

# **EXISTENCE OF FIXED POINTS FOR SOME MAPPINGS IN VARIOUS SPACES**

A

THESIS

SUBMITTED IN FULFILLMENT OF THE  
REQUIREMENT FOR THE AWARD OF THE DEGREE

OF

**DOCTOR OF PHILOSOPHY  
(MATHEMATICS)**

By

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## Candidate's Declaration

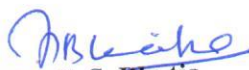
I hereby certify that the work being presented in the thesis entitled "**Existence of Fixed Points for Some Mappings in Various Spaces**", in fulfillment of the requirements for the award of the degree of Doctor of Philosophy, submitted in the School of Mathematics and Computer Applications of Thapar University, Patiala, is an authentic record of my own work carried out during a period from January, 2010 to June, 2014 under the supervision of Dr. S. S. Bhatia and Dr. Jatinderdeep Kaur.

The matter embodied in the thesis has not been submitted by me for the award of any other degree.


Date: 30/09/2014

  
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This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

  
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*I dedicate this work to my parents, who guided and loved me through my life, taught me again and again to never give up and to follow my dreams. And to my Teachers from whom I learned lot of principles and the best ones are the principles of life.*

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# List of symbols

$A = B, A \neq B:$	Equality and Inequality for sets
$[a, b], [a, b)$ etc:	Intervals on the real line
$f(x)$ or $fx:$	Image of $x$ under $f$
iff:	If and only if
sup:	supremum (or least upper bound)
inf:	infimum (or greatest lower bound)
$\lim x_n = x:$	limit of the sequence $x_n$
max:	Maximum
min:	Minimum
$\mathbb{N}:$	The set of all natural numbers
$\mathbb{R}:$	The set of all real numbers
$\mathbb{R}^+:$	The set of all non negative real numbers
$\mathbb{R}^n:$	n-dimensional Euclidean space
$ x :$	Absolute value of $x$
$\rightarrow, \Leftrightarrow:$	Implication and logical equivalence

For a mapping  $T : A \rightarrow A$ ,

$F(T)$ : The set of fixed points of  $T$

$I = I_A$ : The identity mapping

$2^A$ : The collection of all subsets of  $A$

For a metric space  $(A, d)$ ,

$d(a, b)$ : Distance from one point to another

$d(x, B)$ : Distance between the point  $x$  and set  $B$

$B(a, r)$ :  $\{b \in A : d(a, b) < r\}$

$\phi$ : empty set

$\in$ : belongs to, belonging to

$\notin$ : does not belong to

For a complex number  $w$ ,

$\Re$ : The real part of  $w$

$\Im$ : The imaginary part of  $w$

# Abstract

The present thesis entitled “**Existence of Fixed Points for Some Mappings in Various Spaces**” comprises certain investigations carried out by me at the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, under the supervision of Dr. S. S. Bhatia, Professor, SMCA, Thapar University, Patiala and Dr. Jatinderdeep Kaur, Assistant Professor, SMCA, Thapar University, Patiala.

Nonlinear analysis deals with solving nonlinear problems in many branches of mathematics, physics and in industry. Fixed-point theory is an important branch of nonlinear analysis. It is used to investigate the conditions under which single-valued or multivalued mappings have solutions. Numerous problems occurring in different branches of mathematics, such as differential equations, optimization theory and variational analysis can be modeled by the equation

$$x = Tx$$

where  $T$  is a nonlinear operator defined on a metric space. The solutions of this equation are called fixed points of  $T$ . In 1906, the French mathematician Fréchet [69] introduced metric spaces which helped in the development of fixed point theory.

The first fixed point theorem in metric spaces for contraction mappings was proved by a Polish mathematician Stefan Banach in 1922. This theorem is known as Banach’s fixed point theorem or the Banach contraction principle. By now, this result has become one of the most popular and effective tool in solving existence problems in many branches of mathematical analysis. Due to the simplicity and usefulness of this basic theorem, it has become a very popular tool for proving the existence and uniqueness theorems in various

branches of mathematical analysis. This theorem depicts the unifying power of functional analytic methods and their utility in different areas of mathematics. In 1968, Kannan [91] introduced a contractive condition which possessed a unique fixed point like that of Banach. However, unlike the Banach condition, Kannan [91] proved that there are mappings that have a discontinuity in their domain but still have fixed point, although such mappings are continuous at their fixed point. This paper of Kannan [91] lead to lot of improvements and extensions of Banach contraction principle.

A very important mechanism for the progress in the field of science and technology is to generalize the existing ideas. The present thesis has been written on the same basis. In this thesis, various fixed point results in various abstract spaces such as: metric spaces,  $G$ -metric spaces, complex valued metric spaces, partial Hausdorff metric spaces and partially ordered metric spaces have been discussed and thereby many existing results have been extended and generalized.

The thesis embodies eight chapters. **Chapter 1**, is introductory. In this chapter, apart from setting up the notations and terminologies to be used in the sequel, some known results interrelated with the work done in the present thesis have also been presented. Further, a brief plan of the results presented in the subsequent chapters is given towards the end of this chapter.

In **Chapter 2**, a new and simple approach has been introduced in the theory of expansive mappings by presenting a new kind of expansive mappings called  $(\xi, \alpha)$ -expansive mappings in metric spaces. Several fixed point theorems for these mappings have been studied in this chapter. This new concept complements the idea of  $\alpha$ - $\psi$ -contractive type mappings introduced recently by Samet *et al.* [165]. In order to generalize the expansion mappings in a different way, a new type of expansive mappings called  $F$ -expansive mappings has also been introduced in this chapter. This new notion of  $F$ -expansive mappings complements the concept of  $F$ -contraction introduced recently by Wardowski [193].

**Chapter 3** deals with the generalization of the contractive conditions of integral type. In this chapter, a new kind of mapping called  $\alpha$ - $\psi$ -contractive type mapping of integral type has been introduced and some fixed point and coupled fixed point theorems have been

established for such mappings in complete metric spaces. As a consequence of the main results, some fixed point results on partially ordered metric spaces have also been obtained. The results presented in this chapter improve, extend and generalize the results derived by Banach [30], Berinde [31], Branciari [39], Rhoades and Abbas [152], Samet *et al.* [165] and references therein.

In **Chapter 4**, a new class of contractive pair of mappings called generalized  $\alpha$ - $\psi$  contractive pair of mappings has been introduced. Various coincidence and common fixed point results regarding this class of contractive pair of mappings have been studied in this chapter. As consequences of the main results, some common fixed point theorems have also been established for metric spaces endowed with a partial order as well as in respect of cyclic contractive mappings. The theorems presented in this chapter extend and generalize various known relevant results from the current literature. Some illustrative examples are provided to demonstrate the main results. The results presented in this chapter unify and generalize the results derived by Karapinar and Samet [100], Samet *et al.* [165], Ćirić *et al.* [55] and various other related results in the literature.

**Chapter 5** is devoted to the generalization of the results of Ahmed [17] from 2-metric spaces to  $G$ -metric spaces. In order to prove the results, a more generalized concept of weak compatibility in  $G$ -metric spaces have been used instead of compatibility of type(A) used by Ahmed [17] in 2-metric spaces. Also, the results of Şahin and Telci [161] are extended to  $G$ -metric spaces and thereby extending the theorem of Wang *et al.* [192] for a pair of mappings to  $G$ -metric spaces.

In **Chapter 6**, some quadruple coincidence point theorems have been obtained for a mixed  $g$ -monotone mapping satisfying nonlinear contractions in the setting of partially ordered  $G$ -metric spaces. The results presented in this chapter generalize the very recent results of Karapinar [93], Karapinar [94], Karapinar and Berinde [97], Karapinar *et al.* [101] and various other related results in the literature.

The objective of **Chapter 7** is to obtain the fixed point results for  $\alpha_*$ - $\psi$ -contractive multifunctions in a partial Hausdorff metric space. The presented theorems generalize and consolidate many related results in the literature, in particular the results of Aydi *et al.* [24],

Asl *et al.* [23] and the references therein. As a consequence of these results, a homotopy result has been presented as an application.

In **Chapter 8**, the generalized version of  $\alpha$ - $\psi$  contractive type mappings has been presented in complex valued metric spaces by defining the notion of  $(\alpha, \phi)$ -contractive mappings of rational type. Some fixed point theorems for these mappings have been established in the framework of complex valued metric spaces. The results presented in this chapter extend the recent results of Azam *et al.* [29], Rouzkard and Imdad [156] and references therein. Some examples have also been given to verify the effectiveness and applicability of main results. In order to support the usability of these results, an application to integral equations has also been presented. Towards the end of this chapter, some relevant topics for further research have been suggested based on the present study.

The thesis concludes by listing the Bibliography of various publications cited in this work.

# Research Publications

## Published/Accepted Papers in Referred Journals

- (1) Priya Shahi, Jatinderdeep Kaur and S. S. Bhatia, “Fixed point theorems for  $(\xi, \alpha)$ -expansive mappings in complete metric spaces”, **Fixed Point Theory and Applications** **2012**, **2012:157**. (SCI - Impact Factor - 1.87)
- (2) Priya Shahi, Jatinderdeep Kaur and S. S. Bhatia, “Common Fixed Points of Expansive Mappings in Generalized Metric Spaces” accepted in **Journal of Applied Research and Technology**. (SCI - Impact Factor - 0.3)
- (3) Priya Shahi, Jatinderdeep Kaur and S. S. Bhatia, “Fixed Point Theorems For Mappings of Integral Type with Applications” accepted in **Journal of Nonlinear and Convex Analysis**. (SCI - Impact Factor - 0.98)
- (4) Priya Shahi, Jatinderdeep Kaur and S. S. Bhatia, “Quadruple fixed point theorems for nonlinear contractions in partially ordered G-metric spaces”, **Tbilisi Mathematical Journal**, vol. 6, 2013, pp. 29-44.

## Communicated Papers

- (1) Priya Shahi, Jatinderdeep Kaur and S. S. Bhatia, “Fixed Point Theorems for  $(\alpha, \phi)$ -Contractive Mappings of Rational Type in Complex Valued Metric Spaces with Applications”, **Bulletin of the Iranian Mathematical Society**.

- (2) Priya Shahi, Jatinderdeep Kaur and S. S. Bhatia, “Fixed point theorems for F-expansive mappings in complete metric spaces”, **Ciência e Técnica Vitivincola Journal**.
- (3) Priya Shahi, Jatinderdeep Kaur and S. S. Bhatia, “Some Coincidence and Common Fixed Point Theorems for Generalized  $\alpha$ - $\psi$ -Contractive Type Mappings”, **The Bulletin of the Belgian Mathematical Society**.
- (4) Priya Shahi, Jatinderdeep Kaur and S. S. Bhatia, “Existence of fixed points for  $\alpha^*$ - $\psi$ -contractive multifunctions on partial metric spaces with applications”, **Proceedings of the Indian Academy of Sciences - Mathematical Sciences**.

# Table of Contents

<b>Candidate's Declaration</b>	<b>i</b>
<b>Acknowledgements</b>	<b>v</b>
<b>List of Symbols</b>	<b>vii</b>
<b>Abstract</b>	<b>ix</b>
<b>Research Publications</b>	<b>xiii</b>
<b>Table of Contents</b>	<b>xv</b>
<b>1 Introduction</b>	<b>2</b>
1.1 Origin of Fixed Point Theory . . . . .	2
1.2 Metric Fixed Point Theory . . . . .	3
1.3 Various Types of Abstract Spaces . . . . .	11
1.3.1 $G$ -Metric Spaces . . . . .	12
1.3.2 Partial Metric Spaces . . . . .	15
1.3.3 Partial Hausdorff metric spaces . . . . .	17
1.3.4 Complex Valued Metric Spaces . . . . .	20
1.3.5 Partially Ordered Metric Spaces . . . . .	24
1.4 $\alpha$ -admissible mapping, $\alpha$ - $\psi$ -contractive type mapping and its various generalizations . . . . .	33
1.5 $F$ -contraction Mapping . . . . .	39
1.6 Objectives of the study . . . . .	40
1.7 Thesis organization . . . . .	41
<b>2 Fixed Point Theorems for Expansive Mappings in Metric Spaces</b>	<b>45</b>
2.1 Introduction . . . . .	45

2.2	Preliminaries . . . . .	47
2.3	Fixed points for $(\xi, \alpha)$ -expansive mappings in metric spaces . . . . .	52
2.4	Existence of coupled fixed points in metric spaces . . . . .	58
2.5	Fixed points of $F$ -expansive mappings in metric spaces . . . . .	62
<b>3</b>	<b>Fixed Point Theorems For Mappings of Integral Type</b>	<b>67</b>
3.1	Introduction . . . . .	67
3.2	Preliminaries . . . . .	69
3.3	Fixed points for $\alpha$ - $\psi$ -contractive type mappings of integral type . . . . .	71
3.4	Existence of coupled fixed point points in complete metric spaces . . . . .	78
3.5	Consequences . . . . .	83
3.5.1	Some standard fixed point theorems . . . . .	83
3.5.2	Existence of fixed points in partially ordered metric spaces . . . . .	84
<b>4</b>	<b>Some Coincidence and Common Fixed Point Theorems for Generalized <math>\alpha</math>-<math>\psi</math>-Contractive Type Mappings</b>	<b>90</b>
4.1	Introduction . . . . .	90
4.2	Preliminaries . . . . .	92
4.3	Generalized $\alpha$ - $\psi$ contractive pair of mappings and $\alpha$ -admissible mapping w.r.t another mapping . . . . .	94
4.4	Coincidence and Common fixed point results for Generalized $\alpha$ - $\psi$ -contractive pair of mappings . . . . .	96
4.5	Consequences . . . . .	102
4.5.1	Standard Fixed Point Theorems . . . . .	102
4.5.2	Fixed Point Theorems on Metric Spaces Endowed with a Partial Order . . . . .	105
4.5.3	Fixed Point Theorems for Cyclic Contractive Mappings . . . . .	111
<b>5</b>	<b>Some Common Fixed Point Theorems for Expansive Mappings in G-Metric Spaces</b>	<b>118</b>
5.1	Introduction . . . . .	118
5.2	Preliminaries . . . . .	120
5.3	Some common fixed point theorems for expansive mappings in $G$ -metric spaces . . . . .	126
<b>6</b>	<b>Quadruple fixed point theorems for nonlinear contractions in partially ordered <math>G</math>-metric spaces</b>	<b>136</b>
6.1	Introduction . . . . .	136
6.2	Preliminaries . . . . .	137

6.3	Quadruple coincidence point theorems for a mixed $g$ -monotone mapping satisfying nonlinear contractions in partially ordered $G$ -metric spaces . . .	142
<b>7</b>	<b>Fixed Point Results in Partial Hausdorff Metric Spaces with Applications</b>	<b>156</b>
7.1	Introduction . . . . .	156
7.2	Preliminaries . . . . .	158
7.3	Fixed point theorem for $\alpha_*$ - $\psi$ -contractive multifunctions in a partial Hausdorff metric space . . . . .	163
7.4	An application . . . . .	168
<b>8</b>	<b>Existence of Fixed Points in Complex Valued Metric Spaces</b>	<b>173</b>
8.1	Introduction . . . . .	173
8.2	Preliminaries . . . . .	175
8.3	Fixed Points for $(\alpha, \phi)$ -contractive pair of mappings of rational type . . . .	178
8.4	Consequences . . . . .	189
8.5	Application . . . . .	191
8.6	Scope for future work . . . . .	193
	<b>Bibliography</b>	<b>195</b>



# Chapter 1

## Introduction

### 1.1 Origin of Fixed Point Theory

One of the most natural ways to prove that an equation has a solution is to pose it as a fixed point problem, that is, to find a function  $f$  such that  $x$  is a solution of the equation iff  $f(x) = x$ . Then, the fixed point theorems can be used to show that  $f$  has a fixed point. To illustrate this fact, let us consider the simple quadratic equation  $x^2 - 11x + 10 = 0$ . Here,  $x = 1$  and  $x = 10$  are the roots of this equation. Now, we can also write this equation in the following form:

$$x = \frac{x^2 + 10}{11}$$

Taking  $f(x) = \frac{x^2 + 10}{11}$ , we obtain that  $x = 1$  and  $x = 10$  are the two fixed points of  $f$ . So it can be concluded that the problem of finding the solution of an equation  $F(x) = 0$  is the same as deducing the fixed point of a function  $f(x)$ , where  $f(x) = F(x) + x$ . However, the results of fixed point theory are often nonconstructive, that is, they guarantee that a fixed point exists but do not help in finding the fixed point. Generally in mathematics, the problem of solving a system of equations can be converted to the problem of determining

the fixed points of a self-mapping  $T$  on an appropriate space  $X$ .

The first fixed point theorem in topological fixed point theory was proved by Brouwer [40] in 1912 which asserts that *a continuous mapping  $T$  of the closed unit ball in  $\mathbb{R}^n$  has at least one fixed point, that is, a point  $x$  such that  $T(x) = x$* . The Brouwer fixed point theorem is an important fixed point theorem that applies to finite-dimensional spaces and forms the basis for several general fixed point theorems. It is named after Dutch mathematician L. E. J. Brouwer. Later, various authors extended the Brouwer fixed point theorem whose nice illustration can be found in the survey article of Park [136]. After the appearance of Brouwer fixed point theorem and its generalizations, fixed point theory became an effective tool in solving various problems of applied mathematics.

In the present thesis, we will concentrate on Metric fixed point theory. The present chapter is elementary in nature. In this chapter, some fundamental concepts along with relevant results which will be frequently utilized in our forthcoming discussions have been presented. Many interesting theorems and notions related to the fixed point theory can not be mentioned in this chapter due to the limitations of the space. For a complete account of fixed point theory, one can refer to Agarwal, Meehan and Regan [12], Dugundji and Granas [67], Goebel and Kirk [72], Istratescu [81], Kirk and Sims [103], Smart [183] and Takahashi [184].

## 1.2 Metric Fixed Point Theory

The first fixed point result in metric fixed point theory was proved by Polish mathematician Stefan Banach in 1922, popularly referred as Banach contraction principle. This principle states that *a contraction mapping of a complete metric space into itself has a unique fixed point*. The simplicity and utility of this classical and celebrated theorem makes it a

popular tool for proving the existence and uniqueness theorems in different branches of mathematical analysis. This theorem provides a perfect example of the unifying strength of functional analysis and its utility in various areas of mathematics. During the last four decades, Banach contraction principle have been generalized and extended in various ways. In this regard, the survey articles of Istratescu [81], Kirk and Sims [103], Rus [157], Smart [183] need special attention.

**Definition 1.2.1.** [12] Let  $(A, d)$  be a metric space. A mapping  $T : A \rightarrow A$  is called a Lipschitzian mapping if there exists a constant  $\lambda \geq 0$  with

$$d(T(a), T(b)) \leq \lambda d(a, b), \text{ for all } a, b \in A. \quad (1.2.1)$$

Notice that a Lipschitzian map is necessarily continuous. The smallest  $\lambda$  for which (1.2.1) holds is said to be the Lipschitz constant for  $T$  and is denoted by  $L$ . If  $L < 1$ , then  $T$  is said to be a contraction mapping, whereas if  $L = 1$ , then  $T$  is said to be nonexpansive mapping.

The Banach contraction principle was the only main tool to establish the existence and uniqueness of fixed points until 1968. This principle has been considered as the key of metric fixed point theory, but it suffers from one drawback, i.e., it requires the mapping to be continuous at all points of its domain. In 1968, Kannan [91] introduced a contractive condition which possessed a unique fixed point like that of Banach. However, unlike the Banach condition, Kannan [91] proved that there are mappings that have a discontinuity in their domain but still have fixed point, although such mappings are continuous at their fixed point. Following the appearance of Kannan [91] many researchers started working along this line and presented numerous contractive conditions not requiring continuity of the mapping. Various authors have defined number of contractive type mappings on a complete metric space  $A$  which are generalizations of the well-known Banach contraction

and have the property that each of such mappings have a unique fixed point.

In 1968, Kannan [91] established a fixed point theorem for a non-continuous self-mapping  $T : A \rightarrow A$  satisfying the following condition:

$$d(Ta, Tb) \leq \lambda[d(a, Ta) + d(b, Tb)], \text{ for all } a, b \in A$$

where  $0 \leq \lambda < \frac{1}{2}$ .

Later in 1972, Chatterjea [44] obtained the following contraction condition:

$$d(Ta, Tb) \leq \lambda[d(a, Tb) + d(b, Ta)], \text{ for all } a, b \in A$$

where  $0 \leq \lambda < \frac{1}{2}$ .

Motivated by the work of Kannan [91] and Chatterjea [44], Hardy and Rogers [75] in 1973 considered the following contraction condition for proving their result:

$$d(Ta, Tb) \leq \lambda_1[d(a, Ta) + d(b, Tb)] + \lambda_2[d(b, Ta) + d(a, Tb)] + \lambda_3d(a, b), \text{ for all } a, b \in A$$

where  $\lambda_1, \lambda_2, \lambda_3 \geq 0, 2\lambda_1 + 2\lambda_2 + \lambda_3 < 1$ .

In 1974, Ćirić [52] presented the following different contractive condition and proved that  $T$  has a unique fixed point if there exists a number  $\lambda$  with  $0 \leq \lambda < 1$  such that

$$d(Ta, Tb) \leq \lambda m(a, b), \text{ for all } a, b \in A$$

where  $m(a, b) = \max\{d(a, b), d(a, Ta), d(b, Tb), d(a, Tb), d(b, Ta)\}$ .

Various other authors (Edelstein [68], Rakotch [141], Reich [144, 145] etc.) have given number of different contractive conditions on the same lines. In 1977, Rhoades [147] compared these different contractive conditions. In 1976, Jungck [84] initiated a study of common fixed points of commuting mappings. The common fixed point results are the extensions of the fixed point results. There is a growing interest among the researchers of

this field in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems (see, for example ([42], [43])). The following unique common fixed point theorem due to Jungck [84] has inspired various researchers in recent years:

**Theorem 1.2.1.** [84] Let  $T$  and  $S$  be commuting (i.e.  $TS = ST$ ) self mappings of a complete metric space  $(A, d)$  such that  $T(A) \subseteq S(A)$  and  $S$  is continuous. If there exists  $\lambda \in (0, 1)$  satisfying  $d(Ta, Tb) \leq \lambda d(Sa, Sb)$ , for  $a, b \in A$ , then the mappings  $T$  and  $S$  possess a unique common fixed point.

In an attempt to generalize the concept of commutativity, Sessa [170] in 1982 gave the following notion of weakly commuting mappings:

**Definition 1.2.2.** [170] Let  $S$  and  $T$  be self-mappings of a metric space  $(A, d)$ . The pair  $(S, T)$  is said to be weakly commuting if

$$d(STa, T Sa) \leq d(Ta, Sa), \text{ for all } a \in A.$$

Every commuting pair is weakly commuting but converse is not in general true, as shown by the following example:

**Example 1.2.3.** [170] Consider the set  $A = [0, 1]$  equipped with the usual metric. Let self-mappings  $S$  and  $T$  of  $A$  be defined by  $S(a) = \frac{a}{2}$  and  $T(a) = \frac{a}{a+2}$  for every  $a \in A$ . Then,

$$\begin{aligned} d(STa, T Sa) &= \left| \frac{a}{4+2a} - \frac{a}{4+a} \right| \\ &= \left| \frac{a^2}{(4+a)(4+2a)} \right| \\ &\leq \frac{a^2}{4+2a} = \frac{a}{2} - \frac{a}{2+a} = d(Sa, Ta) \end{aligned}$$

thereby implying that  $S$  and  $T$  are weakly commuting but for any  $a \in A$ , we get

$$T(S(a)) = \frac{a}{4+a} > \frac{a}{4+2a} = S(T(a)).$$

Therefore,  $ST \neq TS$ . Thus,  $S$  and  $T$  are not commuting mappings.

Further, Jungck [85] generalized the notion of weakly commuting mappings by presenting the following more generalized form of commuting mappings called as compatible mappings.

**Definition 1.2.4.** A pair of self-mappings  $(S, T)$  of a metric space  $(A, d)$  is called compatible if and only if

$$\lim_{n \rightarrow \infty} d(TSa_n, STa_n) = 0,$$

whenever  $\{a_n\}$  is a sequence in  $A$  such that  $\lim_{n \rightarrow \infty} Aa_n = \lim_{n \rightarrow \infty} Sa_n = t$  for some  $t \in A$ .

In general, weakly commuting mappings are compatible but the converse is not true as shown by the following example:

**Example 1.2.5.** [85] Consider the set  $A = [0, \infty)$  with the usual metric. Let us define self-mappings  $S$  and  $T$  on  $A$  by  $S(a) = a^3$  and  $T(a) = 2a^3$  for every  $a \in A$ . Clearly,  $STa \neq T Sa$  thereby implying that  $S$  and  $T$  are not commuting on  $A$ . Also, for all  $a \in A$ ,

$$|STa - T Sa| > |Sa - Ta|$$

which implies that  $S$  and  $T$  are not weakly commuting mappings on  $A$ . However,  $S$  and  $T$  are compatible mappings on  $A$ .

Further, Cho [46] in 1993, introduced the concept of compatible mappings of type (A) which is equivalent to the concept of compatible mappings under some conditions and proved a common fixed point result related to compatible mappings of type (A) in a metric space. The concept of compatible mappings of type (A) runs as follows:

**Definition 1.2.6.** Let  $S$  and  $T$  be self-mappings of a metric space  $(A, d)$ . The pair of mappings  $(S, T)$  is called compatible of type  $(A)$  if

$$\lim_{n \rightarrow \infty} d(STa_n, TTa_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(TSa_n, SSa_n) = 0$$

whenever  $\{a_n\}$  is a sequence in  $A$  such that  $\lim_{n \rightarrow \infty} Aa_n = \lim_{n \rightarrow \infty} Sa_n = t$  for some  $t \in A$ .

In 1998, Jungck and Rhoades [87] weakened the notion of compatible mappings by giving the concept of weak compatibility as follows:

**Definition 1.2.7.** A pair  $(S, T)$  of self-mappings of a metric space  $(A, d)$  is called weakly compatible if the mappings commute at their coincidence points, i.e., if  $STa = T Sa$  whenever  $Sa = Ta$  for some  $a \in A$ .

**Remark 1.2.1.** Compatible mappings are weakly compatible mappings but the converse is not true as shown by the following example:

**Example 1.2.8.** [106] Let  $A = [2, 20]$  and  $d$  be the usual metric on  $A$ . Define the mappings  $B, T$  from  $A$  to  $A$  by

$$B(a) = \begin{cases} 2 & a = 2 \text{ or } a > 5, \\ 6 & 2 < a \leq 5. \end{cases}$$

and

$$T(a) = \begin{cases} a & a = 2, \\ 12 & 2 < a \leq 5, \\ a - 3 & a > 5. \end{cases}$$

Let us define a sequence  $\{a_n\}$  in  $A$  by  $\{a_n = 5 + (1/n), n \geq 1\}$ . In this example, mappings  $B$  and  $T$  are non compatible and weakly compatible at the coincidence point  $a = 2$ .

Several fixed point results have been obtained for weakly compatible mappings (see [16], [50], [57], [140], [176]). Various other types of weakly commuting mappings have also been introduced by several authors such as: compatible mappings of type  $(B)$ ,  $(C)$ ,  $(P)$ ,  $R$ -weakly commuting mappings, pointwise  $R$ -weakly commuting mappings,  $R$ -weakly commuting mappings of type  $(A_g)$  and  $(A_f)$  etc. In 2001, Murthy [116] gave a nice survey of all these different kinds of weakly commuting mappings.

In order to further generalize the Banach contraction principle, Branciari [39] in 2002, gave a fixed point result for a single mapping satisfying an analogue of Banach contraction principle for an integral-type inequality. The technique of contractive conditions of integral type is one of the interesting techniques for obtaining fixed point results. Branciari [39] established the following result:

**Theorem 1.2.2.** Suppose  $T$  be a given self-mapping of a complete metric space  $(A, d)$  satisfying for each  $a, b \in A$ ,

$$\int_0^{d(Ta, Tb)} \varphi(s) ds \leq c \int_0^{d(a, b)} \varphi(s) ds, \quad (1.2.2)$$

where  $c \in [0, 1)$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable mapping on every compact subset of  $[0, +\infty)$  and such that for all  $\epsilon > 0$ ,

$$\int_0^\epsilon \varphi(s) ds > 0.$$

Then,  $T$  has a unique fixed point  $u \in A$  and for every  $x \in A$ , the sequence  $\{T^n x\}$  converges to  $u$ .

The Banach contraction principle is immediately obtained by taking  $\varphi(t) = 1$ , for each  $t \geq 0$  in Theorem 1.2.2.

Further, Rhoades [151] in 2003 extended the condition (1.2.2) to more general contractive conditions which embraces the Branciari's result as well as the result of Ćirić [53].

Thereafter, many authors like Aliouche [19], Djoudi and Aliouche [66], Rhoades [151], Samet and Vetro [164], Türkoğlu and Altun [187], Vetro [189], Vijayaraju *et al.* [191], and others undertook further investigations in this direction.

The study of the existence of fixed points for a single expansion mapping in a metric space was initiated by Wang *et al.* [192]. Later, using expansion type conditions, several results have been proved for a pair of mappings (see [137], [148], [185]) and two pairs of mappings (see [82], [90], [173]) in metric spaces.

In 1984, Wang *et al.* [192] introduced expansion mappings as follows:

**Theorem 1.2.3.** [192] Suppose  $(A, d)$  be a complete metric space and  $f : A \rightarrow A$  is an onto mapping. Suppose that there exists a constant  $q > 1$  satisfying

$$d(fa, fb) \geq qd(a, b),$$

for every  $a, b \in A$ . Then  $f$  has a unique fixed point in  $A$ .

In 1992, Daffer and Kaneko [59] established the following fixed point theorem for an expansive pair of mappings as follows:

**Theorem 1.2.4.** Suppose  $(A, d)$  be a complete metric space. Let  $S$  be a surjective self mapping of  $A$  and  $T$  be an injective self mapping of  $A$ . If there exists a number  $q > 1$  satisfying

$$d(Sa, Sb) \geq qd(Ta, Tb),$$

for all  $a, b \in A$ . Then  $S$  and  $T$  have a unique common fixed point.

Further, Rhoades [149] generalized the above result of Daffer and Kaneko [59] to compatible mappings as follows.

**Theorem 1.2.5.** Suppose  $(A, d)$  be a complete metric space. Let  $f, g : A \rightarrow A$  be compatible mappings of  $A$  satisfying

- (i)  $d(Sa, Sb) \geq qd(Ta, Tb)$ , for all  $a, b \in A$  and for some  $q > 1$ ,
- (ii)  $g(A) \subseteq f(A)$ ,
- (iii)  $f$  is continuous.

Then  $f$  and  $g$  have a unique common fixed point.

Later in 2008, Kumar [106] extended Theorems 1.2.4 and 1.2.5 to weakly compatible mappings by presenting the following theorem:

**Theorem 1.2.6.** Let  $(A, d)$  be a complete metric space. Let  $f$  and  $g$  be weakly compatible self-mappings of  $A$  satisfying the following conditions:

- (i) there exists a number  $q > 1$  such that  $d(Sa, Sb) \geq qd(Ta, Tb)$ , for all  $a, b \in A$ ,
- (ii)  $g(A) \subseteq f(A)$ .

If one of the subspaces  $g(A)$  or  $f(A)$  is complete, then  $f$  and  $g$  have a unique common fixed point.

In recent years, several authors have worked on expansion mappings e.g. Han and Xu [74], Imdad *et al.* [80], Pathak and Tiwari [138], Shahi *et al.* [172] and several others.

### 1.3 Various Types of Abstract Spaces

Apart from analysis, metric spaces also provide a more general setting for researchers in various other fields such as optimization, mathematical modelling and economic theories.

Motivated by the impact of metric spaces to mathematics in general and to functional analysis in particular, various researchers generalized the concept of metric spaces to various other abstract spaces such as 2-metric spaces,  $D$ -metric spaces,  $G$ -metric spaces,  $K$ -metric spaces, Probabilistic metric spaces, Cone metric spaces, Complex valued metric spaces, Partial metric spaces, Partial Hausdorff metric spaces, Semimetric spaces, Quasimetric spaces etc. and by now there exists considerable literature on all these generalizations of metric spaces. For more details, one can see ([15], [18], [29], [77], [79], [80], [111] etc). We shall discuss some of these abstract spaces in the following subsections:

### 1.3.1 $G$ -Metric Spaces

Generalizations of metric spaces were proposed by Gahler ([70], [71]) (called 2-metric spaces) and Dhage ([61], [63], [65]) (called  $D$ -metric spaces) to extend the known fixed point theorems from metric spaces to these spaces. But later, different authors proved that these attempts are invalid (for detail see [73], [120], [124]). In 2005, Mustafa and Sims [121] introduced  $G$ -metric spaces as a generalization of metric spaces. This new structure helped in developing new fixed point results for various mappings. Mustafa and Sims [121] presented the following definitions and results concerning  $G$ -metric spaces:

**Definition 1.3.1.** ( $G$ -Metric space [121]). Suppose  $A$  be a nonempty set. Let  $G : A \times A \times A \rightarrow \mathbb{R}^+$  be a function satisfying the following conditions:

- (i)  $G(a, b, c) = 0$  if  $a = b = c$ ,
- (ii)  $0 < G(a, a, b)$  for all  $a, b \in A$  with  $a \neq b$ ,
- (iii)  $G(a, a, b) \leq G(a, b, c)$  for all  $a, b, c \in A$  with  $c \neq b$ ,
- (iv)  $G(a, b, c) = G(a, c, b) = G(b, c, a) = \dots$ ( symmetry in all three variables),

(v)  $G(a, b, c) \leq G(a, x, x) + G(x, b, c)$  for all  $a, b, c, x \in A$  (rectangle inequality).

Then the function  $G$  is called a  $G$ -metric on  $A$ . The set  $A$  equipped with a  $G$ -metric is called as  $G$ -metric space.

**Definition 1.3.2.** Suppose  $(A, G)$  be a  $G$ -metric space and  $\{a_n\}$  be a sequence in  $A$ . A point  $a \in A$  is called as the limit of the sequence  $\{a_n\}$  if

$$\lim_{n, m \rightarrow \infty} G(a, a_n, a_m) = 0$$

and the sequence  $\{a_n\}$  is said to be  $G$ -convergent to  $a$ . Thus, if  $\{a_n\} \rightarrow a$  in a  $G$ -metric space  $(A, G)$ , then for any  $\epsilon > 0$ , there exists a positive integer  $N$  such that  $G(a, a_n, a_m) < \epsilon$ , for all  $n, m \geq N$ .

It has been shown in [121] that the  $G$ -metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to one point.

**Proposition 1.3.1.** Consider a  $G$ -metric space  $(A, G)$ . Then the following statements are equivalent:

- (i)  $\{a_n\}$  is  $G$ -convergent to  $a$ .
- (ii)  $G(a_n, a_n, a) \rightarrow 0$ , as  $n \rightarrow \infty$ .
- (iii)  $G(a_n, a, a) \rightarrow 0$ , as  $n \rightarrow \infty$ .
- (iv)  $G(a_m, a_n, a) \rightarrow 0$ , as  $m, n \rightarrow \infty$ .

**Definition 1.3.3.** Suppose  $(A, G)$  is a  $G$ -metric space. A sequence  $\{a_n\}$  is said to be  $G$ -Cauchy if for every  $\epsilon > 0$ , there is a positive integer  $N$  such that  $G(a_n, a_m, a_l) < \epsilon$ , for all  $n, m, l \geq N$ , that is, if  $G(a_n, a_m, a_l) \rightarrow 0$ , as  $n, m, l \rightarrow \infty$ .

**Proposition 1.3.2.** Let  $(A, G)$  be a  $G$ -metric space. Then the following statements are equivalent:

- (i) the sequence  $\{a_n\}$  is  $G$ -Cauchy,
- (ii) for any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $G(a_n, a_m, a_m) < \epsilon$ , for all  $m, n \geq N$ .

**Definition 1.3.4.** Let  $(A, G), (A', G')$  be two  $G$ -metric spaces. A function  $f : A \rightarrow A'$  is said to be  $G$ -continuous at a point  $a \in A$  if and only if it is  $G$  sequentially continuous at  $a$ , that is, whenever  $\{a_n\}$  is  $G$ -convergent to  $a$ ,  $\{f(a_n)\}$  is  $G$ -convergent to  $f(a)$ .

**Definition 1.3.5.** A  $G$ -metric space  $(A, G)$  is called symmetric if  $G(a, b, b) = G(a, a, b)$  for all  $a, b \in A$ .

**Definition 1.3.6.** A  $G$ -metric space  $(A, G)$  is called  $G$ -complete (or complete  $G$ -metric space) if every  $G$ -Cauchy sequence in  $(A, G)$  is convergent in  $A$ .

Mustafa and Sims [121] proved the following results related to  $G$ -metric spaces.

**Proposition 1.3.3.** Suppose  $(A, G)$  be a  $G$ -metric space. The function  $G(a, y, z)$  is jointly continuous in all three of its variables.

**Proposition 1.3.4.** Every  $G$ -metric space  $(A, G)$  induces a metric space  $(A, d_G)$ , where  $d_G(a, b) = G(a, b, b) + G(a, a, b)$ , for all  $a, b \in A$ .

**Proposition 1.3.5.** A  $G$ -metric space  $(A, G)$  is said to be  $G$ -complete iff  $(A, d_G)$  is a complete metric space.

**Proposition 1.3.6.** Suppose  $(A, G)$  be a  $G$ -metric space. The following holds for every  $a, b, c, u \in A$

- (i) If  $G(a, b, c) = 0$ , then  $a = b = c$ ;

- (ii)  $G(a, b, c) \leq G(a, a, b) + G(a, a, c)$ ;
- (iii)  $G(a, b, b) \leq 2G(b, a, a)$ ;
- (iv)  $G(a, b, c) \leq G(a, u, c) + G(u, b, c)$ ;
- (v)  $G(a, b, c) \leq [G(a, u, u) + G(b, u, u) + G(c, u, u)]$ .

In 2005, Mustafa [117] established the well known Banach contraction principle in his Ph. D. thesis in the context of  $G$ -metric spaces in the following way:

**Theorem 1.3.7.** Suppose  $(A, G)$  be a complete  $G$ -metric space and  $f$  be a self-mapping of  $A$  satisfying

$$G(fa, fb, fc) \leq kG(a, b, c), \text{ for all } a, b, c \in A \text{ and } 0 \leq k < 1.$$

Then  $f$  has a unique fixed point.

Motivated by the work of Mustafa and Sims [121], various researchers (see, e. g., [7], [8], [27], [118], [119], [122], [160], [174]) have established several theorems in  $G$ -metric spaces using various contractive conditions.

### 1.3.2 Partial Metric Spaces

The notion of partial metric spaces was presented by Matthews [111, 112] as a branch of the analysis of denotational semantics of dataflow networks. He demonstrated that Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. In fact, it is extensively recognized that partial metric spaces play an important role in constructing models in the theory of computation (see [76],[132],[154],

[155], [159], [163], [195]). The motivation for introducing non-zero distance (i.e., the 'distance'  $p$  where  $p(x, x) = 0$  need not hold) is explained in a presentation by Bukatin *et al.* [41] which also triggered an interesting research in foundations of topology.

Matthews [111] gave the following definitions and results related to partial metric spaces:

**Definition 1.3.7.** Suppose  $A$  be a nonempty set. A mapping  $p : A \times A \rightarrow [0, \infty)$  is called as partial metric on  $A$  if the following conditions hold:

- (i)  $a = b$  if and only if  $p(a, a) = p(b, b) = p(a, b)$ ,
- (ii)  $p(a, a) \leq p(a, b)$ ,
- (iii)  $p(a, b) = p(b, a)$ ,
- (iv)  $p(a, c) \leq p(a, b) + p(b, c) - p(b, b)$ ,

for any  $a, b, c \in A$ . The pair  $(A, p)$  is said to be a partial metric space (in short PMS).

**Definition 1.3.8.** Let  $(A, p)$  be a partial metric space. Then, the functions  $p_s, p_m : A \times A \rightarrow [0, \infty)$  given by

$$p_s(a, b) = 2p(a, b) - p(a, a) - p(b, b)$$

and

$$p_m(a, b) = \max\{p(a, b) - p(a, a), p(a, b) - p(b, b)\}$$

are well-known metrics on  $A$ . It is easy to check that  $p^s$  and  $p^m$  are equivalent. Note that each partial metric  $p$  on  $A$  generates a  $T_0$ -topology  $\tau_p$  with a base of the family of open  $p$ -balls  $\{B_p(a, \epsilon) : a \in A, \epsilon > 0\}$ , where  $B_p(a, \epsilon) = \{b \in A : p(a, b) < p(a, a) + \epsilon\}$ .

**Definition 1.3.9.** Suppose  $(A, p)$  be a partial metric space.

- (i) A sequence  $\{a_n\}$  in  $A$  converges to  $a \in A$  iff  $p(a, a) = \lim_{n \rightarrow \infty} p(a_n, a)$ .
- (ii) A sequence  $\{a_n\}$  in  $A$  is called a Cauchy sequence iff  $\lim_{m, n \rightarrow \infty} p(a_n, a_m)$  exists (and finite).
- (iii) The space  $(A, p)$  is called complete if every Cauchy sequence  $\{a_n\}$  in  $A$  converges to  $a \in A$ .
- (iv) A mapping  $f : A \rightarrow A$  is continuous at  $a_0 \in A$  if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $f(B_p(a_0, \delta)) \subset B_p(f(a_0), \epsilon)$ .

**Example 1.3.10.** Let  $A = [0, +\infty)$  and define  $p(a, b) = \max\{a, b\}$ , for all  $a, b \in A$ . Then  $(A, p)$  is a complete partial metric space. It is clear that  $p$  is not a (usual) metric.

Later on, many researchers obtained several results in partial metric spaces. For more details, the reader can refer to ([20], [22], [28], [56], [96], [131], [135], [153], [159], [163], [190]).

### 1.3.3 Partial Hausdorff metric spaces

Hausdorff metric evaluates the resemblance of two sets (of geometric points). Suppose  $(A, d)$  be a metric space and  $CB(A)$  denotes the collection of all nonempty closed and bounded subsets of  $A$ . Define

$$H(X, Y) = \max\left\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\right\},$$

where  $X, Y \in CB(A)$ . It is known that  $H$  is a metric on  $CB(A)$  called the Hausdorff metric induced by the metric  $d$ .

In the past few years, the theory of multivalued mappings has been developed in variety of ways. Markin [109] and Nadler [123] obtained a multi-valued version of the Banach contraction principle using the notion of a Hausdorff metric.

**Definition 1.3.11.** [123] Consider a nonempty set  $A$ . A point  $a \in A$  is called as fixed point of a multi-valued mapping  $T : A \rightarrow 2^A$  if  $a \in Ta$ .

**Definition 1.3.12.** [123] A multi-valued mapping  $T : A \rightarrow CB(A)$  is called as contraction mapping if

$$H(Ta, Tb) \leq kd(a, b),$$

for every  $a, b \in A$  and for some  $k \in [0, 1)$ .

Afterwards, a number of generalizations were obtained using different contractive conditions (see for e.g., [54], [58], [60], [146], [169]). The study of hybrid type contractive conditions involving single-valued and multivalued mappings is a useful addition to the metric fixed point theory and its applications (for details, see [125], [175], [177], [178], [179]).

Recently, Aydi *et al.* [24] initiated the concept of a partial Hausdorff metric and obtained an analogue of Nadler's fixed point theorem [123] in partial metric spaces. Aydi *et al.* [24] initiated the study of fixed point theory for multi-valued mappings in partial metric spaces.

