{2+t}\right|^2 \leq \left|\frac{-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))}{2+t}\right|$$
$$\left|\frac{-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))}{2+t}\right|^2 \leq \left|\frac{-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))}{2+t}\right|$$

and

$$\left|\frac{-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))}{2+t}\right|^2 \leq \left|\frac{-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))}{2+t}\right|$$

By adding these inequalities and by the definition of $\mathcal{D}_4$, we have

$$\mathcal{D}_4(\mathcal{A},\mathcal{B}) =$$

$$\left[\frac{1}{3n(2+t)^2}\sum_{i=1}^{n}\begin{pmatrix} \big| -t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))\big|^2 \\ +\big| -t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big|^2 \\ +\big| -t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big|^2 \end{pmatrix}\right]^{1/2}$$

$$\leq \left[\frac{1}{3n(2+t)}\sum_{i=1}^{n}\begin{pmatrix} \big| -t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))\big| \\ +\big| -t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big| \\ +\big| -t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\big| \end{pmatrix}\right]^{1/2}$$

$$\leq (\mathcal{D}_2(\mathcal{A},\mathcal{B}))^{1/2}$$

As $\mathcal{A}$ and $\mathcal{B}$ are arbitrary SVNSs, thus we obtain $\mathcal{D}_4 \leq \sqrt{\mathcal{D}_2}$. ☐

Proposition 8.2.8. Measures $\mathcal{D}_6$ and $\mathcal{D}_5$ satisfy the inequality $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_5}$.

Proof. The proof follows from Proposition 8.2.7. ☐

The Hausdroff distance between two non-empty closed and bounded sets is a measure of the resemblance between them. For example, we consider $\mathcal{A} = [x_1, x_2]$ and $\mathcal{B} = [y_1, y_2]$ in the Euclidean domain R; the Hausdroff distance in the additive set environment is given by the following [48]:

$$H(\mathcal{A},\mathcal{B}) = \max\big\{ \mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid \big\}$$

Now, for any two SVNSs $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, we propose the following utmost distance measures:

- Utmost normalized Hamming distance:

$$\mathcal{D}_1^H(\mathcal{A},\mathcal{B}) = \frac{1}{3n(2+t)}\sum_{i=1}^{n}\max_i\begin{pmatrix} \mid -t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))\mid, \\ \mid -t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\mid, \\ \mid -t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\mid \end{pmatrix} \tag{8.7}$$

- Utmost normalized weighted Hamming distance:

$$\mathcal{D}_2^H(\mathcal{A},\mathcal{B}) = \frac{1}{3n(2+t)}\sum_{i=1}^{n}\omega_i\max_i\begin{pmatrix} \mid -t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))\mid, \\ \mid -t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\mid, \\ \mid -t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))\mid \end{pmatrix} \tag{8.8}$$

- Utmost normalized Euclidean distance:

$$\mathcal{D}_3^H(\mathcal{A},\mathcal{B}) = \left\{ \frac{1}{3n(2+t)^2} \sum_{i=1}^{n} \max_i \begin{pmatrix} |-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|^2, \\ |-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2, \\ |-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2 \end{pmatrix} \right\}^{1/2} \tag{8.9}$$

- Utmost normalized weighted Euclidean distance:

$$\mathcal{D}_4^H(\mathcal{A},\mathcal{B}) = \left\{ \frac{1}{3n(2+t)^2} \sum_{i=1}^{n} \omega_i \max_i \begin{pmatrix} |-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|^2, \\ |-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2, \\ |-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))|^2 \end{pmatrix} \right\}^{1/2} \tag{8.10}$$

Proposition 8.2.9. The distance $\mathcal{D}_1^H(\mathcal{A},\mathcal{B})$ defined in Eq. (8.7) for two SVNSs $\mathcal{A}$ and $\mathcal{B}$ is a valid distance measure.

Proof. The above measure satisfies the following properties:

(P1) As $\mathcal{A}$ and $\mathcal{B}$ are SVNSs, so $\mid \zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i) \mid \leq 1$, $\mid \kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i) \mid \leq 1$ and $\mid \varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i) \mid \leq 1$. Thus,

$$\begin{aligned} &\mid (t\zeta_{\mathcal{A}}(x_i) - \varphi_{\mathcal{A}}(x_i) - \kappa_{\mathcal{A}}(x_i)) - (t\zeta_{\mathcal{B}}(x_i) - \varphi_{\mathcal{B}}(x_i) - \kappa_{\mathcal{B}}(x_i)) \mid \leq (2+t) \\ &\mid (t\kappa_{\mathcal{A}}(x_i) + \varphi_{\mathcal{A}}(x_i) - \zeta_{\mathcal{A}}(x_i)) - (t\kappa_{\mathcal{B}}(x_i) + \varphi_{\mathcal{B}}(x_i) - \zeta_{\mathcal{B}}(x_i)) \mid \leq (2+t) \\ &\mid (t\varphi_{\mathcal{A}}(x_i) + \kappa_{\mathcal{A}}(x_i) - \zeta_{\mathcal{A}}(x_i)) - (t\varphi_{\mathcal{B}}(x_i) + \kappa_{\mathcal{B}}(x_i) - \zeta_{\mathcal{B}}(x_i)) \mid \leq (2+t) \end{aligned}$$

Hence, by the definition of $\mathcal{D}_1^H$, we obtain $0 \leq \mathcal{D}_1^H(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) Similar to the proof of Proposition 8.2.1.

(P3) This is clear from Eq. (8.7).

(P4) Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, which implies $\zeta_{\mathcal{A}}(x_i) \leq \zeta_{\mathcal{B}}(x_i) \leq \zeta_{\mathcal{C}}(x_i)$, $\kappa_{\mathcal{A}}(x_i) \geq \kappa_{\mathcal{B}}(x_i) \geq \kappa_{\mathcal{C}}(x_i)$ and $\varphi_{\mathcal{A}}(x_i) \geq \varphi_{\mathcal{B}}(x_i) \geq \varphi_{\mathcal{C}}(x_i)$. Therefore, $|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))| \leq |-t(\zeta_{\mathcal{A}}(x_i)-\zeta_c(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))|$, $|-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))| \leq |-t(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))-(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{C}}(x_i))|$ and $|-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))| \leq |-t(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{C}}(x_i))-(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{C}}(x_i))+(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{C}}(x_i))|$, which implies that $\max_i \big(|-t(\zeta_{\mathcal{A}}(x_i)-\zeta_{\mathcal{B}}(x_i))+(\kappa_{\mathcal{A}}(x_i)-\kappa_{\mathcal{B}}(x_i))+(\varphi_{\mathcal{A}}(x_i)-\varphi_{\mathcal{B}}(x_i))|, |-$

$t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|, |-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|\big) \leq \max_i \big(|-t(\zeta_{\mathcal{A}}(x_i) - \zeta_c(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{C}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{C}}(x_i))|, |-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{C}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{C}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{C}}(x_i))|$ and $|-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{C}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{C}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{C}}(x_i))|\big)$. Hence $\mathcal{D}_1^H(\mathcal{A}, \mathcal{B}) \leq \mathcal{D}_1^H(\mathcal{A}, \mathcal{C})$. Similarly, we obtain $\mathcal{D}_1^H(\mathcal{B}, \mathcal{C}) \leq \mathcal{D}_1^H(\mathcal{A}, \mathcal{C})$.

□

Proposition 8.2.10. For $\mathcal{A}, \mathcal{B} \in \Phi(\mathcal{X})$, $\mathcal{D}_2^H$, $\mathcal{D}_3^H$ and $\mathcal{D}_4^H$ are the distance measures.

Proof. The proof follows from the above Proposition. □

Proposition 8.2.11. The measures $\mathcal{D}_2^H$ and $\mathcal{D}_1^H$ satisfy the following inequality: $\mathcal{D}_2^H \leq \mathcal{D}_1^H$.

Proof. Because $\omega_i \in [0, 1]$, therefore

$$\begin{aligned}
& \mathcal{D}_2^H(\mathcal{A}, \mathcal{B}) \\
= \; & \frac{1}{3n(2+t)} \sum_{i=1}^{n} \omega_i \left(\begin{array}{l} \max\limits_i \Big(|-t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))|, \\ |-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|, \\ |-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| \Big) \end{array} \right) \\
\leq \; & \frac{1}{3n(2+t)} \sum_{i=1}^{n} \max_i \left(\begin{array}{l} |-t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))|, \\ |-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|, \\ |-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| \end{array} \right) \\
= \; & \mathcal{D}_1^H(\mathcal{A}, \mathcal{B})
\end{aligned}$$

Hence, $\mathcal{D}_2^H \leq \mathcal{D}_1^H$. □

Proposition 8.2.12. The measures $\mathcal{D}_3^H$ and $\mathcal{D}_4^H$ satisfy the inequality $\mathcal{D}_4^H(\mathcal{A}, \mathcal{B}) \leq \mathcal{D}_3^H(\mathcal{A}, \mathcal{B})$.

Proof. The proof follows from Proposition 8.2.11. □

Proposition 8.2.13. The measures $\mathcal{D}_3^H$ and $\mathcal{D}_1^H$ satisfy the inequality $\mathcal{D}_3^H \leq \sqrt{\mathcal{D}_1^H}$.

Proof. Because for any $a \in [0, 1]$, $a^2 \leq a \leq a^{1/2}$, the remaining proof follows from Proposition 8.2.7. □

Proposition 8.2.14. The measures $\mathcal{D}_4^H$ and $\mathcal{D}_2^H$ satisfy the inequality $\mathcal{D}_4^H \leq \sqrt{\mathcal{D}_2^H}$.

Proof. The proof follows from Proposition 8.2.13. $\square$

Proposition 8.2.15. The measures $\mathcal{D}_1^H$ and $\mathcal{D}_2$ satisfy the following inequality:

$$\mathcal{D}_1^H \leq \mathcal{D}_2.$$

Proof. For positive numbers a_i, $i = 1, 2, ..., n$, we have $\max\limits_i\{a_i\} \leq \sum\limits_{i=1}^{n} a_i$. Thus, for any two SVNSs $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{D}_1^H(\mathcal{A},\mathcal{B}) = \frac{1}{3n(2+t)} \sum\limits_{i=1}^{n} \max_i \Big(| - t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))|, | - t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|, | - t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| \Big) \leq \frac{1}{3n(2+t)} \sum\limits_{i=1}^{n} | - t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + | - t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| + | - t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| = \mathcal{D}_2(\mathcal{A},\mathcal{B})$. Hence $\mathcal{D}_1^H \leq \mathcal{D}_2$. $\square$

Proposition 8.2.16. The measures $\mathcal{D}_3^H$ and $\mathcal{D}_4$ satisfy the following inequality:

$$\mathcal{D}_3^H \leq \mathcal{D}_4$$

Proof. The proof follows from Proposition 8.2.15. $\square$

Proposition 8.2.17. The measures $\mathcal{D}_2$, $\mathcal{D}_5$ and $\mathcal{D}_1^H$ satisfy the following inequalities:

(i) $\mathcal{D}_2 \geq \frac{\mathcal{D}_5 + \mathcal{D}_1^H}{2}$.

(ii) $\mathcal{D}_2 \geq \sqrt{\mathcal{D}_5 \cdot \mathcal{D}_1^H}$.

Proof. Because $\mathcal{D}_2 \geq \mathcal{D}_5$ and $\mathcal{D}_2 \geq \mathcal{D}_1^H$, by adding these inequalities, we obtain $\mathcal{D}_2 \geq \frac{\mathcal{D}_5 + \mathcal{D}_1^H}{2}$. On the other hand, by multiplying these, we obtain $\mathcal{D}_2 \geq \sqrt{\mathcal{D}_5 \cdot \mathcal{D}_1^H}$. $\square$

8.3 Generalized Distance Measure

The above-defined Hamming and Euclidean distance measures are generalized for the two SVNSs $\mathcal{A}$ and $\mathcal{B}$ on the universal set $\mathcal{X}$ as follows:

$$\mathcal{D}^p(\mathcal{A},\mathcal{B}) = \left\{ \frac{1}{3n(2+t)^p} \sum_{i=1}^{n} \begin{pmatrix} | - t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))|^p \\ + | - t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|^p \\ + | - t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|^p \end{pmatrix} \right\}^{1/p} \tag{8.11}$$

where $p \geq 1$ is an L_p norm and $t \geq 3$ represents the uncertainty index parameters.

In particular, if $p = 1$ and $p = 2$, then the above measure, given in Eq. (8.11), reduces to measures $\mathcal{D}_2$ and $\mathcal{D}_4$ defined in Eqs. (8.2) and (8.4), respectively.

Proposition 8.3.1. The above-defined distance $\mathcal{D}^p(\mathcal{A}, \mathcal{B})$, between SVNSs $\mathcal{A}$ and $\mathcal{B}$, satisfies the following properties (P1)–(P4):

(P1) $0 \leq \mathcal{D}^p(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{D}^p(\mathcal{A}, \mathcal{B}) = 0$, iff $\mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}^p(\mathcal{A}, \mathcal{B}) = \mathcal{D}^p(\mathcal{B}, \mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{D}^p(\mathcal{A}, \mathcal{C}) \geq \mathcal{D}^p(\mathcal{A}, \mathcal{B})$ and $\mathcal{D}^p(\mathcal{A}, \mathcal{C}) \geq \mathcal{D}^p(\mathcal{B}, \mathcal{C})$.

Proof. For $p \geq 1$ and $t \geq 3$, we have the following:

(P1) For SVNSs, $\mid \zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i) \mid \leq 1$, $\mid \kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i) \mid \leq 1$ and $\mid \varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i) \mid \leq 1$. Thus, we obtain

$$
\begin{aligned}
-(2+t) &\leq t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) \leq (2+t) \\
-(2+t) &\leq -t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) \leq (2+t) \\
-(2+t) &\leq -t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)) - \zeta_{\mathcal{B}}(x_i) \leq (2+t)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
0 &\leq \left| t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) \right|^p \leq (2+t)^p \\
0 &\leq \left| -t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) \right|^p \leq (2+t)^p \\
0 &\leq \left| -t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i)) - \zeta_{\mathcal{B}}(x_i) \right|^p \leq (2+t)^p
\end{aligned}
$$

Thus, by adding these inequalities, we obtain $0 \leq \mathcal{D}^p(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) Assuming that $\mathcal{A} = \mathcal{B} \Leftrightarrow \zeta_{\mathcal{A}}(x) = \zeta_{\mathcal{B}}(x_i)$, $\kappa_{\mathcal{A}}(x_i) = \kappa_{\mathcal{B}}(x_i)$, and $\varphi_{\mathcal{A}}(x) = \varphi_{\mathcal{B}}(x_i)$, thus, $\mathcal{D}^p(\mathcal{A}, \mathcal{B}) = 0$.

Conversely, assuming that $\mathcal{D}^p(\mathcal{A},\mathcal{B}) = 0$ implies that

$$\begin{aligned}
&|-t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))| = 0\\
&|-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| = 0\\
&|-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))| = 0
\end{aligned}$$

and hence, after solving, we obtain $\zeta_{\mathcal{A}}(x_i) = \zeta_{\mathcal{B}}(x_i)$, $\kappa_{\mathcal{A}}(x_i) = \kappa_{\mathcal{B}}(x_i)$ and $\varphi_{\mathcal{A}}(x_i) = \varphi_{\mathcal{B}}(x_i)$. Thus, $\mathcal{A} = \mathcal{B}$.

(P3) This is straightforward.

(P4) Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$; then $\zeta_{\mathcal{A}}(x_i) \leq \zeta_{\mathcal{B}}(x_i) \leq \zeta_{\mathcal{C}}(x_i)$, $\kappa_{\mathcal{A}}(x_i) \geq \kappa_{\mathcal{B}}(x_i) \geq \kappa_{\mathcal{C}}(x_i)$ and $\varphi_{\mathcal{A}}(x_i) \geq \varphi_{\mathcal{B}}(x_i) \geq \varphi_{\mathcal{C}}(x_i)$. Thus, $\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i) \geq \zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{C}}(x_i)$, $\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i) \leq \kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{C}}(x_i)$ and $\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i) \leq \varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{C}}(x_i)$. Hence, we obtain

$$\begin{aligned}
&|-t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))|^p\\
&\qquad \leq |-t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{C}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{C}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{C}}(x_i))|^p\\
&|-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|^p\\
&\qquad \leq |-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{C}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{C}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{C}}(x_i))|^p\\
\text{and}\quad &|-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|^p\\
&\qquad \leq |-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{C}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{C}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{C}}(x_i))|^p
\end{aligned}$$

Thus, we obtain $\mathcal{D}^p(\mathcal{A},\mathcal{B}) \leq \mathcal{D}^p(\mathcal{A},\mathcal{C})$. Similarly, $\mathcal{D}^p(\mathcal{B},\mathcal{C}) \leq \mathcal{D}^p(\mathcal{A},\mathcal{C})$.

□

If the weight vector $\omega_i, (i = 1, 2, \ldots, n)$ of each element is considered such that $\omega_i \in [0,1]$ and $\sum\limits_i \omega_i = 1$, then a generalized parametric distance measure between SVNSs $\mathcal{A}$ and $\mathcal{B}$ takes the following form:

$$\mathcal{D}^p(\mathcal{A},\mathcal{B}) = \left\{ \frac{1}{3(2+t)^p} \sum_{i=1}^{n} \omega_i \begin{pmatrix} \Big(|-t(\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i)) + (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i))|^p \\ + |-t(\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) - (\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|^p \\ + |-t(\varphi_{\mathcal{A}}(x_i) - \varphi_{\mathcal{B}}(x_i)) - (\kappa_{\mathcal{A}}(x_i) - \kappa_{\mathcal{B}}(x_i)) + (\zeta_{\mathcal{A}}(x_i) - \zeta_{\mathcal{B}}(x_i))|^p\Big) \end{pmatrix} \right\}^{1/p} \tag{8.12}$$

In particular, if $p = 1$ and $p = 2$, Eq. (8.12) reduces to Eqs. (8.5) and (8.6), respectively.

Proposition 8.3.2. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $x_i, (i = 1, 2, \ldots, n)$ with $\omega_i \geq 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$; then the generalized parametric distance measure between the SVNSs $\mathcal{A}$ and $\mathcal{B}$ defined by Eq. (8.12) satisfies the following:

(P1) $0 \leq \mathcal{D}^p(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{D}^p(\mathcal{A}, \mathcal{B}) = 0$ iff $\mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}^p(\mathcal{A}, \mathcal{B}) = \mathcal{D}^p(\mathcal{B}, \mathcal{A})$;

(P4) $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}^p(\mathcal{A}, \mathcal{C}) \geq \mathcal{D}^p(\mathcal{A}, \mathcal{B})$ and $\mathcal{D}^p(\mathcal{A}, \mathcal{C}) \geq \mathcal{D}^p(\mathcal{B}, \mathcal{C})$.

Proof. The proof follows from Proposition 8.3.1. □

8.4 Illustrative examples

In order to illustrate the performance and validity of the above-proposed distance measures, two examples from the fields of pattern recognition and medical diagnosis have been taken into account.

8.4.1 Application of Distance Measure in Pattern Recognition

Consider three known patterns $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$, which are represented by the following SVNSs in a given universe $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$:

$$\mathcal{A}_1 = \{(x_1, 0.7, 0.0, 0.1), (x_2, 0.6, 0.1, 0.2), (x_3, 0.8, 0.7, 0.6), (x_4, 0.5, 0.2, 0.3)\}$$
$$\mathcal{A}_2 = \{(x_1, 0.4, 0.2, 0.3), (x_2, 0.7, 0.1, 0.0), (x_3, 0.1, 0.1, 0.6), (x_4, 0.5, 0.3, 0.6)\}$$
$$\mathcal{A}_3 = \{(x_1, 0.5, 0.2, 0.2), (x_2, 0.4, 0.1, 0.2), (x_3, 0.1, 0.1, 0.4), (x_4, 0.4, 0.1, 0.2)\}$$

Consider an unknown pattern $\mathcal{B} \in \Phi(\mathcal{X})$, which will be recognized where

$$\mathcal{B} = \{(x_1, 0.4, 0.1, 0.4), (x_2, 0.6, 0.1, 0.1), (x_3, 0.1, 0.0, 0.4), (x_4, 0.4, 0.4, 0.7)\}$$

Then, the target of this problem is to classify the pattern $\mathcal{B}$ in one of the classes $\mathcal{A}_1$, $\mathcal{A}_2$ or $\mathcal{A}_3$. For this, proposed distance measures, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_1^H$ and $\mathcal{D}_3^H$, have been computed from $\mathcal{B}$ to $\mathcal{A}_k (k = 1, 2, 3)$ corresponding to $t = 3$, and the results are given as follows:

$$\begin{aligned}
\mathcal{D}_1(\mathcal{A}_1, \mathcal{B}) &= 0.5600; \quad \mathcal{D}_1(\mathcal{A}_2, \mathcal{B}) = 0.2932; \quad \mathcal{D}_1(\mathcal{A}_3, \mathcal{B}) = 0.4668\\
\mathcal{D}_2(\mathcal{A}_1, \mathcal{B}) &= 0.1400; \quad \mathcal{D}_2(\mathcal{A}_2, \mathcal{B}) = 0.0733; \quad \mathcal{D}_2(\mathcal{A}_3, \mathcal{B}) = 0.1167\\
\mathcal{D}_3(\mathcal{A}_1, \mathcal{B}) &= 0.3499; \quad \mathcal{D}_3(\mathcal{A}_2, \mathcal{B}) = 0.1641; \quad \mathcal{D}_3(\mathcal{A}_3, \mathcal{B}) = 0.3120\\
\mathcal{D}_4(\mathcal{A}_1, \mathcal{B}) &= 0.1749; \quad \mathcal{D}_4(\mathcal{A}_2, \mathcal{B}) = 0.0821; \quad \mathcal{D}_4(\mathcal{A}_3, \mathcal{B}) = 0.1560\\
\mathcal{D}_1^H(\mathcal{A}_1, \mathcal{B}) &= 0.0633; \quad \mathcal{D}_1^H(\mathcal{A}_2, \mathcal{B}) = 0.0300; \quad \mathcal{D}_1^H(\mathcal{A}_3, \mathcal{B}) = 0.0567\\
\mathcal{D}_3^H(\mathcal{A}_1, \mathcal{B}) &= 0.1252; \quad \mathcal{D}_3^H(\mathcal{A}_2, \mathcal{B}) = 0.0560; \quad \mathcal{D}_3^H(\mathcal{A}_3, \mathcal{B}) = 0.1180
\end{aligned}$$

Thus, from these distance measures, we conclude that the pattern $\mathcal{B}$ belongs to the pattern $\mathcal{A}_2$. On the other hand, if we assume that the weights of x_1, x_2, x_3 and x_4 are 0.3, 0.4, 0.2 and 0.1, respectively, then we utilize the distance measures $\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_2^H$ and $\mathcal{D}_4^H$ for obtaining the most suitable pattern as follows:

$$\begin{aligned}
\mathcal{D}_5(\mathcal{A}_1, \mathcal{B}) &= 0.0338; \quad \mathcal{D}_5(\mathcal{A}_2, \mathcal{B}) = 0.0162; \quad \mathcal{D}_5(\mathcal{A}_3, \mathcal{B}) = 0.0233\\
\mathcal{D}_6(\mathcal{A}_1, \mathcal{B}) &= 0.0861; \quad \mathcal{D}_6(\mathcal{A}_2, \mathcal{B}) = 0.0369; \quad \mathcal{D}_6(\mathcal{A}_3, \mathcal{B}) = 0.0604\\
\mathcal{D}_2^H(\mathcal{A}_1, \mathcal{B}) &= 0.0148; \quad \mathcal{D}_2^H(\mathcal{A}_2, \mathcal{B}) = 0.0068; \quad \mathcal{D}_2^H(\mathcal{A}_3, \mathcal{B}) = 0.0117\\
\mathcal{D}_4^H(\mathcal{A}_1, \mathcal{B}) &= 0.0603; \quad \mathcal{D}_4^H(\mathcal{A}_2, \mathcal{B}) = 0.0258; \quad \mathcal{D}_4^H(\mathcal{A}_3, \mathcal{B}) = 0.0464
\end{aligned}$$

Thus, the ranking order of the three patterns is $\mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_1$, and hence $\mathcal{A}_2$ is the most desirable pattern to be classified with $\mathcal{B}$. Furthermore, it can be easily verified that these results validate the above-proposed propositions on the distance measures.

8.4.2 Comparative analysis of the results with existing measures

The above-mentioned measures have been compared with some existing measures under an NS environment for showing the validity of the approach whose results are summarized in Table 8.1. From these results, it has been shown that the final ordering of the pattern coincides with the proposed measures, and hence it shows the conservative nature of the measures.

Table 8.1: Measurement values and ranking order of Pattern recognition Example

Methods	Measure Value of $\mathcal{B}$ from			Ranking Order
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	
D_H (defined in Eq. (2.10)) [17]	0.3250	0.1250	0.2500	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Correlation coefficient [17]	0.7883	0.9675	0.8615	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
D_{NE} (defined in Eq. (2.12)) [95]	0.5251	0.7674	0.6098	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2$
S_{CS1} (defined in Eq. (2.17)) [181]	0.8209	0.9785	0.8992	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
S_{CS2} (defined in Eq. (2.18)) [181]	0.8949	0.9911	0.9695	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
S_{T1} (defined in Eq. (2.19)) [187]	0.7275	0.9014	0.7976	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
S_{T2} (defined in Eq. (2.20)) [187]	0.9143	0.9673	0.9343	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$

8.4.3 Application of Distance Measure in Medical Diagnosis

Consider a set of diseases $\mathcal{Q} = \{\mathcal{Q}_1$(“Viral fever”),$\mathcal{Q}_2$(“Malaria”), $\mathcal{Q}_3$(“Typhoid”), $\mathcal{Q}_4$ (“Stomach Problem”), $\mathcal{Q}_5$ (“Chest problem”)$\}$ and a set of symptoms $S = \{s_1$ (“Temperature”), s_2 (“HeadAche”), s_3 (“Stomach Pain”), s_4 (“Cough”), s_5 (“Chest pain”)$\}$. Suppose a patient, with respect to all the symptoms, can be represented by the following SVNS:

$$\mathcal{P}(\text{Patient}) = \left\{ \begin{array}{c} (s_1, 0.8, 0.2, 0.1), (s_2, 0.6, 0.3, 0.1), (s_3, 0.2, 0.1, 0.8), \\ (s_4, 0.6, 0.5, 0.1), (s_5, 0.1, 0.4, 0.6) \end{array} \right\}$$

and each diseases $\mathcal{Q}_k (k = 1, 2, 3, 4, 5)$ is given in Table 8.2.

Table 8.2: Rating values of diseases in terms of SVNNs

	s_1	s_2	s_3	s_4	s_5
$\mathcal{Q}_1$	(0.4, 0.6, 0.0)	(0.3, 0.2, 0.5)	(0.1, 0.3, 0.7)	(0.4, 0.3, 0.3)	(0.1, 0.2, 0.7)
$\mathcal{Q}_2$	(0.7, 0.3, 0.0)	(0.2, 0.2, 0.6)	(0.0, 0.1, 0.9)	(0.7, 0.3, 0.0)	(0.1, 0.1, 0.8)
$\mathcal{Q}_3$	(0.3, 0.4, 0.3)	(0.6, 0.3, 0.1)	(0.2, 0.1, 0.7)	(0.2, 0.2, 0.6)	(0.1, 0.0, 0.9)
$\mathcal{Q}_4$	(0.1, 0.2, 0.7)	(0.2, 0.4, 0.4)	(0.8, 0.2, 0.0)	(0.2, 0.1, 0.7)	(0.2, 0.1, 0.7)
$\mathcal{Q}_5$	(0.1, 0.1, 0.8)	(0.0, 0.2, 0.8)	(0.2, 0.0, 0.8)	(0.2, 0.0, 0.8)	(0.8, 0.1, 0.1)

Now, the target is to diagnose the disease of patient $\mathcal{P}$ among $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4$ and $\mathcal{Q}_5$. For this, proposed distance measures, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_1^H$ and $\mathcal{D}_3^H$, have been computed from $\mathcal{P}$ to $\mathcal{Q}_k (k = 1, 2, \ldots, 5)$ and are given in Table 8.3. Thus, from these distance

Table 8.3: Measurement values and ranking order of Medical diagnosis example

Measures	Measure value of $\mathcal{P}$ from					Ranking order
	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	
$\mathcal{D}_1$	0.6400	0.9067	0.6333	1.4600	1.6200	$\mathcal{Q}_3 \succ \mathcal{Q}_1 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$\mathcal{D}_2$	0.1280	0.1813	0.1267	0.2920	0.3240	$\mathcal{Q}_3 \succ \mathcal{Q}_1 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$\mathcal{D}_3$	0.3626	0.4977	0.4113	0.7566	0.8533	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$\mathcal{D}_4$	0.1622	0.2226	0.1840	0.3383	0.3816	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$\mathcal{D}_1^H$	0.0613	0.0880	0.0627	0.1320	0.1400	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$\mathcal{D}_3^H$	0.1175	0.1760	0.1373	0.2439	0.2661	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$

measures, we conclude that the patient $\mathcal{P}$ suffers either from the disease $\mathcal{Q}_3$ or $\mathcal{Q}_1$. On the other hand, if we assign weights 0.3, 0.2, 0.2, 0.1 and 0.2 corresponding to $\mathcal{Q}_k (k = 1, 2, \ldots, 5)$, respectively, then we utilize the distance measures $\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_2^H$ and $\mathcal{D}_4^H$ for obtaining the most suitable pattern. the results corresponding to them are summarized in Table 8.4. Thus, on the basis of the ranking order, we conclude that the patient $\mathcal{P}$ suffers either from the disease $\mathcal{Q}_3$ or $\mathcal{Q}_1$.

Table 8.4: Weighted measurement values and ranking order of Medical diagnosis example

Measures	Measure value of $\mathcal{P}$ from					Ranking order
	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	
$\mathcal{D}_5$	0.0284	0.0403	0.0273	0.0625	0.0684	$\mathcal{Q}_3 \succ \mathcal{Q}_1 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$\mathcal{D}_6$	0.0795	0.1101	0.0862	0.1599	0.1781	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$\mathcal{D}_2^H$	0.0135	0.0200	0.0129	0.0276	0.0289	$\mathcal{Q}_3 \succ \mathcal{Q}_1 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$\mathcal{D}_4^H$	0.0572	0.0885	0.0636	0.1139	0.1226	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$

8.4.4 Comparative analysis of the results with some existing approaches

In order to verify the feasibility of the proposed DM approach based on the distance measure, we conducted a comparison analysis based on the same illustrative example. For this, various measures as presented in Eqs. (2.10), (2.11), (2.12), (2.17), (2.18), (2.19) and (2.20) of Chapter 2 are taken, and their corresponding results are summarized in Table 8.5, which shows that the patient $\mathcal{P}$ suffers from the disease $\mathcal{Q}_1$.

Table 8.5: Comparison of diagnosis result using existing measures.

Approach	Ranking Order
D_H (defined in Eq. (2.10)) [17]	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
Correlation [17]	$\mathcal{Q}_1 \succ \mathcal{Q}_2 \succ \mathcal{Q}_3 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
Distance measure [176]	
$p = 1$	$\mathcal{Q}_3 \succ \mathcal{Q}_1 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$p = 2$	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$p = 3$	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
$p = 5$	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
D_{NH} (defined in Eq. (2.11)) [95]	$\mathcal{Q}_3 \succ \mathcal{Q}_1 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
D_{NH} (defined in Eq. (2.12)) [95]	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
S_{CS1} (defined in Eq. (2.17)) [181]	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
S_{CS1} (defined in Eq. (2.18)) [181]	$\mathcal{Q}_1 \succ \mathcal{Q}_2 \succ \mathcal{Q}_3 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
S_{T1} (defined in Eq. (2.19)) [187]	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$
S_{T1} (defined in Eq. (2.20)) [187]	$\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$

8.4.5 Effect of the Parameters p and t on the Ordering

However, in order to analyze the effect of the parameters t and p on the measure values, an experiment was performed by taking different values of p ($p = 1, 1.5, 2, 3, 5, 10$) corresponding to a different value of the uncertainty parameter t ($t = 3, 5, 7$). On the basis of these different pairs of parameters, distance measures were computed, and their results are summarized in Tables 8.6 and 8.7, respectively, for the considered examples corresponding to different criterion weights.

From these, the following observations have been taken:

(i) For a fixed value of p, it has been observed that the measured values corresponding to each alternative increase with the increase in the value of t. On the other hand, by varying the value of t from 3 to 7, corresponding to a fixed value of p, this implies that values of the distance measures of each diagnosis from the patient $\mathcal{P}$ increase.

(ii) It has also been observed from this table that when the weight vector has been

Table 8.6: Results of classification of given sample using proposed distance measure.

		When Equal Importance Is given to Each Criteria				When Weight Vector $(0.3, 0.4, 0.2, 0.1)^T$ Is Taken			
p	t	$\mathcal{D}^p(\mathcal{A}_1, \mathcal{B})$	$\mathcal{D}^p(\mathcal{A}_2, \mathcal{B})$	$\mathcal{D}^p(\mathcal{A}_3, \mathcal{B})$	Ranking	$\mathcal{D}^p(\mathcal{A}_1, \mathcal{B})$	$\mathcal{D}^p(\mathcal{A}_2, \mathcal{B})$	$\mathcal{D}^p(\mathcal{A}_3, \mathcal{B})$	Ranking
1	3	0.1400	0.0733	0.1167	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.0338	0.0162	0.0233	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	5	0.1667	0.0762	0.1214	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.0387	0.0170	0.0248	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	7	0.1815	0.0778	0.1241	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.0414	0.0175	0.0256	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
1.5	3	0.1598	0.0783	0.1374	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.0620	0.0277	0.0426	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	5	0.1924	0.0817	0.1437	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.0723	0.0293	0.0452	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	7	0.2116	0.0838	0.1480	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.0784	0.0304	0.0469	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
2	3	0.1749	0.0821	0.1560	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.0861	0.0369	0.0604	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	5	0.2137	0.0859	0.1646	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.1021	0.0392	0.0644	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	7	0.2374	0.0885	0.1705	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.1120	0.0408	0.0671	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
3	3	0.1970	0.0880	0.1875	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.1229	0.0507	0.0927	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	5	0.2469	0.0929	0.02012	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.1497	0.0543	0.1000	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	7	0.2785	0.0962	0.2098	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.1672	0.0566	0.1046	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
5	3	0.2240	0.0967	0.2314	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.1680	0.0689	0.1469	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	5	0.2902	0.1041	0.2526	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.2128	0.0749	0.1605	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	7	0.3326	0.1087	0.2650	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.2426	0.0786	0.1685	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
10	3	0.2564	0.1107	0.2830	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.2203	0.0939	0.2248	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
	5	0.3421	0.1231	0.3131	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.2915	0.1047	0.2487	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
	7	0.3942	0.1304	0.3301	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$	0.3356	0.1109	0.2622	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$

assigned to each criterion weight, then the measurement values are less than that of an equal weighting case.

(iii) Finally, it is seen from the table that the measured values corresponding to each alternative $\mathcal{Q}_k (k = 1, 2, 3, 4, 5)$ are conservative in nature.

For each pair, the measure values lie between 0 and 1, and hence, on the basis of this, we conclude that the patient $\mathcal{P}$ suffers from the $\mathcal{Q}_1$ disease. The ranking order for the decision-maker is shown in the table as (13245), which indicates that the order of the different attributes is of the form $\mathcal{Q}_1 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_4 \succ \mathcal{Q}_5$. Hence $\mathcal{Q}_1$ is the most desirable, while $\mathcal{Q}_5$ is the least desirable for different values of t and p.

8.4.6 Advantages of the Proposed Method

According to the above comparison analysis, the proposed method for addressing DM problems has the following advantages:

(i) The distance measure under the IFS environment unable to deal with indeterminacy, which commonly occurs in real-life applications. Because SVNSs is a successful tool in handling indeterminacy, the proposed distance measure in the neutrosophic domain can effectively be used in many real applications in DM.

(ii) The proposed distance measure depends upon two parameters p and t, which help in adjusting the hesitation margin in computing data. The effect of hesitation will be diminished or almost neglected if the value of t is taken very large, and for smaller values of t, the effect of hesitation will rise. Thus, according to requirements, the decision-maker can adjust the parameter to handle incomplete as well as indeterminate information. Therefore, this proposed approach is more suitable for engineering, industrial and scientific applications.

(iii) It is seen that some existing measures under NS environments are unable to distinguish the given alternatives; hence their corresponding algorithm may give an irrelevant result. The proposed measure has the ability to overcome these flaws; thus it is a more suitable measure to tackle problems.

8.5 Conclusions

SVNSs are applied to problems with imprecise, uncertain, incomplete and inconsistent information existing in the real world. Some existing measures have in trouble to rank the objects. Here in this chapter, we overcome these flaws by proposing an alternative way to define new generalized distance measures between the two SVNNs. Further, a family of normalized and weighted normalized Hamming and Euclidean distance measures has been proposed for the SVNSs. Some desirable properties and their relations have been studied in detail. Finally, a DM method has been proposed on the basis of these distance measures. To demonstrate the efficiency of the proposed coefficients, numerical examples of pattern recognition as well as medical diagnosis have been taken. A comparative study, as well as the effect of the parameters on the ranking of the alternative, will support the theory and hence demonstrate that the proposed measures are an alternative way to solve the DM problems.

Table 8.7: Diagnosis result on basis of proposed distance measure.

		When Equal Importance Is Given to Each Criteria					When Weight Vector $(0.3, 0.2, 0.2, 0.1, 0.2)^T$ is Taken				
p	t	$\mathcal{D}^p(\mathcal{Q}_1, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_2, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_3, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_4, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_5, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_1, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_2, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_3, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_4, \mathcal{P})$	$\mathcal{D}^p(\mathcal{Q}_5, \mathcal{P})$
1	3	0.1280	0.1813	0.1267	0.2920	0.3240	0.0284	0.0403	0.0273	0.0625	0.0684
	5	0.1410	0.1867	0.1457	0.3076	0.3400	0.0304	0.0413	0.0300	0.0643	0.0700
	7	0.1481	0.1896	0.1563	0.3178	0.3489	0.0315	0.0419	0.0315	0.0656	0.070
1.5	3	0.1465	0.2023	0.1600	0.3175	0.3574	0.0553	0.0768	0.0579	0.1154	0.1282
	5	0.1612	0.2131	0.1794	0.3364	0.3778	0.0598	0.0808	0.0628	0.1202	0.1334
	7	0.1711	0.2205	0.1916	0.3492	0.3913	0.0630	0.0836	0.0658	0.1237	0.1369
2	3	0.1622	0.2226	0.1840	0.3383	0.3816	0.0795	0.1101	0.0862	0.1599	0.1781
	5	0.1787	0.2391	0.2038	0.3609	0.4052	0.0867	0.1183	0.0928	0.1686	0.1872
	7	0.1895	0.2501	0.2168	0.3760	0.4211	0.0914	0.1238	0.0972	0.1744	0.1933
3	3	0.1870	0.2601	0.2163	0.3715	0.4142	0.1182	0.1662	0.1312	0.2276	0.2509
	5	0.2061	0.2876	0.2376	0.4004	0.4421	0.1297	0.1842	0.1409	0.2436	0.2666
	7	0.2175	0.3047	0.2516	0.4185	0.4601	0.1365	0.1954	0.1475	0.2535	0.2765
5	3	0.2185	0.3187	0.2531	0.4170	0.4504	0.1675	0.2471	0.1892	0.3127	0.3354
	5	0.2405	0.3625	0.2782	0.4531	0.4826	0.1841	0.2817	0.2045	0.3384	0.3588
	7	0.2529	0.3877	0.2940	0.4740	0.5023	0.1934	0.3016	0.2145	0.3532	0.3729
10	3	0.2519	0.3980	0.2969	0.4731	0.4896	0.2215	0.3524	0.2599	0.4095	0.4235
	5	0.2771	0.4586	0.3271	0.5170	0.5252	0.2434	0.4063	0.2840	0.4464	0.4547
	7	0.2912	0.4624	0.3451	0.5420	0.5466	0.2556	0.4363	0.2981	0.4675	0.4730

Chapter 9

Logarithmic operational laws for single-valued neutrosophic numbers and their applications to decision making[1]

In this chapter, we discuss the new concept of the logarithm operational laws for the pairs of SVNNs. Further, based on these proposed laws, some new AOs named as logarithmic weighted averaging and geometric operators for SVNSs are defined and investigated their properties. To further strengthen their applicability, we developed an algorithm based on these operators for solving MCDM problems and demonstrate it with a numerical example. Finally, the influence of the logarithm is examined to show the behavior of the decision makers towards their optimal decision.

9.1 Introduction

The literature related to the AOs under the domain of the SVNSs are summarized in Section 1.1.1 of Chapter 1. It is well known that during the aggregation process, the most important process is to define the operational laws. But from the existing literature, it is observed that most of the existing AOs are based on the assumption that weight is a crisp number within [0,1]. However, Ye [184] introduced the exponential operational laws

---

[1] The content of this chapter is published as "New Logarithmic operational laws and their applications to multiattribute decision making for single-valued neutrosophic numbers", *Cognitive Systems Research*, 52, 931 - 946, 2018 (**Impact Factor: 1.425**).

as a supplement of operational laws of SVNSs, where the bases are the real numbers and the exponents are SVNSs. With the growing sound of the SVNS both in depth and scope, different kinds of some new operational laws and the aggregation methods are needed. As a kind of important mathematical operation, the logarithmic operational law of SVNSs is necessary to be developed in the field of the aggregation process.

In the light of the advantages of the SVNSs and in order to consummate the logarithmic operational laws under the SVNS circumstances, we define the logarithmic operational law (LOL) of SVNSs and SVNNs, in which the logarithm base λ is taken as a positive real number. Also, some properties of LOL are discussed. Furthermore, in the field of the aggregation process, the weighted averaging and geometric operators are developed with the help of LOLs under SVNS environment. These operators are named as logarithm single-valued neutrosophic (L-SVN) weighted average (L-SVNWA), L-SVN weighted geometric (L-SVNWG), L-SVN ordered weighted average (L-SVNOWA) and L-SVN ordered weighted geometric (L-SVNOWG). For instance, the basic averaging and geometric operators are the special cases of the proposed operators. Various prominent characteristics of these operators are discussed in details. Then, we utilized these operations and operators to develop an MCDM approach. At last, we influence the selection of the logarithm base and the logarithm operations for SVNNs in practice. Since proposed operators have different forms through choosing different values of λ. Therefore, the decision maker's can take the decision according to their own choice.

9.2 Logarithmic operational laws for SVNSs

In this section, we have introduced some new logarithmic operational laws (LOL) for the SVNNs.

9.2.1 Logarithmic Operational laws

Let $\mathcal{A}$ be SVNNs and $\lambda > 0$ be a real number. Since $\log_\lambda 0$ and $\log_1 x$ is not defined in real numbers, so we assume that $\mathcal{A} \neq \mathbf{0}$ where $\mathbf{0}$ is the zero SVNN, $\mathcal{A} \neq (0, 1, 1)$ and $\lambda \neq 1$ throughout the study.

Definition 9.2.1. Let $\mathcal{X}$ be the non-empty fixed set and $\mathcal{A} = \{(x, \zeta_{\mathcal{A}}(x), \kappa_{\mathcal{A}}(x), \varphi_{\mathcal{A}}(x))|x \in \mathcal{X}\}$ be SVNS, then we can define a logarithm operational laws of SVNS $\mathcal{A}$ as follows:

$$\log_{\lambda} \mathcal{A} = \big\{(x, 1 - \log_{\lambda} \zeta_{\mathcal{A}}(x), \log_{\lambda}(1 - \kappa_{\mathcal{A}}(x)), \log_{\lambda}(1 - \varphi_{\mathcal{A}}(x)))|x \in \mathcal{X}\big\} \tag{9.1}$$

where $0 < \lambda \leq \min\{\zeta_{\mathcal{A}}, 1 - \kappa_{\mathcal{A}}, 1 - \varphi_{\mathcal{A}}\} \leq 1$, $\lambda \neq 1$. It is clearly seen that the $\log_{\lambda} \mathcal{A}$ is also SVNS. As it is clear from the definition of SVNS, for all $x \in \mathcal{X}$, the functions $\zeta_{\mathcal{A}}, \kappa_{\mathcal{A}}$ and $\varphi_{\mathcal{A}}$ satisfy:

$$\zeta_{\mathcal{A}} : \mathcal{X} \to (0, 1], \kappa_{\mathcal{A}} : \mathcal{X} \to [0, 1), \varphi_{\mathcal{A}} : \mathcal{X} \to [0, 1)$$

and $0 \leq \zeta_{\mathcal{A}}(x) + \kappa_{\mathcal{A}}(x) + \varphi_{\mathcal{A}}(x) \leq 3$. If $0 < \lambda \leq \min\{\zeta_{\mathcal{A}}, 1 - \kappa_{\mathcal{A}}, 1 - \varphi_{\mathcal{A}}\} \leq 1$ and $\lambda \neq 1$, then the membership function:

$$1 - \log_{\lambda} \zeta_{\mathcal{A}} : \mathcal{X} \to [0, 1], \ \ \forall \ \ x \in \mathcal{X} \to 1 - \log_{\lambda} \zeta_{\mathcal{A}}(x) \in [0, 1],$$

the indeterminacy function

$$\log_{\lambda}(1 - \kappa_{\mathcal{A}}) : \mathcal{X} \to [0, 1], \ \ \forall \ \ x \in \mathcal{X} \to \log_{\lambda}(1 - \kappa_{\mathcal{A}}(x)) \in [0, 1],$$

and the non-membership function:

$$\log_{\lambda}(1 - \varphi_{\mathcal{A}}) : \mathcal{X} \to [0, 1], \ \ \forall \ \ x \in \mathcal{X} \to \log_{\lambda}(1 - \varphi_{\mathcal{A}}(x)) \in [0, 1],$$

Therefore,

$$\log_{\lambda} \mathcal{A} = \big\{(x, 1 - \log_{\lambda} \zeta_{\mathcal{A}}(x), \log_{\lambda}(1 - \kappa_{\mathcal{A}}(x)), \log_{\lambda}(1 - \varphi_{\mathcal{A}}(x)))|x \in \mathcal{X}\big\},$$

where $0 < \lambda \leq \min\{\zeta_{\mathcal{A}}, 1 - \kappa_{\mathcal{A}}, 1 - \varphi_{\mathcal{A}}\} \leq 1, \lambda \neq 1$ is SVNS.

Definition 9.2.2. Let $\mathcal{A} = (\zeta, \kappa, \varphi)$ be SVNN. If

$$\log_{\lambda} \mathcal{A} = \begin{cases} \Big(1 - \log_{\lambda} \zeta, \log_{\lambda}(1 - \kappa), \log_{\lambda}(1 - \varphi)\Big) & ; 0 < \lambda \leq \min\{\zeta, 1 - \kappa, 1 - \varphi\} < 1 \\ \Big(1 - \log_{\frac{1}{\lambda}} \zeta, \log_{\frac{1}{\lambda}}(1 - \kappa), \log_{\frac{1}{\lambda}}(1 - \varphi)\Big) & ; 0 < \frac{1}{\lambda} \leq \min\{\zeta, 1 - \kappa, 1 - \varphi\} < 1, \lambda \neq 1 \end{cases} \tag{9.2}$$

then the function $\log_{\lambda} \mathcal{A}$ is called a logarithmic operator, and the value $\log_{\lambda} \mathcal{A}$ is called Logarithmic SVNN (L-SVNN). Here, we take $\log_{\lambda} 0 = 0$, $\lambda > 0$, $\lambda \neq 1$.

