STUDIES ON CONVOLUTION AND CORRELATION THEOREMS FOR THE LINEAR CANONICAL TRANSFORM AND THEIR APPLICATIONS IN SIGNAL PROCESSING

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CERTIFICATE

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The results obtained in this thesis have not been submitted in part or full to any other institute or university for the award of degree or diploma.

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“What you make me know,
That alone I know,
What you make me see,
That alone I see.”

(Navdeep Goel)

ABSTRACT

THE MAIN AIM OF THE PROPOSED WORK is to provide an inclusive approach towards the introduction to the principles and applications of the product and convolution theorem in the linear canonical transform (LCT) domain. As a generalization of fractional Fourier transform (FRFT), Fresnel transform (FST) and Fourier transform (FT), the LCT is a three variable class of integral transform and has been used in many fields of optics and signal processing. The LCT has proved to be a powerful tool for the analysis of time-varying signals by representing rotation of a signal in the time-frequency plane. In the applications, where FT and fractional domain concepts are used, the performance can be enhanced through the use of LCT because of its three extra degrees of freedom as compared to one degree of freedom for FRFT and no degree of freedom for FT.

Many properties of the LCT are currently well known, including sampling, uncertainty principle, product and convolution theorems, which are generalization of the corresponding properties of the FT and FRFT. The product and convolution theorems for the LCT available in the literature, however these do not generalize very nicely to the classical result for the FT and FRFT.

The proposed work can be divided into two broader segments. The first segment includes, the efforts made in establishing the LCT a complete integral transform by developing and deriving the weighted convolution and correlation identities. Also the proposed definitions of these theorems are compared with existing ones and their superiority has been determined with the help of some newly devised performance metrics. The second segment comprises of the applications of proposed identities along with some new application areas of the LCT.

In the first phase, a comprehensive closed-form analytical expression of the behavior of Dirichlet, Generalized Hamming and triangular window functions is established, utilizing various special mathematical functions in the LCT domain. It has been shown that the LCT of Dirichlet, Generalized Hamming and triangular window functions is directly dependent on the LCT variables (a,b,c,d), thus exhibiting the flexibility of various applications in signal processing. Based upon the window analysis, by selecting different LCT variables as tuning parameter in the convolution operation between LCT of window function and ideal frequency response, variability in the transition band of the resulting Hamming window based low pass

FIR filter response has been achieved. Then the closed form analytic expression of the pass-stop band filter has been established in the LCT domain.

The beneficial role of the LCT in the filtering application lies in the capability of the LCT in localizing the non-stationary (chirp) signals in time-frequency plane. This ascertains the superiority of LCT domain filtering over frequency domain filtering and fractional domain filtering in the case of overlapping bandlimited signal and noise. To establish the legacy of the LCT under such circumstances, the proposed weighted convolution theorem has been used to carry out the multiplicative filtering. It has been noticed that in the proposed weighted convolution theorem and even in the theorems available in the literature, consist of an undesirable extra chirp function in the derived results. To eradicate the effect of this chirp function, the LCT takes the advantage of its extra three degrees of freedom as compare to one degree of freedom for FRFT and none for FT. With the help of simulation, it has been shown that in LCT domain filtering, mean square error (MSE) is minimum for different values of signal-to-noise ratio (SNR) as compared to fractional domain filtering and frequency domain filtering. Hence the proposed convolution theorem is best suitable for filtering action as compare to fractional domain filtering.

Based upon the proposed methodology used to derive the convolution theorem, an improved correlation theorem has been developed and for different values of LCT variables. The superiority of the proposed correlation theorem has been shown based upon the computational complexity and simulation comparison. The simulation comparison shows that the plot of theorem derived in the literature is more oscillatory because of more chirp functions are included to derive the correlation integral. Thereafter, the proposed weighted auto-correlation theorem for LCT has been applied in power spectral density analysis of the frequency modulated (FM) wave and it has been found that for different values of LCT variables, the bandwidth of the FM wave shrinks and concludes that the same FM signal may be transmitted with less bandwidth requirement.

Finally, a through introduction has been given to a more powerful integral transform, known as offset LCT (OLCT) or Special Affine Fourier Transform (SAFT). As a generalization of LCT, offset FRFT (OFRFT), FRFT, FST and FT, the OLCT is a six variable class of integral transform and has been used in many fields of optics and signal processing. Apart from the LCT variables, it has two extra variables called time-shifting and frequency-modulation variables. These parameters are helpful to move the time-frequency representation in horizontal direction, vertical direction or combination of both. Further it has been clearly shown from the simulation results that time-shifting and frequency-modulation variables play an important role to approach the required results.

Finally, the proposed theorems have proven the efficacy of the LCT and motivated to develop more application in future.

LIST OF PUBLICATIONS

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ACRONYMS AND ABBREVIATIONS

1-D	One Dimensional
Abs	Absolute
AF	Ambiguity Function
AFT	Affine Fourier Transform
BW	Band Width
CCR	Canonical Correlation
CCV	Canonical Convolution
CLT	Central Limit Theorem
DSP	Digital Signal Processing
FIR	Finite Impulse Response
FM	Frequency Modulated
FRFT	Fractional Fourier Transform
FSLL	First Side-Lobe Level
FST	Fresnel Transform
FT	Fourier Transform
HMLW	Half Main-Lobe Width
Im	Imaginary
IIR	Infinite Impulse Response
IWOP	Integration Within Ordered Products
LCST	Linear Canonical S Transform
LCT	Linear Canonical Transform
LHS	Left Hand Side
LTI	Linear Time Invariant
MSE	Mean Square Error
MSLL	Maximum Side Lobe Level
OFT	Offset Fourier Transform
OLCT	Offset Linear Canonical Transform
PSLL	Peak Side-Lobe Level
Re	Real

RHS	Right Hand Side
SAFT	Special Affine Fourier Transform
SLFOR	Side-Lobe Fall-Off Rate
SNR	Signal to Noise Ratio
STFT	Short-Time Fourier Transform
WD	Wigner Distribution
WDF	Wigner Distribution Function
WT	Wavelet Transform
WVD	Wigner Ville Distribution

GLOSSARY OF SYMBOLS

$\otimes$	Convolution Operation
$\star$	Correlation Operation
$\tilde{a}$	Fractional Fourier Order Variable
α	Fractional Fourier Transform Variable
(a,b,c,d)	Linear Canonical Transform Variable
$\langle\alpha\vert$	Bra Vector
$\vert\alpha\rangle$	Ket Vector
(a,b,c,d,m,n)	Offset Linear Canonical Transform Variable
*	Complex Conjugate
erf	Error Function
erfi	Imaginary Error Function
$L_F^{(a,b,c,d)}$	LCT of the Function
$L_F^{(a,b,c,d,m,n)}$	OLCT of the Function
$\otimes_{(a,b,c,d)}$	Canonical Convolution
$\star_{(a,b,c,d)}$	Canonical Correlation

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CHAPTER 1
INTRODUCTION

Science is a powerful way of understanding the natural world through a process of observation, experimentation, and analysis. Science may set limits to knowledge, but should not set limits to imagination. In other words, the key to the growth of the science is the introduction of available literature with their future possibilities.

-Anonymous

1.1 GENERAL

THE transform is a technique to convert a signal from one domain to another domain for extracting the hidden information contained in the signal which cannot be extracted from the signal in first domain. The time-domain representation of a signal gives the information about signal's amplitude variation with respect to time but it tends to hide the information about frequency components present in the signal. When Fourier Transform (FT) is taken of the time-domain signal, the resultant signal is transformed in frequency-domain, known as spectrum, which gives the information about the frequency components present in the signal along with the amplitude associated with each frequency component. One of the important families of transform is integral transform. Integral transform is an operator used to transform a signal into its equivalent form by integrating the product of signal in time-domain with kernel of the transformation. Mathematically-

$$F(u)=\int_{a}^{b} f(t)K(t,u)\,\mathrm{d}t \qquad \textbf{(1.1.1)}$$

This is called a Fredholm equation of the first kind or an integral transform **[47]**. The bivariate function $K(t,u)$ is called the kernel of the integral equation. The value of limits a and b depends upon the definition of the corresponding transform. The word integral in the name "Integral Transform" is because of the integration function is involved mathematically.

The family of integral transform constitutes of many important transforms like: FT, Fractional Fourier Transform (FRFT), Linear Canonical Transform (LCT), Laplace transform, Hartley transform, Mellin transform, Hilbert transform, Hankel transform etc. In the family of integral transform, the FT, given by Jean-Baptiste-Joseph-Fourier (1768-1830), is most widely used technique for the application area of signal processing and communication **[129]**. Any transform to be a complete transform, it should have closed form expression for transform and its inverse transform as well as it should satisfy certain properties and theorems. Convolution is one of the most important operation for any integral transform used mainly for filtering application. Due to the limitation of FT that gives rotation to the time-frequency plot in multiples of $\pi/2$, LCT which has three free variables can be effectively utilized. In the following section, a brief history of the LCT is presented.

1.2 HISTORICAL DEVELOPMENT OF LINEAR CANONICAL TRANSFORM

The LCT **[23, 102-103, 139-140, 148]** of a given function is a three parameter class of linear integral transform. LCT was first introduced in 1970s **[102, 129]**. In 1979, Wolf gave a systematically introduction for LCT and a brief overview of LCT's origin in quantum mechanics may be found in **[79]**. The name "linear canonical transformation" is from canonical transformation, a map that preserves the symplectic structure, as $SL_2(R)$ can also be interpreted as the symplectic group Sp_2, and thus LCTs are the linear maps of the time–frequency domain which preserve the symplectic form **[62, 71, 75]**. Well-known transforms such as the FT, FRFT, and the Fresnel Transform (FST) are some of the special cases of the LCT **[55, 90, 127]** but some special cases of LCT with complex variables were introduced in 1961 **[153]**.

In literature, LCT is also known as Affine Fourier transform (AFT) **[132]**, ABCD transform **[92]**, generalized FST **[27]**, Collins formula **[130]**, quadratic phase systems **[99-101]**, generalized Huygens integral **[14]**, or extended FRFT **[66]**.

LCT was first used for solving differential equations and optical system analysis **[149]**. But now-a-days, LCT has been shown to be a powerful tool for optics and signal processing.

Comparing to the FRFT with one extra degree of freedom and the FT without a parameter, LCT is more flexible for its extra three degrees of freedom, and has found many applications in filter design, signal synthesis, time- frequency analysis, encryption, modulation, capacity analysis, window function analysis and multiplexing in communications etc. **[3, 8, 10, 19, 26, 33, 34, 48, 49, 91, 110, 117, 124, 141, 144]**. These applications demonstrate the ability as well as the potential of LCT in signal processing.

Many properties of the LCT are currently well known **[20, 38, 55, 139, 150]**, including convolution and correlation theorems **[20, 36-38]**, sampling **[9, 15, 16, 20, 21, 68, 125, 161]**, uncertainty principle **[17, 77, 78, 82, 157-159]**, which are generalization of the corresponding properties of the FT and FRFT **[5, 54, 81, 89, 136]**. The basic theories of the LCT have been developed including discrete approximations to the transforms **[22, 45, 69, 134]**, and so on, which enrich the theoretical system of the LCT. The product and convolution theorems for the LCT available in the literature, however these do not generalize very nicely to the classical result for the FT and FRFT.

1.3 CONVOLUTION AND CORRELATION OPERATIONS

The first occurrence of the convolution integral took place in the year 1754 when the mathematician Jean-le-Rond D'Alembert (1717-1783) derived Taylor's expansion theorem **[96]**. Reiff **[122]** and Nielsen **[107]** pointed out that the series given by D'Alembert was without naming the work of Taylor. Nielsen also pointed out that Jean-Antoine-Nicolas Caritat de Condorcet (1743-1794) used to designate series as 'theoreme de D'Alembert'. Finally in 1901-1908, Burkhardt **[52]** gave the convolution expression of the type-

$$f(x)\otimes g(x)=\int f(u)\,g(x-u)\,\mathrm{d}u \qquad \textbf{(1.3.1)}$$

In mathematics and, in particular, functional analysis, convolution is a mathematical operator, on two functions f and g, producing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated. The main use of convolution is in describing the output of a linear time-invariant (LTI) system **[51]**. The input-output behaviour of an LTI system can be characterized with its impulse response, and the output of an LTI system for any input signal can be expressed as the convolution of the input signal with the system's impulse response. In other words, FT of convolution of two signals is the point wise product of FT of respective signals.

The classical definition of the convolution and product theorem of the FT for the signals $f(t)$ and $g(t)$ is given by-

Convolution: $f(t)\otimes g(t)\xleftrightarrow{FT}\sqrt{2\pi}F(\omega)G(\omega)$ **(1.3.2)**

Product: $f(t).g(t)\xleftrightarrow{FT}F(\omega)\otimes G(\omega)$ **(1.3.3)**

where, $F(\omega)$ and $G(\omega)$ are the FT of $f(t)$ and $g(t)$ respectively and the symbol $'\otimes'$ is the convolution operation. The correlation of two functions is no more than their convolution after one of the two functions has been axis-reversed **[2]**. In signal processing, correlation is a measure of degree of relation of two waveforms as a function of a time-lag applied to one of them. This is also known as a sliding dot product or sliding inner-product. It is an important operation in optics as well as in signal processing, pattern recognition and especially in detection applications **[29, 30, 111, 128]**. The classical definition of the correlation theorem of the FT for the signals $f(t)$ and $g(t)$ is given by-

Cross-correlation: $f(t)\star g(t)\xleftrightarrow{FT}\sqrt{2\pi}F(-\omega)G(\omega)$ **(1.3.4)**

Auto-correlation: $f(t)\star f(t)\xleftrightarrow{FT}\sqrt{2\pi}F(-\omega)F(\omega)$ **(1.3.5)**

where,“ $\star$ ” denotes the correlation operation.

1.4 DIRAC’S NOTATIONS FOR QUANTUM MECHANICAL REPRESENTATION

In quantum mechanics, bra-ket notation is a standard notation for describing quantum states, composed of angle brackets and vertical bars. The notation was introduced in 1939 by Paul Dirac **[112]** and is also known as Dirac notation. It can also be used to denote abstract vectors and linear functional in mathematics. Bra-ket notation is widespread in quantum mechanics: almost every phenomenon that is explained using quantum mechanics-including a large portion of modern physics is usually explained with the help of bra-ket notation. The Dirac bra-ket notation is a concise and convenient way to describe quantum states. The symbol

$$|\alpha\rangle$$

represents a quantum state. This is called a ket, or a ket vector. It is an abstract entity, and serves to describe the "state" of the quantum system. A physical system is in quantum state

α when represented by the ket, where, α represents some physical quantity. If there are two distinct quantum states $|\alpha_1\rangle$ and $|\alpha_2\rangle$, then the following ket-

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle$$

where, c_i is a complex number, is also a possible state for the system. In general, the number of linear independent kets required to express any other ket, is called the dimension of the vector space. In quantum mechanics the vector space of kets is usually non-countable infinite, known as Hilbert space.

Dirac defined something called a bra vector, designated by $\langle\alpha|$. This is not a ket, and does not belong in ket space e.g. $|\alpha\rangle + \langle\beta|$ has no meaning. However, for every ket $|\beta\rangle$, there exists a bra labelled $\langle\beta|$. The bra $\langle\gamma|$ is said to be the dual of the ket $|\gamma\rangle$ and the dual of the ket $c_1|\alpha_1\rangle + c_2|\alpha_2\rangle$ is-

$$c_1|\alpha_1\rangle + c_2|\alpha_2\rangle \Leftrightarrow c_1^*\langle\alpha_1| + c_2^*\langle\alpha_2|$$

where, $\Leftrightarrow$ signifies a dual correspondence. This is an anti-linear relation.
Dirac allowed the bra's and ket's to line up back to back, i.e.

$$\langle\alpha|\beta\rangle \equiv (|\alpha\rangle, |\beta\rangle)$$

The symbol $\langle\alpha|\beta\rangle$ represents a complex number that is equal to the value of the inner product of the ket $|\alpha\rangle$ with $|\beta\rangle$. According to the above definition, that-

$$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$$

Dirac also defined something called an outer product

$$|\alpha\rangle\langle\beta|$$

An outer product is allowed to stand next to a ket on its left, or next to a bra on the bra's right. Lets define $X = |\alpha\rangle\langle\beta|$, then if $|\psi\rangle$ is an arbitrary ket, one is allowed to construct

$$X|\psi\rangle = |\alpha\rangle\langle\beta||\psi\rangle$$

It looks like an inner product on the right of this equation. Indeed, according to the associative axiom of multiplication,

$$X|\psi\rangle = |\alpha\rangle\langle\beta||\psi\rangle = c|\alpha\rangle; \qquad c = (\langle\beta|\psi\rangle)$$

The outer product X is an operator in Hilbert space. It acts on ket $|\psi\rangle$ from the left and turns it into another ket $c|\alpha\rangle$. For $|\psi\rangle X$ has no meaning, however $\langle\psi|X$ does

$$\langle\psi|X = \langle\psi||\alpha\rangle\langle\beta| = (\langle\psi|\alpha\rangle)\langle\beta|; \quad d = \langle\alpha|\psi\rangle$$

If the operator A, operate on a ket $A|\alpha\rangle$, then the dual of $A|\alpha\rangle$ is

$$\langle\alpha|A^{\dagger} \Leftrightarrow A|\alpha\rangle$$

where, $A^{\dagger}$ is called the Hermitian conjugate of operator A.

Sometimes $A = A^{\dagger}$, then A is called a Hermitian operator. Hermitian operators play a central role in quantum theory. Consider a Hermitian operator X, whose eigenstate $|a\rangle$ obey the eigenvalue equation-

$$X|a\rangle = a|a\rangle$$

where, a is an eigenvalue. Suppose these eigenvalues are distinct, then set $\{|a_1\rangle|a_2\rangle......|a_n\rangle\}$ are mutually orthonormal and form a set of basis kets in Hilbert space, provided that-

$$\sum_n |a_n\rangle\langle a_n| = I$$

After the invention of Dirac's bra-ket notation, the need arises of operational rules for it. Dirac first used the "symbolic method" to represent the bra-ket notations and the completeness relation of Dirac's coordinate representation is an integration form **[112]**

$$\int_{-\infty}^{\infty} \mathrm{d}q|q\rangle\langle q| = 1, \qquad Q|q\rangle = q|q\rangle,$$

But the "symbolic method" does not tell us how to perform the integration (a ket-bra form). The use of Newton-Leibniz integration rule **[85]** was limited to commutative functions while operators made of Dirac's symbols in quantum mechanics are usually not commutative. Therefore, the invention of an innovative technique of Integration Within Ordered Product (IWOP) **[56]** of operators took place that made the integration of non-commutative operators possible. Therefore, by virtue of the newly developed technique of integration within an ordered product of operators, quantum version of classical transformations such as LCT, optical FST, Hankel transform, FRFT, Wigner transform, wavelet transform and Fresnel-Hadmard combinatorial transform etc. can be explored. With the help of quantum version of the transformation, various properties can be easily derived **[60, 108]**.

1.5 ORGANISATION OF THE THESIS

Seven chapters are presented in this thesis and the chapter-wise summary of the thesis is given below:

Chapter 2: Literature Survey

This chapter gives a comprehensive review of related literatures to provide background information on the issues to be considered in the thesis and to emphasize the relevance of the present study. It presents the preliminaries of convolution and correlation operations, FRFT, LCT and offset LCT (OLCT). It comprises the definitions and results for existing convolution and correlation theorems for FRFT/LCT/OLCT. To discuss LCT as a tool in time-frequency plane, relation between Wigner distribution function and LCT has been studied. It also contains the mathematical proof of special cases and properties satisfied by LCT. It also explores the use of quantum mechanical representation to derive the convolution theorem. Based upon the gaps found in the literature, objectives and methodology for the current work has been decided.

Chapter 3: The LCT of Window Functions and Analysis of FIR Filter

Following the rich literature review in Chapter 2, this chapter explores the LCT of the various window functions and analysis of Finite Impulse Response (FIR) filter. With the help of extensive mathematical computations, an analysis of Dirichlet, Generalized Hamming and Bartlett window functions in the LCT domain has been obtained so that final results are directly applicable for window based filtering and tuning of transition bandwidth (BW) applications. Further, an introduction to the Canonical Convolution (CCV) and Canonical Correlation (CCR) operations as well as relation between Wigner distribution function and mathematical aspect of filter design using CCV has been given. Finally a closed-form analytical expression of actual impulse response for pass-stop band filter has been obtained in LCT domain.

Chapter 4: Convolution and Product Theorem for LCT

In this chapter, first the advantageous role of convolution theorem is discussed followed by the convolution theorem defined for FT. Then the need of convolution theorem of LCT is highlighted for filtering of the non-stationary signals. A brief outline of the existing convolution theorem for LCT is also included. Subsequently, the weighted convolution and product theorems for LCT are proposed and based upon the different performance metrics, a simulation comparative analysis of proposed convolution theorem for LCT with existing theorems has been performed. Making use of proposed convolution theorem for LCT, practical applications of multiplicative filtering in the LCT domain has been discussed. The performance of multiplicative filter in the LCT domain has been compared with fractional domain filtering and frequency domain filtering in terms of mean square error (MSE) versus Signal to Noise Ratio (SNR).

Chapter 5: Correlation Theorem for LCT

In this chapter, the necessity of a correlation theorem for a transform is presented. Here, the existing definitions of correlation theorem defined in LCT domain is discussed in brief. Subsequently, the weighted cross-correlation and auto-correlation theorems for LCT are proposed and the various properties needed to satisfy by these identities are also derived. Then a comparative analysis is performed using simulation between the proposed definition and the definitions in the literature. Finally, power spectral density analysis of Frequency Modulated (FM) wave has been done by using the proposed correlation theorem.

Chapter 6: Convolution and Correlation Theorems for Offset LCT

In this chapter, convolution and correlation theorems for OLCT has been presented. Here, the existing definitions of convolution and correlation function or theorem defined in OLCT domain is discussed in brief. Subsequently, the weighted theorem for OLCT is proposed and the various properties needed to satisfy by these identities are also derived. Then a comparative analysis is performed between the proposed convolution and correlation theorems and existing ones. Further, with the help of simulation, the effect of time-shifting and frequency-shifting variables is clearly shown.

Finally, the conclusion of the thesis along with the possible future scope in the area of work is discussed in Chapter 7.

CHAPTER 2
LITERATURE SURVEY

A scholar needs to understand what has been done before, the strengths, weaknesses of existing studies and what they might mean.

--- *Booker and Beile, 2005*

Many properties of LCT were derived, developed or established earlier as described in previous chapter. This includes multiplication property, differentiation property, shifting property, modulation property, and few more. Since, the convolution theorem of the transform plays an important role in Digital Signal Processing (DSP), so it is extensively investigated always for the refinement to a well-accepted closed-form expression.

2.1 PRELIMINARIES OF CONVOLUTION AND CORRELATION OPERATIONS

In FT, the convolution theorem states that the FT of a convolution of two signals is the point wise product of respective FT of both the signals. In other words, convolution in one domain (e.g., time domain) equals point-wise multiplication in the other domain (e.g., frequency domain). The usefulness of convolution theorem can be best explained by its application in filtering. Since filtering can be performed both way i.e., time domain filtering and frequency domain filtering. Simultaneously, if the computational complexity is a basis parameter then it can be shown that under different input conditions one type of filtering has advantage over other **[28]** and vice-versa.

One of the inferences of convolution is the Central Limit Theorem (CLT). The CLT is an important tool in probability theory because it mathematically explains why Gaussian probability distribution is observed so commonly in nature. For example: the amplitude of thermal noise in electronic circuits follows a Gaussian distribution; the cross-sectional intensity of a laser beam is Gaussian; even the pattern of holes around a dart board bull's eye is Gaussian. In its simplest form, the CLT states that a Gaussian distribution results when the observed variable is the sum of many random processes, each with finite mean and variance. Even if the component processes do not have a Gaussian distribution, the sum of them will behave as Gaussian distribution. The CLT has an interesting implication for convolution. If a pulse-like signal is convolved with itself many times, a Gaussian is produced. Other applications of convolution are:

- In electrical engineering, the convolution of input signal with the impulse response gives the output of a LTI system. At any given moment, the output is an accumulated effect of all the prior values of the input function, with the most recent values typically having the most influence. Convolution amplifies or attenuates each frequency component of the input independent to the other components.
- In statistics, as noted above, a weighted moving average is a convolution.
- In probability theory, the probability distribution of the sum of two independent random variables is the convolution of their individual distributions.
- In optics, many kinds of "blur" are described by convolutions. A shadow (e.g., the shadow on the table when you hold your hand between the table and a light source) is the convolution of the shape of the light source that is casting the shadow and the object whose shadow is being cast. An out-of-focus photograph is the convolution of the sharp image with the shape of the iris diaphragm.
- Similarly, in digital image processing, convolution filtering plays an important role in many important algorithms in edge detection and related processes.
- In linear acoustics, an echo is the convolution of the original sound with a function representing the various objects that are reflecting it.
- In artificial reverberation (DSP, pro-audio), convolution is used to map the impulse response of a real room on a digital audio.
- In time-resolved fluorescence spectroscopy, the excitation signal can be treated as a chain of delta pulses, and the measured fluorescence is a sum of exponential decays from each delta pulse.
- In physics, wherever there is a linear system with a "superposition principle", a convolution operation makes an appearance.

- In computational fluid dynamics, the large eddy simulation turbulence model uses the convolution operation to lower the range of length scales necessary in computation thereby reducing computational cost.

For non-stationary signals and noise, the time and frequency domain filtering both fails because the signal and noise may have their respective Wigner distribution overlapping to each other in time and frequency domains. In this case, LCT based filtering can provide a better solution, where, for a rotated domain in time-frequency plane corresponding to an optimum value of angle parameter, the Wigner distribution of signal and noise may be separated. The filtering of the signal from the noise can be performed by designing a filter in FRFT domain **[123]** with this optimum angle parameter value. The correlation function is a mathematical operator, very similar to the convolution. Just as with convolution, correlation also uses two signals to produce a third signal in its own form. This third signal is called the cross-correlation of the two input signals. In signal processing, cross-correlation is a measure of similarity of two waveforms as a function of a time-lag applied to one of them. This is also known as a sliding dot product or inner-product. The correlation process and convolution process are identical, except for one minor difference. Whereas, the convolution involves reversing a signal, then shifting it and multiplying by another signal, correlation only involves shifting it and multiplying (no reversing). The convolution and correlation operation have some mathematical resemblance, but they have utilized very differently in signal processing applications. The convolution is the relationship between a system's input signal, output signal, and impulse response. On the other hand, the correlation is a way to detect a known waveform in a noisy background. The evaluation of correlation function is the optimal method for detecting a known waveform in random noise.

2.1.1 Convolution Theorem for FT

In mathematics, and, in particular, functional analysis, convolution is a mathematical operation, on two functions f and g, producing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated. In other words, FT of convolution of two signals is the point wise product of FT of respective signals. The classical definition of the convolution and product theorem of the FT for the signals $f(t)$ and $g(t)$ is given by-

Convolution: $$f(t) \otimes g(t) \xleftrightarrow{FT} \sqrt{2\pi}\, F(\omega) G(\omega) \quad \textbf{(2.1.1)}$$

Product: $$f(t).g(t) \xleftrightarrow{FT} F(\omega) \otimes G(\omega) \quad \textbf{(2.1.2)}$$

where, $F(\omega)$ and $G(\omega)$ are the FT of $f(t)$ and $g(t)$ respectively and $"\otimes"$ denotes the linear convolution operation.

2.1.1.1 Properties

The convolution theorem defined for any integral transform has to satisfy a set of properties in order to establish its utility in various application areas. The set of properties needed are

a) *Commutative property*:

The commutative property of convolution states that the order in which two sequences are convolved is not important. Mathematically, the commutative property is-

$$f(t) \otimes g(t) = g(t) \otimes f(t) \quad \textbf{(2.1.3)}$$

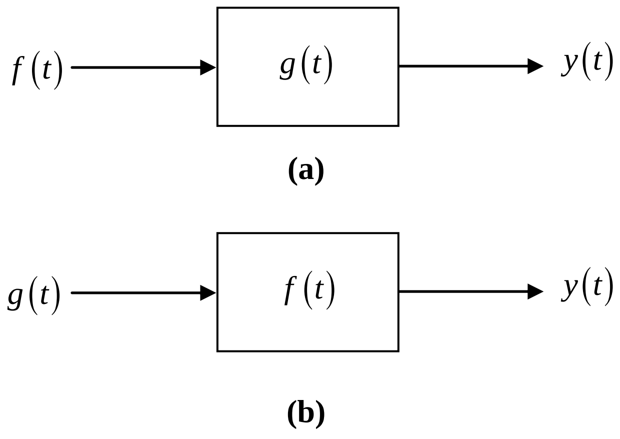


Figure-2.1: The commutative property

From a systems point of view, this property states that a system with a unit sample response $g(t)$ and input $f(t)$ behaves in exactly the same way as a system with unit sample response $f(t)$ and an input $g(t)$. This is illustrated in Figure-2.1.

b) *Associative property*:

The associative property is closely related to the commutative property. The associative property of an expression containing two or more occurrences of the same operator states that the order in which operations are performed does not affect the final result, as long as the order of terms is not changed. In contrast, the commutative property states that the order of the terms does not affect the final result. The associative property is used in system theory to describe how cascaded systems behave, for example, two or more systems are said to be in a cascade if the output of one system is used as the input for the next system.

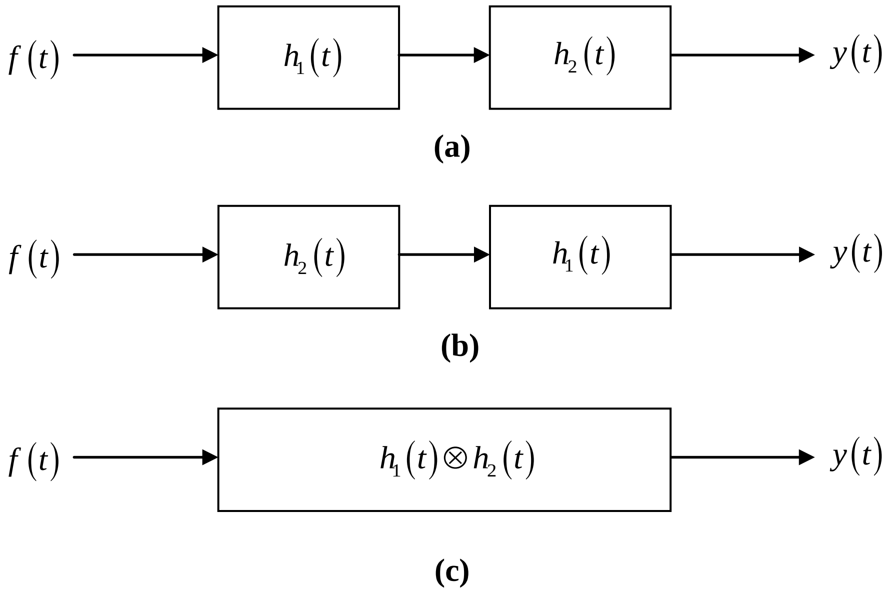


Figure-2.2: The associative property

From the associative property, the order of the systems can be rearranged without changing the overall response of the cascade. Further, any number of cascaded systems can be replaced with a single system. The impulse response of the replacement system is found by convolving the impulse responses of all of the original systems. Mathematically, the commutative property is defined for the two functions $f(t), h_1(t)$ and $h_2(t)$ as-

$$\left[f(t) \otimes h_1(t)\right] \otimes h_2(t) = f(t) \otimes \left[h_1(t) \otimes h_2(t)\right] \qquad \textbf{(2.1.4)}$$

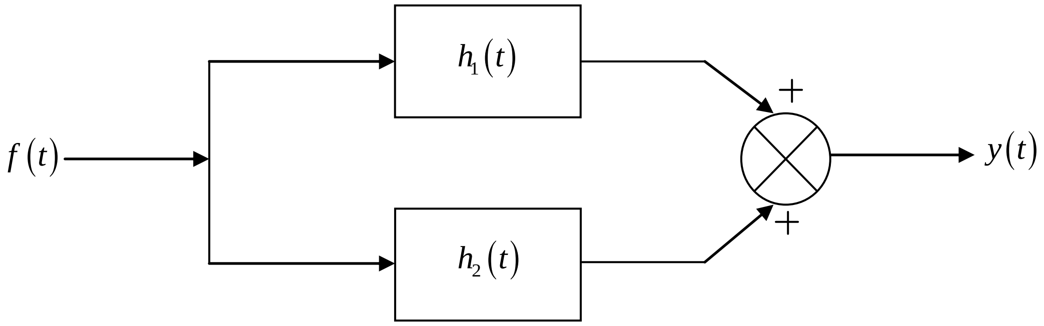
This is described in Figure-2.2

c) *Distributive property*:

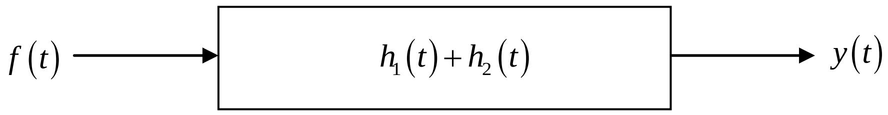
The distributive property describes the operation of parallel systems with added outputs. As shown in Figure-2.3, two or more systems can share the same input, $f(t)$, and have their outputs added to produce $y(t)$. The distributive property allows this combination of systems to be replaced with a single system, having an impulse response equal to the sum of the impulse responses of the original systems.

Mathematically, the distributive property is described as-

$$f(t)\otimes\left[h_1(t)+h_2(t)\right]=f(t)\otimes h_1(t)+f(t)\otimes h_2(t) \qquad \textbf{(2.1.5)}$$



(a)



(b)

Figure-2.3: The distributive property

2.1.2 Correlation Theorem for FT

Correlation, which is similar to convolution, is another important operation in signal processing. The correlation of two functions is no more than their convolution after one of the two functions has been axis-reversed **[2]**.

In signal processing, correlation is a measure of degree of relation of two waveforms as a function of a time-lag applied to one of them. This is also known as a sliding dot product or sliding inner-product. It is an important operation in optics as well as in signal processing, pattern recognition and especially in detection applications **[29, 30, 65, 80, 111, 128, 140, 160]**. The symmetric definition of the correlation theorem of the FT for the signals $f(t)$ and $g(t)$ is given by-

Cross-correlation: $$f(t)\star g(t)=\int_{-\infty}^{\infty} f(\tau)\,g(t+\tau)\,\mathrm{d}\tau \xleftrightarrow{FT} \sqrt{2\pi}\,F(-\omega)G(\omega) \qquad \textbf{(2.1.6)}$$

Auto-correlation: $$f(t)\star f(t)=\int_{-\infty}^{\infty} f(\tau)\,f(t+\tau)\,\mathrm{d}\tau \xleftrightarrow{FT} \sqrt{2\pi}\,F(-\omega)F(\omega). \qquad \textbf{(2.1.7)}$$

where, $F(\omega)$ and $G(\omega)$ are the FTs of $f(t)$ and $g(t)$ respectively and "★" denotes the correlation operation. The FT of auto-correlation of signal $f(t)$ produces power spectral density of signal $f(t)$.

2.1.2.1 Properties

The correlation theorem defined for any integral transform has to satisfy a set of properties in order to establish its utility in various application areas. The set of properties needed are

a) ***Commutative property*:**

The commutative property states that the order in which two sequences are operated is not important. Mathematically, the commutative property is-

$$f(t) \oplus g(t) = g(t) \oplus f(t) \quad \textbf{(2.1.8)}$$

where, '$\oplus$' indicates the mathematical operation, which is commutative in nature. But the cross-correlation function is non-commutative. This is true, as shifting of one function in one direction is equivalent to shifting the other in the opposite direction. Auto-correlation obeys the commutative operation.

b) ***Associative property*:**

The associative property is closely related to the commutative property. The associative property of an expression containing two or more occurrences of the same operator states that the order in which operations are performed does not affect the final result, as long as the order of terms is not changed. In contrast, the commutative property states that the order of the terms does not affect the final result. The associative property is used in system theory to describe how cascaded systems behave, for example, two or more systems are said to be in a cascade if the output of one system is used as the input for the next system.

Mathematically, the associative property is defined for the two functions $f(t), h_1(t)$ as-

$$\left[f(t) \oplus h_1(t)\right] \oplus h_2(t) = f(t) \oplus \left[h_1(t) \oplus h_2(t)\right] \quad \textbf{(2.1.9)}$$

where, '$\oplus$' indicates the mathematical operation, which is associative in nature. But the cross-correlation function is non- associative. This is true, as shifting of one function in one direction is equivalent to shifting the other in the opposite direction. Auto-correlation obeys the associative operation.

c) *Distributive property*:

The distributive property describes the operation of parallel systems with added outputs. Two or more systems can share the same input, $f(t)$ and have their outputs added to produce $y(t)$. The distributive property allows this combination of systems to be replaced with a single system, having an impulse response equal to the sum of the impulse responses of the original systems.

Mathematically, the distributive property is described as-

$$f(t)\oplus\left[h_1(t)+h_2(t)\right]=f(t)\oplus h_1(t)+f(t)\oplus h_2(t) \quad \textbf{(2.1.10)}$$

where, '$\oplus$' indicates the mathematical operation, which is distributive in nature. Distributive property is satisfied by both cross-correlation and auto-correlation theorems.

d) *Even function*:

Even functions are functions for which the left half of the plane looks like the mirror image of the right half of the plane. Let $f(t)$ be a real-valued function of a real variable. Then f is even if the following expression for all *x* and -*x* in the domain of f :

$$f(-t)=f(t) \quad \textbf{(2.1.11)}$$

Geometrically, the graph face of an even function is symmetric with respect to the *y*-axis, meaning that its graph remains unchanged after reflection about the *y*-axis. The auto-correlation function is an even function while cross-correlation function is not an even function.

2.2 PRELIMINARIES OF FRACTIONAL FOURIER TRANSFORM

The FT is undoubtedly one of the most valuable and frequently used tools in signal processing and analysis **[129]**. A generalization of FT, the FRFT has a long history since 1929s **[40, 53, 108]**. It was first introduced in 1980 by Namias as a tool to solve quadratic Hamiltonians in quantum mechanical systems **[154]**. His results were later refined by A. McBride and F. Kerr **[11]**, who, among other things, also developed an operational calculus for the FRFT. The FRFT gained very much popularity in the early 1990s because of its numerous applications in quantum mechanics and quantum optics **[25, 35, 87, 95, 97, 98, 155]**, swept frequency filters **[90]**, optical systems and optical signal processing and time-variant filtering and multiplexing **[54]**, pattern recognition **[29]**, time-frequency distribution **[72]**, Radon transformation of Wigner spectrum **[63, 64, 76]**, wavelet transforms **[54, 135]**, neural network **[135]**, various chirp related operations **[31, 32, 54, 145]**, image compression and encryption **[105, 120, 123]**, watermarking **[86]**, beamforming for mobile antennas **[118, 119]**, window analysis **[142]**, radar and sonar signal processing **[116]** and fractional order differentiator **[143, 144]** and to name a few.

2.2.1 Mathematical Definition of Fractional Fourier Transform

The FRFT is a family of linear transformation generalizing the FT. It can be thought of as the FT to the nth power, where n need not be an integer — thus, it can be interpreted as a rotation of the time-frequency plane and it can transform a function to any intermediate domain between time and frequency. In literature, the FRFT is also known as rotational FT **[24]** or angular FT **[88]**. The mathematical definition of FRFT is given by **[55]-**

$$X_\alpha(u) = X^{\tilde{a}}(u) = \int_{-\infty}^{+\infty} x(t) K_\alpha(t,u)\,\mathrm{d}t \qquad \textbf{(2.2.1)}$$

where, the transform kernel $K_\alpha(t,u)$ of the FRFT is given by-

$$K_\alpha(t,u) = \begin{cases} \sqrt{\dfrac{1-j\cot\alpha}{2\pi}}\, e^{j\frac{u^2}{2}\cot\alpha} \displaystyle\int_{-\infty}^{+\infty} x(t)\, e^{j\frac{t^2}{2}\cot\alpha - jut\csc\alpha}\,\mathrm{d}t & \text{if } \alpha \text{ is not a multiple of } \pi \\ \delta(t) & \text{if } \alpha \text{ is a multiple of } 2\pi \\ \delta(-t) & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi \end{cases} \qquad \textbf{(2.2.2)}$$

and the FRFT is defined by means of the transformation kernel-

$$X_\alpha(u) = X^{\tilde{a}}(u) = \int_{-\infty}^{+\infty} x(t) K_\alpha(t,u)\,\mathrm{d}t$$

$$= \begin{cases} \sqrt{\dfrac{1-j\,cot\,\alpha}{2\pi}}\, e^{j\frac{u^2}{2}cot\,\alpha} \displaystyle\int_{-\infty}^{+\infty} x(t)\, e^{j\frac{t^2}{2}cot\,\alpha - jut\,csc\,\alpha}\,\mathrm{d}t & \text{if } \alpha \text{ is not a multiple of } \pi \\ \delta(t) & \text{if } \alpha \text{ is a multiple of } 2\pi \\ \delta(-t) & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi \end{cases} \qquad \textbf{(2.2.3)}$$

where, $\alpha = \frac{\tilde{a}\pi}{2}$ is interpreted as a rotation angle in the phase plane. The function $x(t)$ can be recovered from a FRFT operation with angular parameter $(-\alpha)$:

$$x(t) = \int_{-\infty}^{\infty} X_\alpha(u) K_{-\alpha}(t,u) \, \mathrm{d}u \tag{2.2.4}$$

where, $K_{-\alpha}(t,u) = K_\alpha^*(t,u)$ and the superscript "*" denotes the complex conjugation. The variable u can be interpreted as some hybrid time-frequency variables. The interpretation of u changes gradually from time to frequency as α takes value from 0 to $\pi/2$. The changed frequency characteristics of the transformed signal are revealed for different values of α.

2.2.2 Time-Frequency Representation of FRFT

The FRFT belongs to the class of time-frequency representations that have been extensively used by the signal processing community. In all the time-frequency representations, one normally uses a plane with two orthogonal axes corresponding to time and frequency. If a signal $x(t)$ to be represented along the time axis and its ordinary FT $X(\omega)$ to be represented along the frequency axis, then the FT operator (denoted by F) can be visualized as a change in representation of the signal corresponding to a counter-clockwise rotation of the axis by an angle π/2. Given that $'I'$ is the identity operator, the FRFT operator $X^{\tilde{a}}(u)$ has the following properties-

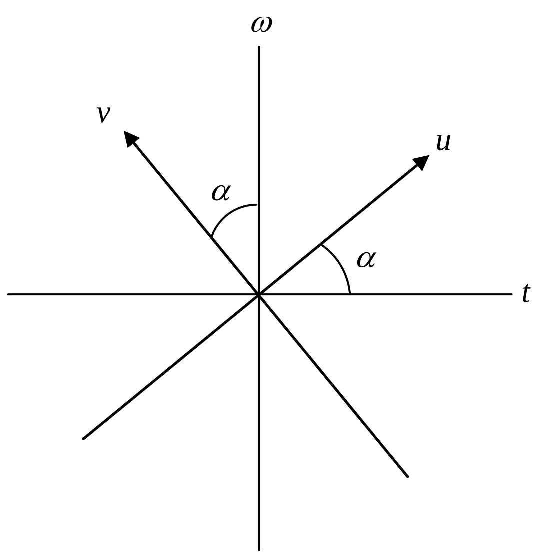


Figure-2.4: Time-frequency plane and a set of coordinates (u,v) rotated by an angle α relative to the original coordinates (t,ω).

Zero Rotation: $X^{0}(u)=I$ **(2.2.5)**

Consistency with FT: $X^{\pi/2}(u)=F$ **(2.2.6)**

Reflection Operator: $X^{\pi}(u)=-I$ **(2.2.7)**

Inverse Fourier operator: $X^{3\pi/2}(u)=F^{-1}$ **(2.2.8)**

2π rotation: $X^{2\pi}(u)=I$ **(2.2.9)**

Additivity of rotations: $X^{\beta}(u)X^{\gamma}(u)=X^{\beta+\gamma}(u)$ **(2.2.10)**

Thus FRFT can be viewed as the generalization of FT. So many properties associated with the FT can be generalized by using the FRFT.

2.2.3 Existing Definitions of Convolution and Correlation Theorems for FRFT

The convolution and product theorems for FRFT have been suggested by many researchers in the literature such as by Almeida **[89]**, Zayed **[6]** and Singh *et al.* **[8]**. In this section, a brief review of these definitions has been presented.

In 1997, Almeida **[89]** gave a definition for convolution theorem for FRFT by considering two signals $x(t)$ and $y(t)$ as-

$$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau \leftrightarrow |\sec\alpha|\,e^{-j\frac{u^2}{2}\tan\alpha}\int_{-\infty}^{\infty} X_{\alpha}(v)\,y\big[(u-v)\sec\alpha\big]e^{j\frac{v^2}{2}\tan\alpha}\mathrm{d}v \qquad \textbf{(2.2.11)}$$

In this definition, the FRFT of a convolution can be obtained by multiplying a chirp $e^{j\frac{v^2}{2}\tan\alpha}$ to the FRFT of one of the signals $X_{\alpha}(v)$ and convolving it with a scaled version of the other signal $y\big[(u-v)\sec\alpha\big]$, subsequently multiplying again by another chirp and a scale factor $|\sec\alpha|\,e^{-j\frac{u^2}{2}\tan\alpha}$, as evident from (2.2.11).

Similarly, the product theorem was given by-

$$x(t)\,y(t) \leftrightarrow \frac{|\csc\alpha|}{\sqrt{2\pi}}e^{j\frac{u^2}{2}\cot\alpha}\int_{-\infty}^{\infty} X_{\alpha}(v)\,Y\big[(u-v)\csc\alpha\big]e^{-j\frac{v^2}{2}\cot\alpha}\mathrm{d}v \qquad \textbf{(2.2.12)}$$

where, $Y(u)$ is the FT of $y(t)$.

After one year, in 1998, Zayed **[6]** introduced a new approach to derive the convolution theorem. In this technique, a chirp signal was multiplied before convolution and the FRFT of the obtained convolution was derived, which was entirely in the FRFT domain. The convolution theorem was given by-

$$\sqrt{\frac{1-j\cot\alpha}{2\pi}}\, e^{-j\frac{t^2}{2}\cot\alpha}\int_{-\infty}^{\infty} x(\tau)e^{j\frac{\tau^2}{2}\cot\alpha}\, y(t-\tau)e^{j\frac{(t-\tau)^2}{2}\cot\alpha}\,\mathrm{d}\tau \leftrightarrow e^{-j\frac{u^2}{2}\cot\alpha} X_\alpha(u)Y_\alpha(u) \qquad \textbf{(2.2.13)}$$

Similarly the product theorem was given by-

$$x(t)\,y(t)e^{-j\frac{t^2}{2}\cot\alpha} \leftrightarrow \sqrt{\frac{1+j\cot\alpha}{2\pi}}\, e^{-j\frac{u^2}{2}\cot\alpha}\int_{-\infty}^{\infty} X_\alpha(v)e^{-j\frac{v^2}{2}\cot\alpha}\, Y_\alpha(u-v)e^{-j\frac{(u-v)^2}{2}\cot\alpha}\,\mathrm{d}v \qquad \textbf{(2.2.14)}$$

It took almost fourteen years, in 2012, Singh *et al.* **[8]** derived the convolution theorem for FRFT. In this theorem again, a chirp signal was multiplied before convolution and the FRFT of the obtained convolution was derived, which was entirely in the FRFT domain. The convolution theorem was given by-

$$\int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)e^{j\tau(\tau-t)\cot\alpha}\,\mathrm{d}\tau \leftrightarrow \sqrt{\frac{2\pi}{1-j\cot\alpha}}\, e^{-\frac{j}{2}u^2\cot\alpha} X_\alpha(u)Y_\alpha(u) \qquad \textbf{(2.2.15)}$$

and the product theorem was given by-

$$x(t)\,y(t)e^{j\frac{t^2}{2}\cot\alpha} \leftrightarrow \sqrt{\frac{1+j\cot\alpha}{2\pi}}\int_{-\infty}^{\infty} X_\alpha(v)Y_\alpha(u-v)e^{\frac{j}{2}v(u-v)\cot\alpha}\,\mathrm{d}v \qquad \textbf{(2.2.16)}$$

The correlation theorem for FRFT have been suggested by many researchers in the literature such as by Akay *et al.* **[111]**, Tao *et al.* **[126]**, Cottone *et al.* **[50]**, Torres *et al.* **[128]** and Singh *et al.* **[7]**. Here a brief review of these definitions have been presented.

In 2001, Akay *et al.* **[111]** defined the correlation theorem in FRFT domain by using the fractional shift operator as given below-

$$x(t)\otimes y(t) = e^{j2\pi\frac{\tau^2}{2}\cos\alpha\sin\alpha}\int_{-\infty}^{\infty} x(t)\,y^*(t-\tau\cos\alpha)e^{-j2\pi t\tau\sin\alpha}\,\mathrm{d}t \qquad \textbf{(2.2.17)}$$

It has been found that if the FRFT of the above correlation definition was determined then it results into very complex calculations. After almost seven years, in 2008 Tao *et al.* **[126]** formulated the correlation theorem for FRFT and the definition was the given by-

$$R_\alpha(\tau) = \int_{-\infty}^{\infty} x(t)\,y(t+\tau)e^{jt\tau\cot\alpha}\,\mathrm{d}t \qquad \textbf{(2.2.18)}$$

In this authors had documented the fractional power spectrum by taking the FRFT of fractional correlation as defined in above expression (2.2.18). As visible from (2.2.18), the correlation integral contains only one chirp which leads to non-compliance of variable dependability parameter. In 2010, Cottone *et al.* **[50]** had showed that the fractional spectral moment function was able to represent both the power spectral density and the correlation function in their whole domains. Actually, the fractional spectral moments are the fractional (complex) moments of the one sided power spectral density. Although, they had defined the fractional correlation and power spectral density but they did not concisely documented the correlation theorem for FRFT. Again in the year 2010, Torres *et al.* **[128]** had introduced another correlation theorem for the FRFT by using fractional translational operator and in 2011 Singh *et al.* **[7]** presented the correlation theorem for FRFT by multiplying a chirp signal before convolution and the FRFT of the obtained convolution was derived, which was entirely in the FRFT domain and is given by-

$$\int_{-\infty}^{\infty} x(\tau)\, y(t+\tau)\, e^{j\tau(\tau+t)\cot\alpha} \mathrm{d}\tau \leftrightarrow \sqrt{\frac{2\pi}{1-j\cot\alpha}}\, e^{-\frac{j}{2}u^2\cot\alpha} X_\alpha(-u)\, Y_\alpha(u) \qquad \textbf{(2.2.19)}$$

FRFT has found numerous application in the signal processing. In the year 2007, Sharma *et al.* **[135]** presented a methodology for on-line tuning of transition BW of window-based FIR filters using FRFT. By selecting FRFT order as a tuning parameter in the convolution operation between FRFT of window function and ideal frequency response, the authors showed the variability in the transition band of the resulting filter response. In 2010, Kumar *et al.* **[142]**, derived a new mathematical model for obtaining the FRFT of Dirichlet and Generalized Hamming window functions. With the help of simulation, the authors showed that there is a variation in the window function parameters with the variation in the FRFT parameter.

2.3 PRELIMINARIES OF LINEAR CANONICAL TRANSFORM

The LCT **[23, 102, 103, 137, 138, 149]** of a given function is a three parameter class of linear integral transform and it has found many applications in filter design, signal synthesis, time-frequency analysis, encryption, modulation, capacity analysis, window function analysis and multiplexing in communications etc. **[3, 8, 10, 19, 26, 48, 49, 91, 110, 117, 124, 139, 142]**. The one-dimensional LCT with (a,b,c,d) variables of a signal $f(t)$ is defined **[101, 149]** as-

$$L_F^{(a,b,c,d)}\left[x(t)\right](u)=\begin{cases}\int_{-\infty}^{+\infty} x(t)K_{(a,b,c,d)}(u,t)\,\mathrm{d}t, & b\neq 0\\ \sqrt{d}\,e^{j(cd/2)u^2} f(d.u), & b=0\end{cases} \quad \textbf{(2.3.1)}$$

where, variables (a,b,c,d) are real numbers and the matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ belongs to SL(2, P). In general, the case of $b\neq 0$ is considered, since LCT is just a chirp multiplication operation if $b=0$. The term $K_{(a,b,c,d)}(u,t)$ represents the integral kernel and is given by-

$$K_{(a,b,c,d)}(u,t)=\sqrt{\frac{1}{j2\pi b}}\exp\left[\frac{j\left(at^2+du^2\right)}{2b}-\frac{jtu}{b}\right] \quad \textbf{(2.3.2)}$$

The inverse of LCT with variables $A=(a,b,c,d)$ is given by LCT with variables $A^{-1}=(d,-b,-c,a)$. The exact inverse LCT expression **[38]** is given by-

$$L_F^{(d,-b,-c,a)}\left[X(u)\right](t)=\int_{-\infty}^{+\infty} X(u)K^{*}_{(a,b,c,d)}(u,t)\,\mathrm{d}u \quad \textbf{(2.3.3)}$$

where, the superscript “*” denotes the complex conjugation.

2.3.1 Special Cases Of LCT

Many important transforms are the special cases of LCT. These are discussed below-

2.3.1.1 Fourier transform

When $(a,b,c,d)=(0,1,-1,0)$, the LCT becomes the FT multiplied by factor $\sqrt{-j}$. From (2.3.1) and (2.3.2), results in-

$$L_F^{(0,1,-1,0)}\left[x(t)\right](u)=\int_{-\infty}^{+\infty} x(t)\sqrt{\frac{1}{j2\pi}}.e^{j\left[(0/2)u^2-(1/1)ut+(0/2)t^2\right]}\,\mathrm{d}t$$

$$=\sqrt{\frac{-j}{2\pi}}\int_{-\infty}^{+\infty} e^{-j.u.t}\,x(t)\,\mathrm{d}t$$

$$=\sqrt{-j}FT\left[x(t)\right]=\sqrt{-j}\,X(\omega) \quad \textbf{(2.3.4)}$$

Equation (2.3.4) represents the FT of the function $x(t)$ multiplied by factor $\sqrt{-j}$. Similarly, when $(a,b,c,d)=(0,-1,1,0)$, the LCT becomes the inverse FT multiplied by the factor $\sqrt{j}$ and is given by-

$$L_F^{(0,-1,1,0)}\left[X(\omega)\right](t)=x(t) \quad \textbf{(2.3.5)}$$

2.3.1.2 Fractional Fourier transform

When $(a,b,c,d)=(\cos\alpha,\sin\alpha,\text{-}\cos\alpha,\sin\alpha)$, the LCT becomes the FRFT with little difference of constant phase. From (2.3.1) and (2.3.2)-

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}\left[x(t)\right](u)=\sqrt{\frac{1}{j2\pi\sin\alpha}}\,e^{(j/2)(\cos\alpha/\sin\alpha)u^2}\int_{-\infty}^{\infty}e^{-j(u/\sin\alpha)t}e^{(j/2)(\cos\alpha/\sin\alpha)t^2}x(t)\,\mathrm{d}t$$

Rearranging, results in-

$$\Rightarrow\sqrt{\frac{1}{j2\pi\sin\alpha}}\,e^{(j/2)(\cot\alpha)u^2}\int_{-\infty}^{\infty}e^{-j(u\,\mathrm{cosec}\,\alpha)t}e^{(j/2)(\cot\alpha)t^2}x(t)\,\mathrm{d}t$$

Multiplying and dividing the above expression by $\sqrt{e^{j\alpha}}$, results in-

$$\Rightarrow\frac{\sqrt{(\cos\alpha+j\sin\alpha)}}{\sqrt{e^{j\alpha}}}\sqrt{\frac{1}{j2\pi\sin\alpha}}\int_{-\infty}^{\infty}e^{\frac{j}{2}\left[u^2\cot\alpha-tu\csc\alpha+t^2\cot\alpha\right]}x(t)\,\mathrm{d}t$$

Rearranging, results in-

$$\Rightarrow\sqrt{e^{-j\alpha}}\sqrt{\frac{(\cos\alpha+j\sin\alpha)}{j2\pi\sin\alpha}}\int_{-\infty}^{\infty}e^{\frac{j}{2}\left[u^2\cot\alpha-tu\csc\alpha+t^2\cot\alpha\right]}x(t)\,\mathrm{d}t$$

$$\Rightarrow\sqrt{e^{-j\alpha}}\sqrt{\frac{(1-j\cot\alpha)}{2\pi}}\int_{-\infty}^{\infty}e^{\frac{j}{2}\left[u^2\cot\alpha-tu\csc\alpha+t^2\cot\alpha\right]}x(t)\,\mathrm{d}t \quad \textbf{(2.3.6)}$$

From (2.3.5), it has been proved that LCT coincides with the FRFT multiplied by the factor $\sqrt{e^{-j\alpha}}$.

2.3.1.3 Fresnel transform

The Fresnel transform is an operation that describes paraxial light propagation and diffraction under the Fresnel approximation **[46]**. The one-dimensional definition of Fresnel transform is given by-

$$O_z^{Fresnel(t)}\left[x(t)\right]=\frac{e^{j\pi z/\lambda}}{\sqrt{j\lambda z}}\int_{-\infty}^{\infty}e^{j(\pi/\lambda z)(u-t)^2}x(t)\,\mathrm{d}t \qquad \textbf{(2.3.7)}$$

When $(a,b,c,d)=(1,\lambda z/2\pi,0,1)$, the LCT becomes the FST with little difference of constant phase, where, z is the propagation distance and λ is its wavelength. From (2.3.1) and (2.3.2)-

$$L_F^{(1,\lambda z/2\pi,0,1)}\left[x(t)\right](u)=\sqrt{\frac{1}{j2\pi(\lambda z/2\pi)}}\,e^{(j/2)(2\pi/\lambda z)u^2}\int_{-\infty}^{\infty}e^{-j(2\pi u/\lambda z)t}e^{(j/2)(2\pi/\lambda z)t^2}x(t)\,\mathrm{d}t$$

Rearranging, results in-

$$L_F^{(1,\lambda z/2\pi,0,1)}\left[x(t)\right](u)=\sqrt{\frac{1}{j\lambda z}}\,e^{(j\pi/\lambda z)u^2}\int_{-\infty}^{\infty}e^{-j(2\pi u/\lambda z)t}e^{(j\pi/\lambda z)t^2}x(t)\,\mathrm{d}t$$

Rearranging, results in-

$$L_F^{(1,\lambda z/2\pi,0,1)}\left[x(t)\right](u)=\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty}e^{(j\pi/\lambda z)(u-t)^2}x(t)\,\mathrm{d}t$$

Multiplying and dividing the above expression by $e^{j\pi z/\lambda}$, results in-

$$L_F^{(1,\lambda z/2\pi,0,1)}\left[x(t)\right](u)=\frac{e^{j\pi z/\lambda}}{e^{j\pi z/\lambda}}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty}e^{(j\pi/\lambda z)(u-t)^2}x(t)\,\mathrm{d}t$$

Rearranging, results in-

$$L_F^{(1,\lambda z/2\pi,0,1)}\left[x(t)\right](u)=e^{-j\pi z/\lambda}O_z^{Fresnel(t)}\left[x(t)\right] \qquad \textbf{(2.3.8)}$$

From (2.3.7), it has been proved that LCT coincides with the FST multiplied by the factor $e^{-j\pi z/\lambda}$.

2.3.1.4 Chirp multiplication

This is another special case of LCT. From (2.3.1), when $(a,b,c,d)=(1,0,\tau,1)$, the LCT becomes the multiplication by Gaussian or chirp multiplication with little difference of constant phase. Mathematically it may be defined as-

$$L_F^{(1,0,\tau,1)}\left[x(t)\right](u)=e^{\frac{j}{2}\tau u^2}\,x(u) \qquad \textbf{(2.3.9)}$$

Substituting the values of $(a,b,c,d)=(1,0,\tau,1)$ in (2.3.1) for $b=0$, results in-

$$L_F^{(1,0,\tau,1)}\left[x(t)\right](u)=\sqrt{1}\,e^{\frac{j}{2}.\tau.1.u^2}\,x(1.u)$$

Rearranging, results in-

$$L_F^{(1,0,\tau,1)}\left[x(t)\right](u)=e^{\frac{j}{2}\tau u^2}\ x(u) \tag{2.3.10}$$

Equation (2.3.9) is the special case of LCT.

