NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS USING HAAR WAVELETS

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Under

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CERTIFICATE

I hereby certify that the dissertation entitled " **NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS USING HAAR WAVELETS** *", which is being submitted by Ms. Anuradha Choudhary (Roll no. 301203002), in the partial fulfillment of the requirements for the award of degree of MASTER OF SCIENCE in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's authentic record of the work studied under the supervision of Dr. Vivek, Assistant Professor, SMCA, Thapar University, Patiala.*

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other university for the award of any degree by the author.

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ABSTRACT

The present dissertation entitled, ”NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS USING HAAR WAVELETS”, embodies a brief account of investigations carried out by various authors on, Haar wavelets under the supervision of Dr.Vivek, Assistant Professor, School of Mathematics and Computer Applications, Thapar University, Patiala. The work presented in this dissertation has been divided into four chapters. The aim of this work is to study some results on Haar wavelets and to solve initial value problem and boundary value problem. In the very first chapter there is brief introduction about the differential equations and Haar wavelets. Differential equations arise in the mathematical modeling of many physical, chemical and biological phenomena and many more areas of science and engineering. Asymptotic and numerical are two principle approaches for solving differential equations. In the Present study, the numerical techniques have been used to approximate the solution. There are many numerical methods to find an approximate solution such as finite difference methods (FDM) ,finite element methods(FEM), finite volume methods(FVM), boundary element methods(BEM), etc. The present work focuses o Haar-wavelet technique for solving differential equations. First chapter, the basic concepts related to Haar wavelets like wavelets, Haar Wavelets, Haar matrix, Haar transform etc. have been elaborated.

In the second chapter a uniform Haar wavelets methods has been presented for solving initial value problems. Firstly the proposed method has been discussed in detail. The general expression of Haar wavelets basis functions have been presented with their properties. Then the algorithm for solving the initial value problem has been presented and explained in details with help of example wherever needed. In the last, the error estimates have been described.

The third chapter focuses on the non-uniform Haar wavelet method. The non-uniform Haar wavelet method is preferred in situations where non-uniform or sharp transitions occur. One such type of problems are singularly perturbed problems where very sharp boundary layer arise as the singular perturbation parameter tends to zero. In this chapter non-uniform Haar wavelet method has been proposed for solving singularly perturbed problems. Because sharp change occurs in the solution and hence special meshes have been proposed to capture these sharp variation. In the last algorithm for non-uniform Haar wavelet method has been discussed in detail over then non-uniform meshes.

The last chapter is concentrated on the application of the Haar wavlet algorithms presented in second and third chapter. The initial value problem has been solved using uniform Haar Wavelet method. The singularly perturbed boundary value problem has been solved using non-uniform Haar wavelet method. The computed solution has been compared with the exact solution. To conclude the Haar wavelet method produces good results even for very small number of collocation points. The method has added advantage over other methods because of its clear and simple structure and less computation cost.

Contents

Chapter 1

INTRODUCTION

1.1 Differential equations

A differential equation relate some function of one or more variables with its derivatives. Differential equations arise in the mathematical modelling of many physical, chemical and biological phenomena and many more areas of science and engineering such as fluid dynamics, electromagnetics, material science, astrophysics, economy etc. We define below some concepts related to differential equations.

Order: The order of a differential equation is the order of the highest order derivatives occuring in the equation.

Degree: The degree of a differential equation is the degree of the highest order derivatives occuring in the equation, after the equation is made free of radicals and fractions in its derivatives.

Solutions of a differential equation: A solution of a differential equation is a function or a relation between the dependent and independent variables that satisfies the differential equation.

$$y' = f(x,y)$$

The general solution of any m^{th} order differential equation

$$\frac{d^m y}{dx^m} + a_1(x)\frac{d^{m-1} y}{dx^{m-1}} + a_2(x)\frac{d^{m-2} y}{dx^{m-2}} + \cdots + a_m(x)y = r(x); \tag{1.1.1}$$

Contains m arbitrary constants. A particular solution is obtained by assigning particular values to these arbitrary constants. Hence, m conditions are needed to obtain a particular solution of (1.1.1). If all these m conditions are prescribed at a single point, say $x = x_0$, then the differential equation together with the conditions is called an initial value problem and the conditions are called the initial conditions. The point x_0 is called the initial point.

If these m conditions are prescribed at more than one point, say $x = a$ and $x = b$, then the differential equation together with these conditions is called a boundary value problem and the conditions are called the boundary conditions.

Types of differential equations:

Differential equations can be categorized in two categories:

(1) **Ordinary differential equations**

(2) **Partial differential equations**

Ordinary Differential Equation: The differential equations which involve only one-independent variable are called ordinary differential equations. A linear ordinary differential equation of order n, can written as

$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots\cdots + a_{n-1}(x)y'(x) + a_n(x)y(x) = r(x)$,

where y is the dependent variable and x is the independent variable and $a_0(x) \neq 0$.

If $r(x) = 0$, then it is called a **homogeneous differential equation**, otherwise it is called non-homogeneous equation.

Partial Differential Equation: A differential equation involving partial derivatives of a function w.r.t. two or more independent variables is known as partial differential equation. If a partial differential equation contains n^{th} and lower order derivatives, it is said to be of n^{th} ordered partial differential equation. The degree of such equation is the greatest exponent of the highest order derivative terms. Further, such equation will be called linear if it is of first degree in the dependent variables and its partial derivatives. A differential equation which is not linear is called a non-linear differential equation [31].

Classification of first order partial differential equations:

(1) **Linear equation**

(2) **Semi-linear equation**

(3) **Quasi-linear equation**

(4) **Non-linear equation**

A first order equation $f(x,y,z,p,q) = 0$ is said to be **linear**, if it can be written as $P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$, where P, Q and R are functions of x and y only and therefore, the partial differential equation is linear in $z, p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

A first order partial differential equation $f(x,y,z,p,q) = 0$ is known as **semi-linear partial differential equation**, if it is linear in p and q. A semi-linear first order partial differential equations can be written as

$P(x,y)p + Q(x,y)q = R(x,y,z)$, where P and Q are functions of x and y only and R is non-linear in z.

A first order partial differential equation $f(x,y,z,p,q) = 0$ is known as **quasi-linear partial differential equation**, if it is linear in p and q and it can be written in the form

$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$, where at least one of P or Q is non-linear in z.

A first order partial differential equation $f(x,y,z,p,q) = 0$ which does not come under the above three types is known as non-linear or fully non-linear partial differential equation.

1.2 Numerical vs Analytical Methods

Suppose we have a mathematical model or differential equation and we want to understand its behaviour, that is, we want to find a solution of the model or differential equation. The best is when we can find out the exact solution using calculus, trigonometry and other techniques. Now using the exact solution we can know how the model will behave under the given circumstances. The techniques used for calculating the exact solution are known as analytic methods because we used the analysis to figure it out. The exact solution is also referred to as a closed form solution or analytical solution.

But this tends to work only for simple models, that is, differential equations with simple co-efficients, but for higher order or non-linear differential equations with complex co-efficients, it becomes very difficult to find exact solution, therefore, we need numerical methods for solving the equations.

Finite difference methods are one of the most widely used numerical schemes to solve differential equations. In finite difference methods, derivatives appearing in the differential equations are replaced by approximations obtained by Taylor series expansions at the grid points. Thus a system of linear equations is obtained which can be solved using direct or iterative methods to give the solution value at the grid points and hence the solution is obtained at grid points. Some of the finite difference methods include forward difference method, backward difference methods, central difference method and upwind methods etc. The general concepts of finite difference methods in details such as stability, convergence conditions, grid independence etc have been presented in literature for solving differential equations.

The finite volume methods are increasingly popular numerical methods for the approximation of solution of partial differential equations. Finite volume methods are sub domain methods with piecewise definition of the field variable in the neighborhood of chosen control volumes. The total solution domain is divided into many small control volumes which are usually rectangular in shape.

Nodal points are used within these control volumes for interpolating the field variables. Usually, a single node at the center of the control volume is used for each control volume. [**?**].

Finite element methods are very powerful for approximating the solution of partial differential equations. Basic idea of finite element methods is to find the solution of a problem by reducing it into variational form. We will be able to find an approximate solution rather than the exact solution. These methods were first proposed in the ninteen fifties. Thereafter, the potentialities of these methods for approximating the solution of different types of applied science and engineering problems were recognized. These methods are widely used for approximating solution of complex engineering problems.

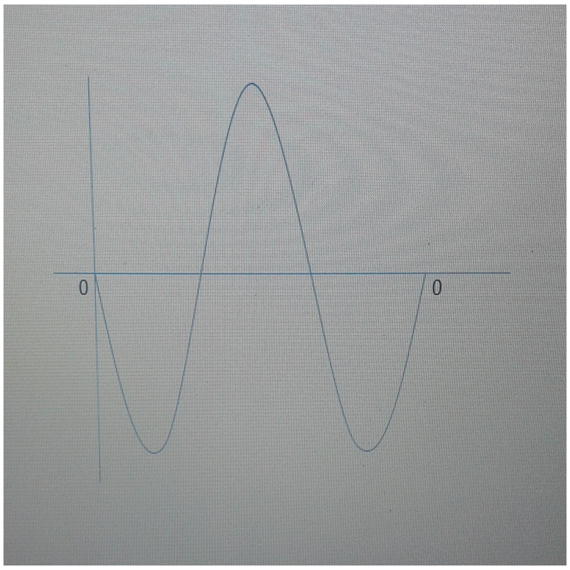
1.3 Haar Wavelets

In this section, we will present some basic definitions and concepts and properties required in the chapters to follow. Haar wavelet method is used for solving both the ordinary differential equations and partial differential equations. This method is prefered for solving the wave like equations with initial and boundary conditions known. It is to convert the differential equation into a group of algebraic equations, which involves a finite number of variables.

In the present study a simple and straight forward numerical technique based on haar wavelets is proposed for solving the differential equations. The main advantage of the haar wavelet method is its simplicity and small computation cost. It is a convenient method for solving variety of initial and boundary value problems, since the initial and boundary conditions in the solution are taken automatically, hence the present method is very fast, simple, reliable, less computationally cost, flexible and convenient alternative method.

1.4 Wavelets:

Wavelets are oscillatory functions starting from zero, increases and decreases back to zero. Wavelets have become a useful tool for having applications in the area of engineering and science such as



edge extraction, image coding, statistical estimation, weather forcasting and numerical simulation of differential equations. There are many families of wavelets such as haar, daubechies, morlet, gaussian, meyers, complex gaussian, coiflets, symlets etc [21]. But the most simple are the haar wavelets. Haar wavelets are very effective in solving the ordinary differential equations and partial differential equations due to its simplicity.

