Effects of a Static Bottleneck on Vehicular Traffic : A Review

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Certificate

It is certified that the work contained in this thesis entitled **”Effects of a Static Bottleneck on Vehicular traffic: A Review ”** in partial fulfillment of the requirements for the award of degree of Master of Science in Mathematics and Computing to the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala is an authentic record of my own work studied under the supervision of Dr. Isha Dhiman.
The matter embodied in this thesis has not been submitted by me for the award of any other degree of this or any other University/Institute.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

In order to understand the complexity of real world processes, one need to model these processes mathematically. In this thesis, we review a second order continuum model of traffic flow. In 2014, Gupta and Dhiman investigated the effects of bottleneck on a vehicular traffic using speed-gradient model under open boundary conditions. The bottleneck situation has been studied using two different approaches-explicit and implicit. The density profiles showing different states are presented. The results of both approaches are compared and we found that implicit approach capture the effect of bottleneck more appropriately.

Keywords: Traffic flow; Static bottleneck; Phase transition; Macroscopic; Second order model.

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Introduction

1.1 Mathematical modeling

When one thinks of modeling, it is natural to visualize the representation of real phenomena in mathematical terms. The significant test of a model is how well it perform when it is applied to the problems. When a model is used, it may lead to incorrect predictions. Ideally, you can correctly predict the behavior of vehicle flow by adjusting the flow in crucial areas. So, a mathematical model is a description of a system using mathematical concepts and language i.e., it is a simplified representation in mathematical terms of the behavior of real world phenomena. It can be done through either equations or by computer code. Mathematical Modeling is the method of changing problems from applications to mathematical formulas, whose numerical analysis provides useful information, which can be used to explore the system properties. Mathematical models are characterized by assumption about:

- Variables (the things which change)
- Parameters (the things which do not change)
- Functional form(the relationship between the two)

1.1.1 Types of Models

Several classifications can be used for mathematical models according to their configuration. Some of them are described as follows.

Linear versus nonlinear: In a mathematical model, if all the operators exhibit linearity, resulting model is defined as linear. So, linear model to be assumed one which describes relationship in linear form in its parameters. On other hand non-linear model is one which describes relationships in non-linear form. So, if all the operators exhibit non-linearity, resulting model is defined as non-linear model. Non linear model represented with a nonlinear equation by one or more constraints and functions; while linear model represents its constraints and function with linear equation.

Static versus dynamic: A dynamic model is one whose nature or behavior changes with time. This model is time-varying or time-dependent model. A dynamic model changes in the state of the system, while a static model examine the system in equilibrium, and thus is time-invariant or do not change with time. Static models are at steady state which means the nature of the model same in one time period.

Explicit versus implicit: If we are given with the input parameters of the model which means the input parameters are known, and the output parameters can be solved or if output parameters can be calculated by a finite series of computations the model is said to be explicit. But sometimes all the input parameters of the model are not known. Then we have to calculate the input parameters by iterative method with output parameters which are known. Then the model is said to be implicit.

Discrete versus continuous: A discrete model uses objects as discrete, such as the particles in a molecular model or the states in a statistical model; while a continuous model treats the objects in a continuous manner, such as the velocity field of fluid in pipe flows, temperatures and stresses in a solid, and electric field that applies continuously over the entire model due to a point charge.

Deterministic versus probabilistic: In a deterministic model, every set of variable states is uniquely represented or determined by parameters and by sets of previous states of these variables. Therefore, a deterministic model always acts the same way for a given set of initial conditions. For eg, a pipe can hold an amount of fluid before breaking. Conversely, a probabilistic model is also called as a statistical model, and variable states are not determined by unique values, but rather by probability distributions. Weather forecasting, rolling of dice are examples of a probabilistic model.

1.2 Traffic flow model

The center of this thesis is around the study of a particular class of models namely traffic flow models. In this chapter, we provide a brief introduction of traffic flow models. Traffic flow theory is concerned with traffic flow theory the interactions of travelers and infrastructures, with the goal of understanding and developing an optimal transport network. From very long distance, if one looks into traffic flow, the flow looks like a fluid. Mainly heavy traffic appears as stream of fluid. There is particular interest in the study of regions of hight traffic density, which may be caused by accidents or closeness of one or more lanes or by traffic lights. Since the presence of traffic jams highly undesirable situation, modeling of traffic flow has been studied by many scientist. Ideally, with the given set of initial data, we can correctly calculate the behavior of vehicle flow. Along a stretch of road, we can maximize the throughput by arranging the flow in particular or crucial areas. Traffic modeling has become a multifaceted field.

The aim of studying traffic flow is to shorten the traffic queueing and then to reduce the flow. There are lot of approaches to analyze the traffic behavior. We can study the traffic behavior by using theoretical as well as experimental approaches. The theoretical approach gives the efficient models to understand the real-world complex traffic phenomena. An example of network of highways is shown in fig. 1.1

Figure 1.1: A complex freeway cloverleaf[14]

1.2.1 Classification of traffic flow models

Microscopic model:

Microscopic models are based on individual behavior of vehicles. The involved parameters are velocity, position and acceleration. In this model, each vehicle has its ordinary differential equation.

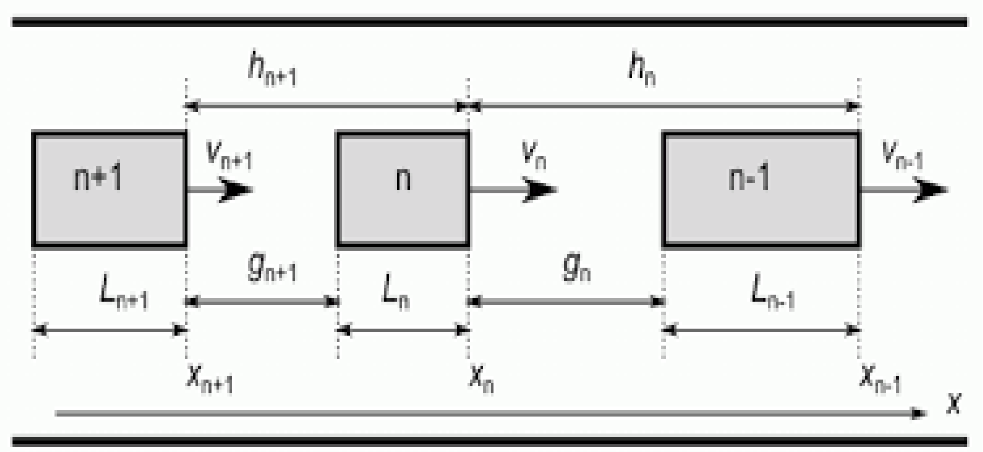


Figure 1.2: Numbering vehicles in car-following models[14]

Car-following model is one of the example of microscopic models. In Car-Following model, driver arrange his or her position according to position(condition) in front as shown in fig.1.2. Another type microscopic model is cellular automata model (also known as CA model). CA models are fully discrete. In this model, a road is divided into cells, each of which is either empty or contains one vehicle.

Macroscopic model:

Macroscopic models were firstly studied in 1950's[13]. Traffic flow appears to be similar as fluid through a pipe system, that is why macroscopic models were chosen. Macroscopic models are attempt to classify average behavior, such as average speed,traffic density and flow rate. These models are mathematically given by partial differential equations which form a hyperbolic system.

LWR model, which is explained ahead in this chapter, is one of the simplest examples of macroscopic model.

Comparison between Macroscopic and Microscopic Models:

The microscopic models are computationally expensive. As each car has its own ODE to be solved at each time step. So, if number of cars increases, the size of the system to be solved also increases, which becomes difficult to solved. As number of molecules/vehicles are very large, the behavior of this model can be analyzed by statistical methods. So, large number of variables/properties are required to describe this model or system.
On the other hand, macroscopic models involved only two or three partial differential equations which makes these models computationally less expensive. Moreover, interaction between vehicles have fewer design limits or terms. Also macroscopic models can be used in online traffic control systems because of less computation time and flexibility. Thus one should prefer macroscopic models for the qualitative study of traffic parameters. From this point we focus only on macroscopic models.
From this point awards, we focus only on macroscopic models.

1.3 Some Definitions for Macroscopic models

Before proceeding, we need to understand following terms in the context of macroscopic traffic flow models.