Theorem 9.2.1. For SVNN $\mathcal{A}$, the value of operator $\log_{\lambda} \mathcal{A}$ is SVNN.

Proof. Let SVNN $\mathcal{A} = (\zeta, \kappa, \varphi)$ satisfies $0 < \zeta \leq 1, 0 \leq \kappa < 1, 0 \leq \varphi < 1$ and $\zeta+\kappa+\varphi \leq 3$. The, following two cases happens.

Case 1: When $0 < \lambda \leq \min\{\zeta, 1-\kappa, 1-\varphi\} < 1, \lambda \neq 1$. Thus, $0 \leq \log_\lambda \zeta, \log_\lambda(1-\kappa), \log_\lambda(1-\varphi) \leq 1$ and hence $0 \leq 1 - \log_\lambda \zeta \leq 1$, $0 \leq \log_\lambda(1-\kappa) \leq 1$, $0 \leq \log_\lambda(1-\varphi) \leq 1$ and $0 \leq 1 - \log_\lambda \zeta + \log_\lambda(1-\kappa) + \log_\lambda(1-\varphi) \leq 3$. Therefore, $\log_\lambda \mathcal{A}$ is SVNN.

Case 2: When $\lambda > 1$ and $0 < \frac{1}{\lambda} < 1$ and $\frac{1}{\lambda} \leq \min\{\zeta, 1-\kappa, 1-\varphi\}$, so it is easy to obtain that $\log_\lambda \mathcal{A}$ is SVNN.

Hence, the operator $\log_\lambda \mathcal{A}$ is SVNN. □

Example 9.2.1. Let $\mathcal{A} = (0.6, 0.4, 0.5)$ be SVNN, $\lambda = 0.3$, then

$$\begin{aligned} \log_\lambda \mathcal{A} &= (1 - \log_{0.3}(0.6), \log_{0.3}(0.6), \log_{0.3}(0.5)) \\ &= (0.5757, 0.4243, 0.5757) \end{aligned}$$

If $\lambda = 3$ then,

$$\begin{aligned} \log_{\frac{1}{\lambda}} \mathcal{A} &= \log_{\frac{1}{3}}(0.6, 0.4, 0.5) \\ &= (1 - \log_{\frac{1}{3}}(0.6), \log_{\frac{1}{3}}(0.6), \log_{\frac{1}{3}}(0.5)) \\ &= (0.5350, 0.4650, 0.6309) \end{aligned}$$

Next, we discuss some basic properties of L-SVNN $\log_\lambda \mathcal{A}$ based on LOL by taking $\lambda \in (0,1)$, while for $\lambda > 1$ it can be obtained analogously.

Theorem 9.2.2. Let $\mathcal{A} = (\zeta, \kappa, \varphi)$ be SVNN. If $0 < \lambda \leq \min\{\zeta, 1-\kappa, 1-\varphi\} \leq 1, \lambda \neq 1$ then

(i) $\lambda^{\log_\lambda \mathcal{A}} = \mathcal{A}$;

(ii) $\log_\lambda \lambda^{\mathcal{A}} = \mathcal{A}$.

Proof. (i) According to the Definitions 2.4.2 of Chapter 2 and Definition 9.2.2, we have

$$\begin{aligned}\lambda^{\log_\lambda \mathcal{A}} &= \left(\lambda^{1-(1-\log_\lambda \zeta)}, 1-\lambda^{\log_\lambda(1-\kappa)}, 1-\lambda^{\log_\lambda(1-\varphi)}\right)\\ &= (\lambda^{\log_\lambda \zeta}, 1-(1-\kappa), 1-(1-\varphi))\\ &= (\zeta, \kappa, \varphi)\\ &= \mathcal{A}\end{aligned}$$

(ii) From Definition 9.2.2, we have

$$\begin{aligned}\log_\lambda \lambda^{\mathcal{A}} &= \log_\lambda \left(\lambda^{1-\zeta}, 1-\lambda^{\kappa}, 1-\lambda^{\varphi}\right)\\ &= \left(1-\log_\lambda \lambda^{1-\zeta}, \log_\lambda(1-(1-\lambda^{\kappa})), \log_\lambda(1-(1-\lambda^{\varphi}))\right)\\ &= (\zeta, \kappa, \varphi)\\ &= \mathcal{A}\end{aligned}$$

□

Theorem 9.2.3. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2)$, be two SVNNs, $0 < \lambda \leq \min_i\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leq 1$ and $\lambda \neq 1$. Then,

(i) $\log_\lambda \mathcal{A}_1 \oplus \log_\lambda \mathcal{A}_2 = \log_\lambda \mathcal{A}_2 \oplus \log_\lambda \mathcal{A}_1$,

(ii) $\log_\lambda \mathcal{A}_1 \otimes \log_\lambda \mathcal{A}_2 = \log_\lambda \mathcal{A}_2 \otimes \log_\lambda \mathcal{A}_1$.

Theorem 9.2.4. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2, 3)$ be three SVNNs, $0 < \lambda \leq \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leq 1$ and $\lambda \neq 1$. Then,

(i) $(\log_\lambda \mathcal{A}_1 \oplus \log_\lambda \mathcal{A}_2) \oplus \log_\lambda \mathcal{A}_3 = \log_\lambda \mathcal{A}_1 \oplus (\log_\lambda \mathcal{A}_2 \oplus \log_\lambda \mathcal{A}_3)$,

(ii) $(\log_\lambda \mathcal{A}_1 \otimes \log_\lambda \mathcal{A}_2) \otimes \log_\lambda \mathcal{A}_3 = \log_\lambda \mathcal{A}_1 \otimes (\log_\lambda \mathcal{A}_2 \otimes \log_\lambda \mathcal{A}_3)$.

Proof. The proof is trial. □

Theorem 9.2.5. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2)$ be two SVNNs, $0 < \lambda \leq \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leq 1$, $\lambda \neq 1$ and $k, k_1, k_2 > 0$ be three real numbers. Then,

(i) $k(\log_\lambda \mathcal{A}_1 \oplus \log_\lambda \mathcal{A}_2) = k \log_\lambda \mathcal{A}_1 \oplus k \log_\lambda \mathcal{A}_2$,

(ii) $(\log_\lambda \mathcal{A}_1 \otimes \log_\lambda \mathcal{A}_2)^k = (\log_\lambda \mathcal{A}_1)^k \otimes (\log_\lambda \mathcal{A}_2)^k$,

(iii) $k_1 \log_\lambda \mathcal{A}_1 \oplus k_2 \log_\lambda \mathcal{A}_1 = (k_1 + k_2) \log_\lambda \mathcal{A}_1$,

(iv) $(\log_\lambda \mathcal{A}_1)^{k_1} \otimes (\log_\lambda \mathcal{A}_1)^{k_2} = (\log_\lambda \mathcal{A}_1)^{k_1+k_2}$,

(v) $\Big((\log_\lambda \mathcal{A}_1)^{k_1}\Big)^{k_2} = (\log_\lambda \mathcal{A}_1)^{k_1 k_2}$.

Proof. For SVNNs $\mathcal{A}_1$, $\mathcal{A}_2$ and by Definition 9.2.2, we get

$$\log_\lambda \mathcal{A}_1 = \big(1 - \log_\lambda \zeta_1, \log_\lambda(1-\kappa_1), \log_\lambda(1-\varphi_1)\big)$$

and

$$\log_\lambda \mathcal{A}_2 = \big(1 - \log_\lambda \zeta_2, \log_\lambda(1-\kappa_2), \log_\lambda(1-\varphi_2)\big)$$

and hence by using the operations laws between two SVNNs, we have

$$\begin{aligned} & \log_\lambda \mathcal{A}_1 \oplus \log_\lambda \mathcal{A}_2 \\ = & \left(\begin{array}{c} 1 - (\log_\lambda \zeta_1)(\log_\lambda \zeta_2), \log_\lambda(1-\kappa_1)\log_\lambda(1-\kappa_2), \\ \log_\lambda(1-\varphi_1)\log_\lambda(1-\varphi_2) \end{array}\right) \end{aligned}$$

and

$$\begin{aligned} & \log_\lambda \mathcal{A}_1 \otimes \log_\lambda \mathcal{A}_2 \\ = & \left(\begin{array}{c} (1 - \log_\lambda \zeta_1)(1 - \log_\lambda \zeta_2), 1 - (1 - \log_\lambda(1-\kappa_1))(1 - \log_\lambda(1-\kappa_2)), \\ 1 - (1 - \log_\lambda(1-\varphi_1))(1 - \log_\lambda(1-\varphi_2)) \end{array}\right) \end{aligned}$$

(i) For a real number $k > 0$, we have

$$\begin{aligned} & k(\log_\lambda \mathcal{A}_1 \oplus \log_\lambda \mathcal{A}_2) \\ = & \Big(1 - \Big(\log_\lambda \zeta_1 \log_\lambda \zeta_2\Big)^k, \Big(\log_\lambda(1-\kappa_1)\log_\lambda(1-\kappa_2)\Big)^k, \\ & \Big(\log_\lambda(1-\varphi_1)\log_\lambda(1-\varphi_2)\Big)^k\Big) \\ = & \Big(1 - \Big(\log_\lambda \zeta_1\Big)^k \Big(\log_\lambda \zeta_2\Big)^k, \Big(\log_\lambda(1-\kappa_1)\Big)^k \Big(\log_\lambda(1-\kappa_2)\Big)^k, \\ & \Big(\log_\lambda(1-\varphi_1)\Big)^k \Big(\log_\lambda(1-\varphi_2)\Big)^k\Big) \end{aligned}$$

$$\begin{aligned}
&= \Big(1-\Big(\log_\lambda \zeta_1\Big)^k, \Big(\log_\lambda(1-\kappa_1)\Big)^k, \Big(\log_\lambda(1-\varphi_1)\Big)^k\Big)\\
&\oplus \Big(1-\Big(\log_\lambda \zeta_2\Big)^k, \Big(\log_\lambda(1-\kappa_2)\Big)^k, \Big(\log_\lambda(1-\varphi_2)\Big)^k\Big)\\
&= k\log_\lambda \mathcal{A}_1 \oplus k\log_\lambda \mathcal{A}_2
\end{aligned}$$

(ii) For a real number $k>0$, we have

$$\begin{aligned}
&(\log_\lambda \mathcal{A}_1 \otimes \log_\lambda \mathcal{A}_2)^k\\
&= \Big(\Big((1-\log_\lambda \zeta_1)(1-\log_\lambda \zeta_2)\Big)^k, 1-\Big((1-\log_\lambda(1-\kappa_1))(1-\log_\lambda(1-\kappa_2))\Big)^k,\\
&\qquad 1-\Big((1-\log_\lambda(1-\varphi_1))(1-\log_\lambda(1-\varphi_2))\Big)^k\Big)\\
&= \Big(\Big(1-\log_\lambda \zeta_1\Big)^k\Big(1-\log_\lambda \zeta_2\Big)^k, 1-\Big(1-\log_\lambda(1-\kappa_1)\Big)^k\Big(1-\log_\lambda(1-\kappa_2)\Big)^k,\\
&\qquad 1-\Big(1-\log_\lambda(1-\varphi_1)\Big)^k\Big(1-\log_\lambda(1-\varphi_2)\Big)^k\Big)\\
&= \Big(\Big(1-\log_\lambda \zeta_1\Big)^k, 1-\Big(1-\log_\lambda(1-\kappa_1)\Big)^k, 1-\Big(1-\log_\lambda(1-\varphi_1)\Big)^k\Big)\\
&\otimes \Big(\Big(1-\log_\lambda \zeta_2\Big)^k, 1-\Big(1-\log_\lambda(1-\kappa_2)\Big)^k, 1-\Big(1-\log_\lambda(1-\varphi_2)\Big)^k\Big)\\
&= \Big(\log_\lambda \mathcal{A}_1\Big)^k \otimes \Big(\log_\lambda \mathcal{A}_2\Big)^k
\end{aligned}$$

(iii) For real positive number k_1 and k_2, we have

$$\begin{aligned}
&k_1\log_\lambda \mathcal{A}_1 \oplus k_2\log_\lambda \mathcal{A}_1\\
&= \Big(1-\Big(\log_\lambda \zeta_1\Big)^{k_1}, \Big(\log_\lambda(1-\kappa_1)\Big)^{k_1}, \Big(\log_\lambda(1-\varphi_1)\Big)^{k_1}\Big)\\
&\oplus \Big(1-\Big(\log_\lambda \zeta_1\Big)^{k_2}, \Big(\log_\lambda(1-\kappa_1)\Big)^{k_2}, \Big(\log_\lambda(1-\varphi_1)\Big)^{k_2}\Big)\\
&= \Big(1-\Big(\log_\lambda \zeta_1\Big)^{k_1+k_2}, \Big(\log_\lambda(1-\kappa_1)\Big)^{k_1+k_2}, \Big(\log_\lambda(1-\varphi_1)\Big)^{k_1+k_2}\Big)\\
&= (k_1+k_2)\log_\lambda \mathcal{A}_1
\end{aligned}$$

(iv) For real positive number k_1 and k_2. Since

$$\Big(\log_\lambda \mathcal{A}_1\Big)^{k_1} = \Big(\big(1-\log_\lambda \zeta_1\big)^{k_1}, 1-\big(1-\log_\lambda(1-\kappa_1)\big)^{k_1}, 1-\big(1-\log_\lambda(1-\varphi_1)\big)^{k_1}\Big)$$

and

$$\Big(\log_\lambda \mathcal{A}_1\Big)^{k_2} = \Big(\big(1-\log_\lambda \zeta_1\big)^{k_2}, 1-\big(1-\log_\lambda(1-\kappa_1)\big)^{k_2}, 1-\big(1-\log_\lambda(1-\varphi_1)\big)^{k_2}\Big)$$

and hence

$$\begin{aligned}
& \left(\log_\lambda \mathcal{A}_1\right)^{k_1} \otimes \left(\log_\lambda \mathcal{A}_1\right)^{k_2} \\
= & \left(\left(1-\log_\lambda \zeta_1\right)^{k_1}, 1-\left(1-\log_\lambda(1-\kappa_1)\right)^{k_1}, 1-\left(1-\log_\lambda(1-\varphi_1)\right)^{k_1}\right) \\
\otimes & \left(\left(1-\log_\lambda \zeta_1\right)^{k_2}, 1-\left(1-\log_\lambda(1-\kappa_1)\right)^{k_2}, 1-\left(1-\log_\lambda(1-\varphi_1)\right)^{k_2}\right) \\
= & \left(\left(1-\log_\lambda \zeta_1\right)^{k_1}\left(1-\log_\lambda \zeta_1\right)^{k_2}, 1-\left(1-\log_\lambda(1-\kappa_1)\right)^{k_1}\left(1-\log_\lambda(1-\kappa_1)\right)^{k_2},\right. \\
& \qquad \left. 1-\left(1-\log_\lambda(1-\varphi_1)\right)^{k_1}\left(1-\log_\lambda(1-\varphi_1)\right)^{k_2}\right) \\
= & \left(\left(1-\log_\lambda \zeta_1\right)^{k_1+k_2}, 1-\left(1-\log_\lambda(1-\kappa_1)\right)^{k_1+k_2}, 1-\left(1-\log_\lambda(1-\varphi_1)\right)^{k_1+k_2}\right) \\
= & \left(\log_\lambda \mathcal{A}_1\right)^{k_1+k_2}
\end{aligned}$$

(v) For positive real numbers k_1 and k_2, we have

$$\begin{aligned}
& \left(\left(\log_\lambda \mathcal{A}_1\right)^{k_1}\right)^{k_2} \\
= & \left(\left(\left(1-\log_\lambda \zeta_1\right)^{k_1}, 1-\left(1-\log_\lambda(1-\kappa_1)\right)^{k_1}, 1-\left(1-\log_\lambda(1-\varphi_1)\right)^{k_1}\right)\right)^{k_2} \\
= & \left(\left(1-\log_\lambda \zeta_1\right)^{k_1k_2}, 1-\left(1-\log_\lambda(1-\kappa_1)\right)^{k_1k_2}, 1-\left(1-\log_\lambda(1-\varphi_1)\right)^{k_1k_2}\right) \\
= & \left(\log_\lambda \mathcal{A}_1\right)^{k_1k_2}
\end{aligned}$$

□

Theorem 9.2.6. Let $\mathcal{A} = (\zeta, \kappa, \varphi)$ be SVNN. If $0 < \lambda_1 \leq \lambda_2 \leq \min\{\zeta, 1-\kappa, 1-\varphi\} \leq 1, \lambda_1, \lambda_2 \neq 1$, then $\log_{\lambda_1} \mathcal{A} \geq \log_{\lambda_2} \mathcal{A}$ and $\log_{\lambda_1} \mathcal{A} \leq \log_{\lambda_2} \mathcal{A}$ for $0 < \frac{1}{\lambda_2} \leq \frac{1}{\lambda_1} \leq \min\{\zeta, 1-\kappa, 1-\varphi\} \leq 1, \lambda_1, \lambda_2 \neq 1$.

Proof. By Definition 9.2.2, we have

$$\log_{\lambda_1} \mathcal{A} = \left(1-\log_{\lambda_1} \zeta, \log_{\lambda_1}(1-\kappa), \log_{\lambda_1}(1-\varphi)\right)$$

$$\text{and} \qquad \log_{\lambda_2} \mathcal{A} = \left(1-\log_{\lambda_2} \zeta, \log_{\lambda_2}(1-\kappa), \log_{\lambda_2}(1-\varphi)\right)$$

If $0 < \lambda_1 \leq \lambda_2 \leq \min\{\zeta, 1-\kappa, 1-\varphi\} \leq 1$ and $\lambda_1, \lambda_2 \neq 1$, then $1-\log_{\lambda_1} \zeta \geq 1-\log_{\lambda_2} \zeta$, $\log_{\lambda_1}(1-\kappa) \leq \log_{\lambda_2}(1-\kappa)$ and $\log_{\lambda_1}(1-\varphi) \leq \log_{\lambda_2}(1-\varphi)$ which implies that $\log_{\lambda_1} \mathcal{A} \geq \log_{\lambda_2} \mathcal{A}$.

On the other hand, when $\lambda_1, \lambda_2 > 1$ and $\lambda_1 \leq \lambda_2$, we get $0 < \frac{1}{\lambda_2} \leq \frac{1}{\lambda_1} \leq \min\{\zeta, 1 - \kappa, 1 - \varphi\} \leq 1, \lambda_1, \lambda_2 \neq 1$. Therefore, as discussed above, we can also obtain $\log_{\lambda_1} \mathcal{A} \leq \log_{\lambda_2} \mathcal{A}$. ☐

Theorem 9.2.7. Let $\mathcal{A}_1 = (\zeta_1, \kappa_1, \varphi_1)$ and $\mathcal{A}_2 = (\zeta_2, \kappa_2, \varphi_2)$ be two SVNNs. If $\zeta_1 \leq \zeta_2$, $\varphi_1 \geq \varphi_2$ and $\kappa_1 \geq \kappa_2$, i.e., $\mathcal{A}_1 \leq \mathcal{A}_2$, $0 < \lambda \leq \min\{\zeta_i, 1 - \kappa_i, 1 - \varphi_i\} \leq 1, \lambda \neq 1$, then $\log_\lambda \mathcal{A}_1 \leq \log_\lambda \mathcal{A}_2$.

Proof. Similar with Theorem 9.2.6. ☐

9.2.2 Aggregation operators

Based on the LOL of SVNNs, we define some weighted AOs as follows.

Definition 9.2.3. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2, \ldots, n)$ be a collection of SVNNs, $0 < \lambda_i \leq \min\{\zeta_i, 1 - \kappa_i, 1 - \varphi_i\} \leq 1, \lambda_i \neq 1$ and let L-SVNWA : $\Phi^n \to \Phi$. If

$$\text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1 \log_{\lambda_1} \mathcal{A}_1 \oplus \omega_2 \log_{\lambda_2} \mathcal{A}_2 \oplus \ldots \oplus \omega_n \log_{\lambda_n} \mathcal{A}_n \tag{9.3}$$

then the function L-SVNWA is called logarithmic SVN weighted averaging operator, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\log_{\lambda_i} \mathcal{A}_i$ with $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$.

Theorem 9.2.8. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2, \ldots, n)$ be a collection of SVNNs. Then, the aggregated value by using L-SVNWA operator is also SVNN and is given by

$$\begin{aligned} &\text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &= \begin{cases} \begin{pmatrix} 1 - \prod\limits_{i=1}^{n} \left(\log_{\lambda_i} \zeta_i\right)^{\omega_i}, \prod\limits_{i=1}^{n} \left(\log_{\lambda_i}(1 - \kappa_i)\right)^{\omega_i}, \\ \prod\limits_{i=1}^{n} \left(\log_{\lambda_i}(1 - \varphi_i)\right)^{\omega_i} \end{pmatrix} & ; 0 < \lambda_i \leq \min \left\{ \begin{array}{r} \zeta_i, \\ 1 - \kappa_i, \\ 1 - \varphi_i \end{array} \right\} \leq 1, \lambda_i \neq 1 \\ \begin{pmatrix} 1 - \prod\limits_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}} \zeta_i\right)^{\omega_i}, \prod\limits_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}}(1 - \kappa_i)\right)^{\omega_i}, \\ \prod\limits_{i=1}^{n} \left(\log_{\frac{1}{\lambda_i}}(1 - \varphi_i)\right)^{\omega_i} \end{pmatrix} & ; 0 < \frac{1}{\lambda_i} \leq \min \left\{ \begin{array}{r} \zeta_i, \\ 1 - \kappa_i, \\ 1 - \varphi_i \end{array} \right\} \leq 1, \lambda_i \neq 1 \end{cases} \end{aligned} \tag{9.4}$$

Proof. We prove the result given in Eq. (9.4) by employing mathematical induction on n for $0 < \lambda_i \leq \min\{\zeta_i, 1 - \kappa_i, 1 - \varphi_i\} \leq 1$, $\lambda_i \neq 1$. Since for each i, $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)$ is SVNN which implies that $\zeta_i, \kappa_i, \varphi_i \in [0,1]$ and $\zeta_i + \kappa_i + \varphi_i \leq 3$. Then the following steps of the mathematical induction are executed.

Step 1: For $n = 2$, we get L-SVNWA$(\mathcal{A}_1, \mathcal{A}_2) = \omega_1\Big(\log_{\lambda_1}\mathcal{A}_1\Big) \oplus \omega_2\Big(\log_{\lambda_2}\mathcal{A}_2\Big)$. Since by Definition 9.2.2, we can see that $\log_{\lambda_1}\mathcal{A}_1$ and $\log_{\lambda_2}\mathcal{A}_2$ are SVNNs and hence $\omega_1\log_{\lambda_1}\mathcal{A}_1 \oplus \omega_2\log_{\lambda_2}\mathcal{A}_2$ is also SVNN. Further, for $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

$$\begin{aligned}
\text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2) &= \omega_1\log_{\lambda_1}\mathcal{A}_1 \oplus \omega_2\log_{\lambda_2}\mathcal{A}_2 \\
&= \Big(1 - \Big(\log_{\lambda_1}\zeta_1\Big)^{\omega_1}, \Big(\log_{\lambda_1}(1-\kappa_1)\Big)^{\omega_1}, \Big(\log_{\lambda_1}(1-\varphi_1)\Big)^{\omega_1}\Big) \\
&\oplus \Big(1 - \Big(\log_{\lambda_2}\zeta_2\Big)^{\omega_2}, \Big(\log_{\lambda_2}(1-\kappa_2)\Big)^{\omega_2}, \Big(\log_{\lambda_2}(1-\varphi_2)\Big)^{\omega_2}\Big) \\
&= \Bigg(1 - \prod_{i=1}^{2}\Big(\log_{\lambda_i}\zeta_i\Big)^{\omega_i}, \prod_{i=1}^{2}\Big(\log_{\lambda_i}(1-\kappa_i)\Big)^{\omega_i}, \\
&\qquad \prod_{i=1}^{2}\Big(\log_{\lambda_i}(1-\varphi_i)\Big)^{\omega_i}\Bigg)
\end{aligned}$$

Thus, result holds for $n = 2$.

Step 2: Assume Eq. (9.4) holds for $n = k$. Now, for $n = k+1$, we have

$$\begin{aligned}
&\text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{k+1}) \\
= {}& \text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k) \oplus \omega_{k+1}\log_{\lambda_{k+1}}\mathcal{A}_{k+1} \\
= {}& \Bigg(1 - \prod_{i=1}^{k}\Big(\log_{\lambda_i}\zeta_i\Big)^{\omega_i}, \prod_{i=1}^{k}\Big(\log_{\lambda_i}(1-\kappa_i)\Big)^{\omega_i}, \prod_{i=1}^{k}\Big(\log_{\lambda_i}(1-\varphi_i)\Big)^{\omega_i}\Bigg) \\
\oplus {}& \Big(1 - \Big(\log_{\lambda_{k+1}}\zeta_{k+1}\Big)^{\omega_{k+1}}, \Big(\log_{\lambda_{k+1}}(1-\kappa_{k+1})\Big)^{\omega_{k+1}}, \Big(\log_{\lambda_{k+1}}(1-\varphi_{k+1})\Big)^{\omega_{k+1}}\Big) \\
= {}& \Bigg(1 - \prod_{i=1}^{k+1}\Big(\log_{\lambda_i}\zeta_i\Big)^{\omega_i}, \prod_{i=1}^{k+1}\Big(\log_{\lambda_i}(1-\kappa_i)\Big)^{\omega_i}, \prod_{i=1}^{k+1}\Big(\log_{\lambda_i}(1-\varphi_i)\Big)^{\omega_i}\Bigg)
\end{aligned}$$

and the aggregated value is also SVNN. Therefore, Eq. (9.4) holds for $n = k+1$ also. Hence, result is true for all positive integer n by the means of principle of mathematical induction.

On the other hand, if $\lambda_i \geq 1$ and $0 < \frac{1}{\lambda_i} \leq \min\{\zeta_i, 1-\kappa_i, 1-\varphi_i\} \leq 1, \lambda_i \neq 1$, we can also get

$$\begin{aligned}\text{L-SVNWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) &= \Bigg(1-\prod_{i=1}^{n}\Big(\log_{\frac{1}{\lambda_i}}\zeta_i\Big)^{\omega_i},\prod_{i=1}^{n}\Big(\log_{\frac{1}{\lambda_i}}(1-\kappa_i)\Big)^{\omega_i},\\ &\qquad \prod_{i=1}^{n}\Big(\log_{\frac{1}{\lambda_i}}(1-\varphi_i)\Big)^{\omega_i}\Bigg)\end{aligned}$$

and aggregated value is SVNN. □

Remark 9.2.1. If $\lambda_1=\lambda_2=\ldots=\lambda_n=\lambda$, $0<\lambda\leq\min_i\{\zeta_i,1-\kappa_i,1-\varphi_i\}\leq 1,\lambda\neq 1$, then L-SVNWA operator reduces to the following

$$\begin{aligned}\text{L-SVNWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) &= \Bigg(1-\prod_{i=1}^{n}\Big(\log_{\lambda}\zeta_i\Big)^{\omega_i},\prod_{i=1}^{n}\Big(\log_{\lambda}(1-\kappa_i)\Big)^{\omega_i},\\ &\qquad \prod_{i=1}^{n}\Big(\log_{\lambda}(1-\varphi_i)\Big)^{\omega_i}\Bigg)\end{aligned}$$

Example 9.2.2. Let $\mathcal{A}_1=(0.5,0.3,0.2)$, $\mathcal{A}_2=(0.2,0.4,0.3)$ and $\mathcal{A}_3=(0.6,0.3,0.5)$ be three SVNNs and $\omega=(0.2,0.5,0.3)^T$ is the weight vector of them. Considering $\lambda_1=\lambda_2=\lambda_3=0.1$ and then

$$\begin{aligned}&\text{L-SVNWA}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)\\ =\ &\Bigg(1-\prod_{i=1}^{3}\Big(\log_{\lambda_i}\zeta_i\Big)^{\omega_i},\prod_{i=1}^{3}\Big(\log_{\lambda_i}(1-\kappa_i)\Big)^{\omega_i},\prod_{i=1}^{3}\Big(\log_{\lambda_i}(1-\varphi_i)\Big)^{\omega_i}\Bigg)\\ =\ &\Bigg(1-\Big(\log_{0.1}(0.5)\Big)^{0.2}\times\Big(\log_{0.1}(0.2)\Big)^{0.5}\times\Big(\log_{0.1}(0.6)\Big)^{0.3},\\ &\Big(\log_{0.1}(1-0.3)\Big)^{0.2}\times\Big(\log_{0.1}(1-0.4)\Big)^{0.5}\times\Big(\log_{0.1}(1-0.3)\Big)^{0.3},\\ &\Big(\log_{0.1}(1-0.2)\Big)^{0.2}\times\Big(\log_{0.1}(1-0.3)\Big)^{0.5}\times\Big(\log_{0.1}(1-0.5)\Big)^{0.3}\Bigg)\\ =\ &\Big(0.5814,0.1854,0.1721\Big)\end{aligned}$$

Next, we give some properties of the proposed L-SVNWA operator for $\lambda_1=\lambda_2=\ldots=\lambda_n=\lambda$, and $0<\lambda\leq\min_i\{\zeta_i,1-\kappa_i,1-\varphi_i\}\leq 1,\lambda_i\neq 1$ and ω_i be the weight vector of SVNN $\mathcal{A}_i$ such that $\omega_i>0$ and $\sum\limits_{i=1}^{n}\omega_i=1$.

Property 9.2.1. If all SVNNs $\mathcal{A}_i=\mathcal{A}(i=1,2,\ldots,n)$, then

$$\text{L-SVNWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)=\log_{\lambda}\mathcal{A}$$

Proof. Let $\mathcal{A} = (\zeta, \kappa, \varphi)$ is SVNN such that $\mathcal{A}_i = \mathcal{A}$ for all i. Then, by Theorem 9.2.8, we get

$$\begin{aligned}
&\text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\\
=\;& \left(1 - \prod_{i=1}^{n} \Big(\log_\lambda \zeta_i\Big)^{\omega_i}, \prod_{i=1}^{n} \Big(\log_\lambda (1-\kappa_i)\Big)^{\omega_i}, \prod_{i=1}^{n} \Big(\log_\lambda (1-\varphi_i)\Big)^{\omega_i}\right)\\
=\;& \left(1 - \prod_{i=1}^{n} \Big(\log_\lambda \zeta\Big)^{\omega_i}, \prod_{i=1}^{n} \Big(\log_\lambda (1-\kappa)\Big)^{\omega_i}, \prod_{i=1}^{n} \Big(\log_\lambda (1-\varphi)\Big)^{\omega_i}\right)\\
=\;& \left(1 - \Big(\log_\lambda \zeta\Big)^{\sum_{i=1}^{n} \omega_i}, \Big(\log_\lambda (1-\kappa)\Big)^{\sum_{i=1}^{n} \omega_i}, \Big(\log_\lambda (1-\varphi)\Big)^{\sum_{i=1}^{n} \omega_i}\right)\\
=\;& \Big(1 - \log_\lambda \zeta, \log_\lambda (1-\kappa), \log_\lambda (1-\varphi)\Big)\\
=\;& \log_\lambda \mathcal{A}
\end{aligned}$$

□

Property 9.2.2. If $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2, \ldots, n)$ with $\mathcal{A}^- = (\min_i\{\zeta_i\}, \max_i\{\kappa_i\}, \max_i\{\varphi_i\})$ and $\mathcal{A}^+ = (\max_i\{\zeta_i\}, \min_i\{\kappa_i\}, \min_i\{\varphi_i\})$, then

$$\log_\lambda \mathcal{A}^- \leq \text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \log_\lambda \mathcal{A}^+$$

Proof. Since, for any i, $\min_i\{\zeta_i\} \leq \zeta_i \leq \max_i\{\zeta_i\}$, $\min_i\{\kappa_i\} \leq \kappa_i \leq \max_i\{\kappa_i\}$, and $\min_i\{\varphi_i\} \leq \varphi_i \leq \max_i\{\varphi_i\}$. This implies that $\mathcal{A}^- \leq \mathcal{A}_i \leq \mathcal{A}^+$. Assume that L-SVNWA($\mathcal{A}_1$, $\mathcal{A}_2$, …, $\mathcal{A}_n$) $= \log_\lambda \mathcal{A} = (\zeta_\mathcal{A}, \kappa_\mathcal{A}, \varphi_\mathcal{A})$, $\log_\lambda \mathcal{A}^- = (\zeta_{\mathcal{A}^-}, \kappa_{\mathcal{A}^+}, \varphi_{\mathcal{A}^+})$, $\log_\lambda \mathcal{A}^+ = (\zeta_{\mathcal{A}^+}, \kappa_{\mathcal{A}^-}, \varphi_{\mathcal{A}^-})$. Then, based on the monotonicity of logarithm function, we have

$$\begin{aligned}
\zeta_\mathcal{A} &= 1 - \prod_{i=1}^{n} \Big(\log_\lambda \zeta_i\Big)^{\omega_i} \geq 1 - \prod_{i=1}^{n} \Big(\log_\lambda \min_i\{\zeta_i\}\Big)^{\omega_i} = 1 - \log_\lambda(\min\{\zeta_i\}) = \zeta_{\mathcal{A}^-},\\
\kappa_\mathcal{A} &= \prod_{i=1}^{n} \Big(\log_\lambda (1-\kappa_i)\Big)^{\omega_i} \geq \prod_{i=1}^{n} \Big(\log_\lambda (1-\min_i\{\kappa_i\})\Big)^{\omega_i} = \log_\lambda(1-\min\{\kappa_i\}) = \kappa_{\mathcal{A}^-}\\
\text{and}\quad \varphi_\mathcal{A} &= \prod_{i=1}^{n} \Big(\log_\lambda (1-\varphi_i)\Big)^{\omega_i} \geq \prod_{i=1}^{n} \Big(\log_\lambda (1-\min_i\{\varphi_i\})\Big)^{\omega_i} = \log_\lambda(1-\min\{\varphi_i\}) = \varphi_{\mathcal{A}^-}
\end{aligned}$$

Also

$$\begin{aligned}
\zeta_\mathcal{A} &= 1 - \prod_{i=1}^{n} \Big(\log_\lambda \zeta_i\Big)^{\omega_i} \leq 1 - \prod_{i=1}^{n} \Big(\log_\lambda \max_i\{\zeta_i\}\Big)^{\omega_i} = 1 - \log_\lambda(\max\{\zeta_i\}) = \zeta_{\mathcal{A}^+},\\
\kappa_\mathcal{A} &= \prod_{i=1}^{n} \Big(\log_\lambda (1-\kappa_i)\Big)^{\omega_i} \leq \prod_{i=1}^{n} \Big(\log_\lambda (1-\max_i\{\kappa_i\})\Big)^{\omega_i} = \log_\lambda(1-\max\{\kappa_i\}) = \kappa_{\mathcal{A}^+}
\end{aligned}$$

and

$$\varphi_{\mathcal{A}} = \prod_{i=1}^{n} \Big(\log_{\lambda}(1-\varphi_i) \Big)^{\omega_i} \leq \prod_{i=1}^{n} \Big(\log_{\lambda}(1-\max_{i}\{\varphi_i\}) \Big)^{\omega_i} = \log_{\lambda}(1-\max\{\varphi_i\}) = \varphi_{\mathcal{A}^+}$$

Based on score function, we get

$$\begin{aligned} \mathcal{S}(\log_{\lambda}\mathcal{A}) &= \zeta_{\mathcal{A}} - \kappa_{\mathcal{A}} - \varphi_{\mathcal{A}} \\ &\leq \zeta_{\mathcal{A}^+} - \kappa_{\mathcal{A}^-} - \varphi_{\mathcal{A}^-} = \mathcal{S}(\log_{\lambda}\mathcal{A}^+) \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}(\log_{\lambda}\mathcal{A}) &= \zeta_{\mathcal{A}} - \kappa_{\mathcal{A}} - \varphi_{\mathcal{A}} \\ &\geq \zeta_{\mathcal{A}^-} - \kappa_{\mathcal{A}^+} - \varphi_{\mathcal{A}^+} = \mathcal{S}(\log_{\lambda}\mathcal{A}^-) \end{aligned}$$

Hence, $\mathcal{S}(\log_{\lambda}\mathcal{A}^-) \leq \mathcal{S}(\log_{\lambda}\mathcal{A}) \leq \mathcal{S}(\log_{\lambda}\mathcal{A}^+)$. Now, we discuss the three cases:

Case 1: If $\mathcal{S}(\log_{\lambda}\mathcal{A}^-) < \mathcal{S}(\log_{\lambda}\mathcal{A}) < \mathcal{S}(\log_{\lambda}\mathcal{A}^+)$, then result holds.

Case 2: If $\mathcal{S}(\log_{\lambda}\mathcal{A}^+) = \mathcal{S}(\log_{\lambda}\mathcal{A})$ then $\zeta_{\mathcal{A}} - \kappa_{\mathcal{A}} - \varphi_{\mathcal{A}} = \zeta_{\mathcal{A}^+} - \kappa_{\mathcal{A}^-} - \varphi_{\mathcal{A}^-}$ which implies that $\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}^+}$, $\kappa_{\mathcal{A}} = \kappa_{\mathcal{A}^-}$ and $\varphi_{\mathcal{A}} = \varphi_{\mathcal{A}^-}$ and hence $\mathcal{H}(\log_{\lambda}\mathcal{A}^+) = \mathcal{H}(\log_{\lambda}\mathcal{A})$.

Case 3: If $\mathcal{S}(\log_{\lambda}\mathcal{A}^-) = \mathcal{S}(\log_{\lambda}\mathcal{A})$ then $\zeta_{\mathcal{A}} - \kappa_{\mathcal{A}} - \varphi_{\mathcal{A}} = \zeta_{\mathcal{A}^-} - \kappa_{\mathcal{A}^+} - \varphi_{\mathcal{A}^+}$ which implies that $\zeta_{\mathcal{A}} = \zeta_{\mathcal{A}^-}$, $\kappa_{\mathcal{A}} = \kappa_{\mathcal{A}^+}$ and $\varphi_{\mathcal{A}} = \varphi_{\mathcal{A}^+}$ and hence $\mathcal{H}(\log_{\lambda}\mathcal{A}^-) = \mathcal{H}(\log_{\lambda}\mathcal{A})$.

Therefore, by combining all these cases, we get

$$\log_{\lambda}\mathcal{A}^- \leq \text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \log_{\lambda}\mathcal{A}^+$$

□

Property 9.2.3. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)$ and $\mathcal{A}_i^* = (\zeta_i^*, \kappa_i^*, \varphi_i^*)(i = 1, 2, \ldots, n)$ be two collections of SVNNs. If $\zeta_i \leq \zeta_i^*$, $\kappa_i \geq \kappa_i^*$, $\varphi_i \geq \varphi_i^*$, then

$$\text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \text{L-SVNWA}(\mathcal{A}_1^*, \mathcal{A}_2^*, \ldots, \mathcal{A}_n^*)$$

Proof. Follows from the above, so we omit here. □

Definition 9.2.4. A L-SVN ordered weighted average (L-SVNOWA) operator is a mapping L-SVNOWA : $\Phi^n \to \Phi$, such that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, with $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$, and

$$\begin{aligned}\text{L-SVNOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \Big(\omega_1 \log_{\lambda_{\sigma(1)}} \mathcal{A}_{\sigma(1)}\Big) \oplus \Big(\omega_2 \log_{\lambda_{\sigma(2)}} \mathcal{A}_{\sigma(2)}\Big) \oplus \\ &\quad \ldots \oplus \Big(\omega_n \log_{\lambda_{\sigma(n)}} \mathcal{A}_{\sigma(n)}\Big) \end{aligned} \tag{9.5}$$

where $0 < \lambda_{\sigma(i)} \le \min\{\zeta_{\sigma(i)}, 1 - \kappa_{\sigma(i)}, 1 - \varphi_{\sigma(i)}\} \le 1, \lambda_{\sigma(i)} \ne 1$ and σ is the permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\sigma(i-1)} \ge \mathcal{A}_{\sigma(i)}$ for $i = 2, 3, \ldots, n$.

Theorem 9.2.9. For a collection of SVNNs $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2, \ldots, n)$, the aggregated value by using L-SVNOWA operator is still SVNN and given by

$$\begin{aligned}&\text{L-SVNOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ &= \begin{cases} \left(\begin{array}{c} 1 - \prod\limits_{i=1}^{n} \left(\log_{\lambda_{\sigma(i)}} \zeta_{\sigma(i)} \right)^{\omega_i}, \prod\limits_{i=1}^{n} \left(\log_{\lambda_{\sigma(i)}} (1 - \kappa_{\sigma(i)}) \right)^{\omega_i}, \\ \prod\limits_{i=1}^{n} \left(\log_{\lambda_{\sigma(i)}} (1 - \varphi_{\sigma(i)}) \right)^{\omega_i} \end{array} \right) & ; 0 < \lambda_{\sigma(i)} \le \min_i \left\{ \begin{array}{r} \zeta_{\sigma(i)} \\ 1 - \kappa_{\sigma(i)} \\ 1 - \varphi_{\sigma(i)} \end{array} \right\} \le 1, \lambda_{\sigma(i)} \ne 1 \\ \left(\begin{array}{c} 1 - \prod\limits_{i=1}^{n} \left(\log_{\frac{1}{\lambda_{\sigma(i)}}} \zeta_{\sigma(i)} \right)^{\omega_i}, \prod\limits_{i=1}^{n} \left(\log_{\frac{1}{\lambda_{\sigma(i)}}} (1 - \kappa_{\sigma(i)}) \right)^{\omega_i}, \\ \prod\limits_{i=1}^{n} \left(\log_{\frac{1}{\lambda_{\sigma(i)}}} (1 - \varphi_{\sigma(i)}) \right)^{\omega_i} \end{array} \right) & ; 0 < \frac{1}{\lambda_{\sigma(i)}} \le \min_i \left\{ \begin{array}{r} \zeta_{\sigma(i)} \\ 1 - \kappa_{\sigma(i)} \\ 1 - \varphi_{\sigma(i)} \end{array} \right\} \le 1, \lambda_{\sigma(i)} \ne 1 \end{cases}\end{aligned} \tag{9.6}$$

Proof. The proof follows from Theorem 9.2.8. □

Definition 9.2.5. Let $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)(i = 1, 2, \ldots, n)$ be a collection of SVNNs, $0 \le \lambda_i \le \min\limits_{i}\{\zeta_i, 1 - \kappa_i, 1 - \varphi_i\} \le 1, \lambda_i \ne 1$ and let L-SVNWG : $\Phi^n \to \Phi$. If

$$\text{L-SVNWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\log_{\lambda_1} \mathcal{A}_1\right)^{\omega_1} \otimes \left(\log_{\lambda_2} \mathcal{A}_2\right)^{\omega_2} \otimes \ldots \otimes \left(\log_{\lambda_n} \mathcal{A}_n\right)^{\omega_n} \tag{9.7}$$

then the function L-SVNWG is called logarithmic SVN weighted geometric operator, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\log_{\lambda_i} \mathcal{A}_i$ with $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$.

Theorem 9.2.10. The aggregated value by L-SVNWG operator for SVNNs $\mathcal{A}_i = (\zeta_i, \kappa_i, \varphi_i)$

is also SVNN and is given by

$$\begin{aligned}&\text{L-SVNWG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\ =&\begin{cases}\begin{pmatrix}\prod\limits_{i=1}^{n}\left(1-\log_{\lambda_i}\zeta_i\right)^{\omega_i},1-\prod\limits_{i=1}^{n}\left(1-\log_{\lambda_i}(1-\kappa_i)\right)^{\omega_i},\\ 1-\prod\limits_{i=1}^{n}\left(1-\log_{\lambda_i}(1-\varphi_i)\right)^{\omega_i}\end{pmatrix} & ;0<\lambda_i\leq \min_i\begin{Bmatrix}\zeta_i\\ 1-\kappa_i\\ 1-\varphi_i\end{Bmatrix}\leq 1,\lambda_i\neq 1\\ \\ \begin{pmatrix}\prod\limits_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_i}}\zeta_i\right)^{\omega_i},1-\prod\limits_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_i}}(1-\kappa_i)\right)^{\omega_i},\\ 1-\prod\limits_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_i}}(1-\varphi_i)\right)^{\omega_i}\end{pmatrix} & ;0<\frac{1}{\lambda_i}\leq \min_i\begin{Bmatrix}\zeta_i\\ 1-\kappa_i\\ 1-\varphi_i\end{Bmatrix}\leq 1,\lambda_i\neq 1\end{cases}\end{aligned}\tag{9.8}$$

Definition 9.2.6. A logarithmic SVN ordered weighted geometric (L-SVNOWG) operator is a mapping L-SVNOWG : $\Phi^n\rightarrow\Phi$, such that $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$, with $\omega_i>0$ and $\sum\limits_{i=1}^{n}\omega_i=1$, and

$$\begin{aligned}\text{L-SVNOWG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) &= \left(\log_{\lambda_{\sigma(1)}}\mathcal{A}_{\sigma(1)}\right)^{\omega_1}\otimes\left(\log_{\lambda_{\sigma(2)}}\mathcal{A}_{\sigma(2)}\right)^{\omega_2}\otimes\\ &\quad \ldots\otimes\left(\log_{\lambda_{\sigma(n)}}\mathcal{A}_{\sigma(n)}\right)^{\omega_n}\end{aligned}\tag{9.9}$$

where $0<\lambda_{\sigma(i)}\leq\min\{\zeta_{\sigma(i)},1-\kappa_{\sigma(i)},1-\varphi_{\sigma(i)}\}\leq 1,\lambda_{\sigma(i)}\neq 1$ and σ is the permutation of $(1,2,\ldots,n)$ such that $\mathcal{A}_{\sigma(i-1)}\geq\mathcal{A}_{\sigma(i)}$ for $i=2,3,\ldots,n$.