Consider the example of a string of numbers: (2,2,2,2) and we want to transmit this over a network. We would like to do this in the fastest possible way, which implies that we want to send the least amount of data possible. One of the simple options is to just send all the four numbers, *i.e.*, send the first $'2'$, then the second $'2'$, then the third, and lastly the fourth [18]. Here we are implicitly choosing the following basis:

$< 1,0,0,0 >$

$< 0,1,0,0 >$

$< 0,0,1,0 >$

$< 0,0,0,1 >$

For the second option, we choose a basis that represents our data efficiently. Since the data is uniform, in fact it is just a constant signal of 2. We would like to exploit this uniformity. If we choose the basis vector $< 1,1,1,1 >$, we can represent our data by just one number. We would only have to send the number 2 over the network, and our entire data string could be reconstructed by just multiplying with the basis vector $< 1,1,1,1 >$. But still three more basis vectors are needed to complete the basis since the space in our example is 4 dimensional. All basis vectors have to be orthogonal (or perpendicular). This means that if we take the dot (or scalar) product of any two basis vectors, the result should be zero. So we want to find a vector that is orthogonal to $< 1,1,1,1 >$. One such vector is $< 1,1,-1,-1 >$. If we take the dot product of these two vectors, the result is indeed zero. Graphically these basis vectors look like waves, hence the name wavelets. Now, we have two basis vectors, we need two more. Haar constructed the remaining basis vectors by a pro-

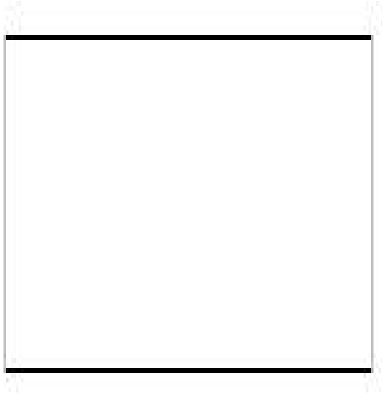


Figure 1.1: $< 1,1,1,1 >$

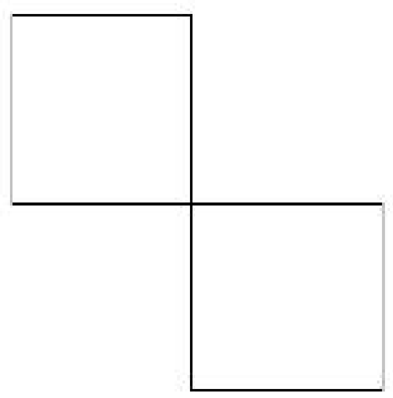


Figure 1.2: $< 1, 1, -1, -1 >$

cess of dilation and shifting. Therefore, the remaining basis vectors were constructed by squeezing and shifting. If we squeeze the vector $< 1, 1, -1, -1 >$, we get $< 1, -1, 0, 0 >$. The 1, 1 pair gets squeezed in to a single 1, and similarly the -1, -1 pair becomes a single -1. Next, we perform a shift on the resultant basis vector and get: $< 0, 0, 1, -1 >$ which is fourth basis vector. Graphically, these two vectors look like this in fig.(1.3) and fig(1.4): Thus, we have a complete basis for the four dimensional space, comprised of the basis vectors as wavelets.

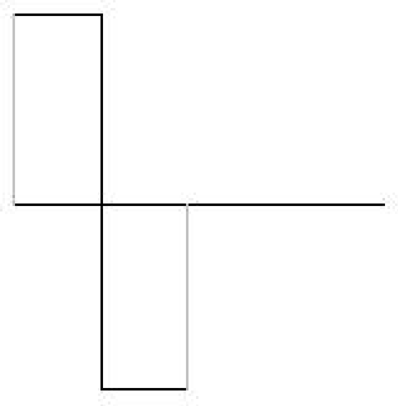


Figure 1.3: $< 1, -1, 0, 0 >$

1.5 Haar wavelets

Haar wavelets is a sequence of rescaled square shaped functions which together form a wavelet family or basis. It is the simplest type of wavelets. It was first proposed by Alfred Haar [29] in 1910. Haar wavelets are not continous, therefore, it is not differentiable. Haar wavelets are the step functions on the real line that can take only 1, -1 and 0 value. Because of the discontinuity, of haar

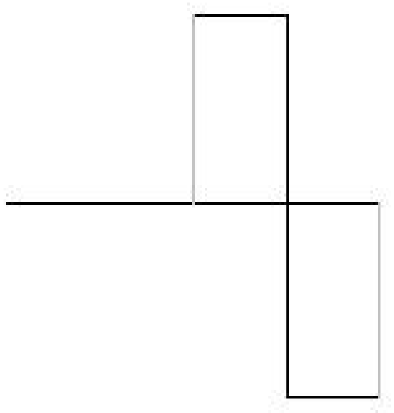
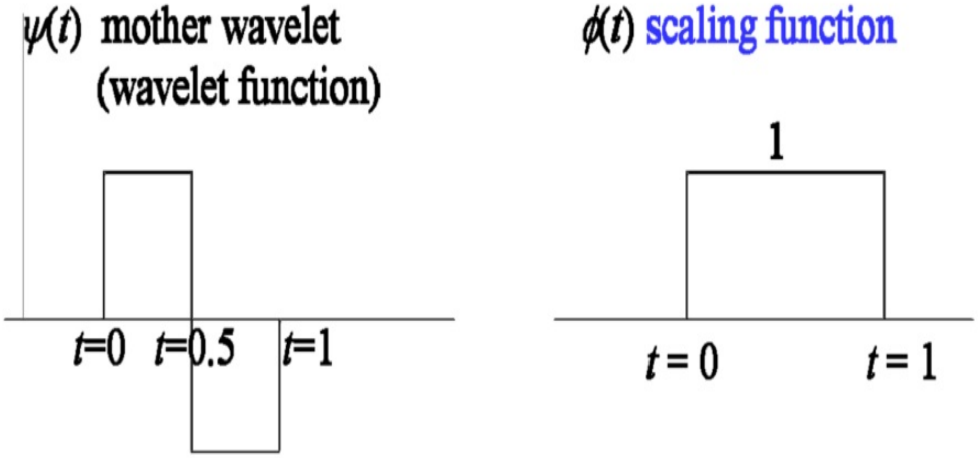


Figure 1.4: $< 0, 0, 1, -1 >$

wavelets are widely used for the analysis of signals with sudden transitions. The simplest wavelet analysis is based on haar scaling function.

The mother wavelet function $\psi(t)$ can be defined as



$$\psi(t) = \begin{cases} 1, 0 \leq t \leq \frac{1}{2} \\ -1, \frac{1}{2} \leq t \leq 0 \\ 0, otherwise \end{cases} \tag{1.5.1}$$

Rest of the Haar wavelet functions can be defined as $\psi(t) = \phi(2t) - \phi(2t-1)$ where the scaling function $\phi(t)$ is defined as

$$\phi(t) = \begin{cases} 1, 0 \leq t < 1 \\ 0, otherwise \end{cases} \tag{1.5.2}$$

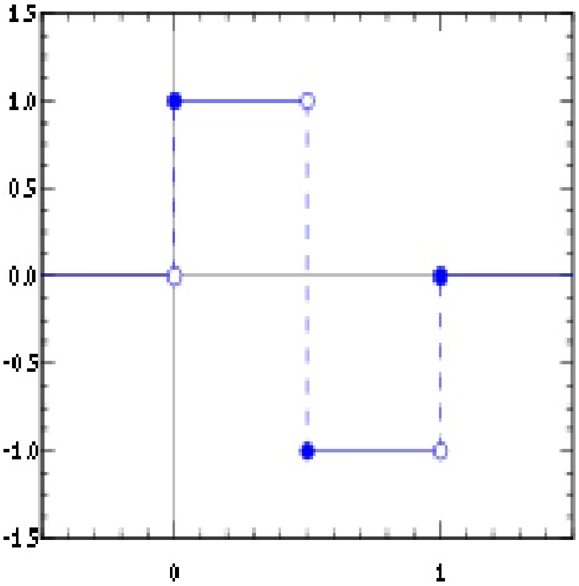


Figure 1.5: Haar wavelet

The haar system denotes the set of haar wavelets, which are defined as:.
$t \mapsto \psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k);\ n \in N,\ 0 \leq k < 2^0 n$
This forms a complete orthogonal basis for the functions on the unit interval.

1.6 Haar Functions

Haar functions have been used from 1910 when they were first introduced by the Hungarian mathematician Alfred Haar [29]. Haar functions are group of square waves with magnitude ± 1 in some intervals and zero elsewhere else. It is an odd rectangular pulse pair and it is the simplest and oldest orthonormal wavelet with compact support. Haar functions are widely used as basis functions for approximating the solution in many applications such as image coding, edge extraction, binary logic design etc. Haar functions possesses ability to detect singularities, irregular structure, transient phenomena, etc exhibited by the analyzed equations.

Properties:

(1) Any continuous real function can be written as a linear combination of $\phi(t), \phi(2t), \phi(4t), ..., \phi(2^k t), ...$ and their shifted functions. This extends the function space where any function can be approximated by continuous functions.

(2) Any continuous real function can be written as a linear combination of the piecewise constant functions $\psi(t), \psi(2t)$, $\psi(4t), ..., \psi(2^k t), ...$ and their shifted functions.

(3) Haar functions possesses orthogonality in the form of

$$\int_{-\infty}^{\infty} 2^{(mm_1)} \psi(2^m t - n) \psi(2^{m_1} t - n_1)\, dt = \delta_{m,m_1} \delta_{n,n_1}. \tag{1.6.1}$$

δ_i, j represents the Kronecker delta.

(4) Scaling functions or wavelet functions with different scale have a functional relationship:

$$\phi(t) = \phi(2t) + \phi(2t-1) \tag{1.6.2}$$

$$\psi(t) = \phi(2t) - \phi(2t-1) \tag{1.6.3}$$

1.6.1 Haar Matrix

The discrete Haar functions form a basis of the Haar matrix H.

A $2n$ - order Haar matrix is defined as:

$$H_{2N} = \begin{bmatrix} H_N \otimes [1,1] \\ I_N \otimes [1,-1] \end{bmatrix}.$$

where $N = 2^k$, $H_0 = 1$ and I_N is the identity matrix,

$$I_N = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

and $\otimes$ the Kronecker product.