1.3.1 Density

The number of vehicles per unit distance is called density. A section of road crowded with vehicles gives an idea of density. The concept of density considers the quantity 'number of vehicles' but this concept totally ignores vehicle lengths and traffic compositions.

$$\rho = \frac{no.ofvehiclesinasection[A,B]}{distancebetweenpoint[A,B]} \tag{1.1}$$

1.3.2 Velocity

Velocity is defined as distance per unit time. In traffic flow theory, number of cars moving in particular distance per unit time is called velocity. The units of speed is the m/s meter per second.

$$v = \frac{distance}{time} \tag{1.2}$$

1.3.3 Flow

Flow rate is the number of vehicles passing through a particular point per unit time. Basically, flow is moving from one place to another in a uniform bombardment(stream). Here we create a close relationship between velocity, density and flow. Let there is a road on which cars are moving. All cars are moving with constant distance such that the car has constant velocity v_0, and constant density ρ_0. Now suppose that the number of cars per unit time t is measure by an observer. Each car has moved $v_0 t$ with t time.The number of cars in $v_0 t$ distance is the number of cars pass through an observer with t time. The traffic flow is given by:

$$f = \rho_0 v_0 \tag{1.3}$$

We can write above equation as:

$$f(\rho, v) = \rho(x,t)v(x,t) \tag{1.4}$$

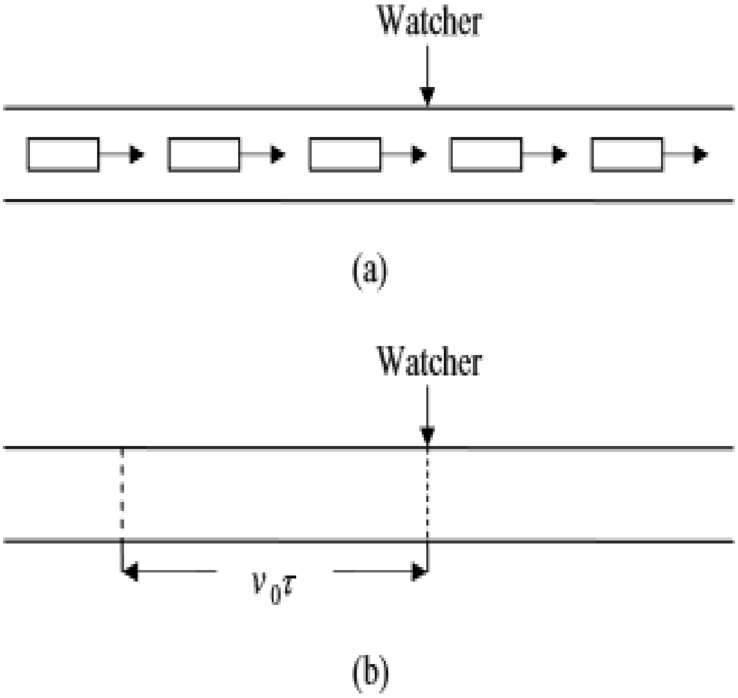


Figure 1.3: (a) Constant flow of cars; (b)Distance traveled in τ hours for single car

1.4 Law of conservation of mass

Suppose any segment on the x axis from distance x_1 to distance x_2. This segment contain fluid with density $\rho(x,t)$. Let us suppose that fluid enters from left side given by the flow $f(x_1,t)$ and it leaves this segment at its right side at x_2 where the flow is $f(x_2,t)$. We know that flux is product of speed and density of flow as given in the equation:

$$f(x,t) = \rho(x,t)v(x,t) \tag{1.5}$$

For mass conservation,in any segment due to density is only due to the fluxes at the boundary, which in this one dimensional case is at x_1 and x_2.
All the traffic flow models are based on the principle of conservation law. These models consist either a one-equation or system of equations. These models always follow principle of conservation law. In principle law of conservation, we use these physical quantities such as velocity, density and flow. When these quantities remain same/conserved within the system or process, these quantities are said to be conserved.
Consider a unidirectional continuous highway or road(no entrance or exit) as shown in fig.1.4.

Consider a stretch of highway or road on which cars are moving from left to right. The number of cars moving from left to right between A and B depend upon the time t. Here the integral of traffic density in the segment[A,B], is equal to the number of cars moving in the segment at given time t.

$$M = \int_A^B \rho(x,t)dx. \tag{1.6}$$

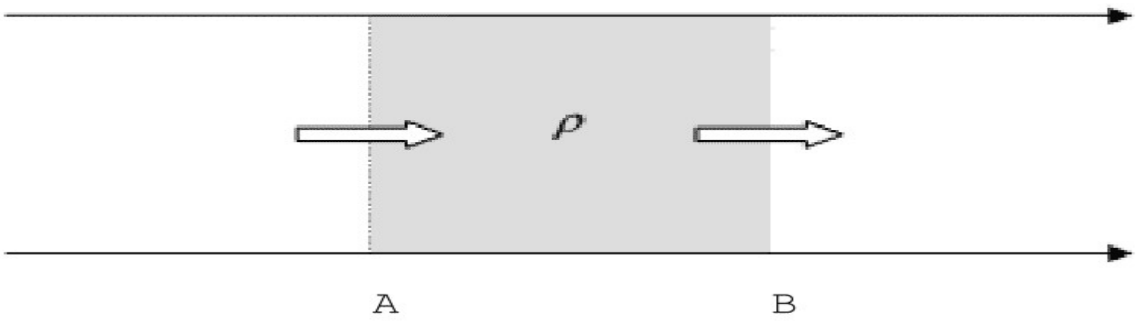


Figure 1.4: Unidirectional flow

The cars are crossing both ends of the segments, so cars can increase or decrease with time. Suppose that the number of cars coming and going is fixed. It means no cars are created or destroyed. There is change in cars(number of cars) with respect to boundaries only. So, rate of change in number of cars is:

$$\frac{dM}{dt} = f_{in}(\rho, v) - f_{out}(\rho, v) \tag{1.7}$$

Using equation 1.6, we get

$$\frac{d}{dt}\int_A^B \rho(x,t)dx = f_{in}(\rho, v) - f_{out}(\rho, v) \tag{1.8}$$

Now suppose that end points are not fixed with time (independent variables). So equation can be rewritten as:

$$\frac{\partial}{\partial t}\int_A^B \rho(x,t)dx = f_{in}(\rho, v) - f_{out}(\rho, v) \tag{1.9}$$

We have considered upto till that the change cars(number of cars) with respect to time. Now we consider the change in the number of cars with respect to distance, given by:

$$f_{in}(\rho, v) - f_{out}(\rho, v) = -\int_A^B \frac{\partial f}{\partial x}(\rho, v)dx \tag{1.10}$$

Using equation 2.33 and equation 1.9, we get

$$\int_A^B \left[\frac{\partial \rho}{\partial t}(x,t) + \frac{\partial f}{\partial x}(\rho, v)\right] dx = 0 \tag{1.11}$$

The above equation signifies that the integral of any abundance(quantity) is always zero. This integral abundance is zero for all values of independence limits of any integral. Assuming that density and flow are both smooth. We get the law of conservation of mass as:

$$\rho_t + f_x(\rho, v) = 0 \tag{1.12}$$

This equation is valid for all physical quantities and traffic.

1.5 First Order traffic flow Model

In this section, we discuss about LWR model. This model is described by a partial differential equation and based on mass of conservation.

LWR Model

The first model which described the traffic flow problem is referred as the Lighthill Whitham Richards model. This model is scalar, non-linear, time independent and hyperbolic PDE(partial differential equation). This model is also known as the simple continuum traffic flow model. This one-dimensional model was developed independently by Lighthill, Whitham and Richards and this model is macroscopic 1-D model. The conservation law can be described as follows:

$$\rho_t(x,t) + f_x(x,t) = 0 \tag{1.13}$$

We can also write above equation as:

$$\rho_t(x,t) + (\rho(x,t)V(rho(x,t)))_x = 0 \tag{1.14}$$

Where ρ is the traffic density and f is flux. Equation (1.14) contains two variables, so we need to know velocity as a function of density $v(\rho)$. Flux is product of density and traffic speed i.e $f = \rho v$.

$$f(x,t) = \rho(x,t)V(\rho(x,t)) \tag{1.15}$$

Here velocity is depend on density only. With the change in density, there will be change in velocity. This model describes the traffic at global level. It describe the individual nature of speed, flow and density are components of LWR model.
Here the function $V_e(\rho)$ is speed density relationship. The speed density functions should satisfy following conditions:

1. $V_e(\rho = 0) = v_{max}$(free velocity)

2. $V_e(\rho = \rho_m) = 0 : \rho_m$ is jam density.

3. $\frac{dV_e(\rho)}{d\rho} < 0$ and $\frac{d^2(\rho V_e(\rho)}{d\rho^2} < 0$

Various researchers presented speed-gradient relationships. Few of them are Greenshield's model [4], Greenberg model [3], Underwood model [19] and Diffusion model. One of the famous model among all the models is Greenshied's model explained below.

Greenshield's Model

In this model, velocity is linearly decreasing function of density as given below:

$$V(\rho) = v_f \left(1 - \frac{\rho}{\rho_m}\right) \tag{1.16}$$

where $v_f$is free flow speed and ρ_m is jam(maximum) density.