Theorem 9.2.11. For a collection of SVNNs $\mathcal{A}_i=(\zeta_i,\kappa_i,\varphi_i)(i=1,2,\ldots,n)$, the aggregated value by using L-SVNOWG operator is still SVNN and given by

$$\begin{aligned}&\text{L-SVNOWG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\ =&\begin{cases}\begin{pmatrix}\prod\limits_{i=1}^{n}\left(1-\log_{\lambda_{\sigma(i)}}\zeta_{\sigma(i)}\right)^{\omega_i},1-\prod\limits_{i=1}^{n}\left(1-\log_{\lambda_{\sigma(i)}}(1-\kappa_{\sigma(i)})\right)^{\omega_i},\\ 1-\prod\limits_{i=1}^{n}\left(1-\log_{\lambda_{\sigma(i)}}(1-\varphi_{\sigma(i)})\right)^{\omega_i}\end{pmatrix} & ;0<\lambda_{\sigma(i)}\leq \min_i\begin{Bmatrix}\zeta_{\sigma(i)}\\ 1-\kappa_{\sigma(i)}\\ 1-\varphi_{\sigma(i)}\end{Bmatrix}\leq 1,\lambda_i\neq 1\\ \\ \begin{pmatrix}\prod\limits_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_{\sigma(i)}}}\zeta_{\sigma(i)}\right)^{\omega_i},1-\prod\limits_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_{\sigma(i)}}}(1-\kappa_{\sigma(i)})\right)^{\omega_i},\\ 1-\prod\limits_{i=1}^{n}\left(1-\log_{\frac{1}{\lambda_{\sigma(i)}}}(1-\varphi_{\sigma(i)})\right)^{\omega_i}\end{pmatrix} & ;0<\frac{1}{\lambda_{\sigma(i)}}\leq \min_i\begin{Bmatrix}\zeta_{\sigma(i)}\\ 1-\kappa_{\sigma(i)}\\ 1-\varphi_{\sigma(i)}\end{Bmatrix}\leq 1,\lambda_i\neq 1\end{cases}\end{aligned}\tag{9.10}$$

Proof. Similar to Theorem 9.2.8. ☐

As similar to L-SVNWA operator, the L-SVNOWA, L-SVNOWG and L-SVNWG operators also have the same properties. Furthermore, if $\lambda_1=\lambda_2=\ldots=\lambda_n=\lambda$,

$0 < \lambda \leq \min\limits_{i}\{\zeta_{\sigma(i)}, 1-\kappa_{\sigma(i)}, 1-\varphi_{\sigma(i)}\} \leq 1$, then L-SVNOWA, LSVNOWG operators becomes

$$\begin{aligned}
\text{L-SVNOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \Bigg(1-\prod_{i=1}^{n}\Big(\log_{\lambda}\zeta_{\sigma(i)}\Big)^{\omega_i}, \prod_{i=1}^{n}\Big(\log_{\lambda}(1-\kappa_{\sigma(i)})\Big)^{\omega_i}, \\
&\qquad \prod_{i=1}^{n}\Big(\log_{\lambda}(1-\varphi_{\sigma(i)})\Big)^{\omega_i}\Bigg) \\
&\text{and} \\
\text{L-SVNOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \Bigg(\prod_{i=1}^{n}\Big(1-\log_{\lambda}\zeta_{\sigma(i)}\Big)^{\omega_i}, 1-\prod_{i=1}^{n}\Big(1-\log_{\lambda}(1-\kappa_{\sigma(i)})\Big)^{\omega_i}, \\
&\qquad 1-\prod_{i=1}^{n}\Big(1-\log_{\lambda}(1-\varphi_{\sigma(i)})\Big)^{\omega_i}\Bigg)
\end{aligned}$$

9.3 Proposed MADM method

The general description of the MADM method is given in Section 2.5 of Chapter 2. Here, the collection information of the given alternatives is taken in the form of the SVN decision matrix $\mathcal{M} = (\alpha_{ij})_{m\times n}$ where $\alpha_{ij} = (\zeta_{ij}, \kappa_{ij}, \varphi_{ij})$ such that $0 \leq \zeta_{ij}, \kappa_{ij}, \varphi_{ij} \leq 1$ and $\zeta_{ij} + \kappa_{ij} + \varphi_{ij} \leq 3$. Corresponding to each SVNN, the logarithm base index for them are denoted by λ_{ij} where $0 < \lambda_{ij} \leq \min\{\zeta_{ij}, 1-\kappa_{ij}, 1-\varphi_{ij}\} \leq 1$, $\lambda_{ij} \neq 1$ for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ and are summarized in the matrix format $\Lambda = (\lambda_{ij})_{m\times n}$. To access the attribute weights of $\mathcal{G}_j$, we convert the matrix $\mathcal{M}$ into its equivalent logarithm score matrix $\mathbb{S}$ as

$$\mathbb{S} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array}\begin{array}{c} \begin{array}{cccc} \quad\mathcal{G}_1 \quad & \quad\mathcal{G}_2\quad & \ldots & \quad\mathcal{G}_n\quad \end{array} \\ \begin{pmatrix} \mathcal{S}(\log_{\lambda_{11}}\alpha_{11}) & \mathcal{S}(\log_{\lambda_{12}}\alpha_{12}) & \ldots & \mathcal{S}(\log_{\lambda_{1n}}\alpha_{1n}) \\ \mathcal{S}(\log_{\lambda_{21}}\alpha_{21}) & \mathcal{S}(\log_{\lambda_{22}}\alpha_{22}) & \ldots & \mathcal{S}(\log_{\lambda_{2n}}\alpha_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{S}(\log_{\lambda_{m1}}\alpha_{m1}) & \mathcal{S}(\log_{\lambda_{m2}}\alpha_{m2}) & \ldots & \mathcal{S}(\log_{\lambda_{mn}}\alpha_{mn}) \end{pmatrix} \end{array} \tag{9.11}$$

where $\mathcal{S}(\log_{\lambda_{ij}}\alpha_{ij}) = 1 - \log_{\lambda_{ij}}\zeta_{ij} - \log_{\lambda_{ij}}(1-\kappa_{ij}) - \log_{\lambda_{ij}}(1-\varphi_{ij})$ is the score function of $\log_{\lambda_{ij}}\alpha_{ij}$. Let $\omega_j > 0$ be the partial unknown weight vector of the attribute $\mathcal{G}_j$ with $\sum\limits_{j=1}^{n}\omega_j = 1$. Then, the weighted sum of each alternative, called as suitability function $Q(\mathcal{A}_i)$, is obtained as

$$Q(\mathcal{A}_i) = \sum_{j=1}^{n}\omega_j\mathbb{S}_{ij} \quad ; \quad i = 1, 2, \ldots, m \tag{9.12}$$

Based on this function, a mathematical programming model for determining the weight vector is formulated as below.

$$\begin{aligned} \max \quad & f = \sum_{i=1}^{m}\sum_{j=1}^{n} \omega_j \mathbb{S}_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} \omega_j = 1 \\ & \omega_j \geq 0 \quad ; \quad \omega \in H \end{aligned} \tag{9.13}$$

where H is the set consisting of the partial information of the attribute weights. After solving this model, we get the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$. Now, based on these weight vectors, aggregate all the preference values either by using L-SVNWA or L-SVNWG or L-SVNOWA or L-SVNOWG and get the collective one. Later, rank the given alternatives based on the score values of obtained SVNN's and choose the desired one(s).

In the nutshells, after combination all the above demonstrations, our proposed DM method under SVNN environment has been summarized as follows.

Step 1: Formulate the neutrosophic decision matrix $\mathcal{M}$ of rating values of the alternative $\mathcal{A}_i$ and the parameters λ_{ij} for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.

Step 2: Compute the score matrix $\mathbb{S}$ as given in Eq. (9.11).

Step 3: Formulate and solve the optimization model (9.13) to determine the attribute weights $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$.

Step 4: Utilize the appropriate AO such as L-SVNWA, or L-SVNOWA, or L-SVNWG or L-SVNOWG to aggregate the different preferences of the decision maker.

Step 5: Rank the alternative by using score function and chose the best alternative(s).

9.4 Illustrative example

The proposed approach is illustrating with a practical example which is stated as below.

9.4.1 Case study

Goods and Services Tax (GST) is an indirect tax which is designed to make India an integrated common market. While GST promises to the user in an era of the unified indirect tax regime, integrating India into a single homogenous market, it comes with certain complications inherited from the legacy tax regime. With the government gearing up to enforce the GST in Punjab from July 1, the issue of traders having limited computer knowledge and poor connectivity. In order to counter this, the state government has planned to train more than 2,000 youths as 'GST Mitra' to cater the traders. Punjab GST Mitra Scheme, which has to be started as a pilot project from Patiala, proposes to assist taxpayers in furnishing the details of outward supplies, inward supplies and returns, filing claims or refunds, filing any other applications, etc., in GST Regime. It aims to create a group of Tax professionals available in the locality or at the doorstep of the taxpayer, at affordable costs throughout the State of Punjab.

The poor internet connectivity in far-flung areas has emerged as a big stumbling block in the success of 'GST Mitra' scheme. In order to provide the online services to run this scheme, state government is planning to give contract combinedly to private mobile service provider along with state-owned BSNL. For this, the Indian government had been issued the global tender to select the contractor for these projects in the newspaper and considered the five attribute required for its namely, Technology Expertise ($\mathcal{G}_1$), Service quality ($\mathcal{G}_2$), Bandwidth ($\mathcal{G}_3$), Internet speed ($\mathcal{G}_4$) and Customer Services ($\mathcal{G}_5$). The importance of these attribute is taken as partially known. The five contractors (i.e. alternatives) namely, "Jaihind Road Builders private (Pvt.) limited (Ltd.)" ($\mathcal{A}_1$), "J.K. Construction" ($\mathcal{A}_2$), "Build quick Infrastructure Pvt. Ltd." ($\mathcal{A}_3$), "Relcon Infra projects Ltd." ($\mathcal{A}_4$), and "Tata Infrastructure Ltd." ($\mathcal{A}_5$) bid for these projects. Then, the aim of the government is to recognize the best internet service to their own citizens.

The approach described in Section 9.3 are executed here step by step to compute the best alternatives.

Step 1: The rating values of the expert towards the given alternatives $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$ are listed as

$$\mathcal{M} = \begin{array}{cccccc} & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 & \mathcal{G}_5 \\ \mathcal{A}_1 & (0.5, 0.3, 0.4) & (0.5, 0.2, 0.3) & (0.2, 0.2, 0.6) & (0.3, 0.2, 0.4) & (0.3, 0.3, 0.4) \\ \mathcal{A}_2 & (0.7, 0.1, 0.3) & (0.7, 0.2, 0.3) & (0.6, 0.3, 0.2) & (0.6, 0.4, 0.2) & (0.7, 0.1, 0.2) \\ \mathcal{A}_3 & (0.5, 0.3, 0.4) & (0.6, 0.2, 0.4) & (0.6, 0.1, 0.2) & (0.5, 0.1, 0.3) & (0.6, 0.4, 0.3) \\ \mathcal{A}_4 & (0.7, 0.3, 0.2) & (0.7, 0.2, 0.2) & (0.4, 0.5, 0.2) & (0.5, 0.2, 0.2) & (0.4, 0.5, 0.4) \\ \mathcal{A}_5 & (0.4, 0.1, 0.3) & (0.5, 0.1, 0.2) & (0.4, 0.1, 0.5) & (0.4, 0.3, 0.6) & (0.3, 0.2, 0.4) \end{array}$$

In this matrix, corresponding to alternative $\mathcal{A}_1$ under criterion $\mathcal{G}_1$, the rating (0.5, 0.3, 0.4) represents that an expert gives the possibility degree in which the statement is good is 0.5, the statement is false is 0.4 and the degree in which he or she is unsure is 0.3. The other values in the matrix have similar meanings. Furthermore, the preferences of the logarithm base λ_{ij} are taken as

$$\Lambda = \begin{pmatrix} 0.4 & 0.3 & 0.1 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0.3 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.3 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.3 & 0.2 \end{pmatrix}$$

Step 2: By using Eq. (9.11), the score matrix $\mathbb{S}$ is

$$\mathbb{S} = \begin{array}{cccccc} & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 & \mathcal{G}_5 \\ \mathcal{A}_1 & -0.7032 & -0.0573 & -0.1938 & -0.2041 & -0.2871 \\ \mathcal{A}_2 & 0.1065 & 0.4181 & 0.0941 & -0.3585 & 0.4309 \\ \mathcal{A}_3 & -0.2962 & -0.3585 & 0.3029 & 0.2822 & -0.5042 \\ \mathcal{A}_4 & -0.0220 & 0.5011 & -0.5221 & -0.2435 & 0.0792 \\ \mathcal{A}_5 & -0.1448 & -0.1150 & -0.0655 & -0.8184 & -0.2041 \end{array}$$

Step 3: Assume that the partial weight information is $H = \{0.15 \leq \omega_1 \leq 0.20,\ 0.2 \leq \omega_2 \leq 0.3,\ 0.2 \leq \omega_3 \leq 0.4,\ 0.22 \leq \omega_4 \leq 0.25,\ 0.15 \leq \omega_5 \leq 0.20\}$ and hence, an optimization model (9.13) becomes

$$\begin{aligned}
\max f &= -1.0598\omega_1 + 0.3884\omega_2 - 0.3844\omega_3 - 1.3423\omega_4 - 0.4854\omega_5 \\
s.t. \quad & 0.15 \leq \omega_1 \leq 0.20 \quad ; \quad 0.20 \leq \omega_2 \leq 0.30 \\
& 0.20 \leq \omega_3 \leq 0.40 \quad ; \quad 0.22 \leq \omega_4 \leq 0.25 \\
& 0.15 \leq \omega_5 \leq 0.20 \quad ; \quad \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = 1 \\
& \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \geq 0
\end{aligned}$$

and hence after solving we get $\omega = (0.15, 0.28, 0.20, 0.22, 0.15)^T$.

Step 4a: Utilize the L-SVNWA aggregation operator, as given in Eq. (9.3), to aggregate all the rating of the matrix $\mathcal{M}$ corresponding to each alternative $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$. The collective values are

$$\begin{aligned}
r_1 &= \text{L-SVNWA}(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}) \\
&= \left(1 - \prod_{j=1}^{5} \Big(\log_{\lambda_{1j}} \zeta_{1j} \Big)^{\omega_j}, \prod_{j=1}^{5} \Big(\log_{\lambda_{1j}} (1 - \kappa_{1j}) \Big)^{\omega_j}, \prod_{j=1}^{5} \Big(\log_{\lambda_{1j}} (1 - \varphi_{1j}) \Big)^{\omega_j} \right) \\
&= \Bigg(1 - \Big(\log_{0.4}(0.5) \Big)^{0.15} \times \Big(\log_{0.3}(0.5) \Big)^{0.28} \times \Big(\log_{0.1}(0.2) \Big)^{0.20} \times \\
&\qquad \times \Big(\log_{0.2}(0.3) \Big)^{0.22} \times \Big(\log_{0.2}(0.3) \Big)^{0.15}, \quad \Big(\log_{0.4}(0.7) \Big)^{0.15} \times \\
&\qquad \times \Big(\log_{0.3}(0.8) \Big)^{0.28} \times \Big(\log_{0.1}(0.8) \Big)^{0.20} \times \Big(\log_{0.2}(0.8) \Big)^{0.22} \times \\
&\qquad \times \Big(\log_{0.2}(0.7) \Big)^{0.15}, \quad \Big(\log_{0.4}(0.6) \Big)^{0.15} \times \Big(\log_{0.3}(0.7) \Big)^{0.28} \times \\
&\qquad \times \Big(\log_{0.1}(0.4) \Big)^{0.20} \times \Big(\log_{0.2}(0.6) \Big)^{0.22} \times \Big(\log_{0.2}(0.6) \Big)^{0.15} \Bigg) \\
&= \Big(0.3130, 0.1753, 0.3544 \Big)
\end{aligned}$$

$$\begin{aligned}
r_2 &= \text{L-SVNWA}(\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}, \alpha_{25}) \\
&= \Bigg(1 - \Big(\log_{0.4}(0.7) \Big)^{0.15} \times \Big(\log_{0.2}(0.7) \Big)^{0.28} \times \Big(\log_{0.3}(0.6) \Big)^{0.20} \times \\
&\qquad \times \Big(\log_{0.4}(0.6) \Big)^{0.22} \times \Big(\log_{0.3}(0.7) \Big)^{0.15}, \Big(\log_{0.4}(0.9) \Big)^{0.15} \times
\end{aligned}$$

$$
\begin{aligned}
& \times\Big(\log_{0.2}(0.8)\Big)^{0.28}\times\Big(\log_{0.3}(0.7)\Big)^{0.20}\times\Big(\log_{0.4}(0.6)\Big)^{0.22}\times\\
& \times\Big(\log_{0.3}(0.9)\Big)^{0.15},\Big(\log_{0.4}(0.7)\Big)^{0.15}\times\Big(\log_{0.2}(0.7)\Big)^{0.28}\times\\
& \times\Big(\log_{0.3}(0.4)\Big)^{0.20}\times\Big(\log_{0.4}(0.8)\Big)^{0.22}\times\Big(\log_{0.3}(0.8)\Big)^{0.15}\Bigg)\\
= \;& \big(0.6486, 0.1989, 0.2313\big)
\end{aligned}
$$

Similarly, we can get $r_3 = \big(0.4989, 0.1733, 0.3321\big)$, $r_4 = \big(0.5585, 0.2736, 0.1942\big)$ and $r_5 = \big(0.2849, 0.1249, 0.3758\big)$.

Step 4b: If we utilize L-SVNWG aggregation operator, given in Eq. (9.8) to aggregate the decision information, then the values are obtained as $r_1 = \big(0.3006, 0.1992, 0.3709\big)$, $r_2 = \big(0.6147,\ 0.2764,\ 0.2421\big)$, $r_3 = \big(0.4899,\ 0.2490,\ 0.3819\big)$, $r_4 = \big(0.4416,\ 0.3313,\ 0.2004\big)$, and $r_5 = \big(0.2724,\ 0.1488,\ 0.4597\big)$.

Step 5: The score values of the aggregated numbers r_i's corresponding to L-SVNWA operator are $\mathcal{S}(r_1) = -0.2168$, $\mathcal{S}(r_2) = 0.2185$, $\mathcal{S}(r_3) = -0.0066$, $\mathcal{S}(r_4) = 0.0907$ and $\mathcal{S}(r_5) = -0.2158$. Thus, the ranking order of the alternatives is $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$. On the other hand, score values of the aggregated number obtained through L-SVNWG operator are $\mathcal{S}(r_1) = -0.2695$, $\mathcal{S}(r_2) = 0.0962$, $\mathcal{S}(r_3) = -0.1410$, $\mathcal{S}(r_4) = -0.0900$ and $\mathcal{S}(r_5) = -0.3362$. Thus, the ranking order of the alternatives is $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_5$ in which $\succ$ means "preferred to".

From these ranking order, we see that the best alternative remains $\mathcal{A}_2$ by both the operators while the alternative $\mathcal{A}_5 \succ \mathcal{A}_1$ for averaging and $\mathcal{A}_1 \succ \mathcal{A}_5$ for geometric AO. Thus, according to the decision maker behavior either as an optimistic or pessimistic, they can select the alternatives accordingly and reach their desired goals.

9.4.2 Validity test

Wang and Triantaphyllou [160] established the following testing criteria to evaluate the validity of MADM methods.

Test criterion 1: "An effective MADM method does not change the index of the best alternative by replacing a non-optimal alternative with a worse alternative without shifting the corresponding importance of every decision attribute".

Test criterion 2: "To an effective MADM method must be satisfied transitive property".

Test criterion 3: "If we decomposed a MADM problem into the sub DM problems and same MADM method is utilized on subproblems to rank alternatives, the collective ranking of alternatives must be identical to ranking of un-decomposed DM problem".

Validity test by criterion 1

Under the test criterion 1, we change the rating value for non-optimal alternative $\mathcal{A}_1$ by an arbitrary worse alternative $\mathcal{A}_1'$ as $\mathcal{A}_1' = \{(\mathcal{G}_1$, 0.4, 0.4, 0.6), ($\mathcal{G}_2$, 0.3, 0.4, 0.5), ($\mathcal{G}_3$, 0.1, 0.5, 0.4), ($\mathcal{G}_4$, 0.2, 0.3, 0.5), ($\mathcal{G}_5$, 0.2, 0.4, 0.4)$\}$ which are summarized as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}_1' \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \mathcal{A}_4 \\ \mathcal{A}_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 & \mathcal{G}_5 \end{array} \\ \begin{pmatrix} (0.4,0.4,0.6) & (0.3,0.4,0.5) & (0.1,0.5,0.4) & (0.2,0.3,0.5) & (0.2,0.4,0.4) \\ (0.7,0.1,0.3) & (0.7,0.2,0.3) & (0.6,0.3,0.2) & (0.6,0.4,0.2) & (0.7,0.1,0.2) \\ (0.5,0.3,0.4) & (0.6,0.2,0.4) & (0.6,0.1,0.2) & (0.5,0.1,0.3) & (0.6,0.4,0.3) \\ (0.7,0.3,0.2) & (0.7,0.2,0.2) & (0.4,0.5,0.2) & (0.5,0.2,0.2) & (0.4,0.5,0.4) \\ (0.4,0.1,0.3) & (0.5,0.1,0.2) & (0.4,0.1,0.5) & (0.4,0.3,0.6) & (0.3,0.2,0.4) \end{pmatrix} \end{array}$$

Then, by utilizing the proposed approach using L-SVNWA operator to this transform data, we get the score values of the alternatives $\mathcal{A}_i (i = 1, 2, 3, 4, 5)$ as -0.7859, 0.2185, -0.0066, 0.0907 and -0.2158. Thus, ranking order of the alternatives is $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1'$ and it validate the test criterion 1.

Validity test by criteria 2 and 3

Under these test, we have decomposed original DM problem into three sub problems with alternatives $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_5\}$, $\{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$ and $\{\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$. Now, by applying proposed MADM approach on these alternatives by using L-SVNWA approach then we get $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$, $\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$ and $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5$ respectively. Therefore, from these, we get the final ranking order as $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$ which is same as the original ranking. Hence, it validates test criteria 2 and 3.

9.4.3 Influence of logarithm operation and λ selection in practice

Here, we have investigated the influence of the logarithm operations for SVNNs and the selection of the logarithm base parameter λ in practice. From SVNN $\mathcal{A}$, the operation of logarithm is defined as $\log_\lambda \mathcal{A} = (1 - \log_\lambda \zeta, \log_\lambda(1-\kappa), \log_\lambda(1-\varphi))$ and $0 < \lambda \leq \min\{\zeta, 1-\kappa, 1-\varphi\} < 1$. When $0 < \lambda < 1$, $\log_\lambda \gamma$ increases as λ increases, however, the larger the value of a real number γ, the smaller the value of $\log_\lambda \gamma$ is.

(P1) From the properties of the logarithm function of real numbers, and from the definition of $\log_\lambda \mathcal{A}$, we observe that

(a) There always exists a real number $\lambda_1 = \zeta^{\frac{1}{1-\zeta}}$ such that

- if $\lambda = \lambda_1$, then $1 - \log_\lambda \zeta = \zeta$;
- if $\lambda > \lambda_1$, then $1 - \log_\lambda \zeta < \zeta$; and
- if $\lambda < \lambda_1$, then $1 - \log_\lambda \zeta > \zeta$.

(b) There always exists a real number $\lambda_2 = (1-\kappa)^{\frac{1}{\kappa}}$ such that

- If $\lambda = \lambda_2$, then $\log_\lambda(1-\kappa) = \kappa$;
- If $\lambda > \lambda_2$, then $\log_\lambda(1-\kappa) > \kappa$;
- If $\lambda < \lambda_2$, then $\log_\lambda(1-\kappa) < \kappa$.

(c) There exists a real number $\lambda_3 = (1-\varphi)^{\frac{1}{\varphi}}$ such that

- If $\lambda = \lambda_3$, then $\log_\lambda(1-\varphi) = \varphi$;
- If $\lambda > \lambda_3$, then $\log_\lambda(1-\varphi) > \varphi$;
- If $\lambda < \lambda_3$, then $\log_\lambda(1-\varphi) < \varphi$.

(P2) If we choose a relative small number λ such that $\lambda < \lambda_1 < \lambda_2 < \lambda_3$ and $\lambda < \zeta$ then by part (P1), we have $1 - \log_\lambda \zeta > \zeta$, $\log_\lambda(1-\kappa) < \kappa$ and $\log_\lambda(1-\varphi) < \varphi$. From this, it implies that $\log_\lambda \mathcal{A} > \mathcal{A}$, i.e., the value of SVNN $\mathcal{A}$ will be increased after applying the logarithmic operator. In other words, the logarithm operator will enhanced the values of SVNN.

(P3) If we choose the parameter λ in such a way that $\lambda_1 < \lambda < \lambda_2 < \lambda_3$, then we get $1 - \log_\lambda \zeta < \zeta$, $\log_\lambda(1-\varphi) < \varphi$ and $\log_\lambda(1-\kappa) < \kappa$ which suggests that the value of the truth, indeterminacy and the falsity degrees are decreased after applying the logarithm operator.

(P4) If we choose the parameter λ in such a way that $\lambda_1 < \lambda_2 < \lambda < \lambda_3$, then we get $1 - \log_\lambda \zeta < \zeta$, $\log_\lambda(1-\kappa) > \kappa$ and $\log_\lambda(1-\varphi) < \varphi$ which suggests that the value of the truth and falsity degrees decreases while indeterminacy degree increases after applying the logarithm operator.

(P5) If we choose a relatively large number λ such that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda$ then we get $\log_\lambda \mathcal{A} < \mathcal{A}$. That is, the logarithm operator will reduce the value of SVNN.

Therefore, based on this comprehensive evaluation, the decision maker can select the desired value of λ for their suitable task. For instance, if we want to enhance the SVNN which is undervalued for poor information, then we can choose the logarithm operator with a small number λ, and vice versa. This task has been illustrated in the following example.

Consider $\mathcal{A}_1$=(0.3, 0.5, 0.3), $\mathcal{A}_2$ =(0.7, 0.2, 0.4), $\mathcal{A}_3$=(0.2, 0.7, 0.1) and $\mathcal{A}_4$=(0.4, 0.2, 0.8) be four SVNNs, which is the achievement of the employ evaluated by the their senior administrator during their promotion interview. In the following, we show the different comprehensive evaluation based on the above results.

(1) If we utilized traditional SVNWA operator with senior administrator weight ω = (0.25, 0.25, 0.25, 0.25)^T towards the rating, then

$$\begin{aligned}
\mathcal{L}_1 &= \text{SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) \\
&= \left(1 - \prod_{j=1}^{4}(1-\zeta_j)^{\omega_j}, \prod_{j=1}^{4}(\kappa_j)^{\omega_j}, \prod_{j=1}^{4}(\varphi_j)^{\omega_j}\right) \\
&= \begin{pmatrix} 1 - (0.7 \times 0.3 \times 0.8 \times 0.6)^{0.25}, (0.5 \times 0.2 \times 0.7 \times 0.2)^{0.25}, \\ (0.3 \times 0.4 \times 0.1 \times 0.8)^{0.25} \end{pmatrix} \\
&= \Big(0.4365, 0.3440, 0.3130\Big)
\end{aligned}$$

(2) Assume that the values of $\mathcal{A}_j (j = 1, 2, 3, 4)$ are all undervalued for evaluator's preferences and have re-evaluated according to (P1). In order to get more reasonable results, expert will agree to adjust their scores by using logarithm operator and then by

(P1), we get $(\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}) = (0.1791, 0.3046, 0.1337, 0.2172)$, $(\lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{24}) =$ (0.2500, 0.3277, 0.1791, 0.3277) and $(\lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{34})=$ (0.3046, 0.2789, 0.3487, 0.1337) be the threshold values of $\mathcal{A}_j (j = 1, 2, 3, 4)$. Here, we assume that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.15$, then

$$\begin{aligned}
\mathcal{L}_2 &= \text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) \\
&= \left(1 - \prod_{j=1}^{4} \Big(\log_{\lambda} \zeta_j\Big)^{\omega_j}, \prod_{j=1}^{4} \Big(\log_{\lambda}(1-\kappa_j)\Big)^{\omega_j}, \prod_{j=1}^{4} \Big(\log_{\lambda}(1-\varphi_j)\Big)^{\omega_j}\right) \\
&= \Bigg(1 - \Big(\log_{0.15} 0.3\Big)^{0.25} \times \Big(\log_{0.15} 0.7\Big)^{0.25} \times \Big(\log_{0.15} 0.2\Big)^{0.25} \times \Big(\log_{0.15} 0.4\Big)^{0.25}, \\
&\quad \Big(\log_{0.15} 0.5 \times \log_{0.15} 0.8 \times \log_{0.15} 0.3 \times \log_{0.15} 0.8\Big)^{0.25}, \\
&\quad \Big(\log_{0.15} 0.7 \times \log_{0.15} 0.6 \times \log_{0.15} 0.9 \times \log_{0.15} 0.2\Big)^{0.25}\Bigg) \\
&= \Big(0.5298, 0.2380, 0.2210\Big)
\end{aligned}$$

(3) In order to induce the values of $\mathcal{A}_j (j = 1, 2, 3, 4)$ reasonable and by taking the different weight of the senior administrator during evaluating the achievement of the employ, we suppose that $\lambda_1 = 0.2$, $\lambda_2 = 0.4$, $\lambda_3 = 0.1$ and $\lambda_4 = 0.2$ and $\omega_j = \frac{\lambda_j}{\sum_{j=1}^{5} \lambda_j}$, we get $\omega_1 = 0.2222$, $\omega_2 = 0.4444$, $\omega_3 = 0.1111$ and $\omega_4 = 0.2223$. Thus, by using L-SVNWA operator to aggregate the different preference, we have

$$\begin{aligned}
\mathcal{L}_3 &= \text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) \\
&= \left(1 - \prod_{j=1}^{4} \Big(\log_{\lambda_j} \zeta_j\Big)^{\omega_j}, \prod_{j=1}^{4} \Big(\log_{\lambda_j}(1-\kappa_j)\Big)^{\omega_j}, \prod_{j=1}^{4} \Big(\log_{\lambda_j}(1-\varphi_j)\Big)^{\omega_j}\right) \\
&= \Bigg(1 - \Big(\log_{0.2} 0.3\Big)^{0.2222} \times \Big(\log_{0.4} 0.7\Big)^{0.4444} \times \Big(\log_{0.1} 0.2\Big)^{0.1111} \times \Big(\log_{0.2} 0.4\Big)^{0.2223}, \\
&\quad \Big(\log_{0.2} 0.5\Big)^{0.2222} \times \Big(\log_{0.4} 0.8\Big)^{0.4444} \times \Big(\log_{0.1} 0.3\Big)^{0.1111} \times \Big(\log_{0.2} 0.8\Big)^{0.2223}, \\
&\quad \Big(\log_{0.2} 0.7\Big)^{0.2222} \times \Big(\log_{0.4} 0.6\Big)^{0.4444} \times \Big(\log_{0.1} 0.9\Big)^{0.1111} \times \Big(\log_{0.2} 0.2\Big)^{0.2223}\Bigg) \\
&= \Big(0.4773, 0.2655, 0.3917\Big)
\end{aligned}$$

(4) Consider the weight of the senior administrator which are going to evaluate the performance of the employs are not equally distributed. For it, we assume that during

the evaluation, the weight of second administrator is double than the third ones and half than the one administrator. On the other hand, the importance of the first administrator is double than the fourth administrator. Thus, it implies that $\omega_1 = 4/9$, $\omega_2 = 2/9$, $\omega_3 = 1/9$ and $\omega_4 = 2/9$ for some acceptable reasons. Further, for the undervalued SVNNs, assume that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.15$, then

$$\begin{aligned}
\mathcal{L}_4 &= \text{L-SVNWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) \\
&= \left(1 - \prod_{j=1}^{4} \left(\log_\lambda \zeta_j\right)^{\omega_j}, \prod_{j=1}^{4} \left(\log_\lambda (1-\kappa_j)\right)^{\omega_j}, \prod_{j=1}^{4} \left(\log_\lambda (1-\varphi_j)\right)^{\omega_j}\right) \\
&= \Bigg(1 - \left(\log_{0.15} 0.3\right)^{4/9} \times \left(\log_{0.15} 0.7\right)^{2/9} \times \left(\log_{0.15} 0.2\right)^{1/9} \times \left(\log_{0.15} 0.4\right)^{2/9}, \\
&\quad \left(\log_{0.15} 0.5\right)^{4/9} \times \left(\log_{0.15} 0.8\right)^{2/9} \times \left(\log_{0.15} 0.3\right)^{1/9} \times \left(\log_{0.15} 0.8\right)^{2/9}, \\
&\quad \left(\log_{0.15} 0.7\right)^{4/9} \times \left(\log_{0.15} 0.6\right)^{2/9} \times \left(\log_{0.15} 0.9\right)^{1/9} \times \left(\log_{0.15} 0.2\right)^{2/9}\Bigg) \\
&= \left(0.5293, 0.2347, 0.2486\right)
\end{aligned}$$

Hence, based on score function, we get $\mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_1 > \mathcal{L}_3$ which is in accordance with our expectation.

9.4.4 Comparison with other existing methods

In order to verify the validity of our method, we make a comparison between our proposed operator with the weighted averaging and geometric AOs as proposed by the authors in [79, 113, 175, 184] for multi attribute DM with SVN information. Under this, if we utilize weighted averaging AOs such as SVNWA [113], SVNOWA[113], NWA[175], SVNHWA[79], and SVNFWA [106] under SVN environment to aggregate the information of each alternative into the collective one, then the aggregated values corresponding to these operators are summarized in Table 9.1 along with the proposed operators. On the other hand, the aggregated values by using some existing geometric AOs which include the NWG[175], SVNWG[113], SVNOWG[113], SVNFWG[106], SNWEA[184] and SVNHWG[79] operators are summarized in Table 9.2 along with the proposed geometric operators. By using these collective values and the score functions, the ranking order of the alternatives are

Table 9.1: Neutrosophic aggregated results by averaging operators

	SVNWA[113]	SVNOWA[113]	NWA[175]	SVNFWA[106]
$\mathcal{A}_1$	$(0.3779, 0.2259, 0.4002)$	$(0.3820, 0.2449, 0.4071)$	$(0.3779, 0.2314, 0.4223)$	$(0.3755, 0.2262, 0.4018)$
$\mathcal{A}_2$	$(0.6615, 0.2052, 0.2381)$	$(0.6663, 0.1801, 0.2430)$	$(0.6615, 0.2426, 0.2446)$	$(0.6611, 0.2072, 0.2385)$
$\mathcal{A}_3$	$(0.5656, 0.1763, 0.3131)$	$(0.5597, 0.1838, 0.3122)$	$(0.5656, 0.2109, 0.3272)$	$(0.5652, 0.1779, 0.3141)$
$\mathcal{A}_4$	$(0.5722, 0.2929, 0.2219)$	$(0.5706, 0.3145, 0.2219)$	$(0.5722, 0.3348, 0.2338)$	$(0.5692, 0.2956, 0.2225)$
$\mathcal{A}_5$	$(0.4165, 0.1413, 0.3607)$	$(0.3960, 0.1373, 0.3696)$	$(0.4165, 0.1633, 0.4131)$	$(0.4159, 0.1422, 0.3646)$
	SVNHWA [79]		L-SVNWA	L-SVNOWA
	$\gamma = 2$	$\gamma = 3$		
$\mathcal{A}_1$	$(0.3725, 0.2264, 0.4033)$	$(0.3693, 0.2266, 0.4048)$	$(0.3130, 0.1753, 0.3544)$	$(0.3229, 0.1926, 0.3607)$
$\mathcal{A}_2$	$(0.6608, 0.2086, 0.2388)$	$(0.6604, 0.2099, 0.2390)$	$(0.6486, 0.1989, 0.2313)$	$(0.6549, 0.1719, 0.2368)$
$\mathcal{A}_3$	$(0.5648, 0.1790, 0.3149)$	$(0.5645, 0.1800, 0.3157)$	$(0.4989, 0.1733, 0.3321)$	$(0.4896, 0.1823, 0.3303)$
$\mathcal{A}_4$	$(0.5663, 0.2978, 0.2230)$	$(0.5635, 0.3000, 0.2234)$	$(0.5585, 0.2736, 0.1942)$	$(0.5561, 0.2975, 0.1942)$
$\mathcal{A}_5$	$(0.4151, 0.1427, 0.3680)$	$(0.4143, 0.1432, 0.3714)$	$(0.2849, 0.1249, 0.3758)$	$(0.2442, 0.1209, 0.3834)$

Table 9.2: Neutrosophic aggregated results by geometric operators

	SVNWG[113]	SVNOWG[113]	NWG [175]	SVNHWG [79]	
				$\gamma = 2$	$\gamma = 3$
$\mathcal{A}_1$	$(0.3446, 0.2314, 0.4223)$	$(0.3516, 0.2517, 0.4222)$	$(0.3446, 0.2259, 0.4002)$	$(0.3491, 0.2305, 0.4183)$	$(0.3511, 0.2300, 0.4162)$
$\mathcal{A}_2$	$(0.6561, 0.2426, 0.2446)$	$(0.6612, 0.2173, 0.2497)$	$(0.6561, 0.2052, 0.2381)$	$(0.6570, 0.2374, 0.2437)$	$(0.6576, 0.2338, 0.2430)$
$\mathcal{A}_3$	$(0.5609, 0.2109, 0.3272)$	$(0.5548, 0.2257, 0.3233)$	$(0.5609, 0.1763, 0.3131)$	$(0.5617, 0.2059, 0.3250)$	$(0.5621, 0.2025, 0.3237)$
$\mathcal{A}_4$	$(0.5344, 0.3348, 0.2338)$	$(0.5320, 0.3524, 0.2338)$	$(0.5344, 0.2929, 0.2219)$	$(0.5406, 0.3275, 0.2316)$	$(0.5439, 0.3232, 0.2302)$
$\mathcal{A}_5$	$(0.4078, 0.1633, 0.4131)$	$(0.3882, 0.1555, 0.4065)$	$(0.4078, 0.1413, 0.3607)$	$(0.4091, 0.1603, 0.4046)$	$(0.4097, 0.1582, 0.3997)$
	SVNFWG [106]	SNWEA [184]	L-SVNOWG	L-SVNWG	
$\mathcal{A}_1$	$(0.3471, 0.2311, 0.4204)$	$(0.0058, 0.8315, 0.9665)$	$(0.2771, 0.2359, 0.4040)$	$(0.3006, 0.1992, 0.3709)$	
$\mathcal{A}_2$	$(0.6565, 0.2405, 0.2442)$	$(0.1399, 0.7168, 0.7589)$	$(0.6159, 0.2422, 0.2635)$	$(0.6147, 0.2764, 0.2421)$	
$\mathcal{A}_3$	$(0.5612, 0.2089, 0.3263)$	$(0.0727, 0.6965, 0.8422)$	$(0.4909, 0.2743, 0.3657)$	$(0.4899, 0.2490, 0.3819)$	
$\mathcal{A}_4$	$(0.5373, 0.3317, 0.2329)$	$(0.0362, 0.9206, 0.8428)$	$(0.4278, 0.3678, 0.2126)$	$(0.4416, 0.3313, 0.2004)$	
$\mathcal{A}_5$	$(0.4085, 0.1622, 0.4092)$	$(0.0184, 0.6522, 0.9338)$	$(0.2727, 0.1333, 0.4240)$	$(0.2724, 0.1488, 0.4597)$	

summarized in Table 9.3 in which $\succ$ means "preferred to". From these results, it has been seen that the best alternative is $\mathcal{A}_2$ by all the operators while the different AOs have different ranking strategies which are slightly different. Thus, we can conclude that decision maker reach different decisions based on their preference in terms of AOs.

Table 9.3: Ordering of the alternatives

Existing operators	Ordering	Proposed operators	Ordering
NWA [175]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SVNWA[113]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$	L-SVNWA	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
SVNOWA[113]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$	L-SVNOWA	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
SVNWG[113]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$	L-SVNWG	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_5$
SVNOWG[113]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$	L-SVNOWG	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
SVNFWA[106]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SVNHWA[79] ($\gamma = 2$)	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SVNHWA[79] ($\gamma = 3$)	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
NWG [175]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SVNFWG[106]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SVNHWG[79] ($\gamma = 2$)	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SVNHWG[79] ($\gamma = 3$)	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$		
SNWEA[184]	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1$		

9.5 Conclusion

In this chapter, we present a novel logarithm operational laws (LOL) of SVNSs with the real base number λ which is a useful supplement to the existing operational laws. Also, we have examined their properties and correlations. Based on these LOLs, we developed the weighted averaging and geometric AOs named as L-SVNWA, L-SVNOWA, L-SVNWG, and L-SVNOWG. Then, we utilized these operators to develop a multiattribute DM approach for solving the practical problem with SVN fuzzy information. The proposed approach has been verified by an illustrative example. A comparative study with several of the existing approaches is presented to show their superiority as well as the validity of the approach. At last, the influence of the logarithm operations, as well as the selection of the logarithm base λ, are discussed. From the study, it is concluded that the proposed operational laws and the AOs can equivalently solve the DM problem in a more efficient manner. Also, by assigning a different parameter to base λ, the decision maker can choose the best alternative according to his or her preferences.

Chapter 10

Linguistic single-valued neutrosophic prioritized aggregation operators and their applications to group decision-making[1]

This chapter deal with DM problems to address qualitative information rather than quantitative information. For it, a concept of linguistic SVNS is utilized with linguistic variables (LVs) to represent the data. Further, to add prioritized factor during analyzing the data, we developed some new laws and hence based on it, some new prioritized weighted averaging and geometric AOs are developing to address the problems with LSVN information. The technique corresponding to them is demonstrated with a numerical example and compared with the existing studies.

10.1 Introduction

In the literature, most of the AOs are based on the assumption that input argument which we want to aggregate are independent but it may be possible that there are interactions among the DM criteria in the neutrosophic domain. To resolve this issue, Liu and Wang

---

[1] The content of this chapter is published as "Linguistic single-valued neutrosophic prioritized aggregation operators and their applications to multiple-attribute group decision-making", *Journal of Ambient Intelligence and Humanized Computing*, 9(6), 1975 - 1997, 2018 (**Impact Factor: 1.423**).

[86] applied the Bonferroni mean to the neutrosophic environment and introduce the SVN normalized weighted Bonferroni mean (SVNNWBM) operator. Also, Yang and Li [170] extends the power operator to NS domain which can consider the attribute values to support each other. Wang, Yang and Li [159] proposed the Maclaurin symmetric mean (MSM) AOs to capture the correlation between the aggregated arguments. These existing operators have not considered the condition in which the criteria are dependent and have precedence relation among them. To overcome these flaws, Wu et al. [166] defined the prioritized weighted average operator (SVNPWA) and prioritized weighted geometric operator (SVNPWG) for SVNs. Further, Liu and Wang [87] developed the prioritized ordered weighted/geometric operators in the neutrosophic environment and Ji et al. [57] established the prioritized Bonferroni mean operator by using the Frank operations. Garg and Kumar [44] presented a linguistic connection number for intuitionistic fuzzy set and implemented on the DM problems.

In the neutrosophic environment, the information which is evaluated is quantitative in nature and is expressed by the means of numeric numbers. But in real applications, the decision-makers' opinions or preferences is usually uncertain due to increase in complexities and the subjective nature of human thoughts. Thus, the exact numbers are not the best option to represent such kind of qualitative information. For this, a new concept, namely, LVs [196] has been established to access the information which cannot estimate by exact numbers. Due to the great importance of LVs, it has been becoming the hot topic of research among the researchers. Based on this idea, Li et al. [71] introduce the linguistic neutrosophic sets(LNSs) in which membership, indeterminacy, and non-membership are expressed as a LV instead of real numbers and also proposed some Heronian mean operators. Fang and Ye [39] introduced most basic AOs, namely, Linguistic neutrosophic number weighted averaging and geometric operators. For more details about the extensive literature related to SVNSs under the linguistic approaches, we refer to read Section 1.1.4 of Chapter 1. During their formulation, it has been assumed that all the arguments have the same priority level. However, these existing operators cannot consider the relationship between the attributes.

In the light of the advantages of the LVs and SVNSs, in this chapter, we utilize linguistic neutrosophic number (LNN) and prioritized operator (PO) to aggregate the different preferences of the decision makers. In the approach, LNN is used to represent the information qualitatively while PO is used to consider the interdependency relationship between the attributes by the power weighting. So far, in the literature, the existing PO cannot process the LNNs during the formulation and hence there is an important significance to study it in the present article. For it, we will propose some new prioritized weighted averaging and geometric AOs for the situations in which different criteria's have different priority level under the LSVN environment. Various desirable relations are also investigated. Then, based on the proposed operators, a group DM approach presents to solve the problems. The developed approach demonstrates with a numerical example and compared their results with some of the existing approaches under the linguistic environment.

10.2 Operational laws for Linguistic SVNS

In this section, we discuss some operational laws and corresponding properties for LSVNNs.

Definition 10.2.1. [39] Let $Q = \{s_0, s_1, \ldots, s_t\}$ is a LTS with odd cardinality $t+1$ and $\bar{Q} = \{s_h \mid s_0 \leq s_h \leq s_t, h \in [0,t]\}$. Then, a LSVNS $\mathcal{A}$ in $\mathcal{X}$ is defined as

$$\mathcal{A} = \Big\{(x, s_\theta(x), s_\psi(x), s_\sigma(x)) \mid x \in \mathcal{X}\Big\}, \tag{10.1}$$

where $s_\theta(x), s_\psi(x), s_\sigma(x) \in \bar{Q}$ represent the linguistic truth, indeterminacy and falsity degrees of x to A, respectively, with condition $0 \leq \theta + \psi + \sigma \leq 3t$. The triple $(s_\theta, s_\psi, s_\sigma)$ is called LSVN number (LSVNN).

Definition 10.2.2. Let $\mathcal{A} = (s_\theta, s_\psi, s_\sigma)$, $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ be three LSVNNs and $\lambda > 0$ be any real number, then

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(s_{t\left(\frac{\theta_1}{t} + \frac{\theta_2}{t} - \frac{\theta_1\theta_2}{t^2}\right)}, s_{t\left(\frac{\psi_1\psi_2}{t^2}\right)}, s_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)}\right)$.

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \left(s_{t\left(\frac{\theta_1\theta_2}{t^2}\right)}, s_{t\left(\frac{\psi_1}{t} + \frac{\psi_2}{t} - \frac{\psi_1\psi_2}{t^2}\right)}, s_{t\left(\frac{\sigma_1}{t} + \frac{\sigma_2}{t} - \frac{\sigma_1\sigma_2}{t^2}\right)}\right)$.

(iii) $\lambda\mathcal{A} = \left(s_{t\left(1-\left(1-\frac{\theta}{t}\right)^\lambda\right)}, s_{t\left(\frac{\psi}{t}\right)^\lambda}, s_{t\left(\frac{\sigma}{t}\right)^\lambda}\right)$.

(iv) $\mathcal{A}^{\lambda} = \left(s_{t\left(\frac{\theta}{t}\right)^{\lambda}}, s_{t\left(1-\left(1-\frac{\psi}{t}\right)^{\lambda}\right)}, s_{t\left(1-\left(1-\frac{\sigma}{t}\right)^{\lambda}\right)} \right)$.