Aydi *et al.* [24] gave the following definition of partial Hausdorff metric as follows:

**Definition 1.3.13.** [24](Partial Hausdorff metric) Suppose  $(X, p)$  is a partial metric space. Denote  $CB^p(X)$  by the family of all nonempty, closed and bounded subsets of the partial metric space  $(X, p)$ , induced by the partial metric  $p$ . Note that Closedness is taken from  $(X, \tau_p)$  ( $\tau_p$  is the topology induced by  $p$ ) and boundedness is given as follows:  $A$  is a bounded subset in  $(X, p)$  if there exist  $a_0 \in X$  and  $M \geq 0$  such that for all  $x \in A$ , we have  $x \in B_p(x_0, M)$ , that is,  $p(a_0, x) < p(x, x) + M$ .

For  $A, B \in CB^p(X)$  and  $x \in X$ , define

$$\begin{aligned} p(x, A) &= \inf\{p(x, a), a \in A\}, \\ \delta_p(A, B) &= \sup\{p(a, B) : a \in A\} \quad \text{and} \\ \delta_p(B, A) &= \sup\{p(b, A) : b \in B\} \end{aligned}$$

**Remark 1.3.1.** [24] Suppose  $(A, p)$  be a partial metric space. If  $B$  is any nonempty set in  $(A, p)$ , then

$$a \in \overline{B} \quad \text{iff} \quad p(a, B) = p(a, a), \quad (1.3.1)$$

where  $\overline{B}$  denotes the closure of  $B$  with respect to the partial metric  $p$ . The set  $B$  is said to be closed in  $(A, p)$  if and only if  $B = \overline{B}$ .

Aydi *et al.* obtained the following results:

**Proposition 1.3.8.** [24] Suppose  $(X, p)$  be a partial metric space. For any  $A, B, C \in CB^p(X)$ , the following holds:

- (i)  $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$ ,
- (ii)  $\delta_p(A, A) \leq \delta_p(A, B)$ ,
- (iii)  $\delta_p(A, B) = 0$  implies that  $A \subseteq B$ ,
- (iv)  $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$ .

Following [24], Consider a partial metric space  $(X, p)$ . Define

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\},$$

where  $A, B \in CB^p(X)$ .

**Proposition 1.3.9.** [24] Suppose  $(A, p)$  is a partial metric space. For all  $A, B, C \in CB^p(X)$ , the following holds

$$(i) \ H_p(A, A) \leq H_p(A, B),$$

$$(ii) \ H_p(A, B) = H_p(B, A),$$

$$(iii) \ H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c).$$

Following are the main results of Aydi *et al.* [24]:

**Theorem 1.3.10.** [24]. Suppose  $(A, p)$  is a complete partial metric space. If the mapping  $T : A \rightarrow CB^p(A)$  satisfies

$$H_p(Ta, Tb) \leq kp(a, b), \text{ for all } a, b \in A$$

where  $k \in (0, 1)$ . Then  $T$  has a fixed point.

Recently, some authors have worked on Partial Hausdorff metric spaces. In 2013, Abbas and Ali [1] proved the coincidence point theorems for hybrid pairs of single-valued and multi-valued mappings in a partial Hausdorff metric space. Also, Ahmad *et al.* [14] established a common fixed point theorem of a pair of multivalued mappings satisfying Mizoguchi and Takahashi's contractive condition [113]. Later, Chen and Karapinar [45] obtained some fixed point results for a multi-valued mapping for  $\alpha$ -Meir-Keeler contractions with respect to the partial Hausdorff metric  $H$  in complete partial metric spaces.

### 1.3.4 Complex Valued Metric Spaces

In 2011, Azam *et al.* [29] extended the notion of metric spaces by introducing complex valued metric spaces. Also, Azam *et al.* [29] proved several fixed point results for mappings

satisfying a rational inequality in complex valued metric spaces.

Azam *et al.* [29] considered the following partial order:

**Definition 1.3.14.** Suppose  $\mathbb{C}$  be the set of complex numbers and  $w_1, w_2 \in \mathbb{C}$ . Consider a partial order  $\preceq$  on  $\mathbb{C}$  as follows:

$$w_1 \preceq w_2 \text{ if and only if } \Re(w_1) \leq \Re(w_2), \Im(w_1) \leq \Im(w_2).$$

**Remark 1.3.2.** So,  $w_1 \preceq w_2$  if one of the following conditions is satisfied:

- (i)  $\Re(w_1) = \Re(w_2), \Im(w_1) < \Im(w_2)$ ,
- (ii)  $\Re(w_1) < \Re(w_2), \Im(w_1) = \Im(w_2)$ ,
- (iii)  $\Re(w_1) < \Re(w_2), \Im(w_1) < \Im(w_2)$ ,
- (iv)  $\Re(w_1) = \Re(w_2), \Im(w_1) = \Im(w_2)$ .

Also,  $w_1 \prec w_2$  if  $w_1 \neq w_2$  and one of (i), (ii) and (iii) is satisfied and  $w_1 \prec w_2$  if only (iii) is satisfied. Also,  $0 \preceq w_1 \prec w_2 \Rightarrow |w_1| < |w_2|$  and  $w_1 \preceq w_2, w_2 \prec w_3 \Rightarrow w_1 \prec w_3$ .

**Definition 1.3.15.** [29] Suppose  $A$  be a nonempty set. If the mapping  $d : A \times A \rightarrow \mathbb{C}$  satisfies the following properties:

- (i)  $0 \preceq d(a, b)$ , for all  $a, b \in A$  and  $d(a, b) = 0$  if and only if  $a = b$ ,
- (ii)  $d(a, b) = d(b, a)$  for all  $a, b \in A$ ,

(iii)  $d(a, b) \lesssim d(a, c) + d(c, b)$ , for all  $a, b, c \in A$ ,

then  $d$  is called a complex valued metric on  $A$ . The space  $(A, d)$  is called a complex valued metric space.

Azam *et al.* gave the following definitions related to complex valued metric spaces:

**Definition 1.3.16.** [29] Let  $(A, d)$  be a complex valued metric space and  $P \subseteq A$ . A point  $a \in P$  is said to be an interior point of a set  $P$  if there exists  $0 \prec r \in \mathbb{C}$  such that  $B(a, r) \subseteq P$ .

**Definition 1.3.17.** [29] Let  $(A, d)$  be a complex valued metric space and  $P \subseteq A$ . A point  $a \in A$  is said to be a limit point of  $P$  if for every  $0 \prec r \in \mathbb{C}$ ,  $B(a, r) \cap (P \setminus a) \neq \phi$

**Definition 1.3.18.** [29] Let  $(A, d)$  be a complex valued metric space. A set  $P \subseteq A$  is said to be an open set if each element of  $P$  is an interior point of  $P$ .

**Definition 1.3.19.** [29] Let  $(A, d)$  be a complex valued metric space. A set  $P \subseteq A$  is said to be a closed set if each limit point of  $P$  belongs to  $P$ .

**Definition 1.3.20.** [29] Let  $(A, d)$  be a complex valued metric space. Suppose  $\{a_n\}$  be a sequence in  $A$  and  $a \in A$ . If for every  $c \in \mathbb{C}$ , with  $0 \prec c$  there is  $n_0 \in \mathbb{N}$  such that

$$d(a_n, a) \prec c, \text{ for every } n \geq n_0,$$

then the sequence  $\{a_n\}$  is said to be convergent sequence and  $a$  is the limit point of  $\{a_n\}$ .

It is denoted by  $\lim_n \{a_n\} = a$ , or  $a_n \rightarrow a$ , as  $n \rightarrow \infty$

**Definition 1.3.21.** [29] Let  $(A, d)$  be a complex valued metric space. A sequence  $\{a_n\}$  is said to be a Cauchy sequence in  $A$  if for every  $c \in \mathbb{C}$  with  $c \succ 0$  there is  $n_0 \in \mathbb{N}$  such that

$$d(a_n, a_{n+m}) \prec c, \text{ for every } n, m > n_0,$$

**Definition 1.3.22.** [29] A complex valued metric space  $(A, d)$  is known as complete complex valued metric space if every Cauchy sequence in  $A$  is convergent in  $A$ .

Azam *et al.* [29] established the following two lemmas for the proof of their theorems:

**Lemma 1.3.11.** Suppose  $(A, d)$  be a complex valued metric space and  $\{a_n\}$  be a sequence in  $A$ . Then the sequence  $\{a_n\}$  is said to be convergent to  $a$  iff  $|d(a_n, a)| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 1.3.12.** Suppose  $(A, d)$  be complex valued metric space and  $\{a_n\}$  be a sequence in  $A$ . Then the sequence  $\{a_n\}$  is said to be a Cauchy sequence iff  $|d(a_n, a_{n+m})| \rightarrow 0$  as  $n \rightarrow \infty$ .

Azam *et al.* [29] obtained the following extension of the Banach contraction principle in complex valued metric spaces:

**Theorem 1.3.13.** Suppose  $(A, d)$  be a complete complex valued metric space. Let the mappings  $S, T : A \rightarrow A$  satisfy:

$$d(Sa, Tb) \lesssim \lambda d(a, b) + \frac{\mu d(a, Sa)d(b, Tb)}{1 + d(a, b)}, \text{ for every } a, b \in A$$

where  $\lambda, \mu$  are nonnegative reals with  $\lambda + \mu < 1$ . Then  $S$  and  $T$  have a unique common fixed point.

Thus, an extension of the Banach contraction principle is obtained in complex valued metric spaces. We can study improvements of a number of results of analysis involving divisions in complex valued metric spaces. Later, Rouzkard and Imdad [156] established some common fixed point theorems satisfying certain rational expressions in complex valued metric spaces which generalize, unify and complement the results of Azam *et al.* [29]. The main result of Rouzkard and Imdad [156] is as follows:

**Theorem 1.3.14.** If  $S$  and  $T$  are self-mappings defined on a complete complex valued metric space  $(A, d)$  satisfying the condition:

$$d(Sa, Tb) \preceq \lambda d(a, b) + \frac{\mu d(a, Sa)d(b, Tb) + \gamma d(b, Sa)d(a, Tb)}{1 + d(a, b)}, \quad (1.3.2)$$

for all  $a, b \in A$ , where  $\lambda, \mu, \gamma$  are non-negative reals with  $\lambda + \mu + \gamma < 1$ , then  $S$  and  $T$  have a unique common fixed point.

Further, Sintunavarat and Kumam [181] obtained common fixed point results by replacing constant of contractive condition to control functions. Recently, Klin-eam and Suanoom [105] extend the concept of complex valued metric spaces and generalized the results of Azam *et al.* [29] and Rouzkard and Imdad [156].

### 1.3.5 Partially Ordered Metric Spaces

**Definition 1.3.23.** (Partial Order) A partial order relation is a binary relation  $\preceq$  on a non-empty set  $A$  satisfying the following conditions for every  $a, b$  and  $c$  in  $A$ :

- (i)  $A$  is reflexive i.e.  $a \preceq a$  for every  $a \in A$ ,
- (ii)  $A$  is antisymmetric i.e. if  $a \preceq b$  and  $b \preceq a$ , then  $a = b$ ,
- (iii)  $A$  is transitive i.e. if  $a \preceq b$  and  $b \preceq c$  then  $a \preceq c$ .

A set equipped with a partial order  $\preceq$  is called a partially ordered set. Some of the examples of partially ordered sets include the set of integers and real numbers with their ordinary ordering, the set of subsets of a given set with partial order " $\subseteq$ " and natural numbers ordered by divisibility.

**Definition 1.3.24.** (Comparable elements) Let  $(A, \preceq)$  be a partially ordered set and  $a, b \in A$ . The elements  $a$  and  $b$  are known as comparable elements of  $A$  if either  $a \preceq b$  or  $b \preceq a$ .

**Definition 1.3.25.** Suppose  $A$  be a non empty set. The space  $(A, d, \preceq)$  is called a partially ordered metric space if:

- (i)  $(A, \preceq)$  is a partially ordered set,
- (ii)  $(A, d)$  is a metric space.

In recent times, there has been an increasing interest in studying the existence of fixed points for mappings satisfying contractive conditions and monotone properties in partially ordered metric spaces. The first fixed point result in a partially ordered metric space was given by Turinici [186]. Further, some applications of Turinici's theorem to matrix equations were presented by Ran and Reurings [142]. Ran and Reurings [142] proved the following Banach contraction principle in ordered metric spaces:

**Theorem 1.3.15.** Suppose  $A$  be a partially ordered set such that each pair  $a, b \in A$  has a lower and an upper bound. Let  $d$  be a metric on  $A$  such that the metric space  $(A, d)$  is complete. Let  $T : A \rightarrow A$  be a continuous and monotone mapping. Suppose that

- (i) there exists  $\alpha \in (0, 1)$  such that  $d(Ta, Tb) \leq \alpha.d(a, b)$  for each  $a, b \in A$  with  $b \preceq a$  and
- (ii) there exists  $a_0 \in A$  such that  $Ta_0 \preceq a_0$  or  $a_0 \preceq Ta_0$ .

Then  $T$  has a unique fixed point  $u \in A$  and for  $z \in A$  the sequence  $\{T^n(z)\}$  ( $n \in \mathbb{N}$ ) of successive approximations of  $T$  starting from  $z$  converges to  $u \in A$ .

Subsequently, Nieto and López [128, 129] improved the result of Ran and Reurings [142] and used these results to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. In 2005, Nieto and Lopez [128] obtained a

modified variant of Theorem 1.3.15, by removing the continuity of  $T$ . The theorem is as follows:

**Theorem 1.3.16.** [128] Let  $A$  be a partially ordered set such that every pair  $a, b \in A$  has a lower or an upper bound. Let  $d$  be a metric on  $A$  such that the metric space  $(A, d)$  is complete. Let  $T : A \rightarrow A$  be an increasing operator. Suppose that

- (i) there exists  $\alpha \in (0, 1)$  satisfying  $d(Ta, Tb) \leq \alpha \cdot d(a, b)$ , for every  $a, b \in A$  with  $b \preceq a$ ,
- (ii) there exists  $a_0 \in A$  such that  $a_0 \preceq Ta_0$  and
- (iii) if an increasing sequence  $\{a_n\}$  converges to  $a$  in  $A$ , then  $a_n \preceq a$  for every  $n \in \mathbb{N}$ .

Then  $T$  has a unique fixed point  $u \in A$  and for each  $z \in A$  the sequence  $\{T^n(z)\}$  ( $n \in \mathbb{N}$ ) of successive approximations of  $T$  starting from  $z$  converges to  $u \in A$ .

In 2007, Nieto and Lopez [129] discussed the case of decreasing operators thereby presenting some interesting applications to ordinary differential equations with periodic boundary conditions. On the other hand, Agarwal *et al.* [11] generalized the results of Ran and Reurings [142] for the case of generalized contractions and showed that the results of Nieto and Lopez [128], Nieto and Lopez [129] and Ran and Reurings [142] follow from their results as particular cases. Agarwal *et al.* [11] proved the following result:

**Theorem 1.3.17.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space. The mapping  $T$  be an increasing self-mapping of  $A$  satisfying the following conditions:

- (i) there exists an increasing mapping  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\lim_{n \rightarrow \infty} \phi^n(t) = 0$  for each  $t > 0$ , such that for each  $a, b \in A$  with  $b \preceq a$  we have

$$d(Ta, Tb) \leq \phi \left( \max\{d(a, b), d(a, Ta), d(b, Tb), \frac{1}{2}[d(a, Tb) + d(b, Ta)]\} \right)$$

- (ii) there exists  $a_0 \in A$  such that  $a_0 \preceq Ta_0$  and
- (iii)  $T$  is continuous or if an increasing sequence  $\{a_n\} \subset A$  converges to  $a$  in  $A$ , then  $a_n \preceq a$  for all  $n \in \mathbb{N}$ .

Then  $T$  has at least one fixed point in  $A$ .

Later, Bhaskar and Lakshmikantham [36] introduced the idea of coupled fixed point for mappings  $F : A \times A \rightarrow A$  satisfying the contractive conditions and mixed monotone property, where  $A$  is a partially ordered metric space and proved some interesting coupled fixed point theorems. Bhaskar and Lakshmikantham [36] gave the following concepts:

**Definition 1.3.26.** Consider a partially ordered set  $(A, \preceq)$  and a mapping  $F : A \times A \rightarrow A$ . The mapping  $F$  is said to have the mixed monotone property if  $F$  is monotone nondecreasing in its first coordinate and is monotone nonincreasing in its second coordinate, that is, for any  $a_1, a_2 \in A$ ,  $a_1 \preceq a_2$  implies  $F(a_1, b) \preceq F(a_2, b)$ , for  $b \in A$  and for all  $b_1, b_2 \in A$ ,  $b_1 \preceq b_2$  implies  $F(a, b_1) \succeq F(a, b_2)$ , for  $a \in A$ .

**Definition 1.3.27.** (Coupled Fixed Point). Let  $A$  be a nonempty set. A point  $(a, b) \in A \times A$  is called a coupled fixed point of the mapping  $F : A \times A \rightarrow A$  if  $F(a, b) = a$  and  $F(b, a) = b$ .

Following theorem is one of the main results of Bhaskar and Lakshmikantham [36]:

**Theorem 1.3.18.** Let  $(A, d, \preceq)$  be a partially ordered complete metric space. Consider a continuous mapping  $F : A \times A \rightarrow A$  having the mixed monotone property on  $A$ . Suppose that there exists a  $k \in [0, 1)$  with

$$d(F(a, b), F(u, v)) \leq \frac{k}{2} [d(a, u) + d(b, v)], \text{ for every } a \geq u, b \leq v.$$

If there exists  $a_0, b_0 \in A$  satisfying

$$a_0 \leq F(a_0, b_0) \text{ and } b_0 \geq F(b_0, a_0),$$

then there exist  $a, b \in A$  such that

$$a = F(a, b) \text{ and } b = F(b, a).$$

**Theorem 1.3.19.** In addition to the hypotheses of Theorem 1.3.18, suppose that for all  $(a, b), (a^*, b^*) \in A \times A$ , there exists a  $(c_1, c_2) \in A \times A$  that is comparable to  $(a, b)$  and  $(a^*, b^*)$ , then  $F$  has a unique coupled fixed point.

Also, Bhaskar and Lakshmikantham [36] obtained the existence and uniqueness of solution for a periodic boundary value problem.

Further, in 2009, Lakshmikantham and Ćirić [107] presented the notion of a mixed  $g$ -monotone mapping and proved coupled coincidence and coupled common fixed point theorems and thereby extending the theorems derived by Bhaskar and Lakshmikantham [36]. Lakshmikantham and Ćirić [107] gave the following concepts:

**Definition 1.3.28.** Consider a partially ordered set  $(A, \preceq)$  and mappings  $F : A \times A \rightarrow A$  and  $g : A \rightarrow A$ . The mapping  $F$  is said to have mixed  $g$ -monotone property if  $F(a, b)$  is monotone  $g$ -nondecreasing in  $a$  and is monotone  $g$ -nonincreasing in  $b$ , that is,

$$\begin{aligned} a_1, a_2 \in A, ga_1 \preceq ga_2 \text{ implies } F(a_1, b) \preceq F(a_2, b), \\ b_1, b_2 \in A, gb_1 \preceq gb_2 \text{ implies } F(a, b_2) \preceq F(a, b_1), \text{ for every } a, b \in A, \end{aligned}$$

**Definition 1.3.29.** Let  $(A, \preceq)$  be a partially ordered set. A point  $(a, b) \in A \times A$  is called a coupled coincidence point of the mappings  $F : A \times A \rightarrow A$  and  $g : A \rightarrow A$  if  $F(a, b) = ga$  and  $F(b, a) = gb$ .

**Definition 1.3.30.** Let  $(A, \preceq)$  be a partially ordered set. The mappings  $F : A \times A \rightarrow A$  and  $g : A \rightarrow A$  are commutative mappings if  $g(F(a, b)) = F(ga, gb)$  for all  $a, b \in A$ .

One of the main results of Lakshmikantham and Ćirić [107] on coupled coincidence points are as follows:

**Theorem 1.3.20.** Let  $(A, d, \leq)$  be a complete partially ordered metric space,  $F : A \times A \rightarrow A$  and  $g : A \rightarrow A$  such that  $F$  has the mixed  $g$ -monotone property. Suppose there is a function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  with  $\varphi(t) < t$  and  $\lim_{r \rightarrow t^+} \varphi(r) < t$  for each  $t > 0$  such that

$$d(F(a, b), F(u, v)) \leq \varphi \left( \frac{d(ga, gu) + d(gb, gv)}{2} \right)$$

holds for all  $a, b, u, v \in A$  with  $ga \leq gu$  and  $gb \geq gv$ . Let  $F(A \times A) \subseteq g(A)$ ,  $g$  is continuous and commutes with  $F$  and also suppose either

(a)  $F$  is continuous or

(b)  $A$  has the following property:

(i) if a nondecreasing sequence  $\{a_n\} \rightarrow a$ , then  $a_n \leq a$  for all  $n$ ,

(ii) if a nonincreasing sequence  $\{b_n\} \rightarrow b$ , then  $b \leq b_n$  for all  $n$ ,

If there exist  $a_0, b_0 \in A$  such that  $ga_0 \leq F(a_0, b_0)$  and  $gb_0 \geq F(b_0, a_0)$ , then there exist  $a, b \in A$  such that  $ga = F(a, b)$  and  $gb = F(b, a)$ , that is,  $F$  and  $g$  have a coupled coincidence.

Motivated by these results of Lakshmikantham and Ćirić [107], many authors have obtained number of coincidence and common fixed point results concerning coupled fixed points in ordered metric spaces (see ([4], [25], [48], [47], [108], [133])).

In 2011, Berinde and Borcut [32] extended the concept of coupled fixed point by introducing the notion of tripled fixed point and established fixed point results for mappings having a monotone property and satisfying a contractive condition in ordered metric spaces.

In 2012, Karapinar and Luong [98] presented the idea of quadruple fixed point and obtained some related new fixed point results.

Karapinar and Luong [98] introduced the following partial order on the product space  $A^4$ :

$$(u, v, r, t) \preceq (a, b, c, d) \text{ iff } a \succeq u, b \preceq v, c \succeq r, d \preceq t,$$

where  $(u, v, r, t), (a, b, c, d) \in A^4$ .

Regarding this partial order, Karapinar and Luong [98] gave the following definitions and results:

**Definition 1.3.31.** Let  $(A, \preceq)$  be a partially ordered set and  $F : A^4 \rightarrow A$ . The mapping  $F$  possess the mixed monotone property if  $F(a, b, c, d)$  is monotone nondecreasing in  $a$  and  $c$ , and is monotone nonincreasing in  $b$  and  $d$ , that is,

$$a_1, a_2 \in A, a_1 \preceq a_2 \text{ implies } F(a_1, b, c, d) \preceq F(a_2, b, c, d),$$

$$b_1, b_2 \in A, b_1 \preceq b_2 \text{ implies } F(a, b_1, c, d) \succeq F(a, b_2, c, d),$$

$$c_1, c_2 \in A, c_1 \preceq c_2 \text{ implies } F(a, b, c_1, d) \preceq F(a, b, c_2, d),$$

and  $d_1, d_2 \in A, d_1 \preceq d_2 \text{ implies } F(a, b, c, d_1) \succeq F(a, b, c, d_2)$ , for every  $a, b, c, d \in A$ .

**Definition 1.3.32.** A point  $(a, b, c, d) \in A^4$  is said to be a quadruple fixed point of  $F : A^4 \rightarrow A$  if  $F(a, b, c, d) = a, F(a, d, c, b) = b, F(c, b, a, d) = c$  and  $F(c, d, a, b) = d$ .

**Definition 1.3.33.** Let  $\kappa$  denote all the functions  $\varphi : [0, \infty) \rightarrow [0, \infty)$  which satisfy that  $\lim_{t \rightarrow r} \varphi(t) > 0$  for all  $r > 0$  and  $\lim_{t \rightarrow 0^+} \varphi(t) = 0$ .

**Definition 1.3.34.** Let  $\iota$  denote all the functions  $\varsigma : [0, \infty) \rightarrow [0, \infty)$  which satisfy (i)  $\varsigma(t) = 0$  iff  $t = 0$ , (ii)  $\varsigma$  is continuous and nondecreasing, (iii)  $\varsigma(s + t) \leq \varsigma(s) + \varsigma(t)$ ,  $\forall s, t \in [0, \infty)$ .

**Theorem 1.3.21.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $F : A^4 \rightarrow A$  be a mapping having the mixed monotone property on  $A$ . Assume that for all  $a \succeq u$ ,

$$b \leq v, c \geq r, d \leq t,$$

$$\begin{aligned} \psi(d(F(a, b, c, d), F(u, v, r, t))) &\leq \frac{1}{4}\zeta(d(a, u) + d(b, v) + d(c, r) + d(d, t)) \\ &\quad -\varphi(d(a, u) + d(b, v) + d(c, r) + d(d, t)) \end{aligned}$$

where  $\varphi \in \kappa$  and  $\zeta \in \iota$ . Suppose that there exist  $a_0, b_0, c_0, d_0 \in A$  such that

$$a_0 \leq F(a_0, b_0, c_0, d_0), b_0 \geq F(b_0, c_0, d_0, a_0), c_0 \leq F(c_0, d_0, a_0, b_0),$$

$$d_0 \geq F(d_0, a_0, b_0, c_0). \text{ Suppose that either (a) } F \text{ is continuous, or}$$

(b)  $A$  satisfies the following property:

(i) if nondecreasing sequence  $a_n$  converges to  $a$ , then  $a_n \leq a$  for all  $n$ ,

(ii) if nonincreasing sequence  $b_n$  converges to  $b$ , then  $b_n \geq b$  for all  $n$ ,

then there exist  $a, b, c, d \in A$  such that  $F(a, b, c, d) = a, F(b, c, d, a) = b, F(c, d, a, b) = c, F(d, a, b, c) = d$ .

**Theorem 1.3.22.** In addition to the hypotheses of Theorem 1.3.21, suppose that for all  $(a, b, c, d), (u, v, r, t) \in A^4$ , there exists  $(x, y, z, w) \in A^4$  that is comparable to  $(a, b, c, d)$  and  $(u, v, r, t)$ , then  $F$  has a unique quadruple fixed point.

Later, Karapinar and Berinde [97] proved that there exists a unique common quadruple fixed point of mappings  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  on a partially ordered complete metric space  $(A, d, \leq)$  under certain appropriate conditions. Karapinar and Berinde [97] presented the following definitions and results:

**Definition 1.3.35.** Let  $(A, \leq)$  be a partially ordered set and  $F : A^4 \rightarrow A$ . The mapping  $F$  is said to have the mixed  $g$ -monotone property if  $F(a, b, c, w)$  is monotone  $g$ -nondecreasing in  $a$  and  $c$ , and is monotone  $g$ -nonincreasing in  $b$  and  $d$ , that is,

$$a_1, a_2 \in A, g(a_1) \preceq g(a_2) \text{ implies } F(a_1, b, c, d) \preceq F(a_2, b, c, d),$$

$b_1, b_2 \in A, g(b_1) \preceq g(b_2)$  implies  $F(a, b_1, c, d) \succeq F(a, b_2, c, d)$ ,  
 $c_1, c_2 \in A, g(c_1) \preceq g(c_2)$  implies  $F(a, b, c_1, d) \preceq F(a, b, c_2, d)$ ,  
 and  $d_1, d_2 \in A, g(d_1) \preceq g(d_2)$  implies  $F(a, b, c, d_1) \succeq F(a, b, c, d_2)$ , for every  
 $a, b, c, d \in A$ .

**Definition 1.3.36.** Let  $(A, \preceq)$  be a partially ordered set. A point  $(a, b, c, d) \in A^4$  is called a quadruple coincidence point of mappings  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  if  $F(a, b, c, d) = g(a)$ ,  $F(a, d, c, b) = g(b)$ ,  $F(c, b, a, d) = g(c)$  and  $F(c, d, a, b) = g(d)$ .

**Definition 1.3.37.** Let  $(A, \preceq)$  be a partially ordered set. The mappings  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  are said to be commutative mappings if  $g(F(a, b, c, d)) = F(ga, gb, gc, gd)$ , for all  $a, b, c, d \in A$ .

**Definition 1.3.38.** [97] Let  $\omega$  be the set of the all functions  $\nu : [0, 1) \rightarrow [0, 1)$  satisfying the following conditions: (i)  $\nu$  is continuous, (ii)  $\nu(t) < t$ , (iii)  $\lim_{r \rightarrow t^+} \nu(r) < t$  for each  $r > 0$ .

**Theorem 1.3.23.** Let  $(A, d, \preceq)$  be a partially ordered complete metric space. Suppose  $F : A^4 \rightarrow A$  and there exists  $\nu \in \omega$  such that  $F$  has the mixed  $g$ -monotone property and

$$d(F(a, b, c, d), F(u, v, r, t)) \leq \phi \left( \frac{d(g(a), g(u)) + d(g(b), g(v)) + d(g(c), g(r)) + d(g(d), g(t))}{4} \right)$$

for all  $a, u, b, v, z, r, w, t$  for which  $g(a) \leq g(u)$ ,  $g(b) \geq g(v)$ ,  $g(c) \leq g(r)$  and  $g(d) \geq g(t)$ .

Suppose there exist  $a_0, b_0, c_0, d_0 \in A$  such that

$$g(a_0) \leq F(a_0, b_0, c_0, d_0), g(b_0) \geq F(a_0, d_0, c_0, b_0), g(c_0) \leq F(c_0, b_0, a_0, d_0), g(d_0) \geq F(c_0, d_0, a_0, b_0).$$

Suppose that either (a)  $F$  is continuous, or

(b)  $A$  has the following property:

(i) if non-decreasing sequence  $a_n$  tends to  $a$ , then  $a_n \leq a$  for all  $n$ ,

(ii) if non-increasing sequence  $b_n$  tends to  $b$ , then  $b_n \geq b$  for all  $n$ .

Assume also that  $F(A^4) \subset g(A)$  and  $g$  commutes with  $F$ . Then there exist  $a, b, c, d \in A$  such that  $F(a, b, c, d) = g(a)$ ,  $F(a, d, c, b) = g(b)$ ,  $F(c, b, a, d) = g(c)$ ,  $F(c, d, a, b) = g(d)$ .

**Theorem 1.3.24.** In addition to the hypotheses of Theorem 1.3.23, assume that for all  $(a, b, c, d), (u, v, r, t) \in A^4$ , there exists  $(x, y, z, w) \in A^4$  such that  $(F(x, y, z, w), F(x, w, z, y), F(z, y, x, w), F(z, w, x, y))$  is comparable to  $(F(a, b, c, d), F(a, d, c, b), F(c, b, a, d), F(c, d, a, b))$  and  $(F(u, v, r, t), F(u, t, r, v), F(t, v, u, r), F(r, t, u, v))$ . Then,  $F$  and  $g$  have a unique quadruple coincidence point, that is, there exists a unique  $(p, q, s, o) \in A^4$  such that  $g(p) = F(p, q, s, o)$ ,  $g(q) = F(p, o, s, q)$ ,  $g(s) = F(s, q, p, o)$ ,  $g(o) = F(s, o, p, q)$ .

Later, various authors established the related fixed point theorems (see [93, 94, 101]). In an attempt to generalize the concepts of coupled fixed point, tripled fixed point and quartet fixed point, Berzig and Samet [35] gave the notion of fixed point of  $N$ -order for  $m$ -mixed monotone mappings and proved the existence of a unique fixed point for these contractive type mappings in complete ordered metric spaces.

## 1.4 $\alpha$ -admissible mapping, $\alpha$ - $\psi$ -contractive type mapping and its various generalizations

In an attempt to generalize the Banach contraction principle, Samet *et al.* [165] in 2012 presented the new ideas of  $\alpha$ - $\psi$ -contractive mappings and  $\alpha$ -admissible mappings by considering the following family of functions:

**Definition 1.4.1.** Let  $\Psi$  denote the family of all functions  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  satisfying the following properties

- (i)  $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$  for each  $t > 0$ , where  $\psi^n$  is the  $n$ -th iterate of  $\psi$ .  
(ii)  $\psi$  is nondecreasing.

**Definition 1.4.2.** [165] Suppose  $(A, d)$  be a metric space. The mapping  $T : A \rightarrow A$  is said to be an  $\alpha$ - $\psi$ -contractive mapping if there exists two functions  $\alpha : A \times A \rightarrow [0, +\infty)$  and  $\psi \in \Psi$  such that

$$\alpha(a, b)d(Ta, Tb) \leq \psi(d(a, b)),$$

for all  $a, b \in A$ .

**Definition 1.4.3.** [165] Let  $T : A \rightarrow A$  and  $\alpha : A \times A \rightarrow [0, +\infty)$ . The mapping  $T$  is called as  $\alpha$ -admissible if

$$\alpha(a, b) \geq 1 \Rightarrow \alpha(Ta, Tb) \geq 1.$$

for all  $a, b \in A$ .

Now, we present some examples of  $\alpha$ -admissible mappings.

**Example 1.4.4.** Consider  $A = \mathbb{R}^+$ . Define the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  by

$$\alpha(a, b) = \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{if } a \geq b. \end{cases}$$

and the mapping  $T : A \rightarrow A$  by  $Ta = a^2$  for all  $a \in A$ . Clearly, the mapping  $T$  is  $\alpha$ -admissible.

**Example 1.4.5.** Consider  $A = \mathbb{R}^+$ . Define the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  by

$$\alpha(a, b) = \begin{cases} 0 & \text{if } a < b, \\ e^{a-b} & \text{if } a \geq b. \end{cases}$$

and the mapping  $T : A \rightarrow A$  by  $Ta = e^a$  for all  $a \in A$ . Clearly, the mapping  $T$  is  $\alpha$ -admissible.

Following are the main results of Samet *et al.* [165]:

**Theorem 1.4.1.** [165] Suppose  $(A, d)$  be a complete metric space and  $T : A \rightarrow A$  be an  $\alpha$ - $\psi$ -contractive mapping satisfying the following conditions:

- (i) the mapping  $T$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, Ta_0) \geq 1$ ,
- (iii) the mapping  $T$  is continuous.

Then,  $T$  has a fixed point.

**Theorem 1.4.2.** [165] Suppose  $(A, d)$  be a complete metric space and  $T : A \rightarrow A$  be an  $\alpha$ - $\psi$ -contractive mapping satisfying the following conditions:

- (i) the mapping  $T$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, Ta_0) \geq 1$ ,
- (iii) if  $\{a_n\}$  is a sequence in  $A$  such that  $\alpha(a_n, a_{n+1}) \geq 1$  for all  $n$  and  $a_n \rightarrow a \in A$  as  $n \rightarrow +\infty$ , then  $\alpha(a_n, a) \geq 1$  for all  $n$ .

Then,  $T$  has a fixed point.

Samet *et al.* [165] added the following condition (H) to the hypotheses of Theorems 1.4.1 and 1.4.2 for the uniqueness of the fixed point:

(H) : For every  $a, b \in A$ , there exists  $c \in A$  such that  $\alpha(a, c) \geq 1$  and  $\alpha(b, c) \geq 1$ .

Further, Samet *et al.* [165] obtained the following coupled fixed point theorems in complete metric spaces:

**Theorem 1.4.3.** Suppose  $(A, d)$  be a complete metric space and  $F : A \times A \rightarrow A$  be a given mapping. Suppose that there exists  $\psi \in \Psi$  and a function  $\alpha : A^2 \times A^2 \rightarrow [0, +\infty)$  such that

$$\alpha((a, b), (u, v))d(F(a, b), F(u, v)) \leq \frac{1}{2}\psi(d(a, u) + d(b, v)),$$

for all  $(a, b), (u, v) \in A \times A$ . Also, suppose that

- (i)  $\alpha((a, b), (u, v)) \geq 1 \Rightarrow \alpha((F(a, b), F(b, a)), (F(u, v), F(v, u))) \geq 1$  for all  $(a, b), (u, v) \in A \times A$ ,
- (ii) there exists  $(a_0, b_0) \in A \times A$  satisfying  $\alpha((a_0, b_0), (F(a_0, b_0), F(b_0, a_0))) \geq 1$  and  $\alpha((F(b_0, a_0), F(a_0, b_0)), (b_0, a_0)) \geq 1$ .
- (iii)  $F$  is continuous.

Then,  $F$  has a coupled fixed point.

**Theorem 1.4.4.** Suppose  $(A, d)$  be a complete metric space and  $F : A \times A \rightarrow A$  be a given mapping. Let there exists  $\psi \in \Psi$  and a function  $\alpha : A^2 \times A^2 \rightarrow [0, +\infty)$  satisfying

$$\alpha((a, b), (u, v))d(F(a, b), F(u, v)) \leq \frac{1}{2}\psi(d(a, u) + d(b, v)),$$

for every  $(a, b), (u, v) \in A \times A$ . Also, suppose that

- (i)  $\alpha((a, b), (u, v)) \geq 1 \Rightarrow \alpha((F(a, b), F(b, a)), (F(u, v), F(v, u))) \geq 1$  for all  $(a, b), (u, v) \in A \times A$ ,

(ii) there exists  $(a_0, b_0) \in A \times A$  such that

$$\alpha((a_0, b_0), (F(a_0, b_0), F(b_0, a_0))) \geq 1 \text{ and } \alpha((F(b_0, a_0), F(a_0, b_0)), (b_0, a_0)) \geq 1.$$

(iii) if  $\{a_n\}$  and  $\{b_n\}$  are sequences in  $A$  such that

$$\alpha((a_n, b_n), (a_{n+1}, b_{n+1})) \geq 1 \text{ and } \alpha((b_{n+1}, a_{n+1}), (b_n, a_n)) \geq 1,$$

$a_n \rightarrow a \in A$  and  $b_n \rightarrow b \in A$  as  $n \rightarrow +\infty$ , then

$$\alpha((a_n, b_n), (a, b)) \geq 1 \text{ and } \alpha((b, a), (b_n, a_n)) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Then,  $F$  has a coupled fixed point.

Samet *et al.* [165] added the following condition  $(H')$  to the hypotheses of Theorems 1.4.3 and 1.4.4 for the uniqueness of the coupled fixed point:

$(H')$ : For every  $(a, b), (u, v) \in A \times A$ , there exists  $(c_1, c_2) \in A \times A$  such that

$$\alpha((a, b), (c_1, c_2)) \geq 1, \quad \alpha((c_2, c_1), (b, a)) \geq 1$$

and

$$\alpha((u, v), (c_1, c_2)) \geq 1, \quad \alpha((c_2, c_1), (v, u)) \geq 1.$$

Recently, Karapinar and Samet [100] introduced the following notion of generalized  $\alpha$ - $\psi$ -contractive type mappings:

**Definition 1.4.6.** Suppose  $(A, d)$  be a metric space and  $T : A \rightarrow A$  be a given mapping. The mapping  $T$  is called a generalized  $\alpha$ - $\psi$ -contractive type mapping if there exists two functions  $\alpha : A \times A \rightarrow [0, \infty)$  and  $\psi \in \Psi$  satisfying

$$\alpha(a, b)d(Ta, Tb) \leq \psi(M(a, b)), \text{ for every } a, b \in A,$$

where  $M(a, b) = \max \left\{ d(a, b), \frac{d(a, Ta) + d(b, Tb)}{2}, \frac{d(a, Tb) + d(b, Ta)}{2} \right\}$ .

Further, Karapinar and Samet [100] established some fixed point results for these new type of contractive mappings. Also, they proved fixed point theorems in partially ordered metric spaces and fixed point theorems for cyclic contractive mappings.

In 2012, motivated by the results of Samet *et al.* [165], Asl *et al.* [23] gave the following concepts of  $\alpha_*$ - $\psi$ -contractive multifunctions and  $\alpha_*$ -admissible mapping thereby presenting some fixed point results for  $\alpha_*$ - $\psi$ -contractive multifunctions.

**Definition 1.4.7.** [23] Suppose  $(A, d)$  be a metric space. Let  $T : A \rightarrow 2^A$  be a closed-valued multifunction,  $\psi \in \Psi$  and  $\alpha : A \times A \rightarrow [0, \infty)$  be a function. The mapping  $T$  is called as  $\alpha_*$ - $\psi$ -contractive multifunction if it satisfies the following condition

$$\alpha_*(Tu, Tv)H(Tu, Tv) \leq \psi(d(u, v)), \text{ for all } u, v \in A$$

where  $H$  is the Hausdorff metric,  $\alpha_*(A, B) = \inf\{\alpha(a, b) : a \in A, b \in B\}$ .

**Definition 1.4.8.** [23] The mapping  $T$  is known as  $\alpha_*$ -admissible whenever  $\alpha(a, b) \geq 1$  implies  $\alpha_*(Ta, Tb) \geq 1$ .

Asl *et al.* [23] established the following theorem:

**Theorem 1.4.5.** [23] Let  $(A, d)$  be a complete metric space,  $\alpha : A \times A \rightarrow [0, \infty)$  be a function and  $\psi \in \Psi$  be a strictly increasing map. Suppose  $T$  be a closed-valued,  $\alpha_*$ -admissible and  $\alpha_*$ - $\psi$ -contractive multifunction on  $A$ . Also, suppose that there exists  $a_0 \in A$  and  $a_1 \in Ta_0$  such that  $\alpha(a_0, a_1) \geq 1$  and if  $\{a_n\}$  is a sequence in  $A$  such that  $\alpha(a_n, a_{n+1}) \geq 1$  for every  $n$  and  $a_n \rightarrow a$ , then  $\alpha(a_n, a) \geq 1$  for every  $n$ . Then  $T$  has a fixed point.

Very recently, Mursaleen *et al.* [114, 115] established some coupled fixed point theorems for  $\alpha$ - $\psi$ -contractive mappings in partially ordered metric spaces.

## 1.5 $F$ -contraction Mapping

Recently, Wardowski [193] introduced a new concept of contraction called  $F$ -contraction and proved a fixed point theorem which generalizes Banach contraction principle in a unique way. Wardowski [193] considered the following family of functions:

**Definition 1.5.1.** Let  $\mathfrak{F}$  be the family of functions  $F : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying the following conditions:

- (i)  $F$  is strictly increasing, i.e. for all  $\alpha, \beta \in F : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that  $\alpha < \beta$ ,  $F(\alpha) < F(\beta)$ ,
- (ii) For each sequence  $\{\alpha_n\}_{n \in \mathbb{N}}$  of positive numbers  $\lim_{n \rightarrow \infty} \alpha_n = 0$  if and only if  $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$ ,
- (iii) There exists  $k \in (0, 1)$  such that  $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$ .

Wardowski [193] presented the concept of  $F$ -contraction as follows:

**Definition 1.5.2.** A mapping  $T : A \rightarrow A$  is known as  $F$ -contraction mapping if there exists  $\tau > 0$  such that

$$\forall_{a, b \in A} (d(Ta, Tb) > 0 \Rightarrow F(d(Ta, Tb)) + \tau \leq F(d(a, b))) \quad (1.5.1)$$

By considering different types of mapping  $F$  in (1.5.1), Wardowski [193] obtained the following variety of contractions, out of which some are already known in the literature:

- (i)  $\frac{d(Ta, Tb)(d(Ta, Tb) + 1)}{d(a, b)(d(a, b) + 1)} \leq e^{-\tau}$ , for every  $a, b \in A, Ta \neq Tb$ ,
- (ii)  $d(Ta, Tb) \leq \frac{1}{(1 + \tau\sqrt{d(a, b)})^2} d(a, b)$ , for every  $a, b \in A, Ta \neq Tb$ ,

$$(iii) \frac{d(Ta, Tb)}{d(a, b)} e^{d(Ta, Tb) - d(a, b)} \leq e^{-\tau}, \text{ for every } a, b \in A, Ta \neq Tb,$$

$$(iv) d(Ta, Tb) \leq e^{-\tau} d(a, b), \text{ for every } a, b \in A, Ta \neq Tb.$$

It is to be noted that all the above contractive conditions are satisfied for  $a, b \in A$ , such that  $Ta = Tb$ .

In what follows, we state the main theorem of Wardowski [193].

