Lie Group Applications to Some Nonlinear Systems

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Certificate

This is to certify that the thesis entitled, **"Lie Group Applications to Some Nonlinear Systems"**, submitted by Ms. Anupma in the fulfillment of the requirements for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidate's own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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Declaration

It is certified that the thesis is entirely my own and that the ideas and references cited herein have been duly acknowledged.

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Abstract

In this thesis, we study the applications of Lie group theory to the nonlinear partial differential equations (PDEs) or their systems which represent some of the important physical phenomena. Our primary objective in this thesis is to identify the symmetries of PDEs in order to obtain exact solutions and a further point is also to discuss the integrability and physical behaviour of the equations. The investigations carried out in this thesis are confined to the applications of Lie group methods to the six nonlinear systems which are the (2+1)-dimensional Calogero Degasperis (CD) equation with its generalized form, the coupled Klein-Gordon-Schrödinger (KGS) equation along its variable coefficients form, the (2+1)-dimensional potential Kadomstev Petviashvili (PKP) equation and its generalized form, the generalized Dullin-Gottwald-Holm (GDGH) equation and the generalized Bretherton equation.

Our thesis comprises of six chapters. In the introductory part some important features of Lie groups and symmetries are demonstrated and the mathematical fundamentals of continuous group theory are reviewed which are of great importance to the work dealt in Chapters 2-6.

In **Chapter 2**, we study the (2+1)-dimensional Calogero Degasperis equation and its generalized form for similarity reductions and exact solutions. For CD equation, we have derived infinite-dimensional symmetries involving two arbitrary functions $Q(t), R(t)$ and for certain specific values of arbitrary functions involved in solutions, we obtain periodic and kink wave solutions. In this Chapter, the CD equation with time dependent coefficients (VCCD) has also been investigated using symmetry approach. The interesting outcome of the study is that the VCCD equation is shown to be Painléve integrable and also yields new physically important solutions.

Chapter 3 is devoted to the use of combination of Lie group method and modified (G'/G)-expansion method to KGS equations. Firstly, we have derived the symmetries of KGS equations and use them to reduce equation to a nonlinear ODE system. Then, with the use of modified (G'/G)-expansion method, more explicit traveling wave solutions involving arbitrary parameters are found out, which are expressed by the hyperbolic functions, the trigonometric functions and rational functions. The variable coefficients KGS equation has also been studied for symmetries and new physically important solutions.

In **Chapter 4**, the (2+1)-dimensional PKP equation has been investigated for symmetries and exact solutions as this equation has many physical applications from water waves to plasma physics and field theories. The symmetries of PKP equation turn out to be infinite dimensional and using the obtained symmetries, we have reduced equation to (1+1)-dimensional PDEs in three different cases. The reduced PDEs are again studied for their reductions to ordinary differential equations (ODEs) and then besides recovering certain available results, some new interesting results are derived. As the variable coefficient generalizations of PKP equation (VCPKP) are able to provide more realistic models in physical situations, so we have also studied the Painléve properties, similarity reductions and invariant solutions of VCPKP equation in this chapter.

Chapter 5 is concerned with a new generalized DGH equation (GDGH) which has been investigated for its classical and nonclassical symmetries. We have also obtained new solutions of GDGH equation corresponding to different choices of arbitrary function involved $f(u)$.

Chapter 6 deals with the investigation of symmetries and exact solutions of generalized Bretherton equation with variable cooefficients which contains some particular important equations such as Duffing equation, Landau-Ginburg-Higgs equation, sin-Gordon equation and ϕ^4 equation. In this Chapter, sech-ansätz method and some other methods are successfully used to obtain solitary wave solutions.

List of Research Papers

1. R.K. Gupta, Anupma, “The Dullin-Gottwald -Holm Equation: Symmetries and Exact Solutions”, *International Journal of Nonlinear Science*, 10 (2010) 146-152.
2. Anupma Bansal, R.K. Gupta, “Modified (G’/G)-Expansion Method for Finding Exact Wave Solutions of the Coupled Klein-Gordon-Schrödinger Equation”, *Mathematical Methods in the Applied Sciences*, 35 (2012) 1175-1187. **(Wiley, Impact Factor 0.753)**
3. Anupma Bansal, R.K. Gupta, “On Certain New Exact Solutions of (2+1)-dimensional Calogero Degasperis Equation via Symmetry Approach”, *International Journal of Nonlinear Science*, 13 (2012) 475-481.
4. Anupma Bansal, R.K. Gupta, “Lie Point Symmetries and Similarity Solutions of the Time Dependent Coefficients Calogero Degasperis Equation”, *Physica Scripta*, 86 (2012) 035005 (11 pages). **(IOP, Impact Factor 1.204)**
5. R.K. Gupta, Anupma Bansal, “Similarity Reductions and Exact Solutions of Generalized Bretherton Equation with Time Dependent Coefficients”, *Nonlinear Dynamics*, 71 (2013) 1-12. **(Springer, Impact Factor 1.247)**
6. R.K. Gupta, Anupma Bansal, “Painlevé Analysis, Lie Symmetries and Invariant Solutions of potential Kadomstev Petviashvili Equation with Time Dependent Coefficients”, *Applied Mathematics and Computation*, 219 (2013) 5290-5302. **(Elsevier, Impact Factor 1.317)**
7. R.K. Gupta, Anupma Bansal, “Symmetry Analysis and Exact Solutions of the New Generalized Dullin-Gottwald-Holm Equation”, *International Journal of Engineering Mathematics* (communicated).

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TO

MY FAMILY

Chapter 1

Introduction

1.1 Literature Survey and Motivation

Problems of physical interest are often translated in terms of differential equations that may turn out to be linear or nonlinear, ordinary or partial. Exact solutions of these resulting equations are of much interest both from mathematical and application points of view. On account of their applications in the disciplines of mathematics, chemistry, engineering and in almost all branches of theoretical physics, including classical mechanics, quantum mechanics and relativity, differential equations and their symmetries have retained their central role. One reason for the overall prominence of the concept of symmetry is its nativeness and its simplicity. Intuitively speaking, a symmetry is a transformation of an object leaving this object invariant. This is clearly such a general property that it can be recovered almost everywhere in nature and, correspondingly, in numerous areas of science and art.

The concept of continuous symmetry is formalized by Lie groups and thus, become a part of the foundations of mathematics. They link with many branches of mathematics, from analysis to number theory, passing through topology, algebraic geometry and so on. The main motivation for investigating symmetries of differential equations is:

- Mapping known solutions to other solutions,
- Extensions to methods of integration of ordinary differential equations,
- Construction of invariant solutions, i.e., solutions which are unaltered under the action of a subgroup of the admitted group, and,
- Detection of linearising transformation.

The symmetry analysis of differential equations was developed and applied by Sophus Lie [130] during the period 1872-1899. Despite of its important features, the Lie's approach to differential equations was not exploited for half a century and only the abstract theory of Lie groups grew. It was in the Forties of last century, with the work of G. Birkhoff [47] and I. Sedov [77] on dimensional analysis, that the theory gave relevant results in concrete applied problems and then, it was developed to an advanced state through the pioneering efforts of Ovsiannikov [83] in the late 1950's. By the late 1960's and early 1970's, the whole field was active again and new applications of group theory were being developed by a number of researchers including Cantwell [24], Bluman and Cole [50], Bluman and Anco [49, 52], Bluman and Kumei [53], Stephani [54], Hydon [109], Olver [111], Ibragimov [101], Grundy [117], Bhutani et al. [104, 105, 106], Hill et al. [31, 62, 63, 64], Clarkson and Mansfield [108], Gagnon and Winternitz [76].

Lie symmetry analysis of differential equations provides a powerful and fundamental framework to the exploitation of systematic procedures leading to the integration by quadrature of ordinary differential equations, to the determination of invariant solutions of initial and boundary value problems, to the derivation of conservation laws, to the construction of links between different differential equations that turn out to be equivalent. Lie has eastablished that if an ODE admits a one-parameter group of transformations, then special solutions called invariant solutions can be constructed without knowledge of the general solution of the ODE. Such solutions are invariant curves of the group. For an exhaustive review of Lie's work on this aspect, we refer to the works of Lie and Engel [131], Cohen [7], Goursat [38], Dickson [75], Ince [40] and Hermann [118].

The key idea of Lie's theory of symmetry analysis of differential equations relies on the invariance of the latter under a transformation of independent and dependent variables. This transformation forms a local group of point transformations establishing a diffeomorphism on the space of independent and dependent variables, mapping solutions of the equations to other solutions. Any transformation of the independent and dependent variables in turn induces a transformation of the derivatives. Lie showed that the problem of finding the Lie group of point transformations leaving invariant a differential equation (ordinary or partial), i.e., a point symmetry of a differential equation (DE), reduced to solving related linear systems of determining equations for its infinitesimal generators. He also showed that a point symmetry of a DE leads, in the case of an ordinary differential equation, to reducing the order of the DE and in the case of a partial differential equation, to finding special solutions called invariant (similarity) solutions of the DE. In this direction, some recent contributions are from Gandarias and Bruzón [93, 94, 97], Nucci [87, 88], Anco and Dennis [126], Biswas et al. [25], Gupta et al. [72, 73, 99, 100, 121, 122, 135], A. Bihlo et al. [1, 2], Sharma et al. [149, 150], Y.K. Gupta et al. [163, 164].

Modern developments in applications of Lie group methods have proceeded in a variety of directions. General theories of infinite-dimensional Lie groups and algebras [152] and Lie pseudo-groups, arising in relativity, field theory, fluid mechanics, solitons, and geometry, remain elusive. Higher order or generalized symmetries, in which the infinitesimal generators also depend upon derivative coordinates, first proposed by Noether [41] have been used to classify integrable (soliton) systems. Recursion operators are used to generate such higher order symmetries, and, via Noether's theorem, higher order conservation laws [111]. Most recursion operators are derived from a pair of compatible Hamiltonian structures, and demonstrate the integrability of biHamiltonian systems. The higher order symmetries also appear in series expansions of Bäcklund transformations in the spectral parameter.

Lie's classical theory is a source for various generalizations. Among these generalizations there are the following techniques:

1. Nonclassical method [51]
2. General method of differential constraints [42, 112]
3. Introduction of approximate symmetries [102, 148]
4. Generalized symmetries [111]
5. Equivalence transformations [56]
6. Nonlocal symmetries [53, 110, 111]

There also exist alternative methods, which are not based on the applications of group theory, such as Direct method [107], Bäcklund transformation [27], Painlevé analysis [66, 67], Inverse scattering transformation [90]. Recently, a variety of powerful methods, such as the tanh-sech method [16, 155], extended tanh method [17, 125], sine-cosine method [6, 18], Hirota method [19, 20], homogeneous balance method [36, 95], Jacobi elliptic function method [26, 37], F-expansion method [61, 86], homotopy perturbation method [28, 146], variational iteration method [58, 59], non-perturbative method [60], extended (G'/G)-expansion method [127], modified (G'/G)-expansion method [160] were developed.

The work comprising this thesis is based on the applications of two techniques viz. Lie classical method and nonclassical method. The prime motivation in carrying out this study has been to demonstrate the importance and efficacy of these methods over various other methods available in literature. Some specific physical sytems, governed by nonlinear partial differential equations have been considered to accomplish the task.

The description of the various systems studied and forming the subject of investigation for different chapters is made in brief in section (1.8). The problems studied are dealt with in two phases - in the first, the symmetries of the sytem under investigation are derived using either of the above mentioned method and then in the second phase, after successful deduction of the reduced systems of ODEs, the efforts are confined to furnish the exact solutions with the help of the following methods:

1. Hyperbolic functions expansion method
2. Jacobi elliptic function method
3. Extended (G'/G)-expansion method
4. Modified (G'/G)-expansion method

In some (2+1)-dimensional problems, we have also studied the Painlevé properties of the equation and used classical method two times to reduce the equation to ODE and then find solutions using above written methods.

After giving a brief survey of the available literature relevant to the work put up in chapters, we reproduce in the following sections, certain characterstic features of the techniques utilized, general notions essential for understanding and carry over of the Lie classical method, nonclassical method to find more solutions, Painlevé analysis to check integrability, Hyperbolic functions expansion method, Jacobi elliptic function method, Extended (G'/G)-expansion method, Modified (G'/G)-expansion method to furnish new solutions.

1.2 Lie Classical Method to Construct Solutions of PDEs

In this section we will show how to find local symmetries of a given PDE system. A local symmetry maps solutions of the PDE system into one-parameter families of solutions. However, there can exist solutions that map into themselves, i.e., are invariant, under the action of a local symmetry of the PDE system. Such solutions are called invariant solutions (similarity solutions) and include the well-known self-similar solutions (automodel solutions) that result from scaling symmetries. The method of finding invariant solutions is commonly referred to as the classical method. To start with classical method, we will firstly present some fundamentals of Lie group theory [24, 49] (refer to sections (1.2.1) to (1.2.7)).

1.2.1 One-Parameter Lie Groups

Let the vector $x = (x^1, x^2, ..., x^n)$ lie in some continous open set D on the n-dimensional Euclidean manifold $\mathbb{R}^n$. Define the transformation

$$T^{\epsilon} : \{x^* = \mathbb{X}(x; \epsilon)\}\,. \tag{1.2.1}$$

The function X is infinitely differentiable with respect to the real variables x and an analytic function of the real continous parameter ϵ, which lies in an open interval, S. The transformation T^{ϵ} is a **one-parameter Lie group** with respect to the binary operation of composition if and only if:

(i) There is an identity element $\epsilon \to \epsilon_0$ such that

$$T^{\epsilon_0} : \{\mathbb{X}(x; \epsilon_0) = x\}\,. \tag{1.2.2}$$

(ii) For every value of ϵ, there is an inverse $\epsilon \to \epsilon_{inv}$ such that

$$T^{\epsilon_{inv}} : \{\mathbb{X}(x^*; \epsilon_{inv}) = x\}\,. \tag{1.2.3}$$

(iii) The binary operation of composition produces a transformation i.e. a member of the group $T^{\epsilon_1}.T^{\epsilon_2} = T^{\epsilon_3}$, i.e., the group is closed. Consider two members of the group,

$$T^{\epsilon_1} : \{x^{**} = \mathbb{X}(x^*; \epsilon_1)\}\,, \tag{1.2.4}$$

and

$$T^{\epsilon_2} : \{x^* = \mathbb{X}(x; \epsilon_2)\}\,. \tag{1.2.5}$$

If we compose $T^{\epsilon_1}, T^{\epsilon_2}$, we get

$$T^{\epsilon_3} : \{x^{**} = \mathbb{X}(x; \epsilon_3)\}\,, \tag{1.2.6}$$

where $\epsilon_3 = \phi(\epsilon_1, \epsilon_2) \in S$. The function ϕ defining the law of composition of T^{ϵ} is an analytic function of $\epsilon_1 \in S$ and $\epsilon_2 \in S$ and is commutative, i.e., $\phi(\epsilon_1, \epsilon_2) = \phi(\epsilon_2, \epsilon_1)$; thus, Lie groups are abelian.

(iv) The group is associative, i.e.

$$(T^{\epsilon_1}.T^{\epsilon_2}).T^{\epsilon_3} = T^{\epsilon_1}.(T^{\epsilon_2}.T^{\epsilon_3}) \tag{1.2.7}$$

1.2.2 Infinitesimal Form of a Lie Group

Consider a one-parameter Lie groups of the form

$$T^\epsilon : \{x^* = \mathbb{X}(x;\epsilon)\}, \tag{1.2.8}$$

where ϵ is the group parameter and assumed to be defined in such a way that the identity element is $\epsilon_0 = 0$. Now, expanding (1.2.8) in a Taylor series about $\epsilon = 0$, we get

$$\begin{aligned} x^* &= x + \epsilon \left(\frac{\partial \mathbb{X}(x;\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} \right) + \frac{\epsilon^2}{2} \left(\frac{\partial^2 \mathbb{X}(x;\epsilon)}{\partial \epsilon^2}\bigg|_{\epsilon=0} \right) + ... \\ &= x + \epsilon \left(\frac{\partial \mathbb{X}(x;\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} \right) + O(\epsilon^2). \end{aligned} \tag{1.2.9}$$

The derivatives of $\mathbb{X}(x;\epsilon)$ with respect to the group parameter ϵ evaluated at $\epsilon = 0$ are called the **infinitesimals** of the group and are denoted by $\xi(x)$, where $\xi(x) = \frac{\partial \mathbb{X}(x;\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}$.

1.2.3 Lie Series, Group Operator and Infinitesimal Invariance Conditions for Function

A function $\psi(x)$ is said to be **invariant** under the Lie group of transformations (1.2.1) if and only if

$$\psi(x^*) = \psi(x) \tag{1.2.10}$$

or equivalently, if and only if $\psi(x)$ satisfies the condition

$$\sum_{i=1}^{n} \xi^i(x) \frac{\partial \psi(x)}{\partial x^i} = 0. \tag{1.2.11}$$

The operator

$$X = X(x) = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i}. \tag{1.2.12}$$

is called the **group operator** or the **infinitesimal generator** and $X\psi$ is called the Lie derivative of ψ. Procceding to higher order derivatives, we can write $\psi(x^*)$ as a Taylor series around $\epsilon = 0$:

$$\psi(x^*) = \psi(x) + \epsilon X\psi(x) + \frac{\epsilon^2}{2} X^2\psi(x) + ... + \frac{\epsilon^n}{n!} X^n\psi(x) + O(\epsilon^{n+1}). \tag{1.2.13}$$

If the series converges, it is termed as **Lie series** and

$$\psi(x^*) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} X^n \psi(x), \tag{1.2.14}$$

which can be written as

$$\psi(x^*) = exp(\epsilon X)\psi(x). \tag{1.2.15}$$

1.2.4 Lie Algebra

For the Lie group of transformations with infinitesimal generators V_1, V_2, the **commutator (Lie bracket)** of V_1, V_2 is first order operator defined by

$$[V_1, V_2] = V_1V_2 - V_2V_1. \tag{1.2.16}$$

The commutator has the following properties:

1. **Bilinearity:**

   $$[aV_1 + bV_2, V_3] = a[V_1, V_3] + b[V_2, V_3], \tag{1.2.17}$$

   $$[V_1, aV_2 + bV_3] = a[V_1, V_2] + b[V_1, V_3], \tag{1.2.18}$$

   where $a, b \in \mathbb{R}$.

2. **Skew-Symmetry:**

   $$[V_1, V_2] = -[V_2, V_1]. \tag{1.2.19}$$

3. **Jacobi Identity:**

   $$[V_1, [V_2, V_3]] + [V_3, [V_1, V_2]] + [V_2, [V_3, V_1]] = 0. \tag{1.2.20}$$

The commutator of two vector fields again is a vector field. Moreover, if V_i and V_j are two infinitesimal generators of a symmetry transformation, the commutator of both generators will again be a generator of a symmetry group [54, 109]. As a consequence, the set of all infinitesimal generators is closed under commutation of vector fields, thus possessing more structure than just that of vector space. This additional closure property endows the space of infinitesimal generators with an additional algebraic structure, the so called **Lie algebra**.

Hence, having found some of the infinitesimal generators V_i of an r-parameter Lie group it may be possible to find new generators by computing the commutators of the known ones. A common way to visualise the structure of a Lie algebra is the commutator table [111]. Let $V_1, V_2, ..., V_r$ be a basis of r-dimensional Lie algebra, then its commutator table has (i, j)-th entry $[V_i, V_j]$. Because the commutator is antisymmetric it suffices to compute just the part above the diagonal, as $[V_i, V_j] = -[V_j, V_i]$. The commutator table therefore reads:

Table 1.1: Commutator Table

	V_1	V_2	...	V_r
V_1	0	$[V_1, V_2]$	...	$[V_1, V_r]$
V_2	$-[V_1, V_2]$	0	...	$[V_2, V_r]$
...	...	...	...	...
V_r	$-[V_1, V_r]$	$-[V_2, V_r]$	...	0

1.2.5 Classification of Subalgebras and Group Invariant Solutions

Classification of subgroups of Lie symmetry groups of differential equations is an essential part in the study of these equations. This is since classification allows for an efficient computation of group-invariant solutions, without the possibility of an occurrence of equivalent solutions. Classifying subgroups may further lead to the construction of simple ansätze for the corresponding equivalence classes of reduced differential equations. Thereby, the classification also provides an important step for further investigations of properties of these reduced equations.

The classification of subgroups of symmetry groups is usually done by the classification of the associated Lie subalgebras with respect to the adjoint representation [83, 111] and to compute the adjoint representation, we use the Lie series

$$Ad(exp(\epsilon v))w_0 = w_0 - \epsilon[v, w_0] + \frac{\epsilon^2}{2}[v, [v, w_0]] + ... \tag{1.2.21}$$

The classification of one-dimensional subalgebras of the whole symmetry algebra is done by an inductive approach [111]. Let $V_1, V_2, ..., V_r$ are basis of Lie algebra, then we start with the most general infinitesimal generator,

$$V = a_1V_1 + a_2V_2 + a_3V_3 + ... + a_rV_r, \tag{1.2.22}$$

and simplify it as much as possible by means of adjoint actions. Depending on the respective values of the coefficients $a_i, i = 1, ..., r$, we will find the list of inequivalent one-dimensional subalgebras. On using the inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra, group invariant reductions can be easily carried out which corresponds to group invariant solutions of studied equations.

1.2.6 Point Transformations and Point Symmetries

Consider a system $R(x;u)$ of N PDEs of order k with n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u(x) = (u^1(x), u^2(x), ..., u^m(x))$ given by

$$R^\sigma(x, u, \partial u, ..., \partial^k u) = 0, \quad \sigma = 1, 2, ...N. \tag{1.2.23}$$

Partial derivatives are denoted by $u^\mu_i = \partial u^\mu(x)/\partial x^i$; the notation

$$\partial u \equiv \partial^1 u = \big(u^1_1(x), ..., u^1_n(x), ..., u^m_1(x), ..., u^m_n(x)\big) \tag{1.2.24}$$

denotes the set of all first-order partial derivatives;

$$\begin{aligned} \partial^p u &= \Big\{u^\mu_{i_1...i_p} \big| \mu = 1, 2, ..., m; i_1, ..., i_p = 1, 2...n\Big\} \\ &= \Big\{\tfrac{\partial^p u^\mu(x)}{\partial x^{i_1},...\partial x^{i_p}} \big| \mu = 1, 2, ..., m; i_1, ..., i_p = 1, 2...n\Big\} \end{aligned} \tag{1.2.25}$$

denotes the set of all partial derivatives of order p.

A point transformation is a one-to-one transformation acting on the $n+m$-dimensional space $(x;u)$. In particular, a point transformation is of the form

$$\begin{aligned} x^* &= f(x, u), \\ u^* &= g(x, u). \end{aligned} \tag{1.2.26}$$

Through invariance of contact conditions, a point transformation (assuming that (1.2.26) is differentiable as needed) naturally extends to a one-to-one transformation acting on $(x, u, \partial u, ..., \partial^p u)$-space for $p = 1, 2, ...$

In particular the pth extended transformation of (1.2.26) is given by

$$\begin{aligned} &(x^*)^i = f^i(x, u), \\ &(u^*)^\mu = g^\mu(x, u), \\ &(u^*)^\mu_i = h^\mu_i(x, u, \partial u), \\ &\cdots \\ &(u^*)^\mu_{i_1...i_p} = (h)^\mu_{i_1...i_p}(x, u, \partial u, ..., \partial^p u), \end{aligned} \tag{1.2.27}$$

where $i, i_1, ..., i_p = 1, 2...n; \mu = 1, 2, ..., m; (u^*)^\mu_i = \partial(u^*)^\mu(x)/\partial(x^*)^i$, etc. In particular, the transformed components of first-order derivatives are determined by

$$\begin{bmatrix} (u^*)^\mu_1 \\ \cdots \\ \cdots \\ (u^*)^\mu_n \end{bmatrix} = \begin{bmatrix} h^\mu_1 \\ \cdots \\ \cdots \\ h^\mu_n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 g^\mu \\ \cdots \\ \cdots \\ D_n g^\mu \end{bmatrix} \tag{1.2.28}$$

where A^{-1} is the inverse of the Jacobian matrix

$$A = \begin{bmatrix} D_1 f^1 & \dots & D_1 f^n \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ D_n f^1 & \dots & D_n f^n \end{bmatrix} \tag{1.2.29}$$

in terms of total derivative operators

$$D_i = \frac{\partial}{\partial x^i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ii_1}^\mu \frac{\partial}{\partial u_{i_1}^\mu} + u_{ii_1i_2}^\mu \frac{\partial}{\partial u_{i_1i_2}^\mu} + \dots, \tag{1.2.30}$$

where $i = 1, 2, ..., n$. The transformed components of higher-order derivatives are determined by

$$\begin{bmatrix} (u^*)^\mu_{i_1 i_2 \dots i_{p-1} 1} \\ \dots \\ \dots \\ (u^*)^\mu_{i_1 i_2 \dots i_{p-1} n} \end{bmatrix} = \begin{bmatrix} h^\mu_{i_1 i_2 \dots i_{p-1} 1} \\ \dots \\ \dots \\ h^\mu_{i_1 i_2 \dots i_{p-1} n} \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 h^\mu_{i_1 i_2 \dots i_{p-1}} \\ \dots \\ \dots \\ D_n h^\mu_{i_1 i_2 \dots i_{p-1}} \end{bmatrix} \tag{1.2.31}$$

Now let us consider the situation where the point transformation (1.2.26) is a one-parameter Lie group of point transformations given by

$$\begin{aligned} (x^*)^i &= f^i(x, u; \epsilon) = x^i + \xi^i(x, u) + O(\epsilon^2), \quad i = 1, 2..., n, \\ (u^*)^\mu &= g^\mu(x, u; \epsilon) = u^\mu + \eta^\mu(x, u) + O(\epsilon^2), \quad \mu = 1, 2..., m, \end{aligned} \tag{1.2.32}$$

with the corresponding infinitesimal generator given by

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\mu(x, u)\frac{\partial}{\partial u^\mu}. \tag{1.2.33}$$

A one-parameter Lie group of point transformations (1.2.32) induces one-parameter Lie groups of point transformations acting on $(x, u, \partial u)$-space,..., $(x, u, \partial u, ..., \partial^k u)$-space, as follows:

$$\begin{aligned} &(u^*)_i^\mu = u_i^\mu + \epsilon \eta_i^{(1)\mu}(x, u, \partial u) + O(\epsilon^2), \\ &\dots \\ &\dots \\ &(u^*)^\mu_{i_1 i_2 \dots i_k} = u^\mu_{i_1 i_2 \dots i_k} + \epsilon \eta^{(k)\mu}_{i_1 i_2 \dots i_k}(x, u, \partial u, ..., \partial^k u) + O(\epsilon^2), \end{aligned} \tag{1.2.34}$$

with the extended infinitesimals given by

$$\begin{aligned} &\eta_i^{(1)\mu} = D_i \eta^\mu - (D_i \xi^j) u_j^\mu, \\ &\dots \\ &\dots \\ &\eta^{(k)\mu}_{i_1 i_2 \dots i_k} = D_{i_k} \eta^{(k-1)\mu}_{i_1 i_2 \dots i_{k-1}} - (D_{i_k} \xi^j) u^\mu_{i_1 i_2 \dots i_{k-1} j}, \end{aligned} \tag{1.2.35}$$

where $\mu = 1, 2, ..., m; i, i_j = 1, 2, ..., n$ for $j = 1, 2, ...k$ with $k = 2, 3, ...$

The kth **extended infinitesimal generator** (kth prolongation of (1.2.33)) is given by

$$\begin{aligned} X^{(k)} &= \xi^i(x,u)\tfrac{\partial}{\partial x^i} + \eta^\mu(x,u)\tfrac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}(x,u,\partial u)\tfrac{\partial}{\partial u_i^\mu} \\ &+ \ldots + \eta_{i_1 i_2 \ldots i_k}^{(k)\mu}(x,u,\partial u,...,\partial^k u)\tfrac{\partial}{\partial u^\mu_{i_1 i_2 \cdots i_k}}. \end{aligned} \tag{1.2.36}$$

Definition 1.2.1. (**Point Symmetry**) A one-parameter Lie group of point transformations (1.2.32) leaves the PDE system $R(x; u)$ (1.2.23) invariant if and only if its kth extension (1.2.36) leaves invariant the solution manifold of $R(x; u)$ in $(x, u, \partial u, ..., \partial^k u)$-space, i.e., it maps any family of solution surfaces of the PDE system (1.2.23) into another family of solution surfaces of PDE system (1.2.9). In this case, the one-parameter Lie group of point transformations (1.2.32) is called a **point symmetry** of the PDE system $R(x; u)$.

Definition 1.2.2. $u = \Theta(x)$, with components $u^\nu = \Theta^\nu(x),\ \ \nu = 1, 2..., m$, is an **invariant solution** of the PDE system $R(x; u)$ (1.2.23) resulting from the point symmetry (1.2.33) if and only if

(i) $u^\nu = \Theta^\nu(x)$ is an invariant surface of the point symmetry (1.2.33) for each $\nu = 1, 2, ..m$.

(ii) $u = \Theta(x)$ is a solution of $R(x; u)$ (1.2.23).

It follows that $u = \Theta(x)$ is an invariant solution of the PDE system $R(x; u)$ resulting from the point symmetry (1.2.33), if and only if $u = \Theta(x)$ satisfies

(i)

$$\begin{aligned} & X(u^\nu - \Theta^\nu(x)) = 0 \ when\ \ u = \Theta(x), \quad \nu = 1, 2, ..., m \\ \leftrightarrow\ & X(u^\nu - \Theta^\nu(x))\Big|_{u=\Theta(x)}, \quad \nu = 1, 2, ..., m \\ \leftrightarrow\ & \eta^\nu(x, \Theta(x)) - \xi^i(x, \Theta(x))\tfrac{\partial \Theta^\nu(x)}{\partial x^i} = 0, \quad \nu = 1, 2, ..., m \\ \leftrightarrow\ & \overline{X}u^\nu\Big|_{u=\Theta(x)} = 0, \quad \nu = 1, 2, ..., m; \end{aligned} \tag{1.2.37}$$

where $\overline{X}$ is the infinitesimal generator (1.2.33) in evolutionary form which is given by

$$\overline{X} = (\eta^\mu(x,u) - \xi^i(x,u)u_i^\mu)\frac{\partial}{\partial u^\mu} \tag{1.2.38}$$

(ii)

$$\begin{aligned} & R^\sigma(x, u, \partial u, ..., \partial^k u) = 0 \ when\ u = \Theta(x), \quad \sigma = 1, 2, ..., N \\ \leftrightarrow\ & R^\sigma(x, \Theta(x), \partial\Theta(x), ..., \partial^k\Theta(x)) = 0, \quad \sigma = 1, 2, ..., N \\ \leftrightarrow\ & R^\sigma(x, u, \partial u, ..., \partial^k u)\Big|_{u=\Theta(x)} = 0, \quad \sigma = 1, 2, ..., N. \end{aligned} \tag{1.2.39}$$

In (1.2.39) $\partial^j\Theta(x)$ denotes the components $\partial^j\Theta^\mu(x)/(\partial x^{i_1}...\partial x^{i_j})$, $\mu = 1,2,...,m$, for $i_j = 1,2,...,n$ with $j = 1,2,...,k$. The solutions of equations (1.2.37) are **invariant surfaces** of the point symmetry (1.2.33). Equations (1.2.37) and (1.2.39) define the **classical method** to obtain particular solutions of a PDE system $R(x;u)$ (1.2.23).

1.2.7 Lie's Algorithm

In summary, $u = \Theta(x)$ is a solution (invariant solution) of the PDE system $R(x;u)$ (1.2.23) obtained through the **classical method** if and only if there exists a Lie group of point transformations with infinitesimal generator X given by (1.2.33) ($\overline{X}$ given by (1.2.38)) with its kth extension given by (1.2.36)

(i)

$$X^{(k)}R^\sigma(x,u,\partial u,...,\partial^k u)\Big|_{R^\lambda(x,u,\partial u,...,\partial^k u)=0_{\lambda=1}^N} = 0, \quad \sigma = 1,2,...,N; \tag{1.2.40}$$

(ii)

$$\overline{X}u^\nu\Big|_{u=\Theta(x)} = 0, \quad \nu = 1,2,...,m; \tag{1.2.41}$$

(iii)

$$R^\sigma(x,u,\partial u,...,\partial^k u)\Big|_{u=\Theta(x)} = 0, \quad \sigma = 1,2,...,N. \tag{1.2.42}$$

Having found a point symmetry with infinitesimal generator X given by (1.2.33) through solving the linear system of determining equations (1.2.40), one can proceed in two ways to solve the systems of equations (1.2.41) and (1.2.42) to find an invariant solution $u = \Theta(x)$, as follows:

Invariant form method: Here one first solves the invariant surface conditions (1.2.41) by explicitly solving the corresponding characteristic equations for $u = \Theta(x)$ given by

$$\frac{dx^1}{\xi^1(x,u)} = \ldots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \ldots = \frac{du^m}{\eta^m(x,u)}. \tag{1.2.43}$$

If $z^1(x,u),\ldots,z^{n-1}(x,u),h^1(x,u),...,h^m(x,u)$, are $n+m-1$ functionally independent constants of integration that arise from solving the characteristic system of ODEs (1.2.43) with the Jacobian $\partial(h^1,h^2,...,h^m)/\partial(u^1,u^2,...,u^m) \neq 0$, then the general solution $u = \Theta(x)$ of the invariant surface condition equations (1.2.41) is given implicitly by the invariant form

$$h^\nu(x,u) = H^\nu(z^1(x,u),...,z^{n-1}(x,u)), \tag{1.2.44}$$

where H^ν is an arbitrary differentiable function of its arguments, $\nu = 1, 2, ..., m$. Note that $z^1(x,u), \dots, z^{n-1}(x,u), h^1(x,u), ..., h^m, (x,u)$ are $n+m-1$ functionally independent invariants of the one-parameter Lie group of point transformations with the infinitesimal generator X given by (1.2.33), and hence are $n+m-1$ canonical coordinates for the one-parameter Lie group of point transformations with the infinitesimal generator X given by (1.2.33). Let $z^n(x,u)$ be the $(n+m)$th canonical coordinate satisfying $Xz^n = 1$. If the PDE system $R(x;u)$ (1.2.23) is transformed by the corresponding invertible point transformation into a PDE system $S(z;h)$ with independent variables $z = (z^1, z^2, ..., z^n)$ and dependent variables $h = (h^1, h^2, ..., h^m)$, then the transformed PDE system $S(z;h)$ has the translation point symmetry given by

$$\begin{aligned} (z^*)^i &= z^i, \quad i = 1, 2, ..., n-1, \\ (z^*)^n &= z^n + \epsilon, \\ (h^*)^\nu &= h^\nu, \quad \nu = 1, 2, ..., m. \end{aligned} \tag{1.2.45}$$

Thus the variable z^n does not appear explicitly in the transformed PDE system $S(z;h)$ and hence the transformed PDE system has particular solutions of the form (1.2.44) that in turn define, implicitly, specific functions $u = \Theta(x)$ which are invariant solutions of the PDE system $R(x;u)$ (1.2.23), i.e., the PDE system $R(x;u)$ (1.2.23) has invariant solutions implicitly given by the invariant form (1.2.44). In particular, these invariant solutions are found by solving a reduced system of DEs with $n-1$ independent variables $z^1, z^2, ..., z^{n-1}$ and m dependent variables $h^1, h^2, ..., h^m$. The variables $z^1, z^2, ..., z^{n-1}$ are commonly called **similarity variables**. The reduced system of DEs is found by substituting the invariant form (1.2.44) into the given PDE system $R(x;u)$ (1.2.23). It is assumed that this substitution does not lead to a DE system with a singular equation. Note that if $\partial\xi/\partial u \equiv 0$, as is commonly the case, then $z^i = z^i(x),\ i = 1, 2, ..., n-1$. In the case when $R(x;u)$ (1.2.23) has two independent variables, i.e., $n = 2$, the reduced system of DEs is an ODE system with independent variable $z = z^1$.

Direct substitution method: This procedure is essential if one is unable to solve explicitly the invariant surface condition equations (1.2.41), i.e., if one is unable to obtain the general solution of the characteristic ODE system (1.2.43). Without loss of generality, one can assume that $\xi^n(x,u) \neq 0$. Then the first-order PDE system (1.2.41) can be written as

$$\frac{\partial u^\nu}{\partial x^n} = \frac{\eta^\nu(x,u)}{\xi^n(x,u)} - \sum_{i=1}^{n-1} \frac{\xi^i(x,u)}{\xi^n(x,u)} \frac{\partial u^\nu}{\partial x^i}, \quad \nu = 1, 2, ..., m. \tag{1.2.46}$$

From (1.2.46) and its differential consequences, it follows that any term involving derivatives of components of u with respect to the independent variable x^n can be expressed in terms of components of x and u as well as derivatives of components of u with respect to the independent variables $x^1, x^2, ..., x^{n-1}$. Hence, after directly substituting (1.2.46) and its differential consequences for any partial derivative with respect to x^n appearing in the given PDE system $R(x; u)$ (1.2.23), one obtains a reduced DE system directly involving the m dependent variables $u^1, u^2, ..., u^m$, the $n-1$ independent variables $x^1, x^2, ..., x^{n-1}$, the derivatives of $u^1, u^2, ..., u^m$ with respect to $x^1, x^2, ..., x^{n-1}$, and the parameter x^n. A solution $u = \Phi(x^1, ..., x^{n-1}; x^n)$ of this reduced DE system yields an invariant solution $u = \Theta(x)$ of the given PDE system $R(x; u)$ (1.2.23) provided that the invariant surface condition equations (1.2.41) or, equivalently, the given PDE system $R(x; u)$ (1.2.23) itself, are also satisfied. In the case when $R(x; u)$ (1.2.23) has two independent variables, i.e., $n = 2$, the reduced system of DEs is an ODE system. Here the constants of integration that appear in the general solution of the reduced ODE system are arbitrary functions of the parameter x^n, and these arbitrary functions are then determined by substituting the general solution into either the invariant surface condition equations (1.2.41) or the given PDE system $R(x; u)$ (1.2.23).

1.3 Nonclassical Method

The classical method to find invariant solutions can be generalized to the nonclassical method. Here one considers an augmented system of PDEs consisting of the given PDE system and an unknown constraint system and seeks symmetries that leave invariant this augmented system such that the invariant surface condition is the unknown constraint system itself. In general, such symmetries do not map solutions of the given PDE system into one-parameter families of solutions but are useful to find further specific solutions beyond those obtained by the classical method. However, in the nonclassical method, the (over-determined) system of determining equations for symmetries is nonlinear unlike the situation in the calculations for local symmetries. By construction, solutions obtained by the nonclassical method include those obtained by the classical method. The nonclassical method was introduced by Bluman [48] and Bluman and Cole [51] with the restriction that the invariant surface condition is of the form arising for point symmetries. For further discussions of the nonclassical method, the readers can refer to Levi and Winternitz [32], Olver and Rosenau [112, 113], Nucci and Clarkson [89] and Clarkson and Mansfield [108], but here we are presenting a brief outline [49] of this method.

As we discussed earlier, the nonclassical method, introduced in Bluman [48] and Bluman and Cole [51], generalizes and includes Lie's classical method for obtaining solutions of PDEs. Here one first seeks functions $\xi^i(x, u), \eta^\mu(x, u),\ i = 1, 2, ..., n,\ \mu = 1, 2, ..., m,$

so that (1.2.33) is a "symmetry" ("nonclassical symmetry") of the augmented PDE system $A(x;u)$ consisting of the given PDE system $R(x;u)$ (1.2.23), the invariant surface condition equations

$$I^{\nu}(x,u,\partial u)=\eta^{\nu}(x,u)-\xi^{i}(x,u)\frac{\partial u^{\nu}}{\partial x^{i}}=0,\ \ \nu=1,2,...,m, \tag{1.3.1}$$

and the differential consequences of (1.3.1). Consequently, one obtains an overdetermined set of nonlinear determining equations for the unknown functions $\xi^{i}(x,u),\eta^{\mu}(x,u)$, $i=1,2,...,n,\ \mu=1,2,...,m$. It is straightforward to show that, for any set of $\xi^{i}(x,u)$, $\eta^{\mu}(x,u),\ i=1,2,...,n,\ \mu=1,2,...,m$, (1.2.33) is a symmetry of the invariant surface condition equations (1.3.1), and from this it follows that the nonclassical method includes Lie's classical method. The resulting set of determining equations is nonlinear due to the substitution of the equations (1.3.1) (each written in solved form with respect to some derivative term) and their differential consequences into the symmetry determining equations (1.2.40) that now hold only for solutions of the augmented PDE system. In the nonclassical method, the invariant surface condition equations (1.3.1) are essentially a set of constraint equations of a specific form. In particular, the nonclassical method is equivalent to seeking all solutions of the PDE system (1.2.23) of the form (1.3.1) for any possible set of $\xi^{i}(x,u),\eta^{\mu}(x,u),\ i=1,2,...,n,\ \mu=1,2,...,m$. The set of determining equations satisfied by $\xi^{i}(x,u),\eta^{\mu}(x,u),\ i=1,2,...,n,\ \mu=1,2,...,m$, are the compatibility conditions for the existence of solutions of the augmented PDE system $A(x;u)$ that includes the PDE system $R(x;u)$ and the constraint equations (1.3.1).

A "nonclassical symmetry" is not a symmetry of a given PDE system $R(x;u)$ (1.2.23) unless the infinitesimals yielding an infinitesimal generator (1.2.33) yield a point symmetry of $R(x;u)$. Otherwise, a mapping resulting from such an infinitesimal generator maps no solution of $R(x;u)$ (1.2.23) into a different solution of $R(x;u)$. It just maps the solution obtained by the nonclassical method into itself. In other words, strictly speaking, the nonclassical method is not a "symmetry" method but an extension of Lie's symmetry method ("classical method") for the purpose of finding specific solutions of PDEs.

Consider the situation of a scalar PDE of order k with two independent variables (x,t) and dependent variable u, given by

$$R(x,t,u,\partial u,...,\partial^{k}u),\ \ k=1,2,...,n, \tag{1.3.2}$$

where n is order of PDEs. Let $\xi^{1}=\xi(x,t,u),\xi^{2}=\tau(x,t,u)$. Then the set of invariant surface condition equations (1.3.1) becomes the invariant surface condition equation

$$\xi(x,t,u)u_{x}+\tau(x,t,u)u_{t}=\eta(x,t,u). \tag{1.3.3}$$

From the nature of the constraint invariant surface condition equation (1.3.3), without loss of generality, in using the nonclassical method, two simplifying cases need only be considered when solving the determining equations for $(\xi(x,t), \tau(x,t), \eta(x,t,u))$, namely, $\tau \equiv 1; \tau \equiv 0, \xi \equiv 1$. This follows from the observations that if $\tau \neq 0$, then the constraint invariant surface condition equation (1.3.3) can be divided through by τ , and hence, without loss of generality, one can set $\tau \equiv 1$, so that there are really only two independent infinitesimals; similarly if $\tau \equiv 0, \xi \neq 0$, then the constraint invariant surface condition equation (1.3.3) can be divided through by ξ, and hence, without loss of generality, one can set $\xi \equiv 1$, so that here there is really only one independent infinitesimal. For a given set of infinitesimals $(\xi(x,t), \tau(x,t), \eta(x,t,u))$ that satisfies the nonlinear determining equations, one can use either the invariant form or direct substitution method to find the resulting solutions of the scalar PDE (1.3.2).

1.4 The Painlevé Test

Painlevé analysis [29] is the study of the singularity structure of differential equations. Specifically, we are concerned with how the singularities of the solutions depend on the initial conditions of the differential equation.

Definition 1.4.1. A differential equation has the Painlevé property if all the movable singularities of all its solutions are poles.

A singularity is **movable** if it depends on the constants of integration of the ODE. For instance, the Riccati equation,

$$w'(z) + w^2(z) = 0, \tag{1.4.1}$$

has the general solution $w(z) = 1/(z-c)$, where c is the constant of integration. Hence, (1.4.1) has a movable simple pole at $z = c$ because it depends on the constant of integration. The solutions of ODE can have various kinds of singularities; including branch points and essential singularities. Weiss et al. (WTC) [66, 67] have defined the Painlevé property for PDE and developed a method for testing a common particular type of movable singularity, without studying any similarity reductions. This Painlevé test has proved to be a useful criterion for the identification of completely integrable PDE. In the following section, for details of the WTC-method for testing PDEs for the Painlevé property, we are presenting some outlines from [29]:

1.4.1 Algorithm

Consider a system of M polynomial differential equations,

$$F(u(z), u'(z), u''(z), ..., u^{(m)}(z)) = 0, \tag{1.4.2}$$

where F has components $F_1, F_2, ..., F_M$, the dependent variable $u(z)$ has components $u_1(z), u_2(z), ..., u_M(z)$, the independent variable z has components $z_1, z_2, ..., z_N$, and $u^{(m_i)}(z)$ denotes the collection of mixed derivative terms of order m_i so that the order of system is $m = \sum_{i=1}^{M} m_i$. If there are any arbitrary coefficients parameterizing the system, we assume they are nonzero.

In general, a function of several complex variables cannot have an isolated singularity. For example, $f(z) = 1/z$ has an isolated singularity at the point $z = 0$, but the function of two complex variables, $w = u + iv, z = x + iy$,

$$f(w, z) = \frac{1}{z}, \tag{1.4.3}$$

has a two-dimensional manifold of singularities in the four-dimensional space of these variables, namely the points $(u, v, 0, 0)$. Therefore, we will define a pole of a function of several complex variables as a point $(a_1, a_2, ..., a_N)$, in whose neighborhood the function can be written in the form

$$f(z) = \frac{h(z)}{g(z)}, \tag{1.4.4}$$

where g and h are both analytic in a region containing $(a_1, a_2, ..., a_N)$ in its interior, and

$$g(a_1, a_2, ..., a_N) = 0, \quad h(a_1, a_2, ..., a_N) \neq 0. \tag{1.4.5}$$

Thus, the WTC-method considers the singularity structure of the solutions around manifolds of the form

$$g(z) = 0, \tag{1.4.6}$$

where $g(z)$ is an analytic function of $z = (z_1, z_2, ..., z_N)$ in a neighborhood of the manifold. Specifically, if the singularity manifold is determined by (1.4.6) and $u(z)$ is a solution of the PDE, then we assume a Laurent series solution

$$u_i(z) = g^{\alpha_i}(z) \sum_{k=0}^{\infty} u_{i,k}(z) g^k(z), \quad i = 1, 2, ..., M, \tag{1.4.7}$$

where the $u_{i,k}(z)$ are analytic functions of z with $u_{i,0}(z) \neq 0$ in a neighborhood of the manifold and α_i is an integer (with at least one $\alpha_i < 0$).

Substituting (1.4.7) into (1.4.2) and equating coefficients of like powers of $g(z)$ determines the possible values of α_i and defines a recursion relation for $u_{i,k}(z)$. The recursion relation is of the form

$$Q_k u_k = G_k(u_0, u_1, ..., u_{k-1}, g, z), \tag{1.4.8}$$

where Q_k is an MXM matrix and $u_k = (u_{1,k}, u_{2,k}, ..., u_{M,k})$.

For (1.4.2) to pass the Painlevé test, the series (1.4.7) should have $m-1$ arbitrary functions and hence corresponds to the general solution of the equation. The $m-1$ arbitrary functions $u_{i,k}(z)$ occur when k is one of the roots of $det(Q_k) = 0$. These roots $r_1 \leq r_2 \leq ... \leq r_m$ are called **resonances**. The algorithm for the Painlevé test is composed of the following three steps:

1. **Determine the dominant behavior:** It is sufficient to substitute

$$u_i(z) = \chi_i g^{\alpha_i}(z), \quad i = 1, 2, ..., M, \tag{1.4.9}$$

where χ_i is a constant, into (1.4.2) to determine the leading exponents $\alpha_i \in \mathbb{Z}$ (one of which must be a negative integer). In the resulting polynomial system, equating every two possible lowest exponents of $g(z)$ in each equation gives a linear system to determine α_i. The linear system is then solved for α_i.

If one or more exponents α_i remain undetermined, we assign integer values to the free α_i so that every equation in (1.4.2) has at least two different terms with equal lowest exponents. Once α_i is known, we substitute

$$u_i(z) = u_{i,0}(z) g^{\alpha_i}(z), \quad i = 1, 2, ..., M, \tag{1.4.10}$$

into (1.4.2). We then solve the (typically) nonlinear equation for $u_{i,0}(z)$, which is found by requiring that the leading terms balance. By leading terms, we mean those terms with the lowest exponent of $g(z)$.

If any of the α_i are non-integer, all the α_i are positive, or any of the $u_{i,0}(z) \equiv 0$, then the algorithm terminates.

2. **Determine the resonances:** For each α_i and $u_{i,0}(z)$, we calculate the integers $r_1 \leq r_2 \leq ... \leq r_m$ for which $u_{i,r_j}(z)$ is an arbitrary function in (1.4.7). To do this, we substitute

$$u_i(z) = u_{i,0}(z) g^{\alpha_i}(z) + u_{i,r}(z) g^{\alpha_i + r}(z) \tag{1.4.11}$$

into (1.4.2). Then, keeping only the terms with the lowest exponents of $g(z)$, we require that the coefficients of $u_{i,r}(z)$ equate to zero. This is done by computing the roots for r of $det(Q_r) = 0$, where the MXM matrix Q_r satisfies

$$Q_r u_r = 0, \quad u_r = (u_{1,r}, u_{2,r}, ..., u_{M,r})^T. \tag{1.4.12}$$

If any of the resonances are non-integer, then the solutions of (1.4.2) have a movable algebraic branch point and the algorithm terminates. If $r_m \notin \mathbb{Z}^+$, then the algorithm terminates; if $r_{m-s+1} = ... = r_m = 0$ and s of the $u_{i,0}(z)$ found in Step 1 are arbitrary, then (1.4.2) has the Painlevé property. If (1.4.2) is parameterized, the values for $r_1 \leq r_2 \leq ... \leq r_m$ may depend on the parameters, and hence restrict the allowable values for the coefficients.

There is always a resonance at -1 which corresponds to the arbitrariness of $g(z)$, and is often called the universal resonance. When there are negative resonances other than -1, then the series solution is not the general solution and further analysis is needed to determine if (1.4.2) passes the Painlevé test.

3. **Find the constants of integration and check compatibility conditions:** For the system to possess the Painlevé property, the arbitrariness of $u_{i,r}(z)$ must be verified up to the highest resonance level. This is done by substituting

$$u_i(z) = g^{\alpha_i}(z) \sum_{k=0}^{r_m} u_{i,k}(z) g^k(z) \tag{1.4.13}$$

into (1.4.2), where r_m is the largest positive integer resonance.

For the (1.4.2) to have the Painlevé property, the $(M+1)XM$ augmented matrix (Q_k/G_k) must have rank M when $k \neq r$ and rank $M - s$ when $k = r$, where s is the algebraic multiplicity of r in $det(Q_r) = 0$, $1 \leq k \leq r_m$, and Q_k and G_k are as defined in (1.4.8). If the augmented matrix (Q_k/G_k) is the correct rank, solve the linear system (1.4.8) for $u_{1,k}(z), ..., u_{M,k}(z)$ and use the results in the linear system at level $k+1$.

If the linear system (1.4.8) does not have a solution, then the solution of (1.4.2) has a movable logarithmic branch point and the algorithm terminates. Often, when (1.4.2) is parameterized, carefully choosing the parameters will resolve the difference in the ranks of Q_k and (Q_k/G_k). If the algorithm does not terminate, then the solutions of (1.4.2) are free of movable algebraic or logarithmic branch points and (1.4.2) has the Painlevé property.

1.5 Hyperbolic Functions Expansion Method

The tanh method or hyperbolic tangent method is a powerful facility for obtaining traveling wave solutions [3] for classes of nonlinear wave equations and nonlinear evolution equations. The tanh method was introduced many years ago and it has become a standard simple wave solution technique. In particular, this method plays an important role in solving the class of problems that exhibit dispersive effects and reaction-diffusion features. Using the tanh method, exact solutions as well as approximate solutions can be obtained for a vast class of nonlinear ordinary equations and partial differential equations. The tanh method is developed by Malfliet [155, 156] where the tanh is introduced as a new variable. This method generates solutions in the form of the various product terms of tanh and sech functions. Here, we are going to present an overview of tanh-function method for a single partial differential equation with two variables and it can be easily extended to a system with more number of differential equations. The method mainly consists following steps [151]:

1. Let us consider a partial differential equation in two independent variables,

   $$F(u, u_x, u_t, u_{xx}, ...) = 0, \tag{1.5.1}$$

   where $u(x,t)$ is function of variables x and t. Since, we are looking for traveling wave solutions, the first step is to introduce the wave transformation, $u(x,t) = u(\xi)$, where $\xi = x \pm ct$ and c is the wave velocity. It transforms the equation (1.5.1) to an ODE as folllows:

   $$F(u, u', u'', u''', ...) = 0, \tag{1.5.2}$$

   where prime ($'$) denotes $\frac{d}{d\xi}$.

2. In this step, we introduce a new independent variable Y and then the following sum is suggested as a solution of reduced ODE:

   $$u(x,t) = u(\xi) = \sum_{r=0}^{R} a_r Y^r, \tag{1.5.3}$$

   where

   $$Y = \tanh(\xi) = \tanh(x \pm ct), \tag{1.5.4}$$

and a_r are constants to be determined later. The value of R can be determined by using the homogeneous balancing method, that is, by balancing the highest order derivative terms with the highest nonlinear terms in the system (1.5.2).

3. The introduction of new independent variable brings out the following changes in derivatives:

$$\begin{array}{l} \frac{d}{d\xi} \rightarrow (1-Y^2)\frac{d}{dY}, \\ \frac{d^2}{d\xi^2} \rightarrow (1-Y^2)(-2Y\frac{d}{dY} + (1-Y^2)\frac{d^2}{dY^2}), \\ \ldots \\ \ldots \end{array} \tag{1.5.5}$$

4. Substituting (1.5.3) in reduced ODE (1.5.2) and on using (1.5.5), we will get an equation in terms of Y^r. Because the coefficients of Y^r have to be vanished, we will get set of algebraic equations comprising the nonlinear equations in a_r involving c. The solutions to the algebraic equations give us various relations among the physical parameters and the undetermined constants in the form (1.5.3). Using these constants in (1.5.3), solutions of the system (1.5.1) can be obtained.

1.6 Jacobi Elliptic Functions Method

As we know, exact solutions may describe not only the propagation of nonlinear waves but also spatially localized structures of permanent shape that may be of interest to experiments and to obtain such special exact solutions of nonlinear evolution equations, direct ansätze methods are always useful. To construct the proper ansätze, a clue may be given from Painlevé analysis, which is based on seeking solutions whose movable critical points are poles only. Thus the use of elliptic function functions in the ansätz is rather natural because they are the most general functions having such singular points and has the relations with nonlinear equations. Some more solutions such as soliton solutions and triangular periodic solutions can also be established as the limits of Jacobi doubly periodic wave solutions [14, 15, 133]. Since it is algorithmic procedure to obtain such solutions, so with the advent of computer software this method is easy to implement. Softwares Mathematica and Maple are used successfully by different authors [39, 172] to obtain variety of periodic solutions including some shock wave solutions and solitary wave solutions.