2.3.1.5 Scaling operation

The scaling operation can be viewed as the special case of the LCT when $(a,b,c,d)=(\sigma,0,0,1/\sigma)$. Mathematically-

$$L_F^{(\sigma,0,0,1/\sigma)}\left[x(t)\right](u)=\sqrt{\sigma^{-1}}e^{\frac{j}{2\sigma}u^2}\ x\left(\sigma^{-1}u\right)$$

Substituting the values of $(a,b,c,d)=(\sigma,0,0,1/\sigma)$ in (2.3.1) for $b=0$, results in-

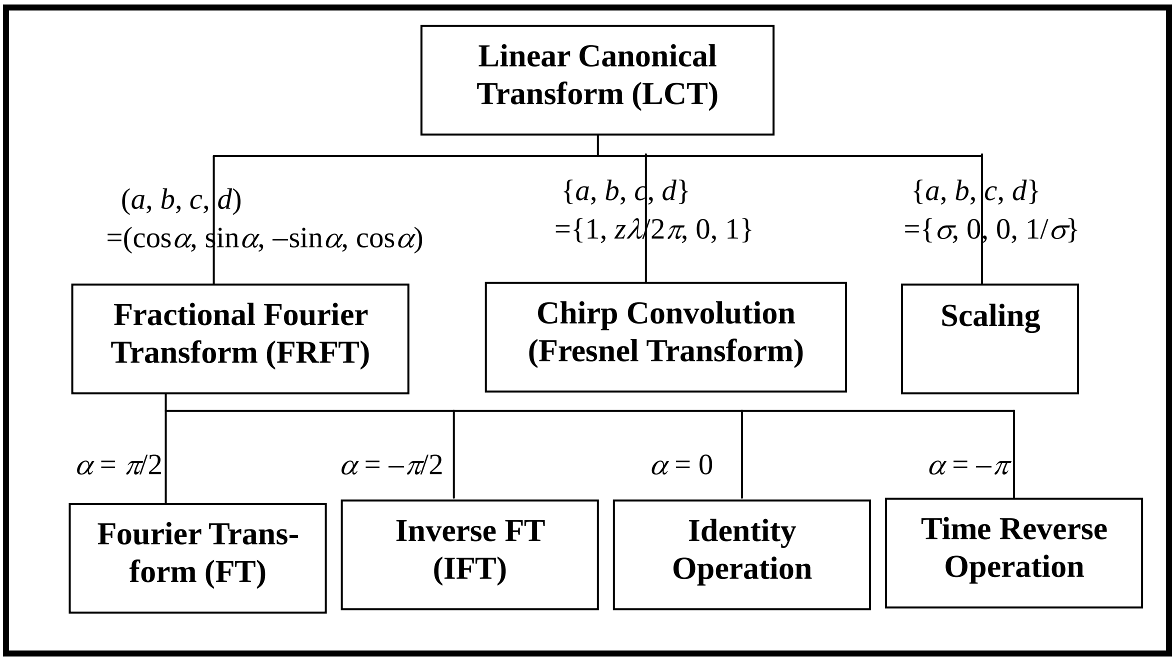
$$L_F^{(\sigma,0,0,1/\sigma)}\left[x(t)\right](u)=\sqrt{\sigma^{-1}}.e^{j\frac{1}{2}0.\sigma^{-1}.u^2}.g\left(\sigma^{-1}u\right)$$

$$L_F^{(\sigma,0,0,1/\sigma)}\left[x(t)\right](u)=\sqrt{\sigma^{-1}}\ g\left(\sigma^{-1}u\right) \tag{2.3.11}$$

From (2.3.10), time scaling operation is achieved from LCT. Therefore, the FT, FRFT, chirp multiplication, FST and scaling operations are all the special cases of the LCT. The LCT can extend their utilities and applications and can solve some problems that cannot be solved well by these operations. The relation between LCT and its special cases is shown in Table-2.1 **[67].**

TABLE 2.1

SPECIAL CASES OF LCT



2.3.2 Basic Properties of LCT

In this subsection, various properties of LCT **[55, 67, 150]** are discussed. Table-2.2 list the properties of LCT **[67]**. The following is the proof of some of the properties of LCT:

2.3.2.1 Multiplication property

If $X_{(a,b,c,d)}(u)$ is the LCT of $x(t)$, then differentiating (2.3.1) and (2.3.2) with respect to *u*, results in-

$$\frac{d}{du}X_{(a,b,c,d)}(u)=\sqrt{\frac{1}{j2\pi b}}\cdot e^{\frac{j}{2}\frac{d}{b}u^2}\int_{-\infty}^{\infty}\left(\frac{jd}{b}u-\frac{j}{b}t\right)e^{\frac{-j}{b}ut}e^{\frac{j}{2}\frac{a}{b}t^2}x(t)\,\mathrm{d}t\,,$$

$$\frac{d}{du}X_{(a,b,c,d)}(u)=\frac{jd}{b}u\cdot X_{(a,b,c,d)}(u)-\frac{j}{b}\cdot\sqrt{\frac{1}{j2\pi b}}\,e^{\frac{j}{2}\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-\frac{j}{b}ut}e^{\frac{j}{2}\frac{a}{b}t^2}t\,x(t)\,\mathrm{d}t$$

Rearranging, results in-

$$\left(\frac{d}{du}-\frac{jd}{b}u\right)X_{(a,b,c,d)}(u)=-\frac{j}{b}\cdot\sqrt{\frac{1}{j2\pi b}}\,e^{\frac{j}{2}\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-\frac{j}{b}ut}e^{\frac{j}{2}\frac{a}{b}t^2}t\,x(t)\,\mathrm{d}t$$

$$\left(b.j\frac{d}{du}+d.u\right).X_{(a,b,c,d)}(u)=\sqrt{\frac{1}{j2\pi b}}\,e^{\frac{j}{2}\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-\frac{j}{b}ut}e^{\frac{j}{2}\frac{a}{b}t^2}t\,x(t)\,\mathrm{d}t$$

Therefore, the multiplication property is proved.

2.3.2.2 Differentiation property

From (2.3.3), the LCT inverse is given by-

$$x(t)=\sqrt{\frac{j}{2\pi b}}.e^{-\frac{ja}{2b}t^2}\int_{-\infty}^{\infty}e^{\frac{j}{b}tu}e^{-\frac{jd}{2b}u^2}X_{(a,b,c,d)}(u)\,\mathrm{d}u \qquad \textbf{(2.3.12)}$$

Taking differentiation of (2.3.12)

$$x'(t)=\sqrt{\frac{j}{2\pi b}}\cdot e^{-\frac{j}{2}\frac{a}{b}t^2}\int_{-\infty}^{\infty}\left(-\frac{a}{b}jt+\frac{j}{b}u\right)e^{\frac{j}{b}tu}e^{-\frac{j}{2}\frac{d}{b}u^2}X_{(a,b,c,d)}(u)\,\mathrm{d}u\,,$$

Rearranging, results in-

$$x'(t)+\frac{a}{b}jt\cdot x(t)=\frac{j}{b}\cdot\sqrt{\frac{j}{2\pi b}}.e^{-\frac{j}{2}\frac{a}{b}t^2}\int_{-\infty}^{\infty}e^{\frac{j}{b}tu}e^{-\frac{j}{2}\frac{d}{b}u^2}u\cdot X_{(a,b,c,d)}(u)\,\mathrm{d}u\,,$$

$$L_F^{(a,b,c,d)}\left(x'(t)+\frac{a}{b}jt\cdot x(t)\right)=\frac{j}{b}\cdot u\cdot X_{(a,b,c,d)}(u)\,.$$

Then, after substituting (2.3.12) into the above expression, the property is proved.

2.3.2.3 Division property

From (2.3.1),

$$L_F^{(a,b,c,d)}\left(\frac{x(t)}{t}\right)=\sqrt{\frac{1}{j2\pi b}}\cdot e^{\frac{j}{2}\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-\frac{j}{b}ut}e^{\frac{j}{2}\frac{a}{b}t^2}\cdot\frac{1}{t}x(t)\,\mathrm{d}t\,.$$

Rearranging, results in-

$$e^{-\frac{j}{2}\frac{d}{b}u^2}L_F^{(a,b,c,d)}\left(\frac{f(t)}{t}\right)=\sqrt{\frac{1}{j2\pi b}}\cdot\int_{-\infty}^{\infty}e^{-\frac{j}{b}ut}e^{\frac{j}{2}\frac{a}{b}t^2}\cdot\frac{1}{t}x(t)\,\mathrm{d}t\,,$$

$$\frac{d}{\mathrm{d}u}\left[e^{-\frac{j}{2}\frac{d}{b}u^2}L_F^{(a,b,c,d)}\left(\frac{x(t)}{t}\right)\right]=-\frac{j}{b}\cdot e^{-\frac{j}{2}\frac{d}{b}u^2}\cdot X_{(a,b,c,d)}(u)\,. \tag{2.3.13}$$

Integrating (2.3.13), results in-

$$L_F^{(a,b,c,d)}\left(\frac{x(t)}{t}\right)=-\frac{j}{b}e^{\frac{j}{2}\frac{d}{b}u^2}\int_{-\infty}^{u}e^{-\frac{j}{2}\frac{d}{b}z^2}X_{(a,b,c,d)}(z)\,\mathrm{d}z$$

Therefore, the division property is proved.

2.3.2.4 Integration property

Substituting $t'=t-\tau$ in the signal of integration property, results in

$$L_F^{(a,b,c,d)}\left(\int_{-\infty}^{t}x(t')\,\mathrm{d}t'\right)=L_F^{(a,b,c,d)}\left(\int_{0}^{\infty}x(t-\tau)\,\mathrm{d}\tau\right),$$

Rearranging, results in-

$$L_F^{(a,b,c,d)}\left(\int_{-\infty}^{t}x(t')\,\mathrm{d}t'\right)=\left(\int_{0}^{\infty}L_F^{(a,b,c,d)}\left[x(t-\tau)\right]\mathrm{d}\tau\right)$$

Then, by using the time shift property, results in-

$$L_F^{(a,b,c,d)}\left(\int_{-\infty}^{t}x(t')\,\mathrm{d}t'\right)=\left(\int_{0}^{\infty}e^{-j\frac{ac}{2}\tau^2}e^{jc\tau u}X_{(a,b,c,d)}(u-a\tau)\,\mathrm{d}\tau\right)$$

Substituting $z=u-a\tau$ and $\mathrm{d}\tau=-\mathrm{d}z/a\,,\tau=(u-z)/a$ in the above expression, results in

$$L_F^{(a,b,c,d)}\left(\int_{-\infty}^{t}x(t')\,\mathrm{d}t'\right)=\left(\int_{0}^{\infty}e^{-j\frac{ac}{2}\left[(u-z)/a\right]^2}e^{jcu(u-z)/a}X_{(a,b,c,d)}(z)\frac{-\mathrm{d}z}{a}\right)$$

Rearranging, the above expression, results in the integration property.

TABLE 2.2

PROPERTIES OF LCT [67]

S. NO.	PROPERTY DESCRIPTION	SIGNAL	LCT OF THE SIGNAL
1.	Time shift	$x(t-\tau)$	$e^{-j\frac{ac}{2}\tau^2}.e^{jc\tau u}.X_{(a,b,c,d)}(u-a\tau)$
2.	Modulation	$e^{j\eta}x(t)$	$e^{-j\frac{bd}{2}\eta^2}.e^{jd\eta u}.X_{(a,b,c,d)}(u-b\eta)$
3.	Time shift and modulation	$e^{j\eta t}x(t-\tau)$	$e^{j\varphi}e^{j(c\tau+d\eta)u}.X_{(a,b,c,d)}(u-a\tau-b\eta)$ where, $\varphi=-(ac/2)\tau^2-bc\tau\eta-(bd/2)\eta^2$
4.	Scaling	$\sqrt{\sigma^{-1}}x(\sigma^{-1}t)$	$L_F^{\left(\sigma a,\frac{b}{\sigma},\sigma c,\frac{d}{\sigma}\right)}\left[x(t)\right]$
5.	Time reverse	$x(-t)$	$X_{(a,b,c,d)}(-u)$
6.	Even/odd input even/odd output	$x(t)=x(-t)$ $x(t)=-x(-t)$	$X_{(a,b,c,d)}(-u)$ $-X_{(a,b,c,d)}(-u)$
7.	Multiplication	$t.x(t)$	$\left(b.j\frac{d}{du}+d.u\right).X_{(a,b,c,d)}(u)$
8.	Differentiation	$x'(t)$	$\left(a\frac{d}{du}+c.ju\right).X_{(a,b,c,d)}(u).$
9.	Division	$\frac{x(t)}{t}$	$-\frac{j}{b}e^{\frac{j}{2}\frac{d}{b}u^2}\int_{-\infty}^{u}e^{-\frac{j}{2}\frac{d}{b}z^2}X_{(a,b,c,d)}(z)\,\mathrm{d}z$
10.	Integration	$\int_{-\infty}^{t}x(t')\,\mathrm{d}t'$	$\frac{e^{\frac{jc}{2a}u^2}}{a}\int_{-\infty}^{u}e^{-\frac{jc}{2a}z^2}X_{(a,b,c,d)}(z)\,\mathrm{d}z$ when $a>0$, $\frac{e^{\frac{jc}{2a}u^2}}{-a}\int_{u}^{\infty}e^{-\frac{jc}{2a}z^2}X_{(a,b,c,d)}(z)\,\mathrm{d}z$ when $a<0$.
11.	Conjugation		$\overline{L_F^{(a,b,c,d)}\left[x(t)\right]}=L_F^{(a,-b,-c,d)}\left[\overline{x(t)}\right]$
12.	DC value	0	$\int_{-\infty}^{\infty}e^{\frac{ja}{2b}t^2}x(t)\,\mathrm{d}t$
13.	Energy Conservation (Parseval's theorem)	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\mathrm{d}t$	$\int_{-\infty}^{\infty}\lvert X_{(a,b,c,d)}(u)\rvert^2\mathrm{d}u$
14.	Generalized Parseval's Theorem	$\int_{-\infty}^{\infty}x(t)\overline{g(t)}\,\mathrm{d}t$	$\int_{-\infty}^{\infty}X_{(a,b,c,d)}(u)\overline{G_{(a,b,c,d)}(u)}\,\mathrm{d}u$

2.3.2.5 Parseval's theorem

As per definition of Parseval's theorem,

$$\int_{-\infty}^{\infty} \left| X_{(a,b,c,d)}(u) \right|^2 \mathrm{d}u = \int_{-\infty}^{\infty} X_{(a,b,c,d)}(u)\, \overline{X_{(a,b,c,d)}(u)}\, \mathrm{d}u$$

Expanding, results in-

$$\Rightarrow \frac{1}{\sqrt{4\pi^2 b^2}} \int_{-\infty}^{\infty} e^{\frac{j}{2}\frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{-\frac{j}{b}ut} e^{\frac{j}{2}\frac{a}{b}t^2} x(t) e^{-\frac{j}{2}\frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{\frac{j}{b}ut'} e^{-\frac{j}{2}\frac{a}{b}t'^2} \overline{x(t')}\, \mathrm{d}t'\, \mathrm{d}t\, \mathrm{d}u$$

Rearranging, results in-

$$\Rightarrow \frac{1}{2\pi|b|} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{j}{2b}\left(at^2 - at'^2\right)} e^{-j\frac{u}{b}(t-t')} x(t)\, \overline{x(t')}\, \mathrm{d}u\, \mathrm{d}t\, \mathrm{d}t'$$

Using, $\int_{-\infty}^{\infty} e^{-j\frac{u}{b}(t-t')} \mathrm{d}u = 2\pi|b| \cdot \delta(t-t')$ in the above expression, results in-

$$\int_{-\infty}^{\infty} \left| F_{(a,b,c,d)}(u) \right|^2 \mathrm{d}u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{j}{2b}\left(at^2 - at'^2\right)} \delta(t-t')\, x(t)\, \overline{x(t')}\, \mathrm{d}t\, \mathrm{d}t' = \int_{-\infty}^{\infty} \left| x(t) \right|^2 \mathrm{d}t$$

Therefore, the theorem is proved.

From the time shifting property, it is found that the shifting operation in time domain will correspond to the combination of shifting and modulation operations in transform domain for LCT. For the conventional FT, the shifting operation in time domain just corresponds to the modulation operation in frequency domain, and the location of spectrum is unchanged. But for LCT, the shifting in time domain will also cause the shifting of spectrum in frequency domain. So, the spectrum of FT is shift-invariant, but the spectrum of LCT is partially shift-variant. This property is useful for the application that using LCT for space-variant pattern recognition.

2.3.2.6 Combination of the time-shifting and modulation property

From time-shifting and modulation property-

$$L_F^{(a,b,c,d)}\left(e^{jn_1 \cdot t} x(t - m_1)\right) = e^{j\varphi} . e^{jn_2 u} . X_{(a,b,c,d)}(u - m_2),$$

where,

$$\begin{bmatrix} m_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \begin{bmatrix} m_1 \\ n_1 \end{bmatrix}, \qquad \varphi = -\frac{ac}{2} m_1^2 - bc m_1 n_1 - \frac{bd}{2} n_1^2$$

That is, the ABCD matrix of LCT can be used to calculate the amount of time-shift and modulation of the transform output. So the time shifting and modulation operations pair has very close relation with the LCT.

2.3.2.7 Combination of the differentiation and multiplication property

From differentiation and multiplication property-

$$L_F^{(a,b,c,d)}\left[\left(g_1\frac{d}{dt}-h_1.jt\right)x(t)\right]=\left(g_2\frac{d}{du}-h_2.ju\right)X_{(a,b,c,d)}(u),$$

where,

$$\begin{bmatrix} g_2 \\ h_2 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}.\begin{bmatrix} g_1 \\ h_1 \end{bmatrix}$$

So the differentiation-multiplication operations pair also has very close relation with LCT.

2.3.3 Transform Results of LCT for Some Special Signals

The transform results of LCT for some special signals are given in Table-2.3 **[67]**. The transform results of FRFT for these signals can be obtained by substituting $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$ and then multiplying the results by $\left(e^{-j\alpha}\right)^{1/2}$.

To the prove the transform results given in Table-2.3, the following result can be used **[94, 104]** as given by-

$$\int_{-\infty}^{\infty} e^{-\left(pt^2+qt\right)}\mathrm{d}t=\sqrt{\pi/p}\,e^{\frac{q^2}{4p}}$$

2.3.4 Relation of LCT to the Wigner Distribution Function

The Wigner Distribution Function (WDF) was first proposed in physics to account for quantum corrections to classical statistical mechanics in 1932 by Eugene Wigner **[39]**. It is a time-varying spectrum analysis technique for non-stationary signals and processes. It is a time-frequency representation of fundamental theoretical importance as well as an eminently practical signal analysis tool.

2.3.4.1 The Wigner distribution function

The WDF **[39, 43, 138, 151, 152]** of $x(t)$ is defined as-

$$W_x(t,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)\cdot e^{-j\omega\tau}\,\mathrm{d}\tau$$

TABLE 2.3

TRANSFORM RESULTS OF LCT FOR SOME SPECIAL SIGNALS [67]

S. No.	SIGNAL	LCT OF THE SIGNAL
1.	$\delta(t-\tau)$	$\left(j2\pi b\right)^{-1/2}\cdot e^{\frac{j}{2}\frac{d}{b}u^2}\cdot e^{-\frac{j}{b}u\tau}\cdot e^{\frac{j}{2}\frac{a}{b}\tau^2}$
2.	$\exp\left[-j\left(qt^2+rt\right)\right]$	$\left(a-2qd\right)^{-1/2}.e^{\frac{jd}{2b}u^2}.e^{-j\frac{(u+rb)^2}{2ab-4qb^2}}$ It can also be rewritten as $\left(a-2qb\right)^{-1/2}\cdot e^{j\frac{(c-2qd)}{2a-4qb}u^2}e^{j\frac{rb}{2qb^2-ab}u}e^{j\frac{r^2b^2}{4qb^2-2ab}}$
3.	1	$\sqrt{a^{-1}}.e^{j\frac{c}{2a}u^2}$
4.	$\exp\left(j\tau t\right)$	$\sqrt{a^{-1}}\cdot e^{j\frac{c}{2a}u^2}\cdot e^{j\frac{\tau}{a}u}\cdot e^{-j\frac{b}{2a}\tau^2}$ $\left(Im\left(\tau\right)\geq 0\right)$
5.	$\exp\left(jht^2/2\right)$	$\left(hb+a\right)^{-\frac{1}{2}}\cdot e^{j\frac{(c+hd)}{2(a+hb)}\cdot u^2}$ $\left(Im\left(h\right)\geq 0\right)$
6.	t	$\sqrt{a^{-3}}\cdot u.e^{j\frac{c}{2a}u^2}$.
7.	Step function $u(t)$	$\frac{1}{2\sqrt{\lvert a\rvert}}\cdot e^{\frac{j}{2}\frac{c}{a}u^2}\cdot erfc\left(\frac{u}{\sqrt{j2ab}}\right)$.

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X\left(\omega+\eta/2\right).X^*\left(\omega-\eta/2\right)e^{jt\eta}\,\mathrm{d}\eta, \qquad \textbf{(2.3.14)}$$

where, $X(\omega)$ is the FT of $x(t)$.

If $W_x(u,v)$ and $W_{X_{(a,b,c,d)}}(u,v)$ be the Wigner distributions of $x(t)$ and $X_{(a,b,c,d)}(u)$ respectively, then the relation between Wigner distribution and LCT is as given below-

$$W_{X_{(a,b,c,d)}}\left(u,v\right)=W_x\left(du-bv,-cu+av\right) \qquad \textbf{(2.3.15)}$$

$$W_x\left(u,v\right)=W_{X_{(a,b,c,d)}}\left(au+bv,cu+dv\right) \qquad \textbf{(2.3.16)}$$

$$\left|X_{(a,b,c,d)}\left(u\right)\right|^2=\int_{-\infty}^{\infty}W_x\left(du-bv,-cu+av\right)\mathrm{d}v \qquad \textbf{(2.3.17)}$$

From (2.3.17), it has found that the square modulus of the LCT equals the marginal integration of the Wigner distribution following the rotation of coordinate. This result is helpful to derive the transfer function of the multiplicative filter in the LCT domain.

2.3.4.2 LCT as a tool in time-frequency plane

The LCT is more general than the FRFT. The FRFT can only do the rotation of the time-frequency distribution but the LCT can do the chirp multiplication, chirp convolution, tilting, dilation etc. along with time-frequency rotation as shown in Figure-2.5 **[67]**.

2.3.4.3 Quantum Mechanical Representation of LCT

To relate the LCT kernel in quantum mechanical representation, the completeness relation of the coherent state with the arguments of z_1 and z_2 is used. The coherent state $|z\rangle$ is defined **[156]** by the eigenstate of the annihilation operator a with the complex eigenvalue z, i.e.

$$a|z\rangle = z|z\rangle$$

and form the completeness relation **[13]**

$$\int \frac{d^2 z}{2\pi j} |z\rangle\langle z| = 1$$

where, $\int d^2 z \equiv \int \mathrm{d}\left[\mathrm{Re}(z)\right]\mathrm{d}\left[\mathrm{Im}(z)\right]$

To relate the LCT kernel in quantum mechanics, the completeness relation of the coherent state with the arguments of z_1 and z_2 can be used-

$$\langle Q|U(t)|q\rangle = \int \frac{d^2 z_1 d^2 z_2}{(2\pi j)^2} \langle Q|z_1\rangle\langle z_1|U(t)|z_2\rangle\langle z_2|q\rangle \qquad \textbf{(2.3.18)}$$

where, the kernel is just the transition amplitude from position q to the position at a later time Q as given by $K_{(a,b,c,d)}(Q,q) = \langle Q|U(t)|q\rangle$, $U(t)$ is the unitary operator and with the aid of the IWOP technique, the coherent state representation of the unitary operator is given by **[12, 59, 60]**

$$\langle z_1|U(t)|z_2\rangle = \frac{1}{\sqrt{s}} \exp\left[-\frac{r}{2s}\left(z_1^*\right)^2 + \frac{z_2 z_1^*}{s} + \frac{r^*}{2s}\left(z_2\right)^2 - \frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} \right] \qquad \textbf{(2.3.19)}$$

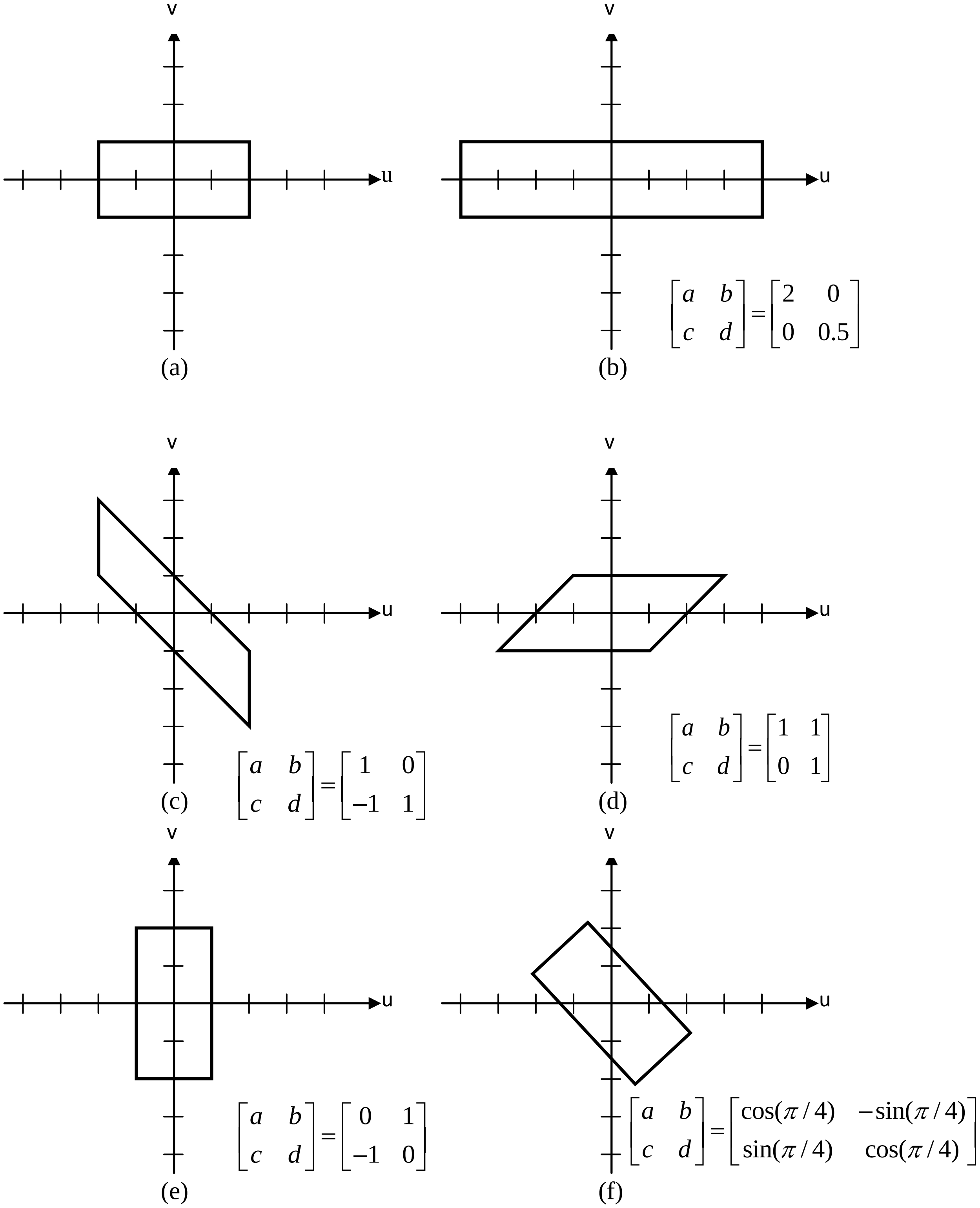


Figure-2.5: (a) Region representing the support of a signal. Effect of (b) scaling (c) chirp multiplication (d) chirp convolution (e) Fourier transform (f) fractional Fourier transform with order 0.5 [67].

where, $(r,s), \left|s^2\right| - \left|r^2\right| = 1$, are related to a classical ray transfer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by-

$$s = \frac{1}{2}\left[a + d + j(b-c)\right], \quad r = \frac{1}{2}\left[d - a - j(b+c)\right], \tag{2.3.20}$$

the uni-modularity condition $ad - bc = 1$ is equivalent to $\left|s^2\right| - \left|r^2\right| = 1$ and the coherent state with coordinate representation is given by **[13]** -

$$\langle z_2 | q \rangle = \frac{1}{\pi^{1/4}} \exp\left[-\frac{q^2}{2} + \sqrt{2} z_2^* q - \frac{\left(z_2^*\right)^2}{2} - \frac{\left|z_2\right|^2}{2} \right] \tag{2.3.21}$$

$$\langle z_1 | Q \rangle = \frac{1}{\pi^{1/4}} \exp\left[-\frac{Q^2}{2} + \sqrt{2} Q z_1^* - \frac{\left(z_1^*\right)^2}{2} - \frac{\left|z_1\right|^2}{2} \right] \tag{2.3.22}$$

By substituting the values of (2.3.19), (2.3.21) and (2.3.22) in (2.3.18), results-

$$\begin{aligned} \langle Q | U(t) | q \rangle &= \int \frac{d^2 z_1 d^2 z_2}{(2\pi j)^2} . \frac{1}{\pi^{1/2}} \frac{1}{\sqrt{s}} \exp\left[-\frac{Q^2}{2} + \sqrt{2} z_1 Q - \frac{z_1^2}{2} - \frac{\left|z_1\right|^2}{2} \right] \\ &\exp\left[-\frac{r}{2s}\left(z_1^*\right)^2 + \frac{z_2 z_1^*}{s} + \frac{r^*}{2s}\left(z_2\right)^2 - \frac{\left|z_1\right|^2}{2} - \frac{\left|z_2\right|^2}{2} \right] \exp\left[-\frac{q^2}{2} + \sqrt{2} z_2^* q - \frac{\left(z_2^*\right)^2}{2} - \frac{\left|z_2\right|^2}{2} \right] \end{aligned} \tag{2.3.23}$$

Solving for z_1 by using the fact **[121]**, if $a_1^2 - 4\left|c_1\right|^2 > 0$, then-

$$\begin{aligned} &\frac{1}{\pi} \int d^2\alpha . \exp\left(-a_1 |\alpha|^2 + b_1 \alpha + b_1^* \alpha^* + c_1 \alpha^2 + c_1^* \left(\alpha^*\right)^2 \right) = \sqrt{\frac{1}{a_1^2 - 4\left|c_1\right|^2}} \exp\left[\frac{b_1^2 c_1^* + b^{*2} c_1 + a_1 \left|b_1\right|^2}{a_1^2 - 4\left|c_1\right|^2} \right] \\ &\Rightarrow \frac{1}{\pi} \int d^2 z_1 \exp\left[-\left|z_1\right|^2 + z_1 \sqrt{2} Q + \frac{z_2 z_1^*}{s} - \frac{z_1^2}{2} - \frac{r}{2s}\left(z_1^*\right)^2 \right] \end{aligned} \tag{2.3.24}$$

where, $a_1 = 1, b_1 = \sqrt{2}Q,\ b_1^* = \frac{z_2}{s}, c_1 = -\frac{1}{2}, c_1^* = -\frac{r}{2s}$ and $a_1^2 - 4\left|c_1\right|^2 > 0$,

$$\therefore \sqrt{\frac{1}{a_1^2 - 4\left|c_1\right|^2}} \exp\left[\frac{b_1^2 c_1^* + b^{*2} c_1 + a_1 \left|b_1\right|^2}{a_1^2 - 4\left|c_1\right|^2} \right] = \sqrt{\frac{s}{s-r}} \exp\left[\frac{-Q^2 r - \frac{z_2^2}{2s} + \sqrt{2} Q z_2}{s-r} \right] \tag{2.3.25}$$

Similarly, solving for z_2 and substituting the value in (2.3.23), finally it results in-

$$\langle Q|U(t)|q\rangle = \sqrt{\frac{1}{\pi\left(s-s^*-r+r^*\right)}}\exp\left[\frac{2qQ}{s-s^*-r+r^*}-\frac{q^2}{2}\frac{s+s^*-r-r^*}{s-s^*-r+r^*}-\frac{Q^2}{2}\frac{s+s^*+r+r^*}{s-s^*-r+r^*}\right] \quad \textbf{(2.3.26)}$$

By substituting the value of r and s from (2.3.20), the LCT kernel is given by-

$$\langle Q|U(t)|q\rangle = \sqrt{\frac{1}{2jb\pi}}\exp\left[\frac{j}{2b}\left(aq^2-2qQ+dQ^2\right)\right] \quad \textbf{(2.3.27)}$$

Equation (2.3.27) represents the LCT kernel derived by representation transformation in quantum mechanics.

2.3.6 Existing Convolution and Correlation Theorems for LCT

The convolution and correlation theorems for LCT have been suggested by many researchers in the past. The chronological development in this dimension is given in the following paragraphs.

In 2006, Deng *et al.* **[20]** proposed two definitions of convolution theorem for LCT and one definition of product theorem for LCT. But, in his paper he agreed that one of his convolution theorem is more complicated, and is inconvenient for the analysis of multiplicative filter in the LCT domain. The convolution theorem given by Deng *et al.* **[20]** is-

If $X_{(a,b,c,d)}(u)=L_{(a,b,c,d)}\left[x(t)\right]$ and $Y_{(a,b,c,d)}(u)=L_{(a,b,c,d)}\left[y(t)\right]$ then

$$\begin{aligned} z(t)=(x\otimes y)(t)=\sqrt{\frac{1}{j2\pi b}}\,e^{-j\frac{a}{2b}t^2}\int_{-\infty}^{\infty}x(\tau)e^{j\frac{a}{2b}t^2}y(t-\tau)e^{j\frac{a}{2b}(t-\tau)^2}\mathrm{d}\tau \\ \xleftrightarrow{LCT} L_{(a,b,c,d)}\left[z(t)\right]=e^{-j\frac{d}{2b}u^2}Y_{(a,b,c,d)}(u)\,X_{(a,b,c,d)}(u) \end{aligned} \quad \textbf{(2.3.28)}$$

where, $'\otimes'$ indicates the convolution operation and the product theorem is given by-

$$s(t)=x(t)y(t)\xleftrightarrow{LCT}S_{(a,b,c,d)}(u)=\left|\frac{1}{2\pi b}\right|e^{j\frac{d}{2b}u^2}\int_{-\infty}^{\infty}X_{(a,b,c,d)}(v)Y_{(a,b,c,d)}\left(\frac{u-v}{b}\right)e^{-j\frac{d}{2b}v^2}\mathrm{d}v \quad \textbf{(2.3.29)}$$

But the theorem given by Deng *et al.* **[20]** with modified convolution operation contains three and seven chirp multiplications on the left-hand side, i.e., the proposed correlation process and right-hand side that represents its transform respectively as compared to two and seven for the proposed convolution theorem.

Later on in year 2009, a different definition of convolution theorem in LCT domain is given by Wei *et al.* **[38]**. Prior to defining convolution theorem, author defines a $\tau -$ generalized translation of signal $y(t)$ denoted by $y(t\theta\tau)$.

where, $$y(t\theta\tau)=\int_{-\infty}^{\infty} Y_{(a,b,c,d)}(u)K_{(a,b,c,d)}(u,\tau)\ K^{*}_{(a,b,c,d)}(u,t)\,\mathrm{d}u \qquad \textbf{(2.3.30)}$$

where, $Y_{(a,b,c,d)}(u)$ is the LCT of $y(t)$. K and K^{*} represents the LCT and inverse LCT kernels respectively. Based on this generalized function convolution theorem is defined as-

$$z(t)=(x\otimes y)(t)=\int_{-\infty}^{\infty} x(\tau)y(t\theta\tau)\,\mathrm{d}\tau \xleftrightarrow{LCT} X_{(a,b,c,d)}(u)Y_{(a,b,c,d)}(u) \qquad \textbf{(2.3.31)}$$

In the definition given by Wei *et al.* **[38]**, the generalized convolution operation defined in time domain is not only dependent on time variable but it also depends on transform domain variable '*u*' in which it has to be transformed.

Later in the year 2012, Wei *et al.* **[36]** again derived a convolution and correlation theorem based upon space and shift property of LCT. If $X_{(a,b,c,d)}(u)=L_F^{(a,b,c,d)}\left[x(t)\right]$ and $Y_{(a,b,c,d)}(u)=L_F^{(a,b,c,d)}\left[y(t)\right]$ then-

$$\left[f\overset{A}{\otimes}g\right](t)=B_A L^{-1}_{(a,b,c,d)}\left[F_{(a,b,c,d)}(u)G_{(a,b,c,d)}(u)e^{-j\frac{d}{2b}u^{2}}\right](t) \qquad \textbf{(2.3.32)}$$

where, $B_A=\sqrt{\dfrac{1}{j2\pi b}}$ and the operator $\overset{A}{\otimes}$ is defined as-

$$\left[f\overset{A}{\otimes}g\right](t)=B_A^{2}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)e^{-ja\tau(t-\tau)/b}\,\mathrm{d}\tau \qquad \textbf{(2.3.33)}$$

In this paper, Wei *et al.* **[36]** used the space and shift properties of the LCT to derive the convolution and correlation operation. In the derived operations, it has been found that the result does not meet the requirement of symmetric definition of FT. As a comparison of computational complexity, it has been found that number of chirp multiplications required by the left hand side i.e. by the correlation function and right hand side that represents its transform are two and ten respectively.

In 2012, again Wei *et al.* **[37]** suggested a convolution theorem especially for translation invariance property. In **[37]**, the author suggested that this theorem can be more useful in practical analog filtering. In this thesis, convolution and correlations have been derived abd compared with the literature in terms of computational complexity, FT conversion and variable dependability. The improvement of the proposed theorems is shown with the help of simulation comparison as well as by comparing the results of multiplicative filtering in LCT domain with that of frequency domain and fractional domain filtering.

2.3.7 Advantages of LCT over FT/FRFT

The FT is one of special cases of LCT. The LCT can solve the problems that the FT cannot solve and LCT can improve the performance. The following are the advantages of LCT contrast with FT/FRFT:

- The LCT is more general and flexible than the FT. So, applications, properties and operations of the FT can be generalized by the LCT. When LCT's $(a,b,c,d)=(0,1,-1,0)$, it becomes the FT.
- The LCT can be applied to partial differential equations (order $n>2$). With the appropriate choice of LCT variables (a,b,c,d), the equation can be reduced to the order of $n-1$. Furthermore, the LCT can be used in optical systems.
- Since the FT only deals with the stationary signals, whereas LCT deals with stationary as well as time-varying signals.
- The LCT is more general than the FRFT. The FRFT can only do the rotation of the time-frequency distribution but the LCT can do chirp multiplication, chirp convolution, tilting, dilation etc. along with time-frequency rotation.
- In the literature, it has been found that the derived convolution theorems for FRFT and LCT contains extra chirp factor $e^{-j\frac{u^2}{2}\cot\alpha}$ and $e^{-j\frac{d}{2b}u^2}$ respectively. In filtering applications for better results, it is desired to diminish this chirp equal to 1. In LCT, it can be easily obtained by substituting LCT variable $d=0$ whereas in FRFT $\cot\alpha$ is zero only when $\alpha=\frac{\tilde{a}\pi}{2}=90^{\circ}$ and $\tilde{a}=1$. But at $\tilde{a}=1$, the fractional domain is converted to frequency domain and results more MSE as compare to fractional domain. Therefore, LCT supersedes the FRFT because of it is more flexible for its extra degree of freedom of three free variables.
- The LCT with complex variables can transform some functions that cannot be

transformed by the LCT with real variables e.g. Laplace transform. Besides this, the LCT with six variables (a,b,c,d,m,n) is very helpful for optical system analysis.

- The one dimensional (1-D) LCT can let filters to reduce the sampling rate and encryption, but the FT cannot do that.
- Using the LCT to design the filters, it can reduce the MSE. Besides, using the LCT, many noises can be filtered out that the FT and FRFT cannot remove in optical system, microwave system, radar system, and acoustics etc.
- In modulation, FT can be only used for bandlimited signals. However, if signals are not bandlimited, LCT is helpful.
- Since, the LCT have more free variables than the FRFT, so, encryption is safer in using the LCT than in using the FRFT.
- In signal synthesis, using the transformed domain of the LCT to analyze some signal is easier than using the time domain or frequency domain to analyze signals.
- To calculate the Ambiguity Function (AF), using LCT, the positions of output sampling points are freely selected.
- In multiplexing, OLCT is very helpful because of it's time-shifting and frequency-shifting property.

2.4 PRELIMINARIES OF OFFSET LINEAR CANONICAL TRANSFORM

The OLCT **[15, 79, 136, 148]** is a six parameter (a,b,c,d,m,n) class of linear integral transform. It was introduced in 1994 **[131]** and found many applications in signal processing, optics and many other areas **[15, 74, 115, 132]**. Well-known transforms such as FT **[42, 73, 74]**, the offset FT **[15, 136, 148]**, FRFT **[90, 138]**, the offset FRFT **[136, 148]**, LCT **[23, 101, 103, 137, 149]**, FST **[27]**, Gaussian Weierstrass **[79]**, time-shifting and scaling, frequency-modulation, pulse chirping, Lorentz squeezing **[132]** and others, can be seen as the special cases of the OLCT **[15, 115, 131]**.

The OLCT is also known as the Special Affine Fourier Transform (SAFT) **[132]**. The name is based upon the properties of the transform. The term AFFINE refers to the fact that under any such transformation every triangular area is mapped to another triangular area. The term SPECIAL refers to the fact that under this transformation it is only affine transforms with determinant equal to unity (area conserving). The SAFT allows shifting/translation, rotating and squeezing of a signal to fit within a fixed window as compared to only rotation in case of FRFT/LCT. The one-dimensional OLCT in matrix form with (a,b,c,d,m,n) variables is given as **[114, 133]**-

$$\begin{bmatrix} l \\ k \end{bmatrix} \xrightarrow{SAFT} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} l_1 \\ k_1 \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix}. \qquad \textbf{(2.4.1)}$$

where, variables (a,b,c,d,m,n) are real numbers with the 'lossless' (area-preserving, or power-preserving) condition $ad-bc=1$, and the coefficients (a,b,c,d,m,n) being independent of the phase space co-ordinate l (position) and k (spatial frequency).

The integral form of one-dimensional OLCT with (a,b,c,d,m,n) variables of a signal $x(t)$ is defined **[15, 148]** as-

$$L_F^{(a,b,c,d,m,n)}\left[x(t)\right](u) = \begin{cases} \sqrt{\dfrac{1}{j2\pi b}} \displaystyle\int_{-\infty}^{+\infty} x(t) K_{(a,b,c,d,m,n)}(u,t)\,\mathrm{d}t, & b \neq 0 \\ \sqrt{d}\, e^{j\frac{cd}{2}(u-m)^2 + jnu} h\left[d.(u-m)\right], & b = 0 \end{cases} \qquad \textbf{(2.4.2)}$$

In this thesis, only the case of $b \neq 0$ is considered, since the definition for case $b=0$ is the limit of the integral in (2.4.2) for the case $b \neq 0$ as $|b| \to 0$. The term $K_{(a,b,c,d,m,n)}(u,t)$ represents the integral kernel and is given by-

$$K_{(a,b,c,d,m,n)}(u,t) = K_A.\exp\left[\frac{j\left(at^2+du^2\right)}{2b} + \frac{jx.(m-u)}{b} - \frac{ju.(dm-bn)}{b}\right] \qquad \textbf{(2.4.3)}$$

where, $K_A = \exp\left[\dfrac{jd}{2b} m^2\right]$

The one-dimensional inverse OLCT in matrix form with $(d,-b,-c,a,bn-dm,cm-an)$ variables is given as **[66, 132]**:

$$\begin{bmatrix} l \\ k \end{bmatrix} \xrightarrow{INVERSE\ SAFT} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} l_1 \\ k_1 \end{bmatrix} + \begin{bmatrix} bn-dm \\ cm-an \end{bmatrix} \qquad \textbf{(2.4.4)}$$

and with integral form is given as-

$$x(t) = \sqrt{\frac{1}{-j2\pi b}} \int_{-\infty}^{+\infty} X_{(a,b,c,d,m,n)}(u) K^*_{(a,b,c,d,m,n)}(u,t)\,\mathrm{d}u, \quad b \neq 0 \qquad \textbf{(2.4.5)}$$

where, $*$ indicates the complex conjugate and the integral kernel is given as-

$$K^{*}_{(a,b,c,d,m,n)}(u,t) = K^{*}_{A}\exp\left[\frac{-j\left(at^{2}+du^{2}\right)}{2b} - \frac{jx.(m-u)}{b} + \frac{ju.(dm-bn)}{b}\right] \tag{2.4.6}$$

where, $K^{*}_{A} = \exp\left[\frac{-jd}{2b}m^{2}\right]$

2.4.1 Special Cases of OLCT

Some of the special cases of OLCT are given in Table-2.4 **[15, 115]** and these cases can be easily verified by substituting the specific values of $A=(a,b,c,d,m,n)$ variables.

TABLE 2.4

SPECIAL CASES OF OLCT [115]

TRANSFORM	PARAMETER
Offset Linear Canonical Transform (OLCT)	$A=(a,b,c,d,m,n)$
Linear Canonical Transform (LCT)	$A=(a,b,c,d,0,0)$
Fourier Transform (FT)	$A=(0,1,-1,0,0,0)$
Offset Fourier Transform (OFT)	$A=(0,1,-1,0,m,n)$
Fractional Fourier Transform (FRFT)	$A=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)$
Offset Fractional Fourier Transform OFRFT)	$A=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,m,n)$
Fresnel Transform (FST)	$A=(1,b,0,1,0,0)$
Time Scaling	$A=(d^{-1},0,0,d,0,0)$
Time Shifting	$A=(1,0,0,1,m,0)$
Frequency Modulation	$A=(1,0,0,1,0,n)$

2.4.2 Basic Properties of OLCT

The two basic properties of OLCT are the additivity and reversibility properties **[148]**, which are helpful to design the multiplicative filter.

Property 1. Additivity property

If $O_F^{(a,b,c,d,m,n)}\left[x(t)\right]$ is the OLCT of $x(t)$, then

$$O_F^{(a_2,b_2,c_2,d_2,m_2,n_2)}\left\{O_F^{(a_1,b_1,c_1,d_1,m_1,n_1)}\left[x(t)\right]\right\}=\exp\left(j\varphi\right)O_F^{(e,f,g,h,r,s)}\left[x(t)\right] \qquad \textbf{(2.4.7)}$$

where,

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix},$$

$$\begin{pmatrix} r \\ s \end{pmatrix}=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{pmatrix} m_1 \\ n_1 \end{pmatrix}+\begin{pmatrix} m_2 \\ n_2 \end{pmatrix},$$

$$\varphi=-\frac{a_2c_2}{2}m_1^2-b_2c_2m_1n_1-\frac{b_2d_2}{2}n_1^2-\left(c_2m_1+n_1d_2\right)m_2$$

Property 2. Reversibility property

If $O_F^{(d,-b,-c,a,-dm+bn,cm-an)}\left[x(t)\right]$ is the inverse OLCT of $x(t)$, then

$$\left\{O_F^{(a,b,c,d,m,n)}\left[x(t)\right]\right\}^{-1}=\exp\left(\frac{j}{2}cdm^2-jadmn+\frac{j}{2}abn^2\right)\times O_F^{(d,-b,-c,a,-dm+bn,cm-an)}\left[x(t)\right] \qquad \textbf{(2.4.8)}$$

The proof of these properties can be obtained similar to proof of LCT properties given in section 2.3.2.

2.4.3 Time-frequency Representation of OLCT

The joint parameter time-frequency representation for a signal $s(t)$ with an analytic associate $z(t)$ obtained through the Wigner-Ville Distribution (WVD) $W(t,\omega)$ is given by **[39]**

$$W(t,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}z\left(t+\frac{\tau}{2}\right)z^*\left(t-\frac{\tau}{2}\right)e^{-j\tau\omega}\mathrm{d}\tau \qquad \textbf{(2.4.9)}$$

In general, the Wigner Distribution (WD) of a signal $z(t)$ is defined as above, while the WVD is defined as the WD where, $z(t)$ is the analytic associate (also "analytic signal" or "complex signal") of $s(t)$. The analytic associate $z(t)$ of a signal $s(t)$ is defined as $x(t)=s(t)+jH\left[s(t)\right]$, where $H\left[s(t)\right]$ is the Hilbert Transform of the signal $s(t)$.

The WVD is related to OLCT as given by

$$W_{F_{(a,b,c,d,m,n)}}(u,v) = W_f\left[d(u-m)-b(v-n), -c(u-m)+a(v-n)\right] \quad \textbf{(2.4.10)}$$

Equation (2.4.10) can be verified directly from the definitions (2.4.2) and (2.4.10) or alternatively, it can be proved by using the respective relation for the WVD of the LCT **[138]** together with the fact that the OLCT can be obtained by shifting the LCT of $z(t)$ by m in the time-domain and modulating with e^{jun} i.e.

$$L_F^{(a,b,c,d,m,n)}\left[h(t)\right](u) = L_F^{(a,b,c,d,0,0)}\left[h(t)\right](u-m)e^{jn(u-m)} \quad \textbf{(2.4.11)}$$

Hence from equation (2.4.10), it can be seen that the OLCT performs the most general linear inhomogeneous linear mapping of the time-frequency domain given by equation (2.4.1), which is recognized as an affine mapping. With only four variables (a,b,c,d) in equation (2.4.10), a homogeneous linear mapping is obtained that represents the LCT. The other two variables (m,n) performs an offset in the time-frequency domain, generalizing the OLCT to include transforms and responses of systems that perform frequency modulation and time delay. The Figure-2.6 shows the WD of several OLCT of $x(t)$.

2.4.4 Existing Convolution and Correlation Theorems for OLCT

In 2012, Xiang *et al.* **[115]** first time presented the convolution and correlation theorems for the OLCT. The author used the space and shift properties of the OLCT to derive the convolution and correlation operation.

If $X_{(a,b,c,d,m,n)}(u) = L_F^{(a,b,c,d,m,n)}\left(x(t)\right)$ and $Y_{(a,b,c,d,m,n)}(u) = L_F^{(a,b,c,d,m,n)}\left(y(t)\right)$ then-

$$\left[f \overset{A}{\otimes} g\right](t) = B_A L_F^{(a,b,c,d,m,n)^{-1}}\left[F_{(a,b,c,d,m,n)}(u)G_{(a,b,c,d,m,n)}(u)e^{\frac{j}{2b}\left[-du^2+2u(dm-bn)\right]}\right](t) \quad \textbf{(2.4.12)}$$

where, $B_A = \sqrt{\frac{1}{j2\pi b}}e^{j\frac{d}{2b}m^2}$ and the operator $\overset{A}{\otimes}$ is the convolution operation and is defined as

$$\left[f \overset{A}{\otimes} g\right](t) = B_A \int_{-\infty}^{\infty} f(\tau)g(t-\tau)e^{-ja\tau(t-\tau)/b}\mathrm{d}\tau \quad \textbf{(2.4.13)}$$

and the correlation operation is defined by-

$$\left[f \overset{A}{\star} g\right](t) = B_A L_F^{(a,b,c,d,m,n)^{-1}}\left[F_{(a,b,c,d,m,n)}(u)G^*_{(a,b,c,d,m,n)}(u)e^{\frac{j}{2b}\left[du^2-2u(dm-bn)\right]}\right](t) \quad \textbf{(2.4.14)}$$

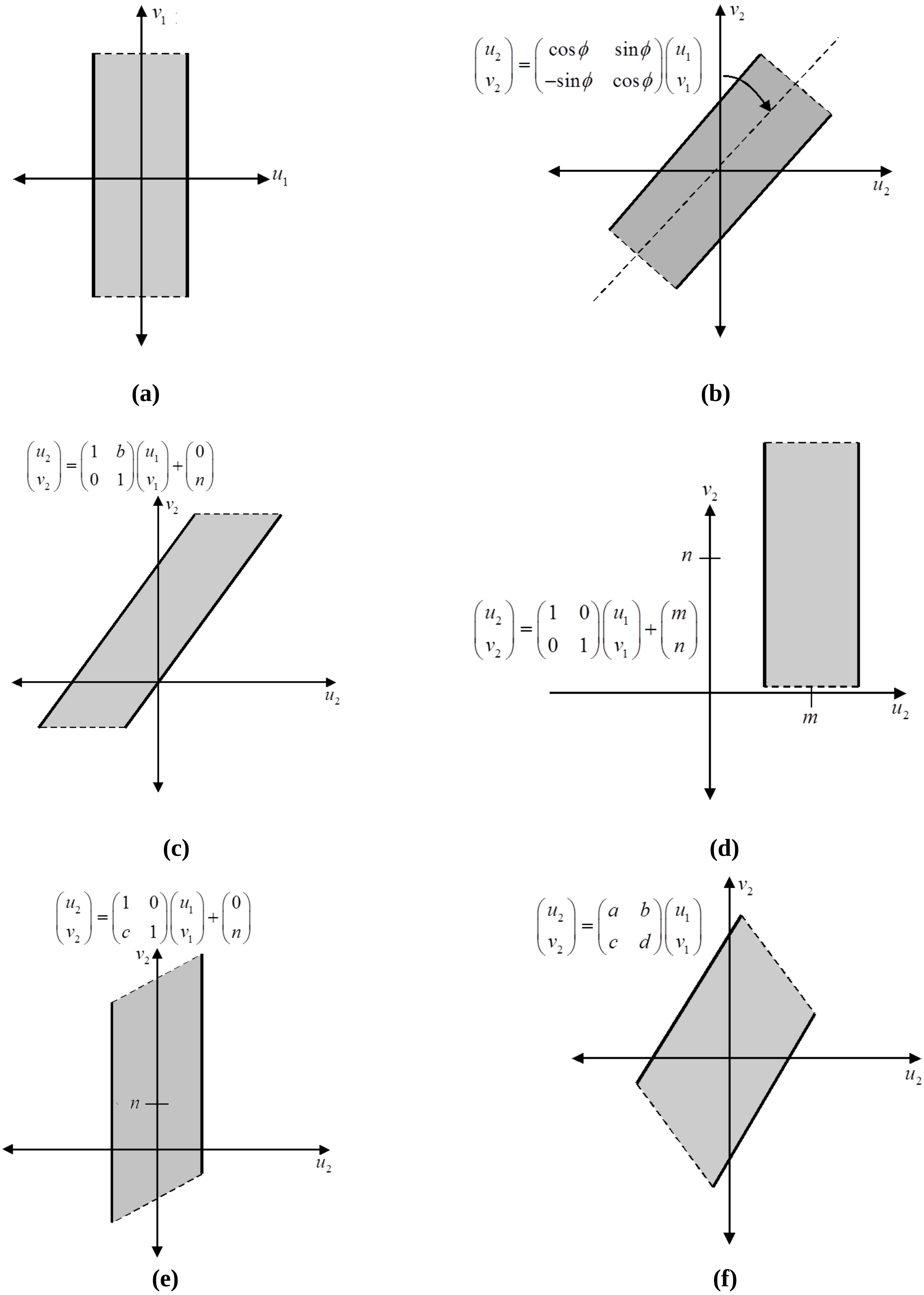


Figure-2.6: (a) The support of $x(t)$ in the Wigner domain. (b)-(f) examples of the support of the WD of several OLCT of $x(t)$.

where, $B_A = \sqrt{\frac{1}{j2\pi b}} e^{j\frac{d}{2b}m^2}$ and the operator $\overset{A}{\star}$ is the correlation operation and is defined as-

$$\left[f \overset{A}{\star} g \right](t) = B_A B_A^* \int_{-\infty}^{\infty} f(\tau) g^*(\tau - t) e^{ja\tau(\tau - t)/b} \mathrm{d}\tau \qquad \textbf{(2.4.15)}$$

In the above derived operations, it has been found that the result does not meet the requirement of symmetric definition of FT.

- A comparative analysis of convolution and product theorems for LCT can be made on some performance metrics as it is not been found.

2.5 MOTIVATION

From the comprehensive study of the available literature, it has been concluded that research has been carried out to derive the convolution and correlation theorems for LCT. There is scope for a possible improvement in these theorems because none of the theorems have received acclamation in terms of FT convertability. The research is motivated to derive the proposed theorems for LCT and then to perform a comparative analysis based upon some perfromane metrics. The theorems have been derived mathematically but until now, these are not been applied for signal processing applications especially in multiplicative filtering. In some of the research papers, a filter has been designed by using the convolution theorem for FRFT but that amounts to be isolation with only one parameter as the degree of freedom as compare to three variables as degree of freedom in the proposed work.

So an interest arises to implement a practical application of multiplicative filtering for LCT domain which has three free variables as compare to one free parameter of FRFT. Based upon the proposed methodology to design the convolution theorem, a research interest develops to define and derive the correlation theorem for LCT as well as convolution and correlation theorem for six-parameter LCT, also known as OLCT.

Based upon the inclusive study of the available literature, efforts has been carried out to outline the gaps in the study and to design the statement of problems for the proposed work.

2.5.1 Gaps in the Study

Keeping in view, the study as reported above, the following observations are made:

- A comparative analysis of convolution and correlation theorems for LCT can be made on some performance metrics as it is not been found.

- Most of the work reported, discussed the use of LCT in optics. Considering the three variables in LCT and the advantages given by FRFT in variety of signal processing applications, there is a need to explore LCT for DSP applications.
- The generalized convolution and correlation theorems can be derived which may be more useful in signal processing applications.
- The analysis of fixed windows in FRFT domain and their usage in FIR filter design has been reported in literature. On the similar lines, the design of FIR filter using window function along with their analysis in LCT domain can be carried out.
- The generalisation of the convolution and correlation theorems derived can be done in OLCT domain, which have been reported very recently.
- To authenticate the usefulness of these theorems in signal processing applications, a framework for the design of multiplicative filter in LCT domain can be established.

2.5.2 Statement of Problems

Based on the initial studies, literature survey and the understanding established, the following objectives are included in this study:

(i) To propose an improved method to derive the convolution and product theorems for LCT.

(ii) On similar lines, the correlation theorem for LCT and the special cases for the proposed theorems can also be derived.

(iii) The proposed theorems will be tested by implementing various applications in signal processing.

2.6 RESEARCH METHODOLOGY

To define and derive the proposed convolution and correlation theorems for LCT is very important for signal processing applications. A number of mathematical algorithms/special functions/theorems etc. will be studied to derive the proposed convolution and correlation theorems. To derive the proposed theorems, the generalization of various existing method for FRFT will be examined broadly. Keeping in view the convolution theorem derived in the literature, proposed theorem will be derived and a comparative analysis will be done based upon some performance metrics. Based upon the results obtained from the proposed convolution theorem, a multiplicative filter will be designed for LCT domain and results will be compared with that of fractional domain filtering and frequency domain filtering. A mathematical analysis of different window functions will be carried out by using extensive mathematical computations. Based upon the anlaysis of the window funcions, a FIR filter will be designed in the LCT domain and considering LCT variables as tuning parameter in the convolution operation between LCT of window function and ideal frequency response, variability in the transition band of the resulting window based low pass FIR filter will be performed.

Based upon the methodology used to derive the convolution theorem, correlation theorem will be derived and will be used to analyze the BW of the FM signal for different combinations of LCT variables. Finally for the proposed theorems, some of the important special cases of LCT will be derived. Finally, the generalisation of the proposed theorems for LCT will be done in the OLCT domain which is more flexible and reliable because of additional two time-shifting and frequency-shifting variables.

The simulation results for comparison of proposed convolution and correlation theorems with that of literature will be simulated on the platform of Wolfram Mathematica® software (version 7.0) and the proposed work model of multiplicative filter will be simulated on the platform of MATLAB software (version 7.8.0, R2009a) on a system having configuration Intel® Core™ i3-330M Processor 2.13 GHz having 3 GB RAM.

CHAPTER 3

THE LCT OF WINDOW FUNCTIONS AND ANALYSIS OF FIR FILTER

The convolution theorem is one of the most important definition of the signal processing in the frequency domain, explains the effects of the transformation from time-domain to the frequency-domain as well as the consequences of applying a filter. Following this, an attempt has been made to understand the filtering using the CCV and windows method.

3.1 ANALYTICAL ASPECTS OF WINDOW FUNCTIONS IN LCT

Window functions, also known as weighting functions, tapering functions, or apodization functions, **[41]**, are explicity or implicity used in many DSP applications **[18, 93, 105]**. These are used in harmonic analysis to reduce the undesirable effects related to the spectral leakage. In signal processing, a window function is a mathematical function that is zero-valued outside of some chosen interval and when a signal is multiplied by a window function, the product is zero-valued outside this interval. These have strong impact on the spectrum of the signal and essentially permit a trade-off between time and frequency resolution. These affect many attributes of a harmonic processor which include detectability, resolution, dynamic range, confidence and ease of implementation **[1]**. Several standard windows are available and the choice is based upon the requirements of a particular application in signal processing **[105]**.