**Theorem 1.5.1.** Suppose  $(A, d)$  is a complete metric space and  $T : A \rightarrow A$  be an  $F$ -contraction. Then the mapping  $T$  has a unique fixed point  $a^* \in A$  and for every  $a_0 \in A$  a sequence  $\{T^n a_0\}_{n \in \mathbb{N}}$  is convergent to  $a^*$ .

Several authors (e.g. [2], [168], [171], [194]) have further improved the results of Wardowski [193] in several ways.

## 1.6 Objectives of the study

In specific terms, the objectives of this study are as follows:

- (i) To study the existence of fixed points and extend various known results to various spaces such as  $G$ -metric, cone metric, partial metric and quasimetric etc.
- (ii) To study the coupled and tripled fixed point theorems using some mappings such as weakly compatible mappings, weakly commuting mappings,  $R$ -weakly commuting mappings and expansion mappings etc in various partially ordered spaces.
- (iii) To derive some applications of fixed point theorems.

In view of these objectives, some results have been obtained in various abstract spaces such as metric spaces,  $G$ -metric spaces, complex valued metric spaces, partial Hausdorff metric spaces and partially ordered spaces.

## 1.7 Thesis organization

The present thesis consists of eight chapters. Each chapter is divided into various sections. The numbers like 6.4.1 indicates Theorem (or Lemma/Proposition/Remark/Corollary/Definition) 1 of Section 4 of Chapter 6. The hypotheses, conditions or conclusions of Chapters 1 to 8 are denoted by (i), (ii), (iii), (iv),... . As usual the numbers in big brackets refer to the references listed in the bibliography. The work carried out in this thesis can be described as follows:

In **Chapter 2**, a new and simple approach has been introduced in the theory of expansive mappings by presenting a new kind of expansive mappings called  $(\xi, \alpha)$ -expansive mappings in metric spaces. Several fixed point theorems for these mappings have been studied in this chapter. This new concept complements the idea of  $\alpha$ - $\psi$ -contractive type mappings introduced recently by Samet *et al.* [165]. In order to generalize the expansion mappings in a different way, a new type of expansive mappings called  $F$ -expansive mappings has also been introduced in this chapter. This new notion of  $F$ -expansive mappings complements the concept of  $F$ -contraction introduced recently by Wardowski [193].

**Chapter 3** deals with the generalization of the contractive conditions of integral type. In this chapter, a new kind of mapping called  $\alpha$ - $\psi$ -contractive type mapping of integral type has been introduced and some fixed point and coupled fixed point theorems have been established for such mappings in complete metric spaces. As a consequence of the main results, some fixed point results on partially ordered metric spaces have also been obtained. The results presented in this chapter improve, extend and generalize the results derived by Banach [30], Berinde [31], Branciari [39], Rhoades and Abbas [152], Samet *et al.* [165] and references therein.

In **Chapter 4**, a new class of contractive pair of mappings called generalized  $\alpha$ - $\psi$  contractive pair of mappings has been introduced. Various coincidence and common fixed point results regarding this class of contractive pair of mappings have been studied in this chapter. As consequences of the main results, some common fixed point theorems have also been established for metric spaces endowed with a partial order as well as in respect of cyclic contractive mappings. The theorems presented in this chapter extend and generalize various known relevant results from the current literature. Some illustrative examples are provided to demonstrate the main results. The results presented in this chapter unify and generalize the results derived by Karapinar and Samet [100], Samet *et al.* [165], Ćirić *et al.* [55] and various other related results in the literature.

**Chapter 5** is devoted to the generalization of the results of Ahmed [17] from 2-metric spaces to  $G$ -metric spaces. In order to prove the results, a more generalized concept of weak compatibility in  $G$ -metric spaces have been used instead of compatibility of type(A) used by Ahmed [17] in 2-metric spaces. Also, the results of Şahin and Telci [161] are extended to  $G$ -metric spaces and thereby extending the theorem of Wang *et al.* [192] for a pair of mappings to  $G$ -metric spaces.

In **Chapter 6**, some quadruple coincidence point theorems have been obtained for a mixed  $g$ -monotone mapping satisfying nonlinear contractions in the setting of partially ordered  $G$ -metric spaces. The results presented in this chapter generalize the very recent results of Karapinar [94], Karapinar and Berinde [97], Karapinar *et al.* [101] and various other related results in the literature.

The objective of **Chapter 7** is to obtain the fixed point results for  $\alpha_*$ - $\psi$ -contractive multifunctions in a partial Hausdorff metric space. The presented theorems generalize and consolidate many related results in the literature, in particular the results of Aydi *et al.* [24],

Asl *et al.* [23] and the references therein. As a consequence of these results, a homotopy result has been presented as an application.

In **Chapter 8**, the generalized version of  $\alpha$ - $\psi$  contractive type mappings has been presented in complex valued metric spaces by defining the notion of  $(\alpha, \phi)$ -contractive mappings of rational type. Some fixed point theorems for these mappings have been established in the framework of complex valued metric spaces. The results presented in this chapter extend the recent results of Azam *et al.* [29], Rouzkard and Imdad [156] and references therein. Some examples have also been given to verify the effectiveness and applicability of main results. In order to support the usability of these results, an application to integral equations has also been presented. Towards the end of this chapter, some relevant topics for further research have been suggested based on the present study.

The thesis concludes by listing the Bibliography of various publications cited in this work.



## Chapter 2

# FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS IN METRIC SPACES

### 2.1 Introduction

In nonlinear functional analysis, metric fixed point theory is one of the most effective tools because of its wide range of applications in fields such as economics (see e.g. [38, 130]), computer science (see e.g. [111, 155, 167]), and many others. The Banach contraction principle [30] is the most fundamental result in metric fixed point theory. This principle is stated as: "Each contraction defined on a complete metric space  $X$  has a unique fixed point".

In 1977, Rhoades [147] studied different types of contractive mappings and investigated the relationships among them. Analogous to some contractive mappings in [147], Wang *et al.* [192] in 1984 introduced expansion mappings in metric spaces. Wang *et al.* [192] gave the concept of expansion mappings in the form of the following theorem:

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Some contents of the work presented in this chapter have been published in *Fixed Point Theory and Applications* 2012, 2012:157 (SCI) and the rest have been communicated in *Ciência e Técnica Vitivinícola Journal*.

**Theorem 2.1.1.** [192] Let  $f : A \rightarrow A$  be an onto mapping of a complete metric space  $(A, d)$ . If there exists a constant  $q > 1$  satisfying

$$d(fa, fb) \geq qd(a, b), \text{ for every } a, b \in A,$$

then  $f$  has a unique fixed point in  $A$ .

Further, Khan *et al.* [102] extended the work of Wang *et al.* [192] by utilizing some functions. Later, Rhoades [148] and Taniguchi [185] generalized the results of Wang [192] for pair of mappings. Kang [88] improved the results of Khan *et al.* [102], Rhoades [148] and Taniguchi [185] for expansion mappings.

In an attempt to generalize the Banach contraction principle and many available results in the literature, Samet *et al.* [165] introduced a new concept of  $\alpha$ - $\psi$ -contractive type mappings via  $\alpha$ -admissible mappings and presented various fixed point theorems for these mappings in complete metric spaces. Recently, Wardowski [193] created an interesting and different type of contraction called  $F$ -contraction and proved a fixed point theorem which generalizes Banach contraction principle in an unusual way.

In this chapter, a unique approach in the theory of expansive mappings has been introduced in the form of  $(\xi, \alpha)$ -expansive mappings. This new idea complements the notion of  $\alpha$ - $\psi$ -contractive type mappings due to Samet *et al.* [165]. Also, a new kind of expansive mappings called  $F$ -expansive mappings has been given in order to complement the concept of  $F$ -contraction introduced by Wardowski [193].

The contents of this chapter have been divided into four sections. Section 2.2 deals with the preliminaries related to this chapter. In Section 2.3, various fixed point results have been established in complete metric spaces for  $(\xi, \alpha)$ -expansive mappings. In Section 2.4, the existence of coupled fixed points have been studied for  $(\xi, \alpha)$ -expansive mappings in complete metric spaces. Section 2.5 concerns with a type of expansive mappings called

$F$ -expansive mappings which generalizes the expansion mappings in a distinct way as compared to the relevant results from the literature. The results presented in this chapter extend many related results of the literature. In order to illustrate these results, some examples are given in this chapter.

## 2.2 Preliminaries

Recently, Samet *et al.* [165] introduced the new notions of  $\alpha$ - $\psi$ -contractive mappings and  $\alpha$ -admissible mappings by considering the following family of functions:

**Definition 2.2.1.** Let  $\Psi$  be the family of all functions  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  satisfying the following properties:

- (i)  $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$  for every  $t > 0$ , where  $\psi^n$  is the  $n$ -th iterate of  $\psi$ .
- (ii) the function  $\psi$  is nondecreasing.

**Lemma 2.2.1.** [165] If  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  is a nondecreasing function, then for each  $a > 0$ ,  $\lim_{n \rightarrow +\infty} \psi^n(a) = 0$  implies  $\psi(a) < a$ .

**Lemma 2.2.2.** [165] If  $\psi \in \Psi$ , then the function  $\psi$  is continuous at 0.

**Definition 2.2.2.** [165] Let  $f$  be a given self-mapping of a metric space  $(A, d)$ . The mapping  $f$  is said to be an  $\alpha$ - $\psi$ -contractive mapping if there exists two functions  $\alpha : A \times A \rightarrow [0, +\infty)$  and  $\psi \in \Psi$  satisfying

$$\alpha(a, b)d(fa, fb) \leq \psi(d(a, b)),$$

for every  $a, b \in A$ .

**Definition 2.2.3.** [165] Let  $f : A \rightarrow A$  and  $\alpha : A \times A \rightarrow [0, +\infty)$ . The mapping  $f$  is known as  $\alpha$ -admissible mapping if

$$\alpha(a, b) \geq 1 \Rightarrow \alpha(fa, fb) \geq 1$$

for every  $a, b \in A$ .

Some of the main results related to this class of mappings have already been presented in Chapter 1.

Now, we consider the following family of functions and introduce a new type of expansive mappings called  $(\xi, \alpha)$ -expansive mappings as follows:

**Definition 2.2.4.** Let  $\chi$  be the set of all functions  $\xi : [0, +\infty) \rightarrow [0, +\infty)$  satisfying the following properties:

- (i) the function  $\xi$  is nondecreasing,
- (ii)  $\sum_{n=1}^{+\infty} \xi^n(u) < +\infty$  for each  $u > 0$ , where  $\xi^n$  is the  $n$ -th iterate of  $\xi$ ,
- (iii)  $\xi(a + b) = \xi(a) + \xi(b)$ , for every  $a, b \in [0, +\infty)$ .

**Definition 2.2.5.** Suppose  $f$  be a self-mapping of a metric space  $(A, d)$ . The mapping  $f$  is said to be an  $(\xi, \alpha)$ -expansive mapping if there exist two functions  $\xi \in \chi$  and  $\alpha : A \times A \rightarrow [0, +\infty)$  satisfying

$$\xi(d(fa, fb)) \geq \alpha(a, b)d(a, b), \quad (2.2.1)$$

for every  $a, b \in A$ .

**Remark 2.2.1.** Every expansive mapping  $f$  is an  $(\xi, \alpha)$ -expansive mapping by taking  $\alpha(a, b) = 1$  for all  $a, b \in A$  and  $\xi(u) = ku$  for all  $u \geq 0$  and some  $k \in [0, 1)$ .

Recently, Wardowski [193] introduced a new type of contraction called  $F$ -contraction in complete metric spaces by considering the following family of functions:

**Definition 2.2.6.** Let  $\mathfrak{F}$  be the family of all functions  $F : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying

- (i)  $F$  is strictly increasing,
- (ii) for every sequence  $\{\alpha_n\}_{n \in \mathbb{N}}$  of positive real numbers,  $\lim_{n \rightarrow \infty} \alpha_n = 0$  if and only if  $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$ ,
- (iii) there exists  $k \in (0, 1)$  satisfying  $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$ .

Wardowski [193] presented the following new concept of  $F$ -contraction:

**Definition 2.2.7.** A self-mapping  $f$  of a metric space  $(A, d)$  is known as  $F$ -contraction if there exists  $\tau > 0$  satisfying

$$d(fa, fb) > 0 \Rightarrow F(d(fa, fb)) + \tau \leq F(d(a, b)) \quad (2.2.2)$$

for every  $a, b \in A$ .

The new notion of  $F$ -expansive mapping has been introduced as follows:

**Definition 2.2.8.** Let  $(A, d)$  be a metric space and  $F \in \mathfrak{F}$ . The self-mapping  $f$  of the space  $A$  is said to be an  $F$ -expansive mapping if there exists  $\tau < 0$  satisfying

$$d(a, b) > 0 \Rightarrow \tau + F(d(fa, fb)) \geq F(d(a, b)) \quad (2.2.3)$$

for every  $a, b \in A$ .

**Remark 2.2.2.** The Definition 2.2.6 and inequality (2.2.3) implies that every  $F$ -expansive mapping  $T$  satisfies

$$d(fa, fb) > d(a, b).$$

By substituting different types of the mapping  $F$  in (2.2.3), the variety of expansions are obtained, out of which some are already known in the literature:

**Example 2.2.9.** Suppose the mapping  $F : \mathbb{R}^+ \rightarrow \mathbb{R}$  be defined by  $F(\alpha) = \ln(\alpha)$ . Clearly,  $F \in \mathfrak{F}$  for any  $k \in (0, 1)$ . The mapping  $f : A \rightarrow A$  satisfying the condition (2.2.3) is an  $F$ -expansive mapping satisfying

$$d(fa, fb) \geq e^{-\tau} d(a, b), \quad (2.2.4)$$

for every  $a, b \in A$ . The condition  $d(fa, fb) \geq hd(a, b)$  for  $h > 1$  of Wang *et al.* [192] holds for every  $a, b \in A$ .

**Example 2.2.10.** Consider  $F(\alpha) = \ln \alpha + \alpha$ ,  $\alpha > 0$ . Clearly,  $F \in \mathfrak{F}$  for  $k \in (0, 1)$ . The condition (2.2.3) reduces to

$$\frac{d(fa, fb)}{d(a, b)} e^{d(fa, fb) - d(a, b)} \geq e^{-\tau}, \quad (2.2.5)$$

for every  $a, b \in A$ ,  $a \neq b$ .

**Example 2.2.11.** Suppose  $F(\alpha) = \ln \alpha - \alpha$ ,  $\alpha > 0$ . Clearly,  $F \in \mathfrak{F}$  for any  $k \in (0, 1)$ . The inequality (2.2.3) reduces to

$$\frac{d(fa, fb)}{d(a, b)} e^{d(a, b) - d(fa, fb)} \geq e^{-\tau}, \quad (2.2.6)$$

for every  $a, b \in A$ ,  $a \neq b$ .

**Example 2.2.12.** Consider  $F(\alpha) = \ln(\alpha^2 - \alpha)$ ,  $\alpha > 0$ . Clearly,  $F \in \mathfrak{F}$  for any  $k \in (0, 1)$ . For this  $F$ -expansion  $f$ , we have

$$\frac{d(fa, fb)(d(fa, fb) - 1)}{d(a, b)(d(a, b) - 1)} \geq e^{-\tau}, \quad (2.2.7)$$

for every  $a, b \in A$ ,  $a \neq b$ .

**Example 2.2.13.** Let  $F(\alpha) = \ln(2^\alpha - 1)$ ,  $\alpha > 0$ . Clearly,  $F \in \mathfrak{F}$  for any  $k \in (0, 1)$ . For this  $F$ -expansion  $f$ , the following inequality holds

$$\frac{2^{d(fa,fb)} - 1}{2^{d(a,b)} - 1} \geq e^{-\tau}, \quad (2.2.8)$$

for every  $a, b \in A$ ,  $a \neq b$ .

Clearly, in all the above examples the expansive conditions are fulfilled for each  $a, b \in A$  such that  $a \neq b$ .

**Remark 2.2.3.** If we take the mappings  $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$ ,  $\alpha > 0$  and  $F(\alpha) = \ln(\alpha^2 + \alpha)$ ,  $\alpha > 0$  as considered by Wardowski [193], then we obtain the following different types of expansions respectively:

$$(a) \ d(fa, fb) \geq \frac{1}{(1 + \tau\sqrt{d(a,b)})^2} d(a, b), \text{ for all } a, b \in A, fa \neq fb$$

$$(b) \ \frac{d(fa, fb)(d(fa, fb) + 1)}{d(a, b)(d(a, b) + 1)} \geq e^{-\tau}, \text{ for all } a, b \in A, fa \neq fb. \text{ It is to be noted that all the}$$

above expansive conditions are satisfied for every  $a, b \in A$  with  $a \neq b$ .

**Remark 2.2.4.** Suppose  $F_1, F_2 \in \mathfrak{F}$  and  $F_1(\alpha) \leq F_2(\alpha)$  for all  $\alpha > 0$ . Clearly, the mapping  $C = B - A$  is nondecreasing. Now, in view of Remark 2.2.2, the following holds

$$C(d(fa, fb)) \geq C(d(a, b)),$$

for every  $a, b \in A$ ,  $a \neq b$ . Therefore,

$$\begin{aligned} \tau + B(d(fa, fb)) &= \tau + A(d(fa, fb)) + C(d(fa, fb)) \\ &\geq A(d(a, b)) + C(d(a, b)) = B(d(a, b)), \end{aligned}$$

for every  $a, b \in A$  with  $a \neq b$ . Thus, every  $A$ -expansive mapping  $f$  is  $B$ -expansive, but the converse need not hold true.

## 2.3 Fixed points for $(\xi, \alpha)$ -expansive mappings in metric spaces

The main results of this section are as follows:

**Theorem 2.3.1.** Let the mapping  $f : A \rightarrow A$  be a bijective,  $(\xi, \alpha)$ -expansive mapping in a complete metric space  $(A, d)$ . Also, suppose that:

- (i) the mapping  $f^{-1}$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, f^{-1}a_0) \geq 1$ ,
- (iii) the mapping  $f$  is continuous.

Then,  $f$  has a fixed point.

*Proof.* Define the sequence  $\{a_n\}$  in  $A$  by  $a_n = f a_{n+1}$  for every  $n \in \mathbb{N}$ . Let  $a_0 \in A$  be such that  $\alpha(a_0, f^{-1}a_0) \geq 1$ . Now, if  $a_n = a_{n+1}$  for each  $n \in \mathbb{N}$ , then  $a_n$  is a fixed point of  $f$ . Without loss of generality, suppose that  $a_n \neq a_{n+1}$  for every  $n \in \mathbb{N}$ .

In view of condition (ii), we have

$$\alpha(a_0, a_1) = \alpha(a_0, f^{-1}a_0) \geq 1.$$

The  $\alpha$ -admissibility of  $f^{-1}$  implies that

$$\alpha(f^{-1}a_0, f^{-1}a_1) = \alpha(a_1, a_2) \geq 1.$$

On making use of mathematical induction, we obtain

$$\alpha(a_n, a_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \quad (2.3.1)$$

The use of inequalities (2.3.1) and (2.2.1) gives

$$d(a_n, a_{n+1}) \leq \alpha(a_n, a_{n+1})d(a_n, a_{n+1}) \leq \xi(d(fa_n, fa_{n+1})) = \xi(d(a_{n-1}, a_n)).$$

Therefore, by repetition of the above process, we get  $d(a_n, a_{n+1}) \leq \xi^n(d(a_0, a_1))$ , for all  $n \in \mathbb{N}$ . Now, for any  $n > m \geq 0$ ,

$$\begin{aligned} d(a_m, a_n) &\leq d(a_m, a_{m+1}) + d(a_{m+1}, a_{m+2}) \\ &\quad + d(a_{m+2}, a_{m+3}) + \dots + d(a_{n-1}, a_n) \\ &\leq \xi^m(d(a_0, a_1)) + \dots + \xi^{n-1}(d(a_0, a_1)). \end{aligned}$$

As  $\sum \xi^n(a) < +\infty$  for each  $a > 0$ , so  $\{a_n\}$  is a Cauchy sequence in the complete metric space  $(A, d)$ . Therefore, there exists a point  $u \in A$  such that  $a_n \rightarrow u$  as  $n \rightarrow +\infty$ .

From the continuity of  $f$ , it follows that

$$a_n = f a_{n+1} \rightarrow f u \text{ as } n \rightarrow +\infty.$$

The uniqueness of the limit implies that  $u = f u$ . This completes the proof.  $\square$

The following lemma due to Samet *et al.* [165] is required in our next theorem:

**Lemma 2.3.2.** [165] If  $\xi \in \chi$ , then  $\xi$  is continuous at 0.

*Proof.* Since  $\sum_{n=1}^{\infty} \xi^n(t) < +\infty$  for all  $t > 0$ , then  $\xi^n(t) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $t > 0$ . By Lemma 2.2.1, since  $\xi$  is nondecreasing, this implies that  $\xi(t) < t$  for all  $t > 0$ . Letting  $t \rightarrow 0+$ , we get  $\lim_{t \rightarrow 0+} \xi(t) \leq 0$ . This implies that  $\lim_{t \rightarrow 0+} \xi(t) = 0$ .

Now suppose that  $\xi(0) = a > 0$ . Since  $\xi$  is nondecreasing, we get

$$\xi(a) \geq \xi(0) = a.$$

On the other hand, we have  $\xi(a) < a$ . Thus,

$$a \leq \xi(a) < a,$$

which is a contradiction. Hence, we have  $\xi(0) = 0$ . Finally,  $\xi(0) = 0 = \lim_{t \rightarrow 0+} \xi(t)$ . This implies the continuity of  $\xi$  at 0.  $\square$

We, now establish that Theorem 2.3.1 holds true even if the continuity requirement of  $f$  is replaced with a new condition as mentioned below:

**Theorem 2.3.3.** Let the mapping  $f : A \rightarrow A$  be a bijective,  $(\xi, \alpha)$ -expansive mapping in a complete metric space  $(A, d)$ . Also, suppose that:

- (i) the mapping  $f^{-1}$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, f^{-1}a_0) \geq 1$ ,
- (iii) if  $\{a_n\}$  is a sequence in  $A$  satisfying  $\alpha(a_n, a_{n+1}) \geq 1$  for every  $n$  and  $\{a_n\} \rightarrow a \in A$  as  $n \rightarrow +\infty$ , then

$$\alpha(f^{-1}a_n, f^{-1}a) \geq 1 \quad (2.3.2)$$

for every  $n$ . Then,  $f$  has a fixed point.

*Proof.* From the proof of Theorem 2.3.1, we infer that  $\{a_n\}$  is a sequence in  $A$  such that  $\alpha(a_n, a_{n+1}) \geq 1$  for all  $n$  and  $\{a_n\} \rightarrow u \in A$  as  $n \rightarrow +\infty$ .

The use of condition (iii) gives

$$\alpha(f^{-1}a_n, f^{-1}u) \geq 1, \text{ for every } n \in \mathbb{N}. \quad (2.3.3)$$

The use of (2.2.1), (2.3.3) and the triangular inequality implies that

$$\begin{aligned} d(f^{-1}u, u) &\leq d(f^{-1}u, a_{n+1}) + d(a_{n+1}, u) \\ &= d(f^{-1}a_n, f^{-1}u) + d(a_{n+1}, u) \\ &\leq \alpha(f^{-1}a_n, f^{-1}u)d(f^{-1}a_n, f^{-1}u) + d(a_{n+1}, u) \\ &\leq \xi(d(a_n, u)) + d(a_{n+1}, u). \end{aligned}$$

Therefore, by Lemma 2.3.2, we get  $d(f^{-1}u, u) = 0$  as  $n \rightarrow +\infty$ . That is,  $f^{-1}u = u$ .

Hence,  $fu = f(f^{-1}u) = (ff^{-1})u = u$ . Hence proved.  $\square$

Following examples demonstrates the validity of the above theorems:

**Example 2.3.1.** Consider the set  $A = [0, +\infty)$  equipped with the standard metric. Consider the mappings  $f : A \rightarrow A$  and  $\alpha : A \times A \rightarrow [0, +\infty)$ , where

$$f(a) = \begin{cases} 2a - \frac{3}{2} & a \geq 1, \\ \frac{a}{2} & a \in [0, 1). \end{cases}$$

and

$$\alpha(a, b) = \begin{cases} 0 & a, b \in [0, 1), \\ 1 & \text{otherwise.} \end{cases}$$

Define  $\xi(a) = a/2$  for all  $a \geq 0$ . Clearly, for all  $a, b \in A$ , we have

$$\frac{1}{2}d(fa, fb) \geq \alpha(a, b)d(a, b).$$

thereby implying that  $f$  is an  $(\xi, \alpha)$ -expansive mapping. Also, there exists  $a_0 = 1 \in A$  such that  $\alpha(a_0, f^{-1}a_0) = \alpha(1, f^{-1}1) = 1$ . Clearly,  $f$  is continuous. For proving the  $\alpha$ -admissible property of  $f^{-1}$ , let  $\alpha(a, b) \geq 1$  for some  $a, b \in A$ . This proves that  $a \geq 1$  and  $b \geq 1$ . The definitions of  $f^{-1}$  and  $\alpha$  implies that

$$f^{-1}a = \frac{a}{2} + \frac{3}{4} \geq 1, f^{-1}b = \frac{b}{2} + \frac{3}{4} \geq 1 \text{ and } \alpha(f^{-1}a, f^{-1}b) = 1,$$

thereby implying that  $f^{-1}$  is  $\alpha$ -admissible. All the conditions of Theorem 2.3.1 are satisfied. Here, the two fixed points of  $f$  are 0 and 3/2.

**Remark 2.3.1.** The expansion mapping theorem due to Wang *et al.* [192] cannot be applied in the above example. Since in this case, we have

$$d(f(\frac{1}{2}), f(0)) = \frac{1}{4} < \frac{1}{2} = d(1/2, 0)$$

In what follows, an example is given involving a discontinuous function  $f$ .

**Example 2.3.2.** Consider a set  $A = [0, +\infty)$  equipped with the standard metric. Consider the mappings  $f : A \rightarrow A$  and  $\alpha : A \times A \rightarrow [0, +\infty)$ , where

$$fa = \begin{cases} a^2 & a \geq 1, \\ \frac{a}{2} & 0 \leq a < 1. \end{cases}$$

and

$$\alpha(a, b) = \begin{cases} 0 & a, b \in [0, 1), \\ 1 & \text{otherwise.} \end{cases}$$

Let  $\xi(a) = a/2$  for all  $a \geq 0$ . Theorem 2.3.1 cannot be applied in this case because of the discontinuity of  $f$  at 1. For all  $a, b \in A$ , we have

$$\frac{1}{2}d(fa, fb) \geq \alpha(a, b)d(a, b),$$

thereby implying that  $f$  is an  $(\xi, \alpha)$ -expansive mapping. Also, there exists  $a_0 = 1 \in X$  such that  $\alpha(a_0, f^{-1}a_0) = \alpha(1, f^{-1}1) = 1$ . To prove that  $f^{-1}$  is  $\alpha$ -admissible, let  $\alpha(a, b) \geq 1$  for some  $a, b \in A$ . This proves that  $a \geq 1, b \geq 1$  and the use of the definition of  $f^{-1}$  and  $\alpha$  gives

$$f^{-1}a = \sqrt{a} \geq 1, f^{-1}b = \sqrt{b} \geq 1 \text{ and } \alpha(f^{-1}a, f^{-1}b) = 1$$

Finally, let  $\{a_n\}$  be a sequence in  $A$  such that  $\alpha(a_n, a_{n+1}) \geq 1$  for all  $n$  and  $\{a_n\} \rightarrow a \in A$  as  $n \rightarrow +\infty$ . Since  $\alpha(a_n, a_{n+1}) \geq 1$  for all  $n$ , so the definition of  $\alpha$  implies for all  $n$  that  $a_n \geq 1$  and  $a \geq 1$ . Thus,  $\alpha(f^{-1}a_n, f^{-1}a) = 1$ .

All the conditions of Theorem 2.3.3 are fulfilled thereby implying that  $f$  has a fixed point. Here, 0 and 1 are two fixed points of  $f$ .

**Remark 2.3.2.** The expansion mapping theorem due to Wang *et al.* [192] cannot be applied in this case also.

For the uniqueness of the fixed point in Theorems 2.3.1 and 2.3.3, the following condition has been taken:

(U): For every  $u, v \in A$ , there exists  $w \in A$  such that  $\alpha(u, w) \geq 1$  and  $\alpha(v, w) \geq 1$ .

**Theorem 2.3.4.** If the condition (U) along with the hypotheses of Theorems 2.3.1 and 2.3.3 holds, then there exists a unique fixed point of  $f$ .

*Proof.* The set of fixed points is non-empty from Theorems 2.3.1 and 2.3.3. Now, we prove that if  $u$  and  $v$  are two fixed points of  $f$ , then  $u = v$ . The condition (U) implies the existence of  $w \in A$  such that

$$\alpha(u, w) \geq 1 \text{ and } \alpha(v, w) \geq 1. \quad (2.3.4)$$

The  $\alpha$ -admissible property of  $f^{-1}$  and inequality (2.3.4) implies that

$$\alpha(u, f^{-1}w) \geq 1 \text{ and } \alpha(v, f^{-1}w) \geq 1, \forall n \in \mathbb{N}. \quad (2.3.5)$$

Therefore, repeated application of  $\alpha$ -admissible property of  $T^{-1}$  gives

$$\alpha(u, f^{-n}w) \geq 1 \text{ and } \alpha(v, f^{-n}w) \geq 1, \forall n \in \mathbb{N}. \quad (2.3.6)$$

Use of the inequalities (2.2.1) and (2.3.6) gives

$$\begin{aligned} d(u, f^{-n}w) &\leq \alpha(u, f^{-n}w)d(u, f^{-n}w) \\ &\leq \xi(d(u, f^{-n+1}w)) \end{aligned}$$

On applying the above inequality  $(n-1)$  times, we infer that

$$d(u, f^{-n}w) \leq \xi^n(d(u, w)), \text{ for every } n \in \mathbb{N}.$$

Thus,  $f^{-n}w \rightarrow u$  as  $n \rightarrow +\infty$ . Similarly, letting  $n \rightarrow +\infty$  gives

$$f^{-n}w \rightarrow v.$$

The uniqueness of the limit of  $f^{-n}w$  implies that  $u = v$ . This completes the proof.  $\square$

## 2.4 Existence of coupled fixed points in metric spaces

In this section, it has been shown that the coupled fixed point results concerning  $(\xi, \alpha)$ -expansive mappings in complete metric spaces can be derived from the results presented in Section 2.3. Before proving the results, the following concept of Bhaskar and Lakshmikantham [36] is recalled:

**Definition 2.4.1.** [36] Let  $A$  be a non-empty set. A point  $(a, b) \in A \times A$  is said to be a coupled fixed point of  $T : A \times A \rightarrow A$  if

$$T(a, b) = a \text{ and } T(b, a) = b.$$

The following lemma of Samet *et al.* [165] has been used for proving our results:

**Lemma 2.4.1.** [165] Let  $T : A \times A \rightarrow A$  be a given mapping. Define the mapping  $f : A \times A \rightarrow A \times A$  by

$$f(a, b) = (T(a, b), T(b, a)), \forall (a, b) \in A \times A. \quad (2.4.1)$$

Then,  $(a, b)$  is a fixed point of  $f$  if and only if  $(a, b)$  is a coupled fixed point of  $T$ .

The main results of this section are given by the following theorems:

**Theorem 2.4.2.** Let  $(A, d)$  be a complete metric space and  $T : A \times A \rightarrow A$  be a given bijective mapping. Suppose that there exists  $\xi \in \mathcal{X}$  and a function  $\alpha : A^2 \times A^2 \rightarrow [0, +\infty)$  such that

$$\xi(d(T(x, y), T(u, v))) \geq \frac{1}{2} \alpha((x, y), (u, v)) [d(x, u) + d(y, v)] \quad (2.4.2)$$

for every  $(x, y), (u, v) \in A \times A$ . Also, suppose that

- (i)  $\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha(T^{-1}(x), T^{-1}(u)) \geq 1$  for all  $(x, y), (u, v) \in A \times A$ ,
- (ii) there exists  $(x_0, y_0) \in A \times A$  such that

$$\alpha((x_0, y_0), (a, b)) \geq 1 \text{ and } \alpha((b, a), (y_0, x_0)) \geq 1,$$

where  $T^{-1}(x_0) = (a, b)$ ,

(iii) the mapping  $T$  is continuous.

Then,  $T$  has a coupled fixed point.

*Proof.* Let the mapping  $f$  defined in (2.4.1) be a bijective mapping satisfying

$$f^{-1}(x, y) = T^{-1}(x).$$

Consider the complete metric space  $(Y, \rho)$ , where  $Y = A \times A$  and

$$\rho((x, y), (u, v)) = d(x, u) + d(y, v) \text{ for all } (x, y), (u, v) \in A \times A.$$

The use of the inequality (2.4.2) gives

$$\xi(d(T(x, y), T(u, v))) \geq \frac{1}{2}\alpha((x, y), (u, v))[d(x, u) + d(y, v)] \quad (2.4.3)$$

and

$$\xi(d(T(v, u), T(y, x))) \geq \frac{1}{2}\alpha((v, u), (y, x))[d(v, y) + d(u, x)] \quad (2.4.4)$$

Define the mapping  $\eta : Y \times Y \rightarrow [0, +\infty)$  by

$$\eta((\mu_1, \mu_2), (\nu_1, \nu_2)) = \min\{\alpha((\mu_1, \mu_2), (\nu_1, \nu_2)), \alpha((\nu_2, \nu_1), (\mu_2, \mu_1))\} \quad (2.4.5)$$

for every  $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in Y$ . Summing up the inequalities (2.4.3)-(2.4.4) and using (2.4.5), we get

$$\xi(d(T(\mu_1, \mu_2), T(\nu_1, \nu_2))) + \xi(d(T(\nu_2, \nu_1), T(\mu_2, \mu_1))) \geq \eta(\mu, \nu)\rho(\mu, \nu), \quad (2.4.6)$$

for every  $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in Y$ . The use of the property  $(\xi(a + b) = \xi(a) + \xi(b))$  gives

$$\xi(\rho(f\mu, f\nu)) \geq \eta(\mu, \nu)\rho(\mu, \nu), \quad (2.4.7)$$

for all  $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in Y$ . Therefore,  $f$  is continuous and  $(\xi, \alpha)$ -expansive mapping.

Let  $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in Y$  be such that  $\eta(\mu, \nu) \geq 1$ . The condition (i) implies that  $\eta(f^{-1}\mu, f^{-1}\nu) \geq 1$ . This proves the  $\eta$ -admissible property of  $f^{-1}$ .

Also, in view of condition (ii), there exists  $(x_0, y_0) \in Y$  such that

$$\eta((x_0, y_0), f^{-1}(x_0, y_0)) \geq 1.$$

So, this problem gets transformed to the complete metric space  $(Y, \rho)$ . Thus, all the conditions of Theorem 2.3.1 are fulfilled thereby deducing the fixed point of  $f$  as well as  $f^{-1}$ .

The use of Lemma 2.4.1 implies that there exists a coupled fixed point of  $T$ .  $\square$

In the next theorem, it has been established that Theorem 2.4.2 holds true even if the continuity requirement of  $T$  is replaced with a new condition as mentioned below:

**Theorem 2.4.3.** Let  $(A, d)$  be a complete metric space and  $T : A \times A \rightarrow A$  a given bijective mapping. Suppose that there exists  $\xi \in \chi$  and a function  $\alpha : A^2 \times A^2 \rightarrow [0, +\infty)$  such that

$$\xi(d(T(x, y), T(u, v))) \geq \frac{1}{2}\alpha((x, y), (u, v))[d(x, u) + d(y, v)] \quad (2.4.8)$$

for every  $(x, y), (u, v) \in A \times A$ . Also, suppose that

- (i)  $\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha(T^{-1}(x), T^{-1}(u)) \geq 1$ , for every  $(x, y), (u, v) \in A \times A$ ,
- (ii) there exists  $(x_0, y_0) \in A \times A$  such that

$$\alpha((x_0, y_0), (a, b)) \geq 1 \text{ and } \alpha((b, a), (y_0, x_0)) \geq 1,$$

where  $T^{-1}(x_0) = (a, b)$ .

- (iii) if  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $A$  such that

$$\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1 \text{ and } \alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \geq 1,$$

$x_n \rightarrow x \in A$  and  $y_n \rightarrow y \in A$  as  $n \rightarrow +\infty$ , then

$\alpha(f^{-1}(x_n, y_n), f^{-1}(x, y)) \geq 1$  and  $\alpha(f^{-1}(y, x), f^{-1}(y_n, x_n)) \geq 1$  for each  $n \in \mathbb{N}$ .

Then,  $T$  has a coupled fixed point.

*Proof.* Here, the same notations of the proof of Theorem 2.4.2 has been used. Let  $\{(x_n, y_n)\}$  be a sequence in  $Y$  such that

$$\eta((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1 \text{ and } (x_n, y_n) \rightarrow (x, y) \text{ as } n \rightarrow +\infty.$$

Using hypothesis (iii), we have

$$\eta(f^{-1}(x_n, y_n), f^{-1}(x, y)) \geq 1.$$

Therefore, the conditions of Theorem 2.3.3 are fulfilled thereby deducing the fixed point of  $T$ . The use of Lemma 2.4.1 implies that there exists a coupled fixed point of  $T$ .  $\square$

For the uniqueness of the coupled fixed point, the following additional condition has been used:

$(U')$ : For every  $(x_1, y_1), (x_2, y_2) \in A \times A$ , there exists  $(x_3, y_3) \in A \times A$  such that

$$\alpha((x_1, y_1), (x_3, y_3)) \geq 1, \quad \alpha((y_3, x_3), (y_1, x_1)) \geq 1 \text{ and}$$

$$\alpha((x_2, y_2), (x_3, y_3)) \geq 1, \quad \alpha((y_3, x_3), (y_2, x_2)) \geq 1$$

**Theorem 2.4.4.** If the condition  $(U')$  along with the hypotheses of Theorems 2.4.2 and 2.4.3 holds, then there exists a unique coupled fixed point of  $F$ .

*Proof.* Clearly, the condition  $(U')$  implies that  $f$  and  $\eta$  satisfy the condition (U). Thus, from Theorem 2.3.4 and Lemma 2.4.1, the result follows.  $\square$

**Example 2.4.2.** Consider the  $A = \mathbb{N}$  equipped with the usual metric. Clearly,  $(A, d)$  is a complete metric space. The mapping  $T : A \times A \rightarrow A$  defined by  $T(a, b) = 2^{a-1} \cdot (2b - 1)$  is clearly a continuous and bijective mapping. Consider the mapping  $\alpha : A^2 \times A^2 \rightarrow [0, +\infty)$  by

$$\alpha((a, b), (u, v)) = \begin{cases} 1 & \text{if } a = u, \\ 0 & \text{otherwise.} \end{cases}$$

For all points  $(a, b), (u, v)$  in  $A \times A$ , we have

$$\xi(d(T(a, b), T(u, v))) \geq \frac{1}{2}\alpha((a, b), (u, v))[d(a, u) + d(b, v)]$$

Thus, inequality (2.4.2) is satisfied with  $\xi(a) = a/2$  for all  $a \geq 0$ . Clearly, condition (i) of Theorem 2.4.2 holds. For  $(a_0, b_0) = (1, 1)$ , condition (ii) of the same theorem is also satisfied. All the conditions of Theorem 2.4.2 are fulfilled thereby deducing the coupled fixed point of  $T$ . In this case,  $(1, 1)$  is a coupled fixed point of  $T$ .

## 2.5 Fixed points of $F$ -expansive mappings in metric spaces

The main result of this section is the following theorem:

**Theorem 2.5.1.** Suppose  $(A, d)$  be a complete metric space. Let the self-mapping  $f$  be an  $F$ -expansive and surjective mapping. Then  $f$  has a unique fixed point.

*Proof.* Suppose  $a_0$  is an arbitrary point of  $A$ . Since  $f$  is surjective, there is  $a_1 \in A$  such that  $a_0 = fa_1$ . Also, there exists  $a_2 \in X$  such that  $a_1 = fa_2$ . Continuing this process, having chosen  $a_n$  in  $A$ , choose  $a_{n+1}$  in  $A$  such that  $a_n = fa_{n+1}$ .

Set

$$\gamma_n = d(a_n, a_{n+1}), n = 0, 1, \dots$$

If there exists  $n_0 \in \mathbb{N}$  with  $a_{n_0+1} = a_{n_0}$ , then  $fa_{n_0} = a_{n_0}$  and the proof is finished. So, let us assume that

$$a_{n+1} \neq a_n, \text{ for every } n \in \mathbb{N}$$

thereby implying that  $\gamma_n > 0$  for all  $n \in \mathbb{N}$ .

Now, using (2.2.3), we obtain for every  $n \in \mathbb{N}$

$$F(\gamma_n) \leq F(\gamma_{n-1}) + \tau \leq F(\gamma_{n-2}) + 2\tau \leq \dots \leq F(\gamma_0) + n\tau \quad (2.5.1)$$

thereby implying  $\lim_{n \rightarrow \infty} F(\gamma_n) = -\infty$ . Now, making use of Definition 2.2.6, we infer that

$$\lim_{n \rightarrow \infty} \gamma_n = 0. \quad (2.5.2)$$

Using Definition 2.2.6, there exists  $k \in (0, 1)$  such that

$$\lim_{n \rightarrow \infty} \gamma_n^k F(\gamma_n) = 0. \quad (2.5.3)$$

From (2.5.1), the following holds:

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leq \gamma_n^k (F(\gamma_0) + n\tau) - \gamma_n^k F(\gamma_0) = \gamma_n^k n\tau \leq 0 \quad (2.5.4)$$

Making use of equations (2.5.2) and (2.5.3), and letting  $n \rightarrow \infty$  in (2.5.4), we get that

$$\lim_{n \rightarrow \infty} n\gamma_n^k = 0. \quad (2.5.5)$$

thereby implying that there exists  $n_1 \in \mathbb{N}$  satisfying  $n\gamma_n^k \leq 1$  for every  $n \geq n_1$ . So,

$$\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}, \quad (2.5.6)$$

for every  $n \geq n_1$ . Now, to prove that  $\{a_n\}$  is a Cauchy sequence, let  $m, n \in \mathbb{N}$  be such that  $m > n \geq n_1$ . Owing to (2.5.6) and triangle inequality, we obtain that

$$d(a_m, a_n) \leq \gamma_{m-1} + \gamma_{m-2} + \dots + \gamma_n < \sum_{i=n}^{\infty} \gamma_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$$

The convergence of the series  $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$  implies that  $\{a_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence. The completeness of  $A$  yields that there exists  $a^* \in A$  such that  $\lim_{n \rightarrow \infty} a_n = a^*$ . The continuity

of  $f$  gives

$$d(fa^*, a^*) = \lim_{n \rightarrow \infty} d(fa_n, a_n) = \lim_{n \rightarrow \infty} d(a_{n+1}, a_n) = 0.$$

Therefore,  $a^*$  is a fixed point of  $f$ . To show the uniqueness of the fixed point of  $f$ , let  $u, v \in A$ ,  $fu = u \neq v = fv$ . So,

$$\tau \geq F(d(u, v)) - F(d(fu, fv)) = 0,$$

which is a contradiction. This completes the proof. □

**Remark 2.5.1.** Consider the mappings  $F_1(\alpha) = \ln(\alpha)$ ,  $\alpha > 0$ ,  $F_2(\alpha) = \ln(\alpha) + \alpha$ ,  $\alpha > 0$ ,  $F_1(\alpha) < F_2(\alpha)$ . Clearly, the mapping  $F_2 - F_1$  is strictly increasing. Therefore, the use of Remark 2.2.4 implies that every expansive mapping (2.2.4) satisfies the expansive condition (2.2.5).