Here, the Jacobi sine function method [13] is presented only for one partial differential equation, the method can be easily extended for system of partial differential equations. Similar procedure can be followed for other Jacobi functions also. Consider a constant coefficients partial differential equation for a function $u(x,t)$ of the form

$$G(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0. \tag{1.6.1}$$

The first step is to unite the independent variables x, t into one particular variable through the new variable

$$\xi = x + ct, \;\; u(x,t) = U(\xi), \tag{1.6.2}$$

where c is the wave speed and it reduces (1.6.1) to an ODE of the following form:

$$G(U, U', U'', U''', ...) = 0. \tag{1.6.3}$$

Our main goal is to find exact or at least approximate solutions, if possible, for this ODE. For this purpose, using the Jacobi elliptic function expansion method, $U(\xi)$ can be expressed as a finite series of Jacobi elliptic function, $sn(\xi, m)$,

$$u(x,t) = U(\xi) = \sum_{i=0}^{N} a_i sn^i(\xi, m), \tag{1.6.4}$$

where $sn(\xi, m)$ is the Jacobi elliptic sine function with argument ξ and modulus m. The parameter N is determined by balancing the linear term(s) of highest order with the nonlinear one(s). And,

$$\begin{array}{l} cn^2(\xi) = 1 - sn^2(\xi), \;\; dn^2(\xi) = 1 - m^2 sn^2(\xi), \\ \frac{d}{d\xi} sn(\xi) = cn(\xi) dn(\xi), \;\; \frac{d}{d\xi} cn(\xi) = -sn(\xi) dn(\xi), \;\; \frac{d}{d\xi} dn(\xi) = -m^2 cn(\xi) sn(\xi), \end{array} \tag{1.6.5}$$

where $cn(\xi)$ and $dn(\xi)$ are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind, respectively, with the modulus $m (0 < m < 1)$. Since the highest degree of $\frac{d^pU}{d\xi^p}$ is taken as

$$\begin{array}{l} O(\frac{d^pU}{d\xi^p}) = N + p, \;\; p = 1, 2, 3, ... \\ O(U^q \frac{d^pU}{d\xi^p}) = (q+1)N + p, \;\; q = 0, 1, 2, ... \;\; p = 1, 2, 3, ... \end{array} \tag{1.6.6}$$

Normally N is a positive integer, so that an analytic solution in closed form may be obtained. Substituting Eqs. (1.6.4)-(1.6..7) into Eq. (1.6.2) and comparing the coefficients of each power of $sn(\xi)$ in both sides, to get an over-determined system of nonlinear algebraic equations with respect to c, a_i. Solving the over-determined system of nonlinear algebraic equations by use of Maple, if there is a real nontrivial solution of these equations, then we will get a solution of the form (1.6.3) for the differential equation. We can also get other kinds of Jacobi doubly periodic wave solutions as when $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions,

$$sn(\xi) \to \tanh(\xi),\;\; cn(\xi) \to \operatorname{sech}(\xi),\;\; dn(\xi) \to \operatorname{sech}(\xi). \tag{1.6.7}$$

and, when $m \to 0$, the Jacobi functions degenerate to the triangular functions,

$$sn(\xi) \to \sin(\xi),\;\; cn(\xi) \to \cos(\xi),\;\; dn(\xi) \to 1. \tag{1.6.8}$$

1.7 The (G'/G)-Expansion Method

In the recent decade, several direct methods for finding the explicit traveling wave solutions to nonlinear evolution equations (NLEEs) have been proposed that have been discussed in earlier sections. Using these methods, many exact solutions including the solitary wave solutions, shock wave solutions and periodic wave solutions are obtained. More recently, a new method which is called the (G'/G)-expansion method [96] has been proposed to construct more explicit traveling wave solutions to many NLEEs. This method can also be applied to some NLEEs with variable coefficients [142]. The key ideas of this method are that the traveling wave solutions of NLEEs can be constructed by means of various solutions of a second order linear ordinary differential equation and the traveling wave solutions are expressed in terms of hyperbolic, trigonometric and rational functions. In the further sections, we are going to describe extended (G'/G)-expansion method [127] and modified (G'/G)-expansion method [160].

1.7.1 Description of an Extended (G'/G)-Expansion Method

Consider the nonlinear partial differential equation in the following form:

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{1.7.1}$$

where $u = u(x, t)$ is unknown functions and F is a polynomial in $u(x, t)$ and its partial derivatives. In the following, we give the main steps [127] for solving (1.7.1) using an extended (G'/G)-expansion method.

1. The traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = x - Vt, \tag{1.7.2}$$

where V is a constant to be determined later, permits us reducing (1.7.1) to an ODE in the form

$$P(u, -Vu', u', V^2u'', -Vu'', u'', ...) = 0, \tag{1.7.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives.

2. Suppose the solution of (1.7.3) can be expressed in (G'/G) as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{n} \left\{ a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{i-1} \sqrt{\nu \left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)} \right\}, \tag{1.7.4}$$

where $G = G(\zeta)$ satisfies the following second-order linear ODE:

$$G''(\zeta) + \mu G(\zeta) = 0, \tag{1.7.5}$$

while $a_i, b_i (i = 1, 2, ..., n)$ and a_0 are constants to be determined, such that $\nu = \pm 1$ and $\mu \neq 0$. The positive integer n can be determined by balancing the highest-order derivatives with the nonlinear terms appearing in (1.7.3).

3. Substituting (1.7.4) into (1.7.3) and using (1.7.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)^k$ and $\left(\frac{G'}{G}\right)^k \sqrt{\nu\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$ together, and equating each coefficient of them to zero, yield a set of following algebraic equations for a_0, a_1, b_1 and V.

4. Since the general solution of (1.7.5) has been well known for us, then substituting a_i, b_i, V and the general solution of (1.7.5) into (1.7.4), we have the traveling wave solutions of the nonlinear partial differential equation (1.7.1).

1.7.2 Description of Modified (G'/G)-Expansion Method

Suppose that a nonlinear partial differential equation is given by

$$F(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0, \tag{1.7.6}$$

where $u = u(x, t)$ is an unknown function and F is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the modified (G'/G)-expansion method [160]:

1. The traveling wave variable

$$u(x,t) = u(\xi), \quad \xi = k(x - ct), \tag{1.7.7}$$

where k and c are constants, permits us to reduce Eq. (1.7.6) into the following ODE:

$$P(u, u', u'', u''', ...) = 0. \tag{1.7.8}$$

2. Suppose that the solution of Eq. (1.7.8) can be expressed by a polynomial (G'/G) as follows:

$$u(\xi) = \alpha_0 + \sum_{i=1}^{m} \left[\alpha_i \left(\frac{G'}{G} \right)^{i} + \alpha_{-i} \left(\frac{G'}{G} \right)^{-i} \right], \tag{1.7.9}$$

where $G = G(\xi)$ satisfies the second order linear ODE

$$G'' + \mu G = 0, \tag{1.7.10}$$

where $\alpha_0, \alpha_i, \alpha_{-i}, \mu$ are constants to be determined.

3. The parameter m in (1.7.9) can be found by balancing the highest order derivative term and the highest nonlinear term in (1.7.6) or (1.7.8) and conclude the following:

(a) If m is a positive integer then go to step 4,

(b) If m is not positive integer, we put $u = v^m$ and then return to step 1.

4. Substituting (1.7.9) into (1.7.8) and using (1.7.10), collecting all terms with the same powers of (G'/G) together and then equating each coefficient of the resulted polynomial to zero, yield a system of algebraic equations for $\alpha_0, \alpha_i, \alpha_{-i}, c, \mu$.

5. Since the general solutions of (1.7.10) are well known to us, then substituting $\alpha_0, \alpha_i, \alpha_{-i}, \mu$ and the general solutions of (1.7.10) into (1.7.9) we have the traveling wave solutions of Eq. (1.7.6).

1.8 Problems To Be Considered

Keeping in view the rich treasure and wide applicability of nonlinear equations in almost every field, we have in this thesis carried out the application of Lie group analysis for obtaining exact solutions to nonlinear partial differential equation and their systems also.

In short, this thesis is devoted to a wide range of applications of continuous symmetry groups to following physically important systems of differential equations:

1. The (2+1)-dimensional Calogero Degasperis equation with its variable coefficients form
2. The coupled Klein-Gordon-Schrödinger equation with its generalized form
3. The (2+1)-dimensional potential Kadomstev Petviashvili equation and its generalized form
4. The variable coefficient Dullin-Gottwald-Holm equation
5. The generalized Bretherton equation with variable coefficients

Chapter 2 deals with the study of following Calogero-Degasperis (CD) equation which is (2+1)-dimensional nonlinear PDE of fourth order:

$$\psi_{xt} - 4\psi_x\psi_{xy} - 2\psi_y\psi_{xx} + \psi_{xxxy} = 0. \tag{1.8.1}$$

On carrying over the Lie group method to CD equation, we have derived the groups of transformations admitted by the equation under consideration. Consequently, by using the symmetries involving arbitrary functions, the equation has been reduced to (1+1)-dimensional PDE with variable coefficients which is again studied for its symmetries, reductions and invariant solutions. The solutions obtained by us are such that they involve certain arbitrary functions such as $Q(t), R(t)$ and some other parameters also. To understand more about physical phenomena of CD equation, we consider the variable coefficient CD (VCCD) equation

$$\psi_{xt} + \alpha(t)\psi_x\psi_{xy} + \beta(t)\psi_y\psi_{xx} + \gamma(t)\psi_{xxxy} = 0, \tag{1.8.2}$$

where $\alpha(t), \beta(t), \gamma(t)$ are arbitrary functions. To study the Painlevé properties, we have performed the Painlevé analysis of VCCD equation and applied the Lie-group formalism to deduce the symmetries and to reduce the (2+1)-dimensional VCCD equation to lower dimensional equations which are again investigated by different methods such as extended (G'/G)-expansion method, Hyperbolic rational function expansion method, Jacobi elliptic function method to obtain certain new exact solutions.

Chapter 3 is devoted to the study of following coupled Klein-Gordon-Schrödinger (KGS) equation:

$$\begin{aligned} &u_{tt} - c^2u_{xx} + u + |v|^2 = 0, \\ &iv_t + v_{xx} + uv = 0, \end{aligned} \tag{1.8.3}$$

where $v(x,t)$ is a complex function, $u(x,t)$ is a real one and $i^2 = -1$. The coupled KGS equation is reduced to a nonlinear ODE by using Lie classical symmetries and various solutions of nonlinear ODE are obtained by the modified (G'/G)-expansion method proposed recently. With the aid of solutions of nonlinear ODE, more explicit traveling wave solutions of the coupled KGS equation are found out and the traveling wave solutions are expressed by the hyperbolic functions, trigonometric functions and rational functions. We have also investigated the symmetries of variable coefficients Klein-Gordon-Schrödinger (VCKGS) equation that will be utilized to reduce studied equation to various system of ODEs and then construct their solutions.

In Chapter 4, we study the following (2+1)-dimensional potential Kadomstev Petviashvili (PKP) equation for invariance under continuous group of transformations via Lie classical approach:

$$\sigma_{xt} + \frac{3}{2}\sigma_x\sigma_{xx} + \frac{1}{4}\sigma_{xxxx} + \frac{3}{4}\sigma_{yy} = 0. \tag{1.8.4}$$

For the PKP equation, we get infinite-dimensional symmetries and using the subalgebras of Lie algebras, it is shown that there are three group theoretic reductions of this equation depending on certain choices of arbitrary functions of time occuring in the symmetries. The reduced PDEs are again investigated by Lie group method to obtain ODEs and solutions of these reduced ODEs have provided us with new solutions of PKP equations such as periodic and kinky periodic solutions involving upto three arbitrary functions $f(t), g(t), h(t)$. The solutions obtained by us are new and more general, more precisely, solutions in literature can be recovered from our general solutions. In recent years, much attention has also been paid to equations with variable coefficients as the physical situations in which nonlinear systems arise tend to be highly idealized due to assumption of constant coefficients. This has led us to undertake the study of following PKP equation with variable coefficients (VCPKP) and to derive the admissible forms of the coefficients along with their exact solutions:

$$\sigma_{xt} + \alpha(t)\sigma_x\sigma_{xx} + \beta(t)\sigma_{xxxx} + \delta(t)\sigma_{yy} = 0, \tag{1.8.5}$$

where $\alpha(t), \beta(t)$ and $\delta(t)$ are arbitrary functions. We have performed the Painlevé analysis to check the integrability of Eq. (1.8.2) and obtained certain conditions on $\alpha(t), \beta(t), \delta(t)$ to pass the Painlevé test. The efforts are then concentrated on finding the symmetries, reductions and exact solutions of VCPKP equation by using various methods including extended (G'/G)-expansion method and others.

In Chapter 5, we study the classical and nonclassical symmetries of Generalized Dullin-Gottwald-Holm (GDGH) equation

$$u_t - \alpha^2 u_{xxt} + 2wu_x + f(u)u_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \tag{1.8.6}$$

for various choices of $f(u)$ and obtained certain new solutions of GDGH equation.

Chapter 6 is devoted to the perturbed nonlinear Klein-Gordon equation with time-dependent coefficients which is the so-called generalized Bretherton equation with variable coefficients (VCGBE)

$$u_{tt} + \alpha(t)u_{xx} + \beta(t)u_{xxxx} + \delta(t)u^m + \theta(t)u^n = 0, \tag{1.8.7}$$

where $u = u(x,t)$ is unknown function to be determined, $\alpha(t), \beta(t), \delta(t), \theta(t)$ are arbitrary time functions and the exponent m is a natural number and for the exponent $n \in \mathbf{N}$ we assume that $n \neq 1$ and $n \neq m$. Particular cases corresponding to certain specific values of the parameters involved and those spatial forms for which the equation can be reduced to ODEs are presented. The sech-ansätze method and other methods are successfully used to derive solutions for VCGBE and we have also plotted some figures to see the propagation and asymptotic characteristics of the solitary waves.

Chapter 2

The (2+1)-Dimensional Calogero Degasperis Equation with its Variable Coefficients Form[1]

2.1 Introduction

The (2+1)-dimensional CD or breaking soliton equation in the form

$$\psi_{xt} - 4\psi_x\psi_{xy} - 2\psi_y\psi_{xx} + \psi_{xxxy} = 0. \tag{2.1.1}$$

was first eastablished by Calogero and Degasperis [43, 44] and is used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis.

The mathematical interest of this equation stems from the fact that it is, in a well defined sense, the generic member of a class of integrable partial differential equations [143], associated with certain infinite-dimensional Lie algebras and groups [152]. Finding special solutions and investigating the corresponding properties of solutions are very important in both practice and theory for understanding these problems. The Hamiltonian structure and the Lax pair of equation (2.1.1) have been given by Li [165]. Multi-soliton solutions and algebra-geometric solutions were also found in [159]. In [157], Ma et al. present a general class of Riemann theta function solution to two similar breaking soliton equations. In [174], using computerized symbolic computation, new families of soliton-like solutions are obtained for (2+1)-dimensional breaking soliton equations using an

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[1] A part of this chapter has been published in **International Journal of Nonlinear Science, 13 (2012) 475-481** and some part has been published in **Physica Scripta, 86 (2012) 035005 (11 pages)**.

ansätz and these solutions contain traveling wave solutions that are of important significance in explaining some physical phenomena. With the help of symbolic computation, sixteen kinds of new special exact soliton-like solutions of (2+1)-dimensional breaking soliton equation are obtained by further generalized projective Riccati equation method in [173]. Zhang and Meng [69] derived a general variable separation solution of the (2+1)-dimensional breaking soliton system and two classes of novel localized coherent structures like both multipeakon-antipeakon solution and multi compacton-anticompacton solutions are found by selecting appropriate functions.

In the study of the (2+1)-dimensional nonlinear physical models, much effort has been focused on the single valued localized excitations, such as solitoffs, dromions, rings, lumps, breathers, instantons, peakons, compactons, localized chaotic and fractal patterns and in this direction Zhang et al. [70] obtained folded solitary waves and foldons in the (2+1)-dimensional breaking soliton equation. Sheng Zhang [141] obtained many new and more general exact non-traveling wave and coefficient function solutions including soliton-like solutions, trigonometric function solutions, exponential solutions and rational solutions using a generalized auxiliary equation method. Radha and Lakshmanan [123] studied the existence of Dromion like structures in the (2+1)-dimensional breaking soliton equation. Quan [158] used the new idea of a combination of Lie group method and homoclinic test technique to seek non-traveling wave solutions.

As we know, the knowledge of point or higher-order symmetries of a differential equation becomes a basis for finding the corresponding solutions known as invariant solutions or exact solutions. Most differential equations (or systems of differential equations) which arise in real life applications contain arbitrary functions of dependent variable or its derivatives and independent variables. These arbitrary functions also called arbitrary elements or parameters can be obtained from experiments or known natural laws. However, in some problems of interest or practical applications they may not be deduced from experiments or physical laws. In such cases we use the method of group classification to specify their forms and also, in multifarious real physical backgrounds, nonlinear partial differential equations with variable coefficients often provide more powerful and realistic models than their constant coefficient counterparts when the inhomogeneities of media is considered. So it is of great importance to find exact solutions of NLPDEs with variable coefficients and recently, many authors have researched in this direction [72, 73, 122]. In this chapter, we also study the following variable coefficient CD equation (VCCD equation):

$$\psi_{xt} + \alpha(t)\psi_x\psi_{xy} + \beta(t)\psi_y\psi_{xx} + \gamma(t)\psi_{xxxy} = 0, \tag{2.1.2}$$

which is an important mathematical model in nonlinear physics and we have studied this for integrability and the exact solutions using combination of Lie classical method and several other methods including extended (G'/G)-expansion method. Variable-coefficient NLEEs are not completely integrable unless the variable coefficients satisfy some specific constraint conditions. Thereby, we will find the conditions for the equation (2.1.2) to pass the Painlevé test firstly, then the symmetries and exact solutions are considered.

In the following sections, with the aid of Lie group theory we reduce the CD equation to partial differential equation in two variables and the reduced PDE is again operated by Lie classical method to get the ordinary differential equations and further solutions of ODEs yields another family of solutions involving upto two arbitrary functions of variable t. In section (2.3), we perform the Painlevé analysis for VCCD equation (2.1.2). We use Lie classical method to generate various symmetries of the VCCD equation and the admissible forms of the coefficients that admit the classical symmetry group corresponding to each member of the optimal system of subalgebras that helps us to reduce VCCD equation to (1+1)-dimensional PDEs and then to ODEs. We also investigate reduced ODEs for their exact solutions.

2.2 The Calogero Degasperis Equation

In this section, we will study the exact solutions of the CD equation with the aid of symmetry group. Quan [158] and Tian [147] have studied the equation (2.1.1) and obtained solutions by assuming particular values for arbitrary constants and functions in symmetries. But, here we are discussing the reductions and solutions of CD equation for the general case.

2.2.1 Lie Symmetries, Classification of One-dimensional Subalgebras and Group Invariant Solution

In order to apply the classical method to the CD equation (2.1.1) with three independent variables and one field, we consider the one parameter Lie group of infinitesimal transformations in x, y, t, ψ given by:

$$\begin{aligned}
\psi^{'} &= \psi + \epsilon\eta(x,y,t,\psi) + O(\epsilon^2) \\
x^{'} &= x + \epsilon\xi(x,y,t,\psi) + O(\epsilon^2) \\
y^{'} &= y + \epsilon\phi(x,y,t,\psi) + O(\epsilon^2) \\
t^{'} &= t + \epsilon\tau(x,y,t,\psi) + O(\epsilon^2),
\end{aligned} \tag{2.2.1}$$

where ϵ is the group parameter. It is therefore, necessary that this one transformation leaves the set of solutions of (2.1.1) invariant. This yields an overdetermined linear system of equations for the infinitesimals $\xi(x,y,t,\psi), \phi(x,y,t,\psi), \tau(x,y,t,\psi), \eta(x,y,t,\psi)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$X = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial \psi}. \tag{2.2.2}$$

By applying the classical method to (2.1.1), we obtain the following system of determining equations:

$$\begin{aligned}
&(i)\ \ \xi_{xx} = 0,\ \ \xi_y = 0,\ \ \xi_\psi = 0,\\
&(ii)\ \ \tau_x = 0,\ \ \tau_y = 0,\ \ \tau_\psi = 0,\\
&(iii)\ \ \phi_x = 0,\ \ \phi_\psi = 0,\\
&(iv)\ \ \eta_{xx} = 0,\ \ \eta_{x\psi} = 0,\ \ \eta_{xt} = 0,\ \ \eta_{y\psi} = 0,\ \ \eta_{\psi\psi} = 0, (v)\ \ -\eta_\psi + \xi_x - \tau_t + \phi_y = 0,\\
&(vi)\ \ 4\eta_x + \phi_t = 0,\\
&(vii)\ \ \xi_t + 2\eta_y = 0,\\
&(viii)\ \ \eta_{t\psi} - 4\eta_{xy} - \xi_{xt} = 0,\\
&(ix)\ \ 2\xi_x - \tau_t + \phi_y = 0.
\end{aligned} \tag{2.2.3}$$

whose solution is:

$$\begin{aligned}
&\eta = \tfrac{-pxy}{2} - cx - pt\psi - 2q\psi - \tfrac{Q'(t)y}{2} + R(t)\\
&\xi = ptx + 2qx + Q(t)\\
&\phi = 2pyt + 4ct\\
&\tau = 2pt^2 + 4qt.
\end{aligned} \tag{2.2.4}$$

Note that the corresponding Lie symmetry algebra depends on two functions $Q(t), R(t)$ of t. Having determined the infinitesimals of (2.1.1), then the symmetry variables are found by solving the invariant surface conditions

$$\Phi \equiv \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} - \eta = 0. \tag{2.2.5}$$

or the corresponding characterstic equations

$$\frac{d\psi}{\frac{-pxy}{2} - cx - pt\psi - 2q\psi - \frac{Q'(t)y}{2} + R(t)} = \frac{dx}{ptx + 2qx + Q(t)} = \frac{dy}{2pyt + 4ct} = \frac{dt}{2pt^2 + 4qt}. \tag{2.2.6}$$

Integration of (2.2.6) provides the reduced variables:

$$\begin{aligned}
\rho &= x\frac{e^{\int \frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}} - \left(\int \frac{Q(t)}{(2pt^2+4qt)^{\frac{5}{4}}}\left(e^{\int \frac{-q}{2pt^2+4qt}dt}\right)dt\right),\\
\sigma &= y\frac{e^{\int \frac{2q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{2}}} - \left(\int \frac{4ct}{(2pt^2+4qt)^{\frac{3}{2}}}\left(e^{\int \frac{2q}{2pt^2+4qt}dt}\right)dt\right).
\end{aligned} \tag{2.2.7}$$

and the following reduction for the field:

$$\psi(x,y,t) = \chi(x,y,t) + \frac{e^{\int \frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}} H(\rho,\sigma), \tag{2.2.8}$$

where

$$\chi(x,y,t) = \frac{-pxyt}{2(2pt^2+4qt)} - \frac{cxt}{(2pt^2+4qt)} - \frac{yQ(t)}{2(2pt^2+4qt)} - \frac{qye^{\int \frac{q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{3}{4}}}(\int \frac{Q(t)}{(2pt^2+4qt)^{\frac{5}{4}}}(e^{\int \frac{-q}{2pt^2+4qt}dt})dt)$$
$$+\frac{e^{\int \frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}}(\int \frac{R(t)}{(2pt^2+4qt)^{\frac{3}{4}}}(e^{\int \frac{q}{2pt^2+4qt}dt})dt). \tag{2.2.9}$$

Substitution of the reduction ansätz (2.2.7)-(2.2.9) gives us:

$$H_{\rho\rho\rho\sigma} - 2H_\sigma H_{\rho\rho} - 4H_\rho H_{\rho\sigma} - 2q\rho H_{\rho\rho} - 4qH_\rho = 0. \tag{2.2.10}$$

Proceeding in the similar manner as earlier, we get the symmetry algebra admitted by (2.2.10) is

$$\begin{aligned} V_1 &= \rho\frac{\partial}{\partial\rho} - 2\sigma\frac{\partial}{\partial\sigma} - H\frac{\partial}{\partial H} \\ V_2 &= \frac{\partial}{\partial\rho} - q\sigma\frac{\partial}{\partial H} \\ V_3 &= \frac{\partial}{\partial\sigma} \\ V_4 &= \frac{\partial}{\partial H}. \end{aligned} \tag{2.2.11}$$

We now present the optimal system of one-dimensional subalgebras for the equation (2.2.10). The method used here for obtaining the one-dimensional optimal system of subalgebras is that given in [83, 111]. This approach, in essence, is taking a general element from the Lie algebra and reducing it to its simplest equivalent form by applying carefully choosen adjoint transformations. The commutator table of the Lie symmetries of equation (2.2.10) and the adjoint representation of the symmetry group of (2.2.10) on its Lie algebra are given in Tables (2.1) and (2.2). For brevity, we only consider the construction of the optimal system of one-dimensional subalgebras for the Eq. (2.2.10) in detail and they will be listed in Table (2.3) as they can be derived in like manner.

TABLE 2.1: Commutator Table

	V_1	V_2	V_3	V_4
V_1	0	$-V_2$	$2V_3$	V_4
V_2	V_2	0	qV_4	0
V_3	$-2V_3$	$-qV_4$	0	0
V_4	$-V_4$	0	0	0

The optimal system for (2.2.11) consists of following vector fields:

$$(i)\ V_1,\ (ii)\ V_2,\ (iii)\ V_2+V_3,\ (iv)\ V_3,\ (v)\ V_4. \tag{2.2.12}$$

Table 2.2: Adjoint Table

	V_1	V_2	V_3	V_4
V_1	V_1	$V_2 e^{\epsilon}$	$V_3 e^{-2\epsilon}$	$V_4 e^{-\epsilon}$
V_2	$V_1 - \epsilon V_2$	V_2	$V_3 - \epsilon q V_4$	V_4
V_3	$V_1 + 2\epsilon V_3$	$V_2 + q\epsilon V_4$	V_3	V_4
V_4	$V_1 + \epsilon V_4$	V_2	V_3	V_4

Because corresponding to vector field V_2 and V_4, PDE (2.2.10) is identically satisfied, that's why we confine ourselves to remaining vector fields.

Table 2.3: Similarity Reductions of PDE (2.2.10) to ODEs

Essential fields	*Similarity variable*(ζ)	*Similarity solution*(H)	*Reduced ODEs*
V_1	$\rho\sigma^{\frac{1}{2}}$	$\sigma^{\frac{1}{2}} F(\zeta)$	$\zeta F''''(\zeta) + 4F'''(\zeta) - 2F(\zeta)F''(\zeta) - 6\zeta F'(\zeta)F''(\zeta)$ $-8(F'(\zeta))^2 - 4q\zeta F''(\zeta) - 8qF'(\zeta) = 0$
$V_2 + V_3$	$\rho - \sigma$	$\frac{-q\sigma^2}{2} + F(\zeta)$	$F''''(\zeta) + 2q\zeta F''(\zeta) - 6F'(\zeta)F''(\zeta) + 4qF'(\zeta) = 0$
V_3	ρ	$F(\zeta)$	$\zeta F''(\zeta) + 2F'(\zeta) = 0$

1. **Vector field** V_1

The reduced ODE corresponding to vector field V_1 (as shown in Table 2.3) has the following solutions:

$$\begin{aligned} &(i)\ \ F(\zeta) = C_0 \\ &(ii)\ \ F(\zeta) = C_0 - q\zeta \\ &(iii)\ \ F(\zeta) = -2\sqrt{q}\coth(C_0 + \sqrt{q}\zeta) \\ &(iv)\ \ F(\zeta) = -2\sqrt{q}\tanh(C_0 + \sqrt{q}\zeta), \end{aligned} \tag{2.2.13}$$

where C_0 is arbitrary constant. The solution (2.2.13) leads by back substitution to the solution of equation (2.1.1) of the form

$$(i)\ \ \psi(x,y,t) = \chi(x,y,t) + \frac{C_0 e^{\int \frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}}\Big(y\frac{e^{\int \frac{2q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{2}}} - \Big(\int \frac{4ct}{(2pt^2+4qt)^{\frac{3}{2}}}\Big(e^{\int \frac{2q}{2pt^2+4qt}dt}\Big)dt\Big)\Big)^{\frac{1}{2}}$$

$$
\begin{aligned}
&(ii)\ \ \psi(x,y,t)=\chi(x,y,t)+\frac{e^{\int\frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}}(y\frac{e^{\int\frac{2q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{2}}}-(\int\frac{4ct}{(2pt^2+4qt)^{\frac{3}{2}}}(e^{\int\frac{2q}{2pt^2+4qt}dt})dt))^{\frac{1}{2}}\\
&(C_0-q((x\frac{e^{\int\frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}}-(\int\frac{Q(t)}{(2pt^2+4qt)^{\frac{5}{4}}}(e^{\int\frac{-q}{2pt^2+4qt}dt})dt))(y\frac{e^{\int\frac{2q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{2}}}\\
&-(\int\frac{4ct}{(2pt^2+4qt)^{\frac{3}{2}}}(e^{\int\frac{2q}{2pt^2+4qt}dt})dt))^{\frac{1}{2}}))\\
&(iii)\ \ \psi(x,y,t)=\chi(x,y,t)+\frac{e^{\int\frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}}(y\frac{e^{\int\frac{2q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{2}}}-(\int\frac{4ct}{(2pt^2+4qt)^{\frac{3}{2}}}(e^{\int\frac{2q}{2pt^2+4qt}dt})dt))^{\frac{1}{2}}\\
&(-2\sqrt{q}\coth(C_0+\sqrt{q}((x\frac{e^{\int\frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}}-(\int\frac{Q(t)}{(2pt^2+4qt)^{\frac{5}{4}}}(e^{\int\frac{-q}{2pt^2+4qt}dt})dt))(y\frac{e^{\int\frac{2q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{2}}}\\
&-(\int\frac{4ct}{(2pt^2+4qt)^{\frac{3}{2}}}(e^{\int\frac{2q}{2pt^2+4qt}dt})dt))^{\frac{1}{2}})))\\
&(iv)\ \ \psi(x,y,t)=\chi(x,y,t)+\frac{e^{\int\frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}}(y\frac{e^{\int\frac{2q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{2}}}-(\int\frac{4ct}{(2pt^2+4qt)^{\frac{3}{2}}}(e^{\int\frac{2q}{2pt^2+4qt}dt})dt))^{\frac{1}{2}}\\
&(-2\sqrt{q}\tanh(C_0+\sqrt{q}((x\frac{e^{\int\frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}}-(\int\frac{Q(t)}{(2pt^2+4qt)^{\frac{5}{4}}}(e^{\int\frac{-q}{2pt^2+4qt}dt})dt))(y\frac{e^{\int\frac{2q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{2}}}\\
&-(\int\frac{4ct}{(2pt^2+4qt)^{\frac{3}{2}}}(e^{\int\frac{2q}{2pt^2+4qt}dt})dt))^{\frac{1}{2}}))),
\end{aligned}
\tag{2.2.14}
$$

where $\chi(x,y,t)$ is given by equation (2.2.9). By the analysis of solutions obtained corresponding to vector field V_1, we conclude that solutions (2.2.14) depend upon four constants p,q,c,C_0 and two arbitrary functions $Q(t),R(t)$ of time. Depending upon these constants and functions of time, we obtain certain periodic and bell profile solutions. For $p=q=1, c=C_0=0$ and $Q(t)=\frac{(2t^2+4t)^{\frac{5}{4}}}{e^{(\frac{1}{4}\log(t+2)-\frac{1}{4}\log(t))}}, R(t)=0$, with $y=\sin(x)$ solution (2.2.14)(iv) takes the form of periodic solutions as shown in Fig. (2.1). By taking the same considerations for constants and arbitrary functions, for $y=\cos(x)$ solution (2.2.14)(iii) takes the form of bell profile solutions and with $Q(t)=0, y=x\sin(x)$ solution (2.2.14)(iv) behaves as kink wave as shown in Fig. (2.2) and Fig. (2.3) respectively.

2. **Vector field** V_2+V_3

For $q\neq 0$, we are able to obtain trivial solutions only for the reduced ODE corresponding to vector field V_2+V_3. For $q=0$, integrating equation once, multiplying integrated equation with $F''(\zeta)$, again integrating and on substituting $F'(\zeta)=J(\zeta)$, we get

$$J'(\zeta)^2-2J(\zeta)^3+2K_1J(\zeta)+2K_2=0, \tag{2.2.15}$$

where K_1, K_2 are arbitrary constants. Solving equation (2.2.15), we get

$$J\left(\zeta\right)=\wp\left(1/2\,2^{2/3}\zeta+C_1, 2\,K_1\,\sqrt[3]{2}, 2\,K_2\right)\sqrt[3]{2}, \tag{2.2.16}$$

where C_1 is arbitrary constant and $\wp$ denotes the WeirstrassP function.

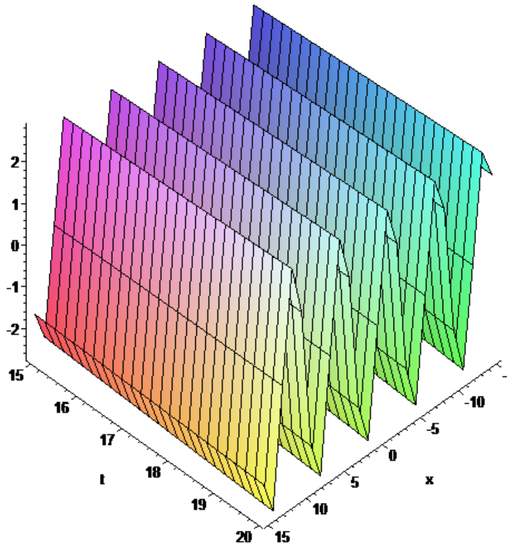

FIGURE 2.1: The periodic solution (2.2.14)(iv) for $p = q = 1, c = C_0 = 0$ and $Q(t) = \frac{(2t^2+4t)^{\frac{5}{4}}}{e^{(\frac{1}{4}\log(t+2)-\frac{1}{4}\log(t))}}, R(t) = 0,\ y = \sin(x)$

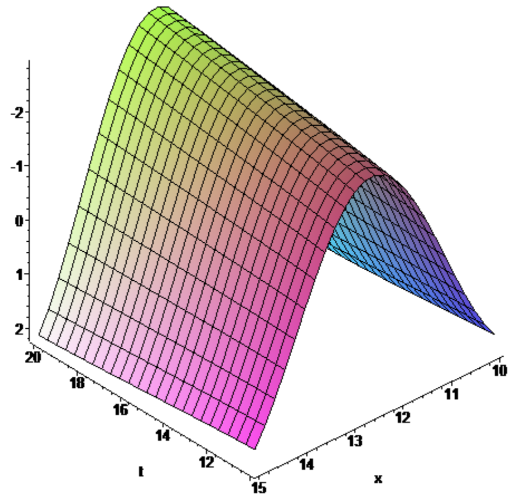

FIGURE 2.2: The bell profile solution (2.2.14)(iii) for $p = q = 1, c = C_0 = 0$ and $Q(t) = \frac{(2t^2+4t)^{\frac{5}{4}}}{e^{(\frac{1}{4}\log(t+2)-\frac{1}{4}\log(t))}}, R(t) = 0,\ y = \cos(x)$

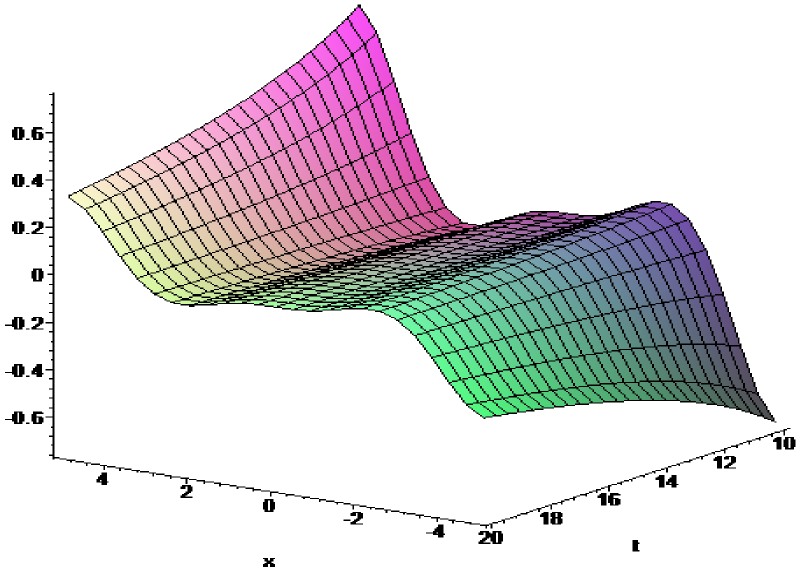

FIGURE 2.3: The kink wave solution (2.2.14)(iv) for $p = q = 1, c = C_0 = 0$ and $Q(t) = 0, R(t) = 0,\ y = x\sin(x)$

3. **Vector field** V_3

The reduced ODE has the following solution:

$$F(\zeta) = C_1 + \frac{C_2}{\zeta}, \tag{2.2.17}$$

where C_1, C_2 are arbitrary constants and solution (2.2.17) corresponds to the following solution of Eq. (2.1.1):

$$\psi(x,y,t) = \chi(x,y,t) + \frac{e^{\int \frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}} \left(C_1 + \frac{C_2}{x\frac{e^{\int \frac{-q}{2pt^2+4qt}dt}}{(2pt^2+4qt)^{\frac{1}{4}}} - (\int \frac{Q(t)}{(2pt^2+4qt)^{\frac{5}{4}}}(e^{\int \frac{-q}{2pt^2+4qt}dt})dt)} \right), \tag{2.2.18}$$

where $\chi(x,y,t)$ is given by (2.2.8).

2.3 The Calogero Degasperis Equation with Variable Coefficients

In this section, we work for integrability, symmetries and exact solutions of VCCD equation.

2.3.1 Painlevé Analysis for VCCD Equation

As we have already discussed the Painlevé analysis method in chapter 1 (section (1.4)) to check the integrability of PDEs, in this section we are applying that method to study the Painlevé properties of VCCD equation. By applying the Painlevé test, Toda and Kobayashi [74, 144, 145] extended integrable equations with variable coeffcients to higher-dimensions and also checked the integrability of generalized KdV-family with variable coefficients in (2+1)-dimensions. Now, we are looking for a solution of equation (2.1.2) in the Laurent series expansion with $g = g(x,y,t)$:

$$\psi(x,y,t) = \psi_0 g^{\alpha} + \sum_{r=1}^{\infty} \psi_r g^{r-1}, \tag{2.3.1}$$

where $\psi_r = \psi_r(x,y,t)$ and $g = g(x,y,t)$ are analytic functions in a neighborhood of $g = 0$. In this case, the leading order $\alpha = -1$ and

$$\psi_0 = \frac{12\gamma(t)g_x}{\alpha(t)+\beta(t)}, \tag{2.3.2}$$

where g_x denotes the partial differentiation of $g(x, y, t)$ with respect to x. For finding the resonances, the full Laurent series (2.3.1) is substituted into (2.1.2) and by equating the coefficients of like terms, the polynomial equation is derived as

$$-(r^2-1)(r-4)(r-6)(\alpha(t)+\beta(t))^2\gamma(t)g^{(r-2)}\psi_r = F(\psi_{r-1}, ..., \psi_0, g_x, g_y, g_t, ...). \tag{2.3.3}$$

Using equation (2.3.3), the resonances are found to be

$$r = -1, 1, 4, 6. \tag{2.3.4}$$

As usual, the resonance at $r = -1$ corresponds to the arbitrariness of the singular manifold $g(x, y, t) = 0$. Therefore there are three compatibility tests at $r = 1, 4$ and 6. When $r = 1$, ψ_1 is arbitrary; $r = 4$, ψ_4 is arbitrary; $r = 6$, we can get the compatibility condition which corresponds to the following relations in arbitrary variable coefficients of Eq. (2.1.2):

$$\begin{aligned} &(i)\ \ \alpha(t) = 2\beta(t),\\ &(ii)\ \ -9\left(\beta\left(t\right)\right)^2 \tfrac{d}{dt}\gamma\left(t\right) + 9\,\beta\left(t\right)\gamma\left(t\right)\tfrac{d}{dt}\beta\left(t\right) = 0. \end{aligned} \tag{2.3.5}$$

Then we obtain the following explicit constraints on the variable coefficients $\alpha(t), \beta(t)$ and $\gamma(t)$ for Eq. (2.1.2) to pass the Painlevé test:

$$\alpha(t) = 2\beta(t),\ \ \gamma(t) = C_1\beta(t), \tag{2.3.6}$$

where C_1 is arbitrary constant.

Remark 2.3.1. In [143], Alagesan et al. performed the Painlevé test for uncoupled breaking soliton equation (2.1.1) and also derived the associated Bäcklund transformation and bilinear form directly from the Painlevé test, but here we have performed the integrability test for breaking soliton equation with variable coefficients and found certain constraints on variables $\alpha(t), \beta(t), \gamma(t)$ given by Eq. (2.3.6) for VCCD equation to be integrable.

2.3.2 Classical Lie Symmetry Analysis

Here, we will study the similarity reductions and exact solutions of the VCCD equation with the aid of symmetry group, i.e., the Lie group of transformations acting on the independent variables (x, y, t) and the dependent variable ψ. By following the same procedure as in section (2.2), we get the following forms for the infinitesimal elements

η, ξ, ϕ and τ:

$$\begin{aligned}
&\xi = a_1 x + a_2 \\
&\tau = \tfrac{1}{\beta(t)}((a_1 - a_3)\textstyle\int \beta(t)dt + a_7) \\
&\phi = a_4 \textstyle\int \alpha(t)dt + a_6 \\
&\eta = a_3\psi + a_4 x + a_5,
\end{aligned} \tag{2.3.7}$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 are arbitrary constants. The functions $\alpha(t), \beta(t)$ and $\gamma(t)$ are governed by the following conditions:

$$\begin{aligned}
&\alpha'(t)\beta(t) - \alpha(t)\beta'(t) = 0, \\
&\gamma(t)\tau_t + \tau\gamma'(t) - 2a_1\gamma(t) = 0.
\end{aligned} \tag{2.3.8}$$

The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned}
&V_1 = x\tfrac{\partial}{\partial x} + \left(\tfrac{\int \beta(t)dt}{\beta(t)}\right)\tfrac{\partial}{\partial t}, \\
&V_2 = \psi\tfrac{\partial}{\partial \psi} - \left(\tfrac{\int \beta(t)dt}{\beta(t)}\right)\tfrac{\partial}{\partial t}, \\
&V_3 = x\tfrac{\partial}{\partial \psi} + \left(\textstyle\int \alpha(t)dt\right)\tfrac{\partial}{\partial y}, \\
&V_4 = \tfrac{\partial}{\partial x}, \\
&V_5 = \tfrac{1}{\beta(t)}\tfrac{\partial}{\partial t}, \\
&V_6 = \tfrac{\partial}{\partial y}, \\
&V_7 = \tfrac{\partial}{\partial \psi}.
\end{aligned} \tag{2.3.9}$$

Here, we can see that the Lie symmetry operators and the Lie algebra depend on the solutions for $\alpha(t), \beta(t), \gamma(t)$. Thus, we can deduce that the VCCD equation has an infinite continous group of transformations which is generated by the infinite-dimensional Lie algebra spanned by the operators (2.3.9). Using (1.2.16) and (1.2.21), we can find commutator and adjoint relations for (2.3.9) which are interpreted in following tables as under:

Table 2.4: Commutator Table

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	0	V_3	$-V_4$	$-V_5$	0	0
V_2	0	0	$-V_3$	0	V_5	0	$-V_7$
V_3	$-V_3$	V_3	0	$-V_7$	$-K_1V_6$	0	0
V_4	V_4	0	V_7	0	0	0	0
V_5	V_5	$-V_5$	K_1V_6	0	0	0	0
V_6	0	0	0	0	0	0	0
V_7	0	V_7	0	0	0	0	0

Table 2.5: Adjoint Table

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	V_1	V_2	$V_3e^{-\epsilon}$	V_4e^{ϵ}	V_5e^{ϵ}	V_6	V_7
V_2	V_1	V_2	V_3e^{ϵ}	V_4	$V_5e^{-\epsilon}$	V_6	V_7e^{ϵ}
V_3	$V_1+\epsilon V_3$	$V_2-\epsilon V_3$	V_3	$V_4+\epsilon V_7$	$V_5+\epsilon V_6$	V_6	V_7
V_4	$V_1-\epsilon V_4$	V_2	$V_3-\epsilon V_7$	V_4	V_5	V_6	V_7
V_5	$V_1-\epsilon V_5$	$V_2+\epsilon V_5$	$V_3-\epsilon K_1V_6$	V_4	V_5	V_6	V_7
V_6	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_7	V_1	$V_2-\epsilon V_7$	V_3	V_4	V_5	V_6	V_7

The optimal system for (2.3.9) consists of following vector fields:

$$\begin{aligned}&(i)\ V_1+aV_2,\ \ (ii)\ V_2+bV_4,\ \ (iii)\ V_3+cV_4+dV_5,\\ &(iv)\ V_4+pV_5,\ \ (v)\ V_5+qV_7,\ \ (vi)\ V_6+sV_7,\ \ (vii)\ V_7,\end{aligned} \tag{2.3.10}$$

where a, b, c, d, p, q and s are arbitrary constants. Since, corresponding to essential vector field V_7, equation (2.1.2) is identically satisfied, Hence in Table 2.6 we confine ourselves to list the remaining six essential fields of the optimal system and in each case, we get $\alpha(t) = K_1\beta(t)$.

2.3.2.1 Group Invariant Reductions with One-Dimensional Subalgebras and Exact Solutions

In this section, corresponding to each essential vector field in the optimal system, the reduced PDEs in two variables ρ and σ are obtained. Since, each reduced PDE can be further studied for symmetry reductions, but to avoid cumbersome results, here the reduced PDEs are studied only for physically significant reductions along with their exact solutions by using extended (G'/G)-expansion method, hyperbolic rational function expansion method and others.

1. **Vector Field** $V_1 + aV_2$

The reduced PDE in this case is given by

$$-(1-a)H_\sigma-\sigma H_{\sigma\sigma}+K_1(1-a)H_\sigma H_{\rho\sigma}+(1-a)H_\rho H_{\sigma\sigma}+K_2(1-a)^{\frac{2}{1-a}}H_{\rho\sigma\sigma\sigma}=0, \tag{2.3.11}$$

where K_1, K_2 are arbitrary constants. To reduce the PDE, let us assume the transformation of the form

$$H(\rho,\sigma)=\frac{1}{\sigma}F(\zeta), \zeta=\rho\sigma^2, \tag{2.3.12}$$

TABLE 2.6: Similarity Variables, Forms and Coefficient Functions of VCCD Equation

Essential fields	*Similarity variables*(ρ, σ)	*Similarity solution*(ψ)	*Coefficient functions*$(\gamma(t))$
$V_1 + aV_2$	y, $\frac{x}{\left(\int \beta(t)dt\right)^{\frac{1}{1-a}}}$	$\left(\int \beta(t)dt\right)^{\frac{a}{1-a}} H(\rho, \sigma)$	$K_2\beta(t)\left((1-a)\int \beta(t)dt\right)^{\frac{1+a}{1-a}}$
$V_2 + bV_4$	y, $x + b\log\left(\int \beta(t)dt\right)$	$\frac{1}{\left(\int \beta(t)dt\right)} H(\rho, \sigma)$	$K_2\beta(t)\left(-\int \beta(t)dt\right)^{-1}$
$V_3 + cV_4 + dV_5$	$x - \frac{c}{d}\left(\int \beta(t)dt\right)$, $y - \frac{K_1}{2d}\left(\int \beta(t)dt\right)^2$	$\frac{x^2}{2c} + H(\rho, \sigma)$	$K_2\beta(t)$
$V_4 + pV_5$	y, $px - \left(\int \beta(t)dt\right)$	$H(\rho, \sigma)$	$K_2\beta(t)$
$V_5 + qV_7$	x, y	$q\left(\int \beta(t)dt\right) + H(\rho, \sigma)$	$K_2\beta(t)$
$V_6 + sV_7$	x, t	$sy + H(\rho, \sigma)$	$K_2\beta(t)$

which reduces the equation (2.3.11) to ODE of the form

$$
\begin{aligned}
&8K_2(1-a)^{\frac{2}{1-a}}\zeta^2 F''''(\zeta) + 24K_2(1-a)^{\frac{2}{1-a}}\zeta F'''(\zeta) + \left(6K_2(1-a)^{\frac{2}{1-a}} - 4\zeta\right) F''(\zeta)\\
&+4(K_1+1)(1-a)\zeta F'(\zeta)F''(\zeta) + 2(1-a)(K_1-1)(F'(\zeta))^2 + (1-a)(2-K_1)\tfrac{F(\zeta)F'(\zeta)}{\zeta}\\
&-2K_1(1-a)F(\zeta)F''(\zeta) - (1+a)\tfrac{F(\zeta)}{\zeta} + 2aF'(\zeta) = 0,
\end{aligned} \tag{2.3.13}
$$

where prime ($'$) denotes the differentiation with respect to the variable ζ. To get a solution for Eq. (2.3.13), let us assume that (2.3.13) admits a solution in the form

$$
F(\zeta) = \frac{b_0}{\zeta} + b_1 + b_2\zeta. \tag{2.3.14}
$$

On Substitution, we get $b_0 = b_1 = 0$ and $b_2 = \frac{1}{K_1}$. Hence, we find the solution of Eq. (2.1.2) as follows:

$$\psi(x, y, t) = \frac{xy}{K_1 \left(\int \beta(t)dt\right)}. \tag{2.3.15}$$

2. **Vector Field** $V_2 + bV_4$

The reduced PDE is

$$-H_\sigma + bH_{\sigma\sigma} + K_1 H_\sigma H_{\rho\sigma} + H_\rho H_{\sigma\sigma} - K_2 H_{\rho\sigma\sigma\sigma} = 0. \tag{2.3.16}$$

Make transformation as follows:

$$H(\rho, \sigma) = H(\zeta), \zeta = \rho - C\sigma, \tag{2.3.17}$$

where C is non-zero constant. Substituting (2.3.17) into Eq. (2.3.16) obtains an ODE to ζ. Integrating it once with respect to ζ and taking integration constant to zero, yield

$$2K_2C^2H'''(\zeta) + C(1 + K_1)(H'(\zeta))^2 + 2bCH'(\zeta) + 2H(\zeta) = 0. \tag{2.3.18}$$

We seek the solution of Eq. (2.3.18) in the following form

$$H(\zeta) = \frac{b_0 \tanh(\zeta) + b_1 + b_2 \text{sech}(\zeta)}{\tanh(\zeta) + b_3 + b_4 \text{sech}(\zeta)}, \tag{2.3.19}$$

where b_0, b_1, b_2, b_3 and b_4 are constants to be found out. The substitution of the form of $H(\zeta)$ in equation (2.3.18) brings forth the several possibilities for values of constants and all these possibilities correspond to the following solutions of Eq. (2.1.2):

$$\begin{aligned}
&(i)\ \ \psi(x,y,t) = b_2 \frac{\text{sech}(y-1/2\,u(x,t))\left(\int\beta(t)dt\right)^{-1}}{(-1+\tanh(y-1/2\,u(x,t)))},\\
&(ii)\ \ \psi(x,y,t) = b_0 \frac{\left(\tanh(y-1/2\,u(x,t))+1-sech(y-1/2\,u(x,t)){b_4}^{-1}\right)\left(\int\beta(t)dt\right)^{-1}}{(-1+\tanh(y-1/2\,u(x,t))+b_4\,sech(y-1/2\,u(x,t)))},
\end{aligned} \tag{2.3.20}$$

with the restriction that $\alpha(t) = -\beta(t)$ and $u(x,t) = \frac{\left(-b+\sqrt{b^2-4\,K_2}\right)\left(x+b\ln\left(\int\beta(t)dt\right)\right)}{K_2}$. By the analysis of solutions obtained in this section, we conclude that solutions (2.3.20) depends upon certain arbitrary constants and function $\beta(t)$. Depending upon these constants and function of time, we obtain certain periodic and kinky wave solutions. For $b_0 = b_4 = K_2 = 1, y = \sin(x), \beta(t) = \cos(t)$ and for $y = \cos(x), \beta(t) = 1$, solution (2.3.20)(ii) behaves as periodic and kinky wave as shown in Fig. (2.4) and Fig. (2.5) respectively.

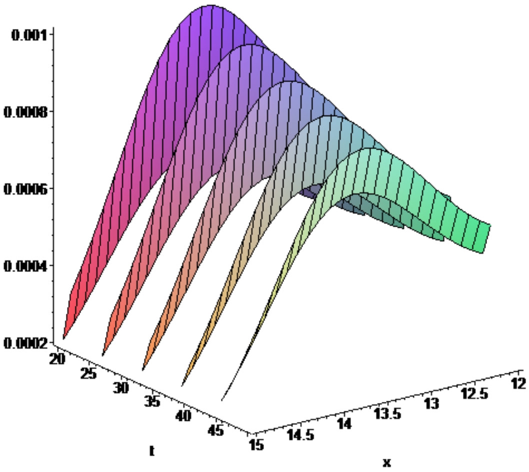

FIGURE 2.4: The figure of periodic solution (2.3.20)(ii) for $b_0 = b_4 = K_2 = 1, y = \sin(x), \beta(t) = \cos(t)$

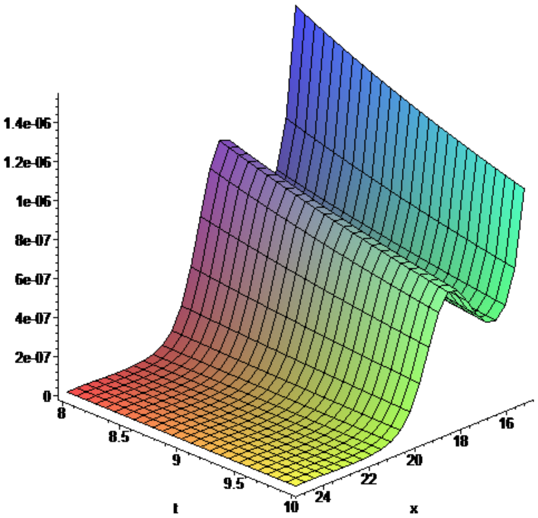

FIGURE 2.5: The figure of kink periodic solution (2.3.20)(ii) for $b_0 = b_4 = K_2 = 1, y = \cos(x), \beta(t) = 1$

3. **Vector Field** $V_3 + cV_4 + dV_5$

In this case, we get the following (1+1)-dimensional nonlinear PDE with variable coefficients:

$$-\frac{c}{d}H_{\rho\rho} + \frac{K_1}{c}\rho H_{\rho\sigma} + K_1 H_\rho H_{\rho\sigma} + \frac{1}{c}H_\sigma + H_{\rho\rho}H_\sigma + K_2 H_{\rho\rho\rho\sigma} = 0. \tag{2.3.21}$$

To reduce the above Eq. to ODE, make the transformation as follows:

$$H(\rho,\sigma) = \frac{c}{d}\sigma - \frac{1}{2c}\rho^2 + \frac{1}{\rho}F(\zeta), \zeta = \sigma\rho^4. \tag{2.3.22}$$

Substituting Eq. (2.3.22) into Eq. (2.3.21) yields nonlinear ODE as follows:

$$\begin{aligned}&K_2\left(6F'(\zeta) + 204\zeta F''(\zeta) + 288\zeta^2 F'''(\zeta) + 64\zeta^3 F''''(\zeta)\right) + 2F(\zeta)F'(\zeta) + 16\zeta^2 F''(\zeta)\\ &+4\zeta(F'(\zeta))^2 + K_1\left(-3F(\zeta)F'(\zeta) - 4\zeta F(\zeta)F''(\zeta) + 12\zeta(F'(\zeta))^2 + 16\zeta^2 F'(\zeta)F''(\zeta)\right) + \tfrac{1}{d} = 0.\end{aligned} \tag{2.3.23}$$

To get a solution for (2.3.23), let us assume that the ODE admits a solution of the form

$$F(\zeta) = \frac{b_0}{\zeta} + b_1 + b_2\zeta. \tag{2.3.24}$$

In this case we get $b_0 = 0, b_1 = \frac{-(6dK_2b_2+1)}{4db_2}$ and $b_2 = b_2$. Hence, we find the solution of Eq. (2.1.2) as follows:

$$\begin{aligned}\psi(x,y,t) &= 1/2\,\tfrac{x^2}{c} + c\left(y + 1/3\,\tfrac{\left(\int\beta(t)dt\right)^2}{d}\right)d^{-1} - 1/2\left(x - \tfrac{c\int\beta(t)dt}{d}\right)^2 c^{-1}\\ &+ \left(1/4\,\tfrac{-6\,dK_2\,b_2-1}{db_2} + b_2\left(y + 1/3\,\tfrac{\left(\int\beta(t)dt\right)^2}{d}\right)\left(x - \tfrac{c\int\beta(t)dt}{d}\right)^4\right)\left(x - \tfrac{c\int\beta(t)dt}{d}\right)^{-1},\end{aligned} \tag{2.3.25}$$

with the restriction that $\alpha(t) = -\frac{2}{3}\beta(t)$.