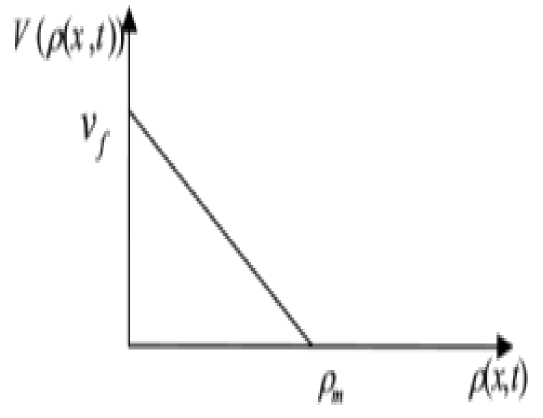


Figure 1.5: Greeshield's model for traffic flow speed[8]

From figure 1.5 we can see that velocity is decreasing function.For maximum density, there is no car in or out and free flow speed v_f is obtained at zero density.

From figure 1.6, there is a Greenshield' model with flux density relationship. In fig, flux increases at peak point(maximum) the goes back to zero.

This LWR model can be solved by methods of characteristics curves. For detailed solution, please ref to [2].

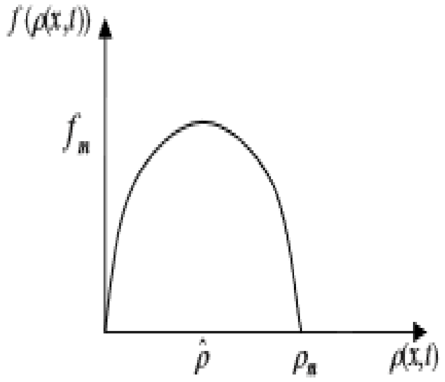


Figure 1.6: Greenshield's model-flux as a function of density of traffic flow [8]

1.5.1 Limitations of LWR Model

LWR model fails to represent/describe the behavior of traffic in light traffic. If we use velocity-density functions, traffic is in equilibrium position. It means that especially for light traffic, for a given traffic density, the velocity will be fixed. So we can say that this model does not recognize the individual or distributions of desired velocities. Stop-and-go traffic condition can't explain of LWR model. Basically LWR model is simplest continuum model among all the models(continuum models) and cannot display traffic sound waves, shocks and refraction waves. To incorporate the physical features of traffic, which LWR model could not explain, one needs to study second order models.

1.6 Momentum Conservation

Momentum is the commodity(product) of the velocity $v(x,t)$ and the density $\rho(x,t)$. Mass conservation clears, the flux for momentum is uses by the product of momentum and velocity i.e $\rho(x,t)v^2(x,t)$. We know that due to Newton's second law, the change in product od velocity and density(momentum) should be equal to the force applied.Basically, Force is the product of pressure and area. Now force to be $\rho(x_1,t)$ on the left side, and $\rho(x_2,t)$ on the right side. Use the Newton's law [9]:

$$[\rho(x,t)v(x,t)]_x + \left[\rho(x,t)v^2(x,t) + \rho(x,t)\right]_x = 0 \tag{1.17}$$

1.7 Second Order Models

The second order models consist of two partial differential equations. The first equation is based on mass conservation and second equation is based on the momentum conservation. Below, we discuss briefly two famous second order models.

Payne Whitham(PW) Model

This is the first system model to be conferred is a two-equation model. This two-equation model was designed in the 1970s[15, 20]. In their model, second equation was treated as a couple velocity dynamics. Their two-equation model was the first model in which second equation has coupled velocity and this model is referred to as PW model. The first equation of this model is the mass conservation as we already used in previous sections:

$$\rho_t + (\rho v)_x = 0 \tag{1.18}$$

where $\rho v = f(\rho, v)$ as the flux function.

Basically, in LWR model velocity is a function of density only where a specific form of v was assumed. But in higher order models, these quantities ρ and v are considered to be independent. In fluid and gas models, the second equation has these quantities as independent. For 1-D compressible flow, by using of Navier-Stokes equation of motion second equation of this model is derived. In this model, the pressure term is replaced by $P = C_0^2$ Where C_0 is anticipation term . For keeps speed concentration in equilibrium, the relaxation term include this model.

$$\frac{V(\rho) - v}{\tau} \tag{1.19}$$

where $V(\rho)$ is the maximum velocity and τ is a relaxation time. The nonconservative form of PW model in second equation is given by:

$$v_t + v v_x = \frac{V(\rho) - v}{\tau} - \frac{C_0^2}{\rho}\rho_x \tag{1.20}$$

The major drawback of this model is that this model follows the fluid flow theory. Researchers says that the model heavily follows the fluid theory. In fluids, surrounding particles can affects the behavior of the particles. It means that vehicles are allowed to move against the flow, i.e., with negative velocity. The eigenvalue, $\lambda_{1,2} = v + C_0$ is always bigger than the vehicle velocity v. This is clear from the eigenvalue that information behind the vehicle can be affected by action of driver. But this is not valid in real life traffic flow. So, This is referred as isotropic property.

AR model

Many researchers are argued to revised on the PW model. The authors says that traffic is affected more by the flow in front rather than behind. This model is looks like a fluid flow models. There is no particular difference between fluids and traffic. Like fluids, traffic is affected maximum by flow in front.Using anisotropic property, authors argued that "Pressure" terms replaced with an anticipation term. This is like as an driver can't fix a difference between any two types of flow. Authors claim that there is derivative of pressure with respect to x of incorrect anticipation term. The correct one contains full derivative is given by:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + (\vec{v}.\delta)\phi \tag{1.21}$$

Where ϕ is a function of $\vec{x}$ and t. Here $\vec{x}$ belongs to n real numbers. We know that driver drivers a car with v.Here the density is increasing with x but decreasing with $(x - vt)$. In this the anisotropic property is preserved.

Basically second order models can be divided into two broad categories:

1.7.1 Density Gradient(DG) Model

Many authors have investigated non-linear characteristic of traffic using DG model [15]. In this model pressure term depend on density only. That is why this model is called density-gradient(DG) model.

$$\rho_t + (\rho v)_x = 0 \tag{1.22}$$

$$v_t + v v_x = -\left(\frac{1}{\rho}\right) P_x + \frac{1}{\tau}(V_e(\rho) - v) \tag{1.23}$$

Where $P = P(\rho, v)$ denotes the traffic pressure term and τ is relaxation time. Here pressure is depend on density only.

1.7.2 Speed Gradient(SG) Model

Jiang et al. proposed a continuum SG model as [7]:

$$\rho_t + (\rho v)_x = 0 \tag{1.24}$$

$$v_t + v v_x = \frac{1}{\tau}(V_e(\rho) - v) + c_0 v_x \tag{1.25}$$

Where c_0 is the propagation speed of a small perturbation. This model is anisotropic model.

2

Review of a continuum second order traffic flow model

Traffic congestion is causing by slowing down of vehicles. Due to traffic congestion, there is no empty space on roads and increment in the vehicular queueing. These factors leads to traffic congestion or jam. Bottleneck situation is one of the commonly arising interruption .

In this chapter, we review a second order continuum model in which Gupta and Dhiman [5] have investigated the effects of a static bottleneck in a second order continuum traffic flow model.

2.1 What is bottleneck?

A disruption of vehicular traffic on road is called traffic bottleneck. Due to presence of bottleneck on roads,the speed of vehicles is affected.

2.2 Background

In literature,the physical fact of bottleneck had been given importance in past [1]. The disruption characterized by a bottleneck had been studied using car-following modeling [17] as well as macroscopic approach [8]. The dynamics properties of traffic were unable

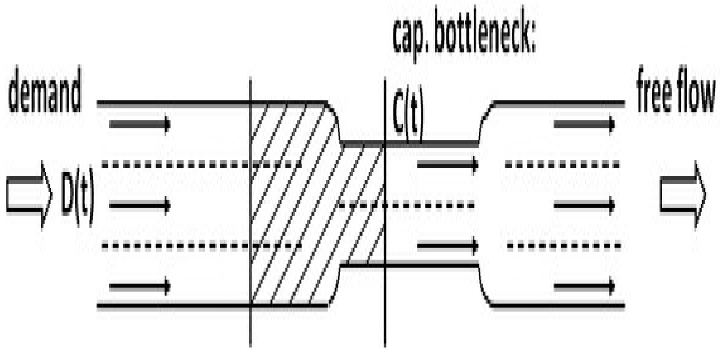


Figure 2.1: Bottleneck is at middle of the road[Source:Google images]

to completely describe by almost all the models because of lack of explicit study of bottleneck into model equations in the presence of bottleneck. Lattanzio et al.[11] introduce a continuum model. This continuum model was incorporating explicitly with moving bottleneck .The conservation of vehicle was not secured or preserve ,described lacking of this model. This model disrupts the anisotropic features of traffic flow. Lattanzio's model had some shortcomings. These shortcomings were improved by Tang el al. and its effect on different traffic flow facts(phenomena) was examined by Tang el al. These traffic flow phenomena were such as refraction waves and shock, in case of static [18] there was equilibrium flux as well as multi-static bottleneck[16]. We can also study the phase diagrams by using traffic patterns on the model.Many authors investigate these phase patterns.