In the following example, a mapping  $f$  is given on a metric space which is not  $F_1$ -expansive (Expansion mapping), but is an  $F_2$ -expansive.

**Example 2.5.1.** Take the sequence  $\{A_n\}_{n \in \mathbb{N}}$  as follows:

$$\begin{aligned} A_1 &= 1, \\ A_2 &= 1 + 2, \\ &\dots \\ A_n &= 1 + 2 + \dots + n = \frac{n(n+1)}{2}, n \in \mathbb{N}, \\ &\dots \end{aligned}$$

Let the set  $A = \{A_n : n \in \mathbb{N}\}$  be equipped with the usual metric  $d$ . Clearly,  $(A, d)$  is a complete metric space. Consider the mapping  $f : A \rightarrow A$  defined by

$$\begin{aligned} f(A_n) &= A_{n+1} \quad \text{for } n > 1, \\ f(A_1) &= A_1. \end{aligned}$$

Now, we proceed to examine the mapping  $F_1$  defined in Example 2.2.9. In this case, the mapping  $f$  is not  $F_1$ -expansive thereby implying that  $f$  is not the expansion mapping principle. Indeed, we obtain

$$\lim_{n \rightarrow \infty} \frac{d(f(A_n), f(A_1))}{d(A_n, A_1)} = \lim_{n \rightarrow \infty} \frac{A_{n+1} - 1}{A_n - 1} = 1.$$

Consider the mapping  $F_2$  as defined in Example 2.2.10. It is to be noted that  $\forall_{m,n \in \mathbb{N}} [f(A_m) \neq f(A_n) \Leftrightarrow ((m \geq 2 \wedge n = 1) \vee (m > n > 1))]$ .

For all  $m \in \mathbb{N}$ ,  $m \geq 2$ , we get

$$\begin{aligned} \frac{d(f(A_m), f(A_1))}{d(A_m, A_1)} e^{d(f(A_m), f(A_1)) - d(A_m, A_1)} &= \frac{A_{m+1} - 1}{A_m - 1} e^{A_{m+1} - A_m} \\ &= \frac{m^2 + 3m}{m^2 + m - 2} e^{m+1} > e^{m+1} > e^3 = e^{(-1)(-3)}. \end{aligned}$$

The following condition holds for every  $m, n \in \mathbb{N}$ ,  $m > n > 1$

$$\begin{aligned} \frac{d(f(A_m), f(A_n))}{d(A_m, A_n)} e^{d(f(A_m), f(A_n)) - d(A_m, A_n)} &= \frac{A_{m+1} - A_{n+1}}{A_m - A_n} e^{A_n - A_{n+1} + A_{m+1} - A_m} \\ &= \frac{m + n + 3}{m + n + 1} e^{m-n} > e^{m-n} \geq e^1 = e^{(-1)(-1)}. \end{aligned}$$

This proves that  $f$  is  $F_2$ -expansive but not  $F_1$ -expansive. Clearly,  $A_1$  is a fixed point of  $f$ .

This example shows that Theorem 2.5.1 presents the family of expansions which are not equivalent in general.



# Chapter 3

## FIXED POINT THEOREMS FOR MAPPINGS OF INTEGRAL TYPE

### 3.1 Introduction

The technique of contractive conditions of integral type is one of the interesting techniques for proving fixed point results. In the recent years, various researchers have shown significant interest in proving the existence of fixed points for mappings satisfying contractive integral type inequalities in metric spaces. The study related to contractive conditions of integral type was initiated by Branciari [39] in 2002 by presenting an integral version of the Banach contraction principle. More precisely, Branciari [39] gave the following result:

**Theorem 3.1.1.** Let  $T$  be a given self-mapping of a complete metric space  $(A, d)$  satisfying for every  $a, b \in A$ ,

$$\int_0^{d(Ta, Tb)} \varphi(s) ds \leq c \int_0^{d(a, b)} \varphi(s) ds, \quad (3.1.1)$$

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where  $c \in [0, 1)$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable function on every compact subset of  $[0, +\infty)$  and

$$\int_0^\epsilon \varphi(s) ds > 0, \text{ for all } \epsilon > 0.$$

Then,  $T$  has a unique fixed point  $u \in A$  and for every  $a \in A$ , the sequence  $\{T^n a\}$  converges to  $u$ .

The Banach contraction principle is obtained by substituting  $\varphi(t) = 1$ , for each  $t \geq 0$  in Theorem 3.1.1.

Further, Rhoades [151] in 2003 extended the condition (3.1.1) to more general contractive conditions which embraces the Branciari's result as well as a result of Ćirić [53]. Thereafter, many authors like Aliouche [19], Djoudi and Aliouche [66], Rhoades [151], Samet and Vetro [164], Türkoğlu and Altun [187], Vetro [189], Vijayaraju *et al.* [191], and others undertook further investigations in this direction.

Motivated by the notion of  $\alpha$ - $\psi$ -contractive type mappings of Samet *et al.* [165], Berzig and Rus [34] introduced the notion of  $\alpha$ -contractive mapping of Meir-Keeler type in complete metric spaces and proved the related theorems for this type of contraction.

In this chapter, a new notion of  $\alpha$ - $\psi$ -contractive type mappings of integral type has been introduced in order to generalize the idea of  $\alpha$ - $\psi$ -contractive type mappings of Samet *et al.* [165].

The contents of this chapter have been divided into four sections. Section 3.2 deals with the preliminaries related to this chapter. In section 3.3, the existence and uniqueness of fixed points for  $\alpha$ - $\psi$ -contractive type mappings of integral type have been studied. Moreover, some examples are provided to show that the presented results are substantial

improvements of some known related results in literature. Section 3.4 deals with the existence of coupled fixed points for  $\alpha$ - $\psi$ -contractive type mappings of integral type in complete metric spaces. As a consequences of the obtained results in Section 3.3, the fixed point theorems in partially ordered metric spaces have been established in Section 3.5. The results presented in this chapter generalize and unify some results proved by Branciari [39], Samet *et al.* in [165] and references therein.

## 3.2 Preliminaries

The definitions and results related to this chapter due to Samet *et al.* [165] have already been discussed in Chapters 1 and 2. We, now present some results of Branciari [39].

**Definition 3.2.1.** [39] Let  $\Phi$  denote the family of all functions  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$  which satisfy

- (i) the mapping  $\varphi$  is nonnegative,
- (ii) the mapping  $\varphi$  is Lebesgue integrable mapping,
- (iii) the mapping  $\varphi$  satisfies the condition

$$\int_0^\epsilon \varphi(t)dt > 0 \text{ for each } \epsilon > 0. \quad (3.2.1)$$

Now, we introduce a new notion of  $\alpha$ - $\psi$ -contractive type mappings of integral type as under:

**Definition 3.2.2.** Let  $T$  be a given self-mapping of a metric space  $(A, d)$ . The mapping  $T$  is said to be an  $\alpha$ - $\psi$ -contractive mapping of integral type if there exist two functions  $\alpha : A \times A \rightarrow [0, +\infty)$  and  $\psi \in \Psi$  satisfying for every  $a, b \in A$

$$\alpha(a, b) \int_0^{d(Ta, Tb)} \varphi(t)dt \leq \psi \left( \int_0^{d(a, b)} \varphi(t)dt \right), \quad (3.2.2)$$

where  $\varphi \in \Phi$ .

**Remark 3.2.1.** Every  $\alpha$ - $\psi$ -contractive mapping  $T$  is an  $\alpha$ - $\psi$ -contractive mapping of integral type by taking  $\varphi(t) = 1$  for each  $t \geq 0$  in equation (3.2.2).

Recently in 2013, Berzig and Rus [34] introduced the notion of  $\alpha$ -contractive mapping of Meir-Keeler type in complete metric spaces and proved theorems which assured the existence, uniqueness and iterative approximation of the fixed point for these type of contractions. Berzig and Rus [34] have used the following definition for proving their main results:

**Definition 3.2.3.** [34] Let  $N \in \mathbb{N}$ . The mapping  $\alpha$  is  $N$ -transitive (on  $A$ ) if

$$a_0, a_1, \dots, a_{N+1} \in A : \alpha(a_i, a_{i+1}) \geq 1$$

for all  $i \in \{0, 1, \dots, N\} \Rightarrow \alpha(a_0, a_{N+1}) \geq 1$ .

In particular, the mapping  $\alpha$  is transitive if it is 1-transitive, that is,

$$\alpha(a, b) \geq 1 \text{ and } \alpha(b, c) \geq 1 \Rightarrow \alpha(a, c) \geq 1,$$

for all  $a, b, c \in A$ .

Berzig and Rus [34] obtained the following remarks from Definition 3.2.3:

**Remark 3.2.2.** Any function  $\alpha : A \times A \rightarrow [0, +\infty)$  is 0-transitive.

**Remark 3.2.3.** Any  $N$  transitive mapping  $\alpha$  is  $kN$ -transitive for every  $k \in \mathbb{N}$ .

**Remark 3.2.4.** Any transitive mapping  $\alpha$  is  $N$ -transitive for every  $N \in \mathbb{N}$ .

**Remark 3.2.5.** Any  $N$ -transitive mapping  $\alpha$  is not necessarily transitive for every  $N \in \mathbb{N}$ .

### 3.3 Fixed points for $\alpha$ - $\psi$ -contractive type mappings of integral type

The main results of this section are as follows:

**Theorem 3.3.1.** Consider a complete metric space  $(A, d)$  and a transitive mapping  $\alpha : A \times A \rightarrow [0, +\infty)$ . Suppose that  $T$  be an  $\alpha$ - $\psi$ -contractive mapping of integral type satisfying the following properties:

- (i) the mapping  $T$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, Ta_0) \geq 1$ ,
- (iii) the mapping  $T$  is continuous.

Then  $T$  has a fixed point.

*Proof.* Let  $a_0 \in A$  be such that  $\alpha(a_0, Ta_0) \geq 1$ . Take the sequence  $\{a_n\}$  in  $A$  defined by  $a_{n+1} = Ta_n$  for every  $n \geq 0$ . If for some  $n$ ,  $a_n = a_{n+1}$ , then  $a_n$  is a fixed point of  $T$ . So, let  $a_n \neq a_{n+1}$  for all  $n$ . Since  $T$  is  $\alpha$ -admissible, we have

$$\alpha(a_0, a_1) = \alpha(a_0, Ta_0) \geq 1 \Rightarrow \alpha(Ta_0, Ta_1) = \alpha(a_1, a_2) \geq 1.$$

By induction, we get

$$\alpha(a_n, a_{n+1}) \geq 1, \text{ for each } n \geq 0. \quad (3.3.1)$$

Applying the inequality (3.2.2) with  $a = a_{n-1}$  and  $y = a_n$  and using (3.3.1), we obtain

$$\begin{aligned} \int_0^{d(a_n, a_{n+1})} \varphi(t) dt &= \int_0^{d(Ta_{n-1}, Ta_n)} \varphi(t) dt \leq \alpha(a_{n-1}, a_n) \int_0^{d(Ta_{n-1}, Ta_n)} \varphi(t) dt \\ &\leq \psi \left( \int_0^{d(a_{n-1}, a_n)} \varphi(t) dt \right). \end{aligned}$$

Therefore, by induction, we get

$$\int_0^{d(a_n, a_{n+1})} \varphi(t) dt \leq \psi^n \left( \int_0^{d(a_0, a_1)} \varphi(t) dt \right) = \psi^n(d) \text{ for all } n \in \mathbb{N}, \quad (3.3.2)$$

where  $d = \int_0^{d(a_0, a_1)} \varphi(t) dt$ .

Now, letting  $n \rightarrow +\infty$ , we obtain

$$\int_0^{d(a_n, a_{n+1})} \varphi(t) dt = 0 \quad (\text{from the property of } \psi)$$

thereby implying from (3.2.1),

$$d(a_n, a_{n+1}) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.3.3)$$

We shall now prove that the sequence  $\{a_n\}$  is a Cauchy sequence. Contrarily, suppose that it is not a Cauchy sequence. This implies that there exists an  $\epsilon > 0$  and subsequences  $\{m(p)\}$  and  $\{n(p)\}$  such that  $m(p) < n(p) < m(p+1)$  satisfying

$$d(a_{m(p)}, a_{n(p)}) \geq \epsilon, \quad d(a_{m(p)}, a_{n(p)-1}) < \epsilon. \quad (3.3.4)$$

The triangular inequality and the inequalities (3.3.3), (3.3.4) implies that

$$\begin{aligned} \epsilon \leq d(a_{m(p)}, a_{n(p)}) &\leq d(a_{m(p)}, a_{n(p)-1}) + d(a_{n(p)-1}, a_{n(p)}) \\ &< \epsilon + d(a_{n(p)-1}, a_{n(p)}). \end{aligned}$$

On taking the limit as  $p \rightarrow \infty$ , we get

$$\lim_p d(a_{m(p)}, a_{n(p)}) = \epsilon.$$

The transitivity of  $\alpha$  implies from (3.3.1) that

$$\alpha(a_{m(p)-1}, a_{n(p)-1}) \geq 1. \quad (3.3.5)$$

The use of inequalities (3.2.2) and (3.3.5) gives

$$\begin{aligned} \int_0^{d(a_{m(p)}, a_{n(p)})} \varphi(t) dt &= \int_0^{d(Ta_{m(p)-1}, Ta_{n(p)-1})} \varphi(t) dt \\ &\leq \alpha(a_{m(p)-1}, a_{n(p)-1}) \int_0^{d(Ta_{m(p)-1}, Ta_{n(p)-1})} \varphi(t) dt \\ &\leq \psi \left( \int_0^{d(a_{m(p)-1}, a_{n(p)-1})} \varphi(t) dt \right). \end{aligned}$$

Letting  $p \rightarrow \infty$  in the above inequality and Lemma 2.2.1 implies that

$$\int_0^\epsilon \varphi(t)dt \leq \psi \left( \int_0^\epsilon \varphi(t)dt \right) < \int_0^\epsilon \varphi(t)dt,$$

a contradiction. Therefore,  $\{a_n\}$  is a Cauchy sequence in  $(A, d)$ . The completeness of the space  $A$  implies the existence of  $z \in A$  such that  $a_n \rightarrow z$  as  $n \rightarrow +\infty$ . The continuity of  $T$  yields that  $Ta_n \rightarrow Tz$  as  $n \rightarrow +\infty$ , that is,  $a_{n+1} \rightarrow Tz$  as  $n \rightarrow +\infty$ . The uniqueness of the limit implies the existence of a fixed point of  $T$ .  $\square$

In what follows, we establish that Theorem 3.3.1 holds true even if the continuity requirement of  $T$  is replaced with a new condition as mentioned below:

**Theorem 3.3.2.** Consider a complete metric space  $(A, d)$  and a transitive mapping  $\alpha : A \times A \rightarrow [0, +\infty)$ . Suppose that  $T$  be an  $\alpha$ - $\psi$ -contractive mapping of integral type satisfying the following properties:

- (i) the mapping  $T$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, Ta_0) \geq 1$ ,
- (iii) if  $\{a_n\}$  is a sequence in  $A$  such that  $\alpha(a_n, a_{n+1}) \geq 1$  for each  $n$  and  $a_n \rightarrow a \in A$  as  $n \rightarrow \infty$ , then there exists a subsequence  $\{a_{n(k)}\}$  of  $\{a_n\}$  satisfying  $\alpha(a_{n(k)}, a) \geq 1$  for every  $k$ . Then  $T$  has a fixed point.

*Proof.* The proof of Theorem 3.3.1 implies that the sequence  $\{a_n\}$  given by  $a_{n+1} = Ta_n$  for all  $n \geq 0$  converges to  $z \in A$ . The condition (iii) and inequality (3.3.1) implies the existence of a subsequence  $\{a_{n(k)}\}$  of  $a_n$  satisfying

$$\alpha(a_{n(k)}, z) \geq 1 \text{ for every } k.$$

The use of the inequality (3.2.2) gives

$$\begin{aligned} \int_0^{d(a_{n(k)+1}, Tz)} \varphi(t) dt &= \int_0^{d(Ta_{n(k)}, Tz)} \varphi(t) dt \leq \alpha(a_{n(k)}, z) \int_0^{d(Ta_{n(k)}, Tz)} \varphi(t) dt \\ &\leq \psi \left( \int_0^{d(a_{n(k)}, z)} \varphi(t) dt \right), \text{ for all } k. \end{aligned}$$

Suppose that  $d(z, Tz) > 0$ . In view of Lemma 2.2.2 and letting  $k \rightarrow +\infty$ , we get

$$\int_0^{d(z, Tz)} \varphi(t) dt = 0,$$

which, in view of inequality (3.2.1) yields  $d(z, Tz) = 0$  or  $Tz = z$ .  $\square$

The following example shows that Theorems 3.3.1 and 3.3.2 do not ensure the uniqueness of the fixed point.

**Example 3.3.1.** Consider the complete metric space  $A = \{(1, 0), (0, 1)\} \subset \mathbb{R}^2$  equipped with the Euclidean metric  $d((a, b), (c, d)) = |a - c| + |b - d|$  for every  $(a, b), (c, d) \in A$ . Define the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  by

$$\alpha((a, b), (c, d)) = \begin{cases} 1 & \text{if } (a, b) = (c, d), \\ 0 & \text{if } (a, b) \neq (c, d) \end{cases}$$

Clearly,  $\alpha$  is transitive mapping.

The mapping  $T(u, v) = (u, v)$  is a continuous mapping satisfying for any  $\psi \in \Psi$  and  $\varphi \in \Phi$

$$\alpha((a, b), (c, d)) \int_0^{d(T(a, b), T(c, d))} \varphi(t) dt \leq \psi \left( \int_0^{d((a, b), (c, d))} \varphi(t) dt \right)$$

for every  $(a, b), (c, d) \in A$ . Thus,  $T$  is an  $\alpha$ - $\psi$ -contractive mapping of integral type.

To prove the  $\alpha$ -admissible property of  $T$ , let  $\alpha((a, b), (c, d)) \geq 1$  for every  $(a, b), (c, d) \in A$ . The definition of  $\alpha$  shows that  $(a, b) = (c, d)$  and thereby implying that  $T(a, b) = T(c, d)$ . Again from the definition of  $\alpha$ , we get that  $\alpha((T(a, b), T(c, d))) = 1$ . This proves

the  $\alpha$ -admissible property of  $T$ .

Also,

$$\alpha((a, b), T(a, b)) \geq 1, \text{ for every } (a, b) \in A.,$$

thereby fulfilling all the conditions of Theorem 3.3.1. Indeed, if  $\{(a_n, b_n)\}$  is a sequence in  $A$  converging to a point  $(a, b) \in A$  with  $\alpha((a_n, b_n), (a_{n+1}, b_{n+1})) \geq 1$  for each  $n$ , then the definition of  $\alpha$  implies that

$$(a_n, b_n) = (a_{n+1}, b_{n+1}) \text{ for all } n,$$

which further implies that  $(a_n, b_n) = (a, b)$  for all  $n$ . Therefore,  $\alpha((a_n, b_n), (a, b)) = 1$  for all  $n$ .

Thus, all the conditions of Theorem 3.3.2 are also satisfied. Consequently,  $T$  has a fixed point namely  $(1, 0)$  and  $(0, 1)$ .

To guarantee the uniqueness of the fixed point, the following condition has been considered:

(H): For every  $a, b \in A$ , there exists  $c \in A$  such that  $\alpha(a, c) \geq 1$  and  $\alpha(b, c) \geq 1$ .

**Theorem 3.3.3.** If the condition (H) along with the hypotheses of Theorems 3.3.1 and 3.3.2 holds, then there exists a unique fixed point of  $T$ .

*Proof.* Let  $a$  and  $b$  be two fixed points of  $T$ . The condition (H) implies the existence of a point  $c \in A$  such that

$$\alpha(a, c) \geq 1 \text{ and } \alpha(b, c) \geq 1. \tag{3.3.6}$$

Utilizing the  $\alpha$ -admissibility of  $T$  and (3.3.6), we get

$$\alpha(a, T^n(c)) \geq 1 \text{ and } \alpha(b, T^n(c)) \geq 1, \text{ for all } n \in \mathbb{N} \tag{3.3.7}$$

The inequalities (3.2.2) and (3.3.7) gives

$$\begin{aligned} \int_0^{d(a, T^n(c))} \varphi(t) dt &= \int_0^{d(Ta, T(T^{n-1}(c)))} \varphi(t) dt \\ &\leq \alpha(a, T^{n-1}(c)) \int_0^{d(Ta, T(T^{n-1}(c)))} \varphi(t) dt \\ &\leq \psi \left( \int_0^{d(a, T^{n-1}(c))} \varphi(t) dt \right). \end{aligned}$$

By induction, we obtain

$$\int_0^{d(a, T^n(c))} \varphi(t) dt \leq \psi^n \left( \int_0^{d(a, c)} \varphi(t) dt \right), \text{ for all } n \in \mathbb{N}.$$

Letting  $n \rightarrow \infty$  in the above inequality implies that

$$\int_0^{d(a, T^n(c))} \varphi(t) dt = 0, \text{ (from the property (ii) of the function } \psi)$$

which, from (3.2.1) implies that

$$T^n c \rightarrow a \text{ as } n \rightarrow \infty. \quad (3.3.8)$$

Similarly, it can be proved that

$$T^n c \rightarrow b \text{ as } n \rightarrow \infty. \quad (3.3.9)$$

From (3.3.8) and (3.3.9), we find that  $a = b$ , due to the uniqueness of the limit. This proves the uniqueness of the fixed point of  $T$ .  $\square$

The following example shows that Theorem 3.3.3 is a generalization of Theorem 2.3 of Samet *et al.* [165].

**Example 3.3.2.** Suppose that  $A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}$  with the usual metric induced by  $\mathbb{R}$ . The space  $A$  is clearly a complete metric space (being a closed subset of  $\mathbb{R}$ ). Consider the mappings  $f : A \rightarrow A$  and  $\psi \in \Psi$  defined by

$$f(a) = \begin{cases} \frac{1}{n+1} & \text{if } a = \frac{1}{n} \\ 0 & \text{if } a = 0 \end{cases}$$

and

$$\psi(a) = \frac{a}{2} \text{ for all } a \geq 0.$$

Consider the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  defined by

$$\alpha(a, b) = \begin{cases} 1 & b = 0 \\ 0 & \text{otherwise} \end{cases}$$

Clearly,  $\alpha$  is transitive.

Define  $\varphi \in \Phi$  by  $\varphi(t) = t^{1/t-2}[1 - \log t]$  for  $t > 0$  and  $\varphi(0) = 0$ . Then, for any  $a > 0$ , we have

$$\int_0^a \varphi(t) dt = a^{1/a}.$$

Consider the mapping  $\psi \in \Psi$  by  $\psi(t) = \frac{t}{2}$  for every  $t \geq 0$ .

This case is not applicable to  $\alpha$ - $\psi$ -contractive mappings since

$$d(f(\frac{1}{2}), f(0)) = \frac{1}{3} > \frac{1}{4} = \frac{1}{2}(d(\frac{1}{2}, 0)) = \psi(d(a, b)).$$

Now, we show that  $f$  is an  $\alpha$ - $\psi$ -contractive mapping of integral type. Using ([39], Example 3.6), we have for  $a = \frac{1}{n}$  and  $b = 0$ ;

$$\begin{aligned} \int_0^{d(fa, f0)} \varphi(t) dt &= d(fa, f0)^{1/d(fa, f0)} \\ &= \left| \frac{1}{n+1} - 0 \right|^{1/|(1/n+1)-0|} \\ &\leq \frac{1}{2} \left| \frac{1}{n} \right|^{1/|(1/n)|} \\ &= \frac{1}{2} d(a, b)^{1/d(a, b)} = \frac{1}{2} \int_0^{d(a, b)} \varphi(t) dt, \end{aligned}$$

Therefore,  $f$  is said to be an  $\alpha$ - $\psi$ -contractive mapping of integral type, that is,

$$\alpha(a, b) \int_0^{d(fa, fb)} \varphi(t) dt \leq \psi \left( \int_0^{d(a, b)} \varphi(t) dt \right)$$

Now, to establish the  $\alpha$ -admissible property of  $f$ , let  $a, b \in A$  satisfying  $\alpha(a, b) \geq 1$ . This implies that  $b = 0$  and the definition of  $f$  and  $\alpha$  gives  $fb = 0$ , that is,  $\alpha(fa, fb) = 1$ . Thus,  $f$  is  $\alpha$ -admissible thereby proving the condition (i) of Theorem 3.3.3. Also, for  $a_0 = 0$ ,

$$\alpha(0, f0) = \alpha(0, 0) = 1,$$

which proves the condition (ii) of Theorem 3.3.3.

In order to prove the condition (iii) of Theorem 3.3.2, let  $\{a_n\}$  be a sequence in  $A$  satisfying for all  $n$  and for some  $a \in A$

$$\alpha(a_n, a_{n+1}) \geq 1 \text{ and } a_n \rightarrow a \text{ as } n \rightarrow +\infty.$$

In view of the definition of  $\alpha$ , we obtain that  $a_n = 0$  for all  $n$ . Consequently,  $a = 0$  and  $\alpha(a_n, a) = 1$  for all  $n \in \mathbb{N}$ . Let  $(a, b) \in A \times A$ . Clearly, for  $z = 0$ , we have

$$\alpha(a, z) = \alpha(b, z) = 1, \text{ for all } (a, b) \in A \times A,$$

which satisfies condition (H). All the conditions of Theorem 3.3.3 are satisfied. Hence,  $T$  has a unique fixed point  $u \in A$ .

### **3.4 Existence of coupled fixed point points in complete metric spaces**

In this section, some coupled fixed point results have been derived in complete metric spaces from the results of Section 3.3. For this, we recall the following definition:

**Definition 3.4.1.** (Bhaskar and Lakshmikantham [36]) Let  $F : A \times A \rightarrow A$  be a given mapping. The point  $(a, b) \in A \times A$  is called as a coupled fixed point of  $F$  if

$$F(a, b) = a \text{ and } F(b, a) = b.$$

Samet *et al.* [165] gave the following useful lemma which states that a coupled fixed point is a fixed point:

**Lemma 3.4.1.** Consider the mappings  $F : A \times A \rightarrow A$  and  $T : A \times A \rightarrow A \times A$  such that

$$T(a, b) = (F(a, b), F(b, a)), \forall (a, b) \in A \times A. \quad (3.4.1)$$

Then,  $(a, b)$  is a coupled fixed point of  $F$  if and only if  $(a, b)$  is a fixed point of  $T$ .

The main results of this section are the following theorems:

**Theorem 3.4.2.** Consider a mapping  $F : A \times A \rightarrow A$ , where  $(A, d)$  is a complete metric space. Suppose that there exists  $\psi \in \Psi$  and a function  $\alpha : A^2 \times A^2 \rightarrow [0, +\infty)$  satisfying for all  $(x, y), (u, v) \in A \times A$ ,

$$\alpha((x, y), (u, v)) \int_0^{d(F(x,y), F(u,v))} \varphi(t) dt \leq \frac{1}{2} \psi \left( \int_0^{d(x,u)+d(y,v)} \varphi(t) dt \right), \quad (3.4.2)$$

where  $\varphi \in \Phi$ . Also, suppose that

(i)  $\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$  for all  $(x, y), (u, v) \in A \times A$ ,

(ii) there exists  $(x_0, y_0) \in A \times A$  such that

$$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \text{ and } \alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1,$$

(iii) The mapping  $F$  is continuous,

(iv)  $\int_0^{a+b} \varphi(t) dt \leq \int_0^a \varphi(t) dt + \int_0^b \varphi(t) dt$ .

Then,  $F$  has a coupled fixed point.

*Proof.* For the proof, this problem has been converted to the complete metric space  $(Y, \delta)$ , where  $Y = A \times A$  and  $\delta((x, y), (u, v)) = d(x, u) + d(y, v)$  for all  $(x, y), (u, v) \in A \times A$ .

The use of inequality(3.4.2) implies that

$$\alpha((x, y), (u, v)) \int_0^{d(F(x,y), F(u,v))} \varphi(t) dt \leq \frac{1}{2} \psi \left( \int_0^{\delta((x,y), (u,v))} \varphi(t) dt \right) \quad (3.4.3)$$

and

$$\alpha((v, u), (y, x)) \int_0^{d(F(v,u), F(y,x))} \varphi(t) dt \leq \frac{1}{2} \psi \left( \int_0^{\delta((x,y), (u,v))} \varphi(t) dt \right). \quad (3.4.4)$$

Adding equations (3.4.3) and (3.4.4) gives

$$\beta((x, y), (u, v)) \left( \int_0^{d(F(x,y), F(u,v))} \varphi(t) dt + \int_0^{d(F(v,u), F(y,x))} \varphi(t) dt \right) \leq \psi \left( \int_0^{\delta((x,y), (u,v))} \varphi(t) dt \right)$$

where  $\beta : Y \times Y \rightarrow [0, +\infty)$  is the function defined by

$$\beta((x, y), (u, v)) = \min\{\alpha((x, y), (u, v)), \alpha((v, u), (y, x))\}$$

which further implies from condition (iv) that

$$\beta((x, y), (u, v)) \left( \int_0^{d(F(x,y), F(u,v)) + d(F(v,u), F(y,x))} \varphi(t) dt \right) \leq \psi \left( \int_0^{\delta((x,y), (u,v))} \varphi(t) dt \right) \quad (3.4.5)$$

Hence,

$$\beta((x, y), (u, v)) \int_0^{\delta(T(x,y), T(u,v))} \varphi(t) dt \leq \psi \left( \int_0^{\delta((x,y), (u,v))} \varphi(t) dt \right), \quad (3.4.6)$$

for every  $(x, y), (u, v) \in A \times A$ , where  $T : Y \rightarrow Y$  is given by (3.4.1). Thus, for all  $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in A \times A$ , we get

$$\beta(\mu, \nu) \int_0^{\delta(T(\mu), T(\nu))} \varphi(t) dt \leq \psi \left( \int_0^{\delta(\mu, \nu)} \varphi(t) dt \right) \quad (3.4.7)$$

Consequently, in view of the inequality (3.4.7) and condition (iii), we obtain the continuity of  $T$  and the fact that  $T$  is a  $\beta$ - $\psi$ -contractive mapping of integral type.

Let  $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in Y$  such that

$$\beta(\mu, \nu) \geq 1.$$

The condition (i) implies that  $\beta(T\mu, T\nu) \geq 1$ . Thus,  $T$  is  $\beta$ -admissible. From condition (ii), we get the existence of  $(x_0, y_0) \in Y$  such that

$$\beta((x_0, y_0), T(x_0, y_0)) \geq 1.$$

Thus, all the conditions of Theorem 3.3.1 are fulfilled thereby implying the existence of fixed point of  $T$ . The use of Lemma 3.4.1 gives the existence of a coupled fixed point of  $F$ .  $\square$

In the next theorem, we establish that Theorem 3.4.2 holds true even if the continuity requirement of  $T$  is replaced with a new condition as mentioned below:

**Theorem 3.4.3.** Let  $F : A \times A \rightarrow A$  be a given mapping, where  $(A, d)$  is a complete metric space. Let there exists  $\psi \in \Psi$  and a function  $\alpha : A^2 \times A^2 \rightarrow [0, +\infty)$  satisfying for all  $(x, y), (u, v) \in A \times A$ ,

$$\alpha((x, y), (u, v)) \int_0^{d(F(x,y), F(u,v))} \varphi(t) dt \leq \frac{1}{2} \psi \left( \int_0^{d(x,u)+d(y,v)} \varphi(t) dt \right),$$

where  $\varphi \in \Phi$ . Also, suppose that

$$(i) \alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$$

for all  $(x, y), (u, v) \in A \times A$ ,

(ii) there exists  $(x_0, y_0) \in A \times A$  such that

$$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \text{ and } \alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1,$$

(iii) if  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $A$  such that

$$\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1 \text{ and } \alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \geq 1 \text{ for all } n \in \mathbb{N}.$$

$x_n \rightarrow x \in A$  and  $y_n \rightarrow y \in A$  as  $n \rightarrow +\infty$ , then

$$\alpha((x_n, y_n), (x, y)) \geq 1 \text{ and } \alpha((y, x), (y_n, x_n)) \geq 1 \text{ for every } n \in \mathbb{N}.$$

Then,  $F$  has a coupled fixed point.

*Proof.* Utilizing the same notations of the proof of previous theorem, let  $\{(x_n, y_n)\}$  be a sequence in  $Y$  such that

$$\beta((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1 \text{ and } (x_n, y_n) \rightarrow (x, y) \text{ as } n \rightarrow +\infty.$$

The use of condition (iii) gives

$$\beta((x_n, y_n), (x, y)) \geq 1.$$

Consequently, all the conditions of Theorem 3.2.2 are fulfilled thereby implying the existence of a fixed point of  $T$ . Further, the use of Lemma 3.4.1 proves the existence of a coupled fixed point of  $F$ .  $\square$

For the uniqueness of the coupled fixed point, the following condition has been taken:

(H'): For every  $(a, b), (c, d) \in A \times A$ , there exists  $(u_1, u_2) \in A \times A$  such that  $\alpha((a, b), (u_1, u_2)) \geq 1, \alpha((u_2, u_1), (b, a)) \geq 1$  and  $\alpha((c, d), (u_1, u_2)) \geq 1, \alpha((u_2, u_1), (d, c)) \geq 1$ .

**Theorem 3.4.4.** If the condition (H') along with the hypotheses of Theorems 3.4.2 and 3.4.3 holds, then there exists a unique fixed point of  $T$ .

*Proof.* The condition (H') shows that  $T$  and  $\beta$  satisfy the condition (H). Therefore, the result follows from Theorem 3.3.3 and Lemma 3.4.1.  $\square$

## 3.5 Consequences

In this section, many relevant results of the literature have been derived from Theorem 3.3.3.

### 3.5.1 Some standard fixed point theorems

By substituting  $\alpha(a, b) = 1$  for every  $a, b \in A$  and  $\psi(t) = kt$  for  $k \in [0, 1)$  in Theorem 3.3.3, the following corollary has been obtained:

**Corollary 3.5.1.** (Branciari [39]) Suppose that  $T$  is a self-mapping of a complete metric space  $(A, d)$  satisfying for every  $a, b \in A$ ,

$$\int_0^{d(Ta, Tb)} \varphi(t) dt \leq k \int_0^{d(a, b)} \varphi(t) dt, \text{ where } k \in [0, 1)$$

where  $\varphi \in \Phi$ . Then  $T$  has a unique fixed point  $u \in A$  such that for each  $a \in A$ ,

$$\lim_{n \rightarrow +\infty} T^n a = u.$$

By taking  $y = Tx$  in Corollary 3.5.1, we obtain:

**Corollary 3.5.2.** (Rhoades and Abbas [152]). Let  $T : A \rightarrow A$  be a mapping of a complete metric space  $(A, d)$  satisfying for all  $a \in A$ ,

$$\int_0^{d(Ta, T^2a)} \varphi(t) dt \leq k \int_0^{d(a, Ta)} \varphi(t) dt, \text{ where } k \in [0, 1)$$

where  $\varphi \in \Phi$ . Then  $T$  has a unique fixed point  $u \in A$ .

By taking  $\varphi(t) = 1$  for all  $t \geq 0$  in Theorem 3.3.1, we obtain the following result:

**Corollary 3.5.3.** (Samet *et al.* [165]). Let  $T$  be an  $\alpha$ - $\psi$ -contractive mapping in a complete metric space  $(A, d)$ . Also, suppose that:

- (i) the mapping  $T$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, Ta_0) \geq 1$ ,
- (iii) the mapping  $T$  is continuous.

Then,  $T$  has a fixed point.

By taking  $\alpha(a, b) = 1$  for every  $a, b \in A$  and  $\varphi(t) = 1$  for each  $t \geq 0$  in Theorem 3.3.3, we obtain the following:

**Corollary 3.5.4.** (Berinde [31]). Let  $T : A \rightarrow A$  be a mapping of a complete metric space  $(A, d)$ . Suppose that there exists a function  $\psi \in \Psi$  satisfying

$$d(Ta, Tb) \leq \psi(d(a, b)),$$

for all  $a, b \in A$ . Then,  $T$  has a unique fixed point.

Similarly, by taking  $\psi(t) = rt$ , for  $r \in [0, 1)$ , we get the Banach Contraction principle.

**Corollary 3.5.5.** (Banach [30]). Let  $T$  be a self-mapping of a complete metric space  $(A, d)$  satisfying

$$d(Ta, Tb) \leq rd(a, b), \text{ for all } a, b \in A,$$

where  $r \in [0, 1)$ . Then  $T$  has a unique fixed point.

### 3.5.2 Existence of fixed points in partially ordered metric spaces

Recently, authors have started a new direction in fixed point theory by analyzing the existence and uniqueness of a fixed point in partially ordered sets. In this direction, the first result was given by Turinici [186] by extending the Banach contraction principle in partially ordered sets. Later, some applications of Turinici's theorem to matrix equations were

given by Ran and Reurings [142]. After this fascinating paper, many useful results have been obtained in this direction(see, for example, [11, 32, 33, 36, 107, 108, 126, 128] and the references cited therein). Various fixed point results have been derived in partially ordered metric spaces in this section. For this, the following concepts are required:

**Definition 3.5.1.** Let  $T$  be a given self-mapping of a partially ordered set  $(A, \preceq)$ . The mapping  $T$  is said to be nondecreasing with respect to  $\preceq$  if

$$a, b \in A, a \preceq b \Rightarrow Ta \preceq Tb.$$

**Definition 3.5.2.** Let  $(A, \preceq)$  be a partially ordered set. A sequence  $\{a_n\} \subset A$  is said to be nondecreasing if  $a_n \preceq a_{n+1}$  for all  $n$ .

**Definition 3.5.3.** [100] Let  $(A, d, \preceq)$  be a partially ordered metric space. The space  $(A, \preceq, d)$  is said to be a regular space if for every nondecreasing sequence  $\{a_n\}$  in  $A$  with  $a_n \rightarrow a \in A$  as  $n \rightarrow \infty$ , there exists a subsequence  $\{a_{n(k)}\}$  of  $\{a_n\}$  such that  $a_{n(k)} \preceq a$  for all  $k$ .

The main result of this subsection is as follows:

**Corollary 3.5.6.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $T : A \rightarrow A$  be a nondecreasing mapping with respect to  $\preceq$ . Assume that there exists a function  $\psi \in \Psi$  satisfying for all  $a, b \in A$  with  $a \preceq b$ ,

$$\int_0^{d(Ta, Tb)} \varphi(t) dt \leq \psi \left( \int_0^{d(a, b)} \varphi(t) dt \right), \quad (3.5.1)$$

where  $\varphi \in \Phi$ . Also, suppose that

- (i) there exists  $a_0 \in A$  such that  $a_0 \preceq Ta_0$ ,
- (ii) the mapping  $T$  is continuous or the space  $(A, d, \preceq)$  is regular.

Then,  $T$  has a fixed point. Also, if there exists  $c \in A$  such that  $a \preceq c$  and  $b \preceq c$  for all  $a, b \in A$ , then  $T$  has a unique fixed point.

*Proof.* Consider the mapping  $\alpha : A \times A \rightarrow [0, \infty)$  defined by

$$\alpha(a, b) = \begin{cases} 1 & \text{if } a \preceq b \\ 0 & \text{otherwise} \end{cases}$$

Clearly,  $\alpha$  is transitive. The inequality (3.5.1) proves that  $T$  is an  $\alpha$ - $\psi$ -contractive mapping of integral type, that is,

$$\alpha(a, b) \int_0^{d(Ta, Tb)} \varphi(t) dt \leq \psi \left( \int_0^{d(a, b)} \varphi(t) dt \right),$$

for all  $a, b \in A$ . From condition (i), we have

$$\alpha(a_0, Ta_0) \geq 1.$$

Now, to prove the  $\alpha$ -admissible property of  $T$ , let  $\alpha(a, b) \geq 1$  for some  $a, b \in A$ . The monotonicity of  $T$  implies for all  $a, b \in A$ ,

$$\alpha(a, b) \geq 1 \Rightarrow a \preceq b \Rightarrow Ta \preceq Tb \Rightarrow \alpha(Ta, Tb) \geq 1.$$

Thus,  $T$  is  $\alpha$ -admissible. Now, if  $T$  is continuous, we get the existence of a fixed point from Theorem 3.3.1.

Now, suppose that  $(A, \preceq, d)$  is regular. Let  $\{a_n\}$  be a sequence in  $A$  such that  $\alpha(a_n, a_{n+1}) \geq 1$  for every  $n$  and  $a_n \rightarrow a \in A$  as  $n \rightarrow \infty$ . The regularity of the space  $A$  shows the existence of a subsequence  $\{a_{n(k)}\}$  of  $\{a_n\}$  such that  $a_{n(k)} \preceq a$  for every  $k$ . The definition of  $\alpha$  implies that  $\alpha(a_{n(k)}, a) \geq 1$  for all  $k$ . Thus, in this case, the existence of a fixed point follows from Theorem 3.3.2.

For the uniqueness of the fixed point, let  $a, b \in A$ . The given condition shows the existence of a point  $c \in A$  such that  $a \preceq c$  and  $b \preceq c$ . Thus, the definition of  $\alpha$  proves that  $\alpha(a, c) \geq 1$  and  $\alpha(b, c) \geq 1$ . Therefore, we deduce the uniqueness of the fixed point from Theorem 3.3.3.  $\square$

The following results can easily be derived from Corollary 3.5.6.

**Corollary 3.5.7.** (Karapinar and Samet [100]). Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $T : A \rightarrow A$  be a nondecreasing mapping with respect to  $\preceq$ . Suppose that there exists a function  $\psi \in \Psi$  such that

$$d(Ta, Tb) \leq \psi(d(a, b)),$$

for all  $a, b \in A$  with  $a \preceq b$ . Also, suppose that the following conditions hold:

- (i) there exists  $a_0 \in A$  satisfying  $a_0 \preceq Ta_0$ ,
- (ii) the mapping  $T$  is continuous or the space  $(A, d, \preceq)$  is regular.

Then,  $T$  has a fixed point. Moreover, if for all  $u, v \in A$  there exists  $w \in A$  such that  $u \preceq w$  and  $v \preceq w$ , we have uniqueness of the fixed point.

*Proof.* By taking  $\varphi(t) = 1$  for all  $t \geq 0$  in Corollary 3.5.6, we get the proof of this corollary. □

**Corollary 3.5.8.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $T : A \rightarrow A$  be a nondecreasing mapping with respect to  $\preceq$ . Suppose that there exists a function  $\psi \in \Psi$  such that for all  $a, b \in A$  with  $a \preceq b$ , we have

$$\int_0^{d(Ta, Tb)} \varphi(t) dt \leq k \int_0^{d(a, b)} \varphi(t) dt,$$

where  $\varphi \in \Phi$ . Also, suppose that

- (i) there exists  $a_0 \in A$  satisfying  $a_0 \preceq Ta_0$ ,
- (ii) the mapping  $T$  is continuous or the space  $(A, d, \preceq)$  is regular.

Then,  $T$  has a fixed point. Moreover, if for all  $u, v \in A$  there exists  $w \in A$  such that  $u \preceq w$  and  $v \preceq w$ , we have uniqueness of the fixed point.

*Proof.* By taking  $\psi(t) = kt$  for all  $t \geq 0$  and some  $k \in [0, 1)$  in Corollary 3.5.6, we get the proof of this corollary. □

**Corollary 3.5.9.** (Ran and Reurings [142], Nieto and Lopez [128]). Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $T : A \rightarrow A$  be a nondecreasing mapping with respect to  $\preceq$ . Suppose that there exists a constant  $k \in [0, 1)$  such that

$$d(Ta, Tb) \leq kd(a, b),$$

for all  $a, b \in A$  with  $a \preceq b$ . Also, suppose that

- (i) there exists  $a_0 \in A$  satisfying  $a_0 \preceq Ta_0$ ,
- (ii) the mapping  $T$  is continuous or the space  $(A, d, \preceq)$  is regular.