2. **Vector Field** $V_4 + pV_5$

The reduced PDE is given as

$$K_2p^2H_{\rho\sigma\sigma\sigma} + pH_\rho H_{\sigma\sigma} + K_1pH_\sigma H_{\rho\sigma} - H_{\sigma\sigma} = 0. \tag{2.3.26}$$

In this section, we seek solutions of Eq. (2.3.26) by extended (G'/G)-expansion method as described in section (1.7.2). The traveling wave variable

$$H(\rho,\sigma) = H(\zeta), \zeta = \rho - C\sigma, \tag{2.3.27}$$

where C is a constant to be determined later, permits us reducing the Eq. (2.3.26) to an ODE in the form

$$-2K_2p^2CH''' + (pK_1 + p)(H')^2 - 2H' + K_3 = 0, \tag{2.3.28}$$

where K_3 is constant of integration. Suppose the solution of (2.3.28) can be expressed in (G'/G) as follows:

$$H(\zeta) = a_0 + \sum_{i=1}^{n}\left\{a_i\left(\frac{G'}{G}\right)^i + b_i\left(\frac{G'}{G}\right)^{i-1}\sqrt{\nu\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}\right\}, \tag{2.3.29}$$

where $G = G(\zeta)$ satisfies the second-order linear ODE (1.7.5), while $a_i, b_i (i = 1, 2, ..., n)$ and a_0 are constants to be determined, such that $\nu = \pm 1$ and $\mu \neq 0$. On balancing the highest-order derivatives with the nonlinear terms appearing in (2.3.28), we get $n = 1$. Substituting (2.3.29) into (2.3.28) and using (1.7.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)^k$ and $\left(\frac{G'}{G}\right)^k\sqrt{\nu\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$ together, and equating each coefficient of them to zero, yield a set of following algebraic equations for a_0, a_1, b_1 and C:

(i) $\frac{K_1\,p{b_1}^2\nu}{\mu} + 12\,K_2\,p^2Ca_1 + p{a_1}^2 + K_1\,p{a_1}^2 + \frac{p{b_1}^2\nu}{\mu} = 0,$

(ii) $2\,K_1\,pa_1\,b_1 + 12\,K_2\,p^2Cb_1 + 2\,pa_1\,b_1 = 0,$

$$
\begin{aligned}
&(iii)\;\; K_1\,p{b_1}^2\nu + 16\,K_2\,p^2Ca_1\,\mu + 2\,K_1\,p{a_1}^2\mu + p{b_1}^2\nu + 2\,p{a_1}^2\mu + 2\,a_1 = 0,\\
&(iv)\;\; 10\,K_2\,p^2Cb_1\,\mu + 2\,K_1\,pa_1\,\mu\,b_1 + 2\,b_1 + 2\,pa_1\,\mu\,b_1 = 0,\\
&(v)\;\; 4\,K_2\,p^2Ca_1\,\mu^2 + K_1\,p{a_1}^2\mu^2 + 2\,a_1\,\mu + p{a_1}^2\mu^2 + K_3 = 0.
\end{aligned}
\tag{2.3.30}
$$

Solving these algebraic equations by Maple, we obtain the following results:

Case 1:

$$
a_0 = a_0,\;\; a_1 = -\frac{3}{p\mu(K_1+1)},\;\; C = \frac{1}{4p^2\mu K_2},\;\; b_1 = K_3 = 0. \tag{2.3.31}
$$

Case 2:

$$
a_0 = a_0,\;\; a_1 = -\frac{6}{p\mu(K_1+1)},\;\; C = \frac{1}{p^2\mu K_2},\;\; b_1 = \pm\frac{6\sqrt{\frac{1}{\mu\nu}}}{p(K_1+1)},\;\; K_3 = 0. \tag{2.3.32}
$$

From (2.3.29) and the general solution of (2.3.30), we deduce the traveling wave solutions of (2.3.28). When $\mu > 0$, then Case 1 gives the exact traveling wave solution as:

$$
H(\rho,\sigma) = a_0 - \frac{3\left(B\cos\left(\sqrt{\mu}\left(\rho-1/4\,\frac{\sigma}{p^2\mu\,K_2}\right)\right)-A\sin\left(\sqrt{\mu}\left(\rho-1/4\,\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)p^{-1}\frac{1}{\sqrt{\mu}}(K_1+1)^{-1}}{\left(A\cos\left(\sqrt{\mu}\left(\rho-1/4\,\frac{\sigma}{p^2\mu\,K_2}\right)\right)+B\sin\left(\sqrt{\mu}\left(\rho-1/4\,\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)}, \tag{2.3.33}
$$

where A, B are arbitrary constants. Case 2 gives the solution of Eq. (2.3.28) as:

$$
\begin{aligned}
H(\rho,\sigma) = a_0 &- \frac{6\left(B\cos\left(\sqrt{\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)-A\sin\left(\sqrt{\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)p^{-1}\frac{1}{\sqrt{\mu}}(K_1+1)^{-1}}{\left(A\cos\left(\sqrt{\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)+B\sin\left(\sqrt{\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)}\\
&\pm 6\sqrt{\frac{1}{\mu\,\nu}}\sqrt{\nu\left(1+\frac{\left(B\cos\left(\sqrt{\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)-A\sin\left(\sqrt{\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)^2}{\left(A\cos\left(\sqrt{\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)+B\sin\left(\sqrt{\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)^2}\right)}\,(K1+1)^{-1}\,p^{-1}.
\end{aligned}
\tag{2.3.34}
$$

When $\mu < 0$, then Case 1 gives the exact traveling wave solution as:

$$
H(\rho,\sigma) = a_0 - 3\sqrt{-\mu}\frac{\left(B\cosh\left(\sqrt{-\mu}\left(\rho-1/4\,\frac{\sigma}{p^2\mu\,K_2}\right)\right)+A\sinh\left(\sqrt{-\mu}\left(\rho-1/4\,\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)p^{-1}\mu^{-1}(K_1+1)^{-1}}{\left(A\cosh\left(\sqrt{-\mu}\left(\rho-1/4\,\frac{\sigma}{p^2\mu\,K_2}\right)\right)+B\sinh\left(\sqrt{-\mu}\left(\rho-1/4\,\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)}. \tag{2.3.35}
$$

and Case 2 gives the solution as:

$$
\begin{aligned}
H(\rho,\sigma) = a_0 &+ 6\sqrt{-\mu}\frac{\left(B\cosh\left(\sqrt{-\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)+A\sinh\left(\sqrt{-\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)p^{-1}\mu^{-1}(K_1+1)^{-1}}{\left(A\cosh\left(\sqrt{-\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)+B\sinh\left(\sqrt{-\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)}\\
&\pm 6\sqrt{\frac{1}{\mu\,\nu}}\sqrt{\nu\left(1-\frac{\left(B\cosh\left(\sqrt{-\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)+A\sinh\left(\sqrt{-\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)^2}{\left(A\cosh\left(\sqrt{-\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)+B\sinh\left(\sqrt{-\mu}\left(\rho-\frac{\sigma}{p^2\mu\,K_2}\right)\right)\right)^2}\right)}\,(K_1+1)^{-1}\,p^{-1}.
\end{aligned}
\tag{2.3.36}
$$

Corresponding to the above solutions (2.3.33)-(2.3.36), solutions of Eq. (2.1.2) are given by the relation

$$
\psi(x,y,t) = H(\rho,\sigma),\;\; \rho = y,\;\; \sigma = px - \left(\int \beta(t)dt\right). \tag{2.3.37}
$$

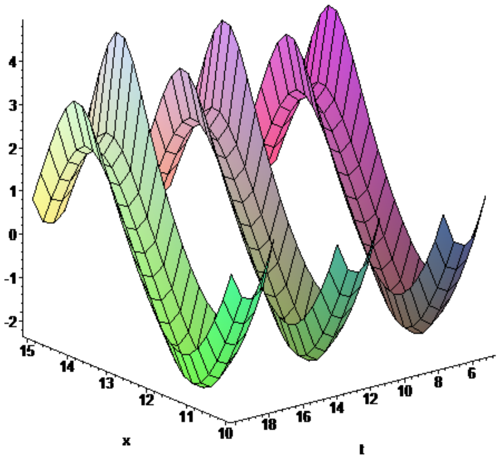

FIGURE 2.6: The figure of periodic solution (2.2.33) for $\mu = \nu = p = K_2 = b_1 = A = 1, K_1 = a_0 = B = 0, y = \sin(x), \beta(t) = \tan(t)$

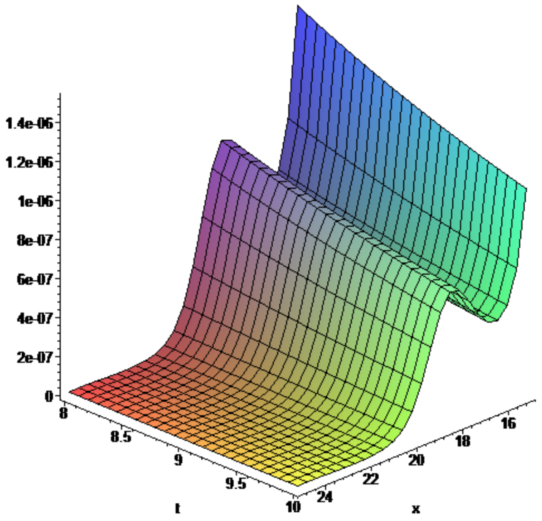

FIGURE 2.7: The figure of periodic solution (2.2.36) for $\mu = \nu = p = K_2 = b_1 = A = 1, K_1 = a_0 = B = 0, y = \sin(x), \beta(t) = 0$

In this section, certain solutions obtained by extended (G'/G) method behaves as periodic solutions for different values of constants and functions involved. For $\mu = \nu = p = K_2 = b_1 = A = 1, K_1 = a_0 = B = 0, y = \sin(x)$ and with $\beta(t) = \tan(t)$ and $\beta(t) = 0$, solutions (2.3.33) and (2.3.36) takes the form of periodic solutions as shown in Fig. (2.6) and Fig. (2.7) respectively.

5. **Vector Field** $V_5 + qV_7$

In this case, we get the following (1+1)-dimensional nonlinear PDE:

$$K_1 H_\rho H_{\rho\sigma} + H_{\rho\rho} H_\sigma + K_2 H_{\rho\rho\rho\sigma} = 0. \tag{2.3.38}$$

Make transformation as follows:

$$H(\rho, \sigma) = H(\zeta),\ \zeta = \rho - C\sigma, \tag{2.3.39}$$

where C is a non-zero constant. Substitute (2.3.39) into Eq. (2.3.38) to obtain an ODE to ζ. Integrating it once with respect to ζ yield

$$(K_1 + 1)(H')^2 + 2K_2 H''' + K_3 = 0, \tag{2.3.40}$$

where K_3 is constant of integration. Multiplying Eq. (2.3.40) by H'' and then once more integrating with respect to ζ, taking integration constant to zero and substituting

$H'(\zeta) = J(\zeta)$, we get

$$\frac{K_1+1}{3}J^3 + K_2(J')^2 + K_3 J = 0. \tag{2.3.41}$$

Here, we will seek the solutions of Eq. (2.3.41) using Jacobi elliptic function method. For this, we consider the solution of (2.3.41) in the form

$$J(\zeta) = a_0 + a_1 sn(\zeta, m) + a_2 sn^2(\zeta, m), \tag{2.3.42}$$

where $sn(\zeta, m)$ is the Jacobi elliptic sine function with modulus m. On substituting Eq. (2.3.42) into (2.3.41) and performing algebraic calculations, we obtain the following results:

Case 1: One has

$$a_0 = 0,\ \ a_1 = 0,\ \ a_2 = -\frac{K_3}{4K_2},\ \ m = i,\ \ K_1 = -\frac{(K_3 + 48K_2^2)}{K_3}, \tag{2.3.43}$$

which corresponds to the following solution of Eq. (2.1.2):

$$\psi(x,y,t) = q\int \beta\left(t\right)dt - 1/4\,\frac{K_3\,\left(-x+Cy-E\left(sn\left(-x+Cy,i\right),i\right)\right)}{K_2}, \tag{2.3.44}$$

where E denotes the elliptic integral of second kind.

Case 2: One has

$$a_0 = -\frac{K_3}{4K_2},\ \ a_1 = 0,\ \ a_2 = \frac{K_3}{2K_2}, m = \sqrt{2},\ \ K_1 = -\frac{(K_3 + 48K_2^2)}{K_3}, \tag{2.3.45}$$

and in this case, the solution of Eq. (2.1.2) takes the following form:

$$\begin{aligned}&\psi(x,y,t) = \\ &q\int\beta\left(t\right)dt - 1/4\,\tfrac{K_3\,(x-Cy)}{K_2} + 1/2\,\tfrac{K_3\,\left(1/2\,x-1/2\,Cy+1/2\,E\left(sn\left(-x+Cy,\sqrt{2}\right),\sqrt{2}\right)\right)}{K_2}.\end{aligned} \tag{2.3.46}$$

Case 3: One has

$$a_0 = -\frac{K_3}{2K_2},\ \ a_1 = 0,\ \ a_2 = \frac{K_3}{2K_2}, m = \frac{\sqrt{2}}{2},\ \ K_1 = -\frac{(12K_2^2 + K_3)}{K_3}, \tag{2.3.47}$$

and equation (2.1.2) possesses the following solution in this case:

$$\begin{aligned}&\psi(x,y,t) = \\ &q\int\beta\left(t\right)dt - 1/2\,\tfrac{K_3\,(x-Cy)}{K_2} + 1/2\,\tfrac{K_3\,\left(2\,x-2\,Cy+2\,E\left(sn\left(-x+Cy,1/2\,\sqrt{2}\right),1/2\,\sqrt{2}\right)\right)}{K_2}.\end{aligned} \tag{2.3.48}$$

6. **Vector Field** $V_6 + sV_7$

The reduced PDE is given by

$$H_{\rho\sigma} + s\beta(\sigma)H_{\rho\rho} = 0. \tag{2.3.49}$$

On solving the equation, we get

$$H(\rho,\sigma) = F_1(\sigma) + F_2(\rho - s\int \beta(\sigma)d\sigma), \tag{2.3.50}$$

where F_1, F_2 are arbitrary functions and the above solution corresponds to the solution of Eq. (2.1.2) as

$$\psi(x,y,t) = sy + F_1(t) + F_2(x - s\int \beta(t)dt). \tag{2.3.51}$$

2.4 Discussions

In this chapter, we have presented the similarity reductions of Eq. (2.1.1) and (2.1.2) by classical Lie group method and some new solutions are given, including the periodic solutions, bell profile solutions etc. The discussions on our results are as follows:

- Firstly, Eq. (2.1.1) has been reduced to (1+1)-dimensional nonlinear Eq. (2.2.10) by means of the classical Lie group method. The (1+1)-dimensional reduced PDE is further reduced to ODEs using several transformations and reduced ODEs are studied to get several exact general solutions which corresponds to solutions of Eq. (2.1.1).
- The solutions of Eq. (2.1.1) obtained by us are such that one can choose the arbitrary functions $Q(t), R(t)$ along with various other parameters, in a suitable manner, to simulate physical situations governed by Eq. (2.1.1) to obtain particular solutions having desired features (as shown in Figures 2.1-2.3). To understand the solutions well, we plot the graphs of the solution surfaces with some special parameters. As shown in Figures 2.1,2.2 and 2.3 the periodic solutions, bell profile solutions and kinky wave solutions can be obtained through solutions (2.2.14).
- In this chapter, we have also checked the Painlevé property of (2+1)-dimensional VCCD equation and prove that VCCD equation is integrable under certain restrictions on coefficients of equation under study.
- The Lie group method is utilized for the purpose of obtaining the group infinitesimals of VCCD equation and the basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations.
- The motivation for the present study lies in the physical importance of VCCD equation (2.1.2) and the need to have exact solutions. The solutions furnished by us brings forth a new class of exact periodic and kinky wave solutions. In almost all the cases, one can choose the arbitrary function $\beta(t)$ along with various

other parameters, in a suitable manner and this provides enough freedom to build solutions that may correspond to particular physical situations.

- If we consider the new time variable as

$$T(t) = \int_0^t \beta(t)dt, \tag{2.4.1}$$

then the VCCD equation can be reduced to the following form:

$$\beta(t)(\psi_{xT} + \alpha'(t)\psi_x\psi_{xy} + \psi_y\psi_{xx} + \gamma'(t)\psi_{xxxy}) = 0, \tag{2.4.2}$$

where $\alpha'(t) = \frac{\alpha(t)}{\beta(t)}$, $\gamma'(t) = \frac{\gamma(t)}{\beta(t)}$. Then, the Lie symmetry operators involving variable t will be transformed to

$$\begin{aligned}
&V_1 = x\tfrac{\partial}{\partial x} + \tfrac{T(t)}{\beta(t)} \equiv x\tfrac{\partial}{\partial x} + T\tfrac{\partial}{\partial T}, \;\; \alpha'(t) = K_1, \;\; \gamma'(t) = K_2T,\\
&V_2 = \psi\tfrac{\partial}{\partial \psi} - \tfrac{T(t)}{\beta(t)}\tfrac{\partial}{\partial t} \equiv \psi\tfrac{\partial}{\partial \psi} - T\tfrac{\partial}{\partial T}, \;\; \alpha'(t) = K_1, \;\; \gamma'(t) = -\tfrac{K_2}{T},\\
&V_3 = x\tfrac{\partial}{\partial \psi} + (\textstyle\int_0^t \alpha(t')dt')\tfrac{\partial}{\partial y} \equiv x\tfrac{\partial}{\partial \psi} + K_1T(t)\tfrac{\partial}{\partial y}, \;\; \alpha'(t) = K_1, \;\; \gamma'(t) = \gamma'(t),\\
&V_5 = \tfrac{1}{\beta(t)}\tfrac{\partial}{\partial t} \equiv \tfrac{\partial}{\partial T}, \;\; \alpha'(t) = K_1, \;\; \gamma'(t) = K_2.
\end{aligned} \tag{2.4.3}$$

Hence, we deduce that the analysis done by us in this manuscript can also be performed by using (2.4.1)-(2.4.3) in more simpler way, but finally we will get the same solutions of VCCD equation.

It is worth to be mentioned that, with the aid of Maple, the exact solutions reported here in this chapter are found to indeed satisfy the VCCD equation.

Chapter 3

The Coupled Klein-Gordon-Schrödinger Equation and its Generalized Form[1]

3.1 Introduction

We have considered the Klein-Gordon-Schrödinger equation [22, 23] in the form

$$\begin{aligned} &u_{tt} - c^2 u_{xx} + u + |v|^2 = 0, \\ &iv_t + v_{xx} + uv = 0, \end{aligned} \tag{3.1.1}$$

where $v(x,t)$ is a complex function, $u(x,t)$ is a real one and $i^2 = -1$. This system is a classical model described the interaction between conservative complex neutron field and neutral meson Yukawa in quantum field theory. It possesses two conserved quantities, which are

$$\begin{aligned} &Q(t) = ||v||^2 = Q(0), \\ &E(t) = \tfrac{1}{2}(||u||^2 + ||u_t||^2 + ||u_x||^2 + ||v_x||^2) - (|v|^2, u) = E(0), \end{aligned} \tag{3.1.2}$$

where $Q(0)$ and $E(0)$ are constants depending only on initial values, and $||.||$ and $(.,.)$ denote the norm and inner product in any Hilbert space, respectively.

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[1] A part of this chapter has been published in **Mathematical Methods in Applied Sciences, 35 (2012) 1175-1187**.

In the literature, the existence, uniqueness, and asymptotic behaviour of global solution of Klein-Gordon-Schrödinger equations are considered in [22, 23] and the exact solitary wave solution is given in [68]. Numerically, two conservative difference schemes are constructed in [84, 85] and the multisymplectic method is considered in [78]. In [139], the modified decomposition method was used for finding the solutions for the coupled KGS equations with initial conditions and the approximate solutions to the equations have been calculated without any need to a transformation techniques and linearization of the equations. Wang [115] established a rational physical model for controlling KGS dynamics system. In the case of perturbation caused by disturbances and uncertainties in control field, numerical quantum approach is presented for finding the quantum optimal control pairing of perturbative system. Assume that the amplitude of perturbed disturbance is bounded, then perturbative optimal quantum control is investigated both in theoretic and computational aspects. Based on a reliable semi-discrete algorithm, numerical simulations interpret the effectiveness of performing control design. In [140], a class of discrete-time orthogonal spline collocation schemes for solving coupled KGS equations with initial and boundary conditions are considered. These schemes are constructed by using piecewise cubic Hermite interpolations in space combined with finite difference methods in time. It is proved that the schemes have the conservation laws of discrete energy, and possess second order accuracy in maximum norm and fourth order in L^2-norm for time and space, respectively. The conservation laws and rate of convergence are also verified in numerical experiments.

Some new generalized solitary solutions of the KGS equations are obtained by Wang et al. [166] using the Exp-function method, which include some known solutions obtained by the F-expansion method and the homogeneous balance method. Darwish et al. [9] devised an algebraic method to uniformly construct a series of explicit exact solutions for the coupled KGS equations. By applying the Jacobi elliptic function expansion method, the periodic solutions for a class of coupled nonlinear Klein-Gordon equations, which include coupled nonlinear Klein-Gordon equation, coupled nonlinear Klein-Gordon-Schrödinger equations and coupled nonlinear Klein-Gordon-Zakharov equations, are obtained in [134]. Anjan Biswas and Houria Triki [5] obtains the 1-soliton solution of the KGS equation with power law nonlinearity. The solitary wave ansatz is used to carry out the integration. The efficient, unconditionally stable and accurate numerical methods for approximations of the KGS equations with/without damping terms are presented in [153]. The key features of methods used are based on:
(i) the application of a time-splitting spectral discretization for a Schrödinger-type equation in KGS,
(ii) the utilization of Fourier pseudospectral discretization for spatial derivatives in the Klein-Gordon equation in KGS,

(iii) the adoption of solving the ordinary differential equations in phase space analytically under appropriate chosen transmission conditions between different time intervals or applying Crank-Nicolson/leap-frog for linear/nonlinear terms for time derivatives.

Because of the wide range of applications in several branches of physics, the generalized Klein-Gordon-Schrödinger equation i.e. variable coefficient Klein-Gordon-Schrödinger (VCKGS) equations are the subject of studies both in physical and mathematical contexts. In this chapter, we also consider the following form of coupled Klein-Gordon-Schrödinger equation [22, 23] with variable coefficients:

$$\begin{aligned} &u_{tt} + f(t)u_{xx} + g(t)u + h(t)|v|^2 = 0, \\ &iv_t + p(t)v_{xx} + q(t)uv = 0, \end{aligned} \tag{3.1.3}$$

where $v(x,t)$ is a complex function, $u(x,t)$ is a real one and $i^2 = -1$. Here, the coefficients $f(t), g(t), h(t), p(t), q(t)$ are the functions of independent variable t that corresponds to new or more realistic physical conditions. In the following sections, we have obtained the symmetries of VCKGS equation and utilized them to obtain new physically important solutions.

The study in this chapter has been structured as follows. In section (3.2), Eq. (3.1.1) and Eq. (3.1.3) has been decomposed into three coupled PDEs and then using the Lie classical approach, symmetries are obtained. On using the obtained symmetries, we reduce the number of independent variables of the coupled K-G-S equation and VCKGS equation and some exact solutions are also obtained. In section (3.3), one of the reduced equation is studied by modified (G'/G)-expansion method to obtain new traveling wave solutions and some solutions are also graphically interpreted. In the last section (3.4), some conclusions are made.

3.2 Classical Lie Symmetry Analysis

In the study of nonlinear partial differential equations, a long established systematic approach to find exact solutions is the method of classical Lie point symmetry analysis. The method of point symmetry analysis involves looking for a Lie group of invertible transformations that map every solution of the differential equation to another solution of the differential equation. In this section, we use Lie classical method (as described in Chapter 1) to investigate the symmetry reductions of the Klein-Gordon-Schrödinger equation (3.1.1) and VCKGS equation (3.1.3).

3.2.1 The Klein-Gordon-Schrödinger Equation

In this subsection, we perform Lie symmetry analysis for the KGS equation. Since $v(x,t)$ is a complex function, so as to separate real and imaginary parts, we take the complex function $v(x,t)$ as

$$v(x,t) = a(x,t) + ib(x,t), \tag{3.2.1}$$

which decomposes the system (3.1.1) into the following system of equtaions:

$$\begin{aligned} &u_{tt} - c^2 u_{xx} + u + a^2 + b^2 = 0,\\ &a_t + b_{xx} + ub = 0,\\ &-b_t + a_{xx} + au = 0. \end{aligned} \tag{3.2.2}$$

To find the symmetries, let us consider the Lie group of point transformations as

$$\begin{aligned} u^* &= u + \epsilon\phi(x,t,u,a,b) + O(\epsilon^2),\\ a^* &= a + \epsilon\psi(x,t,u,a,b) + O(\epsilon^2),\\ b^* &= b + \epsilon\eta(x,t,u,a,b) + O(\epsilon^2),\\ x^* &= x + \epsilon\xi(x,t,u,a,b) + O(\epsilon^2),\\ t^* &= t + \epsilon\tau(x,t,u,a,b) + O(\epsilon^2), \end{aligned} \tag{3.2.3}$$

which leaves the system (3.2.2) invariant. The method for determining the symmetry group of (3.2.2) consists of finding the infinitesimals ϕ, ψ, η, ξ and τ, which are functions of x, t, u, a, b. Assuming that the system (3.2.2) is invariant under the transformations (3.2.3), we get the following relation from the coefficients of the first order of ϵ:

$$\begin{aligned} &\phi^{tt} - c^2\phi^{xx} + \phi + 2a\psi + 2b\eta = 0,\\ &\psi^t + \eta^{xx} + u\eta + b\phi = 0,\\ &-\eta^t + \psi^{xx} + a\phi + u\psi = 0, \end{aligned} \tag{3.2.4}$$

where $\eta^t, \eta^{xx}, \psi^t, \psi^{xx}, \phi^{xx}$ and ϕ^{tt} are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables $b_t, b_{xx}, a_t, a_{xx}, u_{xx}$ and u_{tt} . The infinitesimals are determined from invariance condition (3.2.4), by setting the coefficients of different differentials equal to zero. We obtain a large number of PDEs in ϕ, ψ, η, ξ and τ that need to be satisfied. The general solution of this large system provides following forms for the infinitesimal elements ϕ, ψ, η, ξ and τ:

$$\xi = c_1, \ \tau = c_2, \ \phi = 0, \ \psi = c_3 a, \ \eta = c_3 b, \tag{3.2.5}$$

where c_1, c_2 and c_3 are arbitrary constants. Now we can reduce the system (3.1.1) to system of ODEs using characteristic equation:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi} = \frac{da}{\psi} = \frac{db}{\eta}. \tag{3.2.6}$$

Solving characteristic equation and using (3.2.1) we have the following similarity variables for system (3.1.1)

$$u(x,t) = U(\xi), \;\; v = e^{(i\theta)}V(\xi), \;\; \xi = x + \alpha t, \;\; \theta = px + rt, \tag{3.2.7}$$

where α, p, r are constants, we have a relation $\alpha + 2p = 0$ and reduce system (3.1.1) to the following system of ODEs

$$\begin{aligned} &(\alpha^2 - c^2)U''(\xi) + U(\xi) + V^2(\xi) = 0, \\ &V''(\xi) + U(\xi)V(\xi) - (r + p^2)V(\xi) = 0, \end{aligned} \tag{3.2.8}$$

where the prime means differentiation with respect to ξ. In this case we get only trivial symmetries, so we will get only traveling wave solutions of the equation (3.1.1) and for the traveling wave solutions, we will apply Modified (G'/G)-expansion method to the reduced ODE (3.2.8) as in section (3.3).

3.2.2 The Generalized Klein-Gordon-Schrödinger Equation

In this section, in virtue of classical Lie group method, we will discuss the classical similarity reductions and exact solutions for the VCKGS equation (3.1.3). First of all, we take the complex function $v(x,t)$ as

$$v(x,t) = r(x,t) + is(x,t), \tag{3.2.9}$$

which decomposes the system (3.1.3) into the following system of equtaions:

$$\begin{aligned} &u_{tt} + f(t)u_{xx} + g(t)u + h(t)(r^2 + s^2) = 0, \\ &r_t + p(t)s_{xx} + q(t)us = 0, \\ &-s_t + p(t)r_{xx} + q(t)ur = 0. \end{aligned} \tag{3.2.10}$$

Proceeding in similar manner as mentioned earlier, we get the following forms of the infinitesimal elements ϕ, ψ, η, ξ and τ for VCKGS equation:

$$\xi = c_2 x + c_5, \;\; \tau = c_3 t + c_6, \;\; \phi = c_4 u, \;\; \psi = c_1 r, \;\; \eta = c_1 s, \tag{3.2.11}$$

where $\phi, \psi, \eta, \xi, \tau$ are infinitesimals corresponding to u, r, s, x, t, respectively and time variables $f(t), g(t), h(t), p(t), q(t)$ satisfy the following conditions:

$$\begin{aligned}
&\tau f'(t) - 2f(t)\xi_x + 2\tau_t f(t) = 0, \\
&\tau p'(t) + \tau_t p(t) - 2p(t)\xi_x = 0, \\
&\tau g'(t) + 2\tau_t g(t) = 0, \\
&\tau h'(t) + 2\tau_t h(t) - \phi_u h(t) + 2h(t)\eta_s = 0, \\
&\tau q'(t) + \tau_t q(t) + q(t)\phi_u = 0.
\end{aligned} \tag{3.2.12}$$

The infinitesimal generators of the corresponding Lie algebra are given by

$$Z_1 = s\tfrac{\partial}{\partial s} + r\tfrac{\partial}{\partial r},\ \ Z_2 = x\tfrac{\partial}{\partial x},\ \ Z_3 = t\tfrac{\partial}{\partial t},\ \ Z_4 = u\tfrac{\partial}{\partial u},\ \ Z_5 = \tfrac{\partial}{\partial x},\ \ Z_6 = \tfrac{\partial}{\partial t}. \tag{3.2.13}$$

In general, there are infinite number of subalgebras of this Lie algebra formed from any linear combination of generators $Z_j; j = 1, 2, 3, 4, 5, 6$ and to any sublagebra one can get the reduction using characterstic equations:

$$\tfrac{dx}{\xi} = \tfrac{dt}{\tau} = \tfrac{du}{\phi} = \tfrac{dr}{\psi} = \tfrac{ds}{\eta}. \tag{3.2.14}$$

However, this problem becomes manageable by recognizing that if two algebras are similar, i.e. they are connected to each other by a transformation from the symmetry group, then their corresponding invariant solutions are connected to each other by the same transformation. Therefore, it is sufficient to put all similar subalgebras in to one class and select a representative from each class. The set of all these representatives is called an optimal system as discussed in chapter 1. Thus, the optimal system for (3.2.13) consists of following vector fields:

$$\begin{aligned}
&(i)\ Z_1 + \alpha Z_2 + \beta Z_3 + \gamma Z_4,\ \ (ii)\ Z_2 + pZ_3 + qZ_4,\ \ (iii)\ Z_3 + cZ_4 + Z_5,\ \ (iv)\ Z_3 + dZ_4, \\
&(v)\ Z_4 + aZ_5 + Z_6,\ \ (vi)\ Z_4 + bZ_5,\ \ (vii) Z_5 + Z_6,\ \ (viii) Z_5,\ \ (ix)\ Z_6,
\end{aligned} \tag{3.2.15}$$

3.2.2.1 Symmetry Reductions and Exact Solutions

In this section, we will make use of the optimal system of vector fields (3.2.15) and reduce the Eq. (3.1.3) to ODEs. The similarity variables and the similarity solutions of the Eq. (3.1.3) can be obtained by solving characterstic equation (3.2.14) and the coefficient functions are given by equation (3.2.12). The general solution of these equations involves two constants, one become independent variable ζ and other plays the role of new dependent variables $F(\zeta), G(\zeta), H(\zeta)$.

Vector Field $Z_1 + \alpha Z_2 + \beta Z_3 + \gamma Z_4$

For this vector field, on solving the equations (3.2.12) and (3.2.14), we obtain

$$\begin{aligned}
&u(x,t) = t^{\frac{\gamma}{\beta}} F(\zeta),\ \ v(x,t) = t^{\frac{1}{\beta}} e^{iG(\zeta)} H(\zeta),\ \ \zeta = xt^{-\frac{\alpha}{\beta}},\\
&f(t) = K_1 t^{\frac{2\alpha-2\beta}{\beta}},\ \ g(t) = \tfrac{K_2}{t^2},\ \ h(t) = K_3 t^{\frac{\gamma-2-2\beta}{\beta}},\ \ p(t) = K_4 t^{\frac{2\alpha-\beta}{\beta}},\ \ q(t) = K_5 t^{\frac{-\beta-\gamma}{\beta}},
\end{aligned} \tag{3.2.16}$$

where K_1, K_2, K_3, K_4, K_5 are arbitrary constants. Substituting (3.2.16) into Eq. (3.1.3), we have the functions $F(\zeta), G(\zeta), H(\zeta)$ which must satisfy the following system of ODEs:

$$\begin{aligned}
&\tfrac{\gamma}{\beta}(\tfrac{\gamma}{\beta} - 1)F(\zeta) - \tfrac{\alpha\gamma}{\beta^2}\zeta F'(\zeta) - \tfrac{\alpha}{\beta}(\tfrac{\gamma-\alpha-\beta}{\beta})\zeta F'(\zeta) + \tfrac{\alpha^2}{\beta^2}\zeta^2 F''(\zeta)\\
&+K_1 F''(\zeta) + K_2 F(\zeta) + K_3 H(\zeta)^2 = 0,\\
&\tfrac{1}{\beta}H(\zeta) - \tfrac{\alpha}{\beta}\zeta H'(\zeta) + 2K_4 G'(\zeta)H'(\zeta) + K_4 G''(\zeta)H(\zeta) = 0,\\
&\tfrac{\alpha}{\beta}\zeta G'(\zeta)H(\zeta) + K_4 H''(\zeta) - K_4 G'(\zeta)^2 H(\zeta) + K_5 F(\zeta)H(\zeta) = 0.
\end{aligned} \tag{3.2.17}$$

To construct solutions for above system, let us suppose that system (3.2.17) assumes the solution in the following form:

$$\begin{aligned}
F(\zeta) &= a_0 + a_1\zeta + a_2\zeta^2,\\
G(\zeta) &= b_0 + b_1\zeta + b_2\zeta^2,\\
H(\zeta) &= c_0 + c_1\zeta + c_2\zeta^2,
\end{aligned} \tag{3.2.18}$$

where $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ are constants to be determined. Using (3.2.18) in ODEs (3.2.17), we get a set of algebraic equations which can be solved to get the following solution of eq. (3.1.3):

$$\begin{aligned}
u(x,t) &= -1/9\, t^{\frac{\gamma}{\beta}} \left(-\alpha + 2\,\alpha^2 - 1\right) x^2 \left(t^{-\frac{\alpha}{\beta}}\right)^2 {K_4}^{-1}\beta^{-2}{K_5}^{-1},\\
v(x,t) &= t^{\beta^{-1}} \mathrm{e}^{i\left(b_0 + 1/6\,(\alpha-1)x^2\left(t^{-\frac{\alpha}{\beta}}\right)^2 {K_4}^{-1}\beta^{-1}\right)} c_1\, xt^{-\frac{\alpha}{\beta}},
\end{aligned} \tag{3.2.19}$$

and from (3.2.16), we get $K_1 = 0, K_3 = 1/9\, \frac{(2\,\alpha+1)(\alpha-1)\left(4\,\alpha^2+2\,\alpha\,\beta-4\,\alpha\,\gamma-\gamma\,\beta+K_2\,\beta^2+\gamma^2\right)}{K_5\,\beta^4{c_1}^2K_4}$.

Vector Field $Z_2 + pZ_3 + qZ_4$

For this case, the similarity variable, similarity solution and coefficient functions are as follows:

$$\begin{aligned}
&u(x,t) = t^{\frac{q}{p}} F(\zeta),\ \ v(x,t) = e^{iG(\zeta)} H(\zeta),\ \ \zeta = xt^{-\frac{1}{p}},\\
&f(t) = K_1 t^{\frac{2-2p}{p}},\ \ g(t) = \tfrac{K_2}{t^2},\ \ h(t) = K_3 t^{\frac{q-2p}{p}},\ \ p(t) = K_4 t^{\frac{2-p}{p}},\ \ q(t) = K_5 t^{\frac{-p-q}{p}},
\end{aligned} \tag{3.2.20}$$

Substituting (3.2.20) into Eq. (3.1.3), it follows the corresponding reduced ODEs:

$$\begin{aligned}
&\tfrac{q(q-p)}{p^2}F(\zeta) - \tfrac{q}{p^2}\zeta F'(\zeta) - \tfrac{q-p-1}{p^2}\zeta F'(\zeta) + \tfrac{1}{p^2}\zeta^2 F''(\zeta) + K_1F''(\zeta) + K_2F(\zeta) + K_3H(\zeta)^2 = 0,\\
&-\tfrac{1}{p}\zeta H'(\zeta) + 2K_4G'(\zeta)H'(\zeta) + K_4G''(\zeta)H(\zeta) = 0,\\
&\tfrac{1}{p}\zeta G'(\zeta)H(\zeta) + K_4H''(\zeta) - K_4G'(\zeta)^2H(\zeta) + K_5F(\zeta)H(\zeta) = 0.
\end{aligned} \tag{3.2.21}$$

Solving above system of equations using substitution (3.2.18), we get the following values of $u(x,t), v(x,t)$:

$$\begin{aligned}
u(x,t) &= t^{\frac{q}{p}}\left(\frac{9}{8}\,\frac{K_3\,p^2c_0{}^2}{3-2\,q+p} - \frac{3pc_0{}^2K_3\,xt^{-p^{-1}}}{16(3-2\,q+p)\sqrt{-\frac{K_1}{9\,p+9-18\,q}}} + \frac{9\,K_3\,c_0{}^2(p+1-2\,q)x^2\left(t^{-p^{-1}}\right)^2}{16(3-2\,q+p)K_1}\right),\\
v(x,t) &= \mathrm{e}^{i\left(b_0+2\sqrt{-\frac{K_1}{9\,p+9-18\,q}}\,xt^{-p^{-1}}K_4{}^{-1}+1/6\,x^2\left(t^{-p^{-1}}\right)^2K_4{}^{-1}p^{-1}\right)}\\
&\left(c_0 - 1/12\,c_0\,xt^{-p^{-1}}\frac{1}{\sqrt{-\frac{K_1}{9\,p+9-18\,q}}}p^{-1}\right),
\end{aligned} \tag{3.2.22}$$

and on using (3.2.20), we have $K_2 = \frac{-33+34q-17p-9q^2+9pq}{9p^2}, K_5 = \frac{-32K_1(3-2q+p)}{81K_4(p+1-2q)K_3p^2c_0{}^2}$.

Vector Field $Z_3 + cZ_4 + Z_5$

This case yields the following forms of invariants and coefficient functions:

$$\begin{aligned}
&u(x,t) = t^cF(\zeta),\ \ v(x,t) = e^{iG(\zeta)}H(\zeta),\ \ \zeta = x - \log(t),\\
&f(t) = K_1t^{-2},\ \ g(t) = \tfrac{K_2}{t^2},\ \ h(t) = K_3t^{c-2},\ \ p(t) = K_4t^{-1},\ \ q(t) = K_5t^{-c-1}.
\end{aligned} \tag{3.2.23}$$

On using (3.2.23) into eq. (3.1.3), it corresponds to the following system of reduced ODEs:

$$\begin{aligned}
&c(c-1)F(\zeta) - (2c-1)F'(\zeta) + F''(\zeta) + K_1F''(\zeta) + K_2F(\zeta) + K_3H(\zeta)^2 = 0,\\
&-H'(\zeta) + 2K_4G'(\zeta)H'(\zeta) + K_4G''(\zeta)H(\zeta) = 0,\\
&G'(\zeta)H(\zeta) + K_4H''(\zeta) - K_4H(\zeta)(G'(\zeta))^2 + K_5F(\zeta)H(\zeta) = 0.
\end{aligned} \tag{3.2.24}$$

The variable coefficients Klein-Gordon-Schrödinger equation has the following solution corresponding to above system of ODEs:

$$\begin{aligned}
u(x,t) &= -1/4\,\frac{t^c\left(C_2{}^2+4\,K_4{}^2\right)}{C_2{}^2K_5\,K_4},\\
v(x,t) &= -1/2\,\mathrm{e}^{i\left(1/2\,\frac{x-\log(t)}{K_4}+C_2\right)}\sqrt{-C_4\left(\mathrm{e}^{\frac{x-\log(t)}{C_2}}\right)^{-2}\left(\mathrm{e}^{\frac{C_3}{C_2}}\right)^{-2}}\left(\left(\mathrm{e}^{\frac{x-\log(t)}{C_2}}\right)^2\left(\mathrm{e}^{\frac{C_3}{C_2}}\right)^2+1\right),
\end{aligned} \tag{3.2.25}$$

where C_2, C_3, C_4 are arbitrary constants and $K_1 = -1, K_2 = -c^2 + c, K_3 = 0$.

Vector Field $Z_3 + dZ_4$

On solving the equations (3.2.12) and (3.2.14), we get

$$\begin{aligned} &u(x,t) = t^d F(\zeta),\ \ v(x,t) = e^{iG(\zeta)}H(\zeta),\ \ \zeta = x,\\ &f(t) = K_1 t^{-2},\ \ g(t) = \tfrac{K_2}{t^2},\ \ h(t) = K_3 t^{d-2},\ \ p(t) = K_4 t^{-1},\ \ q(t) = K_5 t^{-d-1}, \end{aligned} \tag{3.2.26}$$

We get the following system of ODEs using above transformations:

$$\begin{aligned} &d(d-1)F(\zeta) + K_1 F''(\zeta) + K_2 F(\zeta) + K_3 H(\zeta)^2 = 0,\\ &2K_4 G'(\zeta)H'(\zeta) + K_4 G''(\zeta)H(\zeta) = 0,\\ &K_4 H''(\zeta) - K_4 (G'(\zeta))^2 H(\zeta) + K_5 F(\zeta)H(\zeta) = 0. \end{aligned} \tag{3.2.27}$$

The above reduced system of ODEs corresponds to following solution of eq. (3.1.3):

$$\begin{aligned} u(x,t) &= -6\,\frac{t^d K_4\, {C_5}^2 \wp(C_4 + C_5\, x, 0, C_1)}{K_5},\\ v(x,t) &= -6\,\frac{\mathrm{e}^{iC_1}\sqrt{K_1\, K_4\, K_3\, K_5}{C_5}^2 \wp(C_4 + C_5\, x, 0, C_1)}{K_3\, K_5}, \end{aligned} \tag{3.2.28}$$

where C_1, C_4, C_5 are arbitrary constants, $K_2 = -d^2 + d$ and $\wp$ denotes WeirstrassP function.

Vector Field $Z_4 + aZ_5 + Z_6$

Following the same way as above, we get

$$\begin{aligned} &u(x,t) = e^t F(\zeta),\ \ v(x,t) = e^{iG(\zeta)}H(\zeta),\ \ \zeta = x - at,\\ &f(t) = K_1,\ \ g(t) = K_2,\ \ h(t) = K_3 e^t,\ \ p(t) = K_4,\ \ q(t) = K_5 e^{-t}, \end{aligned} \tag{3.2.29}$$

Substituting (3.2.29) into eq. (3.1.3), it follows the following system of reduced ODEs:

$$\begin{aligned} &F(\zeta) - 2aF'(\zeta) + a^2 F''(\zeta) + K_1 F''(\zeta) + K_2 F(\zeta) + K_3 H(\zeta)^2 = 0,\\ &-aH'(\zeta) + 2K_4 G'(\zeta)H'(\zeta) + K_4 G''(\zeta)H(\zeta) = 0,\\ &aG'(\zeta)H(\zeta) + K_4 H''(\zeta) - K_4 (G'(\zeta))^2 H(\zeta) + K_5 F(\zeta)H(\zeta) = 0. \end{aligned} \tag{3.2.30}$$

On solving the above system of ODEs, we get the following solution:

$$\begin{aligned} u(x,t) &= -1/4\,\frac{\mathrm{e}^t\left(a^2{C_2}^2 + 4\,{K_4}^2\right)}{{C_2}^2 K_5\, K_4},\\ v(x,t) &= -1/2\,\mathrm{e}^{i\left(1/2\,\frac{a(x-at)}{K_4} + C_2\right)}\sqrt{-C_4\left(\mathrm{e}^{\frac{x-at}{C_2}}\right)^{-2}\left(\mathrm{e}^{\frac{C_3}{C_2}}\right)^{-2}}\left(\left(\mathrm{e}^{\frac{x-at}{C_2}}\right)^2\left(\mathrm{e}^{\frac{C_3}{C_2}}\right)^2 + 1\right), \end{aligned} \tag{3.2.31}$$

where C_2, C_3, C_4 are arbitrary constants and $K_1 = -a^2, K_2 = -1, K_3 = 0$.

Vector Field $Z_4 + bZ_5$

For this case, the similarity variable, similarity solution and coefficient functions are obtained as follows:

$$\begin{aligned} &u(x,t) = e^{\frac{x}{b}} F(\zeta),\ \ v(x,t) = e^{iG(\zeta)} H(\zeta),\ \ \zeta = t, \\ &f(t) = f(t),\ \ g(t) = g(t),\ \ h(t) = 0,\ \ p(t) = p(t),\ \ q(t) = 0, \end{aligned} \tag{3.2.32}$$

On using (3.2.32), eq. (3.1.3) transforms to following system of ODEs:

$$\begin{aligned} &b^2 F''(\zeta) + f(\zeta)F(\zeta) + b^2 g(\zeta)F(\zeta) = 0, \\ &H'(\zeta) = 0, \\ &G'(\zeta)H(\zeta) = 0. \end{aligned} \tag{3.2.33}$$

For $f(\zeta) = \zeta, g(\zeta) = \zeta^2$, we get the following solution of eq. (3.1.3):

$$\begin{aligned} &u(x,t) = \mathrm{e}^{\frac{x}{b}} \left(C_1\, F_1(-1/16\, \tfrac{-4\, b^4 + i}{b^4};\ 1/2;\ \tfrac{1/4\, i\left(2\, b^2 t + 1\right)^2}{b^4}) \mathrm{e}^{\frac{-1/2\, it\left(1 + b^2 t\right)}{b^2}} \right) \\ &+\mathrm{e}^{\frac{x}{b}} \left(C_2\, F_1(-1/16\, \tfrac{-12\, b^4 + i}{b^4};\ 3/2;\ \tfrac{1/4\, i\left(2\, b^2 t + 1\right)^2}{b^4}) \left(2\, b^2 t + 1\right) \mathrm{e}^{\frac{-1/2\, it\left(1 + b^2 t\right)}{b^2}} \right), \\ &v(x,t) = \mathrm{e}^{iC_2} C_1, \end{aligned} \tag{3.2.34}$$

where C_1, C_2 are arbitrary constants and F_1 represents the hypergeometric function. For $f(\zeta) = \sin(\zeta), g(\zeta) = 1$, eq. (3.1.3) possesses the following solution:

$$\begin{aligned} &u(x,t) = \mathrm{e}^{\frac{x}{b}} \left(C_1\, C\left(4, -2\, b^{-2}, -1/4\, \pi + 1/2\, t\right) + C_2\, S\left(4, -2\, b^{-2}, -1/4\, \pi + 1/2\, t\right) \right), \\ &v(x,t) = \mathrm{e}^{iC_2} C_1, \end{aligned} \tag{3.2.35}$$

where C_1, C_2 are arbitrary constants and C, S represents Mathieu cosine and Mathieu sine functions respectively.

Vector Field $Z_5 + Z_6$

For this vector field, solving equations (3.2.12) and (3.2.14), we get

$$\begin{aligned} &u(x,t) = F(\zeta),\ \ v(x,t) = e^{iG(\zeta)} H(\zeta),\ \ \zeta = x - t, \\ &f(t) = K_1,\ \ g(t) = K_2,\ \ h(t) = K_3,\ \ p(t) = K_4,\ \ q(t) = K_5, \end{aligned} \tag{3.2.36}$$

By using above transformations in VCKGS equation, we get the following system of ODEs:

$$\begin{aligned} &F''(\zeta) + K_1 F''(\zeta) + K_2 F(\zeta) + K_3 H(\zeta)^2 = 0, \\ &-H'(\zeta) + 2K_4 G'(\zeta)H'(\zeta) + K_4 G''(\zeta)H(\zeta) = 0, \\ &G'(\zeta)H(\zeta) + K_4 H''(\zeta) - K_4 H(\zeta)(G'(\zeta))^2 + K_5 F(\zeta)H(\zeta) = 0. \end{aligned} \tag{3.2.37}$$

Solving above system of ODEs, we get the following solutions of eq. (3.1.3):

$$\begin{aligned}
&(i)\ \ u(x,t) = -1/4\,\frac{1+16\,{K_4}^2{C_3}^2}{K_5\,K_4},\\
&\quad v(x,t) = \mathrm{e}^{i\left(1/2\,\frac{x-t}{K_4}+C_2\right)}\left(1/2\,C_4 + C_4\,\left(\sinh\left(r(x,t)\right)\right)^2\right),\\
&(ii)\ \ u(x,t) = -1/4\,\frac{1+36\,{K_4}^2{C_3}^2}{K_5\,K_4},\\
&\quad v(x,t) = \mathrm{e}^{i\left(1/2\,\frac{x-t}{K_4}+C_2\right)}\left(-3/4\,C_4\,\cosh\left(r(x,t)\right) + C_4\,\left(\cosh\left(r(x,t)\right)\right)^3\right),
\end{aligned}\tag{3.2.38}$$

where C_2, C_3, C_4 are arbitrary constants and $r(x,t) = C_2 + C_3\,(x-t)\,, K_2 = 0, K_3 = 0$.

Vector Field Z_5

For this vector field, solving the equations (3.2.12) and (3.2.14), we get

$$\begin{aligned}
&u(x,t) = F(\zeta),\ \ v(x,t) = e^{iG(\zeta)}H(\zeta),\ \ \zeta = t,\\
&f(t) = f(t),\ \ g(t) = g(t),\ \ h(t) = h(t),\ \ p(t) = p(t),\ \ q(t) = q(t),
\end{aligned}\tag{3.2.39}$$

The above forms of invariants and coefficient functions transforms VCKGS equation to following system of ODEs:

$$\begin{aligned}
&F''(\zeta) + g(\zeta)F(\zeta) + h(\zeta)H(\zeta)^2 = 0,\\
&-G'(\zeta)H(\zeta) + q(\zeta)F(\zeta)H(\zeta) = 0,\\
&H'(\zeta) = 0.
\end{aligned}\tag{3.2.40}$$

For $h(\zeta) = 0, g(\zeta) = \zeta$, Eq. (3.1.3) yields the following solution:

$$\begin{aligned}
&u(x,t) = C_3\,Ai(-t) + C_4\,Bi(-t),\\
&v(x,t) = \mathrm{e}^{i\left(\int(C_3\,Ai(-t)+C_4\,Bi(-t))q(t)dt+C_1\right)}C_2,
\end{aligned}\tag{3.2.41}$$

where C_1, C_2, C_3, C_4 are arbitrary constants and $Ai(t), Bi(t)$ are Airy wave functions. For $h(\zeta) = 0, g(\zeta) = \zeta^2$, we get the another solution for Eq. (3.1.3).

$$\begin{aligned}
&u(x,t) = C_1\,\sqrt{t}\,\mathrm{J}_{\frac{1}{4}}\left(1/2\,t^2\right) + C_2\,\sqrt{t}\mathrm{Y}_{\frac{1}{4}}\left(1/2\,t^2\right),\\
&v(x,t) = \mathrm{e}^{i\left(\int q(t)\left(C_1\,\sqrt{t}\mathrm{J}_{\frac{1}{4}}\left(1/2\,t^2\right)+C_2\,\sqrt{t}\mathrm{Y}_{\frac{1}{4}}\left(1/2\,t^2\right)\right)dt+C_1\right)}C_2,
\end{aligned}\tag{3.2.42}$$

where C_1, C_2 are arbitrary constants and J and Y are Bessel functions of first and second kind respectively.

Vector Field Z_6

Following the same way as above, we get

$$\begin{aligned} &u(x,t) = F(\zeta),\ \ v(x,t) = e^{iG(\zeta)}H(\zeta),\ \ \zeta = x, \\ &f(t) = K_1,\ \ g(t) = K_2,\ \ h(t) = K_3,\ \ p(t) = K_4,\ \ q(t) = K_5, \end{aligned} \tag{3.2.43}$$

The above tranformations corresponds to the following reductions of eq (3.1.3):

$$\begin{aligned} &K_1F''(\zeta) + K_2F(\zeta) + K_3H(\zeta)^2 = 0, \\ &K_4H''(\zeta) - K_4H(\zeta)(G'(\zeta))^2 + K_5F(\zeta)H(\zeta) = 0, \\ &2G'(\zeta)H'(\zeta) + G''(\zeta)H(\zeta) = 0. \end{aligned} \tag{3.2.44}$$

The above system of ODEs can be solved to get the following solutions:

$$\begin{aligned} &(i)\ \ u(x,t) = -\tfrac{K_4\left(2\,C_4\,(\sin(C_1+C_3\,x))^2{C_3}^2-2\,C_4\,(\cos(C_1+C_3\,x))^2{C_3}^2\right)}{\left(-1/2\,C_4+C_4\,(\cos(C_1+C_3\,x))^2\right)K_5}, \\ &\quad v(x,t) = \mathrm{e}^{iC_1}\left(-1/2\,C_4 + C_4\,\left(\cos\left(C_1+C_3\,x\right)\right)^2\right), \\ &(ii)\ \ u(x,t) = -\tfrac{9\,K_4\,{C_4}^4\left(\mathrm{e}^{C_1+C_3\,x}\right)^{12}{C_3}^2-K_4\,{C_2}^2}{K_5\,{C_4}^4\left(\mathrm{e}^{C_1+C_3\,x}\right)^{12}}, \\ &\quad v(x,t) = \mathrm{e}^{iC_1}C_4\,\left(\mathrm{e}^{C_1+C_3\,x}\right)^3, \\ &(iii)\ \ u(x,t) = \tfrac{K_4\,{C_2}^2}{K_5\,(C_4+C_5\,(C_1+C_3\,x))^4}, \\ &\quad v(x,t) = \mathrm{e}^{iC_1}\left(C_4 + C_5\,\left(C_1+C_3\,x\right)\right), \\ &(iv)\ \ u(x,t) = \\ &-\tfrac{K_4\,{C_3}^2\,\sin(C_1+C_3\,x)\left(3/4\,C_4+6\,C_4(\cos(C_1+C_3\,x))^2-3\,C_4\,(\sin(C_1+C_3\,x))^2\right)\left(v(x,t)/\mathrm{e}^{iC_1}\right)^3-K_4\,{C_2}^2}{K_5\left(v(x,t)/\mathrm{e}^{iC_1}\right)^4}, \\ &\quad v(x,t) = \mathrm{e}^{iC_1}\left(-3/4\,C_4\,\sin\left(C_1+C_3\,x\right) + C_4\,\left(\sin\left(C_1+C_3\,x\right)\right)^3\right), \end{aligned} \tag{3.2.45}$$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants and $K_2 = 0, K_3 = 0$. It is worth mentioning here that the solutions (3.2.45) are functions of x only.