The Kronecker product of $A \otimes B$, where A is an $m \times n$ matrix and B is a $p \times q$ matrix, is expressed as

$$A \otimes B = \begin{bmatrix} a_{11}B & ... & a_{1n}B \\ . & \cdots & . \\ . & \cdots & . \\ a_{m1}B & ... & a_{mn}B \end{bmatrix}$$

The Haar matrix H_n of order $N = 2^k$, associated with Haar functions is defined as

$$H_N = \begin{bmatrix} \phi \\ h_{00} \\ h_{10} \\ h_{11} \\ . \\ . \\ h_{k-1,0} \\ h_{k-1,1} \\ . \\ . \\ h_{k-1,2^{k-1}-1} \end{bmatrix}$$

where ϕ=[1 1 1....1] is a $1 \times N$ matrix, and $h_{p,q}[n]$ is a Haar function.

Using the definitions mentioned above, we can construct Haar matrix as follows:

$$H_2 = \begin{bmatrix} H_1 \otimes [1,1] \\ I_1 \otimes [1,-1] \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} H_2 \otimes [1,1] \\ I_2 \otimes [1,-1] \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$H_8 = \begin{bmatrix} H_4 \otimes [1,1] \\ I_4 \otimes [1,-1] \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The haar matrix is real and orthogonal. *i.e.*

1. $H = H^*$

2. $H^{-1} = H^t H$ i.e. $H^T H = I$

From the definition of the Haar matrix H, we can observe that, unlike the Fourier transform, H matrix has only real elements (*i.e.*, 1, -1 or 0) and is non-symmetric.

The first row of H matrix measures the average value, and the second row of H matrix measures a low frequency component of the input vector. The next two rows are sensitive to the first and second half of the input vector respectively, which corresponds to moderate frequency components. The remaining four rows are sensitive to the four section of the input vector, which corresponds to high frequency components. In the figure, the Haar function at each row of H matrix. Notice that the width and location of the Haar function is changed at every next stage. The Haar function with narrower width is responsible for analyzing the higher frequency content of the input signal.

1.6.2 Haar Transform

The haar transform is the simplest of the wavelet transforms. This transform cross-multiplies a function against the haar wavelet with various shifts and stretches, like the fourier transform cross-multiplies a function against a sine wave with two phases and many stretches.

The haar transform is derived from the haar matrix. An example of a 4 × 4 Haar transformation matrix is shown below.

$$H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

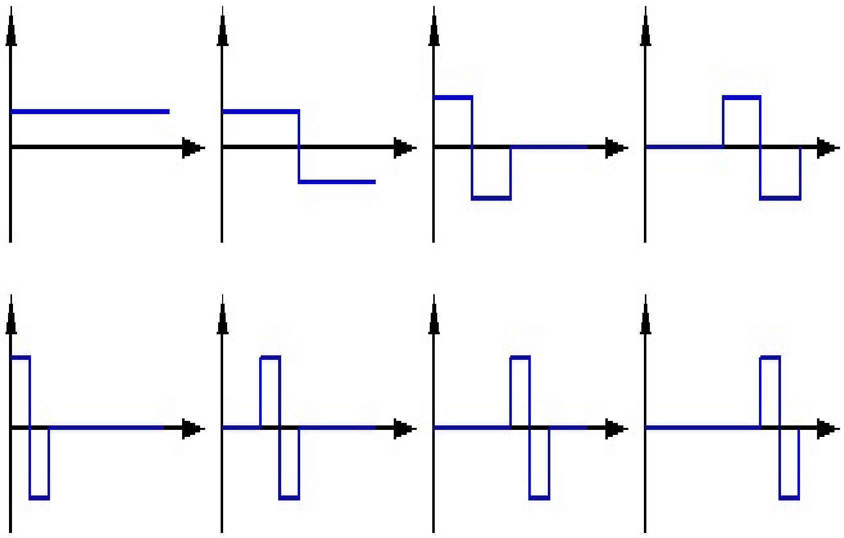


Figure 1.6: Haar functions for composing 8-point Haar transform matrix

The Haar transform can be thought of as a sampling process in which rows of the transformation matrix act as samples of finer and finer resolution.

The Haar transform $HT^n(f)$ of an N-input function $X^n(f)$ is the 2^n element vector [13].

$HT^n(f) = H^n X^n(f)$.

The Haar transform cross multiplies a function with Haar matrix that contains Haar functions with different width at different location.

For example:

$$H_8 \begin{bmatrix} 1.2 \\ 1.2 \\ 1.8 \\ 0.8 \\ 2 \\ 2 \\ 1.9 \\ 2.1 \end{bmatrix} = \begin{bmatrix} 13 \\ -3 \\ -0.2 \\ 0 \\ 0 \\ 1 \\ 0 \\ -0.2 \end{bmatrix}$$

The Haar transform is performed in levels. At each level, the Haar transform decomposes a discrete signal into two components with half of its length: an approximation (or trend) and a detail (or fluctuation) component. The first level of approximation $a^1 = (a_1, a_2, ..., a_{N/2})$ is defined as

$a_m = \frac{X_{2m-1}+X_{2m}}{\sqrt{2}}$

for $m = 1,2,3,...,N/2$, where X is the input signal. The multiplication of $\sqrt{2}$ ensures that the Haar transform preserves the energy of the signal. The values of a^1 represents the average of successive pairs of X value.

The first level detail $d^1 = (d_1, d_2, ..., d_{N/2})$ is defined as

$d_m = \frac{X_{2m-1} - X_{2m}}{\sqrt{2}}$

for $m = 1, 2, 3, ..., N/2$. The values of d^1 represents the difference of successive pairs of X value. The first level Haar transform is denoted as H_1. The inverse of this transformation can be achieved by

$X = \frac{a_1+d_1}{\sqrt{2}}, \frac{a_1-d_1}{\sqrt{2}}, ... \frac{a_{N/2}+d_{N/2}}{\sqrt{2}}, \frac{a_{N/2}-d_{N/2}}{\sqrt{2}}$

The successive level of Haar transform, the approximation and detail component are calculated in the same way, except that these two components are calculated from the previous approximation component only.

An example: X=(4,6,10,12,8,6,5,5), the first level approximation and detail components are

$$
\begin{aligned}
a^1 &= (\frac{10}{\sqrt{2}}, \frac{22}{\sqrt{2}}, \frac{14}{\sqrt{2}}, \frac{10}{\sqrt{2}}), \\
d^1 &= (\frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0), \\
a^2 &= (16, 12), \\
d^2 &= (-6, 2), \\
a^3 &= \frac{28}{\sqrt{2}}, \\
d^3 &= \frac{4}{\sqrt{2}},
\end{aligned}
$$

Chapter 2

HAAR WAVELET ALGORITHM

2.1 Introduction

For dynamical systems which involve abrupt variations or in case of functions having short interval of existence, the global support of orthogonal functions like block pulse, Laguerre, Legendre, Chebyshev and Fourier functions on the whole interval $A \leq t \leq B$ are a drawback. Chen and Hsiao fill the gap by developing Haar based algorithm for solving differential equations in 1997 [3]. In numerical analysis, this technique is used to reduce differential equations to algebraic one in linear matrix system.

The basic idea relies on the integral of a basic vector $\phi(t)$ such that

$$\int_0^x \phi(t)dt \cong P\phi(x), \tag{2.1.1}$$

where $\phi(t) = [\phi_{(0)}(t), \phi_{(1)}(t), \phi_{(2)}(t), \phi_{(m-1)}(t)]^T$ in which elements $\phi_{(0)}(t)$, $i = 0, 1, 2, ..., m-1$ are the basic orthogonal functions defined on $[A, B]$ and the matrix P is uniquely defined.

Wavelet analysis allows representing a function or signal in terms of a set of orthonormal basis functions called wavelets, which are localized both in time and scale. From a continuous function $\psi(x)$, called mother wavelet, wavelet family is formed by translation and dilation of $\psi(x) = 2^{j/2}\psi(2^j x - k)$ where j, k are non-negative integers. Selecting a suitable $\psi(x)$, various wavelet families are obtained. Wavelets introduced by Ingrid Daubechies in 1998 are quite frequently used for solving differential problems, due to differentiability and having minimum size support of these wavelets.

The first attempt to apply the Haar wavelets for solving differential and integral equations was made in 1990s [13, 20]. Based on Haar function, a rectangular pulse pair, introduced by Alfred Haar in (1910), Chen and Hsiao(1997), applied successfully applied the Haar wavelets for solving differential equations. It is the simplest of the orthonormal wavelets with compact support. Haar wavelets are not continuous and their derivatives do not exist at the point of discontinuity. Thereby

direct application of Haar wavelets is not possible in solving differential equations. Two possibilities are generally worked out to have the applications of Haar wavelets:
(i) Regularization of piecewise constant Haar function by interpolation splines as introduced by Cattani(2001). But this complicates the solution process and simplicity of Haar functions is lost.
(ii) Chen and Hsiao(1997) have given a concept in which function (required solution) corresponding to highest derivative of differential equation is expanded into Haar series. The other derivatives are obtained through Haar series. The simplicity of Haar wavelet is preserved in this technique. Since then the solutions of dynamical systems by Haar wavelet took tremendous growth.