2.3 Mathematical Model

In the reviewed paper, two approaches have been used by the authors.

2.3.1 Explicit Approach

Many authors have investigated the situation of bottleneck[1].But they did not investigate the impact of bottleneck directly. Lattanzio et al. [11] introduced a model with moving bottleneck given by:

$$\rho_t + f(x, y(t), \rho)_x = 0 : \rho(0, x) = \rho_0(x) \tag{2.1}$$

$$\dot{y}(t) = \omega(\rho(t, y)), y(0) = y_0 \tag{2.2}$$

where $\omega(\rho)$ is speed of the bottleneck and $y = y(t)$ is position of bottleneck. The flux f is defined as:

$$f(x, y(t), \rho) = \rho v(\rho)\phi(x - y(t)) \tag{2.3}$$

where $\phi(x-y(t))$ is the cutoff function. Due to the presence of bottleneck, cutoff function performing capacity drop. Equation (2.2) perform that flux will be shorten near the bottleneck. Above two equations (2.2) and (2.3) investigate or examine on dynamic properties of traffic flow on moving bottleneck. Tang el al. [18] says that this model didn't satisfy the law of conservation of traffic flow. But this model also fails to describe the anisotropic property. Tang el at. [18] gives a new SG model for incorporating of a static bottleneck given by follows 2.3 equation:

$$\rho_t + (\rho v)_x = 0 \tag{2.4}$$

$$v_t + v v_x = \frac{1}{\tau}(V_e(\rho) - v) + c_0 v_x + F \tag{2.5}$$

Where F denotes the friction term due to bottleneck. This term tells the explicit behavior of bottleneck. It was observed that on-ramp's[17] friction is same as friction yield by bottleneck. So,

$$F = -\mu s(x, t)\rho v \tag{2.6}$$

where μ is friction coefficient and $s(x, t) = \frac{\tilde{q}}{L_0}$, L_0 is the bottleneck's length and

$$\tilde{q} = \rho v(\rho)(1 - \phi(x - y)) \tag{2.7}$$

where y is the location of static bottleneck. Friction F depends upon bottleneck's architect and stability of bottleneck.

2.3.2 Implicit Approach

Lighthill and Withiam [13] identified that maximum value of flow rate shorten or reduces inside the region of bottleneck. If we plot a graph between flow and density, then curve reduces in its vertical scale. The maximum (jam) density also reduces inside the bottleneck. If we consider a region of bottleneck only on a small portion of a long roads or highway, then the reduction of jam density can be ignored. On passing through the bottleneck, each vehicle feels a reduction in its velocity, which further leads to reduction in the magnitude of the free velocity. The reduction in the velocity of the vehicle can occur due to the capacity dropping.

We consider a road which is one-dimensional with static bottleneck. The length of the road denoted by M. Now, static bottleneck is located on a small region between M_1 and M_2 $(0 < M_1 < M_2 < M)$

Suppose x is a union of $[0, M_1)$ and $(M_2, M]$, i.e x belongs to $[0, M_1)U(M_2, M]$. This means that the region outside the bottleneck, Speed-gradient continuum model conducted the traffic dynamics.

$$v_t + vv_x = \frac{1}{\tau}(V_{eb}(\rho) - v) + c_0 v_x \tag{2.8}$$

where $V_{eb} < V_e(\rho)$ is the modified equilibrium peed inside the bottleneck.

2.4 Numerical Scheme

We have two equations:

$$\rho_t + (\rho v)_x = 0 = 0 \tag{2.9}$$

$$v_t + vv_x = \frac{1}{\tau}(V_e(\rho) - v) + c_0 v_x + F \tag{2.10}$$

Where $F = -\mu s(x,t)\rho v$, also $s(x,t) = \tilde{q}/L_0$ and $\tilde{q} = \rho v(\rho)(1 - \phi(x - y))$

We have to find the finite difference of above two equations according as follows: let's take first equation.

$$\rho_t + (\rho v)_x = 0 \tag{2.11}$$

The second term of the above equation can be written as:

$$(\rho v)_x = \rho_x v + v_x \rho \tag{2.12}$$

$$(\rho v)_x = \left(\frac{\rho(i,j) - \rho(i-1,j)}{\partial x}\right) v(i,j) + \left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right) \rho(i,j) \tag{2.13}$$

The first term of the (2.11)equation can be written as:

$$\rho_t = \frac{\rho(i,j+1) - \rho(i,j)}{\partial t} \tag{2.14}$$

Combining these equations (2.11),(2.13) and (2.14), we get as follows:

$$\frac{\rho(i,j+1)-\rho(i,j)}{\partial t}+\left(\frac{\rho(i,j)-\rho(i-1,j)}{\partial x}\right)v(i,j)$$
$$+\left(\frac{v(i+1,j)-v(i,j)}{\partial x}\right)\rho(i,j)=0 \quad (2.15)$$

$$\frac{\rho(i,j+1)-\rho(i,j)}{\partial t}=-\left(\frac{\rho(i,j)-\rho(i-1,j)}{\partial x}\right)v(i,j)$$
$$+\left(\frac{v(i+1,j)-v(i,j)}{\partial x}\right)\rho(i,j) \quad (2.16)$$

$$\frac{\rho(i,j+1)}{\partial t}=\frac{\rho(i,j)}{\partial t}-\frac{\rho(i,j)v(i,j)}{\partial x}+\frac{\rho(i-1,j)v(i,j)}{\partial x}$$
$$-\frac{v(i+1,j)\rho(i,j)}{\partial x}+\frac{v(i,j)\rho(i,j)}{\partial x} \quad (2.17)$$

Now equation can be written as follows:

$$\rho(i,j+1)=\rho(i,j)+\frac{\partial t}{\partial x}\rho(i-1,j)v(i,j)-\frac{\partial t}{\partial x}v(i+1,j)\rho(i,j) \quad (2.18)$$

$$\rho(i,j+1)=\rho(i,j)+\frac{\partial t}{\partial x}\left(\rho(i-1,j)v(i,j)-v(i+1,j)\rho(i,j)\right) \quad (2.19)$$

$$\rho_i^{j+1}=\rho_i^j+\frac{\partial t}{\partial x}\left(\rho_{i-1}^j v_i^j-\rho_i^j v_{i+1}^j\right) \quad (2.20)$$

We pick the second equation of the model:

$$v_t + vv_x = \frac{1}{\tau}(V_e(\rho) - v) + c_0 v_x - \frac{\mu\rho v(\rho)(1-\phi(x-y))\rho v}{L_0} \tag{2.21}$$

The first term of the above equation can be written as:

$$v_t = \frac{v(i,j+1) - v(i,j)}{\partial t} \tag{2.22}$$

The second term of the equation (2.21) can be written as:

$$vv_x = v(i,j)\left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right) \tag{2.23}$$

Combining these equation (2.21), (2.22) and (2.23), we get:

$$\begin{aligned} \frac{v(i,j+1) - v(i,j)}{\partial t} + v(i,j)\left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right) &= \frac{1}{\tau}(V_e(\rho) - v(i,j)) \\ &+ c_0\left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right) - \frac{\mu\rho v(\rho)(1-\phi(x-y))\rho v}{L_0} \end{aligned} \tag{2.24}$$

$$\begin{aligned} \frac{v(i,j+1) - v(i,j)}{\partial t} &= -v(i,j)\left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right) + \frac{1}{\tau}(V_e(\rho) - v(i,j)) \\ &+ c_0\left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right) - \frac{\mu\rho v(\rho)(1-\phi(x-y))\rho v}{L_0} \end{aligned} \tag{2.25}$$

$$\begin{aligned} \frac{v(i,j+1)}{\partial t} = \frac{v(i,j)}{\partial t} - v(i,j)\left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right) &= \frac{1}{\tau}(V_e(\rho) - v) \\ &+ c_0\left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right) - \frac{\mu\rho v(\rho)(1-\phi(x-y))\rho v}{L_0} \end{aligned} \tag{2.26}$$

$$v(i,j+1) = v(i,j) - \frac{\partial t}{\partial x}\left(v(i+1,j) - v(i,j)\right)v(i,j) + \partial t\frac{1}{\tau}(V_e(\rho) - v(i,j))$$

$$+ c_0\left(\frac{v(i+1,j) - v(i,j)}{\partial x}\right)\partial - \frac{\mu\rho v(\rho)(1-\phi(x-y))\rho v}{L_0}\delta t \quad (2.27)$$

$$v_i^{j+1} = v_i^j - \frac{\partial t}{\partial x}\left(v_{i+1}^j - v_i^j\right)v_i^j + \partial t\frac{1}{\tau}(V_e - v_i^j) + c_0\frac{\partial t}{\partial x}\left(v_{i+1}^j - v_i^j\right) - \frac{\partial t}{L_0}(\rho_i^j v_i^j)^2\mu(1-\phi) \quad (2.28)$$

$$v_i^{j+1} = v_i^j - \frac{\partial t}{\partial x}\left(v_{i+1}^j - v_i^j\right)v_i^j + \partial t\frac{1}{\tau}(V_e - v_i^j) + c_0\frac{\partial t}{\partial x}\left(v_{i+1}^j - v_i^j\right) - g_i^j \quad (2.29)$$