3.3 Exact Traveling Wave Solutions for Eq. (3.1.1) Using Modified (G'/G)-Expansion Method to Reduced ODE (3.2.8)

In this section, we utilized the modified (G'/G)-expansion method as discussed in section (1.7). Assume that the solution of Eqs. (3.2.8) can be expressed by a polynomial in (G'/G) as follows:

$$\begin{aligned} U(\xi) &= a_0 + \textstyle\sum_{i=1}^{m}\left\{a_i\left(\tfrac{G'(\xi)}{G(\xi)}\right)^i + b_i\left(\tfrac{G'(\xi)}{G(\xi)}\right)^{-i}\right\}, \\ V(\xi) &= c_0 + \textstyle\sum_{j=1}^{n}\left\{c_j\left(\tfrac{G'(\xi)}{G(\xi)}\right)^j + d_j\left(\tfrac{G'(\xi)}{G(\xi)}\right)^{-j}\right\}, \end{aligned} \tag{3.3.1}$$

where $a_0, a_i, b_i, c_0, c_j, d_j$ are constants and the positive integers m, n can be determined by considering the homogeneous balance of the highest order derivatives and highest order nonlinear appearing in ODEs (3.2.8). The function $G(\xi)$ is the solution of the auxiliary linear ODE

$$G''(\xi) + \mu G(\xi) = 0, \tag{3.3.2}$$

where μ is a constant to be determined and the nonzero parameter μ in Eq. (3.3.2) plays an essential role in the determination of the type of the solutions. Indeed,

1. If $\mu < 0$ then we find the hyperbolic-type solutions and we have

$$\tfrac{G'}{G} = \sqrt{-\mu}\left(\tfrac{A\sinh\sqrt{-\mu}\xi + B\cosh\sqrt{-\mu}\xi}{A\cosh\sqrt{-\mu}\xi + B\sinh\sqrt{-\mu}\xi}\right), \tag{3.3.3}$$

2. If $\mu > 0$ then we find trigonometric-type solutions and we have

$$\tfrac{G'}{G} = \sqrt{\mu}\left(\tfrac{A\cos\sqrt{\mu}\xi - B\sin\sqrt{\mu}\xi}{A\sin\sqrt{\mu}\xi + B\cos\sqrt{\mu}\xi}\right), \tag{3.3.4}$$

where A, B are arbitrary constants. Noticing the homogeneous balance of highest order derivatives and nonlinear terms appearing in Eq. (3.2.8), we get $m = 2, n = 2$. Consequently, it follows from (3.3.1) that

$$\begin{aligned} U(\xi) &= a_0 + a_1\left(\tfrac{G'(\xi)}{G(\xi)}\right) + a_2\left(\tfrac{G'(\xi)}{G(\xi)}\right)^2 + b_1\left(\tfrac{G'(\xi)}{G(\xi)}\right)^{-1} + b_2\left(\tfrac{G'(\xi)}{G(\xi)}\right)^{-2}, \\ V(\xi) &= c_0 + c_1\left(\tfrac{G'(\xi)}{G(\xi)}\right) + c_2\left(\tfrac{G'(\xi)}{G(\xi)}\right)^2 + d_1\left(\tfrac{G'(\xi)}{G(\xi)}\right)^{-1} + d_2\left(\tfrac{G'(\xi)}{G(\xi)}\right)^{-2}. \end{aligned} \tag{3.3.5}$$

On substituting (3.3.5) into the first equation of ODE (3.2.8) and using linear ODE (3.3.2), collecting all terms with the same powers of (G'/G) together and equating their coefficients to zero, yield a system of algebraic equations for $a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, d_1, d_2$, μ, p as follows:

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^{-4} : \; 24\,p^2 b_2\,\mu^2 - 6\,c^2 b_2\,\mu^2 + {d_2}^2 = 0, \\
&\left(\tfrac{G'}{G}\right)^{-3} : \; -2\,c^2 b_1\,\mu^2 + 8\,p^2 b_1\,\mu^2 + 2\,d_1\,d_2 = 0, \\
&\left(\tfrac{G'}{G}\right)^{-2} : \; 32\,p^2 b_2\,\mu - 8\,c^2 b_2\,\mu + {d_1}^2 + 2\,c_0\,d_2 + b_2 = 0, \\
&\left(\tfrac{G'}{G}\right)^{-1} : \; -2\,c^2 b_1\,\mu + b_1 + 2\,c_1\,d_2 + 8\,p^2 b_1\,\mu + 2\,c_0\,d_1 = 0, \\
&\left(\tfrac{G'}{G}\right)^{0} : \; a_0 - 2\,c^2 a_2\,\mu^2 + 8\,p^2 a_2\,\mu^2 + {c_0}^2 + 8\,p^2 b_2 + 2\,c_1\,d_1 + 2\,c_2\,d_2 - 2\,c^2 b_2 = 0, \\
&\left(\tfrac{G'}{G}\right)^{1} : \; 2\,c_2\,d_1 + a_1 + 2\,c_0\,c_1 + 8\,p^2 a_1\,\mu - 2\,c^2 a_1\,\mu = 0, \\
&\left(\tfrac{G'}{G}\right)^{2} : \; {c_1}^2 + 2\,c_0\,c_2 + a_2 - 8\,c^2 a_2\,\mu + 32\,p^2 a_2\,\mu = 0, \\
&\left(\tfrac{G'}{G}\right)^{3} : \; 2\,c_1\,c_2 - 2\,c^2 a_1 + 8\,p^2 a_1 = 0,
\end{aligned}$$

$$\left(\tfrac{G'}{G}\right)^4 : \; -6\,c^2 a_2 + {c_2}^2 + 24\,p^2 a_2 = 0. \tag{3.3.6}$$

Substituting (3.3.5) into the second equation of ODE (3.2.8), yields following system of algebraic equations:

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^{-4} : \; b_2\,d_2 + 6\,d_2\,\mu^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^{-3} : \; b_2\,d_1 + 2\,d_1\,\mu^2 + b_1\,d_2 = 0,\\
&\left(\tfrac{G'}{G}\right)^{-2} : \; -p^2 d_2 - r d_2 + 8\,d_2\,\mu + b_1\,d_1 + a_0\,d_2 + b_2\,c_0 = 0,\\
&\left(\tfrac{G'}{G}\right)^{-1} : \; -r d_1 + a_1\,d_2 + a_0\,d_1 - p^2 d_1 + b_2\,c_1 + 2\,d_1\,\mu + b_1\,c_0 = 0,\\
&\left(\tfrac{G'}{G}\right)^{0} : \; 2\,c_2\,\mu^2 + a_1\,d_1 + a_2\,d_2 + b_1\,c_1 + b_2\,c_2 + a_0\,c_0 + 2\,d_2 - r c_0 - p^2 c_0 = 0,\\
&\left(\tfrac{G'}{G}\right)^{1} : \; 2\,c_1\,\mu - r c_1 + a_1\,c_0 - p^2 c_1 + a_2\,d_1 + b_1\,c_2 + a_0\,c_1 = 0,\\
&\left(\tfrac{G'}{G}\right)^{2} : \; a_2\,c_0 - r c_2 - p^2 c_2 + a_1\,c_1 + 8\,c_2\,\mu + a_0\,c_2 = 0,\\
&\left(\tfrac{G'}{G}\right)^{3} : \; a_2\,c_1 + a_1\,c_2 + 2\,c_1 = 0,\\
&\left(\tfrac{G'}{G}\right)^{4} : \; 6\,c_2 + a_2\,c_2 = 0.
\end{aligned} \tag{3.3.7}$$

Solving the algebraic equations (3.3.6) and (3.3.7), yields the following cases:

Case (i). $a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = \frac{-(4r+c^2)^2}{32},\ c_0 = 0,\ c_1 = 0,\ c_2 = 0,$
$d_1 = \frac{\sqrt{2}(4r+c^2)}{8},\ d_2 = 0,\ \mu = \frac{r}{2} + \frac{c^2}{8},\ p = \frac{c}{2},$

Case (ii). $a_0 = \frac{3r}{4} + \frac{3c^2}{16} - \frac{3\sqrt{16r^2+8rc^2+c^4-16}}{16},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,$
$b_2 = \frac{3}{32} + \frac{3(\frac{-r}{8}-\frac{c^2}{32}+\frac{1}{32}\sqrt{16r^2+8rc^2+c^4-16})r}{2} + \frac{3c^2(\frac{-r}{8}-\frac{c^2}{32}+\frac{1}{32}\sqrt{16r^2+8rc^2+c^4-16})}{8},$
$c_0 = 3\sqrt{\frac{-r}{8} - \frac{c^2}{32} + \frac{\sqrt{16r^2+8rc^2+c^4-16}}{32}},\ c_1 = 0,\ c_2 = 0,\ d_1 = 0,$
$d_2 = 3(\frac{-r}{8} - \frac{c^2}{32} + \frac{\sqrt{16r^2+8rc^2+c^4-16}}{32})^{(3/2)},\ \mu = \frac{-r}{8} - \frac{c^2}{32} + \frac{\sqrt{16r^2+8rc^2+c^4-16}}{32},$
$p = \sqrt{\frac{-r}{2} + \frac{c^2}{8} - \frac{\sqrt{16r^2+8rc^2+c^4-16}}{8}},$

Case (iii). $d_2 = -\frac{3}{32}\left(-1/2\,c^2 - 2\,r + 1/2\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}\right)^{-3/2},$
$a_1 = 0,\ a_2 = -6,\ b_2 = \frac{3}{32}\left(-c^4 + (1/4\,c^2 + r)\left(2\,c^2 - 8\,r + 2\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}\right) - 2 - 4\,rc^2\right)^{-1},$
$b_1 = 0,\ c_0 = 0,\ c_1 = 0,\ c_2 = 6\,\sqrt{-1/2\,c^2 - 2\,r + 1/2\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}},\ d_1 = 0,$
$a_0 = \frac{-3}{2}\,\frac{-c^4+1/4\,c^2\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-2+r\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-4\,rc^2}{-c^6+(1/4\,c^4+2rc^2+4r^2)\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-3\,c^2-12\,r+\sqrt{c^4+8\,rc^2+16\,r^2+8}-8\,rc^4-16\,r^2c^2},$
$\mu = -\left(-4\,c^2 - 16\,r + 4\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}\right)^{-1},\ p = 1/4\,\sqrt{2\,c^2 - 8\,r + 2\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}},$

Case(iv). $a_0 = 0,\ a_1 = 0,\ a_2 = -2,\ b_1 = 0,\ b_2 = 0,\ c_0 = 0,$
$c_1 = \sqrt{2},\ c_2 = 0,\ d_1 = 0,\ d_2 = 0,\ \mu = \frac{r}{2} + \frac{c^2}{8},\ p = \frac{c}{2},$

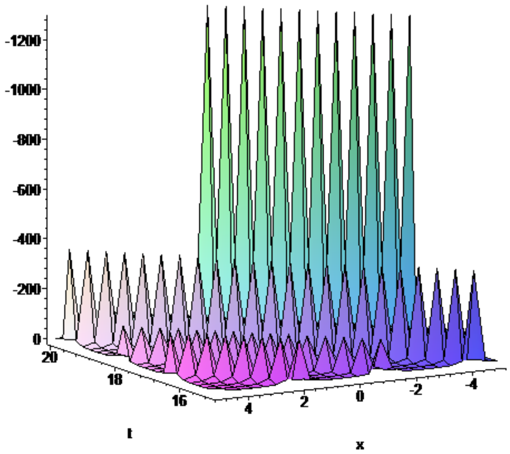

FIGURE 3.1: Traveling Solution of the field $u_1(x,t)$

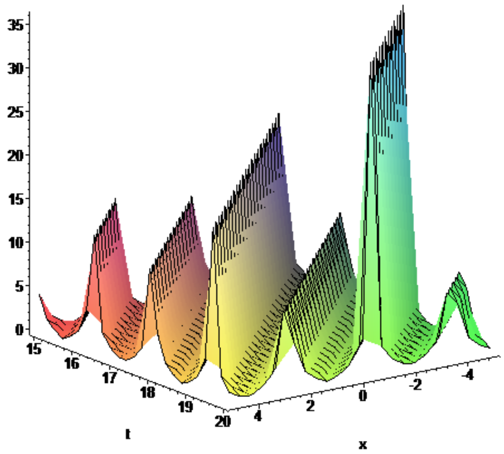

FIGURE 3.2: Traveling Solution of the field $|v_1(x,t)|$

Case (v). $a_0 = r + \frac{c^2}{4},\ a_1 = 0,\ a_2 = -2,\ b_1 = 0,\ b_2 = \frac{-(4r+c^2)^2}{128},\ c_0 = 0,\ c_1 = \sqrt{2},$ $c_2 = 0,\ d_1 = \frac{-\sqrt{2}(4r+c^2)}{16},\ d_2 = 0,\ \mu = \frac{-r}{4} - \frac{c^2}{16},\ p = \frac{c}{2},$

Case (vi). $a_0 = \frac{r}{2} + \frac{c^2}{8},\ a_1 = 0,\ a_2 = -2,\ b_1 = 0,\ b_2 = \frac{-(4r+c^2)^2}{512},\ c_0 = 0,\ c_1 = \sqrt{2},$ $c_2 = 0,\ d_1 = \frac{-\sqrt{2}(4r+c^2)}{32},\ d_2 = 0,\ \mu = \frac{r}{8} + \frac{c^2}{32},\ p = \frac{c}{2}.$

(3.3.8)

Corresponding to the above six cases and on substituting the general solution of (3.3.2) given by (3.3.3) and (3.3.4), we obtained traveling wave solutions of the coupled Klein-Gordon-Schrödinger equation (3.1.1) which have been descibed in following subcases:

Subcase 1. If $\mu > 0$, we obtain

Case (i) gives the following values of $u(x,t)$ and $v(x,t)$:

$$u_1(x,t) = -1/32\,\frac{(4r+c^2)^2\left(A\sin\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)+B\cos\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)\right)^2}{(1/2\,r+1/8\,c^2)\left(A\cos\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)-B\sin\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)\right)^2},$$
$$v_1(x,t) = 1/8\,\frac{e^{i(1/2\,cx+rt)}\sqrt{2}(4\,r+c^2)\left(A\sin\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)+B\cos\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)\right)}{\sqrt{1/2\,r+1/8\,c^2}\left(A\cos\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)-B\sin\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)\right)}.$$

(3.3.9)

On choosing the constants $r = 1, c = 2, A = 1, B = 2$, the traveling waves corresponding for $u_1(x,t)$ and $|v_1(x,t)|$ (where $|v_1(x,t)|$ represents the norm of $v_1(x,t)$) have been shown in Fig. (3.1) and Fig. (3.2) respectively.

Corresponding to Case (ii), we have

$$
\begin{aligned}
&u_2(x,t) = \tfrac{3r}{4} + \tfrac{3c^2}{16} - \tfrac{3\sqrt{16r^2+8rc^2+c^4-16}}{16} + \\
&\left(\tfrac{3}{32} + (3/2r + 3/8c^2)(\tfrac{-r}{8} - \tfrac{c^2}{32} + \tfrac{1}{32}\sqrt{16r^2+8rc^2+c^4-16})\right) \\
&\tfrac{\left(A\sin\left(\sqrt{\mu}(x-2\,pt)\right)+B\cos\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2\mu^{-1}}{\left(A\cos\left(\sqrt{\mu}(x-2\,pt)\right)-B\sin\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}, \\
&v_2(x,t) = e^{i(px+rt)}\left(3\sqrt{-1/8\,r - 1/32\,c^2 + 1/32\,\sqrt{16\,r^2+8\,rc^2+c^4-16}}\right) \\
&+3e^{i(px+rt)}\left(\tfrac{\left(-1/8\,r-1/32\,c^2+1/32\,\sqrt{16\,r^2+8\,rc^2+c^4-16}\right)^{3/2}\left(A\sin\left(\sqrt{\mu}(x-2\,pt)\right)+B\cos\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}{\mu\left(A\cos\left(\sqrt{\mu}(x-2\,pt)\right)-B\sin\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}\right),
\end{aligned}
\tag{3.3.10}
$$

where μ, p are given by $\mu = \frac{-r}{8} - \frac{c^2}{32} + \frac{\sqrt{16r^2+8rc^2+c^4-16}}{32}$, $p = \sqrt{\frac{-r}{2} + \frac{c^2}{8} - \frac{\sqrt{16r^2+8rc^2+c^4-16}}{8}}$.
Case (iii) yields the following solution of Eq. (3.1.1):

$$
\begin{aligned}
&u_3(x,t) = \\
&\tfrac{-3}{2}\,\tfrac{-c^4+1/4\,c^2\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-2+r\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-4\,rc^2}{-c^6+(1/4\,c^4+2rc^2+4r^2)\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-3\,c^2-12\,r+\sqrt{c^4+8\,rc^2+16\,r^2+8}-8\,rc^4-16\,r^2c^2} \\
&-6\,\tfrac{\mu\left(A\cos\left(\sqrt{\mu}(x-2\,pt)\right)-B\sin\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}{\left(A\sin\left(\sqrt{\mu}(x-2\,pt)\right)+B\cos\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2} \\
&+\tfrac{3}{32}\,\tfrac{\left(A\sin\left(\sqrt{\mu}(x-2\,pt)\right)+B\cos\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}{\left(-c^4+(1/4\,c^2+r)\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-2-4\,rc^2\right)\mu\left(A\cos\left(\sqrt{\mu}(x-2\,pt)\right)-B\sin\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}, \\
&v_3(x,t) = e^{i(px+rt)}\left(6\,\tfrac{\sqrt{-1/2\,c^2-2\,r+1/2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}}\,\mu\left(A\cos\left(\sqrt{\mu}(x-2\,pt)\right)-B\sin\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}{\left(A\sin\left(\sqrt{\mu}(x-2\,pt)\right)+B\cos\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}\right) \\
&-e^{i(px+rt)}\left(\tfrac{3}{32}\,\tfrac{\left(A\sin\left(\sqrt{\mu}(x-2\,pt)\right)+B\cos\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}{\left(-1/2\,c^2-2\,r+1/2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)^{3/2}\mu\left(A\cos\left(\sqrt{\mu}(x-2\,pt)\right)-B\sin\left(\sqrt{\mu}(x-2\,pt)\right)\right)^2}\right),
\end{aligned}
\tag{3.3.11}
$$

where μ, p are given by $\mu = -\left(-4\,c^2 - 16\,r + 4\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}\right)^{-1}$,
$p = 1/4\,\sqrt{2\,c^2 - 8\,r + 2\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}}$.
Corresponding to Case (iv), we get

$$
\begin{aligned}
u_4(x,t) &= -2\,\frac{\left(1/2\,r+1/8\,c^2\right)\left(A\cos\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)-B\sin\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)\right)^2}{\left(A\sin\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)+B\cos\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)\right)^2}, \\
v_4(x,t) &= \frac{e^{i(1/2\,cx+rt)}\sqrt{2}\sqrt{1/2\,r+1/8\,c^2}\left(A\cos\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)-B\sin\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)\right)}{A\sin\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)+B\cos\left(\sqrt{1/2\,r+1/8\,c^2}(x-ct)\right)}.
\end{aligned}
\tag{3.3.12}
$$

Case (v) corresponds to the following solution of Eq. (3.1.1):

$$
\begin{aligned}
&u_5(x,t) = r + 1/4\,c^2 \\
&-2\,\frac{\left(-1/4\,r-1/16\,c^2\right)\left(A\cos\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)-B\sin\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)\right)^2}{\left(A\sin\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)+B\cos\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)\right)^2} \\
&-\frac{1}{128}\,\frac{\left(4\,r+c^2\right)^2\left(A\sin\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)+B\cos\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)\right)^2}{(-1/4\,r-1/16\,c^2)\left(A\cos\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)-B\sin\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
&v_5(x,t) = \\
&e^{i(1/2\,cx+rt)} \left(\frac{\sqrt{2}\sqrt{-1/4\,r-1/16\,c^2}\left(A\cos\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)-B\sin\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)\right)}{A\sin\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)+B\cos\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)} \right) \\
&-e^{i(1/2\,cx+rt)}1/16 \left(\frac{\sqrt{2}\left(4\,r+c^2\right)\left(A\sin\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)+B\cos\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)\right)}{\sqrt{-1/4\,r-1/16\,c^2}\left(A\cos\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)-B\sin\left(\sqrt{-1/4\,r-1/16\,c^2}(x-ct)\right)\right)} \right).
\end{aligned}
\tag{3.3.13}
$$

The coupled Klein-Gordon-Schrödinger equation has following solution with respect to case (vi):

$$
\begin{aligned}
&u_6(x,t) = 1/2\,r + 1/8\,c^2 \\
&-2\,\frac{\left(1/8\,r+1/32\,c^2\right)\left(A\cos\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)-B\sin\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)\right)^2}{\left(A\sin\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)+B\cos\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)\right)^2} \\
&-\frac{1}{512}\,\frac{\left(4\,r+c^2\right)^2\left(A\sin\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)+B\cos\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)\right)^2}{\left(1/8\,r+1/32\,c^2\right)\left(A\cos\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)-B\sin\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)\right)^2}, \\
&v_6(x,t) = \\
&e^{i(1/2\,cx+rt)} \left(\frac{\sqrt{2}\sqrt{1/8\,r+1/32\,c^2}\left(A\cos\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)-B\sin\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)\right)}{A\sin\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)+B\cos\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)} \right) \\
&-1/32e^{i(1/2\,cx+rt)} \left(\frac{\sqrt{2}\left(4\,r+c^2\right)\left(A\sin\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)+B\cos\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)\right)}{\sqrt{1/8\,r+1/32\,c^2}\left(A\cos\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)-B\sin\left(\sqrt{1/8\,r+1/32\,c^2}(x-ct)\right)\right)} \right).
\end{aligned}
\tag{3.3.14}
$$

The traveling waves for $u_6(x,t)$ and $|v_6(x,t)|$ for $r = 8, c = 4, A = 1, B = 2$ are shown in Fig. (3.3) and Fig. (3.4) respectively.

Subcase 2. If $\mu < 0$, then we obtain

Case (i) gives the exact traveling wave solution of Eq. (3.1.1) as:

$$
\begin{aligned}
u_7(x,t) &= 1/32\,\frac{\left(4\,r+c^2\right)^2\left(A\cosh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)+B\sinh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)\right)^2}{\left(1/2\,r+1/8\,c^2\right)\left(A\sinh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)+B\cosh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)\right)^2}, \\
v_7(x,t) &= 1/8\,\frac{e^{i(1/2\,cx+rt)}\sqrt{2}\left(4\,r+c^2\right)\left(A\cosh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)+B\sinh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)\right)}{\sqrt{-1/2\,r-1/8\,c^2}\left(A\sinh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)+B\cosh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)\right)}.
\end{aligned}
\tag{3.3.15}
$$

Equation (3.1.1) possesses the following solution corresponding to Case (ii):

$$
\begin{aligned}
&u_8(x,t) = 3/4\,r + 3/16\,c^2 - 3/16\,\sqrt{16\,r^2+8\,rc^2+c^4-16} - \\
&\left(\tfrac{3}{32} + (\tfrac{3r}{2} + \tfrac{3c^2}{8})\ \left(\tfrac{-r}{8} - \tfrac{c^2}{32} + \tfrac{1}{32}\,\sqrt{16\,r^2+8\,rc^2+c^4-16}\right)\right) \\
&\frac{\left(A\cosh(\sqrt{-\mu}(x-2\,pt))+B\sinh(\sqrt{-\mu}(x-2\,pt))\right)^2\mu^{-1}}{\left(A\sinh(\sqrt{-\mu}(x-2\,pt))+B\cosh(\sqrt{-\mu}(x-2\,pt))\right)^2}, \\
&v_8(x,t) = e^{i(px+rt)}\left(3\,\sqrt{-1/8\,r-1/32\,c^2+1/32\,\sqrt{16\,r^2+8\,rc^2+c^4-16}}\right) \\
&-3e^{i(px+rt)}\left(\frac{\left(-1/8\,r-1/32\,c^2+1/32\,\sqrt{16\,r^2+8\,rc^2+c^4-16}\right)^{3/2}\left(A\cosh(\sqrt{-\mu}(x-2\,pt))+B\sinh(\sqrt{-\mu}(x-2\,pt))\right)^2}{\mu\left(A\sinh(\sqrt{-\mu}(x-2\,pt))+B\cosh(\sqrt{-\mu}(x-2\,pt))\right)^2} \right),
\end{aligned}
\tag{3.3.16}
$$

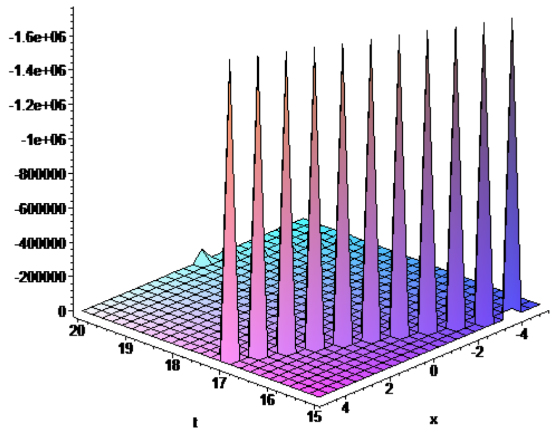

FIGURE 3.3: Traveling Solution of the field $u_6(x,t)$

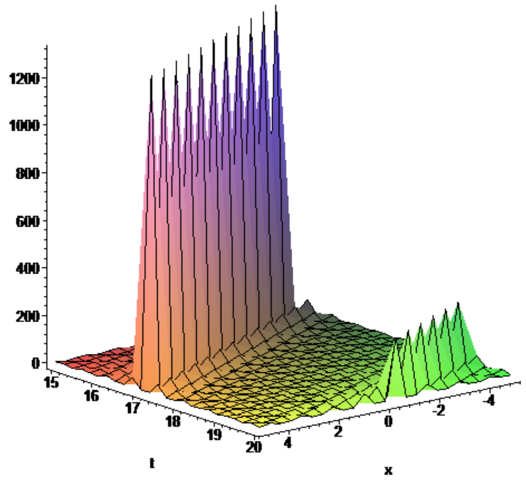

FIGURE 3.4: Traveling Solution of the field $|v_6(x,t)|$

where μ, p are given by $\mu = \frac{-r}{8} - \frac{c^2}{32} + \frac{\sqrt{16r^2+8rc^2+c^4-16}}{32}$, $p = \sqrt{\frac{-r}{2} + \frac{c^2}{8} - \frac{\sqrt{16r^2+8rc^2+c^4-16}}{8}}$. Case (iii) corresponds to the following values of $u(x,t)$ and $v(x,t)$:

$$\begin{aligned}
&u_9(x,t) = \\
&\frac{-3}{2}\,\frac{-c^4+1/4\,c^2\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-2+r\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-4\,rc^2}{-c^6+(1/4\,c^4+2rc^2+4r^2)\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-3\,c^2-12\,r+\sqrt{c^4+8\,rc^2+16\,r^2+8}-8\,rc^4-16\,r^2c^2} \\
&+6\,\frac{\mu\,(A\sinh(\sqrt{-\mu}(x-2\,pt))+B\cosh(\sqrt{-\mu}(x-2\,pt)))^2}{(A\cosh(\sqrt{-\mu}(x-2\,pt))+B\sinh(\sqrt{-\mu}(x-2\,pt)))^2} \\
&-\frac{3}{32}\,\frac{(A\cosh(\sqrt{-\mu}(x-2\,pt))+B\sinh(\sqrt{-\mu}(x-2\,pt)))^2}{\left(-c^4+(1/4\,c^2+r)\left(2\,c^2-8\,r+2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)-2-4\,rc^2\right)\mu\,(A\sinh(\sqrt{-\mu}(x-2\,pt))+B\cosh(\sqrt{-\mu}(x-2\,pt)))^2},
\end{aligned}$$

$$\begin{aligned}
&v_9(x,t) = \\
&e^{i(px+rt)}\left(-6\,\frac{\sqrt{-1/2\,c^2-2\,r+1/2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}}\,\mu\,(A\sinh(\sqrt{-\mu}(x-2\,pt))+B\cosh(\sqrt{-\mu}(x-2\,pt)))^2}{(A\cosh(\sqrt{-\mu}(x-2\,pt))+B\sinh(\sqrt{-\mu}(x-2\,pt)))^2}\right) \\
&+e^{i(px+rt)}\left(\frac{3}{32}\,\frac{(A\cosh(\sqrt{-\mu}(x-2\,pt))+B\sinh(\sqrt{-\mu}(x-2\,pt)))^2}{\left(-1/2\,c^2-2\,r+1/2\,\sqrt{c^4+8\,rc^2+16\,r^2+8}\right)^{3/2}\mu\,(A\sinh(\sqrt{-\mu}(x-2\,pt))+B\cosh(\sqrt{-\mu}(x-2\,pt)))^2}\right),
\end{aligned} \tag{3.3.17}$$

where μ, p are given by $\mu = -\left(-4\,c^2 - 16\,r + 4\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}\right)^{-1}$, $p = 1/4\,\sqrt{2\,c^2 - 8\,r + 2\,\sqrt{c^4 + 8\,rc^2 + 16\,r^2 + 8}}$. The traveling waves for $u_9(x,t)$ and $|v_9(x,t)|$ for $r = 1, c = 2, A = 1, B = 1/2$ are shown in Fig. (3.5) and Fig. (3.6) respectively.

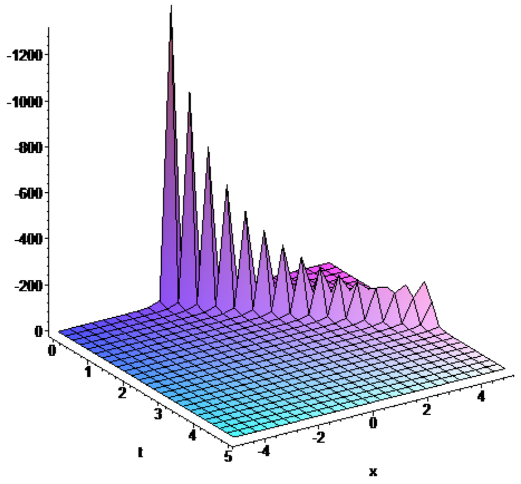

FIGURE 3.5: Traveling Solution of the field $u_9(x,t)$

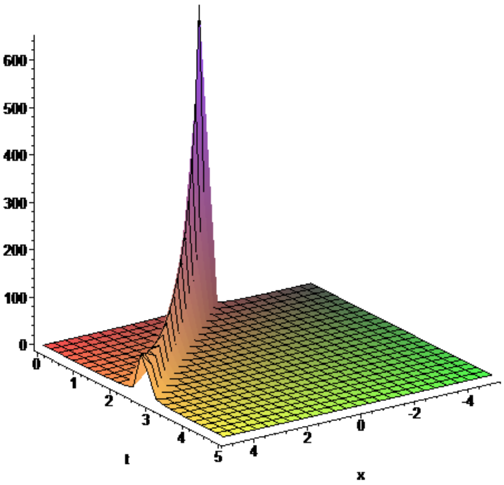

FIGURE 3.6: Traveling Solution of the field $|v_9(x,t)|$

Case (iv) yields the following solution of Eq. (3.1.1):

$$\begin{aligned} u_{10}(x,t) &= 2\,\frac{\left(1/2\,r+1/8\,c^2\right)\left(A\sinh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)+B\cosh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)\right)^2}{\left(A\cosh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)+B\sinh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)\right)^2}, \\ v_{10}(x,t) &= \frac{e^{i(1/2\,cx+rt)}\sqrt{2}\sqrt{-1/2\,r-1/8\,c^2}\left(A\sinh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)+B\cosh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)\right)}{A\cosh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)+B\sinh\left(\sqrt{-1/2\,r-1/8\,c^2}(x-ct)\right)}. \end{aligned} \tag{3.3.18}$$

Case (v) gives the following traveling wave solution:

$$\begin{aligned} u_{11}(x,t) &= r+1/4\,c^2 \\ &+2\,\frac{\left(-1/4\,r-1/16\,c^2\right)\left(A\sinh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)+B\cosh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)\right)^2}{\left(A\cosh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)+B\sinh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)\right)^2} \\ &+\frac{1}{128}\,\frac{\left(4\,r+c^2\right)^2\left(A\cosh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)+B\sinh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)\right)^2}{\left(-1/4\,r-1/16\,c^2\right)\left(A\sinh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)+B\cosh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)\right)^2}, \end{aligned}$$

$$\begin{aligned} &v_{11}(x,t) = \\ &e^{i(1/2\,cx+rt)}\left(\frac{\sqrt{2}\sqrt{1/4\,r+1/16\,c^2}\left(A\sinh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)+B\cosh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)\right)}{A\cosh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)+B\sinh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)}\right) \\ &-1/16e^{i(1/2\,cx+rt)}\left(\frac{\sqrt{2}\left(4\,r+c^2\right)\left(A\cosh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)+B\sinh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)\right)}{\sqrt{1/4\,r+1/16\,c^2}\left(A\sinh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)+B\cosh\left(\sqrt{1/4\,r+1/16\,c^2}(x-ct)\right)\right)}\right). \end{aligned} \tag{3.3.19}$$

The traveling waves for $u_{11}(x,t)$ and $|v_{11}(x,t)|$ for $r=8, c=4, A=1, B=-2$ are shown in Fig. (3.7) and Fig. (3.8) respectively.

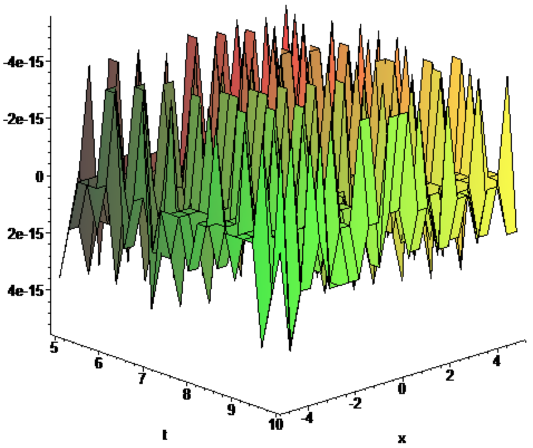

FIGURE 3.7: Traveling Solution of the field $u_{11}(x,t)$

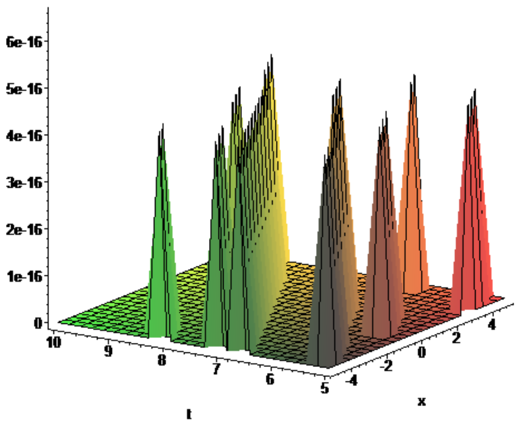

FIGURE 3.8: Traveling Solution of the field $|v_{11}(x,t)|$

Equation (3.1.1) has the following solution corresponds to Case (vi):

$$
\begin{aligned}
&u_{12}(x,t) = 1/2\,r + 1/8\,c^{2}\\
&+2\,\frac{\left(1/8\,r+1/32\,c^{2}\right)\left(A\sinh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)+B\cosh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)\right)^{2}}{\left(A\cosh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)+B\sinh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)\right)^{2}}\\
&+\frac{1}{512}\,\frac{\left(4\,r+c^{2}\right)^{2}\left(A\cosh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)+B\sinh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)\right)^{2}}{(1/8\,r+1/32\,c^{2})\left(A\sinh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)+B\cosh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)\right)^{2}},\\
&v_{12}(x,t) =\\
&e^{i(1/2\,cx+rt)}\left(\frac{\sqrt{2}\sqrt{-1/8\,r-1/32\,c^{2}}\left(A\sinh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)+B\cosh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)\right)}{A\cosh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)+B\sinh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)}\right)\\
&-1/32e^{i(1/2\,cx+rt)}\left(\frac{\sqrt{2}\left(4\,r+c^{2}\right)\left(A\cosh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)+B\sinh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)\right)}{\sqrt{-1/8\,r-1/32\,c^{2}}\left(A\sinh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)+B\cosh\left(\sqrt{-1/8\,r-1/32\,c^{2}}(x-ct)\right)\right)}\right).
\end{aligned}
\tag{3.3.20}
$$

3.4 Discussions and Concluding Remarks

In this chapter, we have established the traveling wave solutions of the Klein-gordon-Schrödinger Eq. (3.1.1) using Lie classical method and the modified (G'/G)-expansion method. These traveling wave solutions are expressed in terms of hyperbolic, trigonometric and rational functions involving arbitrary parameters. When these parameters are taken special values, the solitary waves are derived from the traveling waves. It has been shown that the proposed method is direct, concise, basic and effective and easy to calculate, and it is a powerful mathematical tool for obtaining exact traveling wave solutions of nonlinear evolution equations and can be used to solve other nonlinear PDEs in mathematical physics.

Keeping in view the efficacy and physical importance of the Klein-Gordon-Schrödinger equations, we have studied here the coupled KGS equation in variable form. By the Lie classical approach, we have investigated the symmetries of VCKGS equations and utilized these symmetries for obtaining group infinitesimals that are helpful in the reduction of a system of PDEs to a system of ODEs. After that by solving the reduced ODEs, new exact solutions are obtained.

Remark 3.4.1. By applying the Lie classical approach, we are able to find vector fields that are used to derive exact solutions of the nonlinear system (3.1.3) in variable form. Corresponding to certain vector fields, we have obtained solutions involving special functions such as Airy wave functions, Bessel functions, Mathieu functions etc.

Remark 3.4.2. Here, the variable form of KGS equation has been studied for new symmetry reductions and exact solutions that are not found in the literature.

It is worth mentioning here that the authenticity of all the solutions has been checked with the aid of Maple software.

Chapter 4

The (2+1)-Dimensional potential Kadomstev Petviashvili Equation and its Generalized Form[1]

4.1 Introduction

The (2+1)-dimensional PKP equation [21, 154]

$$\sigma_{xt} + \frac{3}{2}\sigma_x\sigma_{xx} + \frac{1}{4}\sigma_{xxxx} + \frac{3}{4}\sigma_{yy} = 0, \tag{4.1.1}$$

describes the dynamics of 2-dimensional, small, but finite amplitude waves and solitons in a variety of media, for example, in plasma physics, hydrodynamics and solid-state physics. This equation is also derived in various physical contexts assuming that the wave is moving along x and all changes in y are slower than in the direction of motion [91]. It is well known that the PKP equation arises in number of remarkable non-linear problems both in physics and mathematics. The mathematical interest of this equation stems from the fact that it is associated with certain infinite-dimensional Lie algebras and groups [152]. The solutions of PKP equation have been studied extensively in various aspects since they were first found. By using various techniques and methods exact traveling wave solutions, linearly solitary wave solutions, soliton-like solutions and some numerical solutions were obtained. Senthivelan [98] has studied the traveling wave reductions for certain (2+1)-dimensional and (3+1)-dimensional physically important nonlinear evolutionary equations by using the homogeneous balance method. In [33, 34], Li and Zhang obtained some exact solutions by improving on the key steps of homogeneous

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[1] A part of this chapter has been published in **Applied Mathematics and Computation, 219 (2013) 5290-5302**.

balance method. Kaya and El-Sayed [30] have presented an Adomian's decomposition method for obtaining the numerical soliton-like solutions of PKP equation. Inan and Kaya [55] found some exact linear soliton solutions by improved tanh function method. Exact periodic kink-wave solution, periodic soliton and doubly periodic solutions for the PKP equation are obtained using homoclinic test technique and extended homoclinic test technique in [169]. Z. Dai et al. [170, 171] discovered singular periodic soliton and spatial temporal bifurcation for KP equation. Zhu and Geng [71] obtained N-soliton solution of the variable coefficient KP equation by using Pfaffian technique. Here, we obtain the similarity reductions of the PKP equation followed by their exact group-invariant solutions.

The physical situations in which nonlinear equations arise tend to be highly idealized due to assumption of constant coefficients. Due to this, much attention has been paid on study of nonlinear equations with variable coefficients to obtain exact solutions and some recent contributions are in [72, 73, 122]. These exact solutions provide much information about nonlinear phenomena and well descibed various aspects of the physical phenomena and these solutions are also useful to discuss and examine the sensitivity of physical phenomena with several important parameters described by variable coefficients. In this chapter, we will also study the generalized form of PKP equation i.e. the (2+1)-dimensional PKP equation with variable coefficients

$$\sigma_{xt} + \alpha(t)\sigma_x\sigma_{xx} + \beta(t)\sigma_{xxxx} + \delta(t)\sigma_{yy} = 0, \tag{4.1.2}$$

where $\alpha(t), \beta(t)$ and $\delta(t)$ are arbitrary functions.

The outline of this chapter is as follows: In section (4.2), by using the Lie group theory, we find out the symmetries of PKP equation (4.1.1), utilize the symmetries to reduce equation to (1+1)-dimensional linear and nonlinear PDEs and then construct the exact solutions. In section (4.3), we investigate variable coefficient potential Kadomstev Petviashvili (VCPKP) equation (4.1.2) for integrability and for the exact solutions using combination of Lie classical method and several other methods including extended (G'/G)-expansion method and some concluding remarks are given in last section.

4.2 The potential Kadomstev Petviashvili Equation

In this section, we will first determine the Lie point symmetries of (4.1.1) and then use them to construct some exact solutions.

4.2.1 Invariance and Infinitesimal Characterization

To apply the classical method to (2+1)-dimensional PKP equation, we consider the one parameter Lie group of infinitesimal transformations in (x, y, t, σ). The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$V = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial \sigma}. \tag{4.2.1}$$

We then require that the transformations leave the set of solutions of (4.1.1) invariant. This yields an overdetermined following linear system of equations for the infinitesimals $\xi(x, y, t, \sigma), \phi(x, y, t, \sigma), \tau(x, y, t, \sigma)$ and $\eta(x, y, t, \sigma)$:

$$\begin{aligned}
&(i)\ \phi_x = 0,\ \ \phi_\sigma = 0,\\
&(ii)\ \tau_x = 0,\ \ \tau_t = 0,\ \ \tau_\sigma = 0,\\
&(iii)\ \xi_\sigma = 0,\\
&(iv)\ \eta_{\sigma\sigma} = 0,\\
&(v)\ \tfrac{1}{4}\tau_t - \tfrac{3}{4}\xi_x = 0,\\
&(vi)\ \tfrac{-3}{2}\xi_{xx} + \eta_{x\sigma} = 0,\\
&(vii)\ \phi_t + \tfrac{3}{2}\xi_y = 0,\\
&(viii)\ \tfrac{3}{4}\xi_x - \tfrac{3}{2}\phi_y + \tfrac{3}{4}\tau_t = 0,\\
&(ix)\ \tfrac{3}{2}\eta_\sigma + \tfrac{3}{2}\tau_t - 3\xi_x = 0,\\
&(x)\ -\xi_{xxx} - \xi_t + \tfrac{3}{2}\eta_{xx\sigma} + \tfrac{3}{2}\eta_x = 0,\\
&(xi)\ 3\eta_{x\sigma} - \tfrac{3}{2}\xi_{xx} = 0,\\
&(xii)\ -\tfrac{3}{4}\xi_{yy} - \tfrac{1}{4}\xi_{xxxx} + \eta_{xxx\sigma} - \xi_{xt} + \tfrac{3}{2}\eta_{xx} + \eta_{t\sigma} = 0,\\
&(xiii)\ -\tfrac{3}{4}\phi_{yy} + \tfrac{3}{2}\eta_{y\sigma} = 0,\\
&(xiv)\ \eta_{xt} + \tfrac{1}{4}\eta_{xxxx} + \tfrac{3}{4}\eta_{yy} = 0.
\end{aligned} \tag{4.2.2}$$

The general solution of this large system provides following forms for the infinitesimal elements η, ξ, ϕ and τ:

$$\begin{aligned}
\eta &= -f(t)\sigma + \tfrac{x^2}{3}f'(t) + x(-\tfrac{4}{9}y^2 f''(t) - \tfrac{4}{9}yg''(t) + \tfrac{2}{3}h'(t)) + \tfrac{4}{81}y^4 f'''(t)\\
&\qquad + \tfrac{8}{81}y^3 g'''(t) - \tfrac{4}{9}y^2 h''(t),\\
\xi &= xf(t) - \tfrac{2}{3}y^2 f'(t) - \tfrac{2}{3}yg'(t) + h(t),\\
\phi &= 2yf(t) + g(t),\\
\tau &= 3\textstyle\int f(t),
\end{aligned} \tag{4.2.3}$$

where $f(t), g(t)$ and $h(t)$ are arbitrary functions of t in a C^∞ class and the primes denote time derivatives. We find that the symmetry algebra of the PKP equation is infinite dimensional and depends on three arbitrary functions of time. The general element of the symmetry algebra of the PKP equation can be written as $V = X(f) + Y(g) + Z(h)$, where

$$\begin{aligned} X(f) &= 2yf(t)\tfrac{\partial}{\partial y} + 3\textstyle\int f(t)\tfrac{\partial}{\partial t} + xf(t)\tfrac{\partial}{\partial x} - \tfrac{2}{3}y^2 f'(t)\tfrac{\partial}{\partial x} - \sigma f(t)\tfrac{\partial}{\partial \sigma} \\ &\quad + \tfrac{1}{3}x^2 f'(t)\tfrac{\partial}{\partial \sigma} - \tfrac{4}{9}xy^2 f''(t)\tfrac{\partial}{\partial \sigma} + \tfrac{4}{81}y^4 f'''(t)\tfrac{\partial}{\partial \sigma}, \\ Y(g) &= g(t)\tfrac{\partial}{\partial y} - \tfrac{2}{3}yg'(t)\tfrac{\partial}{\partial x} - \tfrac{4}{9}xyg''(t)\tfrac{\partial}{\partial \sigma} + \tfrac{8}{81}y^3 g'''(t)\tfrac{\partial}{\partial \sigma}, \\ Z(h) &= h(t)\tfrac{\partial}{\partial x} + \tfrac{2}{3}xh'(t)\tfrac{\partial}{\partial \sigma} - \tfrac{4}{9}y^2 h''(t)\tfrac{\partial}{\partial \sigma}. \end{aligned} \tag{4.2.4}$$

4.2.2 Similarity Variables, Similarity Forms and Reduction of Independent Variables

Having determined the infinitesimals, we find the symmetry variables by solving the invariant surface condition

$$\Phi \equiv \xi\tfrac{\partial}{\partial x} + \phi\tfrac{\partial}{\partial y} + \tau\tfrac{\partial}{\partial t} - \eta = 0. \tag{4.2.5}$$

The general solution of above equation involves three constants, two become independent variables ρ, ψ and other plays the role of new dependent variable $H(\rho,\psi)$. By solving the equation (4.2.5) we have

$$\sigma(x,y,t) = \chi(x,y,t) + \frac{1}{(\int f(t))^{\frac{1}{3}}}H(\rho,\psi), \tag{4.2.6}$$

where

$$\begin{aligned} \chi(x,y,t) &= \tfrac{1}{9}\tfrac{x^2 f(t)}{(\int f(t))} - \tfrac{4}{27}\tfrac{xy^2 f'(t)}{(\int f(t))} + \tfrac{8}{81}\tfrac{xy^2 f(t)^2}{(\int f(t))^2} + \tfrac{4}{243}\tfrac{y^4 f''(t)}{(\int f(t))} - \tfrac{8}{243}\tfrac{y^4 f(t)f'(t)}{(\int f(t))^2} + \tfrac{40}{2187}\tfrac{y^4 f(t)^3}{(\int f(t))^3} \\ &- \tfrac{4}{27}\tfrac{xyg'(t)}{(\int f(t))} + \tfrac{8}{243}\tfrac{y^3 g''(t)}{(\int f(t))} + \tfrac{2}{9}\tfrac{xh(t)}{(\int f(t))} - \tfrac{4}{27}\tfrac{y^2 h'(t)}{(\int f(t))} - \tfrac{32}{729}\tfrac{y^3 g'(t)f(t)}{(\int f(t))^2} + \tfrac{8}{81}\tfrac{xyf(t)g(t)}{(\int f(t))^2} \\ &+ \tfrac{80}{2187}\tfrac{y^3 g(t)f(t)^2}{(\int f(t))^3} - \tfrac{16}{729}\tfrac{y^3 g(t)f'(t)}{(\int f(t))^2} + \tfrac{8}{81}\tfrac{yg(t)h(t)}{(\int f(t))^2} + \tfrac{2}{81}\tfrac{xg(t)^2}{(\int f(t))^2} - \tfrac{8}{243}\tfrac{y^2 g(t)g'(t)}{(\int f(t))^2} + \tfrac{20}{729}\tfrac{y^2 f(t)g(t)^2}{(\int f(t))^3} \\ &+ \tfrac{20}{2187}\tfrac{yg(t)^3}{(\int f(t))^3} + \tfrac{8}{81}\tfrac{y^2 h(t)f(t)}{(\int f(t))^2}. \end{aligned} \tag{4.2.7}$$

$$\begin{aligned} \rho &= \frac{y}{(\int f(t))^{\frac{2}{3}}} - \frac{1}{3}\int \frac{g(t)}{(\int f(t))^{\frac{5}{3}}}, \\ \psi &= \frac{x}{(\int f(t))^{\frac{1}{3}}} + \frac{2}{9}\frac{yg(t)}{(\int f(t))^{\frac{4}{3}}} + \frac{2}{9}\frac{y^2 f(t)}{(\int f(t))^{\frac{4}{3}}} - \frac{1}{3}\int \frac{h(t)}{(\int f(t))^{\frac{4}{3}}} - \frac{2}{27}\int \frac{g(t)^2}{(\int f(t))^{\frac{7}{3}}}. \end{aligned} \tag{4.2.8}$$

Here $\sigma(x,y,t)$ is a solution of the PKP equation for any sufficiently smooth function $f(t) \neq 0$ if and only if $H(\rho,\psi)$ satisfies the equation

$$\tfrac{3}{4}H_{\rho\rho} + \tfrac{3}{2}H_{\psi}H_{\psi\psi} + \tfrac{1}{4}H_{\psi\psi\psi\psi} = 0. \tag{4.2.9}$$

If $f(t) = 0$, then on solving the characterstic equation (4.2.5) we have

$$\sigma(x,y,t) = \nu(x,y,t) + \frac{1}{\sqrt{g(t)}}H(\rho,\psi), \tag{4.2.10}$$

where

$$\begin{aligned}\nu(x,y,t) = {} & \tfrac{1}{6}\tfrac{x^2g'(t)}{g(t)} + \tfrac{1}{9}\tfrac{xy^2g'(t)^2}{g(t)^2} - \tfrac{1}{3}\tfrac{xyh(t)g'(t)}{g(t)^2} - \tfrac{1}{2}\tfrac{xh(t)^2}{g(t)^2} - \tfrac{2}{9}\tfrac{xy^2g''(t)}{g(t)} - \tfrac{1}{27}\tfrac{y^4g'(t)g''(t)}{g(t)^2} \\ & + \tfrac{2}{27}\tfrac{y^3h(t)g''(t)}{g(t)^2} + \tfrac{2}{81}\tfrac{y^4g'''(t)}{g(t)} + \tfrac{2}{3}\tfrac{xyh'(t)}{g(t)} + \tfrac{4}{27}\tfrac{y^3h'(t)g'(t)}{g(t)^2} + \tfrac{2}{3}\tfrac{y^2h(t)h'(t)}{g(t)^2} - \tfrac{4}{27}\tfrac{y^3h''(t)}{g(t)} + \tfrac{1}{54}\tfrac{y^4g'(t)^3}{g(t)^3} \\ & - \tfrac{1}{9}\tfrac{y^3h(t)g'(t)^2}{g(t)^3} - \tfrac{1}{2}\tfrac{y^2g'(t)h(t)^2}{g(t)^3}.\end{aligned} \tag{4.2.11}$$

$$\begin{aligned}\rho &= \textstyle\int_0^t g(s)^{\frac{-3}{2}}\,ds, \\ \psi &= \tfrac{x}{g(t)^{\frac{1}{2}}} - \tfrac{yh(t)}{g(t)^{\frac{3}{2}}} + \tfrac{1}{3}\tfrac{y^2g'(t)}{g(t)^{\frac{3}{2}}}.\end{aligned} \tag{4.2.12}$$

In this case $\sigma(x,y,t)$ is a solution of the PKP equation for $g(t) \neq 0$ if and only if $H(\rho,\psi)$ satisfies the equation

$$H_{\rho\psi} + \tfrac{3}{2}H_{\psi}H_{\psi\psi} + \tfrac{1}{4}H_{\psi\psi\psi\psi} = 0. \tag{4.2.13}$$

If $f(t) = 0$ and $g(t) = 0$, following the same way as in above cases, we get

$$\sigma(x,y,t) = \tfrac{1}{3}\tfrac{x^2h'(t)}{h(t)} - \tfrac{4}{9}\tfrac{xy^2h''(t)}{h(t)} + \tfrac{4}{81}\tfrac{y^4h'''(t)}{h(t)} + H(y,t). \tag{4.2.14}$$

Here (4.2.14) solves the PKP equation for any sufficiently smooth $h(t) \neq 0$ if and only if $H(y,t)$ solves the linear equation

$$H_{yy} = 0. \tag{4.2.15}$$

Combining the above results, we obtain some reduced equations of Eq. (4.1.1) expressed by Eqs. (4.2.9), (4.2.13) and (4.2.15), respectively. Meanwhile many new solutions of Eq. (4.1.1) from these reduced Eqs. can be achieved (as we have done in the next section).

4.2.3 Solutions of PKP Equation Obtained by Symmetry Reduction

Our main goal is to derive exact solutions of (4.1.1) as exact solutions are helpful for mathematical as well as physical description. In this section, we will look for some exact solutions of Eq. (4.1.1) by the reduced equations in above section.

4.2.3.1 Exact Solutions of Eq. (4.2.9)

Next, we look for the group-invariant solutions to Eq. (4.2.9). The finite-dimensional symmetries of Eq. (4.2.9) form a five-dimensional Lie algebra generated by following vector fields:

$$\begin{aligned} Z_1 &= \psi\frac{\partial}{\partial\psi} + 2\rho\frac{\partial}{\partial\rho} - H\frac{\partial}{\partial H} \\ Z_2 &= \frac{\partial}{\partial\psi} \\ Z_3 &= \frac{\partial}{\partial\rho} \\ Z_4 &= \rho\frac{\partial}{\partial H} \\ Z_5 &= \frac{\partial}{\partial H}. \end{aligned} \tag{4.2.16}$$

The communication relations between these vector fields is given in Table (4.1). To each s-parameter subgroup there corresponds a family of group invariant solutions. So, in general, it is quite impossible to determine all possible group-invariant solutions of a PDE. In order to minimize this search, it is useful to construct the optimal system of solutions. It is well-known that the problem of the construction of the optimal system of solutions is equivalent to that of the construction of the optimal system of subalgebras. Here, we will deal with the construction of the optimal system of one-dimensional subalgebras of (4.2.16). The construction of the one-dimensional optimal system of subalgebras can be carried out by using a global matrix of the adjoint transformations as suggested in [83, 111]. The latter problem, tends to determine a list (that is called an optimal system) of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation. To deal with the optimal system of subalgebras of (4.2.16), the adjoint representation is given in Table (4.2).

TABLE 4.1: Commutator Table

	Z_1	Z_2	Z_3	Z_4	Z_5
Z_1	0	$-Z_2$	$-2Z_3$	$3Z_4$	Z_5
Z_2	Z_2	0	0	0	0
Z_3	$2Z_3$	0	0	Z_5	0
Z_4	$-3Z_4$	0	$-Z_5$	0	0
Z_5	Z_5	0	0	0	0

TABLE 4.2: Adjoint Table

	Z_1	Z_2	Z_3	Z_4	Z_5
Z_1	Z_1	Z_2e^{ϵ}	$Z_3e^{2\epsilon}$	$Z_4e^{-3\epsilon}$	$Z_5e^{-\epsilon}$
Z_2	$Z_1-\epsilon Z_2$	Z_2	Z_3	Z_4	Z_5
Z_3	$Z_1-2\epsilon Z_3$	Z_2	Z_3	$Z_4-\epsilon Z_5$	Z_5
Z_4	$Z_1+3\epsilon Z_4$	Z_2	$Z_3+\epsilon Z_5$	Z_4	Z_5
Z_5	$Z_1+\epsilon Z_5$	Z_2	Z_3	Z_4	Z_5

The optimal system for (4.2.16) consists of following vector fields:

$$(i)\ Z_1,\ \ (ii)\ Z_2+\beta Z_4,\ \ (iii)\ Z_2+Z_3+\beta Z_4,\ \ (iv)\ Z_3+Z_4,\ \ (v)\ Z_3-Z_4,$$
$$(vi)\ Z_3,\ \ (vii)\ Z_4,\ \ (viii)\ Z_5. \tag{4.2.17}$$

In Table (4.3), we list the similarity variable, form and the reduced ODEs corresponding to the optimal system, since corresponding to the vector fields Z_4 and Z_5, PDE (4.2.9) is identically satisfied, so in the table we list the remaining vector fields.