2.2 Multiresolution Analysis (MRA)

Family of Haar wavelets utilizes the concept of MRA.
The increasing sequence $\{V_j\}_{j\in Z}$ of subsets of $L^2(R)$ with scaling function ϕ is called MRA if it satisfies the following conditions

$$\begin{aligned} &(a)\ \ \bigcap_{-\infty}^{\infty} V_j = \{0\},\ \ \bigsqcup_{-\infty}^{\infty} V_j = L^2(R) \\ &(b)\ \ f(x) \in V_0 \ \ iff \ \ f(2^j x) \in V_j \ \ \forall \ \ j \in Z \\ &(c)\ \ \{\phi(x-k),\ k \in Z\} \text{ is an orthonormal basis for } V_0 \end{aligned} \tag{2.2.1}$$

By above definition of MRA, the sequence

$$\{\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)\}_{-\infty}^{\infty} \tag{2.2.2}$$

forms an orthogonal basis for V_j.
For mother wavelet $\psi(x)$

$$\psi(x) = \sum_{k=0}^{N-1} a_k \phi(2x-k)$$

2.3 Haar Wavelets And Operational Matrices

A Haar function, called Haar scaling, is a function with magnitude unity in the interval $[0,1]$. Let $h_1(x) = 1$ for $0 \le x \le 1$. Take the second generated curve $h_2(x)$ as a square pulse obtained from $h_1(x)$ after compression of $[0,1]$ into two halves $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. The curve $h_2(x)$ is called Haar wavelet. All the other subsequent curves are generated from $h_2(x)$ with the two operations of translation and dilation.

$$h_1(x) = \begin{cases} 1, \; for\, x \in [0,1] \\ 0, \; elsewhere \end{cases}$$

In nut shell,

$$h_i(x) = \begin{cases} 1, \; for\, x \in [\alpha_1(i), \alpha_2(i)] \\ -1, \; for\, x \in [\alpha_2(i), \alpha_3(i)] \\ 0, \; elsewhere \end{cases} \tag{2.3.1}$$

where $i = 2^j + k + 1, \;\; j \geq 0, \; 0 \leq k \leq 2^j - 1$

Here $\alpha_1 = \frac{k}{m}, \; \alpha_2 = \frac{k+0.5}{m} \; and \; \alpha_3 = \frac{k+1}{m}, \; m = 2^j, \; j = 0,1,2,...,J$. J is the maximum level of resolution. $k = 0,1,2,....,m-1$, is the translation parameter.

The index $i = m + k + 1$. Maximum of i is $M = 2m = 2^{J+1}$.

The collocation points $x_l = \frac{l-0.5}{2m}, \, l = 1,2,3,....2m$ are obtained by discretizing Haar function $h_i(x)$ by dividing the interval [0,1] into $2m$ parts of equal length $\Delta t = \frac{1}{2m}$ to get coefficient matrix H of order $2m \times 2m$.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Notice that Haar wavelets are orthogonal, *i.e.*

$$\int_0^1 h_i(x) h_1(x) dx == \begin{cases} \frac{1}{m}, \; for\, i = l \\ 0, \; for\, i \neq l \end{cases}$$

The operational matrix P which is a $2m$ square matrix is defined by

$$P_{1,i} = \int_0^x h_i(t) dt \tag{2.3.2}$$

Often, we need the integrals

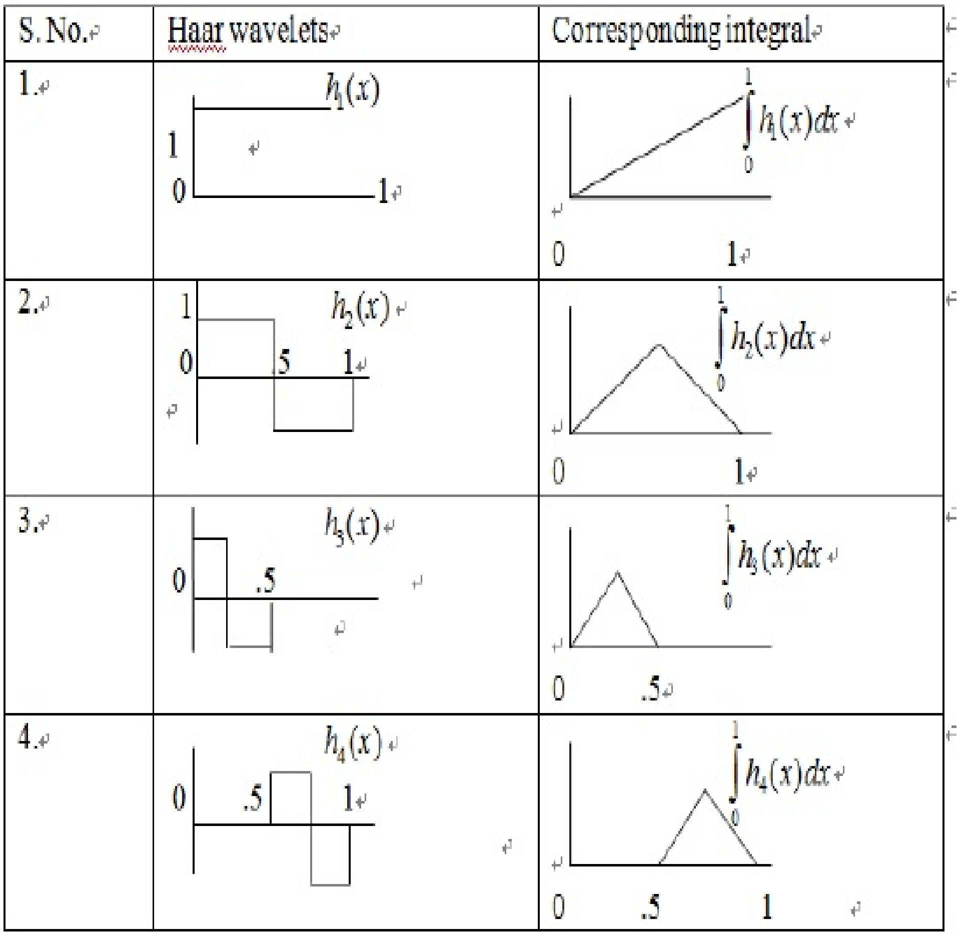
$$P(x) = \underbrace{\int_A^x \int_A^x ... \int_A^x}_{\beta} h_i(t) dt^{\beta} = \frac{1}{(\beta-1)!} \int_A^x (x-t)^{\beta-1} h_i(t) dt \tag{2.3.3}$$

where $\beta = 2,3,....,n$ and $i = 1,2,...2m$.

The case $\beta = 1$, corresponds to function $P_{1,i}(x)$. Taking into account (2.3.1), these integrals can be calculated analytically and we obtain

$$P_{\beta,i}(x) = \begin{cases} 0, \quad for\, x < \alpha_1(i) \\ \frac{1}{\beta!}[x-\alpha_1(i)]^{\beta}, \quad for\, x \in [\alpha_1(i), \alpha_2(i)] \\ \frac{1}{\beta!}[x-\alpha_1(i)]^{\beta} - 2[x-\alpha_2(i)]^{\beta}, \quad for\, x \in [\alpha_2(i), \alpha_3(i)] \\ \frac{1}{\beta!}[x-\alpha_1(i)]^{\beta} - 2[x-\alpha_2(i)]^{\beta} + [x-\alpha_3(i)]^{\beta}, \quad for\, x > \alpha_3(i) \end{cases} \tag{2.3.4}$$

These formulas hold for $i > 1$.
In case $i = 1$ we have $\alpha_1 = A,\ \alpha_2 = \alpha_3 = B$ and
$P_{\beta,1}(x) = \frac{1}{\beta!}[x-A]^{\beta}$.
Here $A = 0, B = 1$.
The resulting $P_{1,i}(x)$ is a triangular shape, whereas $P_{2,i}(x)$ is the parabolic one.
The table provides below gives the graph of first four Haar wavelets and their corresponding integrals.



S. No.	Haar wavelets	Corresponding integral
1.	$h_1(x)$ 1 0 1	$\int_0^1 h_1(x)dx$ 0 1
2.	1 $h_2(x)$ 0 .5 1	$\int_0^1 h_2(x)dx$ 0 1
3.	$h_3(x)$ 0 .5	$\int_0^1 h_3(x)dx$ 0 .5
4.	$h_4(x)$ 0 .5 1	$\int_0^1 h_4(x)dx$ 0 .5 1

The pattern of design of the matrices P is similar to Haar wavelet and is based on the algorithm.

Below we develop an algorithm based on (2.3.4) which would reduce computational and procedural complexity in comparison to what has been developed in Chen and Hsiao (1997) [3].

Algorithm:

(2.3.4) can be expressed in notational form as:

$$\begin{bmatrix} P_{\beta,1} \\ P_{\beta,2} \\ . \\ . \\ P_{\beta,2M} \end{bmatrix}$$

wherein

$$I.\ P_{\beta,1}(x) = \frac{1}{\beta!}x^{\beta} \quad for x \in [0,1] \tag{2.3.5}$$

$$II.\ P_{\beta,i}(x) = \begin{cases} \frac{1}{\beta!}x^{\beta}, & \text{for } x \in [\alpha_1, \alpha_2] \\ \frac{1}{\beta!}(\frac{2l-1}{4M} + \frac{1}{2M})^{\beta} - 2(\frac{2l-1}{4M})^{\beta}, & \text{for } x \in [\alpha_2, \alpha_3] \end{cases} \tag{2.3.6}$$

$M = max\{m\}$ and $l = 1,2,3,..., \frac{M}{m}$, $2 \leq i \leq 2M$.

Explanation of Algorithm

Let $x - \alpha_2 = \frac{2l-1}{4M}$.

We find $x = \frac{2l-1}{4M} + \alpha_2$.

Now $x - \alpha_1 = \frac{2l-1}{4M} + \alpha_2 - \alpha_1 = \frac{2l-1}{4M} + \frac{1}{2M}$.

Using the values of $x - \alpha_1$ and $x - \alpha_2$ in 3^{rd} term of (2.3.4) and taking $\beta = 1$ and $M = 2$. we get

$$P_4 = \frac{1}{8}\begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Corresponding to $M = 4$, the Haar matrix is given by

$$P_8 = \frac{1}{16}\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

If $\beta = 2$ and $M = 2$, then we get

$$P_4 = \frac{1}{2!8^2}\begin{bmatrix} 1 & 3^2 & 5^2 & 7^2 \\ 1 & 3^2 & 23 & 31 \\ 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Corresponding to M=4, the Haar matrix is given by

$$P_8 = \frac{1}{2!16^2}\begin{bmatrix} 1 & 3^2 & 5^2 & 7^2 & 9^2 & 11^2 & 13^2 & 15^2 \\ 1 & 3^2 & 5^2 & 7^2 & 79 & 103 & 119 & 122 \\ 1 & 3^2 & 23 & 31 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3^2 & 23 & 31 \\ 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Remark: Note that Chen and Hsiao [2] considered integral of Haar wavelets as:

$$\int_0^t h_m(t)dt \cong P_{m\times m}h_m(t),\ t \in [0,1) \tag{2.3.7}$$

which is, however, different than in Lepik , m-square matrix P, called operational matrix of integration, is given by

$$P_{m\times m} = \frac{1}{2m}\begin{bmatrix} 2mP_{(\frac{m}{2})\times(\frac{m}{2})} & -H_{(\frac{m}{2})\times(m/2)} \\ H^{-1}_{(\frac{m}{2})\times(\frac{m}{2})} & O_{(\frac{m}{2})\times(\frac{m}{2})} \end{bmatrix}$$

$P_{|\times|} = [\frac{1}{2}]$

where $H_{m\times m} = [h_m(t_0), h_m(t_1), ..., h_m(t_{m-1})]^T$, $\frac{i}{m} \leq t_i \leq \frac{i+1}{m}$.

$$H^{-1}_{m\times m} = \frac{1}{m}H^T_{m\times m}diag(r),\ r = [1.1.2.2.4.4....,\frac{m}{2},\frac{m}{2},....,\frac{m}{2}]^T\ for\ m > 2. \tag{2.3.8}$$

If

$$(QH)_{il} = \int_0^{t_l} \int_0^t h_i(t)dtdt, \tag{2.3.9}$$

then for $j = 1$, $m = 2$, we get

$$QH = \frac{1}{128} \begin{bmatrix} 1 & 9 & 25 & 49 \\ 1 & 9 & 23 & 31 \\ 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

which is the same as obtained through (2.3.3).
Notice that $P^2 \neq Q$.