For $v_i^j < c_0$

$$v_i^{j+1} = v_i^j + \frac{\partial t}{\partial x}(c_0 - v_i^j)\left(v_{i+1}^j - v_i^j\right) + \frac{\partial t}{\tau}((V_e)_i^j - v_i^j) + g_i^j \quad (2.30)$$

For $v_i^j >= c_0$

$$v_i^{j+1} = v_i^j + \frac{\partial t}{\partial x}(c_0 - v_i^j)\left(v_i^j - v_{i+1}^j\right) + \frac{\partial t}{\tau}((V_e)_i^j - v_i^j) + g_i^j \quad (2.31)$$

where

$$g_i^j = -\frac{\partial t}{L_0}(\rho_i^j v_i^j)^2\mu(1-\phi) \quad (2.32)$$

2.5 Results and discussion

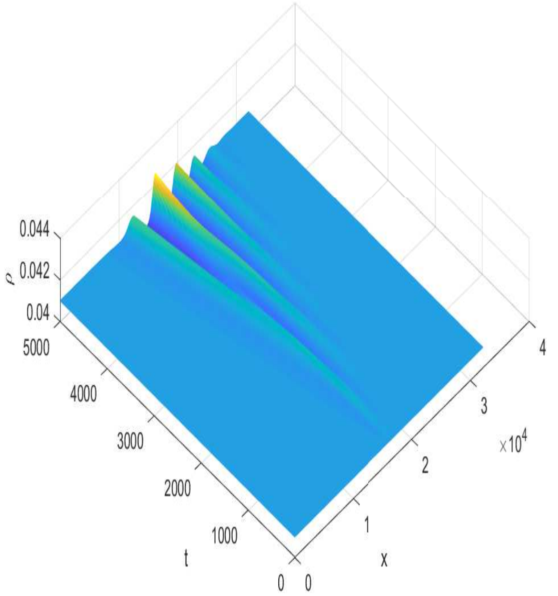
2.5.1 Spatiotemporal evolution with explicit approach

For explicit approach, we use the following equilibrium speed-density relation[10].

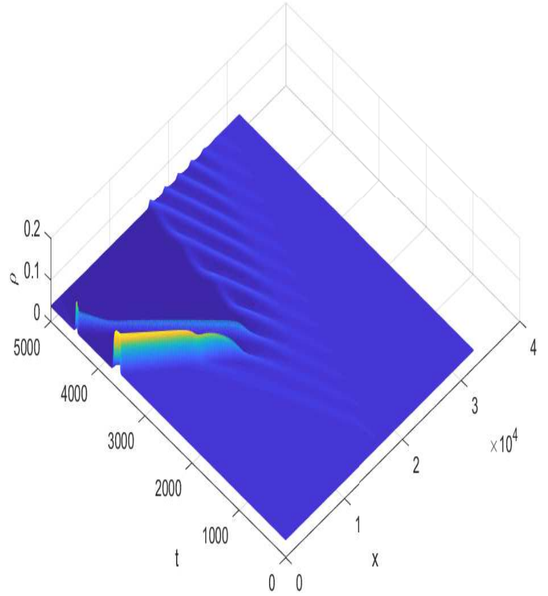
$$V_e(\rho) = v_f\left[\left(1 + exp\left(\frac{\frac{\rho}{\rho_m} - 0.25}{0.06}\right)\right)^{-1} - 3.72 * 10^{-6}\right] \quad (2.33)$$

where ρ_m is maximum density and v_f is free flow speed.
Inside the portion of instability, we review for different phases in order to characterize