Table 4.3: Similarity Reductions of PDE (4.2.9) to ODEs

Essential fields	*Similarity variable*(ζ)	*Similarity solution*(H)	*ReducedODEs*
Z_1	$\frac{\psi}{\sqrt{\rho}}$	$\frac{1}{\sqrt{\rho}}F(\zeta)$	$\frac{9}{16}F(\zeta)+\frac{15}{16}\zeta F'(\zeta)+\frac{3}{16}\zeta^2F''(\zeta)$ $+\frac{3}{2}F'(\zeta)F''(\zeta)+\frac{1}{4}F''''(\zeta)=0$
$Z_2+\beta Z_4$	ρ	$\beta\rho\psi+F(\zeta)$	$F''(\zeta)=0$
$Z_2+Z_3+\beta Z_4$	$\psi-\rho$	$\frac{\beta}{2}\rho^2+F(\zeta)$	$\frac{3\beta}{4}+\frac{3}{4}F''(\zeta)+\frac{3}{2}F'(\zeta)F''(\zeta)+\frac{1}{4}F''''(\zeta)=0$
Z_3+Z_4	ψ	$\frac{\rho^2}{2}+F(\zeta)$	$\frac{3}{4}+\frac{3}{2}F'(\zeta)F''(\zeta)+\frac{1}{4}F''''(\zeta)=0$
Z_3-Z_4	ψ	$\frac{-\rho^2}{2}+F(\zeta)$	$\frac{-3}{4}+\frac{3}{2}F'(\zeta)F''(\zeta)+\frac{1}{4}F''''(\zeta)=0$
Z_3	ψ	$F(\zeta)$	$\frac{3}{2}F'(\zeta)F''(\zeta)+\frac{1}{4}F''''(\zeta)=0$

1. **Vector field Z_1:** To solve the reduced ODE in this case, we seek a special solution in the following form:

$$F(\zeta)=C_0\zeta^{-1}+C_1+C_2\zeta, \tag{4.2.18}$$

where C_0, C_1, C_2 are arbitrary constants. On substituting (4.2.18) in the reduced ODE, we get values of constants as $C_0=2, C_1=0, C_2=0$ and correseponding to values of constants, solution of equation (4.1.1) is given as:

$$\begin{aligned}&\sigma(x,y,t)=\chi(x,y,t)\\&+\tfrac{2}{(\int f(t))^{\frac{1}{3}}}(\tfrac{x}{(\int f(t))^{\frac{1}{3}}}+\tfrac{2}{9}\tfrac{yg(t)}{(\int f(t))^{\frac{4}{3}}}+\tfrac{2}{9}\tfrac{y^2f(t)}{(\int f(t))^{\frac{4}{3}}}-\tfrac{1}{3}\int\tfrac{h(t)}{(\int f(t))^{\frac{4}{3}}}-\tfrac{2}{27}\int\tfrac{g(t)^2}{(\int f(t))^{\frac{7}{3}}})^{-1},\end{aligned} \tag{4.2.19}$$

where $\chi(x,y,t)$ is given by equation (4.2.7).

4. **Vector field $Z_2+\beta Z_4$:** The reduced ODE has a solution

$$F(\zeta)=C_0+C_1\zeta, \tag{4.2.20}$$

where C_0 and C_1 are arbitrary constants. The solution (4.2.20) leads by back substitution to the solution of equation (4.1.1) of the form

$$\begin{aligned}&\sigma(x,y,t)=\chi(x,y,t)+\tfrac{1}{(\int f(t))^{\frac{1}{3}}}(\beta(\tfrac{y}{(\int f(t))^{\frac{2}{3}}}-\tfrac{1}{3}\int\tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}})\\&(\tfrac{x}{(\int f(t))^{\frac{1}{3}}}+\tfrac{2}{9}\tfrac{yg(t)}{(\int f(t))^{\frac{4}{3}}}+\tfrac{2}{9}\tfrac{y^2f(t)}{(\int f(t))^{\frac{4}{3}}}-\tfrac{1}{3}\int\tfrac{h(t)}{(\int f(t))^{\frac{4}{3}}}-\tfrac{2}{27}\int\tfrac{g(t)^2}{(\int f(t))^{\frac{7}{3}}})\\&+(C_0+C_1(\tfrac{y}{(\int f(t))^{\frac{2}{3}}}-\tfrac{1}{3}\int\tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}}))),\end{aligned} \tag{4.2.21}$$

where $\chi(x,y,t)$ is given by equation (4.2.7).

3. **Vector field $Z_2+Z_3+\beta Z_4$:**

Case (i): Integrating the reduced ODE (as shown in Table 4.3) with respect to ζ, we get

$$\frac{3\beta}{4}\zeta+\frac{3}{4}F'(\zeta)+\frac{3}{4}F'(\zeta)^2+\frac{1}{4}F'''(\zeta)+K_1=0, \tag{4.2.22}$$

where K_1 is arbitrary constant. Multiplying Eq. (4.2.22) with $F''(\zeta)$ and again integrating, we get the reduced equation as follows:

$$\frac{3\beta}{4}\zeta F'(\zeta)-\frac{3\beta}{4}F(\zeta)+\frac{3}{8}F'(\zeta)^2+\frac{1}{4}F'(\zeta)^3+\frac{1}{8}F''(\zeta)^2+K_1F'(\zeta)+K_2=0, \tag{4.2.23}$$

where K_2 is arbitrary constant. Let us assume that Eq. (4.2.23) takes the solution in the form

$$F(\zeta)=C_0\zeta^{-1}+C_1+C_2\zeta, \tag{4.2.24}$$

where C_0, C_1, C_2 are constants to be determined. On substituting (4.2.24) in Eq. (4.2.23), we get the values of constants as $C_0=0, C_1=1/6\,\frac{3\,{C_2}^2+2\,{C_2}^3+8\,K_1\,C_2+8\,K_2}{\beta}, C_2=$

C_2, which corresponds to the following solution of Eq. (4.1.1):

$$\begin{aligned}
&\sigma(x,y,t) = \chi(x,y,t) + \tfrac{1}{(\int f(t))^{\frac{1}{3}}}\big(\tfrac{\beta}{2}\big(\tfrac{y}{(\int f(t))^{\frac{2}{3}}} - \tfrac{1}{3}\int \tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}}\big)^2 + 1/6\,\tfrac{3\,{C_2}^2+2\,{C_2}^3+8\,K_1\,C_2+8\,K_2}{\beta}\\
&+C_2\big(\tfrac{x}{(\int f(t))^{\frac{1}{3}}} + \tfrac{2}{9}\tfrac{yg(t)}{(\int f(t))^{\frac{4}{3}}} + \tfrac{2}{9}\tfrac{y^2f(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{1}{3}\int \tfrac{h(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{2}{27}\int \tfrac{g(t)^2}{(\int f(t))^{\frac{7}{3}}}\big)\\
&-C_2\big(\tfrac{y}{(\int f(t))^{\frac{2}{3}}} - \tfrac{1}{3}\int \tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}}\big)\big),
\end{aligned} \tag{4.2.25}$$

where $\chi(x,y,t)$ is given by Eq. (4.2.7).

Case (ii): For $\beta = 0$, the reduced ODE has the following solutions:

$$\begin{aligned}
&(i)\ \ F(\zeta) = C_0 + C_1\zeta\\
&(ii)\ \ F(\zeta) = C_0 + \sqrt{3}\cot(C_1 + \tfrac{\sqrt{3}}{2}\zeta)\\
&(iii)\ \ F(\zeta) = C_0 - \sqrt{3}\tan(C_1 + \tfrac{\sqrt{3}}{2}\zeta),
\end{aligned} \tag{4.2.26}$$

where C_0 and C_1 are arbitrary constants. The solutions (4.2.26) leads by back substitution to the solution of equation (4.1.1) of the form

$$\begin{aligned}
&(i)\ \ \sigma(x,y,t) = \chi(x,y,t) + \tfrac{1}{(\int f(t))^{\frac{1}{3}}}\big(C_0 + C_1\big(\big(\tfrac{x}{(\int f(t))^{\frac{1}{3}}} + \tfrac{2}{9}\tfrac{yg(t)}{(\int f(t))^{\frac{4}{3}}} + \tfrac{2}{9}\tfrac{y^2f(t)}{(\int f(t))^{\frac{4}{3}}}\\
&-\tfrac{1}{3}\int \tfrac{h(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{2}{27}\int \tfrac{g(t)^2}{(\int f(t))^{\frac{7}{3}}}\big) - \tfrac{y}{(\int f(t))^{\frac{2}{3}}} - \tfrac{1}{3}\int \tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}}\big)\big)\\
&(ii)\ \ \sigma(x,y,t) = \chi(x,y,t) + \tfrac{1}{(\int f(t))^{\frac{1}{3}}}\big(C_0 + \sqrt{3}\cot\big(C_1 + \tfrac{\sqrt{3}}{2}\big(\big(\tfrac{x}{(\int f(t))^{\frac{1}{3}}} + \tfrac{2}{9}\tfrac{yg(t)}{(\int f(t))^{\frac{4}{3}}}\\
&+\tfrac{2}{9}\tfrac{y^2f(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{1}{3}\int \tfrac{h(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{2}{27}\int \tfrac{g(t)^2}{(\int f(t))^{\frac{7}{3}}}\big) - \big(\tfrac{y}{(\int f(t))^{\frac{2}{3}}} - \tfrac{1}{3}\int \tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}}\big)\big)\big)\big)\\
&(iv)\ \ \sigma(x,y,t) = \chi(x,y,t) + \tfrac{1}{(\int f(t))^{\frac{1}{3}}}\big(C_0 - \sqrt{3}\tan\big(C_1 + \tfrac{\sqrt{3}}{2}\big(\big(\tfrac{x}{(\int f(t))^{\frac{1}{3}}} + \tfrac{2}{9}\tfrac{yg(t)}{(\int f(t))^{\frac{4}{3}}}\\
&+\tfrac{2}{9}\tfrac{y^2f(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{1}{3}\int \tfrac{h(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{2}{27}\int \tfrac{g(t)^2}{(\int f(t))^{\frac{7}{3}}}\big) - \big(\tfrac{y}{(\int f(t))^{\frac{2}{3}}} - \tfrac{1}{3}\int \tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}}\big)\big)\big)\big),
\end{aligned} \tag{4.2.27}$$

where $\chi(x,y,t)$ is given by equation (4.2.7).

Case (iii): For $\beta = 0$, integrating the reduced ODE with respect to ζ, we get the following equation:

$$3F'(\zeta) + 3F'(\zeta)^2 + F'''(\zeta) + K_1 = 0, \tag{4.2.28}$$

where K_1 is arbitrary constant. Multiplying Eq. (4.2.28) with $F''(\zeta)$, integrating and taking $F'(\zeta) = J(\zeta)$, we get

$$\frac{3}{2}J(\zeta)^2 + J(\zeta)^3 + \frac{1}{2}J'(\zeta)^2 + K_1J(\zeta) + K_2 = 0, \tag{4.2.29}$$

where K_2 is constant of integration. For $K_2 = 0$, we get solution of Eq. (4.2.29) as follows:

$$J(\zeta) = -1/2 - 2\,{C_4}^2\wp\left(C_3 + C_4\,\zeta, -1/4\,\frac{-3+4\,K_1}{{C_4}^4}, -1/8\,\frac{2\,K_1-1}{{C_4}^6}\right), \tag{4.2.30}$$

where C_3, C_4 are arbitrary constants and $\wp$ denotes WeirstrassP function.

4. **Vector field $Z_3 + Z_4$:** Integrating the reduced ODE with respect to ζ, we get

$$\frac{3}{4}\zeta + \frac{3}{4}F'(\zeta)^2 + \frac{1}{4}F'''(\zeta) + K_1 = 0, \tag{4.2.31}$$

where K_1 is arbitrary constant. Multiplying Eq. (4.2.31) with $F''(\zeta)$ and again integrating, we get the reduced equation as follows:

$$\frac{3}{4}\zeta F'(\zeta) - \frac{3}{4}F(\zeta) + \frac{1}{4}F'(\zeta)^3 + \frac{1}{8}F''(\zeta)^2 + K_1 F'(\zeta) + K_2 = 0, \tag{4.2.32}$$

where K_2 is arbitrary constant. On substituting (4.2.24) in Eq. (4.2.32), we get values of constants as $C_0 = 0, C_1 = 1/3\, {C_2}^3 + 4/3\, K_1\, C_2 + 4/3\, K_2, C_2 = C_2$, which corresponds to following solution of Eq. (4.1.1):

$$\begin{aligned}\sigma(x,y,t) = \chi(x,y,t) &+ \tfrac{1}{(\int f(t))^{\frac{1}{3}}}(\tfrac{1}{2}(\tfrac{y}{(\int f(t))^{\frac{2}{3}}} - \tfrac{1}{3}\int \tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}})^2 + 1/3\, {C_2}^3 + 4/3\, K_1\, C_2 \\ &+4/3\, K_2 + C_2(\tfrac{x}{(\int f(t))^{\frac{1}{3}}} + \tfrac{2}{9}\tfrac{yg(t)}{(\int f(t))^{\frac{4}{3}}} + \tfrac{2}{9}\tfrac{y^2 f(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{1}{3}\int \tfrac{h(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{2}{27}\int \tfrac{g(t)^2}{(\int f(t))^{\frac{7}{3}}})),\end{aligned} \tag{4.2.33}$$

where $\chi(x,y,t)$ is given by Eq. (4.2.7).

5. **Vector field $Z_3 - Z_4$:** Integrating the reduced ODE with respect to ζ, we get

$$-\frac{3}{4}\zeta + \frac{3}{4}F'(\zeta)^2 + \frac{1}{4}F'''(\zeta) + K_1 = 0, \tag{4.2.34}$$

where K_1 is arbitrary constant. Multiplying Eq. (4.2.34) with $F''(\zeta)$ and again integrating, we get the reduced equation as follows:

$$-\frac{3}{4}\zeta F'(\zeta) + \frac{3}{4}F(\zeta) + \frac{1}{4}F'(\zeta)^3 + \frac{1}{8}F''(\zeta)^2 + K_1 F'(\zeta) + K_2 = 0, \tag{4.2.35}$$

On using the same substitution as (4.2.24), we get the following solution of Eq. (4.1.1):

$$\begin{aligned}\sigma(x,y,t) = \chi(x,y,t) &+ \tfrac{1}{(\int f(t))^{\frac{1}{3}}}(-\tfrac{1}{2}(\tfrac{y}{(\int f(t))^{\frac{2}{3}}} - \tfrac{1}{3}\int \tfrac{g(t)}{(\int f(t))^{\frac{5}{3}}})^2 - 1/3\, {C_2}^3 - 4/3\, K_1\, C_2 \\ &-4/3\, K_2 + C_2(\tfrac{x}{(\int f(t))^{\frac{1}{3}}} + \tfrac{2}{9}\tfrac{yg(t)}{(\int f(t))^{\frac{4}{3}}} + \tfrac{2}{9}\tfrac{y^2 f(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{1}{3}\int \tfrac{h(t)}{(\int f(t))^{\frac{4}{3}}} - \tfrac{2}{27}\int \tfrac{g(t)^2}{(\int f(t))^{\frac{7}{3}}})),\end{aligned} \tag{4.2.36}$$

where $\chi(x,y,t)$ is given by Eq. (4.2.7).

6. **Vector field Z_3:** Integrating the reduced ODE with respect to ζ, we get

$$3F'(\zeta)^2 + F'''(\zeta) + K_1 = 0, \tag{4.2.37}$$

where K_1 is arbitrary constant. Multiplying (4.2.37) with $F''(\zeta)$, integrating it with respect to ζ, taking integration constant to be zero and assuming $F'(\zeta) = J(\zeta)$, we obtain

$$2J(\zeta)^3 + J'(\zeta)^2 + 2K_1J(\zeta) = 0. \tag{4.2.38}$$

We get the general solution of above ODE as

$$J\left(\zeta\right) = -\wp\left(1/2\,2^{2/3}\zeta + C_1, -2\,K_1\,\sqrt[3]{2}, 0\right)\sqrt[3]{2}, \tag{4.2.39}$$

where C_1 is arbitrary constant and $\wp$ denotes WeirstrassP function.

4.2.3.2 Exact Solutions of Eq. (4.2.13)

Using Lie Classical Approach

In this section, we will apply Lie classical approach to reduce the no of independent variables in Eq. (4.2.13). In this direction, equation (4.2.13) yields the symmetries as under:

$$\begin{aligned} \eta &= b\psi + Q(\rho) \\ \xi &= \tfrac{3}{2}b\rho + c \\ \tau &= a, \end{aligned} \tag{4.2.40}$$

where η, ξ and τ are infinitesimals corresponding to H, ψ and ρ, respectively and a, b and c are arbitrary constants and $Q(\rho)$ is an arbitrary function of ρ. The similarity variable and form can be obtained by solving the characterstic equations

$$\tfrac{d\rho}{\tau} = \tfrac{d\psi}{\xi} = \tfrac{dH}{\eta}. \tag{4.2.41}$$

The general solution of these equations involves two constants, one becomes new independent variable ζ and other F play the role of new dependent variable. Hence, we get

$$\begin{aligned} \zeta &= \psi - \tfrac{3b}{4a}\rho^2 - \tfrac{c}{a}\rho \\ H(\rho, \psi) &= \tfrac{b}{a}\psi\rho + \tfrac{1}{a}R(\rho) + F(\zeta), \end{aligned} \tag{4.2.42}$$

where $\int Q(\rho)d\rho = R(\rho)$, which is an arbitrary function of ρ. On substituting the equation (4.2.42) in equation (4.2.13), we get the reduced ODE as

$$\tfrac{a}{4}F''''(\zeta) + \tfrac{3a}{2}F'(\zeta)F''(\zeta) - cF''(\zeta) + b = 0. \tag{4.2.43}$$

Case (i): Integrating Eq. (4.2.43) with respect to ζ, we get

$$\frac{a}{4}F'''(\zeta) + \frac{3a}{4}F'(\zeta)^2 - cF'(\zeta) + b\zeta + K_1 = 0, \tag{4.2.44}$$

where K_1 is arbitrary constant. Multiplying Eq. (4.2.44) with $F''(\zeta)$ and again integrating, we get

$$\frac{a}{8}F''(\zeta)^2+\frac{a}{4}F'(\zeta)^3-\frac{c}{2}F'(\zeta)^2+b\zeta F'(\zeta)-bF(\zeta)+K_1F'(\zeta)+K_2=0, \tag{4.2.45}$$

where K_2 is constant of integration. On substituting (4.2.24) in Eq. (4.2.45), we get values of constants as $C_0=0, C_1=1/4\,\frac{-2\,cC_2{}^2+aC_2{}^3+4\,K_1\,C_2+4\,K_2}{b}, C_2=C_2$, which corresponds to the following solution of Eq. (4.1.1):

$$\begin{aligned}&\sigma(x,y,t)=\nu(x,y,t)+\tfrac{1}{\sqrt{g(t)}}\big(\tfrac{b}{a}\big(\tfrac{x}{g(t)^{\frac{1}{2}}}-\tfrac{yh(t)}{g(t)^{\frac{3}{2}}}+\tfrac{1}{3}\tfrac{y^2g'(t)}{g(t)^{\frac{3}{2}}}\big)\big(\textstyle\int_0^t g(s)^{\frac{-3}{2}}ds\big)+\tfrac{1}{a}R\big(\textstyle\int_0^t g(s)^{\frac{-3}{2}}ds\big)\\&+1/4\,\tfrac{-2\,cC_2{}^2+aC_2{}^3+4\,K_1\,C_2+4\,K_2}{b}+C_2\big(\big(\tfrac{x}{g(t)^{\frac{1}{2}}}-\tfrac{yh(t)}{g(t)^{\frac{3}{2}}}+\tfrac{1}{3}\tfrac{y^2g'(t)}{g(t)^{\frac{3}{2}}}\big)-\tfrac{3b}{4a}\big(\textstyle\int_0^t g(s)^{\frac{-3}{2}}ds\big)^2\\&-\tfrac{c}{a}\big(\textstyle\int_0^t g(s)^{\frac{-3}{2}}ds\big)\big)\big),\end{aligned} \tag{4.2.46}$$

where $\nu(x,y,t)$ is given by (4.2.11).

Case (ii): For $b=0$, integrating the equation (4.2.43) once and taking the integration constant to be zero, we obtain

$$aF'''(\zeta)+3a(F'(\zeta))^2-4cF'(\zeta)=0. \tag{4.2.47}$$

We assume that the solution of equation (4.2.47) can be expressed in the following form

$$F(\zeta)=\tfrac{a_0+a_1e^{\zeta}+a_2e^{-\zeta}}{b_0+b_1e^{\zeta}+b_2e^{-\zeta}}, \tag{4.2.48}$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are constants to be found out. The substitution of the form of $F(\zeta)$ in equation (4.2.47) brings forth the several possibilities for values of constants and all these possibilities have been described in the following two subcases:

Subcase 1: $a-4c=0$

$$\begin{aligned}&\Big\{a_0=\tfrac{a_2b_0}{b_2},\ a_1=\tfrac{a_2b_1}{b_2},\ a_2=a_2,\ b_0=b_0,\ b_1=b_1,\ b_2=b_2;\\&\Big\{a_0=\tfrac{b_0(-2b_1+a_1)}{b_1},\ a_1=a_1,\ a_2=0,\ b_0=b_0,\ b_1=b_1,\ b_2=0;\\&\Big\{a_0=a_0,\ a_1=\tfrac{-(4a_2{}^2b_0{}^2b_2+2a_0{}^2b_2{}^3-6a_2a_0b_2{}^2b_0+a_2{}^3b_0{}^2+a_2a_0{}^2b_2{}^2-2a_2{}^2a_0b_2b_0-4a_0b_2{}^3b_0+4a_2b_0{}^2b_2{}^2)}{4b_2{}^4},\\&\Big\{a_2=a_2,\ b_0=b_0,\ b_1=\tfrac{-(-2a_0b_2{}^2b_0+a_2{}^2b_0{}^2+a_0{}^2b_2{}^2+2a_2b_0{}^2b_2-2a_2a_0b_2b_0)}{4b_2{}^3},\ b_2=b_2;\end{aligned} \tag{4.2.49}$$

which corresponds to the solution of equation (4.2.47) as

$$
\begin{aligned}
&(i)\ \ F(\zeta) = \tfrac{a_2}{b_2}\\
&(ii)\ \ F(\zeta) = \tfrac{a_1}{b_1} - \tfrac{2b_0}{b_0+b_1e^{\zeta}}\\
&(iii)\ \ F(\zeta) =\\
&\tfrac{4{b_2}^4a_0-(4{a_2}^2{b_0}^2b_2+2{a_0}^2{b_2}^3-6a_2a_0{b_2}^2b_0+{a_2}^3{b_0}^2+a_2{a_0}^2{b_2}^2-2{a_2}^2a_0b_2b_0-4a_0{b_2}^3b_0+4a_2{b_0}^2{b_2}^2)e^{\zeta}+4a_2{b_2}^4e^{-\zeta}}{4{b_2}^4b_0-b_2(-2a_0{b_2}^2b_0+{a_2}^2{b_0}^2+{a_0}^2{b_2}^2+2a_2{b_0}^2b_2-2a_2a_0b_2b_0)e^{\zeta}+4{b_2}^5e^{-\zeta}}.
\end{aligned}
\tag{4.2.50}
$$

Subcase 2: $a - 4c \neq 0$

$$
\begin{cases} a_0 = \frac{b_0a_1}{b_1},\ a_1 = a_1,\ a_2 = \frac{b_2a_1}{b_1},\ b_0 = b_0,\ b_1 = b_1,\ b_2 = b_2 \end{cases}
\tag{4.2.51}
$$

which corresponds to the "constant" solution of equation (4.2.47). The solutions of Eq. (4.1.1) corresponding to subcases 1, 2 are given by relation (4.2.10), (4.2.11), (4.2.12) and (4.2.42).

Case (iii): For $b = 0$, integrating the equation (4.2.43) once with respect to ζ, multilying the integrated equation with $F''(\zeta)$ and again integrate it, assuming $F'(\zeta) = J(\zeta)$, we obtain

$$
a(J'(\zeta))^2 + 2aJ(\zeta)^3 - 4c(J(\zeta))^2 - C_1J(\zeta) + C_2 = 0,
\tag{4.2.52}
$$

where C_1 and C_2 are arbitrary constants. Corresponding to ODE, choosing $C_2 = 0$, solutions of equation (4.2.52) are given as

$$
J(\zeta) = 2/3\,\frac{c}{a} - 2\,{C_4}^2\wp\left(C_3 + C_4\,\zeta, 1/6\,\frac{3\,aC_1+8\,c^2}{{C_4}^4a^2}, -\frac{1}{54}\,\frac{c\left(9\,aC_1+16\,c^2\right)}{{C_4}^6a^3}\right),
\tag{4.2.53}
$$

where C_3, C_4 are arbitrary constants and $\wp$ denotes WeirstrassP function. In this case, the solution of equation (4.2.43) is given by

$$
F(\zeta) = \int J(\zeta)d\zeta + C_5,
\tag{4.2.54}
$$

where C_5 is arbitrary constant.

Using Homoclinic Test Method In this section, Eq. (4.2.13) is considered to obtain more periodic soliton solutions using homoclinic test method. The homoclinic test method [167, 168] is a technique which usually can be used to construct exact homoclinic orbit solution for non-linear integrable equation, it was applied successfully to study spatiotemporal bifurcation of equilibrium solution for some integrable equation in recent years.

Firstly, integrating Eq. (4.2.13) once with respect to ψ and taking integration constant to zero, we obtain

$$H_\rho + \frac{3}{2}(H_\psi)^2 + \frac{1}{4}H_{\psi\psi\psi} = 0. \tag{4.2.55}$$

Making transformation of unknown function in Eq. (4.2.55) as

$$H = 2(\ln f)_\psi. \tag{4.2.56}$$

Substituting (4.2.56) into (4.2.55) and using the bilinear form, we can get

$$(D_\psi D_\rho + \frac{1}{4}{D_\psi}^4)(f.f) = 0, \tag{4.2.57}$$

where the Hirota operator D is defined as [119]. In this case we choose extended homoclinic test function

$$f = e^{-p_1(\psi - w_1\rho)} + c_1\cos(p_2(\psi + w_2\rho)) + c_2 e^{p_1(\psi - w_1\rho)}, \tag{4.2.58}$$

where p_1, p_2, w_1, w_2, c_1 and c_2 are constants to be determined. Substituting (4.2.58) into (4.2.57) yields a set of algebraic equations as follows:

$$\begin{aligned}
&(i)\ 1/2c_1\,{p_2}^4 + 1/2\,c_1\,{p_1}^4 - 2\,c_1\,{p_2}^2w_2 - 2\,c_1\,{p_1}^2w_1 - 3\,{p_1}^2c_1\,{p_2}^2 = 0,\\
&(ii)\ 2\,c_1\,p_2\,{p_1}^3 - 2\,p_1\,w_1\,c_1\,p_2 + 2\,c_1\,p_2\,w_2\,p_1 - 2\,p_1\,c_1\,{p_2}^3 = 0,\\
&(iii)\ 2\,c_2\,p_1\,w_1\,c_1\,p_2 - 2\,c_1\,p_2\,c_2\,{p_1}^3 + 2\,c_2\,p_1\,c_1\,{p_2}^3 - 2\,c_1\,p_2\,w_2\,c_2\,p_1 = 0,\\
&(iv)\ 1/2\,c_1\,c_2\,{p_2}^4 + 1/2\,c_1\,c_2\,{p_1}^4 - 2\,c_1\,c_2\,{p_1}^2w_1 - 3\,c_1\,c_2\,{p_2}^2{p_1}^2 - 2\,c_1\,c_2\,{p_2}^2w_2 = 0,\\
&(v)\ c_2\,\left(-4\,{p_1}^2w_1 + 4\,{p_1}^4\right) + {c_1}^2{p_2}^4 - {c_1}^2{p_2}^2w_2 = 0.
\end{aligned} \tag{4.2.59}$$

Solving the above algebraic equations, we get the following solutions of Eq. (4.2.55):

$$\begin{aligned}
&(i)\ H(\rho,\psi) = 2\,\frac{-p_1\,\mathrm{e}^{-p_1\,\left(\psi - {p_1}^2\rho\right)} + c_2\,p_1\,\mathrm{e}^{p_1\,\left(\psi - {p_1}^2\rho\right)}}{\mathrm{e}^{-p_1\,(\psi - {p_1}^2\rho)} + c_2\,\mathrm{e}^{p_1\,(\psi - {p_1}^2\rho)}},\\
&(ii)\ H(\rho,\psi) =\\
&2\,\frac{\left(-p_1\,\mathrm{e}^{-p_1\,\left(\psi - \left(1/4\,{p_1}^2 - 3/4\,{p_2}^2\right)\rho\right)} - c_1\,\sin\!\left(p_2\,\left(\psi + \left(1/4\,{p_2}^2 - 3/4\,{p_1}^2\right)\rho\right)\right)p_2 - 1/4\,\frac{{c_1}^2{p_2}^2\mathrm{e}^{p_1\,\left(\psi - \left(1/4\,{p_1}^2 - 3/4\,{p_2}^2\right)\rho\right)}}{p_1}\right)}{\left(\mathrm{e}^{-p_1\,(\psi - (1/4\,{p_1}^2 - 3/4\,{p_2}^2)\rho)} + c_1\,\cos(p_2\,(\psi + (1/4\,{p_2}^2 - 3/4\,{p_1}^2)\rho)) - 1/4\,\frac{{c_1}^2{p_2}^2\mathrm{e}^{p_1\,(\psi - (1/4\,{p_1}^2 - 3/4\,{p_2}^2)\rho)}}{{p_1}^2}\right)},\\
&(iii)\ H(\rho,\psi) = -2\,\frac{p_1\,\mathrm{e}^{-p_1\,\left(\psi - 1/4\,{p_1}^2\rho\right)}}{\mathrm{e}^{-p_1\,(\psi - 1/4\,{p_1}^2\rho)} + c_1},\\
&(iv)\ H(\rho,\psi) = 2\,\frac{-ip_2\,\mathrm{e}^{-ip_2\,(\psi - w_1\,\rho)} - c_1\,\sin\!\left(p_2\,\left(\psi + \left(w_1 + 2\,{p_2}^2\right)\rho\right)\right)p_2 + 1/4\,ic_1{}^2p_2\,\mathrm{e}^{ip_2\,(\psi - w_1\,\rho)}}{\mathrm{e}^{-ip_2\,(\psi - w_1\,\rho)} + c_1\,\cos(p_2\,(\psi + (w_1 + 2\,{p_2}^2)\rho)) + 1/4\,{c_1}^2\mathrm{e}^{ip_2\,(\psi - w_1\,\rho)}},\\
&(v)\ H(\rho,\psi) = 2\,\frac{-ip_2\,\mathrm{e}^{-ip_2\,\left(\psi + {p_2}^2\rho\right)} - c_1\,\sin\!\left(p_2\,\left(\psi + {p_2}^2\rho\right)\right)p_2 + ic_2\,p_2\,\mathrm{e}^{ip_2\,\left(\psi + {p_2}^2\rho\right)}}{\mathrm{e}^{-ip_2\,(\psi + {p_2}^2\rho)} + c_1\,\cos(p_2\,(\psi + {p_2}^2\rho)) + c_2\,\mathrm{e}^{ip_2\,(\psi + {p_2}^2\rho)}}.
\end{aligned} \tag{4.2.60}$$

Corresponding to above solutions, solutions of Eq. (4.1.1) are given by the relation (4.2.10), (4.2.11) and (4.2.12).

4.2.3.3 Exact Solutions of Eq. (4.2.15)

Integrating (4.2.15) we obtain a family of solutions of the PKP equation depending on three arbitrary functions of time:

$$\sigma(x,y,t) = \tfrac{1}{3}\tfrac{x^2h'(t)}{h(t)} - \tfrac{4}{9}\tfrac{xy^2h''(t)}{h(t)} + \tfrac{4}{81}\tfrac{y^4h'''(t)}{h(t)} + yq(t) + r(t), \qquad (4.2.61)$$

where $q(t)$ and $r(t)$ are arbitrary functions of time.

4.2.4 Analysis and Discussions of Solutions

Now, we wish to show the major features of exact solutions obtained in section (4.2.3). One of the main feature of solutions is that the form and the behavior of solutions are strongly affected by the arbitrary functions $f(t), g(t)$ and $h(t)$ which are involved in symmetry algebra of PKP equation. As fact, we have freedom in selecting functions $f(t), g(t)$ and $h(t)$ appropriately, according to some actual physical requirements. So, we can choose the $f(t), g(t)$ and $h(t)$ in terms of trigonometric, hyperbolic and jacobi elliptic functions to display physical behavior of solutions.

- By the analysis of solutions obtained in section (4.2.3.1), we conclude that solutions (4.2.27) depend upon two constants C_0, C_1 and three arbitrary functions $f(t), g(t), h(t)$ of time. Depending upon these constants and functions of time, we obtain certain periodic and kinky periodic solutions. For $C_0 = 0, C_1 = 1, x = 1, f(t) = h(t) = 1, g(t) = \sin(t)$ and $C_0 = 0, C_1 = 1, y = \cos(x), f(t) = g(t) = 1, h(t) = \mathrm{sech(t)}t^{\frac{4}{3}}$ solution (4.2.27)(ii) takes the form of periodic solution as shown in Fig. (4.1) and kinky periodic solution as shown in Fig. (4.2).

- We also observe that certain solutions include several properties of kinematics, for example, solution (4.2.50)(ii) represents periodicity on t-axis and paraboloid on y-axis for $a_1 = b_0 = b_1 = 1, c = 1, x = 0, g(t) = 1, h(t) = \cos(t), R(t) = \mathrm{sech(t)}$ as shown in Fig. (4.3) and also solution (4.2.50)(ii) takes the form of kinky periodic solution (Fig. (4.4)) for $a_1 = 0, b_0 = b_1, c = 1, x = 1, g(t) = 1, h(t) = \sin(t), R(t) = \cos(t)$.

- We also analyze that the solution (4.2.60)(ii) behaves as periodic solution for $p_1 = 1, p_2 = 2, c_1 = 1, h(t) = (\cos(t))^2, y = \sin(x), g(t) = 1$ and (4.2.60)(i) represents the kinky solution for $g(t) = 1, y = sn(x,2), h(t) = cn(t,1), p_1 = 1, c_2 = 2$, where $sn(x,2), cn(t,1)$ represents jacobi elliptic functions with modulus $2, 1$ respectively.

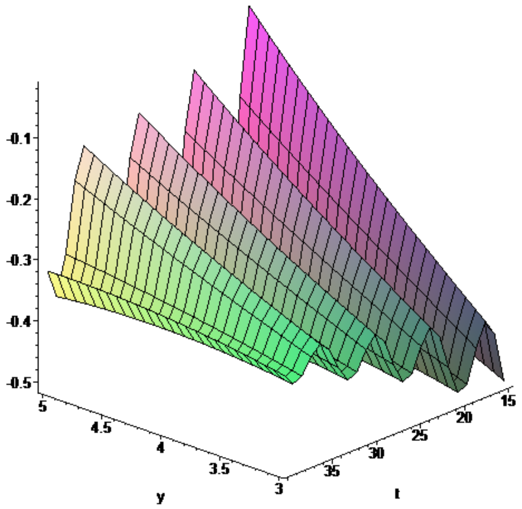

FIGURE 4.1: The form of periodic solution (4.2.27)(ii) for $C_0 = 0, C_1 = 1, x = 1, f(t) = h(t) = 1, g(t) = \sin(t)$

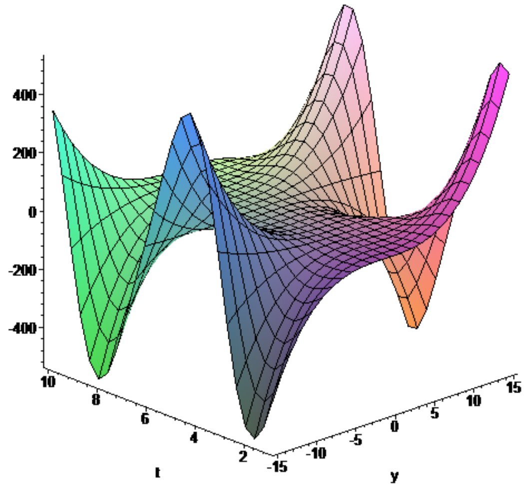

FIGURE 4.2: The kinky periodic solution (4.2.27)(ii) for $C_0 = 0, C_1 = 1, y = \cos(x), f(t) = g(t) = 1, h(t) = \text{sech(t)}t^{\frac{4}{3}}$

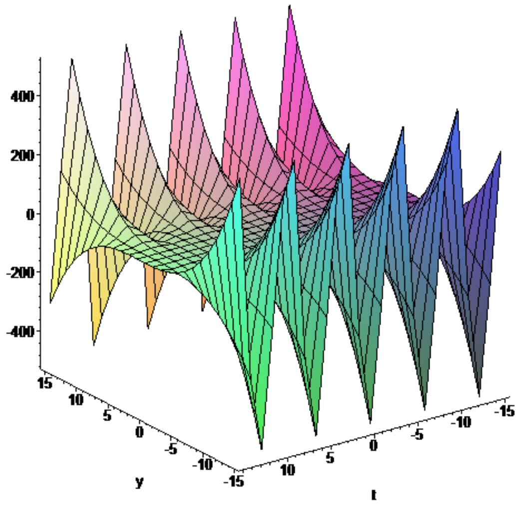

FIGURE 4.3: The periodic solution (4.2.50)(ii) for $a_1 = b_0 = b_1 = 1, c = 1, x = 0, g(t) = 1, h(t) = \cos(t), R(t) = \text{sech(t)}$

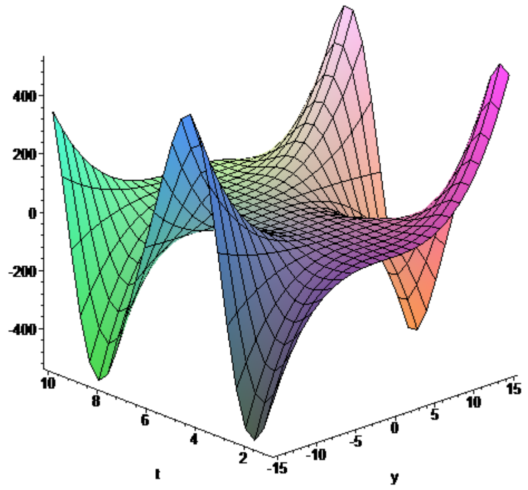

FIGURE 4.4: The kinky periodic form of solution (4.2.50)(ii) for $a_1 = 0, b_0 = b_1, c = 1, x = 1, g(t) = 1, h(t) = \sin(t), R(t) = \cos(t)$

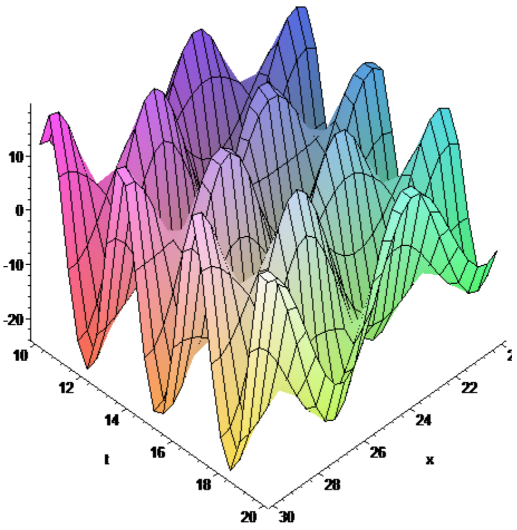

FIGURE 4.5: The periodic solution (4.2.60)(ii) for $p_1 = 1, p_2 = 2, c_1 = 1, h(t) = (\cos(t))^2, y = \sin(x), g(t) = 1$

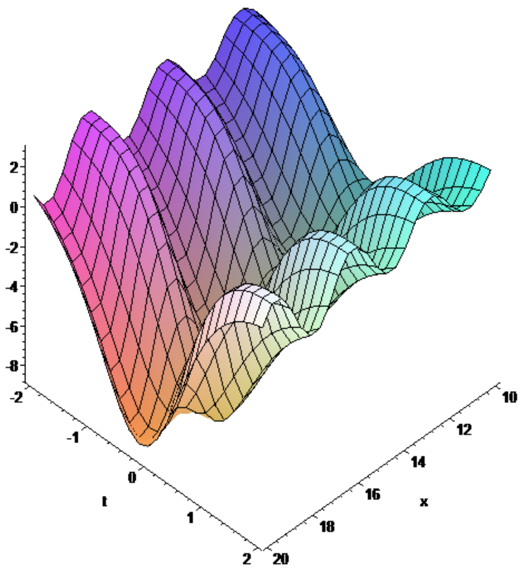

FIGURE 4.6: The kink solution (4.2.60)(i) for $g(t) = 1, y = sn(x, 2), h(t) = cn(t, 1), p_1 = 1, c_2 = 2$

4.3 The Generalized potential Kadomstev Petviashvili Equation

In this section, we work for integrability, symmetries and exact solutions of VCPKP equation.

4.3.1 Painlevé Analysis for VCPKP Equation

A PDE is said to possess the Painlevé property if the only singularities of the general integral which can live on arbitrary non-characteristic (movable) hypersurfaces are poles as discussed in section (1.4) of this thesis. This Painlevé test has proved to be a useful criterion for the identification of completely integrable PDE. The leading order of the solution of equation (4.1.2) is assumed as

$$\sigma(x, y, t) \approx \sigma_0 g^{\alpha}, \tag{4.3.1}$$

where $\sigma_0 = \sigma_0(x, y, t)$ and $g = g(x, y, t)$ are analytic functions of x, y and t. On substituting equation (4.3.1) into (4.1.2) and equating the most dominant terms, the following results are obtained:

$$\alpha = -1, \sigma_0 = \frac{12\beta(t)g_x}{\alpha(t)}, \tag{4.3.2}$$

where g_x denotes the partial differentiation of $g(x, y, t)$ with respect to x. For finding the resonances, the full Laurent series

$$\sigma(x, y, t) = \sigma_0 g^{-1} + \sum \sigma_r g^{r-1}, \tag{4.3.3}$$

where $\sigma_r = \sigma_r(x, y, t)$, is substituted into (4.1.2) and by equating the coefficients of like terms, the polynomial equation is derived as

$$\begin{gathered}\beta\left(t\right)\left(\alpha\left(t\right)\right)^{2}\sigma_{r}\left(x,y,t\right)\left(g\left(x,y,t\right)\right)^{r-2}r^{4}+23\,\beta\left(t\right)\left(g\left(x,y,t\right)\right)^{r-2}\sigma_{r}\left(x,y,t\right)r^{2}\left(\alpha\left(t\right)\right)^{2}\\+10\,\beta\left(t\right)\left(g\left(x,y,t\right)\right)^{r-2}\sigma_{r}\left(x,y,t\right)r\left(\alpha\left(t\right)\right)^{2}-24\,\beta\left(t\right)\left(g\left(x,y,t\right)\right)^{r-2}\sigma_{r}\left(x,y,t\right)\left(\alpha\left(t\right)\right)^{2}\\-10\,\beta\left(t\right)\left(\alpha\left(t\right)\right)^{2}\sigma_{r}\left(x,y,t\right)\left(g\left(x,y,t\right)\right)^{r-2}r^{3}.\end{gathered} \tag{4.3.4}$$

Using equation (4.3.4), the resonances are found to be

$$r = -1, 1, 4, 6. \tag{4.3.5}$$

As usual, the resonance at $r = -1$ corresponds to the arbitrariness of the singular manifold $g(x, y, t) = 0$. Therefore there are three compatibility tests at $r = 1, 4$ and 6. When $r = 1$, σ_1 is arbitrary; $r = 4$, σ_4 is arbitrary; $r = 6$, we can get the compatibility condition which corresponds to the following relations in arbitrary variable coefficients of Eq. (4.1.2):

$$\begin{aligned}&(i)\ 3\,\beta\left(t\right)\left(\tfrac{d}{dt}\alpha\left(t\right)\right)^{2}-3\,\left(\tfrac{d}{dt}\beta\left(t\right)\right)\alpha\left(t\right)\tfrac{d}{dt}\alpha\left(t\right)-\beta\left(t\right)\alpha\left(t\right)\tfrac{d^{2}}{dt^{2}}\alpha\left(t\right)+\left(\alpha\left(t\right)\right)^{2}\tfrac{d^{2}}{dt^{2}}\beta\left(t\right)=0,\\&(ii)\ 3\,\left(\tfrac{d}{dt}\beta\left(t\right)\right)\left(\alpha\left(t\right)\right)^{2}\delta\left(t\right)-4\,\beta\left(t\right)\left(\tfrac{d}{dt}\alpha\left(t\right)\right)\delta\left(t\right)\alpha\left(t\right)+\beta\left(t\right)\left(\alpha\left(t\right)\right)^{2}\tfrac{d}{dt}\delta\left(t\right)=0,\\&(iii)\ -6\,\left(\tfrac{d}{dt}\beta\left(t\right)\right)\left(\alpha\left(t\right)\right)^{2}\delta\left(t\right)-2\,\beta\left(t\right)\left(\alpha\left(t\right)\right)^{2}\tfrac{d}{dt}\delta\left(t\right)+8\,\beta\left(t\right)\left(\tfrac{d}{dt}\alpha\left(t\right)\right)\delta\left(t\right)\alpha\left(t\right)=0.\end{aligned} \tag{4.3.6}$$

Then we obtain the explicit constraints on the variable coefficients $\alpha(t), \beta(t)$ and $\delta(t)$ for Eq. (4.1.2) to pass the Painlevé test

$$\begin{aligned}\beta(t) &= C_1\,\alpha\left(t\right)+C_2\,\left(\int\alpha\left(t\right)dt\right)\alpha\left(t\right),\\ \delta(t) &= C_3\,\mathrm{e}^{\int-\frac{-C_1\,\frac{d}{dt}\alpha(t)+3\,C_2\,(\alpha(t))^{2}-C_2\,\int\alpha(t)dt\,\frac{d}{dt}\alpha(t)}{\alpha(t)(C_1+C_2\,\int\alpha(t)dt)}dt},\end{aligned} \tag{4.3.7}$$

where C_1, C_2, C_3 are arbitrary constants.

4.3.2 Lie Point Symmetries and Classification of One-Dimensional Subalgebras

In this section, we intend to deduce the one-dimensional optimal system of vector fields corresponding to Eq. (4.1.2). To this end, first we utilize the classical Lie group method to construct the infinitesimals admitted by (4.1.2). By calculation, we show that the symmetries of Eq. (4.1.2) form a seven-dimensional Lie algebra generated by the following vector fields:

$$\begin{aligned} &V_1 = x\frac{\partial}{\partial x} + \left(\frac{2\int \alpha(t)dt}{\alpha(t)}\right)\frac{\partial}{\partial t},\ V_2 = \sigma\frac{\partial}{\partial \sigma} - \left(\frac{\int \alpha(t)dt}{\alpha(t)}\right)\frac{\partial}{\partial t},\ V_3 = x\frac{\partial}{\partial \sigma} + \left(\int \alpha(t)dt\right)\frac{\partial}{\partial x},\\ &V_4 = \frac{\partial}{\partial y},\ V_5 = \frac{1}{\alpha(t)}\frac{\partial}{\partial t},\ V_6 = \frac{\partial}{\partial x},\ V_7 = \frac{\partial}{\partial \sigma}. \end{aligned} \tag{4.3.8}$$

and the functions $\alpha(t), \beta(t)$ and $\delta(t)$ are governed by the following conditions:

$$\begin{aligned} \beta(t)\tau_t + \tau\beta'(t) - 3a_1\beta(t) = 0,\\ \delta(t)\tau_t + \tau\delta'(t) + a_1\delta(t) = 0. \end{aligned} \tag{4.3.9}$$

Then the non zero commutation relations of the Lie algebra are as follows:

$$[V_1, V_3] = V_3,\ [V_1, V_5] = -2V_5,\ [V_1, V_6] = -V_6,\ [V_2, V_3] = -V_3,\ [V_2, V_5] = V_5,$$

$$[V_2, V_7] = -V_7,\ [V_3, V_1] = -V_3,\ [V_3, V_2] = V_3,\ [V_3, V_5] = -V_6,\ [V_3, V_6] = -V_7,$$

$$[V_5, V_1] = 2V_5,\ [V_5, V_2] = -V_5,\ [V_5, V_3] = V_6,\ [V_6, V_1] = V_6,$$

$$[V_6, V_3] = V_7,\ [V_7, V_2] = V_7, \tag{4.3.10}$$

where the Lie brackets are obtained using the expression $[V_i, V_j] = V_i(V_j) - V_j(V_i)$. As we know, classification of subgroups of Lie symmetry groups of differential equations is an essential part in study of equations. This is since classification allows for an efficient computation of group-invariant solutions, without the possibility of an occurence of equivalent solutions as discussed in section (1.2). The classification of subgroups of symmetry groups is usually done by the classification of the associated Lie subalgebras w.r.t. the adjoint representation and the adjoint action for the Lie algebra for (4.1.2) is given as follows:

$$Adj(exp(\epsilon V_1)V_3) = e^{-\epsilon}V_3,\ Adj(exp(\epsilon V_1)V_5) = e^{2\epsilon}V_5,\ Adj(exp(\epsilon V_1)V_6) = e^{\epsilon}V_6,$$

$$Adj(exp(\epsilon V_2)V_3) = e^{\epsilon}V_3,\ Adj(exp(\epsilon V_2)V_5) = e^{-\epsilon}V_5,\ Adj(exp(\epsilon V_2)V_7) = e^{\epsilon}V_7,$$

$$Adj(exp(\epsilon V_3)V_1) = V_1 - \epsilon V_3,\ Adj(exp(\epsilon V_3)V_2) = V_2 - \epsilon V_3,$$

$$Adj(exp(\epsilon V_3)V_5) = V_5 + \epsilon V_6 + \frac{\epsilon^2}{2}V_7,\ Adj(exp(\epsilon V_3)V_6) = V_6 + \epsilon V_7,$$

$$Adj(exp(\epsilon V_5)V_1) = V_1 - 2\epsilon V_5,\ \ Adj(exp(\epsilon V_5)V_2) = V_2 + \epsilon V_5,$$

$$Adj(exp(\epsilon V_5)V_3) = V_3 - \epsilon V_6,\ \ Adj(exp(\epsilon V_6)V_1) = V_1 - \epsilon V_6,$$

$$Adj(exp(\epsilon V_6)V_3) = V_3 - \epsilon V_7,\ \ Adj(exp(\epsilon V_7)V_2) = V_2 - \epsilon V_7, \tag{4.3.11}$$

where the adjoint action is given by the Lie series (1.2.21) and we have shown the cases here for which $Adj(exp(\epsilon V_i)V_j) \neq V_j$. A one-dimensional optimal system for (4.1.2) is given as follows:

$$\begin{aligned}&(i)\ \ V_1 + aV_2 + bV_4,\ (ii)\ \ V_2 + cV_4,\ (iii)\ \ V_3 + dV_4 + V_5,\ (iv)\ \ V_3 + pV_4,\\ &(v)\ \ V_4 + V_5 + qV_7,\ (vi)\ \ V_4 + rV_7,\ (vii)\ \ V_5 + sV_7,\ (viii)\ \ V_6,\\ &(ix)\ \ V_7,\end{aligned} \tag{4.3.12}$$

where a, b, c, d, p, q, r and s are arbitrary constants.

4.3.3 Group Invariant Solutions

In this subsection, we construct the group invariant solutions to eq. (4.1.2) according to the classification of one-dimensional subalgebras. With those Lie algebras, we reduce equation (4.1.2) to two independent variables partial differential equations and further to ordinary differential equations. These reduced ODEs are further studied for some exact solutions of Eq. (4.1.2).

4.3.3.1 Vector Field $V_1 + aV_2 + bV_4$

For this vector field, on solving the equations (4.3.9) and (4.2.5) we obtain

$$\begin{aligned}&\sigma(x,y,t) = \left(\textstyle\int \alpha(t)dt\right)^{\frac{a}{2-a}} H(\rho,\psi),\ \ \rho = \frac{x}{\left(\int \alpha(t)dt\right)^{\frac{1}{2-a}}},\ \ \psi = y - \tfrac{b}{2-a}\log\left(\textstyle\int \alpha(t)dt\right),\\ &\beta(t) = K_1\alpha(t)\left((2-a)\textstyle\int \alpha(t)dt\right)^{\frac{1+a}{2-a}},\ \ \delta(t) = K_2\alpha(t)\left((2-a)\textstyle\int \alpha(t)dt\right)^{\frac{a-3}{2-a}}.\end{aligned} \tag{4.3.13}$$

Substituting (4.3.13) into Eq. (4.1.2), we have the function $H(\rho,\psi)$ which must satisfy the following PDE:

$$\begin{aligned}&(a-1)H_\rho - \rho H_{\rho\rho} - bH_{\rho\psi} + (2-a)H_\rho H_{\rho\rho} + K_1(2-a)^{\frac{3}{2-a}}H_{\rho\rho\rho\rho}\\ &+K_2(2-a)^{\frac{-1}{2-a}}H_{\psi\psi} = 0.\end{aligned} \tag{4.3.14}$$

To reduce the PDE (4.3.14), let us assume the transformation of the form

$$H(\rho,\psi) = \rho\psi - (2-a)^{\frac{1}{2-a}}(a-1)\frac{\psi^3}{6K_2} + \frac{\psi^2}{2} + F(\zeta),\ \ \zeta = \rho - (2-a)\psi, \tag{4.3.15}$$

which reduces the equation (4.3.14) to ODE of the form

$$\begin{aligned}&(a-1)F'(\zeta)-\zeta F''(\zeta)-b+b(2-a)F''(\zeta)+(2-a)F'(\zeta)F''(\zeta)\\&+K_1(2-a)^{\frac{3}{2-a}}F''''(\zeta)+K_2(2-a)^{\frac{1}{a-2}}+K_2(2-a)^{\frac{2a-3}{a-2}}F''(\zeta)=0,\end{aligned}\tag{4.3.16}$$

where prime ($'$) denotes the differentiation with respect to the variable ζ and K_1, K_2 are arbitrary constants. To get a solution for Eq. (4.3.16), let us assume that (4.3.16) admits a solution in the form

$$F(\zeta)=\frac{b_0}{\zeta}+b_1+b_2\zeta.\tag{4.3.17}$$

Substituting this in Eq. (4.3.16), we get $b_0=0, b_1=b_1, b_2=\frac{-(-b+K_2(2-a)^{\frac{1}{a-2}})}{(a-1)}$. Hence, we find the solution of Eq. (4.1.2) as follows:

$$\begin{aligned}\sigma(x,y,t)=&\left(\int\alpha(t)dt\right)^{\frac{a}{2-a}}\left(\left(\frac{x}{\left(\int\alpha(t)dt\right)^{\frac{1}{2-a}}}\right)(r(y,t))-(2-a)^{\frac{1}{2-a}}(a-1)\frac{(r(y,t))^3}{6K_2}\right)\\&+\left(\int\alpha(t)dt\right)^{\frac{a}{2-a}}\left(b_1+\frac{-(-b+K_2(2-a)^{\frac{1}{a-2}})}{(a-1)}\left(\frac{x}{\left(\int\alpha(t)dt\right)^{\frac{1}{2-a}}}-(2-a)(r(y,t))\right)+\left(\frac{(r(y,t))^2}{2}\right)\right),\end{aligned}\tag{4.3.18}$$

where $r(y,t)=y-\frac{b}{2-a}\log\left(\int\alpha(t)dt\right)$.