2.4 Function Approximation and Applications

Since Haar wavelets are orthogonal, this means that any square integrable function over $[0, 1]$ can be expanded into Haar wavelets series as:

$$y(x) = \sum_{i=1}^{\infty} a_i h_i(x) \tag{2.4.1}$$

where a_i's are Haar wavelet coefficients.
If y(x) be piecewise constant, then the sum can be terminated to finite number of terms, that is

$$\begin{aligned} y(x_l) &= \sum_{i=1}^{2M} a_i h_i(x_l) = a^T H \\ a^T &= \{a_1, a_2, ..., a_2M\}, H = \{h_1(x), h_2(x), ..., h_{2M}(x)\}^T \end{aligned} \tag{2.4.2}$$

Solutions of boundary value problems can be considered as approximations (2.4.2).
Norm of error function $v(l) = y_{app}(x_l) - y_{ex}(x_l)$ is defined by

$$\|v\|_p = (\sum_{i=1}^{2M} |v(l)|^p)^{1/p} \tag{2.4.3}$$

Following two error estimates can easily be calculated
(i) Local estimates $\delta_j = \|\frac{v}{y_{ex}}\|_\infty = Max_{1 \leq l \leq 2M} |\frac{y(x_l)}{y_{ex}(x_l)} - 1|$
(ii) Global estimates $\sigma_j = \frac{\|v\|_2}{2M}$
However, we prefer the absolute error estimation

$$e_j = Max_{1 \leq l \leq 2M} |y_{app}(x_l) - y_{ex}(x_l)|. \tag{2.4.4}$$

2.5 Application of Haar Wavelets in Solving Linear ODEs

Consider nth order linear ODE:

$$Ly(x) = f(x) \tag{2.5.1}$$

$A \leq x \leq B$, where L is a differential operator Step 1:

$$y^n(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

Step 2: For $\beta < n$

$$y^n(x) = \sum_{i=1}^{2M} a_i P_{n-\beta,i}(x) + \sum_{\sigma=1}^{n-\beta-1} \frac{1}{\sigma!}(x-A)^{\sigma} y_0^{(\beta+\sigma)}$$

Step 3: Substitute various derivatives as obtained in steps 1 & 2 in the equation (2.4.1) and calculate $a_i^{,}s$ to get the numerical solution.

Example 3.1 Consider the second order initial value linear ODE:

$$y''(x) + y(x) = u(x),\ y(0) = y'(0) = 0,\ x \in [0,1]$$

The exact solution of this problem is $1 - cos(x)$
This problem has been solved in chapter 4.

2.6 Conclusion

The sparseness in Haar wavelets based operational matrices gives precise accuracy in solving numerical equations by Haar collocation method. The algorithm developed facilitates for less time consuming. Moreover, the method has an advantage over wavelet-Galerkin [3] procedure being computationally complicated. Also, for small value of resolution a comparatively better solution is obtained. The results are comparable to analytic ones. Better results are expected for comparatively higher values of level of resolution j.

Chapter 3

NUMERICAL SOLUTION OF SINGULARLY PERTURBED TWO-POINT BVPs USING NONUNIFORM HAAR WAVELETS

3.1 Introduction

In this chapter, we consider linear and nonlinear singularly perturbed two-point boundary-value problems of the form:

$$-\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x);$$

$$-\varepsilon y''(x) = f(x,y,y''),\ 0 < x < 1,$$

$y(0) = \beta_1,\ y(1) = \beta_2,$
where β_1, β_2 are real constants and ε is a small positive parameter such that $0 < \varepsilon \leq 1$ and p and q are sufficiently smooth real-valued functions.
Singular perturbation problems (SPPs) have applications in various disciplines, for instance, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, convection-diffusion processes, optimal control, etc. The solution of SPPs exhibits a multi- scale character; *i.e.* there are thin layer(s) where the solution varies rapidly, while away from the layer(s) the solution behaves regularly and varies slowly. So the numerical treatment of SPPs gives major computational difficulties and the standard methods do not yield accurate results when ε is very small. A variety of numerical methods are available in the literature to solve SPPs for second- order ordinary

differential equations.

Haar wavelets are the simplest orthonormal wavelets with compact support and have been used in different numerical approximation problems. The Haar method with a variable stepsize, called nonuniform Haar method, is suitable in the case of problems where abrupt changes in the solution take place.

In the present study to widen the scope of Haar wavelets for numerical solution of SPPs, we propose a new variant of the nonuniform Haar wavelet method has been proposed and this approach has been applied on a variety of single and double layer linear and non-linear SPPs. [1]

3.2 Non-Uniform Haar Wavelets

The family of Haar wavelets can be defined over any interval (a,b). For nonuniform Haar wavelets, the interval (a,b) is divided into $2M$ subintervals of unequal lengths using grid points $x(l),\ l = 0,1,...,2M$ where $M = 2^J$ and J is maximal level of resolution. Each member in the Haar wavelets family is characterized by two integers known as dilation and translation parameters denoted by j and k, respectively, where $j = 0,1,...,J$ and $k = 0,1,...,m-1$ and $m = 2^j$. The index i of the Haar wavelet is calculated as $i = m+k+1$. The Haar wavelets family consists of functions of the form. [26, 27]

$$h_i(x) = \begin{cases} 1,\ for\, x \in [\alpha_1(i), \alpha_2(i)), \\ -c_i,\ for\, x \in [\alpha_2(i), \alpha_3(i)), \\ 0,\ elsewhere, \end{cases} \tag{3.2.1}$$

for $i = 2,3,...,2M$, and where
$\alpha_1(i) = x(2k\mu),\ \alpha_2(i) = x[(2k+1)\mu]$,
$\alpha_3(i) = x[(2k+2)\mu],\ \mu = \frac{M}{m}$.

The coefficients c_i are calculated from the requirement

$$\int_a^b h_i(x)dx = 0 \tag{3.2.2}$$

The values of c_i are calculated as

$$c_i = \frac{\alpha_2(i) - \alpha_1(i)}{\alpha_3(i) - \alpha_2(i)}$$

For $i = 1$, we assume that

$$h_1(x) = \begin{cases} 1,\ for\, x \in [a,b) \\ 0,\ elsewhere \end{cases} \tag{3.2.3}$$

The following notations are introduced:

$$p_{i,1}(x) = \int_a^x h_i(z)dz, \tag{3.2.4}$$

$$p_{i,v+1}(x) = \int_a^x p_{i,v}(z)dz, \quad i = 2,3,\cdots \tag{3.2.5}$$

These integrals can be evaluated using (3.2.1) and are given by

$$p_{i,1}(x) = \begin{cases} x - \alpha_1, \ for\, x \in [\alpha_1, \alpha_2) \\ c_i(\alpha_3 - x), \ for\, x \in [\alpha_2, \alpha_3) \\ 0, \ elsewhere \end{cases} \tag{3.2.6}$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \alpha_1)^2, \ for\, x \in [\alpha_1, \alpha_2), \\ k - \frac{1}{2}c_i(\alpha_3 - x)^2, \ for\, x \in [\alpha_2, \alpha_3), \\ k, \ for\, x \in [\alpha_3, b), \\ 0, elsewhere \end{cases} \tag{3.2.7}$$

where $K = \frac{1}{2}(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)$.
For $i = 1$, we have
$p_{1,1}(x) = x - a, \quad p_{2,1}(x) = \frac{1}{2}(x - a)^2$.
We shall apply the collocation method and to this end the collocation points are defined as

$$x_c(l) = \frac{x(l-1) + x(l)}{2}, \ l = 1,2,...,2M. \tag{3.2.8}$$

3.3 Numerical Method

Lepik [6] has chosen subintervals such that $\Delta x_i = q\Delta x_{i-1}$. Due to this approach most of the grid points are accumulated exactly at the right side boundary of the interval for $q < 1$. We have adopted a new approach, which ensures a denser grid in the layer regions either left or right by taking $\Delta x_i = \Delta x_{i-1} - (i-1)\delta$, $i = 2,3,...,2M$, where $\Delta x_1 = \rho = \frac{5-\varepsilon}{5M}$. The value of δ can be calculated from the requirement [6]

$$\sum_{i=1}^{2M} \Delta x_i = 1 \tag{3.3.1}$$

and is given by

$$\delta = \frac{2M\rho - 1}{M(2M-1)} \tag{3.3.2}$$

If the boundary layer lies on right, we will assume the following grid points:

$$x(l) = l\rho - \frac{1}{2}l(l-1)\delta, \tag{3.3.3}$$

whereas for problems with a boundary layer on the left end, the grid points will be taken as

$$x(l) = 1-(2M-l)\rho+\delta(\frac{2M-l}{2})(2M-l-1). \tag{3.3.4}$$

This approach has positive impact on the accuracy of method when $M \geq 128$. Finally, if there are boundary layers on both ends, the interval is divided into two equal subintervals and both the above schemes will be applied accordingly. We first describe the method for second-order linear perturbation problems with the general form

$$-\varepsilon y''(x)+p(x)y'(x)+q(x)y(x) = f(x); \tag{3.3.5}$$

$$y(0) = \alpha_1,\ y(1) = \alpha_2,$$

We assume that

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{3.3.6}$$

Integrating (3.3.6) from 0 to x, the derivative $y'(x)$ can be expressed as

$$y'(x) = y'(0)+\sum_{i=1}^{2M} a_i p_{i,1}(x). \tag{3.3.7}$$

(3.3.7) contains unknown quantity $y'(0)$, which can be calculated by integrating (3.3.7) and using the boundary conditions, and is given by

$$y'(0) = \alpha_2-\alpha_1+\sum_{i=1}^{2M} a_i C_{i,1}, \tag{3.3.8}$$

where

$$C_i = \int_0^1 p_{i,1}(x)dx. \tag{3.3.9}$$

Substituting this value of $y'(0)$ in (3.3.7), we obtain

$$y'(x) = \alpha_2-\alpha_1+\sum_{i=1}^{2M} a_i(p_{i,1}(x)-C_{i,1}). \tag{3.3.10}$$