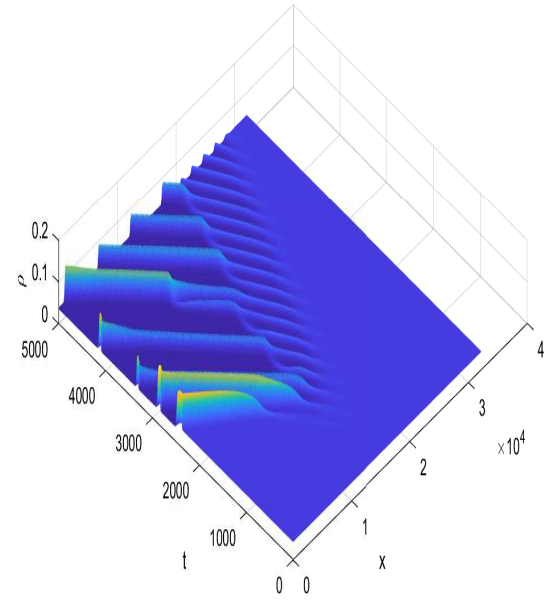
(a) ρ_0 = 0.041 veh/m



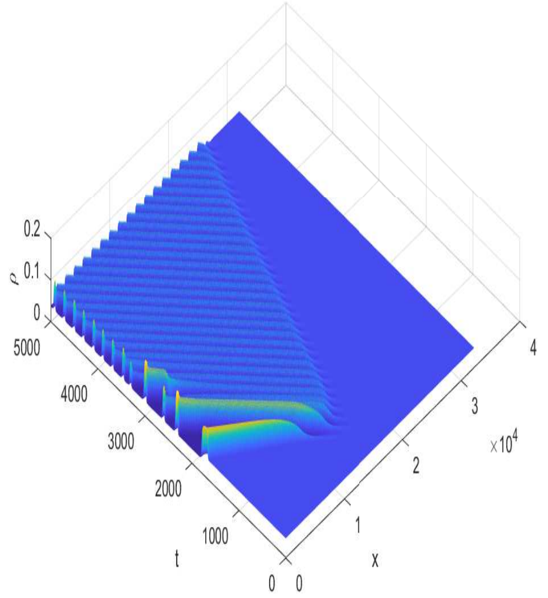
(b) ρ_0 = 0.043 veh/m



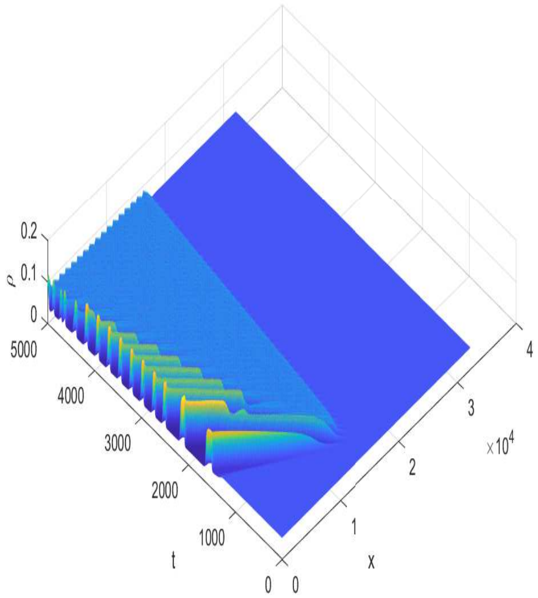
(c) ρ_0 = 0.045 veh/m



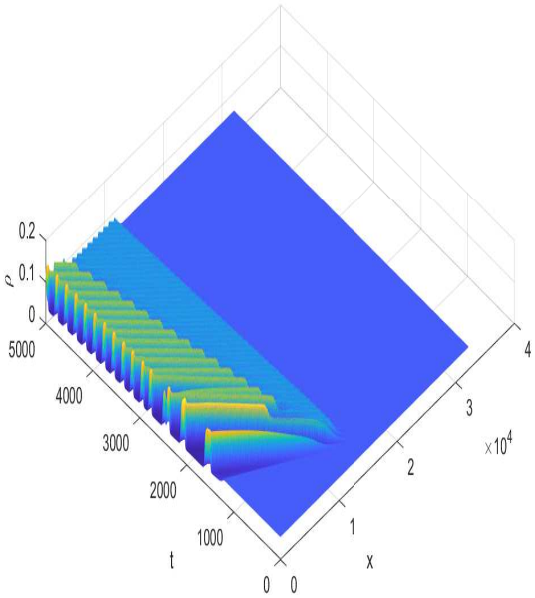
(d) ρ_0 = 0.049 veh/m



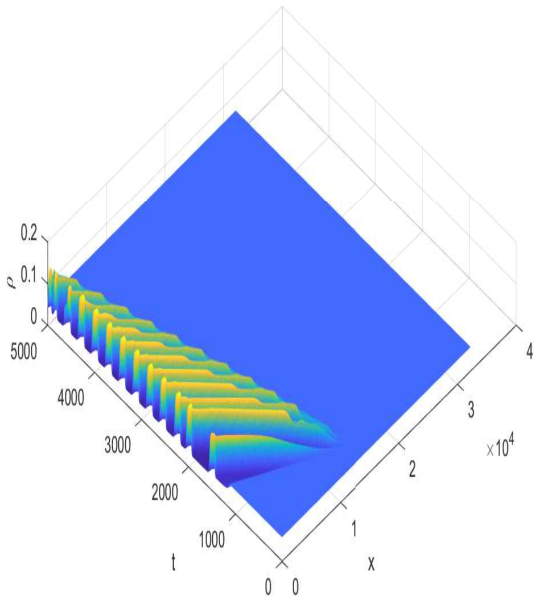
(e) ρ_0 = 0.054 veh/m



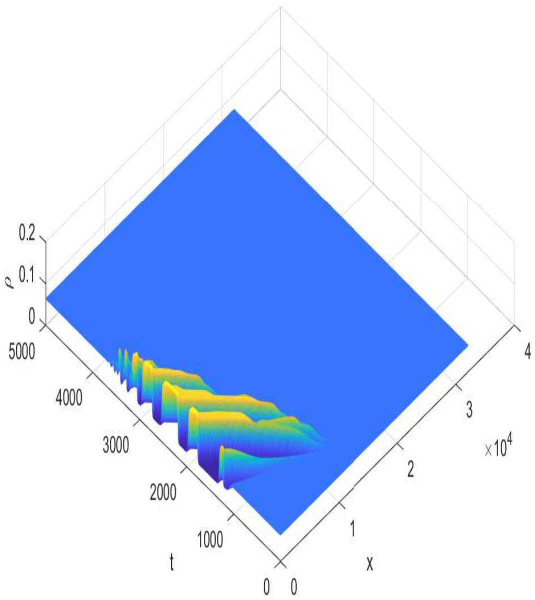
(f) ρ_0 = 0.056 veh/m

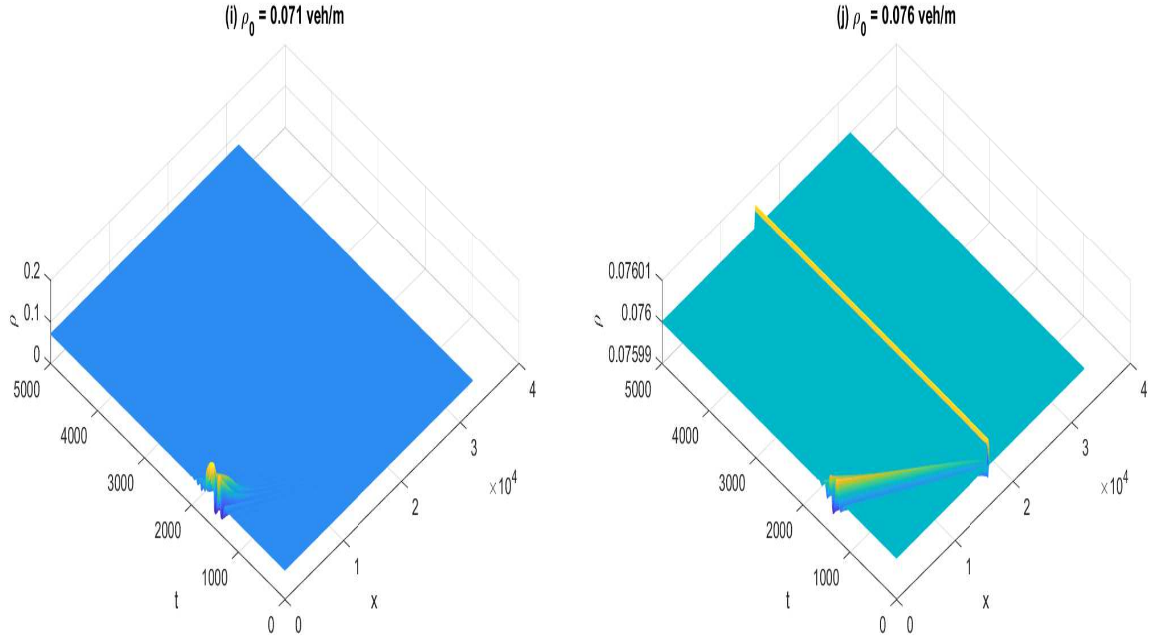


(g) ρ_0 = 0.06 veh/m



(h) ρ_0 = 0.064 veh/m





(i) ρ_0 = 0.071 veh/m

(j) ρ_0 = 0.076 veh/m

Figure 2.2: Spatiotemporal evolution for various states

the different pattern of traffic flow.It is assumed that presence of bottleneck for a small portion will not affect the region of instability.The stability criteria for speed-gradient model without bottleneck has been found as [7]

$$\rho_0 V_0'(\rho) < -c_0 \tag{2.34}$$

According as parameters used in explicit approach, the instability found to be $0.03 \leq \rho_0 \leq 0.085$. Different traffic pattern occurs for different values of initial density are shown in fig(2.2). The spatiotemporal evolution in fig. (2.2) shows that the affect on traffic flow due to presence of bottleneck on the road.

We look that there are different phases due to bottleneck which affects the traffic flow. The spatiotemporal evolution of various patterns considered because of bottleneck on the middle of the road.

It is observed that for smaller values of density (even inside the region of instability) i.e., $0.03 < \rho_0 < 0.039$ veh/m, the bottleneck has no significant effect on traffic flow. For these densities, amplitude is small because perturbation is induced by bottleneck itself. The density in both upstream and downstream remains same. Therefore, there is a small transition layer only at the position of bottleneck. So this state is called free-flow(FF) state.

It is to be considered that even though vehicles are inside the region of instability, there is no interruption in traffic flow even in the presence of bottleneck. For low density, vehicle will move with desired velocity and there is sufficient space even inside the bottleneck. Due to an increase in density, the amplitude of perturbation also increases. As density reaches to $\rho_0 = 0.041 veh/m$ disturbance due to bottleneck increases the amplitude slowly, but remains localized at bottleneck. After a long time, the small amplitude perturbation occurs in the downstream direction. At density $\rho_0 = 0.043 veh/m$, two localized perturbation move backward and increases the amplitude with time (2.2(b)) This state is called moving localized clusters (MLC) state. Further as increase in density , amplitude increases, perturbation emerges and moves backward. We find two states MLC and oscillatory congested traffic (OCT)

At density $\rho_0 = 0.054 veh/m$, we get another phase known as mixed congested traffic(MCT). In figure (2.2(e)) there are two types of subregions. The region which is upstream to the bottleneck having density higher than reliable density. Also downstream region having backward moving clusters. Here, note that there exit a particular value of initial density which is ρ_b. For $\rho_0 \leq \rho_b$ bottleneck spread impact on both upstream and downstream directions. There is a restriction on the affected region up to upstream of the bottleneck for values $\rho_b \leq \rho_0$. We note that our ρ_b is 0.054 veh/m. Mixed congested traffic remains same till $\rho_0 = 0.058 veh/m$. At this points both the subregions merge into one single region. From figure (2.2(e)) to (2.2(g)), we can see that two subregions merge into one region.