Definition 10.2.3. Let $\mathcal{A} = (s_{\theta}, s_{\psi}, s_{\sigma})$, $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ be three LSVNNs, then

(i) $\mathcal{A}^c = (s_{\sigma}, s_{\psi}, s_{\theta})$;

(ii) $\mathcal{A}_1 \cup \mathcal{A}_2 = \Big(\max(s_{\theta_1}, s_{\theta_2}), \min(s_{\psi_1}, s_{\psi_2}), \min(s_{\sigma_1}, s_{\sigma_2}) \Big)$;

(iii) $\mathcal{A}_1 \cap \mathcal{A}_2 = \Big(\min(s_{\theta_1}, s_{\theta_2}), \max(s_{\psi_1}, s_{\psi_2}), \max(s_{\sigma_1}, s_{\sigma_2}) \Big)$;

(iv) $\mathcal{A}_1 = \mathcal{A}_2$ if $s_{\theta_1} = s_{\theta_2}$, $s_{\psi_1} = s_{\psi_2}$ and $s_{\sigma_1} = s_{\sigma_2}$;

(v) $\mathcal{A}_1 \geq \mathcal{A}_2$ if $s_{\theta_1} \geq s_{\theta_2}$, $s_{\psi_1} \leq s_{\psi_2}$ and $s_{\sigma_1} \leq s_{\sigma_2}$.

Theorem 10.2.1. All the operational laws for LSVNNs given in Definition 10.2.2, i.e., $\mathcal{A}_1 \oplus \mathcal{A}_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \lambda\mathcal{A}$ and $\mathcal{A}^{\lambda}$ are also LSVNNs.

Proof. For two LSVNNs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

$$\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(s_{t\left(\frac{\theta_1}{t} + \frac{\theta_2}{t} - \frac{\theta_1\theta_2}{t^2}\right)}, s_{t\left(\frac{\psi_1\psi_2}{t^2}\right)}, s_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)} \right)$$

Since we have, $0 \leq \theta_1, \theta_2 \leq t \Rightarrow 0 \leq \theta_1/t, \theta_2/t \leq 1 \Rightarrow 0 \leq (1 - \theta_1/t)(1 - \theta_2/t) \leq 1 \Rightarrow 0 \leq 1 - (1 - \theta_1/t)(1 - \theta_2/t) \leq 1$. Thus, we get $0 \leq t\big(1 - (1 - \theta_1/t)(1 - \theta_2/t)\big) \leq t$. Also, $0 \leq \sigma_1, \sigma_2 \leq t \Rightarrow 0 \leq \sigma_1/t, \sigma_2/t \leq 1 \Rightarrow 0 \leq \sigma_1\sigma_2/t^2 \leq 1 \Rightarrow 0 \leq t\big(\sigma_1\sigma_2/t^2\big) \leq t$. Further, $0 \leq t\big(1 - (1 - \theta_1/t)(1 - \theta_2/t)\big) + t\big(\sigma_1\sigma_2/t^2\big) + t\big(\psi_1\psi_2/t^2\big) \leq 3t$. Therefore, $\mathcal{A}_1 \oplus \mathcal{A}_2$ is a LSVNN. Similarly, we can prove for the other parts. □

Theorem 10.2.2. Let $\mathcal{A} = (s_{\theta}, s_{\psi}, s_{\sigma})$, $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ be three LSVNNs and $\lambda, \lambda_1, \lambda_2 > 0$ be three real numbers, then we have

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$.

(ii) $\lambda(\mathcal{A}_1 \oplus \mathcal{A}_2) = \lambda\mathcal{A}_1 \oplus \lambda\mathcal{A}_2$.

(iii) $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = (\lambda_1 + \lambda_2)\mathcal{A}$.

(iv) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$.

(v) $\mathcal{A}_1^\lambda \otimes \mathcal{A}_2^\lambda = (\mathcal{A}_1 \otimes \mathcal{A}_2)^\lambda$.

(vi) $\mathcal{A}^{\lambda_1} \otimes \mathcal{A}^{\lambda_2} = \mathcal{A}^{\lambda_1+\lambda_2}$.

(vii) $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}_2 \cup \mathcal{A}_1$.

(viii) $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{A}_2 \cap \mathcal{A}_1$.

(ix) $(\mathcal{A}_1 \cup \mathcal{A}_2) \cap \mathcal{A}_2 = \mathcal{A}_2$.

(x) $(\mathcal{A}_1 \cap \mathcal{A}_2) \cup \mathcal{A}_2 = \mathcal{A}_2$.

(xi) $\lambda(\mathcal{A}_1 \cup \mathcal{A}_2) = \lambda\mathcal{A}_1 \cup \lambda\mathcal{A}_2$.

(xii) $\mathcal{A}_1^\lambda \cup \mathcal{A}_2^\lambda = (\mathcal{A}_1 \cup \mathcal{A}_2)^\lambda$.

(xiii) $\mathcal{A}_1^c \oplus \mathcal{A}_2^c = (\mathcal{A}_1 \otimes \mathcal{A}_2)^c$.

(xiv) $\mathcal{A}_1^c \otimes \mathcal{A}_2^c = (\mathcal{A}_1 \oplus \mathcal{A}_2)^c$.

(xv) $\mathcal{A}_1^c \cup \mathcal{A}_2^c = (\mathcal{A}_1 \cap \mathcal{A}_2)^c$.

(xvi) $\mathcal{A}_1^c \cap \mathcal{A}_2^c = (\mathcal{A}_1 \cup \mathcal{A}_2)^c$.

(xvii) $(\mathcal{A}^c)^\lambda = (\lambda\mathcal{A})^c$.

(xviii) $\lambda(\mathcal{A}^c) = (\mathcal{A}^\lambda)^c$.

Proof. We will prove the parts (ii), (iii) and the proofs of others are similar.

(ii) For $\lambda > 0$,

$$\begin{aligned}
\lambda(\mathcal{A}_1 \oplus \mathcal{A}_2) &= \lambda\left(s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)\left(1-\frac{\theta_2}{t}\right)\right)}, s_{t\left(\frac{\psi_1\psi_2}{t^2}\right)}, s_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)}\right) \\
&= \left(s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^\lambda\left(1-\frac{\theta_2}{t}\right)^\lambda\right)}, s_{t\left(\frac{\psi_1}{t}\right)^\lambda\left(\frac{\psi_2}{t}\right)^\lambda}, s_{t\left(\frac{\sigma_1}{t}\right)^\lambda\left(\frac{\sigma_2}{t}\right)^\lambda}\right) \\
&= \lambda\mathcal{A}_1 \oplus \lambda\mathcal{A}_2
\end{aligned}$$

(iii) For $\lambda_1, \lambda_2 > 0$ be real numbers,

$$\lambda_1\mathcal{A} = \left(s_{t\left(1-\left(1-\frac{\theta}{t}\right)^{\lambda_1}\right)}, s_{t\left(\frac{\psi}{t}\right)^{\lambda_1}}, s_{t\left(\frac{\sigma}{t}\right)^{\lambda_1}}\right)$$

$$\text{and} \qquad \lambda_2\mathcal{A} = \left(s_{t\left(1-\left(1-\frac{\theta}{t}\right)^{\lambda_2}\right)}, s_{t\left(\frac{\psi}{t}\right)^{\lambda_2}}, s_{t\left(\frac{\sigma}{t}\right)^{\lambda_2}}\right)$$

Thus,

$$\begin{aligned}
\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} &= \Big(s_{t\left(1-\left(1-\frac{\theta}{t}\right)^{\lambda_1}+1-\left(1-\frac{\theta}{t}\right)^{\lambda_2}-1+\left(1-\frac{\theta}{t}\right)^{\lambda_1}+\left(1-\frac{\theta}{t}\right)^{\lambda_2}-\left(1-\frac{\theta}{t}\right)^{\lambda_1+\lambda_2}\right)}, \\
&\qquad s_{t\left(\frac{\psi}{t}\right)^{\lambda_1+\lambda_2}}, s_{t\left(\frac{\sigma}{t}\right)^{\lambda_1+\lambda_2}} \Big) \\
&= \Big(s_{t\left(1-\left(1-\frac{\theta}{t}\right)^{\lambda_1+\lambda_2}\right)}, s_{t\left(\frac{\psi}{t}\right)^{\lambda_1+\lambda_2}}, s_{t\left(\frac{\sigma}{t}\right)^{\lambda_1+\lambda_2}} \Big) \\
&= (\lambda_1+\lambda_2)\mathcal{A}
\end{aligned}$$

□

Theorem 10.2.3. Let $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ be two LSVNNs, then

(i) $(\mathcal{A}_1 \cup \mathcal{A}_2) \oplus (\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{A}_1 \oplus \mathcal{A}_2$.

(ii) $(\mathcal{A}_1 \cup \mathcal{A}_2) \otimes (\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{A}_1 \otimes \mathcal{A}_2$.

Proof. We will prove part (i), proof of other is similar.

(i) As we have,

$$\begin{aligned}
\mathcal{A}_1 \cup \mathcal{A}_2 &= \Big(\max(s_{\theta_1}, s_{\theta_2}), \min(s_{\psi_1}, s_{\psi_2}), \min(s_{\sigma_1}, s_{\sigma_2}) \Big) \\
&= \Big(s_{\max(\theta_1,\theta_2)}, s_{\min(\psi_1,\psi_2)}, s_{\min(\sigma_1,\sigma_2)} \Big) \\
\text{and} \qquad \mathcal{A}_1 \cap \mathcal{A}_2 &= \Big(\min(s_{\theta_1}, s_{\theta_2}), \max(s_{\psi_1}, s_{\psi_2}), \max(s_{\sigma_1}, s_{\sigma_2}) \Big) \\
&= \Big(s_{\min(\theta_1,\theta_2)}, s_{\max(\psi_1,\psi_2)}, s_{\max(\sigma_1,\sigma_2)} \Big)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\mathcal{A}_1 \cup \mathcal{A}_2) \oplus (\mathcal{A}_1 \cap \mathcal{A}_2) \\
= &\Bigg(s_{t\left(\frac{\max(\theta_1,\theta_2)}{t}+\frac{\min(\theta_1,\theta_2)}{t}-\frac{\max(\theta_1,\theta_2)\min(\theta_1,\theta_2)}{t^2}\right)}, s_{t\left(\frac{\min(\psi_1,\psi_2)}{t}\frac{\max(\psi_1,\psi_2)}{t}\right)}, \\
&\quad s_{t\left(\frac{\min(\sigma_1,\sigma_2)}{t}\frac{\max(\sigma_1,\sigma_2)}{t}\right)} \Bigg) \\
= &\Big(s_{t\left(\theta_1/t+\theta_2/t-\theta_1\theta_2/t^2\right)}, s_{t\left(\psi_1\psi_2/t^2\right)}, s_{t\left(\sigma_1\sigma_2/t^2\right)} \Big) \\
= &\ \mathcal{A}_1 \oplus \mathcal{A}_2
\end{aligned}$$

□

Theorem 10.2.4. Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be three LSVNNs, then

(i) $(\mathcal{A}_1 \cup \mathcal{A}_2) \cap \mathcal{A}_3 = (\mathcal{A}_1 \cap \mathcal{A}_3) \cup (\mathcal{A}_2 \cap \mathcal{A}_3)$.

(ii) $(\mathcal{A}_1 \cap \mathcal{A}_2) \cup \mathcal{A}_3 = (\mathcal{A}_1 \cup \mathcal{A}_3) \cap (\mathcal{A}_2 \cup \mathcal{A}_3)$.

(iii) $(\mathcal{A}_1 \cup \mathcal{A}_2) \oplus \mathcal{A}_3 = (\mathcal{A}_1 \oplus \mathcal{A}_3) \cup (\mathcal{A}_2 \oplus \mathcal{A}_3)$.

(iv) $(\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \mathcal{A}_3 = (\mathcal{A}_1 \oplus \mathcal{A}_3) \cap (\mathcal{A}_2 \oplus \mathcal{A}_3)$.

(v) $(\mathcal{A}_1 \cup \mathcal{A}_2) \otimes \mathcal{A}_3 = (\mathcal{A}_1 \otimes \mathcal{A}_3) \cup (\mathcal{A}_2 \otimes \mathcal{A}_3)$.

(vi) $(\mathcal{A}_1 \cap \mathcal{A}_2) \otimes \mathcal{A}_3 = (\mathcal{A}_1 \otimes \mathcal{A}_3) \cap (\mathcal{A}_2 \otimes \mathcal{A}_3)$.

Theorem 10.2.5. Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be three LSVNNs, then

(i) $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 = \mathcal{A}_1 \cup \mathcal{A}_3 \cup \mathcal{A}_2$.

(ii) $\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 = \mathcal{A}_1 \cap \mathcal{A}_3 \cap \mathcal{A}_2$.

(iii) $\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 = \mathcal{A}_1 \oplus \mathcal{A}_3 \oplus \mathcal{A}_2$.

(iv) $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3 = \mathcal{A}_1 \otimes \mathcal{A}_3 \otimes \mathcal{A}_2$.

Proof of Theorems 10.2.4 and 10.2.5 are straightforward, so we omit here.

10.3 Prioritized AO for linguistic single-valued neutrosophic sets

In this section, for collection of "n" LSVNNs $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$ denoted by Ω, we presents some new prioritized AOs for LSVNss, namely, LSVN prioritized weighted/ordered weighted averaging and geometric AOs.

10.3.1 Prioritized weighted averaging operator

Definition 10.3.1. A LSVNPWA operator is a mapping LSVNPWA : $\Omega^n \to \Omega$, be defined as

$$\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigoplus_{i=1}^{n} \xi_i \mathcal{A}_i$$

where $\xi_i = \frac{T_i}{\sum_{i=1}^{n} T_i}$, $T_1 = 1$ and $T_i = \prod\limits_{k=1}^{i-1} \frac{I\big(\mathcal{S}(\mathcal{A}_k)\big)}{t} (i = 2, 3, \ldots, n)$. Here $I\big(\mathcal{S}(\mathcal{A}_k)\big)$ represents the subscript of score function, $\mathcal{S}(\mathcal{A}_k)$, of the LSVNN $\mathcal{A}_k$.

Theorem 10.3.1. For a collection of LSVNNs $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})(i = 1, 2, \ldots, n)$, the aggregated value by using proposed LSVNPWA operator is still a LSVNN and is given by:

$$\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(s_{t\left(1-\prod\limits_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\xi_i}\right)} \right) \tag{10.2}$$

Proof. Firstly, we will prove Eq. (10.2) by mathematical induction on n. Since for all $i, \mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$ is LSVNN so we have $\theta_i + \psi_i + \sigma_i \leq 3t$. Then the following steps of the mathematical induction have been followed.

Step 1: For $n = 2$, we have

$$\begin{aligned}
\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2) &= \xi_1 \mathcal{A}_1 \oplus \xi_2 \mathcal{A}_2 \\
&= \begin{pmatrix} s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\xi_1}\right)}, \\ s_{t\left(\left(\frac{\psi_1}{t}\right)^{\xi_1}\right)}, s_{t\left(\left(\frac{\sigma_1}{t}\right)^{\xi_1}\right)} \end{pmatrix} \oplus \begin{pmatrix} s_{t\left(1-\left(1-\frac{\theta_2}{t}\right)^{\xi_2}\right)}, \\ s_{t\left(\left(\frac{\psi_2}{t}\right)^{\xi_2}\right)}, s_{t\left(\left(\frac{\sigma_2}{t}\right)^{\xi_2}\right)} \end{pmatrix} \\
&= \left(s_{t\left(1-\prod\limits_{i=1}^{2}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{2}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{2}\left(\frac{\sigma_i}{t}\right)^{\xi_i}\right)} \right)
\end{aligned}$$

Hence result is true for $n = 2$.

Step 2: Assume the Eq. (10.2) is true for $n = z$, then for $n = z + 1$, we have

$$\begin{aligned}
&\bigoplus_{i=1}^{z+1} \frac{T_i}{\sum_{i=1}^{z+1} T_i} \mathcal{A}_i = \bigoplus_{i=1}^{z} \frac{T_i}{\sum_{i=1}^{z+1} T_i} \mathcal{A}_i \oplus \frac{T_{z+1}}{\sum_{i=1}^{z+1} T_i} \mathcal{A}_{z+1} \\
= \quad & \left(s_{t\left(1-\prod\limits_{i=1}^{z}\left(1-\frac{\theta_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{z+1} T_i}}\right)}, s_{t\left(\prod\limits_{i=1}^{z}\left(\frac{\psi_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{z+1} T_i}}\right)}, s_{t\left(\prod\limits_{i=1}^{z}\left(\frac{\sigma_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{z+1} T_i}}\right)} \right) \\
& \oplus \left(s_{t\left(1-\left(1-\frac{\theta_{z+1}}{t}\right)^{\frac{T_{z+1}}{\sum_{i=1}^{z+1} T_i}}\right)}, s_{t\left(\left(\frac{\psi_{z+1}}{t}\right)^{\frac{T_{z+1}}{\sum_{i=1}^{z+1} T_i}}\right)}, s_{t\left(\left(\frac{\sigma_{z+1}}{t}\right)^{\frac{T_{z+1}}{\sum_{i=1}^{z+1} T_i}}\right)} \right) \\
= \quad & \left(s_{t\left(1-\prod\limits_{i=1}^{z+1}\left(1-\frac{\theta_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{z+1} T_i}}\right)}, s_{t\left(\prod\limits_{i=1}^{z+1}\left(\frac{\psi_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{z+1} T_i}}\right)}, s_{t\left(\prod\limits_{i=1}^{z+1}\left(\frac{\sigma_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{z+1} T_i}}\right)} \right)
\end{aligned}$$

Hence the result given in Eq. (10.2), is true for all $n \in \mathbb{Z}^+$

Further, $0 \leq \frac{\theta_i}{t} \leq 1 \Rightarrow 0 \leq 1 - \frac{\theta_i}{t} \leq 1 \Rightarrow 0 \leq 1 - \prod\limits_{i=1}^{n}\left(1 - \frac{\theta_i}{t}\right)^{\xi_i} \leq 1 \Rightarrow 0 \leq t\left(1 - \prod\limits_{i=1}^{n}\left(1 - \frac{\theta_i}{t}\right)^{\xi_i}\right) \leq t$. Similarly, we get

$$0 \leq t\left(\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right), t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\xi_i}\right) \leq t.$$

Also,

$$t\left(1 - \prod_{i=1}^{n}\left(1 - \frac{\theta_i}{t}\right)^{\xi_i}\right) + t\left(\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right) + t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\xi_i}\right) \leq 3t$$

Thus, Eq. (10.2) is still LSVNN, which completes the proof of the theorem.

□

Example 10.3.1. Let $\mathcal{A}_1 = (s_1, s_3, s_4)$, $\mathcal{A}_2 = (s_3, s_2, s_1)$ and $\mathcal{A}_3 = (s_4, s_1, s_2)$ be three LSVNNs. Based on the score function of LSVNNs, we get $I(\mathcal{S}(\mathcal{A}_1)) = 3.3333$, $I(\mathcal{S}(\mathcal{A}_2)) = 5.3333$ and $I(\mathcal{S}(\mathcal{A}_3)) = 5.6667$ and hence $T_1 = 1$, $T_2 = 0.4167$ and $T_3 = 0.2778$. By utilizing this information, we get

$$\begin{aligned}
& \text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \\
= \quad & \left(s_{8\left(1-\prod\limits_{i=1}^{3}(1-\theta_i/8)^{\xi_i}\right)}, s_{8\left(\prod\limits_{i=1}^{3}\left(\psi_i/8\right)^{\xi_i}\right)}, s_{8\left(\prod\limits_{i=1}^{3}\left(\sigma_i/8\right)^{\xi_i}\right)} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} s_{8\left(1-\left(1-\frac{1}{8}\right)^{0.5902}\times\left(1-\frac{3}{8}\right)^{0.2459}\times\left(1-\frac{4}{8}\right)^{0.1639}\right)}, s_{8\left(\left(\frac{3}{8}\right)^{0.5902}\times\left(\frac{2}{8}\right)^{0.2459}\times\left(\frac{1}{8}\right)^{0.1639}\right)}, \\ s_{8\left(\left(\frac{4}{8}\right)^{0.5902}\times\left(\frac{1}{8}\right)^{0.2459}\times\left(\frac{2}{8}\right)^{0.1639}\right)} \end{pmatrix} \\
&= (s_{2.1207}, s_{2.2678}, s_{2.5390})
\end{aligned}
$$

Theorem 10.3.2. Let $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$ $(i = 1, 2, \ldots, n)$ be the collection of LSVNNs, then LSVNPWA operator satisfies the following properties:

(P1): (Idempotency) If all $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$ are equal, i.e. $\mathcal{A}_i = \mathcal{A}$, then

$$\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$$

(P2): (Monotonicity) If $\mathcal{A}'_i = (s_{\theta'_i}, s_{\psi'_i}, s_{\sigma'_i})$ and $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$ be two collections of LSVNNs such that $\theta'_i \geq \theta_i$, $\psi'_i \leq \psi_i$ and $\sigma'_i \leq \sigma_i$ for all i, then

$$\text{LSVNPWA}(\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_n) \geq \text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

(P3): (Boundedness) Let $\mathcal{A}^- = (\min_i\{s_{\theta_i}\}, \max_i\{s_{\psi_i}\}, \max_i\{s_{\sigma_i}\})$ and $\mathcal{A}^+ = (\max_i\{s_{\theta_i}\}, \min_i\{s_{\psi_i}\}, \min_i\{s_{\sigma_i}\})$, then

$$\mathcal{A}^- \leq \text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

Proof. For a collection of LSVNNs, we have

(P1) Since $\mathcal{A}_i = \mathcal{A}$ for all i, then we have

$$\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigoplus_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i}\mathcal{A} = \sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i}\mathcal{A} = \mathcal{A}$$

(P2) Since $\theta'_i \geq \theta_i$ and $\xi_i = \frac{T_i}{\sum_{i=1}^{n} T_i} \in [0, 1]$. Hence, we have

$$
\begin{aligned}
& 0 \leq \left(1 - \frac{\theta'_i}{t}\right) \leq \left(1 - \frac{\theta_i}{t}\right) \leq 1 \\
\Rightarrow\; & \prod_{i=1}^{n}\left(1 - \frac{\theta'_i}{t}\right) \leq \prod_{i=1}^{n}\left(1 - \frac{\theta_i}{t}\right) \\
\Rightarrow\; & 1 - \prod_{i=1}^{n}\left(1 - \frac{\theta'_i}{t}\right)^{\xi_i} \geq 1 - \prod_{i=1}^{n}\left(1 - \frac{\theta_i}{t}\right)^{\xi_i}
\end{aligned}
$$

which implies

$$t\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i'}{t}\right)^{\xi_i}\right) \geq t\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right) \tag{10.3}$$

Similarly, for $\psi_i' \leq \psi_i$ and $\sigma_i' \leq \sigma_i$, we can get

$$t\left(\prod_{i=1}^{n}\left(\frac{\psi_i'}{t}\right)^{\xi_i}\right) \leq t\left(\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right) \tag{10.4}$$

and

$$t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i'}{t}\right)^{\xi_i}\right) \leq t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\xi_i}\right) \tag{10.5}$$

Thus, from inequalities (10.3), (10.4), (10.5) and Definition 2.1.11 of Chapter 2, we have

$$\text{LSVNPWA}(\mathcal{A}_1', \mathcal{A}_2', \ldots, \mathcal{A}_n') \geq \text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

(P3) Since $\min_i\{s_{\theta_i}\} \leq s_{\theta_i} \leq \max_i\{s_{\theta_i}\}$, $\min_i\{s_{\psi_i}\} \leq s_{\psi_i} \leq \max_i\{s_{\psi_i}\}$ and $\min_i\{s_{\sigma_i}\} \leq s_{\sigma_i} \leq \max_i\{s_{\sigma_i}\}$, thus from above property, we have

$$\mathcal{A}^- \leq \text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+$$

□

Next, we introduce the AO which unifies the concept of prioritized weighted average with the ordered weighted operator and named as LSVN prioritized ordered weighted averaging (LSVNPOWA) operator.

Definition 10.3.2. A LSVNPOWA operator is a mapping LSVNPOWA : $\Omega^n \to \Omega$, defined as follows:

$$\text{LSVNPOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigoplus_{i=1}^{n} \xi_i \mathcal{A}_{\delta(i)} \tag{10.6}$$

where $T_1 = 1$ and $T_i = \prod_{k=1}^{i-1} \frac{I\left(\mathcal{S}(\mathcal{A}_{\delta(k)})\right)}{t} (i = 2, 3, \ldots, n)$, δ is permutation of $(1, 2, \ldots, n)$ such that $\delta(i) \geq \delta(i-1)$ for $i = 2, 3, \ldots, n$.

Theorem 10.3.3. The aggregated value of all "n" LSVNNs $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$ by using LSVNPOWA operator is still a LSVNN and is given by:

$$\begin{aligned} &\text{LSVNPOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = &\left(s_{t\left(1-\prod\limits_{i=1}^{n}\left(1-\frac{\theta_{\delta(i)}}{t}\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(\frac{\psi_{\delta(i)}}{t}\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(\frac{\sigma_{\delta(i)}}{t}\right)^{\xi_i}\right)} \right) \end{aligned} \tag{10.7}$$

Proof. The proof is same as of Theorem 10.3.1, so we omit here. □

Example 10.3.2. Consider the data as given in Example 10.3.1, we get $T_1 = 1$, $T_2 = 0.4167$ and $T_3 = 0.2778$. Based on the score function of LSVNNs, we get $I(\mathcal{S}(\mathcal{A}_1)) = 3.3333$, $I(\mathcal{S}(\mathcal{A}_2)) = 5.3333$ and $I(\mathcal{S}(\mathcal{A}_3)) = 5.6667$. Thus, $I(\mathcal{S}(\mathcal{A}_3)) > I(\mathcal{S}(\mathcal{A}_2)) > I(\mathcal{S}(\mathcal{A}_1))$ and hence $\mathcal{A}_{\delta(1)} = \mathcal{A}_3, \mathcal{A}_{\delta(2)} = \mathcal{A}_2$ and $\mathcal{A}_{\delta(3)} = \mathcal{A}_1$. Therefore, by using the Eq. (10.7) and $t = 8$, we get

$$\begin{aligned} &\text{LSVNPOWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \\ = &\left(s_{8\left(1-\prod\limits_{i=1}^{3}\left(1-\frac{\theta_{\delta(i)}}{8}\right)^{\xi_i}\right)}, s_{8\left(\prod\limits_{i=1}^{3}\left(\frac{\psi_{\delta(i)}}{8}\right)^{\xi_i}\right)}, s_{8\left(\prod\limits_{i=1}^{3}\left(\frac{\sigma_{\delta(i)}}{t}\right)^{\xi_i}\right)} \right) \\ = &\left(\begin{array}{l} s_{8\left(1-\left(1-\frac{4}{8}\right)^{0.5902}\times\left(1-\frac{3}{8}\right)^{0.2459}\times\left(1-\frac{1}{8}\right)^{0.1639}\right)}, s_{8\left(\left(\frac{1}{8}\right)^{0.5902}\times\left(\frac{2}{8}\right)^{0.2459}\times\left(\frac{3}{8}\right)^{0.1639}\right)}, \\ s_{8\left(\left(\frac{2}{8}\right)^{0.5902}\times\left(\frac{1}{8}\right)^{0.2459}\times\left(\frac{4}{8}\right)^{0.1639}\right)} \end{array} \right) \\ = &(s_{3.3684}, s_{1.4198}, s_{1.8895}) \end{aligned}$$

As similar to LSVNPWA operator, the proposed LSVNPOWA operator also satisfies the properties as mentioned in Theorem 10.3.2.

10.3.2 Prioritized weighted geometric operator

Here, we have presented some geometric AOs under LSVN environment.

Definition 10.3.3. Let $\mathcal{A}_i, (i = 1, 2, \ldots, n)$ be a collection of LSVNNs. The LSVNPWG operator is a mapping LSVNPWG : $\Omega^n \to \Omega$, such that

$$\text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigotimes_{i=1}^{n} \mathcal{A}_i^{\xi_i}$$

where $T_1 = 1$ and $T_i = \prod_{k=1}^{i-1} \frac{I\big(\mathcal{S}(\mathcal{A}_k)\big)}{t} (i = 2, 3, \ldots, n)$.

Theorem 10.3.4. For a collection of LSVNNs $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i}), (i = 1, 2, \ldots, n)$, the aggregated value by using proposed LSVNPWG operator is still a LSVNN and is given by:

$$\begin{aligned}
&\text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
&= \left(s_{t\left(\prod_{i=1}^{n} \left(\frac{\theta_i}{t}\right)^{\xi_i}\right)}, s_{t\left(1-\prod_{i=1}^{n} \left(1-\frac{\psi_i}{t}\right)^{\xi_i}\right)}, s_{t\left(1-\prod_{i=1}^{n} \left(1-\frac{\sigma_i}{t}\right)^{\xi_i}\right)} \right)
\end{aligned} \tag{10.8}$$

Proof. It is similar to Theorem 10.3.1 and hence we omit them here. □

Example 10.3.3. Let $\mathcal{A}_1 = (s_2, s_1, s_4)$, $\mathcal{A}_2 = (s_1, s_2, s_3)$ and $\mathcal{A}_3 = (s_4, s_5, s_1)$ be three LSVNNs. Based on the score function of LSVNNs, we get $I(\mathcal{S}(\mathcal{A}_1)) = 4.3333$, $I(\mathcal{S}(\mathcal{A}_2)) = 4$ and $I(\mathcal{S}(\mathcal{A}_3)) = 4.6667$ and hence $T_1 = 1$, $T_2 = 0.5417$ and $T_3 = 0.2708$. By utilizing this information and $t = 8$, we get

$$\begin{aligned}
&\text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \\
= &\left(s_{8\prod_{i=1}^{3} \left(\frac{\theta_i}{8}\right)^{\xi_i}}, \quad s_{8\left(1-\prod_{i=1}^{3} \left(1-\frac{\psi_i}{8}\right)^{\xi_i}\right)}, s_{8\left(1-\prod_{i=1}^{3} \left(1-\frac{\sigma_i}{8}\right)^{\xi_i}\right)} \right) \\
= &\left(\begin{array}{c} s_{8\left(\left(\frac{2}{8}\right)^{0.5517} \times \left(\frac{1}{8}\right)^{0.2989} \times \left(\frac{4}{8}\right)^{0.1494}\right)}, \quad s_{8\left(1-\left(1-\frac{1}{8}\right)^{0.5517} \times \left(1-\frac{2}{8}\right)^{0.2989} \times \left(1-\frac{5}{8}\right)^{0.1494}\right)}, \\ s_{8\left(1-\left(1-\frac{4}{8}\right)^{0.5517} \times \left(1-\frac{3}{8}\right)^{0.2989} \times \left(1-\frac{1}{8}\right)^{0.1494}\right)} \end{array} \right) \\
= &\left(s_{2.0865}, s_{1.5646}, s_{2.9837}\right)
\end{aligned}$$

Here, we extend the prioritized weighted geometric operator to prioritized ordered weighted geometric operator for the LSVNNs.

Definition 10.3.4. Let $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$,$(i = 1, 2, \ldots, n)$ be LSVNNs. Then a LSVN prioritized ordered weighted geometric(LSVNPOWG) is defined as follows:

$$\text{LSVNPOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigotimes_{i=1}^{n} \mathcal{A}_{\delta(i)}^{\xi_i} \tag{10.9}$$

where $T_1 = 1$ and $T_i = \prod_{k=1}^{i-1} \frac{I\big(\mathcal{S}(\mathcal{A}_{\delta(k)})\big)}{t} (i = 2, 3, \ldots, n)$, and δ is permutation of $(1, 2, \ldots, n)$ such that $\delta(i) \geq \delta(i-1)$ for $i = (2, 3, \ldots, n)$.

Theorem 10.3.5. The aggregated value of all LSVNNs $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i}), (i = 1, 2, \ldots, n)$, by using LSVNPOWG operator is still a LSVNN and is given by:

$$\begin{aligned}&\text{LSVNPOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\\ &= \left(s_{t\left(\prod\limits_{i=1}^{n}\left(\frac{\theta_{\delta(i)}}{t}\right)^{\xi_i}\right)}, s_{t\left(1-\prod\limits_{i=1}^{n}\left(1-\frac{\psi_{\delta(i)}}{t}\right)^{\xi_i}\right)} s_{t\left(1-\prod\limits_{i=1}^{n}\left(1-\frac{\sigma_{\delta(i)}}{t}\right)^{\xi_i}\right)} \right)\end{aligned} \tag{10.10}$$

Proof. The proof of this theorem is similar to above Theorem. □

Example 10.3.4. Consider the data as given in Example 10.3.3, we get $T_1 = 1$, $T_2 = 0.5417$ and $T_3 = 0.2708$. Based on the score function of LSVNNs, we get $\mathcal{S}(\mathcal{A}_1) = s_{4.3333}$, $\mathcal{S}(\mathcal{A}_2) = s_4$ and $\mathcal{S}(\mathcal{A}_3) = s_{4.6667}$. Thus, $I(\mathcal{S}(\mathcal{A}_3)) > I(\mathcal{S}(\mathcal{A}_2)) > I(\mathcal{S}(\mathcal{A}_1))$ and hence $\mathcal{A}_{\delta(1)} = \mathcal{A}_3, \mathcal{A}_{\delta(2)} = \mathcal{A}_2$ and $\mathcal{A}_{\delta(3)} = \mathcal{A}_1$. Therefore, by using the Eq. (10.7), we get

$$\begin{aligned}&\text{LSVNPOWG}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)\\ = \;& \left(s_{8\left(\prod\limits_{i=1}^{3}\left(\frac{\theta_{\delta(i)}}{8}\right)^{\xi_i}\right)}, s_{8\left(1-\prod\limits_{i=1}^{3}\left(1-\frac{\psi_{\delta(i)}}{8}\right)^{\xi_i}\right)}, s_{8\left(1-\prod\limits_{i=1}^{3}\left(1-\frac{\sigma_{\delta(i)}}{8}\right)^{\xi_i}\right)} \right)\\ = \;& \left(\begin{array}{l} s_{8\left(\left(\frac{4}{8}\right)^{0.5517}\times\left(\frac{1}{8}\right)^{0.2989}\times\left(\frac{2}{8}\right)^{0.1494}\right)}, \quad s_{8\left(1-\left(1-\frac{5}{8}\right)^{0.5517}\times\left(1-\frac{2}{8}\right)^{0.2989}\times\left(1-\frac{1}{8}\right)^{0.1494}\right)}, \\ \quad s_{8\left(1-\left(1-\frac{1}{8}\right)^{0.5517}\times\left(1-\frac{3}{8}\right)^{0.2989}\times\left(1-\frac{4}{8}\right)^{0.1494}\right)} \end{array} \right)\\ = \;& \left(s_{3.0909}, s_{2.6954}, s_{1.7833}\right)\end{aligned}$$

As similar to LSVNPWA operator, the developed operators, LSVNPWG and LSVNPOWG also satisfies the properties as mentioned in Theorem 10.3.2.

10.3.3 Fundamental properties of the developed operators

Lemma 10.3.1. Let $x_i \geq 0$, $\omega_i > 0$,$i = 1, 2, \ldots, n$, $\sum\limits_{i=1}^{n} \omega_i = 1$, then

$$\prod_{i=1}^{n} x_i^{\omega_i} \leq \sum_{i=1}^{n} \omega_i x_i \tag{10.11}$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$

Theorem 10.3.6. For a collection of LSVNNs, $\mathcal{A}_i; (i = 1, 2, \ldots, n)$, we have

$$\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \geq \text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

Proof. Let $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$, $(i = 1, 2, \ldots, n)$ be the collection of LSVNNs and $0 \leq 1 - (\theta_i/t) \leq 1$. Then by using Lemma 10.3.1 for $\xi_i > 0$, we have

$$\begin{aligned}
&\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i} \leq \sum_{i=1}^{n}\xi_i\left(1-\frac{\theta_i}{t}\right)\\
\Rightarrow\ &\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i} \leq \sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\xi_i\left(\frac{\theta_i}{t}\right)\\
\Rightarrow\ &1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i} \geq \sum_{i=1}^{n}\xi_i\left(\frac{\theta_i}{t}\right)\\
\Rightarrow\ &1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i} \geq \prod_{i=1}^{n}\left(\frac{\theta_i}{t}\right)^{\xi_i}.
\end{aligned}$$

Thus, we have

$$t\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right) \geq t\left(\prod_{i=1}^{n}\left(\frac{\theta_i}{t}\right)^{\xi_i}\right) \tag{10.12}$$

Similarly, we get

$$t\left(\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right) \leq t\left(1-\prod_{i=1}^{n}\left(1-\frac{\psi_i}{t}\right)^{\xi_i}\right) \tag{10.13}$$

$$\text{and}\qquad t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\xi_i}\right) \leq t\left(1-\prod_{i=1}^{n}\left(1-\frac{\sigma_i}{t}\right)^{\xi_i}\right) \tag{10.14}$$

Thus, from the inequalities (10.12), (10.13), (10.14) and Definition 2.1.11 of Chapter 2, we get

$$\begin{aligned}
&\left(s_{t\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right)}, s_{t\left(\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right)}, s_{t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\xi_i}\right)}\right)\\
&\qquad \geq \left(s_{t\left(\prod_{i=1}^{n}\left(\frac{\theta_i}{t}\right)^{\xi_i}\right)}, s_{t\left(1-\prod_{i=1}^{n}\left(1-\frac{\psi_i}{t}\right)^{\xi_i}\right)} s_{t\left(1-\prod_{i=1}^{n}\left(1-\frac{\sigma_i}{t}\right)^{\xi_i}\right)}\right).
\end{aligned}$$

Hence,

$$\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \geq \text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$$

□

Lemma 10.3.2. For any real numbers $a, b \in [0,1]$, we have

$$a + b - ab \geq ab \quad \text{and} \quad 1 - (1-a)b \geq a$$

Theorem 10.3.7. Let $\mathcal{A}$ and $\mathcal{A}_i; (i = 1, 2, \ldots, n)$ be a collection of LSVNNs then

(i) $\text{LSVNPWA}(\mathcal{A}_1 \oplus \mathcal{A}, \mathcal{A}_2 \oplus \mathcal{A}, \ldots, \mathcal{A}_n \oplus \mathcal{A}) \geq \text{LSVNPWA}(\mathcal{A}_1 \otimes \mathcal{A}, \mathcal{A}_2 \otimes \mathcal{A}, \ldots, \mathcal{A}_n \otimes \mathcal{A})$;

(ii) $\text{LSVNPWG}(\mathcal{A}_1 \oplus \mathcal{A}, \mathcal{A}_2 \oplus \mathcal{A}, \ldots, \mathcal{A}_n \oplus \mathcal{A}) \geq \text{LSVNPWG}(\mathcal{A}_1 \otimes \mathcal{A}, \mathcal{A}_2 \otimes \mathcal{A}, \ldots, \mathcal{A}_n \otimes \mathcal{A})$;

(iii) $\text{LSVNPWA}(\mathcal{A}_1 \oplus \mathcal{A}, \mathcal{A}_2 \oplus \mathcal{A}, \ldots, \mathcal{A}_n \oplus \mathcal{A}) \geq \text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \otimes \mathcal{A}$;

(iv) $\text{LSVNPWG}(\mathcal{A}_1 \oplus \mathcal{A}, \mathcal{A}_2 \oplus \mathcal{A}, \ldots, \mathcal{A}_n \oplus \mathcal{A}) \geq \text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \otimes \mathcal{A}$;

(v) $\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \oplus \mathcal{A} \geq \text{LSVNPWA}(\mathcal{A}_1 \otimes \mathcal{A}, \mathcal{A}_2 \otimes \mathcal{A}, \ldots, \mathcal{A}_n \otimes \mathcal{A})$;

(vi) $\text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \oplus \mathcal{A} \geq \text{LSVNPWG}(\mathcal{A}_1 \otimes \mathcal{A}, \mathcal{A}_2 \otimes \mathcal{A}, \ldots, \mathcal{A}_n \otimes \mathcal{A})$;

(vii) $\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \oplus \mathcal{A} \geq \text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \otimes \mathcal{A}$;

(viii) $\text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \oplus \mathcal{A} \geq \text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \otimes \mathcal{A}$.

Proof. We will prove the parts (i) & (iii) and the proofs of remaining are similar.

(i) Since $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$, $(i = 1, 2, \ldots, n)$, $\mathcal{A} = (s_\theta, s_\psi, s_\sigma)$ are LSVNNs and $\lambda > 0$, be a real number, then

$$\begin{aligned}
& \text{LSVNPWA}(\mathcal{A}_1 \oplus \mathcal{A}, \mathcal{A}_2 \oplus \mathcal{A}, \ldots, \mathcal{A}_n \oplus \mathcal{A}) \\
= & \left(s_{t\left(1-\prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)\left(1-\frac{\theta}{t}\right)\right)^{\xi_i}\right)}, s_{t\left(\prod_{i=1}^{n}\left(\frac{\psi_i \psi}{t^2}\right)^{\xi_i}\right)}, s_{t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i \sigma}{t^2}\right)^{\xi_i}\right)} \right) \\
\text{and} \quad & \text{LSVNPWA}(\mathcal{A}_1 \otimes \mathcal{A}, \mathcal{A}_2 \otimes \mathcal{A}, \ldots, \mathcal{A}_n \otimes \mathcal{A}) \\
= & \left(s_{t\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i \theta}{t^2}\right)^{\xi_i}\right)}, s_{t\left(\prod_{i=1}^{n}\left(1-\left(1-\frac{\psi_i}{t}\right)\left(1-\frac{\psi}{t}\right)\right)^{\xi_i}\right)}, s_{t\left(\prod_{i=1}^{n}\left(1-\left(1-\frac{\sigma_i}{t}\right)\left(1-\frac{\sigma}{t}\right)\right)^{\xi_i}\right)} \right)
\end{aligned}$$

By using Lemma 10.3.2, we have $\theta_i/t + \theta/t - (\theta_i/t)(\theta/t) \geq (\theta_i/t)(\theta/t)$ which implies that

$$\begin{aligned}
& 1-\theta_i/t-\theta/t+\theta_i\theta/t^2 \leq 1-(\theta_i/t)(\theta/t) \\
\Rightarrow\ & \left(1-\frac{\theta_i}{t}\right)\left(1-\frac{\theta}{t}\right) \leq 1-\frac{\theta_i\theta}{t^2} \\
\Rightarrow\ & \prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)\left(1-\frac{\theta}{t}\right)\right)^{\xi_i} \leq \prod_{i=1}^{n}\left(1-\frac{\theta_i\theta}{t^2}\right)^{\xi_i} \\
\Rightarrow\ & 1-\prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)\left(1-\frac{\theta}{t}\right)\right)^{\xi_i} \geq 1-\prod_{i=1}^{n}\left(1-\frac{\theta_i\theta}{t^2}\right)^{\xi_i}
\end{aligned}$$

Thus,

$$t\left(1-\prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)\left(1-\frac{\theta}{t}\right)\right)^{\xi_i}\right) \geq t\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i\theta}{t^2}\right)^{\xi_i}\right)$$

Also, we have $1-(1-\psi_i/t)\,(1-\psi/t) \geq \psi_i\psi/t^2$, which implies,

$$t\left(\prod_{i=1}^{n}\left(1-\left(1-\frac{\psi_i}{t}\right)\left(1-\frac{\psi}{t}\right)\right)^{\xi_i}\right) \geq t\left(\prod_{i=1}^{n}\left(\frac{\psi_i\psi}{t^2}\right)^{\xi_i}\right)$$

Similarly, we get

$$t\left(\prod_{i=1}^{n}\left(1-\left(1-\frac{\sigma_i}{t}\right)\left(1-\frac{\sigma}{t}\right)\right)^{\xi_i}\right) \geq t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i\sigma}{t^2}\right)^{\xi_i}\right)$$

Therefore, by using Definition 2.1.11 of Chapter 2, we get

$$\text{LSVNPWA}(\mathcal{A}_1\oplus\mathcal{A},\mathcal{A}_2\oplus\mathcal{A},\ldots,\mathcal{A}_n\oplus\mathcal{A}) \geq \text{LSVNPWA}(\mathcal{A}_1\otimes\mathcal{A},\mathcal{A}_2\otimes\mathcal{A},\ldots,\mathcal{A}_n\otimes\mathcal{A})$$

Hence the result.

(iii) By using Lemma 10.3.2, we have

$$\begin{aligned}
\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right)+\left(\frac{\theta}{t}\right)-\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)\right)\left(\frac{\theta}{t}\right) \\
\geq \left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right)\left(\frac{\theta}{t}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& 1-\prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)\left(1-\frac{\theta}{t}\right)\right)^{\xi_i} \geq \left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right)\left(\frac{\theta}{t}\right) \\
\Rightarrow\ & t\left(1-\prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)\left(1-\frac{\theta}{t}\right)\right)^{\xi_i}\right) \geq t\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{t}\right)^{\xi_i}\right)\left(\frac{\theta}{t}\right)
\end{aligned}$$

Also, we have

$$\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}+\left(\frac{\psi}{t}\right)-\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)\left(\frac{\psi}{t}\right)\geq\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\left(\frac{\psi}{t}\right)$$

$$\Rightarrow\ 1-\left(1-\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right)\left(1-\frac{\psi}{t}\right)\geq\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\left(\frac{\psi}{t}\right)$$

$$\Rightarrow\ t\left(1-\left(1-\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\xi_i}\right)\left(1-\frac{\psi}{t}\right)\right)\geq t\left(\prod_{i=1}^{n}\left(\frac{\psi_i\psi}{t}\right)^{\xi_i}\right)$$

Similarly, we can get

$$t\left(1-\left(1-\prod_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\xi_i}\right)\left(1-\frac{\sigma}{t}\right)\right)\geq t\left(\prod_{i=1}^{n}\left(\frac{\sigma_i\sigma}{t}\right)^{\xi_i}\right)$$

Hence, by Definition 2.1.11 of Chapter 2, we get

$$\text{LSVNPWA}(\mathcal{A}_1\oplus\mathcal{A},\mathcal{A}_2\oplus\mathcal{A},\ldots,\mathcal{A}_n\oplus\mathcal{A})\geq\text{LSVNPWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\otimes\mathcal{A}.$$

□

Lemma 10.3.3. For any two numbers, $0\leq a,b\leq 1$ and for $\lambda>0$ we have

$$1-(1-a)^{\lambda}(1-b)\geq(a)^{\lambda}b$$

and

$$(1-a)^{\lambda}b\geq -a^{\lambda}(1-b)$$

Theorem 10.3.8. Let $\mathcal{A}_i=(s_{\theta_i},s_{\psi_i},s_{\sigma_i})$, $(i=1,2,\ldots,n)$ be a collection of LSVNNs and $\mathcal{A}=(s_\theta,s_\psi,s_\sigma)$ be any other LSVNN, then for real number $\lambda>0$, we have

(i) $\text{LSVNPWA}(\lambda\mathcal{A}_1\oplus\mathcal{A},\lambda\mathcal{A}_2\oplus\mathcal{A},\ldots,\lambda\mathcal{A}_n\oplus\mathcal{A})\geq\text{LSVNPWA}(\mathcal{A}_1^{\lambda}\otimes\mathcal{A},\mathcal{A}_2^{\lambda}\otimes\mathcal{A},\ldots,\mathcal{A}_n^{\lambda}\otimes\mathcal{A})$

(ii) $\text{LSVNPWG}(\lambda\mathcal{A}_1\oplus\mathcal{A},\lambda\mathcal{A}_2\oplus\mathcal{A},\ldots,\lambda\mathcal{A}_n\oplus\mathcal{A})\geq\text{LSVNPWG}(\mathcal{A}_1^{\lambda}\otimes\mathcal{A},\mathcal{A}_2^{\lambda}\otimes\mathcal{A},\ldots,\mathcal{A}_n^{\lambda}\otimes\mathcal{A})$

(iii) $\text{LSVNPWA}(\mathcal{A}_1^{\lambda}\oplus\mathcal{A},\mathcal{A}_2^{\lambda}\oplus\mathcal{A},\ldots,\mathcal{A}_n^{\lambda}\oplus\mathcal{A})\geq\text{LSVNPWA}(\lambda\mathcal{A}_1\otimes\mathcal{A},\lambda\mathcal{A}_2\otimes\mathcal{A},\ldots,\lambda\mathcal{A}_n\otimes\mathcal{A})$

(iv) $\text{LSVNPWG}(\mathcal{A}_1^{\lambda}\oplus\mathcal{A}, \mathcal{A}_2^{\lambda}\oplus\mathcal{A}, \ldots, \mathcal{A}_n^{\lambda}\oplus\mathcal{A}) \geq \text{LSVNPWG}(\lambda\mathcal{A}_1\otimes\mathcal{A}, \lambda\mathcal{A}_2\otimes\mathcal{A}, \ldots, \lambda\mathcal{A}_n\otimes\mathcal{A})$

Proof. We will prove part (i) & (iii) and the proofs of remaining parts are similar.