Then,  $T$  has a fixed point. Moreover, if for all  $u, v \in A$  there exists  $w \in A$  such that  $u \preceq w$  and  $v \preceq w$ , then  $T$  has a unique fixed point.

*Proof.* Taking  $\varphi(t) = 1$  for all  $t \geq 0$  in Corollary 3.5.8, we get the proof of this corollary.

□



## Chapter 4

# SOME COINCIDENCE AND COMMON FIXED POINT THEOREMS FOR GENERALIZED $\alpha$ - $\psi$ -CONTRACTIVE TYPE MAPPINGS

### 4.1 Introduction

The Banach contraction principle [30] is a very powerful tool in solving existence problems in many areas of mathematical analysis. Due to the simplicity and usefulness of this fundamental theorem, it has become a very popular source of existence and uniqueness theorems in different branches of mathematical analysis. The existing literature of fixed point theory contains a large number of generalizations of Banach contraction principle by using different form of contraction condition in various spaces. Various researchers have presented a number of contractive type conditions on a complete metric space  $X$  which are

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the generalizations of well-known Banach contraction principle and have the property that each of such mappings have a unique fixed point. In 1977, Rhoades [147] compared these different contractive conditions.

A new approach in the theory of contractive mappings was recently given by Samet *et al.* [165] by using the concepts of  $\alpha$ - $\psi$ -contractive type mappings and  $\alpha$ -admissible mappings in metric spaces. In 2012, Karapinar and Samet [100] further generalized the  $\alpha$ - $\psi$ -contractive type mappings and obtained various fixed point theorems for this generalized class of  $\alpha$ - $\psi$ -contractive mappings. The study concerning common fixed points of mappings satisfying certain contractive conditions has been at the center of attraction for various authors working in the field of fixed point theory.

The aim of this chapter is to introduce a new class of contractive pair of mappings called generalized  $\alpha$ - $\psi$  contractive pair of mappings and to study various coincidence and common fixed point theorems for these contractive pair of mappings. Also, we introduce a new notion of  $\alpha$ -admissible mapping w.r.t an another mapping  $g$  which in turn generalizes the concept of  $g$ -monotone mapping recently introduced by Ćirić *et al.* [55]. As an application of the main results, common fixed point theorems in partially ordered metric spaces as well as in respect of cyclic contractive mappings have been established. The presented theorems extend various known comparable results from the current literature. Some illustrative examples are provided to demonstrate the main results. The results given in this chapter extend the results derived by Karapinar and Samet [100], Samet *et al.* [165], Ćirić *et al.* [55] and various other related results in the literature.

The contents of this chapter have been divided into four sections. Section 6.2 deals with the preliminaries related to this chapter. In section 6.3, we present a new notion of generalized  $\alpha$ - $\psi$  contractive pair of mappings. Also, the concept of  $\alpha$ -admissible mapping has

been extended to the notion of  $\alpha$ -admissible mapping w.r.t an another mapping  $g$ . Section 6.4 concerns with some coincidence and common fixed point results for this generalized class of  $\alpha$ - $\psi$ -contractive pair of mappings. Section 6.5 presents the consequences of our main results.

## 4.2 Preliminaries

Before presenting the main results, some of the related definitions and results are recalled as uner. Some of the definitions and results concerning this chapter due to Samet *et al.* [165] have already been mentioned in Chapters 1 and 2.

**Definition 4.2.1.** [165]. Let  $\Psi$  denote the family of all functions  $\psi : [0, +\infty) \rightarrow [0, +\infty)$

which satisfy

- (i)  $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$  for every  $t > 0$ , where  $\psi^n$  is the  $n$ -th iterate of  $\psi$ ,
- (ii) the function  $\psi$  is nondecreasing.

Recently, Karapinar and Samet [100] introduced the notion of generalized  $\alpha$ - $\psi$ -contractive type mappings as follows:

**Definition 4.2.2.** Let  $(A, d)$  be a metric space and  $T$  be a given self-mapping of  $A$ . The mapping  $T$  is a generalized  $\alpha$ - $\psi$ -contractive type mapping if there exists two functions  $\alpha : A \times A \rightarrow [0, \infty)$  and  $\psi \in \Psi$  such that for all  $a, b \in A$ ,

$$\alpha(a, b)d(Ta, Tb) \leq \psi(M(a, b)),$$

where  $M(a, b) = \max \left\{ d(a, b), \frac{d(a, Ta) + d(b, Tb)}{2}, \frac{d(a, Tb) + d(b, Ta)}{2} \right\}$ .

The main results of Karapinar and Samet [100] are as follows:

**Theorem 4.2.1.** Let  $(A, d)$  be a complete metric space and  $T : A \rightarrow A$  is a generalized  $\alpha$ - $\psi$ -contractive mapping satisfying the following conditions:

- (i) the mapping  $T$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, Ta_0) \geq 1$ ,
- (iii) the mapping  $T$  is continuous.

Then  $T$  has a fixed point.

**Theorem 4.2.2.** Let  $(A, d)$  be a complete metric space and  $T : A \rightarrow A$  is a generalized  $\alpha$ - $\psi$ -contractive mapping satisfying the following conditions:

- (i) the mapping  $T$  is  $\alpha$ -admissible,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(a_0, Ta_0) \geq 1$ ,
- (iii) if  $\{a_n\}$  is a sequence in  $A$  such that  $\alpha(a_n, a_{n+1}) \geq 1$  for all  $n$  and  $a_n \rightarrow a \in A$  as  $n \rightarrow \infty$ , then there exists a subsequence  $\{a_{n(k)}\}$  of  $\{a_n\}$  such that  $\alpha(a_{n(k)}, a) \geq 1$  for all  $k$ . Then  $T$  has a fixed point.

Karapinar and Samet [100] added the following condition (H) to the hypotheses of Theorems 4.2.1 and 4.2.2 to assure the uniqueness of the fixed point:

(H) : For every  $a, b \in A$ , there exists  $c \in A$  such that  $\alpha(a, c) \geq 1$  and  $\alpha(b, c) \geq 1$ .

**Definition 4.2.3.** [26] Let  $A$  be a non-empty set,  $N$  is a natural number ( $N \geq 2$ ) and  $T_1, T_2, \dots, T_N : A \rightarrow A$  are mappings. If  $t = T_1u = T_2u = \dots = T_Nu$  for some  $u \in A$ , then  $u$  is called a coincidence point of  $T_1, T_2, \dots, T_{N-1}$  and  $T_N$ , and  $t$  is called a point of coincidence of  $T_1, T_2, \dots, T_{N-1}$  and  $T_N$ . If  $t = u$ , then  $u$  is called a common fixed point of  $T_1, T_2, \dots, T_{N-1}$  and  $T_N$ .

Consider two self-mappings  $f_1$  and  $f_2$  of  $A$ . Let the set  $C(f_1, f_2)$  be the set of all coincidence points of  $f_1$  and  $f_2$ , that is,

$$C(f_1, f_2) = \{z \in A : f_1 z = f_2 z\}$$

### 4.3 Generalized $\alpha$ - $\psi$ contractive pair of mappings and $\alpha$ -admissible mapping w.r.t another mapping

In this section, we first present the following new class of contractive pair of mappings:

**Definition 4.3.1.** Let  $(A, d)$  be a metric space and  $f, g : A \rightarrow A$  be given mappings. The pair of mappings  $(f, g)$  is said to be a generalized  $\alpha$ - $\psi$  contractive pair of mappings if there exists two functions  $\alpha : A \times A \rightarrow [0, +\infty)$  and  $\psi \in \Psi$  such that for all  $a, b \in A$ ,

$$\alpha(ga, gb)d(fa, fb) \leq \psi(M(ga, gb)), \quad (4.3.1)$$

where  $M(ga, gb) = \max \left\{ d(ga, gb), \frac{d(ga, fa) + d(gb, fb)}{2}, \frac{d(ga, fb) + d(gb, fa)}{2} \right\}$ .

Now, in order to generalize the concept of  $\alpha$ -admissible mappings due to Samet *et al.* [165], we introduce a new type of mapping called  $\alpha$ -admissible mapping w.r.t an another mapping  $g$  as follows:

**Definition 4.3.2.** Let  $f, g : A \rightarrow A$  and  $\alpha : A \times A \rightarrow [0, \infty)$ . The mapping  $f$  is said to be  $\alpha$ -admissible w.r.t  $g$  if for all  $a, b \in A$ , we have

$$\alpha(ga, gb) \geq 1 \Rightarrow \alpha(fa, fb) \geq 1.$$

The following examples are furnished in order to demonstrate the preceding definition:

**Example 4.3.3.** Let  $A$  be the set of all non-negative real numbers. Consider the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  given by

$$\alpha(a, b) = \begin{cases} 1 & \text{if } a \geq b, \\ 0 & \text{if } a < b. \end{cases}$$

and define the mappings  $f, g : A \rightarrow A$  by  $f(a) = e^a$  and  $g(a) = a^2$  for all  $a \in A$ . Thus, the mapping  $f$  is  $\alpha$ -admissible w.r.t  $g$ .

**Example 4.3.4.** Let  $A = [1, \infty)$ . Consider the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  given by

$$\alpha(a, b) = \begin{cases} 3 & \text{if } a, b \in [0, 1], \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

and define the mappings  $f, g : A \rightarrow A$  by  $f(a) = \ln\left(1 + \frac{a}{3}\right)$  and  $g(a) = \sqrt{a}$  for all  $a \in A$ . Thus, the mapping  $f$  is  $\alpha$ -admissible w.r.t  $g$ .

**Remark 4.3.1.** Clearly, every  $\alpha$ -admissible mapping is  $\alpha$ -admissible w.r.t mapping  $g$  when  $g = I$ .

The following example shows that a mapping which is  $\alpha$ -admissible w.r.t  $g$  may not be  $\alpha$ -admissible.

**Example 4.3.5.** Let  $A = [1, \infty)$ . Consider the mapping  $\alpha : A \times A \rightarrow [0, \infty)$  given by

$$\alpha(a, b) = \begin{cases} 2 & \text{if } a > b \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

Define the mappings  $f, g : A \rightarrow A$  by  $f(a) = \frac{1}{a}$  and  $g(a) = e^{-a}$  for all  $a \in A$ .

Suppose that  $\alpha(a, b) \geq 1$ . The definition of  $\alpha$  proves that  $a > b$  which further implies that  $\frac{1}{a} < \frac{1}{b}$ . Thus,  $\alpha(fa, fb) \not\geq 1$ , that is,  $f$  is not  $\alpha$ -admissible.

Now, we prove that  $f$  is  $\alpha$ -admissible w.r.t  $g$ . Let us suppose that  $\alpha(ga, gb) \geq 1$ . So,

$$\alpha(ga, gb) \geq 1 \Rightarrow ga > gb \Rightarrow e^{-a} > e^{-b} \Rightarrow \frac{1}{a} > \frac{1}{b} \Rightarrow \alpha(fa, fb) \geq 1$$

Therefore,  $f$  is  $\alpha$ -admissible w.r.t  $g$ .

## 4.4 Coincidence and Common fixed point results for Generalized $\alpha$ - $\psi$ -contractive pair of mappings

The main results of this section are as follows:

**Theorem 4.4.1.** Let  $(A, d)$  be a complete metric space and  $f, g : A \rightarrow A$  be such that  $f(A) \subseteq g(A)$ . Suppose that the pair  $(f, g)$  is a generalized  $\alpha$ - $\psi$  contractive pair of mappings satisfying the following conditions:

- (i) the mapping  $f$  is  $\alpha$ -admissible w.r.t.  $g$ ,
- (ii) there exists  $a_0 \in A$  satisfying  $\alpha(ga_0, fa_0) \geq 1$ ,
- (iii) if  $\{ga_n\}$  is a sequence in  $A$  such that  $\alpha(ga_n, ga_{n+1}) \geq 1$  for all  $n$  and  $ga_n \rightarrow gz \in g(A)$  as  $n \rightarrow \infty$ , then there exists a subsequence  $\{ga_{n(k)}\}$  of  $\{ga_n\}$  satisfying  $\alpha(ga_{n(k)}, gz) \geq 1$  for every  $k$ ,
- (iv)  $g(A)$  is closed.

Then,  $f$  and  $g$  have a coincidence point.

*Proof.* In view of condition (ii) implies the existence of a point  $a_0 \in A$  be such that  $\alpha(ga_0, fa_0) \geq 1$ . Since  $f(A) \subseteq g(A)$ , choose a point  $a_1 \in A$  such that  $fa_0 = ga_1$ . Continuing this process having chosen  $a_1, a_2, \dots, a_n$ , we choose  $a_{n+1}$  in  $A$  satisfying

$$fa_n = ga_{n+1}, n = 0, 1, 2, \dots \quad (4.4.1)$$

Since  $f$  is  $\alpha$ -admissible w.r.t  $g$ , we have

$$\alpha(ga_0, fa_0) = \alpha(ga_0, ga_1) \geq 1 \Rightarrow \alpha(fa_0, fa_1) = \alpha(ga_1, ga_2) \geq 1.$$

Using mathematical induction, we get

$$\alpha(ga_n, ga_{n+1}) \geq 1, \forall n = 0, 1, 2, \dots \quad (4.4.2)$$

If  $fa_{n+1} = fa_n$  for some  $n$ , then by (4.4.1),

$$fa_{n+1} = ga_{n+1}, n = 0, 1, 2, \dots$$

that is,  $f$  and  $g$  have a coincidence point at  $a = a_{n+1}$ . So, proof is over. Otherwise, suppose that  $d(fa_n, fa_{n+1}) > 0$  for all  $n$ . Applying the inequality (4.3.1) and using (4.4.2), we obtain

$$\begin{aligned} d(fa_n, fa_{n+1}) &\leq \alpha(ga_n, ga_{n+1})d(fa_n, fa_{n+1}) \\ &\leq \psi(M(ga_n, ga_{n+1})). \end{aligned} \quad (4.4.3)$$

On the other hand, we have in view of (4.4.1)

$$\begin{aligned} M(ga_n, ga_{n+1}) &= \max \left\{ d(ga_n, ga_{n+1}), \frac{d(ga_n, fa_n) + d(ga_{n+1}, fa_{n+1})}{2}, \right. \\ &\quad \left. \frac{d(ga_n, fa_{n+1}) + d(ga_{n+1}, fa_n)}{2} \right\} \\ &\leq \max \{ d(fa_{n-1}, fa_n), d(fa_n, fa_{n+1}) \} \end{aligned}$$

Owing to monotonicity of the function  $\psi$  and using the inequality (4.4.3), we have for all  $n \geq 1$

$$d(fa_n, fa_{n+1}) \leq \psi(\max \{ d(fa_{n-1}, fa_n), d(fa_n, fa_{n+1}) \}). \quad (4.4.4)$$

If  $d(fa_{n-1}, fa_n) \leq d(fa_n, fa_{n+1})$ , for some  $n \geq 1$ , then inequality (4.4.4) implies that

$$d(fa_n, fa_{n+1}) \leq \psi(d(fa_n, fa_{n+1})) < d(fa_n, fa_{n+1}),$$

which is a contradiction. Thus, for all  $n \geq 1$ ,

$$\max \{ d(fa_{n-1}, fa_n), d(fa_n, fa_{n+1}) \} = d(fa_{n-1}, fa_n). \quad (4.4.5)$$

From (4.4.4) and (4.4.5), we get

$$d(fa_n, fa_{n+1}) \leq \psi(d(fa_{n-1}, fa_n)), \forall n \geq 1. \quad (4.4.6)$$

Continuing this process inductively, we obtain

$$d(fa_n, fa_{n+1}) \leq \psi^n(d(fa_0, fa_1)), \quad \forall n \geq 1. \quad (4.4.7)$$

The triangular inequality and (4.4.7) implies for all  $k \geq 1$ ,

$$\begin{aligned} d(fa_n, fa_{n+k}) &\leq d(fa_n, fa_{n+1}) + \dots + d(fa_{n+k-1}, fa_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^p(d(fa_1, fa_0)) \\ &\leq \sum_{p=n}^{+\infty} \psi^p(d(fa_1, fa_0)). \end{aligned} \quad (4.4.8)$$

Letting  $p \rightarrow \infty$  in (4.4.8), we obtain that  $\{fa_n\}$  is a Cauchy sequence in  $(A, d)$ . Since by inequality (4.4.1), we have  $\{fa_n\} = \{ga_{n+1}\} \subseteq g(A)$  and  $g(A)$  is closed, therefore there exists  $z \in A$  such that

$$\lim_{n \rightarrow \infty} ga_n = gz. \quad (4.4.9)$$

Now, we show that  $z$  is a coincidence point of  $f$  and  $g$ . On contrary, let  $d(fz, gz) > 0$ . By condition (iii) and (4.4.9),

$$\alpha(ga_{n(k)}, gz) \geq 1 \text{ for all } k,$$

Therefore, by the use of triangle inequality and (4.3.1),

$$\begin{aligned} d(gz, fz) &\leq d(gz, fa_{n(k)}) + d(fa_{n(k)}, fz) \\ &\leq d(gz, fa_{n(k)}) + \alpha(ga_{n(k)}, gz)d(fa_{n(k)}, fz) \\ &\leq d(gz, fa_{n(k)}) + \psi(M(ga_{n(k)}, gz)). \end{aligned} \quad (4.4.10)$$

On the other hand,

$$M(ga_{n(k)}, gz) = \max \left\{ d(ga_{n(k)}, gz), \frac{d(ga_{n(k)}, fa_{n(k)}) + d(gz, fz)}{2}, \frac{d(ga_{n(k)}, fz) + d(gz, fa_{n(k)})}{2} \right\}.$$

In view of the above equality and the inequality (4.4.10) implies that,

$$\begin{aligned} d(gz, fz) &\leq d(gz, fa_{n(k)}) + \psi(M(ga_{n(k)}, gz)) \\ &\leq d(gz, fa_{n(k)}) + \psi\left(\max\left\{d(ga_{n(k)}, gz), \frac{d(ga_{n(k)}, fa_{n(k)}) + d(gz, fz)}{2}, \right. \right. \\ &\quad \left. \left. \frac{d(ga_{n(k)}, fz) + d(gz, fa_{n(k)})}{2}\right\}\right). \end{aligned}$$

Letting  $k \rightarrow \infty$  in the above inequality yields  $d(gz, fz) \leq \psi\left(\frac{d(fz, gz)}{2}\right) < \frac{d(fz, gz)}{2}$ , which is a contradiction. Hence, our supposition is wrong thereby implying that  $d(fz, gz) = 0$ , that is,  $fz = gz$ . This shows that  $f$  and  $g$  have a coincidence point.  $\square$

In what follows, we furnish an illustrative example which demonstrates Theorem 4.4.1.

**Example 4.4.1.** Consider the set  $A = [0, +\infty)$  equipped with the usual metric  $d$ . Define the mappings  $f : A \rightarrow A$  and  $g : A \rightarrow A$  by

$$f(a) = \begin{cases} 2a - 3 & \text{if } a \geq 2, \\ \frac{a}{3} & \text{if } 0 \leq a < 2. \end{cases}$$

and

$$g(a) = \frac{a}{2} \forall a \in A.$$

Now, we define the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  by

$$\alpha(a, b) = \begin{cases} 1 & \text{if } (a, b) \in [0, 1/2], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the pair  $(f, g)$  is a generalized  $\alpha$ - $\psi$  contractive pair of mappings with  $\psi(t) = \frac{4}{5}t$  for all  $t \geq 0$ . Also,

$$\begin{aligned} \alpha(ga, gb).d(fa, fb) &= 1. \left| \frac{a}{3} - \frac{b}{3} \right| \leq \frac{4}{5} \left| \frac{a}{2} - \frac{b}{2} \right| \\ &= \frac{4}{5}d(ga, gb) \\ &\leq \frac{4}{5}M(ga, gb) = \psi(M(ga, gb)), \text{ for all } a, b \in A, \end{aligned}$$

Moreover, there exists  $a_0 \in A$  such that  $\alpha(ga_0, fa_0) \geq 1$ . Infact, for  $a_0 = 1$ , we have  $\alpha\left(\frac{1}{2}, \frac{1}{3}\right) = 1$ .

Now, it remains to show that  $f$  is  $\alpha$ -admissible w.r.t  $g$ . For this, let  $\alpha(ga, gb) \geq 1$  for some  $a, b \in A$ . This implies that  $ga, gb \in [0, 1/2]$  and by the definition of  $g$ , we have  $a, b \in [0, 1]$ .

Therefore, by the definition of  $f$  and  $\alpha$ , we have

$$f(a) = \frac{a}{3} \in [0, 1/2], f(b) = \frac{b}{3} \in [0, 1/2] \text{ and } \alpha(fa, fb) = 1.$$

Thus,  $f$  is  $\alpha$ -admissible w.r.t  $g$ . Clearly,  $f(A) \subseteq g(A)$  and  $g(A)$  is closed.

Finally, let  $\{ga_n\}$  be a sequence in  $A$  such that  $\alpha(ga_n, ga_{n+1}) \geq 1$  for all  $n$  and  $ga_n \rightarrow gz \in g(A)$  as  $n \rightarrow +\infty$ . Since  $\alpha(ga_n, ga_{n+1}) \geq 1$  for all  $n$ , by the definition of  $\alpha$ , we have  $ga_n \in [0, 1/2]$  for all  $n$  and  $gz \in [0, 1/2]$ . Then,  $\alpha(ga_n, gz) \geq 1$ . Now, all the conditions of Theorem 4.4.1 are satisfied. Consequently,  $f$  and  $g$  have coincidence point. Here, 0 and 2 are coincidence points of  $f$  and  $g$ .

**Theorem 4.4.2.** In addition to the hypotheses of Theorem 4.4.1, assume that for all  $u, v \in C(g, f)$ , there exists  $w \in A$  such that  $\alpha(gu, gw) \geq 1$  and  $\alpha(gv, gw) \geq 1$  and  $f, g$  commute at their coincidence points. Then  $f$  and  $g$  have a unique common fixed point.

*Proof.* We need to consider three steps:

Step 1. We claim that if  $u, v \in C(g, f)$ , then  $gu = gv$ . By given condition, there exists  $w \in A$  such that

$$\alpha(gu, gw) \geq 1, \alpha(gv, gw) \geq 1 \tag{4.4.11}$$

Since  $f(A) \subseteq g(A)$ , define the sequence  $\{w_n\}$  in  $A$  by  $gw_{n+1} = fw_n$  for all  $n \geq 0$  and  $w_0 = w$ . As  $f$  is  $\alpha$ -admissible w.r.t  $g$ , the inequality (4.4.11) implies that

$$\alpha(gu, gw_n) \geq 1, \alpha(gv, gw_n) \geq 1, \tag{4.4.12}$$

for all  $n \geq 0$ . Applying inequality (4.3.1) and using (4.4.12), we obtain

$$\begin{aligned} d(gu, gw_{n+1}) &= d(fu, fw_n) \\ &\leq \alpha(gu, gw_n)d(fu, fw_n) \\ &\leq \psi(M(gu, gw_n)). \end{aligned} \quad (4.4.13)$$

On the other hand,

$$\begin{aligned} M(gu, gw_n) &= \max \left\{ d(gu, gw_n), \frac{d(gu, fu) + d(gw_n, fw_n)}{2}, \frac{d(gu, fw_n) + d(gw_n, fu)}{2} \right\} \\ &\leq \max \{d(gu, gw_n), d(gu, gw_{n+1})\}, \end{aligned} \quad (4.4.14)$$

Using the inequality (4.4.13) and in view of monotone property of  $\psi$ , we get

$$d(gu, gw_{n+1}) \leq \psi(\max \{d(gu, gw_n), d(gu, gw_{n+1})\}), \quad (4.4.15)$$

for all  $n$ . Without restriction to the generality, suppose that  $d(gu, gw_n) > 0$  for all  $n$ . If  $\max\{d(gu, gw_n), d(gu, gw_{n+1})\} = d(gu, gw_{n+1})$ , then from (4.4.15), we obtain

$$d(gu, gw_{n+1}) \leq \psi(d(gu, gw_{n+1})) < d(gu, gw_{n+1}), \quad (4.4.16)$$

which is a contradiction. Thus,  $\max\{d(gu, gw_n), d(gu, gw_{n+1})\} = d(gu, gw_n)$ , and  $d(gu, gw_{n+1}) \leq \psi(d(gu, gw_n))$  for all  $n$ . This implies that

$$d(gu, gw_n) \leq \psi^n(d(gu, gw_0)), \quad \forall n \geq 1. \quad (4.4.17)$$

Letting  $n \rightarrow \infty$  in the above inequality, we infer that

$$\lim_{n \rightarrow \infty} d(gu, gw_n) = 0. \quad (4.4.18)$$

Similarly,

$$\lim_{n \rightarrow \infty} d(gv, gw_n) = 0. \quad (4.4.19)$$

It follows from (4.4.18) and (4.4.19) that  $gu = gv$ .

Step 2. Existence of a common fixed point: Let  $u \in C(g, f)$ , that is,  $gu = fu$ . The commutativity of  $f$  and  $g$  at their coincidence points implies that

$$g^2u = gfu = fgu \quad (4.4.20)$$

Denote  $gu = z$ , then from (4.4.20),  $gz = fz$ . Thus,  $z$  is a coincidence point of  $f$  and  $g$ . Now, from Step 1, we have  $gu = gz = z = fz$ . Then,  $z$  is a common fixed point of  $f$  and  $g$ .

Step 3. Uniqueness: Assume that  $z^*$  is another common fixed point of  $f$  and  $g$ . Then  $z^* \in C(g, f)$ . By Step 1, we have  $z^* = gz^* = gz = z$ . This completes the proof.  $\square$

## 4.5 Consequences

In this section, we show that many related results of the literature can easily be obtained from Theorem 4.4.2.

### 4.5.1 Standard Fixed Point Theorems

By taking  $\alpha(u, v) = 1$  for all  $u, v \in A$  in Theorem 4.4.2, we obtain the following result:

**Corollary 4.5.1.** Let  $(A, d)$  be a complete metric space and  $f, g : A \rightarrow A$  be such that  $f(A) \subseteq g(A)$ . Suppose that there exists a function  $\psi \in \Psi$  satisfying

$$d(fa, fb) \leq \psi(M(ga, gb)), \quad (4.5.1)$$

where  $M(ga, gb) = \max \left\{ d(ga, gb), \frac{d(ga, fa) + d(gb, fb)}{2}, \frac{d(ga, fb) + d(gb, fa)}{2} \right\}$  for all  $a, b \in A$ . Also, suppose  $g(A)$  is closed. Then,  $f$  and  $g$  have a coincidence point. Further, if  $f, g$  commute at their coincidence points, then  $f$  and  $g$  have a common fixed point.

By taking  $g = I$  in Corollary 4.5.1, we obtain immediately the following fixed point theorem of Karapinar and Samet: [100]:

**Corollary 4.5.2.** Let  $f : A \rightarrow A$  be a given mapping of a complete metric space  $(A, d)$ . Suppose that there exists a function  $\psi \in \Psi$  satisfying

$$d(fa, fb) \leq \psi(M(a, b)), \quad (4.5.2)$$

where  $M(a, b) = \max \left\{ d(a, b), \frac{d(a, Ta) + d(b, Tb)}{2}, \frac{d(a, Tb) + d(b, Ta)}{2} \right\}$  for all  $a, b \in A$ . Then  $f$  has a unique fixed point.

By taking  $M(ga, gb) = d(ga, gb)$  in Corollary 4.5.1, the following corollary has been obtained:

**Corollary 4.5.3.** Let  $(A, d)$  be a complete metric space and  $f, g : A \rightarrow A$  be such that  $f(A) \subseteq g(A)$ . Suppose that there exists a function  $\psi \in \Psi$  satisfying

$$d(fa, fb) \leq \psi(d(ga, gb)), \quad (4.5.3)$$

for all  $a, b \in A$ . Also, suppose  $g(A)$  is closed. Then,  $f$  and  $g$  have a coincidence point. Further, if  $f, g$  commute at their coincidence points, then  $f$  and  $g$  have a common fixed point.

By taking  $g = I$  in Corollary 4.5.3, we get the following corollary:

**Corollary 4.5.4.** (Berinde [31]). Let  $(A, d)$  be a complete metric space and  $f : A \rightarrow A$ . Suppose that there exists a function  $\psi \in \Psi$  satisfying

$$d(fa, fb) \leq \psi(d(a, b)), \quad (4.5.4)$$

for all  $a, b \in A$ . Then  $f$  has a unique fixed point.

**Corollary 4.5.5.** (Ćirić [51]). Let  $f$  be a given self-mapping of a complete metric space  $(A, d)$ . Suppose that there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq \lambda \max \left\{ d(a, b), \frac{d(a, fa) + d(b, fb)}{2}, \frac{d(a, fb) + d(b, fa)}{2} \right\}, \quad (4.5.5)$$

for all  $a, b \in A$ . Then  $f$  has a unique fixed point.

**Corollary 4.5.6.** (Hardy and Rogers [75]). Let  $f$  be a given self-mapping of a complete metric space  $(A, d)$ . Suppose that there exists constants  $A, B, C \geq 0$  with  $(A + 2B + 2C) \in (0, 1)$  such that

$$d(fa, fb) \leq Ad(a, b) + B[d(a, fa) + d(b, fb)] + C[d(a, fb) + d(b, fa)], \quad (4.5.6)$$

for all  $a, b \in A$ . Then  $f$  has a unique fixed point.

By taking  $\psi(t) = \lambda t$ , where  $\lambda \in [0, 1)$  in Corollary 4.5.4, we get the following corollary:

**Corollary 4.5.7.** (Banach Contraction Principle [30]). Let  $f$  be a given self-mapping of a complete metric space  $(A, d)$ . Suppose that there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq \lambda d(a, b), \quad (4.5.7)$$

for all  $a, b \in A$ . Then  $f$  has a unique fixed point.

**Corollary 4.5.8.** (Kannan [91]) Let  $f$  be a given self-mapping of a complete metric space  $(A, d)$ . Suppose that there exists a constant  $\lambda \in (0, 1/2)$  satisfying

$$d(fa, fb) \leq \lambda[d(a, fa) + d(b, fb)], \quad (4.5.8)$$

for all  $a, b \in A$ . Then  $f$  has a unique fixed point.

**Corollary 4.5.9.** (Chatterjee [44]) Let  $f$  be a given self-mapping of a complete metric space  $(A, d)$ . Suppose that there exists a constant  $\lambda \in (0, 1/2)$  satisfying

$$d(fa, fb) \leq \lambda[d(a, fb) + d(b, fa)], \quad (4.5.9)$$

for all  $a, b \in A$ . Then  $f$  has a unique fixed point.

## 4.5.2 Fixed Point Theorems on Metric Spaces Endowed with a Partial Order

Recently, there have been enormous developments in the study of fixed point problems of contractive mappings in metric spaces endowed with a partial order. The first result in this direction was given by Turinici [186], where the Banach contraction principle was extended in partially ordered sets. Some applications of Turinici's theorem to matrix equations were presented by Ran and Reurings [142]. Many useful results have been obtained by Bhaskar and Lakshmikantham [36], Nieto and Lopez [128, 129], Agarwal *et al.* [11], Lakshmikantham and Ćirić [107] and Samet [162] concerning the existence of fixed points for contraction type mappings in partially ordered metric spaces. In this section, various fixed point results have been derived in partially ordered metric spaces. For proving these results, the following concepts have been utilized:

**Definition 4.5.1.** [100] Let  $(A, \preceq)$  be a partially ordered set and  $T$  be a given self-mapping on  $A$ . The mapping  $T$  is said to be nondecreasing with respect to  $\preceq$  if

$$a, b \in A, a \preceq b \Rightarrow Ta \preceq Tb.$$

**Definition 4.5.2.** [100] Let  $(A, \preceq)$  be a partially ordered set. A sequence  $\{a_n\} \subset A$  is said to be nondecreasing with respect to  $\preceq$  if  $a_n \preceq a_{n+1}$  for all  $n$ .

**Definition 4.5.3.** [100] Let  $(A, d, \preceq)$  be a partially ordered metric space. The space  $(A, d, \preceq)$  is said to be regular space if for every nondecreasing sequence  $\{a_n\} \subset A$  such that  $a_n \rightarrow t \in A$  as  $n \rightarrow \infty$ , there exists a subsequence  $\{a_{n(k)}\}$  of  $\{a_n\}$  such that  $a_{n(k)} \preceq t$  for all  $k$ .

**Definition 4.5.4.** [55] Suppose  $(A, \preceq)$  is a partially ordered set and  $F, g : A \rightarrow A$  are mappings. The mapping  $F$  is said to be  $g$ -nondecreasing if for  $a, b \in A$ ,

$$ga \preceq gb \quad \text{implies} \quad Fa \preceq Fb. \quad (4.5.10)$$

**Definition 4.5.5.** Let  $(A, d, \preceq)$  be a partially ordered metric space and  $g$  be a self-mapping on  $A$ . The space  $(A, d, \preceq)$  is said to be  $g$ -regular if for every nondecreasing sequence  $\{ga_n\} \subset A$  such that  $ga_n \rightarrow gz \in A$  as  $n \rightarrow \infty$ , there exists a subsequence  $\{ga_{n(k)}\}$  of  $\{ga_n\}$  such that  $ga_{n(k)} \preceq gz$  for every  $k$ .

The main result of this subsection is the following corollary:

**Corollary 4.5.10.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space. Suppose that  $f, g : A \rightarrow A$  with  $f(A) \subseteq g(A)$  and  $f$  be a  $g$ -nondecreasing mapping w.r.t  $\preceq$ . Suppose there exists a function  $\psi \in \Psi$  such that for all  $a, b \in A$  with  $ga \preceq gb$ , we have

$$d(fa, fb) \leq \psi(M(ga, gb)), \quad (4.5.11)$$

where  $M(ga, gb) = \max \left\{ d(ga, gb), \frac{d(ga, fa) + d(gb, fb)}{2}, \frac{d(ga, fb) + d(gb, fa)}{2} \right\}$ . Also,

the following conditions hold: (i) there exists  $a_0 \in A$  s.t  $ga_0 \preceq fa_0$ ,

(ii) the space  $(A, d, \preceq)$  is  $g$ -regular,

(iii)  $g(A)$  is closed.

Then,  $f$  and  $g$  have a coincidence point. Moreover, if for every pair  $(a, b) \in C(g, f) \times C(g, f)$  there exists  $z \in A$  such that  $ga \preceq gz$  and  $gb \preceq gz$ , and if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point.

*Proof.* Consider the mapping  $\alpha : A \times A \rightarrow [0, \infty)$  given by

$$\alpha(a, b) = \begin{cases} 1 & \text{if } a \preceq b \text{ or } a \succeq b, \\ 0 & \text{otherwise.} \end{cases} \quad (4.5.12)$$

Clearly,

$$\alpha(ga, gb)d(fa, fb) \leq \psi(M(ga, gb)),$$

for every  $a, b \in A$ . The condition (i) implies that  $\alpha(ga_0, fa_0) \geq 1$ . Also, the  $g$ -monotone property of  $f$  implies for all  $a, b \in A$ ,

$$\begin{aligned} \alpha(ga, gb) \geq 1 &\Rightarrow ga \preceq gb \text{ or } ga \succeq gb \\ &\Rightarrow fa \preceq fb \text{ or } fa \succeq fb \\ &\Rightarrow \alpha(fa, fb) \geq 1, \end{aligned} \quad (4.5.13)$$

which amounts to say that  $f$  is  $\alpha$ -admissible w.r.t  $g$ . Now, let  $\{ga_n\}$  be a sequence in  $A$  such that  $\alpha(ga_n, ga_{n+1}) \geq 1$ , for all  $n$  and  $ga_n \rightarrow gz \in A$  as  $n \rightarrow \infty$ . Due to the  $g$ -regularity hypothesis, there exists a subsequence  $\{ga_{n(k)}\}$  of  $\{ga_n\}$  such that  $ga_{n(k)} \preceq gz$  for all  $k$ . So, the definition of  $\alpha$  proves that  $\alpha(ga_{n(k)}, gz) \geq 1$ . All the conditions of Theorem 4.4.1 are satisfied thereby implying that  $f$  and  $g$  have a coincidence point  $z$ .

Now, we need to show the existence of a unique common fixed point. For this, let  $a, b \in A$ . The given condition implies the existence of a point  $z \in A$  such that  $ga \preceq gz$  and  $gb \preceq gz$ , which further implies from the definition of  $\alpha$  that  $\alpha(ga, gz) \geq 1$  and  $\alpha(gb, gz) \geq 1$ . Thus, we deduce a unique common fixed point by Theorem 4.4.2.  $\square$

The following results can be obtained from Corollary 4.5.10.

**Corollary 4.5.11.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $f, g$  are given self-mappings of  $A$ . Suppose that the mapping  $f$  be a  $g$ -nondecreasing mapping w.r.t

$\preceq$  and there exists a function  $\psi \in \Psi$  satisfying

$$d(fa, fb) \leq \psi(d(ga, gb)),$$

for every  $a, b \in A$  such that  $ga \preceq gb$ . Also, the following conditions hold:

- (i) there exists  $a_0 \in A$  s.t  $ga_0 \preceq fa_0$ ,
- (ii) the space  $(A, d, \preceq)$  is  $g$ -regular,
- (iii)  $g(A)$  is closed.

Then,  $f$  and  $g$  have a coincidence point. Moreover, if for every pair  $(a, b) \in C(g, f) \times C(g, f)$  there exists  $c \in A$  such that  $ga \preceq gc$  and  $gb \preceq gc$ , and if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point.

**Corollary 4.5.12.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $f, g$  are given self-mappings of  $A$ . Suppose that the mapping  $f$  be a  $g$ -nondecreasing mapping w.r.t  $\preceq$  and there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq \lambda \max \left\{ d(ga, gb), \frac{d(ga, fa) + d(gb, fb)}{2}, \frac{d(ga, fb) + d(gb, fa)}{2} \right\},$$

for every  $a, b \in A$  such that  $ga \preceq gb$ . Also, the following conditions hold:

- (i) there exists  $a_0 \in A$  s.t  $ga_0 \preceq fa_0$ ,
- (ii) the space  $(A, d, \preceq)$  is  $g$ -regular,
- (iii)  $g(A)$  is closed.

Then,  $f$  and  $g$  have a coincidence point. Moreover, if for every pair  $(a, b) \in C(g, f) \times C(g, f)$  there exists  $c \in A$  such that  $ga \preceq gc$  and  $gb \preceq gc$ , and if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point.

**Corollary 4.5.13.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $f, g$  are given self-mappings of  $A$ . Suppose that the mapping  $f$  be a  $g$ -nondecreasing mapping w.r.t

$\preceq$  and there exists constants  $A, B, C \geq 0$  with  $(A + 2B + 2C) \in [0, 1)$  satisfying

$$d(fa, fb) \leq Ad(ga, gb) + B[d(ga, fa) + d(gb, fb)] + C[d(ga, fb) + d(gb, fa)],$$

for every  $a, b \in A$  such that  $ga \preceq gb$ . Also, the following conditions hold:

- (i) there exists  $a_0 \in A$  s.t  $ga_0 \preceq fa_0$ ,
- (ii) the space  $(A, d, \preceq)$  is  $g$ -regular,
- (iii)  $g(A)$  is closed.

Then,  $f$  and  $g$  have a coincidence point. Moreover, if for every pair  $(a, b) \in C(g, f) \times C(g, f)$  there exists  $c \in A$  such that  $ga \preceq gc$  and  $gb \preceq gc$ , and if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point.

**Corollary 4.5.14.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space. Assume that  $f, g : A \rightarrow A$  and  $f$  be a  $g$ -nondecreasing mapping w.r.t  $\preceq$ . Suppose there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq \lambda(d(ga, gb)), \tag{4.5.14}$$

for every  $a, b \in A$  such that  $ga \preceq gb$  and also the following conditions hold:

- (i) there exists  $a_0 \in A$  s.t  $ga_0 \preceq fa_0$ ,
- (ii) the space  $(A, d, \preceq)$  is  $g$ -regular,
- (iii)  $g(A)$  is closed.

Then,  $f$  and  $g$  have a coincidence point. Moreover, if for every pair  $(a, b) \in C(g, f) \times C(g, f)$  there exists  $c \in A$  such that  $ga \preceq gc$  and  $gb \preceq gc$ , and if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point.

**Corollary 4.5.15.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $f, g$  are given self-mappings of  $A$ . Suppose that the mapping  $f$  be a  $g$ -nondecreasing mapping w.r.t

$\preceq$  and there exists constants  $A, B, C \geq 0$  with  $(A + 2B + 2C) \in (0, 1)$  satisfying

$$d(fa, fb) \leq \lambda[d(ga, fa) + d(gb, fb)],$$

for every  $a, b \in A$  such that  $ga \preceq gb$ . Also, the following conditions hold:

- (i) there exists  $a_0 \in A$  s.t  $ga_0 \preceq fa_0$ ,
- (ii) the space  $(A, \preceq, d)$  is  $g$ -regular,
- (iii)  $g(A)$  is closed.

Then,  $f$  and  $g$  have a coincidence point. Moreover, if for every pair  $(a, b) \in C(g, f) \times C(g, f)$  there exists  $c \in A$  such that  $ga \preceq gc$  and  $gb \preceq gc$ , and if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point.

**Corollary 4.5.16.** Let  $(A, d, \preceq)$  be a complete partially ordered metric space and  $f, g$  are given self-mappings of  $A$ . Suppose that  $f$  be a  $g$ -nondecreasing mapping w.r.t  $\preceq$  and there exists constants  $A, B, C \geq 0$  with  $(A + 2B + 2C) \in (0, 1)$  such that

$$d(fa, fb) \leq \lambda[d(ga, fb) + d(gb, fa)],$$

for every  $a, b \in A$  such that  $ga \preceq gb$ . Also, the following conditions hold:

- (i) there exists  $a_0 \in A$  s.t  $ga_0 \preceq fa_0$ ,
- (ii) the space  $(A, \preceq, d)$  is  $g$ -regular,
- (iii)  $g(A)$  is closed.

Then,  $f$  and  $g$  have a coincidence point. Moreover, if for every pair  $(a, b) \in C(g, f) \times C(g, f)$  there exists  $c \in A$  such that  $ga \preceq gc$  and  $gb \preceq gc$ , and if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point.

**Remark 4.5.1.** Letting  $g = I_A$  in Corollaries (4.5.11 - 4.5.16), we get the corresponding Corollaries (3.12 - 3.17) of Karapinar and Samet [100].

### 4.5.3 Fixed Point Theorems for Cyclic Contractive Mappings

As a generalization of the Banach contraction mapping principle, Kirk *et al.* [104] in 2003 introduced cyclic representations and cyclic contractions. A mapping  $T : A \cup B \rightarrow A \cup B$  is called a cyclic mapping if  $T(A) \subseteq B$  and  $T(B) \subseteq A$ , where  $A, B$  are nonempty subsets of a metric space  $(X, d)$ . The mapping  $T$  is called a cyclic contraction if there exists  $k \in (0, 1)$  such that  $d(Ta, Ty) \leq kd(a, y)$  for all  $a \in A$  and  $y \in B$ . A contraction mapping is continuous but the cyclic contractions need not be. This is one of the importance of the results of Kirk *et al.* [104]. In the last decade, several authors ([10, 95, 99, 134, 139, 158]) have used the cyclic representations and cyclic contractions to obtain various fixed point results.