Finally, integrating (3.3.10) from 0 to x, we can express the approximate solution as

$$y(x) = \alpha_1+(\alpha_2-\alpha_1)x+\sum_{i=1}^{2M} a_i(p_{i,2}(x)-xC_{i,1}). \tag{3.3.11}$$

Substituting the values in (3.3.5) and applying discretization using collocation points given in (3.2.8), we obtain the system

$$\begin{aligned}-\varepsilon\sum_{i=1}^{2M} a_i h_i(x_c(l)) + p(x)\{\alpha_2-\alpha_1+\sum_{i=1}^{2M} a_i(p_{i,1}(x_c(l))-C_{i,1})\} + q(x_c(l))\{\alpha_1+(\alpha_2-\alpha_1)x_c(l)\\ +\sum_{i=1}^{2M} a_i(p_{i,2}(x_c(l))-x_c(l)C_{i,1})\} = f(x_c(l)), \quad l=1,2,...,2M. \qquad (3.3.12)\end{aligned}$$

The system (3.3.12) contains $2M$ equations in $2M$ unknowns. Solving (3.3.12), we obtain the Haar coefficients a_i, $i=1,2,...,2M$, which are used to find the approximate solution [6].

ALGORITHM

To approximate the solution of the singularly perturbed two-point boundary-value problem

$$-\varepsilon y''(x) = f(x,y,y'), \;\; 0<x<1,$$

subject to boundary conditions

$$y(0)=\alpha_1, \;\; y(1)=\alpha_2.$$

INPUT: boundary conditions α, β; value of ε; level of resolution M.

OUTPUT: approximations $y(x_j)$ for each $j=1,2,...,2M$.

Step 1

Case 1: When the boundary layer lies on the right of the interval,

```
Set $\rho = \frac{5-\varepsilon}{5M}$
Set $\delta = \frac{2M\rho-1}{M(2M-1)}$
Set $x_0 = 0.$

For $j = 1,2,...,2M$
    Set $x_j = x_0 + j\rho - \delta\frac{j(j-1)}{2}.$
For $j = 1,2,...,2M$
    Set $xc_j = \frac{x_{j-1}+x_j}{2}$
```

Case 2: When the boundary layer lies on the left of the interval,

```
Set $\rho = \frac{5-\varepsilon}{5M}$
Set $\delta = \frac{2M\rho - 1}{M(2M-1)}$
Set $x_0 = 0.$
For $j = 1, 2, ..., 2M$
    Set $x_j = 1 - (2M-j)\rho + \delta\frac{2M-j}{2}(2M-j+1).$
For $j = 1, 2, ..., 2M$
    Set $xc_j = \frac{x_{j-1}+x_j}{2}$
```

Case 3: When the boundary layer lies in the middle of the interval,

```
Set $M' = \frac{M}{2}$
Set $\rho = \frac{5-\varepsilon}{5M'}$
Set $\delta = \frac{2M'\rho - 1}{M'(2M'-1)}$
Set $x_0 = 0.$
```

```
For $j = 1,2,...,2M'$
    Set $x_j = \frac{1}{2}(x_0 + j\rho - \delta\frac{j(j-1)}{2}).$
For $j = 2M'+1, 2M'+2, ...., 4M'$
    Set $x_j = 1 - x_{2M'-j}$
For $j = 1,2,...,2M$
    Set $xc_j = \frac{x_{j-1}+x_j}{2}$
```

Case 4: When the boundary layers are at both ends,

```
Set $M' = \frac{M}{2}$
Set $\rho = \frac{5-\varepsilon}{5M'}$
Set $\delta = \frac{2M'\rho - 1}{M'(2M'-1)}$
Set $x'_0 = 0.$
For $j = 1,2,...,2M'$
    Set $x'_j = x'_0 + j\rho - \delta\frac{j(j-1)}{2}.$
    Set $x_0 = 0.$
For $j = 1,2,...,2M'$
    Set $x_j = \frac{1 - x'_{2M'-j}}{2}$
For $j = 2M'+1, 2M'+2, ...., 4M'$
    Set $x_j = 1 - x_{4M'-j}$
For $j = 1,2,...,2M$
    Set $xc_j = \frac{x_{j-1} + x_j}{2}$
```

Step 2 For $j = 1,2,...,2M$

```
    Set $C_j = \int_0^1 p_{j,1}(x)dx.$
```

Step 2 Apply Newton method to system

$$\begin{aligned}-\varepsilon\sum_{i=1}^{2M} a_i h_i(xc_j) =& f(xc_j, \alpha_1 + (\alpha_2-\alpha_1)xc_j \\ &+\sum_{i=1}^{2M} a_i(p_{i,2}(xc_j) - xc_jC_i), \alpha_2-\alpha_1 \\ &+\sum_{i=1}^{2M} a_i(p_{i,1}(xc_j) - C_i)), j = 1,2,...,2M,\end{aligned}$$

with unknowns $a_1, a_2, ..., a_{2M}$.

Step 3 For $j = 1,2,...,2M$,

$$Set\ y(x_j) = \alpha + (\beta-\alpha)x_j + \sum_{i=1}^{2M} a_i(p_{i,2}(x_j) - x_jC_i);$$

OUTPUT: $(y(x_j))$, $j = 1,2,\cdots,2M$.

NUMERICAL EXAMPLES

Example 1: We consider the nonhomogeneous equation

$$\begin{aligned}-\varepsilon y''(x) + y'(x) + y = 1,\ (0 \le x \le 1), \\ y(0) = 0,\ y(1) = 0.\end{aligned}$$

we solve this given example in chapter 4.

3.4 Conclusion

A new variant of nonuniform Haar wavelets is used to develop a numerical method for solving singularly perturbed boundary-value problems. The method is computationally efficient and the algorithm can easily be implemented on a computer. Comparison with other existing methods demonstrates the superiority of the proposed method.

Chapter 4

SOLUTION METHODOLOGY FOR ORDINARY DIFFERENTIAL EQUATIONS

4.1 Introduction

Numerical analysis plays a significant role when we face difficulties in finding the exact solution of an equation using a direct method when it becomes very difficult to apply theoretical methods proposed earlier to find the exact solution. Concerning which numerical technique to be proposed for solving a problem, it is widely accepted that the numerical scheme should be less complex and also it should take computationally less time, while producing more accurate results at the same time. Haar wavelet methods are one such type of methods for solving differential equations which are simple in structure, also computationally efficient and provides good approximate results. The idea of wavelets can be summarised as a family of functions constructed from transformation and dilation of a single function called mother wavelet. From various type of continuous and discrete wavelets, Haar Wavelet is the discrete type of wavelet which was $1st$ proposed and the $1st$ orthonormal wavelet basis is the Haar basis [13].

In the present chapter, we will apply the Haar wavelet method to approximate the solution of initial value problems and boundary value problems.

4.2 Haar Wavelets and Integration of Haar Wavelets

The Haar wavelet family for $x \in [0,1)$ is defined by

$$h_i(x) = \begin{cases} 1, \; for\, x \in [\alpha, \beta) \\ -1, \; for\, x \in [\beta, \gamma) \\ 0, \; elsewhere \end{cases} \tag{4.2.1}$$

where $\alpha = \frac{k}{m}, \; \beta = \frac{k+0.5}{m}, \; \gamma = \frac{k+1}{m}$
and $m = 2^j$, $j = 0,1,2,....J$, $k = 0,1,....,m-1$, where j indicates the level of wavelet, k denotes translation parameter and J denotes the maximum level of resolution. The index i in $h_i(x)$ is determined by $i = m+k+1$ where the minimum values of k and m are 0 and 1 respectively.
The maximum value of $i = 2^{J+1} = 2M$
We introduce the following notations or definitions:

$$p_i(x) = \int_0^x h_i(x')dx'$$
$$q_i(x) = \int_0^x p_i(x')dx'$$

In particular when $i = 1$

$$h_1(x) = \begin{cases} 1, \; for\, x \in [0,1) \\ 0, \; elsewhere \end{cases}$$

$$p_1(x) = \begin{cases} x, \; for\, x \in [0,1) \\ 0, \; elsewhere \end{cases}$$

$$q_1(x) = \begin{cases} \frac{x^2}{2}, \; for\, x \in [0,1) \\ 0, \; elsewhere \end{cases}$$

and for $i > 1$, we define $h_i(x)$, $p_i(x)$ and $q_i(x)$ as follows:

$$h_i(x) = \begin{cases} 1, \; for\, x \in [\alpha,\beta) \\ -1, \; for\, x \in [\beta,\gamma) \\ 0, \; elsewhere \end{cases}$$

$$p_i(x) = \begin{cases} x-\alpha, \; for\, x \in [\alpha,\beta) \\ \gamma - x, \; for\, x \in [\beta,\gamma) \\ 0, \; elsewhere \end{cases}$$

$$q_i(x) = \begin{cases} \frac{(x-\alpha)^2}{2}, \; for\, x \in [\alpha,\beta) \\ \frac{(\alpha-\beta)^2+(\beta-\gamma)^2-(\gamma-x)^2}{2}, \; for\, x \in [\beta,\gamma) \\ \frac{(\alpha-\beta)^2+(\beta-\gamma)^2}{2}, \; for\, x \in [\gamma,1) \\ 0, \; elsewhere \end{cases}$$

4.3 Solution Methodology

Form the property of the Haar wavelet transformation [13], $y''(x)$ which is a function of x, can be approximated by the Haar Wavelet functions like wise

$$y''(x) = \phi(x, y(x), y'(x)) = \sum_{i=1}^{2M} a_i h_i(x) \tag{4.3.1}$$

It is difficult to find the solution $y(x)$ by direct integration when the differential equation is nonlinear or has complicated co-efficients. But approximating the $y''(x)$ with Haar wavelet functions, it is quite easier to get $y''(x)$ or $y'(x)$ explicitly in terms of x.
Hence, integrating (4.3.1), we get

$$y'(x) = \int y''(x)dx = \int [\sum_{i=1}^{2M} a_i h_i(x)]dx = \sum_{i=1}^{2M} \int a_i h_i(x)dx \tag{4.3.2}$$

and

$$y(x) = \int y'(x)dx = \int\int [\sum_{i=1}^{2M} a_i h_i(x)]dxdx = \sum_{i=1}^{2M} \int\int a_i h_i(x)dx \tag{4.3.3}$$