The outcome of this chapter has been published in Research Journal as per following detail: N. Goel, K.Singh, Analysis of Dirichlet, Generalized Hamming and Triangular window functions in the linear canonical transform domain, Springer-Signal Image and Video Processing, vol. 7, no. 5, pp. 911-923, 2013.

Window functions have been successfully used in various areas of signal processing and communications such as, spectral analysis, radar signal processing, digital filter design, biomedical engineering and audio, speech, and image processing **[84]**. A complete harmonic analysis of many window functions and their properties was represented by Harris **[44]** by using Discrete Fourier Transform (DFT). Since a window function is time-limited, its FT is a damped oscillation waveform over the frequency domain, and is classified into mainlobe near the origin, and other infinitely ranging sidelobes. One important characteristic of a window is the resulting sidelobe levels. If a window is designed to give very low sidelobes, it is expected that this can be achieved only at the cost of some other characteristics of the window. This makes selection of the windows difficult because the choice of low peak sidelobe level is beneficial in terms of the resolution of weaker signals in the power spectrum, and a choice of narrow mainlobe width is beneficial in terms of the resolution of two adjacently spaced tones having large amplitude differences.

These window functions are usually characterized by the parameters like Half Main-Lobe Width (HMLW), First Side-Lobe Level (FSLL), Peak Side-Lobe Level (PSLL), 3dB-Band Width (3db-BW), 6dB-Band Width (6dB-BW) and the Side-Lobe Fall-Off Rate (SLFOR). However, all of the window families have a trade-off between the HMLW and the PSLL **[141]**. The smaller is the HMLW, better is the resolution of the estimates. Smaller is the PSLL, smaller is the leakage through the near side lobes. Larger is the SLFOR, smaller is the leakage through the far side lobes **[83]**. In general, a window which yields smaller values of HMLW, FSLL, PSLL, 3dB-BW, 6dB-BW, and large SLFOR is desirable in most of the applications in DSP **[4]**.

In this section, the mathematical model of Dirichlet (Rectagular), Generalised Hamming and Bartlett (Triangular) window functions has been derived using LCT along with its special cases.

3.1.1 Dirichlet Window Function

This window function is also known as rectangular window function. In this section, the mathematical analysis of this window function is carried out by using LCT domain. Without loss of generality, let $x(t)$ be unity at the origin, and time limited to the interval $|t| \le 1/2$, i.e.

$$x(t)=\begin{cases}1 & |t|\le 1/2\\ 0 & \text{elsewhere}\end{cases} \quad \textbf{(3.2.1)}$$

Therefore, the LCT of $x(t)$ can be written as-

$$L_F^{(a,b,c,d)}(u)=\frac{1}{\sqrt{j2\pi b}}\left[\int_{-1/2}^{1/2}1.e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t\right] \quad \textbf{(3.2.2)}$$

By substituting $\left(t-\frac{u}{a}\right)^2=R$ and changing the limit of integration in (3.2.2)-

$$L_F^{(a,b,c,d)}(u)=\frac{1}{\sqrt{j2\pi b}}.e^{\frac{ju^2}{2b}\left(d-\frac{1}{a}\right)}\left[\int_{-\frac{1}{2}-\frac{u}{a}}^{\frac{1}{2}-\frac{u}{a}}1.e^{\frac{ja}{2b}R^2}\mathrm{d}t\right] \quad \textbf{(3.2.3)}$$

By applying $\int_{z_1}^{z_2}e^{at^2}\mathrm{d}t=\frac{\sqrt{\pi}}{2\sqrt{a}}\left[erfi\left(\sqrt{a}\,z_2\right)-erfi\left(\sqrt{a}\,z_1\right)\right]$ **[79]** to (3.2.3) and rearranging, the following expression results-

$$L_F^{(a,b,c,d)}(u)=\frac{1}{\sqrt{j2\pi b}}.e^{\frac{ju^2}{2b}\left(d-\frac{1}{a}\right)}\times\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{ja}{2b}}}\left\{erfi\left(\sqrt{\frac{ja}{2b}}\left(\frac{1}{2}-\frac{u}{a}\right)\right)-erfi\left(\sqrt{\frac{ja}{2b}}\left(-\frac{1}{2}-\frac{u}{a}\right)\right)\right\}\right]$$

(3.2.4)

where, $erfi(z)$ is imaginary error function of z which is defined in the whole complex z-plane and $erfi(z)=-jerf\,(jz)$ **[94]**. Equation (3.2.4) gives the LCT of a Dirichlet function. As a special case of LCT, Substituting $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, in (3.2.4) for FRFT, results in-

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(u)=$$

$$\frac{1}{2\sqrt{\cos\alpha}}.e^{\frac{-ju^2}{2}(\tan\alpha)}\times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha-2u)}{\sqrt{\sin\alpha\cos\alpha}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha+2u)}{\sqrt{\sin\alpha\cos\alpha}}\right)\right\} \quad \textbf{(3.2.5)}$$

Thus from (3.2.5), it can be seen that the LCT for $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$ of Dirichlet window function is directly dependent on the angle α.

Now, substituting $(a,b,c,d)=(0,1,-1,0)$, the LCT results

$$L_F^{(0,1,-1,0)}(u)=\frac{(1-j)\,sin\left(\frac{u}{2}\right)}{u\sqrt{\pi}} \quad \textbf{(3.2.6)}$$

Thus from (3.2.6), it can be seen that the LCT of Dirichlet window function for $(a,b,c,d)=(0,1,-1,0)$ is equal to FT of the function multiplied by factor $\sqrt{-j}$.

3.1.2 Generalized Hamming Window Function

The expression for the Generalized Hamming window function in time domain is given as **[93]**-

$$x(t)=\begin{cases}\beta+(1\text{-}\beta)\cos(2\pi t) & |t|\le 1/2\\ 0 & \text{elsewhere}\end{cases} \qquad \textbf{(3.2.7)}$$

For $\beta=0.54$, Hamming window results and for $\beta=0.50$, Hanning window results.

Rewriting (3.2.7) by using Euler's formulae-

$$x(t)=\beta+(1-\beta)\left(\frac{e^{j2\pi t}+e^{-j2\pi t}}{2}\right) \qquad \textbf{(3.2.8)}$$

Taking LCT of (3.2.8), results in-

$$L_F^{(a,b,c,d)}(u)=\frac{1}{\sqrt{j2\pi b}}\left[\int_{-1/2}^{1/2}\left\{\beta+\left(e^{j2\pi t}+e^{-j2\pi t}\right)\left(\frac{1-\beta}{2}\right)\right\}\times e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t\right] \qquad \textbf{(3.2.9)}$$

Rewriting (3.2.9), results in-

$$L_F^{(a,b,c,d)}(u)=I_1+I_2+I_3 \qquad \textbf{(3.2.10)}$$

where,

$$I_1=\frac{1}{\sqrt{j2\pi b}}\int_{-1/2}^{1/2}\beta.e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t \qquad \textbf{(3.2.11)}$$

$$I_2=\frac{1}{\sqrt{j2\pi b}}\int_{-1/2}^{1/2}\left(\frac{1-\beta}{2}\right).e^{\frac{j}{2}\left(\frac{a}{b}t^2-\left(\frac{2u}{b}-4\pi\right)t+\frac{d}{b}u^2\right)}\mathrm{d}t \qquad \textbf{(3.2.12)}$$

$$I_3=\frac{1}{\sqrt{j2\pi b}}\int_{-1/2}^{1/2}\left(\frac{1-\beta}{2}\right).e^{\frac{j}{2}\left(\frac{a}{b}t^2-\left(\frac{2u}{b}+4\pi\right)t+\frac{d}{b}u^2\right)}\mathrm{d}t \qquad \textbf{(3.2.13)}$$

Now, solving (3.2.11) for I_1 in the same manner as (3.2.2)-

$$I_1=\frac{\beta}{2\sqrt{a}}.e^{\frac{ju^2}{2b}\left(d-\frac{1}{a}\right)}\times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a-2u)}{\sqrt{ab}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a+2u)}{\sqrt{ab}}\right)\right\} \qquad \textbf{(3.2.14)}$$

Now, solving (3.2.12) for I_2 -

$$I_2 = \frac{1-\beta}{4\sqrt{a}}.e^{\frac{j}{2ab}\left((ad-1)u^2-4\pi^2b^2+4b\pi u\right)} \times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a+4\pi b-2u)}{\sqrt{ab}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a-4\pi b+2u)}{\sqrt{ab}}\right)\right\}$$

(3.2.15)

Now, solving (3.2.13) for I_3 -

$$I_3 = \frac{1-\beta}{4\sqrt{a}}.e^{\frac{j}{2ab}\left((ad-1)u^2-4\pi^2b^2-4b\pi u\right)} \times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a+4\pi b+2u)}{\sqrt{ab}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a-4\pi b-2u)}{\sqrt{ab}}\right)\right\}$$

(3.2.16)

Therefore, the LCT of the Generalized Hamming window function can be obtained by adding (3.2.14), (3.2.15) and (3.2.16), results in-

$$L_F^{(a,b,c,d)}(u) = \frac{\beta}{2\sqrt{a}}.e^{\frac{ju^2}{2b}\left(d-\frac{1}{a}\right)} \times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a-2u)}{\sqrt{ab}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a+2u)}{\sqrt{ab}}\right)\right\}+$$
$$\frac{1-\beta}{4\sqrt{a}}.e^{\frac{j}{2ab}\left((ad-1)u^2-4\pi^2b^2+4b\pi u\right)} \times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a+4\pi b-2u)}{\sqrt{ab}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a-4\pi b+2u)}{\sqrt{ab}}\right)\right\}$$
$$\frac{1-\beta}{4\sqrt{a}}.e^{\frac{j}{2ab}\left((ad-1)u^2-4\pi^2b^2-4b\pi u\right)} \times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a+4\pi b+2u)}{\sqrt{ab}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a-4\pi b-2u)}{\sqrt{ab}}\right)\right\}$$

(3.2.17)

Equation (3.2.17) gives the LCT of a Generalized Hamming window function. Substituting $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$ (3.2.17), results in

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(u)=$$
$$\frac{\beta}{2\sqrt{\cos\alpha}}.e^{\frac{-ju^2}{2}(\tan\alpha)} \times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha-2u)}{\sqrt{\cos\alpha\sin\alpha}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha+2u)}{\sqrt{\cos\alpha\sin\alpha}}\right)\right\}+$$
$$\frac{1-\beta}{4\sqrt{\cos\alpha}}.e^{j\left(2\pi u\sec\alpha-2\pi^2\tan\alpha-\frac{1}{2}u^2\tan\alpha\right)} \times\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha-2u+4\pi\sin\alpha)}{\sqrt{\cos\alpha\sin\alpha}}\right)+\right. \quad \textbf{(3.2.18)}$$
$$\left.erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha+2u-4\pi\sin\alpha)}{\sqrt{\cos\alpha\sin\alpha}}\right)\right\}+\frac{1-\beta}{4\sqrt{\cos\alpha}}.e^{-j\left(2\pi u\sec\alpha+2\pi^2\tan\alpha+\frac{1}{2}u^2\tan\alpha\right)} \times$$
$$\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha+2u+4\pi\sin\alpha)}{\sqrt{\cos\alpha\sin\alpha}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha-2u-4\pi\sin\alpha)}{\sqrt{\cos\alpha\sin\alpha}}\right)\right\}$$

Thus from (3.2.18), FRFT as a special case of LCT, it can be seen that the LCT for $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$ of Generalized Hamming window function is directly dependent on the angle α. Also, for $\beta = 0.54$ and 0.50, Hamming and Hanning window functions are obtained respectively.

Now, substituting $(a,b,c,d)=(0,1,-1,0)$, in (3.2.17), results in-

$$L_F^{(0,1,-1,0)}(u)=\frac{(1-j)\left[u^2(1-2\beta)+4\pi^2\beta\right]\sin\left(\frac{u}{2}\right)}{u\sqrt{\pi}\left(4\pi^2-u^2\right)} \tag{3.2.19}$$

Thus from (3.2.19), it can be seen that the LCT of Generalized Hamming window function for $(a,b,c,d)=(0,1,-1,0)$ is equal to FT of the function multiplied by factor $\sqrt{-j}$.

3.1.3 Triangular (Bartlett) Window Function

The expression for the triangular window function in time domain is given as **[93]-**

$$x(t)=\begin{cases}1-2|t| & |t|\le 1/2\\ 0 & \text{elsewhere}\end{cases} \tag{3.2.20}$$

Therefore, LCT of $x(t)$ can be written as-

$$L_F^{(a,b,c,d)}(u)=\frac{1}{\sqrt{j2\pi b}}\left[\int_{-1/2}^{1/2}\left[1-2|t|\right].e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t\right] \tag{3.2.21}$$

Rewriting (3.2.21), results in-

$$L_F^{(a,b,c,d)}(u)=\frac{1}{\sqrt{j2\pi b}}\left[\int_{-1/2}^{1/2}1.e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t+2\int_{-1/2}^{0}t.e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t-2\int_{0}^{1/2}t.e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t\right] \tag{3.2.22}$$

Rewriting (3.2.22), results in-

$$L_F^{(a,b,c,d)}(u)=I_4+I_5+I_6 \tag{3.2.23}$$

where,

$$I_4=\frac{1}{\sqrt{j2\pi b}}\int_{-1/2}^{1/2}1.e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t \tag{3.2.24}$$

$$I_5=\frac{2}{\sqrt{j2\pi b}}\int_{-1/2}^{0}t.e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t \tag{3.2.25}$$

$$I_6=\frac{-2}{\sqrt{j2\pi b}}\int_{0}^{1/2}t.e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t \tag{3.2.26}$$

Now, solving (3.2.24) for I_4 in the same manner as (3.2.2), results in-

$$I_4 = \frac{1}{2\sqrt{a}}.e^{\frac{ju^2}{2b}\left(d-\frac{1}{a}\right)} \times \left\{ erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a-2u)}{\sqrt{ab}} \right) + erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a+2u)}{\sqrt{ab}} \right) \right\} \quad \textbf{(3.2.27)}$$

Now, solving (3.2.25) for I_5. Integrating by parts

$$I_5 = \frac{2}{\sqrt{j2\pi b}}\left[t.\underbrace{\int e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t}_{y} - \int 1.\left(\underbrace{\int e^{\frac{i}{2}\left(\frac{a}{b}t^2-\frac{2u}{b}t+\frac{d}{b}u^2\right)}\mathrm{d}t}_{y} \right)\mathrm{d}t \right]_{-1/2}^{0} \quad \textbf{(3.2.28)}$$

Solving (3.2.28) for y, by applying $erfi(z) = -erf(jz)$ and $\int e^{j\left(at^2+bt+c\right)}\mathrm{d}t = \dfrac{j\sqrt{j\pi}.e^{-j\left(\frac{b^2}{a}-c\right)} erfi\left[\frac{j(b+at)}{\sqrt{a}}\right]}{2\sqrt{a}}$ **[94]**, results in-

$$y = \sqrt{\frac{jb\pi}{2a}}.e^{\frac{j(-1+ad)u^2}{2ab}}.erf\left[-\frac{\left(\frac{1}{2}-\frac{j}{2}\right)(-at+u)}{\sqrt{ab}} \right] \quad \textbf{(3.2.29)}$$

Solving (3.2.28) for $\int y\,\mathrm{d}t$, by applying $\int erf(b+az)\mathrm{d}z = \dfrac{b\,erf(b+az)}{a} + z\,erf(b+az) + \dfrac{e^{-\left(a^2z^2+2abz+b^2\right)}}{a\sqrt{\pi}}$ **[94]**, results in-

$$\int y\,\mathrm{d}t = \sqrt{\frac{jb\pi}{2a}}.e^{\frac{j(-1+ad)u^2}{2ab}}.\left[\left(t-\frac{u}{a}\right) erf\left\{ \frac{-\left(\frac{1}{2}-\frac{j}{2}\right)(-at+u)}{\sqrt{ab}} \right\} \right] + \frac{e^{\frac{j}{2ab}(-at+u)^2}.\sqrt{b}(1+j)}{\sqrt{a\pi}} \quad \textbf{(3.2.30)}$$

By using (3.2.29) and (3.2.30), solving (3.2.28) for I_5, results in-

$$I_5 = \frac{-1}{a^{\frac{3}{2}}.\sqrt{\pi}} e^{\frac{j(-1+ad)u^2}{2ab}} \times$$

$$\left[u\sqrt{\pi}\left\{ erf\left(\frac{\left(\frac{1}{2}-\frac{j}{2}\right)u}{\sqrt{ab}} \right) - erf\left(\frac{\left(\frac{1}{2}-\frac{j}{2}\right)\left(\frac{a}{2}+u\right)}{\sqrt{ab}} \right) \right\} + (1+j)\sqrt{ab}\left\{ e^{\frac{ju^2}{2ab}} - e^{\frac{j(a+2u)^2}{8ab}} \right\} \right] \quad \textbf{(3.2.31)}$$

Similarly, solving (3.2.26) for I_6, results in -

$$I_6 = \frac{-1}{a^{3/2}.\sqrt{j2\pi b}}\left[j2b\sqrt{a}.e^{\frac{ju(-1+du)}{2b}}\left(e^{-\frac{ja}{8b}}+e^{\frac{ja}{2b}}\right)+\right.$$

$$\left.u(1+j)\sqrt{b\pi}\times e^{\frac{ju^2(-1+da)}{2ab}}\left\{erf\left(\frac{\left(\frac{1}{2}-\frac{j}{2}\right)\left(\frac{a}{2}-u\right)}{\sqrt{ab}}\right)+erf\left(\frac{\left(\frac{1}{2}-\frac{j}{2}\right)u}{\sqrt{ab}}\right)\right\}\right] \quad \textbf{(3.2.32)}$$

Therefore,, the LCT of the triangular window function can be obtained by adding (3.2.27), (3.2.31) and (3.2.32) , results in-

$$L_F^{(a,b,c,d)}(u) = \frac{1}{2\sqrt{a}}.e^{\frac{ju^2}{2b}\left(d-\frac{1}{a}\right)}\left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a-2u)}{\sqrt{ab}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(a+2u)}{\sqrt{ab}}\right)\right\}-\frac{1}{a^{\frac{3}{2}}.\sqrt{\pi}}$$

$$e^{\frac{j(-1+ad)u^2}{2ab}}\times\left[u\sqrt{\pi}\left\{erf\left(\frac{\left(\frac{1}{2}-\frac{j}{2}\right)u}{\sqrt{ab}}\right)-erf\left(\frac{\left(\frac{1}{2}-\frac{j}{2}\right)\left(\frac{a}{2}+u\right)}{\sqrt{ab}}\right)\right\}-\frac{1}{a^{3/2}.\sqrt{j2\pi b}}\times\left\{j2b\sqrt{a}.\right.\right.$$

$$\left.\left.e^{\frac{ju(-1+du)}{2b}}\left(e^{-\frac{ja}{8b}}+e^{\frac{ju}{2b}}\right)+u(1+j)\sqrt{b\pi}\times e^{\frac{iu^2(-1+da)}{2ab}}+\left(erf\left[\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\left(\frac{a}{2}-u\right)}{\sqrt{ab}}\right]+erf\left[\frac{\left(\frac{1}{2}-\frac{i}{2}\right)u}{\sqrt{ab}}\right]\right)\right\}\right]$$

(3.2.33)

Equation (3.2.33) gives the LCT of a triangular window function. Substituting $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$ in (3.2.33), results in-

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(u) = \left\{erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha-2u)}{\sqrt{\cos\alpha\sin\alpha}}\right)+erf\left(\left(\frac{1}{4}-\frac{j}{4}\right)\frac{(\cos\alpha+2u)}{\sqrt{\cos\alpha\sin\alpha}}\right)\right\}$$

$$\frac{1}{2\sqrt{\cos\alpha}}.e^{\frac{-ju^2}{2}\tan\alpha}-\frac{1}{(\cos\alpha)^{\frac{3}{2}}.\sqrt{\pi}}\left[u\sqrt{\pi}\left\{erf\left[\frac{\left(\frac{1}{2}-\frac{j}{2}\right)u}{\sqrt{\cos\alpha\sin\alpha}}\right]-erf\left[\frac{\left(\frac{1}{2}-\frac{j}{2}\right)\left(\frac{\cos\alpha}{2}+u\right)}{\sqrt{\cos\alpha\sin\alpha}}\right]\right\}\right]$$

$$e^{\frac{-ju^2}{2}\tan\alpha}-\frac{1}{(\cos\alpha)^{3/2}.\sqrt{j2\pi\sin\alpha}}\left[j2\sin\alpha\sqrt{\cos\alpha}.e^{\frac{ju(-1+u\cos\alpha)}{2\sin\alpha}}\left(e^{-\frac{j\cot\alpha}{8}}+e^{\frac{ju\csc\alpha}{2}}\right)+u(1+j)\right.$$

$$\left.\sqrt{\pi\sin\alpha}\times e^{\frac{-ju^2\tan\alpha}{2}}\left[erf\left[\frac{\left(\frac{1}{2}-\frac{j}{2}\right)\left(\frac{\cos\alpha}{2}-u\right)}{\sqrt{\sin\alpha\cos\alpha}}\right]+erf\left[\frac{\left(\frac{1}{2}-\frac{j}{2}\right)u}{\sqrt{\sin\alpha\cos\alpha}}\right]\right]\right]$$

(3.2.34)

Thus from (3.2.34), it can be seen that for $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$ the LCT of triangular window function is directly dependent on the angle α. Now, substituting $(a,b,c,d)=(0,1,-1,0)$, in (3.2.33), results in

$$L_F^{(0,1,-1,0)}(u)=\frac{4(1-j)}{u^2\sqrt{\pi}}\sin^2\left(\frac{u}{4}\right) \qquad \textbf{(3.2.35)}$$

Thus from (3.2.35), it can be seen that the LCT of Generalized Hamming window function for $(a,b,c,d)=(0,1,-1,0)$ is equal to FT of the function multiplied by factor $\sqrt{-j}$.

The plots of Dirichlet, Hamming, Hanning and Bartlett windows as a function of time for calculating -3 dB BW, -6 dB BW, maximum side lobe level (MSLL), HMLW and SLFOR by varying different values of LCT variables (a,b,c,d) as functions of gain (dB) versus bins are shown in Figure-3.1 and Figure-3.2.

The values of MSLL, HMLW, -3 dB BW, -6 dB BW and SLFOR for Dirichlet, Hamming, Hanning and triangular window functions for different values of LCT variables are tabulated in Tables-3.1, 3.2, 3.3 and 3.4 respectively and shown graphically in Figures 3.3, 3.4, 3.5 and 3.6 respectively.

From these figures, it has been observed that for all the window functions, the window parameters HMLW, -3dB BW and -6dB BW increases as the value of LCT variables approaches the special case FT. Whereas, there are little oscillations in the value of SLFOR and MSLL and these remains almost constant. Further, it has been observed that there is a trade-off between the values of HMLW and MSLL. Smaller the value of HMLW, more is the value of MSLL and vice versa. To reduce the effect of near side lobes, Dirichlet window function is preferred because it has smaller values of MSLL and to reduce the effect of far side lobes, Hanning window is preferred because it has highest SLFOR. Hence a fixed window function for FT can be converted into variable window function by using different values of LCT variables. However, as per the application requirement, the window function may be selected but in general, a window which yields smaller values of HMLW, FSLL, PSLL, 3dB-BW, 6dB-BW, and large SLFOR is desirable in most of the applications in DSP **[4]**.

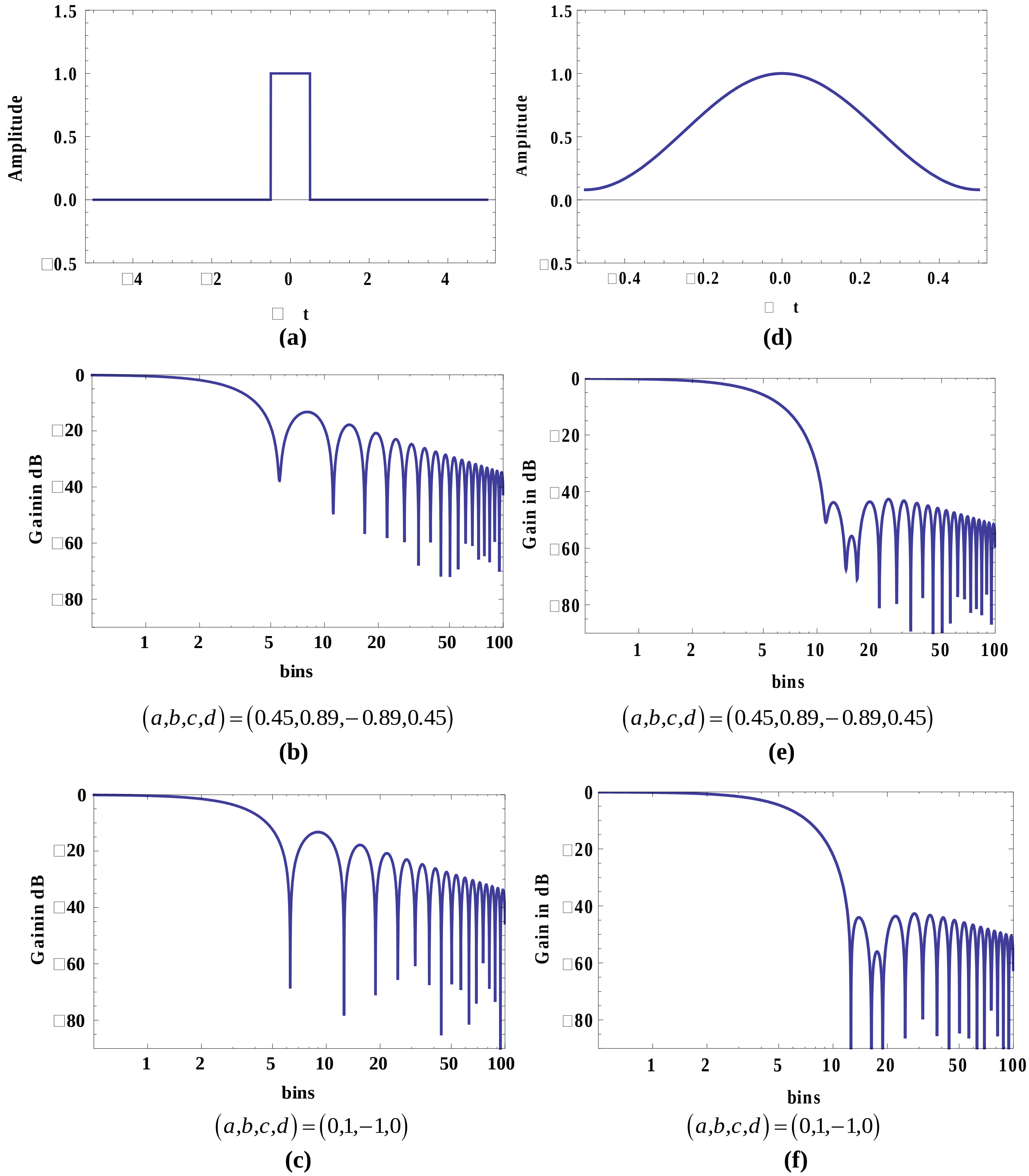


Figure-3.1: (a) Dirichlet window function (b)-(c) Log-magnitude plots to find MSLL, HMLW, -3dB, -6dB and SLFOR values of Dirichlet window function (d) Hamming window function (e)-(f) Log-magnitude plots to find MSLL, HMLW, -3dB, -6dB and SLFOR values of Hamming window function.

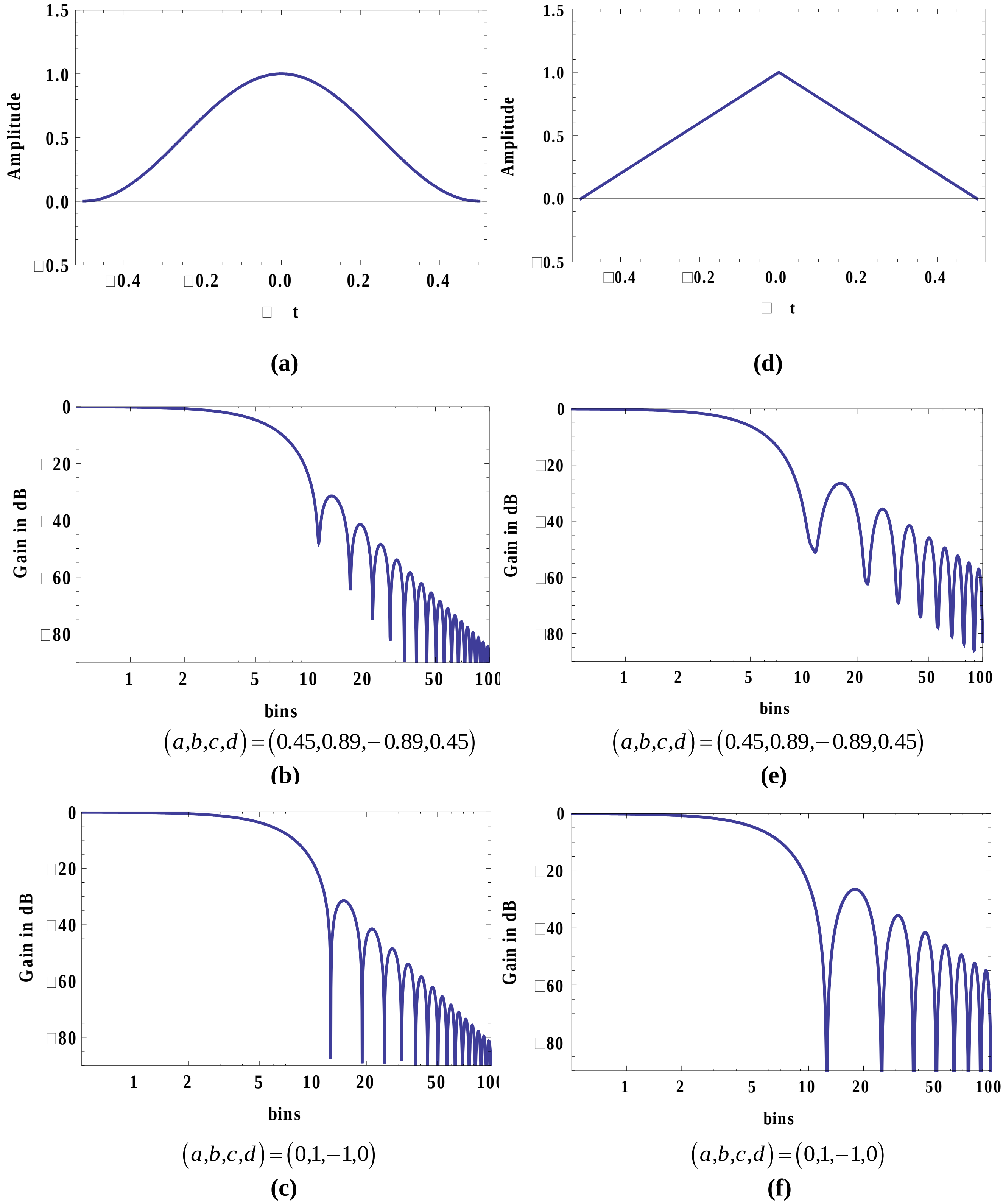


Figure-3.2: (a) Hanning window function (b)-(c) Log-magnitude plots to find MSLL, HMLW, -3dB, -6dB and SLFOR values of Hanning window function (d) Triangular window function (e)-(f) Log-magnitude plots to find MSLL, HMLW, -3dB, -6dB and SLFOR values of Triangular window function.

TABLE 3.1

PARAMETERS OF HAMMING WINDOW FUNCTION WITH VARIATIONS IN LCT VARIABLES

S. No.	*a*	*b*	*c*	*d*	MSLL (dB)	HMLW (bins)	SLFOR (dB/octave)	3-dB BW (bins)	6-dB BW (bins)
1	0.99	0.16	-0.16	0.99	-44.02	0.44	-5.70	0.20	0.29
2	0.95	0.31	-0.31	0.95	-43.63	0.72	-5.47	0.40	0.56
3	0.89	0.45	-0.45	0.89	-43.64	1.04	-5.22	0.59	0.81
4	0.81	0.59	-0.59	0.81	-43.48	1.32	-5.46	0.77	1.05
5	0.71	0.71	-0.71	0.71	-43.19	1.38	-5.88	0.92	1.27
6	0.59	0.81	-0.81	0.59	-43.61	1.56	-5.10	1.03	1.46
7	0.45	0.89	-0.89	0.45	-43.76	1.72	-5.11	1.14	1.62
8	0.31	0.95	-0.95	0.31	-43.84	1.83	-5.10	1.22	1.72
9	0.16	0.99	-0.99	0.16	-43.98	1.89	-4.89	1.25	1.79
10	0.00	1.00	-1.00	0.00	-43.98	1.92	-6.00	1.30	1.81

TABLE 3.2

PARAMETERS OF DIRICHLET WINDOW FUNCTION WITH VARIATIONS IN LCT VARIABLES

S. No.	*a*	*b*	*c*	*d*	MSLL (dB)	HMLW (bins)	SLFOR (dB/octave)	3-dB BW (bins)	6-dB BW (bins)
1	0.99	0.16	-0.16	0.99	-11.91	0.13	-6.30	0.14	0.19
2	0.95	0.31	-0.31	0.95	-12.85	0.25	-6.07	0.27	0.37
3	0.89	0.45	-0.45	0.89	-13.07	0.37	-6.07	0.40	0.55
4	0.81	0.59	-0.59	0.81	-13.10	0.47	-5.97	0.51	0.70
5	0.71	0.71	-0.71	0.71	-13.23	0.57	-5.97	0.61	0.85
6	0.59	0.81	-0.81	0.59	-13.11	0.65	-6.08	0.71	0.97
7	0.45	0.89	-0.89	0.45	-13.23	0.72	-5.97	0.77	1.06
8	0.31	0.95	-0.95	0.31	-13.22	0.77	-5.97	0.83	1.14
9	0.16	0.99	-0.99	0.16	-13.33	0.79	-6.07	0.86	1.18
10	0.00	1.00	-1.00	0.00	-13.14	0.81	-6.00	0.89	1.21

TABLE 3.3

PARAMETERS OF HANNING WINDOW FUNCTION WITH VARIATIONS IN LCT VARIABLES

S. No.	*a*	*b*	*c*	*d*	MSLL (dB)	HMLW (bins)	SLFOR (dB/octave)	3-dB BW (bins)	6-dB BW (bins)
1	0.99	0.16	-0.16	0.99	-39.89	0.45	-18.21	0.23	0.31
2	0.95	0.31	-0.31	0.95	-30.55	0.59	-18.36	0.44	0.62
3	0.89	0.45	-0.45	0.89	-30.96	0.85	-18.21	0.66	0.90
4	0.81	0.59	-0.59	0.81	-31.20	1.10	-18.09	0.84	1.17
5	0.71	0.71	-0.71	0.71	-31.34	1.32	-18.34	1.00	1.41
6	0.59	0.81	-0.81	0.59	-31.39	1.51	-18.72	1.14	1.61
7	0.45	0.89	-0.89	0.45	-31.43	1.66	-18.67	1.25	1.77
8	0.31	0.95	-0.95	0.31	-31.36	1.77	-18.48	1.35	1.90
9	0.16	0.99	-0.99	0.16	-31.43	1.84	-18.33	1.40	1.97
10	0.00	1.00	-1.00	0.00	-31.36	1.87	-18.00	1.44	2.00

TABLE 3.4

PARAMETERS OF TRIANGULAR WINDOW FUNCTION WITH VARIATIONS IN LCT VARIABLES

S. No.	*a*	*b*	*c*	*d*	MSLL (dB)	HMLW (bins)	SLFOR (dB/octave)	3-dB BW (bins)	6-dB BW (bins)
1	0.99	0.16	-0.16	0.99	-26.94	0.30	-12.03	0.20	0.28
2	0.95	0.31	-0.31	0.95	-26.94	0.52	-12.00	0.39	0.55
3	0.89	0.45	-0.45	0.89	-26.55	0.75	-11.89	0.57	0.80
4	0.81	0.59	-0.59	0.81	-26.47	0.96	-12.14	0.74	1.03
5	0.71	0.71	-0.71	0.71	-26.43	1.15	-11.89	0.88	1.25
6	0.59	0.81	-0.81	0.59	-26.38	1.27	-12.03	1.01	1.43
7	0.45	0.89	-0.89	0.45	-26.52	1.44	-11.92	1.13	1.57
8	0.31	0.95	-0.95	0.31	-26.47	1.54	-11.91	1.21	1.65
9	0.16	0.99	-0.99	0.16	-26.39	1.60	-11.99	1.24	1.73
10	0.00	1.00	-1.00	0.00	-26.41	1.62	-12.00	1.28	1.78

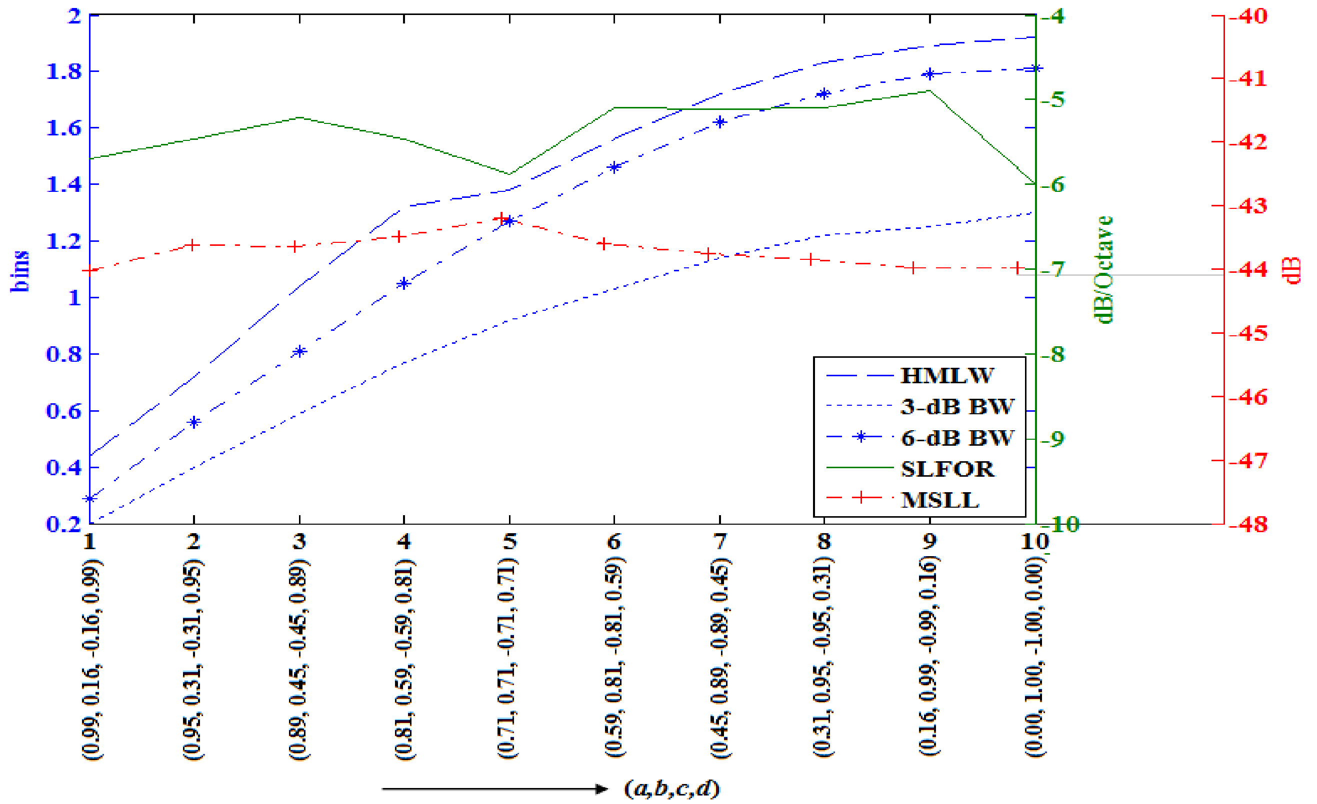


Figure-3.3: MSLL, HMLW,-3dB, -6dB and SLFOR plot for Hamming window function.

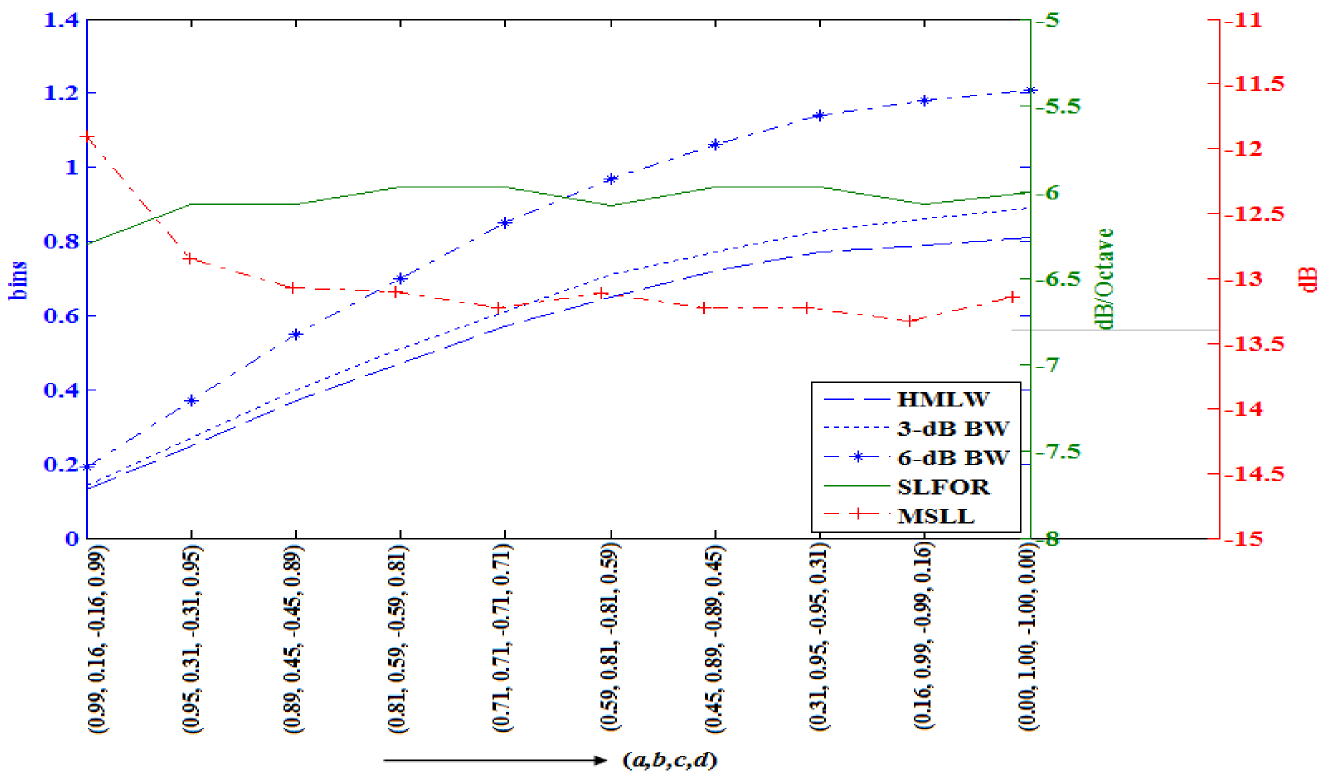


Figure-3.4: MSLL, HMLW, -3dB, -6dB and SLFOR plot for Dirichlet window function.

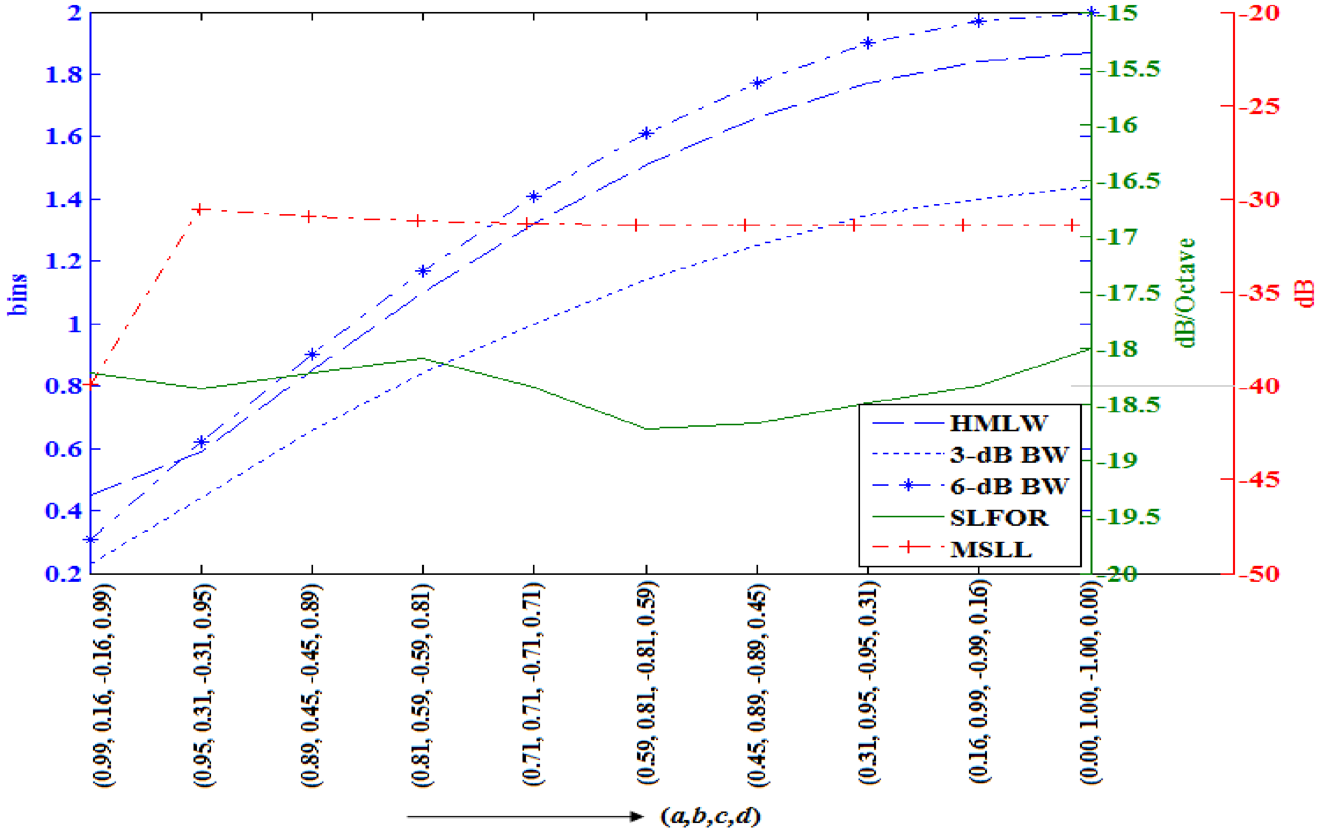


Figure-3.5: MSLL, HMLW, -3dB, -6dB and SLFOR plot for Hanning window function.

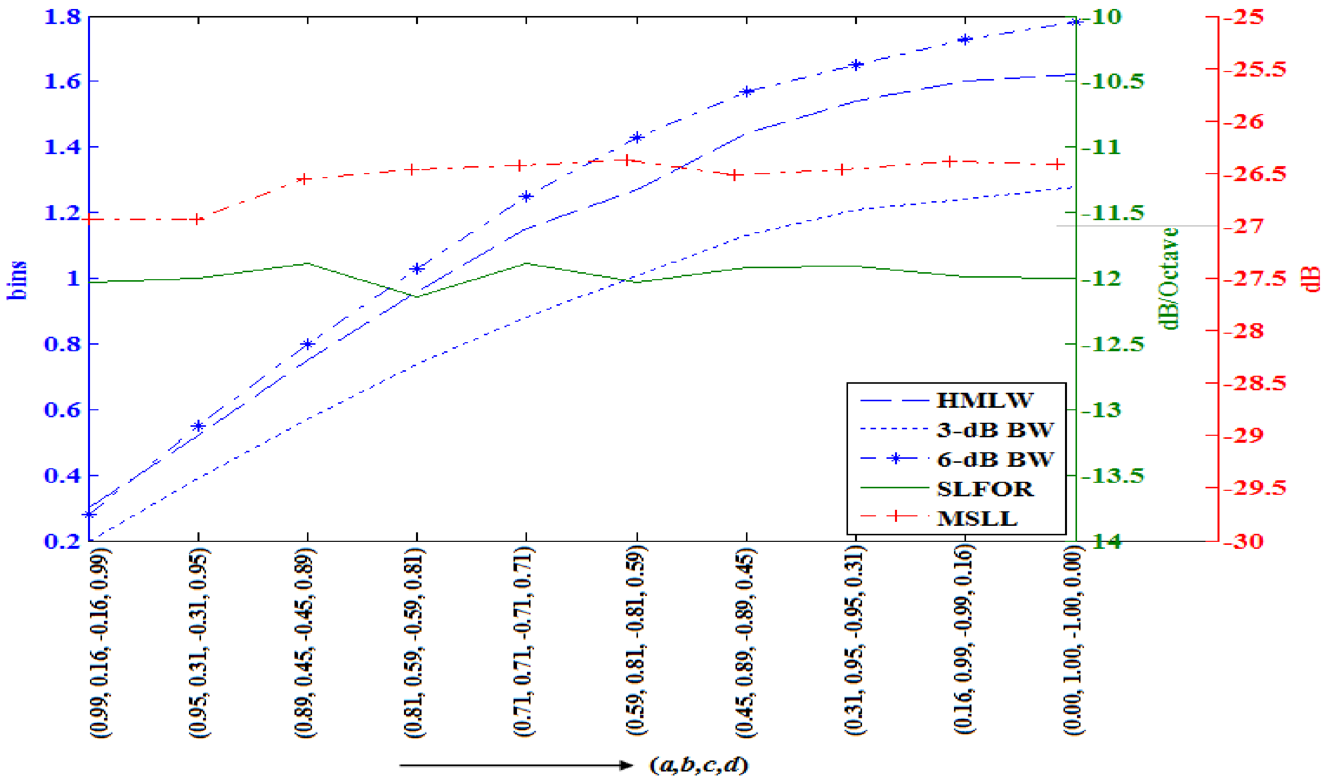


Figure-3.6: MSLL, HMLW, -3dB, -6dB and SLFOR plot for triangular window function.

3.1.4 Transition BW Tuning using Linear Canonical Transform

The transition BW of window-based finite impulse response (FIR) filters is proportional to the window main-lobe width, which in turn is proportional to the length of the window function. As such, transition BW of FIR filters can be directly tuned by varying window length for online tuning applications. In this section, an alternate methodology to tune the transition BW based on LCT is discussed. A Hamming window-based low-pass FIR filter satisfying the following design specifications is simulated, and frequency response is shown in Figure-3.7.

Pass-band ripple ≤ 0.1 dB, Stop band attenuation = 45 dB

Sampling frequency = 500 Hz Normalized cut-off frequency = 0.25

A plot of low pass FIR filter by using the same filter specifications except the normalized cut-off frequency = 0.2 is shown in Figure-3.8. The simulation results suggest that a fixed window filter for FT can be converted into variable window function by using different values of LCT variables. Further, by selecting different LCT variables as tuning parameters in the convolution operation between LCT of window function and ideal frequency response, variability in the transition band of the resulting window based low pass FIR filter response has been achieved.

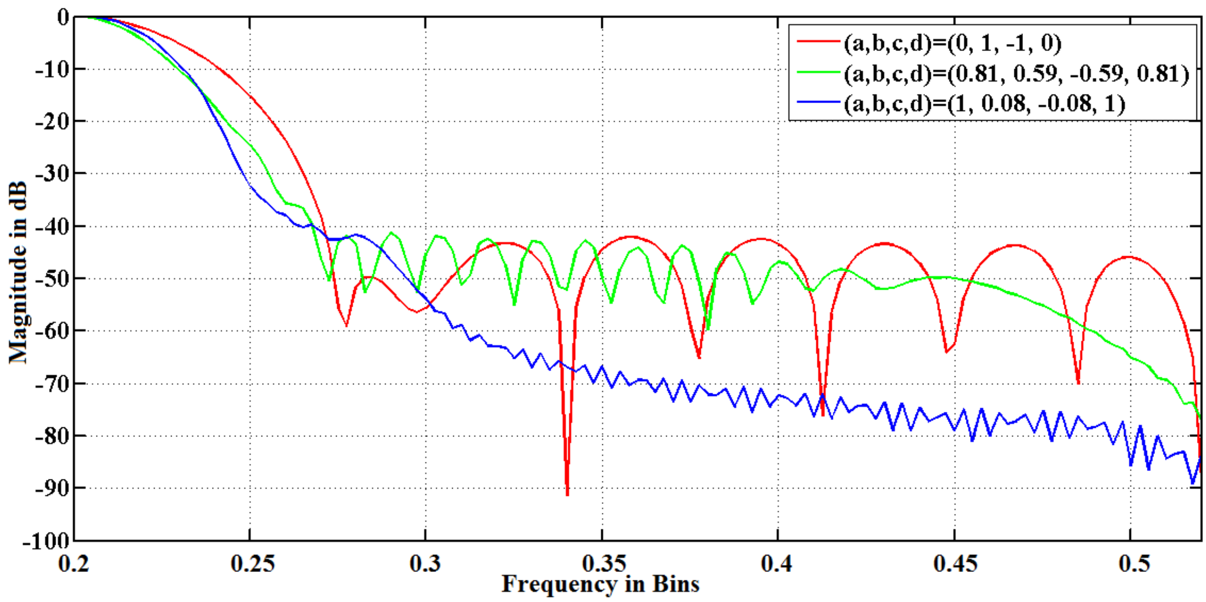


Figure-3.7: The frequency response of FIR low-pass filter designed with Hamming window for different values of (a,b,c,d) with normalized cut-off frequency 0.25.

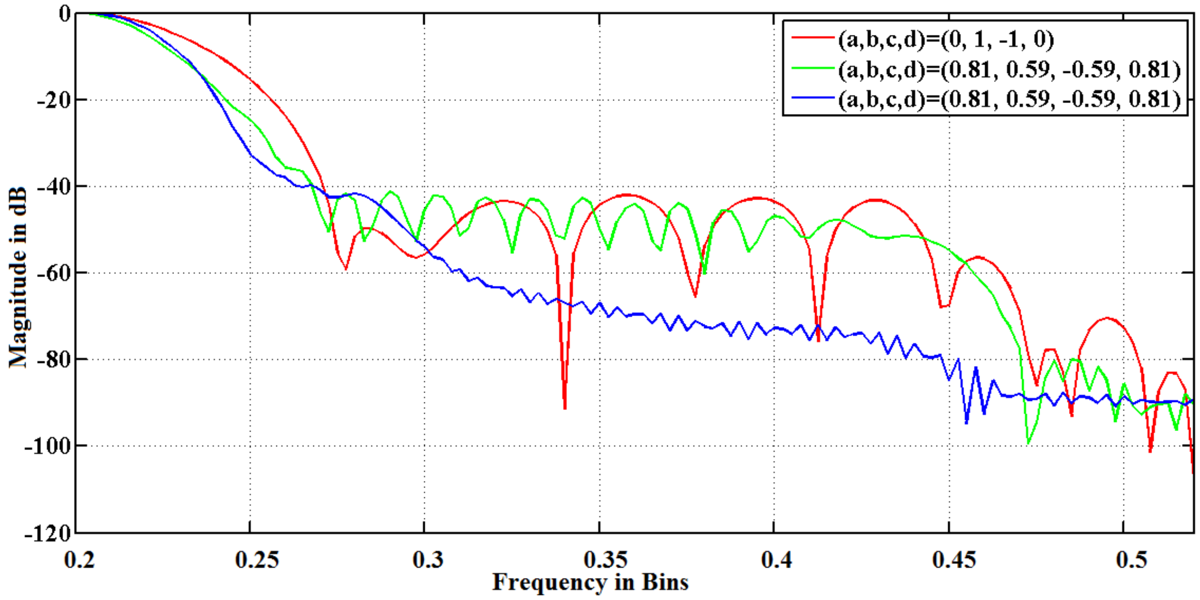


Figure-3.8: The frequency response of FIR low-pass filter designed with Hamming window for different values of (a,b,c,d) with normalized cut-off frequency = 0.2.

3.2 CANONICAL CONVOLUTION (CCV) AND CORRELATION OPERATIONS (CCR)

The canonical operations related to LCT are the CCV and CCR **[67, 138]**.

Canonical Convolution (CCV):

If $G_{(a,b,c,d)}(u)$ is the LCT of $g(t)$, $F_{(a,b,c,d)}(u)$ is the LCT of $f(t)$ and $H_{(a,b,c,d)}(u)$ is the LCT of $h(t)$, then mathematically

$$g(t)=L_F^{(d,-b,-c,a)}\left[L_F^{(a,b,c,d)}\{f(t)\}.L_F^{(a,b,c,d)}\{h(t)\}\right] \quad \textbf{(3.2.1)}$$

where, $g(t)=f(t)\otimes_{(a,b,c,d)}h(t)$

In the transformed domain, the CCV can be expressed as-

$$G_{(a,b,c,d)}(u)=F_{(a,b,c,d)}(u).H_{(a,b,c,d)}(u)$$

In time domain, the CCV can be expressed as-

CCV for $b\neq 0$, $d\neq 0$ (integration form)

$$g(t)=\frac{1}{2\pi|b|\sqrt{d}}.e^{-j\frac{ad+1}{2db}t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j\frac{t(k+s)-ks}{db}}e^{j\frac{c}{2d}(k^2+s^2)}f(k)h(s)\,ds\,dk \quad \textbf{(3.2.2)}$$

CCV for $b\neq 0$, $d=0$ (integration form)

$$z(t)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty} e^{j\frac{a}{b}k(t-k)}x(k)\,y(t-k)\,\mathrm{d}k \qquad \textbf{(3.2.3)}$$

Thus, when $b \neq 0$ and $d \neq 0$, two integration operations are required to express the CCV. But when $b \neq 0$ and $d = 0$, it require only one integration operation to express the CCV. So in the case that $d = 0$, the relation between the output and the inputs of CCV is very simple, and the effects of $f(t)$ and $h(t)$ on the output $g(t)$ are easier to analyse. Setting $d = 0$ for CCV also has an advantage i.e. three chirp multiplication operations can be avoided (one for the LCT of $f(t)$, other one for the LCT of $h(t)$, and another one for the inverse LCT) but these chirp multiplication operations cannot be avoided in case of FRFT because for FRFT $\cos\alpha \neq 0$. Thus, when CCV is used for practical applications (such as filter design), usually it is useful to set $d = 0$.

Mathematically the CCR function is given as **[67, 138]-**

Canonical Correlation (CCR):

If $G_{(a,b,c,d)}(u)$ is the LCT of $g(t)$, $F_{(a,b,c,d)}(u)$ is the LCT of $f(t)$ and $H_{(a,b,c,d)}(u)$ is the LCT of $h(t)$, then mathematically

$$g(t)=L_F^{(d,-b,-c,a)}\left[L_F^{(a,b,c,d)}\left\{f(t)\right\}.L_F^{(a,b,c,d)}\left\{h^*(t)\right\}\right] \qquad \textbf{(3.2.4)}$$

where, $g(t)= f(t)\bigstar_{(a,b,c,d)}h(t)$

In the transformed domain, the CCR can be expressed as-

$$G_{(a,b,c,d)}(u)=F_{(a,b,c,d)}(u).H^*_{(a,b,c,d)}(u) \qquad \textbf{(3.2.5)}$$

3.2.1 Relation between WDF and CCV

The relation between WDF and CCV is given as below **[67, 138]-**

$$W_z(t,\omega)=\int_{-\infty}^{\infty} W_x(adt+bd\omega-b\rho,-act-bc\omega+a\rho)W_y(t+b\rho,\omega-a\rho)\,\mathrm{d}\rho \qquad \textbf{(3.2.6)}$$

Rewriting (3.2.5), results in-

$$W_Z(\sigma t_1+\mu w_1)=\int_{-\infty}^{\infty} W_x(\sigma t_1+\rho w_1).W_y(\sigma t_1+\mu w_1-\rho w_1)\,\mathrm{d}\rho \qquad \textbf{(3.2.7)}$$

where,

$$\sigma=at+b\omega,\quad \mu=ct+d\omega,\quad t_1=(d,-c),\quad w_1=(-b,a) \qquad \textbf{(3.2.8)}$$

For WDF, the CCV corresponds to the convolution along the line that intersects the ω-axis with the angle of $\tan^{-1}(b/a)$ in the counter clockwise direction, as shown in Figure-3.9.