4.3.3.2 Vector Field V_2+cV_4

Following the same way as above, we get

$$\begin{aligned}&\sigma(x,y,t)=\frac{1}{\left(\int\alpha(t)dt\right)}H(\rho,\psi),\ \rho=x,\ \psi=y+c\log\left(\int\alpha(t)dt\right),\\&\beta(t)=K_1\alpha(t)\left(-\int\alpha(t)dt\right)^{-1},\ \delta(t)=K_2\alpha(t)\left(-\int\alpha(t)dt\right)^{-1}.\end{aligned}\tag{4.3.19}$$

Substituting (4.3.19) into Eq. (4.1.2) yields a reduced nonlinear PDE with constant coefficients:

$$-H_\rho+cH_{\rho\psi}+H_\rho H_{\rho\rho}-K_1H_{\rho\rho\rho\rho}-K_2H_{\psi\psi}=0.\tag{4.3.20}$$

Make transformation as follows:

$$H(\rho,\psi)=H(\zeta),\zeta=\psi-w\rho,\tag{4.3.21}$$

where w is a non-zero constant. Substituting (4.3.21) into Eq. (4.3.20) obtains an ODE to ζ. Integrating it once with respect to ζ and taking integration constant to zero, yield

$$2wH(\zeta)-2cwH'(\zeta)-w^3(H'(\zeta))^2-2K_1w^4H'''(\zeta)-2K_2H'(\zeta)=0.\tag{4.3.22}$$

We seek the solution of Eq. (4.3.22) in the following form

$$H(\zeta) = b_0 + b_1\zeta + b_2\zeta^2, \tag{4.3.23}$$

where b_0, b_1 and b_2 are constants to be found out. On substitution of the form of $H(\zeta)$ in equation (4.3.22), we get $b_0 = \frac{{b_1}^2{K_2}^2}{2c^2}, b_1 = b_1, b_2 = \frac{c^2}{2{K_2}^2}$ with $w = \frac{-K_2}{c}$. Then corresponding solution of Eq. (4.1.2) is as follows:

$$\sigma(x,y,t) = \frac{\left(1/2\,\frac{{b_1}^2{K_2}^2}{c^2} + b_1\,(r(x,y,t)) + 1/2\,c^2\,(r(x,y,t))^2\,{K_2}^{-2}\right)}{\left(\int \alpha\,(t)\,dt\right)}, \tag{4.3.24}$$

where $r(x,y,t) = y + c\ln\left(\int \alpha\,(t)\,dt\right) + \frac{K_2\,x}{c}$

4.3.3.3 Vector Field $V_3 + dV_4 + V_5$

Solving the equations (4.2.5) and (4.3.9), we get

$$\begin{gathered}\sigma(x,y,t) = x\big(\textstyle\int \alpha(t)dt\big) - \frac{1}{3}\big(\textstyle\int \alpha(t)dt\big)^3 + H(\rho,\psi),\ \rho = y - d\textstyle\int \alpha(t)dt,\\ \psi = x - \frac{1}{2}\big(\textstyle\int \alpha(t)dt\big)^2,\ \beta(t) = K_1\alpha(t),\ \delta(t) = K_2\alpha(t).\end{gathered} \tag{4.3.25}$$

Substituting (4.3.25) into Eq. (4.1.2) yields a reduced nonlinear PDE with constant coefficients:

$$K_1H_{\psi\psi\psi\psi} + H_\psi H_{\psi\psi} + K_2H_{\rho\rho} - dH_{\rho\psi} + 1 = 0. \tag{4.3.26}$$

To reduce the Eq. (4.3.26) to ODE, make the transformation as follows:

$$H(\rho,\psi) = \frac{d^2}{4K_2}\psi + \frac{d^3}{24{K_2}^2}\rho - \frac{1}{2K_2}\rho^2 + \frac{2}{3}\rho + F(\zeta), \zeta = \psi + \frac{d}{2K_2}\rho. \tag{4.3.27}$$

Substituting Eq. (4.3.27) into Eq. (4.3.26) yields nonlinear ODE as follows:

$$K_1F''''(\zeta) + F'(\zeta)F''(\zeta) = 0. \tag{4.3.28}$$

Integrating it once with respect to ζ yield

$$K_1F'''(\zeta) + \frac{1}{2}(F')^2(\zeta) + K_3 = 0, \tag{4.3.29}$$

where K_3 is constant of integration. Multiplying Eq. (4.3.29) by F'' and then integrating with respect to ζ, taking integration constant to zero and substituting $F'(\zeta) = J(\zeta)$, we get

$$K_1(J'(\zeta))^2 + \frac{1}{3}J(\zeta)^3 + K_3J(\zeta) = 0. \tag{4.3.30}$$

Solving the above ODE, we get the solution of (4.3.30) as follows:

$$J(\zeta) = -\wp\left(1/6\,\frac{2^{2/3}\sqrt{3}\zeta}{\sqrt{K_1\,2^{2/3}}} + C_1, -3\,K_3\,2^{2/3}, 0\right)2^{2/3}, \tag{4.3.31}$$

where C_1 is arbitrary constant, $\wp$ denotes WeirstrassP function and final solution of Eq. (4.1.2) is given by relations (4.3.25) and (4.3.27).

4.3.3.4 Vector Field $V_3 + pV_4$

This case yields the following forms of invariants and coefficient functions:

$$\begin{aligned}&\sigma(x,y,t) = \tfrac{x^2}{2\left(\int \alpha(t)dt\right)} + H(\rho,\psi),\ \rho = t,\ \psi = x - \tfrac{y\left(\int \alpha(t)dt\right)}{p},\ \alpha(t) = \alpha(t),\\ &\beta(t) = \beta(t),\ \delta(t) = \delta(t).\end{aligned} \tag{4.3.32}$$

Substituting (4.3.32) into Eq. (4.1.2), the VCPKP equation transforms to following equation:

$$H_{\rho\psi} + \frac{\alpha(\rho)}{\left(\int \alpha(\rho)d\rho\right)}(\psi H_{\psi\psi} + H_\psi) + \alpha(\rho)H_\psi H_{\psi\psi} + \beta(\rho)H_{\psi\psi\psi\psi} + \frac{1}{p^2}\delta(\rho)\left(\int \alpha(\rho)d\rho\right)^2 H_{\psi\psi} = 0. \tag{4.3.33}$$

Eq. (4.3.33) can be integrated with respect to ψ to get the following form:

$$H_\rho + \frac{\alpha(\rho)}{\left(\int \alpha(\rho)d\rho\right)}(\psi H_\psi) + \alpha(\rho)\frac{{H_\psi}^2}{2} + \beta(\rho)H_{\psi\psi\psi} + \frac{1}{p^2}\delta(\rho)\left(\int \alpha(\rho)d\rho\right)^2 H_\psi = J(\rho), \tag{4.3.34}$$

where $J(\rho)$ denotes the arbitrary function of ρ. Due to complexity of the above integrated equation (4.3.34), we are unable to obtain nontrivial solutions for it.

4.3.3.5 Vector Field $V_4 + V_5 + qV_7$

For this case, the similarity variable, similarity solution and coefficient functions are as follows:

$$\sigma(x,y,t) = qy + H(\rho,\psi),\ \rho = x,\ \psi = y - \left(\int \alpha(t)dt\right),\ \beta(t) = K_1\alpha(t),\ \delta(t) = K_2\alpha(t). \tag{4.3.35}$$

Substituting (4.3.35) into Eq. (4.1.2), we get

$$K_1 H_{\rho\rho\rho\rho} + H_\rho H_{\rho\rho} + K_2 H_{\psi\psi} - H_{\rho\psi} = 0. \tag{4.3.36}$$

The traveling wave variable

$$H(\rho,\psi) = H(\zeta), \zeta = \rho - w\psi, \tag{4.3.37}$$

where w is a constant to be determined later, permits us reducing the Eq. (4.3.36) to an ODE in the form

$$2K_1H'''(\zeta) + 2(w + K_2w^2)H'(\zeta) + (H'(\zeta))^2 + K_3 = 0, \tag{4.3.38}$$

where K_3 is constant of integration. The ODE (4.3.38) possesses the following solutions:

Case 1: Multiplying Eq. (4.3.38) by $H''(\zeta)$ and then once more integrating with respect to ζ, taking integration constant to zero and substituting $H'(\zeta) = J(\zeta)$, we get

$$K_1(J'(\zeta))^2 + (w + K_2w^2)J(\zeta)^2 + \frac{1}{3}J(\zeta)^3 + K_3J(\zeta) = 0. \tag{4.3.39}$$

Solving the above ODE, we get the solution of (4.3.39) as follows:

$$J(\zeta) = -w - K_2w^2 - 12{C_2}^2K_1\wp\left(C_1 + C_2\zeta, C_3, C_4\right), \tag{4.3.40}$$

where C_1, C_2 are arbitrary constants, $\wp$ denotes WeirstrassP function and $C_3 = \frac{w^2+{K_2}^2w^4+2K_2w^3-K_3}{12{K_1}^2{C_2}^4}, C_4 = \frac{w\left(2{K_2}^3w^5+6{K_2}^2w^4+6K_2w^3+2w^2-3K_2wK_3-3K_3\right)}{432{K_1}^3{C_2}^6}$.

Case 2: In this case, we seek solutions of Eq. (4.3.38) by extended (G'/G)-expansion method as desribed in section (1.7.2). Suppose the solution of (4.3.38) can be expressed in (G'/G) as Eq. (1.7.4) where $G = G(\zeta)$ satisfies the second-order linear ODE (1.7.5). On balancing the highest-order derivatives with the nonlinear terms appearing in (4.3.38), we get $n = 1$. Substituting (1.7.4) into (4.3.38) and using (1.7.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)^k$ and $\left(\frac{G'}{G}\right)^k\sqrt{\nu\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$ together, and equating each coefficient of them to zero, yield a set of following algebraic equations for a_0, a_1, b_1 and w:

$$\begin{aligned}
&(i) \quad -12\,K_1\,a_1 + \tfrac{{b_1}^2\nu}{\mu} + {a_1}^2 = 0,\\
&(ii) \quad -12\,K_1\,b_1 + 2\,a_1\,b_1 = 0,\\
&(iii) \;\; 2\,{a_1}^2\mu - 2\,wa_1 + {b_1}^2\nu - 2\,K_2\,w^2a_1 - 16\,K_1\,a_1\,\mu = 0,\\
&(iv) \;\; 2\,a_1\,\mu\,b_1 - 2\,wb_1 - 10\,K_1\,b_1\,\mu - 2\,K_2\,w^2b_1 = 0,\\
&(v) \quad -2\,wa_1\,\mu + K_3 + {a_1}^2\mu^2 - 2\,K_2\,w^2a_1\,\mu - 4\,K_1\,a_1\,\mu^2 = 0.
\end{aligned} \tag{4.3.41}$$

Solving these algebraic equations by Maple, we obtain the following results:

Subcase 1:

$$a_0 = a_0, \;\; a_1 = 12K_1, \;\; w = \frac{-1+\sqrt{1+16K_2K_1\mu}}{2K_2}, \;\; b_1 = K_3 = 0. \tag{4.3.42}$$

Subcase 2:

$$a_0 = a_0, \;\; a_1 = 6K_1, \;\; w = \frac{-1+\sqrt{1+4K_2K_1\mu}}{2K_2}, \;\; b_1 = \pm 6\sqrt{\frac{\mu}{\nu}}K_1, \;\; K_3 = 0. \tag{4.3.43}$$

From (1.7.4) and the general solution of (1.7.5), we deduce the traveling wave solutions of (4.1.2) as follows:

When $\mu > 0$, then Subcase 1 gives the exact traveling wave solution as follows:

$$\sigma(x,y,t) = qy + a_0 + 12K_1 \frac{\sqrt{\mu}\left(B\cos\left(\sqrt{\mu}(u(x,y,t))\right) - A\sin\left(\sqrt{\mu}(u(x,y,t))\right)\right)}{\left(A\cos\left(\sqrt{\mu}(u(x,y,t))\right) + B\sin\left(\sqrt{\mu}(u(x,y,t))\right)\right)}, \tag{4.3.44}$$

Subcase 2 gives the solution of Eq. (4.1.2) as follows:

$$\begin{aligned}\sigma(x,y,t) &= qy + a_0 + 6K_1 \frac{\sqrt{\mu}\left(B\cos\left(\sqrt{\mu}(v(x,y,t))\right) - A\sin\left(\sqrt{\mu}(v(x,y,t))\right)\right)}{\left(A\cos\left(\sqrt{\mu}(v(x,y,t))\right) + B\sin\left(\sqrt{\mu}(v(x,y,t))\right)\right)} \\ &\pm 6\sqrt{\tfrac{\mu}{\nu}}K_1\sqrt{\nu\left(1 + \frac{\left(B\cos\left(\sqrt{\mu}(v(x,y,t))\right) - A\sin\left(\sqrt{\mu}(v(x,y,t))\right)\right)^2}{\left(A\cos\left(\sqrt{\mu}(v(x,y,t))\right) + B\sin\left(\sqrt{\mu}(v(x,y,t))\right)\right)^2}\right)}.\end{aligned} \tag{4.3.45}$$

When $\mu < 0$, then Subcase 1 gives the exact traveling wave solution as follows:

$$\sigma(x,y,t) = qy + a_0 + 12K_1 \frac{\sqrt{-\mu}(B\cosh(\sqrt{-\mu}(u(x,y,t))) + A\sinh(\sqrt{-\mu}(u(x,y,t))))}{(A\cosh(\sqrt{-\mu}(u(x,y,t))) + B\sinh(\sqrt{-\mu}(u(x,y,t))))}. \tag{4.3.46}$$

and Subcase 2 gives the solution as follows:

$$\begin{aligned}\sigma(x,y,t) &= qy + a_0 + 6K_1 \frac{\sqrt{-\mu}(B\cosh(\sqrt{-\mu}(v(x,y,t))) + A\sinh(\sqrt{-\mu}(v(x,y,t))))}{(A\cosh(\sqrt{-\mu}(v(x,y,t))) + B\sinh(\sqrt{-\mu}(v(x,y,t))))} \\ &\pm 6\sqrt{\tfrac{\mu}{\nu}}K_1\sqrt{\nu\left(1 - \frac{(B\cosh(\sqrt{-\mu}(v(x,y,t))) + A\sinh(\sqrt{-\mu}(v(x,y,t))))^2}{(A\cosh(\sqrt{-\mu}(v(x,y,t))) + B\sinh(\sqrt{-\mu}(v(x,y,t))))^2}\right)},\end{aligned} \tag{4.3.47}$$

where A, B are arbitrary constants and $u(x,y,t) = x - \frac{\left(-1+\sqrt{1+16K_2K_1\mu}\right)\left(y-\int\alpha(t)dt\right)}{2K_2}$, $v(x,y,t) = x - \frac{\left(-1+\sqrt{1+4\,K_2\,K_1\,\mu}\right)\left(y-\int\alpha(t)dt\right)}{2K_2}$. In this case, certain solutions obtained by extended (G'/G) method behaves as periodic solutions for different values of constants and functions involved. For $q = a_0 = 0, \mu = \nu = K_1 = K_2 = A = B = 1, y = \sin(x)$ and with $\alpha(t) = 0$ and $\alpha(t) = \tan(t)$, solutions (4.3.44) and (4.3.45) takes the form of periodic solutions as shown in Fig. (4.7) and Fig. (4.8) respectively.

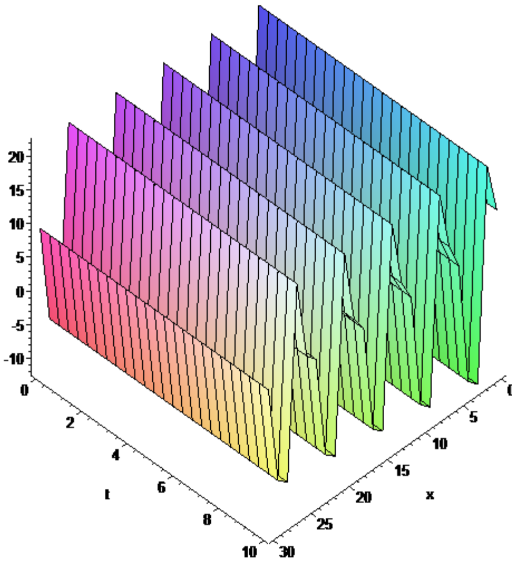

FIGURE 4.7: The figure of periodic solution (4.3.44) for $q = a_0 = 0, \mu = \nu = K_1 = K_2 = A = B = 1, y = \sin(x), \alpha(t) = 0$

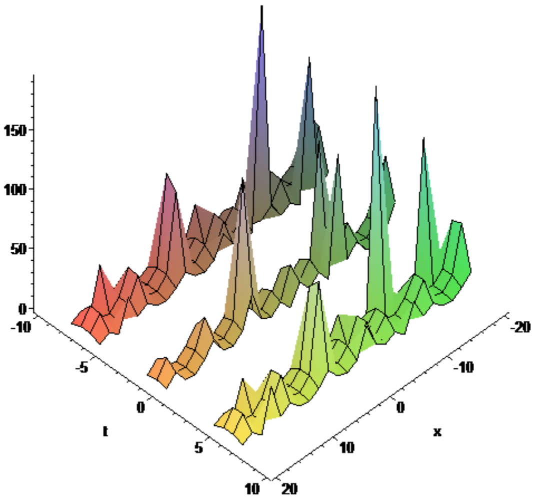

FIGURE 4.8: The figure of periodic solution (4.3.45) for $q = a_0 = 0, \mu = \nu = K_1 = K_2 = A = B = 1, y = \sin(x), \alpha(t) = \tan(t)$

4.3.3.6 Vector Field $V_4 + rV_7$

For this vector field, on Solving the equations (4.2.5) and (4.3.9) we are able to obtain

$$\sigma(x,y,t) = ry + H(\rho,\psi),\ \ \rho = x,\ \ \psi = t,\ \ \alpha(t) = \alpha(t),\ \ \beta(t) = \beta(t),\ \ \delta(t) = \delta(t). \tag{4.3.48}$$

On using (4.3.48), the VCPKP equation transforms to:

$$\beta(\psi)H_{\rho\rho\rho\rho} + \alpha(\psi)H_\rho H_{\rho\rho} + H_{\rho\psi} = 0. \tag{4.3.49}$$

After applying the Lie symmetry method, equation (4.3.49) yields the symmetries as under:

$$\begin{aligned} \xi^1 &= a_1\rho + a_4, \\ \tau^1 &= \tfrac{1}{\alpha(\psi)}((2a_1 - a_2)\int(\alpha(\psi)\,d\psi) + a_3), \\ \eta^1 &= a_2H + a_5, \end{aligned} \tag{4.3.50}$$

where a_1, a_2, a_3, a_4, a_5 are arbitrary constants and ξ^1, τ^1, η^1 are infinitesimals corresponding to ρ, ψ, H. The function $\beta(\psi)$ is governed by the following condition:

$$\beta(\psi)\tau_\psi + \tau\beta'(\psi) - 3a_1\beta(\psi) = 0. \tag{4.3.51}$$

The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned} &Z_1 = \rho\frac{\partial}{\partial\rho} + \left(\frac{2\int\alpha(\psi)d\psi}{\alpha(\psi)}\right)\frac{\partial}{\partial\psi},\ Z_2 = H\frac{\partial}{\partial H} - \left(\frac{\int\alpha(\psi)d\psi}{\alpha(\psi)}\right)\frac{\partial}{\partial\psi},\ Z_3 = \frac{1}{\alpha(\psi)}\frac{\partial}{\partial\psi},\\ &Z_4 = \frac{\partial}{\partial\rho},\ Z_5 = \frac{\partial}{\partial H}. \end{aligned} \tag{4.3.52}$$

A one-dimensional optimal system for (4.3.49) is given as follows:

$$(i)\ Z_1 + d_1Z_2,\ (ii)\ Z_2,\ (iii)\ Z_2 + Z_4,\ (iv)\ Z_3 + d_2Z_4 + d_3Z_5, \tag{4.3.53}$$

where d_1, d_2, d_3 are arbitrary constants. In Table (4.4), we list the similarity variable, similarity solution and forms of coefficient functions corresponding to the optimal system (4.3.53).

TABLE 4.4: Similarity Variables, Forms and Coefficient Functions of Equation (4.3.49)

Essential fields	*Similarity variables*(ζ)	*Similarity solution*(H)	*Coefficient functions*$(\beta(\psi))$
$Z_1 + d_1Z_2$	$\frac{\rho}{\left(\int\alpha(\psi)d\psi\right)^{\frac{1}{2-d_1}}}$	$\left(\int\alpha(\psi)d\psi\right)^{\frac{d_1}{2-d_1}}F(\zeta)$	$K_1\alpha(\psi)\left(\int\alpha(\psi)d\psi\right)^{\frac{1+d_1}{2-d_1}}$
Z_2	ρ	$\frac{1}{\left(\int\alpha(\psi)d\psi\right)}F(\zeta)$	$-K_1\alpha(\psi)\left(\int\alpha(\psi)d\psi\right)^{-1}$
$Z_2 + Z_4$	$\rho + \log\left(\int\alpha(\psi)d\psi\right)$	$\frac{1}{\left(\int\alpha(\psi)d\psi\right)}F(\zeta)$	$-K_1\alpha(\psi)\left(\int\alpha(\psi)d\psi\right)^{-1}$
$Z_3 + d_2Z_4 + d_3Z_5$	$\rho - d_2\left(\int\alpha(\psi)d\psi\right)$	$d_3\left(\int\alpha(\psi)d\psi\right) + F(\zeta)$	$K_1\alpha(\psi)$

Now, we will discuss the reductions corresponding to above mentioned vector fields.

1. The reduced ODE corresponding to vector field $Z_1 + d_1Z_2$ is given as follows:

$$(d_1 - 1)F'(\zeta) - \zeta F''(\zeta) + (2 - d_1)F'(\zeta)F''(\zeta) + (2 - d_1)K_1F''''(\zeta) = 0. \tag{4.3.54}$$

Integrating Eq. (4.3.54) with respect to ζ, taking integration constant to be zero, we get:

$$d_1F(\zeta) - \zeta F'(\zeta) + \frac{(2 - d_1)}{2}(F'(\zeta))^2 + (2 - d_1)K_1F'''(\zeta) = 0. \tag{4.3.55}$$

We assume that the solution of Eq. (4.3.55) takes the form

$$F(\zeta) = \frac{b_0}{\zeta} + b_1 + b_2\zeta, \tag{4.3.56}$$

where b_0, b_1, b_2 are arbitrary constants. On substituting (4.3.56) in Eq. (4.3.55), we get $b_0 = 0, b_1 = -1/2b_2^2, b_2 = b_2$ with the restriction that $d_1 = 1$. Hence, the corresponding solution of (4.1.2) is as follows:

$$\sigma(x, y, t) = ry - 1/2\,{b_2}^2 \int \alpha\,(t)\,dt + b_2\,x. \tag{4.3.57}$$

2. The vector field Z_2 reduces Eq. (4.3.49) to the following ODE:

$$-F'(\zeta) + F'(\zeta)F''(\zeta) - K_1F''''(\zeta) = 0. \tag{4.3.58}$$

Integrating Eq. (4.3.58) with respect to ζ and taking integration constant to be zero, we get:

$$-F(\zeta) + (F'(\zeta))^2 - K_1F'''(\zeta) = 0. \tag{4.3.59}$$

Solving above ODE, we get solution of Eq. (4.1.2) as follows:

$$\sigma(x, y, t) = ry + \left(1/2\,{C_2}^2{C_3}^2 + C_3\,(C_1 + C_2\,x) + 1/2\,\frac{(C_1 + C_2\,x)^2}{{C_2}^2}\right)\left(\int \alpha\,(t)\,dt\right)^{-1}, \tag{4.3.60}$$

where C_1, C_2, C_3 are arbitrary constants.

3. The vector field $Z_2 + Z_4$ corresponds to the following ODE:

$$F''(\zeta) - F'(\zeta) + F'(\zeta)F''(\zeta) - K_1F''''(\zeta) = 0. \tag{4.3.61}$$

On solving Eq. (4.3.61) for $K_1 = 0$, we get the following solution:

$$F(\zeta) = W\left(e^{\zeta}C_1\right) + 1/2\,\left(W\left(e^{\zeta}C_1\right)\right)^2 + C_2, \tag{4.3.62}$$

where C_1, C_2 are arbitrary constants and W denotes the LambertW function. Hence, the solution (4.3.62) gives the following solution of Eq. (4.1.2):

$$\sigma(x, y, t) = ry + \frac{W\left(e^{x+\log\left(\int \alpha(t)dt\right)}C_1\right) + 1/2\,\left(W\left(e^{x+\log\left(\int \alpha(t)dt\right)}C_1\right)\right)^2 + C_2}{\int \alpha\,(t)\,dt}. \tag{4.3.63}$$

4. The reduced ODE corresponding to vector field $Z_3 + d_2Z_4 + d_3Z_5$ is as follows:

$$-d_2F''(\zeta) + F'(\zeta)F''(\zeta) + K_1F''''(\zeta) = 0. \tag{4.3.64}$$

Integrating above ODE with respect to ζ, we get

$$-d_2F'(\zeta)+1/2F'(\zeta)^2+K_1F'''(\zeta)+K_2=0, \tag{4.3.65}$$

where K_2 is arbitrary constant. Multiplying Eq. (4.3.65) by $F''(\zeta)$ and then once more integrating with respect to ζ, taking integration constant to zero and substituting $F'(\zeta)=J(\zeta)$, we get

$$-d_2/2J(\zeta)^2+1/6J(\zeta)^3+K_1/2J'(\zeta)^2+K_2J(\zeta)=0. \tag{4.3.66}$$

On solving Eq. (4.3.66), we get

$$J(\zeta)=d_2-12\,K_1\,{C_4}^2\wp\left(C_3+C_4\,\zeta,-1/12\,\frac{-{d_2}^2+2\,K_2}{{K_1}^2{C_4}^4},\frac{1}{216}\,\frac{d_2\,\left(3\,K_2-{d_2}^2\right)}{{K_1}^3{C_4}^6}\right), \tag{4.3.67}$$

where C_3, C_4 are arbitrary constants.

4.3.3.7 Vector Field V_5+sV_7

Solving the equations (4.2.5) and (4.3.9), we get

$$\sigma(x,y,t)=s\big(\textstyle\int\alpha(t)dt\big)+H(\rho,\psi),\ \rho=x,\ \psi=y,\ \beta(t)=K_1\alpha(t),\ \delta(t)=K_2\alpha(t). \tag{4.3.68}$$

Substituting (4.3.68) into the VCPKP equation, we obtain

$$K_1H_{\rho\rho\rho\rho}+H_\rho H_{\rho\rho}+K_2H_{\psi\psi}=0. \tag{4.3.69}$$

The traveling wave variable

$$H(\rho,\psi)=H(\zeta),\zeta=\rho-w\psi, \tag{4.3.70}$$

where w is a constant to be determined later, permits us reducing the Eq. (4.3.69) to an ODE in the form

$$2K_1H'''(\zeta)+2K_2w^2H'(\zeta)+(H'(\zeta))^2+K_3=0, \tag{4.3.71}$$

where K_3 is constant of integration. The ODE (4.3.71) possesses the following solutions:

Case 1: Multiplying Eq. (4.3.71) by $H''(\zeta)$ and then once more integrating with respect to ζ, taking integration constant to zero and substituting $H'(\zeta) = J(\zeta)$, we get

$$K_1(J'(\zeta))^2 + K_2 w^2 J(\zeta)^2 + \frac{1}{3} J(\zeta)^3 + K_3 J(\zeta) = 0. \tag{4.3.72}$$

Solving the above ODE, we get the solution of (4.3.72) as follows:

$$J(\zeta) = -K_2\, w^2 - 12\, {C_2}^2 K_1\, \wp\left(C_1 + C_2\, \zeta, 1/12\, \frac{-K_3 + {K_2}^2 w^4}{{C_2}^4 {K_1}^2}, \frac{1}{432}\, \frac{K_2\, w^2 \left(2\, {K_2}^2 w^4 - 3\, K_3\right)}{{C_2}^6 {K_1}^3}\right), \tag{4.3.73}$$

where C_1, C_2 are arbitrary constants and $\wp$ denotes WeirstrassP function.

Case 2: In this case, we seek solutions of Eq. (4.3.71) by extended (G'/G)-expansion method as discussed in section (1.7). By applying this method, we get a set of following algebraic equations for a_0, a_1, b_1 and w:

$$\begin{aligned}
&(i)\ \ \tfrac{{b_1}^2 \nu}{\mu} + {a_1}^2 - 12\, K_1\, a_1 = 0,\\
&(ii)\ \ -12\, K_1\, b_1 + 2\, a_1\, b_1 = 0,\\
&(iii)\ \ 2\, {a_1}^2 \mu + {b_1}^2 \nu - 2\, K_2\, w^2 a_1 - 16\, K_1\, a_1\, \mu = 0,\\
&(iv)\ \ -10\, K_1\, b_1\, \mu - 2\, K_2\, w^2 b_1 + 2\, a_1\, \mu\, b_1 = 0,\\
&(v)\ \ -4\, K_1\, a_1\, \mu^2 + {a_1}^2 \mu^2 - 2\, K_2\, w^2 a_1\, \mu = 0.
\end{aligned} \tag{4.3.74}$$

Solving these algebraic equations by Maple, we obtain the following results:

Subcase 1:

$$a_0 = a_0,\ \ a_1 = 12 K_1,\ \ w = 2\sqrt{\frac{K_1 \mu}{K_2}},\ \ b_1 = K_3 = 0. \tag{4.3.75}$$

Subcase 2:

$$a_0 = a_0,\ \ a_1 = 6 K_1,\ \ w = \sqrt{\frac{K_1 \mu}{K_2}},\ \ b_1 = \pm 6\sqrt{\frac{\mu}{\nu}} K_1,\ \ K_3 = 0. \tag{4.3.76}$$

From (1.7.4) and the general solution of (1.7.5), we deduce the traveling wave solutions of (4.1.2) as follows:

When $\mu > 0$, then Subcase 1 gives the exact traveling wave solution as:

$$\begin{aligned}
&\sigma(x, y, t) = s \textstyle\int \alpha\,(t)\, dt + a_0\\
&+12\, K_1\, \sqrt{\mu} \frac{\left(B \cos\left(\sqrt{\mu}\left(x - 2\sqrt{\frac{K_1\, \mu}{K_2}} y\right)\right) - A \sin\left(\sqrt{\mu}\left(x - 2\sqrt{\frac{K_1\, \mu}{K_2}} y\right)\right)\right)}{\left(A \cos\left(\sqrt{\mu}\left(x - 2\sqrt{\frac{K_1\, \mu}{K_2}} y\right)\right) + B \sin\left(\sqrt{\mu}\left(x - 2\sqrt{\frac{K_1\, \mu}{K_2}} y\right)\right)\right)},
\end{aligned} \tag{4.3.77}$$

Subcase 2 gives the solution of Eq. (4.1.2) as:

$$\begin{aligned}\sigma(x,y,t) &= s\int\alpha\,(t)\,dt + a_0 + 6\,K_1\,\sqrt{\mu}\frac{\left(B\cos\left(\sqrt{\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)-A\sin\left(\sqrt{\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)}{\left(A\cos\left(\sqrt{\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)+B\sin\left(\sqrt{\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)}\\ &\pm 6\,\sqrt{\frac{\mu}{\nu}}K_1\,\sqrt{(\nu)\,\left(1+\frac{\left(B\cos\left(\sqrt{\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)-A\sin\left(\sqrt{\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)^2}{\left(A\cos\left(\sqrt{\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)+B\sin\left(\sqrt{\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)^2}\right)}.\end{aligned} \tag{4.3.78}$$

When $\mu < 0$, then Subcase 1 gives the exact traveling wave solution as:

$$\begin{aligned}\sigma(x,y,t) &= s\int\alpha\,(t)\,dt + a_0\\ &+12\,K_1\,\sqrt{-\mu}\frac{\left(B\cosh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)+A\sinh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)}{\left(A\cosh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)+B\sinh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)}.\end{aligned} \tag{4.3.79}$$

and Subcase 2 gives the solution as:

$$\begin{aligned}\sigma(x,y,t) &= s\int\alpha\,(t)\,dt + a_0\\ &+6\,K_1\,\sqrt{-\mu}\frac{\left(B\cosh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)+A\sinh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)}{\left(A\cosh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)+B\sinh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)}\\ &\pm 6\,\sqrt{\frac{\mu}{\nu}}K_1\,\sqrt{\nu\,\left(1-\frac{\left(B\cosh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)+A\sinh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)^2}{\left(A\cosh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)+B\sinh\left(\sqrt{-\mu}\left(x-2\,\sqrt{\frac{K_1\,\mu}{K_2}}y\right)\right)\right)^2}\right)},\end{aligned} \tag{4.3.80}$$

where A, B are arbitrary constants. Certain solutions obtained by extended (G'/G) method behaves as kink wave solutions for different values of constants and functions involved. For $s = 1, a_0 = 0, K_1 = 2, \mu = \nu = -1, K_2 = 3, y = \tan(x)$ and with $\alpha(t) = \text{sech(t)}$ and $\alpha(t) = \cosh(t)$, solutions (4.3.79) and (4.3.80) takes the form of kink wave solutions as shown in Fig. (4.9) and Fig. (4.10) respectively.

4.3.3.8 Vector Field V_6

Following the same way, we get

$$\sigma(x,y,t) = H(\rho,\psi),\ \rho = y,\ \psi = t,\ \alpha(t) = \alpha(t),\ \beta(t) = \beta(t),\ \delta(t) = \delta(t). \tag{4.3.81}$$

Using (4.3.81), Eq. (4.1.2) transforms to following linear PDE:

$$H_{\rho\rho} = 0. \tag{4.3.82}$$

On solving the equation (4.3.82), we get

$$\sigma(x,y,t) = yF_1(t) + F_2(t). \tag{4.3.83}$$

where F_1, F_2 are arbitrary functions.

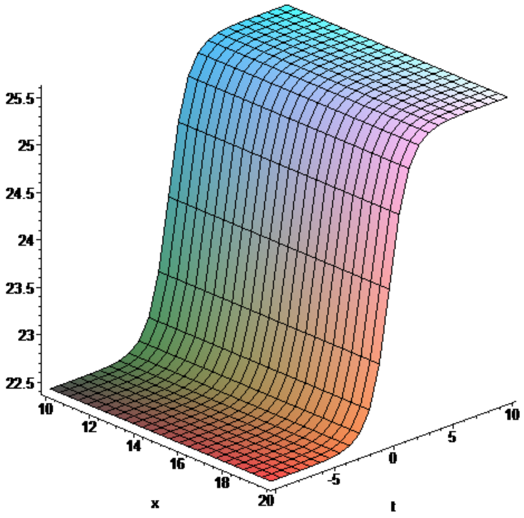

FIGURE 4.9: The figure of kink wave solution (4.3.79) for $s = 1, a_0 = 0, K_1 = 2, \mu = \nu = -1, K_2 = 3, y = \tan(x), \alpha(t) = \text{sech(t)}$

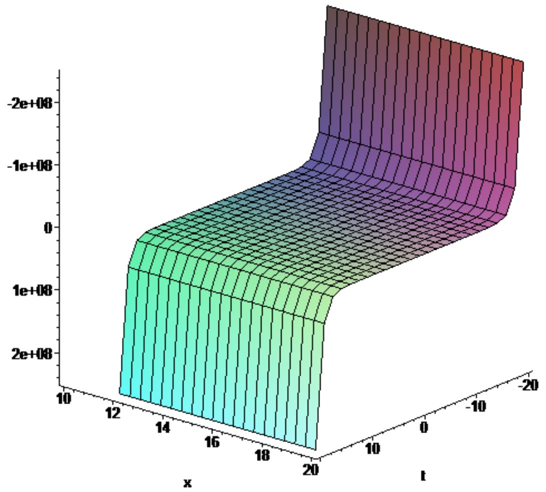

FIGURE 4.10: The figure of kink wave solution (4.3.79) for $s = 1, a_0 = 0, K_1 = 2, \mu = \nu = -1, K_2 = 3, y = \tan(x), \alpha(t) = \cosh(t)$

4.4 Conclusion and Discussions

In this chapter, we have investigated the symmetries and invariant solutions of PKP and VCPKP equation. Firstly, the Lie group method is utilized for obtaining the group infinitesimals of PKP equation. This equation turn out to have infinite-dimensional symmetry group, the Lie algebras of which involve arbitrary functions. Using subalgebras of PKP algebra, we have reduced the PKP equation in PDEs in two variables and then they are again applied by Lie symmetry method to reduce them to ODEs and for each element in the optimal system, certain exact solutions are obtained. One of the reduced PDE is again investigated with the help of Homoclinic test method to obtain more solutions. To the best of our knowledge, reduction of (2+1)-dimensional PKP equation to two dimensional PDE's using infinite dimensional symmetry algebra are new. Due to dependence of solutions on arbitrary functions, the solutions of PKP equation obtained by us are new and have not been reported earlier in literature. The solutions obtained by us are such that one can choose the arbitrary functions $f(t), g(t), h(t)$ along with various other parameters, in a suitable manner, to simulate physical situations governed by Eq. (4.1.1) to obtain particular solutions having desired features (as shown in Figures (4.1)-(4.6)). By using Lie Symmetry method, we obtain new type of special solutions of PKP equation, which includes exact periodic and periodic kinky solutions.

With symbolic computation in section (4.3), we have performed the Painlevé analysis for VCPKP equation (4.1.2) and obtained its explicit Painlevé-integrable condition. We have presented the similarity reductions of Eq. (4.1.2) by classical Lie group method and plenty of new solutions are given, including the periodic solutions and kink wave solutions. The availability of mathematical computer software like Maple facilitates the tedious algebraic calculations. It is worth to mention here that the correctness of the solutions has been checked with the aid of software Maple.

Chapter 5

The Variable Coefficient Dullin-Gottwald-Holm Equation[1]

5.1 Introduction

Dullin et al. [116] derived the following equation:

$$m_t + c_0 u_x + u m_x + 2 m u_x = -\gamma u_{xxx}, \quad x \in \mathbb{R}, t \in \mathbb{R}. \tag{5.1.1}$$

which is called Dullin-Gottwald-Holm (DGH) equation, by using asymptotic expansions directly in the Hamiltonian for Euler equations in the shallow water regime and thereby is shown to be bi-Hamiltonian and has a Lax pair formulation. DGH equation combines the linear dispersion of Korteweg-de Vries (KdV) equation with the nonlinear/nonlocal dispersion of the Camassa-Holm (CH) equation, yet still preserves integrability via the inverse scattering transform (IST) method. Eq. (5.1.1) in dimensionless time-space variables (t, x) models the unidirectional propagation of two dimensional waves in shallow water over a flat bottom. In (5.1.1), $u(t, x)$ represents the horizontal component of the fluid velocity, $m = u - \alpha^2 u_{xx}$ is a momentum variable, the constants α^2 and γ/c_0 are squares of length scales, and $c_0 = \sqrt{gh}$ (where $c_0 = 2w$) is the linear wave speed for undisturbed water at rest at spatial infinity. Using the notation $m = u - \alpha^2 u_{xx}$, we rewrite the DGH equation as

$$u_t - \alpha^2 u_{xxt} + 2 w u_x + 3 u u_x + \gamma u_{xxx} = \alpha^2 (2 u_x u_{xx} + u u_{xxx}). \tag{5.1.2}$$

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[1] A part of this chapter has been published in **International Journal of Nonlinear Science, 10 (2010) 146-152** and some part has been communicated to **International Journal of Engineering Mathematics**.

Recently, the authors [35, 79, 80, 81] studied the well-posedness of Cauchy problem and the scattering problem for the DGH equation. Moreover, the issue of passing to the limit as the dispersive parameter tends to zero for the solution of DGH equation was investigated, and the scattering data of the scattering problem for the equation were explicitly expressed in [79]. Octavian G. Mustafa [103] investigated the low regularity conditions needed for the Cauchy problem of DGH equation via the semigroup approach of quasilinear hyperbolic equations of evolution and the viscosity method. Yue Liu [162] investigated the problems of the existence of global solutions and the formation of singularities for the DGH equation. In [82], Gui et al. studied the limit behaviour of the solutions to a class of nonlinear dispersive wave equations, which can be seen as some extension of DGH equation. Y. Li and P. Olver [161] studied the well-posedness, blow-up and the low regular solutions for an integrable nonlinearly dispersive model wave equation. In [8], Adrian Constantin and Jonatan Lenells presented a simple algorithm for the inverse scattering approach to the Camassa-Holm equation.

In this chapter, the following form of Generalized Dullin-Gottwald-Holm equation (GDGH) i.e. DGH equation with variable coefficient is going to be studied:

$$u_t - \alpha^2 u_{xxt} + 2wu_x + f(u)u_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \tag{5.1.3}$$

where the first term is the evolution term and the fourth term is the nonlinear term, while the rest of the terms account for dispersion. This equation with $f(u) = 3u$ appeared in [114] in connection with study of periodic waves such as peakon-like periodic waves, compacton-like periodic waves and singular periodic waves, their dynamical behaviors and certain strange phenomena using semi-inverse method and integral bifurcation method. The 1-soliton solution of the GDGH were obtained in [4] when $f(u) = cu^m$.

The study in this chapter is structured as follows. In Sec. (5.2), we study the Lie symmetries of the GDGH equation, its Lie algebra, and also the corresponding optimal system. We also present the reduction obtained from the optimal system of subalgebras. These equations admit symmetries that lead to further reductions. In search of more symmetries, in Sec. (5.3), we apply the nonclassical method to Eq. (5.1.3).

5.2 Classical Symmetries

As we have already mentioned in Chapter 1 that there is no existing general theory for solving nonlinear PDEs. The classical method for finding symmetry reductions of PDEs is the Lie group method. The fundamental basis of this method is that when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. For PDEs with two independent variables, a single group reduction transforms the PDEs into ODEs, which are generally easier to solve. Motivated by the fact that symmetry reductions for many PDEs cannot be obtained by using classical symmetries, there have been several generalizations of the classical Lie group method for symmetry reductions, so we have also investigated its nonclassical symmetries (refer to section (1.3)). Here, we study Eq. (5.1.3), where u is a function of two real variables x and t and $f(u)$ is an arbitrary function of u, from the standpoint of the theory of symmetry reductions in PDEs. Using this theory, we find that for the GDGH equation, some similarity solutions are solutions with physical interest. A vector field

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}, \tag{5.2.1}$$

is a generator of point symmetry of equation (5.1.3) if

$$X^{[3]}(u_t - \alpha^2 u_{xxt} + 2wu_x + f(u)u_x + \gamma u_{xxx} - \alpha^2(2u_x u_{xx} + uu_{xxx}))\Big|_{(5.1.3)} = 0, \tag{5.2.2}$$

where the operator $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{xxt}\frac{\partial}{\partial u_{xxt}}, \tag{5.2.3}$$

where $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ andη^{xxt} are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables $u_t, u_x, u_{xx}, u_{xxx}$ and u_{xxt}. The set of determining equations for the group infinitesimals η, ξ and τ which we get from (5.2.2) after equating the coefficients of various derivative terms to zero, is as follows:

$$\begin{aligned}
&(i)\ \tau_x = 0 = \tau_u,\\
&(ii)\ \xi_u = 0,\\
&(iii)\ \eta_{uu} = 0\\
&(iv)\ \xi_{xx} - 2\eta_{xu} = 0,\\
&(v)\ 2\xi_x - \alpha^2\eta_{xxu} = 0,\\
&(vi)\ -2\alpha^2\eta_u - 2\alpha^4\eta_{xxu} + 6\alpha^2\xi_x - 2\alpha^2\tau_t = 0,\\
&(vii)\ -\alpha^2\eta + \gamma\tau_t + \alpha^2\gamma\eta_{xxu} - \alpha^2 u\tau_t - \alpha^4 u\eta_{xxu} - 3\gamma\xi_x + \alpha^2\xi_t + 3\alpha^2 u\xi_x = 0,\\
&(viii)\ -2\alpha^2\eta_x + 3\gamma\eta_{xu} - \alpha^2\eta_{tu} - 3\gamma\xi_{xx} + 2\alpha^2\xi_{xt} + 3\alpha^2 u\xi_{xx} - 3\alpha^2 u\eta_{xu} = 0,\\
&(ix)\ 3\gamma\eta_{xxu} - 2\alpha^2\eta_{xx} + \tau_t f(u) - f(u)\xi_x - 2\alpha^2\eta_{xtu} + \alpha^2\xi_{xxt} - 2w\xi_x + 2w\tau_t + \eta f'(u)\\
&-\gamma\xi_{xxx} + \alpha^2 u\xi_{xxx} + 2\alpha^2 w\eta_{xxu} - 3\alpha^2 u\eta_{xxu} - \xi_t + \alpha^2\eta_{xxu}f(u) = 0,\\
&(x)\ f(u)\eta_x - \alpha^2 u\eta_{xxx} + \eta_t - \alpha^2\eta_{xxt} + 2w\eta_x + \gamma\eta_{xxx} = 0.
\end{aligned} \tag{5.2.4}$$

The structure of the determining equations prompts the selection of the following forms of infinitesimals:

$$\eta = C_1 u + C_2, \qquad \tau = -C_1 t + C_3, \qquad \xi = (C_2 + \tfrac{\gamma}{\alpha^2} C_1)t + C_4, \tag{5.2.5}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants and f must satisfy the equation

$$f(u)\tau_t + 2w\tau_t + \eta f'(u) - \xi_t = 0. \tag{5.2.6}$$

The Lie algebra associated with equation (5.1.3) consists of following four vector fields:

$$V_1 = -t\tfrac{\partial}{\partial t} + \tfrac{\gamma}{\alpha^2} t\tfrac{\partial}{\partial x} + u\tfrac{\partial}{\partial u}, \qquad V_2 = \tfrac{\partial}{\partial u} + t\tfrac{\partial}{\partial x}, \qquad V_3 = \tfrac{\partial}{\partial t}, \qquad V_4 = \tfrac{\partial}{\partial x}. \tag{5.2.7}$$

5.2.1 Optimal System

In this section, we intend to deduce the one-dimensional optimal system (as discussed in section (1.2)) of eq. (5.1.3). By calculation, we show that the symmetries of eq. (5.1.3) form a four-dimensional Lie algebra generated by vector fields (5.2.7). To find an optimal system, we require nonzero Lie brackets and adjoint representations for the Lie algebra which are shown as below:

The nonzero Lie brackets or commutator relations of the Lie algebra for (5.1.3) are as follows:

$$\begin{array}{lll} [V_1, V_2] = -V_2, & [V_1, V_3] = V_3 - \frac{\gamma}{\alpha^2} V_4, & [V_2, V_1] = V_2, \\ [V_2, V_3] = -V_4, & [V_3, V_1] = \frac{\gamma}{\alpha^2} V_4 - V_3, & [V_3, V_2] = V_4, \end{array} \tag{5.2.8}$$

where the Lie brackets are obtained using the expression $[V_i, V_j] = V_iV_j - V_jV_i$ and the adjoint relations are given by using the Lie series

$$Ad(exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2}[V_i, [V_i, V_j]] - ..., \tag{5.2.9}$$

Here, we are showing the only cases where $Ad(exp(\epsilon V_i)V_j) \neq V_j$. Thus the adjoint relations for the Lie algebra for (5.1.3) are calculated as follows:

$$\begin{array}{ll} Ad(exp(\epsilon V_1)V_2) = V_2 e^{\epsilon}, & Ad(exp(\epsilon V_1)V_3) = \frac{\gamma}{\alpha^2} V_4 + e^{-\epsilon}(V_3 - \frac{\gamma}{\alpha^2} V_4) \\ Ad(exp(\epsilon V_2)V_1) = V_1 - \epsilon V_2, & Ad(exp(\epsilon V_2)V_3) = V_3 + \epsilon V_4, \\ Ad(exp(\epsilon V_3)V_1) = V_1 - \epsilon(\frac{\gamma}{\alpha^2} V_4 - V_3), & Ad(exp(\epsilon V_3)V_2) = V_2 - \epsilon V_4. \end{array} \tag{5.2.10}$$

A one-dimensional optimal system for (5.1.3) is given as follows:

$$V_1 + aV_4, \qquad V_2 + bV_3, \qquad V_3, \qquad V_4, \tag{5.2.11}$$

where a, b are arbitrary constants.

5.2.2 Symmetry Reductions and Exact Solutions

The similarity variables and forms can be obtained by solving following characterstic equations:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{5.2.12}$$

The general solution of these equations involves two constants; one becomes the new independent variable ζ and the other, say F, plays the role of new dependent variable. On substituting these solutions of (5.2.12) in equation (5.1.3), one gets the reduced ordinary differential equation.

1. **Vector field $V_1 + aV_4$:** For this vector field, on using the characterstic equations (5.2.12), the similarity variable, the form of similarity solution and coefficient function is as follows:

$$\zeta(x,t) = x + a\log t + \tfrac{\gamma}{\alpha^2}t, \qquad u(x,t) = \tfrac{1}{t}F(\zeta), \qquad f(u) = cu - (2w + \tfrac{\gamma}{\alpha^2}), \tag{5.2.13}$$

where c is any constant. On using these in equation (5.1.3), the reduced ODE is given by

$$\begin{aligned} &-F(\zeta) + aF'(\zeta) + \alpha^2 F''(\zeta) - a\alpha^2 F'''(\zeta) + cF(\zeta)F'(\zeta) \\ &-2\alpha^2 F'(\zeta)F''(\zeta) - \alpha^2 F(\zeta)F'''(\zeta) = 0, \end{aligned} \tag{5.2.14}$$

where prime ($'$) denotes the differentiation with respect to the variable ζ. We solved the reduced ODE (5.2.14) and obtained the following solutions:

$$\begin{aligned} &(i)\ \ u(x,t) = \frac{C_2\, \mathrm{e}^{C_1 + 1/3\,\sqrt{3}\sqrt{c}\left(x + \frac{\sqrt{3}\alpha\,\log(t)}{\sqrt{c}} + \frac{\gamma\, t}{\alpha^2}\right)\alpha^{-1}}}{t}, \\ &(ii)\ \ u(x,t) = \frac{C_2\left(\mathrm{e}^{C_1 + 1/6\,\sqrt{3}\sqrt{c}\left(x + \frac{\sqrt{3}\alpha\,\log(t)}{\sqrt{c}} + \frac{\gamma\, t}{\alpha^2}\right)\alpha^{-1}}\right)^2}{t}, \\ &(iii)\ \ u(x,t) = \frac{C_2\left(\mathrm{e}^{C_1 + 1/9\,\sqrt{3}\sqrt{c}\left(x + \frac{\sqrt{3}\alpha\,\log(t)}{\sqrt{c}} + \frac{\gamma\, t}{\alpha^2}\right)\alpha^{-1}}\right)^3}{t}, \end{aligned} \tag{5.2.15}$$

where C_1 and C_2 are arbitrary constants and $a = \frac{\sqrt{3}\alpha}{\sqrt{c}}$. The evolution of solution (5.2.15)(i) with $C_2 = 1, C_1 = 2, c = 1, \alpha = 2, \gamma = 3$ has been shown in Fig. 1 for $t = 1$

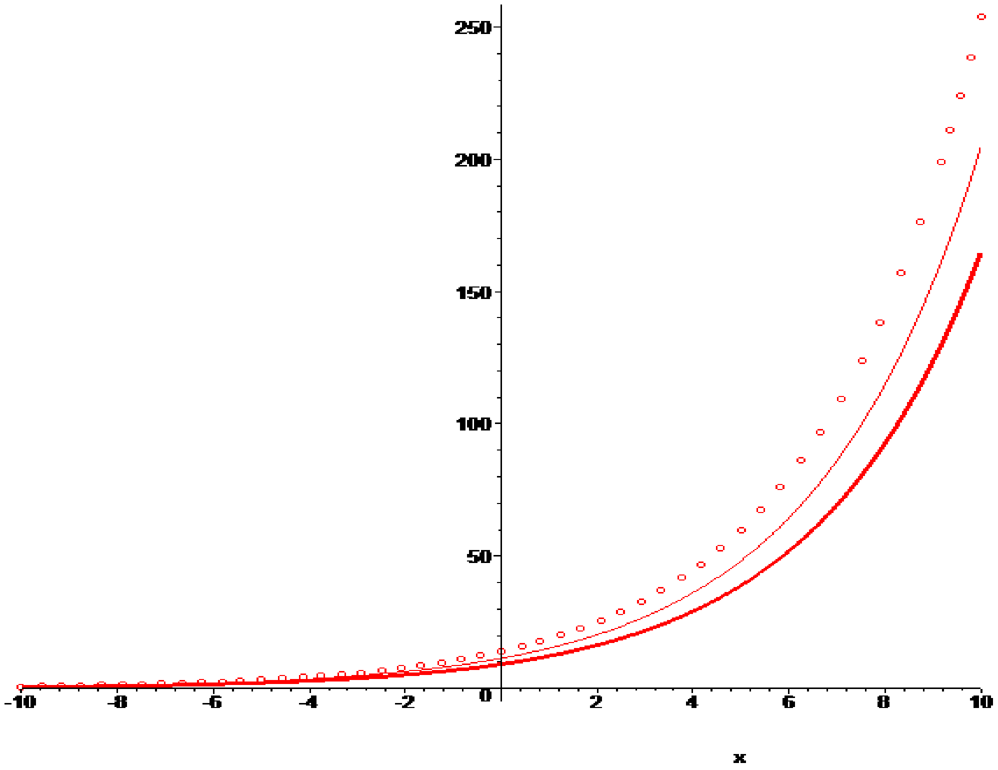

FIGURE 5.1: The solution (5.2.15)(i) for $t = 1$ (thick line), $t = 2$ (thin line), $t = 3$ (dotted line)

(thick line), $t = 2$ (thin line), $t = 3$ (dotted line). We have obtained some more solutions of (5.2.14) with $c = 3$ that have been mentioned below:

$$
\begin{aligned}
&(i)\ \ u(x,t) = \frac{C_2\,\cosh\left(C_1 - \left(x + a\log(t) + \frac{\gamma\,t}{\alpha^2}\right)\alpha^{-1}\right)}{t},\\
&(ii)\ \ u(x,t) = \frac{\left(-3/4\,C_2\,\cosh\left(C_1 - 1/3\left(x + a\log(t) + \frac{\gamma\,t}{\alpha^2}\right)\alpha^{-1}\right) + C_2\left(\cosh\left(C_1 - 1/3\left(x + a\log(t) + \frac{\gamma\,t}{\alpha^2}\right)\alpha^{-1}\right)\right)^3\right)}{t},\\
&(iii)\ \ u(x,t) = \frac{C_2\,\sinh\left(C_1 - \left(x + a\log(t) + \frac{\gamma\,t}{\alpha^2}\right)\alpha^{-1}\right)}{t},\\
&(iv)\ \ u(x,t) = \frac{\left(3/4\,C_2\,\sinh\left(C_1 - 1/3\left(x + a\log(t) + \frac{\gamma\,t}{\alpha^2}\right)\alpha^{-1}\right) + C_2\left(\sinh\left(C_1 - 1/3\left(x + a\log(t) + \frac{\gamma\,t}{\alpha^2}\right)\alpha^{-1}\right)\right)^3\right)}{t},
\end{aligned}
\tag{5.2.16}
$$

where C_1 and C_2 are arbitrary constants. By the analysis of solutions (5.2.16), we conclude that these solutions depend upon constants $C_1, C_2, a, \gamma, \alpha$. Depending upon these constants, we obtain certain kinky solutions as shown in Fig. 5.2 and Fig. 5.3 for solutions (5.2.16)(i) and (5.2.16)(iv) with $C_1 = 1, C_2 = 1, a = 2, \gamma = 3, \alpha = 4$.