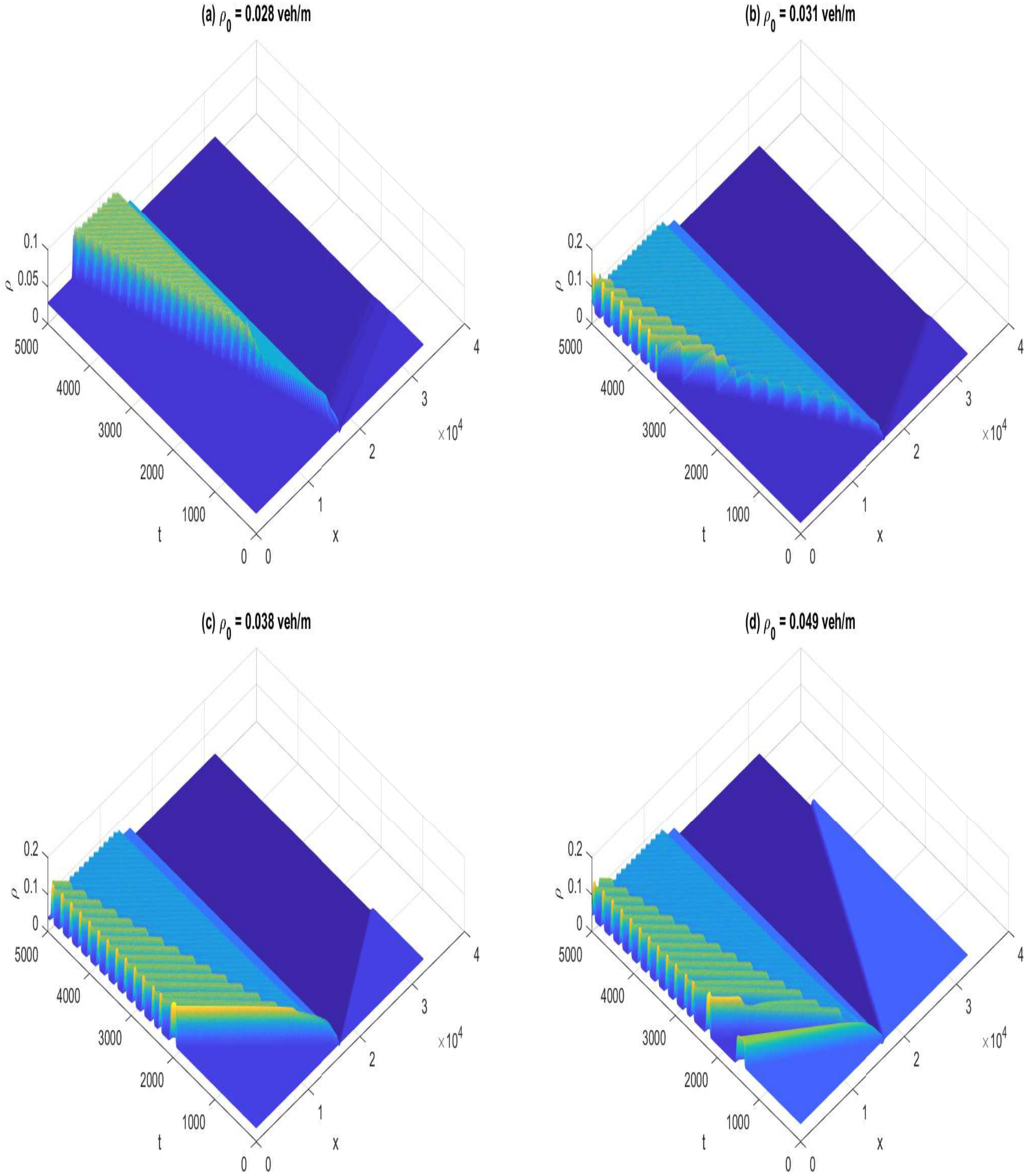
Further, with the increases in density , the perturbation starts shrinking. For ρ_0 greater than 0.064 figures (2.2(h) ,2.2(i) and 2.2(j)) This value is much lesser than $\rho_c = 0.085 veh/m$, which is critical value.

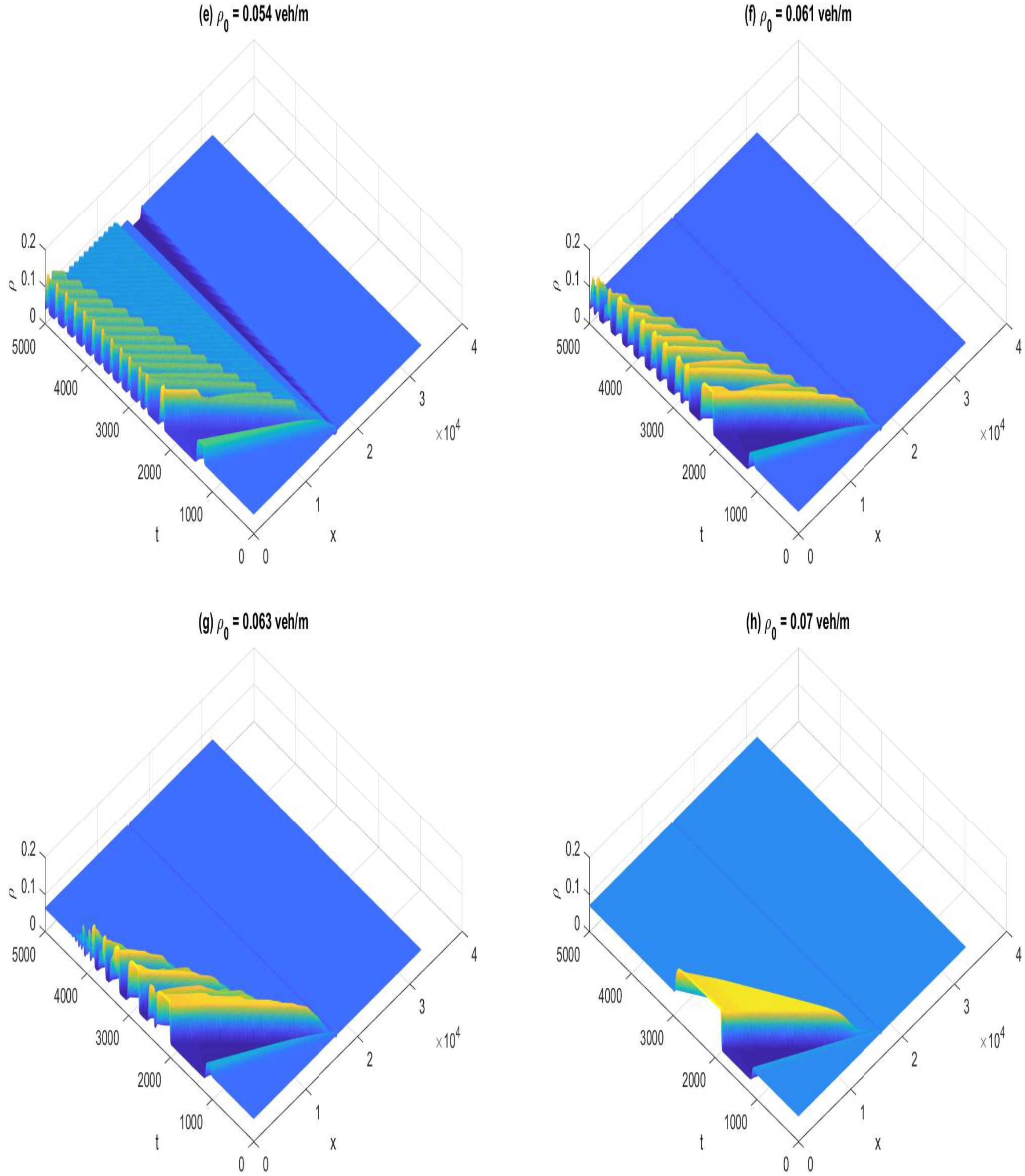
2.5.2 Spatiotemporal evolution with implicit approach

For implicit approach, we use the modified equilibrium speed inside the bottleneck is given as:

$$V_{eb} = v_b \left[\left(1 + exp \left(\frac{\frac{\rho}{\rho_m} - 0.25}{0.06} \right) \right)^{-1} - 3.72 * 10^{-6} \right] \tag{2.35}$$

where v_b is reduced free velocity. The only difference which differentiate this approach from explicit approach is the equilibrium speed-density relation. The vehicle's free velocity v_f reduces to v_b,when vehicle is entering from the region L_1 into the bottleneck. This reduction remain same throughout the bottleneck region. When vehicle exits the


(a) ρ_0 = 0.028 veh/m
(b) ρ_0 = 0.031 veh/m
(c) ρ_0 = 0.038 veh/m
(d) ρ_0 = 0.049 veh/m
ρ
0
0.05
0.1
0.2
t
0
1000
2000
3000
4000
5000
x
0
1
2
3
4
$\times 10^4$


(e) ρ_0 = 0.054 veh/m
(f) ρ_0 = 0.061 veh/m
(g) ρ_0 = 0.063 veh/m
(h) ρ_0 = 0.07 veh/m
ρ
0
0.1
0.2
t
0
1000
2000
3000
4000
5000
x
0
1
2
3
4
$\times 10^4$