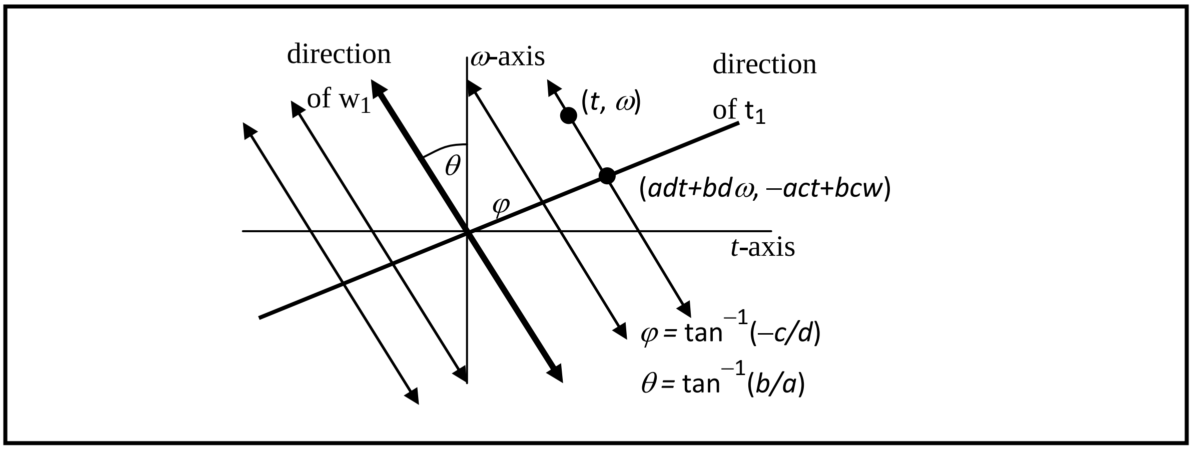


Figure-3.9: The direction of convolution on WDF for CCV

From (3.2.7), it is found that the variables of LCT will affect the CCV from the following ways:

- The value of b/a will affect the direction of convolution. This is because $w_1 = (-b, a)$ is the direction of the convolution.
- The value of c/d will shear the WDF of the result. This is because the direction of $t_1 = (d, -c)$ in Figure-3.9 is determined by c/d.

For example, suppose the same values of *a*, *b* is used as given in Figure-3.9, but the value of *c*, *d* is changed as *p*, *q*. Then the direction of t_1 is changed (let the new direction (p, q) is denoted as t_2, but the direction of w_1 (i.e., the direction of convolution) will remain unchanged. To see why the WDF is sheared, let us suppose that

$$z(t) = x(t) \otimes_{(a,b,c,d)} y(t), \qquad z_1(t) = x(t) \otimes_{(a,b,p,q)} y(t), \tag{3.2.9}$$

then the WDF of $z(t)$ is the same as (3.2.7) and (3.2.8), but the WDF of $z_1(t)$ is

$$W_{z1}(\sigma t_2 + \mu_2 w_1) = \int_{-\infty}^{\infty} W_x(\sigma t_2 + \rho w_1).W_y(\sigma t_2 + \mu_2 w_1 - \rho w_1)\,\mathrm{d}\rho \tag{3.2.10}$$

where, $\sigma = at + b\omega$, $\mu_2 = pt + q\omega$, $t_2 = (q, -p)$, $w_1 = (-b, a)$ **(3.2.11)**

Then, from (3.2.6) and (3.2.11)

$$\sigma t_1 + \mu w_1 = \sigma t_2 + \mu_2 w_2 \tag{3.2.12}$$

after some computation, it can be proved that

$$W_{z_1}(\sigma t_2 + (2\mu_2 - \mu) w_1) = W_z(\sigma t_1 + \mu w_1) \tag{3.2.13}$$

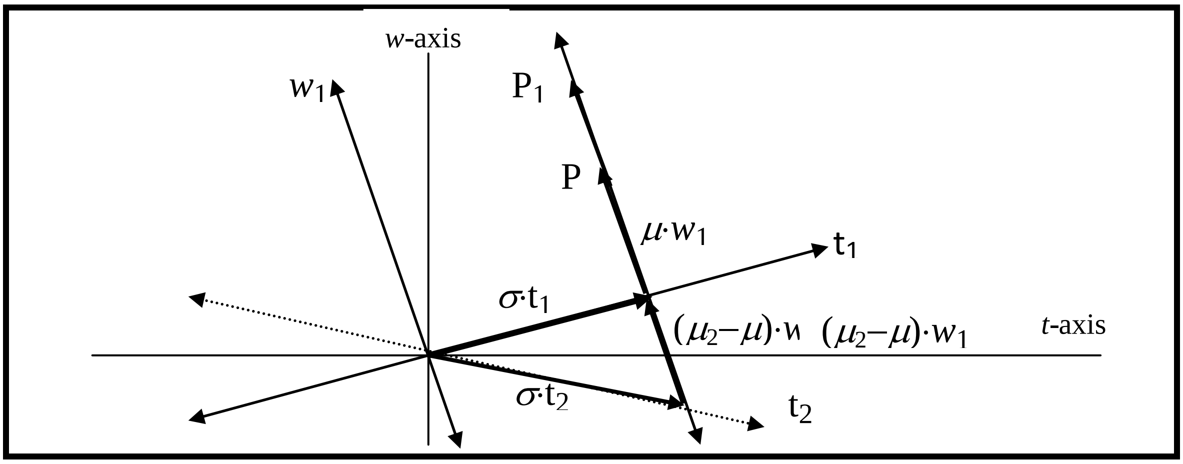


Figure-3.10: The value of $W_{z_1}(t,\omega)$ (defined as (3.2.9)) at the location P_1 will be the same as the value $W_z(t,\omega)$ at the location P.

Thus, if b/a is unchanged, and only the value of c/d is changed, then the WDF of the CCV results in just sheared along the direction of $w_1 = (-b,a)$ as shown in Figure-3.10.

Most of the applications of LCT, such as the filter design, are associated with the CCV and from the above discussion, only the values of b/a and c/d will affect the results of CCV, so only the values of b/a and c/d are important for the LCT. Besides, since the value of b/a affects the direction of convolution for the WDF, and the value of c/d just shears the WDF, so it can be concluded that the value of b/a is more important than the value of c/d. The LCT has four variables, and for the applications related to LCT, it seems that all the four variables have to be adjusted. But in fact,

- By just controlling the value of b/a always enough to satisfy the requirement of filter.
- Sometimes, for other applications, the value of c/d may also be controlled.
- It is always unnecessary to control other variables.

3.2.2 Filter Design using the relation between WDF and CCV

From the relations between WDF and CCV, the design method of filter applications associated with the CCV has been discussed **[67, 138]**.

The effect of the filter designed by LCT can be written in the following expression-

$$y(t) = L_F^{(d,-b,-c,a)}\left[H_{(a,b,c,d)}(u).X_{(a,b,c,d)}(u)\right] \quad \textbf{(3.2.14)}$$

It is just a special case of CCV. There are many possible types of canonical filter. The simplest is the pass-stop band canonical filter. The pass-stop band filter may be a band pass filter, low pass filter, hagh pass filter or a band reject filter. The transform function of the pass-stop band filter is given by-

$$H(u) = \Pi\left[(u-u_0)/B\right] \quad \textbf{(3.2.15)}$$

where, Π indicates the rectangular window, and $H(u)$ may be written as

$$H(u) = \begin{cases} 1 & \text{for } u_0 - B/2 < u < u_0 + B/2 \text{ (Pass Band)} \\ 0 & \text{otherwise (Stop Band)} \end{cases} \quad \textbf{(3.2.16)}$$

The WDF of $H(u)$ is-

$$W_H(u,v) = (\pi v)^{-1}\sin\left(v\cdot\left(2B - \left|4(u-u_0)\right|\right)\right)\cdot\Pi\left((u-u_0)/B\right) \quad \textbf{(3.2.17)}$$

and if $h(t) = L_F^{(d,-b,-c,a)}\left[H(u)\right]$, then the WDF of $h(t)$ is-

$$W_h(t,\omega) = \frac{\sin\left((ct+dw)\left(2B - \left|4(at+b\omega-u_0)\right|\right)\right)}{\pi(ct+d\omega)}\cdot\Pi\left(\frac{at+b\omega-u_0}{B}\right) \quad \textbf{(3.2.18)}$$

and from (3.2.6), the WDF of the filter output $y(t)$ is-

$$W_y(t,\omega) = \int_{-\infty}^{\infty} W_x(adt+bd\omega-b\rho,-act-bc\omega+a\rho) \quad \textbf{(3.2.19)}$$
$$\frac{\sin\left((ct+d\omega-\rho)\left(2B-\left|4(at+b\omega-u_0)\right|\right)\right)}{\pi(ct+d\omega-\rho)}\cdot\Pi\left(\frac{(at+b\omega-u_0)}{B}\right)\cdot \mathrm{d}\rho$$

$$= \int_{-\infty}^{\infty} W_x(d\eta-b\rho,-c\eta+a\rho)\frac{\sin\left((ct+d\omega-\rho)\left(2B-\left|4(\eta-u_0)\right|\right)\right)}{\pi(ct+d\omega-\rho)}\Pi\left(\frac{(\eta-u_0)}{B}\right)d\rho \quad \textbf{(3.2.20)}$$

where, $\eta = at + b\omega$. Since

$$\Pi\left((\eta-u_0)/B\right) = 0 \qquad \text{when } \eta < u_0 - B/2 \text{ or } \eta > u_0 + B/2 \quad \textbf{(3.2.21)}$$

therefore, $W_x(t,\omega)$ will have no effects on the WDF of $y(t)$ when-

$$u_0 - B/2 < at + b\omega < u_0 + B/2 \quad \textbf{(3.2.22)}$$

That is, if the WDF of the undesired part of $x(t)$ is outside the region of (3.2.22), then the undesired part can be filtered out by the method given in (3.2.14) and (3.2.15). Thus, from the WDF of the received signal as shown in Figure-3.11, the pass-stop band canonical filter can be designed. In fact,

- The parameter b/a can control the slope of the cut-off line on WDF.
- The variables u_0, B can control the location of the pass region on WDF.

For example, suppose the WDF of the received signal can be drawn as shown in Figure-3.11. The dotted regions mean the locations that

$$W_x(t,\omega) > T, \qquad \text{where, } T \text{ is the threshold.} \qquad \textbf{(3.2.23)}$$

Suppose that the desired and undesired parts can be separated by two parallel cut-off lines. Then, the variables (a,b,c,d,u_0,B) may be obtained from (3.2.13) and (3.2.14) as:

$$\frac{a}{b} = \frac{\omega_1}{t_1} \text{ and } \quad u_0 = a(t_0+t_1)/2, \qquad B = |a(t_1-t_0)| \qquad \textbf{(3.2.24)}$$

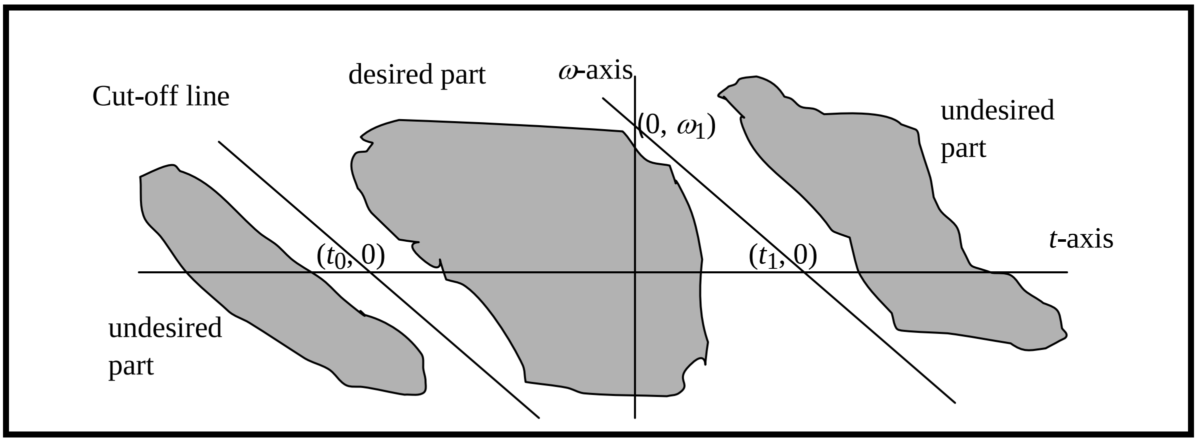


Figure-3.11: Using WDF to filter out the undesired signal by pass-stop band canonical filter.

3.3 CLOSED-FORM ANALYTICAL EXPRESSION OF PASS-STOP BAND FILTER IN LCT DOMAIN

This section derives the closed-form analytical expression of pass-stop band filter that behaves in the LCT domain. The known desired frequency response of filter is given by **[67, 138]**-

$$H(u) = \begin{cases} 1 & \text{for } u_0 - B/2 < u < u_0 + B/2 \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{(3.3.1)}$$

where, $u_0 = a(t_0+t_1)/2, \qquad B = |a(t_1-t_0)|$

where, $H(u)$ is the desired frequency response of the pass-stop band filter in the LCT domain and a is one of the LCT variables, t_0 and t_1 can be obtained from Figure-3.11.

Therefore the ideal impulse response of the pass-stop band filter using the inverse LCT **[38]** is given by-

$$h(n)=\sqrt{\frac{1}{-j2\pi b}}\int_{u_0-B/2}^{u_0+B/2}H(u)\exp\left[\frac{-j\left(an^2+du^2\right)}{2b}+\frac{jnu}{b}\right]\mathrm{d}u \quad \textbf{(3.3.2)}$$

where, $u_0-B/2=at_0$ and $u_0+B/2=at_1$

Substituting the value of $H(u)$ in (3.3.2), results in-

$$h(n)=\sqrt{\frac{1}{-j2\pi b}}\int_{at_0}^{at_1}1.\exp\left[\frac{-j\left(an^2+du^2\right)}{2b}+\frac{jnu}{b}\right]\mathrm{d}u \quad \textbf{(3.3.3)}$$

Solving (3.3.3) by applying $\int e^{-\left(xt^2+2yt+z\right)}\mathrm{d}t=\dfrac{\sqrt{\pi}\,e^{\left(\frac{y^2}{x}-z\right)}erf\left[\frac{(y+xt)}{\sqrt{x}}\right]}{2\sqrt{x}}$ **[94]**, results in-

here, $x=j\dfrac{d}{2b}$, $y=-j\dfrac{n}{2b}$, and $z=j\dfrac{an^2}{2b}$, therefore-

$$h(n)=\sqrt{\frac{1}{-j2\pi b}}\times\left.\frac{\sqrt{\pi}\;e^{\left(\frac{n^2}{4b^2}\cdot\frac{j2b}{d}-j\frac{an^2}{2b}\right)}erf\left[\frac{\left(-j\frac{n}{2b}+j\frac{du}{2b}\right)}{\sqrt{j\frac{d}{2b}}}\right]}{2\sqrt{j\frac{d}{2b}}}\right|_{at_0}^{at_1} \quad \textbf{(3.3.4)}$$

Rearranging (3.3.4), results in-

$$h(n)=\frac{1}{2\sqrt{d}}e^{-j\left(\frac{n^2(ad-1)}{2db}\right)}\left[erf\left\{\frac{\left(\frac{1}{2}+\frac{j}{2}\right)\left(adt_1-n\right)}{\sqrt{db}}\right\}-erf\left\{\frac{\left(\frac{1}{2}+\frac{j}{2}\right)\left(adt_0-n\right)}{\sqrt{db}}\right\}\right] \quad \textbf{(3.3.5)}$$

By using the De Moivre's theorem **[94]**, (3.3.5) can be further simplified as-

$$h(n)=\frac{1}{2\sqrt{d}}e^{-j\left(\frac{n^2(ad-1)}{2db}\right)}\left[erf\left\{\frac{\sqrt{j}\left(adt_1-n\right)}{\sqrt{2db}}\right\}-erf\left\{\frac{\sqrt{j}\left(adt_0-n\right)}{\sqrt{2db}}\right\}\right] \quad \textbf{(3.3.6)}$$

Thus (3.3.6) represents the closed-form analytical expression of the impulse response for the pass-stop band filter in the LCT domain. It is clear from the above analytical expression that the coefficients of the desired impulse response of pass-stop band filter in the LCT domain are dependent on the LCT variables.

The coefficients of $h(n)$, obtained in (3.3.6) are of infinite in duration, as n ranging from $-\infty$ to ∞. So to obtain the actual response from the desired Infinite Impulse Response (IIR) $h(n)$ of (3.3.6), the ideal impulse response $h(n)$ must be truncated using the window function as:

$$h_{actual}(n) = h(n)\,w(n) \qquad \textbf{(3.3.7)}$$

$$h_{actual}(n) = \underbrace{\frac{1}{2\sqrt{d}} e^{-j\left(\frac{n^2(ad-1)}{2db}\right)}}_{f_1(n)} \left[\underbrace{erf\left\{\frac{\sqrt{j}\,(adt_1 - n)}{\sqrt{2db}}\right\}}_{f_2(n)} \underbrace{- erf\left\{\frac{\sqrt{j}\,(adt_0 - n)}{\sqrt{2db}}\right\}}_{f_3(n)} \right] w(n) \qquad \textbf{(3.3.8)}$$

Thus the actual impulse response of pass-stop band filter in the LCT domain is represented by (3.3.8) and is a function of LCT variables, threshold points (t_0, t_1) and the window function $w(n)$.

It can be seen from (3.3.8) that, the actual impulse response is a function of linear relation of two error functions i.e. $f_2(n) + f_3(n)$ multiplied by an exponential signal denoted by $f_1(n)$. In mathematics, error function is a special function of “S” shape, called the sigmoid curve **[94].** The plot of functions $f_2(n)$ and $f_3(n)$ is shown in Figure-3.12 and 3.13 respectively.

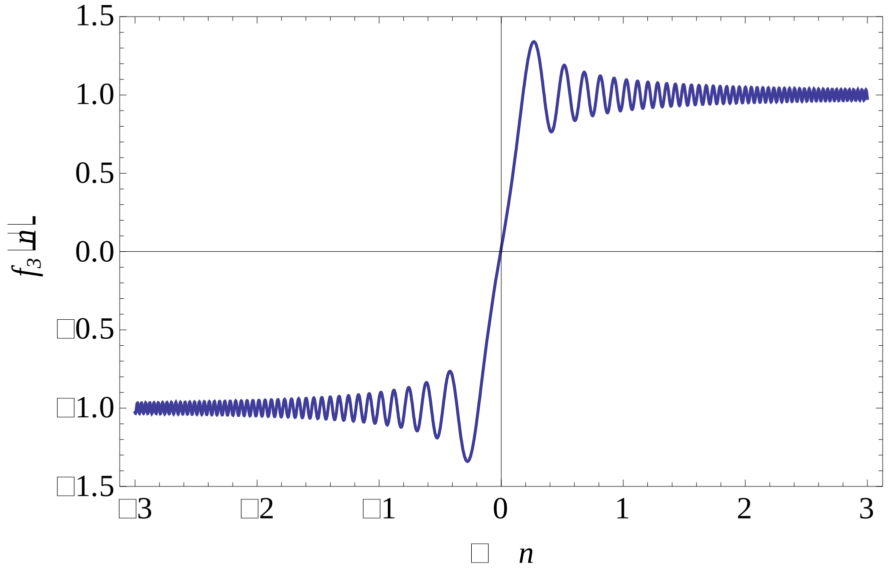


Figure-3.12: The plot of function $f_3(n)$.

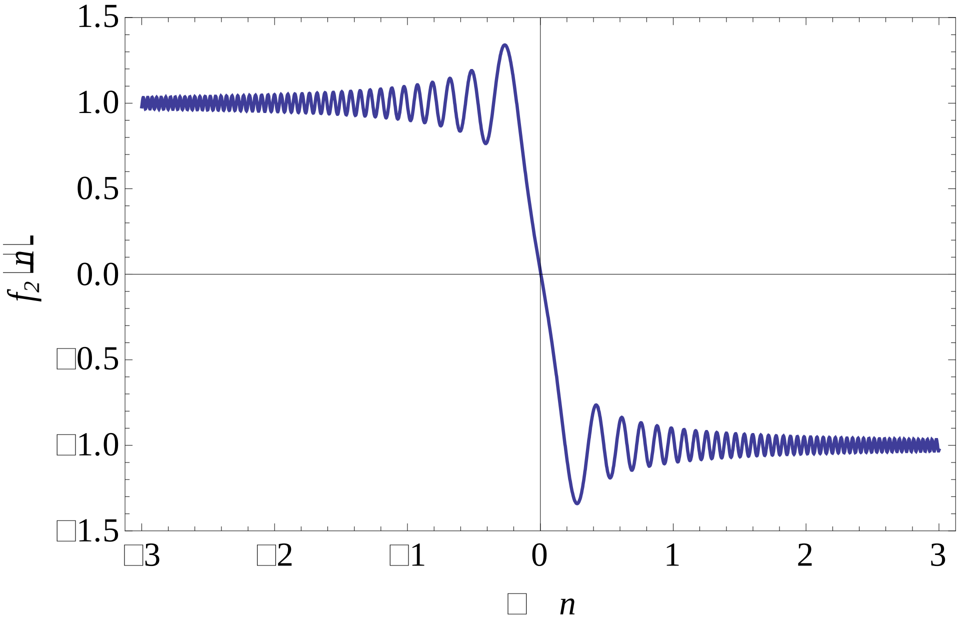


Figure-3.13: The plot of function $f_2(n)$.

As seen from the Figure-3.12, the function $f_3(n)$ has the shape that is very similar to the Sigmoid curve. Since the threshold points (t_0, t_1) are the mirror image of each other with respect to y-axis, the function $f_2(n)$ is the inverted plot of $f_3(n)$ as shown in Figure-3.13. The plot of linear relation between $f_2(n)$ and $f_3(n)$ is shown in Figure-3.14.

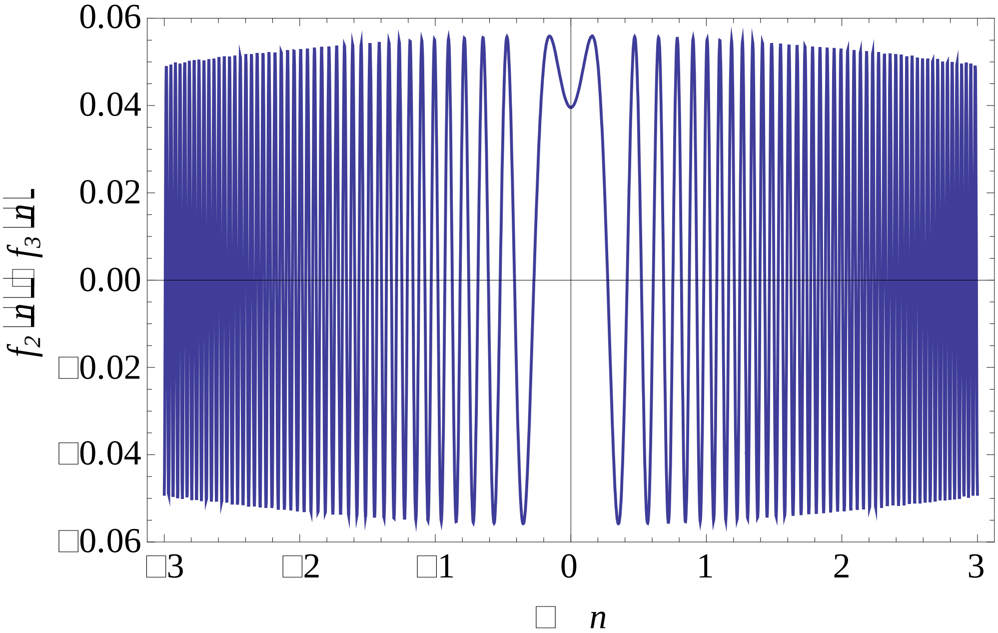


Figure-3.14: The plot of function $f_2(n)+ f_3(n)$.

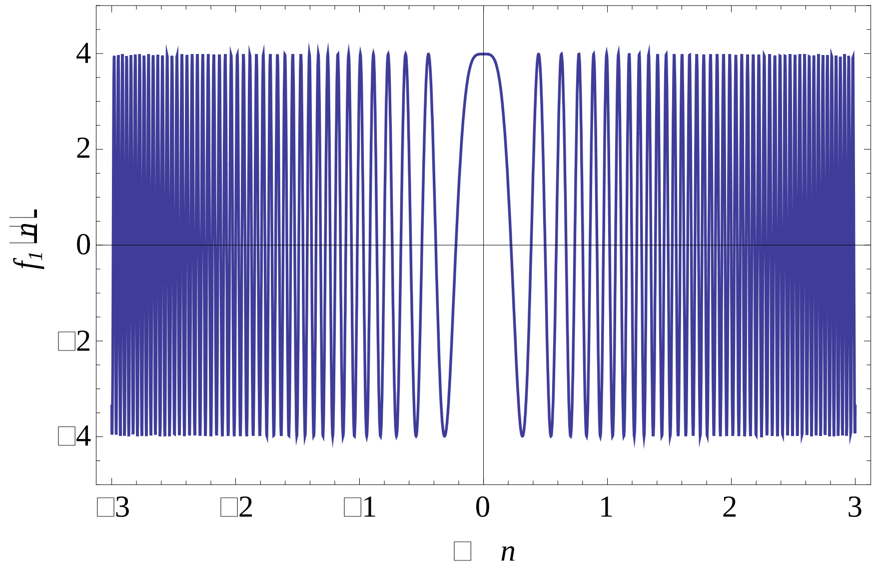
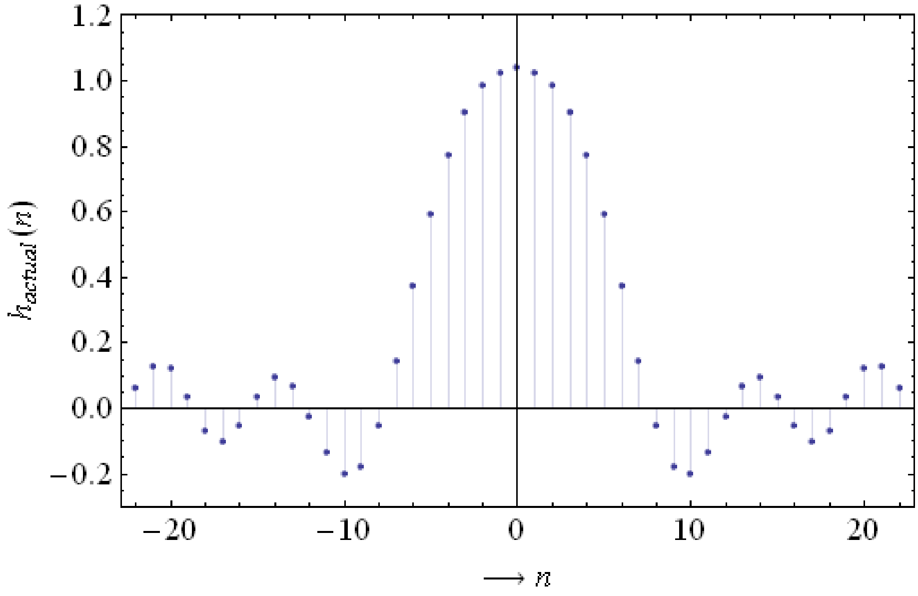


Figure-3.15: The plot of function $f_1(n)$.

As seen from (3.3.8), apart from the linear relation of two error functions, it consists of function $f_1(n)$ which is in exponential form. The plot of function $f_1(n)$ is shown in Figure-3.15.



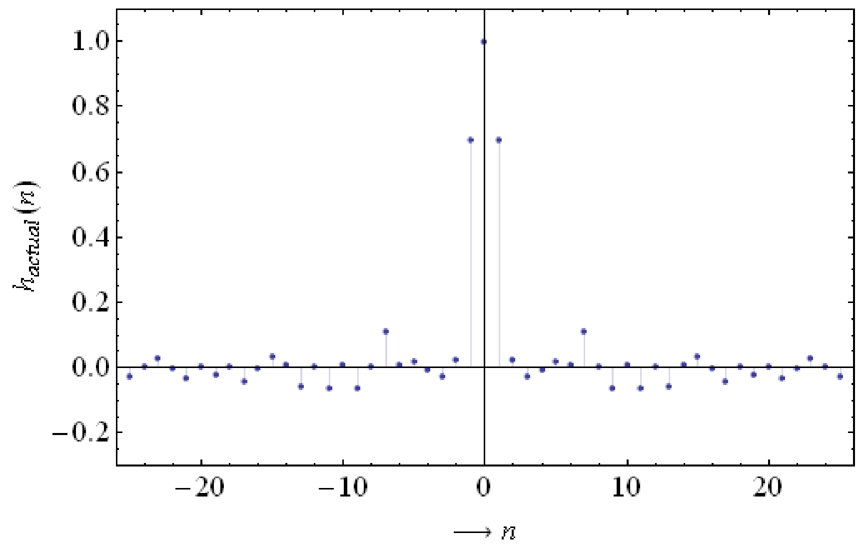
$(a,b,c,d) = (0.0314, 0.999, -0.999, 0.0314)$ and $(t_0, t_1) = (-13.0, 13.0)$

Figure-3.16: The plot of actual impulse response $h_{actual}(n)$.

It can be observed from Figure-3.14 that the linear relation between two functions results in the main lobe of the sinc pulse and the exponential function results in the side lobes. Hence from Figure-3.16, it has been concluded that for different combinations of LCT variables and threshold points, the actual impulse response will result in a sinc like pulse.

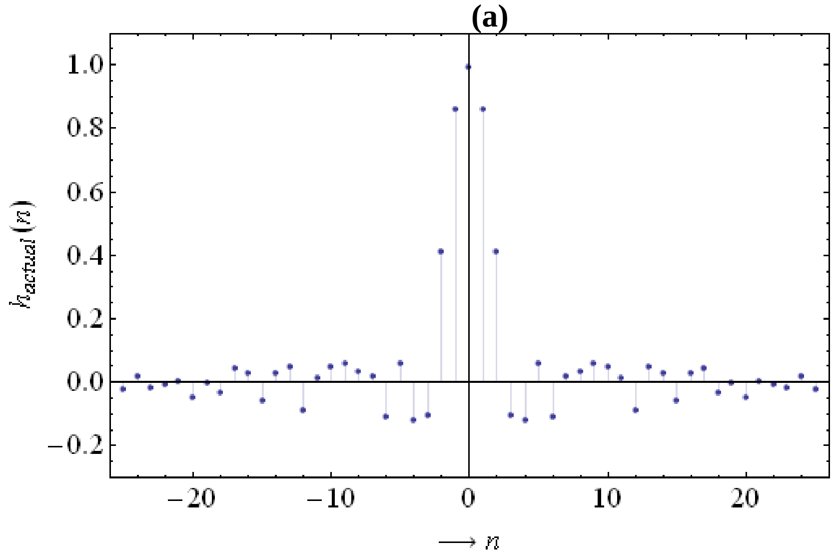
3.3.1 Simulation Results

Figure-3.17 below shows various discrete plots of the impulse response of the pass-stop band filter in the LCT domain for rectangular window function and for the variation of threshold points (t_0,t_1) and LCT variables (a,b,c,d).



$(a,b,c,d)=(0.38,0.92,-0.92,0.38)$ and $(t_0,t_1)=(-3.8,3.8)$

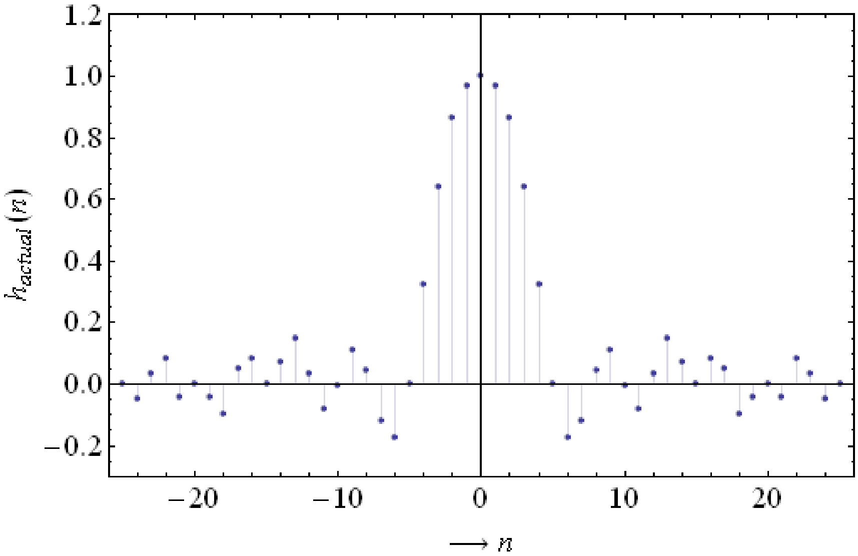
(a)



$(a,b,c,d)=(0.22,0.97,-0.97,0.22)$ and $(t_0,t_1)=(-5.15,5.15)$

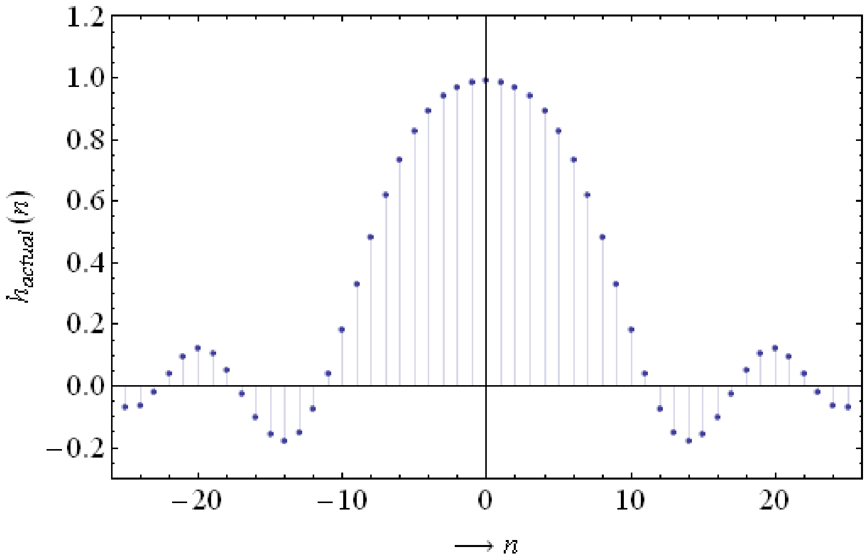
(b)

Figure-3.17: Plots of impulse response co-efficients of pass-stop band filter in the LCT domain for Dirichlet window function.



$(a,b,c,d)=(0.08,0.99,-0.99,0.08)$ and $(t_0,t_1)=(-8,8)$

(c)



$(a,b,c,d)=(0,1,-1,0)$ and $(t_0,t_1)=(-17.7,17.7)$

(d)

Figure-3.17 (*Continued*)

It has been observed from the simulation results for different combinations of LCT variables and threshold points, the impulse response gives the impression of sinc pulse and finally it approaches to perfect sinc pulse when the LCT reduces to FT for $(a,b,c,d)=(0,1,-1,0)$.

3.4 DISCUSSIONS

In this chapter, the mathematical analysis of various window functions has been carried out in the LCT domain and simulated accordingly. The variation of different window parameters like HMLW, MSLL, SLFOR, 3-dB BW and 6-dB BW of these functions with the LCT variables (a,b,c,d) has been observed. Based upon the mathematical analysis of the window functions, variability in the transition band of the resulting window based low pass FIR filter response has been achieved.

Then the concept of CCV and correlation alongwith the analysis of window functions with LCT is focused upon. Firstly, the relation between WDF and CCV has been dealt with. Then based upon this relation, the procedure to design the pass-stop band canonical filter has been established.

Lastly, a mathematical expression of the closed-form analytical relation for the finite impulse response of pass-stop band canonical filter is established in the LCT domain and simulated for different values of LCT variables and threshold points. It has been observed from the simulation results that the actual impulse response approaches to sinc pulse.

CHAPTER 4

CONVOLUTION AND PRODUCT THEOREMS FOR LCT

Many properties of LCT were derived, developed or established earlier as described in chapter 2. This includes multiplication property, differentiation property, shifting property, modulation property, and few more. Since, the convolution theorem of the transform plays an important role in DSP, so it is extensively investigated always for the refinement to a well-accepted closed-form expression.

4.1 INTRODUCTION

Convolution is a mathematical way of combining two signals to form a third signal. It is the single most important technique in DSP. Using the strategy of impulse decomposition, systems are described by a signal called the impulse response. Convolution is important because it relates the three signals of interest: the input signal, the output signal, and the impulse response.

4.2 PERFORMANCE METRICS FOR EVALUATION OF EXISTING CONVOLUTION THEOREMS

The following are the various performance metrics for evaluation of existing convolution theorems.

The outcome of this chapter has been published in Research Journal as per following detail: N. Goel, K.Singh, A modified convolution and product theorem for the linear Canonical transform derived by representation Transformation in quantum mechanics, International Journal of Applied Mathematics and Computer Science, vol. 23, no. 3, pp. 685-695, 2013.

4.2.1 Variable Dependability

The convolution defined in one domain and its transformed counterpart in transformed domain should have mathematical expressions in terms of respective domain variables only. For example, when the time domain convolution of rectangular window function $(r \otimes r)(t)$ is Fourier transformed, it is having a closed form expression in frequency domain $\text{sinc}^2(\pi f/2)$ dependent upon frequency as variable only. This variable is assumed in order to assure that a quantity defined in one domain when transformed will result in an equivalent quantity in transformed domain.

4.2.2 Equivalent FT Conversion

The proposed convolution theorem should be converted into classical convolution theorem for FT with LCT variables $(a,b,c,d)=(0,1,-1,0)$, it is due to the FT as a special case of LCT. This variable has given a prime importance because LCT is assumed as the generalization of FRFT and FT. Therefore, any property defined for LCT should be converted to the analogous property for FRFT and FT.

4.2.3 Hardware Complexity

Hardware complexity is defined by the total number of chirp signals required by the convolution process (LHS) and its transform (RHS) to derive the proposed theorem. From realization point of view, requirement of more chirp signals would impose difficulty because in communication systems it is nearly impossible to generate a chirp signal accurately.

4.2.4 Simulation Comparison

When a unity rectangular function is convolved linearly with itself, it gives a triangular function of double duration to it, in the time domain. Due to this reason, the proposed convolution theorem is compared with the literature by evaluating the convolution of rectangular function in the LCT domain with the LCT of triangular function. The simulation results which are more near to the LCT of triangular function is considered as a better approach for convolution theorem for LCT.

4.3 PROPOSED CONVOLUTION AND PRODUCT THEOREM

In this section, proposed convolution and product theorem has been derived. In the derivations, instead of using time parameter t and LCT domain parameter u, the position-momentum representations have been used to overcome the interpretation problems as time is not an observable in classic quantum mechanics (one cannot measure the time of the particle).

4.3.1 Proposed Convolution Theorem

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$ and $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$, then $\sqrt{j2\pi b}\,F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(p)e^{-j\frac{dp^2}{2b}}$ is the LCT of $h(x)$ i.e.

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=H(p)=\sqrt{j2\pi b}\,F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(p)e^{-j\frac{dp^2}{2b}} \tag{4.3.1}$$

where, $h(x)=(f\otimes g)(x)=\int_{-\infty}^{\infty} f(y)g(x-y)\exp\left\{-j\frac{a}{b}y.(x-y)\right\}\mathrm{d}y$, is the weighted convolution operation and the role of g and f can be interchanged.

Proof. Using the quantum mechanical notation **[57, 61]**, the one-dimensional LCT of the signal $h(x)$ reads as-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\langle x|h\rangle=\langle p|K|h\rangle=\langle p|H\rangle=H_{(a,b,c,d)}(p) \tag{4.3.2}$$

where, the notation $\langle x|h\rangle=h(x)$ and $\langle p|K|x\rangle=K_{(a,b,c,d)}(p,x)$ gives the representation of LCT kernel in quantum mechanics. K is named as the LCT operator and

$$\langle x|h\rangle=\int_{-\infty}^{\infty}\mathrm{d}y\langle y|f\rangle\langle x-y|g\rangle\exp\left\{-j\frac{a}{b}y.(x-y)\right\} \tag{4.3.3}$$

Substituting the value of $\langle x|h\rangle$ in (4.3.2) results-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\int_{-\infty}^{\infty}\mathrm{d}y\langle y|f\rangle\langle x-y|g\rangle.\exp\left\{-j\frac{a}{b}y.(x-y)\right\} \tag{4.3.4}$$

Rearranging (4.3.4), results in-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x\right\rangle\left\langle x-y\,|\,g\right\rangle\left\langle y\,|\,f\right\rangle.\exp\left\{-j\frac{a}{b}y.(x-y)\right\}$$

(4.3.5)

Substituting $x-y=x'$ i.e. $x=x'+y$ and $y=y$ in (4.3.5), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x+y\right\rangle\left\langle x\,|\,g\right\rangle\left\langle y\,|\,f\right\rangle\exp\left\{-j\frac{a}{b}yx\right\} \quad \textbf{(4.3.6)}$$

Rewriting $\left\langle p\,|\,K\,|\,x+y\right\rangle$ explicitly **[61],** results in-

$$\left\langle p\,|\,K\,|\,x+y\right\rangle\exp\left\{-j\frac{a}{b}yx\right\}=\exp j\left\{\frac{a(x+y)^2+dp^2-2p(x+y)-2ayx}{2b}\right\} \quad \textbf{(4.3.7)}$$

Multiplying and dividing (4.3.7) by $\exp j\left[\frac{d}{2b}p^2\right]$ results in-

$$\Rightarrow\exp j\left\{\frac{a(x+y)^2+dp^2-2p(x+y)-2ayx+dp^2}{2b}\right\}.\ \exp j\left\{\frac{-d}{2b}p^2\right\} \quad \textbf{(4.3.8)}$$

Rearranging (4.3.8), results-

$$\Rightarrow\exp j\left\{\frac{ax^2+dp^2-2px}{2b}\right\}.\exp j\left\{\frac{ay^2+dp^2-2py}{2b}\right\}.\exp j\left\{\frac{-d}{2b}p^2\right\} \quad \textbf{(4.3.9)}$$

Substituting the value of (4.3.9) in (4.3.6)-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle x\,|\,g\right\rangle\left\langle y\,|\,f\right\rangle\left[\exp j\left\{\frac{ax^2+dp^2-2px}{2b}\right\}\right.$$
$$\left.\exp j\left\{\frac{ay^2+dp^2-2py}{2b}\right\}.\exp j\left\{\frac{-d}{2b}p^2\right\}\right] \quad \textbf{(4.3.10)}$$

Rewriting (4.3.10) with the aid of quantum mechanics **[60]**, results-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x\right\rangle\left\langle x\,|\,g\right\rangle\left\langle p\,|\,K\,|\,y\right\rangle\left\langle y\,|\,f\right\rangle.\exp j\left\{\frac{-d}{2b}p^2\right\}$$

(4.3.11)

Rearranging (4.3.11), results-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\left\langle p\,|\,K\,|\,x\right\rangle\left\langle x\,|\,g\right\rangle\int_{-\infty}^{\infty}\mathrm{d}y\left\langle p\,|\,K\,|\,y\right\rangle\left\langle y\,|\,f\right\rangle.\exp j\left\{\frac{-d}{2b}p^2\right\}$$

(4.3.12)

Multiplying and dividing (4.3.12) by $\sqrt{j2\pi b}$ and rearranging, results-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{j2\pi b}\,F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(p).e^{-j\frac{d}{2b}p^2} \quad \textbf{(4.3.13)}$$

This is just a new approach for convolution theorem under LCT, derived by representation transformation in quantum mechanics. Therefore, (4.3.1) is obtained and the theorem is proved. The block diagram representation of the proposed convolution theorem is shown in Figure-4.1 and the description of blocks used in the block diagram is shown in Figure-4.2.

The reciprocal transform of (4.3.13) can be obtained by writing the definition of inverse LCT and is given by-

$$L_F^{(d,-c,-b,a)}\left[H(p)\right](x)=\sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}p\left\langle p\,|\,K\,|\,x\right\rangle^{*}.H_{(a,b,c,d)}(p) \quad \textbf{(4.3.14)}$$

where, * indicates the complex conjugate. By using the theory of representation in quantum mechanics **[60, 61]**, (4.3.13) results-

$$\begin{aligned} L_F^{(a,b,c,d)^{-1}}\left[H(p)\right]&=\sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}p\,\left\langle x\,|\,K^{\dagger}\,|\,p\right\rangle.\left\langle p\,|\,H\right\rangle \\ &=\sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}p\,\left\langle x\,|\,K^{\dagger}K\,|\,h\right\rangle=\left\langle x\,|\,h\right\rangle=h(x) \end{aligned} \quad \textbf{(4.3.15)}$$

4.3.1.1 Special cases of LCT for proposed convolution theorem

FT as a special case of LCT, when $(a,b,c,d)=(0,1,-1,0)$, (4.3.13) becomes-

$$L_F^{(0,1,-1,0)}\left[h(x)\right](p)=\sqrt{j2\pi}F_{(0,1,-1,0)}(p)G_{(0,1,-1,0)}(p) \quad \textbf{(4.3.16)}$$

Similarly, FRFT as a special case of LCT, when $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, (4.3.12) becomes-

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}\left[h(x)\right](p)=\sqrt{j2\pi\sin\alpha}\,e^{-\frac{j}{2}p^2\cot\alpha}F_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(p)G_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(p)$$

(4.3.17)

where, $L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}\left[h(x)\right](p)$ indicates the FRFT of $h(x)$ and FT as a special case of FRFT when $\alpha=\pi/2$, (4.3.17) becomes-

$$L_F^{(0,1,-1,0)}\left[h(x)\right](p)=\sqrt{j2\pi}F_{(0,1,-1,0)}(p)G_{(0,1,-1,0)}(p) \quad \textbf{(4.3.18)}$$

Equations (4.3.16) and (4.3.17) are some of the special cases of LCT.

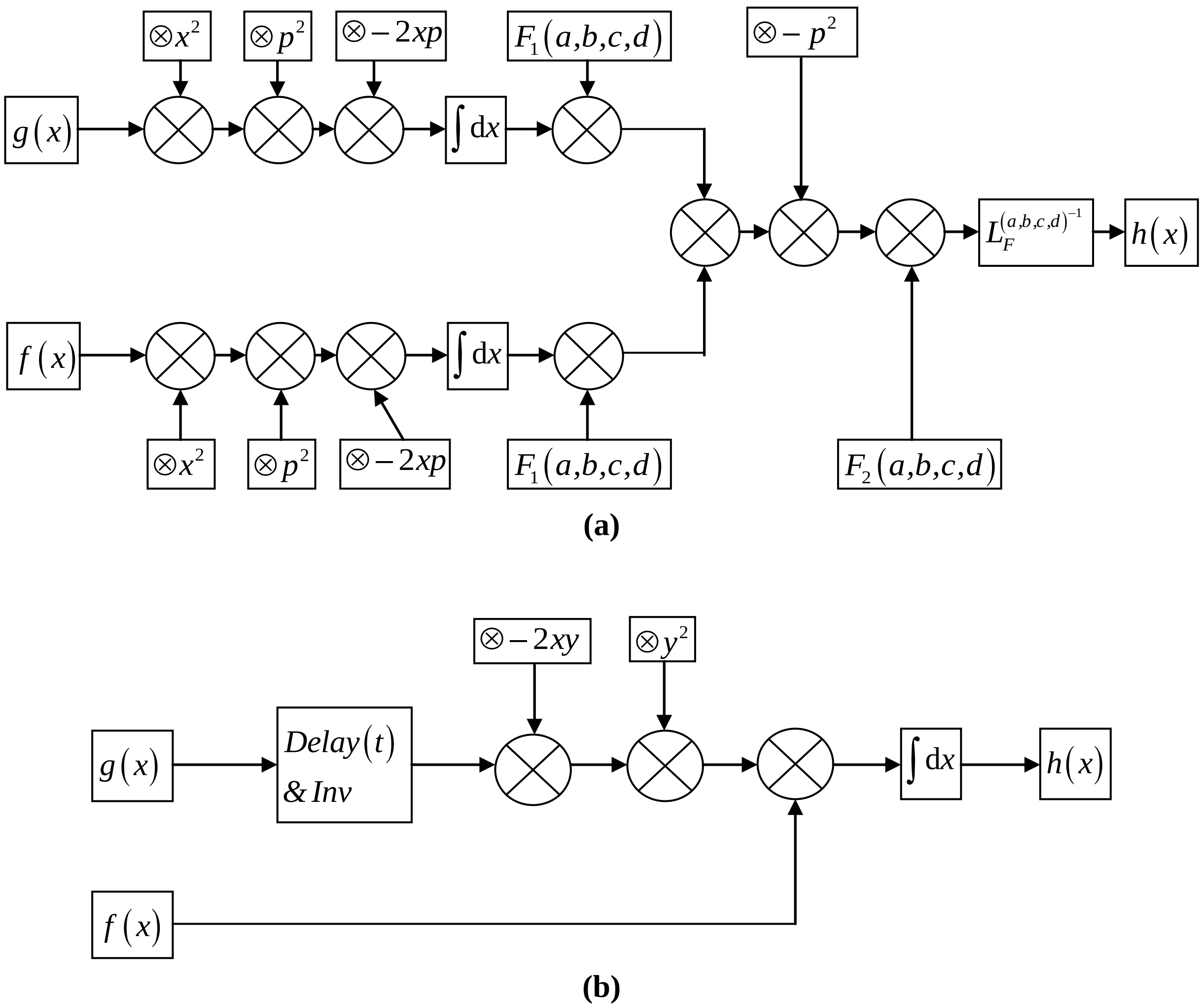


Figure-4.1: Block diagram representation of proposed convolution theorem (a) RHS (b) LHS

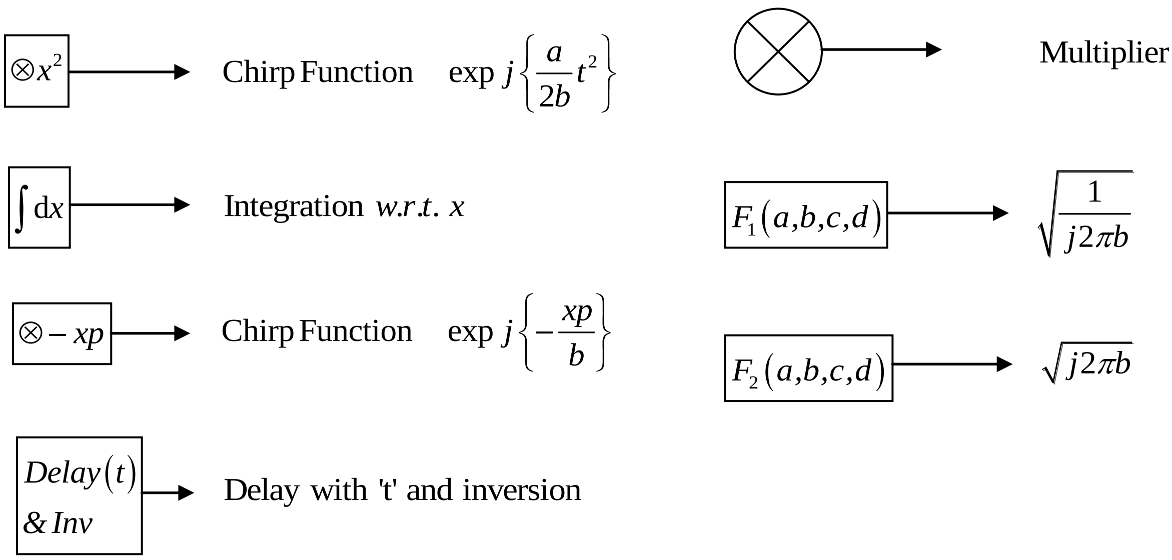


Figure-4.2: Description of blocks used in the block diagram of Figure-4.1.

4.3.1.2 Properties satisfied by proposed convolution theorem

The following are the properties that are satisfied by the proposed convolution theorem:

a) *Commutative property*:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$ and $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$, then as per the commutative property-

$$(f \otimes g)(x) \xleftrightarrow{LCT} \sqrt{j2\pi b}\, e^{-j\frac{dp^2}{2b}} F_{(a,b,c,d)}(p) G_{(a,b,c,d)}(p) \quad \textbf{(4.3.18)}$$

and

$$(g \otimes f)(x) \xleftrightarrow{LCT} \sqrt{j2\pi b}\, e^{-j\frac{dp^2}{2b}} G_{(a,b,c,d)}(p) F_{(a,b,c,d)}(p) \quad \textbf{(4.3.19)}$$

i.e. $(f \otimes g)(x) = (g \otimes f)(x)$ **(4.3.20)**

Proof: Considering the L.H.S. of (4.3.19)-

$$s(x) = (g \otimes f)(x) = \int_{-\infty}^{\infty} g(y)\, f(x-y) \exp\left\{-j\frac{a}{b} y.(x-y)\right\} \mathrm{d}y \quad \textbf{(4.3.21)}$$

Using the quantum mechanical notation **[57, 61]**, the one-dimensional LCT of the signal $s(x)$ reads as-

$$L_F^{(a,b,c,d)}\left[s(x)\right](p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \mathrm{d}x \langle p|K|x\rangle \langle x|s\rangle = \langle p|K|s\rangle = \langle p|S\rangle = S_{(a,b,c,d)}(p) \quad \textbf{(4.3.22)}$$

where, the notation $\langle x|s\rangle = s(x)$ and $\langle p|K|x\rangle = K_{(a,b,c,d)}(p,x)$ gives the representation of LCT kernel in quantum mechanics. K is named as the LCT operator and

$$\langle x|s\rangle = \int_{-\infty}^{\infty} \mathrm{d}y \langle y|g\rangle \langle x-y|f\rangle \exp\left\{-j\frac{a}{b} y.(x-y)\right\}$$

Substituting the value of $\langle x|s\rangle$ in (4.3.22) results-

$$L_F^{(a,b,c,d)}\left[s(x)\right](p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \mathrm{d}x \langle p|K|x\rangle \int_{-\infty}^{\infty} \mathrm{d}y \langle y|g\rangle \langle x-y|f\rangle . \exp\left\{-j\frac{a}{b} y.(x-y)\right\}$$

(4.3.23)

Rearranging (4.3.23), results in-

$$L_F^{(a,b,c,d)}\left[s(x)\right](p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}y \langle p|K|x\rangle \langle x-y|f\rangle \langle y|g\rangle . \exp\left\{-j\frac{a}{b} y.(x-y)\right\}$$

(4.3.24)

Substituting $x-y=x'$ i.e. $x=x'+y$ and $y=y$ in (4.3.24), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$L_F^{(a,b,c,d)}\left[s(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x+y\right\rangle\left\langle x\,|\,f\right\rangle\left\langle y\,|\,g\right\rangle\exp\left\{-j\frac{a}{b}yx\right\} \quad \textbf{(4.3.25)}$$

Rewriting $\left\langle p\,|\,K\,|\,x+y\right\rangle$ explicitly **[61]** ,results in-

$$\left\langle p\,|\,K\,|\,x+y\right\rangle\exp\left\{-j\frac{a}{b}yx\right\}=\exp j\left\{\frac{a(x+y)^2+dp^2-2p(x+y)-2ayx}{2b}\right\} \quad \textbf{(4.3.26)}$$

Multiplying and dividing (4.3.26) by $\exp j\left[\frac{d}{2b}p^2\right]$ results in-

$$\begin{aligned}&\left\langle p\,|\,K\,|\,x+y\right\rangle\exp\left\{-j\frac{a}{b}yx\right\}\\&=\exp j\left\{\frac{a(x+y)^2+dp^2-2p(x+y)-2ayx+dp^2}{2b}\right\}.\exp j\left\{\frac{-d}{2b}p^2\right\}\end{aligned} \quad \textbf{(4.3.27)}$$

Rearranging (4.3.27), results-

$$\begin{aligned}&\left\langle p\,|\,K\,|\,x+y\right\rangle\exp\left\{-j\frac{a}{b}yx\right\}\\&=\exp j\left\{\frac{ax^2+dp^2-2px}{2b}\right\}.\exp j\left\{\frac{ay^2+dp^2-2py}{2b}\right\}.\exp j\left\{\frac{-d}{2b}p^2\right\}\end{aligned} \quad \textbf{(4.3.28)}$$

Substituting the value of (4.3.28) in (4.3.25), results in-

$$\begin{aligned}&L_F^{(a,b,c,d)}\left[s(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\\&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle x\,|\,f\right\rangle\left\langle y\,|\,g\right\rangle\left[\exp j\left\{\frac{ax^2+dp^2-2px}{2b}\right\}.\exp j\left\{\frac{ay^2+dp^2-2py}{2b}\right\}.\exp j\left\{\frac{-d}{2b}p^2\right\}\right]\end{aligned}$$

(4.3.29)

Rewriting (4.3.29) with the aid of quantum mechanics **[60]**, results-

$$L_F^{(a,b,c,d)}\left[s(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x\right\rangle\left\langle x\,|\,f\right\rangle\left\langle p\,|\,K\,|\,y\right\rangle\left\langle y\,|\,g\right\rangle.\exp j\left\{\frac{-d}{2b}p^2\right\}$$

(4.3.30)

Rearranging (4.3.30), results-

$$L_F^{(a,b,c,d)}\left[s(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\left\langle p\,|\,K\,|\,x\right\rangle\left\langle x\,|\,f\right\rangle.\int_{-\infty}^{\infty}\mathrm{d}y\left\langle p\,|\,K\,|\,y\right\rangle\left\langle y\,|\,g\right\rangle.\exp j\left\{\frac{-d}{2b}p^2\right\}$$

(4.3.31)

Multiplying and dividing (4.3.31) by $\sqrt{j2\pi b}$ and rearranging results-

$$L_F^{(a,b,c,d)}\left[s(x)\right](p) = \sqrt{j2\pi b}.e^{-j\frac{d}{2b}p^2}.G_{(a,b,c,d)}(p)F_{(a,b,c,d)}(p) \tag{4.3.32}$$

Therefore, from (4.3.18) and (4.3.21), it conclude that-

$$(f\otimes g)(x) = (g\otimes f)(x)$$

This proves that the proposed convolution theorem for LCT satisfies the commutative law.

b) ***Associative property*:**

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$, $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$, and $M_{(a,b,c,d)}(p)$ is the LCT of $m(x)$ then as per the associative property-

$$((f\otimes g)\otimes m)(x) \xleftrightarrow{LCT} (j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(p)M_{(a,b,c,d)}(p) \tag{4.3.33}$$

$$(f\otimes (g\otimes m))(x) \xleftrightarrow{LCT} (j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(p)M_{(a,b,c,d)}(p) \tag{4.3.34}$$

i.e. $$((f\otimes g)\otimes m) = (f\otimes (g\otimes m)) \tag{4.3.35}$$

Proof: Considering the L.H.S. of (4.3.33)-

i.e. $$z_1(x) = ((f\otimes g)\otimes m)(x) = (h\otimes m)(x) \tag{4.3.36}$$

where, $$h(x) = \langle x\,|\,h\rangle = (f\otimes g)(x) = \int_{-\infty}^{\infty} \mathrm{d}y\langle y\,|\,f\rangle\langle x-y\,|\,g\rangle \exp\left\{-j\frac{a}{b}y.(x-y)\right\} \tag{4.3.37}$$

and $$z_1(x) = ((f\otimes g)\otimes m)(x) = (h\otimes m)(x) = \int_{-\infty}^{\infty} \mathrm{d}\beta\langle \beta\,|\,h\rangle\langle x-\beta\,|\,m\rangle \exp\left\{-j\frac{a}{b}\beta.(x-\beta)\right\} \tag{4.3.38}$$

Substituting the value of $\langle x|h\rangle$ from (4.3.37) to (4.3.38), results in-

$$z_1(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y\langle y\,|\,f\rangle\langle \beta-y\,|\,g\rangle \exp\left\{-j\frac{a}{b}y.(\beta-y)\right\}.\langle x-\beta\,|\,m\rangle \exp\left\{-j\frac{a}{b}\beta.(x-\beta)\right\} \tag{4.3.39}$$

If $Z_{1(a,b,c,d)}(p)$ is the LCT of $z_1(x)$, then taking LCT of $z_1(x)$ results in-

$$Z_{1(a,b,c,d)}(p) = \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,z_1\rangle \tag{4.3.40}$$

Substituting the value of $z_1(x)$ in (4.3.40) results in-

$$Z_{1(a,b,c,d)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\,\mathrm{d}y\,\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle y\,|\,f\,\rangle.$$
$$\langle\beta-y\,|\,g\rangle\langle x-\beta\,|\,m\rangle\exp\left\{-j\frac{a}{b}\beta.(x-\beta)\right\}.\exp\left\{-j\frac{a}{b}y.(\beta-y)\right\} \quad \textbf{(4.3.41)}$$