Solution Methodology for Initial Value Problem:
Consider the second order Initial Value Problem as

$$y''(x) = \phi(x, y, y'),$$

with initial conditions $y(a) = \alpha,\ y'(a) = \beta$

If $a \in [0,1)$, then

$$\begin{aligned}
y''(x) &= \sum_{i=1}^{2M} a_i h_i(x)\\
\int_a^x y''(x) &= \int_a^x \sum_{i=1}^{2M} a_i h_i(x)\\
y'(x) - y'(a) &= \int_a^x \sum_{i=1}^{2M} a_i h_i(x')dx'\\
&= \sum_{i=1}^{2M} a_i p_i(x) - \int_0^a \sum_{i=1}^{2M} a_i h_i(x')dx'\\
y(x) - y(a) &= \int_a^x \sum_{i=1}^{2M} a_i p_i(x')dx'\\
&= (x-a)y'(a) + \sum_{i=1}^{2M} a_i q_i(x) - \int_0^a \sum_{i=1}^{2M} a_i p_i(x')dx' - (x-a)A\\
\text{where }\ A &= \int_0^a \sum_{i=1}^{2M} a_i h_i(x)dx
\end{aligned}$$

When $a = 0$, these equation becomes

$$y'(x) = y'(0) + \sum_{i=1}^{2M} a_i p_i(x) \tag{4.3.4}$$

$$y(x) = y(0) + xy'(0) + \sum_{i=1}^{2M} a_i q_i(x) \tag{4.3.5}$$

Thus to get $y(x)$ we have to first find the unknowns $a_i s$ by solving $2M$ system of equations. These $2M$ equations are generated from the relation

i.e. $y'' = \phi(x,y,y')$ at x_js which are the collocation points; and $j = 1,2,\cdots.2M$. Now, using (4.3.4) and (4.3.5) in (4.3.1), we get

$$\sum_{i=1}^{2M} a_i h_i(x_j) = \phi(x_j, y(a) + (x_j - a)y'(a) + \sum_{i=1}^{2M} a_i q_i(x_j) - \int_0^a \sum_{i=1}^{2M} a_i p_i(x'_j)dx' \tag{4.3.6}$$

$$-(x_j - a)A, y'(a) + \sum_{i=1}^{2M} a_i p_i(x_j) - \int_0^a \sum_{i=1}^{2M} a_i h_i(x'_j)dx') \tag{4.3.7}$$

Example 3.1 Consider the second order linear initial value problem:

$y''(x) + y(x) = u(x),\ y(0) = 0,\ y'(0) = 0,\ x \in [0,1]$

The exact solution for the given problem is $1 - cos(x)$.

To start with, we consider only four collocation points.

therefore, $J = 4$ and $M = 2$. Hence, we need to define $h_i(x), p_i(x), q_i(x)$ for $i = 1,2,3,4$. Now,

$i = m + k + 1$

when $i = 1$ then, we get $m = 0$, $k = 0$

$$h_1(x) = \begin{cases} 1, \; for\, x \in [0,1) \\ 0, \; elsewhere \end{cases}$$

$$p_1(x) = \begin{cases} x, \; for\, x \in [0,1) \\ 0, \; elsewhere \end{cases}$$

$$q_1(x) = \begin{cases} \frac{x^2}{2}, \; for\, x \in [0,1) \\ 0, \; elsewhere \end{cases}$$

When $i = 2$, then, we get $m = 1$, $k = 0$.

$\implies \alpha = 0, \; \beta = \frac{1}{2}, \; \gamma = 1$

$$h_2(x) = \begin{cases} 1, \; for\, x \in [0,\frac{1}{2}) \\ -1, \; for\, x \in [\frac{1}{2},1) \\ 0, \; elsewhere \end{cases}$$

$$p_2(x) = \begin{cases} x - 0, \; for\, x \in [0,\frac{1}{2}) \\ 1 - x, \; for\, x \in [\frac{1}{2},1) \\ 0, \; elsewhere \end{cases}$$

$$q_2(x) = \begin{cases} \frac{x^2}{2}, \; for\, x \in [0,\frac{1}{2}) \\ \frac{-2x^2+4x-1}{4}, \; for\, x \in [\frac{1}{2},1) \\ \frac{1}{4}, \; for\, x = 1 \\ 0, \; elsewhere \end{cases}$$

when $i = 3$, then, we get $m = 2$, $k = 0$

$\implies \alpha = 0,\ \beta = \frac{1}{4},\ \gamma = \frac{1}{2}$

$$h_3(x) = \begin{cases} 1,\ for\, x \in [0, \frac{1}{4}) \\ -1,\ for\, x \in [\frac{1}{4}, \frac{1}{2}) \\ 0,\ elsewhere \end{cases}$$

$$p_3(x) = \begin{cases} x - 0,\ for\, x \in [0, \frac{1}{4}) \\ \frac{1}{2} - x,\ for\, x \in [\frac{1}{4}, \frac{1}{2}) \\ 0,\ elsewhere \end{cases}$$

$$q_3(x) = \begin{cases} \frac{x^2}{2},\ for\, x \in [0, \frac{1}{4}) \\ \frac{8x-8x^2-1}{16},\ for\, x \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{1}{16},\ for\, x \in [\frac{1}{2}, 1) \\ 0,\ elsewhere \end{cases}$$

When $i = 4$, then, we get $m = 2$, $k = 1$

$\implies \alpha = \frac{1}{2},\ \beta = \frac{3}{4},\ \gamma = 1$

$$h_4(x) = \begin{cases} 1,\ for\, x \in [\frac{1}{2}, \frac{3}{4}) \\ -1,\ for\, x \in [\frac{3}{4}, 1) \\ 0,\ elsewhere \end{cases}$$

$$p_4(x) = \begin{cases} x - \frac{1}{2},\ for\, x \in [\frac{1}{2}, \frac{3}{4}) \\ 1 - x,\ for\, x \in [\frac{3}{4}, 1) \\ 0,\ elsewhere \end{cases}$$

$$q_4(x) = \begin{cases} \frac{4x^2-4x+1}{8},\ for\, x \in [\frac{1}{2}, \frac{3}{4}) \\ \frac{-7-8x^2+16x}{16},\ for\, x \in [\frac{3}{4}, 1) \\ \frac{1}{16},\ for\, x = 1 \\ 0,\ elsewhere \end{cases}$$

For the given problem, $u(x) = 1$

Hence, the given equation becomes

$$y''(x) + y(x) = 1 \tag{4.3.8}$$

$$y''(x) = \phi(x, y, y') = 1 - y(x) \tag{4.3.9}$$

From (4.3.6)

$$\sum_{i=1}^{2M} a_i h_i(x_j) = \phi(x_j, y(a) + (x_j - a)y'(a) + \sum_{i=1}^{2M} a_i q_i(x_j) - \int_0^a \sum_{i=1}^{2M} a_i p_i(x_j')dx'$$
$$-(x_j - a)A, y'(a) + \sum_{i=1}^{2M} a_i p_i(x_j) - \int_0^a \sum_{i=1}^{2M} a_i h_i(x_j')dx')$$

From (4.3.8)

$$\phi(x,y,y') = 1 - [y(a) + (x_j - a)y'(a) + \sum_{i=1}^{4} a_i q_i(x_j)$$
$$-\int_0^a \sum_{i=1}^{4} a_i p_i(x_j')dx' - (x_j - a)A]$$

Now $a = 0$ (given), thus.

$$\phi(x,y,y') = 1 - [y(0) + (x_j - 0)y'(0) + \sum_{i=1}^{4} a_i q_i(x_j)]$$
$$\sum_{i=1}^{4} a_i h_i(x_j) = 1 - \sum_{i=1}^{4} a_i q_i(x_j)$$
$$\sum_{i=1}^{4} a_i h_i(x_j) + \sum_{i=1}^{4} a_i q_i(x_j) = 1$$
$$\sum_{i=1}^{4} a_i (h_i(x_j) + q_i(x_j)) = 1$$
$$a_1(h_1(x_j) + q_1(x_j)) + a_2(h_2(x_j) + q_2(x_j)) + a_3(h_3(x_j) + q_3(x_j)) + a_4(h_4(x_j) + q_4(x_j)) = 1 \tag{4.3.10}$$

Now we put the values of $h_i's$ and $q_i's$ in (4.3.10) where $i = 1,2,3,4$
When $x_j = 0.125$,

$$a_1 + a_2 + a_3 = 0.992248$$

when $x_j = 0.375$,

$$a_1 + a_2 - 0.88321a_3 = 0.9343066$$

when $x_j = 0.625$,

$$1.1953125a_1 - 0.8203125a_2 + 0.0625a_3 + 1.0078125a_4 = 1$$

when $x_j = 0.875$,

$$1.3828125a_1 - 0.7578125a_2 + 0.0625a_3 - 0.9453125a_4 = 1$$

Solving these above equations, we get the values of a_1, a_2, a_3, a_4 given by

$$a_1 = 0.844076138$$

$$a_2 = 0.117404501$$

$$a_3 = 0.0307673599$$

$$a_4 = 0.0847882533$$

Now using these values of $a_i's$ in the relation

$$y(x) = y(0) + xy'(0) + \sum_{i=1}^{4} a_i q_i(x)$$

we get

$$y(x) = \sum_{i=1}^{2M} a_i q_i(x)$$

$$y(x) = 0.844076138(\frac{x^2}{2}) + 0.117404501(\frac{4x - 2x^2 - 1}{4})$$

$$+ 0.0307673599(\frac{1}{16}) + 0.0847882533(\frac{16x - 8x^2 - 7}{16})$$

Next, we compare the Haar Wavelet solution with the exact solution at different grid points.