The main result is the following theorem:

**Corollary 4.5.17.** Let  $(A, d)$  be a complete metric space,  $A_1$  and  $A_2$  are two nonempty closed subsets of  $A$  and  $f, g : Y \rightarrow Y$  be two mappings, where  $Y = A_1 \cup A_2$ . Assume that the following conditions hold:

- (i)  $g(A_1)$  and  $g(A_2)$  are closed,
- (ii)  $f(A_1) \subseteq g(A_2)$  and  $f(A_2) \subseteq g(A_1)$ ,
- (iii)  $g$  is one-to-one,
- (iv) there exists a function  $\psi \in \Psi$  satisfying

$$d(fa, fb) \leq \psi(M(ga, gb)), \quad \forall (a, b) \in A_1 \times A_2, \quad (4.5.15)$$

where  $M(ga, gb) = \max \left\{ d(ga, gb), \frac{d(ga, fa) + d(gb, fb)}{2}, \frac{d(ga, fb) + d(gb, fa)}{2} \right\}$  for all  $a, b \in A$ . Then,  $f$  and  $g$  have a coincidence point  $z \in A_1 \cap A_2$ . Further, if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point that belongs to  $A_1 \cap A_2$ .

*Proof.* Due to the fact that  $g$  is one-to-one, condition (iv) is equivalent to

$$d(fa, fb) \leq \psi(M(ga, gb)), \forall (ga, gb) \in g(A_1) \times g(A_2). \quad (4.5.16)$$

Now, the space  $(Y, d)$  is complete due to the fact that  $A_1$  and  $A_2$  are closed subsets of the complete metric space  $(A, d)$ .

Consider the mapping  $\alpha : Y \times Y \rightarrow [0, \infty)$  given by

$$\alpha(a, b) = \begin{cases} 1 & \text{if } (a, b) \in (g(A_1) \times g(A_2)) \cup (g(A_2) \times g(A_1)) \\ 0 & \text{otherwise} \end{cases} \quad (4.5.17)$$

In view of definition of  $\alpha$  and condition (iv), we can write

$$\alpha(ga, gb)d(fa, fb) \leq \psi(M(ga, gb)) \quad (4.5.18)$$

for all  $ga \in g(A_1)$  and  $gb \in g(A_2)$ . Thus, the pair  $(f, g)$  is a generalized  $\alpha$ - $\psi$  contractive pair of mappings.

By using condition (ii),  $f(Y) \subseteq g(Y)$ . Moreover,  $g(Y)$  is closed also.

Next, we proceed to show that  $f$  is  $\alpha$ -admissible w.r.t  $g$ . Let  $(ga, gb) \in Y \times Y$  such that  $\alpha(ga, gb) \geq 1$ , that is,

$$(ga, gb) \in (g(A_1) \times g(A_2)) \cup (g(A_2) \times g(A_1)) \quad (4.5.19)$$

Since  $g$  is one-to-one, therefore (4.5.19) implies that

$$(a, b) \in (A_1 \times A_2) \cup (A_2 \times A_1) \quad (4.5.20)$$

So, from condition (ii), we have

$$(fa, fb) \in (g(A_2) \times g(A_1)) \cup (g(A_1) \times g(A_2)) \quad (4.5.21)$$

that is,  $\alpha(fa, fb) \geq 1$ . This implies that  $f$  is  $\alpha$ -admissible w.r.t  $g$ .

Now, let  $\{ga_n\}$  be a sequence in  $A$  such that  $\alpha(ga_n, ga_{n+1}) \geq 1$  for all  $n$  and  $ga_n \rightarrow gz \in$

$g(A)$  as  $n \rightarrow \infty$ . From the definition of  $\alpha$ , we get

$$(ga_n, ga_{n+1}) \in (gA_1 \times gA_2) \cup (gA_2 \times gA_1) \quad (4.5.22)$$

Since  $(gA_1 \times gA_2) \cup (gA_2 \times gA_1)$  is a closed set in the Euclidean metric, so

$$(gz, gz) \in (gA_1 \times gA_2) \cup (gA_2 \times gA_1), \quad (4.5.23)$$

thereby implying that  $gz \in g(A_1) \cap g(A_2)$ . Therefore, the definition of  $\alpha$  implies that  $\alpha(ga_n, gz) \geq 1$  for all  $n$ .

Now, let  $a$  be an arbitrary point in  $A_1$ . We need to show that  $\alpha(ga, fa) \geq 1$ . Indeed, from condition (ii), we have  $fa \in g(A_2)$ . Since  $ga \in g(A_1)$ , so we get  $(ga, fa) \in g(A_1) \times g(A_2)$ , which in turn implies that  $\alpha(ga, fa) \geq 1$ .

Thus, all the conditions of Theorem 4.4.1 are fulfilled. Hence,  $f$  and  $g$  have a coincidence point  $z \in A_1 \cup A_2$ , that is,  $fz = gz$ . If  $z \in A_1$ , from (ii),  $fz \in g(A_2)$ . On the other hand,  $fz = gz \in g(A_1)$ . Thus, we have  $gz \in g(A_1) \cap g(A_2)$ , which implies from the one-to-one property of  $g$  that  $z \in A_1 \cap A_2$ . Similarly, if  $z \in A_2$ , we obtain that  $z \in A_1 \cap A_2$ .

Notice that, if  $a$  is a coincidence point of  $f$  and  $g$ , then  $a \in A_1 \cap A_2$ . Finally, let  $a, b \in C(g, f)$ , that is,  $a, b \in A_1 \cap A_2$ ,  $ga = fa$  and  $gb = fb$ . So, for any  $z \in Y$ , we have  $g(z) \in g(A_1) \cup g(A_2)$  thereby implying that  $\alpha(ga, gz) \geq 1$  and  $\alpha(gb, gz) \geq 1$ . All the conditions of Theorem 4.4.2 are fulfilled thereby implying that  $z = A_1 \cap A_2$  is the unique common fixed point of  $f$  and  $g$ .  $\square$

The following results can be easily derived from Corollary 4.5.17.

**Corollary 4.5.18.** Let  $(A, d)$  be a complete metric space. Suppose  $A_1$  and  $A_2$  are two nonempty closed subsets of  $A$  and  $f, g : Y \rightarrow Y$  be two mappings, where  $Y = A_1 \cup A_2$ . Assume that the following conditions hold:

- (i)  $g(A_1)$  and  $g(A_2)$  are closed,
- (ii)  $f(A_1) \subseteq g(A_2)$  and  $f(A_2) \subseteq g(A_1)$ ,
- (iii)  $g$  is one-to-one,
- (iv) there exists a function  $\psi \in \Psi$  such that

$$d(fa, fb) \leq \psi(d(ga, gb)), \forall (a, b) \in A_1 \times A_2.$$

Then,  $f$  and  $g$  possess a coincidence point  $c \in A_1 \cap A_2$ . Further, if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point belonging to  $A_1 \cap A_2$ .

**Corollary 4.5.19.** Let  $(A, d)$  be a complete metric space. Suppose  $A_1$  and  $A_2$  are two nonempty closed subsets of  $A$  and  $f, g : Y \rightarrow Y$  be two mappings, where  $Y = A_1 \cup A_2$ . Assume that the following conditions hold:

- (i)  $g(A_1)$  and  $g(A_2)$  are closed,
- (ii)  $f(A_1) \subseteq g(A_2)$  and  $f(A_2) \subseteq g(A_1)$ ,
- (iii)  $g$  is one-to-one,
- (iv) there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq \lambda \max \left\{ d(ga, gb), \frac{d(ga, fa) + d(gb, fb)}{2}, \frac{d(ga, fb) + d(gb, fa)}{2} \right\} \forall (a, b) \in A_1 \times A_2.$$

Then,  $f$  and  $g$  possess a coincidence point  $c \in A_1 \cap A_2$ . Further, if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point belonging to  $A_1 \cap A_2$ .

**Corollary 4.5.20.** Let  $(A, d)$  be a complete metric space. Suppose  $A_1$  and  $A_2$  are two nonempty closed subsets of  $A$  and  $f, g$  be two self-mappings on  $Y$ , where  $Y = A_1 \cup A_2$ . Assume that the following conditions hold:

- (i)  $g(A_1)$  and  $g(A_2)$  are closed,
- (ii)  $f(A_1) \subseteq g(A_2)$  and  $f(A_2) \subseteq g(A_1)$ ,

(iii)  $g$  is one-to-one,

(iv) there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq Ad(ga, gb) + B[d(ga, fa) + d(gb, fb)] + C[d(ga, fb) + d(gb, fa)], \forall (a, b) \in A_1 \times A_2.$$

Then,  $f$  and  $g$  possess a coincidence point  $c \in A_1 \cap A_2$ . Further, if  $f, g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point belonging to  $A_1 \cap A_2$ .

**Corollary 4.5.21.** Let  $(A, d)$  be a complete metric space. Suppose  $A_1$  and  $A_2$  are two nonempty closed subsets of  $A$  and  $f, g$  two self-mappings on  $Y$ , where  $Y = A_1 \cup A_2$ .

Assume that the following conditions hold:

(i)  $g(A_1)$  and  $g(A_2)$  are closed,

(ii)  $f(A_1) \subseteq g(A_2)$  and  $f(A_2) \subseteq g(A_1)$ ,

(iii)  $g$  is one-to-one,

(iv) there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq \lambda(d(ga, gb)), \forall (a, b) \in A_1 \times A_2.$$

Then,  $f$  and  $g$  possess a coincidence point  $c \in A_1 \cap A_2$ . Further, if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point belonging to  $A_1 \cap A_2$ .

**Corollary 4.5.22.** Let  $(A, d)$  be a complete metric space. Suppose  $A_1$  and  $A_2$  are two nonempty closed subsets of  $A$  and  $f, g$  be two self-mappings on  $Y$ , where  $Y = A_1 \cup A_2$ .

Assume that the following conditions hold:

(i)  $g(A_1)$  and  $g(A_2)$  are closed,

(ii)  $f(A_1) \subseteq g(A_2)$  and  $f(A_2) \subseteq g(A_1)$ ,

(iii)  $g$  is one-to-one,

(iv) there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq \lambda[d(ga, fa) + d(gb, fb)], \forall (a, b) \in A_1 \times A_2.$$

Then,  $f$  and  $g$  possess a coincidence point  $c \in A_1 \cap A_2$ . Further, if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point belonging to  $A_1 \cap A_2$ .

**Corollary 4.5.23.** Let  $(A, d)$  be a complete metric space. Suppose  $A_1$  and  $A_2$  are two nonempty closed subsets of  $A$  and  $f, g$  be two self-mappings on  $Y$ , where  $Y = A_1 \cup A_2$ . Assume that the following conditions hold:

- (i)  $g(A_1)$  and  $g(A_2)$  are closed,
- (ii)  $f(A_1) \subseteq g(A_2)$  and  $f(A_2) \subseteq g(A_1)$ ,
- (iii)  $g$  is one-to-one,
- (iv) there exists a constant  $\lambda \in [0, 1)$  satisfying

$$d(fa, fb) \leq \lambda[d(ga, fb) + d(gb, fa)], \forall (a, b) \in A_1 \times A_2.$$

Then,  $f$  and  $g$  possess a coincidence point  $c \in A_1 \cap A_2$ . Further, if  $f$  and  $g$  commute at their coincidence points, then  $f$  and  $g$  have a unique common fixed point belonging to  $A_1 \cap A_2$ .

**Remark 4.5.2.** Letting  $g = I_A$  in Corollaries (4.5.18- 4.5.23), we obtain Corollaries (3.19- 3.24) of Karapinar and Samet [100].



# Chapter 5

## SOME COMMON FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS IN G-METRIC SPACES

### 5.1 Introduction

Fixed point theory is a very important area in mathematics and has wide range of applications in various fields. The impact of metric spaces to mathematics in general and to functional analysis in particular has inspired several authors to generalize and extend the notion of metric spaces to various other abstract spaces. In 1963, Gähler [70, 71] investigated the concept of 2-metric space and claimed that a 2-metric is a generalization of the usual notion of a metric. In 1992, Dhage [62] extended the concept of metric spaces by introducing D-metric spaces. Later, Dhage ([61], [63], [64], [65]) developed topological structures in D-metric spaces and presented several fixed point results in these spaces. Motivated by the work of Dhage, some other authors presented various fixed point results in

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D-metric spaces (see, for example, [13], [143], [150], [188] etc).

But, in 2003, Mustafa and Sims [120] demonstrated that most of the claims concerning the fundamental topological properties of D-metric spaces were incorrect. These considerations led researchers to seek a more appropriate notion of generalized metric space. To overcome fundamental flaws in Dhage's theory of generalized metric spaces, Mustafa and Sims [121] introduced an alternatively more robust generalization of metric spaces known as G-metric spaces and proved some results related to these spaces.

In 2007, Huang and Zhang [77] generalized metric spaces in the form of cone metric spaces. Some fixed point theorems have been proved by Huang and Zhang [77] in cone metric spaces showing that metric spaces really do not provide enough space for the fixed point theory.

In 1984, Wang *et al.* [192] initiated the study of metric fixed point theory for expansion mappings. Khan *et al.* [102] in 1986 further generalized the result of Wang *et al.* [192]. Also, Rhoades [148] and Taniguchi [185] generalized the results of Wang [192] for pair of mappings. Kang [88] generalized the result of Khan *et al.* [102], Rhoades [148] and Taniguchi [185] for expansion mappings. In 2009, Ahmed [17] proved a common fixed point result for expansive mappings in 2-metric spaces by using the concept of compatibility of type(A) and thereby generalized the result of Kang *et al.* [89]. Later in 2010, Şahin and Telci [161] obtained a common fixed point theorem for expansive mappings in complete cone metric spaces which generalizes the result of Wang *et al.* [192] for a pair of mappings.

The aim of this chapter is to generalize the results of Ahmed [17] to  $G$ -metric spaces by removing the condition of sequential continuity of the mappings. In order to prove the results, a more generalized concept of weak compatibility in  $G$ -metric spaces have been

used instead of compatibility of type(A) used by Ahmed [17] in 2-metric spaces. Also, we extend the results of Şahin and Telci [161] to  $G$ -metric spaces thereby extending the theorem of Wang *et al.* [192] for a pair of mappings to  $G$ -metric spaces.

The contents of this chapter have been divided into two sections. Section 4.2 deals with the preliminaries related to this chapter. In Section 4.3, some common fixed point theorems have been proved for expansive mappings in  $G$ -metric spaces.

## 5.2 Preliminaries

Before presenting the main results, some of the definitions and results related to this chapter have been presented.

In 1963, Gähler [70, 71] gave the following concept of 2-metric spaces:

**Definition 5.2.1.** Let  $A$  be a nonempty set. A function  $d : A \times A \times A \rightarrow \mathbb{R}^+$  satisfying the following properties:

- (i) For distinct points  $a, b \in A$ , there is  $c \in A$ , such that  $d(a, b, c) \neq 0$ ,
- (ii)  $d(a, b, c) = 0$  if two of the triple  $a, b, c \in A$  are equal,
- (iii)  $d(a, b, c) = d(a, c, b) = \dots$  (symmetry in all three variables),
- (iv)  $d(a, b, c) \leq d(a, b, u) + d(a, u, c) + d(u, b, c)$  for all  $a, b, c, u \in A$ .

is said to be a 2-metric on  $A$ . The set  $A$  equipped with a 2-metric is called a 2-metric space.

Geometrically,  $d(a, b, c)$  demonstrates the area of the triangle with vertices  $a, b$  and  $c$ .

In 2006, Mustafa and Sims [121] introduced the following notion of  $G$ -metric spaces which is a generalization of 2-metric spaces:

**Definition 5.2.2.** Suppose  $A$  be a nonempty set and  $G : A \times A \times A \rightarrow \mathbb{R}^+$  be a function satisfying the following properties:

- (i)  $G(a, b, c) = 0$  if  $a = b = c$ ,
- (ii)  $0 < G(a, b, b)$  for all  $a, b \in A$  with  $a \neq b$ ,
- (iii)  $G(a, a, b) \leq G(a, b, c)$  for all  $a, b, c \in A$  with  $c \neq b$ ,
- (iv)  $G(a, b, c) = G(a, c, b) = G(b, c, a) = \dots$  (symmetry in all three variables),
- (v)  $G(a, b, c) \leq G(a, u, u) + G(u, b, c)$  for all  $a, b, c, u \in A$  (rectangle inequality).

Then the function  $G$  is called a  $G$ -metric on  $A$ , and the pair  $(A, G)$  is called a  $G$ -metric space.

It is known from [121] that the function  $G(a, b, c)$  on a  $G$ -metric space  $A$  is jointly continuous in all three of its variables, and  $G(a, b, c) = 0$  if and only if  $a = b = c$ ; for more details, see [121] and the references therein.

Mustafa and Sims [121] gave the following definitions:

**Definition 5.2.3.** Let  $(A, G)$  be a  $G$ -metric space and  $\{a_n\}$  be a sequence in  $A$ . A point  $a \in A$  is said to be the limit of the sequence  $\{a_n\}$  if

$$\lim_{n, m \rightarrow \infty} G(a, a_n, a_m) = 0,$$

then the sequence  $\{a_n\}$  is said to be  $G$ -convergent to  $a$ . Thus, if  $\{a_n\} \rightarrow a$  in a  $G$ -metric space  $(A, G)$ , then for any  $\epsilon > 0$ , there exists a positive integer  $N$  such that  $G(a, a_n, a_m) < \epsilon$ , for every  $n, m \geq N$ .

Following [121],  $G$ -metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to one point.

**Definition 5.2.4.** Let  $(A, G)$  be a  $G$ -metric space. A sequence  $\{a_n\}$  is said to be  $G$ -Cauchy sequence if for every  $\epsilon > 0$ , there is a positive integer  $N$  such that  $G(a_n, a_m, a_l) < \epsilon$ , for every  $n, m, l \geq N$ , that is, if  $G(a_n, a_m, a_l) \rightarrow 0$ , as  $n, m, l \rightarrow \infty$ .

**Definition 5.2.5.** A  $G$ -metric space  $(A, G)$  is said to be  $G$ -complete (or complete  $G$ -metric space) if every  $G$ -Cauchy sequence in  $(A, G)$  is convergent in  $A$ .

**Definition 5.2.6.** Suppose  $(A, G)$  and  $(A', G')$  be two  $G$ -metric spaces. A function  $f : A \rightarrow A'$  is said to be  $G$ -continuous at a point  $a \in A$  if and only if it is  $G$  sequentially continuous at  $a$ , that is, whenever  $\{a_n\}$  is  $G$ -convergent to  $a$ ,  $\{f(a_n)\}$  is  $G$ -convergent to  $f(a)$ .

**Definition 5.2.7.** A  $G$ -metric space  $(A, G)$  is called symmetric  $G$ -metric space if  $G(a, b, b) = G(b, a, a)$  for every  $a, b \in A$ .

Below, we record some useful lemmas due to Mustafa and Sims [121] which are relevant to our results.

**Lemma 5.2.1.** Let  $(A, G)$  be a  $G$ -metric space. The following statements are equivalent:

- (i)  $\{a_n\}$  is  $G$ -convergent to  $a$ .
- (ii)  $G(a_n, a_n, a) \rightarrow 0$ , as  $n \rightarrow \infty$ .
- (iii)  $G(a_n, a, a) \rightarrow 0$ , as  $n \rightarrow \infty$ .
- (iv)  $G(a_m, a_n, a) \rightarrow 0$ , as  $m, n \rightarrow \infty$ .

**Lemma 5.2.2.** If  $(A, G)$  is a  $G$ -metric space, then  $G(a, b, b) \leq 2G(b, a, a)$  for every  $a, b \in A$ .

**Lemma 5.2.3.** The following statements are equivalent in a  $G$ -metric space  $(A, G)$ :

- (i) The sequence  $\{a_n\}$  is  $G$ -Cauchy.
- (ii) For any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $G(a_n, a_m, a_m) < \epsilon$ , for every  $m, n \geq N$ .

Jungck [85] generalized the notion of weakly commuting mappings by presenting the following concept of compatible mappings:

**Definition 5.2.8.** Two self mappings  $f$  and  $g$  of a metric space  $(A, d)$  are said to be compatible if  $\lim_{n \rightarrow \infty} d(fga_n, gfa_n) = 0$ , whenever  $a_n$  is a sequence in  $A$  such that  $\lim_{n \rightarrow \infty} fa_n = \lim_{n \rightarrow \infty} ga_n = t$  for some  $t \in A$ .

Further, Jungck [86] in 1993 defined the following concept of compatible mappings of type (A) in metric spaces:

**Definition 5.2.9.** Two self mappings  $f$  and  $g$  of a metric space  $(A, d)$  are said to be compatible of type (A) if

$$\lim_{n \rightarrow \infty} d(fga_n, gga_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(gfa_n, ffa_n) = 0,$$

whenever  $a_n$  is a sequence in  $A$  such that  $\lim_{n \rightarrow \infty} fa_n = \lim_{n \rightarrow \infty} ga_n = t$  for some  $t \in A$ .

Subsequently, Cho [46] extended the concept of compatible mappings of type (A) in the framework of 2-metric spaces as follows:

**Definition 5.2.10.** Let  $S$  and  $T$  be self-mappings of a 2-metric space  $(A, d)$ . The mappings  $S$  and  $T$  are said to be compatible of type (A) if

$$\lim_{n \rightarrow \infty} d(TSa_n, SSa_n, u) = 0 \text{ and } \lim_{n \rightarrow \infty} d(STa_n, TTa_n, u) = 0 \text{ for all } u \in A,$$

whenever  $\{a_n\}$  is a sequence in  $A$  such that  $\lim_{n \rightarrow \infty} Sa_n = \lim_{n \rightarrow \infty} Ta_n = t$  for some  $t \in A$ .

In 1998, Jungck and Rhoades [87] introduced the notion of weak compatibility in metric spaces as follows:

**Definition 5.2.11.** Two self mappings  $f$  and  $g$  of a metric space  $(A, d)$  are said to be weakly compatible if they commute at coincidence points.

The concept of weak compatibility is more general than the concept of compatibility of type (A) as shown in the following lemma:

**Lemma 5.2.4.** [86] Let  $(A, d)$  be a metric space and  $F, G : A \rightarrow A$  be compatible mappings of type(A). If  $a \in A$  is a coincidence point of the mappings  $F$  and  $G$ , then  $FGa = GFa$ .

In 2011, Abbas *et al.* [6] gave the following concept of compatible mappings in  $G$ -metric spaces:

**Definition 5.2.12.** Let  $f$  and  $g$  be two self mappings of a  $G$ -metric space  $(A, G)$ . The mappings  $f$  and  $g$  are said to be compatible if

$$G(fga_n, fga_n, gfa_n) = 0$$

whenever  $\{a_n\}$  is a sequence in  $A$  such that  $\lim_{n \rightarrow \infty} fa_n = \lim_{n \rightarrow \infty} ga_n = z$  for some  $z \in A$ .

**Definition 5.2.13.** Two self mappings  $f$  and  $g$  of a  $G$ -metric space  $(A, d)$  are said to be weakly compatible if they commute at coincidence points.

Matkowski [110] considered the set  $\Phi$  of all functions  $\phi : [0, \infty) \rightarrow [0, \infty)$  satisfying the following conditions:

- (i)  $\phi$  is non-decreasing and upper-semicontinuous from the right at 0,
- (ii)  $\phi(t) < t$ , for each  $t > 0$ ,
- (iii)  $\phi(t) = 0 \Leftrightarrow t = 0$ .

The following lemma due to Matkowski [110] will be useful for the proof of the main results:

**Lemma 5.2.5.** [110] Let  $\phi : [0, \infty) \rightarrow [0, \infty)$  be a function satisfying the conditions (i) and (ii). Then  $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ , where  $\phi^n(t)$  denotes the composition of  $\phi(t)$  with  $n$ -times.

Ahmed [17] presented the following common fixed point theorem for expansive mappings in 2-metric spaces:

**Theorem 5.2.6.** Let  $F, G, S$  and  $T$  be mappings of a complete 2-metric space  $(A, d)$  into itself such that  $F$  and  $G$  are surjective. Suppose that one of the mappings  $F, G, S, T$  is sequentially continuous and the pairs  $\{F, S\}$  and  $\{G, T\}$  are compatible mappings of type (A). If there exists  $\phi \in \Phi$  such that the inequality

$$\phi(d(Fa, Gb, u)) \geq d(Sa, Tb, u),$$

holds, then the mappings  $F, G, S$  and  $T$  have a unique common fixed point.

Huang and Zhang [77] in 2007 generalized the concept of metric spaces by presenting the following concept of cone metric spaces:

**Definition 5.2.14.** Let  $A$  be a nonempty set. Suppose that  $E$  is a real Banach space equipped with partial ordering  $\leq$  with respect to the cone  $P \subseteq E$ . The mapping  $d : A \times A \rightarrow E$  satisfying the following properties:

- (i)  $0 \leq d(a, b)$  for all  $a, b \in A$  and  $d(a, b) = 0$  if and only if  $a = b$ ,
- (ii)  $d(a, b) = d(b, a)$  for all  $a, b \in A$ ,
- (iii)  $d(a, b) \leq d(a, c) + d(c, b)$  for all  $a, b, c \in A$ .

is said to be a cone metric on  $A$  and the space  $(A, d)$  is called a cone metric space.

Later in 2010, Şahin and Telci [161] established the following common fixed point result in complete cone metric spaces for expansive mappings:

**Theorem 5.2.7.** Let  $(A, d)$  be a complete cone metric space and  $P$  be a cone. Suppose  $f$  and  $g$  be surjective self-mappings of  $A$  satisfying the following inequalities:

$$\begin{aligned}d(gfa, fa) &\geq md(fa, a), \\d(fga, ga) &\geq nd(ga, a),\end{aligned}$$

for all  $a$  in  $A$ , where  $m, n > 1$ . If either  $f$  or  $g$  is continuous, then  $f$  and  $g$  have a common fixed point.

### 5.3 Some common fixed point theorems for expansive mappings in $G$ -metric spaces

Consider the self-mappings  $F, B, S$  and  $T$  of a  $G$ -metric space  $(A, G)$  satisfying the following conditions:

$$F \text{ and } B \text{ are surjective,} \tag{5.3.1}$$

$$\phi(G(Fa, Bb, Bb)) \geq G(Sa, Tb, Tb) \text{ for each } a, b \in A, \tag{5.3.2}$$

where  $\phi \in \Phi$ . Since,  $F$  and  $B$  are surjective, choose a point  $a_1$  in  $A$  for an arbitrary point  $a_0$  in  $A$  such that

$$Fa_1 = Ta_0 = b_0$$

For a point  $a_1$ , there exists a point  $a_2$  in  $A$  s.t  $Ba_2 = Sa_1 = b_1$ . Inductively, one can define a sequence  $\{b_n\}$  in  $A$  s.t

$$\begin{aligned} Fa_{2n+1} &= Ta_{2n} = b_{2n}, \\ Ba_{2n+2} &= Sa_{2n+1} = b_{2n+1}, \forall n \in \mathbb{N} \cup \{0\} \end{aligned} \quad (5.3.3)$$

where  $\mathbb{N}$  is the set of all positive integers.

We, now establish a common fixed point result in  $G$ -metric spaces for expansive mappings by using the concept of weak compatibility.

**Theorem 5.3.1.** Let  $F, B, S$  and  $T$  be self-mappings of a complete symmetric  $G$ -metric space  $(A, G)$  satisfying the condition (5.3.1). Suppose that the pairs  $\{F, S\}$  and  $\{B, T\}$  of mappings are weakly compatible. If there exists  $\phi \in \Phi$  satisfying the inequality (5.3.2), then  $F, B, S$  and  $T$  have a unique common fixed point.

*Proof.* First, we have to prove that the sequence  $\{b_n\}$  defined in equation (5.3.3) is a  $G$ -Cauchy sequence. Clearly,

$$G(b_n, b_{n+1}, b_{n+1}) \leq \phi^n(G(b_0, b_1, b_1)) \forall n$$

The use of Lemma 5.2.5 gives,

$$G(b_n, b_{n+1}, b_{n+1}) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.3.4)$$

Consider,

$$\begin{aligned} G(b_n, b_m, b_m) &\leq G(b_n, b_{n+1}, b_{n+1}) + G(b_{n+1}, b_{n+2}, b_{n+2}) + \dots + G(b_{m-1}, b_m, b_m) \\ &\leq \phi^n(G(b_0, b_1, b_1)) + \phi^{n+1}(G(b_0, b_1, b_1)) + \dots + \phi^{m-1}(G(b_0, b_1, b_1)) \end{aligned}$$

Using Lemma 5.2.5, we obtain  $G(b_n, b_m, b_m) \rightarrow 0$  as  $m, n \rightarrow \infty$ . Thus,  $\{b_n\}$  is a  $G$ -Cauchy sequence. The completeness of the space  $(A, G)$  implies that the sequence  $\{b_n\}$  and hence any subsequence of  $\{b_n\}$  converges to  $c \in A$ . So,  $\{Fa_n\}$ ,  $\{Ba_n\}$ ,  $\{Sa_n\}$  and  $\{Ta_n\}$  converges to  $c \in A$ . Due to the completeness of the space  $F(A)$ , there exists a point  $p \in A$  such that  $Ap = c$ .

Now, by using inequality (5.3.2), we get

$$\phi(G(Fp, Ba_{2n+2}, Ba_{2n+2})) \geq G(Sp, Ta_{2n+2}, Ta_{2n+2})$$

Letting  $n \rightarrow \infty$  implies that

$$\phi(G(Fp, c, c)) \geq G(Sp, c, c).$$

That is,  $0 = \phi(0) \geq G(Sp, c, c)$ . This proves that  $Sp = c$ . The weakly compatible property of the mappings  $F$  and  $S$  implies that  $FSp = SFp \Rightarrow Fc = Sc$ .

Since  $B(A)$  is also a complete  $G$ -metric space. So, there exists a point  $p_1 \in A$  such that  $Bp_1 = c$ .

Now, consider

$$\phi(G(Fa_{2n+1}, Bp_1, Bp_1)) \geq G(Sa_{2n+1}, Tp_1, Tp_1)$$

Letting  $n \rightarrow \infty$ , we get

$$\phi(G(c, Bp_1, Bp_1)) \geq G(c, Tp_1, Tp_1)$$

Recalling that  $Bp_1 = c$ , we obtain

$$0 = \phi(0) \geq G(c, Tp_1, Tp_1)$$

This implies that  $Tp_1 = c$ . Since  $B$  and  $T$  are weakly compatible mappings, therefore,  $BTp_1 = TBp_1 \Rightarrow Bc = Tc$ .

Now, it is required to show that  $Fc = c$  and  $Bc = c$ . Suppose that  $G(Fc, c, c) > 0$ . Using the inequality (5.3.2) and the fact that  $\phi(t) < t$  for all  $t > 0$ ,

$$G(Fc, c, c) > \phi(G(Fc, c, c)) \geq G(Sc, c, c) = G(Fc, c, c),$$

which is a contradiction. So,  $G(Fc, c, c) = 0$ . That is,  $Fc = Sc = c$ .

Similarly, suppose that  $G(Bc, c, c) > 0$ . Therefore, by using the fact that the  $G$ -metric space  $A$  is symmetric and  $Fc = Sc = c$ , we get

$$\begin{aligned} G(Bc, c, c) &> \phi(G(Bc, c, c)) \\ &= \phi(G(Fc, Bc, Bc)) \\ &\geq G(Sc, Tc, Tc) = G(Fc, Bc, Bc) = G(Bc, c, c), \end{aligned}$$

which is a contradiction. So,  $G(Bc, c, c) = 0$ . This proves that  $Bc = Tc = c$ . Thus,  $Fc = Bc = Sc = Tc = c$  thereby implying that  $c$  is the unique common fixed point of  $F, B, S$  and  $T$ .  $\square$

As a corollary of the previous theorem, we have the following results:

**Corollary 5.3.2.** Let  $F, B, S$  and  $T$  be mappings of a complete symmetric  $G$ -metric space  $(A, G)$  into itself satisfying the condition (5.3.1). Suppose that the pairs  $\{F, S\}$  and  $\{B, T\}$  of mappings are weakly compatible. Assume that there exists  $k > 1$  s.t

$$G(Fa, Bb, Bb) \geq kG(Sa, Tb, Tb) \tag{5.3.5}$$

for every  $a, b \in A$ . Then,  $F, B, S$  and  $T$  have a unique common fixed point.

*Proof.* Taking  $\phi(t) = \frac{t}{k}$ , where  $k > 1$  in Theorem 5.3.1, the proof of the Corollary 5.3.2 follows.  $\square$

**Corollary 5.3.3.** Let  $S$  and  $T$  be self-mappings of a complete symmetric  $G$ -metric space  $(A, G)$  satisfying the condition (5.3.1). If there exists  $\phi \in \Phi$  such that the inequality

$$\phi(G(Sa, Tb, Tb)) \geq G(a, b, b)$$

holds, then  $S$  and  $T$  have a unique common fixed point.

*Proof.* By taking  $S = T = I_X$  in Theorem 5.3.1, we obtain the proof of the Corollary 5.3.3. □

**Corollary 5.3.4.** Let  $S$  and  $T$  be self-mappings of a complete symmetric  $G$ -metric space  $(A, G)$  satisfying the condition (5.3.1). Assume that there exists  $k > 1$  s.t

$$G(Sa, Tb, Tb) \geq k(G(a, b, b)) \quad \forall a, b \in X$$

Then  $S$  and  $T$  have a unique common fixed point.

*Proof.* By taking  $\phi(t) = \frac{t}{k}$  where  $k > 1$  in Corollary 5.3.3, we obtain the proof of the Corollary 5.3.4. □

In what follows, we furnish an illustrative example which demonstrates Theorem 5.3.1.

**Example 5.3.1.** Consider the set  $A = \{(a, b) : a, b \in [0, 1]\}$  and  $G$ -metric on  $A$  given by

$$G(a, b, c) = \max\{d(a, b), d(a, c), d(b, c)\},$$

where  $d$  is the metric on  $A$  defined by  $d((a_1, b_1), (a_2, b_2)) = |a_1 - a_2| + |b_1 - b_2|$  for all  $(a_1, b_1), (a_2, b_2) \in A$ . Define  $F, B, S, T : A \rightarrow A$  by

$$F(a, b) = B(a, b) = (a, 0), \quad S(a, b) = T(a, b) = \left(a - \frac{1}{3}a^2, 0\right).$$

for each  $(a, b) \in A$ . Clearly,  $F$  and  $S$  are weakly compatible mappings. Consider

$$\phi(t) = \begin{cases} t - \frac{1}{3}t^2 & \text{if } 0 \leq t \leq 1, \\ \frac{1}{3}t & \text{if } t > 1. \end{cases},$$

then  $\phi \in \Phi$ . Further, for all  $a = (a_1, a_2)$ ,  $b = (b_1, b_2)$ , we have

$$\begin{aligned} \phi(G(Fa, Bb, Bb)) &= G(Fa, Bb, Bb)\left(1 - \frac{1}{3}G(Fa, Bb, Bb)\right) \\ &= |a_1 - b_1|\left(1 - \frac{1}{3}|a_1 - b_1|\right) \\ &\geq |a_1 - b_1|\left(1 - \frac{1}{3}|a_1 + b_1|\right) = G(Sa, Tb, Tb) \end{aligned}$$

Therefore, all the conditions of Theorem 5.3.1 are satisfied.

However, Condition (5.3.5) is not satisfied. Indeed, for  $a = (0, 0)$ ,  $b = (u, 0)$ ,  $0 < u \leq 1$  and  $h > 1$

$$G(Fa, Bb, Bb) = u \geq hG(Sa, Tb, Tb) = h\left(u - \frac{1}{3}u^2\right).$$

This implies that  $h \leq 1$ , which yields a contradiction.

Now, we give a common fixed point result in  $G$ -metric spaces for expansive mappings, which extends the results of Şahin and Telci [161].

**Theorem 5.3.5.** Let  $f$  and  $g$  be surjective self-mappings of a complete  $G$ -metric space  $(A, G)$  satisfying for all  $a \in A$ ,

$$\begin{aligned} \phi(G(gfa, fa, fa)) &\geq G(fa, a, a), \\ \phi(G(fga, ga, ga)) &\geq G(ga, a, a), \end{aligned} \tag{5.3.6}$$

where  $\phi \in \Phi$ . Suppose that either  $f$  or  $g$  is continuous, then  $f$  and  $g$  have a common fixed point.

*Proof.* Let  $a_0 \in A$  be an arbitrary point of  $A$ . Due to the fact that the mappings  $f$  and  $g$  are surjective, there exists points  $a_1 = f^{-1}(a_0)$  and  $a_2 = g^{-1}(a_1)$ . Continuing this process, we

obtain the sequence  $\{a_n\}$  with  $a_{2n+1} = f^{-1}(a_{2n})$  and  $a_{2n+2} = g^{-1}(a_{2n+1})$ .

If  $a_n = a_{n+1}$  for some  $n$ , then  $a_n$  is a common fixed point of  $f$  and  $g$ . Indeed, if  $a_{2n} = a_{2n+1}$  for some  $n \geq 0$ , then  $a_{2n}$  is a fixed point of  $f$ . On the other hand, the inequality (5.3.6) implies that

$$\begin{aligned} 0 = \phi(G(a_{2n}, a_{2n+1}, a_{2n+1})) &= \phi(G(fa_{2n+1}, ga_{2n+2}, ga_{2n+2})) \\ &= \phi(G(fga_{2n+2}, ga_{2n+2}, ga_{2n+2})) \\ &\geq G(ga_{2n+2}, a_{2n+2}, a_{2n+2}), \end{aligned}$$

which further shows that  $G(a_{2n+1}, a_{2n+2}, a_{2n+2}) = 0$ . So, by the property of a  $G$ -metric, we have  $a_{2n+1} = a_{2n+2}$ . Therefore,  $a_{2n}$  is a common fixed point of  $f$  and  $g$ . So, assume that  $a_n \neq a_{n+1}$  for all  $n$ .

Clearly,

$$G(a_n, a_{n+1}, a_{n+1}) \leq \phi^n(G(a_0, a_1, a_1)) \quad \forall n.$$

In order to prove that  $\{a_n\}$  is a  $G$ -Cauchy sequence. For this, it is required to show that  $G(a_n, a_m, a_m) \rightarrow 0$  as  $m, n \rightarrow \infty$ . Let  $n, m \in \mathbb{N}$  with  $m > n$ ,

$$\begin{aligned} G(a_n, a_m, a_m) &\leq G(a_n, a_{n+1}, a_{n+1}) + G(a_{n+1}, a_{n+2}, a_{n+2}) + \dots + G(a_{m-1}, a_m, a_m) \\ &\leq \phi^n(G(a_0, a_1, a_1)) + \phi^{n+1}(G(a_0, a_1, a_1)) + \dots + \phi^{m-1}(G(a_0, a_1, a_1)) \end{aligned}$$

Using Lemma 5.2.5,  $G(a_n, a_m, a_m) \rightarrow 0$  as  $m, n \rightarrow \infty$ . Hence,  $\{a_n\}$  is a  $G$ -Cauchy sequence in  $A$ . Since  $A$  is  $G$ -complete, there exists a point  $c \in A$  such that  $\lim_{n \rightarrow \infty} a_n = c$ .

Now, assume that  $f$  is continuous. Since  $a_{2n} = fa_{2n+1}$ , so

$$c = \lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} fa_{2n+1} = fc,$$

and  $c$  is a fixed point of  $f$ . Since the mapping  $g$  is surjective, there exists a point  $b \in A$  such that  $gb = c$ . Now, in view of Inequality (5.3.6),

$$\begin{aligned} 0 = \phi(G(fc, gb, gb)) &= \phi(G(fgb, gb, gb)) \\ &\geq G(gb, b, b) = G(c, b, b). \end{aligned}$$

This implies that  $G(c, b, b) = 0$  thereby implying that  $c = b$ . Therefore,  $c$  is a common fixed point of  $f$  and  $g$ . Similarly, it can be proved that  $f$  and  $g$  have a common fixed point by using the continuity of  $g$ . This completes the proof.  $\square$

**Corollary 5.3.6.** Let  $f, g$  be surjective self-mappings of a complete  $G$ -metric space  $(A, G)$  satisfying for all  $a \in A$ ,

$$\begin{aligned} G(gfa, fa, fa) &\geq mG(fa, a, a), \\ G(fga, ga, ga) &\geq nG(ga, a, a), \end{aligned}$$

where  $m, n > 1$ . Suppose that either  $f$  or  $g$  is continuous, then  $f$  and  $g$  have a common fixed point.

*Proof.* By taking  $\phi(t) = \frac{t}{h}$ , where  $h = \max\{m, n\} > 1$  in inequality (5.3.6) of Theorem 5.3.5, we get the proof of this corollary.  $\square$

**Corollary 5.3.7.** Let  $f, g$  be surjective self-mappings of a complete  $G$ -metric space  $(A, G)$  satisfying for all  $a \in A$ ,

$$G(f^2a, fa, fa) \geq kG(fa, a, a),$$

where  $k > 1$ . If  $f$  is continuous, then  $f$  has a fixed point.

*Proof.* Taking  $f = g$  and  $k = \min\{m, n\}$  in Corollary 5.3.6, we get Corollary 5.3.7.  $\square$

**Remark 5.3.1.** Taking  $G(a, b, c) = \max\{d(a, b), d(b, c), d(c, a)\}$  for usual metric space  $(A, d)$  in the Corollary 5.3.7, we obtain the following result of Wang *et al.*[192].

**Corollary 5.3.8.** Let  $f$  be a continuous and surjective self-mapping of a complete metric space  $(A, d)$  satisfying for every  $a \in A$

$$d(f^2a, fa) \geq kd(fa, a),$$

where  $k > 1$ . Then  $f$  has a fixed point.

The following example is given in support of Theorem 5.3.5:

**Example 5.3.2.** Let  $A = [0, \infty)$  and  $\phi : [0, \infty) \rightarrow [0, \infty)$  defined by  $\phi(t) = \frac{t}{2}$ . Consider a  $G$ -metric on  $A$  given by

$$G(a, b, c) = \max\{|a - b|, |a - c|, |b - c|\}$$

Define the surjective self mappings  $f, g : A \rightarrow A$  by

$$f(a) = 8a \text{ and } g(a) = 6a$$

for all  $a$  in  $A$ . Then we have

$$\begin{aligned} \phi(G(gfa, fa, fa)) &= \phi(G(48a, 8a, 8a)) = 20a \\ &\geq 7a = G(fa, a, a) \end{aligned}$$

and

$$\begin{aligned} \phi(G(fga, ga, ga)) &= \phi(G(48a, 6a, 6a)) = 21a \\ &\geq 5a = G(ga, a, a) \end{aligned}$$

hold for every  $a \in A$ . Thus, Inequality (5.3.6) is satisfied and hence all the conditions of Theorem 5.3.5 are fulfilled. Clearly,  $a = 0$  is a common fixed point of  $f$  and  $g$ .



## Chapter 6

# QUADRUPLE FIXED POINT THEOREMS FOR NONLINEAR CONTRACTIONS IN PARTIALLY ORDERED G-METRIC SPACES

### 6.1 Introduction

In recent years, authors have started a new direction in fixed point theory by analyzing the existence and uniqueness of fixed points in partially ordered metric spaces. In 2003, Ran and Reurings [142] proved an analogue of Banach contraction principle in partially ordered metric spaces and thereby presenting some applications to linear and nonlinear matrix equations. After this fascinating paper, many useful results have been obtained in this direction(see, [11, 126, 128] and the references cited therein).

In 2006, Bhaskar and Lakshmikantham [36] introduced the concept of coupled fixed

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point and mixed monotone property and proved some related fixed point theorems in partially ordered metric spaces. In an attempt to generalize the results of Bhaskar and Lakshmikantham [36], Lakshmikantham and Ćirić [107] defined the notion of mixed  $g$ -monotone property. After that, many results appeared on coupled fixed point theory (see e.g. [33], [47], [49], [92], [108], [162]). In 2011, Berinde and Borcut [32] introduced tripled fixed points and proved some related results in partially ordered metric spaces. In 2012, Karapinar and Luong [98] introduced the notion of quadruple fixed point and proved several related fixed point theorems. Later, Karapinar and Berinde [97] obtained the existence of a unique common quadruple fixed point of  $F : X^4 \rightarrow X$  and  $g : X \rightarrow X$  in the framework of partially ordered complete metric spaces.