Substituting $x-\beta=x'$ i.e. $x=x'+\beta$ and $y=y$ in (4.3.41), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$Z_{1(a,b,c,d)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\,\mathrm{d}y\,\mathrm{d}x\langle p\,|\,K\,|\,x+\beta\rangle.$$
$$\langle y\,|\,f\,\rangle\langle\beta-y\,|\,g\rangle\langle x\,|\,m\rangle.\exp\left\{-j\frac{a}{b}\beta x\right\}.\exp\left\{-j\frac{a}{b}y.(\beta-y)\right\} \quad \textbf{(4.3.42)}$$

Rewriting $\langle p\,|\,K\,|\,x+\beta\rangle$ explicitly **[61]**, results in-

$$\langle p\,|\,K\,|\,x+\beta\rangle\exp\left\{-j\frac{a}{b}\beta x\right\}=\exp j\left\{\frac{a(x+\beta)^2+dp^2-2p(x+\beta)-2a\beta x}{2b}\right\} \quad \textbf{(4.3.43)}$$

Substituting the value of (4.3.43) in (4.3.42), results in-

$$Z_{1(a,b,c,d)}(p)=$$
$$M_{(a,b,c,d)}(p)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\,\mathrm{d}y\langle y\,|\,f\,\rangle\langle\beta-y\,|\,g\,\rangle.\exp j\left\{\frac{a\beta^2-2p\beta}{2b}\right\}\exp\left\{-j\frac{a}{b}y.(\beta-y)\right\} \quad \textbf{(4.3.44)}$$

Multiplying and dividing (4.3.44) by $\exp j\left\{\frac{d}{2b}p^2\right\}$ results-

$$Z_{1(a,b,c,d)}(p)=$$
$$M_{(a,b,c,d)}(p)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\,\mathrm{d}y\langle y\,|\,f\,\rangle\langle\beta-y\,|\,g\rangle.\langle p\,|\,K\,|\,\beta\rangle\exp j\left\{-\frac{d}{2b}p^2\right\}\exp\left\{-j\frac{a}{b}y.(\beta-y)\right\}$$

(4.3.45)

Substituting $\beta-y=\beta'$ i.e. $\beta=\beta'+y$ and $y=y$ in (4.3.45), then $\mathrm{d}\beta\mathrm{d}y=\mathrm{d}\beta'\mathrm{d}y$ **[70]** and then replacing β' by β, results-

$$Z_{1(a,b,c,d)}(p)=$$
$$M_{(a,b,c,d)}(p)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\,\mathrm{d}y\langle y\,|\,f\,\rangle\langle\beta\,|\,g\rangle.\langle p\,|\,K\,|\,\beta+y\rangle\exp j\left\{-\frac{d}{2b}p^2\right\}\exp\left\{-j\frac{a}{b}y\beta\right\} \quad \textbf{(4.3.46)}$$

Rewriting $\langle p\,|\,K\,|\,\beta+y\rangle$ explicitly **[61]**, results-

$$\langle p\,|\,K\,|\,\beta+y\rangle\exp\left\{-j\frac{a}{b}y\beta\right\}=\exp j\left\{\frac{a(\beta+y)^2+dp^2-2p(\beta+y)-2a\beta y}{2b}\right\} \quad \textbf{(4.3.47)}$$

Substituting the value of (4.3.47) in (4.3.46), results in-

$$Z_{1(a,b,c,d)}(p) = \sqrt{j2\pi b}\, G_{(a,b,c,d)}(p) M_{(a,b,c,d)}(p) \int_{-\infty}^{\infty} \mathrm{d}y \langle y \,|\, f \rangle . \exp j\left\{\frac{ay^2 - 2py}{2b}\right\} \exp j\left\{-\frac{d}{2b}p^2\right\} \quad \textbf{(4.3.48)}$$

Multiplying and dividing (4.3.48) by $\exp j\left\{\frac{d}{2b}p^2\right\} . \sqrt{\frac{1}{j2\pi b}}$ results-

$$Z_{1(a,b,c,d)}(p) = (j2\pi b)\exp j\left\{-\frac{d}{b}p^2\right\} F_{(a,b,c,d)}(p) G_{(a,b,c,d)}(p) M_{(a,b,c,d)}(p) \quad \textbf{(4.3.49)}$$

Similarly it can proved that the LCT of $z_2(x) = (f \otimes (g \otimes m))$ is

$$Z_{1(a,b,c,d)}(p) = (j2\pi b)\exp j\left\{-\frac{d}{b}p^2\right\} F_{(a,b,c,d)}(p) G_{(a,b,c,d)}(p) M_{(a,b,c,d)}(p) \quad \textbf{(4.3.50)}$$

Therefore, from (4.3.49) and (4.3.50), it has been proved that-

$$((f \otimes g) \otimes m) = (f \otimes (g \otimes m)) \quad \textbf{(4.3.51)}$$

This proves that the proposed convolution theorem for LCT satisfies the associative law.

c) *Distributive property*:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$, $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$ and $M_{(a,b,c,d)}(p)$ is the LCT of $m(x)$ then as per the distributive property-

$$(f \otimes (g+m))(x) \xleftrightarrow{LCT} (j2\pi b) e^{-j\frac{d}{b}p^2} F_{(a,b,c,d)}(p)\left\{G_{(a,b,c,d)}(p) + M_{(a,b,c,d)}(p)\right\} \quad \textbf{(4.3.52)}$$

and

$$(f \otimes g + f \otimes m)(x) \xleftrightarrow{LCT} (j2\pi b) e^{-j\frac{d}{b}p^2} \left\{F_{(a,b,c,d)}(p) G_{(a,b,c,d)}(p) + F_{(a,b,c,d)}(p) M_{(a,b,c,d)}(p)\right\} \quad \textbf{(4.3.53)}$$

i.e. $(f \otimes (g+m))(x) = (f \otimes g + f \otimes m)(x)$ **(4.3.54)**

Proof: Let $z_3(x) = (f \otimes g + f \otimes m)(x)$ and considering the L.H.S. of (4.3.53)-

$$z_3(x) = \int_{-\infty}^{\infty} \mathrm{d}y \langle y \,|\, f \rangle \langle x-y \,|\, g \rangle \exp\left\{-j\frac{a}{b}y.(x-y)\right\} + \int_{-\infty}^{\infty} \mathrm{d}y \langle y \,|\, f \rangle \langle x-y \,|\, m \rangle \exp\left\{-j\frac{a}{b}y.(x-y)\right\} \quad \textbf{(4.3.55)}$$

If $Z_{3(a,b,c,d)}(p)$ is the LCT of $z_3(x)$, then taking LCT of (4.3.55)-

$$Z_{3(a,b,c,d)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,z_3\rangle \qquad \textbf{(4.3.56)}$$

Substituting the value of $z_3(x)$ in (4.3.56), results in-

$$Z_{3(a,b,c,d)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle$$
$$\left[\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x-y\,|\,g\rangle\exp\left\{-j\frac{a}{b}y.(x-y)\right\}+\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x-y\,|\,m\rangle\exp\left\{-j\frac{a}{b}y.(x-y)\right\}\right]$$

(4.3.57)

Substituting $x-y=x'$ i.e. $x=x'+y$ and $y=y$ in (4.3.57), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$Z_{3(a,b,c,d)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x+y\rangle$$
$$\left[\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x\,|\,g\rangle\exp\left\{-j\frac{a}{b}yx\right\}+\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x\,|\,m\rangle\exp\left\{-j\frac{a}{b}yx\right\}\right] \qquad \textbf{(4.3.58)}$$

Rearranging (4.3.58) results in-

$$Z_{3(a,b,c,d)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x+y\rangle\langle y\,|\,f\rangle\langle x\,|\,g\rangle\exp\left\{-j\frac{a}{b}yx\right\}+$$
$$\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x+y\rangle\langle y\,|\,f\rangle\langle x\,|\,m\rangle\exp\left\{-j\frac{a}{b}yx\right\} \qquad \textbf{(4.3.59)}$$

Rewriting $\langle p\,|\,K\,|\,x+y\rangle$ explicitly **[61]**, results in-

$$\langle p\,|\,K\,|\,x+y\rangle\exp\left\{-j\frac{a}{b}yx\right\}=\exp j\left\{\frac{a(x+y)^2+dp^2-2p(x+y)-2ayx}{2b}\right\} \qquad \textbf{(4.3.60)}$$

Substituting the value of (4.3.60) in (4.3.59), results in-

$$Z_{3(a,b,c,d)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\ \langle y\,|\,f\rangle\langle x\,|\,g\rangle.\exp j\left\{\frac{ax^2+ay^2+dp^2-2p(x+y)}{2b}\right\}+$$
$$\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\ \langle y\,|\,f\rangle\langle x\,|\,m\rangle.\exp j\left\{\frac{ax^2+ay^2+dp^2-2p(x+y)}{2b}\right\}$$

(4.3.61)

Multiplying and dividing (4.3.61) by $\exp j\left\{\frac{d}{2b}p^2\right\}.\sqrt{\frac{1}{j2\pi b}}$ results-

$$Z_{3(a,b,c,d)}(p) = (j2\pi b)e^{-j\frac{d}{b}p^2}\left\{F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(p) + F_{(a,b,c,d)}(p)M_{(a,b,c,d)}(p)\right\} \quad \textbf{(4.3.62)}$$

Utilizing the linearity property of LCT, (4.3.62) can be written as-

$$Z_{3(a,b,c,d)}(p) = (j2\pi b)e^{-j\frac{d}{b}p^2}F_{(a,b,c,d)}(p)\left\{G_{(a,b,c,d)}(p) + M_{(a,b,c,d)}(p)\right\} \quad \textbf{(4.3.63)}$$

Similarly it can proved that the LCT of $z_4(x) = (f \otimes (g+m))(x)$ is-

$$Z_{4(a,b,c,d)}(p) = (j2\pi b)e^{-j\frac{d}{b}p^2}F_{(a,b,c,d)}(p)\left\{G_{(a,b,c,d)}(p) + M_{(a,b,c,d)}(p)\right\} \quad \textbf{(4.3.64)}$$

Therefore, from (4.3.63) and (4.3.64), it has been proved that-

$$(f \otimes (g+m))(x) = (f \otimes g + f \otimes m)(x) \quad \textbf{(4.3.65)}$$

This proves that the proposed convolution theorem for LCT satisfies the distributive law.

4.3.1.3 Comparative analysis of the proposed convolution theorem to existing theorems

A comparative analysis of available definitions of convolution function on the following parameters is presented in this section.

a) *FT convertibility*:

The proposed convolution theorem should be converted into classical convolution theorem for FT with variables $(a,b,c,d) = (0,1,-1,0)$. In Table-4.1, 'Satisfying' is entered for the method where, the relation is converted into classical convolution theorem of FT at $(a,b,c,d) = (0,1,-1,0)$ and 'Not Satisfying' is mentioned otherwise. In the existing definitions

TABLE 4.1

FT CONVERTIBILITY FOR PROPOSED CONVOLUTION THEOREM

Name of Methods	Performance Index – FT convertibility
Deng *et al.* **[20]**	Satisfying
Wei *et al.* **[36, 38]**	Satisfying
Proposed Method	Satisfying

b) *Variable dependability*:

The convolution defined in one domain and its transformed counterpart in transformed domain should have mathematical expressions in terms of respective domain variables only. This parameter is assumed in order to assure that a quantity defined in one domain when transformed will result in an equivalent quantity in transformed domain. In Table-4.2, 'Satisfying' is included for the method, which transform a convolution function defined in one domain variable into equivalent function of transform domain variable and 'Not Satisfying' is for the method in which either convolution function is dependent on both variable or transformed equivalent quantity is function of both variable.

TABLE 4.2

VARIABLE DEPENDABILITY FOR PROPOSED CONVOLUTION THEOREM

Name of Methods	Performance Index – FT convertibility
Deng *et al.* **[20]**	Satisfying
Wei *et al.* **[38]**	Not Satisfying
Wei *et al.* **[36]**	Satisfying
Proposed Method	Satisfying

c) *Hardware complexity*:

As a comparison of computational complexity of the proposed convolution theorem with that of the theorems proposed in the literature, the numbers of chirp multiplications are calculated for each of the methods and resulting analysis is shown in the Table-4.3 (LHS represents the defined convolution process by different methods and RHS represents their transforms).

TABLE 4.3

HARDWARE COMPLEXITY FOR PROPOSED CONVOLUTION THEOREM

Parameter	Deng *et al.* [20]		Wei *et al.* [38]		Wei *et al.* [36]		Proposed	
Hardware Complexity (No. of Chirp Functions)	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS
	3	7	7	6	2	10	2	7

d) *Simulation comparison*:

The simulation comparison of the proposed convolution theorem in LCT domain has been done with the theorems given by Deng *et al.* **[20]** and Wei *et al.* **[36, 38]** as shown in Figures-4.3-4.5.

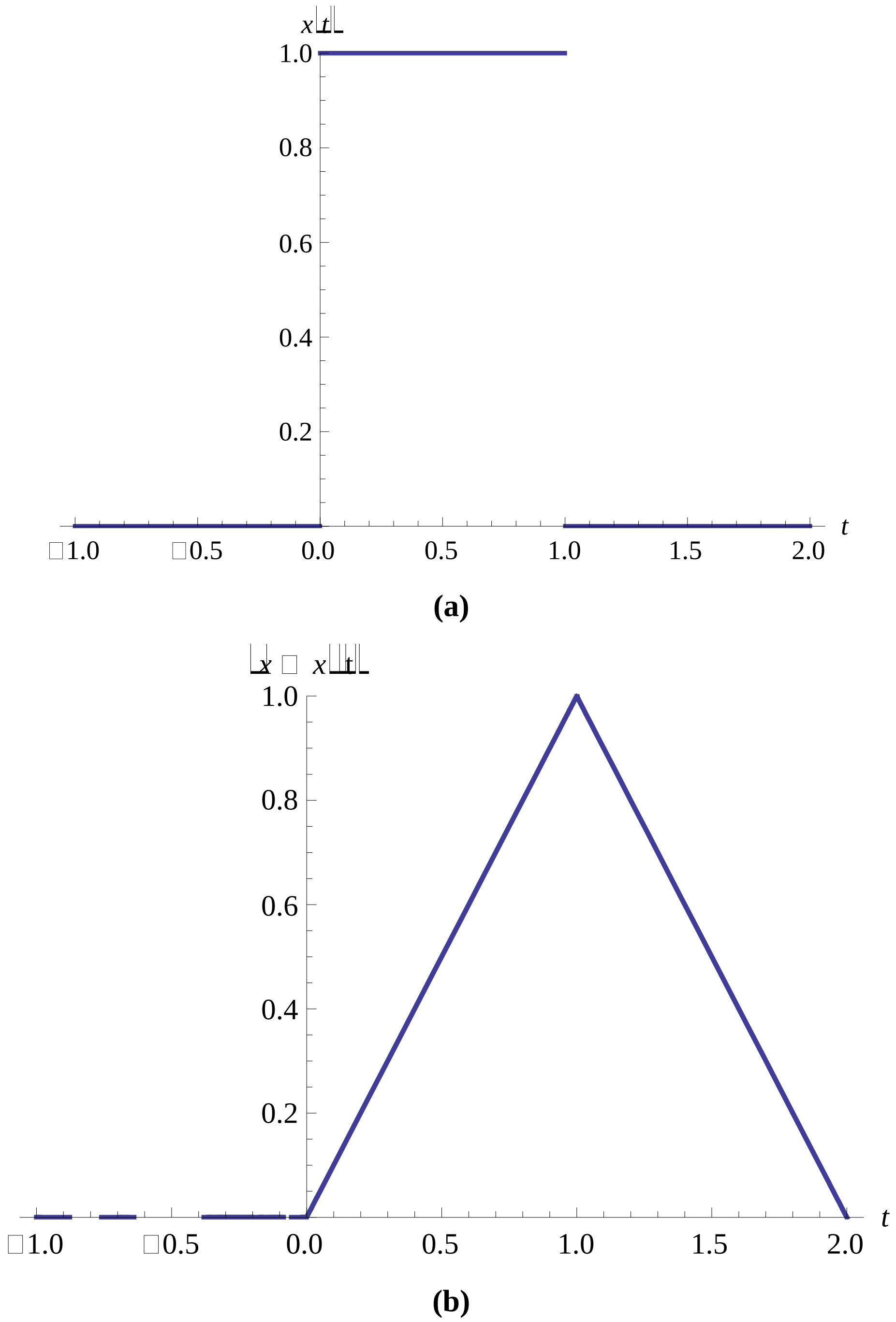
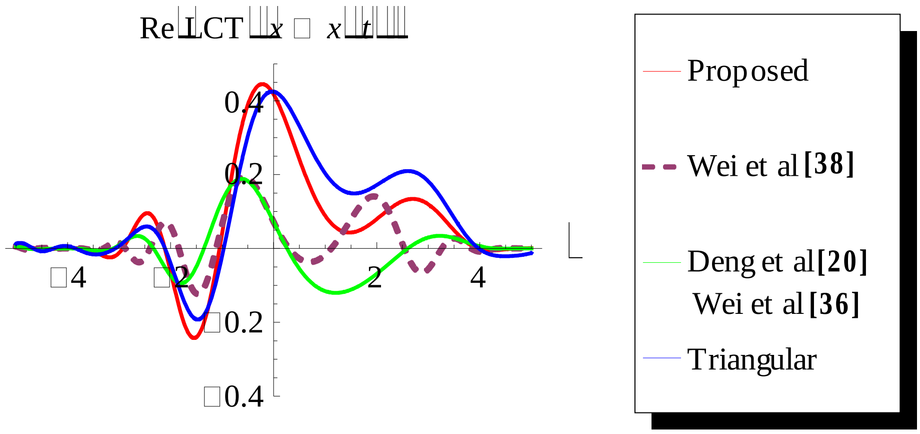
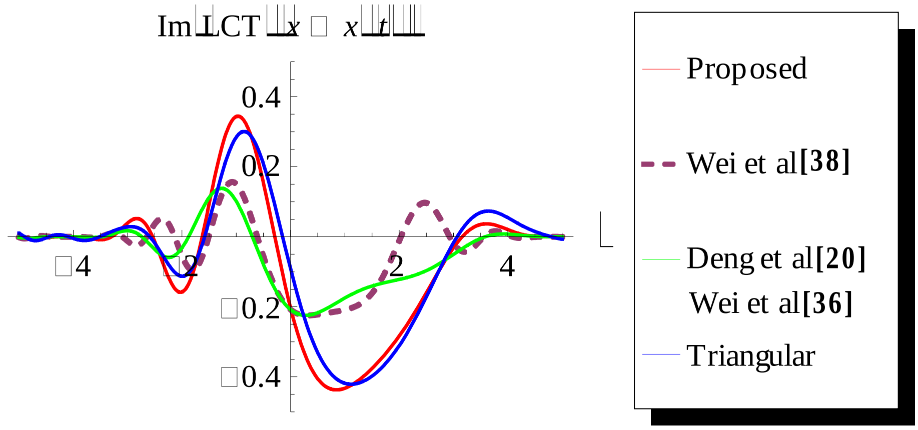


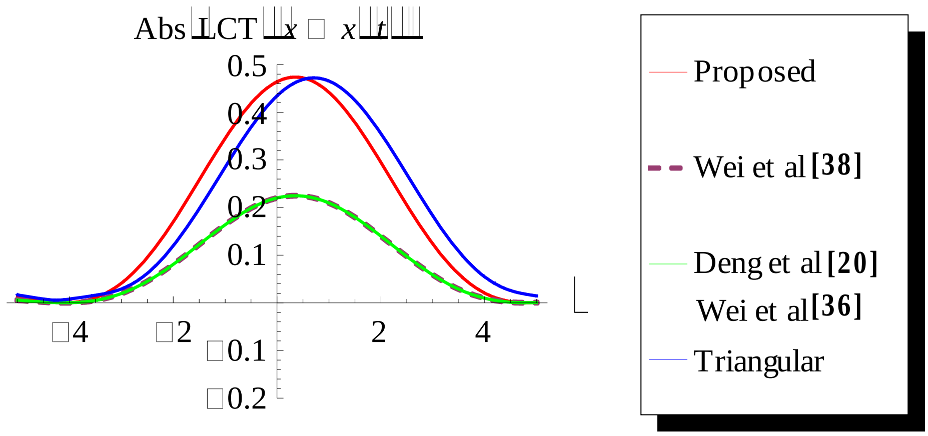
Figure-4.3: (a) Rectangular function $x(t)$ and (b) Convolved signal $(x \otimes x)(t)$ i.e. triangular function



(a)

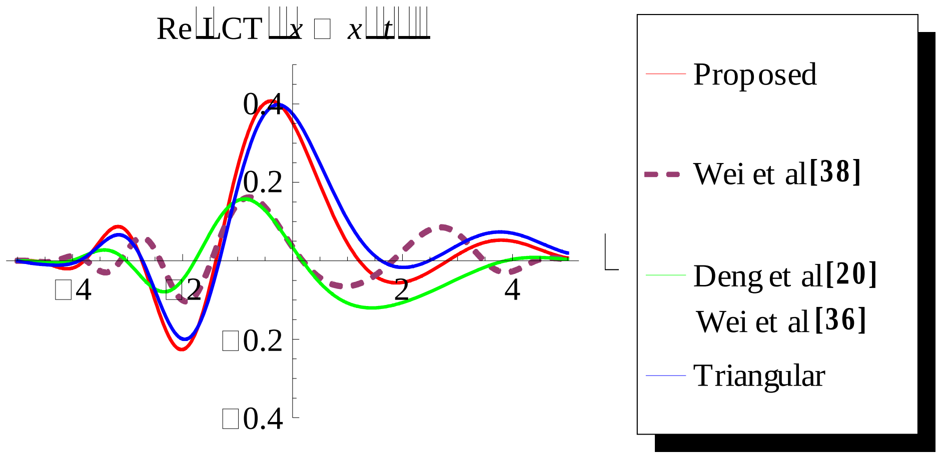


(b)

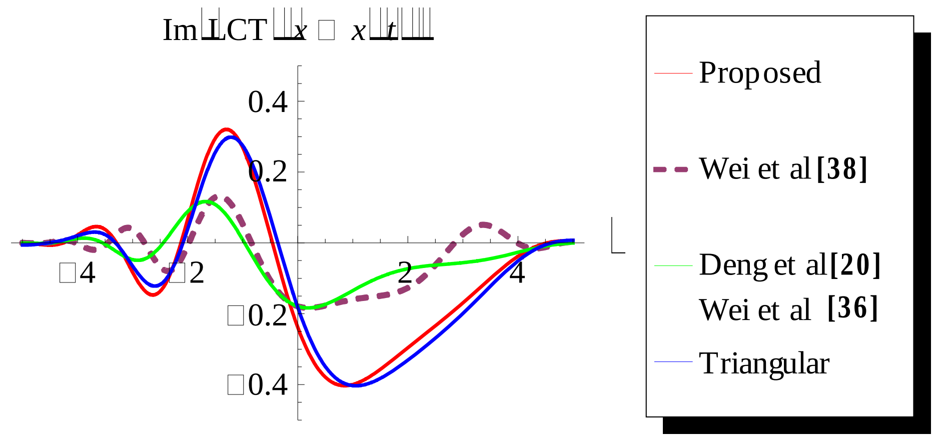


(c)

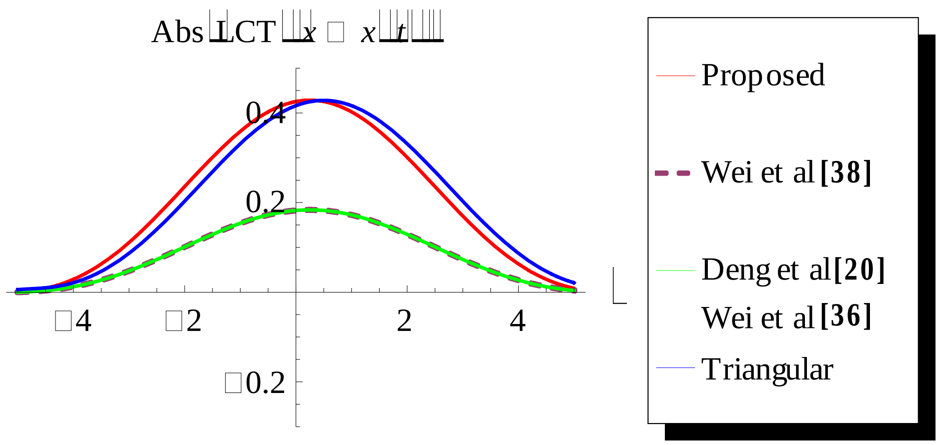
Figure-4.4: (a) Real value (b) Imaginary value (c) Absolute value: of LCT of $(x \otimes x)(t)$ for triangular function, Wei *et al.* [36, 38], Deng *et al.* [20], and proposed methods for $(a,b,c,d) = (0.707, 0.707, -0.707, 0.707)$.



(a)



(b)



(c)

Figure-4.5: (a) Real value (b) Imaginary value (c) Absolute value: of LCT of $(x \otimes x)(t)$ for triangular function, Wei *et al.* [36, 38], Deng *et al.* [20], and proposed methods for $(a,b,c,d) = (0.5, 0.866, -0.866, 0.5)$.

4.3.1 Proposed Product Theorem

For any two functions $f(x)$ and $g(x)$, the definition of product theorem for LCT is given by-

$$q(x) = g(x).f(x).\exp\left\{j\frac{a}{2b}x^2\right\} \tag{4.3.65}$$

and

$$L_F^{(a,b,c,d)}\left[q(x)\right](p) = G_{(a,b,c,d)}(p) \otimes \left(F_{(a,b,c,d)}(p).\exp\left[j\frac{d}{b}vp\right]\right) \tag{4.3.66}$$

Proof. The one-dimensional LCT of $q(x)$ for proposed identities of product theorem is given by-

$$L_F^{(a,b,c,d)}\left[q(x)\right](p) = \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}x\langle x|g\rangle\langle x|f\rangle\langle p|K|x\rangle.\exp\left\{j\frac{a}{2b}x^2\right\} \tag{4.3.67}$$

The one-dimensional inverse LCT in the context of quantum mechanics is given as-

$$\langle x|g\rangle = \sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}v\,\langle v|K|x\rangle^{*}\langle v|G\rangle \tag{4.3.68}$$

where, * indicates the complex conjugate. Rewriting (4.3.68) results-

$$\langle x|g\rangle = \sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}v\,\langle x|K^{\dagger}|v\rangle\langle v|G\rangle \tag{4.3.69}$$

where, $K^{\dagger}$ indicates the Hermitian conjugate of the operator K. Substituting the value of (4.3.69) in (4.3.67), results-

$$L_F^{(a,b,c,d)}\left[q(x)\right](p) = \left|\frac{1}{2\pi b}\right|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}v\,\mathrm{d}x\langle v|G\rangle\langle x|f\rangle\langle x|K^{\dagger}|v\rangle\langle p|K|x\rangle.\exp\left\{j\frac{a}{2b}x^2\right\} \tag{4.3.70}$$

Solving for $\langle x|K^{\dagger}|v\rangle\langle p|K|x\rangle$ results-

$$\langle x|K^{\dagger}|v\rangle\langle p|K|x\rangle = \exp j\left[\frac{ax^2+dp^2-2xp}{2b}\right].\exp j\left[\frac{-ax^2-dv^2+2xv}{2b}\right] \tag{4.3.71}$$

Substituting the value of (4.3.71) in (4.3.70), results-

$$L_F^{(a,b,c,d)}\left[q(x)\right](p) = \left|\frac{1}{2\pi b}\right|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}v\,\mathrm{d}x\langle v|G\rangle\langle x|f\rangle\exp j\left[\frac{ax^2+dp^2-2xp}{2b}\right]. \exp j\left[\frac{-ax^2-dv^2+2xv}{2b}\right].\exp\left\{j\frac{a}{2b}x^2\right\} \tag{4.3.72}$$

Multiplying and dividing (4.3.72) by $\exp\left[-\frac{j}{2}\frac{d}{b}v\left(v-2p\right)\right]$, results-

$$L_F^{(a,b,c,d)}\left[q(x)\right](p)=\left|\frac{1}{2\pi b}\right|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}v\,\mathrm{d}x\langle v|G\rangle\langle x|f\rangle.$$
$$\exp j\left[\frac{ax^2+dp^2-2xp-dv^2+2vx}{2b}\right]\exp j\left[\frac{dv^2-2dpv-dv^2+2dpv}{2b}\right] \quad \textbf{(4.3.73)}$$

Rearranging (4.3.73), results-

$$L_F^{(a,b,c,d)}\left[q(x)\right](p)=\left|\frac{1}{2\pi b}\right|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}v\,\mathrm{d}x\langle v|G\rangle\langle x|f\rangle.$$
$$\exp\left[\frac{j}{2}\left\{\frac{d}{b}\left(p^2+v^2-2pv\right)-\frac{2x}{b}\left(p-v\right)+\frac{ax^2}{b}\right\}\right].\exp j\left[-\frac{d}{b}v^2+\frac{d}{b}pv\right] \quad \textbf{(4.3.74)}$$

Rearranging (4.3.74), results-

$$L_F^{(a,b,c,d)}\left[q(x)\right](p)=\left|\frac{1}{2\pi b}\right|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}v\,\mathrm{d}x\langle v|G\rangle\langle x|f\rangle.$$
$$\exp\left[\frac{j}{2}\left\{\frac{d}{b}\left(p-v\right)^2-\frac{2x}{b}\left(p-v\right)+\frac{ax^2}{b}\right\}\right]\exp j\left[\frac{d}{b}v\left(p-v\right)\right] \quad \textbf{(4.3.75)}$$

From the representation theory of quantum mechanics, (4.3.75) can be written as-

$$L_F^{(a,b,c,d)}\left[q(x)\right](p)=\sqrt{\frac{-1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}v\langle v|G\rangle\langle p-v|f\rangle.\exp j\left[\frac{d}{b}v\left(p-v\right)\right] \quad \textbf{(4.3.76)}$$

Equation (4.3.76) represents the convolution operation as-

$$L_F^{(a,b,c,d)}\left[q(x)\right](p)=G_{(a,b,c,d)}(p)\otimes\left(F_{(a,b,c,d)}(p).\exp\left[j\frac{d}{b}vp\right]\right) \quad \textbf{(4.3.77)}$$

This is just a new approach for product theorem under LCT, derived by representation transformation in quantum mechanics. Therefore, (4.3.66) is proved. All the properties that are satisfied by proposed convolution theorem, are also satisfied by proposed product theorem.

4.4 FILTERING USING PROPOSED CONVOLUTION THEOREM FOR LCT

Filtering is a process of removing some of the frequency components of a signal in order to suppress interfering signals and reduce background noise. Actually, filtering is required to perform many tasks like - signal restoration, signal reconstruction and signal synthesis. Although, all these processes are different in nature, but as a fact, all requires filtering of an incoming signal to produce the desired signal. Both time domain filtering and frequency domain filtering are utilized and superiority of one above other depends on the computational complexity required under different input conditions **[28]**. For non-stationary signals and noise, the time and frequency domain filtering both fails because the signal and noise may have their respective WD overlapping to each other in time and frequency domains.

In this case, fractional Fourier based filtering and LCT based filtering can provide a better solution, where, for a rotated domain corresponding to an optimum value of angle parameter/LCT variables in time-frequency plane, the WD of signal and noise may be separated and filtering of the signal from the noise can be performed by designing a filter in FRFT/LCT domain corresponding to this optimum angle/LCT variables. But from the literature **[8, 20, 36-38]**, it has been found that convolution theorem for FRFT and LCT contains extra chirp function, which is always undesirable. For LCT the chirp function is $e^{-j\frac{d}{b}u^2}$ and for FRFT the chirp function is $e^{-ju^2\cot\alpha}$. Therefore, LCT domain filtering is advantageous over FRFT domain because in LCT domain filtering, the exponential term can be reduced to unity by setting $d=0$, whereas for FRFT, making $\cot\alpha=0$ for $\alpha=\frac{\pi}{2}$ which results in frequency domain filtering. In this section, a filtering exercise is performed to compare the performance of frequency-domain, fractional-domain and LCT domain filtering. The criterion used for optimal filtering is MSE between the original signal and filtered signal for different values of SNR. For making situation more complex, desired signal is assumed to be second-order chirp signal whereas noise signal is assumed to be Additive White Gaussian Noise (AWGN), having overlapping frequency bands.

4.4.1 Design Model of Multiplicative Filter

Many papers discuss the use of multiplicative filter designed by FRFT/LCT **[16, 23, 125, 134]** to remove distortion or noise. In this section a design method of multiplicative filter in LCT domain is discussed based upon the proposed convolution operation. The simple model of multiplicative filter in LCT domain is shown in Figure-4.6. Let the received signal comprises of the desired signal $f(t)$ and noise $n(t)$ such that

$r_{in}(t) = f(t) + n(t)$

Many models of multiplicative filter can be achieved by designing $\tilde{G}_{(a,b,c,d)}(u)$ such as low pass, high pass, band pass, band stop, and so on. If $F_{(a,b,c,d)}(u)$ and $N_{(a,b,c,d)}(u)$ be the LCT components of the desired signal and noise respectively, having no overlapping or minimal overlapping, then the desired signal can be recovered and noise can be discarded to a large extent for increasing SNR through a multiplicative filter in the LCT domain.

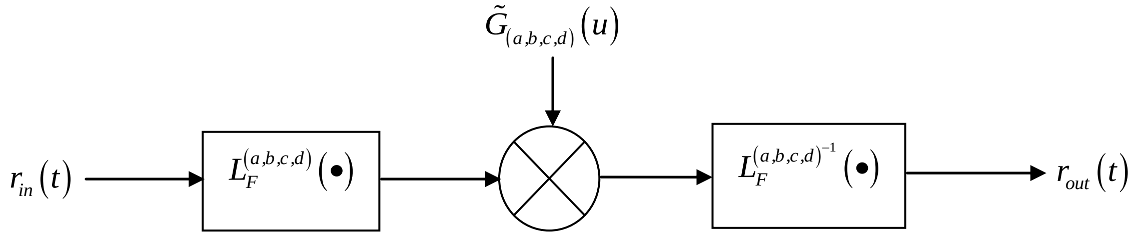


Figure-4.6: The multiplicative filter in the LCT domain

For example, if only the frequency spectrum of the LCT in the region $[u_1, u_2]$ of the desired signal $f(t)$ is of interest, then according to convolution theorem, the transfer function of the multiplicative filter is $\tilde{G}_{(a,b,c,d)}(u) = \sqrt{j2\pi b}\,.\exp\left\{-\frac{j}{2}\left(\frac{d}{b}u^2\right)\right\}G_{(a,b,c,d)}(u)$ such that $\tilde{G}_{(a,b,c,d)}(u)$ is constant over $[u_1, u_2]$ and zero or of rapid delay outside that region. By inverse LCT, the desired signal $f(t)$ can be obtained.

4.4.1.1 Multiplicative filter in LCT domain

If $F_{(a,b,c,d)}(u)$ is the LCT of $f(t)$ and $G_{(a,b,c,d)}(u)$ is the LCT of $g(t)$, then $\sqrt{j2\pi b}\,.\exp\left\{-\frac{j}{2}\left(\frac{d}{b}u^2\right)\right\}F_{(a,b,c,d)}(u)G_{(a,b,c,d)}(u)$ is the LCT of $h(t)$ by the proposed convolution theorem i.e.

$$L_F^{(a,b,c,d)}\left[h(t)\right](u) = \sqrt{j2\pi b}\,.\exp\left\{-\frac{j}{2}\left(\frac{d}{b}u^2\right)\right\}F_{(a,b,c,d)}(u)G_{(a,b,c,d)}(u) \quad \textbf{(4.4.1)}$$

where, $h(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)y(t,\tau)\,\mathrm{d}\tau$ is the weighted convolution operation and the role of g and f can be interchanged and the weighting function is $y(t,\tau)=\exp j\left\{-\frac{a}{b}\tau(t-\tau)\right\}$.

Let us consider an example in which received signal consist of desired signal $f(t)$ and the noise signals $n_1(t)$ and $n_2(t)$ are present. The time-frequency distribution of the received signal is shown in Figure-4.7.

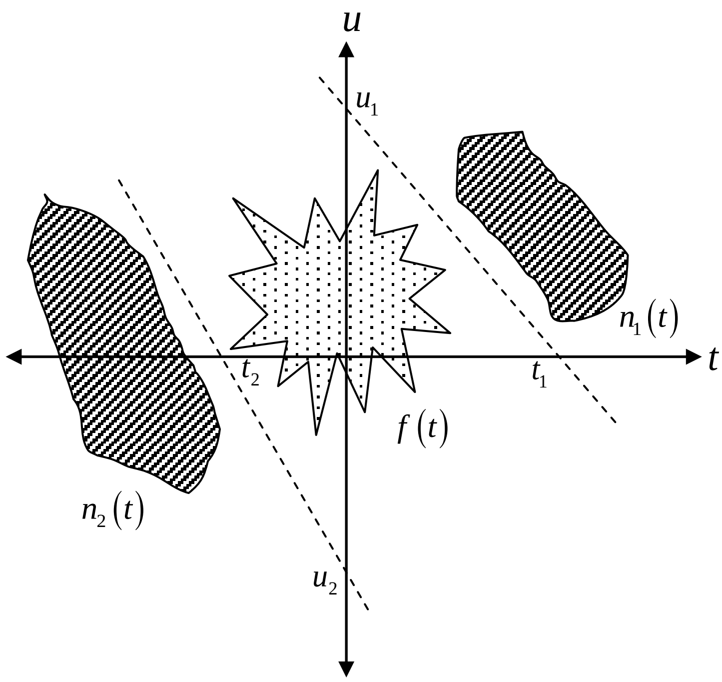


Figure-4.7: Time-frequency distribution of the received signal

The relationship between LCT and the time-frequency distribution **[138]** depicts that there should be greater overlap between the desired signal and noise in time-frequency domain whereas lesser overlap in LCT domain. For different values of (a,b,c,d), different slopes of passband lines $\left(\frac{a_1}{b_1}=\frac{u_1}{t_1}, \frac{a_2}{b_2}=\frac{u_2}{t_2}\right)$ can be achieved as shown in Figure-4.7 and the noise can be filtered out completely keeping the desired signal undistorted through the two consecutive multiplicative filter operations in the time-frequency plane as shown in Figure-4.8. Therefore, one LCT and one inverse LCT can be merged into one operation for the realization of cascade operations making use of additivity property of LCT as shown in Figure-4.8. To validate the proposed model of multiplicative filtering shown in Figure-4.8, an application of filtering has been presented in this section for LCT domain, fractional domain and frequency domain filtering. Consider an original signal-

$$f(t) = 2\exp\left[-\frac{1}{10}(t+2)^2 + j2t^2\right] \qquad \textbf{(4.4.2)}$$

The plot of real and imaginary part of the original signal is shown in Figure-4.9(a). The time-frequency plot of the original signal using WD **[101]** is shown in Figure-4.9(b). Let this signal is corrupted by an AWGN of 5 dB SNR i.e. $n(t)$. Therefore, the input signal is-

$$r_{in}(t) = f(t) + n(t) \qquad \textbf{(4.4.3)}$$

The plot of $r_{in}(t)$ and its time-frequency distribution is shown in Figure-4.9(c) and Figure-4.9(d) respectively. Since the noise is random, so it is distributed in all directions. To filter out the original signal by using the proposed multiplicative filter, two different optimal filtering domains are considered as shown in Figure-4.9(d). Following the Pei and Deng **[137]**, the optimal filtering domain 1 is found by using the procedure shown in Figure-4.10. There are many possible types of canonical filter. The simplest is the pass-stop band filter. The transfer function of the pass-stopband filter is **[138]-**

$$H(u) = \Pi\left((u-u_0)/B\right) \qquad \textbf{(4.4.4)}$$

i.e.

$$H(u) = \begin{cases} 1 & \text{when } u_0 - B/2 < u < u_0 + B/2 \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{(4.4.5)}$$

and, from additivity property of LCT

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{bmatrix}$$

and the LCT variables can be calculated as below (refer Figure-4.10)

$$\frac{a}{b} = \frac{\omega_1}{t_1} \quad \text{and} \quad u_0 = a(t_0 + t_1)/2, \quad B = a(t_1 - t_0)$$

It has been found that-

$$\frac{a}{b} = \frac{-0.3854}{0.606} = -0.636 \qquad \textbf{(4.4.6)}$$

For the relation of LCT variables a and b obtained in (4.4.6), the repeated filtering is performed for different combinations of a and b so that ratio a/b could be maintained. It has been found that for $(a,b,c,d) = (-0.743, 1.168, -0.856, 0)$, minimum MSE is obtained. For these values of LCT variables, the transfer function $G_{(a_1,b_1,c_1,d_1)}(u_1)$ is obtained.

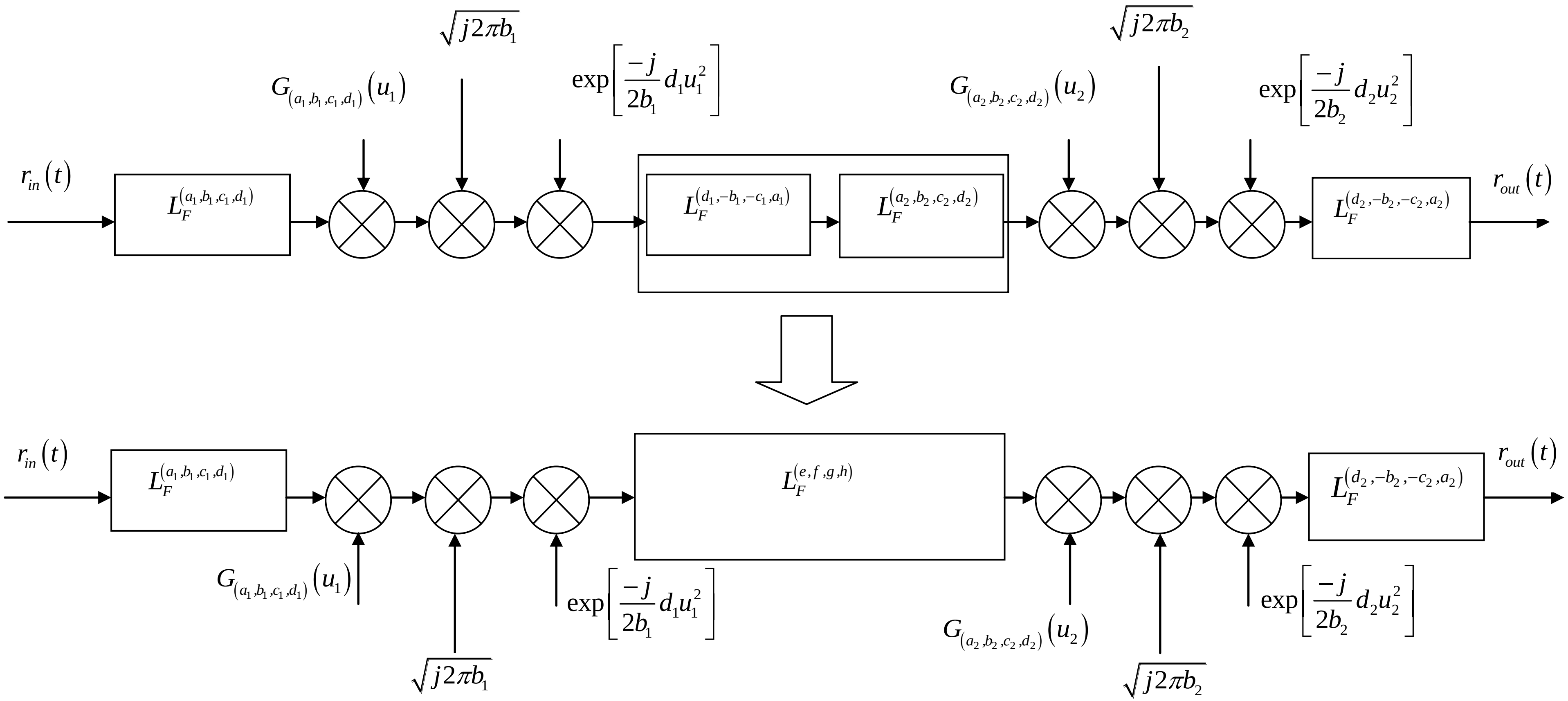


Figure-4.8: Multiplicative Filtering: Cascade operations using additivity property of LCT.

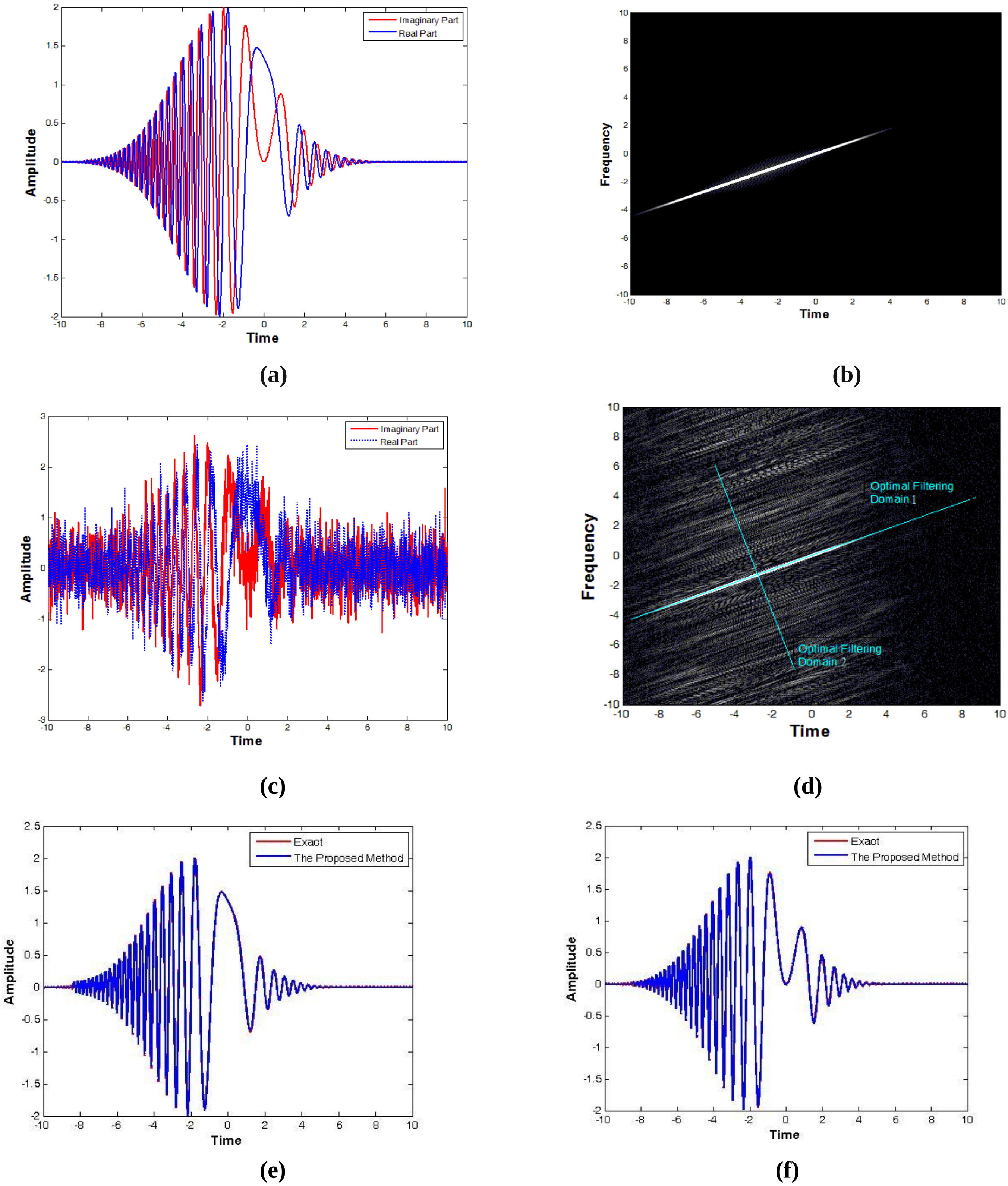


Figure-4.9: (a) Original Signal (b) WD of original signal (c) Corrupted Signal (d) WD of corrupted signal (e) Real part: comparison of original signal and recovered signal (f) Imaginary part: comparison of original signal and recovered signal.

To obtain the optimal filtering domain 2 as shown in Figure 4.10, the WD plot is rotated for different values of LCT variables and it has been found that for $(a,b,c,d)=(0.8443,0.5358,-0.5358,0.8443)$, the desired rotation is obtained and filtering is performed. For these values of LCT variables, the transfer function $G_{(a_2,b_2,c_2,d_2)}(u_2)$ is obtained. As shown in Figure 4.8, by using the additivity property of LCT, first LCT inverse and second LCT operations can be reduced to one LCT operation i.e.

$$\begin{bmatrix} 0.4586 & -1.3842 \\ -0.7227 & 0 \end{bmatrix} \cong \begin{bmatrix} 0.8443 & 0.5358 \\ -0.5358 & -0.8443 \end{bmatrix} \begin{bmatrix} 0 & -1.168 \\ 0.856 & -0.743 \end{bmatrix}$$

Figures 4.9(e) and 4.9(f) shows the comparison between real part and imaginary part of the original signal and the filtered signal with MSE equal to 3.817155×10^{-4}. The same process is repeated for fractional domain filtering. For fractional domain filtering, same code has been used as a special case of LCT. Substituting $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$ in (4.4.1) results in convolution theorem for FRFT, given by-

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}\left[h(t)\right](u)= \sqrt{e^{j\alpha}}.\sqrt{\frac{2\pi}{1-j\cot\alpha}}\,e^{-\frac{j}{2}u^2\cot\alpha}F_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(u)\,G_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(u) \quad \textbf{(4.4.7)}$$

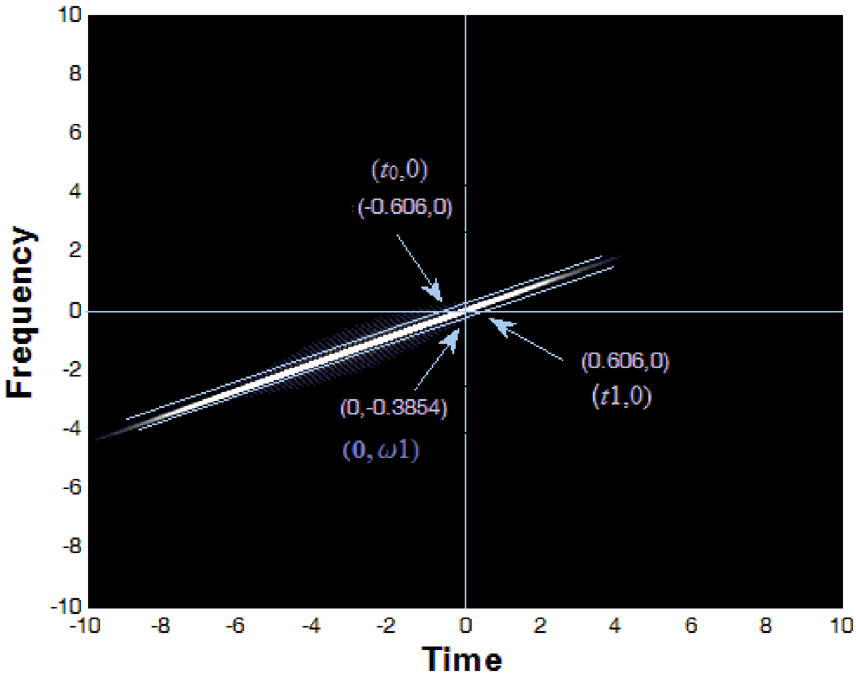


Figure-4.10: Calculation of LCT variables from Wigner distribution plot of the original signal.

where, $\alpha = \frac{\tilde{a}\pi}{2}$. For the optimum domain filtering 1 and 2, the value of fractional variables $\tilde{a}$ is found to be -0.649 and 0.0806 respectively. The frequency domain filtering is performed as a special case of LCT for $(a,b,c,d) = (0,1,-1,0)$. The comparison plot of MSE versus SNR for LCT domain filtering, fractional domain filtering and frequency domain filtering is shown in Figure-4.11.

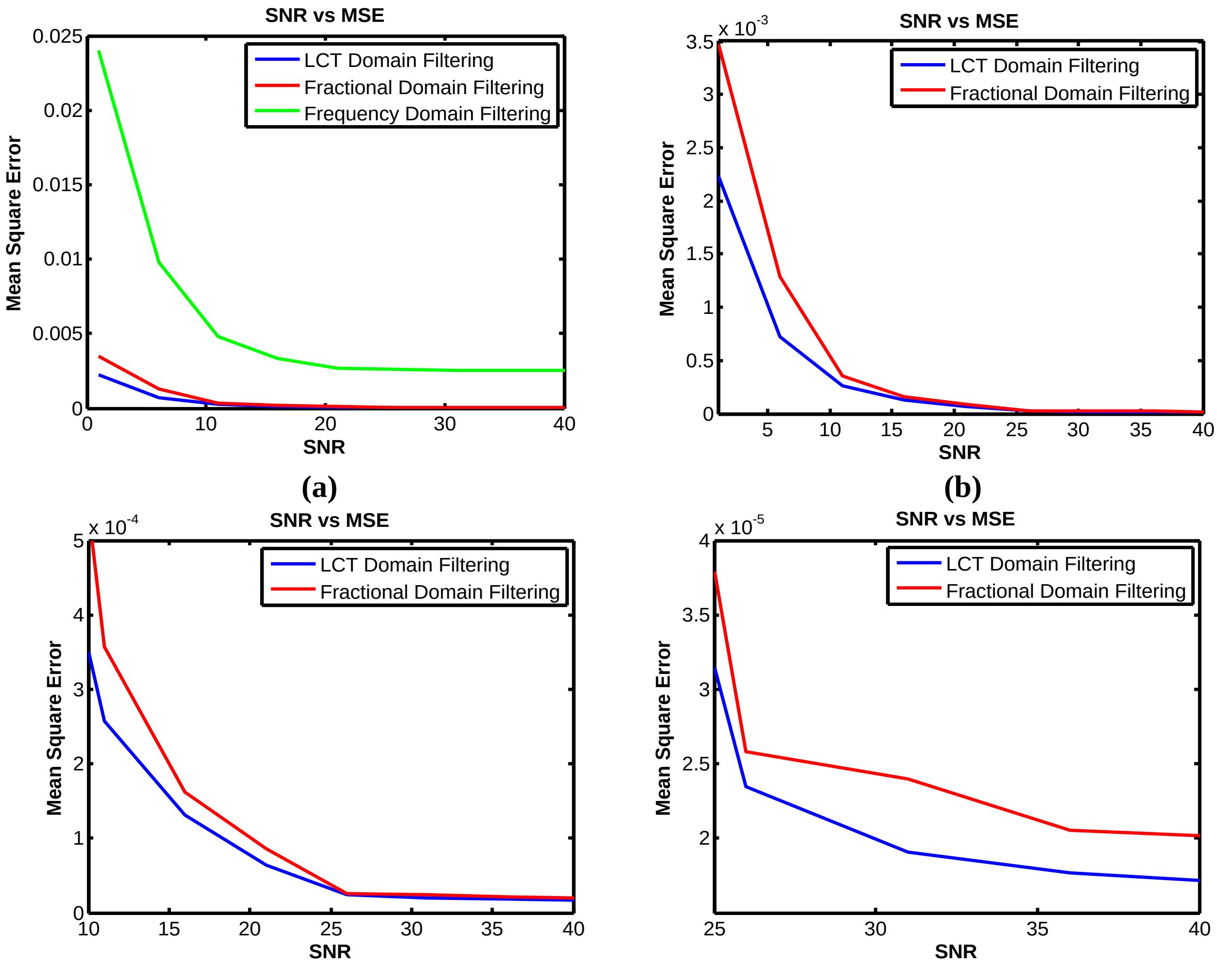


Figure-4.11: (a) MSE versus SNR for LCT domain, fractional domain and frequency domain low pass filtering (b)-(d) Zoomed view of LCT domain filtering versus fractional domain filtering.

From Figure-4.11, it has been found that LCT takes the advantage of three free variables as compare to one free variable for FRFT and no free variable for FT. From (4.4.1), it has been found that a chirp function $\exp\left\{-\frac{j}{2}\left(\frac{d}{b}u^2\right)\right\}$ is involved in the proposed convolution theorem. To make this chirp function equal to 1, $d=0$ is substituted and other variables are responsible for rotation. Whereas from (4.4.7), convolution theorem for FRFT results in $\exp\left\{-j\frac{u^2}{2}\cot\alpha\right\}$ chirp function. To make this chirp function equal to 1, one has to substitute $\alpha=\frac{\pi}{2}$ that means $\tilde{a}=1$ which results in FT i.e. frequency domain filtering.

This is the reason that LCT takes the advantage of its three free variables and it gives minimum MSE for different values of SNR as compared to fractional domain filtering and frequency domain filtering.

4.5 DISCUSSIONS

The proposed convolution theorem along with better performance in features is included in this chapter. The work reported in this chapter has been published in the month of August 2013. However the proposed convolution theorem for LCT was communicated to the journal in the month of July 2012. At later stage, it was found that a matching work has been published by Wei *et al.* **[37]** in Optik-International Journal for Light and Electron Optics (Elsevier Publication) in the month of August 2012. This is a shear co-incidence that to separate groups are working on the same problem and converged in a similar manner with some time lag unknowingly and unnoticeably. The performance and complexity is at par of both the approaches as given below

Parameter	**Wei *et al.* [37]**		**Proposed**	
Hardware Complexity	LHS	RHS	LHS	RHS
(No. of Chirp Functions)	2	7	2	7
Varaible Dependability	Satisfying		Satisfying	
FT Convertbility	Satisfying		Satisfying	

To validate the improvement of the proposed convolution theorem in comparison with FRFT and FT, a comparative analysis of LCT domain multiplicative filtering has been included. The philosophy of multiplicative filtering has been introduced for the first time. It has been found that the multiplicative filtering is the true and best application of LCT to extract the advantage of constituting three variables in contrast to one variable and no variable of FRFT and FT respectively. The multiplicative filtering also demonstrates the power associated with LCT in comparison to FRFT and FT. These statements are made on the basis of MSE determined in different cases under numerous conditions.

CHAPTER 5
CORRELATION THEOREM FOR LCT

The correlation function is a mathematical operator, very similar to the convolution. The correlation process and convolution process are identical, except for one minor difference. Whereas, the convolution involves reversing a signal, then shifting it and multiplying by another signal, correlation only involves shifting it and multiplying (no reversing. In statistics, correlation determines the degree of similarity between two signals. If the signals are identical, then the correlation coefficient is 1; if they are totally different, the correlation coefficient is 0, and if they are identical except that the phase is shifted by 180^o exactly (i.e. mirrored), then the correlation coefficient is -1. When two independent signals are compared, the procedure is known as *cross-correlation*, and when the same signal is compared to phase shifted copies of itself, the procedure is known as *auto-correlation* **[18]**.

In signal processing, cross-correlation is a measure of similarity of two waveforms as a function of a time-lag applied to one of them. This is also known as a sliding dot product or inner-product **[2]**.

5.1 INTRODUCTION

If a signal is correlated with itself, the resulting signal is called the autocorrelation. Informally, it is the similarity between observations as a function of the time separation between them. There are various applications of cross- and auto-correlation functions in the areas of signal processing, pattern recognition, radar communication, optics, statistics, single

The outcome of this chapter has been published in Research Journal as per following detail: N. Goel, K.Singh, Modified correlation theorem for the linear canonical transform with representation transformation in quantum mechanics, Springer-Signal Image and Video Processing, vol. 8, no. 5, pp. 595-601, 2014.

particle analysis, electron tomographic averaging, cryptanalysis, astrophysics and neurophysiology **[29, 30, 111, 128]**. Some of the applications are listed below:

- In signal processing, auto-correlation function can give information about repeating events like musical beats (for example, to determine tempo) or pulsar frequencies, though it cannot tell the position in time of the beat. It can also be used to estimate the pitch of a musical tone.
- In radar signal processing, the received echo from a target is correlated with the transmitted signal to estimate the distance, velocity and acceleration of the target with respect to the receiver.
- Correlation function is used in power spectrum analysis.
- Matched filtering used in many communication systems is based on the concept of correlation function.
- Auto-correlation in space rather than time, via the Patterson function, is used by X-ray diffractionists to help recover the "Fourier phase information" on atom positions not available through diffraction alone.
- One application of auto-correlation is the measurement of optical spectra and the measurement of very-short-duration light pulses produced by lasers, both using optical auto-correlators. Also normalized autocorrelations and cross-correlations give the degree of coherence of an electromagnetic field.
- In statistics, spatial auto-correlation between sample locations also helps one estimate mean value uncertainties when sampling a heterogeneous population.

In this chapter, a brief description of correlation theorem of FT along with basic properties satisfied by this identity is given. Subsequently, the need of correlation theorem in LCT domain is highlighted followed by the existing methods for the same. To remove the shortcomings inherent in previous definitions, weighted cross- correlation and auto-correlation theorems for LCT are proposed along with various properties being satisfied by these identities are presented. A comparative study for auto-correlation theorem of LCT has also been made between proposed and existing definitions.