$$y_c(0.1) = 0.00496124,\ y_e(0.1) = 0.00499583$$
$$error = |y_c(0.1) - y_e(0.1)| = 0.00003459$$
$$y_c(0.2) = 0.01984495998,\ y_e(0.2) = 0.01984495998$$
$$error = |y_c(0.2) - y_e(0.2)| = 0.000088462$$
$$y_c(0.3) = 0.04457424,\ y_e(0.3) = 0.0446635$$
$$error = |y_c(0.3) - y_e(0.3)| = 0.000089269$$
$$y_c(0.4) = 0.07868757,\ y_e(0.4) = 0.07868757$$
$$error = |y_c(0.4) - y_e(0.4)| = 0.000025143168$$
$$y_c(0.5) = 0.122108039,\ y_e(0.5) = 0.12241744$$
$$error = |y_c(0.5) - y_e(0.5)| = 0.000309398$$
$$y_c(0.6) = 0.17423937,\ y_e(0.6) = 0.174664385$$
$$error = |y_c(0.6) - y_e(0.6)| = 0.0004250138$$
$$y_c(0.7) = 0.25179194,\ y_e(0.7) = 0.2351578$$
$$error = |y_c(0.7) - y_e(0.7)| = 0.016634128$$
$$y_c(0.8) = 0.545661177,\ y_e(0.8) = 0.30329329$$
$$error = |y_c(0.8) - y_e(0.8)| = 0.2423678866$$
$$y_c(0.9) = 0.37741322,\ y_e(0.9) = 0.3783$$
$$error = |y_c(0.9) - y_e(0.9)| = 0.000886778$$

Solution Methodology for Boundary Value Problem:

Case:1

$$y''(x) = \phi(x, y, y') \tag{4.3.11}$$

with $y'(0) = \alpha, y'(1) = \beta$

Now proceeding in the similar fashion as in the case if initial value problem, we get

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

$$\int_0^x y''(x)dx = \int_0^x \sum_{i=1}^{2M} a_i h_i(x)dx$$

$$y'(x) - y'(0) = \sum_{i=1}^{2M} a_i p_i(x)$$

$$y'(x) - \alpha = \sum_{i=1}^{2M} a_i p_i(x)$$

$$\int_x^1 y''(x) = \int_x^1 \sum_{i=1}^{2M} a_i h_i(x) = \sum_{i=1}^{2M} [\int_0^1 a_i h_i(x)dx - \int_0^x a_i h_i(x)dx]$$

$$y'(1) - y'(x) = a_1 - \sum_{i=1}^{2M} a_i p_i(x)$$

$$\beta - y'(x) = a_1 - \sum_{i=1}^{2M} a_i p_i(x)$$

$$a_1 = \beta - \alpha$$

$$y''(x) = a_1 h_1(x) + \sum_{i=2}^{2M} a_i h_i(x)$$

$$y''(x) = (\beta - \alpha) h_1(x) + \sum_{i=1}^{2M} a_i h_i(x)$$

$$y'(x) = \alpha + \sum_{i=1}^{2M} a_i p_i(x)$$

$$y'(x) = \alpha + a_1 p_1(x) + \sum_{i=2}^{2M} a_i p_i(x)$$

$$y'(x) = \alpha + (\beta - \alpha) p_1(x) + \sum_{i=2}^{2M} a_i p_i(x)$$

$$\int_0^x y'(x)dx = \int_0^x \alpha dx + \int_0^x (\beta - \alpha) p_1(x)dx + \int_0^x \sum_{i=2}^{2M} a_i p_i(x)dx$$

$$y(x) = y(0) + \alpha x + (\beta - \alpha) q_1(x) + \sum_{i=2}^{2M} a_i q_i(x)$$

$$(\beta - \alpha) h_1(x_j) + \sum_{i=1}^{2M} a_i h_i(x_j) = \phi(x_j, y(0) + \alpha x_j + (\beta - \alpha) q_1(x_j) + \sum_{i=2}^{2M} a_i$$

$$q_i(x_j), \alpha + (\beta - \alpha) p_1(x_j) + \sum_{i=2}^{2M} a_i p_i(x_j))$$

Case:2

$$y''(x) = \phi(x,y,y') \; with \; y(0) = \alpha_1, y(1) = \beta_1$$

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

$$\int_0^x y''(x)dx = \int_0^x \sum_{i=1}^{2M} a_i h_i(x)dx$$

$$y'(x) = y'(0) + \sum_{i=1}^{2M} a_i p_i(x)$$

$$y(x) = \alpha_1 + xy'(0) + \sum_{i=1}^{2M} a_i q_i(x)$$

$$y'(0) = \beta_1 - \alpha_1 - \sum_{i=1}^{2M} a_i C_i$$

$$y(x) = \alpha_1 + x[\beta_1 - \alpha_1 - \sum_{i=1}^{2M} a_i C_i] + \sum_{i=1}^{2M} a_i q_i(x)$$

$$y'(x) = \beta_1 - \alpha_1 - \sum_{i=1}^{2M} a_i C_i + \sum_{i=1}^{2M} a_i p_i(x) \tag{4.3.12}$$

$$\sum_{i=1}^{2M} a_i h_i(x_j) = \phi(x_j, \alpha_1 + x_j[\beta_1 - \alpha_1 - \sum_{i=1}^{2M} a_i C_i] + \sum_{i=1}^{2M} a_i q_i(x_j), \beta_1 - \alpha_1 - \sum_{i=1}^{2M} a_i C_i + \sum_{i=1}^{2M} a_i p_i(x_j))$$

Now this relation will produce $2M$ equation for $j = 1,2,\cdots,2M$. These 2M equations can be solved for a_i; $i = 1,2,\cdot\cdot,2M$. Using these a_i values in (4.3.12), we can get the Haar Wavelet solution. **Example 3.2** We consider the nonhomogeneous equation

$$-\varepsilon y''(x) + y'(x) + y = 1, \; (0 \leq x \leq 1), \tag{4.3.13}$$

$$y(0) = 0, \; y(1) = 0. \tag{4.3.14}$$

$$y''(x) = \phi(x,y,y') \tag{4.3.15}$$

$$-\varepsilon y''(x) + y'(x) + y = 1$$

$$-\varepsilon y''(x) = 1 - y - y'(x)$$

$$\varepsilon y''(x) = y'(x) + y - 1$$

$$taking \; \varepsilon = 0.1$$

$$y''(x) - 10y'(x) - 10y = -10 \tag{4.3.16}$$

$$y = -x\sum_{i=1}^{4} a_i C_i + \sum_{i=1}^{4} a_i q_i(x) \tag{4.3.17}$$

$$y' = -\sum_{i=1}^{4} a_i C_i + \sum_{i=1}^{4} a_i p_i(x) \tag{4.3.18}$$

Now putting the values of y and y'' in (4.3.16), we obtain,

$$\sum_{i=1}^{4} a_i h_i(x) + 10(\sum_{i=1}^{4} a_i C_i - \sum_{i=1}^{4} a_i p_i(x)) + 10x \sum_{i=1}^{4} a_i C_i - 10 \sum_{i=1}^{4} a_i q_i(x) = -10$$

$$\sum_{i=1}^{4} [h_i(x_j) + 10C_i - 10p_i(x_j) + 10x_j C_i - 10q_i(x_j)] a_i = -10$$

$$\sum_{i=1}^{4} [h_i(x_j) + 10(1 + x_j) C_i - 10p_i(x_j) - 10q_i(x_j)] a_i = -10 \tag{4.3.19}$$

$$C_1 = 0,\ C_2 = \frac{1}{4},\ C_3 = 0,\ C_4 = \frac{1}{16} \tag{4.3.20}$$

Using these values of $C_1,\ C_2,\ C_3,\ C_4$ in (4.3.19), we get

$$\begin{aligned} \{h_1(x_j) - 10p_1(x_j) - 10q_1(x_j)\} a_1 + \{h_2(x_j) + 2.5(1 + x_j) - 10p_2(x_j) - 10q_2(x_j)\} a_2 \\ + \{h_3(x_j) - 10p_3(x_j) - 10q_3(x_j)\} a_3 + \{h_4(x_j) + 0.625(1 + x_j) \\ - 10p_4(x_j) - 10q_4(x_j)\} a_4 = -10 \end{aligned} \tag{4.3.21}$$

when $x = 0.3$

$$-2.45a_1 + 0.8a_2 - 3.425a_3 + 0.8125 = -10$$

when $x = 0.6$

$$-6.8a_1 - 2.7a_2 - 0.625a_3 + 0.95a_4 = -10$$

when $x = 0.8$

$$-10.2a_1 - 0.8a_2 - 0.625a_3 - 2.3a_4 = -10$$

when $x = 0.9$

$$-12.05a_1 + 0.3a_2 - 0.625a_3 - 1.3875a_4 = -10$$

solving these equations, we get

$$a_1 = 0.72517516$$

$$a_2 = 1.24510885$$

$$a_3 = 2.68450679$$

$$a_4 = -0.030735$$

$$y(x_j) = -x_j \sum_{i=1}^{4} a_i C_i + \sum_{i=1}^{4} a_i q_i(x)$$

(4.3.22)

$$y(x_j) = -x_j(C_2a_2 + C_4a_4) + a_1\frac{x^2}{2} + a_2\begin{cases} \frac{x^2}{2},\ x \in [0,\frac{1}{2}) \\ \frac{-2x^2+4x-1}{4},\ x \in [\frac{1}{2},1) \\ \frac{1}{4},\ x = 1 \\ 0,\ elsewhere \end{cases}$$

$$+a_3\begin{cases} \frac{x^2}{2},\ x \in [0,\frac{1}{4}) \\ \frac{-8x^2+8x-1}{16},\ x \in [\frac{1}{4},\frac{1}{2}) \\ \frac{1}{16},\ x \in [\frac{1}{2},1) \\ 0,\ elsewhere \end{cases}$$

$$+a_4\begin{cases} \frac{4x^2-4x+1}{8},\ x \in [\frac{1}{2},\frac{3}{4}) \\ \frac{-8x^2+16x-7}{16},\ x \in [\frac{3}{4},1) \\ \frac{1}{16},\ x = 1 \\ 0,\ elsewhere \end{cases} \tag{4.3.23}$$

Next, we compute the Haar wavelet solution at different grid points. at $x = 0.3$

$$y(0.3) = 0.295561119,$$

at $x = 0.6$

$$y(0.6) = 0.3242142677,$$

at $x = 0.8$

$$y(0.8) = 0.4374215036,$$

at $x = 0.9$, we get

$$y(0.9) = 0.4863413725.$$

4.4 Conclusion

Haar Wavelet transform method is best shoot for the initial value problems, boundary value problems with less error. It is easy to extend the numerical scheme in two dimensions also. The methods

has added advantage over all other methods because of its simple structure, and less compatationally cost.
It is observed that if the level of resolution is more *i.e.* if the collocation points are more then we can get a better solution with lesser error. It can be concluded that this method is quite suitable, accurate, and efficient in comparison to other classical methods.

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