(i) Since $\mathcal{A}_i (i = 1, 2, \ldots, n)$ and $\mathcal{A}$ are the LSVNNs then for $\lambda > 0$ we have

$$
\begin{aligned}
&\text{LSVNPWA}(\lambda\mathcal{A}_1 \oplus \mathcal{A}, \lambda\mathcal{A}_2 \oplus \mathcal{A}, \ldots, \lambda\mathcal{A}_n \oplus \mathcal{A})\\
&= \begin{pmatrix} s_{t\left(1-\prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)^{\lambda}\left(1-\frac{\theta}{t}\right)\right)^{\xi_i}\right)}, s_{t\left(\prod_{i=1}^{n}\left(\left(\frac{\psi_i}{t}\right)^{\lambda}\left(\frac{\psi}{t}\right)\right)^{\xi_i}\right)}, \\ s_{t\left(\prod_{i=1}^{n}\left(\left(\frac{\sigma_i}{t}\right)^{\lambda}\left(\frac{\sigma}{t}\right)\right)^{\xi_i}\right)} \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
&\text{LSVNPWA}(\mathcal{A}_1^{\lambda} \otimes \mathcal{A}, \mathcal{A}_2^{\lambda} \otimes \mathcal{A}, \ldots, \mathcal{A}_n^{\lambda} \otimes \mathcal{A})\\
&= \begin{pmatrix} s_{t\left(1-\prod_{i=1}^{n}\left(1-\left(\frac{\theta_i}{t}\right)^{\lambda}\left(\frac{\theta}{t}\right)\right)^{\xi_i}\right)}, s_{t\left(\prod_{i=1}^{n}\left(1-\left(1-\frac{\psi_i}{t}\right)^{\lambda}\left(1-\frac{\psi}{t}\right)\right)^{\xi_i}\right)}, \\ s_{t\left(\prod_{i=1}^{n}\left(1-\left(1-\frac{\sigma_i}{t}\right)^{\lambda}\left(1-\frac{\sigma}{t}\right)\right)^{\xi_i}\right)} \end{pmatrix}
\end{aligned}
$$

By Lemma 10.3.3 , we have

$$
\begin{aligned}
& 1-(1-\theta_i/t)^{\lambda}(1-\theta/t) \geq (\theta_i/t)^{\lambda}(\theta/t)\\
\Rightarrow\ & (1-\theta_i/t)^{\lambda}(1-\theta/t) \leq 1-(\theta_i/t)^{\lambda}(\theta/t)\\
\Rightarrow\ & \prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)^{\lambda}\left(1-\frac{\theta}{t}\right)\right)^{\xi_i} \leq \prod_{i=1}^{n}\left(1-\left(\frac{\theta_i}{t}\right)^{\lambda}\left(\frac{\theta}{t}\right)\right)^{\xi_i}\\
\Rightarrow\ & t\left(1-\prod_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)^{\lambda}\left(1-\frac{\theta}{t}\right)\right)^{\xi_i}\right) \geq t\left(1-\prod_{i=1}^{n}\left(1-\left(\frac{\theta_i}{t}\right)^{\lambda}\left(\frac{\theta}{t}\right)\right)^{\xi_i}\right)
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
& \left(\frac{\psi_i}{t}\right)^{\lambda}\left(\frac{\psi}{t}\right) \leq 1-\left(1-\frac{\psi_i}{t}\right)^{\lambda}\left(1-\frac{\psi}{t}\right)\\
\Rightarrow\ & t\prod_{i=1}^{n}\left(\left(\frac{\psi_i}{t}\right)^{\lambda}\left(\frac{\psi}{t}\right)\right)^{\xi_i} \leq t\prod_{i=1}^{n}\left(1-\left(1-\frac{\psi_i}{t}\right)^{\lambda}\left(1-\frac{\psi}{t}\right)\right)^{\xi_i}
\end{aligned}
$$

Similarly, we have

$$
t\left(\prod_{i=1}^{n}\left(\left(\frac{\sigma_i}{t}\right)^{\lambda}\left(\frac{\sigma}{t}\right)\right)^{\xi_i}\right) \leq t\left(\prod_{i=1}^{n}\left(1-\left(1-\frac{\sigma_i}{t}\right)^{\lambda}\left(1-\frac{\sigma}{t}\right)\right)^{\xi_i}\right)
$$

Therefore, by Definition 2.1.11 of Chapter 2, we get

$$\begin{aligned}&\text{LSVNPWA}(\lambda\mathcal{A}_1 \oplus \mathcal{A}, \lambda\mathcal{A}_2 \oplus \mathcal{A}, \ldots, \lambda\mathcal{A}_n \oplus \mathcal{A})\\ &\qquad \geq \text{LSVNPWA}(\mathcal{A}_1^\lambda \otimes \mathcal{A}, \mathcal{A}_2^\lambda \otimes \mathcal{A}, \ldots, \mathcal{A}_n^\lambda \otimes \mathcal{A})\end{aligned}$$

Hence the result.

(iii) Since $\mathcal{A}_i (i = 1, 2, \ldots, n)$ and $\mathcal{A}$ are the LSVNNs then for $\lambda > 0$ we have

$$\begin{aligned}&\text{LSVNPWA}(\mathcal{A}_1^\lambda \oplus \mathcal{A}, \mathcal{A}_2^\lambda \oplus \mathcal{A}, \ldots, \mathcal{A}_n^\lambda \oplus \mathcal{A})\\ &= \left(\begin{array}{c} s_{t\left(1-\prod\limits_{i=1}^{n}\left(\left(1-\left(\frac{\theta_i}{t}\right)^{\lambda}\right)\left(1-\frac{\theta}{t}\right)\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(\left(1-\left(1-\frac{\psi_i}{t}\right)^{\lambda}\right)\left(\frac{\psi}{t}\right)\right)^{\xi_i}\right)}, \\ s_{t\left(\prod\limits_{i=1}^{n}\left(\left(1-\left(1-\frac{\sigma_i}{t}\right)^{\lambda}\right)\left(\frac{\sigma}{t}\right)\right)^{\xi_i}\right)} \end{array}\right)\end{aligned}$$

and

$$\begin{aligned}&\text{LSVNPWA}(\lambda\mathcal{A}_1 \otimes \mathcal{A}, \lambda\mathcal{A}_2 \otimes \mathcal{A}, \ldots, \lambda\mathcal{A}_n \otimes \mathcal{A})\\ &= \left(\begin{array}{c} s_{t\left(1-\prod\limits_{i=1}^{n}\left(1-\left(1-\left(1-\frac{\theta_i}{t}\right)^{\lambda}\right)\left(\frac{\theta}{t}\right)\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(1-\left(1-\left(\frac{\psi_i}{t}\right)^{\lambda}\right)\left(1-\frac{\psi}{t}\right)\right)^{\xi_i}\right)}, \\ s_{t\left(\prod\limits_{i=1}^{n}\left(1-\left(1-\left(\frac{\sigma_i}{t}\right)^{\lambda}\right)\left(1-\frac{\sigma}{t}\right)\right)^{\xi_i}\right)} \end{array}\right)\end{aligned}$$

From Lemma 10.3.3, $\big(1-(\theta_i/t)\big)^{\lambda}\big(\theta/t\big) \geq -\big(\theta_i/t\big)^{\lambda}\big(1-(\theta/t)\big)$, which implies, $1-(\theta/t)+\big(1-(\theta_i/t)\big)^{\lambda}\big(\theta/t\big) \geq 1-(\theta/t)-\big(\theta_i/t\big)^{\lambda}\big(1-(\theta/t)\big) \Rightarrow 1-(\theta/t)\big(1-\big(1-(\theta_i/t)\big)^{\lambda}\big) \geq \big(1-(\theta/t)\big)\big(1-(\theta_i/t)^{\lambda}\big) \Rightarrow \prod\limits_{i=1}^{n}\Big(1-(\theta/t)\big(1-\big(1-(\theta_i/t)\big)^{\lambda}\big)\Big)^{\xi_i} \geq \prod\limits_{i=1}^{n}\Big(\big(1-(\theta/t)\big)\big(1-(\theta_i/t)^{\lambda}\big)\Big)^{\xi_i} \Rightarrow 1-\prod\limits_{i=1}^{n}\Big(1-(\theta/t)\big(1-\big(1-(\theta_i/t)\big)^{\lambda}\big)\Big)^{\xi_i} \leq 1-\prod\limits_{i=1}^{n}\Big(\big(1-(\theta/t)\big)\big(1-(\theta_i/t)^{\lambda}\big)\Big)^{\xi_i}$, which implies

$$\begin{aligned}&t\left(1-\prod_{i=1}^{n}\left(\left(1-\frac{\theta}{t}\right)\left(1-\left(\frac{\theta_i}{t}\right)^{\lambda}\right)\right)^{\xi_i}\right)\\ &\qquad \geq t\left(1-\prod_{i=1}^{n}\left(1-\left(\frac{\theta}{t}\right)\left(1-\left(1-\frac{\theta_i}{t}\right)^{\lambda}\right)\right)^{\xi_i}\right)\end{aligned}$$

Also, we have $\big(1-\psi_i/t\big)^{\lambda}\big(\psi/t\big) \geq -\big(\psi_i/t\big)^{\lambda}\big(1-(\psi/t)\big)$, which implies, $\big(\psi/t\big)-\big(1-\psi_i/t\big)^{\lambda}\big(\psi/t\big) \leq \big(\psi/t\big)+\big(\psi_i/t\big)^{\lambda}\big(1-(\psi/t)\big) \Rightarrow \left(\frac{\psi}{t}\right)\left(1-\left(1-\frac{\psi}{t}\right)^{\lambda}\right) \leq 1-$

$\left(1-\left(\frac{\psi_i}{t}\right)^{\lambda}\right)\left(1-\frac{\psi}{t}\right)$ implies,

$$t\prod_{i=1}^{n}\left(\frac{\psi}{t}\left(1-\left(1-\frac{\psi}{t}\right)^{\lambda}\right)\right)^{\xi_i} \leq t\prod_{i=1}^{n}\left(1-\left(1-\left(\frac{\psi_i}{t}\right)^{\lambda}\right)\left(1-\frac{\psi}{t}\right)\right)^{\xi_i}$$

Similarly, we have

$$t\left(\prod_{i=1}^{n}\left(\frac{\sigma}{t}\left(1-\left(1-\frac{\sigma}{t}\right)^{\lambda}\right)\right)^{\xi_i}\right) \leq t\left(\prod_{i=1}^{n}\left(1-\left(1-\left(\frac{\sigma_i}{t}\right)^{\lambda}\right)\left(1-\frac{\sigma}{t}\right)\right)^{\xi_i}\right)$$

Hence, we get

$$\begin{aligned}&\mathrm{LSVNPWA}(\mathcal{A}_1^{\lambda}\oplus\mathcal{A},\mathcal{A}_2^{\lambda}\oplus\mathcal{A},\ldots,\mathcal{A}_n^{\lambda}\oplus\mathcal{A})\\ &\qquad\geq \mathrm{LSVNPWA}(\lambda\mathcal{A}_1\otimes\mathcal{A},\lambda\mathcal{A}_2\otimes\mathcal{A},\ldots,\lambda\mathcal{A}_n\otimes\mathcal{A})\end{aligned}$$

□

Theorem 10.3.9. Let $\mathcal{A}_i=(s_{\theta_i},s_{\psi_i},s_{\sigma_i})$, $(i=1,2,\ldots,n)$ be a collection of LSVNNs and $\mathcal{A}=(s_{\theta},s_{\psi},s_{\sigma})$ is also LSVNN, then for $\lambda>0$, we have

(i) $\mathrm{LSVNPWA}(\lambda\mathcal{A}_1\otimes\mathcal{A},\lambda\mathcal{A}_2\otimes\mathcal{A},\ldots,\lambda\mathcal{A}_n\otimes\mathcal{A}) \leq \mathrm{LSVNPWA}(\lambda\mathcal{A}_1,\lambda\mathcal{A}_2,\ldots,\lambda\mathcal{A}_n)$ $\leq \mathrm{LSVNPWA}(\lambda\mathcal{A}_1\oplus\mathcal{A},\lambda\mathcal{A}_2\oplus\mathcal{A},\ldots,\lambda\mathcal{A}_n\oplus\mathcal{A})$

(ii) $\mathrm{LSVNPWG}(\lambda\mathcal{A}_1\otimes\mathcal{A},\lambda\mathcal{A}_2\otimes\mathcal{A},\ldots,\lambda\mathcal{A}_n\otimes\mathcal{A}) \leq \mathrm{LSVNPWG}(\lambda\mathcal{A}_1,\lambda\mathcal{A}_2,\ldots,\lambda\mathcal{A}_n) \leq$ $\mathrm{LSVNPWG}(\lambda\mathcal{A}_1\oplus\mathcal{A},\lambda\mathcal{A}_2\oplus\mathcal{A},\ldots,\lambda\mathcal{A}_n\oplus\mathcal{A})$

(iii) $\mathrm{LSVNPWA}(\mathcal{A}_1^{\lambda}\otimes\mathcal{A},\mathcal{A}_2^{\lambda}\otimes\mathcal{A},\ldots,\mathcal{A}_n^{\lambda}\otimes\mathcal{A}) \leq \mathrm{LSVNPWA}(\mathcal{A}_1^{\lambda},\mathcal{A}_2^{\lambda},\ldots,\mathcal{A}_n^{\lambda})$

$\leq \mathrm{LSVNPWA}(\mathcal{A}_1^{\lambda}\oplus\mathcal{A},\mathcal{A}_2^{\lambda}\oplus\mathcal{A},\ldots,\mathcal{A}_n^{\lambda}\oplus\mathcal{A})$

(iv) $\mathrm{LSVNPWG}(\mathcal{A}_1^{\lambda}\otimes\mathcal{A},\mathcal{A}_2^{\lambda}\otimes\mathcal{A},\ldots,\mathcal{A}_n^{\lambda}\otimes\mathcal{A}) \leq \mathrm{LSVNPWG}(\mathcal{A}_1^{\lambda},\mathcal{A}_2^{\lambda},\ldots,\mathcal{A}_n^{\lambda})$

$\leq \mathrm{LSVNPWG}(\mathcal{A}_1^{\lambda}\oplus\mathcal{A},\mathcal{A}_2^{\lambda}\oplus\mathcal{A},\ldots,\mathcal{A}_n^{\lambda}\oplus\mathcal{A})$

(v) $\lambda\ \mathrm{LSVNPWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\otimes\mathcal{A} \leq \lambda\mathrm{LSVNPWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)$

$\leq \lambda\mathrm{LSVNPWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\oplus\mathcal{A}$

(vi) $\lambda\ \mathrm{LSVNPWG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\otimes\mathcal{A} \leq \lambda\mathrm{LSVNPWG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)$

$\leq \lambda\mathrm{LSVNPWG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\oplus\mathcal{A}$

Proof. We will prove part (i) and remaining parts are similar.

(i) For any $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$ and $\mathcal{A} = (s_\theta, s_\psi, s_\sigma)$, we have

$$\text{LSVNPWA}(\lambda\mathcal{A}_1 \otimes \mathcal{A}, \lambda\mathcal{A}_2 \otimes \mathcal{A}, \ldots, \lambda\mathcal{A}_n \otimes \mathcal{A})$$

$$= \begin{pmatrix} s_{t\left(1-\prod\limits_{i=1}^{n}\left(1-\left(1-\left(1-\frac{\theta_i}{t}\right)^{\lambda}\right)\left(1-\frac{\theta}{t}\right)\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(1-\left(1-\left(\frac{\psi_i}{t}\right)^{\lambda}\right)\left(1-\frac{\psi}{t}\right)\right)^{\xi_i}\right)}, \\ s_{t\left(\prod\limits_{i=1}^{n}\left(1-\left(1-\left(\frac{\sigma_i}{t}\right)^{\lambda}\right)\left(1-\frac{\sigma}{t}\right)\right)^{\xi_i}\right)} \end{pmatrix};$$

and

$$\text{LSVNPWA}(\lambda\mathcal{A}_1, \lambda\mathcal{A}_2, \ldots, \lambda\mathcal{A}_n)$$

$$= \begin{pmatrix} s_{t\left(1-\prod\limits_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)^{\lambda}\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(\left(\frac{\psi_i}{t}\right)^{\lambda}\right)^{\xi_i}\right)}, \\ s_{t\left(\prod\limits_{i=1}^{n}\left(\left(\frac{\sigma_i}{t}\right)^{\lambda}\right)^{\xi_i}\right)} \end{pmatrix};$$

and

$$\text{LSVNPWA}(\lambda\mathcal{A}_1 \oplus \mathcal{A}, \lambda\mathcal{A}_2 \oplus \mathcal{A}, \ldots, \lambda\mathcal{A}_n \oplus \mathcal{A})$$

$$= \begin{pmatrix} s_{t\left(1-\prod\limits_{i=1}^{n}\left(\left(1-\frac{\theta_i}{t}\right)^{\lambda}\left(1-\frac{\theta}{t}\right)\right)^{\xi_i}\right)}, s_{t\left(\prod\limits_{i=1}^{n}\left(\left(\frac{\psi_i}{t}\right)^{\lambda}\left(\frac{\psi}{t}\right)\right)^{\xi_i}\right)}, \\ s_{t\left(\prod\limits_{i=1}^{n}\left(\left(\frac{\sigma_i}{t}\right)^{\lambda}\left(\frac{\sigma}{t}\right)\right)^{\xi_i}\right)} \end{pmatrix}$$

We shall prove this part in two steps :

First, we first prove $\text{LSVNPWA}(\lambda\mathcal{A}_1 \otimes \mathcal{A}, \lambda\mathcal{A}_2 \otimes \mathcal{A}, \ldots, \lambda\mathcal{A}_n \otimes \mathcal{A}) \leq \text{LSVNPWA}(\lambda\mathcal{A}_1, \lambda\mathcal{A}_2, \ldots, \lambda\mathcal{A}_n)$.

Since, $0 \leq 1 - (1-\theta_i/t)^{\lambda} \leq 1, 0 \leq (\theta/t) \leq 1$, then based on the Lemma 10.3.4, we have

$$\begin{aligned} & 1 - \left(1 - \left(1-\theta_i/t\right)^{\lambda}\right)\left(1-\theta/t\right) \geq \left(1-\theta_i/t\right)^{\lambda} \\ \Rightarrow\ & \prod_{i=1}^{n}\left(1 - \left(1 - \left(1-\theta_i/t\right)^{\lambda}\right)\left(1-\theta/t\right)\right)^{\xi_i} \geq \prod_{i=1}^{n}\left(\left(1-\tfrac{\theta_i}{t}\right)^{\lambda}\right)^{\xi_i} \\ \Rightarrow\ & t\left(1 - \prod_{i=1}^{n}\left(1 - \left(1 - \left(1-\tfrac{\theta_i}{t}\right)^{\lambda}\right)\left(1-\tfrac{\theta}{t}\right)\right)^{\xi_i}\right) \\ & \qquad \leq t\left(1 - \prod_{i=1}^{n}\left(\left(1-\tfrac{\theta_i}{t}\right)^{\lambda}\right)^{\xi_i}\right) \end{aligned} \tag{10.15}$$

Now, we have

$$\begin{aligned}
&1-\left(1-\left(\tfrac{\psi_i}{t}\right)^{\lambda}\right)\left(1-\tfrac{\psi}{t}\right)\geq\left(\tfrac{\psi_i}{t}\right)^{\lambda}\\
\Rightarrow\quad &t\prod_{i=1}^{n}\left(1-\left(1-\left(\tfrac{\psi_i}{t}\right)^{\lambda}\right)\left(1-\tfrac{\psi}{t}\right)\right)^{\xi_i}\geq t\prod_{i=1}^{n}\left(\left(\tfrac{\psi_i}{t}\right)^{\lambda}\right)^{\xi_i}
\end{aligned}\tag{10.16}$$

Similarly,

$$t\left(\prod_{i=1}^{n}\left(1-\left(1-\left(\frac{\sigma_i}{t}\right)^{\lambda}\right)\left(1-\frac{\sigma}{t}\right)\right)^{\xi_i}\right)\geq t\left(\prod_{i=1}^{n}\left(\left(\frac{\sigma_i}{t}\right)^{\lambda}\right)^{\xi_i}\right)\tag{10.17}$$

Therefore, from Eqs. (10.15), (10.16) and (10.17), we get

$\mathrm{LSVNPWA}(\lambda\mathcal{A}_1\otimes\mathcal{A},\lambda\mathcal{A}_2\otimes\mathcal{A},\ldots,\lambda\mathcal{A}_n\otimes\mathcal{A})\leq\mathrm{LSVNPWA}(\lambda\mathcal{A}_1,\lambda\mathcal{A}_2,\ldots,\lambda\mathcal{A}_n)$

Later, we prove $\mathrm{LSVNPWA}(\lambda\mathcal{A}_1,\lambda\mathcal{A}_2,\ldots,\lambda\mathcal{A}_n)\leq\mathrm{LSVNPWA}(\lambda\mathcal{A}_1\oplus\mathcal{A},\lambda\mathcal{A}_2\oplus\mathcal{A},\ldots,\lambda\mathcal{A}_n\oplus\mathcal{A})$.

Since, $0\leq\left(1-\tfrac{\theta_i}{t}\right)^{\lambda}\leq 1$, $0\leq\left(1-\tfrac{\theta}{t}\right)\leq 1$, therefore we have

$$\begin{aligned}
&\left(1-\tfrac{\theta_i}{t}\right)^{\lambda}\left(1-\tfrac{\theta}{t}\right)\leq\left(1-\tfrac{\theta_i}{t}\right)^{\lambda}\\
\Rightarrow\quad &\prod_{i=1}^{n}\left(\left(1-\tfrac{\theta_i}{t}\right)^{\lambda}\left(1-\tfrac{\theta}{t}\right)\right)^{\xi_i}\leq\prod_{i=1}^{n}\left(1-\tfrac{\theta_i}{t}\right)^{\lambda\xi_i}\\
\Rightarrow\quad &t\left(1-\prod_{i=1}^{n}\left(\left(1-\tfrac{\theta_i}{t}\right)^{\lambda}\left(1-\tfrac{\theta}{t}\right)\right)^{\xi_i}\right)\geq t\left(1-\prod_{i=1}^{n}\left(1-\tfrac{\theta_i}{t}\right)^{\lambda\xi_i}\right)
\end{aligned}\tag{10.18}$$

Similarly, we get

$$t\left(\prod_{i=1}^{n}\left(\tfrac{\psi_i}{t}\right)^{\lambda}\left(\tfrac{\psi}{t}\right)^{\xi_i}\right)\leq t\left(\prod_{i=1}^{n}\left(\left(\tfrac{\psi_i}{t}\right)^{\lambda}\right)^{\xi_i}\right)\tag{10.19}$$

and

$$t\left(\prod_{i=1}^{n}\left(\tfrac{\sigma_i}{t}\right)^{\lambda}\left(\tfrac{\sigma}{t}\right)^{\xi_i}\right)\leq t\left(\prod_{i=1}^{n}\left(\left(\tfrac{\sigma_i}{t}\right)^{\lambda}\right)^{\xi_i}\right)\tag{10.20}$$

Therefore, from inequalities (10.18), (10.19) and (10.20), we get

$\mathrm{LSVNPWA}(\lambda\mathcal{A}_1,\lambda\mathcal{A}_2,\ldots,\lambda\mathcal{A}_n)\leq\mathrm{LSVNPWA}(\lambda\mathcal{A}_1\oplus\mathcal{A},\lambda\mathcal{A}_2\oplus\mathcal{A},\ldots,\lambda\mathcal{A}_n\oplus\mathcal{A})$.

Hence, based on these steps, we get the result.

□

Lemma 10.3.4. If $\mathcal{A}_1, \mathcal{A}_2 \in [0,1]$ and a real number $\lambda > 0$, we have

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 \geq \mathcal{A}_1 \otimes \mathcal{A}_2$;

(ii) $\lambda \mathcal{A}_1 \geq \mathcal{A}_1^{\lambda}$.

Theorem 10.3.10. Let $\mathcal{A}_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$, $(i = 1, 2, \ldots, n)$ be a collection of LSVNNs and $\mathcal{A} = (s_\theta, s_\psi, s_\sigma)$ is also LSVNN, then for $\lambda > 0$, we have

(i) $\lambda \text{ LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \oplus \mathcal{A} \geq (\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n))^{\lambda} \otimes \mathcal{A}$

(ii) $\lambda \text{ LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \oplus \mathcal{A} \geq (\text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n))^{\lambda} \otimes \mathcal{A}$

(iii) $(\text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n))^{\lambda} \oplus \mathcal{A} \geq \lambda \text{LSVNPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \otimes \mathcal{A}$

(iv) $(\text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n))^{\lambda} \oplus \mathcal{A} \geq \lambda \text{LSVNPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \otimes \mathcal{A}$

Proof. Proof is trivially follows from above Lemma and Theorems. □

10.4 An approach based on proposed operators for solving DM problems

In this section, we present a DM approach based on proposed operators under linguistic neutrosophic set environment.

10.4.1 Proposed approach

The description of the MADM is given in Section 2.5 of Chapter 2. Suppose the decision matrix given by the decision-maker $\mathcal{DM}^{(q)}$ is denoted by $r_{ij}^{(q)} = (s_{\theta_{ij}^{(q)}}, s_{\psi_{ij}^{(q)}}, s_{\sigma_{ij}^{(q)}})$ and hence formulated the LSVN decision matrix $\mathcal{R}^{(q)} = \left(r_{ij}^{(q)}\right)_{n \times m}$. Based on these the following steps have been summarized for describing the DM approach based on the proposed operators as:

Step 1: Calculate $T_{ij}^{(q)} (q = 1, 2, \ldots, l)$, as follows

$$T_{ij}^{(1)} = 1; \tag{10.21}$$

$$T_{ij}^{(q)} = \prod_{\gamma=1}^{q-1} \frac{I\big(\mathcal{S}(r_{ij}^{\gamma})\big)}{t}; \tag{10.22}$$

Step 2: Aggregate all the decision matrices into a collective one $R = (r_{ij})_{n\times m}$ by utilizing the LSVNPWA or LSVNPWG operator which is as follows:

$$\begin{aligned} r_{ij} &= \text{LSVNPWA}(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(l)}) \\ &= \left(s_{t\left(1-\prod\limits_{q=1}^{l}\left(1-\frac{\theta_{ij}^{(q)}}{t}\right)^{\frac{T_{ij}^{(q)}}{\sum_{q=1}^{l} T_{ij}^{(q)}}}\right)}, s_{t\left(\prod\limits_{q=1}^{l}\left(\frac{\psi_{ij}^{(q)}}{t}\right)^{\frac{T_{ij}^{(q)}}{\sum_{q=1}^{l} T_{ij}^{(q)}}}\right)}, s_{t\left(\prod\limits_{q=1}^{l}\left(\frac{\sigma_{ij}^{(q)}}{t}\right)^{\frac{T_{ij}^{(q)}}{\sum_{q=1}^{l} T_{ij}^{(q)}}}\right)} \right) \end{aligned}$$

or

$$\begin{aligned} r_{ij} &= \text{LSVNPWG}(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(l)}) \\ &= \left(s_{t\left(\prod\limits_{q=1}^{l}\left(\frac{\theta_{ij}^{(q)}}{t}\right)^{\frac{T_{ij}^{(q)}}{\sum_{q=1}^{l} T_{ij}^{(q)}}}\right)}, s_{t\left(1-\prod\limits_{q=1}^{l}\left(1-\frac{\psi_{ij}^{(q)}}{t}\right)^{\frac{T_{ij}^{(q)}}{\sum_{q=1}^{l} T_{ij}^{(q)}}}\right)}, s_{t\left(1-\prod\limits_{q=1}^{l}\left(1-\frac{\sigma_{ij}^{(q)}}{t}\right)^{\frac{T_{ij}^{(q)}}{\sum_{q=1}^{l} T_{ij}^{(q)}}}\right)} \right) \end{aligned}$$

Step 3: Calculate the values of $T_{ij} (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ as follows

$$T_{i1} = 1; \tag{10.23}$$

$$T_{ij} = \prod_{\nu=1}^{j-1} \frac{I\big(\mathcal{S}(r_{\nu j})\big)}{t}; \tag{10.24}$$

Step 4: Aggregate these LSVNNs r_{ij} into collective overall LSVNNs $r_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i})$, $i = 1, 2, \ldots, n$ either by using LSVNPOWA operator:

$$\begin{aligned} r_i &= \text{LSVNPOWA}(r_{i1}, r_{i2}, \ldots, r_{im}) \qquad (10.25) \\ &= \left(s_{t\left(1-\prod\limits_{j=1}^{m}\left(1-\frac{\theta_{\delta(ij)}}{t}\right)^{\frac{T_{ij}}{\sum_{j=1}^{m} T_{ij}}}\right)}, s_{t\left(\prod\limits_{j=1}^{m}\left(\frac{\psi_{\delta(ij)}}{t}\right)^{\frac{T_{ij}}{\sum_{j=1}^{m} T_{ij}}}\right)}, s_{t\left(\prod\limits_{j=1}^{m}\left(\frac{\sigma_{\delta(ij)}}{t}\right)^{\frac{T_{ij}}{\sum_{j=1}^{m} T_{ij}}}\right)} \right) \end{aligned}$$

or by using LSVNPOWG operator

$$\begin{aligned} r_i &= \text{LSVNPOWG}(r_{i1}, r_{i2}, \ldots, r_{im}) \qquad (10.26) \\ &= \left(s_{t\left(\prod\limits_{j=1}^{m}\left(\frac{\theta_{\delta(ij)}}{t}\right)^{\frac{T_{ij}}{\sum_{j=1}^{m} T_{ij}}}\right)}, s_{t\left(1-\prod\limits_{j=1}^{m}\left(1-\frac{\psi_{\delta(ij)}}{t}\right)^{\frac{T_{j}}{\sum_{j=1}^{m} T_{ij}}}\right)}, s_{t\left(1-\prod\limits_{j=1}^{m}\left(1-\frac{\sigma_{\delta(ij)}}{t}\right)^{\frac{T_{ij}}{\sum_{j=1}^{m} T_{ij}}}\right)} \right) \end{aligned}$$

Step 5: Calculate the subscript value of score function for each $r_i (i = 1, 2, \ldots, n)$, by using

$$I(\mathcal{S}(r_i)) = (2t + \theta_i - \psi_i - \sigma_i)/3 \tag{10.27}$$

Step 6: Rank the alternatives $\mathcal{A}_i (i = 1, 2, \ldots, n)$ according to the descending order of score values and hence choose the best one(s).

10.4.2 Illustrative example

In this section, the proposed approach has been illustrative by a numerical example.

Goods and Services Tax (GST) is one indirect tax for the whole nation, which will make India a unified common market. While GST promises to the user in an era of the unified indirect tax regime, integrating India into a single homogenous market, it comes with certain complications inherited from the legacy tax regime. With the government gearing up to enforce the GST in Punjab from July 1, the issue of traders having limited computer knowledge and poor connectivity. In order to counter this, the state government has planned to train more than 2,000 youths as 'GST Mitra' to cater the traders. Punjab GST Mitra Scheme, which has to be started as a pilot project from Patiala, proposes to assist taxpayers in furnishing the details of outward supplies, inward supplies and returns, filing claims or refunds, filing any other applications, etc., in GST Regime. It aims to create a group of Tax professionals available in the locality or at the doorstep of the tax payer, at affordable costs throughout the State of Punjab.

The poor internet connectivity in far-flung areas has emerged as a big stumbling block in the success of 'GST Mitra' scheme. In order to provide the online services to run this scheme, state government is planning to give contract combinedly to private mobile service provider along with state-owned BSNL. For this purpose, committee formed by the government short-listed the four internet service provider, namely, Bharti Airtel ($\mathcal{A}_1$), Relience Communications ($\mathcal{A}_2$), vodafone India ($\mathcal{A}_3$) and Mahanagar Telecom Nigam ($\mathcal{A}_4$) under the criteria: Customer Services ($\mathcal{G}_1$), Bandwidth ($\mathcal{G}_2$), Pacakage Deal ($\mathcal{G}_3$), Total Cost ($\mathcal{G}_4$). The prioritization relationship for the criterion are $\mathcal{G}_1 \succ \mathcal{G}_2 \succ \mathcal{G}_3 \succ \mathcal{G}_4$. The decision-makers have evaluated these different alternative $\mathcal{A}_i (i = 1, 2, 3, 4)$ under the set of criteria $\mathcal{G}_j (j = 1, 2, 3, 4)$ and give their decision preferences in terms of LSVNNs on the basis of the linguistic term set: $Q = \{s_0 =$ "extremely poor", $s_1 =$ "very poor", $s_2 =$ "poor", $s_3 =$ "slightly poor", $s_4 =$ "fair", $s_5 =$ "slightly good", $s_6 =$ "good", $s_7 =$ "very good", $s_8 =$ "extremely good" and obtains three decision matrices $\mathcal{M}^{(q)}$ shown in Table 10.1 which are given by three decision-makers $\mathcal{DM}^{(1)}$, $\mathcal{DM}^{(2)}$ and $\mathcal{DM}^{(3)}$ respectively.

Then, the steps of the developed algorithms are executed as follows.

Step 1: By utilizing the Eqs. (10.21) and (10.22), we get

$$T^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad T^{(2)} = \begin{bmatrix} 0.5416 & 0.5417 & 0.4167 & 0.3750 \\ 0.6250 & 0.5000 & 0.7083 & 0.5833 \\ 0.6250 & 0.4583 & 0.4583 & 0.4167 \\ 0.5417 & 0.5833 & 0.5833 & 0.7083 \end{bmatrix}$$

$$T^{(3)} = \begin{bmatrix} 0.2708 & 0.3385 & 0.2431 & 0.2188 \\ 0.4688 & 0.2500 & 0.2951 & 0.3646 \\ 0.2865 & 0.1910 & 0.2674 & 0.2778 \\ 0.2708 & 0.3160 & 0.4375 & 0.4427 \end{bmatrix}$$

Step 2: Based on the information given in Table 10.1, the aggregated value obtained by LSVNPWA operator using Eq. (10.2) are summarized in Table 10.2.

Step 3: The values of T_{ij} for the data given in Table 10.2 are computed as

$$T = \begin{bmatrix} 1 & 0.6006 & 0.3924 & 0.2242 \\ 1 & 0.6910 & 0.3644 & 0.2170 \\ 1 & 0.5677 & 0.2482 & 0.1262 \\ 1 & 0.5683 & 0.3218 & 0.2071 \end{bmatrix}$$

Step 4: By Eq. (10.7), the overall collective values of the alternatives $\mathcal{A}_i$ are computed as $r_1 = (s_{3.7123}, s_{2.2050}, s_{2.7154})$, $r_2 = (s_{3.9492}, s_{1.9681}, s_{2.6355})$, $r_3 = (s_{3.0489}, s_{2.9908}, s_{2.8874})$, and $r_4 = (s_{3.6694}, s_{2.3764}, s_{2.1958})$.

Step 5: The subscript of the score functions for these numbers are $I(\mathcal{S}(r_1)) = 4.9307$, $I(\mathcal{S}(r_2)) = 5.1152$, $I(\mathcal{S}(r_3)) = 4.3902$ and $I(\mathcal{S}(r_4)) = 5.0324$.

Step 6: Thus, the ranking order of the four alternatives, based on the score values, is obtained as $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$ and hence $\mathcal{A}_2$ is the best internet provider company .

On the other hand, if we utilize the LSVNPWG and LSVNPOWG operators instead of averaging operators to find the best alternative(s) then the following steps has been executed as follows,

Step 1: Same as above.

Step 2: By applying the LSVNPWG operator, defined in (10.8), we get the aggregated preference values, which are summarized in Table 10.3.

Step 3: Based on the information in Table 10.3, calculate the values of $T_{ij}(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$.

$$T = \begin{bmatrix} 1.0000 & 0.5601 & 0.3227 & 0.1600 \\ 1.0000 & 0.6772 & 0.3322 & 0.1717 \\ 1.0000 & 0.5287 & 0.2245 & 0.1097 \\ 1.0000 & 0.4800 & 0.2562 & 0.1557 \end{bmatrix}$$

Step 4: By Eq. (10.10), we get the aggregated LSVNNs $r_i; (i = 1, 2, 3, 4)$ as $r_1 = (s_{3.3612}, s_{2.8224}, s_{3.6029})$, $r_2 = (s_{3.5786}, s_{2.4942}, s_{2.9723})$, $r_3 = (s_{2.3486}, s_{3.2682}, s_{3.0002})$, and $r_4 = (s_{3.2664}, s_{2.7254}, s_{2.8026})$.

Step 5: By Eq. (10.27), we get $I(\mathcal{S}(r_1)) = 4.3120$, $I(\mathcal{S}(r_2)) = 4.7041$, $I(\mathcal{S}(r_3)) = 4.0267$ and $I(\mathcal{S}(r_4)) = 4.5795$.

Step 6: According to the score values, the ranking order of alternatives is $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$ and hence $\mathcal{A}_2$ provides best internet service.

10.4.3 Comparative Analysis

To further prove the validity and effectiveness of the proposed operator, we conduct the comparative analysis by applying the LNNWAA operator [39] on the same illustrative example by taking same importance to the decision makers and the criteria's. The following steps shows the procedure of DM process by means of LNNWAA operator as

Step 1: The aggregated matrix is obtained by using the LNNWAA operator as shown in Table 10.4.

Step 2: By using LNNWAA operator, we get the collective overall values $r_i(i = 1, 2, 3, 4)$ as $r_1 = (s_{4.3729},\ s_{1.9421},\ s_{2.2972})$, $r_2 = (s_{3.4867}, s_{2.2748}, s_{2.7737})$, $r_3 = (s_{2.7316},\ s_{2.8845},\ s_{2.9676})$ and $r_4 = (s_{3.2377},\ s_{2.3175},\ s_{2.6180})$.

Step 3: The score values are $\mathcal{S}(r_1) = s_{5.3779}$, $\mathcal{S}(r_2) = s_{4.8127}$, $\mathcal{S}(r_3) = s_{4.2932}$ and $\mathcal{S}(r_4) = s_{4.7674}$.

Step 4: According to the score values, the ranking order of alternatives is $\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$ and hence $\mathcal{A}_1$ is best one.

It is clear that ranking as well the best alternative obtained by this method and the method proposed in this work is slightly different. The possible reason for this difference is that while aggregating the three decision matrix, the proposed operator consider the priority relationship among the decision maker and taking weights according to the priority relationship. Also, while aggregating the final aggregated matrix, the proposed operator take the priority relationship into account with the ordered operators. But, these concepts are not involved during the aggregation process done by the existing LNNWAA operator.

10.5 Conclusion

The linguistic approach represents the qualitative aspects, which can provide a different way to express the uncertainty in the real world. The purpose of the work is to develop a DM method which combines the concept of prioritization among the criteria's and AOs to aggregate the set of alternatives in order to reach a suitable and best-qualified alternative. In this chapter, some linguistic neutrosophic prioritized operators have been developed from both arithmetic and geometric points of view which consider the priority relationship among the attributes. Further, these operators were employed to establish an innovative MCGDM approach for evaluating the best alternative for the 'GST Mitra' scheme. Based on the comparison analysis, we can conclude that the proposed method can be successfully utilized in evaluating the best alternative and is more robust than simple algebraic and geometric operators because of the involvement of the priority levels of different criteria.

Table 10.1: Linguistic neutrosophic decision making matrices given by decision makers

		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{DM}^{(1)}$	$\mathcal{A}_1$	(s_4, s_4, s_3)	(s_3, s_1, s_5)	(s_2, s_3, s_5)	(s_2, s_3, s_6)
	$\mathcal{A}_2$	(s_4, s_3, s_2)	(s_1, s_2, s_3)	(s_5, s_1, s_3)	(s_3, s_1, s_4)
	$\mathcal{A}_3$	(s_5, s_3, s_3)	(s_2, s_3, s_4)	(s_2, s_4, s_3)	(s_1, s_4, s_3)
	$\mathcal{A}_4$	(s_3, s_5, s_1)	(s_2, s_2, s_2)	(s_3, s_2, s_3)	(s_5, s_2, s_2)
$\mathcal{DM}^{(2)}$	$\mathcal{A}_1$	(s_3, s_5, s_2)	(s_4, s_4, s_1)	(s_3, s_1, s_4)	(s_2, s_1, s_3)
	$\mathcal{A}_2$	(s_5, s_1, s_2)	(s_2, s_5, s_1)	(s_3, s_5, s_4)	(s_5, s_4, s_2)
	$\mathcal{A}_3$	(s_2, s_4, s_3)	(s_2, s_3, s_5)	(s_4, s_2, s_4)	(s_3, s_1, s_2)
	$\mathcal{A}_4$	(s_2, s_1, s_5)	(s_3, s_3, s_3)	(s_5, s_1, s_2)	(s_4, s_2, s_1)
$\mathcal{DM}^{(3)}$	$\mathcal{A}_1$	(s_6, s_1, s_1)	(s_5, s_2, s_1)	(s_7, s_2, s_1)	(s_6, s_1, s_2)
	$\mathcal{A}_2$	(s_5, s_2, s_2)	(s_2, s_2, s_3)	(s_1, s_2, s_6)	(s_3, s_4, s_5)
	$\mathcal{A}_3$	(s_1, s_2, s_3)	(s_2, s_6, s_4)	(s_3, s_4, s_3)	(s_4, s_2, s_1)
	$\mathcal{A}_4$	(s_3, s_5, s_4)	(s_2, s_1, s_6)	(s_4, s_5, s_2)	(s_1, s_2, s_6)

Table 10.2: Aggregated Linguistic neutrosophic matrix by using LSVNPOWA operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$\big(s_{4.1449}, s_{3.4758}, s_{2.2553}\big)$	$\big(s_{3.7233}, s_{1.6891}, s_{2.3537}\big)$	$\big(s_{3.5912}, s_{2.1456}, s_{3.7349}\big)$	$\big(s_{2.5076}, s_{2.2438}, s_{4.8350}\big)$
$\mathcal{A}_2$	$\big(s_{4.5581}, s_{1.9737}, s_{2.0000}\big)$	$\big(s_{1.4475}, s_{2.5985}, s_{2.1918}\big)$	$\big(s_{3.9284}, s_{1.9564}, s_{3.6782}\big)$	$\big(s_{3.6032}, s_{1.8018}, s_{3.4919}\big)$
$\mathcal{A}_3$	$\big(s_{3.7274}, s_{3.1016}, s_{3.0000}\big)$	$\big(s_{2.0000}, s_{3.2507}, s_{4.2559}\big)$	$\big(s_{2.7626}, s_{3.3274}, s_{3.2382}\big)$	$\big(s_{1.9990}, s_{2.6282}, s_{2.3552}\big)$
$\mathcal{A}_4$	$\big(s_{2.7200}, s_{3.0909}, s_{1.9900}\big)$	$\big(s_{2.3267}, s_{2.0185}, s_{2.7195}\big)$	$\big(s_{3.8890}, s_{1.9966}, s_{2.4444}\big)$	$\big(s_{4.0427}, s_{2.4937}, s_{2.0400}\big)$

Table 10.3: Aggregated Linguistic neutrosophic matrix by using LSVNPOWG operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$\big(s_{3.8997}, s_{4.0094}, s_{2.4478}\big)$	$\big(s_{3.5732}, s_{2.2053}, s_{3.5395}\big)$	$\big(s_{2.6602}, s_{2.4121}, s_{4.3493}\big)$	$\big(s_{2.3255}, s_{2.3323}, s_{5.1150}\big)$
$\mathcal{A}_2$	$\big(s_{4.4945}, s_{2.2414}, s_{2.0000}\big)$	$\big(s_{1.3459}, s_{3.0780}, s_{2.4955}\big)$	$\big(s_{3.2928}, s_{2.9285}, s_{3.9628}\big)$	$\big(s_{3.4959}, s_{2.6688}, s_{3.7203}\big)$
$\mathcal{A}_3$	$\big(s_{2.9114}, s_{3.2231}, s_{3.0000}\big)$	$\big(s_{2.0000}, s_{3.5033}, s_{4.3073}\big)$	$\big(s_{2.5601}, s_{3.5452}, s_{3.2877}\big)$	$\big(s_{1.6445}, s_{3.0944}, s_{2.4742}\big)$
$\mathcal{A}_4$	$\big(s_{2.6576}, s_{4.1355}, s_{3.0018}\big)$	$\big(s_{2.2652}, s_{2.1794}, s_{3.2744}\big)$	$\big(s_{3.7001}, s_{2.6011}, s_{2.5176}\big)$	$\big(s_{3.3358}, s_{2.7499}, s_{2.9650}\big)$

Table 10.4: Aggregated Linguistic neutrosophic decision matrix by existing LNNWAA operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$\big(s_{4.5800}, s_{2.7144}, s_{1.8171}\big)$	$\big(s_{4.0851}, s_{2.0000}, s_{1.7100}\big)$	$\big(s_{4.8928}, s_{1.8171}, s_{2.7144}\big)$	$\big(s_{3.8398}, s_{1.4422}, s_{3.3019}\big)$
$\mathcal{A}_2$	$\big(s_{4.6981}, s_{1.8171}, s_{2.0000}\big)$	$\big(s_{1.6836}, s_{2.7144}, s_{2.0801}\big)$	$\big(s_{3.2823}, s_{2.1544}, s_{4.1602}\big)$	$\big(s_{3.7828}, s_{2.5198}, s_{3.4200}\big)$
$\mathcal{A}_3$	$\big(s_{2.9867}, s_{2.8845}, s_{3.0000}\big)$	$\big(s_{2.0000}, s_{3.7798}, s_{4.3089}\big)$	$\big(s_{3.0676}, s_{3.1748}, s_{3.3019}\big)$	$\big(s_{2.8075}, s_{2.0000}, s_{1.8171}\big)$
$\mathcal{A}_4$	$\big(s_{2.6867}, s_{2.9240}, s_{2.7144}\big)$	$\big(s_{2.3538}, s_{1.8171}, s_{3.3019}\big)$	$\big(s_{4.0851}, s_{2.1544}, s_{2.2894}\big)$	$\big(s_{3.6205}, s_{2.5198}, s_{2.2894}\big)$

Chapter 11

Possibility linguistic single-valued neutrosophic sets and its application[1]

In this chapter, we develop a new concept named as possibility linguistic single-valued neutrosophic set (PLSVNS) for better dealing with the imprecise and uncertain information during the DM process. The presented idea considered the degree of the possibility value towards each linguistic features of SVNS. The prominent characteristics of this set are that it considers two distinctive sorts of information such as the membership, indeterminacy, non-membership degrees, and their corresponding possibility degree. Based on the features of PLSVNS, theories of COPRAS method and weighted averaging and geometric AOs have been defined where the weight vector of the attributes are computed with some information measures. The applicability, as well as the feasibility of the developed methods, are explained with a numerical example. Finally, the advantages of the presented concepts are explained.