5.2 PROPOSED CORRELATION THEOREM FOR LCT

5.2.1 Proposed Cross-Correlation Theorem for LCT

The proposed definition of cross-correlation theorem for LCT is as follows:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$ and $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$, then $\sqrt{j2\pi b}\exp j\left(-\frac{d}{2b}p^2\right)F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p)$ is the LCT of $h(x)$ i.e.

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{j2\pi b}\exp j\left(-\frac{d}{2b}p^2\right)F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p) \quad \textbf{(5.2.1)}$$

where, $h(x)=(f\star g)(x)=\int_{-\infty}^{\infty} f(y)g(x+y)\exp\left(j\frac{a}{b}y.(x+y)\right)\mathrm{d}y$, is the weighted cross-correlation operation and the role of g and f can be interchanged.

Proof. Using the quantum mechanical notation **[57, 61]**, the one-dimensional LCT of the signal $h(x)$ reads as-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\langle x|h\rangle=\langle p|K|h\rangle=\langle p|H\rangle=H_{(a,b,c,d)}(p) \quad \textbf{(5.2.2)}$$

where, the notation $\langle x|h\rangle=h(x)$ and $\langle p|K|x\rangle=K_{(a,b,c,d)}(p,x)$ gives the representation of LCT kernel in quantum mechanics. K is named as the LCT operator and

$$\langle x|h\rangle=\int_{-\infty}^{\infty}\mathrm{d}y\langle y|f\rangle\langle x+y|g\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\} \quad \textbf{(5.2.3)}$$

Substituting the value of $\langle x|h\rangle$ in (5.2.2), results in-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\int_{-\infty}^{\infty}\mathrm{d}y\langle y|f\rangle\langle x+y|g\rangle.\exp\left\{j\frac{a}{b}y.(x+y)\right\}$$

(5.2.4)

Rearranging (5.2.4), results in-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p|K|x\rangle\langle x+y|g\rangle\langle y|f\rangle.\exp\left\{j\frac{a}{b}y.(x+y)\right\} \quad \textbf{(5.2.5)}$$

Substituting $x+y=x'$ i.e. $x=x'-y$ and $y=y$ in (5.2.5), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p|K|x-y\rangle\langle x|g\rangle\langle y|f\rangle.\exp\left\{j\frac{a}{b}yx\right\} \quad \textbf{(5.2.6)}$$

Rewriting $\langle p|K|x-y\rangle$ explicitly, results in-

$$\langle p|K|x-y\rangle\sqrt{\frac{1}{j2\pi b}}\exp\left\{j\frac{a}{b}yx\right\}=\sqrt{\frac{1}{j2\pi b}}\exp j\left\{\frac{a(x-y)^2+dp^2-2p(x-y)+2ayx}{2b}\right\} \quad \textbf{(5.2.7)}$$

Multiplying and dividing (5.2.7) by $\frac{1}{\sqrt{j2\pi b}}\exp\left[j\frac{d}{2b}p^2\right]$ results-

$$\langle p|K|x-y\rangle\sqrt{\frac{1}{j2\pi b}}\exp\left\{j\frac{a}{b}yx\right\}=\frac{\sqrt{j2\pi b}}{j2\pi b}\exp j\left\{\frac{ax^2+ay^2+dp^2-2px+2py+dp^2-dp^2}{2b}\right\}$$

(5.2.8)

Rearranging (5.2.8) and simplifying-

$$\Rightarrow\frac{\sqrt{j2\pi b}}{j2\pi b}\left[\exp j\left\{\frac{dp^2-2px+ax^2}{2b}\right\}\exp j\left\{\frac{dp^2+2py+ay^2}{2b}\right\}\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.9)}$$

Substituting the value of (5.2.9) in (5.2.6) and simplifying-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle x|g\rangle\langle y|f\rangle\left[\exp j\left\{\frac{dp^2-2px+ax^2}{2b}\right\}.\right.$$
$$\left.\exp j\left\{\frac{dp^2+2py+ay^2}{2b}\right\}.\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.10)}$$

Rewriting (5.2.10), results in-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p|K|x\rangle\langle x|g\rangle\langle p|K|-y\rangle\langle y|f\rangle\ \exp\left\{-j\frac{d}{2b}p^2\right\}$$

(5.2.11)

Rearranging (5.2.11), results in-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{j2\pi b}\left[\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}y\langle p|K|-y\rangle\langle y|f\rangle\right.$$
$$\left.\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\langle x|g\rangle.\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.12)}$$

Rewriting (5.2.12), results-

$$L_F^{(a,b,c,d)}\left[h(x)\right](p)=\sqrt{j2\pi b}.\exp\left\{-j\frac{d}{2b}p^2\right\}F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p) \quad \textbf{(5.2.13)}$$

This is just a new correlation theorem under LCT, derived by representation in quantum mechanics. The reciprocal transform of (5.2.13) can be obtained by using the definition of inverse LCT given as-

$$L_F^{(d,-c,-b,a)}\left[H(p)\right](x)=\sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}p\langle p|K|x\rangle^*.H_{(a,b,c,d)}(p) \quad \textbf{(5.2.14)}$$

where, * indicates the complex conjugate. By using the theory of representation in quantum mechanics, (5.2.14) results-

$$\begin{aligned} L_F^{(d,-c,-b,a)}\left[H(p)\right](x) &= \sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}p \left\langle x \,|\, K^{\dagger} \,|\, p\right\rangle . \left\langle p \,|\, H\right\rangle \\ &= \sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}p \left\langle x \,|\, K^{\dagger}K \,|\, h\right\rangle = \left\langle x \,|\, h\right\rangle = h(x) \end{aligned} \quad \textbf{(5.2.15)}$$

5.2.1.1 Special cases of LCT for proposed cross-correlation theorem

The following are the special cases of LCT for the proposed cross-correlation theorem:

FT as a special case of LCT, when $(a,b,c,d)=(0,1,-1,0)$, (5.2.15) becomes-

$$L_F^{(0,1,-1,0)}\left[h(x)\right](p) = \sqrt{j2\pi}.F_{(0,1,-1,0)}(-p)G_{(0,1,-1,0)}(p) \quad \textbf{(5.2.16)}$$

Similarly, FRFT as a special case of LCT, when $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, (5.2.16) becomes-

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}\left[h(x)\right](p) = \sqrt{j2\pi\sin\alpha}\, e^{-\frac{j}{2}p^2\cot\alpha} F_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(-p)G_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(p) \quad \textbf{(5.2.17)}$$

where, $L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}\left[h(x)\right](p)$ indicates the FRFT of $h(x)$ and FT as a special case of FRFT when $\alpha=\frac{\pi}{2}$, (5.2.17) becomes-

$$L_F^{(0,1,-1,0)}\left[h(x)\right](p) = \sqrt{j2\pi}.F_{(0,1,-1,0)}(-p)G_{(0,1,-1,0)}(p) \quad \textbf{(5.2.18)}$$

Equations (5.2.16) and (5.2.17) are some of the special cases of LCT for the proposed correlation theorem.

5.2.1.2 Properties satisfied by proposed cross-correlation theorem

The following are the properties that are satisfied by the proposed cross-correlation theorem:

a) ***Commutative property*****:**

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$ and $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$, then as per the commutative property

$$(f\star g)(x) \xleftrightarrow{LCT} \sqrt{j2\pi b}\, e^{-j\frac{dp^2}{2b}} F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p) \quad \textbf{(5.2.19)}$$

and

$$\left(g\star f\right)(x)\xleftrightarrow{LCT}\sqrt{j2\pi b}\,e^{-j\frac{dp^2}{2b}}G_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p) \quad \textbf{(5.2.20)}$$

i.e. $(f\star g)(x)\neq(g\star f)(x)$ **(5.2.21)**

Proof: Considering the L.H.S. of (5.2.20)

$$r_{gf}(x)=\left(g\star f\right)(x)=\int_{-\infty}^{\infty}g(y)\,f(x+y)\exp\left\{j\frac{a}{b}y.(x+y)\right\}\,\mathrm{d}y \quad \textbf{(5.2.22)}$$

Using the quantum mechanical notation **[57, 61]**, the one-dimensional LCT of the signal $r_{gf}(x)$ reads as-

$$L_F^{(a,b,c,d)}\left[r_{gf}(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,r_{gf}\rangle=\langle p\,|\,K\,|\,r_{gf}\rangle=\langle p\,|\,R_{gf}\rangle=R_{gf\,(a,b,c,d)}(p)$$

(5.2.23)

where, the notation $\langle x\,|\,r_{gf}\rangle=r_{gf}(x)$ and $\langle p\,|\,K\,|\,x\rangle=K_{(a,b,c,d)}(p,x)$ gives the representation of LCT kernel in quantum mechanics. K is named as the LCT operator and

$$\langle x\,|\,r_{gf}\rangle=\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,g\rangle\langle x+y\,|\,f\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\}$$

Substituting the value of $\langle x|r_{gf}\rangle$ in (5.2.23) results-

$$\langle p\,|\,R_{gf}\rangle=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,g\rangle\langle x+y\,|\,f\rangle.\exp\left\{j\frac{a}{b}y.(x+y)\right\} \quad \textbf{(5.2.24)}$$

where, $\langle p\,|\,R_{gf}\rangle$ is the LCT of $\langle x\,|\,r_{gf}\rangle$. Rearranging (5.2.24), results in-

$$\langle p\,|\,R_{gf}\rangle=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x\rangle\langle x+y\,|\,f\rangle\langle y\,|\,g\rangle.\exp\left\{j\frac{a}{b}y.(x+y)\right\} \quad \textbf{(5.2.25)}$$

Substituting $x+y=x'$ i.e. $x=x'-y$ and $y=y$ in (5.26), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$\langle p\,|\,R_{gf}\rangle=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x-y\rangle\langle x\,|\,f\rangle\langle y\,|\,g\rangle\exp\left\{j\frac{a}{b}yx\right\} \quad \textbf{(5.2.26)}$$

Rewriting $\langle p\,|\,K\,|\,x-y\rangle$ explicitly **[60]**, results in-

$$\langle p\,|\,K\,|\,x-y\rangle\frac{1}{\sqrt{j2\pi b}}\exp\left\{j\frac{a}{b}yx\right\}=\frac{1}{\sqrt{j2\pi b}}\exp j\left\{\frac{dp^2-2p.(x-y)+a.(x-y)^2+2axy}{2b}\right\} \quad \textbf{(5.2.27)}$$

Multiplying and dividing (5.2.27) by $\frac{1}{\sqrt{j2\pi b}}\exp\left[j\frac{d}{2b}p^2\right]$, results-

$$\langle p|K|x-y\rangle\frac{1}{\sqrt{j2\pi b}}\exp\left\{j\frac{a}{b}yx\right\}=\frac{\sqrt{j2\pi b}}{j2\pi b}.\exp j\left\{\frac{dp^2-2px+2py+ax^2+ay^2+dp^2-dp^2}{2b}\right\}$$

(5.2.28)

Rearranging (5.2.28) and simplifying-

$$\langle p|K|x-y\rangle\frac{1}{\sqrt{j2\pi b}}\exp\left\{j\frac{a}{b}yx\right\}=\frac{\sqrt{j2\pi b}}{j2\pi b}\left[\exp j\left\{\frac{dp^2-2px+ax^2}{2b}\right\}\right.$$
$$\left.\exp j\left\{\frac{dp^2+2py+ay^2}{2b}\right\}.\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.29)}$$

Substituting the value of (5.30) in (5.27) and simplifying-

$$\langle p|R_{gf}\rangle=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle x|g\rangle\langle y|f\rangle\left[\exp j\left\{\frac{dp^2-2px+ax^2}{2b}\right\}\right.$$
$$\left.\exp j\left\{\frac{dp^2+2py+ay^2}{2b}\right\}.\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.30)}$$

Rewriting (5.2.30) , results in-

$$\langle p|R_{gf}\rangle=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dx\,dy\langle p|K|x\rangle\langle x|g\rangle\langle p|K|-y\rangle\langle y|f\rangle.\exp\left\{-j\frac{d}{2b}p^2\right\} \quad \textbf{(5.2.31)}$$

Rearranging (5.2.31) , results in-

$$\langle p|R_{gf}\rangle=\sqrt{j2\pi b}\left[\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}y\langle p|K|-y\rangle\langle y|f\rangle.\right.$$
$$\left.\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\langle x|g\rangle.\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.32)}$$

Rewriting (5.2.32), results-

$$\langle p|R_{gf}\rangle=\sqrt{j2\pi b}\,e^{-\frac{j}{2}\left(\frac{d}{b}p^2\right)}F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p) \quad \textbf{(5.2.33)}$$

Therefore, from (5.2.19) and (5.2.33), it concludes that

$$(f\star g)(x)\neq(g\star f)(x)$$

This proves that the proposed cross-correlation theorem for LCT does not satisfy the commutative law.

b) *Associative property*:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$, $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$ and $M_{(a,b,c,d)}(p)$ is the LCT of $m(x)$, then as per the associative property

$$\big((f\star g)\star m\big)(x)\xleftrightarrow{LCT}(j2\pi b)e^{-j\frac{d}{b}p^2}F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(-p)M_{(a,b,c,d)}(p) \quad \textbf{(5.2.34)}$$

$$\big(f\star(g\star m)\big)(x)\xleftrightarrow{LCT}(j2\pi b)e^{-j\frac{d}{b}p^2}F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(-p)M_{(a,b,c,d)}(p) \quad \textbf{(5.2.35)}$$

i.e. $\big((f\star g)\star m\big)(x)\neq\big(f\star(g\star m)\big)(x)$ **(5.2.36)**

Proof: Considering the L.H.S. of (5.2.34)

i.e. $r_{(fg)m}(x)=\big((f\star g)\star m\big)(x)=\big(r_{fg}\star m\big)(x)$ **(5.2.37)**

where, $r_{fg}(x)=\langle x\,|\,r_{fg}\rangle=(f\star g)(x)=\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,g\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\}$ **(5.2.38)**

and $r_{(fg)m}(x)=\big((f\star g)\star m\big)(x)=\big(r_{fg}\star m\big)(x)=\int_{-\infty}^{\infty}\mathrm{d}\beta\langle\beta\,|\,r_{fg}\rangle\langle x+\beta\,|\,m\rangle\exp\left\{j\frac{a}{b}\beta.(x+\beta)\right\}$

(5.2.39)

Substituting the value of $\langle x\,|\,r_{fg}\rangle$ from (5.2.38) to (5.2.39), results in-

$$r_{(fg)m}(x)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\mathrm{d}y\langle y\,|\,f\rangle\langle\beta+y\,|\,g\rangle\exp\left\{j\frac{a}{b}y.(\beta+y)\right\}.\langle x+\beta\,|\,m\rangle\exp\left\{j\frac{a}{b}\beta.(x+\beta)\right\}$$

(5.2.40)

Taking LCT of $r_{(fg)m}(x)$, results in-

$$\langle p\,|\,R_{(fg)m}\rangle=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,r_{(fg)m}\rangle \quad \textbf{(5.2.41)}$$

Substituting the value of $r_{(fg)m}(x)$ in (5.2.41), results in-

$$\langle p\,|\,R_{(fg)m}\rangle=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\mathrm{d}y\,\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle y\,|\,f\rangle\langle\beta+y\,|\,g\rangle\langle x+\beta\,|\,m\rangle. \\ \exp\left\{j\frac{a}{b}\beta.(x+\beta)\right\}.\exp\left\{j\frac{a}{b}y.(\beta+y)\right\} \quad \textbf{(5.2.42)}$$

Substituting $x+\beta=x'$ i.e. $x=x'-\beta$ and $y=y$ in (5.2.42), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$\left\langle p \mid R_{(fg)m} \right\rangle = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \, \mathrm{d}x \left\langle p \mid K \mid x-\beta \right\rangle \left\langle y \mid f \right\rangle \left\langle \beta + y \mid g \right\rangle \left\langle x \mid m \right\rangle . \exp\left\{ j\frac{a}{b}\beta x \right\} . \exp\left\{ j\frac{a}{b} y.\left(\beta + y\right) \right\} \tag{5.2.43}$$

Rewriting $\left\langle p \mid K \mid x-\beta \right\rangle$ explicitly, results in-

$$\left\langle p \mid K \mid x-\beta \right\rangle \exp\left\{ j\frac{a}{b}\beta x \right\} = \exp j\left\{ \frac{a\left(x-\beta\right)^2 + dp^2 - 2p\left(x-\beta\right) + 2a\beta x}{2b} \right\} \tag{5.2.44}$$

Substituting the value of (5.2.44) in (5.2.43), results in-

$$\left\langle p \mid R_{(fg)m} \right\rangle = M_{(a,b,c,d)}\left(p\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \left\langle y \mid f \right\rangle \left\langle \beta + y \mid g \right\rangle . \exp j\left\{ \frac{a\beta^2 + 2p\beta}{2b} \right\} \exp\left\{ j\frac{a}{b} y.\left(\beta + y\right) \right\} \tag{5.2.45}$$

Multiplying and dividing (5.2.45) by $\exp j\left\{ \frac{d}{2b} p^2 \right\}$, results-

$$\left\langle p \mid R_{(fg)m} \right\rangle = M_{(a,b,c,d)}(p) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \left\langle y \mid f \right\rangle \left\langle \beta + y \mid g \right\rangle . \left\langle p \mid K \mid -\beta \right\rangle \exp j\left\{ -\frac{d}{2b} p^2 \right\} \exp\left\{ j\frac{a}{b} y.\left(\beta + y\right) \right\} \tag{5.2.46}$$

Substituting $\beta + y = \beta'$ i.e. $\beta = \beta' - y$ and $y = y$ in (5.2.46), then $\mathrm{d}\beta \mathrm{d}y = \mathrm{d}\beta' \mathrm{d}y$ **[70]** and then replacing β' by β, results-

$$\left\langle p \mid R_{(fg)m} \right\rangle = M_{(a,b,c,d)}(p) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \left\langle y \mid f \right\rangle \left\langle \beta \mid g \right\rangle . \left\langle p \mid K \mid -\beta + y \right\rangle \exp j\left\{ -\frac{d}{2b} p^2 \right\} \exp\left\{ j\frac{a}{b} y.\beta \right\} \tag{5.2.47}$$

Rewriting $\left\langle p \mid K \mid -\beta + y \right\rangle$ explicitly **[60]**, results in-

$$\left\langle p \mid K \mid -\beta + y \right\rangle \exp\left\{ j\frac{a}{b} y\beta \right\} = \exp j\left\{ \frac{a\left(-\beta + y\right)^2 + dp^2 - 2p\left(-\beta + y\right) + 2a\beta y}{2b} \right\} \tag{5.2.48}$$

Substituting the value of (5.2.48) in (5.2.47), results-

$$\left\langle p \mid R_{(fg)m} \right\rangle = \sqrt{j2\pi b}\, G_{(a,b,c,d)}\left(-p\right) M_{(a,b,c,d)}\left(p\right) \int_{-\infty}^{\infty} \mathrm{d}y \left\langle y \mid f \right\rangle . \exp j\left\{ \frac{ay^2 - 2py}{2b} \right\} \exp j\left\{ -\frac{d}{2b} p^2 \right\} \tag{5.2.49}$$

Multiplying and dividing (5.2.49) by $\exp j\left\{\frac{d}{2b}p^2\right\}.\sqrt{\frac{1}{j2\pi b}}$, results-

$$\left\langle p\middle|R_{(fg)m}\right\rangle=\left(j2\pi b\right)\exp j\left\{-\frac{d}{b}p^2\right\}F_{(a,b,c,d)}\left(p\right)G_{(a,b,c,d)}\left(-p\right)M_{(a,b,c,d)}\left(p\right) \quad \textbf{(5.2.50)}$$

Similarly it can proved that the LCT of $r_{f(gm)}(x)=\left(f\star(g\star m)\right)(x)$ is-

$$\left\langle p\,|\,R_{f(gm)}\right\rangle=\left(j2\pi b\right)\exp j\left\{-\frac{d}{b}p^2\right\}F_{(a,b,c,d)}\left(-p\right)G_{(a,b,c,d)}\left(-p\right)M_{(a,b,c,d)}\left(p\right) \quad \textbf{(5.2.51)}$$

Therefore, from (5.2.50) and (5.2.51), it has been proved that-

$$\left((f\star g)\star m\right)(x)\neq\left(f\star(g\star m)\right)(x) \quad \textbf{(5.2.52)}$$

This proves that the proposed cross-correlation theorem for LCT does not satisfies the associative law.

c) *Distributive property*:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$, $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$ and $M_{(a,b,c,d)}(p)$ is the LCT of $m(x)$ then as per the distributive property

$$\left(f\star(g+m)\right)(x)\xleftrightarrow{LCT}\left(j2\pi b\right)e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}\left(-p\right)\left\{G_{(a,b,c,d)}\left(p\right)+M_{(a,b,c,d)}\left(p\right)\right\} \quad \textbf{(5.2.53)}$$

and

$$\left(f\star g+f\star m\right)(x)\xleftrightarrow{LCT}$$
$$\left(j2\pi b\right)e^{-j\frac{dp^2}{2b}}\left\{F_{(a,b,c,d)}\left(-p\right)G_{(a,b,c,d)}\left(p\right)+F_{(a,b,c,d)}\left(-p\right)M_{(a,b,c,d)}\left(p\right)\right\} \quad \textbf{(5.2.54)}$$

i.e. $\left(f\star(g+m)\right)(x)=\left(f\star g+f\star m\right)(x)$ **(5.2.55)**

Proof: Considering the L.H.S. of (5.2.54)

$$r_{fg+fm}(x)=\left(f\star g+f\star m\right)(x)=\int_{-\infty}^{\infty}\mathrm{d}y\left\langle y\,|\,f\right\rangle\left\langle x+y\,|\,g\right\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\}$$
$$+\int_{-\infty}^{\infty}\mathrm{d}y\left\langle y\,|\,f\right\rangle\left\langle x+y\,|\,m\right\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\} \quad \textbf{(5.2.56)}$$

Taking LCT of (5.2.56)

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\left\langle p\,|\,K\,|\,x\right\rangle\left\langle x\,|\,r_{fg+fm}\right\rangle \quad \textbf{(5.2.57)}$$

Substituting the value of $r_{fg+fm}(x)$ in (5.2.57), results in-

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\left[\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,g\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\}\right.$$
$$\left.+\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,m\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\}\right] \qquad \textbf{(5.2.58)}$$

Substituting $x+y=x'$ i.e. $x=x'-y$ and $y=y$ in (5.2.58), then $\mathrm{d}x\,\mathrm{d}y=\mathrm{d}x'\,\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x-y\rangle\left[\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x\,|\,g\rangle\exp\left\{j\frac{a}{b}yx\right\}\right.$$
$$\left.+\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x\,|\,m\rangle\exp\left\{j\frac{a}{b}yx\right\}\right] \qquad \textbf{(5.2.59)}$$

Rearranging (5.2.59), results in-

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x-y\rangle\langle y\,|\,f\rangle\langle x\,|\,g\rangle\exp\left\{j\frac{a}{b}yx\right\}+$$
$$\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x-y\rangle\langle y\,|\,f\rangle\langle x\,|\,m\rangle\exp\left\{j\frac{a}{b}yx\right\} \qquad \textbf{(5.2.60)}$$

Rewriting $\langle p\,|\,K\,|\,x-y\rangle$ explicitly **[60]**, results in-

$$\langle p\,|\,K\,|\,x-y\rangle\exp\left\{j\frac{a}{b}yx\right\}=\exp j\left\{\frac{a(x-y)^2+dp^2-2p(x-y)+2ayx}{2b}\right\} \qquad \textbf{(5.2.61)}$$

Substituting the value of (5.2.61) in (5.2.60) results-

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\ \langle y\,|\,f\rangle\langle x\,|\,g\rangle.\exp j\left\{\frac{ax^2+ay^2+dp^2-2p(x-y)}{2b}\right\}+$$
$$\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\ \langle y\,|\,f\rangle\langle x\,|\,m\rangle.\exp j\left\{\frac{ax^2+ay^2+dp^2-2p(x-y)}{2b}\right\}$$

(5.2.62)

Multiplying and dividing (5.2.62) by $\exp j\left\{\frac{d}{2b}p^2\right\}.\sqrt{\frac{1}{j2\pi b}}$ results-

$$R_{fg+fm}(p)=$$
$$(j2\pi b)e^{-j\frac{dp^2}{b}}\left\{F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p)+F_{(a,b,c,d)}(-p)M_{(a,b,c,d)}(p)\right\} \qquad \textbf{(5.2.63)}$$

Utilizing the linearity property of LCT, (5.2.63) can be written as-

$$R_{fg+fm}(p)=(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(-p)\left\{G_{(a,b,c,d)}(p)+M_{(a,b,c,d)}(p)\right\} \quad \textbf{(5.2.64)}$$

Similarly it can proved that the LCT of $r_{f(g+m)}(p)=(f\star(g+m))(x)$ is-

$$R_{f(g+m)}(p)=(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(-p)\left\{G_{(a,b,c,d)}(p)+M_{(a,b,c,d)}(p)\right\} \quad \textbf{(5.2.65)}$$

Therefore, from (5.2.65) and (5.2.53), it has been proved that-

$$(f\star(g+m))(x)=(f\star g+f\star m)(x) \quad \textbf{(5.2.66)}$$

This proves that the proposed cross-correlation theorem for LCT satisfies the distributive law.

d) ***Even Function*:**

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$ and $G_{(a,b,c,d)}(p)$ is the LCT of $g(x)$ then-

$$(f\star g)(x)\xleftrightarrow{LCT}\sqrt{j2\pi b}\,e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p) \quad \textbf{(5.2.67)}$$

$$(f\star g)(-x)\xleftrightarrow{LCT}\sqrt{j2\pi b}\,e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(-p) \quad \textbf{(5.2.68)}$$

$$(g\star f)(-x)\xleftrightarrow{LCT}\sqrt{j2\pi b}\,e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p) \quad \textbf{(5.2.69)}$$

and as per evenness property

$$(f\star g)(x)\neq(f\star g)(-x) \quad \textbf{(5.2.70)}$$

$$(f\star g)(x)=(g\star f)(-x) \quad \textbf{(5.2.71)}$$

Proof: Considering the L.H.S. of (5.2.69)

$$r_{gf}(-x)=(g\star f)(-x)=\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,g\rangle\langle y-x\,|\,f\rangle\exp\left\{j\frac{a}{b}y.(y-x)\right\} \quad \textbf{(5.2.72)}$$

Taking LCT of (5.2.72)-

$$R_{gf1}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle -x\,|\,r_{gf}\rangle \quad \textbf{(5.2.73)}$$

Substituting the value of $\langle -x|r_{gf}\rangle$ in (5.2.73) results-

$$R_{gf1}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,g\rangle\langle y-x\,|\,f\rangle\exp\left\{j\frac{a}{b}y.(y-x)\right\} \quad \textbf{(5.2.74)}$$

Substituting $y-x=-x'$ i.e. $x=x'+y$ and $y=y$ in (5.2.74), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$R_{gf1}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}x\langle p\,|\,K\,|\,x+y\rangle\langle y\,|\,g\rangle\langle -x\,|\,f\rangle\exp\left\{-j\frac{a}{b}yx\right\} \tag{5.2.75}$$

Rewriting $\langle p\,|\,K\,|\,x+y\rangle$ explicitly **[60]**, results in-

$$\langle p\,|\,K\,|\,x+y\rangle\exp\left\{-j\frac{a}{b}yx\right\}=\exp j\left\{\frac{a(x+y)^2+dp^2-2p(x+y)-2ayx}{2b}\right\} \tag{5.2.76}$$

Substituting the value of (5.2.76) in (5.2.75) results-

$$R_{gf1}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}x\,\langle y\,|\,g\rangle\langle -x\,|\,f\rangle\exp j\left\{\frac{ax^2+ay^2+dp^2-2p(x+y)}{2b}\right\} \tag{5.2.77}$$

Multiplying and dividing (5.2.77) by $\exp j\left\{\frac{d}{2b}p^2\right\}.\sqrt{\frac{1}{j2\pi b}}$ results-

$$R_{gf1}(p)=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}x\,\langle y\,|\,g\rangle\langle -x\,|\,f\rangle\langle p\,|\,K\,|\,x\rangle\langle p\,|\,K\,|\,y\rangle\exp j\left\{-\frac{d}{2b}p^2\right\} \tag{5.2.78}$$

Rearranging (5.2.78), results-

$$R_{gf1}(p)=\sqrt{j2\pi b}\,\exp j\left\{-\frac{d}{2b}p^2\right\}G_{(a,b,c,d)}(p)F_{(a,b,c,d)}(-p) \tag{5.2.79}$$

From (5.2.67) and (5.2.79), it concludes that-

$$(f\star g)(x)=(g\star f)(-x)$$

From (5.2.67) and (5.2.68), it concludes that-

$$(f\star g)(x)\neq(f\star g)(-x)$$

This proves that the proposed cross-correlation theorem for LCT is not an even function of delay.

5.2.2 Proposed Auto-Correlation Theorem for LCT

The proposed definition of auto-correlation theorem for LCT is as follows:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$ then $\sqrt{j2\pi b}\,\exp j\left(-\frac{d}{2b}p^2\right)F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p)$ is the LCT of $r(x)$ i.e.

$$L_F^{(a,b,c,d)}\left[r(x)\right](p)=\sqrt{j2\pi b}\exp j\left\{-\frac{d}{2b}p^2\right\}F_{(a,b,c,d)}(p)F_{(a,b,c,d)}(-p) \quad \textbf{(5.2.80)}$$

where, $r(x)=(f\star f)(x)=\int_{-\infty}^{\infty} f(y)f(x+y)\exp\left(j\frac{a}{b}y.(x+y)\right)\mathrm{d}y,$ is the weighted auto-correlation operation.

Proof. Using the quantum mechanical notation **[57, 61]**, the one-dimensional LCT of the signal $r(x)$ reads as-

$$L_F^{(a,b,c,d)}\left[r(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,r\rangle=\langle p\,|\,K\,|\,r\rangle=\langle p\,|\,R\rangle=R_{(a,b,c,d)}(p)$$

(5.2.81)

where, the notation $\langle x\,|\,r\rangle=r(x)$ and $\langle p\,|\,K\,|\,x\rangle=K_{(a,b,c,d)}(p,x)$ gives the representation of LCT kernel in quantum mechanics. K is named as the LCT operator and

$$\langle x\,|\,r\rangle=\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,f\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\}$$

Substituting the value of $\langle x|r\rangle$ in (5.2.81) results-

$$L_F^{(a,b,c,d)}\left[r(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,f\rangle.\exp\left\{j\frac{a}{b}y.(x+y)\right\}$$

(5.2.82)

Rearranging (5.2.82), results in-

$$L_{(a,b,c,d)}(r(x))(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x\rangle\langle x+y\,|\,f\rangle\langle y\,|\,f\rangle.\exp\left\{j\frac{a}{b}y.(x+y)\right\} \quad \textbf{(5.2.83)}$$

Substituting $x+y=x'$ i.e. $x=x'-y$ and $y=y$ in (5.2.83), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$L_F^{(a,b,c,d)}\left[r(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x-y\rangle\langle x\,|\,f\rangle\langle y\,|\,f\rangle.\exp\left\{j\frac{a}{b}yx\right\} \quad \textbf{(5.2.84)}$$

Rewriting $\langle p|K|x-y\rangle$ explicitly **[60]**, results in-

$$\langle p\,|\,K\,|\,x-y\rangle\sqrt{\frac{1}{j2\pi b}}\exp\left\{j\frac{a}{b}yx\right\}=\sqrt{\frac{1}{j2\pi b}}\exp j\left\{\frac{a(x-y)^2+dp^2-2p(x-y)+2ayx}{2b}\right\}$$

(5.2.85)

Multiplying and dividing (5.2.85) by $\frac{1}{\sqrt{j2\pi b}}\exp\left[j\frac{d}{2b}p^2\right]$ results-

$$\langle p\,|\,K\,|\,x-y\rangle\sqrt{\frac{1}{j2\pi b}}\exp\left\{j\frac{a}{b}yx\right\}=\frac{\sqrt{j2\pi b}}{j2\pi b}\exp j\left\{\frac{ax^2+ay^2+dp^2-2px+2py+dp^2-dp^2}{2b}\right\}$$

(5.2.86)

Rearranging (5.2.86) and simplifying-

$$\Rightarrow\frac{\sqrt{j2\pi b}}{j2\pi b}\left[\exp j\left\{\frac{dp^2-2px+ax^2}{2b}\right\}.\exp j\left\{\frac{dp^2+2py+ay^2}{2b}\right\}\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.87)}$$

Substituting the value of (5.2.87) in (5.2.84) and simplifying-

$$L_F^{(a,b,c,d)}\left[r(x)\right](p)=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle x\,|\,f\rangle\langle y\,|\,f\rangle\left[\exp j\left\{\frac{dp^2-2px+ax^2}{2b}\right\}.\right.$$
$$\left.\exp j\left\{\frac{dp^2+2py+ay^2}{2b}\right\}.\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.88)}$$

Rewriting (5.2.88) , results in-

$$L_F^{(a,b,c,d)}\left[r(x)\right](p)=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x\rangle\langle x\,|\,f\rangle\langle p\,|\,K\,|-y\rangle\langle y\,|\,f\rangle\ \exp\left\{-j\frac{d}{2b}p^2\right\}$$

(5.2.89)

Rearranging (5.2.89), results in-

$$L_F^{(a,b,c,d)}\left[r(x)\right](p)=\sqrt{j2\pi b}\left[\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}y\langle p\,|\,K\,|-y\rangle\langle y\,|\,f\rangle\right.$$
$$\left.\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,f\rangle.\exp\left\{-j\frac{d}{2b}p^2\right\}\right] \quad \textbf{(5.2.90)}$$

Rewriting (5.2.90), results-

$$L_F^{(a,b,c,d)}\left[r(x)\right](p)=\sqrt{j2\pi b}.\exp\left\{-j\frac{d}{2b}p^2\right\}F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p) \quad \textbf{(5.2.91)}$$

This is just a new auto-correlation theorem under LCT, derived by representation transformation in quantum mechanics.

5.2.2.1 Special cases of LCT for proposed auto-correlation theorem

The following are the special cases of LCT for the proposed auto-correlation theorem:

FT as a special case of LCT, when $(a,b,c,d)=(0,1,-1,0)$, (5.2.91) becomes-

$$L_F^{(0,1,-1,0)}\left[r(x)\right](p)=\sqrt{j2\pi}\,F_{(0,1,-1,0)}(-p)\,F_{(0,1,-1,0)}(p) \quad \textbf{(5.2.92)}$$

Similarly, FRFT as a special case of LCT, when $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, (5.2.91) becomes-

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}\left[r(x)\right](p)= \sqrt{j2\pi\sin\alpha}.\exp\left(-\frac{j}{2}p^2\cot\alpha\right)F_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(p)\,F_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}(-p) \quad \textbf{(5.2.93)}$$

where, $L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)}\left[r(x)\right](p)$ indicates the FRFT of $r(x)$ and FT as a special case of FRFT, when $\alpha=\pi/2$, (5.2.93) becomes-

$$L_F^{(0,1,-1,0)}\left[r(x)\right](p)=\sqrt{j2\pi}\,F_{(0,1,-1,0)}(-p)\,F_{(0,1,-1,0)}(p) \quad \textbf{(5.2.94)}$$

Equations (5.2.92) and (5.2.93) are the special cases of LCT.

5.2.2.2 Properties satisfied by proposed auto-correlation theorem

The following properties are satisfied by the proposed auto-correlation theorem:

a) *Commutative property*:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$, then as per the commutative property-

$$(f\star f)(x)\xleftrightarrow{LCT}\sqrt{j2\pi b}\,e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}(-p)\,F_{(a,b,c,d)}(p) \quad \textbf{(5.2.95)}$$

and

$$(f\star f)(x)\xleftrightarrow{LCT}\sqrt{j2\pi b}\,e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}(p)\,F_{(a,b,c,d)}(-p) \quad \textbf{(5.2.96)}$$

i.e. $(f\star f)(x)=(f\star f)(x)$ **(5.2.97)**

Proof: From (5.2.33)

$$\left\langle p\,|\,R_{gf}\right\rangle=\sqrt{j2\pi b}\,e^{-\frac{j}{2}\left(\frac{d}{b}p^2\right)}F_{(a,b,c,d)}(-p)\,G_{(a,b,c,d)}(p) \quad \textbf{(5.2.98)}$$

where, $\left\langle p\,|\,R_{gf}\right\rangle$ represents the LCT of the cross-correlation of $\langle y\,|\,f\rangle$ and $\langle x\,|\,g\rangle$ i.e. $(f\star g)(x)$. Replacing $\langle x\,|\,g\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d)}(p)$ with $F_{(a,b,c,d)}(p)$ in (5.2.98) results LCT of auto-correlation operation as-

$$\left\langle p\,|\,R_{ff}\right\rangle=\sqrt{j2\pi b}\,e^{-\frac{j}{2}\left(\frac{d}{b}p^2\right)}F_{(a,b,c,d)}(-p)\,F_{(a,b,c,d)}(p) \quad \textbf{(5.2.99)}$$

This expression does not change by changing the order of commutating the function. Therefore, auto-correlation operation satisfies the commutative property.

b) *Associative property*:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$, then as per the associative property-

$$\left((f\star f)\star f\right)(x)\xleftrightarrow{LCT}(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(p)F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p) \qquad \textbf{(5.2.100)}$$

$$\left(f\star(f\star f)\right)(x)\xleftrightarrow{LCT}(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p) \qquad \textbf{(5.2.101)}$$

i.e. $\left((f\star f)\star f\right)(x)\neq\left(f\star(f\star f)\right)(x)$ **(5.2.102)**

Proof: From (5.2.51)

$$\left\langle p\,|\,R_{f(gm)}\right\rangle=(j2\pi b)\exp j\left\{-\frac{d}{b}p^2\right\}F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(-p)M_{(a,b,c,d)}(p) \qquad \textbf{(5.2.103)}$$

where, $\left\langle p\,|\,R_{f(gm)}\right\rangle$ represents the LCT of the correlation operation $\left(f\star(g\star m)\right)(x)$. Replacing $\langle x\,|\,g\rangle$ and $\langle x\,|\,m\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d)}(p)$ and $M_{(a,b,c,d)}(p)$ with $F_{(a,b,c,d)}(p)$ in (5.104) results LCT of correlation operation as-

$$\left\langle p\,|\,R_{f(ff)}\right\rangle=(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p) \qquad \textbf{(5.2.104)}$$

where, $\left\langle p\,|\,R_{f(ff)}\right\rangle$ represents the LCT of correlation operation $\left(f\star(f\star f)\right)(x)$.

From (5.2.50)

$$\left\langle p\,|\,R_{(fg)m}\right\rangle=(j2\pi b)\exp j\left\{-\frac{d}{b}p^2\right\}F_{(a,b,c,d)}(p)G_{(a,b,c,d)}(-p)M_{(a,b,c,d)}(p) \qquad \textbf{(5.2.105)}$$

where, $\left\langle p\,|\,R_{(fg)m}\right\rangle$ represents the LCT of the correlation operation $\left((f\star g)\star m\right)(x)$. Replacing $\langle x\,|\,g\rangle$ and $\langle x\,|\,m\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d)}(p)$ and $M_{(a,b,c,d)}(p)$ with $F_{(a,b,c,d)}(p)$ in (5.2.105) results LCT of correlation operation as-

$$\left\langle p\,|\,R_{(ff)f}\right\rangle=(j2\pi b)\exp j\left\{-\frac{d}{b}p^2\right\}F_{(a,b,c,d)}(p)F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p) \qquad \textbf{(5.2.106)}$$

where, $\left\langle p\,|\,R_{(ff)f}\right\rangle$ represents the LCT of correlation operation $\left((f\star f)\star f\right)(x)$.

Therefore, from (5.2.104) and (5.2.106), it has been proved that-

$$((f\star f)\star f)(x)\neq(f\star(f\star f))(x)$$

This confirms that the proposed auto-correlation theorem for LCT is not satisfying associativity property in conformity of its claim of generalization of analogous identity of FT.

c) *Distributive property*:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$, then as per the distributive property

$$(f\star(f+f))(x)\xleftrightarrow{LCT}(j2\pi b)e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}(-p)\left\{F_{(a,b,c,d)}(p)+F_{(a,b,c,d)}(p)\right\} \quad \textbf{(5.2.107)}$$

and

$$(f\star f+f\star f)(x)\xleftrightarrow{LCT}$$
$$(j2\pi b)e^{-j\frac{dp^2}{2b}}\left\{F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p)+F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p)\right\} \quad \textbf{(5.2.108)}$$

i.e. $(f\star(f+f))(x)=(f\star f+f\star f)(x)$ **(5.2.109)**

Proof: From (5.2.63)

$$R_{fg+fm}(p)=(j2\pi b)e^{-j\frac{dp^2}{b}}\left\{F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(-p)+F_{(a,b,c,d)}(-p)M_{(a,b,c,d)}(p)\right\}$$

(5.2.110)

Utilizing the linearity property of LCT, (5.2.110) can be written as-

$$R_{fg+fm}(p)=(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(-p)\left\{G_{(a,b,c,d)}(p)+M_{(a,b,c,d)}(p)\right\} \quad \textbf{(5.2.111)}$$

Replacing $\langle x\,|\,g\rangle$ and $\langle x\,|\,m\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d)}(p)$ and $M_{(a,b,c,d)}(p)$ with $F_{(a,b,c,d)}(p)$ in (5.2.111) results LCT of correlation operation as-

$$R_{ff+ff}(p)=(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(-p)\left\{F_{(a,b,c,d)}(p)+F_{(a,b,c,d)}(p)\right\} \quad \textbf{(5.2.112)}$$

From (5.2.65)

$$R_{f(g+m)}(p)=(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(-p)\left\{G_{(a,b,c,d)}(p)+M_{(a,b,c,d)}(p)\right\} \quad \textbf{(5.2.113)}$$

Replacing $\langle x\,|\,g\rangle$ and $\langle x\,|\,m\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d)}(p)$ and $M_{(a,b,c,d)}(p)$ with $F_{(a,b,c,d)}(p)$ in (5.2.113) results LCT of correlation operation as-

$$R_{f(f+f)}(p)=(j2\pi b)e^{-j\frac{dp^2}{b}}F_{(a,b,c,d)}(-p)\left\{F_{(a,b,c,d)}(p)+F_{(a,b,c,d)}(p)\right\} \quad \textbf{(5.2.114)}$$

Therefore, from (5.2.112) and (5.2.114), it has been proved that-

$$\left(f\star\left(f+f\right)\right)(x)=\left(f\star f+f\star f\right)(x) \quad \textbf{(5.2.115)}$$

This proves that the proposed auto-correlation theorem for LCT satisfies the distributive law.

d) *Even function*:

If $F_{(a,b,c,d)}(p)$ is the LCT of $f(x)$ then-

$$\left(f\star f\right)(x)\xleftrightarrow{LCT}\sqrt{j2\pi b}\,e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}(-p)F_{(a,b,c,d)}(p) \quad \textbf{(5.2.116)}$$

$$\left(f\star f\right)(-x)\xleftrightarrow{LCT}\sqrt{j2\pi b}\,e^{-j\frac{dp^2}{2b}}F_{(a,b,c,d)}(p)F_{(a,b,c,d)}(-p) \quad \textbf{(5.2.117)}$$

and as per evenness property

$$\left(f\star f\right)(x)=\left(f\star f\right)(-x) \quad \textbf{(5.2.118)}$$

Proof: From (5.2.79)

$$R_{gf1}(p)=\sqrt{j2\pi b}\exp j\left\{-\frac{d}{2b}p^2\right\}G_{(a,b,c,d)}(p)F_{(a,b,c,d)}(-p) \quad \textbf{(5.2.119)}$$

where, $\langle p\,|\,R_{gf1}\rangle$ represents the LCT of the correlation operation $\left(g\star f\right)(-x)$. Replacing $\langle x\,|\,g\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d)}(p)$ with $F_{(a,b,c,d)}(p)$ in (5.2.119) results LCT of correlation operation as-

$$R_{ff1}(p)=\sqrt{j2\pi b}\exp j\left\{-\frac{d}{2b}p^2\right\}F_{(a,b,c,d)}(p)F_{(a,b,c,d)}(-p) \quad \textbf{(5.2.120)}$$

where, $R_{ff1}(p)$ represents the LCT of $\left(f\star f\right)(-x)$.

Similarly, from (5.2.19)

$$R_{fg}(p)=\sqrt{j2\pi b}\exp j\left\{-\frac{d}{2b}p^2\right\}F_{(a,b,c,d)}(-p)G_{(a,b,c,d)}(p) \quad \textbf{(5.2.121)}$$

where, $\langle p\,|\,R_{fg}\rangle$ represents the LCT of the correlation operation $\left(f\star g\right)(x)$. Replacing $\langle x\,|\,g\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d)}(p)$ with $F_{(a,b,c,d)}(p)$ in (5.2.121) results LCT of correlation operation as-

$$R_{ff}(p)=\sqrt{j2\pi b}\exp j\left\{-\frac{d}{2b}p^2\right\}F_{(a,b,c,d)}(p)F_{(a,b,c,d)}(-p) \quad \textbf{(5.2.122)}$$

where, $R_{ff}(p)$ represents the LCT of $\left(f\star f\right)(x)$. Therefore, from (5.2.120) and (5.2.122), it has been proved that-

$$\left(f\star f\right)\left(-x\right)=\left(f\star f\right)\left(x\right)$$

Therefore, it can be summarized that the proposed auto-correlation theorem for LCT is an even function. Therefore, the proposed auto-correlation theorem for LCT is satisfying commutative and distributive law whereas associativity is not being satisfied by this theorem.

5.2.3 Comparative Analysis of the Proposed Correlation theorem With the Existing Theorems

A comparative analysis of available definitions of correlation function on the following parameters is presented in this section.

5.2.3.1 FT convertibility

The proposed convolution theorem should be converted into classical correlation theorem for FT with variables $\left(a,b,c,d\right)=\left(0,1,-1,0\right)$. In Table-5.1, 'Satisfying' is entered for the method where, the relation is converted into classical correlation theorem of FT at $\left(a,b,c,d\right)=\left(0,1,-1,0\right)$. In the existing definitions

TABLE 5.1

FT CONVERTIBILITY FOR PROPOSED CORRELATION THEOREM

Name of Methods	Performance Index – FT convertibility
Wei *et al.* [36]	Satisfying
Proposed Method	Satisfying

5.2.3.2 Variable dependability

The correlation defined in one domain and its transformed counterpart in transformed domain should have mathematical expressions in terms of respective domain variables only. This parameter is assumed in order to assure that a quantity defined in one domain when transformed will result in an equivalent quantity in transformed domain. In Table-5.2, 'Satisfying' is included for the method, which transform a convolution function defined in one domain variable into equivalent function of transform domain variable.

TABLE 5.2

VARIABLE DEPENDABILITY FOR PROPOSED CORRELATION THEOREM

Name of Methods	Performance Index – FT convertibility
Wei *et al.* **[36]**	Satisfying
Proposed Method	Satisfying

5.2.3.3 Hardware complexity

As a comparison of computational complexity of the proposed correlation theorem with that of the theorems proposed in the literature, the numbers of chirp multiplications are calculated for each of the methods and resulting analysis is shown in the Table-5.3 (LHS represents the defined convolution process by different methods and RHS represents their transforms).

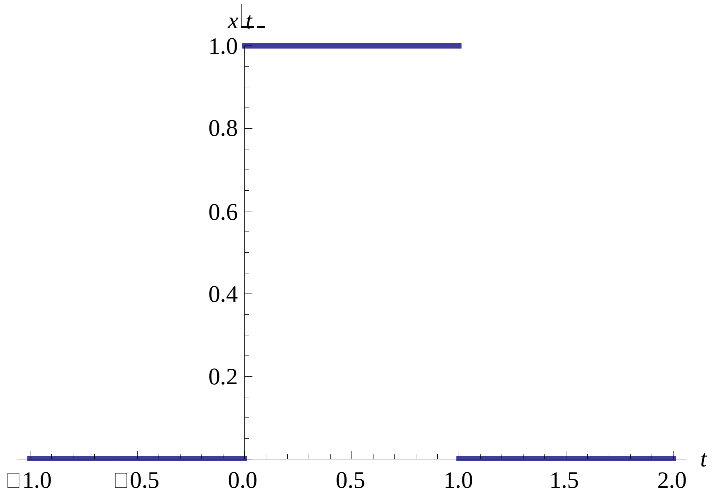
TABLE 5.3

HARDWARE COMPLEXITY FOR PROPOSED CORRELATION THEOREM

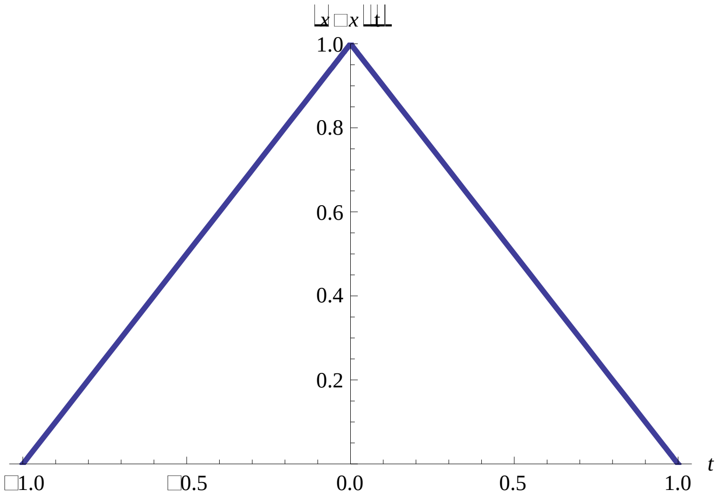
Parameter	Wei *et al.* [36]		Proposed	
Hardware Complexity	LHS	RHS	LHS	RHS
No. of Chirp Functions	2	10	2	7

5.2.3.4 Simulation comparison

The correlation theorem in LCT domain given by Wei *et al.* **[36]** is compared with the proposed correlation theorem by simulation. The correlation operation of a rectangular function $x(t)$ of unit amplitude is performed with itself i.e. $(x \star x)(t)$. As a result of the correlation operation, a triangular function is obtained with double duration of that of the rectangular function as shown in Figure-5.1. Then the LCT of correlation operation defined by Wei *et al.* **[36]** is compared with the LCT of the proposed correlation operation for $(a,b,c,d)=(0.707,0.707,-0.707,0.707)$ and $(a,b,c,d)=(0.5,0.866,-0.866,0.5)$ as shown in Figures-5.2 and 5.3 respectively. Simultaneously, the LCT of the triangular function is also evaluated for the same values of (a,b,c,d) to make a comparison. It has been shown that the real (Re), imaginary (Im) and absolute (Abs) components of the LCT to the proposed correlation theorem resemble maximally to the different components of the LCT of triangular function for $(a,b,c,d)=(0.707,0.707,-0.707,0.707)$ and $(a,b,c,d)=(0.5,0.866,-0.866,0.5)$.

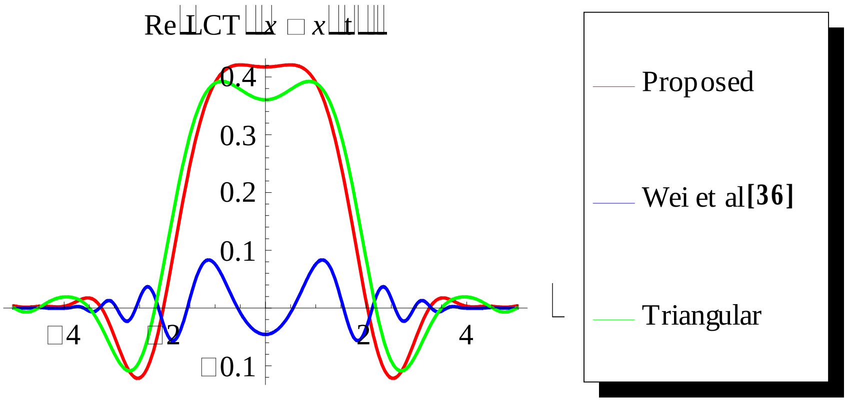


(a)



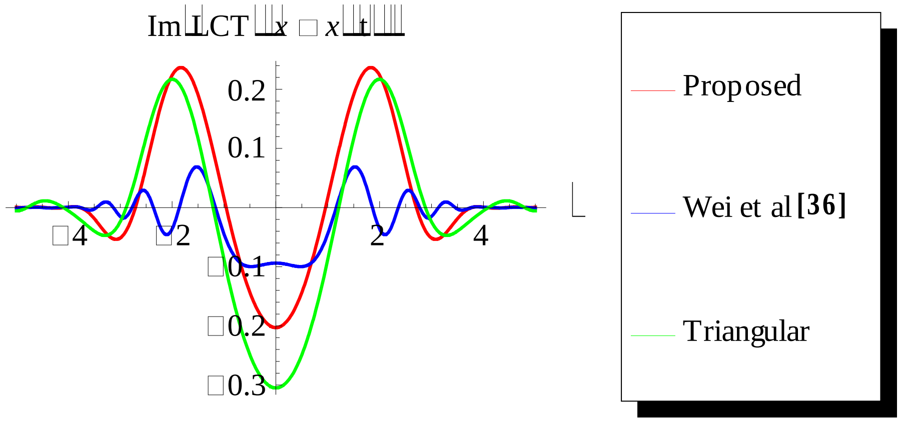
(b)

Figure-5.1: (a) Rectangular function $x(t)$ and (b) Correlated signal $(x \star x)(t)$ i.e. triangular function

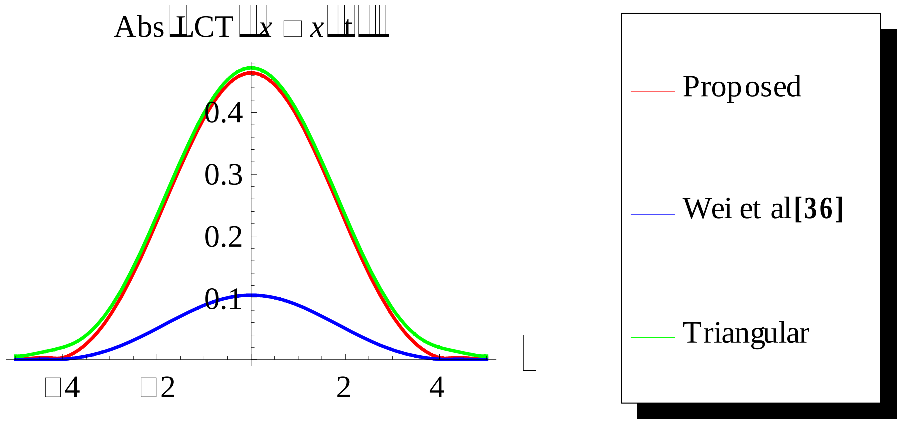


(a)

Figure-5.2: (a) Real value (b) Imaginary value (c) Absolute value: of LCT of $(x \star x)(t)$ for triangular function, Wei *et al.* [36] method, and proposed method for $(a,b,c,d) = (0.707, 0.707, -0.707, 0.707)$.

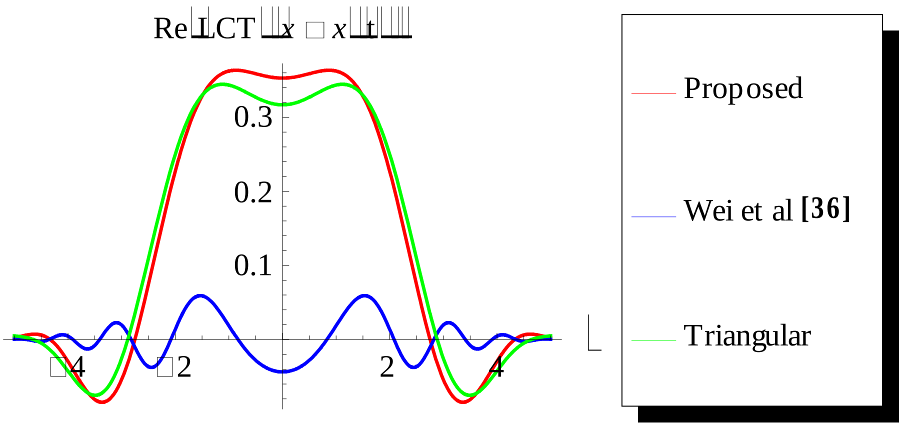


(b)



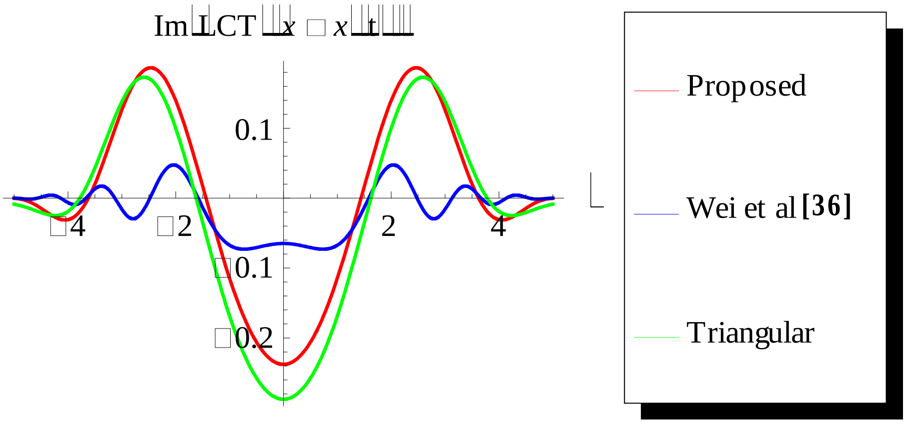
(c)

Figure-5.2 (*Continued*)

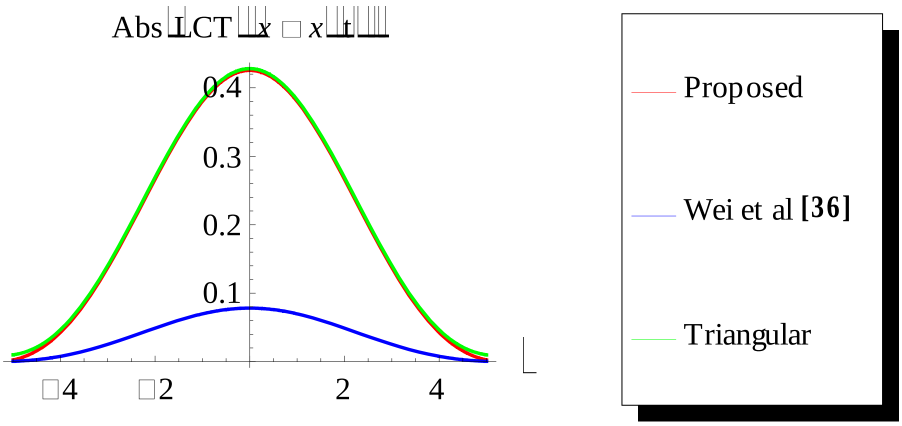


(a)

Figure-5.3: (a) Real value (b) Imaginary value (c) Absolute value: of LCT of $(x \star x)(t)$ for triangular function, Wei *et al.* [36] method, and proposed method for $(a,b,c,d) = (0.5, 0.866, -0.866, 0.5)$.



(b)



(c)

Figure-5.3 (*Continued*)

5.3 Power Spectral Density Analysis of FM Signal by Proposed Auto-Correlation Theorem

As an application, the power spectral density analysis of the FM signal has been done with the help of proposed correlation theorem. It has been shown that proposed auto-correlation theorem can be very useful in power spectrum analysis of FM signal. A frequency modulated signal having carrier frequency of 250 Hz, modulating signal frequency of 10 Hz and modulation index as 5 is chosen as test signal **[7]**. As shown in Figure-5.4, the power spectral density of FM signal is obtained for different combinations of LCT variables. With the help of simulation, it has been shown that as compare to FT as a special case of LCT when $(a,b,c,d)=(0,1,-1,0)$, the LCT of weighted auto-correlation function of FM wave shrinks when transform is taken at different value of LCT variables, which implies that the same signal can be transmitted with less BW requirement.