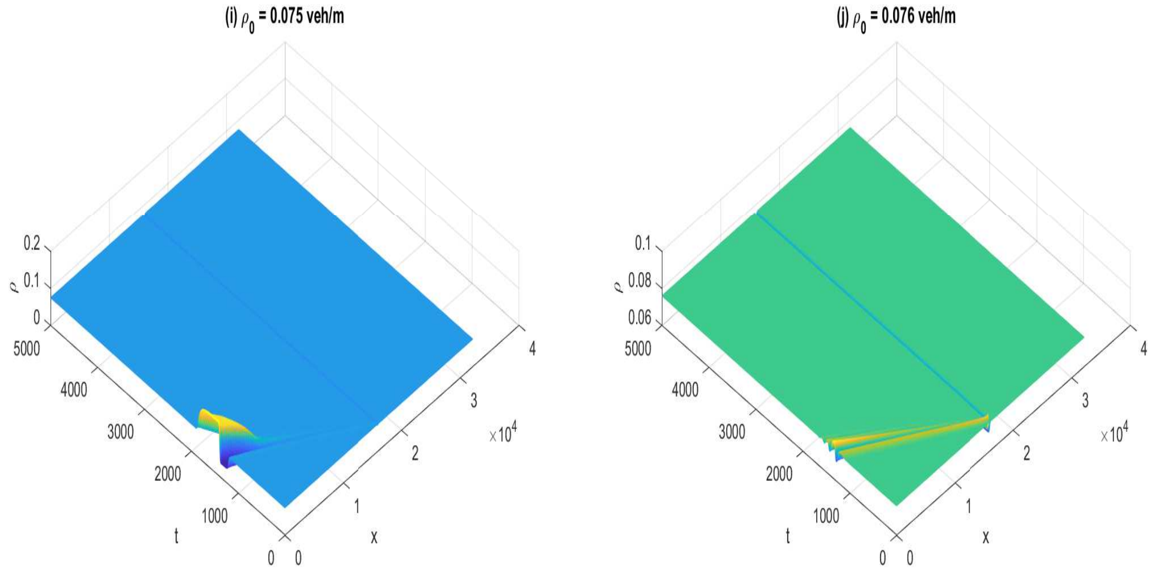


Figure 2.3: Spatiotemporal evolution for various states with $v_b = 20 m/s$

bottleneck region, they will try to obtain the same free velocity v_f . Spatiotemporal evolution of various states for various densities is shown in figure (2.3), which is obtained using implicit approach. There is interruption in traffic flow at the position of bottleneck due to small increases in density such as for $\rho_0 \leq 0.027$. In this region, transition layer is effectively confined. This layer doesn't affect on downstream as well as upstream. At $\rho_0 \leq 0.028 veh/m$, no interruption in traffic flow are found expect with a small rise in the density at the position of bottleneck. The transition layer is completely localized and does not affect upstream and downstream regions.

At $\rho_0 = 0.028 veh/m$, we observe a drastic change in traffic pattern consisting of two subregions in the upstream direction.

A small subpart of homogeneous high density (HCT) is formed just upstream to the bottleneck while the other subpart comprises of high density uniformly spaced clusters moving backwards (OCT). Such state referred as OCT-HCT (2.3(a)) .The HCT area remains fixed in its width as time increases while the OCT area grows in upstream direction with time. Furthermore, one finds a homogeneous low-density area widely spread in downstream to the bottleneck. This can be cleared as when a vehicle comes out of bottleneck, its velocity will increase while the flow rate at this position still equals to the dropped capacity inside the bottleneck. Due to this, the vehicle will observe a decrease in the density which leads to formation of a low-density region in downstream.

This is same as in LWR[13] model. Further, increases in density till $\rho_0 = 0.031 veh/m$, subregion OCT spreads in upstream direction. In figures (2.3(b)) ,(2.3(c)) and (2.3(d)), we observe the phase MCT-HCT In this phase HCT region(upstream) has not changed as previous approach but OCT region divide into two subregions which shows MCT pattern. So this phase is referred as MCT-HCT.

Furthermore, with the increases in density ,the downstream region will shrink but upstream region remains same. Here the critical density is $\rho_d = 0.054veh/m$ in figure (2.3(e)). After this the downstream region wholy disappears.Only small layer shown in upstream region 2.3(h).

On increasing in ρ_0 ,due to perturbation amplitude will also increases. Also after this the perturbations combines and making one with higher amplitude (fig 2.3(f)).

2.6 Comparison between both the approaches

The spatiotemporal evolution figures with implicit approach shows the phase transitions from one traffic state to another traffic state. Here, phase transitions shows the states on varying the magnitude of reduced free velocity. The number of phases decreases with decrease in bottleneck strength. Hence, it is feasible to conclude that strength of bottleneck affects the traffic states considerably. In the explicit case, we also finds the effect of the strength parameter and scope of bottleneck on traffic patterns and found that the traffic patterns do not change, which shows that these parameters do not play significant role. The effect of strength of bottleneck is concealed in explicit approach, which seems unrealistic. From comparison in our opinion, further modifications are required to explicitly model the bottleneck which can describe the complex traffic phenomena more realistically.

The cluster effect in explicit approach is found to emerge strictly within the region of instability, while in implicit approach even inside the stability region,we found large amplitude perturbations for lower values of v_b. In explicit case, the perturbations induced by bottleneck also affect the downstream direction ,while in implicit approach,the homogeneous low-density region exists in the downstream. This difference persists for $v_b \leq 25m/s$, after which we find consistency in the downstream pattern of both the approaches. This situation is fair from fig (2.4)

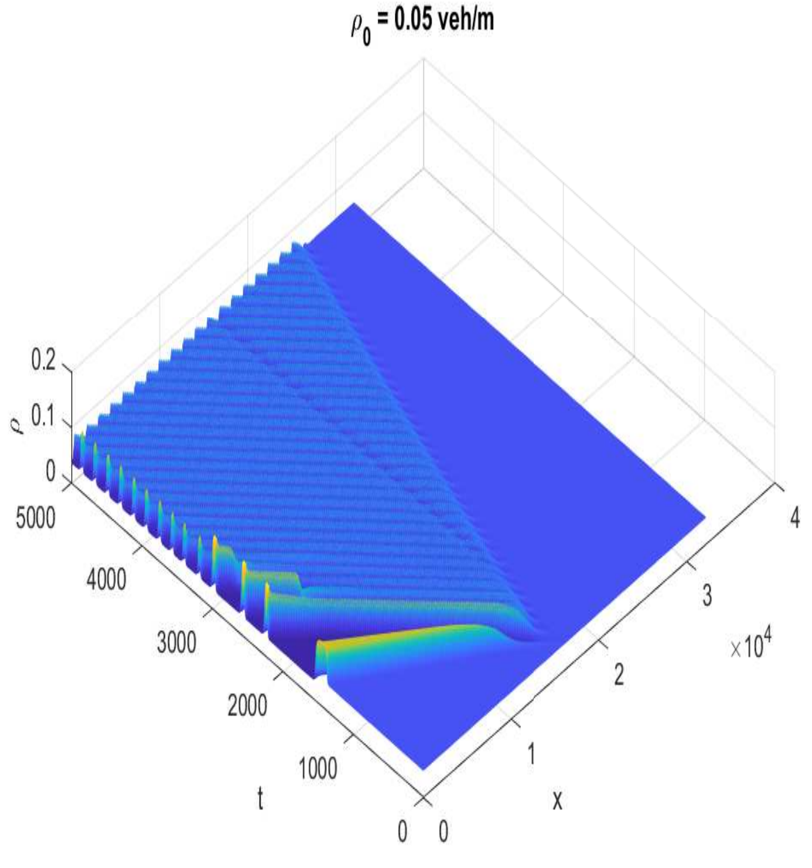


Figure 2.4: Traffic pattern for $v_b = 29m/s$

3

Conclusion

In this thesis, we have studied the effects of bottleneck on vehicular traffic. A through review has been done on the work by Gupta and Dhiman in their paper "Phase diagram of a continuum traffic flow model with a static bottleneck". The authors have considered a second order speed gradient continuum model. The bottleneck has been fixed in middle of the road of length 32.2 km. The effect of bottleneck has been studied by two approaches-explicit and implicit.

In explicit approach, authors have investigated on various phase transitions from one traffic state to another state. The cluster effect emerges within the region of instability and the parameters do not play significant role. This approach seems unrealistic. Further modifications are required in this approach so that this approach can describe the traffic flow more realistically.

In implicit approach, the traffic state varying the magnitude of reduced free velocity. In this approach we found higher amplitude in the region of instability.

Various traffic states MCT, OCT, HCT , OCT-HCT, MCT-HCT are found by variation in value of initial density within the range $0.03 \leq \rho_0 \leq 0.085 veh/m$

It is found that the implicit approach captures the effect of bottleneck more effectively than the explicit approach.Therefore, the results in implicit approach are more sensitive to bottleneck than the explicit approach

3.1 Future directions

The future work will focus on the study of the effect of bottleneck on multi-lane roads and multi-class vehicles. One can also study the effect of two bottleneck on the road. Moreover, one can also explore the role of bottleneck in presence of other physical situations such as on-ramp[12, 6] and off-ramp etc.

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