11.1 Introduction

In the neutrosophic environment, the information which is evaluated is quantitative in nature and is expressed by the means of numeric numbers. But in a real scenario, most

---

[1] The content of this chapter is published as "Algorithms for possibility linguistic single-valued neutrosophic decision-making based on COPRAS and aggregation operators with new information measures", *Measurement*, 138, pp. 278 - 290, 2019 (**Impact Factor: 2.218**).

of the times the uncertain or imprecise data evaluated by the decision maker has the qualitative aspects. Like, while figure out the level of ‘performance’ of any distributor company, decision maker wants to convey his/her assessment by using the labels such as “extremely poor”, “poor”, “fair”, “slightly good”, “very good”, “extremely good”, etc. In these situations, the linguistic variables [196] are used to access the information and deal with the qualitative data. In the field of NS environment, Li et al. [71] introduce the concept of linguistic neutrosophic sets (LNSs) in which membership, indeterminacy, and non-membership are expressed as a linguistic variable instead of real numbers. In recent years, multicriteria methods such as Analytic hierarchy process (AHP), VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method, TOPSIS and COPRAS method have been increasingly used for quantitative and qualitatively evaluation of complicated economic or social processes. The aim of the evaluation is to choose the best alternatives, ranking the alternatives in the order of their significance. Among these, COPRAS method was firstly introduced by Zavadskas et al. [198] in 1994. This method compares the alternatives and determines their priorities under the conflicting criteria by taking into account the criteria weights. Chatterjee et al. [23] have done comparative analysis on different methods such as AHP, VIKOR, TOPSIS, COPRAS with regards to a computational procedure, effortlessness, probability of visual understanding and kind of the data and concluded that COPRAS strategy shows outperform among them. In the literature, there are many applications of COPRAS method. For instance, Razavi Hajiagha et al. [126] presented the COPRAS method for the information in the intuitionistic terms. Rathi and Balamohan [125] used the COPRAS method to solve the group DM problem under fuzzy environment. Zolfani et al. [203] applies the complex assessment strategy to the environmental issues by taking the decision problem. Bausys et al. [11] gives its contribution to the COPRAS method in a neutrosophic domain.

Since all the above-stated studies are widely used into the different environment, but under some certain cases, these existing approaches fail to be utilized for the issues where the linguistic neutrosophic data is given by the experts over various criteria. Furthermore, in the definition of the LSVNS, the possibility of each element of universal set related to each criterion is considered as 1. This poses a limitation in the modeling of some problems.

However, in some practical situations, the possibility of each element related to each object may be different from 1. For instance, consider the linguistic term "intelligence" and three experts evaluate the candidate. The possibility of the intelligence of a candidate by the first expert can be 0.8. However, the linguistic rating values in terms of SVNSs corresponding to a candidate is (s_3, s_2, s_4) where s_3 represent the degree of agreement towards the statement, s_2 represent the degree of indeterminacy and s_4 represent a degree of falsity towards the statement. Based on it, the other experts can be expressed with some different possibility values. Thus, to represent such information more clearly, there is a need to introduce a set which represents all of the corresponding possibility neutrosophic values.

From this point of view, we introduce the concept of possibility linguistic single-valued neutrosophic set (PLSVNS) based on an idea that each of the elements of the universe has got a possibility degree related to each element of the neutrosophic set. Also, we define some basic operational laws, score and accuracy functions, comparison laws to rank the different PLSVNSs. Based on the operational laws of possibility linguistic single-valued numbers (PLSVNs), we stated some weighted averaging and geometric AOs. Furthermore, this chapter extends the COPRAS strategy to the PLSVNS environment. The important highlights of the COPRAS method are: (1) it consider both the angles of the criteria, namely benefit and cost ones, based on the complex proportional assessment; (2) this strategy is compelled to get the DM outcomes in a more convenient way; (3) this strategy makes conceivable to figure out the gap between each alternative and the best one by evaluating the utility degree. Owing to the advantages of both the AOs and COPRAS method, the aim of this chapter is to tackle the challenges under the PLSVNS by developing two MCGDM approaches to manage the information for PLSVNSs, which not only have a great power in distinguishing the optimal alternative, but also can obtain met the optimal conditions as per our real-life situations. Therefore, motivated from the features of AOs, COPRAS method and LSVNS, the following are the fundamental targets for this work:

i) to present a new concept of possibility LSVNS and their associated score/accuracy function, comparison laws and basic operational laws;

ii) to propose different weighted averaging and geometric AOs under PLSVNS environment where the information related to each object is represented in terms of PLSVNNs;

iii) to develop some new distance measures for PLSVNSs to catch the closeness and the discrimination among the sets;

iv) to establish the COPRAS method to rank the PLSVNSs;

v) to create two different algorithms based on AOs and COPRAS method to illuminate group DM issues;

vi) to exhibit an illustration where significance of preferences based on PLSVNS decision problems has been clarified.

11.2 Possibility linguistic SVN sets

In the section, we introduce the concept of PLSVNSs and hence define its some operational laws.

Definition 11.2.1. A PLSVNS "$\mathcal{A}$" in universal set $\mathcal{X}$ is defined as follows:

$$\mathcal{A} = \big\{(s_\theta(x), s_\psi(x), s_\sigma(x); p(x)) \mid x \in \mathcal{X}\big\} \tag{11.1}$$

which consists of two kind of information: one is LSVN value $(s_\theta(x), s_\psi(x), s_\sigma(x))$ and other one is possibility degree, $p(x)$, of existence of this LSVN value for any $x \in \mathcal{X}$ and $p(x) \in (0, 1]$. For convenience, we denote $\mathcal{A}$ as $(s_\theta, s_\psi, s_\sigma; p)$ and named as possibility LSVN number (PLSVNN).

To compare two PLSVNNs, the score and the accuracy functions are stated as below:

Definition 11.2.2. Let $\mathcal{A} = (s_\theta, s_\psi, s_\sigma; p)$ is PLSVNN. Then the score function (Sc) is defined as

$$\mathrm{Sc}(\mathcal{A}) = \frac{p\,(2t + \theta - \psi - \sigma)}{3t} \in [0, 1] \tag{11.2}$$

and the accuracy function (H) is defined as

$$H(\mathcal{A}) = \frac{p\,(\theta + \psi + \sigma)}{3t} \in [0, 1] \tag{11.3}$$

Definition 11.2.3. For two PLSVNNs $\mathcal{A}$ and $\mathcal{B}$, an order relation for ranking them is defined as

(i) if $\mathrm{Sc}(\mathcal{A}) > \mathrm{Sc}(\mathcal{B})$, then $\mathcal{A} \succ \mathcal{B}$, where "$\succ$" means "preferred to";

(ii) if $\mathrm{Sc}(\mathcal{A}) = \mathrm{Sc}(\mathcal{B})$, and $H(\mathcal{A}) > H(\mathcal{B})$, then $\mathcal{A} \succ \mathcal{B}$.

Next, we present some of basic operations laws between the two PLSVNNs and their relations are studied.

Definition 11.2.4. For two PLSVNNs $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1}; p_1)$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2}; p_2)$, we have

(i) $\mathcal{A}_1^c = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1}; 1 - p_1)$.

(ii) $\mathcal{A}_1 \leq \mathcal{A}_2$ if $\theta_1 \leq \theta_2$, $\psi_1 \geq \psi_2$, $\sigma_1 \geq \sigma_2$ and $p_1 \leq p_2$.

(iii) $\mathcal{A}_1 \cup \mathcal{A}_2 = (\max\{s_{\theta_1}, s_{\theta_2}\}, \min\{s_{\psi_1}, s_{\psi_2}\}, \min\{s_{\sigma_1}, s_{\sigma_2}\}; \max\{p_1, p_2\})$.

(iv) $\mathcal{A}_1 \cap \mathcal{A}_2 = \Big(\min\{s_{\theta_1}, s_{\theta_2}\}, \max\{s_{\psi_1}, s_{\psi_2}\}, \max\{s_{\sigma_1}, s_{\sigma_2}\}; \min\{p_1, p_2\}\Big)$.

Based on the Archimedean t-norms and conorms defined in Table 2.1 of Chapter 2, the basic operational laws are as:

Definition 11.2.5. Let $\mathcal{A}_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j}; p_j)$, $j = 1, 2$ be two PLSVNNs and $\lambda > 0$, then

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \big(s_{t(1-(1-\theta_1/t)(1-\theta_2/t))}, s_{t(\psi_1\psi_2/t^2)}, s_{t(\sigma_1\sigma_2/t^2)}; 1 - (1 - p_1)\,(1 - p_2)\big)$.

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \big(s_{t(\theta_1\theta_2/t^2)}, s_{t(1-(1-\psi_1/t)(1-\psi_2/t))}, s_{t(1-(1-\sigma_1/t)(1-\sigma_2/t))}; p_1 p_2\big)$.

(iii) $\lambda\mathcal{A}_1 = \Big(s_{t\left(1-(1-\theta_1/t)^\lambda\right)}, s_{t(\psi_1/t)^\lambda}, s_{t(\sigma_1/t)^\lambda}; 1 - (1 - p_1)^\lambda\Big)$.

(iv) $\mathcal{A}_1^\lambda = \Big(s_{t(\theta_1/t)^\lambda}, s_{t\left(1-(1-\psi_1/t)^\lambda\right)}, s_{t\left(1-(1-\sigma_1/t)^\lambda\right)}; (p_1)^\lambda\Big)$.

Theorem 11.2.1. If $\mathcal{A}_1$ and $\mathcal{A}_2$ be any two PLSVNNs and $\lambda > 0$ be any real number, then $\mathcal{A}_1 \oplus \mathcal{A}_2$, $\mathcal{A}_1 \otimes \mathcal{A}_2$, $\lambda\mathcal{A}_1$ and $\mathcal{A}_1^\lambda$ are also PLSVNNs.

Proof. Let $\mathcal{A}_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j}; p_j)$, $j = 1, 2$ be two PLSVNNs then by the definition of PLSVNNs, we have, $0 \leq \theta_1, \theta_2 \leq t$ which implies that $0 \leq \frac{\theta_1}{t}, \frac{\theta_2}{t} \leq 1$ and hence $0 \leq 1 - \left(1 - \frac{\theta_1}{t}\right)\left(1 - \frac{\theta_2}{t}\right) \leq 1 \Rightarrow 0 \leq t\left(1 - \left(1 - \frac{\theta_1}{t}\right)\left(1 - \frac{\theta_2}{t}\right)\right) \leq t$.

Also, for $0 \leq \sigma_1, \sigma_2 \leq t \Rightarrow 0 \leq \frac{\sigma_1}{t}, \frac{\sigma_2}{t} \leq 1 \Rightarrow 0 \leq \frac{\sigma_1\sigma_2}{t^2} \leq 1 \Rightarrow 0 \leq t\left(\frac{\sigma_1\sigma_2}{t^2}\right) \leq t$. Similarly, we can get $0 \leq t\left(\frac{\psi_1\psi_2}{t^2}\right) \leq t$.

Further, since each component is independent and hence we get

$$0 \leq t\left(1 - \left(1 - \frac{\theta_1}{t}\right)\left(1 - \frac{\theta_2}{t}\right)\right) + t\left(\frac{\sigma_1\sigma_2}{t^2}\right) + t\left(\frac{\psi_1\psi_2}{t^2}\right) \leq 3t.$$

Also, $0 \leq 1 - (1 - p_1)(1 - p_2) \leq 1$. Thus, $\mathcal{A}_1 \oplus \mathcal{A}_2$ is a PLSVNN. Similarly, we can prove that $\mathcal{A}_1 \otimes \mathcal{A}_2$, $\lambda\mathcal{A}_1$ and $\mathcal{A}_1^\lambda$ are also PLSVNNs. $\square$

Theorem 11.2.2. Let $\mathcal{A}_1, \mathcal{A}_2$ be two PLSVNNs and $\lambda, \lambda_1, \lambda_2 > 0$ be real numbers, then

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$.

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$.

(iii) $\lambda(\mathcal{A}_1 \oplus \mathcal{A}_2) = \lambda\mathcal{A}_1 \oplus \lambda\mathcal{A}_2$.

(iv) $(\mathcal{A}_1 \otimes \mathcal{A}_2)^\lambda = \mathcal{A}_1^\lambda \otimes \mathcal{A}_2^\lambda$.

(v) $\lambda_1\mathcal{A}_1 \oplus \lambda_2\mathcal{A}_1 = (\lambda_1 + \lambda_2)\mathcal{A}_1$.

(vi) $\mathcal{A}_1^{\lambda_1} \otimes \mathcal{A}_1^{\lambda_2} = \mathcal{A}_1^{\lambda_1 + \lambda_2}$.

Proof. Here, we shall prove (i), (iii) and (v) parts while the other will prove in same manner. For two PLSVNNs $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1}; p_1)$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2}; p_2)$, we have

(i) It is straightforward.

(iii) By Definition 11.2.5, we have

$$\begin{aligned}
\lambda(\mathcal{A}_1 \oplus \mathcal{A}_2) &= \lambda\begin{pmatrix} s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)\left(1-\frac{\theta_2}{t}\right)\right)}, s_{t\left(\frac{\psi_1\psi_2}{t^2}\right)}, \\ s_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)}; 1 - (1 - p_1)(1 - p_2) \end{pmatrix} \\
&= \begin{pmatrix} s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^\lambda\left(1-\frac{\theta_2}{t}\right)^\lambda\right)}, s_{t\left(\frac{\psi_1\psi_2}{t^2}\right)^\lambda}, \\ s_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)^\lambda}; 1 - (1 - p_1)^\lambda(1 - p_2)^\lambda \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} s_{1-\left(1-\frac{\theta_1}{t}\right)^{\lambda}+1-\left(1-\frac{\theta_2}{t}\right)^{\lambda}-\left(1-\left(1-\frac{\theta_1}{t}\right)^{\lambda}\right)\left(1-\left(1-\frac{\theta_2}{t}\right)^{\lambda}\right)}, \\ s_{t\left(\frac{\psi_1}{t}\right)^{\lambda}\left(\frac{\psi_2}{t}\right)^{\lambda}}, s_{t\left(\frac{\sigma_1}{t}\right)^{\lambda}\left(\frac{\sigma_2}{t}\right)^{\lambda}}; 1-(1-p_1)^{\lambda}(1-p_2)^{\lambda} \end{pmatrix} \\
&= \left(s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\lambda}\right)}, s_{t\left(\frac{\psi_1}{t}\right)^{\lambda}}, s_{t\left(\frac{\sigma_1}{t}\right)^{\lambda}}; 1-(1-p_1)^{\lambda}\right) \\
&\oplus \left(s_{t\left(1-\left(1-\frac{\theta_2}{t}\right)^{\lambda}\right)}, s_{t\left(\frac{\psi_2}{t}\right)^{\lambda}}, s_{t\left(\frac{\sigma_2}{t}\right)^{\lambda}}; 1-(1-p_2)^{\lambda}\right) \\
&= \lambda\mathcal{A}_1 \oplus \lambda\mathcal{A}_2
\end{aligned}$$

Hence, $\lambda\left(\mathcal{A}_1 \oplus \mathcal{A}_2\right) = \lambda\mathcal{A}_1 \oplus \lambda\mathcal{A}_2$.

(v) Since $\mathcal{A}_1$ is PLSVNN and $\lambda_1, \lambda_2 > 0$ are real numbers, so

$$\begin{aligned}
\lambda_1\mathcal{A}_1 \oplus \lambda_2\mathcal{A}_1 &= \left(s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\lambda_1}\right)}, s_{t\left(\frac{\psi_1}{t}\right)^{\lambda_1}}, s_{t\left(\frac{\sigma_1}{t}\right)^{\lambda_1}}; 1-(1-p_1)^{\lambda_1}\right) \\
&\oplus \left(s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\lambda_2}\right)}, s_{t\left(\frac{\psi_1}{t}\right)^{\lambda_2}}, s_{t\left(\frac{\sigma_1}{t}\right)^{\lambda_2}}; 1-(1-p_1)^{\lambda_2}\right) \\
&= \begin{pmatrix} s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\lambda_1}\left(1-\frac{\theta_1}{t}\right)^{\lambda_2}\right)}, s_{t\left(\frac{\psi_1}{t}\right)^{\lambda_1}\left(\frac{\psi_1}{t}\right)^{\lambda_2}}, \\ s_{t\left(\frac{\sigma_1}{t}\right)^{\lambda_1}\left(\frac{\sigma_1}{t}\right)^{\lambda_2}}; 1-(1-p_1)^{\lambda_1+\lambda_2} \end{pmatrix} \\
&= \begin{pmatrix} s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\lambda_1+\lambda_2}\right)}, s_{t\left(\frac{\psi_1}{t}\right)^{\lambda_1+\lambda_2}}, \\ s_{t\left(\frac{\sigma_1}{t}\right)^{\lambda_1+\lambda_2}}; 1-(1-p_1)^{\lambda_1+\lambda_2} \end{pmatrix} \\
&= \left(\lambda_1+\lambda_2\right)\mathcal{A}_1
\end{aligned}$$

Hence, $\lambda_1\mathcal{A}_1 \oplus \lambda_2\mathcal{A}_1 = \left(\lambda_1+\lambda_2\right)\mathcal{A}_1$.

□

Theorem 11.2.3. For three PLSVNNs $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$, we have

(i) $(\mathcal{A}_1 \cup \mathcal{A}_2) \oplus (\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{A}_1 \oplus \mathcal{A}_2$.

(ii) $(\mathcal{A}_1 \cup \mathcal{A}_2) \otimes (\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{A}_1 \otimes \mathcal{A}_2$.

(iii) $(\mathcal{A}_1 \cup \mathcal{A}_2) \cap \mathcal{A}_3 = (\mathcal{A}_1 \cap \mathcal{A}_3) \cup (\mathcal{A}_2 \cap \mathcal{A}_3)$.

(iv) $(\mathcal{A}_1 \cap \mathcal{A}_2) \cup \mathcal{A}_3 = (\mathcal{A}_1 \cup \mathcal{A}_3) \cap (\mathcal{A}_2 \cup \mathcal{A}_3)$.

(v) $(\mathcal{A}_1 \cup \mathcal{A}_2) \oplus \mathcal{A}_3 = (\mathcal{A}_1 \oplus \mathcal{A}_3) \cup (\mathcal{A}_2 \oplus \mathcal{A}_3)$.

(vi) $(\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \mathcal{A}_3 = (\mathcal{A}_1 \oplus \mathcal{A}_3) \cap (\mathcal{A}_2 \oplus \mathcal{A}_3)$.

(vii) $(\mathcal{A}_1 \cup \mathcal{A}_2) \otimes \mathcal{A}_3 = (\mathcal{A}_1 \otimes \mathcal{A}_3) \cup (\mathcal{A}_2 \otimes \mathcal{A}_3)$.

(viii) $(\mathcal{A}_1 \cap \mathcal{A}_2) \otimes \mathcal{A}_3 = (\mathcal{A}_1 \otimes \mathcal{A}_3) \cap (\mathcal{A}_2 \otimes \mathcal{A}_3)$.

Proof. We shall proof the parts (i) and (ii) only, the rest will proved similarly.

(i) If $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1}; p_1)$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2}; p_2)$ be two PLSVNNs. Then

$$\begin{aligned}
& (\mathcal{A}_1 \cup \mathcal{A}_2) \oplus (\mathcal{A}_1 \cap \mathcal{A}_2) \\
= \; & \Big(\max\{s_{\theta_1}, s_{\theta_2}\}, \min\{s_{\psi_1}, s_{\psi_2}\}, \min\{s_{\sigma_1}, s_{\sigma_2}\}; \max\{p_1, p_2\} \Big) \\
\oplus \; & \Big(\min\{s_{\theta_1}, s_{\theta_2}\}, \max\{s_{\psi_1}, s_{\psi_2}\}, \max\{s_{\sigma_1}, s_{\sigma_2}\}; \min\{p_1, p_2\} \Big) \\
= \; & \Big(s_{t\left(1-\left(1-\frac{\max\{s_{\theta_1}, s_{\theta_2}\}}{t}\right)\left(1-\frac{\min\{s_{\theta_1}, s_{\theta_2}\}}{t}\right)\right)}, s_{t\left(\frac{\min\{s_{\psi_1}, s_{\psi_2}\}\max\{s_{\psi_1}, s_{\psi_2}\}}{t^2}\right)}, \\
& s_{t\left(\frac{\min\{s_{\sigma_1}, s_{\sigma_2}\}\max\{s_{\sigma_1}, s_{\sigma_2}\}}{t^2}\right)}; 1 - (1 - \max\{p_1, p_2\})\,(1 - \min\{p_1, p_2\}) \Big) \\
= \; & \Big(s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)\left(1-\frac{\theta_2}{t}\right)\right)}, s_{t\left(\frac{\psi_1\psi_2}{t^2}\right)}, s_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)}; 1 - (1 - p_1)\,(1 - p_2) \Big) \\
= \; & \mathcal{A}_1 \oplus \mathcal{A}_2
\end{aligned}$$

(iii) If $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1}; p_1)$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2}; p_2)$ and $\mathcal{A}_3 = (s_{\theta_3}, s_{\psi_3}, s_{\sigma_3}; p_1)$ be three PLSVNNs. Then

$$\begin{aligned}
& (\mathcal{A}_1 \cup \mathcal{A}_2) \cap \mathcal{A}_3 \\
= \; & \Big(\max\{s_{\theta_1}, s_{\theta_2}\}, \min\{s_{\psi_1}, s_{\psi_2}\}, \min\{s_{\sigma_1}, s_{\sigma_2}\}; \max\{p_1, p_2\} \Big) \cap (s_{\theta_3}, s_{\psi_3}, s_{\sigma_3}; p_1) \\
= \; & \Big(\min\{\max\{s_{\theta_1}, s_{\theta_2}\}, s_{\theta_3}\}, \max\{\min\{s_{\psi_1}, s_{\psi_2}\}, s_{\psi_3}\}, \max\{\min\{s_{\sigma_1}, s_{\sigma_2}\}, s_{\sigma_3}\}; \\
& \quad \min\{\max\{p_1, p_2\}, p_3\} \Big) \\
= \; & \Big(\min\{s_{\theta_1}, s_{\theta_3}\}, \max\{s_{\psi_1}, s_{\psi_3}\}, \max\{s_{\sigma_1}, s_{\sigma_3}\}; \min\{p_1, p_3\} \Big) \cap \\
& \quad \Big(\min\{s_{\theta_2}, s_{\theta_3}\}, \max\{s_{\psi_2}, s_{\psi_3}\}, \max\{s_{\sigma_2}, s_{\sigma_3}\}; \min\{p_2, p_3\} \Big) \\
= \; & (\mathcal{A}_1 \cap \mathcal{A}_3) \cup (\mathcal{A}_2 \cap \mathcal{A}_3)
\end{aligned}$$

□

11.3 Information measures and AOs for the proposed PLSVNSs

This section presents the different measures and the AOs for PLSVNSs as follows.

11.3.1 Distance measures

Definition 11.3.1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two PLSVNSs and $\tau \geq 1$, then the generalized normalized distance between them is stated as

$$\mathcal{D}_\tau(\mathcal{A}_1, \mathcal{A}_2) = \left\{ \frac{1}{3n} \sum_{j=1}^{n} \left(\begin{array}{c} \left| p_1(x_j) \frac{\theta_1(x_j)}{t} - p_2(x_j) \frac{\theta_2(x_j)}{t} \right|^\tau + \left| p_1(x_j) \frac{\psi_1(x_j)}{t} - p_2(x_j) \frac{\psi_2(x_j)}{t} \right|^\tau \\ + \left| p_1(x_j) \frac{\sigma_1(x_j)}{t} - p_2(x_j) \frac{\sigma_2(x_j)}{t} \right|^\tau \end{array} \right) \right\}^{\frac{1}{\tau}} \tag{11.4}$$

Theorem 11.3.1. The measure defined in Definition 11.3.1 satisfies the following axioms

(P1) $0 \leq \mathcal{D}_\tau(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) $\mathcal{D}_\tau(\mathcal{A}_1, \mathcal{A}_2) = 0$ if $\mathcal{A}_1 = \mathcal{A}_2$.

(P3) $\mathcal{D}_\tau(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{D}_\tau(\mathcal{A}_2, \mathcal{A}_1)$.

Proof. For PLSVNSs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

(P1) $0 \leq \frac{\theta_1(x_j)}{t}, \frac{\theta_2(x_j)}{t} \leq 1$, $0 \leq \frac{\psi_1(x_j)}{t}, \frac{\psi_2(x_j)}{t} \leq 1$, $0 \leq \frac{\sigma_1(x_j)}{t}, \frac{\sigma_2(x_j)}{t} \leq 1$ and $0 \leq p_1(x_j), p_2(x_j) \leq 1$. Therefore, we get, $-1 \leq p_1(x_j)\frac{\theta_1(x_j)}{t} - p_2(x_j)\frac{\theta_2(x_j)}{t} \leq 1$, $-1 \leq p_1(x_j)\frac{\psi_1(x_j)}{t} - p_2(x_j)\frac{\psi_2(x_j)}{t} \leq 1$, $-1 \leq p_1(x_j)\frac{\sigma_1(x_j)}{t} - p_2(x_j)\frac{\sigma_2(x_j)}{t} \leq 1$ and $0 \leq \left|p_1(x_j)\frac{\theta_1(x_j)}{t} - p_2(x_j)\frac{\theta_2(x_j)}{t}\right|, \left|p_1(x_j)\frac{\psi_1(x_j)}{t} - p_2(x_j)\frac{\psi_2(x_j)}{t}\right|, \left|p_1(x_j)\frac{\sigma_1(x_j)}{t} - p_2(x_j)\frac{\sigma_2(x_j)}{t}\right| \leq 1$, which implies

$$0 \leq \sum_{j=1}^{n} \left(\begin{array}{c} \left| p_1(x_j) \frac{\theta_1(x_j)}{t} - p_2(x_j) \frac{\theta_2(x_j)}{t} \right|^\tau + \left| p_1(x_j) \frac{\psi_1(x_j)}{t} - p_2(x_j) \frac{\psi_2(x_j)}{t} \right|^\tau \\ + \left| p_1(x_j) \frac{\sigma_1(x_j)}{t} - p_2(x_j) \frac{\sigma_2(x_j)}{t} \right|^\tau \end{array} \right) \leq 3n.$$

Thus, we have

$$0 \leq \left\{ \frac{1}{3n} \sum_{j=1}^{n} \left(\begin{array}{c} \left| p_1(x_j) \frac{\theta_1(x_j)}{t} - p_2(x_j) \frac{\theta_2(x_j)}{t} \right|^\tau + \left| p_1(x_j) \frac{\psi_1(x_j)}{t} - p_2(x_j) \frac{\psi_2(x_j)}{t} \right|^\tau \\ + \left| p_1(x_j) \frac{\sigma_1(x_j)}{t} - p_2(x_j) \frac{\sigma_2(x_j)}{t} \right|^\tau \end{array} \right) \right\}^{\frac{1}{\tau}} \leq 1.$$

Hence, $0 \leq \mathcal{D}_\tau(\mathcal{A}_1, \mathcal{A}_2) \leq 1$.

(P2) If $\mathcal{A}_1 = \mathcal{A}_2$, then $\theta_1(x_j) = \theta_2(x_j)$, $\psi_1(x_j) = \psi_2(x_j)$, $\sigma_1(x_j) = \sigma_2(x_j)$ and $p_1(x_j) = p_2(x_j)$ for all x_j, which implies

$$\begin{aligned}
&\mathcal{D}_\tau\left(\mathcal{A}_1, \mathcal{A}_2\right) \\
=\;& \left(\frac{1}{3n}\sum_{j=1}^{n}\left(\begin{array}{c}\left|p_1(x_j)\frac{\theta_1(x_j)}{t} - p_1(x_j)\frac{\theta_1(x_j)}{t}\right|^\tau + \left|p_1(x_j)\frac{\psi_1(x_j)}{t} - p_1(x_j)\frac{\psi_1(x_j)}{t}\right|^\tau \\ + \left|p_1(x_j)\frac{\sigma_1(x_j)}{t} - p_1(x_j)\frac{\sigma_1(x_j)}{t}\right|^\tau\end{array}\right)\right)^{\frac{1}{\tau}} \\
=\;& 0
\end{aligned}$$

(P3) This property is trivial.

□

Remark 11.3.1. From the Definition 11.3.1, it is stated that when $\tau = 1, 2$, Eq. (11.4) reduces to normalized Hamming and Euclidean distances, respectively.

Definition 11.3.2. For two PLSVNSs $\mathcal{A}_1$ and $\mathcal{A}_2$, the generalized normalized similarity between them is defined as

$$\begin{aligned}
&\mathcal{S}_\tau\left(\mathcal{A}_1, \mathcal{A}_2\right) \\
=\;& 1 - \left\{\frac{1}{3n}\sum_{j=1}^{n}\left(\begin{array}{c}\left|p_1(x_j)\frac{\theta_1(x_j)}{t} - p_2(x_j)\frac{\theta_2(x_j)}{t}\right|^\tau + \left|p_1(x_j)\frac{\psi_1(x_j)}{t} - p_2(x_j)\frac{\psi_2(x_j)}{t}\right|^\tau \\ + \left|p_1(x_j)\frac{\sigma_1(x_j)}{t} - p_2(x_j)\frac{\sigma_2(x_j)}{t}\right|^\tau\end{array}\right)\right\}^{\frac{1}{\tau}} \qquad (11.5)
\end{aligned}$$

Theorem 11.3.2. The similarity $\mathcal{S}_\tau\left(\mathcal{A}_1, \mathcal{A}_2\right)$ defined in Definition 11.3.2 satisfies the following properties:

(S1) $0 \leq \mathcal{S}_\tau\left(\mathcal{A}_1, \mathcal{A}_2\right) \leq 1$.

(S2) $\mathcal{S}_\tau\left(\mathcal{A}_1, \mathcal{A}_2\right) = 1$ if $\mathcal{A}_1 = \mathcal{A}_2$.

(S3) $\mathcal{S}_\tau\left(\mathcal{A}_1, \mathcal{A}_2\right) = \mathcal{S}_\tau\left(\mathcal{A}_2, \mathcal{A}_1\right)$.

Proof. Similar to above, so we omit here. □

11.3.2 Aggregation operators for PLSVNSs

For a collection of PLSVNNs $\mathcal{A}_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j}; p_j)$, we define the some weighted averaging and geometric AOs. Let $F(\mathcal{X})$ be the collections of PLSVNN over the universal set $\mathcal{X}$.

Definition 11.3.3. For "n" PLSVNNs $\mathcal{A}_j; j = 1, 2, \ldots, n$, the PLSVN weighted averaging and geometric operators, denoted by PLSVNWA and PLSVNWG respectively, are defined as:

$$\text{PLSVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigoplus_{j=1}^{n} \omega_j \mathcal{A}_j \tag{11.6}$$

and

$$\text{PLSVNWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigotimes_{j=1}^{n} \mathcal{A}_j^{\omega_j} \tag{11.7}$$

where ω_j is the associated weight of $\mathcal{A}_j$ with $\omega_j > 0$ and $\sum\limits_{j=1}^{n} \omega_j = 1$.

Theorem 11.3.3. The aggregated value of the collection of PLSVNNs $\mathcal{A}_j; j = 1, 2, \ldots, n$, by using PLSVNWA operator is a still PLSVNN and is defined as

$$\begin{aligned} &\text{PLSVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = &\left(s_{t\left(1-\prod_{j=1}^{n}\left(1-\frac{\theta_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}\left(\frac{\psi_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}\left(\frac{\sigma_j}{t}\right)^{\omega_j}\right)}; 1 - \prod_{j=1}^{n} (1 - p_j)^{\omega_j} \right) \end{aligned} \tag{11.8}$$

Proof. We will prove Eq. (11.8) by applying the induction on n, which involves the following steps:

Step 1: For $n = 2$, we have $\mathcal{A}_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1}; p_1)$ and $\mathcal{A}_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2}; p_2)$. Thus, by the operational laws of PLSVNNs, we have

$$\omega_1 \mathcal{A}_1 = \left(s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\omega_1}\right)}, s_{t\left(\frac{\psi_1}{t}\right)^{\omega_1}}, s_{t\left(\frac{\sigma_1}{t}\right)^{\omega_1}}; 1 - (1 - p_1)^{\omega_1} \right)$$

and

$$\omega_2 \mathcal{A}_2 = \left(s_{t\left(1-\left(1-\frac{\theta_2}{t}\right)^{\omega_2}\right)}, s_{t\left(\frac{\psi_2}{t}\right)^{\omega_2}}, s_{t\left(\frac{\sigma_2}{t}\right)^{\omega_2}}; 1 - (1 - p_2)^{\omega_2} \right)$$

Then, by using addition law, we get

$$\begin{aligned} &\text{PLSVNWA}\left(\mathcal{A}_1, \mathcal{A}_2\right) = \omega_1 \mathcal{A}_1 \oplus \omega_2 \mathcal{A}_2 \\ = &\left(s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\omega_1}\right)}, s_{t\left(\frac{\psi_1}{t}\right)^{\omega_1}}, s_{t\left(\frac{\sigma_1}{t}\right)^{\omega_1}}; 1 - (1 - p_1)^{\omega_1} \right) \\ \oplus &\left(s_{t\left(1-\left(1-\frac{\theta_2}{t}\right)^{\omega_2}\right)}, s_{t\left(\frac{\psi_2}{t}\right)^{\omega_2}}, s_{t\left(\frac{\sigma_2}{t}\right)^{\omega_2}}; 1 - (1 - p_2)^{\omega_2} \right) \end{aligned}$$

$$= \begin{pmatrix} s_{t\left(1-\left(1-\frac{\theta_1}{t}\right)^{\omega_1}\left(1-\frac{\theta_2}{t}\right)^{\omega_2}\right)}, s_{t\left(\frac{\psi_1}{t}\right)^{\omega_1}\left(\frac{\psi_2}{t}\right)^{\omega_2}}, \\ s_{t\left(\frac{\sigma_1}{t}\right)^{\omega_1}\left(\frac{\sigma_2}{t}\right)^{\omega_2}}; 1-(1-p_1)^{\omega_1}(1-p_2)^{\omega_2} \end{pmatrix}$$

$$= \begin{pmatrix} s_{t\left(1-\prod\limits_{j=1}^{2}\left(1-\frac{\theta_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod\limits_{j=1}^{2}\left(\frac{\psi_j}{t}\right)^{\omega_j}\right)}, \\ s_{t\left(\prod\limits_{j=1}^{2}\left(\frac{\sigma_j}{t}\right)^{\omega_j}\right)}; 1-\prod\limits_{j=1}^{2}(1-p_j)^{\omega_j} \end{pmatrix}$$

Thus, the result is true for $n = 2$.

Step 2: Assume that the Eq. (11.8) holds for $n = r$, then for $n = r + 1$, we have

$$\begin{aligned} & \text{PLSVNWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{r+1}\right) \\ = \; & \bigoplus_{j=1}^{r} \omega_j \mathcal{A}_j \oplus \omega_{r+1}\mathcal{A}_{r+1} \\ = \; & \left(s_{t\left(1-\prod\limits_{j=1}^{r}\left(1-\frac{\theta_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod\limits_{j=1}^{r}\left(\frac{\psi_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod\limits_{j=1}^{r}\left(\frac{\sigma_j}{t}\right)^{\omega_j}\right)}; 1-\prod_{j=1}^{r}(1-p_j)^{\omega_j} \right) \\ \oplus \; & \left(s_{t\left(1-\left(1-\frac{\theta_{r+1}}{t}\right)^{\omega_{r+1}}\right)}, s_{t\left(\frac{\psi_{r+1}}{t}\right)^{\omega_{r+1}}}, s_{t\left(\frac{\sigma_{r+1}}{t}\right)^{\omega_{r+1}}}; 1-(1-p_{r+1})^{\omega_{r+1}} \right) \\ = \; & \left(s_{t\left(1-\prod\limits_{j=1}^{r+1}\left(1-\frac{\theta_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod\limits_{j=1}^{r+1}\left(\frac{\psi_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod\limits_{j=1}^{r+1}\left(\frac{\sigma_j}{t}\right)^{\omega_j}\right)}; 1-\prod_{j=1}^{r+1}(1-p_j)^{\omega_j} \right) \end{aligned}$$

Thus, the result holds for $n = r + 1$, then by principle of mathematical induction, the result holds for all positive integers n. □

Theorem 11.3.4. The aggregated value for a collection of "n" PLSVNNs $\mathcal{A}_j$ by using PLSVNWG operator is still PLSVNN and is given as

$$\begin{aligned} & \text{PLSVNWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\ = \; & \left(s_{t\left(\prod\limits_{j=1}^{n}\left(\frac{\theta_j}{t}\right)^{\omega_j}\right)}, s_{t\left(1-\prod\limits_{j=1}^{n}\left(1-\frac{\psi_j}{t}\right)^{\omega_j}\right)}, s_{t\left(1-\prod\limits_{j=1}^{n}\left(1-\frac{\sigma_j}{t}\right)^{\omega_j}\right)}; \prod_{j=1}^{n} p_j^{\omega_j} \right) \end{aligned} \qquad (11.9)$$

where ω_j is the associated weight of $\mathcal{A}_j$ with $\omega_j > 0$ and $\sum\limits_{j=1}^{n} \omega_j = 1$.

Proof. As similar to Theorem 11.3.3, so we omit here. □

As an example of PLSVNWA operator, it is concluded that they satisfies the certain characteristics which are given as below.

Theorem 11.3.5. For PLSVNNs $\mathcal{A}_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j}; p_j); j = 1, 2, \ldots, n$, with weights vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, the proposed operator satisfies the following properties:

(P1) (Idempotency) If all $\mathcal{A}_j = \mathcal{A}$; $\forall j$ are equal where $\mathcal{A}$ is another PLSVNN, then

$$\text{PLSVNWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \mathcal{A}$$

(P2) (Monotonicity) Let $\mathcal{B}_j = (s_{\theta'_j}, s_{\psi'_j}, s_{\sigma'_j}; p'_j)$ be another collection of PLSVNNs such that $\theta'_j \leq \theta_j$; $\psi'_j \geq \psi_j$, $\sigma'_j \geq s_{\sigma_j}$ and $p'_j \leq p_j$. Then,

$$\text{PLSVNWA}\left(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\right) \leq \text{PLSVNWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right)$$

(P3) (Boundedness) If $\mathcal{A}^- = (\min_j(s_{\theta_j}), \max_j(s_{\psi_j}), \max_j(s_{\sigma_j}); \min_j(p_j))$ and $\mathcal{A}^+ = (\max_j(s_{\theta_j}),$ $\min_j(s_{\psi_j}),$ $\min_j(s_{\sigma_j}); \max_j(p_j))$ be two PLSVNNs, then

$$\mathcal{A}^- \leq \text{PLSVNWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) \leq \mathcal{A}^+$$

Proof. (P1) If $\mathcal{A}_j = \mathcal{A} = (s_\theta, s_\psi, s_\sigma; p)$ for all j, then

$$\begin{aligned}
& \text{PLSVNWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) \\
= & \left(s_{t\left(1-\prod_{j=1}^{n}\left(1-\frac{\theta_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}\left(\frac{\psi_j}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}\left(\frac{\sigma_j}{t}\right)^{\omega_j}\right)}; 1 - \prod_{j=1}^{n}\left(1-p_j\right)^{\omega_j} \right) \\
= & \left(s_{t\left(1-\prod_{j=1}^{n}\left(1-\frac{\theta}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}\left(\frac{\psi}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}\left(\frac{\sigma}{t}\right)^{\omega_j}\right)}; 1 - \prod_{j=1}^{n}\left(1-p\right)^{\omega_j} \right) \\
= & \left(s_{t\left(1-\left(1-\frac{\theta}{t}\right)^{\sum_{j=1}^{n}\omega_j}\right)}, s_{t\left(\left(\frac{\psi}{t}\right)^{\sum_{j=1}^{n}\omega_j}\right)}, s_{t\left(\left(\frac{\sigma}{t}\right)^{\sum_{j=1}^{n}\omega_j}\right)}; 1 - \left(1-p\right)^{\sum_{j=1}^{n}\omega_j} \right) \\
= & \left(s_{t\left(1-\left(1-\frac{\theta}{t}\right)\right)}, s_{t\left(\frac{\psi}{t}\right)}, s_{t\left(\frac{\sigma}{t}\right)}; 1-\left(1-p\right) \right) \\
= & \left(s_\theta, s_\psi, s_\sigma; p\right) \\
= & \mathcal{A}
\end{aligned}$$

(P2) If $\theta_j' \leq \theta_j$ for all j, then we have

$$\begin{aligned}
\theta_j' \leq \theta_j \quad &\Rightarrow \quad 1-\frac{\theta_j'}{t} \geq 1-\frac{\theta_j}{t}\\
&\Rightarrow \quad \left(1-\frac{\theta_j'}{t}\right)^{\omega_j} \geq \left(1-\frac{\theta_j}{t}\right)^{\omega_j}\\
&\Rightarrow \quad \prod_{j=1}^{n}\left(1-\frac{\theta_j'}{t}\right)^{\omega_j} \geq \prod_{j=1}^{n}\left(1-\frac{\theta_j}{t}\right)^{\omega_j}\\
&\Rightarrow \quad 1-\prod_{j=1}^{n}\left(1-\frac{\theta_j'}{t}\right)^{\omega_j} \leq 1-\prod_{j=1}^{n}\left(1-\frac{\theta_j}{t}\right)^{\omega_j}\\
&\Rightarrow \quad t\left(1-\prod_{j=1}^{n}\left(1-\frac{\theta_j'}{t}\right)^{\omega_j}\right) \leq t\left(1-\prod_{j=1}^{n}\left(1-\frac{\theta_j}{t}\right)^{\omega_j}\right)
\end{aligned}$$

On the other hand, $\psi_j' \geq \psi_j$ for all j, which implies $t\left(\prod_{j=1}^{n}\left(\frac{\psi_j'}{t}\right)^{\omega_j}\right) \geq t\left(\prod_{j=1}^{n}\left(\frac{\psi_j}{t}\right)^{\omega_j}\right)$. Similarly, we can prove $t\left(\prod_{j=1}^{n}\left(\frac{\sigma_j'}{t}\right)^{\omega_j}\right) \geq t\left(\prod_{j=1}^{n}\left(\frac{\sigma_j}{t}\right)^{\omega_j}\right)$ and $1-\prod_{j=1}^{n}\left(1-\frac{p_j'}{t}\right)^{\omega_j} \leq 1-\prod_{j=1}^{n}\left(1-\frac{p_j}{t}\right)^{\omega_j}$ Hence, by Definition 11.2.4, we can easily deduce that

$$\text{PLSVNWA}\left(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\right) \leq \text{PLSVNWA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right).$$

(P3) As we know, $\min_j\left(s_{\theta_j}\right) \leq s_{\theta_j} \leq \max_j\left(s_{\theta_j}\right)$, $\min_j\left(s_{\psi_j}\right) \leq s_{\psi_j} \leq \max_j\left(s_{\psi_j}\right)$, $\min_j\left(s_{\sigma_j}\right) \leq s_{\sigma_j} \leq \max_j\left(s_{\sigma_j}\right)$ and $\min_j\left(p_j\right) \leq p_j \leq \max_j\left(p_j\right)$. Then by using idempotency and monotonicity properties , we can get PLSVNWA$(\mathcal{A}^-, \mathcal{A}^-, \ldots, \mathcal{A}^-) \leq$ PLSVNWA$(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq$ PLSVNWA$(\mathcal{A}^+, \mathcal{A}^+, \ldots, \mathcal{A}^+)$ which implies

$$\mathcal{A}^- \leq \text{PLSVNWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \mathcal{A}^+.$$

□

11.4 COPRAS method with PLSVN information

COPRAS method is mainly developed by Zavadskas et al. [198], which consists of the direct and relative dependencies of the importance and the utility degree of the alternatives under the contrary criteria values. This method takes into consideration the given choices with regards to the distinct criteria and their corresponding weights. This method has the capability to consider both beneficial and non-beneficial criteria, which can be evaluated independently during the assessment process. The main advantage of this method is that it can be utilized to calculate the utility degree of the given choices showing the degree to which one alternative is way better or more awful than other opinions taken for comparison.

Consider a DM problem whose description is given in Section 2.5 of Chapter 2. Assume that rating information towards the assessment of the alternatives is provided by an expert in terms of PLSVNNs $\alpha_{ij} = (s_{\theta_{ij}}, s_{\psi_{ij}}, s_{\sigma_{ij}}; p_{ij})$, where $s_{\theta_{ij}}$, $s_{\psi_{ij}}$, $s_{\sigma_{ij}}$ represents the satisfactory, indeterminant, non-satisfactory degree in linguistic term, respectively, and p_{ij} is the corresponding possibility of the LSVNN. Then the procedure to solve the MADM problem by using proposed COPRAS method is summarized in the following steps.

Step 1: Summarize the collection information in terms of decision matrix $\mathcal{K} = (\alpha_{ij})_{m\times n}$.