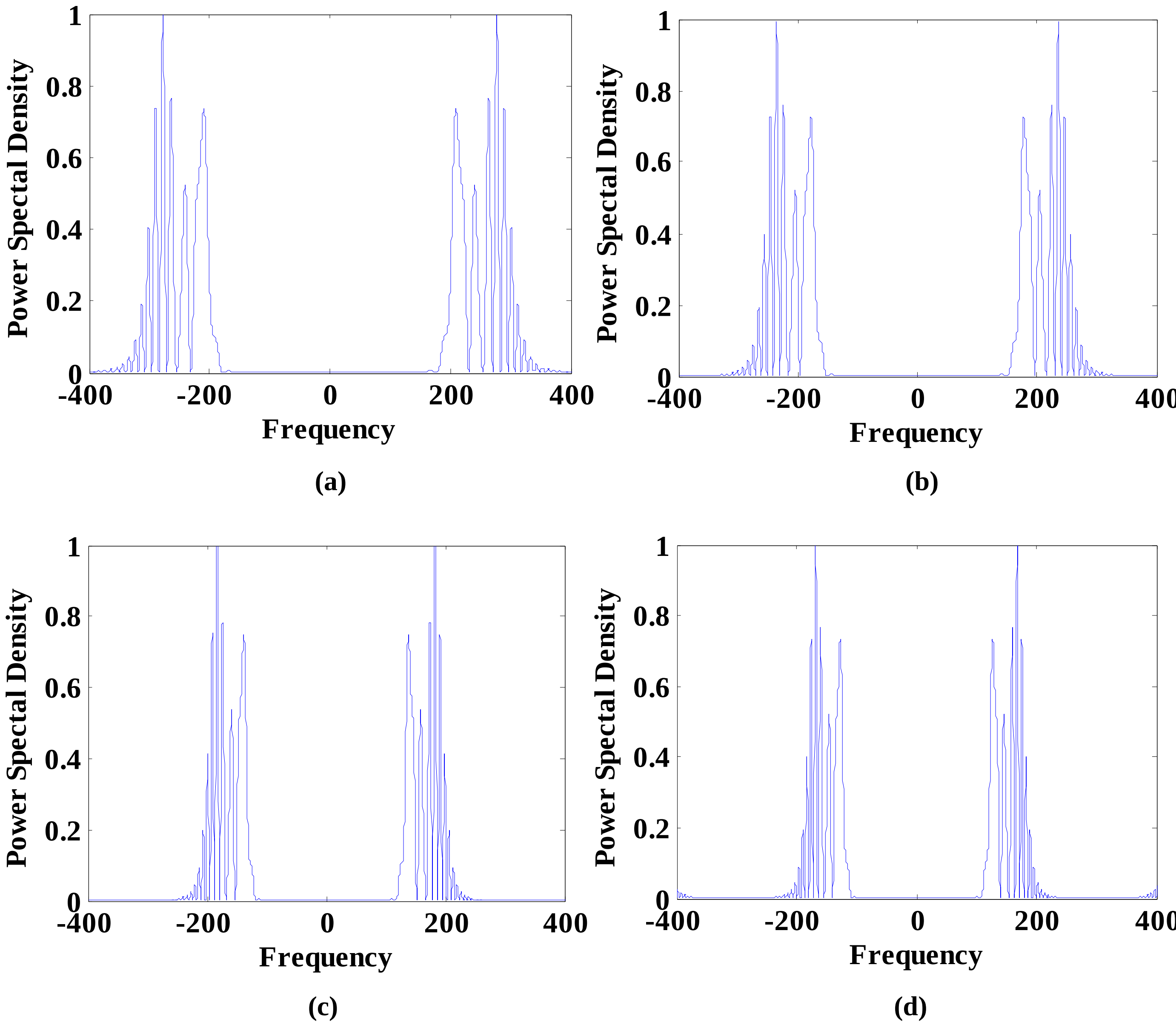


Figure-5.4: LCT of weighted auto-correlation function of FM signal at (a) $(a,b,c,d)=(0,1,-1,0)$ (b) $(a,b,c,d)=(0,0.85,-1.1764,0)$ (c) $(a,b,c,d)=(0,0.66,-1.5,0)$ (d) $(a,b,c,d)=(0,0.606,-1.65,0)$.

5.4 DISCUSSIONS

A definition for cross-correlation theorem and auto-correlation theorem is proposed in LCT domain, which clearly converts into classical definition of cross-correlation theorem and auto-correlation theorem for FT as a special case when value of LCT variables is

$(a,b,c,d)=(0,1,-1,0)$ This definition satisfies all the properties of cross-correlation and auto-correlation functions of classical case. Subsequently, a comparative study is also included to establish the superiority of the proposed definition of cross-correlation and auto-correlation theorems. In the context of computational complexity, the comparison has been established in terms of the number of chirp multiplications performed in realizing the different correlation theorems as shown in Table-5.3. Finally, comparing the proposed theorem with the literature, it has found that number of chirp multiplications are lesser in the case of proposed method.

It has been observed from the simulation results of Figures-5.2 and 5.3 that the proposed weighted correlation theorem gives better results than the correlation theorem given by Wei *et al.* **[36]**. The results determined by the proposed theorem are closer in shape and of matching values to the LCT of a triangular function. The results determined by the correlation expression of Wei *et al.* **[36]** have more oscillations in both the real and imaginary components, as it is visible from the Figures-5.2 and 5.3 for different value of (a,b,c,d) variables. These oscillations are significant and present due to the chirp signal included in the calculation of correlation integral by Wei *et al.* **[36]**. Therefore, proposed modified definition of correlation theorem is found to be a better proposition to other definitions given in the literature. This type of correlation is termed as weighted correlation. Finally by using the proposed correlation theorem, an application of power spectral density analysis of FM signal has been discussed and it has been found that the same FM signal can be transmitted with less BW.

CHAPTER 6

CONVOLUTION AND CORRELATION THEOREMS FOR THE OFFSET LCT

The OLCT **[15, 79, 136, 148]** is a six variable (a,b,c,d,m,n) class of linear integral transform. The OLCT allows shifting/translation, rotating and squeezing of a signal to fit within a fixed window as compared to only rotation in case of FRFT/LCT.

6.1 PROPOSED CONVOLUTION AND PRODUCT THEOREM FOR OLCT

The proposed definition of convolution theorem for OLCT is as follows:

6.1.1 Proposed Convolution Theorem

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$ and $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$, then $\sqrt{j2\pi b}\,F_{(a,b,c,d,m,n)}(p)G_{(a,b,c,d,m,n)}(p)e^{-\frac{j}{2b}\left(dp^2-2p(dm-bn)\right)}$ is the OLCT of $s(x)$ i.e

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)=\sqrt{j2\pi b}\,F_{(a,b,c,d,m,n)}(p)G_{(a,b,c,d,m,n)}(p)e^{-\frac{j}{2b}\left(dp^2-2p(dm-bn)\right)} \quad \textbf{(6.1.1)}$$

The outcome of this chapter has been communicated in the research journals as per following detail: N. Goel, K. Singh, "Convolution and correlation theorems for the offset fractional Fourier transform," Elsevier- Optik-International Journal for Light and Electron Optics.

N. Goel, K. Singh, "Convolution and correlation theorems for the offset linear canonical transform," Iranian Journal of Science and Technology.

where, $s(x)=\left(f\overset{A}{\otimes}g\right)(x)=\int_{-\infty}^{\infty}f(y)g(x-y)\tilde{y}(x,y)\mathrm{d}y$, is the weighted convolution operation and the weight function is $\tilde{y}(x,y)=e^{-j\frac{a}{b}y(x-y)}$ and the role of g and f can be interchanged. The operation $\overset{A}{\otimes}$ indicates the proposed convolution operation.

Proof. The one-dimensional OLCT of $s(x)$ with representation transformation in the context of quantum mechanics **[57, 61]** is given as-

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\langle x|s\rangle=\langle p|K|s\rangle=\langle p|S\rangle=S_{(a,b,c,d,m,n)}(p)$$

(6.1.2)

where, the notation $\langle x|s\rangle=s(x)$ and $\langle p|K|x\rangle=K_{(a,b,c,d,m,n)}(p,x)$ gives the representation of OLCT kernel in quantum mechanics. K is named as the OLCT operator and

$$\langle x|s\rangle=K_A\int_{-\infty}^{\infty}\mathrm{d}y\langle y|f\rangle\langle x-y|g\rangle\exp\left\{-j\frac{a}{b}y.(x-y)\right\}$$

where, $K_A=\exp\left[\frac{jd}{2b}m^2\right]$. Substituting the value of $\langle x|s\rangle$ in (6.1.2) results in-

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)=K_A\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\int_{-\infty}^{\infty}\mathrm{d}y\langle y|f\rangle\langle x-y|g\rangle.\exp\left\{-j\frac{a}{b}y.(x-y)\right\}$$

(6.1.3)

Rearranging (6.1.3), results in-

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)=K_A\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p|K|x\rangle\langle x-y|g\rangle\langle y|f\rangle.\exp\left\{-j\frac{a}{b}y.(x-y)\right\}$$

(6.1.4)

Substituting $x-y=x'$ i.e. $x=x'+y$ and $y=y$ in (6.1.4), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)=K_A\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p|K|x+y\rangle\langle x|g\rangle\langle y|f\rangle\exp\left\{-j\frac{a}{b}yx\right\} \quad \textbf{(6.1.5)}$$

Rewriting $\langle p|K|x+y\rangle$ explicity **[60],** results

$$\langle p|K|x+y\rangle\exp\left\{-j\frac{a}{b}yx\right\}=$$

$$K_A\exp\left\{\frac{ja(x+y)^2+jdp^2+2j(x+y)(m-p)-2jp(dm-bn)-2jayx}{2b}\right\} \quad \textbf{(6.1.6)}$$

Multiplying and dividing (6.1.6) by $\exp\left[\frac{jdp^2-2jp(dm-bn)}{2b}\right]$, results-

$$\Rightarrow K_A \exp\left\{\frac{ja(x+y)^2+jdp^2+2j(x+y)(m-p)-2jp(dm-bn)-2jayx}{2b}\right\}. \exp\left\{\frac{jdp^2-2jp(dm-bn)}{2b}\right\}\exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\} \tag{6.1.7}$$

Rearranging (6.1.7), results in-

$$\Rightarrow K_A\left[\exp\left\{\frac{jax^2+jdp^2+2jx(m-p)-2jp(dm-bn)}{2b}\right\}. \exp\left\{\frac{jay^2+2jy(m-p)+jdp^2-2jp(dm-bn)}{2b}\right\}.\exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\}\right] \tag{6.1.8}$$

Substituting the value of (6.1.8) in (6.1.5), results in-

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)=K_A^2\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle x\,|\,g\rangle\langle y\,|\,f\rangle \left[\exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\}\exp\left\{\frac{jax^2+jdp^2+2jx(m-p)-2jp(dm-bn)}{2b}\right\}. \exp\left\{\frac{jay^2+2jy(m-p)+jdp^2-2jp(dm-bn)}{2b}\right\}\right] \tag{6.1.9}$$

Rewriting (6.1.9) with the aid of quantum mechanics, results in-

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)= K_A^2\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x\rangle\langle x\,|\,g\rangle\langle p\,|\,K\,|\,y\rangle\langle y\,|\,f\rangle.\exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\} \tag{6.1.10}$$

Rearranging (6.1.10), results in-

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)=K_A\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,g\rangle.K_A\int_{-\infty}^{\infty}\mathrm{d}y\langle p\,|\,K\,|\,y\rangle\langle y\,|\,f\rangle .\exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\} \tag{6.1.11}$$

Multiplying and dividing the (6.1.11) by $\sqrt{j2\pi b}$ and rearranging, results in-

$$L_F^{(a,b,c,d,m,n)}\left[s(x)\right](p)=\sqrt{j2\pi b}\,F_{(a,b,c,d,m,n)}(p)G_{(a,b,c,d,m,n)}(p).e^{-\frac{j}{2b}\left(dp^2-2p(dm-bn)\right)} \tag{6.1.12}$$

This is just a new approach for convolution theorem under OLCT, derived by representation transformation in quantum mechanics. Therefore, (6.1.1) is obtained and the theorem is proved.

The reciprocal transform of (6.1.12) can be obtained by writing the definition of inverse OLCT and is given by

$$L_F^{(a,b,c,d,m,n)^{-1}}\left[H(p)\right](x)=\sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}p\langle p\,|\,K\,|\,x\rangle^{*}.H_{(a,b,c,d,m,n)}(p) \tag{6.1.13}$$

where, * indicates the complex conjugate. By using the theory of representation in quantum mechanics **[61]**, the (6.1.13) results-

$$\begin{aligned}L_F^{(a,b,c,d,m,n)^{-1}}\left[S(p)\right](x)&=\sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}p\,\langle x\,|\,K^{\dagger}\,|\,p\rangle.\langle p\,|\,S\rangle\\&=\sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}p\,\langle x\,|\,K^{\dagger}K\,|\,s\rangle=\langle x\,|\,s\rangle=s(x)\end{aligned} \tag{6.1.14}$$

6.1.1.1 Special cases of OLCT for proposed convolution theorem

LCT as a special case of OLCT, when $(a,b,c,d,m,n)=(a,b,c,d,0,0)$, (6.1.12) becomes

$$L_F^{(a,b,c,d,0,0)}\left[s(x)\right](p)=\sqrt{j2\pi b}\,F_{(a,b,c,d,0,0)}(p)G_{(a,b,c,d,0,0)}(p) \tag{6.1.15}$$

FT as a special case of OLCT, when $(a,b,c,d,m,n)=(0,1,-1,0,0,0)$, (6.1.12) becomes

$$L_F^{(0,1,-1,0,0,0)}\left[s(x)\right](p)=\sqrt{j2\pi}\,F_{(0,1,-1,0,0,0)}(p)G_{(0,1,-1,0,0,0)}(p) \tag{6.1.16}$$

Similarly, FRFT as a special case of OLCT, when $(a,b,c,d,m,n)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,$ $0,0)$, (6.1.12) becomes-

$$\begin{aligned}&L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)}\left[s(x)\right](p)=\\&\sqrt{j2\pi\sin\alpha}.\exp\left(-\frac{j}{2}p^{2}\cot\alpha\right)F_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)}(p)G_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)}(p)\end{aligned} \tag{6.1.17}$$

where, $L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)}\left[s(x)\right](p)$ indicates the FRFT of $s(x)$ and FT as a special case of FRFT, when $\alpha=\pi/2$, (6.1.17) becomes-

$$L_F^{(0,1,-1,0,0,0)}\left[s(x)\right](p)=\sqrt{j2\pi}\,F_{(0,1,-1,0,0,0)}(p)G_{(0,1,-1,0,0,0)}(p) \tag{6.1.18}$$

Equations (6.1.15-6.1.18) are some of the special cases of proposed convolution theorem for OLCT.

6.1.1.2 Properties satisfied by proposed convolution theorem for OLCT

The following are some of the properties that are satisfied by the proposed convolution theorem.

a) *Commutative property*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$ and $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$, then as per the commutative property-

$$\left(f\overset{A}{\otimes}g\right)(x)\xleftrightarrow{OLCT} \sqrt{j2\pi b}\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}F_{(a,b,c,d,m,n)}(p)G_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.1.19)}$$

and

$$\left(g\overset{A}{\otimes}f\right)(x)\xleftrightarrow{OLCT} \sqrt{j2\pi b}\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}G_{(a,b,c,d,m,n)}(p)F_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.1.20)}$$

i.e. $\left(f\overset{A}{\otimes}g\right)(x)=\left(g\overset{A}{\otimes}f\right)(x)$ **(6.1.21)**

Proof: Considering the L.H.S. of (6.1.20)

$$q(x)=\left(g\overset{A}{\otimes}f\right)(x)=\int_{-\infty}^{\infty}g(y)f(x-y)\exp\left\{-j\frac{a}{b}y.(x-y)\right\}\mathrm{d}y \quad \textbf{(6.1.22)}$$

Using the quantum mechanical notation **[57, 61]**, the one-dimensional LCT of the signal $q(x)$ reads as-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\langle x|q\rangle=\langle p|K|q\rangle=\langle p|Q\rangle=Q_{(a,b,c,d,m,n)}(p)$$

(6.1.23)

where, the notation $\langle x|q\rangle=q(x)$ and $\langle p|K|x\rangle=K_{(a,b,c,d,m,n)}(p,x)$ gives the representation of OLCT kernel in quantum mechanics. K is named as the OLCT operator and

$$\langle x|q\rangle=\int_{-\infty}^{\infty}\mathrm{d}y\langle y|g\rangle\langle x-y|f\rangle\exp\left\{-j\frac{a}{b}y.(x-y)\right\}$$

Substituting the value of $\langle x|q\rangle$ in (6.1.23), results in-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\left\langle p\,|\,K\,|\,x\right\rangle\int_{-\infty}^{\infty}\mathrm{d}y\left\langle y\,|\,g\right\rangle\left\langle x-y\,|\,f\right\rangle.\exp\left\{-j\frac{a}{b}y.(x-y)\right\}$$

(6.1.24)

Rearranging (6.1.24), results in-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x\right\rangle\left\langle x-y\,|\,f\right\rangle\left\langle y\,|\,g\right\rangle.\exp\left\{-j\frac{a}{b}y.(x-y)\right\}$$

(6.1.25)

Substituting $x-y=x'$ i.e. $x=x'+y$ and $y=y$ in (6.1.4), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x+y\right\rangle\left\langle x\,|\,f\right\rangle\left\langle y\,|\,g\right\rangle\exp\left\{-j\frac{a}{b}yx\right\} \quad \textbf{(6.1.26)}$$

Rewriting $\left\langle p\,|\,K\,|\,x+y\right\rangle$ explicitly **[60]**, results in-

$$\left\langle p\,|\,K\,|\,x+y\right\rangle\exp\left\{-j\frac{a}{b}yx\right\}= K_A\exp j\left\{\frac{a(x+y)^2+dp^2+2(m-p)(x+y)-2ayx-2p(dm-bn)}{2b}\right\} \quad \textbf{(6.1.27)}$$

Multiplying and dividing (6.1.27) by $K_A\exp j\left[\frac{dp^2-2p(dm-bn)}{2b}\right]$ results-

$$\left\langle p\,|\,K\,|\,x+y\right\rangle\exp\left\{-j\frac{a}{b}yx\right\}=\frac{K_A^2}{K_A}\exp\frac{j}{2b}\left\{a(x+y)^2+dp^2+2(m-p)(x+y)\right.$$
$$\left.-2ayx-2p(dm-bn)+dp^2-dp^2+2p(dm-bn)-2p(dm-bn)\right\} \quad \textbf{(6.1.28)}$$

Rearranging (6.1.28), results in-

$$\left\langle p\,|\,K\,|\,x+y\right\rangle\exp\left\{-j\frac{a}{b}yx\right\}=\frac{K_A^2}{K_A}\exp j\left\{\frac{ax^2+dp^2+2(m-p)x-2p(dm-bn)}{2b}\right\}.$$
$$\exp j\left\{\frac{ay^2+dp^2+2(m-p)y-2p(dm-bn)}{2b}\right\}.\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\} \quad \textbf{(6.1.29)}$$

Substituting the value of (6.1.29) in (6.1.26), results in-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=\frac{1}{K_A}\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle x\,|\,f\rangle\langle y\,|\,g\rangle.\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\}$$
$$K_A\exp j\left\{\frac{ax^2+dp^2+2(m-p)x-2p(dm-bn)}{2b}\right\}$$
$$K_A\exp j\left\{\frac{ay^2+dp^2+2(m-p)y-2p(dm-bn)}{2b}\right\}$$

(6.1.30)

Rewriting (6.1.30) with the aid of quantum mechanical representation, results-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=\frac{1}{K_A}\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x\rangle\langle x\,|\,f\rangle\langle p\,|\,K\,|\,y\rangle\langle y\,|\,g\rangle.$$
$$\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\} \qquad \textbf{(6.1.31)}$$

Rearranging (6.1.31), results in-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,f\rangle\int_{-\infty}^{\infty}\mathrm{d}y\langle p\,|\,K\,|\,y\rangle\langle y\,|\,g\rangle.$$
$$\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} \qquad \textbf{(6.1.32)}$$

Multiplying and dividing (6.1.32) by $\sqrt{j2\pi b}$ and rearranging, results-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=$$
$$\sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}.G_{(a,b,c,d,m,n)}(p)F_{(a,b,c,d,m,n)}(p) \qquad \textbf{(6.1.33)}$$

Therefore, from (6.1.20) and (6.1.33), it concludes that-

$$(f\otimes g)(x)=(g\otimes f)(x)$$

This proves that the proposed convolution theorem for OLCT satisfies the commutative law.

b) ***Associative property*:**

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$ and $M_{(a,b,c,d,m,n)}(p)$ is the OLCT of $m(x)$ then as per the associative property-

$$\left(\left(f \overset{A}{\otimes} g\right) \overset{A}{\otimes} m\right)(x) \xleftrightarrow{OLCT} (j2\pi b) \exp j\left[\frac{-dp^2 + 2p(dm - bn) - dm^2}{b}\right] F_{(a,b,c,d,m,n)}(p) G_{(a,b,c,d,m,n)}(p) M_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.1.34)}$$

and

$$\left(f \overset{A}{\otimes}\left(g \overset{A}{\otimes} m\right)\right)(x) \xleftrightarrow{OLCT} (j2\pi b) \exp j\left[\frac{-dp^2 + 2p(dm - bn) - dm^2}{b}\right] F_{(a,b,c,d,m,n)}(p) G_{(a,b,c,d,m,n)}(p) M_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.1.35)}$$

i.e. $$\left(\left(f \overset{A}{\otimes} g\right) \overset{A}{\otimes} m\right)(x) = \left(f \overset{A}{\otimes}\left(g \overset{A}{\otimes} m\right)\right)(x) \quad \textbf{(6.1.36)}$$

Proof: Considering the L.H.S. of (6.1.35)

$$q_1(x) = \left(\left(f \overset{A}{\otimes} g\right) \overset{A}{\otimes} m\right)(x) = \left(s \overset{A}{\otimes} m\right)(x) \quad \textbf{(6.1.37)}$$

where, $$s(x) = \langle x | s\rangle = \left(f \overset{A}{\otimes} g\right)(x) = \int_{-\infty}^{\infty} \mathrm{d}y \langle y | f\rangle \langle x - y | g\rangle \exp\left\{-j\frac{a}{b} y.(x-y)\right\} \quad \textbf{(6.1.38)}$$

and

$$q_1(x) = \left(\left(f \overset{A}{\otimes} g\right) \overset{A}{\otimes} m\right)(x) = \left(s \overset{A}{\otimes} m\right)(x) = \int_{-\infty}^{\infty} \mathrm{d}\beta \langle \beta | h\rangle \langle x - \beta | m\rangle \exp\left\{-j\frac{a}{b}\beta.(x-\beta)\right\} \quad \textbf{(6.1.39)}$$

Substituting the value of $\langle x | s\rangle$ from (6.1.38) to (6.1.39) results in-

$$q_1(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \langle y | f\rangle \langle \beta - y | g\rangle \exp\left\{-j\frac{a}{b} y.(\beta - y)\right\}.\langle x - \beta | m\rangle \exp\left\{-j\frac{a}{b}\beta.(x-\beta)\right\} \quad \textbf{(6.1.40)}$$

Taking OLCT of $q_1(x)$ results in

$$Q_{1(a,b,c,d,m,n)}(p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \mathrm{d}x \langle p | K | x\rangle \langle x | q_1\rangle \quad \textbf{(6.1.41)}$$

Substituting the value of $q_1(x)$ in (6.1.41), results in-

$$Q_{1(a,b,c,d,m,n)}(p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \mathrm{d}x \langle p | K | x\rangle \langle y | f\rangle \langle \beta - y | g\rangle \langle x - \beta | m\rangle. \exp\left\{-j\frac{a}{b}\beta.(x-\beta)\right\}.\exp\left\{-j\frac{a}{b} y.(\beta - y)\right\} \quad \textbf{(6.1.42)}$$

Substituting $x-\beta=x'$ i.e. $x=x'+\beta$ and $y=y$ in (6.1.42), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x , results-

$$Q_{1(a,b,c,d,m,n)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\,\mathrm{d}y\,\mathrm{d}x\langle p\,|\,K\,|\,x+\beta\rangle\langle y\,|\,f\rangle\langle\beta-y\,|\,g\rangle\langle x\,|\,m\rangle.$$
$$\exp\left\{-j\frac{a}{b}\beta x\right\}.\exp\left\{-j\frac{a}{b}y.(\beta-y)\right\} \tag{6.1.43}$$

Rewriting $\langle p\,|\,K\,|\,x+\beta\rangle$ explicitly, results in-

$$\langle p\,|\,K\,|\,x+\beta\rangle\exp\left\{-j\frac{a}{b}\beta x\right\}=$$
$$K_A\exp j\left\{\frac{a(x+\beta)^2+dp^2+2(m-p)(x+\beta)-2a\beta x-2p(dm-bn)}{2b}\right\} \tag{6.1.44}$$

Rearranging the (6.1.44)-

$$\langle p\,|\,K\,|\,x+\beta\rangle\exp\left\{-j\frac{a}{b}\beta x\right\}=$$
$$K_A\exp j\left\{\frac{ax^2+a\beta^2+dp^2+2x(m-p)+2\beta(m-p)-2p(dm-bn)}{2b}\right\} \tag{6.1.45}$$

Substituting the value of (6.1.45) in (6.1.43) results-

$$Q_{1(a,b,c,d,m,n)}(p)=M_{(a,b,c,d,m,n)}(p)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\,\mathrm{d}y\langle y\,|\,f\rangle\langle\beta-y\,|\,g\rangle.$$
$$\exp j\left\{\frac{a\beta^2+2\beta(m-p)}{2b}\right\}.\exp\left\{-j\frac{a}{b}y.(\beta-y)\right\} \tag{6.1.46}$$

Multiplying and dividing (6.1.46) by $\exp j\left[\frac{dp^2-2p(dm-bn)}{2b}\right]$ results-

$$Q_{1(a,b,c,d,m,n)}(p)=\frac{1}{K_A}M_{(a,b,c,d,m,n)}(p)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\beta\,\mathrm{d}y\langle y\,|\,f\rangle\langle\beta-y\,|\,g\rangle.\langle p\,|\,K\,|\,\beta\rangle$$
$$\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\}.\exp\left\{-j\frac{a}{b}y.(\beta-y)\right\} \tag{6.1.47}$$

Substituting $\beta-y=\beta'$ i.e. $\beta=\beta'+y$ and $y=y$ in (6.1.47), then $\mathrm{d}\beta\mathrm{d}y=\mathrm{d}\beta'\mathrm{d}y$ **[70]** and then replacing β' by β, results-

$$Q_{1(a,b,c,d,m,n)}(p) = \frac{1}{K_A} M_{(a,b,c,d,m,n)}(p) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \langle y \mid f \rangle \langle \beta \mid g \rangle . \langle p \mid K \mid \beta + y \rangle$$
$$\exp j\left\{\frac{-dp^2 + 2p(dm - bn)}{2b}\right\}.\exp\left\{-j\frac{a}{b}y\beta\right\} \quad \textbf{(6.1.48)}$$

Rewriting $\langle p \mid K \mid \beta + y \rangle$ explicitly **[60]**, results in-

$$\langle p \mid K \mid \beta + y \rangle \exp\left\{-j\frac{a}{b}\beta y\right\} =$$
$$K_A \exp j\left\{\frac{a(\beta + y)^2 + dp^2 + 2(m - p)(\beta + y) - 2ay\beta - 2p(dm - bn)}{2b}\right\} \quad \textbf{(6.1.49)}$$

Rearranging (6.1.49)

$$\langle p \mid K \mid \beta + y \rangle \exp\left\{-j\frac{a}{b}\beta y\right\} =$$
$$K_A \exp j\left\{\frac{a\beta^2 + ay^2 + dp^2 + 2\beta(m - p) + 2y(m - p) - 2p(dm - bn)}{2b}\right\} \quad \textbf{(6.1.50)}$$

Substituting the value of (6.1.50) in (6.1.48) results-

$$Q_{1(a,b,c,d,m,n)}(p) = \frac{1}{K_A}\sqrt{j2\pi b}\, G_{(a,b,c,d,m,n)}(p) M_{(a,b,c,d,m,n)}(p) \int_{-\infty}^{\infty} \mathrm{d}y \langle y \mid f \rangle .$$
$$\exp j\left\{\frac{ay^2 + 2y(m - p)}{2b}\right\}.\exp j\left\{\frac{-dp^2 + 2p(dm - bn)}{2b}\right\} \quad \textbf{(6.1.51)}$$

Multiplying and dividing (6.58) by $\exp j\left[\frac{dp^2 - 2p(dm - bn)}{2b}\right].\sqrt{\frac{1}{j2\pi b}}$ results-

$$Q_{1(a,b,c,d,m,n)}(p) = (j2\pi b)$$
$$\exp j\left[\frac{-dp^2 + 2p(dm - bn) - dm^2}{b}\right] F_{(a,b,c,d,m,n)}(p) G_{(a,b,c,d,m,n)}(p) M_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.1.52)}$$

Similarly it can proved that the LCT of $q_2(x) = (f \otimes (g \otimes m))$ is

$$Q_{2(a,b,c,d,m,n)}(p) = (j2\pi b)$$
$$\exp j\left[\frac{-dp^2 + 2p(dm - bn) - dm^2}{b}\right] F_{(a,b,c,d,m,n)}(p) G_{(a,b,c,d,m,n)}(p) M_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.1.53)}$$

Therefore, from (6.1.52) and (6.1.53), it has been proved that-

$$\left(\left(f \overset{A}{\otimes} g\right) \overset{A}{\otimes} m\right)(x) = \left(f \overset{A}{\otimes}\left(g \overset{A}{\otimes} m\right)\right)(x) \quad \textbf{(6.1.54)}$$

This proves that the proposed convolution theorem for OLCT satisfies the associative law.

c) *Distributive property*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$ and $M_{(a,b,c,d,m,n)}(p)$ is the OLCT of $m(x)$ then as per the distributive property-

$$\left(f\overset{A}{\otimes}(g+m)\right)(x)\xleftrightarrow{OLCT}(j2\pi b)\ exp\ j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} F_{(a,b,c,d,m,n)}(p)\left\{G_{(a,b,c,d,m,n)}(p)+M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.1.55)}$$

and

$$\left(f\overset{A}{\otimes}g+f\overset{A}{\otimes}m\right)(x)\xleftrightarrow{OLCT}(j2\pi b)\ \exp\ j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} \left\{F_{(a,b,c,d,m,n)}(p)G_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(p)M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.1.56)}$$

i.e. $$\left(f\overset{A}{\otimes}(g+m)\right)(x)=\left(f\overset{A}{\otimes}g+f\overset{A}{\otimes}m\right)(x) \quad \textbf{(6.1.57)}$$

Proof: Considering the L.H.S. of (6.1.56)

$$q_3(x)=\left(f\overset{A}{\otimes}g+f\overset{A}{\otimes}m\right)(x)=\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x-y\,|\,g\rangle\exp\left\{-j\frac{a}{b}y.(x-y)\right\} +\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x-y\,|\,m\rangle\exp\left\{-j\frac{a}{b}y.(x-y)\right\} \quad \textbf{(6.1.58)}$$

Taking OLCT of (6.1.58)

$$Q_{3(a,b,c,d,m,n)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,q_3\rangle \quad \textbf{(6.1.59)}$$

Substituting the value of $q_3(x)$ in (6.1.59) results in-

$$Q_{3(a,b,c,d,m,n)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\left[\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x-y\,|\,g\rangle \exp\left\{-j\frac{a}{b}y.(x-y)\right\}+\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x-y\,|\,m\rangle\exp\left\{-j\frac{a}{b}y.(x-y)\right\}\right] \quad \textbf{(6.1.60)}$$

Substituting $x-y=x'$ i.e. $x=x'+y$ and $y=y$ in (6.1.60), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x , results-

$$Q_{3(a,b,c,d,m,n)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x+y\rangle\left[\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x\,|\,g\rangle\exp\left\{-j\frac{a}{b}yx\right\}+\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x\,|\,m\rangle\exp\left\{-j\frac{a}{b}yx\right\}\right] \qquad \textbf{(6.1.61)}$$

Rearranging (6.1.61) results in-

$$Q_{3(a,b,c,d,m,n)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x+y\rangle\langle y\,|\,f\rangle\langle x\,|\,g\rangle\exp\left\{-j\frac{a}{b}yx\right\}+\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x+y\rangle\langle y\,|\,f\rangle\langle x\,|\,m\rangle\exp\left\{-j\frac{a}{b}yx\right\} \qquad \textbf{(6.1.62)}$$

Rewriting $\langle p\,|\,K\,|\,x+y\rangle$ explicitly **[60]**, results in-

$$\langle p\,|\,K\,|\,x+y\rangle\exp\left\{-j\frac{a}{b}xy\right\}=K_A\exp j\left\{\frac{a(x+y)^2+dp^2+2(m-p)(x+y)-2ayx-2p(dm-bn)}{2b}\right\} \qquad \textbf{(6.1.63)}$$

Rearranging (6.1.63) results

$$\langle p\,|\,K\,|\,x+y\rangle\exp\left\{-j\frac{a}{b}xy\right\}=K_A\exp j\left\{\frac{ax^2+ay^2+dp^2+2x(m-p)+2y(m-p)-2p(dm-bn)}{2b}\right\} \qquad \textbf{(6.1.64)}$$

Substituting the value of (6.1.64) in (6.1.62) results

$$Q_{3(a,b,c,d,m,n)}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\;\langle y\,|\,f\rangle\langle x\,|\,g\rangle.\;K_A\exp j\left\{\frac{ax^2+ay^2+dp^2+2x(m-p)+2y(m-p)-2p(dm-bn)}{2b}\right\}+\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\;\langle y\,|\,f\rangle\langle x\,|\,m\rangle.\;K_A\exp j\left\{\frac{ax^2+ay^2+dp^2+2x(m-p)+2y(m-p)-2p(dm-bn)}{2b}\right\} \qquad \textbf{(6.1.65)}$$

Multiplying and dividing (6.1.65) by $K_A\exp j\left[\frac{dp^2-2p(dm-bn)}{2b}\right].\sqrt{\frac{1}{j2\pi b}}$ results-

$$Q_{3(a,b,c,d,m,n)}(p)=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\,\langle y\mid f\rangle\langle x\mid g\rangle.\frac{1}{K_A}\exp j\left[\frac{-dp^2+2p(dm-bn)}{2b}\right]$$
$$K_A^2\exp j\left\{\frac{ax^2+ay^2+2dp^2+2x(m-p)+2y(m-p)-4p(dm-bn)}{2b}\right\}$$
$$+\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\,\langle y\mid f\rangle\langle x\mid m\rangle.\frac{1}{K_A}\exp j\left[\frac{-dp^2+2p(dm-bn)}{2b}\right] \quad \textbf{(6.1.66)}$$
$$K_A^2\exp j\left\{\frac{ax^2+ay^2+2dp^2+2x(m-p)+2y(m-p)-4p(dm-bn)}{2b}\right\}$$

Rearranging (6.1.66) results-

$$Q_{3(a,b,c,d,m,n)}(p)=(j2\pi b)\exp j\left[\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right] \quad \textbf{(6.1.67)}$$
$$\left\{F_{(a,b,c,d,m,n)}(p)G_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(p)M_{(a,b,c,d,m,n)}(p)\right\}$$

Utilizing the linearity property of OLCT, (6.1.67) can be written as-

$$Q_{3(a,b,c,d,m,n)}(p)=(j2\pi b)\exp j\left[\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right] \quad \textbf{(6.1.68)}$$
$$F_{(a,b,c,d,m,n)}(p)\left\{G_{(a,b,c,d,m,n)}(p)+M_{(a,b,c,d,m,n)}(p)\right\}$$

Similarly it can proved that the OLCT of $q_4(x)=(f\otimes(g+m))(x)$ is

$$Q_{4(a,b,c,d,m,n)}(p)=(j2\pi b)\exp j\left[\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right] \quad \textbf{(6.1.69)}$$
$$F_{(a,b,c,d,m,n)}(p)\left\{G_{(a,b,c,d,m,n)}(p)+M_{(a,b,c,d,m,n)}(p)\right\}$$

Therefore, from (6.1.68) and (6.1.69), it has been proved that-

$$\left(f\overset{A}{\otimes}(g+m)\right)(x)=\left(f\overset{A}{\otimes}g+f\overset{A}{\otimes}m\right)(x) \quad \textbf{(6.1.70)}$$

This proves that the proposed convolution theorem for OLCT satisfies the distributive law.

6.1.1.3 Comparative analysis of the proposed convolution for OLCT theorem with existing theorems

The convolution theorem in OLCT domain given by Xiang *et al.* **[115]** is compared with the proposed convolution theorem by using simulation. The convolution operation of a rectangular function $x(t)$ of unit amplitude is performed with it and as a result of convolution operation, triangular function is obtained of double duration from the rectangular window function as shown in Figure-6.1. Then the OLCT of the triangular function is evaluated for different values of (a,b,c,d,m,n) with and without considering the effect of time-shifting and frequency modulation as shown in Figure-6.2 and 6.3 respectively. Simultaneously, the OLCT of the triangular function is also evaluated for the same values of (a,b,c,d,m,n) to make a comparison. It has been shown in Figure-6.2 and 6.3 that how time-shifting and frequency-modulation variables help to approach the real (Re), imaginary (Im) and absolute (Abs) values of the OLCT of triangular function. Also the convolution operation defined by Xiang *et al.* **[115]** is compared with the OLCT of the proposed convolution operation for $(a,b,c,d,m,n)=(0.707,0.707,-0.707,0.707,0.12,0.14)$ and $(a,b,c,d,m,n)=(0.707,0.707,-0.707,0.707,-0.19,-0.19)$ and it has been shown that the real, imaginary and absolute components to the OLCT of the proposed convolution theorem resembles maximally to the different components of the OLCT of triangular function.

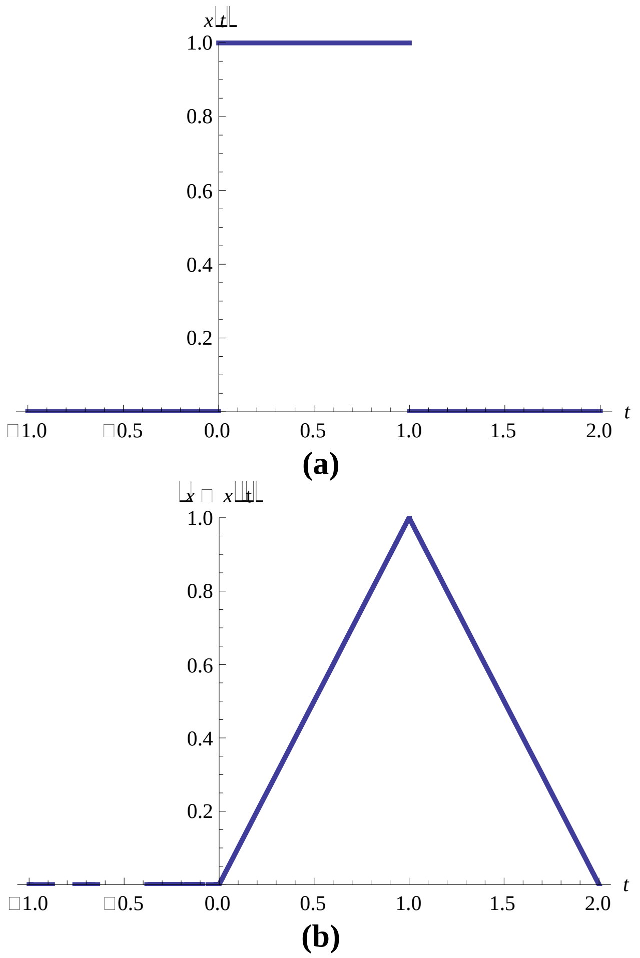
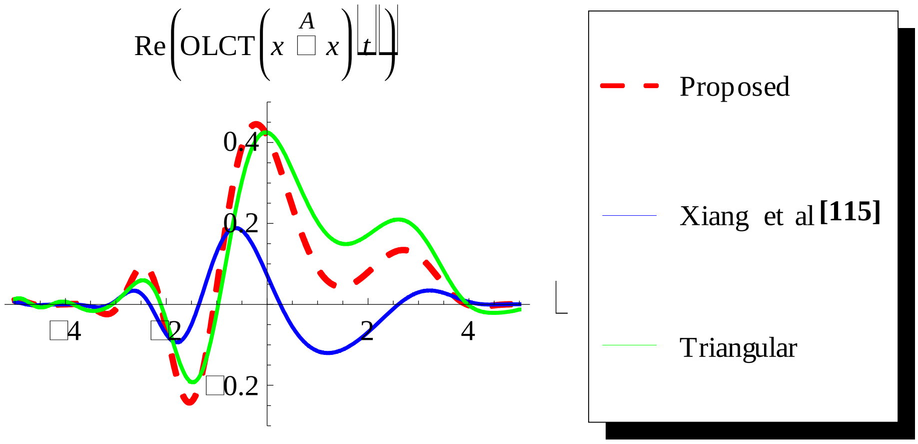
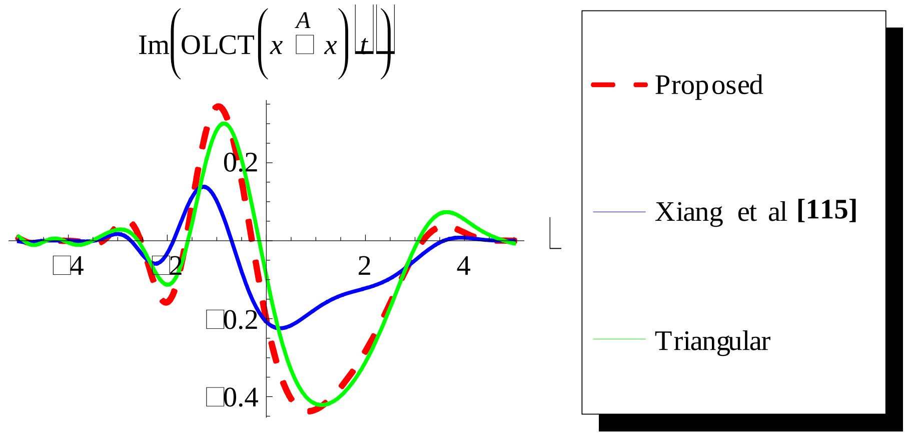


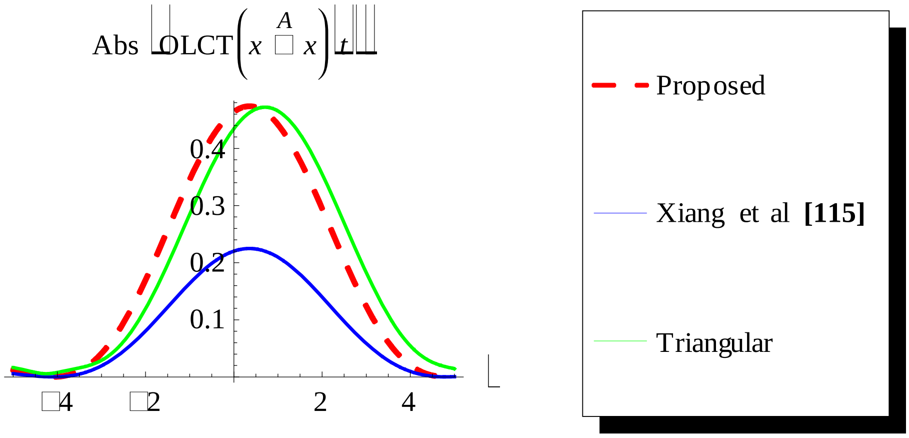
Figure-6.1: (a) Rectangular function $x(t)$ and (b) Convolved signal $(x\otimes x)(t)$ i.e. triangular function.



(a)

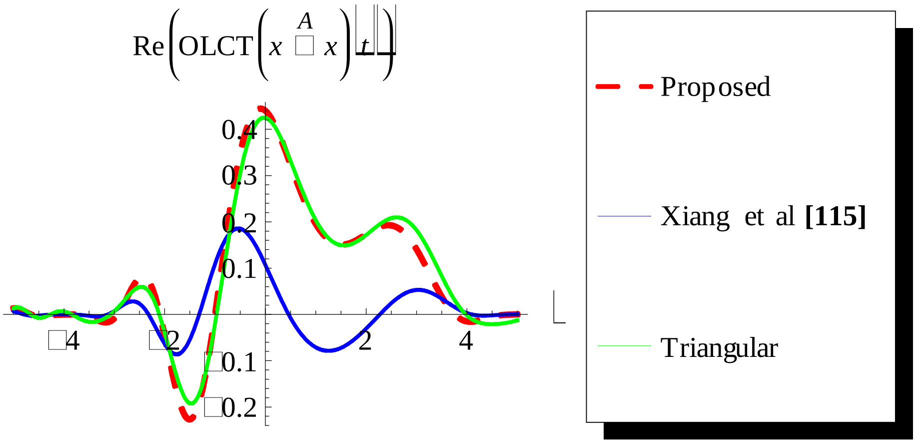


(b)

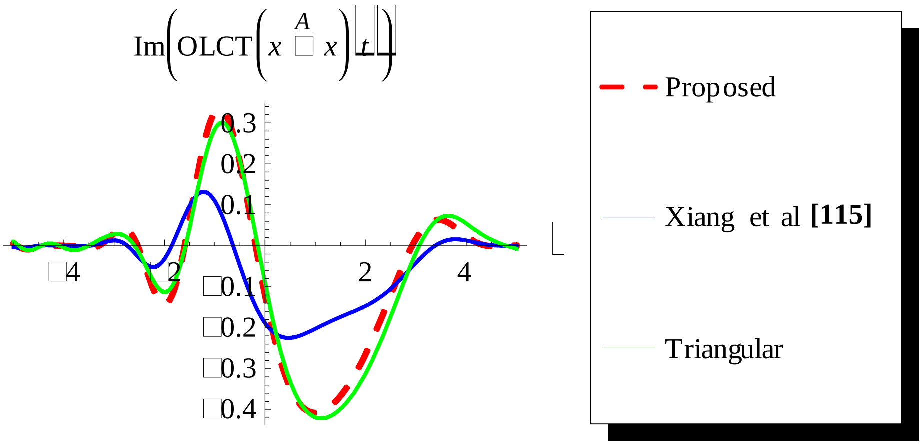


(c)

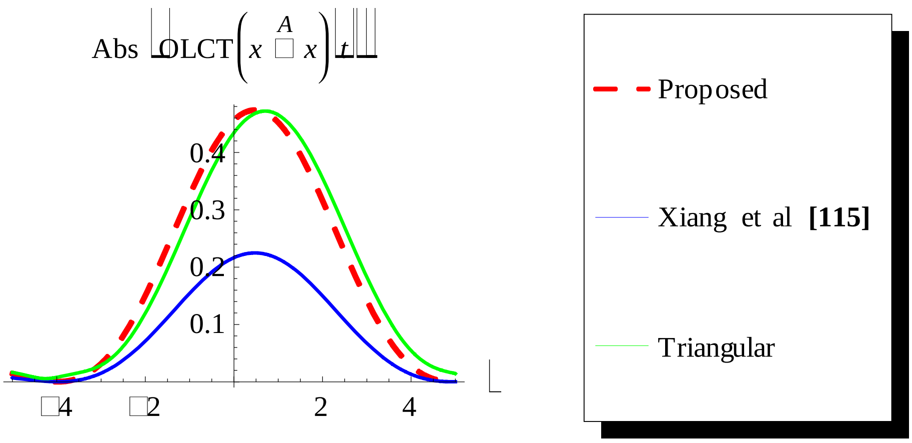
Figure-6.2: (a) Real value (b) Imaginary value (c) Absolute value: of OLCT of $\left(x \overset{A}{\otimes} x\right)(t)$ for triangular function, Xiang *et al.* [115] method, and proposed method for $(a,b,c,d,m,n) = (0.707, 0.707, -0.707, 0.707, 0, 0)$.



(a)



(b)



(c)

Figure-6.3: Effect of time-shifting and frequency-modulation variables on (a) Real value (b) Imaginary value and (c) Absolute value: of OLCT of $\left(x \overset{A}{\otimes} x\right)(t)$ by using proposed methods for $(a,b,c,d,m,n) = (0.707, 0.707, -0.707, 0.707, 0.12, 0.14)$.

6.1.2 Proposed Product Theorem

For any two functions $f(x)$ and $g(x)$, the definition of product theorem for OLCT is given by

$$q(x) = g(x).f(x).\exp\left\{ j\frac{a}{2b}x^2 \right\} \qquad \text{and}$$

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p) = G_{(a,b,c,d,m,n)}(p)\overset{A}{\otimes}\left(F_{(a,b,c,d,m,n)}(p).\exp\left[j\frac{d}{b}vp \right]\right).\exp\left[-j\frac{x}{b}m \right] \quad \textbf{(6.1.71)}$$

Proof. The one-dimensional LCT of $q(x)$ for proposed identities of product theorem is given by-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p) = \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}x \langle x\,|\,g\rangle\langle x\,|\,f\rangle\langle p\,|\,K\,|\,x\rangle.\exp\left\{ j\frac{a}{2b}x^2 \right\} \quad \textbf{(6.1.72)}$$

The one-dimensional inverse LCT in the context of quantum mechanics is given as-

$$\langle x\,|\,g\rangle = \sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}v\,\langle v\,|\,K\,|\,x\rangle^{*}\langle v\,|\,G\rangle \quad \textbf{(6.1.73)}$$

where, * indicates the complex conjugate. Rewriting the (6.1.73) results-

$$\langle x\,|\,g\rangle = \sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}v\,\langle x\,|\,K^{\dagger}\,|\,v\rangle\langle v\,|\,G\rangle \quad \textbf{(6.1.74)}$$

where, $K^{\dagger}$ indicates the Hermitian conjugate of the operator K. Substituting the value of (6.1.74) in (6.1.72), results-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p) = \left|\frac{1}{2\pi b}\right|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}v\,\mathrm{d}x\langle v\,|\,G\rangle\langle x\,|\,f\rangle\langle x\,|\,K^{\dagger}\,|\,v\rangle\langle p\,|\,K\,|\,x\rangle.\exp\left\{ j\frac{a}{2b}x^2 \right\}$$

(6.1.75)

Solving for $\langle x\,|\,K^{\dagger}\,|\,v\rangle\langle p\,|\,K\,|\,x\rangle$ results-

$$\Rightarrow |K_A|^2 \exp j\left[\frac{ax^2 + dp^2 - 2xp + 2xm - 2pdm + 2pbn}{2b} \right].$$

$$\exp j\left[\frac{-ax^2 - dv^2 + 2xv - 2xm + 2vdm - 2vbn}{2b} \right] \quad \textbf{(6.1.76)}$$

Substituting the value of (6.1.76) in (6.1.75), results-

$$\Rightarrow K_A^{*}\sqrt{\frac{1}{-j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}v\,\mathrm{d}x\langle v\,|\,G\rangle\langle p-v\,|\,F\rangle \exp j\left[\frac{d}{b}v(p-v) - \frac{x}{b}m \right] \quad \textbf{(6.1.77)}$$

and (6.1.77) can be written as-

$$L_F^{(a,b,c,d,m,n)}\left[q(x)\right](p)=G_{(a,b,c,d,m,n)}(p)\overset{A}{\otimes}\left(F_{(a,b,c,d,m,n)}(p).\exp\left[j\frac{d}{b}vp\right]\right).\exp\left[-j\frac{x}{b}m\right]$$

(6.1.78)

This is just a new approach for product theorem under OLCT, derived by representation transformation in quantum mechanics. Therefore, (6.1.71) is proved.

6.2 PROPOSED CORRELATION THEOREMS FOR OLCT

6.2.1 Proposed Cross-Correlation Theorem for OLCT

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$ and $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$, then $\sqrt{j2\pi b}\,F_{(a,b,c,d,m,n)}(-p)\,G_{(a,b,c,d,m,n)}(p)\,e^{-\frac{j}{2b}\left(dp^2-2p(dm-bn)\right)}$ is the OLCT of $n(x)$ i.e.

$$L_F^{(a,b,c,d,m,n)}\left[n(x)\right](p)=\sqrt{j2\pi b}\,F_{(a,b,c,d,m,n)}(-p)G_{(a,b,c,d,m,n)}(p)e^{-\frac{j}{2b}\left(dp^2-2p(dm-bn)\right)} \quad \textbf{(6.2.1)}$$

where, $n(x)=\left(f\overset{A}{\star}g\right)(x)=\int_{-\infty}^{\infty}f(y)g(x+y)\tilde{y}(x,y)\mathrm{d}y$, is the weighted cross-correlation and the weight function is $\tilde{y}(x,y)=K_A e^{j\frac{a}{b}y.(x+y)}$ and the role of g and f can be interchanged. The operation $\overset{A}{\star}$ indicates the proposed correlation operation.

Proof. The one-dimensional OLCT of $n(x)$ with representation transformation in the context of quantum mechanics **[57, 61]** is given as-

$$L_F^{(a,b,c,d,m,n)}\left[n(x)\right](p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,n\rangle=\langle p\,|\,K\,|\,n\rangle=\langle p\,|\,N\rangle=N_{(a,b,c,d,m,n)}(p)$$

(6.2.2)

From the definition of proposed identities of cross-correlation theorem, (6.2.2) results-

$$L_F^{(a,b,c,d,m,n)}\left[n(x)\right](p)=K_A\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,g\rangle.\exp\left\{j\frac{a}{b}y.(x+y)\right\}$$

(6.2.3)

Rearranging (6.2.3), results in-

$$L_F^{(a,b,c,d,m,n)}\left[n(x)\right](p)=K_A\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x\rangle\langle x+y\,|\,g\rangle\langle y\,|\,f\rangle.\exp\left\{j\frac{a}{b}y.(x+y)\right\}$$

(6.2.4)

Substituting $x+y=x'$ i.e. $x=x'-y$ and $y=y$ in (6.2.4), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$L_F^{(a,b,c,d,m,n)}\left[n(x)\right](p)=K_A\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x-y\right\rangle\left\langle x\,|\,g\right\rangle\left\langle y\,|\,f\right\rangle\exp\left\{j\frac{a}{b}yx\right\} \quad \textbf{(6.2.5)}$$

Rewriting $\left\langle p\,|\,K\,|\,x-y\right\rangle$ explicitly **[60]**, results in-

$$\left\langle p\,|\,K\,|\,x-y\right\rangle\exp\left\{j\frac{a}{b}yx\right\}= K_A\exp\left\{\frac{ja(x-y)^2+jdp^2+2j(x-y)(m-p)-2jp(dm-bn)+2jayx}{2b}\right\} \quad \textbf{(6.2.6)}$$

Multiplying and dividing (6.2.6) by $\exp\left\{\frac{jdp^2-2jp(dm-bn)}{2b}\right\}$ results-

$$\Rightarrow K_A\exp\left\{\frac{ja(x-y)^2+jdp^2+2j(x-y)(m-p)-2jp(dm-bn)+2jayx}{2b}\right\}. \exp\left\{\frac{jdp^2-2jp(dm-bn)}{2b}\right\}.\exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\} \quad \textbf{(6.2.7)}$$

Rearranging (6.2.7), results in-

$$\Rightarrow K_A\left[\exp\left\{\frac{jax^2+jdp^2+2jx(m-p)-2jp(dm-bn)}{2b}\right\}.\exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\}. .\exp\left\{\frac{jay^2-2jy(m-p)+jdp^2-2jp(dm-bn)}{2b}\right\}\right] \quad \textbf{(6.2.8)}$$

Substituting the value of (6.2.8) in (6.2.5), results-

$$L_F^{(a,b,c,d,m,n)}\left[n(x)\right](p)= K_A^2\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle x\,|\,g\right\rangle\left\langle y\,|\,f\right\rangle\left[\exp\left\{\frac{jax^2+jdp^2+2jx(m-p)-2jp(dm-bn)}{2b}\right\}. \exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\}.\exp\left\{\frac{jay^2-2jy(m-p)+jdp^2-2jp(dm-bn)}{2b}\right\}\right] \quad \textbf{(6.2.9)}$$

Rewriting (6.2.9) with the aid of quantum mechanics representation, results-

$$L_F^{(a,b,c,d,m,n)}\left[n(x)\right](p)=K_A^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\left\langle p\,|\,K\,|\,x\right\rangle\left\langle x\,|\,g\right\rangle\left\langle p\,|\,K\,|-y\right\rangle\left\langle y\,|\,f\right\rangle .\exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\} \quad \textbf{(6.2.10)}$$

Multiplying and dividing the (6.2.10) by $\sqrt{j2\pi b}$ and rearranging-

$$\Rightarrow K_A\sqrt{j2\pi b}\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,g\rangle.K_A\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}y\langle p\,|\,K\,|-y\rangle\langle y\,|\,f\rangle. \exp\left\{\frac{-jdp^2+2jp(dm-bn)}{2b}\right\} \quad \textbf{(6.2.11)}$$

Rewriting (6.2.11), results-

$$L_F^{(a,b,c,d,m,n)}\left[n(x)\right](p)=\sqrt{j2\pi b}\,F_{(a,b,c,d,m,n)}(-p)G_{(a,b,c,d,m,n)}(p)e^{-\frac{j}{2b}\left(dp^2-2p(dm-bn)\right)} \quad \textbf{(6.2.12)}$$

This is just a new approach for cross-correlation theorem under OLCT, derived by representation transformation in quantum mechanics. Therefore, (6.2.1) is obtained and the theorem is proved.

6.2.1.1 Special cases of OLCT for proposed cross-correlation theorem

LCT as a special case of OLCT, when $(a,b,c,d,m,n)=(a,b,c,d,0,0)$, (6.2.12) becomes-

$$L_F^{(a,b,c,d,0,0)}\left[n(x)\right](p)=\sqrt{j2\pi b}\,F_{(a,b,c,d,0,0)}(-p)G_{(a,b,c,d,0,0)}(p) \quad \textbf{(6.2.13)}$$

FT as a special case of OLCT, when $(a,b,c,d,m,n)=(0,1,-1,0,0,0)$, (6.2.12) becomes-

$$L_F^{(0,1,-1,0,0,0)}\left[n(x)\right](p)=\sqrt{j2\pi}\,F_{(0,1,-1,0,0,0)}(-p)G_{(0,1,-1,0,0,0)}(p) \quad \textbf{(6.2.14)}$$

Similarly, FRFT as a special case of OLCT, when $(a,b,c,d,m,n)=$ $(\cos\alpha,\sin\alpha,-\sin\alpha,$ $\cos\alpha,0,0)$, (6.2.12) becomes-

$$L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)}\left[n(x)\right](p)= \sqrt{j2\pi\sin\alpha}.\exp\left(-\frac{j}{2}p^2\cot\alpha\right)F_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)}(p)G_{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)}(p) \quad \textbf{(6.2.15)}$$

where, $L_F^{(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha,0,0)}\left[n(x)\right](p)$ indicates the FRFT of $s(x)$ and FT as a special case of FRFT, when $\alpha=\pi/2$, (6.2.15) becomes-

$$L_F^{(0,1,-1,0,0,0)}\left[n(x)\right](p)=\sqrt{j2\pi}\,F_{(0,1,-1,0,0,0)}(-p)G_{(0,1,-1,0,0,0)}(p) \quad \textbf{(6.2.16)}$$

Equations (6.2.13-6.2.16) are some of the special cases of proposed OLCT.