The motive of this chapter is to prove the quadruple coincidence point theorems regarding a mixed  $g$ -monotone mapping satisfying a nonlinear contractive condition in partially ordered  $G$ -metric spaces. These results improved the very recent results of Karapinar and Berinde [97] from metric spaces to  $G$ -metric spaces and also several other related results in the literature.

The contents of this chapter have been divided into two sections. Section 4.2 deals with the preliminaries related to this chapter. In Section 4.3, some quadruple coincidence point results have been obtained regarding a mixed  $g$ -monotone mapping satisfying a nonlinear contractive condition in partially ordered  $G$ -metric spaces.

## 6.2 Preliminaries

Before presenting our results, we recall some notations, definitions and results required in our subsequent discussions. Some of the definitions and results related to  $G$ -metric spaces have already been discussed in Chapter 5.

**Definition 6.2.1.** Consider a partially ordered set  $(A, \preceq)$ . Then

- (i) the elements  $a, b \in A$  are called comparable with respect to " $\preceq$ " if either  $a \preceq b$  or  $b \preceq a$  and
- (ii) a mapping  $T : A \rightarrow A$  is called nondecreasing with respect to " $\preceq$ " if  $a \preceq b$  implies  $Ta \preceq Tb$ .

The concepts of mixed monotone property and coupled fixed point due to Bhaskar and Lakshmikantham [36] are as follows:

**Definition 6.2.2.** Let  $(A, \preceq)$  be a partially ordered set and  $F : A \times A \rightarrow A$  be a mapping. The mapping  $F$  has the mixed monotone property if  $F$  is monotone nondecreasing in its first coordinate and is monotone nonincreasing in its second coordinate, that is,

$$\text{for all } a_1, a_2 \in A, a_1 \preceq a_2 \Rightarrow F(a_1, b) \preceq F(a_2, b), \text{ for } b \in A$$

and

$$\text{for all } b_1, b_2 \in A, b_1 \preceq b_2 \Rightarrow F(a, b_1) \succeq F(a, b_2), \text{ for } a \in A.$$

**Definition 6.2.3.** Let  $A$  be any non-empty set. An element  $(a, b) \in A \times A$  is called a coupled fixed point of the mapping  $F : A \times A \rightarrow A$  if  $F(a, b) = a$  and  $F(b, a) = b$ .

In 2009, Lakshmikantham and Ćirić [107] presented the following concepts:

**Definition 6.2.4.** ([107]). Consider the mappings  $F : A \times A \rightarrow A$  and  $g : A \rightarrow A$  of a partially ordered set  $(A, \preceq)$ . The mapping  $F$  has the mixed  $g$ -monotone property if  $F$  is monotone  $g$ -nondecreasing in  $a$  and is monotone  $g$ -nonincreasing in  $b$ , that is, for every  $a, b \in A$ ,

$$a_1, a_2 \in A, ga_1 \preceq ga_2 \Rightarrow F(a_1, b) \preceq F(a_2, b),$$

$$b_1, b_2 \in A, gb_1 \preceq gb_2 \Rightarrow F(a, b_2) \preceq F(a, b_1).$$

**Definition 6.2.5.** ([107]). Let  $A$  be any non-empty set. An element  $(a, b) \in A \times A$  is said to be a coupled coincidence point of the mappings  $F : A \times A \rightarrow A$  and  $g : A \rightarrow A$  if  $F(a, b) = ga$  and  $F(b, a) = gb$ .

**Definition 6.2.6.** ([107]). The mappings  $F : A \times A \rightarrow A$  and  $g : A \rightarrow A$  are said to be commutative if  $F(ga, gb) = g(F(a, b))$  for every  $a, b \in A$ .

Further, Berinde and Borcut [32] in 2011 presented the notion of tripled fixed point and proved some related theorems. Motivated by these notions, Karapinar and Luong [98] introduced the quadruple fixed point and proved some fixed point theorems concerning quartet fixed points. The following partial order has been defined by Karapinar and Luong [98] on the product space  $A^4 = A \times A \times A \times A$  :

$$(a, b, c, d) \preceq (x, y, z, w) \text{ if and only if } x \succeq a, y \preceq b, z \succeq c, w \preceq d,$$

where  $(a, b, c, d), (x, y, z, w) \in A^4$ .

**Definition 6.2.7.** ([98]). Let  $F : A^4 \rightarrow A$  be a mapping, where  $(A, \preceq)$  is a partially ordered set. The mapping  $F$  is said to have the mixed monotone property if  $F$  is monotone nondecreasing in  $a$  and  $c$ , and is monotone nonincreasing in  $b$  and  $d$ , that is, for any  $a, b, c, d \in A$ ,

$$a_1, a_2 \in A, a_1 \preceq a_2 \Rightarrow F(a_1, b, c, d) \preceq F(a_2, b, c, d),$$

$$b_1, b_2 \in A, b_1 \preceq b_2 \Rightarrow F(a, b_1, c, d) \succeq F(a, b_2, c, d),$$

$$c_1, c_2 \in A, c_1 \preceq c_2 \Rightarrow F(a, b, c_1, d) \preceq F(a, b, c_2, d)$$

and

$$d_1, d_2 \in A, d_1 \preceq d_2 \Rightarrow F(a, b, c, d_1) \succeq F(a, b, c, d_2).$$

**Definition 6.2.8.** ([98] Quadruple fixed point). Let  $A$  be any non-empty set. An element  $(a, b, c, d) \in A^4$  is said to be a quadruple fixed point of the mapping  $F : A^4 \rightarrow A$  if  $F(a, b, c, d) = a, F(a, d, c, b) = b, F(d, b, a, c) = c$  and  $F(c, d, a, b) = d$ .

Later, Karapinar and Berinde [97] proved the existence of a unique common quadruple fixed point of mappings  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  in partially ordered complete metric space  $(A, d, \leq)$  under certain appropriate conditions.

**Definition 6.2.9.** ([97]). Let  $(A, \preceq)$  be a partially ordered set and  $F : A^4 \rightarrow A$ . The mapping  $F$  is said to have the mixed  $g$ -monotone property if  $F(a, b, c, d)$  is monotone  $g$ -nondecreasing in  $a$  and  $c$ , and is monotone  $g$ -nonincreasing in  $b$  and  $d$ , that is, for any  $a, b, c, d \in A$ ,

$$a_1, a_2 \in A, g(a_1) \preceq g(a_2) \Rightarrow F(a_1, b, c, d) \preceq F(a_2, b, c, d),$$

$$b_1, b_2 \in A, g(b_1) \preceq g(b_2) \Rightarrow F(a, b_1, c, d) \succeq F(a, b_2, c, d),$$

$$c_1, c_2 \in A, g(c_1) \preceq g(c_2) \Rightarrow F(a, b, c_1, d) \preceq F(a, b, c_2, d)$$

and

$$d_1, d_2 \in A, g(d_1) \preceq g(d_2) \Rightarrow F(a, b, c, d_1) \succeq F(a, b, c, d_2).$$

**Definition 6.2.10.** ([97]). Let  $A$  be any non-empty set. An element  $(a, b, c, d) \in A^4$  is called a quadruple coincidence point of mappings  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  if  $F(a, b, c, d) = g(a), F(a, d, c, b) = g(b), F(c, b, a, d) = g(c)$  and  $F(c, d, a, b) = g(d)$ .

**Definition 6.2.11.** ([97]). Let  $A$  be any non-empty set. The mappings  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  are said to be commutative mappings if  $g(F(a, b, c, d)) = F(ga, gb, gc, gd)$ , for every  $a, b, c, d \in A$ .

**Definition 6.2.12.** Consider a mapping  $F : A^4 \rightarrow A$ , where  $(A, G)$  be a  $G$ -metric space. The function  $F$  is said to be continuous if for any  $G$ -convergent sequences  $\{a_n\}, \{b_n\}, \{c_n\}$

and  $\{d_n\}$  converging to  $a, b, c$  and  $d$  respectively,  $\{F(a_n, b_n, c_n, d_n)\}$  is  $G$ -convergent to  $F(a, b, c, d)$ .

In what follows, we present the main result of Karapinar and Berinde [97].

**Definition 6.2.13.** [97] Let  $\omega$  be the set of the all functions  $\nu : [0, 1) \rightarrow [0, 1)$  having the following properties:

- (i)  $\nu$  is continuous,
- (ii)  $\nu(t) < t$ ,
- (iii)  $\lim_{r \rightarrow t^+} \nu(r) < t$  for each  $r > 0$ .

**Theorem 6.2.1.** Let  $(A, d, \leq)$  be a complete partially ordered metric space. Suppose  $F : A^4 \rightarrow A$  and there exists  $\nu \in \omega$  such that  $F$  possess the mixed  $g$ -monotone property and

$$d(F(a, b, c, d), F(u, v, r, t)) \leq \nu\left(\frac{1}{4}(d(g(a), g(u)) + d(g(b), g(v)) + d(g(c), g(r)) + d(g(d), g(t)))\right)$$

for all  $a, u, b, v, c, r, d, t$  for which  $g(a) \leq g(u), g(b) \geq g(v), g(c) \leq g(r)$  and  $g(d) \geq g(t)$ .

Suppose that there exist  $a_0, b_0, c_0, d_0 \in A$  such that  $g(a_0) = F(a_0, b_0, c_0, d_0), g(b_0) = F(a_0, d_0, c_0, b_0), g(c_0) = F(c_0, b_0, a_0, d_0), g(d_0) = F(c_0, d_0, a_0, b_0)$ . Assume that  $F$  is continuous or  $A$  has the following property:

- (i) if a nondecreasing sequence  $a_n \rightarrow a$ , then  $a_n \leq a$  for all  $n$ ,
- (ii) if a nonincreasing sequence  $b_n \rightarrow b$ , then  $b_n \geq b$  for all  $n$ .

Also, suppose that  $F(A^4) \subset g(A)$  and  $g$  commutes with  $F$ . Then,  $F$  and  $g$  have a quadruple coincidence point.

### 6.3 Quadruple coincidence point theorems for a mixed $g$ -monotone mapping satisfying nonlinear contractions in partially ordered $G$ -metric spaces

The main results of this section are given by the following theorems:

**Theorem 6.3.1.** Let  $(A, \preceq)$  be a partially ordered set and  $G$  be a  $G$ -metric on  $A$  such that  $(A, G)$  is a complete  $G$ -metric space. Suppose that there exists  $\nu \in \omega$ ,  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  such that  $F$  and  $g$  are commuting mappings satisfying

$$\begin{aligned} & G(F(a_1, b_1, c_1, w_1), F(a_2, b_2, c_2, w_2), F(a_3, b_3, c_3, w_3)) \\ & \leq \nu \left( \frac{1}{4} (G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) + G(gc_1, gc_2, gc_3) + G(gw_1, gw_2, gw_3)) \right) \end{aligned} \quad (6.3.1)$$

for all  $a_i, b_i, c_i, w_i \in A$  where  $1 \leq i \leq 3$  for which  $ga_3 \preceq ga_2 \preceq ga_1$ ,  $gb_1 \preceq gb_2 \preceq gb_3$ ,  $gc_3 \preceq gc_2 \preceq gc_1$  and  $gw_1 \preceq gw_2 \preceq gw_3$ . Also, suppose that  $F$  is continuous and has the mixed  $g$ -monotone property,  $F(A^4) \subseteq g(A)$  and  $g$  is continuous. Assume that there exist  $a_0, b_0, c_0, w_0 \in A$  such that  $ga_0 \preceq F(a_0, b_0, c_0, w_0)$ ,  $gb_0 \succeq F(a_0, w_0, c_0, b_0)$ ,  $gc_0 \preceq F(c_0, b_0, a_0, w_0)$  and  $gw_0 \succeq F(c_0, w_0, a_0, b_0)$ . Then,  $F$  and  $g$  have a quadruple coincidence point in  $A$ .

*Proof.* Assume  $a_0, b_0, c_0, w_0 \in A$  be such that  $ga_0 \preceq F(a_0, b_0, c_0, w_0)$ ,  $gb_0 \succeq F(a_0, w_0, c_0, b_0)$ ,  $gc_0 \preceq F(c_0, b_0, a_0, w_0)$  and  $gw_0 \succeq F(c_0, w_0, a_0, b_0)$ . Since  $F(A^4) \subseteq g(A)$ , select  $a_1, b_1, c_1, w_1 \in A$  such that

$$\begin{aligned} ga_1 &= F(a_0, b_0, c_0, w_0), \quad gb_1 = F(a_0, w_0, c_0, b_0), \\ gc_1 &= F(c_0, b_0, a_0, w_0) \quad \text{and} \quad gw_1 = F(c_0, w_0, a_0, b_0). \end{aligned}$$

Similarly, we can again choose  $a_2, b_2, c_2, d_2 \in A$  as

$$\begin{aligned} ga_2 &= F(a_1, b_1, c_1, d_1), gb_2 = F(a_1, d_1, c_1, b_1), \\ gc_2 &= F(c_1, b_1, a_1, d_1) \text{ and } gd_2 = F(c_1, d_1, a_1, b_1). \end{aligned}$$

The mixed  $g$ -monotone property of  $F$  implies that

$$\begin{aligned} ga_0 &\preceq ga_1 \preceq ga_2, gb_2 \preceq gb_1 \preceq gb_0, \\ gc_0 &\preceq gc_1 \preceq gc_2 \text{ and } gd_2 \preceq gd_1 \preceq gd_0. \end{aligned}$$

Continuing this process, define sequences  $\{a_n\}, \{b_n\}, \{c_n\}$  and  $\{d_n\}$  in  $A$  in such a manner that

$$\begin{aligned} ga_n &= F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}) \preceq ga_{n+1} = F(a_n, b_n, c_n, d_n), \\ gb_{n+1} &= F(a_n, d_n, c_n, b_n) \preceq gb_n = F(a_{n-1}, d_{n-1}, c_{n-1}, b_{n-1}), \\ gc_n &= F(c_{n-1}, b_{n-1}, a_{n-1}, d_{n-1}) \preceq gc_{n+1} = F(c_n, b_n, a_n, d_n) \\ &\text{and} \\ gd_{n+1} &= F(c_n, d_n, a_n, b_n) \preceq gd_n = F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}). \end{aligned}$$

If  $(ga_{n+1}, gb_{n+1}, gc_{n+1}, gd_{n+1}) = (ga_n, gb_n, gc_n, gd_n)$  for some  $n$ , then there exists a quadruple coincidence point of mappings  $F$  and  $g$ .

So, let  $(ga_{n+1}, gb_{n+1}, gc_{n+1}, gd_{n+1}) \neq (ga_n, gb_n, gc_n, gd_n)$  for all  $n \in \mathbb{N}$ .

For  $n \in \mathbb{N}$ , let

$$\begin{aligned} t_n &= G(ga_{n+1}, ga_{n+1}, ga_n) + G(gb_{n+1}, gb_{n+1}, gb_n) + G(gc_{n+1}, gc_{n+1}, gc_n) + \\ &G(gd_{n+1}, gd_{n+1}, gd_n). \end{aligned} \quad (6.3.2)$$

Since, for all  $a, b \in A$  with  $a \neq b$ ,

$$G(a, a, b) > 0,$$

so  $t_n > 0$  for every  $n \in \mathbb{N}$ . The inequality (6.3.1) gives

$$\begin{aligned} G(ga_{n+1}, ga_{n+1}, ga_n) &= G(F(a_n, b_n, c_n, d_n), F(a_n, b_n, c_n, d_n), F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1})) \\ &\leq \nu \left( \frac{1}{4} (G(ga_n, ga_n, ga_{n-1}) + G(gb_n, gb_n, gb_{n-1}) \right. \\ &\quad \left. + G(gc_n, gc_n, gc_{n-1}) + G(gd_n, gd_n, gd_{n-1})) \right) \end{aligned}$$

$$\begin{aligned} G(gb_{n+1}, gb_{n+1}, gb_n) &= G(F(a_n, d_n, c_n, b_n), F(a_n, d_n, c_n, b_n), F(a_{n-1}, d_{n-1}, c_{n-1}, b_{n-1})) \\ &\leq \nu \left( \frac{1}{4} (G(ga_n, ga_n, ga_{n-1}) + G(gd_n, gd_n, gd_{n-1}) \right. \\ &\quad \left. + G(gc_n, gc_n, gc_{n-1}) + G(gb_n, gb_n, gb_{n-1})) \right) \end{aligned}$$

$$\begin{aligned} G(gc_{n+1}, gc_{n+1}, gc_n) &= G(F(c_n, b_n, a_n, d_n), F(c_n, b_n, a_n, d_n), F(c_{n-1}, b_{n-1}, a_{n-1}, d_{n-1})) \\ &\leq \nu \left( \frac{1}{4} (G(gc_n, gc_n, gc_{n-1}) + G(gb_n, gb_n, gb_{n-1}) \right. \\ &\quad \left. + G(ga_n, ga_n, ga_{n-1}) + G(gd_n, gd_n, gd_{n-1})) \right) \end{aligned}$$

and

$$\begin{aligned} G(gd_{n+1}, gd_{n+1}, gd_n) &= G(F(c_n, d_n, a_n, b_n), F(c_n, d_n, a_n, b_n), F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1})) \\ &\leq \nu \left( \frac{1}{4} (G(gc_n, gc_n, gc_{n-1}) + G(gd_n, gd_n, gd_{n-1}) \right. \\ &\quad \left. + G(ga_n, ga_n, ga_{n-1}) + G(gb_n, gb_n, gb_{n-1})) \right) \end{aligned}$$

Adding the above four inequalities, we get

$$t_n \leq 4\nu \left( \frac{t_{n-1}}{4} \right) \tag{6.3.3}$$

The sequence  $\{t_n\}$  is monotone decreasing as  $\nu(t) < t$  for all  $t > 0$ . So, there exists a  $\delta \geq 0$  such that  $\lim_{n \rightarrow +\infty} t_n = \delta^+$ . We now assert that  $\delta = 0$ . However, contradictorily let us suppose that  $\delta > 0$ . Letting  $n \rightarrow +\infty$  on both sides of (6.3.3) and in view of the properties of the map  $\nu$ , we get

$$\delta = \lim_{n \rightarrow +\infty} t_n \leq 4 \lim_{n \rightarrow +\infty} \nu \left( \frac{t_{n-1}}{4} \right) = 4 \lim_{t \rightarrow (\frac{\delta}{4})^+} \nu(t) < \delta,$$

which is a contradiction. Thus  $\delta = 0$ . Therefore,

$$\lim_{n \rightarrow +\infty} t_n = 0. \quad (6.3.4)$$

Next, we prove that the sequences  $\{ga_n\}$ ,  $\{gb_n\}$ ,  $\{gc_n\}$  and  $\{gd_n\}$  are Cauchy sequences in  $A$ . Contrarily, suppose that at least one of  $\{ga_n\}$ ,  $\{gb_n\}$ ,  $\{gc_n\}$  and  $\{gd_n\}$  are not a Cauchy sequence in  $(A, G)$ . So, there exist  $\epsilon > 0$  and sequences of natural numbers  $(p(r))$  and  $(q(r))$  such that for every natural number  $r$ ,  $p(r) > q(r) \geq r$  and

$$\begin{aligned} l_r &= G(ga_{p(r)}, ga_{p(r)}, ga_{q(r)}) + G(gb_{p(r)}, gb_{p(r)}, gb_{q(r)}) + \\ &G(gc_{p(r)}, gc_{p(r)}, gc_{q(r)}) + G(gd_{p(r)}, gd_{p(r)}, gd_{q(r)}) \geq \epsilon. \end{aligned} \quad (6.3.5)$$

Now, corresponding to  $q(r)$  choose  $p(r)$  to be the smallest number for which equation (6.3.5) holds. So,

$$\begin{aligned} &G(ga_{p(r)-1}, ga_{p(r)-1}, ga_{q(r)}) + G(gb_{p(r)-1}, gb_{p(r)-1}, gb_{q(r)}) + \\ &G(gc_{p(r)-1}, gc_{p(r)-1}, gc_{q(r)}) + G(gd_{p(r)-1}, gd_{p(r)-1}, gd_{q(r)}) < \epsilon. \end{aligned} \quad (6.3.6)$$

Making use of the rectangle inequality property of a  $G$ -metric, we have

$$\begin{aligned} \epsilon &\leq l_r \\ &\leq G(ga_{p(r)}, ga_{p(r)}, ga_{p(r)-1}) + G(ga_{p(r)-1}, ga_{p(r)-1}, ga_{q(r)}) + G(gb_{p(r)}, gb_{p(r)}, gb_{p(r)-1}) \\ &\quad + G(gb_{p(r)-1}, gb_{p(r)-1}, gb_{q(r)}) + G(gc_{p(r)}, gc_{p(r)}, gc_{p(r)-1}) + G(gc_{p(r)-1}, gc_{p(r)-1}, gc_{q(r)}) \\ &\quad + G(gd_{p(r)}, gd_{p(r)}, gd_{p(r)-1}) + G(gd_{p(r)-1}, gd_{p(r)-1}, gd_{q(r)}) \\ &= G(ga_{p(r)-1}, ga_{p(r)-1}, ga_{q(r)}) + G(gb_{p(r)-1}, gb_{p(r)-1}, gb_{q(r)}) + G(gc_{p(r)-1}, gc_{p(r)-1}, gc_{q(r)}) \\ &\quad + G(gd_{p(r)-1}, gd_{p(r)-1}, gd_{q(r)}) + t_{p(r)-1}. \end{aligned}$$

Using equation (6.3.4), (6.3.6) and letting  $r \rightarrow +\infty$  in the above inequality, we get

$$\lim_{r \rightarrow +\infty} l_r = \epsilon^+ \quad (6.3.7)$$

The use of rectangle inequality property gives,

$$\begin{aligned}
l_r &= G(ga_{p(r)}, ga_{p(r)}, ga_{q(r)}) + G(gb_{p(r)}, gb_{p(r)}, gb_{q(r)}) + G(gc_{p(r)}, gc_{p(r)}, gc_{q(r)}) \\
&\quad + G(gd_{p(r)}, gd_{p(r)}, gd_{q(r)}) \\
&\leq G(ga_{p(r)}, ga_{p(r)}, ga_{p(r)+1}) + G(ga_{p(r)+1}, ga_{p(r)+1}, ga_{q(r)+1}) + G(ga_{q(r)+1}, ga_{q(r)+1}, ga_{q(r)}) \\
&\quad + G(gb_{p(r)}, gb_{p(r)}, gb_{p(r)+1}) + G(gb_{p(r)+1}, gb_{p(r)+1}, gb_{q(r)+1}) + G(gb_{q(r)+1}, gb_{q(r)+1}, gb_{q(r)}) \\
&\quad + G(gc_{p(r)}, gc_{p(r)}, gc_{p(r)+1}) + G(gc_{p(r)+1}, gc_{p(r)+1}, gc_{q(r)+1}) + G(gc_{q(r)+1}, gc_{q(r)+1}, gc_{q(r)}) \\
&\quad + G(gd_{p(r)}, gd_{p(r)}, gd_{p(r)+1}) + G(gd_{p(r)+1}, gd_{p(r)+1}, gd_{q(r)+1}) \\
&\quad + G(gd_{q(r)+1}, gd_{q(r)+1}, gd_{q(r)}) \\
&= t_{q(r)} + G(ga_{p(r)}, ga_{p(r)}, ga_{p(r)+1}) + G(gb_{p(r)}, gb_{p(r)}, gb_{p(r)+1}) + G(gc_{p(r)}, gc_{p(r)}, gc_{p(r)+1}) \\
&\quad + G(gd_{p(r)}, gd_{p(r)}, gd_{p(r)+1}) + G(ga_{p(r)+1}, ga_{p(r)+1}, ga_{q(r)+1}) + G(gb_{p(r)+1}, gb_{p(r)+1}, gb_{q(r)+1}) \\
&\quad + G(gc_{p(r)+1}, gc_{p(r)+1}, gc_{q(r)+1}) + G(gd_{p(r)+1}, gd_{p(r)+1}, gd_{q(r)+1})
\end{aligned}$$

Using Lemma 5.2.2, we obtain

$$\begin{aligned}
l_r &\leq t_{q(r)} + 2G(ga_{p(r)}, ga_{p(r)+1}, ga_{p(r)+1}) + 2G(gb_{p(r)}, gb_{p(r)+1}, gb_{p(r)+1}) \\
&\quad + 2G(gc_{p(r)}, gc_{p(r)+1}, gc_{p(r)+1}) + 2G(gd_{p(r)}, gd_{p(r)+1}, gd_{p(r)+1}) \\
&\quad + G(ga_{p(r)+1}, ga_{p(r)+1}, ga_{q(r)+1}) + G(gb_{p(r)+1}, gb_{p(r)+1}, gb_{q(r)+1}) \\
&\quad + G(gc_{p(r)+1}, gc_{p(r)+1}, gc_{q(r)+1}) + G(gd_{p(r)+1}, gd_{p(r)+1}, gd_{q(r)+1}) \\
&= t_{q(r)} + 2t_{p(r)} + G(ga_{p(r)+1}, ga_{p(r)+1}, ga_{q(r)+1}) \\
&\quad + G(gb_{p(r)+1}, gb_{p(r)+1}, gb_{q(r)+1}) + G(gc_{p(r)+1}, gc_{p(r)+1}, gc_{q(r)+1}) \\
&\quad + G(gd_{p(r)+1}, gd_{p(r)+1}, gd_{q(r)+1}). \tag{6.3.8}
\end{aligned}$$

Making use of the inequality (6.3.1), we have

$$\begin{aligned} G(ga_{p(r)+1}, ga_{p(r)+1}, ga_{q(r)+1}) &= G(F(a_{p(r)}, b_{p(r)}, c_{p(r)}, d_{p(r)}), F(a_{p(r)}, b_{p(r)}, c_{p(r)}, d_{p(r)}), \\ &\quad F(a_{q(r)}, b_{q(r)}, c_{q(r)}, d_{q(r)})) \\ &\leq \nu \left( \frac{1}{4} (G(ga_{p(r)}, ga_{p(r)}, ga_{q(r)}) + G(gb_{p(r)}, gb_{p(r)}, gb_{q(r)}) \right. \\ &\quad \left. + G(gc_{p(r)}, gc_{p(r)}, gc_{q(r)}) + G(gd_{p(r)}, gd_{p(r)}, gd_{q(r)})) \right), \end{aligned}$$

$$\begin{aligned} G(gb_{p(r)+1}, gb_{p(r)+1}, gb_{q(r)+1}) &= G(F(a_{p(r)}, d_{p(r)}, c_{p(r)}, b_{p(r)}), F(a_{p(r)}, d_{p(r)}, c_{p(r)}, b_{p(r)}), \\ &\quad F(a_{q(r)}, d_{q(r)}, c_{q(r)}, b_{q(r)})) \\ &\leq \nu \left( \frac{1}{4} (G(ga_{p(r)}, ga_{p(r)}, ga_{q(r)}) + G(gd_{p(r)}, gd_{p(r)}, gd_{q(r)}) \right. \\ &\quad \left. + G(gc_{p(r)}, gc_{p(r)}, gc_{q(r)}) + G(gb_{p(r)}, gb_{p(r)}, gb_{q(r)})) \right), \end{aligned}$$

$$\begin{aligned} G(gc_{p(r)+1}, gc_{p(r)+1}, gc_{q(r)+1}) &= G(F(c_{p(r)}, b_{p(r)}, a_{p(r)}, d_{p(r)}), F(c_{p(r)}, b_{p(r)}, a_{p(r)}, d_{p(r)}), \\ &\quad F(c_{q(r)}, b_{q(r)}, a_{q(r)}, d_{q(r)})) \\ &\leq \nu \left( \frac{1}{4} (G(gc_{p(r)}, gc_{p(r)}, gc_{q(r)}) + G(gb_{p(r)}, gb_{p(r)}, gb_{q(r)}) \right. \\ &\quad \left. + G(ga_{p(r)}, ga_{p(r)}, ga_{q(r)}) + G(gd_{p(r)}, gd_{p(r)}, gd_{q(r)})) \right) \end{aligned}$$

and

$$\begin{aligned} G(gd_{p(r)+1}, gd_{p(r)+1}, gd_{q(r)+1}) &= G(F(c_{p(r)}, d_{p(r)}, a_{p(r)}, b_{p(r)}), F(c_{p(r)}, d_{p(r)}, a_{p(r)}, b_{p(r)}), \\ &\quad F(c_{q(r)}, d_{q(r)}, a_{q(r)}, b_{q(r)})) \\ &\leq \nu \left( \frac{1}{4} (G(gc_{p(r)}, gc_{p(r)}, gc_{q(r)}) + G(gd_{p(r)}, gd_{p(r)}, gd_{q(r)}) \right. \\ &\quad \left. + G(ga_{p(r)}, ga_{p(r)}, ga_{q(r)}) + G(gb_{p(r)}, gb_{p(r)}, gb_{q(r)})) \right) \end{aligned}$$

Summing up the above four inequalities, we get

$$\begin{aligned} G(ga_{p(r)+1}, ga_{p(r)+1}, ga_{q(r)+1}) + G(gb_{p(r)+1}, gb_{p(r)+1}, gb_{q(r)+1}) + G(gc_{p(r)+1}, gc_{p(r)+1}, gc_{q(r)+1}) \\ + G(gd_{p(r)+1}, gd_{p(r)+1}, gd_{q(r)+1}) \leq 4\nu \left( \frac{l_r}{4} \right) \end{aligned} \quad (6.3.9)$$

Now, it follows from inequalities (6.3.8) and (6.3.9) that

$$l_r \leq t_{q(r)} + 2t_{p(r)} + 4\nu\left(\frac{l_r}{4}\right). \quad (6.3.10)$$

Utilicing the properties of the function  $\nu$ , inequalities (6.3.4), (6.3.7), and letting  $r \rightarrow +\infty$  in the above inequality, we have

$$\epsilon \leq 4 \lim_{r \rightarrow +\infty} \nu\left(\frac{l_r}{4}\right) = 4 \lim_{t \rightarrow (\epsilon/4)^+} \nu(t) < \epsilon,$$

which is a contradiction. Therefore, the sequences  $\{ga_n\}$ ,  $\{gb_n\}$ ,  $\{gc_n\}$  and  $\{gd_n\}$  are Cauchy sequences in  $A$ . The completeness of the space  $A$  implies the existence of  $a, b, c, d \in A$  such that  $\{ga_n\}$ ,  $\{gb_n\}$ ,  $\{gc_n\}$  and  $\{gd_n\}$  are  $G$ -convergent to  $a, b, c$  and  $d$  respectively, that is, from Lemma 5.2.1, we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} G(ga_n, ga_n, a) &= \lim_{n \rightarrow +\infty} G(ga_n, a, a) = 0, \\ \lim_{n \rightarrow +\infty} G(gb_n, gb_n, b) &= \lim_{n \rightarrow +\infty} G(gb_n, b, b) = 0, \\ \lim_{n \rightarrow +\infty} G(gc_n, gc_n, c) &= \lim_{n \rightarrow +\infty} G(gc_n, c, c) = 0, \\ \lim_{n \rightarrow +\infty} G(gd_n, gd_n, d) &= \lim_{n \rightarrow +\infty} G(gd_n, d, d) = 0. \end{aligned} \quad (6.3.11)$$

The continuity of  $g$  and the Definition 5.2.6 implies that

$$\begin{aligned} \lim_{n \rightarrow +\infty} G(g(ga_n), g(ga_n), ga) &= \lim_{n \rightarrow +\infty} G(g(ga_n), ga, ga) = 0, \\ \lim_{n \rightarrow +\infty} G(g(gb_n), g(gb_n), gb) &= \lim_{n \rightarrow +\infty} G(g(gb_n), gb, gb) = 0, \\ \lim_{n \rightarrow +\infty} G(g(gc_n), g(gc_n), gc) &= \lim_{n \rightarrow +\infty} G(g(gc_n), gc, gc) = 0, \\ \lim_{n \rightarrow +\infty} G(g(gd_n), g(gd_n), gd) &= \lim_{n \rightarrow +\infty} G(g(gd_n), gd, gd) = 0. \end{aligned} \quad (6.3.12)$$

Since  $ga_{n+1} = F(a_n, b_n, c_n, w_n)$ ,  $gb_{n+1} = F(a_n, w_n, c_n, b_n)$ ,  $gc_{n+1} = F(c_n, b_n, a_n, w_n)$  and  $gd_{n+1} = F(c_n, d_n, a_n, b_n)$ .

The commutativity of mappings  $F$  and  $g$  gives

$$\begin{aligned}
g(ga_{n+1}) &= g(F(a_n, b_n, c_n, w_n)) = F(ga_n, gb_n, gc_n, gw_n) \\
g(gb_{n+1}) &= g(F(a_n, w_n, c_n, b_n)) = F(ga_n, gw_n, gc_n, gb_n) \\
g(gc_{n+1}) &= g(F(c_n, b_n, a_n, w_n)) = F(gc_n, gb_n, ga_n, gw_n) \\
g(gd_{n+1}) &= g(F(c_n, d_n, a_n, b_n)) = F(gc_n, gd_n, ga_n, gb_n)
\end{aligned} \tag{6.3.13}$$

Now, we show the existence of the quadruple coincidence point of the mappings  $F$  and  $g$ . Due to the fact that the sequences  $\{ga_n\}$ ,  $\{gb_n\}$ ,  $\{gc_n\}$  and  $\{gd_n\}$  are respectively  $G$ -convergent to  $a, b, c$  and  $d$ , so Definition 6.2.12 implies that the sequence  $(F(ga_n, gb_n, gc_n, gd_n))$  is  $G$ -convergent to  $F(a, b, c, d)$ . The inequality (6.3.13) yields that  $(g(ga_{n+1}))$  is  $G$ -convergent to  $F(a, b, c, d)$ . The uniqueness of the limit and (6.3.12) yields that  $F(a, b, c, d) = ga$ . Similarly, we can show that  $F(a, d, c, b) = gb$ ,  $F(c, b, a, d) = gc$  and  $F(c, d, a, b) = gd$ . Hence,  $(a, b, c, d)$  is a quadruple coincidence point of  $F$  and  $g$ .  $\square$

**Theorem 6.3.2.** Let  $(A, \preceq)$  be a partially ordered set and  $G$  be a metric on  $A$  such that  $(A, G)$  is a complete  $G$ -metric space having the following properties:

- (i) if a nondecreasing sequence  $\{a_n\} \rightarrow a$ , then  $a_n \preceq a$  for every  $n$ ,
- (ii) if a nonincreasing sequence  $\{b_n\} \rightarrow b$ , then  $b \preceq b_n$  for every  $n$ .

Suppose that there exists a function  $\nu \in \omega$  such that the mappings  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  satisfies

$$\begin{aligned}
&G(F(a_1, b_1, c_1, d_1), F(a_2, b_2, c_2, d_2), F(a_3, b_3, c_3, d_3)) \\
&\leq \nu \left( \frac{G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) + G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3)}{4} \right) \tag{6.3.14}
\end{aligned}$$

for all  $a_i, b_i, c_i, d_i \in A$  where  $1 \leq i \leq 3$  with  $ga_3 \preceq ga_2 \preceq ga_1$ ,  $gb_1 \preceq gb_2 \preceq gb_3$ ,  $gc_3 \preceq gc_2 \preceq gc_1$  and  $gd_1 \preceq gd_2 \preceq gd_3$ . Also, suppose that  $(g(A), G)$  is complete,  $F$

possess the mixed  $g$ -monotone property and  $F(A^4) \subseteq g(A)$ . If there exists  $a_0, b_0, c_0, d_0 \in A$  such that  $ga_0 \preceq F(a_0, b_0, c_0, d_0)$ ,  $gb_0 \succeq F(a_0, d_0, c_0, b_0)$ ,  $gc_0 \preceq F(c_0, b_0, a_0, d_0)$  and  $gd_0 \succeq F(c_0, d_0, a_0, b_0)$ , then  $F$  and  $g$  have a quadruple coincidence point.

*Proof.* The proof of Theorem 6.3.2 yields that  $\{ga_n\}$ ,  $\{gb_n\}$ ,  $\{gc_n\}$  and  $\{gd_n\}$  are Cauchy sequences in the complete  $G$ -metric space  $(g(A), G)$ . This implies the existence of points  $a, b, c, d \in A$  such that  $ga_n \rightarrow ga$ ,  $gb_n \rightarrow gb$ ,  $gc_n \rightarrow gc$  and  $gd_n \rightarrow gd$ . In view of conditions (i) and (ii) and the fact that  $\{ga_n\}$ ,  $\{gc_n\}$  are nondecreasing,  $\{gb_n\}$ ,  $\{gd_n\}$  are nonincreasing, we obtain  $ga_n \preceq ga$ ,  $gb_n \succeq gb$ ,  $gc_n \preceq gc$  and  $gd_n \succeq gd$  for every  $n \geq 0$ . If  $ga_n = ga$ ,  $gb_n = gb$ ,  $gc_n = gc$  and  $gd_n = gd$  for some  $n \geq 0$ , then

$$\begin{aligned} ga &= ga_n \preceq ga_{n+1} \preceq ga = ga_n, \\ gb &\preceq gb_{n+1} \preceq gb_n = gb, \\ gc &= gc_n \preceq gc_{n+1} \preceq gc = gc_n \quad \text{and} \\ gd &\preceq gd_{n+1} \preceq gd_n = gd, \end{aligned}$$

and thereby implying that the mappings  $F$  and  $g$  have a quadruple coincidence point  $(a_n, b_n, c_n, d_n)$ . Assume that  $(ga_n, gb_n, gc_n, gd_n) \neq (ga, gb, gc, gd)$  for all  $n \geq 0$ . From the rectangle inequality, (6.3.14) and the property  $\nu(t) < t$  for all  $t > 0$ , we infer

$$\begin{aligned} G(F(a, b, c, d), g(a), g(a)) &= G(F(a, b, c, d), g(a_{n+1}), g(a_{n+1})) + G(g(a_{n+1}), g(a), g(a)) \\ &= G(F(a, b, c, d), F(a_n, b_n, c_n, d_n), F(a_n, b_n, c_n, d_n)) + G(g(a_{n+1}), g(a), g(a)) \\ &\leq \nu \left( \frac{G(ga, ga_n, ga_n) + G(gb, gb_n, gb_n) + G(gc, gc_n, gc_n) + G(gd, gd_n, gd_n)}{4} \right) \\ &\quad + G(g(a_{n+1}), g(a), g(a)) \\ &< \left( \frac{G(ga, ga_n, ga_n) + G(gb, gb_n, gb_n) + G(gc, gc_n, gc_n) + G(gd, gd_n, gd_n)}{4} \right) \\ &\quad + G(g(a_{n+1}), g(a), g(a)) \end{aligned}$$

On letting  $n \rightarrow \infty$ , the above inequality implies that  $G(F(a, b, c, d), g(a), g(a)) \leq 0$ . Hence,  $g(a) = F(a, b, c, d)$ .

Analogously,

$$g(b) = F(a, d, c, b), g(c) = F(c, b, a, d) \text{ and } g(d) = F(c, d, a, b).$$

Thus, mappings  $F$  and  $g$  have a quadruple coincidence point  $(a, b, c, d)$ .  $\square$

**Corollary 6.3.3.** Let  $(A, \preceq)$  be a partially ordered set and  $G$  be a  $G$ -metric on  $A$  such that  $(A, G)$  is a complete  $G$ -metric space. Suppose that the mappings  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  are such that  $F$  has the mixed  $g$ -monotone property satisfying

$$G(F(a_1, b_1, c_1, d_1), F(a_2, b_2, c_2, d_2), F(a_3, b_3, c_3, d_3)) \leq \frac{k}{4}[G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) + G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3)], \text{ where } k \in [0, 1] \quad (6.3.15)$$

for all  $a_i, b_i, c_i, d_i \in A$  with  $1 \leq i \leq 3$  for which  $ga_3 \preceq ga_2 \preceq ga_1$ ,  $gb_1 \preceq gb_2 \preceq gb_3$ ,  $gc_3 \preceq gc_2 \preceq gc_1$  and  $gd_1 \preceq gd_2 \preceq gd_3$ . Also, suppose that  $F$  is continuous, commutes with  $F$ ,  $F(A^4) \subseteq g(A)$  and  $g$  is continuous. If there exist  $a_0, b_0, c_0, d_0 \in A$  such that  $ga_0 \preceq F(a_0, b_0, c_0, d_0)$ ,  $gb_0 \succeq F(a_0, d_0, c_0, b_0)$ ,  $gc_0 \preceq F(c_0, b_0, a_0, d_0)$  and  $gd_0 \succeq F(c_0, d_0, a_0, b_0)$ , then  $F$  and  $g$  have a quadruple coincidence point in  $A$ .

*Proof.* By substituting  $\nu(t) = kt$  for  $k \in [0, 1)$  in Theorem 6.3.1, the proof of Corollary 6.3.3 is obtained.  $\square$

**Corollary 6.3.4.** Let  $(A, \preceq)$  be a partially ordered set and  $(A, G)$  be a complete  $G$ -metric space satisfying the following properties:

- (i) if a nondecreasing sequence  $\{a_n\} \rightarrow a$ , then  $a_n \preceq a$  for all  $n$ ,
- (ii) if a nonincreasing sequence  $\{b_n\} \rightarrow b$ , then  $b \preceq b_n$  for all  $n$ ,

Suppose that there exists  $k \in [0, 1)$ ,  $F : A^4 \rightarrow A$  and  $g : A \rightarrow A$  such that  $F$  has the mixed

$g$ -monotone property satisfying

$$G(F(a_1, b_1, c_1, d_1), F(a_2, b_2, c_2, d_2), F(a_3, b_3, c_3, d_3)) \leq \frac{k}{4}[G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) + G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3)] \quad (6.3.16)$$

for all  $a_i, b_i, c_i, d_i \in A$  where  $1 \leq i \leq 3$  with  $ga_3 \preceq ga_2 \preceq ga_1$ ,  $gb_1 \preceq gb_2 \preceq gb_3$ ,  $gc_3 \preceq gc_2 \preceq gc_1$  and  $gd_1 \preceq gd_2 \preceq gd_3$ . Also, suppose that  $(g(A), G)$  is complete,  $F(A^4) \subseteq g(A)$ . If there exists  $a_0, b_0, c_0, d_0 \in A$  such that  $ga_0 \preceq F(a_0, b_0, c_0, d_0)$ ,  $gb_0 \succeq F(a_0, d_0, c_0, b_0)$ ,  $gc_0 \preceq F(c_0, b_0, a_0, d_0)$  and  $gd_0 \succeq F(c_0, d_0, a_0, b_0)$ , then  $F$  and  $g$  have a quadruple coincidence point.

*Proof.* By substituting  $\nu(t) = kt$  for  $k \in [0, 1)$  in Theorem 6.3.2, the proof of Corollary 6.3.4 is obtained.  $\square$

**Remark 6.3.1.** [37] Some of the fixed point theorems in G-metric spaces can be derived from fixed point results of metric spaces (see, e.g., [83, 166]). But these results are quite clear due to the strong connection between the usual metric and G-metric (see, e.g., [117, 121, 122]). The originality of a G-metric space comes from the fact that the G-metric space tells us about the distance of three points instead of distance between two points. We also accentuate that the methods used in [83, 166] cannot be applied to our main result since we are dealing with the nonlinear contractive conditions.