2. **Vector field $V_2 + bV_3$:** From vector field $V_2 + bV_3$, we obtain the transformation and coefficient function as:

$$\zeta(x,t) = x - \frac{t^2}{2b}, \qquad u(x,t) = \frac{t}{b} + F(\zeta), \qquad f(u) = u + c, \tag{5.2.17}$$

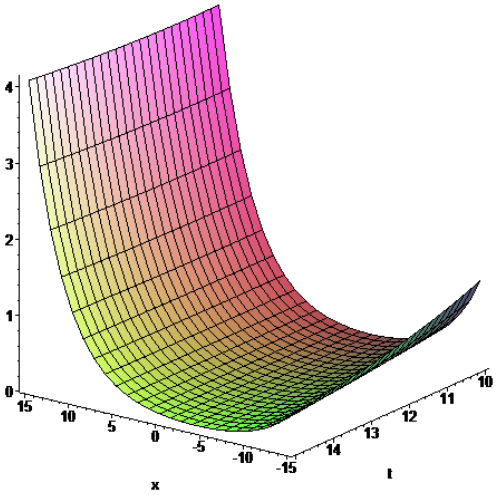

FIGURE 5.2: The figure of kink solution (5.2.16)(i)

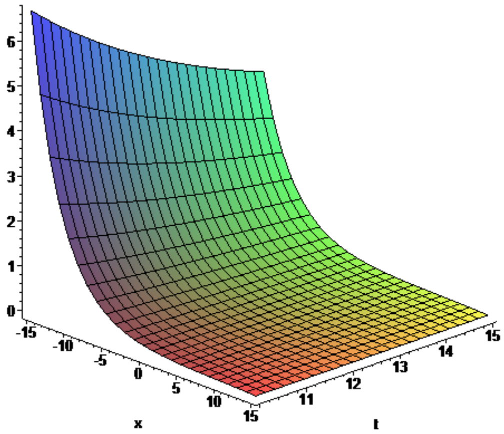

FIGURE 5.3: The figure of kink solution (5.2.16)(iv)

where c is arbitrary constant and $F(\zeta)$ satisfies the following ODE:

$$\tfrac{1}{b} + 2wF'(\zeta) + F(\zeta)F'(\zeta) + cF'(\zeta) + \gamma F'''(\zeta) - 2\alpha^2 F'(\zeta)F''(\zeta) - \alpha^2 F(\zeta)F'''(\zeta) = 0. \tag{5.2.18}$$

The above ODE can be integrated to reduce the order but due to complexity, more solutions cannot be obtained.

3. **Vector field** V_3**:** In this case, the form of transformation is as follows:

$$\zeta(x,t) = x, \qquad u(x,t) = F(\zeta), \qquad f(u) = f(u), \tag{5.2.19}$$

where $F(\zeta)$ satisfies the following ODE:

$$2wF'(x) + f(u)F'(x) + \gamma F'''(x) - 2\alpha^2 F'(x)F''(x) - \alpha^2 F(x)F'''(x) = 0. \tag{5.2.20}$$

To solve the above ODE, we have considered following different cases for values of $f(u)$:

Case 1: If $f(u) = u$, then the solution of Eq. (5.2.20) corresponds to the following solution of GDGH equation:

$$u(x,t) = \frac{\left(\gamma + 6\,w\alpha^2\right)}{\alpha^2\left(-1+\left(\tanh\left(1/6\,\frac{-6\,C_1\,\alpha+\sqrt{3}x}{\alpha}\right)\right)^2\right)}, \tag{5.2.21}$$

where C_1 is arbitrary constant.

Case 2: If $f(u) = u^2$, then we get the following solution of Eq. (5.1.3):

$$u(x,t) = -1/3\,\tfrac{\gamma}{\alpha^2} + 24\,\alpha^2 {C_4}^2 \wp\left(C_3 + C_4\,x, -\tfrac{1}{216}\,\tfrac{18\,\alpha^4 w + \gamma^2}{{C_4}^4 \alpha^8}, C_1\right), \tag{5.2.22}$$

where C_1, C_3, C_4 are arbitrary constants and $\wp$ denotes the WeirstrassP function.

4. **Vector field V_4:** From vector field V_4, we obtain the following transformation:

$$\zeta = t, \qquad u(x,t) = F(\zeta), \qquad f(u) = f(u), \tag{5.2.23}$$

where $F(\zeta)$ satisfies the following ODE:

$$F'(\zeta) = 0, \tag{5.2.24}$$

which corresponds to the "constant" solution of Eq. (5.1.3).

5.3 Nonclassical Method

Bluman and Cole [51] made an important breakthrough in the theory of symmetry reductions and it is their technique that we shall be using here. In their study of symmetry reductions of the linear heat equation, they proposed the so-called nonclassical method of group-invariant solutions, which is a generalization of the classical Lie method. The basic idea of the nonclassical method [97] is that PDE (5.1.3) is augmented with the invariant surface condition

$$\xi u_x + \tau u_t - \eta = 0, \tag{5.3.1}$$

which is associated with the vector field

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{5.3.2}$$

Requiring that both (5.1.3) and (5.3.1) be invariant under the transformation with infinitesimal generator (5.3.2), we can obtain an overdetermined system of nonlinear equations for the infinitesimals $\xi(x,t,u), \tau(x,t,u)$ and $\eta(x,t,u)$. The number of determining equations arising is smaller in the nonclassical method than in the classical method because there are fewer linearly independent expressions in the derivatives. In addition to the nonclassical method, the determining equations are usually highly nonlinear unlike the determining equations for the classical method which are linear. Because all solutions of the classical determining equations necessarily satisfy the nonclassical determining equations, the solution set can be larger in the nonclassical case. We can distinguish two different cases.

Case 1: In the case $\tau \neq 0$, we can set $\tau = 1$ without loss of generality. The nonclassical method applied to (5.1.3) yields following eight determining equations for the infinitesimals:

$$
\begin{aligned}
&(i)\ \ \xi_u = 0,\\
&(ii)\ \ \eta_{uu} = 0,\\
&(iii)\ \ -\alpha^2\eta + \alpha^4\eta_{xxu}\xi + 3\alpha^2 u\xi_x - 3\gamma\xi_x + \alpha^2\xi_t - 2\alpha^2\xi_x\xi + \alpha^2\gamma\eta_{xxu} - \alpha^4 u\eta_{xxu} = 0,\\
&(iv)\ \ -2\alpha^2\eta_u - 2\alpha^4\eta_{xxu} + 6\alpha^2\xi_x = 0,\\
&(v)\ \ 3\alpha^2 u\xi_{xx} - 3\gamma\xi_{xx} - 4\alpha^2{\xi_x}^2 - \alpha^4\eta_{xxu}\eta_u - \alpha^2\eta_{tu} - 3\alpha^2 u\eta_{xu} + 2\alpha^4\eta_{xxu}\xi_x + 2\alpha^2\xi_x\eta_u\\
&+2\alpha^2\xi_{xt} - 2\alpha^2\eta_x + 3\gamma\eta_{xu} - \alpha^2\xi\xi_{xx} + 2\alpha^2\xi\eta_{xu} = 0,\\
&(vi)\ \ -4\alpha^2\eta_{xu} + 2\alpha^2\xi_{xx} = 0,\\
&(vii)\ \ \alpha^2\eta_{xxu}f(u) - \gamma\xi_{xxx} + \eta f'(u) - 2w\xi_x + 3\gamma\eta_{xxu} - 3\alpha^2\xi_x\xi_{xx} + \alpha^2\xi_{xxt} + \alpha^2 u\xi_{xxx}\\
&-2\alpha^2\eta_{xx} - \xi_t + 6\alpha^2\xi_x\eta_{xu} - 2\alpha^2\eta_{xtu} - f(u)\xi_x + \alpha^4\eta_{xxu}\xi_{xx} + \alpha^2\xi_{xx}\eta_u + 2\alpha^2 w\eta_{xxu}\\
&-3\alpha^2 u\eta_{xxu} - 2\alpha^4\eta_{xu}\eta_{xxu} - 2\alpha^2\eta_u\eta_{xu} = 0,\\
&(viii)\ \ -\alpha^4\eta_{xxu}\eta_{xx} + 2\alpha^2\xi_x\eta_{xx} + \eta_t - \alpha^2 u\eta_{xxx} + f(u)\eta_x - \alpha^2\eta_{xxt} - 2\alpha^2\eta_x\eta_{xu}\\
&+2w\eta_x + \gamma\eta_{xxx} + \alpha^2\eta_x\xi_{xx} = 0.
\end{aligned}
\tag{5.3.3}
$$

Solving above system of equations, we obtain

$$
\xi = C_1 t + C_2 \qquad \eta = C_1, \qquad f(u) = u + c, \tag{5.3.4}
$$

where c, C_1, C_2 are arbitrary constants. We can assert that for $\tau \neq 0$, we recover only the classical symmetries and that the nonclassical symmetries of the GDGH equation have been completely classified. We can state that Eq. (5.1.3) does not admit proper nonclassical symmetries with $\tau = 1$ for any function $f(u)$.

Case 2: In this case, we set $\tau = 0, \xi = 1$ and so the invariant surface condition reduces to $u_x = \eta(x,t,u)$. Hence we obtain the differential consequences as follows:

$$
\begin{aligned}
&u_{xx} = \eta_x + \eta_u u_x, \qquad u_{xxx} = \eta_{xx} + 2\eta\eta_{xu} + \eta^2\eta_{uu} + \eta_u u_{xx},\\
&u_{xxt} = \eta_{xt} + \eta_{xu}u_t + \eta\eta_{tu} + \eta\eta_{uu}u_t + \eta_u(\eta_t + \eta_u u_t).
\end{aligned}
\tag{5.3.5}
$$

Applying nonclassical method to Eq. (5.1.3) and equating coefficients of powers of u_t to zero then generates the following determining equations:

$$
\begin{aligned}
&(i)\ \ -\alpha^4\eta_{uu}\eta_{xu}\eta_x - \eta\alpha^4\eta_{uu}^2\eta_x - \eta\alpha^4\eta_{xxu}\eta_{uu} - 2\,\eta\alpha^4\eta_{xuu}\eta_{xu} - \alpha^4\eta_{uu}\eta_u^2\eta_x - \eta\alpha^4\eta_{uu}\eta_{xu}\eta_u\\
&-\alpha^4\eta_{xxu}\eta_{xu} - \eta^3\alpha^4\eta_{uuu}\eta_{uu} - 2\eta\alpha^2\eta_{uu}\eta_u - \eta\alpha^4\eta_{uu}\eta_u^3 - \eta^2\alpha^4\eta_{uuu}\eta_u^2 - 2\alpha^2\eta_{xu}\eta_u\\
&-\eta^2\alpha^4\eta_{uuu}\eta_{xu} - 2\eta^2\alpha^4\eta_{xuu}\eta_{uu} - \eta^2\alpha^4\eta_{uu}^2\eta_u - \alpha^4\eta_{xxu}\eta_u^2 - 2\eta\alpha^4\eta_{xuu}\eta_u^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
(ii)\quad & -\alpha^2 u\eta_{xxx} - 3\,\eta^2\alpha^2\eta_u^2 + \eta^2\tfrac{d}{du}f\left(u\right) - 2\,\eta\alpha^2\eta_{xtu} + \eta^3\gamma\,\eta_{uuu} - 6\,\eta^2\alpha^2\eta_{xu} - 3\,\alpha^2\eta\eta_{xx} \\
& -\alpha^2\eta_{ut}\eta_x + 3\eta\gamma\eta_{xxu} - 2\,\alpha^2\eta_{xu}\eta_t + \eta_t + 3\gamma\eta_{xu}\eta_x + 3\eta^2\gamma\eta_{xuu} - \alpha^4\eta_{xxu}\eta_{xt} - 3\eta^3\alpha^2\eta_{uu} \\
& +2\,w\eta_x - 2\,\alpha^2\eta_x^2 + \gamma\,\eta_{xxx} + f(u)\eta_x - \alpha^2\eta_{xxt} - \eta^3\alpha^4\eta_{uuu}\eta_{ut} - \eta^3\alpha^2 u\eta_{uuu} + \eta^3\alpha^2\eta_{uuu}f(u) \\
& -\eta^2\alpha^4\eta_{uuu}\eta_{xt} - \eta\alpha^4\eta_{xxu}\eta_{ut} + \eta\alpha^2\eta_{xxu}f(u) - \eta\alpha^2\eta_{ut}\eta_u - \alpha^4\eta_{uu}\eta_{xt}\eta_x - \alpha^4\eta_{xxu}u\eta_{xx} \\
& +\alpha^2\eta_{xxu}\gamma\,\eta_{xx} - \alpha^4\eta_{xxu}\eta_u\eta_t - 4\eta^3\alpha^4\eta_{xuu}\eta_u - 2\eta^3\alpha^4\eta_{uuu}\eta_x - 2\eta^3\alpha^4\eta_{uu}\eta_u^2 - 3\alpha^2 u\eta_{xu}\eta_x \\
& -5\,\alpha^2\eta\eta_u\eta_x + 3\eta\gamma\eta_{uu}\eta_x - 3\eta\alpha^2 u\eta_{xxu} - 2\eta\alpha^4\eta_{xxu}\eta_x + 3\eta\gamma\eta_{xu}\eta_u - 2\eta\alpha^4\eta_{uu}\eta_x^2 - 2\eta\alpha^4\eta_{xuu}\eta_{xt} \\
& -2\,\eta\alpha^2\eta_{uu}\eta_t + 2\,\eta\alpha^2\eta_{xxu}w - 3\eta^2\alpha^2 u\eta_{xuu} + 2\eta^2\alpha^2\eta_{xuu}f(u) - 4\eta^2\alpha^4\eta_{xuu}\eta_x - 2\eta^2\alpha^4\eta_{xuu}\eta_{ut} \\
& -2\eta^2\alpha^4\eta_{xxu}\eta_u + 3\eta^2\gamma\eta_{uu}\eta_u + 2\eta^3\alpha^2\eta_{uuu}w - 2\eta^4\alpha^4\eta_{uuu}\eta_u + 4\eta^2\alpha^2\eta_{xuu}w - \eta\alpha^4\eta_{uu}\eta_u^2\eta_t \\
& -\eta^3\alpha^4\eta_{uuu}u\eta_u^2 + \eta^2\alpha^2\eta_{uuu}\gamma\eta_{xx} - \eta^4\alpha^4\eta_{uuu}u\eta_{uu} - \eta^3\alpha^4\eta_{uu}^2 u\eta_u + \eta^4\alpha^2\eta_{uuu}\gamma\eta_{uu} - \eta^2\alpha^2\eta_{uut} \\
& +\eta^3\alpha^2\eta_{uu}^2\gamma\eta_u + \eta^3\alpha^2\eta_{uuu}\gamma\,\eta_u^2 - \eta\alpha^4\eta_{uu}\eta_{ut}\eta_x - \eta\alpha^4\eta_{uu}\eta_{xt}\eta_u - \eta\alpha^4\eta_{xxu}u\eta_u^2 - \eta^2\alpha^4\eta_{uuu}\eta_u\eta_t \\
& +\eta^2\alpha^2\eta_{xxu}\gamma\eta_{uu} - \eta^2\alpha^4\eta_{uuu}u\eta_{xx} + \eta^2\alpha^2\eta_{uu}f(u)\eta_u + \eta^2\alpha^2\eta_{uu}^2\gamma\eta_x + \eta^2\alpha^2\eta_{uu}\gamma\eta_u^3 + \eta\alpha^2\eta_{uu}f(u)\eta_x \\
& +\eta\alpha^2\eta_{xxu}\gamma\,\eta_u^2 + \eta\alpha^2\eta_{uu}\gamma\,\eta_{xx}\eta_u - \eta\alpha^4\eta_{uu}u\eta_{xx}\eta_u + \eta^2\alpha^2\eta_{uuu}\gamma\,\eta_u\eta_x - \eta^2\alpha^4\eta_{uuu}u\eta_u\eta_x \\
& -\eta^2\alpha^4\eta_{uu}u\eta_u^3 + 2\,\eta^3\alpha^2\eta_{xuu}\gamma\,\eta_{uu} + 2\,\eta^3\alpha^2\eta_{uuu}\gamma\,\eta_{xu} - 2\eta^3\alpha^4\eta_{xuu}u\eta_{uu} - 2\eta^3\alpha^4\eta_{uuu}u\eta_{xu} \\
& -3\,\eta\alpha^2 u\eta_{uu}\eta_x + 2\,\eta\alpha^2\eta_{uu}w\eta_x - 2\,\eta\alpha^4\eta_{xuu}u\eta_{xx} + 2\,\eta\alpha^2\eta_{uu}\gamma\,\eta_{xu}\eta_x - 3\,\eta\alpha^2 u\eta_{xu}\eta_u \\
& +2\,\eta^2\alpha^2\eta_{uu}\gamma\,\eta_{xu}\eta_u + 2\,\eta\alpha^2\eta_{xuu}\gamma\,\eta_{xx} + 4\,\eta^2\alpha^2\eta_{xuu}\gamma\,\eta_{xu} - 4\,\eta^2\alpha^4\eta_{xuu}u\eta_{xu} + 2\,\eta\alpha^2\eta_{uu}\gamma\eta_u^2\eta_x \\
& +2\,\eta\alpha^2\eta_{xuu}\gamma\,\eta_u\eta_x - 2\,\eta\alpha^4\eta_{xuu}u\eta_u\eta_x - 2\,\eta\alpha^4\eta_{uu}u\eta_{xu}\eta_x - 2\,\eta\alpha^4\eta_{xuu}\eta_u\eta_t - 2\,\eta\alpha^4\eta_{uu}u\eta_u^2\eta_x \\
& +2\,\eta^2\alpha^2\eta_{xuu}\gamma\,\eta_u^2 - 2\,\eta^2\alpha^4\eta_{xuu}u\eta_u^2 - 4\,\eta^2\alpha^4\eta_{uu}\eta_x\eta_u + 2\,\eta\alpha^2\eta_{xxu}\gamma\,\eta_{xu} - 2\,\eta\alpha^4\eta_{xxu}u\eta_{xu} \\
& +2\,\eta^2\alpha^2\eta_{uu}w\eta_u - 2\,\eta^2\alpha^4\eta_{uu}u\eta_{xu}\eta_u - \alpha^4\eta_{uu}\eta_u\eta_t\eta_x + \alpha^2\eta_{uu}\gamma\,\eta_u\eta_x^2 - 3\,\eta^2\alpha^2 u\eta_{uu}\eta_u - \eta^2\alpha^4\eta_{xxu}u\eta_{uu} \\
& -\eta^2\alpha^4\eta_{uu}\eta_{ut}\eta_u - \eta^2\alpha^4\eta_{uu}^2 u\eta_x + \alpha^2\eta_{uu}\gamma\,\eta_{xx}\eta_x - \alpha^4\eta_{uu}u\eta_u\eta_x^2 + \alpha^2\eta_{xxu}\gamma\,\eta_u\eta_x - \alpha^4\eta_{xxu}u\eta_u\eta_x \\
& -\alpha^4\eta_{uu}u\eta_{xx}\eta_x = 0.
\end{aligned}
\tag{5.3.6}
$$

The complexity of above system of equations is the reason why we cannot solve them in general. We hence proceed by making several ansätz on the form of $\eta(x,t,u)$:

1. For an arbitrary $f(u)$, we obtain an arbitrary $\eta(x,t,u) = \eta(u)$.
2. For $f(u) = 3\alpha^2{k_1}^2u + C_1$, then we get $\eta(u) = k_1u + k_2$. On using the characterstic equations (5.2.12), the similarity variable and the form of similarity solution is as follows:

$$
\zeta(x,t) = t, \qquad u(x,t) = -\frac{k_2}{k_1} + \mathrm{e}^{k_1\,x}F\left(\zeta\right). \tag{5.3.7}
$$

On using these in (5.1.3), we get the reduced ODE as

$$
-F'(\zeta) + \alpha^2{k_1}^2F'(\zeta) - 2wk_1F(\zeta) + 2\alpha^2{k_1}^2k_2F(\zeta) - k_1C_1F(\zeta) - \gamma{k_1}^3F(\zeta) = 0. \tag{5.3.8}
$$

On solving the above ODE, we get the following solution of Eq. (5.1.3):

$$
u(x,t) = -\tfrac{k_2}{k_1} + \mathrm{e}^{k_1\,x}C_2\,\mathrm{e}^{\frac{k_1\,tC_1}{(\alpha\,k_1+1)(\alpha\,k_1-1)}}\,\mathrm{e}^{\frac{{k_1}^3t\gamma}{(\alpha\,k_1+1)(\alpha\,k_1-1)}}\left(\mathrm{e}^{\frac{k_1\,tw}{(\alpha\,k_1+1)(\alpha\,k_1-1)}}\right)^2\left(\mathrm{e}^{\frac{{k_1}^2t\alpha^2k_2}{(\alpha\,k_1+1)(\alpha\,k_1-1)}}\right)^{-2}. \tag{5.3.9}
$$

Remark 5.3.1. In the case when $f(u) = cu^m$ that have discussed in [4], using conditions (5.2.4) and (5.2.6) we get trivial symmetries so we will get only traveling wave solutions of the equation (5.1.3). Therefore, for $f(u) = cu^m$, we conclude that the non-constant similarity reduction of the GDGH equation obtainable using classical Lie method is the traveling wave solution.

5.4 Summary

In this chapter, we have obtained the complete Lie group classification for GDGH equation (5.1.3). We have constructed the optimal system that lead to reductions of equation (5.1.3) to ODEs and for each element in the optimal system some solutions are also obtained. We have also applied the nonclassical method to investigate some more smmetries.

Chapter 6

The Generalized Bretherton Equation with Variable Coefficients[1]

6.1 Introduction

It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations, which are involved in many fields from physics to biology, chemistry, mechanics, plasma physics, optical fibers, chemical kinematics, chemical physics and geochemistry, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help one to understand these phenomena better. In this chapter, we will study perturbed nonlinear Klein-Gordon equation with time-dependent coefficients which is the so-called generalized Bretherton equation with variable coefficients

$$u_{tt} + \alpha(t)u_{xx} + \beta(t)u_{xxxx} + \delta(t)u^m + \theta(t)u^n = 0, \tag{6.1.1}$$

where $u = u(x,t)$ is unknown function to be determined, $\alpha(t), \beta(t), \delta(t), \theta(t)$ are arbitrary time functions and the exponent m is a natural number and for the exponent $n \in \mathbf{N}$ we assume that $n \neq 1$ and $n \neq m$ [10, 45, 124]. This equation is a generalization of the Bretherton equation

$$u_{tt} + u_{xx} + u_{xxxx} + u - u^2 = 0, \tag{6.1.2}$$

which was first introduced by Bretherton [46] as a model of a dispersive wave system to investigate the resonant nonlinear interaction between three linear modes. When

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[1] A part of this chapter has been published in **Nonlinear dynamics, 71 (2013) 1-12**.

$\alpha(t) = 0$ and $\beta(t), \delta(t), \theta(t)$ are constants, Eq. (6.1.1) can be considered as a formal fourth-order extension of the classical Klein-Gordon equation, but it also inherits a Schrödinger structure; however, it can be noted that the equation satisfies neither finite speed propagation nor mass conservation. Eq. (6.1.1) is a generalized form of the equation that arises in particle field theory and this equation contains some particular important equations such as Duffing equation [92], Landau-Ginburg-Higgs equation [120], sin-Gordon equation [132] and ϕ^4 equation [12]. The constant coefficients version of Eq. (6.1.1) (with $m = 1$) was studied by Levandosky [136, 137] who investigated the stability and instability of the solitary waves of this equation. Levandosky and Strauss [138] also established time decay of the solutions of this equation. Recently, Romeiras [45] obtained periodic and solitary traveling wave solutions by a truncated Painlevé analysis. More precisely, for $m = 1$ and $n = 2, 3, 5$, he used a truncated Painlevé expansion method and obtained some kink solutions and (periodic) singular solutions. He also applied the unified algebraic method and obtained various periodic solutions in terms of elliptic functions.

Our interest in the present chapter is to search for the exact solutions for Eq. (6.1.1). In this direction, the layout of our chapter is as follows. Section 6.2 is devoted to the outline of Lie classical method to generate various symmetries of the Eq. (6.1.1), the admissible forms of the coefficients that admit the classical symmetry group and reduced ODEs corresponding to each member of the optimal system of subalgebras. Section 6.3 contains the solutions of variable cofficient version of Generalized Bretherton equation corresponding to the solutions of reduced ODEs. Some discussion is given in section 6.4.

6.2 Classical Lie Symmetry Analysis

In this section, we first determine the Lie point symmetries of (6.1.1) and then use them to reduce it to lower dimension equations.

6.2.1 Lie Point Symmetries and Optimal System

Let us consider the equation (6.1.1) is invariant under following transformation:

$$\begin{bmatrix} u \\ x \\ t \end{bmatrix} \longrightarrow \begin{bmatrix} u \\ x \\ t \end{bmatrix} + \epsilon \begin{bmatrix} \eta(x,t,u) \\ \xi(x,t,u) \\ \tau(x,t,u) \end{bmatrix}, \tag{6.2.1}$$

where ϵ is an infinitesimal parameter.

On invoking the invariance criterion, the following relation from the coefficients of the first order of ϵ is deduced:

$$\begin{aligned}&\eta^{tt}+\alpha(t)\eta^{xx}+\tau\alpha'(t)u_{xx}+\beta(t)\eta^{xxxx}+\tau\beta'(t)u_{xxxx}+m\delta(t)u^{m-1}\eta+\tau\delta'(t)u^{m}\\&+n\theta(t)u^{n-1}\eta+\tau\theta'(t)u^{n}=0,\end{aligned}\tag{6.2.2}$$

where prime ($'$) denotes the derivatives with respect to t and η^{tt},η^{xx} and η^{xxxx} are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables u_{tt},u_{xx} and u_{xxxx}. The infinitesimals are determined from invariance condition (6.2.2), by setting the coefficients of different differentials equal to zero. We obtain a large number of following PDEs in η,ξ and τ that need to be satisfied:

$$\begin{aligned}&(i)\ \tau_x=0,\ \ \tau_u=0,\\&(ii)\ \xi_t=0,\ \ \xi_u=0,\\&(iii)\ \eta_{uu}=0,\\&(iv)\ 2\tau_t\beta(t)-4\beta(t)\xi_x+\tau\beta'(t)=0,\\&(v)\ 4\beta(t)\eta_{xu}-6\beta(t)\xi_{xx}=0,\\&(vi)\ \tau\alpha'(t)-4\beta(t)\xi_{xxx}-2\alpha(t)\xi_x+2\tau_t\alpha(t)+6\beta(t)\eta_{xxu}=0,\\&(vii)\ 2\eta_{tu}-\tau_{tt}=0,\\&(viii)\ 2\alpha(t)\eta_{xu}-\alpha(t)\xi_{xx}+4\beta(t)\eta_{xxxu}-\beta(t)\xi_{xxxx}=0,\\&(ix)\ u^n\tau\theta'(t)+\eta_{tt}+2\tau_t\theta(t)u^n+\alpha(t)\eta_{xx}-\eta_u\delta(t)u^m+\beta(t)\eta_{xxxx}+u^m\tau\delta'(t)\\&+2\tau_t\delta(t)u^m+m\delta(t)\eta u^{m-1}-\eta_u\theta(t)u^n+n\theta(t)u^{n-1}\eta=0.\end{aligned}\tag{6.2.3}$$

The general solution of this large system provides following forms for the infinitesimal elements η,ξ and τ:

$$\xi=a_1x+a_5,\ \tau=a_2t^2+a_3t+a_6,\ \eta=(a_2t+a_4)u,\tag{6.2.4}$$

where a_1,a_2,a_3,a_4,a_5 and a_6 are arbitrary constants. The functions $\alpha(t),\beta(t),\delta(t)$ and $\theta(t)$ are governed by the following conditions:

$$\begin{aligned}&2\alpha(t)\tau_t+\tau\alpha'(t)-2a_1\alpha(t)=0,\\&2\beta(t)\tau_t+\tau\beta'(t)-4a_1\beta(t)=0,\\&2\delta(t)\tau_t+\tau\delta'(t)+(m-1)(a_3t+a_4)\delta(t)=0,\\&2\theta(t)\tau_t+\tau\theta'(t)+(n-1)(a_3t+a_4)\theta(t)=0.\end{aligned}\tag{6.2.5}$$

The infinitesimal generators of the corresponding Lie algebra are given by

$$V_1=x\tfrac{\partial}{\partial x},\ V_2=t^2\tfrac{\partial}{\partial t}+ut\tfrac{\partial}{\partial u},\ V_3=t\tfrac{\partial}{\partial t},\ V_4=u\tfrac{\partial}{\partial u},\ V_5=\tfrac{\partial}{\partial x},\ V_6=\tfrac{\partial}{\partial t}.\tag{6.2.6}$$

In general, there are infinite number of subalgebras of this Lie algebra formed from any linear combination of generators V_j; $j=1,2,3,4,5,6$ and to any sublagebra one can get the reduction using characterstic equations:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{6.2.7}$$

To find the optimal system, we have to firstly compute the commutator and adjoint relations. Using (1.2.16) and (1.2.21), we can find commutator and adjoint relations for (6.2.6) which are interpreted in following tables as under:

Table 6.1: Commutator Table

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	0	$-V_5$	0
V_2	0	0	$-V_2$	0	0	$-2V_3 - V_4$
V_3	0	V_2	0	0	0	$-V_6$
V_4	0	0	0	0	0	0
V_5	V_5	0	0	0	0	0
V_6	0	$2V_3 + V_4$	V_6	0	0	0

Table 6.2: Adjoint Table

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	V_1	V_2	V_3	V_4	$V_5 e^{\epsilon}$	V_6
V_2	V_1	V_2	$V_3 + \epsilon V_2$	V_4	V_5	$V_6 + \epsilon(2V_3 + V_4) + \epsilon^2 V_2$
V_3	V_1	$V_2 e^{-\epsilon}$	V_3	V_4	V_5	$V_6 e^{\epsilon}$
V_4	V_1	V_2	V_3	V_4	V_5	V_6
V_5	$V_1 - \epsilon V_5$	V_2	V_3	V_4	V_5	V_6
V_6	V_1	$V_2 - \epsilon(2V_3 + V_4) + \epsilon^2 V_6$	$V_3 - \epsilon V_6$	V_4	V_5	V_6

The optimal system for (6.2.6) consists of following vector fields:

$$\begin{aligned}
&(i(a))\ V_1 + aV_2 + bV_4 + V_6,\ \ (i(b))\ V_1 + aV_2 + bV_4 - V_6,\ \ (ii)\ V_1 + cV_2 + dV_4,\\
&(iii(a))\ V_2 + hV_4 + jV_5 + V_6,\ \ (iii(b))\ V_2 + hV_4 + jV_5 - V_6,\ \ (iv)\ V_2 + kV_4 + lV_5,\\
&(v(a))\ V_3 + qV_4 + V_5,\ \ (v(b))\ V_3 + qV_4 - V_5,\ \ (vi)\ V_3 + rV_4,\\
&(vii(a))\ V_4 + sV_5 + V_6,\ \ (vii(b))\ V_4 + sV_5 - V_6,\ \ (viii)\ V_4 + wV_5,\\
&(ix(a))\ V_5 + V_6,\ \ (ix(b))\ V_5 - V_6\ \ (x)\ V_5\ \ (xi) V_6,
\end{aligned} \tag{6.2.8}$$

where $a, b, c, d, h, j, k, l, q, r, s$ and w are arbitrary constants.

6.2.2 Symmetry Reductions

In this subsection, we will make use of the optimal system of vector fields (6.2.8) and reduce the Eq. (6.1.1) to ODEs. The similarity variables and the similarity solutions of the Eq. (6.1.1) can be obtained by solving characterstic equation (6.2.7) and the coefficient functions are given by equation (6.2.5). The general solution of these equations involves two constants, one become independent variable ζ and other plays the role of new dependent variable $F(\zeta)$. Because, the discrete symmetry $(t, x, u) \to (t, -x, u)$ will map (v(a)) to (v(b)) and $(t, x, u) \to (-t, x, u)$ will map (vii(a)), (ix(a)) to (vii(b)), (ix(b)) respectively, and therefore, in the optimal system, only thirteen generators are considered, while neglecting the other three.

Type (i(a)) $V_1 + aV_2 + bV_4 + V_6$

For this vector field, on solving the equations (6.2.5) and (6.2.7) we obtain

$$\begin{aligned}
&u(x,t) = \sqrt{at^2+1}e^{\frac{b}{\sqrt{a}}\tan^{-1}(\sqrt{a}t)}F(\zeta),\ \zeta = xe^{\frac{-1}{\sqrt{a}}\tan^{-1}(\sqrt{a}t)},\\
&\alpha(t) = \tfrac{K_1}{(at^2+1)^2}e^{\frac{2}{\sqrt{a}}\tan^{-1}(\sqrt{a}t)},\ \beta(t) = \tfrac{K_2}{(at^2+1)^2}e^{\frac{4}{\sqrt{a}}\tan^{-1}(\sqrt{a}t)},\\
&\delta(t) = K_3(at^2+1)^{\frac{-m-3}{2}}e^{\frac{-(m-1)b}{\sqrt{a}}\tan^{-1}(\sqrt{a}t)},\ \theta(t) = K_4(at^2+1)^{\frac{-n-3}{2}}e^{\frac{-(n-1)b}{\sqrt{a}}\tan^{-1}(\sqrt{a}t)},
\end{aligned} \tag{6.2.9}$$

where K_1, K_2, K_3, K_4 are arbitrary constants. Substituting (6.2.9) into Eq. (6.1.1), we have the function $F(\zeta)$ which must satisfy the following ODE:

$$\begin{aligned}
&(b^2+a)F(\zeta) + (1-2b)\zeta F'(\zeta) + \zeta^2 F''(\zeta) + K_1F''(\zeta) + K_2F''''(\zeta)\\
&+K_3(F(\zeta))^m + K_4(F(\zeta))^n = 0.
\end{aligned} \tag{6.2.10}$$

Type (i(b)) $V_1 + aV_2 + bV_4 - V_6$

Following the same way as above, we get

$$\begin{aligned}
&\sigma(x,y,t) = \sqrt{at^2-1}\left(\tfrac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{\frac{b}{2\sqrt{a}}}F(\zeta),\ \zeta = x\left(\tfrac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{\frac{-1}{2\sqrt{a}}},\\
&\alpha(t) = \tfrac{K_1}{(at^2-1)^2}\left(\tfrac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{\frac{1}{\sqrt{a}}},\ \beta(t) = \tfrac{K_2}{(at^2-1)^2}\left(\tfrac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{\frac{2}{\sqrt{a}}},\\
&\delta(t) = K_3(at^2-1)^{\frac{-m-3}{2}}\left(\tfrac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{\frac{-(m-1)b}{2\sqrt{a}}},\ \theta(t) = K_4(at^2-1)^{\frac{-n-3}{2}}\left(\tfrac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{\frac{-(n-1)b}{2\sqrt{a}}}.
\end{aligned} \tag{6.2.11}$$

Substituting (6.2.11) into Eq. (6.1.1) yields a reduced ODE as follows:

$$(b^2-a)F(\zeta)+(1-2b)\zeta F'(\zeta)+\zeta^2F''(\zeta)+K_1F''(\zeta)+K_2F''''(\zeta)+K_3(F(\zeta))^m+K_4(F(\zeta))^n = 0. \tag{6.2.12}$$

Type (ii) $V_1 + cV_2 + dV_4$

Solving the equations (6.2.5) and (6.2.7), we get

$$\begin{aligned}
&u(x,t) = te^{\frac{-d}{ct}}F(\zeta),\ \zeta = xe^{\frac{1}{ct}},\ \alpha(t) = \tfrac{K_1}{t^4}e^{\frac{-2}{ct}},\\
&\beta(t) = \tfrac{K_2}{t^4}e^{\frac{-4}{ct}},\ \delta(t) = K_3(t)^{(-m-3)}e^{\frac{(m-1)dt}{c}},\ \theta(t) = K_4(t)^{(-n-3)}e^{\frac{(n-1)dt}{c}}.
\end{aligned} \tag{6.2.13}$$

Substituting (6.2.13) into Eq. (6.1.1) yields a reduced ODE with variable coefficients:

$$d^2F(\zeta)+(1-2d)\zeta F'(\zeta)+\zeta^2F''(\zeta)+c^2K_1F''(\zeta)+c^2K_2F''''(\zeta)+c^2K_3(F(\zeta))^m+c^2K_4(F(\zeta))^n=0. \tag{6.2.14}$$

Type (iii(a)) $V_2+hV_4+jV_5+V_6$

For this case, the similarity variable, similarity solution and coefficient functions are as follows:

$$\begin{aligned}
&u(x,t)=\sqrt{t^2+1}e^{h\tan^{-1}t}F(\zeta),\ \zeta=x-j\tan^{-1}t,\ \alpha(t)=\tfrac{K_1}{(t^2+1)^2},\\
&\beta(t)=\tfrac{K_2}{(t^2+1)^2},\ \delta(t)=K_3(t^2+1)^{\frac{(-m-3)}{2}}e^{-(m-1)h\tan^{-1}t},\\
&\theta(t)=K_4(t^2+1)^{\frac{(-n-3)}{2}}e^{-(n-1)h\tan^{-1}t}.
\end{aligned} \tag{6.2.15}$$

Substituting (6.2.15) into Eq. (6.1.1), it follows the corresponding reduced ODE:

$$(h^2+1)F(\zeta)-2hjF'(\zeta)+j^2F''(\zeta)+K_1F''(\zeta)+K_2F''''(\zeta)+K_3(F(\zeta))^m+K_4(F(\zeta))^n=0. \tag{6.2.16}$$

Type (iii(b)) $V_2+hV_4+jV_5-V_6$

This case yields the following forms of invariants and coefficient functions:

$$\begin{aligned}
&u(x,t)=\sqrt{t^2-1}\left(\tfrac{t-1}{t+1}\right)^{\frac{h}{2}}F(\zeta),\ \zeta=x-\tfrac{j}{2}\log\left(\tfrac{t-1}{t+1}\right),\ \alpha(t)=\tfrac{K_1}{(1-t^2)^2},\\
&\beta(t)=\tfrac{K_2}{(1-t^2)^2},\ \delta(t)=K_3(t^2-1)^{\frac{(-m-3)}{2}}\left(\tfrac{t-1}{t+1}\right)^{\frac{-(m-1)h}{2}},\\
&\theta(t)=K_4(t^2-1)^{\frac{(-n-3)}{2}}\left(\tfrac{t-1}{t+1}\right)^{\frac{-(n-1)h}{2}}.
\end{aligned} \tag{6.2.17}$$

Substituting (6.2.17) into Eq. (6.1.1), the Generalized bretherton equation with time-dependent coefficients transforms to the following ODE :

$$(h^2-1)F(\zeta)-2hjF'(\zeta)+j^2F''(\zeta)+K_1F''(\zeta)+K_2F''''(\zeta)+K_3(F(\zeta))^m+K_4(F(\zeta))^n=0. \tag{6.2.18}$$

Type (iv) $V_2+kV_4+lV_5$

For this case, the similarity variable, similarity solution and coefficient functions are as follows:

$$\begin{aligned}
&u(x,t)=te^{\frac{-k}{t}}F(\zeta),\ \zeta=x+\tfrac{l}{t},\ \alpha(t)=\tfrac{K_1}{t^4},\\
&\beta(t)=\tfrac{K_2}{t^4},\ \delta(t)=K_3t^{(-m-3)}e^{\frac{(m-1)k}{t}},\ \theta(t)=K_4t^{(-n-3)}e^{\frac{(n-1)k}{t}}.
\end{aligned} \tag{6.2.19}$$

Substituting (6.2.19) into Eq. (6.1.1), it follows the corresponding reduced ODE:

$$k^2F(\zeta)-2klF'(\zeta)+l^2F''(\zeta)+K_1F''(\zeta)+K_2F''''(\zeta)+K_3(F(\zeta))^m+K_4(F(\zeta))^n=0. \tag{6.2.20}$$

Type (v(a)) $V_3 + qV_4 + V_5$

Following the same way as above, we get

$$\begin{aligned} &u(x,t) = t^q F(\zeta),\ \zeta = x - \log t,\ \alpha(t) = \tfrac{K_1}{t^2},\ \beta(t) = \tfrac{K_2}{t^2},\\ &\delta(t) = K_3 t^{(-2-(m-1)q)},\ \theta(t) = K_4 t^{(-2-(n-1)q)}. \end{aligned} \tag{6.2.21}$$

Substituting (6.2.21) into Eq. (6.1.1), it follows the corresponding reduced ODE:

$$q(q-1)F(\zeta) - (2q-1)F'(\zeta) + F''(\zeta) + K_1 F''(\zeta) + K_2 F''''(\zeta) + K_3 (F(\zeta))^m + K_4 (F(\zeta))^n = 0. \tag{6.2.22}$$

Type (vi) $V_3 + rV_4$

On Solving the equations (6.2.5) and (6.2.7), we get

$$\begin{aligned} &u(x,t) = t^r F(\zeta),\ \zeta = x,\ \alpha(t) = \tfrac{K_1}{t^2},\ \beta(t) = \tfrac{K_2}{t^2},\\ &\delta(t) = K_3 t^{(-2-(m-1)r)},\ \theta(t) = K_4 t^{(-2-(n-1)r)}. \end{aligned} \tag{6.2.23}$$

On using (6.2.23), Eq. (6.1.1) transforms to following ODE:

$$r(r-1)F(\zeta) + K_1 F''(\zeta) + K_2 F''''(\zeta) + K_3 (F(\zeta))^m + K_4 (F(\zeta))^n = 0. \tag{6.2.24}$$

Type (vii(a)) $V_4 + sV_5 + V_6$

Solving the equations (6.2.5) and (6.2.7), we get

$$\begin{aligned} &u(x,t) = e^t F(\zeta),\ \zeta = x - st,\ \alpha(t) = K_1,\ \beta(t) = K_2,\\ &\delta(t) = K_3 e^{(1-m)t},\ \theta(t) = K_4 e^{(1-n)t}. \end{aligned} \tag{6.2.25}$$

Substituting (6.2.25) into (6.1.1), we obtain

$$F(\zeta) - 2sF'(\zeta) + s^2 F''(\zeta) + K_1 F''(\zeta) + K_2 F''''(\zeta) + K_3 (F(\zeta))^m + K_4 (F(\zeta))^n = 0. \tag{6.2.26}$$

Type (viii) $V_4 + wV_5$

Following the same way, we get

$$u(x,t) = e^{\frac{x}{w}} F(\zeta),\ \zeta = t,\ \alpha(t) = \alpha(t),\ \beta(t) = \beta(t),\ \delta(t) = 0,\ \theta(t) = 0. \tag{6.2.27}$$

Using (6.2.27), Eq. (6.1.1) transforms to following ODE:

$$F''(\zeta) + \left(\frac{\alpha(\zeta)}{w^2} + \frac{\beta(\zeta)}{w^4}\right) F(\zeta) = 0. \tag{6.2.28}$$

Type (ix(a)) $V_5 + V_6$

For this vector field, on solving the equations (6.2.5) and (6.2.7) we obtain

$$u(x,t) = F(\zeta),\ \zeta = x - t,\ \alpha(t) = K_1,\ \beta(t) = K_2,\ \delta(t) = K_3,\ \theta(t) = K_4, \qquad (6.2.29)$$

Substituting (6.2.29) into Eq. (6.1.1), we have the function $F(\zeta)$ which must satisfy the following ODE:

$$F''(\zeta) + K_1F''(\zeta) + K_2F''''(\zeta) + K_3(F(\zeta))^m + K_4(F(\zeta))^n = 0. \qquad (6.2.30)$$

Type (x) V_5

For this case, the similarity variable, similarity solution and coefficient functions are as follows:

$$u(x,t) = F(\zeta),\ \zeta = t,\ \alpha(t) = \alpha(t),\ \beta(t) = \beta(t),\ \delta(t) = \delta(t),\ \theta(t) = \theta(t). \qquad (6.2.31)$$

Substituting (6.2.31) into Eq. (6.1.1), it follows the corresponding reduced ODE:

$$F''(\zeta) + \delta(\zeta)(F(\zeta))^m + \theta(\zeta)(F(\zeta))^n = 0. \qquad (6.2.32)$$

Type (xi) V_6

Following the same way, we get

$$u(x,t) = F(\zeta),\ \zeta = x,\ \alpha(t) = K_1,\ \beta(t) = K_2,\ \delta(t) = K_3,\ \theta(t) = K_4. \qquad (6.2.33)$$

Using (6.2.33), Eq. (6.1.1) transforms to following ODE:

$$K_1F''(\zeta) + K_2F''''(\zeta) + K_3(F(\zeta))^m + K_4(F(\zeta))^n = 0. \qquad (6.2.34)$$

Combining the above results, we obtain some reduced equations of Eq. (6.1.1) expressed by Eqs. (6.2.10), (6.2.12), (6.2.14), (6.2.16), (6.2.18), (6.2.20), (6.2.22), (6.2.24), (6.2.26), (6.2.28), (6.2.30), (6.2.32) and (6.2.34), respectively. Meanwhile many new solutions of Eq. (6.1.1) from these reduced Eqs. can be achieved (as we have done in the next section).

6.3 Exact Solutions of Reduced ODEs

Our main goal is to derive exact solutions of (6.1.1) and for this purpose, we will look for some exact solutions of the reduced ordinary differential equations in section (6.2).

6.3.1 Solutions of Eqs. (6.2.10), (6.2.12) and (6.2.14)

Case I: $m \neq 1$

We seek the solutions of Eqs. (6.2.10), (6.2.12) and (6.2.14) in the following form

$$F(\zeta) = A\zeta^p, \tag{6.3.1}$$

where A and p are constants to be found out. We need to equate the exponents of ζ suitably such that their respective coefficients become zero. By equating the exponents $p-4$ and pn, we have $p = \frac{4}{1-n}$ and also, by equating the exponents $p-2$ and pm, we get $p = \frac{2}{1-m}$. Hence, we should have $n = 2m-1$ (where $m \neq 1$). On substituting (6.3.1) into Eq. (6.2.10), we get the following solution of Eq. (6.1.1):

$$u(x,t) = \sqrt{at^2+1}\mathrm{e}^{\frac{b\tan^{-1}(\sqrt{a}t)}{\sqrt{a}}} \left(\left(2\sqrt{-\frac{-K_2\,m+2\,K_2\,m^2+3\,K_2\,m^3}{K_4}}\,(-1+m)^{-2}\right)^{(-1+m)^{-1}}\right)\left(x\mathrm{e}^{-\frac{\tan^{-1}(\sqrt{a}t)}{\sqrt{a}}}\right)^{\frac{2}{(1-m)}}, \tag{6.3.2}$$

where $K_1 = -\sqrt{-\frac{-K_2\,m+2\,K_2\,m^2+3\,K_2\,m^3}{K_4}}K_3\,(1+m)^{-1}$ and $a = -\frac{b^2-2\,b^2m+b^2m^2-4\,b+4\,bm+4}{1-2\,m+m^2}$.

On using (6.3.1) into Eq. (6.2.12), Eq. (6.1.1) has the following solution:

$$u(x,t) = \sqrt{at^2-1}\left(\frac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{\frac{b}{2\sqrt{a}}} \left(\left(2\sqrt{-\frac{-K_2\,m+2\,K_2\,m^2+3\,K_2\,m^3}{K_4}}\,(-1+m)^{-2}\right)^{(-1+m)^{-1}}\right)\left(x\left(\frac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{-\frac{1}{2\sqrt{a}}}\right)^{\frac{2}{(1-m)}}, \tag{6.3.3}$$

where $K_1 = -\sqrt{-\frac{-K_2\,m+2\,K_2\,m^2+3\,K_2\,m^3}{K_4}}K_3\,(1+m)^{-1}$ and $a = \frac{b^2-2\,b^2m+b^2m^2-4\,b+4\,bm+4}{1-2\,m+m^2}$.

On using the relation (6.3.1) in (6.2.14) corresponds to the following solution of Eq. (6.1.1):

$$u(x,t) = t\mathrm{e}^{2\,\frac{1}{(-1+m)ct}} \left(\left(2\sqrt{-\frac{-K_2\,m+2\,K_2\,m^2+3\,K_2\,m^3}{K_4}}\,(-1+m)^{-2}\right)^{(-1+m)^{-1}}\right)\left(x\mathrm{e}^{\frac{1}{ct}}\right)^{\frac{2}{(1-m)}}, \tag{6.3.4}$$

where $K_1 = -\sqrt{-\frac{-K_2\,m+2\,K_2\,m^2+3\,K_2\,m^3}{K_4}}K_3\,(1+m)^{-1}$ and $d = -2\,(-1+m)^{-1}$.

Case II: $m = 1$

In this case by equating the exponents $p-4$ and pn, we get $p = \frac{4}{1-n}$. Hence, on substituting (6.3.1) into Eq. (6.2.10), we get the following solution of Eq. (6.1.1):

$$u(x,t) = \sqrt{at^2+1}\mathrm{e}^{\frac{b\tan^{-1}(\sqrt{a}t)}{\sqrt{a}}} \left(\left(-8\,\frac{K_2\,(3+n)(3\,n+1)(n+1)}{K_4\,(-1+n)^4}\right)^{(-1+n)^{-1}}\right)\left(x\mathrm{e}^{-\frac{\tan^{-1}(\sqrt{a}t)}{\sqrt{a}}}\right)^{4\,(1-n)^{-1}}, \tag{6.3.5}$$

where $K_1 = 0$ and $a = -\frac{b^2-2\,b^2n+b^2n^2-8\,b+8\,bn+16+K_3-2\,K_3\,n+K_3\,n^2}{1-2\,n+n^2}$.

On using (6.3.1) in Eq. (6.2.12), Eq. (6.1.1) has the following solution:

$$u(x,t) = \sqrt{at^2-1}\left(\frac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{1/2\,\frac{b}{\sqrt{a}}} \left(\left(-8\,\frac{K_2\,(3+n)(3\,n+1)(n+1)}{K_4\,(-1+n)^4}\right)^{(-1+n)^{-1}}\right)\left(x\left(\frac{\sqrt{a}t-1}{\sqrt{a}t+1}\right)^{-1/2\,\frac{1}{\sqrt{a}}}\right)^{4\,(1-n)^{-1}}, \tag{6.3.6}$$

where $K_1 = 0$ and $a = \frac{b^2-2\,b^2n+b^2n^2-8\,b+8\,bn+16+K_3-2\,K_3\,n+K_3\,n^2}{1-2\,n+n^2}$.

Eq. (6.2.14) corresponds to the following solution of Eq. (6.1.1):

$$u(x,t) = t\mathrm{e}^{-\frac{b}{at}}\left(\left(-8\,\frac{K_2\,(3+n)(3\,n+1)(n+1)}{K_4\,(-1+n)^4}\right)^{(-1+n)^{-1}}\right)\left(x\mathrm{e}^{\frac{1}{at}}\right)^{4\,(1-n)^{-1}}, \tag{6.3.7}$$

where $K_1 = 0$ and $a = \sqrt{-{K_3}^{-1}}\,(bn-b+4)\,(-1+n)^{-1}$. Finally, in all the above solutions, it is necessary to have $K_2K_4 < 0$.

6.3.2 Solutions of Eqs. (6.2.16), (6.2.18), (6.2.20), (6.2.22) and (6.2.26)

We seek the solutions of Eqs. in the following form

$$F(\zeta) = A\zeta^p, \tag{6.3.8}$$

for $m = 1$, where A and p are constants to be found out. We need to equate the exponents of ζ suitably such that their respective coefficients become zero. By equating the exponents $p-1$ and pn, we have $p = \frac{1}{1-n}$. On substituting (6.3.8) into Eq. (6.2.16), we get the following solution of Eq. (6.1.1):

$$u(x,t) = \sqrt{t^2+1}\mathrm{e}^{h\tan^{-1}(t)}\left(\left(-2\,\frac{hj}{K_4\,(-1+n)}\right)^{(-1+n)^{-1}}\right)\left(x-j\tan^{-1}(t)\right)^{\frac{1}{(1-n)}}, \tag{6.3.9}$$

where $K_1 = -j^2, K_2 = 0, K_3 = -h^2 - 1, hjK_4 < 0$.

On using the relation (6.3.8) in (6.2.18) corresponds to the following solution of Eq. (6.1.1):

$$u(x,t) = \sqrt{t^2-1}\left(\frac{t-1}{t+1}\right)^{1/2\,h}\left(\left(-2\,\frac{hj}{K_4\,(-1+n)}\right)^{(-1+n)^{-1}}\right)\left(x - 1/2\,j\log\left(\frac{t-1}{t+1}\right)\right)^{\frac{1}{(1-n)}}, \tag{6.3.10}$$

where $K_1 = -j^2, K_2 = 0, K_3 = -h^2 + 1, hjK_4 < 0$.

Eq. (6.1.1) possesses the following solution corresponding to Eq. (6.2.20):

$$u(x,t) = t\mathrm{e}^{-\frac{k}{t}}\left(\left(-2\,\frac{kl}{K_4\,(-1+n)}\right)^{(-1+n)^{-1}}\right)\left(x + \frac{l}{t}\right)^{\frac{1}{(1-n)}}, \tag{6.3.11}$$

where $K_1 = -l^2, K_2 = 0, K_3 = -k^2, klK_4 < 0$.

Eq. (6.2.22) corresponds to the following solution of Eq. (6.1.1):

$$u(x,t) = t^q\left(\left(-\frac{2\,q-1}{K_4\,(-1+n)}\right)^{(-1+n)^{-1}}\right)\left(x - \log\left(t\right)\right)^{\frac{1}{(1-n)}}, \tag{6.3.12}$$

where $K_1 = -1, K_2 = 0, K_3 = -q^2 + q, (2q-1)K_4 < 0$.

We get the following solution of Eq. (6.1.1) corresponding to Eq. (6.2.26):

$$u(x,t) = \mathrm{e}^t\left(\left(-2\,\frac{s}{K_4\,(-1+n)}\right)^{(-1+n)^{-1}}\right)\left(x - st\right)^{\frac{1}{(1-n)}}, \tag{6.3.13}$$

where $K_1 = -s^2, K_2 = 0, K_3 = -1, sK_4 < 0$.

6.3.3 Solutions of Eqs. (6.2.24), (6.2.30) and (6.2.34)

Case I: $m \neq 1$

Let us consider the solutions of Eq. (6.2.24) in the following form:

$$F(\zeta) = A\mathrm{sech}(\zeta)^p. \tag{6.3.14}$$

We need to equate the exponents of $\mathrm{sech}(\zeta)$ suitably such that their respective coefficients become zero. By equation the exponents $pn - 4$, p and $pm - 4$, $p - 2$, we get $p = \frac{4}{n-1}$

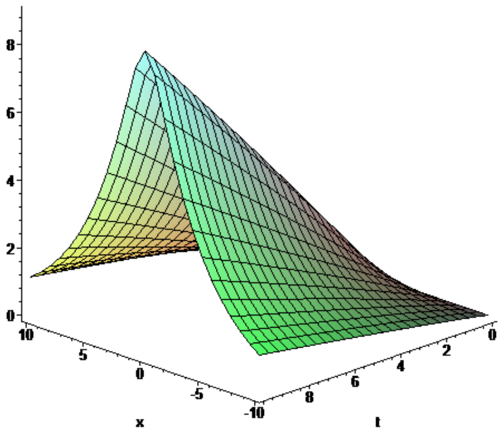

FIGURE 6.1: The figure of the soliton profiles of (6.3.15) with $K_1 = 1/2, K_2 = 20, K_4 = -10, m = 10$

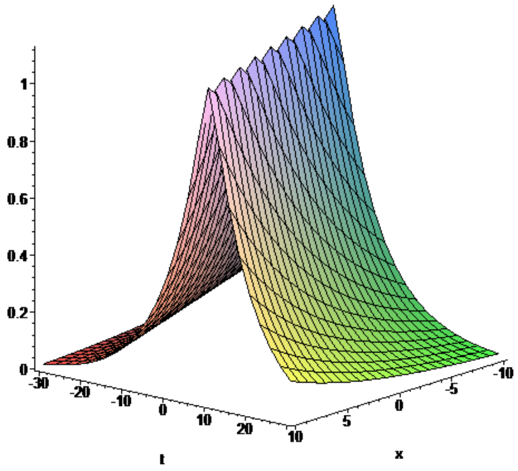

FIGURE 6.2: The wave profile for solution (6.3.16) with $K_3 = 10, K_4 = -2, m = 20$

and $p = \frac{2}{m-1}$ respectively, hence, we should have $n = 2m - 1$. On substituting (6.3.14) into Eq. (6.2.24), we get the following solitary wave solutions of Eq. (6.1.1):

$$u(x,t) = t^r \left(\left(2\sqrt{-\frac{2K_2m^2-K_2m+3K_2m^3}{K_4}}\,(m-1)^{-2} \right)^{(m-1)^{-1}} \right) (\operatorname{sech}(x))^{\frac{2}{(m-1)}}, \tag{6.3.15}$$

where $K_3 = \frac{(4K_2m^3+4K_2m^2+4K_2m+4K_2+K_1m^3-K_1m^2-K_1m+K_1)}{(m^2-2m+1)\sqrt{-\frac{2K_2m^2-K_2m+3K_2m^3}{K_4}}}$,
$r = \frac{1/2m^2-m+1/2+1/2\sqrt{m^4-4m^3+6m^2-4m+1-64K_2-16K_1m^2+32K_1m-16K_1}}{(m-1)^2}$ and $K_2K_4 < 0$. The figure 6.1 shows the soliton profiles of (6.3.15) with $K_1 = 1/2, K_2 = 20, K_4 = -10, m = 10$.