Step 2: Construct the weighted decision matrix $\mathcal{K}_\omega = (\widetilde{\alpha}_{ij})_{m\times n}$ where $\widetilde{\alpha}_{ij} = (s_{\widetilde{\theta}_{ij}}, s_{\widetilde{\psi}_{ij}}, s_{\widetilde{\sigma}_{ij}}; \widetilde{p}_{ij})$ is computed as

$$\begin{aligned}\widetilde{\alpha}_{ij} &= \omega_j \alpha_{ij} \\ &= \left(s_{t\left(1-\left(1-\frac{\theta_{ij}}{t}\right)^{\omega_j}\right)}, s_{t\left(\frac{\psi_{ij}}{t}\right)^{\omega_j}}, s_{t\left(\frac{\sigma_{ij}}{t}\right)^{\omega_j}}; 1-(1-p_{ij})^{\omega_j} \right) \qquad (11.10)\end{aligned}$$

Step 3: Assume that out of n attributes, k are of the benefit types ($\mathcal{F}_1$) and the remaining $n-k$ are the cost types ($\mathcal{F}_2$). In $\mathcal{F}_1$, the higher is the rating values, the better is the fulfillment of the goal. Thus, their rating values are aggregated by utilizing PLSVNWA operator and get

$$\begin{aligned}Q_i &= \bigoplus_{j=1}^{k} \widetilde{\alpha}_{ij} \qquad (11.11) \\ &= \left(s_{t\left(1-\prod_{j=1}^{k}\left(1-\frac{\widetilde{\theta}_{ij}}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{k}\left(\frac{\widetilde{\psi}_{ij}}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{k}\left(\frac{\widetilde{\sigma}_{ij}}{t}\right)^{\omega_j}\right)}; 1-\prod_{j=1}^{k}(1-\widetilde{p}_{ij})^{\omega_j} \right)\end{aligned}$$

Step 4: For $\mathcal{F}_2$, the lower is the values, the better is the fulfillment of our goal. Thus, their values are again aggregated by utilizing PLSVNWA operator and get

$$
\begin{aligned}
T_i &= \bigoplus_{j=k+1}^{n-k} \widetilde{\alpha}_{ij} \\
&= \left(s_{t\left(1-\prod\limits_{j=k+1}^{n-k}\left(1-\frac{\widetilde{\theta}_{ij}}{t}\right)^{\omega_j}\right)}, s_{t\prod\limits_{j=k+1}^{n-k}\left(\frac{\widetilde{\psi}_{ij}}{t}\right)^{\omega_j}}, s_{t\prod\limits_{j=k+1}^{n-k}\left(\frac{\widetilde{\sigma}_{ij}}{t}\right)^{\omega_j}}; 1-\prod_{j=k+1}^{n-k}(1-\widetilde{p}_{ij})^{\omega_j} \right)
\end{aligned}
\tag{11.12}
$$

Step 5: Determine the minimal value of T_i as:

$$T_{\min} = \min_i \{T_i\} \tag{11.13}$$

where $\min\limits_i T_i = \min\{T_1, T_2, \ldots, T_m\}$.

Step 6: Calculate the relative significance value or priority value of the alternatives by using Eq. (11.14) as

$$V_i = \mathrm{Sc}(Q_i) + \frac{\mathrm{Sc}(T_{\min}) \sum\limits_{i=1}^{m} \mathrm{Sc}(T_i)}{\mathrm{Sc}(T_i) \sum\limits_{i=1}^{m} \frac{\mathrm{Sc}(T_{\min})}{\mathrm{Sc}(T_i)}}; \qquad \text{provided} \quad \mathrm{Sc}(T_i) \neq 0 \tag{11.14}$$

The value of V_i expresses the degree of satisfaction earned by the alternative. The maximum value of V_i represents the higher priority of that alternative i.e. $V_{\max}$ is the best choice among the given choices.

Step 7: Compute the quantitative utility for each alternative as

$$U_i = \left(\frac{V_i}{V_{\max}}\right) \times 100\% \tag{11.15}$$

where $V_{\max} = \max\limits_i \{V_i\}$ is non-zero. It can be easily seen that utility degree is directly related with the relative priorities of the alternatives.

Hence, this COPRAS method permits for assessing the direct and relative dependence of priorities and utility degree of the considered alternatives in a DM problem including different criteria, their weights and the rating values of alternative with regard to all the criteria.

11.5 Approaches for solving DM problems

This section offers two different DM approaches which utilize based on the COPRAS technique and the AOs with some new information measures under the PLSVNNs. The presented approaches are illustrated with a numerical example.

11.5.1 Determination of the weight information

Consider a group DM problem whose description is given in Section 2.5 of Chapter 2 with "l" decision makers $\mathcal{DM}^{(1)}$, $\mathcal{DM}^{(2)}$, …, $\mathcal{DM}^{(l)}$. Each decision makers evaluates the given "m" alternatives in the form of PLSVNNs $\alpha_{ij}^{(q)} = (s_{\theta_{ij}^{(q)}}, s_{\psi_{ij}^{(q)}}, s_{\sigma_{ij}^{(q)}}; p_{ij}^{(q)})$. The complete information of the decision maker ratings is summarized in matrix $\mathcal{M}^{(q)}$.

Assume that the importance of each decision maker and criterion are given in the form of weights as $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ and $(\omega_1, \omega_2, \ldots, \omega_n)$ respectively such that $0 < \lambda^{(q)}, \omega_j \leq 1$ and $\sum\limits_{q=1}^{l} \lambda^{(q)} = 1; \sum\limits_{j=1}^{n} \omega_j = 1$. If the weights of such criteria and decision makers are known a priori then we can easily utilized. On the other hand, if the information about them are completely unknown then we utilize the following entropy measure procedure to compute the weight vector of each decision maker and criteria.

(i) **Weight vector for decision maker $\mathcal{DM}^{(q)}, q = 1, 2, \ldots, l$.**

For the collection of 'l' decision maker $\mathcal{DM}^{(1)}, \mathcal{DM}^{(2)}, \ldots, \mathcal{DM}^{(l)}$, the entropy $E^{(q)}, (q = 1, 2, \ldots, l)$ is defined as

$$E^{(q)} = \frac{1}{m\left(\sqrt{(2)} - 1\right)} \sum_{i=1}^{m} \left[\sin\left(\frac{\pi \gamma_i^{(q)}}{2}\right) + \sin\left(\frac{\pi\left(1 - \gamma_i^{(q)}\right)}{2}\right) - 1 \right] \qquad (11.16)$$

where $\gamma_i^{(q)} = \frac{1}{3tn} \sum\limits_{j=1}^{n} p_{ij}^{(q)} \left(2t + \theta_{ij}^{(q)} - \psi_{ij}^{(q)} - \sigma_{ij}^{(q)}\right)$.

In DM problems, if decision makers have comparable almost distinctive choices for different alternatives, the entropy value will be bigger. That is, the more prominent value of entropy for the decision maker, the smaller is the differences between the choices. According to entropy theory, if the entropy value is smaller over the choices then more valuable information is provided by the decision maker in the DM procedure. Therefore, a decision maker with small entropy value will be given a higher priority than others. Then, by combining the above aspects, the weights for decision maker $\mathcal{DM}^{(q)}; (q = 1, 2, \ldots, l)$ based on the entropy $E^{(q)}$ is defined as:

$$\lambda^{(q)} = \frac{1 - E^{(q)}}{l - \sum\limits_{q=1}^{l} E^{(q)}} \tag{11.17}$$

where $\lambda^{(q)} \in (0, 1]$ and $\sum\limits_{q=1}^{l} \lambda^{(q)} = 1$. From the Eq. (11.17), it can easily conclude that smaller the value of $E^{(q)}$, the greater weight will be assigned to the decision maker $\mathcal{DM}^{(q)}$.

(ii) **Weight vector for each criterion $\mathcal{G}_j (j = 1, 2, \ldots, n)$**

For the set of 'n' criteria $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ of the collective decision matrix, the entropy $F_j, (j = 1, 2, \ldots, n)$ is defined as

$$F_j = \frac{1}{m\left(\sqrt{2} - 1\right)} \sum_{i=1}^{m} \left[\sin\left(\frac{\pi \zeta_{ij}}{2}\right) + \sin\left(\frac{\pi\left(1 - \zeta_{ij}\right)}{2}\right) - 1\right] \tag{11.18}$$

where

$$\begin{aligned} \zeta_{ij} &= 1 - \mathcal{D}_2\left(\alpha_{ij}^{(q)}, \alpha_{ij}^{+}\right) \\ &= \frac{1}{l} \sum_{q=1}^{l} \left[1 - \left(\tfrac{1}{3}\left|p_{ij}\tfrac{\theta_{ij}^{(q)}}{t} - p_{ij}^{+}\tfrac{\theta_{ij}^{+}}{t}\right|^2 + \left|p_{ij}\tfrac{\psi_{ij}^{(q)}}{t} - p_{ij}^{+}\tfrac{\psi_{ij}^{-}}{t}\right|^2 + \left|p_{ij}\tfrac{\sigma_{ij}^{(q)}}{t} - p_{ij}^{+}\tfrac{\sigma_{ij}^{-}}{t}\right|^2\right)^{\frac{1}{2}}\right] \end{aligned} \tag{11.19}$$

Here $\alpha_{ij}^{+} = (s_{\theta_{ij}^{+}}, s_{\psi_{ij}^{-}}, s_{\sigma_{ij}^{-}}, p_{ij}^{+})$ is a positive ideal solution and $\mathcal{D}_2\left(\alpha_{ij}^{(q)}, \alpha_{ij}^{+}\right)$ is the Euclidean distance between the each preference value of decision matrices and the positive ideal preference values α_{ij}^{+}. In order to evaluate the information by considering all the aspects, we define the weights of criteria by using entropy F_j as follows:

$$\omega_j = \frac{1 - F_j}{n - \sum\limits_{j=1}^{n} F_j} \tag{11.20}$$

where $\omega_j \in (0, 1]$ and $\sum\limits_{j=1}^{n} \omega_j = 1$. From the Eq. (11.20), it can easily conclude that smaller the value of F_j, the greater weight will be assigned to the criteria ω_j.

Now, based on the collective information under PLSVN environment and the weight vector, we develop two new approaches based on COPRAS and the AOs to find the most accessible alternative(s). The procedure steps fall under these approaches are described below.

11.5.2 Approach based on COPRAS method

In this section, we present a DM approach to rank the given alternative(s) by using the proposed COPRAS method under PLSVN environment. The steps involved in solving the group DM problems under this approach are summarized as below, where the flow chart is presented in Fig. 11.1.

Step 1: Aggregate the preferences of each decision matrix given by the different decision makers $\mathcal{DM}^{(q)}$ in the form of the decision matrices $\mathcal{M}^{(q)} = (\alpha_{ij}^{(q)})$ into a single decision matrix $\mathcal{M} = (\alpha_{ij})$ by utilizing the PLSVNWA operator.

Step 2: Compute the weighted decision matrix $\mathcal{K}_\omega$ by using Eq. (11.10).

Step 3: By using Eq. (11.11) to compute the aggregated values of the weighted decision matrix for benefit type criteria.

Step 4: Utilize Eq. (11.12) to aggregate the rating values of the cost type criteria into a single ones.

Step 5: Compute the relative priority values of each alternative by using Eqs. (11.13) and (11.14).

Step 6: Determine the utility degree for each alternative by using Eq. (11.15).

Step 7: Rank the alternative according to decreasing order of the utility degree.

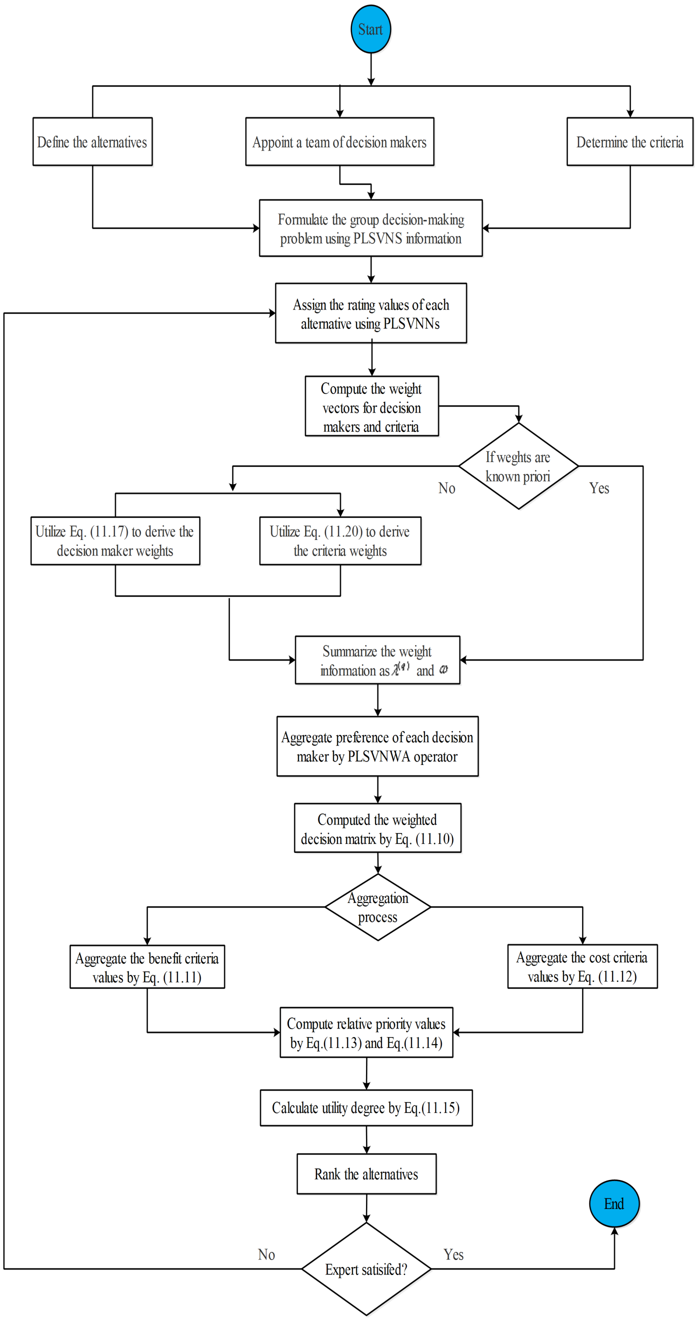

Figure 11.1: Flowchart of the proposed approach based on COPRAS method

11.5.3 Approach based on the AOs

In this section, we present a DM approach to rank the given alternative(s) by using the proposed weighted averaging or geometric AOs. The steps followed under this approach are summarized as below, where the flow chart is presented in Fig. 11.2.

Step 1: In the decision matrix $\mathcal{M}^{(q)}$, transform the rating values $(\alpha_{ij}^{(q)})$ of each cost type criteria into the benefit type by using the normalizing formula as

$$r_{ij}^{(q)} = \begin{cases} \left(s_{\theta_{ij}^{(q)}}, s_{\psi_{ij}^{(q)}}, s_{\sigma_{ij}^{(q)}}; p_{ij}^{(q)}\right) & ; \quad \text{for benefit type criteria} \\ \left(s_{\sigma_{ij}^{(q)}}, s_{\psi_{ij}^{(q)}}, s_{\theta_{ij}^{(q)}}; 1 - p_{ij}^{(q)}\right) & ; \quad \text{for cost type criteria} \end{cases} \tag{11.21}$$

and hence obtain the normalized decision matrix $\mathcal{H}^{(q)} = (r_{ij}^{(q)})$ corresponding to each decision maker.

Step 2: Aggregate the preferences of each decision maker towards the each alternative into the collective one. For it, the preferences values $r_{ij}^{(q)}, q = 1, 2, \ldots, l$ are aggregated either by using PLSVNWA or PLSVNWG operator. For instance, if we utilize PLSVNWA operator and weight vector $\lambda^{(q)}$ to aggregate each decision maker preferences corresponding to each alternative then the overall values of alternative $\mathcal{A}_i$ under criteria $\mathcal{G}_j$, denoted by $\alpha_{ij} = (s_{\theta_{ij}}, s_{\psi_{ij}}, s_{\sigma_{ij}}; p_{ij})$ is given by

$$\begin{aligned} \alpha_{ij} &= \text{PLSVNWA}\left(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(l)}\right) \qquad (11.22) \\ &= \left(s_{t\left(1-\prod\limits_{q=1}^{l}\left(1-\frac{\theta_{ij}^{(q)}}{t}\right)^{\lambda^{(q)}}\right)}, s_{t\left(\prod\limits_{q=1}^{l}\left(\frac{\psi_{ij}^{(q)}}{t}\right)^{\lambda^{(q)}}\right)}, s_{t\left(\prod\limits_{q=1}^{l}\left(\frac{\sigma_{ij}^{(q)}}{t}\right)^{\lambda^{(q)}}\right)}; 1 - \prod_{q=1}^{l}\left(1 - p_{ij}^{(q)}\right)^{\lambda^{(q)}}\right) \end{aligned}$$

On the other hand, if we utilize PLSVNWG operator to aggregate the preference values of each decision maker then the aggregated values are computed as

$$\begin{aligned} \alpha_{ij} &= \text{PLSVNWG}\left(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(l)}\right) \qquad (11.23) \\ &= \left(s_{t\left(\prod\limits_{q=1}^{l}\left(\frac{\theta_{ij}^{(q)}}{t}\right)^{\lambda^{(q)}}\right)}, s_{t\left(1-\prod\limits_{q=1}^{l}\left(1-\frac{\psi_{ij}^{(q)}}{t}\right)^{\lambda^{(q)}}\right)}, s_{t\left(1-\prod\limits_{q=1}^{l}\left(1-\frac{\sigma_{ij}^{(q)}}{t}\right)^{\lambda^{(q)}}\right)}, \prod_{q=1}^{l}\left(p_{ij}^{(q)}\right)^{\lambda^{(q)}}\right) \end{aligned}$$

Step 3: Aggregate the values $\alpha_{ij} (j = 1, 2, \ldots, n)$ by utilizing the weight vector of criteria $\omega_j (j = 1, 2, \ldots, n)$ and the appropriate averaging or geometric AO to get the collective values of each alternative $\alpha_i (i = 1, 2, \ldots, m)$. For instance, if we take PLSVNWA operator to aggregate each value then we get the collective value $\alpha_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i}; p_i)$ as

$$\begin{aligned} \alpha_i &= \text{PLSVNWA}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \qquad (11.24) \\ &= \left(s_{t\left(1-\prod_{j=1}^{n}\left(1-\frac{\theta_{ij}}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}\left(\frac{\psi_{ij}}{t}\right)^{\omega_j}\right)}, s_{t\left(\prod_{j=1}^{n}\left(\frac{\sigma_{ij}}{t}\right)^{\omega_j}\right)}; 1 - \prod_{j=1}^{n} (1 - p_{ij})^{\omega_j} \right) \end{aligned}$$

On the other hand, by taking PLSVNWG operator to aggregate the values, we get

$$\begin{aligned} \alpha_i &= \text{PLSVNWG}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \qquad (11.25) \\ &= \left(s_{t\left(\prod_{j=1}^{n}\left(\frac{\theta_{ij}}{t}\right)^{\omega_j}\right)}, s_{t\left(1-\prod_{j=1}^{n}\left(1-\frac{\psi_{ij}}{t}\right)^{\omega_j}\right)}, s_{t\left(1-\prod_{j=1}^{n}\left(1-\frac{\sigma_{ij}}{t}\right)^{\omega_j}\right)}; \prod_{j=1}^{n} (p_{ij})^{\omega_j} \right) \end{aligned}$$

Step 4: Compute the score value of the aggregate number $\alpha_i = (s_{\theta_i}, s_{\psi_i}, s_{\sigma_i}; p_i), i = 1, 2, \ldots, n$ as

$$\text{Sc}(\alpha_i) = \frac{p_i(2t + \theta_i - \psi_i - \sigma_i)}{3t} \qquad (11.26)$$

However, if the score values of two aggregated PLSVNNs are equal then, calculate their accuracy values as:

$$H(\alpha_i) = \frac{p_i(\theta_i + \psi_i + \sigma_i)}{3t} \qquad (11.27)$$

Step 5: Obtain the ranking order of alternatives using Definition 11.2.3 and hence select the most desirable one(s).

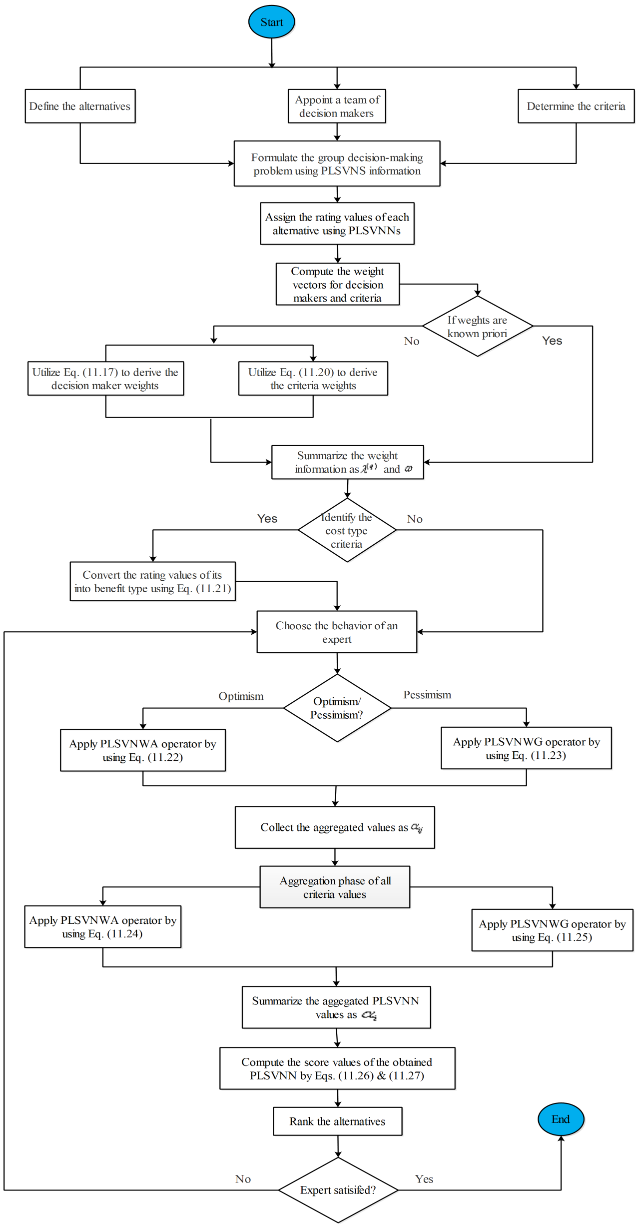

Figure 11.2: Flowchart of the proposed approach based on AOs

11.6 Numerical example

The above presented approaches are illustrated with a numerical example related to outsourcing supplier selection problem and compared their results with some of the existing approaches.

In the following case study, we discuss the IT Outsourcing Selection problem. Millennium semiconductors (MS), established in October 1995, is an ISO $9001-2015$ organization with the distribution of electronic components as its core expertise. MS is the leading distributor of electronic components in India is synonymous with innovation and today it is one of the most reputed names in the market. MS has established roots in almost every region of India with catering more than 1500 customers all segments from the last two decades. It participates in innovative work, creation and showcasing of items, for example, full shading ultra brilliance LED epitaxial items, chips, compound sun-powered cells and high power concentrating sun oriented items. The branch workplaces of MS is situated in Delhi, Bangalore, Hyderabad, Ahmedabad, Chennai and Mumbai in India and abroad workplaces in Singapore and Shenzhen (China). MS contributes the extraordinary layer part of labour and financial resources to its competition rather than IT. The outsourcing of IT is a better option for MS as of its lack of ability to do it efficiently. Therefore, MS selects the following outsourcing providers: Tata Consultancy Services ($\mathcal{A}_1$), Infosys ($\mathcal{A}_2$), Wipro ($\mathcal{A}_3$), HCL ($\mathcal{A}_4$) and TatvaSoft ($\mathcal{A}_5$) under the four criteria namely, Design development ($\mathcal{G}_1$), Quality product ($\mathcal{G}_2$), Delivery time ($\mathcal{G}_3$) and Cost ($\mathcal{G}_4$). Now to find the more suitable or best outsourcing provider among the above choices, MS hires the three decision makers $\mathcal{DM}^{(1)}$, $\mathcal{DM}^{(2)}$ and $\mathcal{DM}^{(3)}$ who have the responsibilities to evaluate the given alternatives and rate their preferences in terms of PLSVNN on the basis of linguistic term set with $t = 8$. The rating values of these decision makers are summarized in the form of the matrices in Table 11.1.

In order to compute the importance of each decision maker and criterion, we find the weight vector associated with them by using rating values of each decision maker summarized in Table 11.1. For it, by using Eqs. (11.16) and (11.17), we compute the weight vector to the decision maker $\mathcal{DM}^{(q)}, q = 1, 2, 3$ as $\lambda^{(1)} = 0.3333$, $\lambda^{(2)} = 0.3337$ and

$\lambda^{(3)} = 0.3330$. Similarly, the weight vectors corresponding to each criteria are computed by utilizing Eqs. (11.18) and (11.20) and hence we get $\omega_1 = 0.3188$, $\omega_2 = 0.2345$, $\omega_3 = 0.1666$ and $\omega_4 = 0.2801$. Then, based on these information, we applied the above developed two approaches to find the most desirable alterative(s).

11.6.1 Results based on COPRAS method

The following steps of the proposed approach are executed on to the considered data to find the most suitable alternative(s).

Step 1: Utilize PLSVNWA operator to aggregate the values of Table 11.1 and represent the collective values in Table 11.2.

Step 2: By using Eq. (11.10), the weighted decision matrix $\mathcal{K}_\omega$ is represented in Table 11.3.

Step 3: By using Eq. (11.11), the values of Q_i's are computed as $Q_1 = (s_{3.9969}, s_{3.4511}, s_{3.7607}; 0.2726)$, $Q_2 = (s_{3.8201}, s_{3.1345}, s_{3.5190}; 0.2320)$, $Q_3 = (s_{2.7389}, s_{4.0031}, s_{4.0034}; 0.3567)$, $Q_4 = (s_{3.0427}, s_{3.8353}, s_{3.4440}; 0.3466)$ and $Q_5 = (s_{2.5758}, s_{4.2611}, s_{3.7606}; 0.2721)$.

Step 4: By Eq. (11.12), we get $T_1 = (s_{2.8214}, s_{3.8850}, s_{4.7547}; 0.2262)$, $T_2 = (s_{2.5772}, s_{5.0721}, s_{4.0373}; 0.2125)$, $T_3 = (s_{2.3203}, s_{5.0720}, s_{4.7723}; 0.2804)$, $T_4 = (s_{3.3650}, s_{5.2943}, s_{4.6455}; 0.2256)$ and $T_5 = (s_{2.7551}, s_{4.3069}, s_{3.2846}; 0.2137)$.

Step 5: By using Eq. (11.14), we get the priority values as $V_1 = 0.2357$, $V_2 = 0.2308$, $V_3 = 0.2472$, $V_4 = 0.2678$ and $V_5 = 0.2070$.

Step 6: Thus, the utility degrees are computed by Eq. (11.15) for each alternative and get $U_1 = 87.9963$, $U_2 = 86.1720$, $U_3 = 92.2850$, $U_4 = 100$ and $U_5 = 77.2776$.

Step 7: Based on the values of U_i's, we get the ranking order as $\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_5$ and hence the best one is $\mathcal{A}_4$.

11.6.2 Results based on AOs

The steps of the approach mentioned in Section 11.5.3 are executed as below:

Step 1: As attributes $\mathcal{G}_3$ and $\mathcal{G}_4$ are the cost types so the normalized decision matrices are obtained by using Eq. (11.21). The updated decision matrices are summarized in Table 11.4.

Step 2: With weights $\lambda^{(q)}, q = 1, 2, 3$ and the data given in Table 11.4, we aggregate preference values by PLSVNWA and PLSVNWG operators. The aggregated values corresponding to them are summarized in Tables 11.5 and 11.6 respectively.

Step 3a: By taking weight $\omega_j (j = 1, 2, 3, 4)$ of the attribute and preference value summarized in Table 11.5, the collective values of each alternative are obtained by using PLSVNWA operator. The obtained values are $\alpha_1 = (s_{4.6502}, s_{1.6759}, s_{2.8675}; 0.5346)$, $\alpha_2 = (s_{4.2708}, s_{1.9873}, s_{2.6949}; 0.4826)$, $\alpha_3 = (s_{3.9177}, s_{2.5380}, s_{2.9814}; 0.5806)$, $\alpha_4 = (s_{3.8576}, s_{2.5382}, s_{2.8849}; 0.6080)$ and $\alpha_5 = (s_{2.9333}, s_{2.2940}, s_{2.9741}; 0.5278)$.

Step 3b: On the other hand, by utilizing PLSVNWG operator to values of Table 11.6, we get $\alpha_1 = (s_{3.9210}, s_{1.8807}, s_{3.8801}; 0.4893)$, $\alpha_2 = (s_{3.2653}, s_{2.3319}, s_{3.3192}; 0.4396)$, $\alpha_3 = (s_{3.6117}, s_{2.6842}, s_{3.3163}; 0.5199)$, $\alpha_4 = (s_{3.3501}, s_{2.8224}, s_{4.1309}; 0.5920)$ and $\alpha_5 = (s_{2.2265}, s_{2.4149}, s_{3.7717}; 0.4811)$.

Step 4: By applying Eq. (11.26) to the values obtained during Step 3a, we get $\mathrm{Sc}(\alpha_1) = 0.3588$, $\mathrm{Sc}(\alpha_2) = 0.3134$, $\mathrm{Sc}(\alpha_3) = 0.3483$, $\mathrm{Sc}(\alpha_4) = 0.3657$ and $\mathrm{Sc}(\alpha_5) = 0.3005$, while score values for the values obtained from Step 3b are $\mathrm{Sc}(\alpha_1) = 0.2887$, $\mathrm{Sc}(\alpha_2) = 0.2494$, $\mathrm{Sc}(\alpha_3) = 0.2948$, $\mathrm{Sc}(\alpha_4) = 0.3058$ and $\mathrm{Sc}(\alpha_5) = 0.2413$.

Step 5: The final ranking order of the given alternatives are obtained as $\mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_5$ corresponding to PLSVNWA operator while $\mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_5$ to the PLSVNWG operator. Here $\succ$ refers "preferred to". Thus, we conclude that $\mathcal{A}_4$ is the best alternative among the given ones.

11.6.3 Comparative analysis

In this section, the proposed approaches are compared with the various existing studies in order to defend its dominance in DM problems. It is noticeable that the PLSVN environment reduces to LSVN environment if we take the possibility of each element in the PLSVNS as one. So, to analyze the behavior of our proposed approaches, we compare them to the existing theories [39, 73] by taking the same numerical example. But for this comparative study, we firstly convert the PLSVNNs into LSVNNs by fixing values of $p_{ij}^{(q)} = 1; (i = 1, 2, 3, 4, 5; j = 1, 2, 3, 4; q = 1, 2, 3)$.

(i) Based on the reduction of PLSVNNs to LSVNNs, we apply the approach given by Fang and Ye [39], on the considered example. Under it, we aggregate the given preference values by using the existing linguistic neutrosophic number weighted averaging operator and hence obtain the score values of each alternative $\mathcal{A}_i (i = 1, 2, \ldots, 5)$ as

$$\begin{aligned} &\mathrm{Sc}(\mathcal{A}_1) = 0.7194; \qquad \mathrm{Sc}(\mathcal{A}_2) = 0.7423; \qquad \mathrm{Sc}(\mathcal{A}_3) = 0.6554; \\ &\mathrm{Sc}(\mathcal{A}_4) = 0.6968; \qquad \mathrm{Sc}(\mathcal{A}_5) = 0.6945. \end{aligned}$$

Thus, the ranking order of the given alternatives is obtained as $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_3$ and hence conclude that $\mathcal{A}_2$ is the best alternative.

On the other hand, if we utilize linguistic neutrosophic number weighted geometric AO to aggregate the different rating values then the score values corresponding to each alternative are obtained as

$$\begin{aligned} &\mathrm{Sc}(\mathcal{A}_1) = 0.7111; \qquad \mathrm{Sc}(\mathcal{A}_2) = 0.7321; \qquad \mathrm{Sc}(\mathcal{A}_3) = 0.6513; \\ &Sc(\mathcal{A}_4) = 0.6732; \qquad \mathrm{Sc}(\mathcal{A}_5) = 0.6838. \end{aligned}$$

Based on these score values, we obtain the final ranking order of the alternatives as $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_3$.

It can be easily seen that that ranking results are given by these operators and the proposed approach based on operators totally vary from each other. This fluctuation in results is due to various reasons. The approach used by the existing ones didn't normalize the given data but our proposed approach do so. Also, the AOs for

PLSVNSs takes the possibility of each LSVNS but in the aggregate process of the LSVNSs, the possibility didn't come into play for numerical evaluations. Thus, we can conclude that the ignorance of the one factor, that is, possibility degree in the data shows the great divergence in final results. But it is clear that involvement of the possibility of each member of the PLSVNS gives more reliable information in case of complex uncertainties. Moreover, LSVNS is a particular case of PLSVNS. Thus, we can say the proposed approach based on operators give more genuine results than the existing one.

(ii) If we utilize the prioritized weighted averaging AO, as given in Chapter 10 to the considered data then the final score values of each alternative $\mathcal{A}_i (i = 1, 2, \ldots, 5)$ are obtained as

$$\begin{aligned} &\mathrm{Sc}(\mathcal{A}_1) = 0.7006; \qquad \mathrm{Sc}(\mathcal{A}_2) = 0.7002; \qquad \mathrm{Sc}(\mathcal{A}_3) = 0.6143; \\ &\mathrm{Sc}(\mathcal{A}_4) = 0.6529; \qquad \mathrm{Sc}(\mathcal{A}_5) = 0.6226. \end{aligned}$$

Thus, the ranking order of the alternatives is obtained as $\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_3$ and get $\mathcal{A}_1$ is the most desirable alternative.

On the other hand, if we utilize the prioritized weighted geometric AO to the given information then the final scores are obtained corresponding to the given alternatives are

$$\begin{aligned} &\mathrm{Sc}(\mathcal{A}_1) = 0.6341; \qquad \mathrm{Sc}(\mathcal{A}_2) = 0.6356; \qquad \mathrm{Sc}(\mathcal{A}_3) = 0.5917; \\ &\mathrm{Sc}(\mathcal{A}_4) = 0.5771; \qquad \mathrm{Sc}(\mathcal{A}_5) = 0.5503. \end{aligned}$$

Based on these score values, the final ranking order of the given alternatives is obtained as $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5$. From this study, it is observed that the final ranking order of the given alternatives obtained through the proposed approach and the existing approaches is entirely different. This is due to the fact that the computational procedure to find the weights of the factors of the proposed approach is different from the existing studies. In the existing studies, the weights of each decision maker and criteria are obtained by taking account the priority of each criterion while in the proposed approach, these weights vectors are computed by using the entropy measures.

(iii) If we apply the TOPSIS method, as proposed by Liang et al. [73], under the LSVN environment to the normalized decision matrices obtained by taking all the possibility degree to be one, then the final ranking order of the given alternatives is obtained as $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_5$. Compared this ranking order with the proposed ranking order by COPRAS method, we found that ranking results by the TOPSIS method are far apart from ranking results of COPRAS method. This difference is due to the fact that both ranking methods adopt different strategies. In TOPSIS, calculation and final ranking are obtained by involving the concept of worst and the best ideal solutions of the DM problems under consideration. In contrast to this, COPRAS method gives the ranking results without finalizing the ideal solutions and use the procedure of aggregating the benefit and cost criteria individually which is not done in TOPSIS method.

11.6.4 Advantages of proposed approaches

This section explores the advantages of the proposed work as follows.

(i) Real life DM problems are often possibilistic and as well as qualitative in nature. This chapter caught the significance of taking the idea of possibility along with the LSVNS in demonstrating the present practical circumstances. As the assigned possibility degree to the element of LSVNS represents the possibility of occurrence of linguistic membership, indeterminacy, and non-membership, therefore, this combination has high potential in true representation in the field of computational intelligence.

(ii) From the presented study, it is observed that the proposed operator is a generalization of the existing ones in the LSVNS environment if we put the value of the possibility of each element in LSVNS equals to 1. As we take possibility as 1, so the proposed PLSVNWA and PLSVNWG AOs reduce to existing Linguistic neutrosophic number weighted averaging and geometric operators [39], respectively. This shows the proposed concept is much more generalized than existing ones.

(iii) In this chapter, we use the COPRAS method as the ranking method which is a suitable strategy to prepare the information in a sensible and effective way. The strategy used by the COPRAS can process the criterion information from distinctive points based on the complex proportional calculation, which contains more accurate information compared with other strategies basically dealing with the benefit criteria or the cost criteria.

(iv) This chapter introduces a new entropy measure that not only provides the overall information about the amount of uncertainty imbued in the specific structure but also used as an effective tool in the DM process. In the DM process, the allocation of weights to the decision maker as well as criteria in order to signify the preference of both, the proposed entropy measure has been utilized.

11.7 Conclusion

In this chapter, we have introduced the new concept of PLSVNSs along with information measures and AOs under the same scenario. The current speculations just manage the qualitative aspects without incorporation of possibility and hence the final outcomes are sometimes inadmissible in grabbing the best choice. To resolve this, this chapter presents the LSVNSs which are imbued with the possibility in the decision process and are named as PLSVNSs. Also, in this chapter, we give the operational laws for this proposed set-theoretic structure with uncertainties and then explore the various relationships among these operations. Further, some weighted averaging and geometric AOs for the PLSVNSs are proposed based on the averaging and geometric conception for assembled the PLSVNSs in the single value. Also, the two approaches are given for evaluating the information with completely unknown weights and these weights are calculated by utilizing the entropy measure. The illustrative example demonstrates all the concepts of the proposed work in this chapter.

Table 11.1: PLSVN decision matrix given by decision makers

		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{DM}^{(1)}$	$\mathcal{A}_1$	$(s_6, s_2, s_1; 0.3)$	$(s_6, s_2, s_6; 0.4)$	$(s_6, s_2, s_4; 0.7)$	$(s_3, s_2, s_2; 0.6)$
	$\mathcal{A}_2$	$(s_5, s_1, s_2; 0.4)$	$(s_5, s_1, s_1; 0.6)$	$\Big(s_4, s_2, s_2; 0.7)$	$(s_6, s_1, s_2; 0.5)$
	$\mathcal{A}_3$	$(s_4, s_2, s_3; 0.5)$	$(s_5, s_2, s_3; 0.7)$	$(s_4, s_4, s_2; 0.8)$	$(s_5, s_2, s_4; 0.4)$
	$\mathcal{A}_4$	$(s_6, s_3, s_1; 0.6)$	$(s_3, s_2, s_5; 0.7)$	$(s_3, s_2, s_2; 0.6)$	$(s_6, s_3, s_3; 0.5)$
	$\mathcal{A}_5$	$(s_4, s_2, s_4; 0.4)$	$(s_5, s_3, s_2; 0.3)$	$(s_5, s_2, s_1; 0.5)$	$(s_4, s_1, s_1; 0.6)$
$\mathcal{DM}^{(2)}$	$\mathcal{A}_1$	$(s_6, s_2, s_2; 0.4)$	$(s_6, s_1, s_2; 0.8)$	$(s_6, s_2, s_3; 0.6)$	$(s_6, s_4, s_3; 0.5)$
	$\mathcal{A}_2$	$(s_5, s_1, s_2; 0.4)$	$(s_6, s_2, s_3; 0.2)$	$(s_4, s_3, s_2; 0.3)$	$(s_5, s_1, s_2; 0.4)$
	$\mathcal{A}_3$	$(s_4, s_2, s_2; 0.8)$	$(s_3, s_2, s_2; 0.9)$	$(s_4, s_2, s_2; 0.6)$	$(s_5, s_2, s_3; 0.7)$
	$\mathcal{A}_4$	$(s_6, s_2, s_1; 0.7)$	$(s_3, s_1, s_2; 0.6)$	$(s_5, s_4, s_2; 0.2)$	$(s_6, s_2, s_2; 0.5)$
	$\mathcal{A}_5$	$(s_4, s_4, s_1; 0.4)$	$(s_3, s_3, s_3; 0.3)$	$(s_5, s_2, s_1; 0.2)$	$(s_4, s_2, s_1; 0.5)$
$\mathcal{DM}^{(3)}$	$\mathcal{A}_1$	$(s_5, s_1, s_2; 0.5)$	$(s_5, s_4, s_2; 0.7)$	$(s_2, s_1, s_2; 0.6)$	$(s_5, s_2, s_2; 0.2)$
	$\mathcal{A}_2$	$(s_6, s_2, s_2; 0.2)$	$(s_6, s_3, s_2; 0.5)$	$(s_3, s_4, s_2; 0.5)$	$(s_4, s_2, s_1; 0.5)$
	$\mathcal{A}_3$	$(s_5, s_3, s_2; 0.6)$	$(s_4, s_3, s_2; 0.4)$	$(s_4, s_3, s_2; 0.7)$	$(s_3, s_1, s_2; 0.7)$
	$\mathcal{A}_4$	$(s_4, s_3, s_2; 0.7)$	$(s_3, s_2, s_2; 0.7)$	$(s_6, s_4, s_2; 0.6)$	$(s_6, s_3, s_3; 0.6)$
	$\mathcal{A}_5$	$(s_4, s_2, s_1; 0.8)$	$(s_4, s_2, s_4; 0.6)$	$(s_5, s_2, s_2; 0.5)$	$(s_6, s_3, s_1; 0.5)$

Table 11.2: Aggregated decision matrix by PLSVNWA operator used in COPRAS method

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$(s_{5.7109}, s_{1.5878}, s_{1.5871}; 0.4056)$	$(s_{5.7108}, s_{1.9991}, s_{2.8845}; 0.6699)$	$(s_{5.116}, s_{1.5878}, s_{2.8849}; 0.6366)$	$(s_{4.8932}, s_{2.5204}, s_{2.2897}; 0.4572)$
$\mathcal{A}_2$	$(s_{5.3789}, s_{1.2596}, s_{2.0000}; 0.4233)$	$(s_{5.7106}, s_{1.8169}, s_{1.5874}; 0.4570)$	$(s_{3.6914}, s_{2.8842}, s_{2.0000}; 0.5281)$	$(s_{5.1158}, s_{1.2596}, s_{1.5878}; 0.4686)$
$\mathcal{A}_3$	$(s_{4.3654}, s_{2.2891}, s_{2.2894}; 0.6581)$	$(s_{4.0848}, s_{2.2891}, s_{2.2894}; 0.7382)$	$(s_{4.0000}, s_{2.8841}, s_{2.0000}; 0.7115)$	$(s_{4.4437}, s_{1.5878}, s_{2.8849}; 0.6220)$
$\mathcal{A}_4$	$(s_{5.4807}, s_{2.6204}, s_{1.2596}; 0.6698)$	$(s_{3.0000}, s_{1.5870}, s_{2.7144}; 0.6698)$	$(s_{4.8924}, s_{3.1748}, s_{2.0000}; 0.4959)$	$(s_{6.0000}, s_{2.6204}, s_{2.6204}; 0.5358)$
$\mathcal{A}_5$	$(s_{4.0000}, s_{2.5204}, s_{1.5874}; 0.5838)$	$(s_{4.0848}, s_{2.6211}, s_{2.8842}; 0.4190)$	$(s_{5.0000}, s_{2.0000}, s_{1.2596}; 0.4151)$	$(s_{4.8245}, s_{1.8169}, s_{1.0000}; 0.5358)$

Table 11.3: Weighted decision matrix

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$(s_{2.6315}, s_{4.7775}, s_{4.7772}; 0.1316)$	$(s_{2.0346}, s_{5.7788}, s_{6.2977}; 0.1624)$	$(s_{1.2506}, s_{6.1108}, s_{6.7500}; 0.1097)$	$(s_{1.8619}, s_{5.0860}, s_{5.6353}; 0.1308)$
$\mathcal{A}_2$	$(s_{2.3946}, s_{4.4376}, s_{5.1423}; 0.1347)$	$(s_{2.0344}, s_{5.6507}, s_{5.4746}; 0.1096)$	$(s_{0.7836}, s_{6.7497}, s_{6.3504}; 0.0905)$	$(s_{1.9884}, s_{6.0116}, s_{5.0860}; 0.1341)$
$\mathcal{A}_3$	$(s_{1.7790}, s_{5.3685}, s_{5.3687}; 0.2160)$	$(s_{1.2344}, s_{5.9654}, s_{5.9656}; 0.1796)$	$(s_{0.8724}, s_{6.7496}, s_{6.3504}; 0.1232)$	$(s_{1.6251}, s_{6.0115}, s_{6.0120}; 0.1793)$
$\mathcal{A}_4$	$(s_{2.4650}, s_{5.6049}, s_{4.4376}; 0.2199)$	$(s_{0.8350}, s_{5.4743}, s_{6.2086}; 0.1624)$	$(s_{1.1659}, s_{6.8585}, s_{6.3504}; 0.0848)$	$(s_{2.5743}, s_{6.1754}, s_{5.8522}; 0.1538)$
$\mathcal{A}_5$	$(s_{1.5861}, s_{5.5358}, s_{4.7772}; 0.1909)$	$(s_{1.2344}, s_{6.1579}, s_{6.2976}; 0.1003)$	$(s_{1.2059}, s_{6.3504}, s_{5.8796}; 0.0707)$	$(s_{1.8242}, s_{5.4257}, s_{4.4683}; 0.1538)$

Table 11.4: Normalized PLSVN decision matrix for decision makers

		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{DM}^{(1)}$	$\mathcal{A}_1$	$(s_6, s_2, s_1; 0.3)$	$(s_6, s_2, s_6; 0.4)$	$(s_4, s_2, s_6; 0.3)$	$(s_2, s_2, s_3; 0.4)$
	$\mathcal{A}_2$	$(s_5, s_1, s_2; 0.4)$	$(s_5, s_1, s_1; 0.6)$	$(s_2, s_2, s_4; 0.3)$	$(s_2, s_1, s_6; 0.5)$
	$\mathcal{A}_3$	$(s_4, s_2, s_3; 0.5)$	$(s_5, s_2, s_3; 0.7)$	$(s_2, s_4, s_4; 0.2)$	$(s_4, s_2, s_5; 0.6)$
	$\mathcal{A}_4$	$(s_6, s_3, s_1; 0.6)$	$(s_3, s_2, s_5; 0.7)$	$(s_2, s_2, s_3; 0.4)$	$(s_3, s_3, s_6; 0.5)$
	$\mathcal{A}_5$	$(s_4, s_2, s_4; 0.4)$	$(s_5, s_3, s_2; 0.3)$	$(s_1, s_2, s_5; 0.5)$	$(s_1, s_1, s_4; 0.4)$
$\mathcal{DM}^{(2)}$	$\mathcal{A}_1$	$(s_6, s_2, s_2; 0.4)$	$(s_6, s_1, s_2; 0.8)$	$(s_3, s_2, s_6; 0.4)$	$(s_3, s_4, s_6; 0.5)$
	$\mathcal{A}_2$	$(s_5, s_1, s_2; 0.4)$	$(s_6, s_2, s_3; 0.2)$	$(s_2, s_3, s_4; 0.7)$	$(s_2, s_1, s_5; 0.6)$
	$\mathcal{A}_3$	$(s_4, s_2, s_2; 0.8)$	$(s_3, s_2, s_2; 0.9)$	$(s_2, s_2, s_4; 0.4)$	$(s_3, s_2, s_5; 0.3)$
	$\mathcal{A}_4$	$(s_6, s_2, s_1; 0.7)$	$(s_3, s_1, s_2; 0.6)$	$(s_2, s_4, s_5; 0.8)$	$(s_2, s_2, s_6; 0.5)$
	$\mathcal{A}_5$	$(s_4, s_4, s_1; 0.4)$	$(s_3, s_3, s_3; 0.3)$	$(s_1, s_2, s_5; 0.8)$	$(s_1, s_2, s_4; 0.5)$
$\mathcal{DM}^{(3)}$	$\mathcal{A}_1$	$(s_5, s_1, s_2; 0.5)$	$(s_5, s_4, s_2; 0.7)$	$(s_2, s_1, s_2; 0.4)$	$(s_2, s_2, s_5; 0.8)$
	$\mathcal{A}_2$	$(s_6, s_2, s_2; 0.2)$	$(s_6, s_3, s_2; 0.5)$	$(s_2, s_4, s_3; 0.5)$	$(s_1, s_2, s_4; 0.5)$
	$\mathcal{A}_3$	$(s_5, s_3, s_2; 0.6)$	$(s_4, s_3, s_2; 0.4)$	$(s_2, s_3, s_4; 0.3)$	$(s_2, s_1, s_3; 0.3)$
	$\mathcal{A}_4$	$(s_4, s_3, s_2; 0.7)$	$(s_3, s_2, s_2; 0.7)$	$(s_2, s_4, s_6; 0.4)$	$(s_3, s_3, s_6; 0.4)$
	$\mathcal{A}_5$	$(s_4, s_2, s_1; 0.8)$	$(s_4, s_2, s_4; 0.6)$	$(s_2, s_2, s_5; 0.5)$	$(s_1, s_3, s_6; 0.5)$

Table 11.5: Aggregated decision matrix by PLSVNWA operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$(s_{5.7109}, s_{1.5878}, s_{1.5874}; 0.4056)$	$(s_{5.7109}, s_{1.9990}, s_{2.8844}; 0.6699)$	$(s_{3.0678}, s_{1.5878}, s_{4.1617}; 0.3684)$	$(s_{2.3542}, s_{2.5205}, s_{4.4818}; 0.6084)$
$\mathcal{A}_2$	$(s_{5.3789}, s_{1.2596}, s_{2.0000}; 0.4232)$	$(s_{5.7106}, s_{1.8169}, s_{1.5874}; 0.4570)$	$(s_{2.0000}, s_{2.8843}, s_{3.6346}; 0.5283)$	$(s_{1.6840}, s_{1.2596}, s_{4.9328}; 0.5359)$
$\mathcal{A}_3$	$(s_{4.3654}, s_{2.2891}, s_{2.2894}; 0.6581)$	$(s_{4.0848}, s_{2.2891}, s_{2.2894}; 0.7381)$	$(s_{2.7593}, s_{2.8840}, s_{4.0000}; 0.3048)$	$(s_{3.8408}, s_{1.5878}, s_{4.2179}; 0.4191)$
$\mathcal{A}_4$	$(s_{5.4807}, s_{2.6204}, s_{1.2596}; 0.6698)$	$(s_{3.0000}, s_{1.5870}, s_{2.7143}; 0.6698)$	$(s_{2.0000}, s_{3.1749}, s_{4.4812}; 0.5842)$	$(s_{3.0000}, s_{2.6204}, s_{6.0000}; 0.4687)$
$\mathcal{A}_5$	$(s_{4.0000}, s_{2.5205}, s_{1.5873}; 0.5838)$	$(s_{4.0848}, s_{2.6211}, s_{2.8843}; 0.4190)$	$(s_{1.3503}, s_{2.0000}, s_{5.0000}; 0.6317)$	$(s_{1.0000}, s_{1.8169}, s_{4.5782}; 0.4687)$

Table 11.6: Aggregated decision matrix by PLSVNWG operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$(s_{5.6466}, s_{1.6840}, s_{1.6837}; 0.3915)$	$(s_{5.6467}, s_{2.4811}, s_{3.8397}; 0.6074)$	$(s_{2.8849}, s_{1.6840}, s_{5.1166}; 0.3634)$	$(s_{2.2898}, s_{2.7593}, s_{4.8933}; 0.5428)$
$\mathcal{A}_2$	$(s_{5.3130}, s_{1.3503}, s_{2.0000}; 0.3636)$	$(s_{5.6463}, s_{2.0577}, s_{1.6837}; 0.3914)$	$(s_{2.0000}, s_{3.0672}, s_{3.6915}; 0.4718)$	$(s_{1.5878}, s_{1.3503}, s_{5.1157}; 0.5314)$
$\mathcal{A}_3$	$(s_{4.3085}, s_{2.3534}, s_{2.3537}; 0.6215)$	$(s_{3.9144}, s_{2.3534}, s_{2.3537}; 0.6318)$	$(s_{2.5205}, s_{3.0672}, s_{4.0000}; 0.2885)$	$(s_{3.4210}, s_{1.6840}, s_{4.4437}; 0.3780)$
$\mathcal{A}_4$	$(s_{5.2422}, s_{2.6864}, s_{1.3503}; 0.6649)$	$(s_{3.0000}, s_{1.6833}, s_{3.2377}; 0.6649)$	$(s_{2.0000}, s_{3.4212}, s_{4.8924}; 0.5041)$	$(s_{3.0000}, s_{2.6864}, s_{6.0000}; 0.4642)$
$\mathcal{A}_5$	$(s_{4.0000}, s_{2.7593}, s_{2.1911}; 0.5039)$	$(s_{3.9144}, s_{2.6870}, s_{3.0672}; 0.3779)$	$(s_{1.2596}, s_{2.0000}, s_{5.0000}; 0.5849)$	$(s_{1.0000}, s_{2.0557}, s_{4.8245}; 0.4642)$

Bibliography

[1] Abdel-Baset, M., Chang, V., Gamal, A. and Smarandache, F.: 2019, An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field, *Computers in Industry* **106**, 94–110.