6.2.1.2. Properties satisfied by proposed cross-correlation theorem for OLCT

The following are some of the properties that are satisfied by the proposed cross-correlation theorem.

a) *Commutative property*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$ and $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$, then as per the commutative property

$$\left(f \overset{A}{\star} g\right)(x) \xleftrightarrow{OLCT} \sqrt{j2\pi b}\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}F_{(a,b,c,d,m,n)}(-p)G_{(a,b,c,d,m,n)}(p)$$

(6.2.17)

and

$$\left(g \overset{A}{\star} f\right)(x) \xleftrightarrow{OLCT} \sqrt{j2\pi b}\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}G_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)$$

(6.2.18)

i.e. $\left(f \overset{A}{\star} g\right)(x) \neq \left(g \overset{A}{\star} f\right)(x)$ **(6.2.19)**

Proof: Considering the L.H.S. of (6.2.18)

$$r_{gf}(x) = \left(g \overset{A}{\star} f\right)(x) = \int_{-\infty}^{\infty} g(y)\, f(x+y)\exp\left\{j\frac{a}{b}y.(x+y)\right\}\,\mathrm{d}y \quad \textbf{(6.2.20)}$$

Using the quantum mechanical notation **[57, 61]**, the one-dimensional OLCT of the signal $r_{gf}(x)$ reads as-

$$L_F^{(a,b,c,d,m,n)}\left[r_{gf}(x)\right](p) = \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p|K|x\rangle\langle x|r_{gf}\rangle = \langle p|K|r_{gf}\rangle = \langle p|R_{gf}\rangle = R_{gf(a,b,c,d,m,n)}(p)$$

(6.2.21)

where, the notation $\langle x|r_{gf}\rangle = r_{gf}(x)$ and $\langle p|K|x\rangle = K_{(a,b,c,d,m,n)}(p,x)$ gives the representation of OLCT kernel in quantum mechanics. K is named as the OLCT operator and

$$\langle x|r_{gf}\rangle = \int_{-\infty}^{\infty}\mathrm{d}y\langle y|g\rangle\langle x+y|f\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\}$$

Substituting the value of $\langle x|r_{gf}\rangle$ in (6.2.22) results-

$$\left\langle p \mid R_{gf} \right\rangle = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \mathrm{d}x \left\langle p \mid K \mid x \right\rangle \int_{-\infty}^{\infty} \mathrm{d}y \left\langle y \mid g \right\rangle \left\langle x+y \mid f \right\rangle . \exp\left\{ j \frac{a}{b} y.\left(x+y\right) \right\} \tag{6.2.22}$$

Rearranging (6.2.22), results in-

$$\left\langle p \mid R_{gf} \right\rangle = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y \left\langle p \mid K \mid x \right\rangle \left\langle x+y \mid f \right\rangle \left\langle y \mid g \right\rangle . \exp\left\{ j \frac{a}{b} y.\left(x+y\right) \right\} \tag{6.2.23}$$

Substituting $x+y=x'$ i.e. $x=x'-y$ and $y=y$ in (6.2.23), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$\left\langle p \mid R_{gf} \right\rangle = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y \left\langle p \mid K \mid x-y \right\rangle \left\langle x \mid f \right\rangle \left\langle y \mid g \right\rangle \exp\left\{ j \frac{a}{b} yx \right\} \tag{6.2.24}$$

Rewriting $\left\langle p \mid K \mid x-y \right\rangle$ explicitly, results in-

$$\begin{aligned} &\left\langle p \mid K \mid x-y \right\rangle \exp\left\{ j \frac{a}{b} yx \right\} = \\ &\qquad K_A \exp j \left\{ \frac{a\left(x-y\right)^2 + dp^2 + 2\left(m-p\right)\left(x-y\right) + 2ayx - 2p\left(dm-bn\right)}{2b} \right\} \end{aligned} \tag{6.2.25}$$

Multiplying and dividing (6.2.25) by $K_A \exp j\left[\frac{dp^2 - 2p\left(dm-bn\right)}{2b}\right]$ results-

$$\begin{aligned} \left\langle p \mid K \mid x-y \right\rangle \exp\left\{ j \frac{a}{b} yx \right\} &= \frac{K_A^2}{K_A} \exp \frac{j}{2b} \left\{ a\left(x-y\right)^2 + dp^2 + 2\left(m-p\right)\left(x-y\right) \right. \\ &\left. + 2ayx - 2p\left(dm-bn\right) + dp^2 - dp^2 + 2p\left(dm-bn\right) - 2p\left(dm-bn\right) \right\} \end{aligned} \tag{6.2.26}$$

Rearranging (6.2.26) results in-

$$\begin{aligned} \left\langle p \mid K \mid x-y \right\rangle \exp\left\{ j \frac{a}{b} yx \right\} &= \frac{K_A^2}{K_A} \exp j \left\{ \frac{ax^2 + dp^2 + 2\left(m-p\right)x - 2p\left(dm-bn\right)}{2b} \right\}. \\ \exp j &\left\{ \frac{ay^2 + dp^2 - 2\left(m-p\right)y - 2p\left(dm-bn\right)}{2b} \right\}. \exp j \left\{ \frac{-dp^2 + 2p\left(dm-bn\right)}{2b} \right\} \end{aligned} \tag{6.2.27}$$

Substituting the value of (6.2.27) in (6.2.24), results-

$$\begin{aligned} \left\langle p \mid R_{gf} \right\rangle = \frac{1}{K_A} \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y \left\langle x \mid f \right\rangle \left\langle y \mid g \right\rangle . \exp j \left\{ \frac{-dp^2 + 2p\left(dm-bn\right)}{2b} \right\} \\ K_A \exp j \left\{ \frac{ax^2 + dp^2 + 2\left(m-p\right)x - 2p\left(dm-bn\right)}{2b} \right\} \\ K_A \exp j \left\{ \frac{ay^2 + dp^2 - 2\left(m-p\right)y - 2p\left(dm-bn\right)}{2b} \right\} \end{aligned} \tag{6.2.28}$$

Rewriting (6.2.28) with the aid of quantum mechanical representation, results-

$$\left\langle p \mid R_{gf} \right\rangle = \frac{1}{K_A}\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y \left\langle p \mid K \mid x\right\rangle \left\langle x \mid f\right\rangle \left\langle p \mid K \mid -y\right\rangle \left\langle y \mid g\right\rangle . \exp j\left\{\frac{-dp^2 + 2p\left(dm - bn\right)}{2b}\right\} \tag{6.2.29}$$

Rearranging (6.2.29), results in-

$$\left\langle p \mid R_{gf} \right\rangle = \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty} \mathrm{d}x \left\langle p \mid K \mid x\right\rangle \left\langle x \mid f\right\rangle \int_{-\infty}^{\infty} \mathrm{d}y \left\langle p \mid K \mid -y\right\rangle \left\langle y \mid g\right\rangle . \exp j\left\{\frac{-dp^2 + 2p\left(dm - bn\right) - dm^2}{2b}\right\} \tag{6.2.30}$$

Multiplying and dividing (6.2.30) by $\sqrt{j2\pi b}$ and rearranging, results in-

$$\left\langle p \mid R_{gf} \right\rangle = \sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2 + 2p\left(dm - bn\right) - dm^2}{2b}\right\}.G_{(a,b,c,d,m,n)}\left(-p\right)F_{(a,b,c,d,m,n)}\left(p\right) \tag{6.2.31}$$

Therefore, from (6.2.17) and (6.2.31), it concludes that-

$$\left(f \overset{A}{\star} g\right)(x) \neq \left(g \overset{A}{\star} f\right)(x) \tag{6.2.32}$$

This proves that the proposed cross-correlation theorem for OLCT does not satisfy the commutative law.

b) *Associative property*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$ and $M_{(a,b,c,d,m,n)}(p)$ is the OLCT of $m(x)$, then as per the associative property

$$\left(\left(f \overset{A}{\star} g\right) \overset{A}{\star} m\right)(x) \xleftrightarrow{OLCT} F_{(a,b,c,d,m,n)}\left(p\right)G_{(a,b,c,d,m,n)}\left(-p\right)M_{(a,b,c,d,m,n)}\left(p\right) \left(j2\pi b\right)\exp j\left\{\frac{-dp^2 + 2p\left(dm - bn\right) - dm^2}{b}\right\} \tag{6.2.33}$$

$$\left(f \overset{A}{\star} \left(g \overset{A}{\star} m\right)\right)(x) \xleftrightarrow{OLCT} F_{(a,b,c,d,m,n)}\left(-p\right)G_{(a,b,c,d,m,n)}\left(-p\right)M_{(a,b,c,d,m,n)}\left(p\right) \left(j2\pi b\right)\exp j\left\{\frac{-dp^2 + 2p\left(dm - bn\right) - dm^2}{b}\right\} \tag{6.2.34}$$

i.e. $$\left(\left(f \overset{A}{\star} g\right) \overset{A}{\star} m\right)(x) \neq \left(f \overset{A}{\star}\left(g \overset{A}{\star} m\right)\right)(x) \qquad \textbf{(6.2.35)}$$

Proof: Considering the L.H.S. of (6.2.33)

$$r_{(fg)m}(x) = \left(\left(f \overset{A}{\star} g\right) \overset{A}{\star} m\right)(x) = \left(r_{fg} \overset{A}{\star} m\right)(x) \qquad \textbf{(6.2.36)}$$

where,

$$r_{fg}(x) = \left\langle x \mid r_{fg} \right\rangle = \left(f \overset{A}{\star} g\right)(x) = \int_{-\infty}^{\infty} \mathrm{d}y \left\langle y \mid f \right\rangle \left\langle x+y \mid g \right\rangle \exp\left\{ j\frac{a}{b} y.(x+y)\right\} \qquad \textbf{(6.2.37)}$$

and

$$r_{(fg)m}(x) = \left(\left(f \overset{A}{\star} g\right) \overset{A}{\star} m\right)(x) = \left(r_{fg} \overset{A}{\star} m\right)(x) = \int_{-\infty}^{\infty} \mathrm{d}\beta \left\langle \beta \mid r_{fg} \right\rangle \left\langle x+\beta \mid m \right\rangle \exp\left\{ j\frac{a}{b} \beta.(x+\beta)\right\}$$

(6.2.38)

Substituting the value of $\left\langle x \mid r_{fg} \right\rangle$ from (6.2.37) to (6.2.38), results in-

$$r_{(fg)m}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \left\langle y \mid f \right\rangle \left\langle \beta+y \mid g \right\rangle \exp\left\{ j\frac{a}{b} y.(\beta+y)\right\}.\left\langle x+\beta \mid m \right\rangle \exp\left\{ j\frac{a}{b} \beta.(x+\beta)\right\}$$

(6.2.39)

Taking OLCT of $r_{(fg)m}(x)$, results in-

$$\left\langle p \mid R_{(fg)m} \right\rangle = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \mathrm{d}x \left\langle p \mid K \mid x \right\rangle \left\langle x \mid r_{(fg)m} \right\rangle \qquad \textbf{(6.2.40)}$$

Substituting the value of $r_{(fg)m}(x)$ in (6.2.40), results in-

$$\left\langle p \mid R_{(fg)m} \right\rangle = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \mathrm{d}x \left\langle p \mid K \mid x \right\rangle \left\langle y \mid f \right\rangle \left\langle \beta+y \mid g \right\rangle \left\langle x+\beta \mid m \right\rangle. \exp\left\{ j\frac{a}{b} \beta.(x+\beta)\right\}.\exp\left\{ j\frac{a}{b} y.(\beta+y)\right\} \qquad \textbf{(6.2.41)}$$

Substituting $x+\beta = x'$ i.e. $x = x'-\beta$ and $y = y$ in (6.2.41), then $\mathrm{d}x\mathrm{d}y = \mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$\left\langle p \mid R_{(fg)m} \right\rangle = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}y \mathrm{d}x \left\langle p \mid K \mid x-\beta \right\rangle \left\langle y \mid f \right\rangle \left\langle \beta+y \mid g \right\rangle \left\langle x \mid m \right\rangle. \exp\left\{ j\frac{a}{b} \beta x\right\}.\exp\left\{ j\frac{a}{b} y.(\beta+y)\right\} \qquad \textbf{(6.2.42)}$$

Rewriting $\langle p\,|\,K\,|\,x-\beta\rangle$ explicitly **[60]**, results in-

$$\langle p\,|\,K\,|\,x-\beta\rangle \exp\left\{ j\frac{a}{b}\beta x\right\} = K_A \exp j\left\{\frac{a(x-\beta)^2 + dp^2 + 2(m-p)(x-\beta) + 2a\beta x - 2p(dm-bn)}{2b}\right\} \quad \textbf{(6.2.43)}$$

Rearranging (6.2.43), results in-

$$\langle p\,|\,K\,|\,x-\beta\rangle \exp\left\{ j\frac{a}{b}\beta x\right\} = K_A \exp j\left\{\frac{ax^2 + dp^2 + 2(m-p)x - 2p(dm-bn)}{2b}\right\}. \exp j\left\{\frac{a\beta^2 - 2(m-p)\beta}{2b}\right\} \quad \textbf{(6.2.44)}$$

Substituting the value of (6.2.44) in (6.2.42)

$$\langle p\,|\,R_{(fg)m}\rangle = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}y\,\mathrm{d}x\,\langle y\,|\,f\rangle\langle \beta+y\,|\,g\rangle\langle x\,|\,m\rangle.\exp\left\{ j\frac{a}{b}y.(\beta+y)\right\} \exp j\left\{\frac{a\beta^2 - 2(m-p)\beta}{2b}\right\}.K_A \exp j\left\{\frac{ax^2 + dp^2 + 2(m-p)x - 2p(dm-bn)}{2b}\right\} \quad \textbf{(6.2.45)}$$

Rearranging (6.2.45) results-

$$\langle p\,|\,R_{(fg)m}\rangle = M_{(a,b,c,d,m,n)}(p) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}y\,\langle y\,|\,f\rangle\langle \beta+y\,|\,g\rangle.\exp\left\{ j\frac{a}{b}y.(\beta+y)\right\} \exp j\left\{\frac{a\beta^2 - 2(m-p)\beta}{2b}\right\} \quad \textbf{(6.2.46)}$$

Multiplying and dividing (6.2.46) by $K_A \exp j\left[\frac{dp^2 - 2p(dm-bn)}{2b}\right]$ results-

$$\langle p\,|\,R_{(fg)m}\rangle = \frac{1}{K_A} M_{(a,b,c,d,m,n)}(p) K_A \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}y\,\langle y\,|\,f\rangle\langle \beta+y\,|\,g\rangle.\exp\left\{ j\frac{a}{b}y.(\beta+y)\right\} \exp j\left\{\frac{a\beta^2 - 2(m-p)\beta + dp^2 - 2p(dm-bn)}{2b}\right\} \exp j\left\{\frac{-dp^2 + 2p(dm-bn)}{2b}\right\} \quad \textbf{(6.2.47)}$$

Rearranging (6.2.47) results-

$$\langle p\,|\,R_{(fg)m}\rangle = M_{(a,b,c,d,m,n)}(p) \frac{1}{K_A} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}y\,\langle y\,|\,f\rangle\langle \beta+y\,|\,g\rangle.\langle p\,|\,K\,|\,-\beta\rangle. \exp\left\{ j\frac{a}{b}y.(\beta+y)\right\} \exp j\left\{\frac{-dp^2 + 2p(dm-bn)}{2b}\right\} \quad \textbf{(6.2.48)}$$

Substituting $\beta + y = \beta'$ i.e. $\beta = \beta' - y$ and $y = y$ in (6.2.48), then $\mathrm{d}\beta\mathrm{d}y = \mathrm{d}\beta'\mathrm{d}y$ **[70]** and then replacing β' by β, results-

$$\left\langle p \mid R_{(fg)m} \right\rangle = M_{(a,b,c,d,m,n)}(p)\frac{1}{K_A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta\mathrm{d}y \left\langle y \mid f \right\rangle\left\langle \beta \mid g \right\rangle.\left\langle p \mid K \mid -\beta + y \right\rangle$$
$$\exp\left\{ j\frac{a}{b} y\beta \right\}.\exp j\left\{ \frac{-dp^2 + 2p(dm - bn)}{2b} \right\} \tag{6.2.49}$$

Rewriting $\langle p \mid K \mid y - \beta \rangle$ explicitly, results in-

$$\left\langle p \mid K \mid y-\beta \right\rangle \exp\left\{ j\frac{a}{b}\beta y \right\} =$$
$$K_A \exp j\left\{ \frac{a(y-\beta)^2 + dp^2 + 2(m-p)(y-\beta) + 2a\beta y - 2p(dm-bn)}{2b} \right\} \tag{6.2.50}$$

Rearranging (6.2.50), results in-

$$\left\langle p \mid K \mid y-\beta \right\rangle \exp\left\{ j\frac{a}{b}\beta y \right\} = K_A \exp j\left\{ \frac{a\beta^2 - 2(m-p)\beta + dp^2 - 2p(dm-bn)}{2b} \right\}$$
$$\exp j\left\{ \frac{ay^2 + 2(m-p)y}{2b} \right\} \exp j\left\{ \frac{-dp^2 + 2p(dm-bn)}{2b} \right\} \tag{6.2.51}$$

Substituting the value of (6.2.51) in (6.2.49) results-

$$\left\langle p \mid R_{(fg)m} \right\rangle = \sqrt{j2\pi b} M_{(a,b,c,d,m,n)}(p) G_{(a,b,c,d,m,n)}(-p) \frac{1}{K_A}\int_{-\infty}^{\infty} \mathrm{d}y \left\langle y \mid f \right\rangle$$
$$\exp j\left\{ \frac{ay^2 + 2(m-p)y}{2b} \right\} \exp j\left\{ \frac{-dp^2 + 2p(dm-bn)}{2b} \right\} \tag{6.2.52}$$

$$\left\langle p \mid R_{(fg)m} \right\rangle = (j2\pi b)$$
$$M_{(a,b,c,d,m,n)}(p) G_{(a,b,c,d,m,n)}(-p) F_{(a,b,c,d,m,n)}(p) \exp j\left\{ \frac{-dp^2 + 2p(dm-bn) - dm^2}{b} \right\} \tag{6.2.53}$$

Similarly it can proved that the OLCT of $r_{f(gm)}(x) = (f \star (g \star m))$ is-

$$\left\langle p \mid R_{f(gm)} \right\rangle = (j2\pi b)$$
$$\exp j\left[\frac{-dp^2 + 2p(dm-bn) - dm^2}{b} \right] F_{(a,b,c,d,m,n)}(-p) G_{(a,b,c,d,m,n)}(-p) M_{(a,b,c,d,m,n)}(p) \tag{6.2.54}$$

Therefore, from (6.2.53) and (6.2.54), it has been proved that-

$$\left(\left(f \overset{A}{\star} g\right) \overset{A}{\star} m\right)(x) \neq \left(f \overset{A}{\star} \left(g \overset{A}{\star} m\right)\right)(x) \tag{6.2.55}$$

This proves that the proposed cross-correlation theorem for OLCT does not satisfy the associative law.

c) *Distributive property*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$ and $M_{(a,b,c,d,m,n)}(p)$ is the OLCT of $m(x)$ then as per the distributive property

$$\left(f\overset{A}{\star}(g+m)\right)(x)\xleftrightarrow{OLCT}(j2\pi b)\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} F_{(a,b,c,d,m,n)}(-p)\left\{G_{(a,b,c,d,m,n)}(p)+M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.55)}$$

and

$$\left(f\overset{A}{\star}g+f\overset{A}{\star}m\right)(x)\xleftrightarrow{OLCT}(j2\pi b)\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} \left\{F_{(a,b,c,d,m,n)}(-p)G_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(-p)M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.56)}$$

i.e. $$\left(f\overset{A}{\star}(g+m)\right)(x)=\left(f\overset{A}{\star}g+f\overset{A}{\star}m\right)(x) \quad \textbf{(6.2.57)}$$

Proof: Considering the L.H.S. of (6.2.56)

$$r_{fg+fm}(x)=\left(f\overset{A}{\star}g+f\overset{A}{\star}m\right)(x)=\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,g\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\} +\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,m\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\} \quad \textbf{(6.2.58)}$$

Taking OLCT of (6.2.58)

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x\rangle\langle x\,|\,r_{fg+fm}\rangle \quad \textbf{(6.2.59)}$$

Substituting the value of $r_{fg+fm}(x)$ in (6.2.59) results in-

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}dx\langle p\,|\,K\,|\,x\rangle\left[\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,g\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\} +\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x+y\,|\,m\rangle\exp\left\{j\frac{a}{b}y.(x+y)\right\}\right] \quad \textbf{(6.2.60)}$$

Substituting $x+y=x'$ i.e. $x=x'-y$ and $y=y$ in (6.2.23), then $\mathrm{d}x\mathrm{d}y=\mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\mathrm{d}x\langle p\,|\,K\,|\,x-y\rangle\left[\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x\,|\,g\rangle\exp\left\{j\frac{a}{b}yx\right\} +\int_{-\infty}^{\infty}\mathrm{d}y\langle y\,|\,f\rangle\langle x\,|\,m\rangle\exp\left\{j\frac{a}{b}yx\right\}\right] \quad \textbf{(6.2.61)}$$

Rearranging (6.2.61) results in-

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x-y\rangle\langle y\,|\,f\rangle\langle x\,|\,g\rangle\exp\left\{j\frac{a}{b}yx\right\}+ \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\langle p\,|\,K\,|\,x-y\rangle\langle y\,|\,f\rangle\langle x\,|\,m\rangle\exp\left\{j\frac{a}{b}yx\right\} \quad \textbf{(6.2.62)}$$

Rewriting $\langle p\,|\,K\,|\,x-y\rangle$ explicitly, results in-

$$\langle p\,|\,K\,|\,x-y\rangle\exp\left\{j\frac{a}{b}xy\right\}= K_A\exp j\left\{\frac{a(x-y)^2+dp^2+2(m-p)(x-y)+2ayx-2p(dm-bn)}{2b}\right\} \quad \textbf{(6.2.63)}$$

Rearranging (6.2.63) results-

$$\langle p\,|\,K\,|\,x-y\rangle\exp\left\{j\frac{a}{b}xy\right\}= K_A\exp j\left\{\frac{ax^2+ay^2+dp^2+2x(m-p)-2y(m-p)-2p(dm-bn)}{2b}\right\} \quad \textbf{(6.2.64)}$$

Substituting the value of (6.2.64) in (6.2.62) results-

$$R_{fg+fm}(p)=\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\,\langle y\,|\,f\rangle\langle x\,|\,g\rangle. K_A\exp j\left\{\frac{ax^2+ay^2+dp^2+2x(m-p)-2y(m-p)-2p(dm-bn)}{2b}\right\}+ \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\,\langle y\,|\,f\rangle\langle x\,|\,m\rangle. K_A\exp j\left\{\frac{ax^2+ay^2+dp^2+2x(m-p)-2y(m-p)-2p(dm-bn)}{2b}\right\} \quad \textbf{(6.2.65)}$$

Multiplying and dividing (6.2.65) by $K_A\exp j\left[\frac{dp^2-2p(dm-bn)}{2b}\right].\sqrt{\frac{1}{j2\pi b}}$ results-

$$R_{fg+fm}(p)=\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\,\langle y\,|\,f\rangle\langle x\,|\,g\rangle.\frac{1}{K_A}\exp j\left[\frac{-dp^2+2p(dm-bn)}{2b}\right] K_A^2\exp j\left\{\frac{ax^2+ay^2+2dp^2+2x(m-p)-2y(m-p)-4p(dm-bn)}{2b}\right\} +\frac{\sqrt{j2\pi b}}{j2\pi b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}y\,\langle y\,|\,f\rangle\langle x\,|\,m\rangle.\frac{1}{K_A}\exp j\left[\frac{-dp^2+2p(dm-bn)}{2b}\right] K_A^2\exp j\left\{\frac{ax^2+ay^2+2dp^2+2x(m-p)-2y(m-p)-4p(dm-bn)}{2b}\right\} \quad \textbf{(6.2.66)}$$

Rearranging (6.2.66) results-

$$R_{fg+fm}(p) = (j2\pi b)\exp j\left[\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right] \left\{F_{(a,b,c,d,m,n)}(-p)G_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(-p)M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.67)}$$

Utilizing the linearity property of OLCT, (6.2.67) can be written as-

$$R_{fg+fm}(p) = (j2\pi b)\exp j\left[\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right] F_{(a,b,c,d,m,n)}(-p)\left\{G_{(a,b,c,d,m,n)}(p)+M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.68)}$$

Similarly it can proved that the OLCT of $r_{f(g+m)}(x) = (f\star(g+m))(x)$ is

$$R_{f(g+m)}(p) = (j2\pi b) \exp j\left[\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right]F_{(a,b,c,d,m,n)}(-p)\left\{G_{(a,b,c,d,m,n)}(p)+M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.69)}$$

Therefore, from (6.2.68) and (6.2.69), it has been proved that-

$$\left(f\overset{A}{\star}(g+m)\right)(x) = \left(f\overset{A}{\star}g + f\overset{A}{\star}m\right)(x) \quad \textbf{(6.2.70)}$$

This proves that the proposed cross-correlation theorem for OLCT satisfies the distributive law.

d) *Even function*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$ and $G_{(a,b,c,d,m,n)}(p)$ is the OLCT of $g(x)$ then

$$\left(f\overset{A}{\star}g\right)(x) \xleftrightarrow{OLCT} \sqrt{j2\pi b}\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}F_{(a,b,c,d,m,n)}(-p)G_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.2.71)}$$

$$\left(f\overset{A}{\star}g\right)(-x) \xleftrightarrow{OLCT} \sqrt{j2\pi b}\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}F_{(a,b,c,d,m,n)}(p)G_{(a,b,c,d,m,n)}(-p) \quad \textbf{(6.2.72)}$$

$$\left(g\overset{A}{\star}f\right)(-x) \xleftrightarrow{OLCT} \sqrt{j2\pi b}\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}F_{(a,b,c,d,m,n)}(-p)G_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.2.73)}$$

and as per evenness property

$$\left(f \overset{A}{\star} g\right)(x) \neq \left(f \overset{A}{\star} g\right)(-x) \tag{6.2.74}$$

$$\left(f \overset{A}{\star} g\right)(x) = \left(g \overset{A}{\star} f\right)(-x) \tag{6.2.75}$$

Proof: Considering the L.H.S. of (6.2.73)

$$r_{gf}(-x) = \left(g \overset{A}{\star} f\right)(-x) = \int_{-\infty}^{\infty} \mathrm{d}y \langle y \mid g\rangle \langle y-x \mid f\rangle \exp\left\{j\frac{a}{b}y.(y-x)\right\} \tag{6.2.76}$$

Taking LCT of (6.2.76)

$$R_{gf1}(p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \mathrm{d}x \langle p \mid K \mid x\rangle \langle -x \middle| r_{gf}\rangle \tag{6.2.77}$$

Substituting the value of $\langle -x | r_{gf}\rangle$ in (6.2.77) results-

$$R_{gf1}(p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} \mathrm{d}x \langle p \mid K \mid x\rangle \int_{-\infty}^{\infty} \mathrm{d}y \langle y \mid g\rangle \langle y-x \mid f\rangle \exp\left\{j\frac{a}{b}y.(y-x)\right\} \tag{6.2.78}$$

Substituting $x+y=x'$ i.e. $x=x'-y$ and $y=y$ in (6.2.23), then $\mathrm{d}x\mathrm{d}y = \mathrm{d}x'\mathrm{d}y$ **[70]** and then replacing x' by x, results-

$$R_{gf1}(p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}x \langle p \mid K \mid x+y\rangle \langle y \mid g\rangle \langle -x \mid f\rangle \exp\left\{-j\frac{a}{b}yx\right\} \tag{6.2.79}$$

Rewriting $\langle p \mid K \mid x+y\rangle$ explicitly, results in-

$$\langle p \mid K \mid x+y\rangle \exp\left\{-j\frac{a}{b}xy\right\} = K_A \exp j\left\{\frac{a(x+y)^2 + dp^2 + 2(m-p)(x+y) - 2ayx - 2p(dm-bn)}{2b}\right\} \tag{6.2.80}$$

Substituting the value of (6.2.80) in (6.2.79) results-

$$R_{gf1}(p) = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}x \langle y \mid g\rangle \langle -x \mid f\rangle K_A \exp j\left\{\frac{a(x+y)^2 + dp^2 + 2(m-p)(x+y) - 2ayx - 2p(dm-bn)}{2b}\right\} \tag{6.2.81}$$

Multiplying and dividing (6.2.81) by $\exp j\left\{\frac{-dp^2 + 2p(dm-bn)}{2b}\right\}.\sqrt{\frac{1}{j2\pi b}}$ results-

$$R_{gf1}(p)=\frac{\sqrt{j2\pi b}}{j2\pi b}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}x\,\langle y\,|\,g\rangle\langle -x\,|\,f\rangle\langle p\,|\,K\,|\,x\rangle\langle p\,|\,K\,|\,y\rangle.\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\}$$

(6.2.82)

Rearranging (6.2.82) results-

$$R_{gf1}(p)=\sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\}G_{(a,b,c,d,m,n)}(p)F_{(a,b,c,d,m,n)}(-p) \quad \textbf{(6.2.83)}$$

From (6.2.74) and (6.2.83), it concludes that-

$$\left(f\overset{A}{\star}g\right)(x)=\left(g\overset{A}{\star}f\right)(-x)$$

From (6.2.71) and (6.2.72), it concludes that-

$$\left(f\overset{A}{\star}g\right)(x)\neq\left(f\overset{A}{\star}g\right)(-x)$$

This proves that the proposed cross-correlation theorem for OLCT is not an even function of delay.

6.2.2 Proposed Auto-Correlation Theorem for OLCT

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, then $\sqrt{j2\pi b}\,F_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)$ $e^{-\frac{j}{2b}\left(dp^2-2p(dm-bn)\right)}$ is the OLCT of $m(x)$ i.e.

$$L_F^{(a,b,c,d,m,n)}\left[m(x)\right](p)=\sqrt{j2\pi b}\,F_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)e^{-\frac{j}{2b}\left(dp^2-2p(dm-bn)\right)} \quad \textbf{(6.2.84)}$$

where, $m(x)=\left(f\overset{A}{\star}f\right)(x)=\int_{-\infty}^{\infty}f(y)\,f(x+y)\,\tilde{y}(x,y)\mathrm{d}y$, is the weighted cross-correlation and the weight function is $\tilde{y}(x,y)=K_A\,e^{j\frac{a}{b}y.(x+y)}$ and. The operation $\overset{A}{\star}$ indicates the proposed correlation operation.

Proof. The proof is obvious. Auto-correlation is the cross-correlation of a signal with itself.

6.2.2.1 Properties satisfied by proposed auto-correlation theorem for OLCT

The following properties are satisfied by the proposed auto-correlation theorem for OLCT

a) *Commutative property*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, then as per the commutative property-

$$\left(f\overset{A}{\star}f\right)(x)\xleftrightarrow{OLCT}\sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}F_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)$$

(6.2.85)

and

$$\left(f\overset{A}{\star}f\right)(x)\xleftrightarrow{OLCT}\sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}F_{(a,b,c,d,m,n)}(p)F_{(a,b,c,d,m,n)}(-p)$$

(6.2.86)

i.e. $\left(f\overset{A}{\star}f\right)(x)=\left(f\overset{A}{\star}f\right)(x)$ **(6.2.87)**

Proof: From (6.2.31)

$$\langle p\,|\,R_{gf}\rangle=\sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}.F_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)$$

(6.2.88)

where, $\langle p\,|\,R_{gf}\rangle$ represents the OLCT of the cross-correlation of $\langle y\,|\,f\rangle$ and $\langle x\,|\,g\rangle$ i.e. $\left(f\overset{A}{\star}g\right)(x)$. Replacing $\langle x\,|\,g\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d,m,n)}(p)$ with $F_{(a,b,c,d,m,n)}(p)$ in (6.2.88) results OLCT of auto-correlation operation as-

$$\langle p\,|\,R_{ff}\rangle=\sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}.F_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)$$

(6.2.89)

This expression does not change by changing the order of commutating the function. Therefore, auto-correlation operation satisfies the commutative property.

b) *Associative property*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, then as per the associative property-

$$\left(\left(f\overset{A}{\star}f\right)\overset{A}{\star}f\right)(x)\xleftrightarrow{OLCT}(j2\pi b)$$
$$\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{b}\right\}F_{(a,b,c,d,m,n)}(p)F_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)$$

(6.2.90)

$$\left(f \overset{A}{\star}\left(f \overset{A}{\star} f\right)\right)(x) \xleftrightarrow{OLCT} (j2\pi b)$$
$$.\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{b}\right\} F_{(a,b,c,d,m,n)}(p)\,F_{(a,b,c,d,m,n)}(-p)\,F_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.2.91)}$$

i.e.
$$\left(\left(f \overset{A}{\star} f\right) \overset{A}{\star} f\right)(x) \neq \left(f \overset{A}{\star}\left(f \overset{A}{\star} f\right)\right)(x) \quad \textbf{(6.2.92)}$$

Proof: From (6.2.54)

$$\left\langle p \mid R_{f(gm)} \right\rangle = (j2\pi b)$$
$$\exp j\left[\frac{-dp^2+2p(dm-bn)-dm^2}{b}\right] F_{(a,b,c,d,m,n)}(-p)\,G_{(a,b,c,d,m,n)}(-p)\,M_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.2.93)}$$

where, $\left\langle p \mid R_{f(gm)} \right\rangle$ represents the OLCT of the correlation operation $\left(f \overset{A}{\star}\left(g \overset{A}{\star} m\right)\right)(x)$.

Replacing $\langle x \mid g \rangle$ and $\langle x \mid m \rangle$ with $\langle x \mid f \rangle$ and $G_{(a,b,c,d,m,n)}(p)$ and $M_{(a,b,c,d,m,n)}(p)$ with $F_{(a,b,c,d,m,n)}(p)$ in (6.2.93) results OLCT of correlation operation as-

$$\left\langle p \mid R_{f(ff)} \right\rangle = (j2\pi b)$$
$$\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{b}\right\} F_{(a,b,c,d,m,n)}(-p)\,F_{(a,b,c,d,m,n)}(-p)\,F_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.2.94)}$$

where, $\left\langle p \mid R_{f(ff)} \right\rangle$ represents the OLCT of correlation operation $\left(f \overset{A}{\star}\left(f \overset{A}{\star} f\right)\right)(x)$.

From (6.2.53)

$$\left\langle p \mid R_{(fg)m} \right\rangle = (j2\pi b)$$
$$M_{(a,b,c,d,m,n)}(p)\,G_{(a,b,c,d,m,n)}(-p)\,F_{(a,b,c,d,m,n)}(p) \exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{b}\right\} \quad \textbf{(6.2.95)}$$

where, $\left\langle p \mid R_{(fg)m} \right\rangle$ represents the OLCT of the correlation operation $\left(\left(f \overset{A}{\star} g\right) \overset{A}{\star} m\right)(x)$.

Replacing $\langle x \mid g \rangle$ and $\langle x \mid m \rangle$ with $\langle x \mid f \rangle$ and $G_{(a,b,c,d,m,n)}(p)$ and $M_{(a,b,c,d,m,n)}(p)$ with $F_{(a,b,c,d,m,n)}(p)$ in (6.2.95) results OLCT of correlation operation as-

$$\left\langle p \mid R_{(ff)f} \right\rangle = (j2\pi b)\,F_{(a,b,c,d,m,n)}(p)\,F_{(a,b,c,d,m,n)}(-p)\,F_{(a,b,c,d,m,n)}(p)$$
$$\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{b}\right\} \quad \textbf{(6.2.96)}$$

where, $\left\langle p\,|\,R_{(ff)f}\right\rangle$ represents the OLCT of correlation operation $\left(\left(f\overset{A}{\star}f\right)\overset{A}{\star}f\right)(x)$.

Therefore, from (6.2.94) and (6.2.96) it has been proved that-

$$\left(\left(f\overset{A}{\star}f\right)\overset{A}{\star}f\right)(x)\neq\left(f\overset{A}{\star}\left(f\overset{A}{\star}f\right)\right)(x)$$

This confirms that the proposed auto-correlation theorem for OLCT is not satisfying associativity property in conformity of its claim of generalization of analogous identity of FT.

c) *Distributive property*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, then as per the distributive property-

$$\left(f\overset{A}{\star}(f+f)\right)(x)\xleftrightarrow{OLCT}(j2\pi b)\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}$$
$$F_{(a,b,c,d,m,n)}(-p)\left\{F_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.97)}$$

and

$$\left(f\overset{A}{\star}f+f\overset{A}{\star}f\right)(x)\xleftrightarrow{OLCT}(j2\pi b).\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}$$
$$\left\{F_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(-p)F_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.98)}$$

i.e. $$\left(f\overset{A}{\star}(f+f)\right)(x)=\left(f\overset{A}{\star}f+f\overset{A}{\star}f\right)(x) \quad \textbf{(6.2.99)}$$

Proof: From (6.2.68)

$$R_{fg+fm}(p)=(j2\pi b)\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}$$
$$\left\{F_{(a,b,c,d,m,n)}(-p)G_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(-p)M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.100)}$$

Utilizing the linearity property of OLCT, (6.2.100) can be written as-

$$R_{fg+fm}(p)=(j2\pi b)\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\}$$
$$F_{(a,b,c,d,m,n)}(-p)\left\{G_{(a,b,c,d,m,n)}(p)+M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.101)}$$

Replacing $\langle x\,|\,g\rangle$ and $\langle x\,|\,m\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d,m,n)}(p)$ and $M_{(a,b,c,d,m,n)}(p)$ with $F_{(a,b,c,d,m,n)}(p)$ in (6.2.101) results OLCT of correlation operation as-

$$R_{ff+ff}(p) = (j2\pi b)\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} F_{(a,b,c,d,m,n)}(-p)\left\{F_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.102)}$$

From (6.2.68)

$$R_{fg+fm}(p) = (j2\pi b)\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} F_{(a,b,c,d,m,n)}(-p)\left\{G_{(a,b,c,d,m,n)}(p)+M_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.103)}$$

Replacing $\langle x\,|\,g\rangle$ and $\langle x\,|\,m\rangle$ with $\langle x\,|\,f\rangle$ and $G_{(a,b,c,d,m,n)}(p)$ and $M_{(a,b,c,d,m,n)}(p)$ with $F_{(a,b,c,d,m,n)}(p)$ in (6.2.103) results OLCT of correlation operation as-

$$R_{ff+ff}(p) = (j2\pi b)\exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} F_{(a,b,c,d,m,n)}(-p)\left\{F_{(a,b,c,d,m,n)}(p)+F_{(a,b,c,d,m,n)}(p)\right\} \quad \textbf{(6.2.104)}$$

Therefore, from (6.2.102) and (6.2.104), it has been proved that-

$$\left(f \overset{A}{\star}(f+f)\right)(x) = \left(f \overset{A}{\star} f + f \overset{A}{\star} f\right)(x) \quad \textbf{(6.2.105)}$$

This proves that the proposed auto-correlation theorem for OLCT satisfies the distributive law.

d) *Even function*:

If $F_{(a,b,c,d,m,n)}(p)$ is the OLCT of $f(x)$, then-

$$\left(f \overset{A}{\star} f\right)(x) \xleftrightarrow{OLCT} \sqrt{j2\pi b}\, \exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} F_{(a,b,c,d,m,n)}(-p)\,F_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.2.106)}$$

$$\left(f \overset{A}{\star} f\right)(-x) \xleftrightarrow{OLCT} \sqrt{j2\pi b}\, \exp j\left\{\frac{-dp^2+2p(dm-bn)-dm^2}{2b}\right\} F_{(a,b,c,d,m,n)}(p)\,F_{(a,b,c,d,m,n)}(-p) \quad \textbf{(6.2.107)}$$

and as per evenness property-

$$\left(f \overset{A}{\star} f\right)(x) = \left(f \overset{A}{\star} f\right)(-x) \quad \textbf{(6.2.108)}$$

Proof: From (6.2.83)

$$\left\langle p \mid R_{gf1} \right\rangle = \sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\} G_{(a,b,c,d,m,n)}(p)\, F_{(a,b,c,d,m,n)}(-p) \quad \textbf{(6.2.109)}$$

where, $\left\langle p \mid R_{gf1} \right\rangle$ represents the OLCT of the correlation operation $\left(g \overset{A}{\star} f\right)(-x)$. Replacing $\langle x \mid g \rangle$ with $\langle x \mid f \rangle$ and $G_{(a,b,c,d,m,n)}(p)$ with $F_{(a,b,c,d,m,n)}(p)$ in (6.2.109) results OLCT of correlation operation as-

$$\left\langle p \mid R_{ff1} \right\rangle = \sqrt{j2\pi b}.\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\} F_{(a,b,c,d,m,n)}(p) F_{(a,b,c,d,m,n)}(-p) \quad \textbf{(6.2.110)}$$

where, $R_{ff1}(p)$ represents the OLCT of $\left(f \overset{A}{\star} f\right)(-x)$.

Similarly, from (6.2.72)

$$\left\langle p \mid R_{fg} \right\rangle = \sqrt{j2\pi b}\,.\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\} F_{(a,b,c,d,m,n)}(-p) G_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.2.111)}$$

where, $\left\langle p \mid R_{fg} \right\rangle$ represents the LCT of the correlation operation $\left(f \overset{A}{\star} g\right)(x)$. Replacing $\langle x \mid g \rangle$ with $\langle x \mid f \rangle$ and $G_{(a,b,c,d,m,n)}(p)$ with $F_{(a,b,c,d,m,n)}(p)$ in (6.2.111) results OLCT of correlation operation as-

$$\left\langle p \mid R_{ff} \right\rangle = \sqrt{j2\pi b}\,.\exp j\left\{\frac{-dp^2+2p(dm-bn)}{2b}\right\} F_{(a,b,c,d,m,n)}(-p) F_{(a,b,c,d,m,n)}(p) \quad \textbf{(6.2.112)}$$

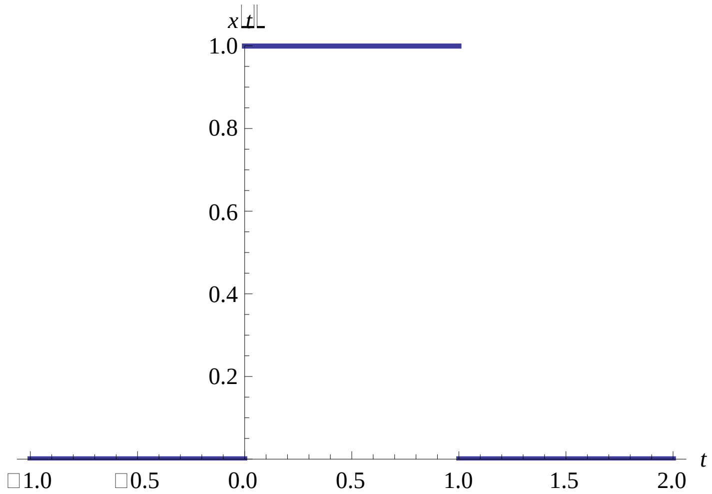
where, $R_{ff}(p)$ represents the OLCT of $\left(f \overset{A}{\star} f\right)(x)$. Therefore, from (6.2.110) and (6.2.112), it has been proved that-

$$\left(f \overset{A}{\star} f\right)(-x) = \left(f \overset{A}{\star} f\right)(x)$$

Therefore, it can be summarized that the proposed auto-correlation theorem for OLCT is an even function. Therefore, the proposed auto-correlation theorem for OLCT is satisfying commutative and distributive law whereas associativity is not being satisfied by this theorem.

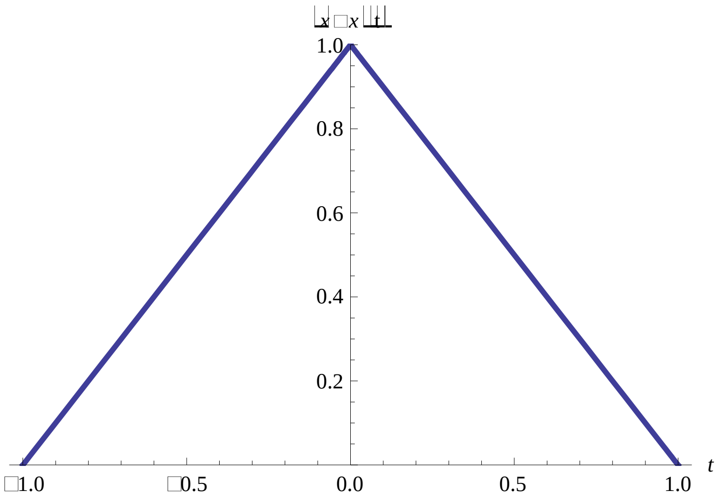
6.2.3 Comparative Analysis of Proposed Correlation Theorem with the existing theorems for OLCT

The correlation theorem in OLCT domain given by Xiang *et al.* **[115]** is compared with the proposed correlation theorem by using simulation. The correlation operation of a rectangular function $x(t)$ of unit amplitude is performed with it and as a result of correlation operation, triangular function is obtained of double duration from the rectangular function as shown in Figures-6.1 and 6.4 respectively. Then the OLCT of the triangular function is evaluated for different values of (a,b,c,d,m,n) with and without considering the effect of time-shifting and frequency-modulation as shown in Figures-6.2 and 6.5 respectively. Simultaneously, the OLCT of the triangular function is also evaluated for the same values of (a,b,c,d,m,n) to make a comparison. It has been shown in Figures-6.3 and 6.6 that how time-shifting and frequency-modulation variables help to approach the real (Re), imaginary (Im) and absolute (Abs) values of the OLCT of triangular function. Also the correlation operation defined by Xiang *et al.* **[115]** is compared with the OLCT of the proposed correlation operation for $(a,b,c,d,m,n)=(0.707,0.707,-0.707,0.707,0.12,0.14)$ and $(a,b,c,d,m,n)=(0.707,0.707,-0.707,0.707,0.12,0.14)$ as shown in Figures-6.3 and 6.6 respectively. From Figures-6.3 and 6.6, it has been shown that the real, imaginary and absolute components to the OLCT of the proposed correlation theorem resembles maximally to the different components of the OLCT of triangular function.



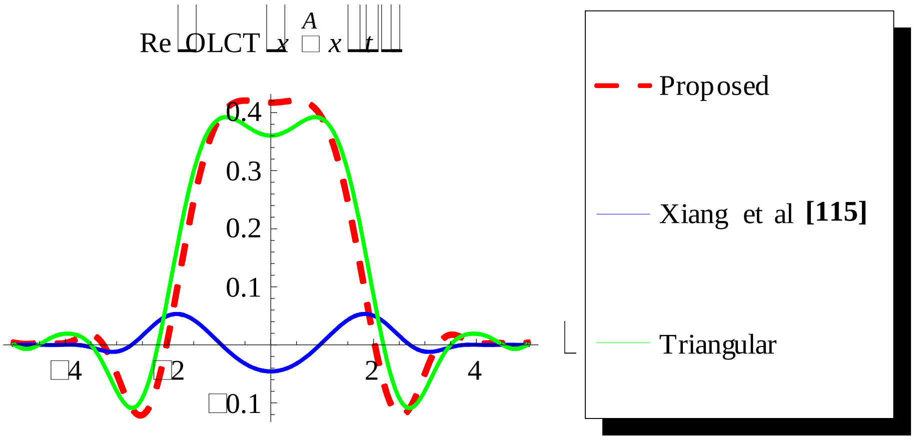
(a)

Figure-6.4: (a) Rectangular function $x(t)$ and (b) Correlated signal $\left(x \overset{A}{\star} x\right)(t)$ i.e. triangular function.

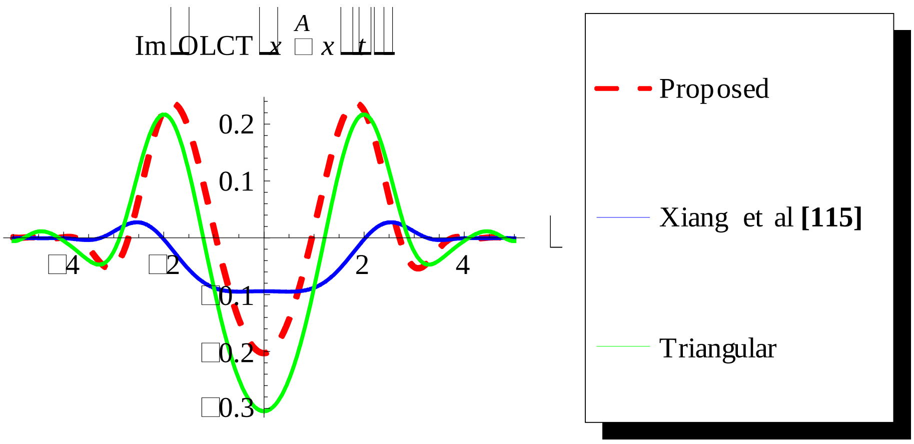


(b)

Figure-6.4 (*Continued*)

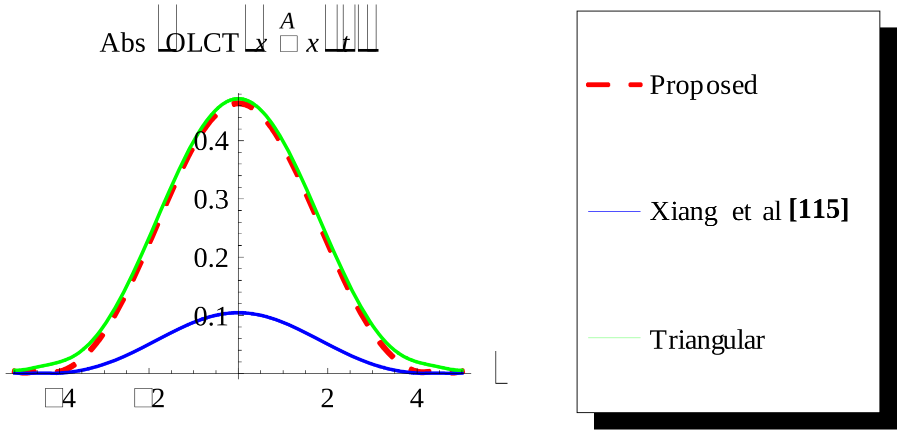


(a)



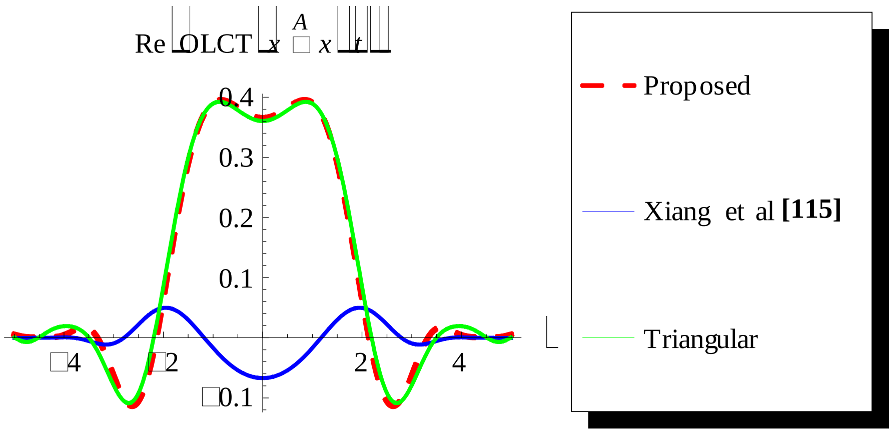
(b)

Figure-6.5 (a) Real value (b) Imaginary value (c) Absolute value: of OLCT of $\left(x \overset{A}{\star} x\right)(t)$ for triangular function, Xiang *et al.* [115] method, and the proposed method for $(a,b,c,d,m,n) = (0.707, 0.707, -0.707, 0.707, 0, 0)$.

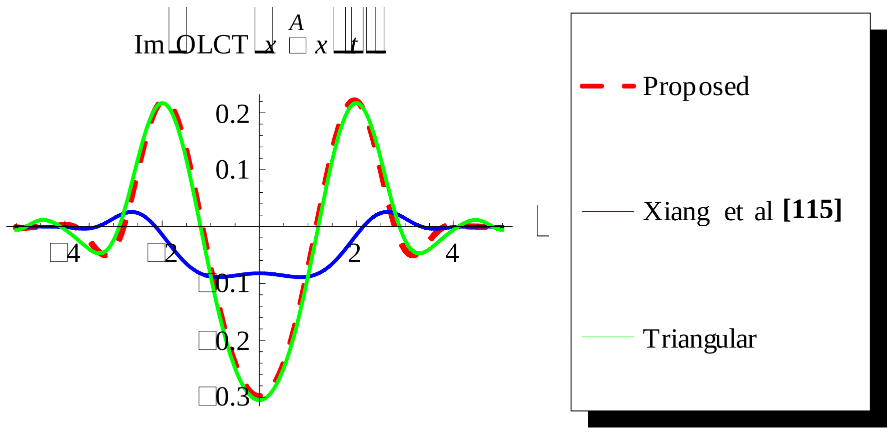


(c)

Figure-6.5 (*Continued*)

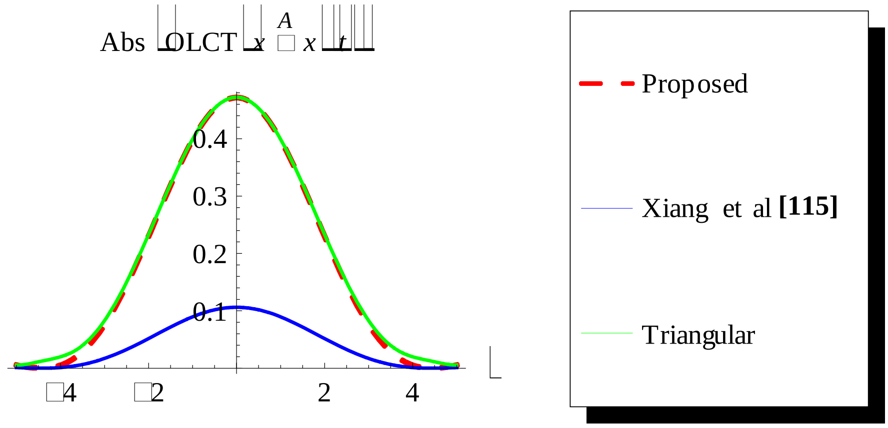


(a)



(b)

Figure-6.6: Effect of time-shifting and frequency-modulation variables on (a) Real value (b) Imaginary value and (c) Absolute value: of OLCT of $\left(x\overset{A}{\star}x\right)(t)$ by using proposed method for $(a,b,c,d,m,n)=(0.707,0.707,-0.707,0.707,-0.19,-0.19)$.



(c)

Figure-6.6 (*Continued*)

6.3 DISCUSSIONS

In this chapter, initially a brief introduction to OLCT/SAFT and its properties has been given. OLCT is different from FRFT/LCT as it allows shifting/translation, rotating and squeezing of a signal to fit within a fixed window as compared to only rotation in case of FRFT/LCT.

A modified expression for the convolution and correlation integral for OLCT has been introduced after successful derivation with quantum mechanical representation. This can be treated as convolution and correlation theorems for OLCT and enhances the support for OLCT for its consideration as an integral transform.

The proposed definition satisfies all the properties of classical convolution theorem for FT, FRFT and LCT, i.e. the commutative property, associative property and distributive property. As can be seen from the simulation results of Figures-6.2 and 6.5, the proposed weighted convolution theorem is giving results better than the convolution theorem given by Xiang *et al.* **[115]**. The results determined by the proposed theorem are closer in shape and of matching values to the OLCT of a triangular function. The results determined by the convolution and correlations expression of Xiang *et al.* **[115]** have more oscillations in both real and imaginary components as it is visible from the Figures-6.2 and 6.5 for different value of OLCT variables. These oscillations are significant and present due to the chirp signal included in the calculation of convolution and correlation integral by Xiang *et al.* **[115]**, which also contains variable of the transformed domain. The effect of time-shifting and frequency-modulation can be seen from Figures-6.3 and 6.6. By changing the value of time-shifting and frequency-modulation variable results are almost exact match to the OLCT of triangular function as compare to LCT.

CHAPTER 7

CONCLUSIONS AND FUTURE SCOPE OF WORK

Reasoning draws a conclusion, but does not make the conclusion certain, unless the mind discovers it by the path of experience.

- *Roger Bacon (1214-1294)*

7.1 CONCLUSION

This study derives the convolution and correlation theorems in the LCT domain and implement these theorems in the signal processing applications. The derivation is taken out by using the Dirac's representation theorem of quantum mechanics. Quantum version of LCT is direct and concise because various properties of LCT can be derived directly. Based upon the proposed convolution theorem, an application of multiplicative filter for LCT domain has been implemented and found to be better MSE as compared to fractional domain filtering and frequency domain filtering. Based upon the proposed methodology used to derive the convolution and product theorem, an improved correlation theorem has been derived for LCT and power spectrum density analysis of FM wave has been analyzed for different values of LCT variables. Finally, weighted convolution and correlation operations has been derived for the OLCT.

Before implementing the multiplicative filter based upon the proposed convolution theorem, an examination is done (both analytically and through simulation) for the behavior of different signal processing window functions in the time-frequency plane of the LCT domain. The closed-form analytical expression of the behavior of Dirichlet, Generalised Hamming and Bartlett window functions is established, utilizing various special mathematical functions in the LCT domain. It has been shown that the LCT of Dirichlet, Generalised Hamming and Bartlett window functions is directly dependent on the LCT variables (a,b,c,d), thus exhibiting the flexibility of various applications in signal processing.

It is also observed that for Dirichlet window function, for different values of LCT variables, the MSLL decreases from -11.91 dB to -13.14 dB, half main lobe width (HMLW) increase from 0.13 bins to 0.81 bins, -3 dB BW increase from 0.14 bins to 0.89 bins, -6 dB BW increase from 0.19 bins to 1.21 bins and SLFOR starts increasing from -6.30 dB/octave to -6.00 dB/octave. For Hamming window function, the MSLL decreases from -44.02 dB to -43.98 dB, HMLW increase from 0.44 bins to 1.92 bins, -3 dB BW increase from 0.20 bins to 1.30 bins, -6 dB BW increase from 0.29 bins to 1.81 bins and SLFOR starts decreasing from -5.70 dB/octave to -6.00 dB/octave. Similarly for triangular window function, the MSLL decreases from -26.94 dB to -26.41 dB, HMLW increase from 0.30 bins to 1.62 bins and SLFOR starts increasing from -12.03 dB/octave to -12.00 dB/octave.

Thus, it is revealed that there is a variation in the window function parameters with the variation in the LCT variables (a,b,c,d) and a best optimal solution can be obtained for the variety of practical applications such as, image compression etc. Efforts have also been made to choose the most convenient parameter adjustment to reduce the side lobe effect and to increase the intensity of the main lobe. Also the results discussed in the above technique can be beneficial to reduce the undesirable effects of the spectral leakage.

In literature, many existing definitions of the convolution and correlation theorems for LCT has been documented as illustrated in Chapter-2. To remove the various ambiguities of existing methods, a weighted convolution theorem along with weighted product theorem for LCT has been established. Subsequently, it has also been shown that the proposed definition of convolution theorem satisfies the basic properties which are being observed compulsorily by a convolution theorem. To validate the improvement of the proposed convolution theorem, a multiplicative filter has been implemented and a comparative analysis of LCT domain filtering has been performed with that of fractional domain filtering and frequency domain filtering. It has been observed that LCT takes the advantage of its three free variables as compare to one free variable of FRFT and no free variable of FT and gives minimum MSE for different values of SNR.

Based upon the proposed methodology to derive the proposed convolution theorem, a weighted cross-correlation and auto-correlation theorem for the LCT has been derived to remove the various discrepancies in the existing definitions. To validate the improvement of the proposed correlation theorem, an analysis of number of chirp functions in the context of hardware complexity has been done and it has been found that only nine chirp functions are required as compared to twelve for the theorem proposed in the literature. Further with the help of simulation, it has been shown that results obtained from the proposed theorems are closer to the required shape as compared to that of literature. The simulation results obtained from the correlation theorem derived in the literature are more oscillatory because of the presence of more chirp signals included in the calculation of correlation integral. Thereafter, the basic properties that must be observed by cross-correlation and auto-correlation theorems are discussed and derived in detail for the first time for LCT. Subsequently, the utility of proposed definition of weighted correlation theorem for the LCT is established successfully in power spectrum density analysis of FM signal and it has been found that same FM signal can be transmitted with less BW.

Finally, the proposed technique used to derive the convolution theorem for LCT is used to derive convolution and correlation theorem for OLCT. OLCT is a time-shifted and frequency-modulated version of the LCT. It is a six variable class of integral transform with one constraint, which plays an important role in many fields of signal processing. Further a comparison of the proposed convolution and correlation theorems is done with that of the existing definitions. It has been found that the proposed theorems are better and befitting proposition. By making use of time-shifting and frequency-modulation variables of OLCT, it has been shown that one can shift the time-frequency plot of a signal in horizontal (time-shifting) and vertical direction (frequency-modulation) and with the help of simulation, it has been shown that the proposed results exactly maps the desired results.

"The signal corrupted by many noise sources with different orientations can be recovered successfully using multiplicative filtering which is generated with the help of convolution and correlation theorems developed for OLCT."

7.2 FUTURE SCOPE OF WORK

"No one 'discovers' the future. The future is not a discovery. The future is not a destiny. The future is a decision based on a continuous effort starting from the past, motivated by the present which certainly leads success thereafter. In the long run, it is the cumulative effect that matters. One can do much. And one and one and one and one can move mountains". Initiated by this thought, all the reported work in this thesis leads to their respective extensions that will strengthen their impact and role in real life applications. The desired and expected extensions are termed here as the future scope of the reported work which can be outlined as –

(i) An efficient algorithm for canonical filter design, especially the algorithm to choose the optimal value of LCT variables (a,b,c,d) could be designed.

(ii) The design model of multiplicative filter based upon the proposed convolution theorem, could be simulated for a more powerful transform i.e. OLCT, which is a six-variable transform and has the flexibility of time-shifting and frequency-modulation.

(iii) The proposed methodology to derive the convolution and correlation theorems could be used to derive theorems that can be applicable for two-dimensional applications of LCT.

(iv) Further work to develop the discrete LCT that has additivity property, less constraints, easy to implement, and has good performance for practical application could be considered. Some of the existed discrete LCT's have no additivity property, some have too many constraints, and some are hard to implement, and some have worse performance for practical application.

(v) Since LCT is more flexible that FRFT because it has three free variables as compare to one free variable of FRFT. More work has to be done in the field of image processing such image compression, image enhancement, watermarking and encryption etc.

(vi) S transform as a time-frequency distribution is a generalization of the Short-Time Fourier transform (STFT), extending the continuous Wavelet Transform (WT) and overcoming some of its disadvantages. Based upon the S transform, Linear Canonical S Transform (LCST) has been introduced and it is an open area for research.

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