On using (6.3.14) into Eq. (6.2.30), Eq. (6.1.1) has the following solution:

$$u(x,t) = \left(\left(-1/2\,\frac{K_3(3m-1)}{K_4m} \right)^{(m-1)^{-1}} \right) (\operatorname{sech}(-x+t))^{\frac{2}{(m-1)}}, \tag{6.3.16}$$

where $K_1 = 1/4\,\frac{-4m^4K_4+3K_3^2m^3-7K_3^2m^2-4m^3K_4+5K_3^2m-K_3^2}{m^3K_4(m+1)}$,
$K_2 = -1/16\,\frac{K_3^2(3m^5-13m^4+22m^3-18m^2+7m-1)}{m^3K_4(m+1)}$ and $K_3K_4 < 0$. The wave profile for solution (6.3.16) can be imagined similar to profile of (6.3.15) with $K_3 = 10, K_4 = -2, m = 20$.

On using the relation (6.3.14) in (6.2.34) corresponds to the following solution of Eq. (6.1.1):

$$u(x,t) = \left(\left(-1/2\,\frac{K_3\,(3\,m-1)}{K_4\,m}\right)^{(m-1)^{-1}}\right)(\mathrm{sech}\,(x))^{\frac{2}{(m-1)}}, \tag{6.3.17}$$

where $K_1 = 1/4\,\frac{{K_3}^2(3\,m-1)\left(m^2-2\,m+1\right)}{K_4\,m^3(m+1)}$, $K_2 = -1/16\,\frac{{K_3}^2(3\,m-1)\left(m^2-2\,m+1\right)^2}{K_4\,m^3(m+1)}$ and $K_3K_4 < 0$.

Case II: $m = 1$

In this case by equating the exponents $pn-4$ and p, we get $p = \frac{4}{n-1}$. Hence, on substituting (6.3.14) into Eq. (6.2.24), we get the following solution of Eq. (6.1.1):

$$u(x,t) = t^r\left(\left(-8\,\frac{K_2\,(n+3)(3\,n+1)(n+1)}{K_4\,(n-1)^4}\right)^{(n-1)^{-1}}\right)(\mathrm{sech}\,(x))^{4\,(n-1)^{-1}}, \tag{6.3.18}$$

where $r = \frac{1/2n^2-n+1/2+1/2\sqrt{n^4-4n^3+6n^2-4n+1+512K_2n+256K_2n^2-4K_3n^4+16K_3n^3+16K_3n-24K_3n^2-4K_3+256K_2}}{(-1+n)^2}$, $K_1 = -4\frac{\left(n^2+2n+5\right)K_2}{1-2n+n^2}$ and $K_2K_4 < 0$.

Eq. (6.2.30) corresponds to the following solution of Eq. (6.1.1):

$$u(x,t) = \left(\left(-1/8\,\frac{(n+3)(3\,n+1)K_3}{K_4\,(n+1)}\right)^{(n-1)^{-1}}\right)(\mathrm{sech}\,(-x+t))^{4\,(n-1)^{-1}}, \tag{6.3.19}$$

where $K_1 = -1/16\,\frac{K_3\,n^4+2\,K_3\,n^2+16\,n^2-8\,K_3\,n+32\,n+5\,K_3+16}{n^2+2\,n+1}$, $K_2 = \frac{1}{64}\,\frac{K_3\,\left(n^4-4\,n^3+6\,n^2-4\,n+1\right)}{n^2+2\,n+1}$, and $K_3K_4 < 0$. The evolution of solution (6.3.19) with $K_3 = 20, K_4 = -5, n = 10$ at $t = 0$, $t = 1$ and $t = 2$ has been shown in Fig. (6.3), Fig. (6.4) and Fig. (6.5) respectively.

Eq. (6.1.1) possesses the following solution corresponding to Eq. (6.2.34):

$$u(x,t) = \left(\left(-1/8\,\frac{(n+3)(3\,n+1)K_3}{K_4\,(n+1)}\right)^{(n-1)^{-1}}\right)(\mathrm{sech}\,(x))^{4\,(n-1)^{-1}}, \tag{6.3.20}$$

where $K_1 = -1/16\,\frac{\left(n^4+2\,n^2-8\,n+5\right)K_3}{n^2+2\,n+1}$, $K_2 = \frac{1}{64}\,\frac{K_3\,\left(n^4-4\,n^3+6\,n^2-4\,n+1\right)}{n^2+2\,n+1}$ and $K_3K_4 < 0$.

6.3.4 Solutions of Eqs. (6.2.28) and (6.2.32)

We assume that the solution of equation (6.2.28) can be expressed in the following form:

$$F(\zeta) = Ae^{p\zeta}. \tag{6.3.21}$$

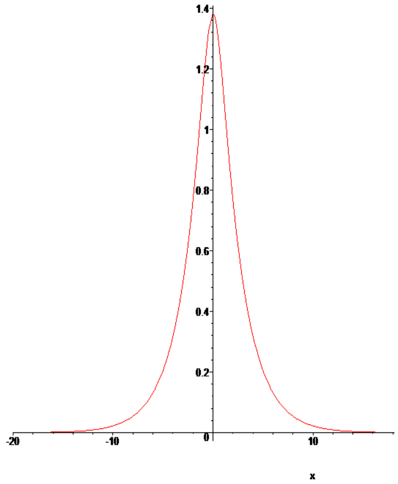

FIGURE 6.3: The figure of solution (6.3.19) with $K_3 = 20, K_4 = -5, n = 10$ at $t = 0$

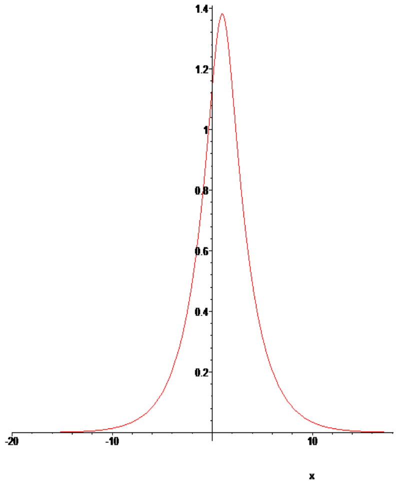

FIGURE 6.4: The figure of solution (6.3.19) with $K_3 = 20, K_4 = -5, n = 10$ at $t = 1$

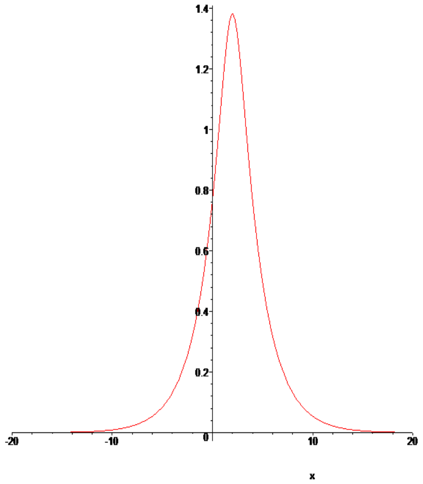

FIGURE 6.5: The figure of solution (6.3.19) with $K_3 = 20, K_4 = -5, n = 10$ at $t = 2$

On substituting (6.3.21) into (6.2.28), we get the following solution of Eq. (6.1.1):

$$u(x,t) = A\mathrm{e}^{\frac{x+ptw}{w}}, \tag{6.3.22}$$

where $\alpha(t) = -\frac{p^2w^4+\beta(t)}{w^2}$.

Let us consider the solutions of Eq. (6.2.32) in the following form:

$$F(\zeta) = Asn(\zeta, q)^p, \tag{6.3.23}$$

where $sn(\zeta, q)$ is the Jacobi elliptic sine function with modulus q. We need to equate the exponents of $sn(\zeta, q)$ suitably such that their respective coefficients become zero. By equation the exponents $pn + 2$, $p + 4$ and $pm + 2$, p, we get $p = \frac{2}{n-1}$ and $p = \frac{2}{1-m}$ respectively, hence, we should have $n = 2-m$. On substituting (6.3.23) into Eq. (6.2.32), we get the following solution of Eq. (6.1.1):

$$u(x,t) = A\left(sn\left(t, i\right)\right)^{2\,(1-m)^{-1}}, \tag{6.3.24}$$

where $\delta(t) = -2\,\frac{A(1+m)}{A^m(1-2\,m+m^2)}$ and $\theta(t) = -2\,\frac{A^{-1+m}(-3+m)}{1-2\,m+m^2}$.

6.4 Conclusion

This chapter obtains the Lie symmetries and similarity reductions of variable coefficients generalized Bretherton equation and further reduced ODEs have been studied to obtain certain exact solitary wave solutions. The sech-ansätze method and other methods were successfully used to derive these solutions.It needs to be noted that however some figures were plotted to see the propagation and asymptotic characteristics of the solitary waves. We should notice that the obtained solutions in this chapter play a crucial role in numerical simulation and the understanding of solitons dynamics of variants of studied equation where they facilitate the verification of numerical solvers.

Summary

In the study of nonlinear partial differential equations, the discovery of symmetries and exact solutions has great theoretical and practical importance. These exact solutions for nonlinear systems are used as models for physical or numerical investigations and often reflect qualitatively on the behaviour of more complicated solutions. In this thesis entitled **“Lie Group Applications to Some Nonlinear Systems”**, our prime focus is to obtain symmetries and exact solutions of certain nonlinear PDEs representing important physical phenomenon which are the (2+1)-dimensional Calogero Degasperis (CD) equation with its generalized form, the coupled Klein-Gordon-Schrödinger (KGS) equation along its variable coefficients form, the (2+1)-dimensional potential Kadomstev Petviashvili (PKP) equation and its generalized form, the generalized Dullin-Gottwald-Holm (GDGH) equation and the generalized Bretherton equation. To determine the admissible symmetries, the methods - Lie classical approach and Nonclassical method have been utilized. Both these techniques consist of steps which can be applied in a very systematic manner. After obtaining the point symmetries of the system under investigation, our attempt is to reduce the number of independent variables of the system corresponding to each element of the optimal system and then reduced equations have been studied by several methods including extended (G'/G)-expansion method, modified (G'/G)-expansion method, Jacobi elliptic function method and several methods and also, it may be noted that the solutions obtained for various systems in the thesis include periodic solutions, traveling wave solutions, kink wave solutions, solitons etc.

In chapters 2, 3 and 4, Lie group method has been applied on constant coefficients as well as on variable coefficients versions of equations that helps us to reduce (2+1)-dimensional equations to (1+1)-dimensional equations and (1+1)-dimensional equations to ODEs. In case of variable coefficients equations, most of the solutions involve an arbitrary coefficient function and it enables us to control and discuss the behaviour of solutions as governed by the choice of this arbitrary function. Another important aspect achieved in this thesis is to check whether studied equation is integrable or not by using Painlevé property. We have performed Painlevé test on variable coefficient equations studied in chapters 2 and 4 and obtained constraints on coefficients for equation to be Painlevé integrable.

Motivated by the fact that symmetry reductions for many PDEs cannot be obtained by using classical symmetries, there have been several generalizations of the classical Lie group method for symmetry reductions, so we have also investigated nonclassical symmetries in chapter 5. We have obtained the complete Lie group classification for the studied equation in this chapter and obtained new results.

We have also eastablished the traveling wave solutions of generalized Bretherton equation with variable coefficients in chapter 6 by using sech-ansätze and some other methods. We have also plotted some figures to see the propagation and asymptotic characteristics of the solitary wave solutions obtained here.

In short, we can say that work in our thesis is devoted to investigating a wide range of applications of continous symmetry groups to physically important systems of partial differential equations. It is worth mentioning that in spite of the focus on symmetries and exact solutions, the author found it really difficult at times to find symmetries for variable coefficient equations as symmetries for constant coefficient equations can be obtained by some mathematical softwares. Keeping in view this limitation, it will be really interesting if such softwares can be developed. In this thesis, it is also a herculean task to extract solutions of PDEs from reduced ODEs and to fulfill this aim, some specific forms of solutions of ODEs have been assumed. Thus, the general solution of reduced ODEs, their physical interpretation and to study the nonclassical symmetries of variable coefficient equations brings forth tremendous scope of future work.

Appendix A

Extended Infinitesimals

A.1 Extended infinitesimals for (2+1)-dimensional PDEs

The following expressions for extended infinitesimals $\eta^x, \eta^{xx}, \eta^{xt}, \eta^{yy}, \eta^{xxxx}$ for (2+1)-dimensional PDEs can be easily obtained from equation (1.2.35) for the space $(x, y, t, \sigma(x, y, t))$ where ξ, ϕ, τ, η are infinitesimals corresponding to x, y, t, σ:

$$\eta^x = -\sigma_x\xi_x - \xi_\sigma{\sigma_x}^2 + \eta_x - \sigma_t\tau_\sigma\sigma_x - \sigma_y\phi_x + \eta_\sigma\sigma_x - \sigma_t\tau_x - \sigma_y\phi_\sigma\sigma_x \tag{A.1.1}$$

$$\begin{gathered}\eta^{xx} = -\sigma_y{\sigma_x}^2\phi_{\sigma\sigma} - \sigma_t{\sigma_x}^2\tau_{\sigma\sigma} - {\sigma_x}^3\xi_{\sigma\sigma} - \sigma_y\phi_{xx} - 2\sigma_{xy}\phi_x - 2\sigma_{xy}\phi_\sigma\sigma_x + 2\eta_{x\sigma}\sigma_x - 2\sigma_{xt}\tau_x \\ -3\xi_\sigma\sigma_x\sigma_{xx} + \eta_{\sigma\sigma}{\sigma_x}^2 - \sigma_x\xi_{xx} + \eta_{xx} - \sigma_t\tau_\sigma\sigma_{xx} - 2\sigma_t\sigma_x\tau_{x\sigma} - 2\sigma_{xx}\xi_x - \sigma_y\phi_\sigma\sigma_{xx} - \sigma_t\tau_{xx} \\ +\eta_\sigma\sigma_{xx} - 2\sigma_{xt}\tau_\sigma\sigma_x - 2\sigma_y\phi_{x\sigma}\sigma_x - 2\xi_{x\sigma}{\sigma_x}^2\end{gathered} \tag{A.1.2}$$

$$\begin{gathered}\eta^{xt} = -\sigma_y\sigma_x\phi_{\sigma\sigma}\sigma_t - \sigma_{xt}\tau_t - \sigma_y\phi_{xt} - \tau_{x\sigma}{\sigma_t}^2 - {\sigma_x}^2\xi_{t\sigma} - \sigma_x\xi_{xt} + \eta_\sigma\sigma_{xt} - \sigma_{tt}\tau_x - \sigma_{xx}\xi_\sigma\sigma_t \\ -\sigma_y\phi_\sigma\sigma_{xt} - \sigma_t\sigma_x\tau_{t\sigma} - \sigma_x\xi_{x\sigma}\sigma_t - {\sigma_t}^2\sigma_x\tau_{\sigma\sigma} - \sigma_{xy}\phi_\sigma\sigma_t - {\sigma_x}^2\xi_{\sigma\sigma}\sigma_t - \sigma_{xx}\xi_t - \sigma_t\tau_{xt} + \eta_{xt} \\ +\eta_{x\sigma}\sigma_t + \sigma_x\eta_{t\sigma} - 2\sigma_t\tau_\sigma\sigma_{xt} - \sigma_y\sigma_x\phi_{t\sigma} - \sigma_{yt}\phi_\sigma\sigma_x - \sigma_y\phi_{x\sigma}\sigma_t + \sigma_x\eta_{\sigma\sigma}\sigma_t - \sigma_{xt}\xi_x - \sigma_{xy}\phi_t \\ -\sigma_{tt}\tau_\sigma\sigma_x - 2\xi_\sigma\sigma_x\sigma_{xt} - \sigma_{yt}\phi_x\end{gathered} \tag{A.1.3}$$

$$\begin{gathered}\eta^{yy} = -\sigma_t\tau_\sigma\sigma_{yy} - \sigma_x\xi_\sigma\sigma_{yy} - 2\sigma_t\sigma_y\tau_{y\sigma} - 3\phi_\sigma\sigma_y\sigma_{yy} - 2\phi_{y\sigma}{\sigma_y}^2 - \sigma_x{\sigma_y}^2\xi_{\sigma\sigma} - \sigma_t\tau_{yy} \\ -2\sigma_x\xi_{y\sigma}\sigma_y + \eta_{\sigma\sigma}{\sigma_y}^2 - \sigma_y\phi_{yy} - \sigma_t{\sigma_y}^2\tau_{\sigma\sigma} - 2\sigma_{yt}\tau_\sigma\sigma_y + \eta_\sigma\sigma_{yy} - 2\sigma_{xy}\xi_\sigma\sigma_y - 2\sigma_{yy}\phi_y \\ -\sigma_x\xi_{yy} + \eta_{yy} + 2\eta_{y\sigma}\sigma_y - {\sigma_y}^3\phi_{\sigma\sigma} - 2\sigma_{xy}\xi_y - 2\sigma_{yt}\tau_y\end{gathered} \tag{A.1.4}$$

$$\begin{gathered}\eta^{xxxx} = -4\sigma_{xy}{\sigma_x}^3\phi_{\sigma\sigma\sigma} - 12\sigma_{xxt}\sigma_x\tau_{x\sigma} + 6{\sigma_x}^2\eta_{\sigma\sigma\sigma}\sigma_{xx} - 6\sigma_{xxt}{\sigma_x}^2\tau_{\sigma\sigma} - 4\sigma_t\sigma_x\tau_{xxx\sigma} \\ -6\sigma_{xxy}{\sigma_x}^2\phi_{\sigma\sigma} - 10{\sigma_x}^2\xi_{\sigma\sigma}\sigma_{xxx} - 4\sigma_{xy}\phi_\sigma\sigma_{xxx} - 4\sigma_t\sigma_x\tau_{\sigma\sigma}\sigma_{xxx} - 12\sigma_{xy}\sigma_x\phi_{\sigma\sigma}\sigma_{xx} \\ -12\sigma_{xt}\sigma_x\tau_{\sigma\sigma}\sigma_{xx} - 6\sigma_t{\sigma_x}^2\tau_{\sigma\sigma\sigma}\sigma_{xx} - 12\sigma_t\sigma_x\tau_{x\sigma\sigma}\sigma_{xx} - 12\sigma_y\sigma_x\phi_{x\sigma\sigma}\sigma_{xx} \\ -6\sigma_y{\sigma_x}^2\phi_{\sigma\sigma\sigma}\sigma_{xx} - 4\sigma_y\sigma_x\phi_{\sigma\sigma}\sigma_{xxx} - 6\sigma_y\phi_{xx\sigma}\sigma_{xx} - 12\sigma_{xy}{\sigma_x}^2\phi_{x\sigma\sigma} - 6\sigma_{xxy}\phi_\sigma\sigma_{xx} \\ -24{\sigma_x}^2\xi_{x\sigma\sigma}\sigma_{xx} - 12\sigma_{xt}\sigma_{xx}\tau_{x\sigma} - 18\xi_{xx\sigma}\sigma_x\sigma_{xx} + 4\eta_{\sigma\sigma}\sigma_x\sigma_{xxx} - 4\sigma_{xt}{\sigma_x}^3\tau_{\sigma\sigma\sigma}\end{gathered}$$

$$\begin{aligned}
&-4\sigma_{xt}\tau_\sigma\sigma_{xxx} - 6\sigma_t\tau_{xx\sigma}\sigma_{xx} - 12\sigma_{xt}\tau_{xx\sigma}\sigma_x - 12\sigma_{xy}\phi_{xx\sigma}\sigma_x - 12\sigma_{xt}{\sigma_x}^2\tau_{x\sigma\sigma} + 12\eta_{x\sigma\sigma}\sigma_x\sigma_{xx}\\
&-16\xi_{x\sigma}\sigma_x\sigma_{xxx} - 15\sigma_x\xi_{\sigma\sigma}{\sigma_{xx}}^2 - \sigma_y\phi_\sigma\sigma_{xxxx} - \sigma_t\tau_\sigma\sigma_{xxxx} - \sigma_t{\sigma_x}^4\tau_{\sigma\sigma\sigma\sigma} - \sigma_y{\sigma_x}^4\phi_{\sigma\sigma\sigma\sigma}\\
&-4\sigma_y{\sigma_x}^3\phi_{x\sigma\sigma\sigma} - 4\sigma_{xxxt}\tau_\sigma\sigma_x - 12\sigma_{xxy}\phi_{x\sigma}\sigma_x - 4\sigma_t{\sigma_x}^3\tau_{x\sigma\sigma\sigma} - 3\sigma_t\sigma_{xx}^2\tau_{\sigma\sigma} - 5\xi_\sigma\sigma_x\sigma_{xxxx}\\
&-4\sigma_t\sigma_{xxx}\tau_{x\sigma} - 6\sigma_{xxt}\tau_\sigma\sigma_{xx} - 6\sigma_t{\sigma_x}^2\tau_{xx\sigma\sigma} - 10\xi_\sigma\sigma_{xx}\sigma_{xxx} - 6\sigma_y{\sigma_x}^2\phi_{xx\sigma\sigma} - 10{\sigma_x}^3\xi_{\sigma\sigma\sigma}\sigma_{xx}\\
&-12\sigma_{xy}\phi_{x\sigma}\sigma_{xx} - 3\sigma_y{\sigma_{xx}}^2\phi_{\sigma\sigma} - 4\sigma_y\phi_{x\sigma}\sigma_{xxx} - 4\sigma_y\sigma_x\phi_{xxx\sigma} - 4\sigma_{xxxy}\phi_\sigma\sigma_x - \sigma_y\phi_{xxxx} + \eta_\sigma\sigma_{xxxx}\\
&-\sigma_t\tau_{xxxx} - \sigma_x\xi_{xxxx} + {\sigma_x}^4\eta_{\sigma\sigma\sigma\sigma} - 6\sigma_{xxy}\phi_{xx} - 4\sigma_{xxxt}\tau_x + 4{\sigma_x}^3\eta_{x\sigma\sigma\sigma} - 4\sigma_{xxxy}\phi_x - 12\xi_{x\sigma}{\sigma_{xx}}^2\\
&-4\xi_{xxx\sigma}{\sigma_x}^2 - 6\sigma_{xxx}\xi_{xx} + 6\eta_{xx\sigma\sigma}{\sigma_x}^2 - 4\sigma_{xx}\xi_{xxx} + 4\eta_{x\sigma}\sigma_{xxx} - 6\sigma_{xxt}\tau_{xx} - 4\sigma_{xt}\tau_{xxx} + 3\eta_{\sigma\sigma}{\sigma_{xx}}^2\\
&-4\sigma_{xxxx}\xi_x + 6\eta_{xx\sigma}\sigma_{xx} + 4\eta_{xxx\sigma}\sigma_x - 6{\sigma_x}^3\xi_{xx\sigma\sigma} - {\sigma_x}^5\xi_{\sigma\sigma\sigma\sigma} - 4{\sigma_x}^4\xi_{x\sigma\sigma\sigma} - 4\sigma_{xy}\phi_{xxx} + \eta_{xxxx}
\end{aligned}\tag{A.1.5}$$

From the equations (1.2.35), one may easily obtain the following expressions for the extended infinitesimals $\eta^x, \eta^y, \eta^{xx}, \eta^{xy}, \eta^{xt}, \eta^{xxxy}$, where ψ is dependent on x, y, t and ξ, ϕ, τ, η are infinitesimals corresponding to x, y, t, ψ:

$$\eta^x = -\psi_x\xi_x - \xi_\psi{\psi_x}^2 + \eta_x - \psi_t\tau_\psi\psi_x - \psi_y\phi_x + \eta_\psi\psi_x - \psi_t\tau_x - \psi_y\phi_\psi\psi_x \tag{A.1.6}$$

$$\eta^y = -\psi_y\phi_y - \phi_\psi{\psi_y}^2 + \eta_y - \psi_t\tau_\psi\psi_y - \psi_x\xi_y + \eta_\psi\psi_y - \psi_t\tau_y - \psi_x\xi_\psi\psi_y \tag{A.1.7}$$

$$\begin{aligned}
\eta^{xx} = &-2\psi_y\phi_{x\psi}\psi_x - 2\psi_{xt}\tau_x - {\psi_x}^3\xi_{\psi\psi} - \psi_y\phi_\psi\psi_{xx} - 2\xi_{x\psi}{\psi_x}^2 - \psi_t\tau_{xx} - \psi_x\xi_{xx} - 2\psi_{xy}\phi_x\\
&-\psi_y\phi_{xx} - \psi_y{\psi_x}^2\phi_{\psi\psi} + \eta_\psi\psi_{xx} - 3\xi_\psi\psi_x\psi_{xx} - 2\psi_{xy}\phi_\psi\psi_x + \eta_{\psi\psi}{\psi_x}^2 + 2\eta_{x\psi}\psi_x - 2\psi_t\psi_x\tau_{x\psi}\\
&-\psi_t\tau_\psi\psi_{xx} + \eta_{xx} - 2\psi_{xx}\xi_x - 2\psi_{xt}\tau_\psi\psi_x - \psi_t{\psi_x}^2\tau_{\psi\psi}
\end{aligned}\tag{A.1.8}$$

$$\begin{aligned}
\eta^{xy} = &-\psi_x\xi_{x\psi}\psi_y - \psi_t\psi_x\tau_{\psi\psi}\psi_y - \psi_{xy}\phi_y + \eta_{x\psi}\psi_y - \psi_{xy}\xi_x - {\psi_x}^2\xi_{y\psi} - \psi_{xx}\xi_y - \phi_{x\psi}{\psi_y}^2\\
&+\psi_x\eta_{y\psi} - \psi_y\phi_{xy} - \psi_{yy}\phi_x + \eta_\psi\psi_{xy} - \psi_{yt}\tau_x - \psi_x\xi_{xy} - \psi_t\tau_{xy} - \psi_{xt}\tau_\psi\psi_y - \psi_t\psi_x\tau_{y\psi}\\
&+\psi_x\eta_{\psi\psi}\psi_y - {\psi_y}^2\psi_x\phi_{\psi\psi} - \psi_t\tau_{x\psi}\psi_y - \psi_{xx}\xi_\psi\psi_y - \psi_{yy}\phi_\psi\psi_x - {\psi_x}^2\xi_{\psi\psi}\psi_y - \psi_{yt}\tau_\psi\psi_x\\
&-\psi_t\tau_\psi\psi_{xy} - 2\psi_y\phi_\psi\psi_{xy} - 2\xi_\psi\psi_x\psi_{xy} - \psi_{xt}\tau_y - \psi_y\psi_x\phi_{y\psi} + \eta_{xy}
\end{aligned}\tag{A.1.9}$$

$$\begin{aligned}
\eta^{xt} = &-\psi_y\psi_x\phi_{\psi\psi}\psi_t - \tau_{x\psi}{\psi_t}^2 - \psi_{xy}\phi_t - \psi_x\xi_{xt} - \psi_t\tau_{xt} - \psi_{xx}\xi_t - {\psi_x}^2\xi_{t\psi} - \psi_{yt}\phi_x\\
&-\psi_{tt}\tau_x - \psi_y\phi_{xt} + \psi_x\eta_{t\psi} + \eta_{x\psi}\psi_t + \eta_\psi\psi_{xt} - \psi_{xt}\xi_x - \psi_{xt}\tau_t - {\psi_x}^2\xi_{\psi\psi}\psi_t - \psi_x\xi_{x\psi}\psi_t\\
&-\psi_{xy}\phi_\psi\psi_t - \psi_y\psi_x\phi_{t\psi} - \psi_{xx}\xi_\psi\psi_t + \psi_x\eta_{\psi\psi}\psi_t - \psi_t\psi_x\tau_{t\psi} - \psi_{tt}\tau_\psi\psi_x - \psi_y\phi_{x\psi}\psi_t\\
&-\psi_{yt}\phi_\psi\psi_x - {\psi_t}^2\psi_x\tau_{\psi\psi} - \psi_y\phi_\psi\psi_{xt} - 2\xi_\psi\psi_x\psi_{xt} - 2\psi_t\tau_\psi\psi_{xt} + \eta_{xt}
\end{aligned}\tag{A.1.10}$$

$$\begin{aligned}
\eta^{xxxy} = &-3\psi_{yy}{\psi_x}^2\phi_{x\psi\psi} - 3\psi_y\psi_x\phi_{xxy\psi} - \psi_t{\psi_x}^3\tau_{y\psi\psi\psi} - 4\xi_\psi\psi_{xy}\psi_{xxx} - 3{\psi_y}^2\psi_{xx}\phi_{x\psi\psi}\\
&-6\psi_y\psi_{xy}\phi_{xx\psi} - 3\psi_{xxt}\tau_{x\psi}\psi_y - 3\psi_y{\psi_x}^2\phi_{xy\psi\psi} - 3\psi_{xt}\tau_\psi\psi_{xxy} + 3\psi_x\eta_{xx\psi\psi}\psi_y - 3{\psi_y}^2{\psi_x}^2\phi_{x\psi\psi\psi}\\
&-3\psi_t{\psi_x}^2\tau_{xy\psi\psi} - 6{\psi_x}^2\xi_{\psi\psi}\psi_{xxy} - 3\psi_{xx}\xi_{xx\psi}\psi_y + 3\psi_x\psi_{xx}\eta_{y\psi\psi} - 6\psi_x\xi_{xx\psi}\psi_{xy} - 6\psi_y\phi_{x\psi}\psi_{xxy}\\
&+3\eta_{\psi\psi}\psi_{xy}\psi_{xx} - 3\psi_{xt}\psi_{xx}\tau_{y\psi} - 6\psi_{xt}\psi_x\tau_{xy\psi} - 6\psi_{xyy}\phi_{x\psi}\psi_x - 3{\psi_{xx}}^2\xi_{\psi\psi}\psi_y - 3\psi_{xyt}\tau_{xx}\\
&-3\psi_{xy}\psi_{xx}\phi_{y\psi} - 3\psi_{xt}{\psi_x}^2\tau_{y\psi\psi} - 3\psi_{xt}\tau_{xx\psi}\psi_y - 3\psi_{xy}{\psi_x}^2\phi_{y\psi\psi} - 6\psi_{xyt}\psi_x\tau_{x\psi} - 3\psi_{xxyy}\phi_\psi\psi_x\\
&-3\psi_{yt}\psi_x\tau_{xx\psi} - 3\psi_{yy}\psi_x\phi_{xx\psi} + 3\psi_{xx}\eta_{x\psi\psi}\psi_y - 3\psi_t\psi_{xxy}\tau_{x\psi} - 3{\psi_y}^2\psi_x\phi_{xx\psi\psi} - 3\psi_{xxt}\tau_\psi\psi_{xy}\\
&-6\psi_{xy}\psi_x\phi_{xy\psi} - 3\psi_y\psi_{xx}\phi_{xy\psi} - 3\psi_t\psi_{xx}\tau_{xy\psi} - 6{\psi_{xy}}^2\psi_x\phi_{\psi\psi} - 2\psi_y\phi_\psi\psi_{xxxy} - 4{\psi_x}^3\xi_{\psi\psi\psi}\psi_{xy}\\
&-3\psi_{xxt}\psi_x\tau_{y\psi} + 3{\psi_x}^2\eta_{\psi\psi\psi}\psi_{xy} - 6{\psi_x}^2\psi_{xx}\xi_{y\psi\psi} - 3\psi_t\psi_{xy}\tau_{xx\psi} - 3\psi_{xyt}\tau_\psi\psi_{xx} - 3\psi_{xxyt}\tau_\psi\psi_x
\end{aligned}$$

$$
\begin{aligned}
&+3{\psi_x}^2\eta_{x\psi\psi\psi}\psi_y - 6\psi_{xy}\phi_\psi\psi_{xxy} - 3\psi_{yy}\phi_{x\psi}\psi_{xx} - 9{\psi_x}^2\xi_{x\psi\psi}\psi_{xy} - 3\psi_{yt}{\psi_x}^2\tau_{x\psi\psi}\\
&-3\psi_{xyt}{\psi_x}^2\tau_{\psi\psi} - 3\psi_{xyy}\phi_\psi\psi_{xx} + {\psi_x}^3\eta_{y\psi\psi\psi} - \psi_t\tau_{xxxy} - \psi_{xxxt}\tau_y - \psi_y\phi_{xxxy} - {\psi_x}^4\xi_{y\psi\psi\psi}\\
&-\psi_{xxxy}\phi_y + \eta_{xxx\psi}\psi_y + \psi_{xxx}\eta_{y\psi} - \phi_{xxx\psi}{\psi_y}^2 - \psi_t{\psi_x}^3\tau_{\psi\psi\psi\psi}\psi_y - \psi_t\psi_{xxx}\tau_{\psi\psi}\psi_y - 4\xi_\psi\psi_x\psi_{xxxy}\\
&-3\psi_{xyy}{\psi_x}^2\phi_{\psi\psi} - 9\xi_{x\psi}\psi_{xy}\psi_{xx} - 3\psi_{xxy}\psi_x\phi_{y\psi} - \psi_{yy}\phi_\psi\psi_{xxx} - \psi_y\psi_{xxx}\phi_{y\psi} - 3\psi_{xy}\phi_{xxy}\\
&-\psi_{yt}{\psi_x}^3\tau_{\psi\psi\psi} - \psi_{yy}{\psi_x}^3\phi_{\psi\psi\psi} - \psi_{xxxt}\tau_\psi\psi_y - \psi_t\tau_{xxx\psi}\psi_y - \psi_y{\psi_x}^3\phi_{y\psi\psi\psi} - 6\xi_\psi\psi_{xx}\psi_{xxy}\\
&-{\psi_y}^2{\psi_x}^3\phi_{\psi\psi\psi\psi} - \psi_x\xi_{xxx\psi}\psi_y + \psi_{xxx}\eta_{\psi\psi}\psi_y + {\psi_x}^3\eta_{\psi\psi\psi\psi}\psi_y - {\psi_x}^4\xi_{\psi\psi\psi\psi}\psi_y - 3\psi_{yt}\psi_{xx}\tau_{x\psi}\\
&-\psi_t\psi_{xxx}\tau_{y\psi} - {\psi_y}^2\psi_{xxx}\phi_{\psi\psi} - \psi_t\tau_\psi\psi_{xxxy} - \psi_{yt}\tau_\psi\psi_{xxx} + 6\psi_x\eta_{x\psi\psi}\psi_{xy} - 4\psi_x\psi_{xxx}\xi_{y\psi}\\
&+\eta_\psi\psi_{xxxy} - \psi_{xxxx}\xi_y - \psi_{xxxx}\xi_\psi\psi_y + \eta_{xxxy} - 3\psi_{xyy}\phi_{xx} - 3\psi_{xxyy}\phi_x - 3\psi_{xxy}\xi_{xx} - 3\psi_{xxxy}\xi_x\\
&-3{\psi_x}^2\xi_{xx\psi\psi}\psi_y + 3\eta_{xx\psi}\psi_{xy} - 3\psi_{xxyt}\tau_x + 3{\psi_x}^2\eta_{xy\psi\psi} - 3{\psi_{xx}}^2\xi_{y\psi} - 3\psi_{xxx}\xi_{xy} - 3\psi_{xx}\xi_{xxy}\\
&+3\eta_{x\psi}\psi_{xxy} - 3{\psi_x}^2\xi_{xxy\psi} - \psi_x\xi_{xxxy} - \psi_{yy}\phi_{xxx} - \psi_{yt}\tau_{xxx} - \psi_{xy}\xi_{xxx} + 3\psi_{xx}\eta_{xy\psi} - 3\psi_{xt}\tau_{xxy}\\
&-12\psi_y\psi_x\phi_{x\psi\psi}\psi_{xy} - 12\psi_x\xi_{\psi\psi}\psi_{xx}\psi_{xy} - 6\psi_t\psi_x\tau_{x\psi\psi}\psi_{xy} - 3{\psi_y}^2\psi_x\psi_{xx}\phi_{\psi\psi\psi} - 6{\psi_{xy}}^2\phi_{x\psi}\\
&-3\psi_y\psi_x\psi_{xx}\phi_{y\psi\psi} - 3\psi_t\psi_{xx}\tau_{x\psi\psi}\psi_y - 3\psi_t{\psi_x}^2\tau_{x\psi\psi\psi}\psi_y - 3\psi_t\psi_x\tau_{xx\psi\psi}\psi_y - 9\psi_x\psi_{xx}\xi_{xy\psi}\\
&-3\psi_t\psi_x\psi_{xx}\tau_{y\psi\psi} - 6{\psi_x}^2\psi_{xx}\xi_{\psi\psi\psi}\psi_y - 9\psi_x\psi_{xx}\xi_{x\psi\psi}\psi_y - 3\psi_{xxt}\psi_x\tau_{\psi\psi}\psi_y + 3\eta_{\psi\psi}\psi_x\psi_{xxy}\\
&+3\psi_x\psi_{xx}\eta_{\psi\psi\psi}\psi_y - 4\psi_x\psi_{xxx}\xi_{\psi\psi}\psi_y - 3\psi_{yy}\psi_x\phi_{\psi\psi}\psi_{xx} - 6\psi_y\psi_{xy}\phi_{\psi\psi}\psi_{xx} + 3\psi_x\eta_{xxy\psi}\\
&-6\psi_y\psi_x\phi_{\psi\psi}\psi_{xxy} - 3\psi_{yt}\psi_x\tau_{\psi\psi}\psi_{xx} - 3\psi_t\psi_{xy}\tau_{\psi\psi}\psi_{xx} - 3\psi_t\psi_x\tau_{\psi\psi}\psi_{xxy} - 9\xi_{x\psi}\psi_x\psi_{xxy}\\
&-3\psi_t{\psi_x}^2\tau_{\psi\psi\psi}\psi_{xy} - 6\psi_y{\psi_x}^2\phi_{\psi\psi\psi}\psi_{xy} - 6\psi_{xt}\psi_x\tau_{\psi\psi}\psi_{xy} - 6\psi_{xt}\psi_x\tau_{x\psi\psi}\psi_y - 3\psi_{xxt}\tau_{xy}\\
&-3\psi_{xt}{\psi_x}^2\tau_{\psi\psi\psi}\psi_y - 3\psi_{xt}\psi_{xx}\tau_{\psi\psi}\psi_y - 3\psi_{xxy}\phi_{xy} - 3\psi_t\psi_x\tau_{xxy\psi} - 3\psi_{xxx}\xi_{x\psi}\psi_y - 3{\psi_x}^3\xi_{x\psi\psi\psi}\psi_y\\
&-\psi_t\psi_x\psi_{xx}\tau_{\psi\psi\psi}\psi_y - 3{\psi_x}^3\xi_{xy\psi\psi} - 6\psi_{xt}\psi_{xy}\tau_{x\psi}
\end{aligned}
\tag{A.1.11}
$$

A.2 Extended Infinitesimals for (1+1)-dimensional PDEs

The following extended infinitesimals for $\phi^{xx}, \phi^{tt}, \eta^t, \eta^{xx}, \psi^t, \psi^{xx}$ can be obtained by using the equations (1.2.35) acting on space $(x, t, u(x,t), a(x,t), b(x,t))$ and $\xi, \tau, \phi, \psi, \eta$ are infinitesimals corresponding to x, t, u, a, b:

$$
\begin{aligned}
\phi^{xx} &= -3\xi_u u_x u_{xx} - 2u_t b_x\tau_{xb} - u_t {b_x}^2\tau_{bb} - 2{u_x}^2 b_x\xi_{ub} - 2u_x a_x\xi_{xa} - 2u_{xt}\tau_b b_x + 2u_x\phi_{ua}a_x\\
&+2b_x\phi_{ab}a_x - 2u_t u_x\tau_{xu} - 2u_{xt}\tau_a a_x - u_t\tau_a a_{xx} - u_t {a_x}^2\tau_{aa} - 2{u_x}^2 a_x\xi_{ua} - u_t{u_x}^2\tau_{uu} - u_t\tau_b b_{xx}\\
&-2u_{xx}\xi_b b_x - 2u_x b_x\xi_{xb} - u_x\xi_b b_{xx} - u_x\xi_a a_{xx} - 2u_{xx}\xi_a a_x - u_t\tau_u u_{xx} + 2b_x\phi_{ub}u_x - 2u_t a_x\tau_{xa}\\
&-u_t\tau_{xx} + \phi_{xx} - {u_x}^3\xi_{uu} + \phi_{bb}{b_x}^2 - u_x\xi_{xx} + \phi_{aa}{a_x}^2 + 2\phi_{xu}u_x + 2\phi_{xa}a_x + \phi_{uu}{u_x}^2 - 2u_{xt}\tau_x\\
&-2u_t u_x\tau_{ub}b_x - 2u_x b_x\xi_{ab}a_x - 2u_t a_x\tau_{ab}b_x + \phi_b b_{xx} + \phi_u u_{xx} + \phi_a a_{xx} - 2u_{xx}\xi_x - u_x{a_x}^2\xi_{aa}\\
&+2\phi_{xb}b_x - u_x{b_x}^2\xi_{bb} - 2{u_x}^2\xi_{xu} - 2u_t u_x\tau_{ua}a_x - 2u_{xt}\tau_u u_x
\end{aligned}
\tag{A.2.1}
$$

$$\begin{aligned}\phi^{tt} &= -2u_{tt}\tau_a a_t - 2u_x u_t \xi_{tu} - u_x {a_t}^2 \xi_{aa} + 2b_t \phi_{ub} u_t - 2u_{xt}\xi_a a_t - u_x \xi_a a_{tt} - 3\tau_u u_t u_{tt} \\ &-u_x {u_t}^2 \xi_{uu} - u_t \tau_a a_{tt} - 2u_t b_t \tau_{tb} - 2u_x a_t \xi_{ta} - u_x \xi_u u_{tt} - 2u_{tt}\tau_b b_t - 2{u_t}^2 b_t \tau_{ub} - u_t {b_t}^2 \tau_{bb} \\ &-2u_t a_t \tau_{ta} - u_x \xi_b b_{tt} - u_x {b_t}^2 \xi_{bb} - u_t \tau_b b_{tt} + 2a_t \phi_{ua} u_t - u_t {a_t}^2 \tau_{aa} - 2{u_t}^2 a_t \tau_{ua} - 2u_{xt}\xi_b b_t \\ &+2b_t \phi_{ab} a_t - 2u_x b_t \xi_{tb} + \phi_{tt} - 2u_t b_t \tau_{ab} a_t - 2u_x u_t \xi_{ua} a_t - 2u_x u_t \xi_{ub} b_t - 2u_x b_t \xi_{ab} a_t - u_t \tau_{tt} \\ &+\phi_{bb} {b_t}^2 + \phi_{aa} {a_t}^2 + \phi_{uu} {u_t}^2 - u_x \xi_{tt} - 2u_{xt}\xi_t + 2\phi_{tb} b_t + 2\phi_{ta} a_t + 2\phi_{tu} u_t - 2\tau_{tu} {u_t}^2 - 2u_{tt}\tau_t \\ &+\phi_u u_{tt} + \phi_b b_{tt} - 2u_{xt}\xi_u u_t + \phi_a a_{tt} - {u_t}^3 \tau_{uu}\end{aligned} \tag{A.2.2}$$

$$\begin{aligned}\eta^t &= \eta_b b_t - b_t \tau_a a_t - b_t \tau_t + \eta_t - \tau_b {b_t}^2 + \eta_u u_t - b_x \xi_u u_t - b_x \xi_a a_t + \eta_a a_t - b_x \xi_b b_t \\ &-b_x \xi_t - b_t \tau_u u_t\end{aligned} \tag{A.2.3}$$

$$\begin{aligned}\eta^{xx} &= -2b_x a_x \xi_{xa} - 2b_t a_x \tau_{xa} + 2u_x \eta_{ub} b_x - 2b_{xt}\tau_u u_x - 2b_{xt}\tau_b b_x - 2b_t u_x \tau_{xu} + \eta_{xx} - 3\xi_b b_x b_{xx} \\ &-b_t \tau_b b_{xx} - 2b_{xt}\tau_a a_x + 2a_x \eta_{ua} u_x - b_x \xi_u u_{xx} + 2a_x \eta_{ab} b_x - 2b_{xx}\xi_a a_x - b_x \xi_a a_{xx} - 2b_t b_x \tau_{xb} \\ &-2{b_x}^2 u_x \xi_{ub} - 2b_x u_x \xi_{xu} - 2b_t u_x \tau_{ua} a_x - 2b_t u_x \tau_{ub} b_x - 2b_x a_x \xi_{ua} u_x - 2b_t b_x \tau_{ab} a_x - 2b_{xx}\xi_x \\ &+\eta_a a_{xx} + \eta_b b_{xx} - b_x \xi_{xx} - 2b_{xt}\tau_x - 2\xi_{xb} {b_x}^2 + 2\eta_{xu} u_x + \eta_{aa} {a_x}^2 + 2\eta_{xa} a_x - {b_x}^3 \xi_{bb} + \eta_{uu} {u_x}^2 \\ &+\eta_{bb} {b_x}^2 + 2\eta_{xb} b_x - b_t \tau_{xx} - 2{b_x}^2 a_x \xi_{ab} - b_x {u_x}^2 \xi_{uu} - b_t {b_x}^2 \tau_{bb} - b_x {a_x}^2 \xi_{aa} - b_t {a_x}^2 \tau_{aa} \\ &-b_t {u_x}^2 \tau_{uu} - b_t \tau_u u_{xx} - b_t \tau_a a_{xx} - 2b_{xx}\xi_u u_x + \eta_u u_{xx}\end{aligned} \tag{A.2.4}$$

$$\begin{aligned}\psi^t &= -a_t \tau_t - a_x \xi_t + \psi_t + \psi_u u_t - a_x \xi_u u_t - a_x \xi_a a_t - a_x \xi_b b_t + \psi_a a_t + \psi_b b_t - a_t \tau_b b_t \\ &-\tau_a {a_t}^2 - a_t \tau_u u_t\end{aligned} \tag{A.2.5}$$

$$\begin{aligned}\psi^{xx} &= -2a_{xt}\tau_b b_x - a_t \tau_u u_{xx} - a_x \xi_b b_{xx} - a_x {u_x}^2 \xi_{uu} - a_t \tau_a a_{xx} - a_t \tau_b b_{xx} - a_x \xi_u u_{xx} \\ &-2a_{xx}\xi_u u_x + \psi_{xx} - 2a_{xx}\xi_x + \psi_a a_{xx} + \psi_u u_{xx} - a_x \xi_{xx} + \psi_{aa} {a_x}^2 - {a_x}^3 \xi_{aa} + \psi_{bb} {b_x}^2 + \psi_{uu} {u_x}^2 \\ &-a_t \tau_{xx} + 2\psi_{xu} u_x - 2\xi_{xa} {a_x}^2 - 2a_{xt}\tau_x + 2\psi_{xa} a_x + 2\psi_{xb} b_x + \psi_b b_{xx} - 2a_x b_x \xi_{ub} u_x - 2a_t b_x \tau_{ub} u_x \\ &-2a_t b_x \tau_{ab} a_x - 2a_t a_x \tau_{ua} u_x - 2a_{xt}\tau_u u_x - 2a_{xx}\xi_b b_x - a_t {a_x}^2 \tau_{aa} - {u_x}^2 a_t \tau_{uu} - a_t {b_x}^2 \tau_{bb} \\ &-2a_{xt}\tau_a a_x - a_x {b_x}^2 \xi_{bb} - 2u_x a_t \tau_{xu} + 2u_x \psi_{ub} b_x + 2b_x \psi_{ab} a_x - 2{a_x}^2 u_x \xi_{ua} - 2a_t b_x \tau_{xb} - 2a_x \xi_{xb} b_x \\ &-2a_x \xi_{xu} u_x + 2u_x \psi_{ua} a_x - 2a_t a_x \tau_{xa} - 2{a_x}^2 b_x \xi_{ab} - 3\xi_a a_x a_{xx}\end{aligned} \tag{A.2.6}$$

On using the equations (1.2.35), the following extended infinitesimals for $\eta^t, \eta^x, \eta^{xx}, \eta^{tt}$, $\eta^{xxx}, \eta^{xxt}, \eta^{xxxx}$ can be obtained for dependent variable $u(x,t)$ and ξ, τ, η are infinitesimals corresponding to x, t, u:

$$\eta^t = \eta_t + (\eta_u - \tau_t)\,u_t - \xi_t u_x - \tau_u {u_t}^2 - \xi_u u_x u_t \tag{A.2.7}$$

$$\eta^x = \eta_x + (\eta_u - \xi_x)\,u_x - \tau_x u_t - \xi_u {u_x}^2 - \tau_u u_x u_t \tag{A.2.8}$$

$$\begin{aligned}\eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})\,u_x - \tau_{xx} u_t + (\eta_{uu} - 2\xi_{xu})\,{u_x}^2 - 2\tau_{xu} u_x u_t + (\eta_u - 2\xi_x)\,u_{xx} \\ &-2\tau_x u_{tx} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{tx} - \tau_{uu} {u_x}^2 u_t - \xi_{uu} {u_x}^3\end{aligned} \tag{A.2.9}$$

$$\begin{aligned}\eta^{tt} &= \eta_u u_{tt} - 2\tau_{tu} {u_t}^2 - 2u_{tt}\tau_t - u_x \xi_{tt} + 2\eta_{tu} u_t - u_t \tau_{tt} - 3\tau_u u_t u_{tt} + \eta_{uu} {u_t}^2 - {u_t}^2 u_x \xi_{uu} \\ &-{u_t}^3 \tau_{uu} - 2u_{xt}\xi_u u_t - 2u_x \xi_{tu} u_t - 2u_{xt}\xi_t - u_x \xi_u u_{tt} + \eta_{tt}\end{aligned} \tag{A.2.10}$$

$$\begin{aligned}
\eta^{xxx} &= \eta_{xxx} + \left(3\eta_{xxu} - \xi_{xxx}\right) u_x - \tau_{xxx} u_t - 3\tau_{xxu} u_x u_t + 3\left(\eta_{xuu} - \xi_{xxu}\right) {u_x}^2 - 3\tau_{xuu} u_t {u_x}^2 \\
&+ \left(\eta_{uuu} - 3\xi_{xuu}\right) {u_x}^3 - \tau_{uuu} {u_x}^3 u_t - \xi_{uuu} {u_x}^4 - 3\tau_{xx} u_{tx} + 3\ \left(\eta_{xu} - \xi_{xx}\right) u_{xx} - 3\tau_{xu} u_t u_{xx} \\
&+3\ \left(\eta_{uu} - 3\,\xi_{xu}\right) u_x u_{xx} - 6\tau_{xu} u_x u_{tx} - 6\xi_{uu} {u_x}^2 u_{xx} - 3\tau_{uu} u_{xt} {u_x}^2 - 3\tau_{uu} u_{xx} u_x u_t \\
&-3\,\xi_u {u_{xx}}^2 - 3\,\tau_u u_{xx} u_{tx} - 3\,\tau_x u_{txx} + \left(\eta_u - 3\,\xi_x\right) u_{xxx} - 4\xi_u u_x u_{xxx} - \tau_u u_{xxx} u_t - 3\tau_u u_{xxt} u_x
\end{aligned} \tag{A.2.11}$$

$$\begin{aligned}
\eta^{xxt} &= -3\xi_u u_{tx} u_{xx} - 2\tau_{xu} u_{tt} u_x - \tau_{uu} u_{xx} {u_t}^2 - 3\xi_{tu} u_x u_{xx} + \left(\eta_{uu} - 2\xi_{xu} - \tau_{tu}\right) u_{xx} u_t \\
&-\tau_{xx} \phi_{tt} - 3\xi_u u_x u_{txx} - 2\tau_x u_{ttx} + \left(2\eta_{xtu} - \xi_{xxt}\right) u_x - \xi_t u_{xxx} - 3\xi_{uu} u_t u_x u_{xx} - 4\tau_{uu} u_x u_t u_{tx} \\
&-\tau_{xxu} {u_t}^2 - \tau_{xuu} u_x {u_t}^2 + \left(\eta_u - 2\xi_x - \tau_t\right) u_{txx} - 2\,\tau_u {u_{xt}}^2 + \eta_{xxt} + \left(2\eta_{xuu} - \xi_{xxu} - 2\tau_{xtu}\right) u_x u_t \\
&-\tau_{uuu} {u_x}^2 {u_t}^2 - 2\tau_u u_t u_{txx} - \xi_u u_t u_{xxx} - 2\tau_u u_x u_{ttx} - \tau_{uu} {u_x}^2 u_{tt} + \left(2\eta_{uu} - \xi_{xx} - 2\tau_{tx}\right) u_{tx} \\
&-\tau_u u_{xx} u_{tt} + \left(\eta_{xxu} - \tau_{xxt}\right) u_t + \left(\eta_{uuu} - 2\xi_{xuu} - \tau_{tuu}\right) {u_x}^2 u_t - \xi_{uuu} {u_x}^3 u_t - 4\tau_{xu} u_t u_{xt} \\
&+ \left(\eta_{uut} - 2\xi_{xtu}\right) {u_x}^2 - \xi_{uut} {u_x}^3 + \left(\eta_{tu} - 2\xi_{tx}\right) u_{xx} + 2\left(\eta_{uu} - 2\xi_{xu} - \tau_{tu}\right) u_x u_{tx}
\end{aligned} \tag{A.2.12}$$

$$\begin{aligned}
\eta^{xxxx} &= 4\eta_{uu} u_x u_{xxx} - 4u_t {u_x}^3 \tau_{xuuu} - 4u_{xxxt} \tau_u u_x - 5\xi_u u_x u_{xxxx} + 6{u_x}^2 \eta_{uuu} u_{xx} - 10\xi_u u_{xx} u_{xxx} \\
&-4u_{xt} {u_x}^3 \tau_{uuu} - 6u_t u_{xx} \tau_{xxu} - 24{u_x}^2 \xi_{xuu} u_{xx} - 4u_t u_x \tau_{xxxu} - 6u_t {u_x}^2 \tau_{xxuu} - 4u_{xt} \tau_u u_{xxx} \\
&+\eta_{xxxx} - 12u_{xt} u_x \tau_{xxu} + 12u_x \eta_{xuu} u_{xx} - 12u_t u_x \tau_{xuu} u_{xx} - 12u_{xt} u_x \tau_{uu} u_{xx} - 4u_t u_x \tau_{uu} u_{xxx} \\
&-3u_t {u_{xx}}^2 \tau_{uu} - 10{u_x}^2 \xi_{uu} u_{xxx} - 10{u_x}^3 \xi_{uuu} u_{xx} - u_t \tau_u u_{xxxx} - 6u_{xxt} {u_x}^2 \tau_{uu} - 12u_{xt} \tau_{xu} u_{xx} \\
&-18u_x \xi_{xxu} u_{xx} - 6u_t {u_x}^2 \tau_{uuu} u_{xx} - 12u_{xt} {u_x}^2 \tau_{xuu} - 12\xi_{xu} {u_{xx}}^2 - 4{u_x}^4 \xi_{xuuu} \\
&-u_x \xi_{xxxx} + 4\eta_{xxxu} u_x - 4u_{xt} \tau_{xxx} + 4{u_x}^3 \eta_{xuuu} - 4\xi_{xxxu} {u_x}^2 - 6{u_x}^3 \xi_{xxuu} + 6{u_x}^2 \eta_{xxuu} \\
&-4u_{xx} \xi_{xxx} - 6u_{xxt} \tau_{xx} - 4u_{xxxt} \tau_x - 6u_{xxt} \tau_u u_{xx} + {u_x}^4 \eta_{uuuu} - {u_x}^5 \xi_{uuuu} - u_t \tau_{xxxx} \\
&-6u_{xxx} \xi_{xx} - 4u_{xxxx} \xi_x + 6\eta_{xxu} u_{xx} + \eta_u u_{xxxx} + 4\eta_{xu} u_{xxx} + 3\eta_{uu} {u_{xx}}^2 - 12u_{xxt} \tau_{xu} u_x \\
&-15u_x \xi_{uu} {u_{xx}}^2 - 16\xi_{xu} u_x u_{xxx} - u_t {u_x}^4 \tau_{uuuu} - 4u_t \tau_{xu} u_{xxx}
\end{aligned} \tag{A.2.13}$$

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