[2] Abdel-Basset, M., Manogaran, G., Gamal, A. and Smarandache, F.: 2018, A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria, *Design Automation for Embedded Systems* pp. 1–22.

[3] Abdel-Basset, M., Mohamed, M. and Smarandache, F.: 2018, A hybrid neutrosophic group ANP-TOPSIS framework for supplier selection problems, *Symmetry* **10**(6), 226–247.

[4] Abu-Faty, H. G.: 2017, Neutrosophic TOPSIS based game theory for solving MCGDM problems, *Australian Journal of Basic and Applied Sciences* **11**(13), 29–38.

[5] Aiwu, Z., Jianguo, D. and Hongjun, G.: 2015, Interval valued neutrosophic sets and multi-attribute decision-making based on generalized weighted aggregation operator, *Journal of Intelligent and Fuzzy Systems* **29**, 2697 – 2706.

[6] Atanassov, K. and Gargov, G.: 1989, Interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **31**, 343 – 349.

[7] Atanassov, K. T.: 1986, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20**, 87 – 96.

[8] Awang, A., Ab Ghani, A., Abdullah, L. and Ahmad, M.: 2018, The Shapley weighting

vector-based neutrosophic aggregation operator in DEMATEL method, *Journal of Physics: Conference Series*, Vol. 1132, IOP Publishing, p. 012059.

[9] Awang, A., Ghani, A. T. A., Abdullah, L. and Ahmad, M. F.: 2018, A DEMATEL method with single valued neutrosophic set (SVNS) in identifying the key contribution factors of setiu wetlands coastal erosion, *AIP Conference Proceedings*, Vol. 1974, AIP Publishing, p. 020011.

[10] Aydin, S.: 2018, Augmented reality goggles selection by using neutrosophic MULTIMOORA method, *Journal of Enterprise Information Management* **31**(4), 565–576.

[11] Bausys, R., Zavadskas, E. K. and Kaklauskas, A.: 2015, Application of neutrosophic set to multicriteria decision making by COPRAS, *Economic Computation And Economic Cybernetics Studies And Research* **49**(2), 91 – 105.

[12] Benayoun, R., Roy, B. and Sussman, N.: 1966, Manual de reference du programme manual de reference du programme ELECTRE, *Note de synthese et Formation* **25**, 79.

[13] Biswas, P., Pramanik, S. and Giri, B. C.: 2014, A new methodology for neutrosophic multi-attribute decision making with unknown weight information, *Neutrosophic Sets and Systems* **3**, 42–52.

[14] Biswas, P., Pramanik, S. and Giri, B. C.: 2016, TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment, *Neural Computing and Applications* **27**(3), 727 – 737.

[15] Biswas, P., Pramanik, S. and Giri, B. C.: 2019, Neutrosophic TOPSIS with group decision making, *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets, Springer,*, pp. 543–585.

[16] Boltürk, E. and Karaşan, A.: 2018, Interval valued neutrosophic CODAS method for renewable energy selection, *Data Science and Knowledge Engineering for Sensing Decision Support* pp. 1026–1033.

[17] Broumi, S. and Smarandache, F.: 2013, Correlation coefficient of interval neutrosophic set, *Applied Mechanics and Material* **436**, 511 – 517.

[18] Broumi, S. and Smarandache, F.: 2014a, Cosine similarity measure of interval valued neutrosophic sets, *Neutrosophic Sets and Systems* **5**, 15 – 20.

[19] Broumi, S. and Smarandache, F.: 2014b, New distance and similarity measures of interval neutrosophic sets, *Neutrosophic Theory and Its Applications* **1**, 249 – 255.

[20] Broumi, S. and Smarandache, F.: 2014c, Single valued neutrosophic trapezoid linguistic aggregation operators based multi-attribute decision making, *Bulletin of Pure & Applied Sciences-Mathematics and Statistics* **33**(2), 135–155.

[21] Cao, Q., Wu, J. and Liang, C.: 2015, An intuitionsitic fuzzy judgement matrix and TOPSIS integrated multi-criteria decision making method for green supplier selection, *Journal of Intelligent & Fuzzy Systems* **28**(1), 117–126.

[22] Chatterjee, K. and Kar, S.: 2016, Multi-criteria analysis of supply chain risk management using interval valued fuzzy TOPSIS, *Opsearch* **53**(3), 474–499.

[23] Chatterjee, P., Athawale, V. M. and Chakraborty, S.: 2011, Materials selection using complex proportional assessment and evaluation of mixed data methods, *Materials & Design* **32**(2), 851–860.

[24] Chen, J. and Ye, J.: 2017, Some single-valued neutrosophic dombi weighted aggregation operators for multiple attribute decision-making, *Symmetry* **9**(6), 82.

[25] Chen, S. M. and Chang, C. H.: 2015, A novel similarity measure between Atanassov's intuitionistic fuzzy sets based on transformation techniques with applications to pattern recognition, *Information Sciences* **291**, 96 – 114.

[26] Chen, S. M., Cheng, S. H. and Lan, T. C.: 2016, Multicriteria decision making based on the TOPSIS method and similarity measures between intuitionistic fuzzy values, *Information Sciences* **367 - 368**, 279 – 295.

[27] Chen, T. Y.: 2015, The inclusion-based TOPSIS method with interval -valued intuitionistic fuzzy sets for multiple criteria group decision making, *Applied Soft Computing* **26**, 57 – 73.

[28] Chi, P. and Liu, P.: 2013, An extended TOPSIS method for the multiple attribute decision making problems based on interval neutrosophic set, *Neutrosophic Sets and Systems* **1**(1), 63 – 70.

[29] Cui, W.-H. and Ye, J.: 2018a, Generalized distance-based entropy and dimension root entropy for simplified neutrosophic sets, *Entropy* **20**(11), 844.

[30] Cui, W. and Ye, J.: 2018b, Improved symmetry measures of simplified neutrosophic sets and their decision-making method based on a sine entropy weight model, *Symmetry* **10**(6), 225.

[31] Cui, W., Ye, J. and Shi, L.: 2019, Linguistic neutrosophic uncertain numbers and their multiple attribute group decision-making method, *Journal of Intelligent & Fuzzy Systems* **36**(1), 649 – 660.

[32] Dammak, F., Baccour, L. and Alimi, A. M.: 2015, The impact of criterion weights techniques in TOPSIS method of multi-criteria decision making in crisp and intuitionistic fuzzy domains, *2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, IEEE, pp. 1–8.

[33] Das, S., Kar, S. and Pal, T.: 2017, Robust decision making using intuitionistic fuzzy numbers, *Granular Computing* **2**(1), 41 – 54.

[34] Deschrijver, G. and Kerre, E.: 2002, A generalization of operators on intuitionistic fuzzy sets using triangular fuzzy t - norms and t - conorms, *IEEE Transactions on Fuzzy Systems* **8**(1), 19 –27.

[35] Ding, Q. and Wang, Y.-M.: 2019, Intuitionistic fuzzy TOPSIS multi-attribute decision making method based on revised scoring function and entropy weight method, *Journal of Intelligent & Fuzzy Systems* (Preprint), 1–11.

[36] Düğenci, M.: 2016, A new distance measure for interval valued intuitionistic fuzzy sets and its application to group decision making problems with incomplete weights information, *Applied Soft Computing* **41**, 120–134.

[37] Dung, V., Thuy, L. T., Mai, P. Q., Van Dan, N. and Lan, N. T. M.: 2018, TOPSIS approach using interval neutrosophic sets for personnel selection, *Asian Journal of Scientific Research* **11**(3), 434–440.

[38] Fan, C., Ye, J., Hu, K. and Fan, E.: 2017, Bonferroni mean operators of linguistic neutrosophic numbers and their multiple attribute group decision-making methods, *Information* **8**(3), 107.
URL: *10.3390/info8030107.*

[39] Fang, Z. and Ye, J.: 2017, Multiple attribute group decision - making method based on linguistic neutrosophic numbers, *Symmetry* **9**, 111.
URL: *10.3390/sym9070111*

[40] Frank, M. J.: 1979, On the simultaneous associativity of $F(x, y)$ and $x+y-F(x, y)$, *Aequationes Mathematicae* **19**(1), 194 – 226.

[41] Fu, J. and Ye, J.: 2017, Simplified neutrosophic exponential similarity measures for the initial evaluation/diagnosis of benign prostatic hyperplasia symptoms, *Symmetry* **9**(8), 154.
URL: *10.3390/sym9080154*

[42] Garg, H. and Kaur, G.: 2018, Extended TOPSIS method for multi-criteria group decision-making problems under cubic intuitionistic fuzzy environment, *Scientia Iranica* pp. 1–18.
URL: *http://dx.doi.org/10.24200/sci.2018.5307.1194*

[43] Garg, H. and Kumar, K.: 2018a, A novel exponential distance and its based TOPSIS method for interval-valued intuitionistic fuzzy sets using connection number of SPA theory, *Artificial Intelligence Review* pp. 1–30.
URL: *10.1007/s10462-018-9668-5*

[44] Garg, H. and Kumar, K.: 2018b, Some aggregation operators for linguistic intuitionistic fuzzy set and its application to group decision-making process using the set pair analysis, *Arabian Journal for Science and Engineering* **43**(6), 3213 – 3227.

[45] Garg, H. and Kumar, K.: 2019, An extended technique for order preference by similarity to ideal solution group decision making method with linguistic interval-valued intuitionistic fuzzy information, *Journal of Multi-Criteria Decision Analysis* **26**(1-2), 16 – 26.
URL: *10.1002/mcda.1654*

[46] Garg, H. and Singh, S.: 2018, A novel triangular interval type-2 intuitionistic fuzzy sets and their aggregation operators, *Iranian Journal of Fuzzy Systems* **15**(5), 69 – 93.

[47] Geng, Y., Wang, X., Li, X., Yu, K. and Liu, P.: 2018, Some interval neutrosophic linguistic Maclaurin symmetric mean operators and their application in multiple attribute decision making, *Symmetry* **10**(4), 127.

[48] Grzegorzewski, P.: 2004, Distance between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the hausdorff metric, *Fuzzy Sets and Systems* **148**(2), 319 – 328.

[49] Guh, Y. Y., Hon, C. C. and Lee, E. S.: 2001, Fuzzy weighted average: The linear programming approach via Charnes and Cooper's rule, *Fuzzy Sets and Systems* **117**(1), 157–160.

[50] Gupta, P., Mehlawat, M. K. and Grover, N.: 2018, A generalized TOPSIS method for intuitionistic fuzzy multiple attribute group decision making considering different scenarios of attributes weight information, *International Journal of Fuzzy Systems* pp. 1–19.

[51] Herrera, F. and Herrera-Viedma, E.: 1996, A model of consensus in group decision making under linguistic assessments, *Fuzzy Sets and Systems* **78**(1), 73 – 87.

[52] Herrera, F. and Herrera-Viedma, E.: 2000, Linguistic decision analysis: steps for solving decision problems under linguistic information, *Fuzzy Sets and Systems* **115**(1), 67 – 82.

[53] Herrera, F. and Martínez, L.: 2001, A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making, *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)* **31**(2), 227 – 234.

[54] Huang, H. L.: 2016, New distance measure of single-valued neutrosophic sets and its application, *International Journal of Intelligent Systems* **31**(10), 1021 – 1032.

[55] Huang, Y. H., Wei, G. W. and Wei, C.: 2017, VIKOR method for interval neutrosophic multiple attribute group decision-making, *Information* **8**(4), 144.
URL: *10.3390/info8040144*

[56] Hwang, C. L. and Yoon, K.: 1981, *Multiple Attribute Decision Making Methods and Applications A State-of-the-Art Survey*, Springer-Verlag Berlin Heidelberg.

[57] Ji, P., Wang, J. Q. and Zhang, H. Y.: 2018, Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers, *Neural Computing and Applications* **30**(3), 799 – 823.

[58] Ji, P. and Zhang, H. Y.: 2016, A subsethood measure with the hausdorff distance for interval neutrosophic sets and its relations with similarity and entropy measures, *2016 Chinese Control and Decision Conference (CCDC), IEEE*, pp. 4152–4157.

[59] Joshi, D. and Kumar, S.: 2014, Intuitionistic fuzzy entropy and distance measure based TOPSIS method for multi-criteria decision making, *Egyptian Informatics Journal* **15**(2), 97 – 104.

[60] Khan, Q., Liu, P., Mahmood, T., Smarandache, F. and Ullah, K.: 2018, Some interval neutrosophic dombi power bonferroni mean operators and their application in multi–attribute decision–making, *Symmetry* **10**(10), 459.

[61] Klir, G. J. and Yuan, B.: 2005, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall of India Private Limited, New Delhi.

[62] Kong, L., Wu, Y. and Ye, J.: 2015, Misfire fault diagnosis method of gasoline engines using the cosine similarity measure of neutrosophic numbers, *Neutrosophic Sets and Systems* **8**, 42 – 45.

[63] Krishankumar, R., Ravichandran, K., Premaladha, J., Kar, S., Zavadskas, E. K. and Antucheviciene, J.: 2018, A decision framework under a linguistic hesitant fuzzy set for solving multi-criteria group decision making problems, *Sustainability* **10**(8), 2608.

[64] Kucukvar, M., Gumus, S., Egilmez, G. and Tatari, O.: 2014, Ranking the sustainability performance of pavements: An intuitionistic fuzzy decision making method, *Automation in Construction* **40**, 33–43.

[65] Kumar, K. and Garg, H.: 2018a, Connection number of set pair analysis based TOPSIS method on intuitionistic fuzzy sets and their application to decision making, *Applied Intelligence* **48**(8), 2112 – 2119.

[66] Kumar, K. and Garg, H.: 2018b, TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment, *Computational and Applied Mathematics* **37**(2), 1319 – 1329.

[67] Li, B., Wang, J., Yang, L. and Li, X.: 2018, A novel generalized simplified neutrosophic number Einstein aggregation operator, *IAENG International Journal of Applied Mathematics* **48**(1), 67 – 72.

[68] Li, M., Jin, L. and Wang, J.: 2014, A new MCDM method combining QFD with TOPSIS for knowledge management system selection from the user's perspective in intuitionistic fuzzy environment, *Applied soft computing* **21**, 28–37.

[69] Li, Y., Liu, P. and Chen, Y.: 2016, Some single valued neutrosophic number Heronian mean operators and their application in multiple attribute group decision making, *Informatica* **27**(1), 85 – 110.

[70] Li, Y. Y., Wang, J. Q. and Wang, T. L.: 2018, A linguistic neutrosophic multi-criteria group decision-making approach with EDAS method, *Arabian Journal for*

Science and Engineering pp. 1–13.
URL: *10.1007/s13369-018-3487-5*

[71] Li, Y. Y., Zhang, H. Y. and Wang, J. Q.: 2017, Linguistic neutrosophic sets and its application to multi-criteria decision-making problems, *International Journal for Uncertainty Quantification* **7**(2), 135 – 154.

[72] Liang, R. X., Jiang, Z. B. and Wang, J. Q.: 2018, A linguistic neutrosophic multi-criteria group decision-making method to university human resource management, *Symmetry* **10**(9), 364.
URL: *10.3390/sym10090364*

[73] Liang, W., Zhao, G. and Wu, H.: 2017, Evaluating investment risks of metallic mines using an extended TOPSIS method with linguistic neutrosophic numbers, *Symmetry* **9**, 149; doi: 10.3390/sym9080149.
URL: *10.3390/sym9080149*

[74] Liu, C. and Luo, Y.: 2016a, Correlated aggregation operators for simplified neutrosophic set and their application in multi-attribute group decision making, *Journal of Intelligent & Fuzzy Systems* **30**(3), 1755 – 1761.

[75] Liu, C. and Luo, Y.: 2016b, The weighted distance measure based method to neutrosophic multiattribute group decision making, *Mathematical Problems in Engineering* **2016**, Article ID 3145341.

[76] Liu, C. and Luo, Y.: 2019, Power aggregation operators of simplified neutrosophic sets and their use in multi-attribute group decision making, *IEEE/CAA Journal of Automatica Sinica* **6**(2), 575 – 583.

[77] Liu, F., Aiwu, G., Lukovac, V. and Vukic, M.: 2018, A multicriteria model for the selection of the transport service provider: A single valued neutrosophic DEMATEL multicriteria model, *Decision Making: Applications in Management and Engineering* **1**(2), 121–130.

[78] Liu, P.: 2016, The aggregation operators based on archimedean t - conorm and t - norm for single-valued neutrosophic numbers and their application to decision making, *International Journal of Fuzzy Systems* **18**(5), 849 – 863.

[79] Liu, P., Chu, Y., Li, Y. and Chen, Y.: 2014, Some generalized neutrosophic number hamacher aggregation operators and their application to group decision making, *International Journal of Fuzzy Systems* **16**(2), 242 – 255.

[80] Liu, P., Khan, Q. and Mahmood, T.: 2018, Multiple-attribute decision making based on single-valued neutrosophic schweizer-sklar prioritized aggregation operator, *Cognitive Systems Research* .
URL: *https://doi.org/10.1016/j.cogsys.2018.10.005*

[81] Liu, P. and Liu, X.: 2016, The neutrosophic number generalized weighted power averaging operator and its application in multiple attribute group decision making, *International Journal of Machine Learning and Cybernetics* **-**, 1–12.

[82] Liu, P., Mahmood, T. and Khan, Q.: 2018, Group decision making based on power heronian aggregation operators under linguistic neutrosophic environment, *International Journal of Fuzzy Systems* **20**(3), 970 – 985.

[83] Liu, P. and Qin, X.: 2017, An extended VIKOR method for decision making problem with interval-valued linguistic intuitionistic fuzzy numbers based on entropy, *Informatica* **28**(4), 665 – 685.

[84] Liu, P. and Tang, G.: 2016, Some power generalized aggregation operators based on the interval neutrosophic sets and their application to decision making, *Journal of Intelligent & Fuzzy Systems* **30**(5), 2517–2528.

[85] Liu, P., Tang, G. and Liu, W.: 2017, Induced generalized interval neutrosophic shapley hybrid operators and their application in multi - attribute decision making, *Scientia Iranica* **24**(4), 2164 – 2181.

[86] Liu, P. and Wang, Y.: 2014, Multiple attribute decision-making method based on

single-valued neutrosophic normalized weighted Bonferroni mean, *Neural Computing & Applications* **25**(7-8), 2001 – 2010.

[87] Liu, P. and Wang, Y.: 2016, Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making, *Journal of Science Science and Complexity* **29**(3), 681 – 697.

[88] Liu, P. and You, X.: 2017, Interval neutrosophic Muirhead mean operators and their application in multiple attribute group decision-making, *International Journal for Uncertainty Quantification* **7**(4), 303 – 334.

[89] Liu, P. and You, X.: 2018, Some linguistic neutrosophic Hamy mean operators and their application to multi-attribute group decision making, *PloS one* **13**(3), e0193027.

[90] Liu, P. and You, X.: 2019, Bidirectional projection measure of linguistic neutrosophic numbers and their application to multi-criteria group decision making, *Computers & Industrial Engineering* **128**, 447–457.

[91] Liu, P. and Zhang, X.: 2018, Some intuitionistic uncertain linguistic Bonferroni mean operators and their application to group decision making, *Soft Computing* pp. 1–18.

[92] Lu, Z. and Ye, J.: 2017, Exponential operations and an aggregation method for single - valued neutrosophic numbers in decision making, *Information* **8**(2), 62.

[93] Ma, H., Zhu, H., Hu, Z., Li, K. and Tang, W.: 2017, Time-aware trustworthiness ranking prediction for cloud services using interval neutrosophic set and ELECTRE, *Knowledge-Based Systems* **138**, 27–45.

[94] Ma, Y. X., Wang, J. Q., Wang, J. and Wu, X. H.: 2017, An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options, *Neural Computing and Applications* **28**(9), 2745–2765.

[95] Majumdar, P.: 2015, *Neutrosophic Sets and Its Applications to Decision Making*, Computational Intelligence for Big Data Analysis: Frontier Advances and Applications, Springer, Cham, pp. 97–115.

[96] Majumdar, P. and Samant, S. K.: 2014, On similarity and entropy of neutrosophic sets, *Journal of Intelligent and Fuzzy Systems* **26**(3), 1245 – 1252.

[97] Maldonado-Macias, A., Alvarado, A., Garcia, J. L. and Balderrama, C. O.: 2014, Intuitionistic fuzzy TOPSIS for ergonomic compatibility evaluation of advanced manufacturing technology, *The International Journal of Advanced Manufacturing Technology* **70**(9-12), 2283–2292.

[98] Mandal, K. and Basu, K.: 2016, Improved similarity measure in neutrosophic environment and its application in finding minimum spanning tree, *Journal of Intelligent & Fuzzy Systems* **31**(3), 1721–1730.

[99] Mareschal, B., Brans, J. P. and Vincke, P.: 1984, PROMETHEE: A new family of outranking methods in multicriteria analysis, *Technical report*, ULB–Universite Libre de Bruxelles.

[100] Mendel, J. M. and John, R. B.: 2002, Type-2 fuzzy sets made simple, *IEEE Transactions on Fuzzy Systems* **10**(2), 117–127.

[101] Mizumoto, M.: 1989, Pictorial representations of fuzzy connectives, part I: cases of t-norms, t-conorms and averaging operators, *Fuzzy sets and systems* **31**(2), 217–242.

[102] Mondal, K., Pramanik, S. and Giri, B. C.: 2018, Hybrid binary logarithm similarity measure for MAGDM problems under SVNS assessments, *Neutrosophic Sets and Systems* **20**, 12 – 25.

[103] Mukherjee, A. and Sarkar, S.: 2014a, Several similarity measures of interval valued neutrosophic soft sets and their application in pattern recognition problems, *Neutrosophic sets and systems* **6**, 55–61.

[104] Mukherjee, A. and Sarkar, S.: 2014b, Similarity measures for interval-valued intuitionistic fuzzy soft sets and its application in medical diagnosis problem, *New Trends in Mathematical Sciences* **2**(3), 159 – 165.

[105] Mukherjee, A. and Sarkar, S.: 2015, A new method of measuring similarity between two neutrosophic soft sets and its application in pattern recognition problems, *Neutrosophic Sets and Systems* **8**, 63–68.

[106] Nancy and Garg, H.: 2016, Novel single-valued neutrosophic decision making operators under Frank norm operations and its application, *International Journal for Uncertainty Quantification* **6**(4), 361 – 375.

[107] Nie, R., Wang, J. Q. and Zhang, H. Y.: 2017, Solving solar-wind power station location problem using an extended WASPAS technique with interval neutrosophic sets, *Symmetry* **9**(7), 106.
URL: *10.3390/sym9070106*

[108] Onat, N. C., Gumus, S., Kucukvar, M. and Tatari, O.: 2016, Application of the TOPSIS and intuitionistic fuzzy set approaches for ranking the life cycle sustainability performance of alternative vehicle technologies, *Sustainable Production and Consumption* **6**, 12 – 25.

[109] Opricovic, S.: 1998, Multicriteria optimization of civil engineering systems, *Faculty of Civil Engineering, Belgrade* **2**(1), 5–21.

[110] Ortega, R. G., Vázquez, M. L., Figueiredo, J. A. and Guijarro-Rodríguez, A.: 2018, Sinos river basin social-environmental prospective assessment of water quality management using fuzzy cognitive maps and neutrosophic AHP-TOPSIS., *Neutrosophic Sets & Systems* **23**, 160–171.

[111] Otay, I. and Kahraman, C.: 2017, Six sigma project selection using interval neutrosophic TOPSIS, *Advances in Fuzzy Logic and Technology 2017, Springer*, pp. 83–93.

[112] Peng, H. G., Wang, J. Q. and Cheng, P. F.: 2018, A linguistic intuitionistic multicriteria decision-making method based on the frank heronian mean operator and its application in evaluating coal mine safety, *International Journal of Machine Learning and Cybernetics* **9**, 1053 – 1068.

[113] Peng, J. J., Wang, J. Q., Wang, J., Zhang, H. Y. and Chen, Z. H.: 2016, Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems, *International Journal of System Science* **47**(10), 2342 – 2358.

[114] Peng, X. D. and Dai, J. G.: 2018a, Approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function, *Neural Computing and Applications* **29**(10), 939 – 954.

[115] Peng, X. D. and Dai, J. G.: 2018b, A bibliometric analysis of neutrosophic set: Two decades review from 1998-2017, *Artificial Intelligence Review* pp. 1 – 57.
URL: *10.1007/s10462-018-9652-0*

[116] Peng, X. D. and Liu, C.: 2017, Algorithms for neutrosophic soft decision making based on EDAS, new similarity measure and level soft set, *Journal of Intelligent & Fuzzy Systems* **32**(1), 955 – 968.

[117] Pouresmaeil, H., Shivanian, E., Khorram, E. and Fathabadi, H. S.: 2017, An extended method using TOPSIS and VIKOR for multiple attribute decision making with multiple decision makers and single valued neutrosophic numbers, *Advances and Applications in Statistics* **50**, 261–292.

[118] Pramanik, S., Biswas, P. and Giri, B. C.: 2017, Hybrid vector similarity measures and their applications to multi-attribute decision making under neutrosophic environment, *Neural Computing and Applications* **28**(5), 1163–1176.

[119] Pramanik, S., Dalapati, S., Alam, S., Smarandache, F. and Roy, T. K.: 2018a, NC-cross entropy based MADM strategy in neutrosophic cubic set environment, *Mathematics* **6**(5), 67.

[120] Pramanik, S., Dalapati, S., Alam, S., Smarandache, F. and Roy, T. K.: 2018b, NS-cross entropy-based MAGDM under single-valued neutrosophic set environment, *Information* **9**(2), 37.

[121] Qin, J. and Liu, X.: 2014, Frank aggregation operators for triangular interval type-2

fuzzy set and its application in multiple attribute group decision making, *Journal of Applied Mathematics* **2014**, Article ID 923213 24 pages.

[122] Qin, J., Liu, X. and Pedrycz, W.: 2016, Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making, *Applied Soft Computing* **41**, 428 – 452.

[123] Radwan, N. M., Senousy, M. B. and Din, M. R. A. E.: 2016, Neutrosophic AHP multi criteria decision making method applied on the selection of learning management system, *International Journal of Advancements in Computing Technology* **8**(5), 95–105.

[124] Rani, D. and Garg, H.: 2019, Some modified results of the subtraction and division operations on interval neutrosophic sets, *Journal of Experimental & Theoretical Artificial Intelligence* **31**(4), 677 – 698.

[125] Rathi, K. and Balamohan, S.: 2017, A mathematical model for subjective evaluation of alternatives in fuzzy multi-criteria group decision making using COPRAS method, *International Journal of Fuzzy Systems* **19**(5), 1290–1299.

[126] Razavi Hajiagha, S. H., Hashemi, S. S. and Zavadskas, E. K.: 2013, A complex proportional assessment method for group decision making in an interval-valued intuitionistic fuzzy environment, *Technological and Economic Development of Economy* **19**(1), 22–37.

[127] Reddy, P. R., Raju, I. N., Reddy, V. D. and Krishnaiah, G.: 2016, An integrated MCGDM approach based lean faclitator selection under neutrosophic environment, *International Journal of Scientific & Engineering Research* **7**(3), 566–572.

[128] Reddy, Y. R. and Reddy, B. C. M.: 2018, Optimization of WEDM parameters for super Ni-718 using neutrosophic sets and TOPSIS method, *International Journal of Applied Engineering Research* **13**(12), 10924–10930.

[129] Ren, H., Xiao, S. and Zhou, H.: 2019, A Chi-square distance-based similarity measure of single-valued neutrosophic set and applications., *International Journal of Computers, Communications & Control* **14**(1), 78 – 89.

[130] Roy, J., Das, S., Kar, S. and Pamučar, D.: 2019, An extension of the CODAS approach using interval-valued intuitionistic fuzzy set for sustainable material selection in construction projects with incomplete weight information, *Symmetry* **11**(3), 393.

[131] Şahin, R.: 2014, Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment, *arXiv preprint arXiv:1412.5202* .

[132] Sahin, R.: 2017, Cross-entropy measure on interval neutrosophic sets and its applications in multicriteria decision making, *Neural Computing and Applications* **28**(5), 1177 – 1187.

[133] Sahin, R. and Liu, P.: 2017, Some approaches to multi criteria decision making based on exponential operations of simplified neutrosophic numbers, *Journal of Intelligent & Fuzzy Systems* **32**(3), 2083 – 2099.

[134] Sahin, R. and Yiğider, M.: 2014, A multi-criteria neutrosophic group decision making method based TOPSIS for supplier selection, *arXiv preprint arXiv:1412.5077* .

[135] Sahin, R. and Zhang, H. Y.: 2018, Induced simplified neutrosophic correlated aggregation operators for multi-criteria group decision-making, *Journal of Experimental & Theoretical Artificial Intelligence* **30**(2), 279 – 292.

[136] Sarkar, M. and Roy, T. K.: 2018, Optimization of welded beam structure using neutrosophic optimization technique: a comparative study, *International Journal of Fuzzy Systems* **20**(3), 847–860.

[137] Selvachandran, G. and Peng, X.: 2018, A modified TOPSIS method based on vague parameterized vague soft sets and its application to supplier selection problems, *Neural Computing and Applications* pp. 1–16.
URL: *10.1007/s00521-018-3409-1*

[138] Shi, L. and Ye, J.: 2017, Cosine measures of linguistic neutrosophic numbers and their application in multiple attribute group decision- making, *Information* **8**(4), 117. **URL:** *10.3390/info8040117*

[139] Shi, L. and Ye, J.: 2018, Multiple attribute group decision-making method using correlation coefficients between linguistic neutrosophic numbers, *Journal of Intelligent & Fuzzy Systems* **35**(1), 917–925.

[140] Smarandache, F.: 1998, *Neutrosophy. Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning*, Ann Arbor, Michigan, USA.

[141] Smarandache, F.: 1999, *A unifying field in logics. Neutrosophy: neutrosophic probability, set and logic*, American Research Press, Rehoboth.

[142] Sun, H. X., Yang, H. X., Wu, J. Z. and Ouyang, Y.: 2015, Interval neutrosophic numbers choquet integral operator for multi-criteria decision making, *Journal of Intelligent & Fuzzy Systems* **28**(6), 2443–2455.

[143] Sun, R., Hu, J. and Chen, X.: 2019, Novel single-valued neutrosophic decision-making approaches based on prospect theory and their applications in physician selection, *Soft Computing* **23**(1), 211–225.

[144] Tan, R., Zhang, W. and Chen, S.: 2018, Exponential aggregation operator of interval neutrosophic numbers and its application in typhoon disaster evaluation, *Symmetry* **10**(6), 196.

[145] Thanh, N. D., Son, L. H. and Ali, M.: 2017, Neutrosophic recommender system for medical diagnosis based on algebraic similarity measure and clustering, *Fuzzy Systems (FUZZ-IEEE), 2017 IEEE International Conference on*, IEEE, pp. 1–6.

[146] Tian, Z. P., Wang, J. Q. and Zhang, H. Y.: 2018, Hybrid single-valued neutrosophic MCGDM with QFD for market segment evaluation and selection, *Journal of Intelligent & Fuzzy Systems* **34**(1), 177–187.

[147] Tian, Z. P., Wang, J., Wang, J. Q. and Zhang, H. Y.: 2017a, An improved MULTIMOORA approach for multi-criteria decision-making based on interdependent inputs

of simplified neutrosophic linguistic information, *Neural Computing and Applications* **28**(1), 585–597.

[148] Tian, Z. P., Wang, J., Wang, J. Q. and Zhang, H. Y.: 2017b, Simplified neutrosophic linguistic multi-criteria group decision-making approach to green product development, *Group Decision and Negotiation* **26**(3), 597–627.

[149] Tian, Z. P., Wang, J., Zhang, H. Y., Chen, X. H. and Wang, J. Q.: 2016, Simplified neutrosophic linguistic normalized weighted Bonferroni mean operator and its application to multi-criteria decision-making problems, *Filomat* **30**(12), 3339–3360.

[150] Tian, Z. P., Zhang, H. Y., Wang, J., Wang, J. Q. and Chen, X. H.: 2016, Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets, *International Journal of Systems Science* **47**(15), 3598–3608.

[151] Torra, V.: 2010, Hesitant fuzzy sets, *International Journal of Intelligent Systems* **25**(6), 529 – 539.

[152] Tu, A., Ye, J. and Wang, B.: 2018, Symmetry measures of simplified neutrosophic sets for multiple attribute decision-making problems, *Symmetry* **10**(5), 144.

[153] Wang, C. Y. and Chen, S. M.: 2017, Multiple attribute decision making based on interval-valued intuitionistic fuzzy sets, linear programming methodology, and the extended TOPSIS method, *Information Sciences* **397**, 155–167.

[154] Wang, H., Smarandache, F., Zhang, Y. Q. and Smarandache, R.: 2005, *Interval neutrosophic sets and logic: theory and applications in computing*, Hexis, Phoenix, AZ.

[155] Wang, H., Smarandache, F., Zhang, Y. Q. and Sunderraman, R.: 2010, Single valued neutrosophic sets, *Multispace Multistructure* **4**, 410 – 413.

[156] Wang, J., Gao, H. and Wei, G.: 2018, Some 2-tuple linguistic neutrosophic number muirhead mean operators and their applications to multiple attribute decision making, *Journal of Experimental & Theoretical Artificial Intelligence* pp. 1 – 31.
URL: *10.1080/0952813X.2018.1552320*

[157] Wang, J. and Li, J.: 2009, The multi-criteria group decision making method based on multi-granularity intuitionistic two semantics,, *Science & Technology Information* **33**(1), 8 – 9.

[158] Wang, J. Q., Tian, C. Q., Zhang, X., Zhang, H. Y. and Wang, T. L.: 2018, Multi-criteria decision-making method based on simplified neutrosophic linguistic information with cloud model, *Symmetry* **10**, 197.
URL: *10.3390/sym10060197*

[159] Wang, J. Q., Yang, Y. and Li, L.: 2018, Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators, *Neural Computing and Applications* **30**(5), 1529 – 1547.

[160] Wang, X. and Triantaphyllou, E.: 2008, Ranking irregularities when evaluating alternatives by using some ELECTRE methods, *Omega - International Journal of Management Science* **36**, 45 – 63.

[161] Wang, Y. and Liu, P.: 2018, Linguistic neutrosophic generalized partitioned Bonferroni mean operators and their application to multi-attribute group decision making, *Symmetry* **10**(5), 160.

[162] Wang, Y., Xi, C., Zhang, S., Zhang, W. and Yu, D.: 2015, Combined approach for government e-tendering using GA and TOPSIS with intuitionistic fuzzy information, *PloS one* **10**(7), e0130767.

[163] Wei, G. and Wei, Y.: 2018, Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making, *Journal of Intelligent & Fuzzy Systems* **35**(2), 2001–2013.

[164] Wei, G. and Zhang, Z.: 2019, Some single-valued neutrosophic Bonferroni power aggregation operators in multiple attribute decision making, *Journal of Ambient Intelligence and Humanized Computing* **10**(3), 863 – 882.

[165] Wu, H., Yuan, Y., Wei, L. and Pei, L.: 2018, On entropy, similarity measure

and cross-entropy of single-valued neutrosophic sets and their application in multi-attribute decision making, *Soft Computing* **22**(22), 7367–7376.

[166] Wu, X. H., Wang, J. Q., Peng, J. J. and Chen, X. H.: 2016, Cross - entropy and prioritized aggregation operator with simplified neutrosophic sets and their application in multi-criteria decision-making problems, *International Journal of Fuzzy Systems* **18**(6), 1104 – 1116.

[167] Xu, G., Wang, S., Yang, T. and Jiang, W.: 2018, A neutrosophic approach based on TOPSIS method to image segmentation., *International Journal of Computers, Communications & Control* **13**(6), 1047–1061.

[168] Xu, Z. S.: 2004, A method based on linguistic aggregation operators for group decision making under linguistic preference relations, *Information Sciences* **166**(1-4), 19 – 30.

[169] Yang, H., Wang, X. and Qin, K.: 2019, New similarity and entropy measures of interval neutrosophic sets with applications in multi-attribute decision-making, *Symmetry* **11**(3), 370.

[170] Yang, L. and Li, B.: 2016, A multi-criteria decision-making method using power aggregation operators for single-valued neutrosophic sets, *International Journal of Database and theory and application* **9**(2), 23 – 32.

[171] Yang, L. and Li, B.: 2018, An extended single-valued neutrosophic normalized weighted Bonferroni mean Einstein aggregation operator, *International Journal of Applied Mathematics* **48**(4), 6 – 12.

[172] Yang, L., Li, B. and Xu, H.: 2018, Novel power aggregation operators based on Einstein operations for interval neutrosophic linguistic sets, *International Journal of Applied Mathematics* **48**(4), 475 – 484.

[173] Ye, J.: 2013, Multicriteria decision making method using the correlation coefficient under single-value neutrosophic environment, *International Journal of General Systems* **42**(4), 386 – 394.

[174] Ye, J.: 2014a, Clustering methods using distance-based similarity measures of single-valued neutrosophic sets, *Journal of Intelligent Systems* **23**(4), 379 – 389.

[175] Ye, J.: 2014b, A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, *Journal of Intelligent and Fuzzy Systems* **26**(5), 2459 – 2466.

[176] Ye, J.: 2014c, Multiple attribute group decision-making method with completely unknown weights based on similarity measures under single valued neutrosophic environment, *Journal of Intelligent and Fuzzy Systems* **27**(6), 2927 – 2935.

[177] Ye, J.: 2014d, Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making, *International Journal of Intelligent Fuzzy Systems* **26**(1), 165 – 172.

[178] Ye, J.: 2014e, Single valued neutrosophic cross-entropy for multicriteria decision making problems, *Applied Mathematical Modelling* **38**(3), 1170 – 1175.

[179] Ye, J.: 2014f, Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making, *Journal of Intelligent & Fuzzy Systems* **27**(5), 2231 – 2241.

[180] Ye, J.: 2015a, An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers, *Journal of Intelligent & Fuzzy Systems* **28**(1), 247 – 255.

[181] Ye, J.: 2015b, Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses, *Artificial Intelligence in Medicine* **63**, 171 – 179.

[182] Ye, J.: 2015c, Multiple attribute decision-making method based on the possibility degree ranking method and ordered weighted aggregation operators of interval neutrosophic numbers, *Journal of Intelligent and Fuzzy Systems* **28**(3), 1307 – 1317.

[183] Ye, J.: 2016a, Aggregation operators of neutrosophic linguistic numbers for multiple attribute group decision making, *SpringerPlus* **5**(1), 1691–1701.

[184] Ye, J.: 2016b, Exponential operations and aggregation operators of interval neutrosophic sets and their decision making methods, *SpringerPlus* **5**, 1488.
URL: *10.1186/s40064-016-3143-z*

[185] Ye, J.: 2016c, The generalized dice measures for multiple attribute decision making under simplified neutrosophic environments, *Journal of Intelligent & Fuzzy Systems* **31**(1), 663–671.

[186] Ye, J.: 2016d, Interval neutrosophic multiple attribute decision-making method with credibility information, *International Journal of Fuzzy Systems* **18**(5), 914–923.

[187] Ye, J.: 2016e, A netting method for clustering-simplified neutrosophic information, *Soft Computing* pp. 1–7.
URL: *10.1007/s00500-016-2310-z*

[188] Ye, J.: 2017, Single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine, *Soft Computing* **21**(3), 817–825.

[189] Ye, J. and Cui, W.: 2018a, Exponential entropy for simplified neutrosophic sets and its application in decision making, *Entropy* **20**(5), 357.

[190] Ye, J. and Cui, W.: 2018b, Operations and aggregation methods of single-valued linguistic neutrosophic interval linguistic numbers and their decision making method, *Information* **9**(8), 196.

[191] Ye, J. and Du, S.: 2019, Some distances, similarity and entropy measures for interval-valued neutrosophic sets and their relationship, *International Journal of Machine Learning and Cybernetics* **10**(2), 347 – 355.

[192] Ye, J. and Fu, J.: 2016, Multi-period medical diagnosis method using a single valued neutrosophic similarity measure based on tangent function, *Computer Methods and Programs in Biomedicine* **123**, 142 – 149.

[193] Ye, J. and Zhang, Q.: 2014, Single valued neutrosophic similarity measures for multiple attribute decision making, *Neutrosophic Sets and Systems* **2**, 48–54.

[194] Yue, Z.: 2014, TOPSIS-based group decision-making methodology in intuitionistic fuzzy setting, *Information Sciences* **277**, 141–153.

[195] Zadeh, L. A.: 1965, Fuzzy sets, *Information and Control* **8**, 338–353.

[196] Zadeh, L. A.: 1975, The concept of a linguistic variable and its application to approximate reasoning: Part-1, *Information Science* **8**, 199–251.

[197] Zavadskas, E. K., Bausys, R., Stanujkic, D. and Magdalinovic-Kalinovic, M.: 2016, Selection of lead-zinc flotation circuit design by applying WASPAS method with single-valued neutrosophic set, *Acta Montanistica Slovaca* **21**(2), 85–92.

[198] Zavadskas, E. K., Kaklauskas, A. and Sarka, V.: 1994, The new method of multicriteria complex proportional assessment of projects, *Technological and economic development of economy* **1**(3), 131–139.

[199] Zhang, H., Wang, J. and Chen, X.: 2016, An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets, *Neural Computing and Applications* **27**, 615–627.

[200] Zhang, H. Y., Wang, J. Q. and Chen, X. H.: 2014, Interval neutrosophic sets and their application in multicriteria decision making problems, *The Scientific World Journal* **Volume 2014**, Article ID 645953, 15 pages.

[201] Zhang, X. and Xu, Z.: 2015, Soft computing based on maximizing consensus and fuzzy TOPSIS approach to interval-valued intuitionistic fuzzy group decision making, *Applied Soft Computing* **26**, 42–56.

[202] Zhang, Z. and Wu, C.: 2014, A novel method for single-valued neutrosophic multi-criteria decision making with incomplete weight information, *Neutrosophic Sets and Systems* **4**, 35 – 49.

[203] Zolfani, S. H., Pourhossein, M., Yazdani, M. and Zavadskas, E. K.: 2018, Evaluating construction projects of hotels based on environmental sustainability with MCDM framework, *Alexandria Engineering Journal* **57**(1), 357–365.