The following example has been presented to illustrate our main result:

**Example 6.3.1.** Consider the set  $A = \mathbb{R}$  equipped with a usual ordering. Define the function  $G : A \times A \times A \rightarrow A$  by  $G(a, b, c) = \max\{|a - b|, |b - c|, |a - c|\}$ . Suppose that the mappings  $g : A \rightarrow A$  and  $F : A \times A \times A \times A \rightarrow A$  are defined by

$$g(a) = \frac{5}{6}a, F(a, b, c, d) = \frac{a - b + c - d}{24}$$

for every  $a, b, c, d \in A$ . Take  $\nu \in \omega$  be given by  $\nu(t) = \frac{4}{5}t$  for all  $t \in [0, +\infty)$ . Clearly,  $(A, G, \leq)$  is a complete ordered G-metric space. Let  $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3 \in A$  with  $ga_1 \geq ga_2 \geq ga_3, gb_3 \geq gb_2 \geq gb_1, gc_1 \geq gc_2 \geq gc_3$  and  $gd_3 \geq gd_2 \geq gd_1$ . Then

$$\begin{aligned}
|F(a_1, b_1, c_1, d_1) - F(a_2, b_2, c_2, d_2)| &= \frac{1}{24}((a_1 - a_2) + (b_2 - b_1) + (c_1 - c_2) + (d_2 - d_1)) \\
&\leq \frac{125}{1296}(\max\{|a_1 - a_2|, |a_2 - a_3|, |a_3 - a_1|\} + \\
&\quad \max\{|b_1 - b_2|, |b_2 - b_3|, |b_3 - b_1|\} \\
&\quad + \max\{|c_1 - c_2|, |c_2 - c_3|, |c_3 - c_1|\} \\
&\quad + \max\{|d_1 - d_2|, |d_2 - d_3|, |d_3 - d_1|\}) \\
&= \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{5}{6} \max\{|a_1 - a_2|, |a_2 - a_3|, |a_3 - a_1|\} + \\
&\quad \frac{5}{6} \max\{|b_1 - b_2|, |b_2 - b_3|, |b_3 - b_1|\} \\
&\quad + \frac{5}{6} \max\{|c_1 - c_2|, |c_2 - c_3|, |c_3 - c_1|\} \\
&\quad + \frac{5}{6} \max\{|d_1 - d_2|, |d_2 - d_3|, |d_3 - d_1|\}) \\
&= \frac{4}{5} \frac{1}{4} (G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) + \\
&\quad G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3))
\end{aligned}$$

So,

$$\begin{aligned}
|F(a_1, b_1, c_1, d_1) - F(a_2, b_2, c_2, d_2)| &\leq \nu \left( \frac{1}{4} (G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) \right. \\
&\quad \left. + G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3)) \right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
|F(a_2, b_2, c_2, d_2) - F(a_3, b_3, c_3, d_3)| &\leq \nu \left( \frac{1}{4} (G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) \right. \\
&\quad \left. + G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3)) \right)
\end{aligned}$$

and\*

$$|F(a_3, b_3, c_3, d_3) - F(a_1, b_1, c_1, d_1)| \leq \nu \left( \frac{1}{4} (G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) + G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3)) \right)$$

Therefore,

$$\begin{aligned} & \max\{|F(a_1, b_1, c_1, d_1) - F(a_2, b_2, c_2, d_2)|, |F(a_2, b_2, c_2, d_2) - F(a_3, b_3, c_3, d_3)|, \\ & |F(a_3, b_3, c_3, d_3) - F(a_1, b_1, c_1, d_1)|\} \leq \nu \left( \frac{1}{4} (G(ga_1, ga_2, ga_3) + G(gb_1, gb_2, gb_3) \right. \\ & \left. + G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3)) \right) \end{aligned}$$

Hence,

$$\begin{aligned} G(F(a_1, b_1, c_1, d_1), F(a_2, b_2, c_2, d_2), F(a_3, b_3, c_3, d_3)) & \leq \nu \left( \frac{1}{4} (G(ga_1, ga_2, ga_3) \right. \\ & \left. + G(gb_1, gb_2, gb_3) + G(gc_1, gc_2, gc_3) + G(gd_1, gd_2, gd_3)) \right) \end{aligned}$$

Now, we proceed to show that  $F$  has the mixed  $g$ -monotone property. Consider the points  $a, b, c, d \in A$ . For proving that  $F(a, b, c, d)$  is  $g$ -monotone nondecreasing in  $a$ , let  $a_1, a_2 \in A$  with  $ga_1 \leq ga_2$ . Then  $a_1 \leq a_2$ , and so  $a_1 - b + c - d \leq a_2 - b + c - d$  thereby implying that  $F(a_1, b, c, d) \leq F(a_2, b, c, d)$ . So,  $F(a, b, c, d)$  is  $g$ -monotone nondecreasing in  $a$ . Similarly, it can be proved that  $F(a, b, c, d)$  is  $g$ -monotone nondecreasing in  $c$ .

Now, we have to show that  $F(a, b, c, d)$  is  $g$ -monotone nonincreasing in  $b$ , let  $b_1, b_2 \in A$  with  $gb_1 \leq gb_2$ , then  $b_1 \leq b_2$ . Hence,  $a - b_2 + c - d \leq a - b_1 + c - d$ , so  $F(a, b_2, c, d) \leq F(a, b_1, c, d)$ . Thus,  $F(a, b, c, d)$  is  $g$ -monotone nonincreasing in  $b$ . Similarly, it can be showed that  $F(a, b, c, d)$  is  $g$ -monotone nonincreasing in  $d$ .

Let  $a_0 = b_0 = c_0 = d_0 = 0$ . Clearly, all the other conditions of Theorem 6.3.1 are satisfied.

Thus,  $F$  and  $g$  have a quadruple coincidence point  $(0, 0, 0, 0)$  in  $A$ .



# Chapter 7

## FIXED POINT RESULTS IN PARTIAL HAUSDORFF METRIC SPACES WITH APPLICATIONS

### 7.1 Introduction

Von Neumann [127] in 1937 originated the fixed point theory for multivalued mappings in the analysis of game theory. Indeed, the fixed point results related to multivalued mappings are quite beneficial in solving many problems of economics, game theory and in the field of control theory. Consecutively, Nadler [123] in 1969 introduced the geometric fixed point theory for multivalued mappings. To establish the multivalued contraction principle, Nadler [123] utilized the notion of the Hausdorff metric and proved the Banach contraction principle as a special case. Consequently, a rich and fascinating fixed point theory developed. The theory concerning multifunctions has applications in control theory, convex optimization, differential equations and economics.

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In 2012, Samet *et al.* [165] generalized the Banach contraction principle with the help of the notions of  $\alpha$ -admissible mappings and  $\alpha$ - $\psi$ -contractive type mappings. Motivated by this new concept, Asl *et al.* [23] introduced a new and simple approach to the theory of multivalued mappings by presenting a new category of multifunctions called as  $\alpha_*$ - $\psi$ -contractive multifunctions.

The notion of partial metric spaces was presented by Matthews [111, 112] as a branch of the analysis of denotational semantics of dataflow networks. He demonstrated that Banach contraction mapping theorem can be generalized in context of the partial metric space for applications in program verification. In fact, it is extensively recognized that partial metric spaces play an important role in constructing models in the theory of computation (see [76],[132],[154], [155], [195]). The motivation for introducing non-zero distance (i.e., the 'distance'  $p$  where  $p(x, x) = 0$  need not hold) is explained in a presentation by Bukatin *et al.* [41] which also triggered an interesting research in foundations of topology.

Recently, Aydi *et al.* [24] gave the concept of a partial Hausdorff metric thereby extending the Nadler's fixed point theorem. Aydi *et al.* [24] triggered the study of existence of fixed points for multi-valued mappings in the framework of partial metric spaces.

Motivated by the above concepts, the motive of this chapter is to study the fixed point results for  $\alpha_*$ - $\psi$ -contractive multifunctions in a partial Hausdorff metric space. The theorems established in this chapter generalize and consolidate many relevant results of the literature, in particular the results of Aydi [24], Asl *et al.* [23] and the references therein.

The results established in this chapter are presented in three sections. Section 7.2 deals with the preliminaries related to this chapter. In Section 7.3, we present the new notions of strictly  $\alpha_*$ -admissible and  $\alpha_*$ - $\psi$ -contractive multifunctions and establish a fixed point result for these multifunctions. In the concluding section, a homotopy result has been presented

as an application of the results obtained in Section 7.3.

## 7.2 Preliminaries

Before presenting our results, we collect relevant concepts and results which will be required in the proof of the main results. Some of the definitions and results related to this chapter due to Samet *et al.* [165] have already been discussed in Chapters 1 and 2.

Denote by  $CB(A)$  the collection of all nonempty closed and bounded subsets of a metric space  $(A, d)$ . Define

$$H(X, Y) = \max\left\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\right\},$$

where  $X, Y \in CB(A)$ . The metric  $H$  on  $CB(A)$  is called the Hausdorff metric.

**Definition 7.2.1.** An element  $a \in A$  is said to be fixed point of a multi-valued mapping  $T : A \rightarrow 2^A$ , where  $A$  is any non-empty set if  $a \in Ta$ .

**Definition 7.2.2.** A multi-valued mapping  $T : A \rightarrow CB(A)$  is called a contraction mapping if

$$H(Ta, Tb) \leq kd(a, b), \text{ where } k \in [0, 1),$$

for all  $a, b \in A$ .

**Theorem 7.2.1.** [123](Nadler's fixed point theorem) Let  $(A, d)$  be a complete metric space and  $T : A \rightarrow CB(A)$  be a contraction mapping. Then,  $T$  has a fixed point.

Matthews [111] gave the following definitions and results:

**Definition 7.2.3.** Consider a non-empty set  $A$ . The mapping  $p : A \times A \rightarrow [0, \infty)$  is said to be a partial metric on  $A$  if the following conditions are satisfied for all  $a, b \in A$ :

- (i)  $a = b$  if and only if  $p(a, a) = p(b, b) = p(a, b)$ ,
- (ii)  $p(a, a) \leq p(a, b)$ ,
- (iii)  $p(a, b) = p(b, a)$ ,
- (iv)  $p(a, c) \leq p(a, b) + p(b, c) - p(b, b)$ .

The set  $A$  equipped with a partial metric  $p$  is called a partial metric space (in short PMS).

**Definition 7.2.4.** Let  $(A, p)$  be a partial metric space. The functions  $p_s, p_m : A \times A \rightarrow [0, \infty)$  given by

$$p_s(a, b) = 2p(a, b) - p(a, a) - p(b, b)$$

and

$$p_m(a, b) = \max\{p(a, b) - p(a, a), p(a, b) - p(b, b)\}$$

are well-known metrics on  $A$ . Clearly, the metrics  $p_s$  and  $p_m$  are equivalent. Every partial metric  $p$  on  $A$  induces a  $T_0$ -topology  $\tau_p$  with a base of the family of open  $p$ -balls  $\{B_p(a, \epsilon) : a \in A, \epsilon > 0\}$ , where  $B_p(a, \epsilon) = \{b \in A : p(a, b) < p(a, a) + \epsilon\}$ .

**Definition 7.2.5.** ([9, 78]). For a partial metric space  $(A, p)$ , we have

- (i) A sequence  $\{a_n\}$  in  $A$  converges to  $a \in A$  if and only if  $p(a, a) = \lim_{n \rightarrow \infty} p(a_n, a)$ .
- (ii) A sequence  $\{a_n\}$  in  $A$  is called a Cauchy sequence if and only if  $\lim_{m, n \rightarrow \infty} p(a_n, a_m)$  exists (and finite).
- (iii) The space  $(A, p)$  is said to be complete if every Cauchy sequence  $\{a_n\}$  in  $A$  converges to  $a \in A$ .

- (iv) A mapping  $f : A \rightarrow A$  is said to be continuous at  $a_0 \in A$  if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $f(B_p(a_0, \delta)) \subset B_p(f(a_0), \epsilon)$ .

**Example 7.2.6.** Let  $A = [0, +\infty)$  and  $p(a, b) = \max\{a, b\}$ , for every  $a, b \in A$ . The space  $(A, p)$  is a complete partial metric space. Clearly,  $p$  is not a (usual) metric.

The following lemmas have an important role in the proof of our results:

**Lemma 7.2.2.** ([21]) For a partial metric space  $(A, p)$ , we have

- (i) A sequence  $\{a_n\}$  is a Cauchy sequence in  $(A, p)$  if and only if  $\{a_n\}$  is a Cauchy sequence in  $(A, p_s)$ .
- (ii) The space  $(A, p)$  is said to be a complete space iff  $(A, p_s)$  is a complete space.

Moreover,

$$\lim_{n \rightarrow \infty} p_s(a_n, a) = 0 \Leftrightarrow p(a, a) = \lim_{n \rightarrow \infty} p(a_n, a) = \lim_{n, m \rightarrow \infty} p(a_m, a_n). \quad (7.2.1)$$

**Lemma 7.2.3.** ([9]) Assume that  $a_n \rightarrow t$  as  $n \rightarrow \infty$  in a PMS  $(A, p)$  such that  $p(t, t) = 0$ .

Then  $\lim_{n \rightarrow \infty} p(a_n, a) = p(t, a)$  for every  $a \in A$ .

Aydi *et al.* [24] presented the following definitions and results:

**Definition 7.2.7.** Let  $CB^p(A)$  be the set of all non-empty closed and bounded subsets of the partial metric space  $(A, p)$ . The closed sets of the space  $(A, p)$  are same as closed sets of the space  $(A, \tau_p)$ , where  $\tau_p$  is the topology induced by  $p$ . The bounded sets are defined as:  $A$  is a bounded subset in  $(A, p)$  if there exist  $a_0 \in A$  and  $M \geq 0$  such that for all  $x \in A$ , we have  $x \in B_p(a_0, M)$ , that is,  $p(a_0, x) < p(x, x) + M$ .

**Definition 7.2.8.** For  $A, B \in CB^p(X)$  and  $x \in X$ , define

$$\begin{aligned} p(x, A) &= \inf\{p(x, a), a \in A\}, \\ \delta_p(A, B) &= \sup\{p(a, B) : a \in A\} \quad \text{and} \\ \delta_p(B, A) &= \sup\{p(b, A) : b \in B\} \end{aligned}$$

**Remark 7.2.1.** [24] Suppose  $(A, p)$  be a partial metric space. If  $B$  is any nonempty set in  $(A, p)$ , then

$$a \in \overline{B} \quad \text{iff} \quad p(a, B) = p(a, a), \quad (7.2.2)$$

where  $\overline{B}$  denotes the closure of  $B$  with respect to the partial metric  $p$ . The set  $B$  is said to be closed in  $(A, p)$  if and only if  $B = \overline{B}$ .

In what follows, we present the results of Aydi *et al.*[24].

**Proposition 7.2.4.** [24] Suppose  $(X, p)$  be a partial metric space. For any  $A, B, C \in CB^p(X)$ , the following holds:

- (i)  $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$ ,
- (ii)  $\delta_p(A, A) \leq \delta_p(A, B)$ ,
- (iii)  $\delta_p(A, B) = 0$  implies that  $A \subseteq B$ ,
- (iv)  $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$ .

**Definition 7.2.9.** Let  $(A, p)$  be a partial metric space. The metric  $H_p$  is defined by

$$H_p(S, T) = \max\{\delta_p(S, T), \delta_p(T, S)\},$$

where  $S, T \in CB^p(A)$ .

**Proposition 7.2.5.** Suppose  $(A, p)$  is a partial metric space. For all  $A, B, C \in CB^p(X)$ , the following holds

$$(i) \ H_p(A, A) \leq H_p(A, B),$$

$$(ii) \ H_p(A, B) = H_p(B, A),$$

$$(iii) \ H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c).$$

**Lemma 7.2.6.** Let  $(A, p)$  be a partial metric space,  $S, T \in CB^p(A)$  and  $t > 1$ . For every  $a \in S$ , there exists  $b = b(a) \in T$  such that

$$p(a, b) \leq tH_p(S, T).$$

**Theorem 7.2.7.** Consider a complete partial metric space  $(A, p)$  and a multi-valued mapping  $T : A \rightarrow CB^p(A)$ . If there exists a constant  $k \in (0, 1)$  such that

$$H_p(Ta, Tb) \leq kp(a, b), \text{ for all } a, b \in A,$$

then  $T$  has a fixed point.

In 2012, Asl *et al.* [23] considered the following family of functions and gave the notions of  $\alpha_*$ -admissible mappings and  $\alpha_*$ - $\psi$ -contractive multifunctions as follows:

**Definition 7.2.10.** Let  $\Psi$  be the set of all functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  satisfying the following properties:

(i) the function  $\psi$  is nondecreasing,

(ii)  $\sum_{n=1}^{\infty} \psi^n(t) < \infty$  for all  $t > 0$ , where  $\psi^n$  is the  $n$ th iterate of  $\psi$ .

Clearly,  $\psi(t) < t$  for every  $t > 0$  and  $\psi \in \Psi$ .

**Definition 7.2.11.** Consider a metric space  $(A, d)$  and a closed-valued multifunction  $T : A \rightarrow 2^A$ . The mapping  $T$  is said to be an  $\alpha_*$ - $\psi$ -contractive multifunction if

$$\alpha_*(Ta, Tb)H(Ta, Tb) \leq \psi(d(a, b)), \text{ for every } a, b \in A,$$

where  $\psi \in \Psi$ ,  $\alpha : A \times A \rightarrow [0, \infty)$ ,  $H$  is the Hausdorff metric and  $\alpha_*(S, T) = \inf\{\alpha(s, t) : s \in S, t \in T\}$ .

**Definition 7.2.12.** The mapping  $T$  is said to be  $\alpha_*$ -admissible whenever

$$\alpha(a, b) \geq 1 \Rightarrow \alpha_*(Ta, Tb) \geq 1.$$

In what follows, we present the main result of Asl *et al.* [23].

**Theorem 7.2.8.** [23] Let the mapping  $T$  be an  $\alpha_*$ - $\psi$ -contractive multifunction on a complete metric space  $(A, d)$ . Also, suppose that

- (i) the mapping  $T$  is  $\alpha_*$ -admissible and closed valued,
- (ii) the mapping  $\psi \in \Psi$  is a strictly increasing mapping,
- (iii) there exists  $a_0 \in A$  and  $a_1 \in Ta_0$  satisfying  $\alpha(a_0, a_1) \geq 1$ ,
- (iv) if  $\alpha(a_n, a_{n+1}) \geq 1$  for all  $n$  and  $a_n \rightarrow a$  for a sequence  $\{a_n\}$  in  $A$ , then  $\alpha(a_n, a) \geq 1$  for all  $n$ .

Then  $T$  has a fixed point.

### 7.3 Fixed point theorem for $\alpha_*$ - $\psi$ -contractive multifunctions in a partial Hausdorff metric space

We first introduce here new concepts of strictly  $\alpha_*$ -admissible mappings and  $\alpha_*$ - $\psi$ -contractive multifunction in partial metric spaces as follows:

**Definition 7.3.1.** Let  $(A, p)$  be a partial metric space,  $T : A \rightarrow CB^p(A)$  and  $\alpha : A \times A \rightarrow \mathbb{R}^+$ . The mapping  $T$  is said to be strictly  $\alpha_*$ -admissible if

$$\alpha(a, b) > 1 \text{ implies that } \alpha_*(Ta, Tb) > 1 \text{ for all } a, b \in A.$$

**Definition 7.3.2.** Let  $(A, p)$  be a partial metric space and  $T : A \rightarrow 2^A$  be a closed-valued multifunction. The mapping  $T$  is said to be an  $\alpha_*$ - $\psi$ -contractive multifunction if

$$\alpha_*(Ta, Tb)H_p(Ta, Tb) \leq \psi(p(a, b)), \text{ for all } a, b \in A, \quad (7.3.1)$$

where  $\psi \in \Psi$ ,  $\alpha : A \times A \rightarrow \mathbb{R}^+$  be a function,  $H_p$  is the partial Hausdorff metric and  $\alpha_*(S, T) = \inf\{\alpha(s, t) : s \in S, t \in T\}$ .

The main result of this section is given by the following theorem:

**Theorem 7.3.1.** Let  $(A, p)$  be a complete partial metric space and  $T : A \rightarrow CB^p(A)$  be an  $\alpha_*$ - $\psi$ -contractive multifunction on  $A$ . Also, suppose that

- (i) the mapping  $T$  is a closed-valued and strictly  $\alpha_*$ -admissible mapping,
- (ii) there exists  $a_0 \in A$  and  $a_1 \in Ta_0$  satisfying  $\alpha(a_0, a_1) > 1$ ,
- (iii) the mapping  $\psi \in \Psi$  is a strictly increasing mapping,
- (iv) if  $\alpha(a_n, a_{n+1}) > 1$  for all  $n$  and  $a_n \rightarrow a$  for a sequence  $\{a_n\}$  in  $A$ , then  $\alpha(a_n, a) > 1$  for all  $n$ .

Then  $T$  has a fixed point.

*Proof.* If  $a_1 = a_0$ , then  $a_0 \in Ta_0$ . So, we have nothing to prove. For this, suppose  $a_1 \neq a_0$ . Further, if  $a_1 \in Ta_1$ , then again there is nothing to prove as  $Ta_1 = a_1$ . For this, assume  $a_1 \notin Ta_1$ . Now, we need to prove that  $p(a_1, Ta_1) > 0$ . Suppose that it is not. Then,

$$p(a_1, Ta_1) = 0 \Rightarrow p_s(a_1, Ta_1) = 0 \Rightarrow a_1 \in Ta_1,$$

a contradiction thereby implying that  $p(a_1, Ta_1) > 0$ . Put  $\alpha(a_0, a_1) = u > 1$ . The use of Lemma 7.2.6 with  $a_1 \in Ta_0$  and  $u > 1$  implies the existence of a point  $a_2 \in Ta_1$  satisfying

$$p(a_1, a_2) \leq uH_p(Ta_0, Ta_1) \quad (7.3.2)$$

Due to the fact that  $\alpha(a_0, a_1) > 1$  and  $T$  is strictly  $\alpha_*$ -admissible, we have

$$\alpha_*(Ta_0, Ta_1) > 1 \quad (7.3.3)$$

Therefore, we have from inequality (7.3.2) and (7.3.3)

$$\begin{aligned} 0 < p(a_1, Ta_1) &\leq p(a_1, a_2) \\ &\leq u.H_p(Ta_0, Ta_1) \\ &< u.\alpha_*(Ta_0, Ta_1).H_p(Ta_0, Ta_1) \\ &\leq u.\psi(p(a_0, a_1)) \end{aligned}$$

Now, clearly,  $a_1 \neq a_2$  and  $\alpha(a_1, a_2) > 1$ . Thus,

$$\alpha_*(Ta_1, Ta_2) > 1. \quad (7.3.4)$$

Now, put  $q_0 = p(a_0, a_1)$ . Then  $q_0 > 0$  and  $p(a_1, a_2) < u(\psi(q_0))$ . The strictly increasing property of  $\psi$  implies that ,

$$\psi(p(a_1, a_2)) < \psi(u(\psi(q_0))).$$

Put  $u_1 = \frac{\psi(u(\psi(q_0)))}{\psi(p(a_1, a_2))}$ . Then, clearly  $u_1 > 1$ . If  $a_2 \in Ta_2$ , then  $a_2$  is a fixed point of  $T$ .

Let us suppose that  $a_2 \notin Ta_2$ . The use of Lemma 7.2.6 with  $a_2 \in Ta_1$  and  $u_1 > 1$  implies the existence of a point  $a_3 \in Ta_2$  satisfying

$$p(a_2, a_3) \leq u_1H_p(Ta_1, Ta_2). \quad (7.3.5)$$

So, the inequalities (7.3.4) and (7.3.5) implies that

$$\begin{aligned}
0 < p(a_2, Ta_2) &\leq p(a_2, a_3) \\
&\leq u_1 \cdot H_p(Ta_1, Ta_2) \\
&< u_1 \cdot \alpha_*(Ta_1, Ta_2) \cdot H_p(Ta_1, Ta_2) \\
&\leq u_1 \cdot \psi(p(a_1, a_2)) \\
&= \psi(u(\psi(q_0)))
\end{aligned}$$

By repeatedly applying the above inequality, we obtain a sequence  $\{a_n\}$  in  $A$  such that  $a_n \in Ta_{n-1}$ ,  $a_n \neq a_{n-1}$ ,  $\alpha(a_n, a_{n+1}) > 1$  and  $p(a_n, a_{n+1}) \leq \psi^{n-1}(u(\psi(q_0)))$  for all  $n$ .

From the definition of  $p^s$ , we get for any  $m \in \mathbb{N}^*$

$$\begin{aligned}
p^s(a_n, a_{n+m}) &\leq 4p(a_n, a_{n+m}) \\
&\leq 4 \sum_{i=n}^{m-1} p(a_i, a_{i+1}) \\
&\leq 4 \sum_{i=n}^{m-1} \psi^{i-1}(u(\psi(q_0)))
\end{aligned}$$

Hence, from the property of  $\psi$ , we conclude that for an arbitrary  $\epsilon > 0$  there is a positive integer  $n_0$  such that for all  $m \in \mathbb{N}$ , we have

$$p^s(a_n, a_{n+m}) < \epsilon$$

for all  $m, n \geq n_0$ . This proves that  $\{a_n\}$  is a Cauchy sequence in  $(A, p^s)$ . Since  $(A, p)$  is complete, then from Lemma 7.2.2,  $(A, p^s)$  is a complete metric space. Therefore, the sequence  $\{a_n\}$  converges to some  $a^* \in A$  w.r.t metric  $p_s$ , that is,  $\lim_{n \rightarrow +\infty} p_s(a_n, a^*) = 0$ .

Again, from equation (7.2.1), we have

$$p(a^*, a^*) = \lim_{n \rightarrow \infty} p(a_n, a^*) = \lim_{n \rightarrow \infty} p(a_n, a_m) = 0 \quad (7.3.6)$$

Due to the fact that  $T$  is  $\alpha_*$ -admissible and  $\alpha(a_n, a^*) > 1$  for all  $n$ , we obtain that  $\alpha_*(Ta_n, Ta^*) > 1$  for all  $n$ . Thus, in view of equation (7.2.1), we get for all  $n \in \mathbb{N}$ , we have in view of (7.3.1)

$$\begin{aligned} p(a^*, Ta^*) &\leq H_p(Ta^*, Ta_n) + p(a_{n+1}, a^*) \\ &< \alpha_*(Ta_n, Ta^*)H_p(Ta_n, Ta^*) + p(a_{n+1}, a^*) \\ &\leq \psi(p(a_n, a^*)) + p(a_{n+1}, a^*) \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain  $p(a^*, Ta^*) = 0$  due to the continuity of  $\psi$  at 0. Also, from (7.3.6), we obtain that  $p(a^*, a^*) = 0$ . That is,  $p(a^*, a^*) = p(a^*, Ta^*)$ , which from (7.2.1) implies that  $a^* \in \overline{Ta^*} = Ta^*$ . Hence proved.  $\square$

**Example 7.3.3.** Let  $p : A \times A \rightarrow \mathbb{R}^+$  be defined by

$$\begin{aligned} p(0, 0) = p(1, 1) = 0, p(0, 1) = p(1, 0) = \frac{1}{4}, p(0, 2) = p(2, 0) = \frac{2}{5} \\ p(1, 2) = p(2, 1) = \frac{11}{15} \text{ and } p(2, 2) = \frac{1}{3}, \end{aligned}$$

where  $A = \{0, 1, 2\}$ . Then  $p$  is a partial metric on  $A$ . Let us define  $T : A \rightarrow CB^p(A)$  by

$$T(2) = \{0, 1\} \text{ and } T(0) = T(1) = \{0\}.$$

Clearly,  $Tx$  is closed and bounded for all  $a \in A$  in partial metric space  $(A, p)$ . Define the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  by

$$\alpha(a, b) = \begin{cases} 2 & a, b \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

and  $\psi \in \Psi$  by  $\psi(t) = \frac{9}{10}t$ . For all  $x, y \in A$ , the following holds:

$$\alpha_*(Ta, Tb)H_p(Ta, Tb) \leq \psi(p(a, b)),$$

thereby implying that  $T$  is an  $\alpha_*$ - $\psi$ -contractive multifunction. Clearly,  $T$  is strictly  $\alpha_*$ -admissible.

For  $a_0 = 0$  and  $0 = a_1 \in T(0) = 0$ , we have

$$\alpha(a_0, a_1) > 1.$$

Also, if  $\alpha(a_n, a_{n+1}) > 1$  for all  $n$  and  $a_n \rightarrow a$  for a sequence  $\{a_n\}$  in  $A$ , then by the definition of  $\alpha$  we obtain that  $a_n \in [0, 1]$  for all  $n$ . Consequently, we have  $a \in [0, 1]$  thereby implying that  $\alpha(a_n, a) > 1$  for all  $n$ . Thus, all the conditions of Theorem 7.3.1 are satisfied and  $a = 0$  is the only fixed point of  $T$ .

## 7.4 An application

In this section, we derive a homotopy result as an application of our main result.

**Theorem 7.4.1.** Let  $(A, p)$  be a complete partial metric space,  $B$  be an open subset of  $A$  and  $C$  be a closed subset of  $A$  such that  $B \subset C$ . Let  $F : C \times [0, 1] \rightarrow CB^p(A)$  be an operator satisfying the following conditions:

(i) for every  $a \in C \setminus B$  and  $t \in [0, 1]$ ,  $a \notin F(a, t)$ ,

(ii) there exists  $\psi \in \Psi$  satisfying

$$H_p(F(a, t), F(b, t)) \leq \psi(p(a, b)), \text{ for every } t \in [0, 1] \text{ and } a, b \in C, \quad (7.4.1)$$

(iii) there exists a continuous function  $\eta : [0, 1] \rightarrow \mathbb{R}$  such that for all  $t, s \in [0, 1]$  and every  $a \in C$ ,

$$H_p(F(a, t), F(a, s)) \leq \psi(|\eta(t) - \eta(s)|), \quad (7.4.2)$$

- (iv) If  $a \in F(a, t)$  then  $F(a, t) = \{a\}$ ,
- (v) there exists  $a_0 \in A$  such that  $a_0 \in F(a_0, t)$ ,
- (vi) the function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is continuous and strictly non-decreasing (here  $\varphi(a) = a - \psi(a)$ ).

Then  $H(., 1)$  has a fixed point iff  $H(., 0)$  has a fixed point.

*Proof.* Let us define the mapping  $\alpha : A \times A \rightarrow [0, \infty)$  by

$$\alpha(a, b) = \begin{cases} 2 & a \in F(a, t), b \in F(b, t), \\ 0 & \text{otherwise.} \end{cases}$$

for each  $t \in [0, 1]$ . Now, we proceed to show that  $F$  is  $\alpha_*$ -admissible. For this, let  $\alpha(a, b) > 1$ . This implies that  $a \in F(a, t)$  and  $b \in F(b, t)$  for all  $t$  thereby implying that  $F(a, t) = \{a\}$  and  $F(b, t) = \{b\}$ . Clearly,  $\alpha_*(F(a, t), F(b, t)) > 1$  for all  $t$ . By (e), we have  $a_0 \in F(a_0, t)$  for all  $t$ . That is,  $\alpha(a_0, a_0) > 1$ . Assume that if  $\alpha(a_n, a_{n+1}) > 1$  for all  $n$  and  $a_n \rightarrow a$  for a sequence  $\{a_n\}$  in  $A$ , then  $a_n \in F(a_n, t)$  and  $a_{n+1} \in F(a_{n+1}, t)$  for all  $n$  and  $t$ . This implies that  $a_n \in F(a_n, t)$  for all  $n$  and  $t$ . Therefore,  $\alpha(a_n, a) > 1$  for all  $n$  and  $a \in F(a, t)$ . Take

$$Q = \{t \in [0, 1] | a \in F(a, t) \text{ for some } a \in B\}.$$

In view of condition (i) and the existence of a fixed point of  $F(., 0)$ , we obtain that  $0 \in Q$  thereby implying that  $Q$  is a nonempty set. Now, we show that the set  $Q$  is both closed and open set in  $[0, 1]$ .

Firstly, assume that the set  $Q$  is an open set in  $[0, 1]$ . Suppose  $a_0 \in F(a_0, t_0)$  for  $t_0 \in Q$  and  $a_0 \in B$ . The set  $B$  being an open set in  $A$  implies the existence of  $r > 0$  such that  $B_p(a_0, r) \subseteq B$ . Let  $\epsilon = r + p(a_0, a_0) - \psi(r + p(a_0, a_0)) > 0$ . Due to the continuity of  $\eta$  on  $t_0$ , there exists  $\alpha(\epsilon) > 0$  such that  $|\eta(t) - \eta(t_0)| < \epsilon$  for all  $t \in (t_0 - \alpha(\epsilon), t_0 + \alpha(\epsilon))$ .

Suppose  $t \in (t_0 - \alpha(\epsilon), t_0 + \alpha(\epsilon))$ , for  $a \in B_p(a_0, r) = \{a \in A | p(a_0, a) \leq p(a_0, a_0) + r\}$ , we get

$$\begin{aligned}
p(F(a, t), a_0) &\leq H_p(F(a, t), F(a_0, t_0)) \\
&\leq H_p(F(a, t), F(a, t_0)) + H_p(F(a, t_0), F(a_0, t_0)) \\
&\leq \psi(|\eta(t) - \eta(t_0)|) + \psi(p(a, a_0)) \text{ ( by condition (b) )} \\
&\leq \psi(\epsilon) + \psi(p(a_0, a_0) + r) \\
&= \psi(r + p(a_0, a_0) - \psi(r + p(a_0, a_0))) + \psi(p(a_0, a_0) + r) \\
&< r + p(a_0, a_0) - \psi(r + p(a_0, a_0)) + \psi(p(a_0, a_0) + r) \\
&= r + p(a_0, a_0)
\end{aligned}$$

Then, for each fixed  $t \in (t_0 - \alpha(\epsilon), t_0 + \alpha(\epsilon))$ ,  $F(., t) : \overline{B_p(a_0, r)} \rightarrow CB^p(A)$  fulfils all the conditions of Theorem 7.3.1 and so  $F(., t)$  has a fixed point in  $\overline{B_p(a_0, r)} \subset C$ . As, the condition (a) holds, this fixed point belongs to  $B$ . Hence,  $(t_0 - \alpha(\epsilon), t_0 + \alpha(\epsilon)) \subseteq Q$  implying thereby that  $Q$  is an open set in  $[0, 1]$ .

Next, we prove that  $Q$  is a closed set in  $[0, 1]$ . For this, consider a sequence  $\{t_n\}_{n \in \mathbb{N}^*}$  in  $Q$  with  $t_n \rightarrow t^* \in [0, 1]$  as  $n \rightarrow +\infty$ . It is required to show that  $t^* \in Q$ . The definition of  $Q$  implies the existence of  $a_n \in B$  with  $a_n \in F(a_n, t_n)$  for all  $n \in \mathbb{N}^*$ . In view of conditions (iii), (iv),

$$\begin{aligned}
p(a_n, a_m) &= H_p(F(a_n, t_n), F(a_m, t_m)) \\
&\leq H_p(F(a_n, t_n), F(a_n, t_m)) + H_p(F(a_n, t_m), F(a_m, t_m)) \\
&\leq \psi(|\eta(t_n) - \eta(t_m)|) + \psi(p(a_n, a_m)), \text{ for all } m, n \in \mathbb{N}^*,
\end{aligned}$$

This implies that

$$\begin{aligned}\varphi(p(a_n, a_m)) &\leq \psi(|\eta(t_n) - \eta(t_m)|) \\ \Rightarrow \varphi(p(a_n, a_m)) &< |\eta(t_n) - \eta(t_m)|\end{aligned}$$

and we obtain

$$p(a_n, a_m) < \varphi^{-1}(|\eta(t_n) - \eta(t_m)|)$$

From the continuity of  $\varphi^{-1}$ ,  $\eta$  and convergence of the sequence  $\{t_n\}_{n \in \mathbb{N}^*}$ , we get

$$\lim_{n, m \rightarrow +\infty} p(a_n, a_m) = 0.$$

The completeness of the space  $(A, p)$  implies the existence of a point  $a^* \in C$  with  $p(a^*, a^*) =$

$$\lim_{n \rightarrow +\infty} p(a^*, a_n) = \lim_{n, m \rightarrow +\infty} p(a_n, a_m) = 0.$$

Consider,

$$\begin{aligned}p(a_n, F(a^*, t^*)) &\leq H_p(F(a_n, t_n), F(a^*, t^*)) \\ &\leq H_p(F(a_n, t_n), F(a_n, t^*)) + H_p(F(a_n, t^*), F(a^*, t^*)) \\ &\leq \psi(|\eta(t_n) - \eta(t^*)|) + \psi(p(a_n, a^*)).\end{aligned}$$

Letting  $n \rightarrow +\infty$  in the above inequality, we obtain  $\lim_{n \rightarrow +\infty} p(a_n, F(a^*, t^*)) = 0$  and

$$p(a^*, F(a^*, t^*)) = \lim_{n \rightarrow +\infty} p(a_n, F(a^*, t^*)) = 0,$$

thereby implying that  $a^* \in F(a^*, t^*)$  and using (i), we have  $a^* \in B$ . Thus,  $t^* \in Q$  and  $Q$  is a closed set in  $[0, 1]$ . Similarly, we can prove the reverse implication. Thus, the connectedness of the space  $[0, 1]$  shows that  $Q = [0, 1]$ .  $\square$



# Chapter 8

## FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES

### 8.1 Introduction

In 2011, Azam *et al.* [29] introduced the concept of a complex valued metric spaces as a generalization of the classical metric spaces and proved various fixed point results for a pair of mappings satisfying contraction condition involving a rational expression in these spaces. The class of complex valued metric spaces is a special class of cone metric space. The fixed point theorems regarding rational contractive conditions cannot be extended in cone metric spaces as the definition of a cone metric space involves the Banach space which is not a division Ring. However, the extensions of a number of fixed point results involving divisions can be studied in complex valued metric spaces.

Later, Rouzkard and Imdad [156] obtained some common fixed point results satisfying certain rational expressions in complex valued metric spaces and thereby generalizing the results of Azam *et al.* [29]. Recently, Sintunavarat and Kumam [181] obtained common

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The contents of this chapter have been communicated in *Bulletin of the Iranian Mathematical Society*.

fixed point results by replacing constant of contractive condition with control functions. For more details, one can see ([3], [5], [105], [180], [182]).

In an attempt to present the generalized version of  $\alpha$ - $\psi$  contractive type mappings in complex valued metric spaces, the notion of  $(\alpha, \phi)$ -contractive pair of mappings of rational type has been introduced. Also, various fixed point theorems for such mappings in the framework of complex valued metric spaces have been established. The results given in this chapter extend the recent results of Azam *et al.* [29], Rouzkard and Imdad [156] and references therein. Some examples are also presented to verify the effectiveness and applicability of our main results. Further, an application to integral equations has been presented to support the usability of the so obtained results. The presented theorems extend, generalize and improve many related results of the literature from metric spaces to complex valued metric spaces.

The contents of this chapter have been divided into five sections. In Section 8.2, some definitions and results relevant to the results of this chapter have been presented. In Section 8.3, a new type of mappings called as  $(\alpha, \phi)$ -contractive pair of mappings of rational type has been introduced. Also, some common fixed point results concerning  $(\alpha, \phi)$ -contractive pair of mappings of rational type have been obtained in complete complex valued metric spaces besides furnishing several illustrative examples to demonstrate the so obtained results. Section 8.4 concerns with the consequences of the results obtained in previous section. In section 8.5, an application to integral equations has been given to support the usability of the obtained results.

## 8.2 Preliminaries

Some of the definitions and results related to this chapter due to Samet *et al.* [165] and Karapinar and Samet [100] have already been discussed in Chapters 1, 2 and 4.

Azam *et al.* [29] introduced the following partial order on the set of complex numbers:

**Definition 8.2.1.** Denote by  $\mathbb{C}$  the set of complex numbers. Let  $w_1, w_2 \in \mathbb{C}$ . The partial order  $\lesssim$  on  $\mathbb{C}$  is defined by

$$w_1 \lesssim w_2 \text{ if and only if } \Re(w_1) \leq \Re(w_2), \Im(w_1) \leq \Im(w_2).$$

Consequently,  $w_1 \lesssim w_2$  if one of the following conditions is satisfied:

- (i)  $\Re(w_1) = \Re(w_2), \Im(w_1) < \Im(w_2)$ ,
- (ii)  $\Re(w_1) < \Re(w_2), \Im(w_1) = \Im(w_2)$ ,
- (iii)  $\Re(w_1) < \Re(w_2), \Im(w_1) < \Im(w_2)$ ,
- (iv)  $\Re(w_1) = \Re(w_2), \Im(w_1) = \Im(w_2)$ .

Particularly,  $w_1 \lesssim w_2$  if  $w_1 \neq w_2$  and one of (i), (ii) or (iii) is satisfied and  $w_1 \prec w_2$  if only (iii) is satisfied. Also,  $0 \lesssim w_1 \lesssim w_2 \Rightarrow |w_1| < |w_2|$  and  $w_1 \lesssim w_2, w_2 \prec w_3 \Rightarrow w_1 \prec w_3$ .

The concept of complex valued metric spaces defined by Azam *et al.* is as follows:

**Definition 8.2.2.** Let  $A$  be a nonempty set. The mapping  $d : A \times A \rightarrow \mathbb{C}$  satisfying the following properties:

- (i)  $0 \lesssim d(a, b)$ , for all  $a, b \in A$  and  $d(a, b) = 0$  if and only if  $a = b$ ,
- (ii)  $d(a, b) = d(b, a)$  for all  $a, b \in A$ ,
- (iii)  $d(a, b) \lesssim d(a, c) + d(c, b)$ , for all  $a, b, c \in A$ .

is called a complex valued metric on  $A$  and the space  $(A, d)$  is called a complex valued metric space.

The following definitions and results due to Azam *et al.* will be used in our results:

**Definition 8.2.3.** [29] Let  $(A, d)$  be a complex valued metric space and  $P \subseteq A$ . A point  $a \in P$  is said to be an interior point of a set  $P$  if there exists  $0 \prec r \in \mathbb{C}$  such that  $B(a, r) \subseteq P$ .

**Definition 8.2.4.** [29] Let  $(A, d)$  be a complex valued metric space and  $P \subseteq A$ . A point  $a \in A$  is said to be a limit point of  $P$  if for every  $0 \prec r \in \mathbb{C}$ ,  $B(a, r) \cap (P \setminus a) \neq \emptyset$

**Definition 8.2.5.** [29] Let  $(A, d)$  be a complex valued metric space. A set  $P \subseteq A$  is said to be an open set if each element of  $P$  is an interior point of  $P$ .

**Definition 8.2.6.** [29] Let  $(A, d)$  be a complex valued metric space. A set  $P \subseteq A$  is said to be a closed set if each limit point of  $P$  belongs to  $P$ .

**Definition 8.2.7.** [29] Let  $(A, d)$  be a complex valued metric space. Suppose  $\{a_n\}$  be a sequence in  $A$  and  $a \in A$ . If for every  $c \in \mathbb{C}$ , with  $0 \prec c$  there is  $n_0 \in \mathbb{N}$  such that

$$d(a_n, a) \prec c, \text{ for every } n \geq n_0,$$

then the sequence  $\{a_n\}$  is said to be convergent sequence and  $a$  is the limit point of  $\{a_n\}$ .

It is denoted by  $\lim_n \{a_n\} = a$ , or  $a_n \rightarrow a$ , as  $n \rightarrow \infty$

**Definition 8.2.8.** [29] Let  $(A, d)$  be a complex valued metric space. A sequence  $\{a_n\}$  is said to be a Cauchy sequence in  $A$  if for every  $c \in \mathbb{C}$  with  $c \succ 0$  there is  $n_0 \in \mathbb{N}$  such that

$$d(a_n, a_{n+m}) \prec c, \text{ for every } n, m > n_0,$$

**Definition 8.2.9.** [29] A complex valued metric space  $(A, d)$  is known as complete complex valued metric space if every Cauchy sequence in  $A$  is convergent in  $A$ .

Azam *et al.* [29] established the following two lemmas for the proof of their theorems:

**Lemma 8.2.1.** Suppose  $(A, d)$  be a complex valued metric space and  $\{a_n\}$  be a sequence in  $A$ . Then the sequence  $\{a_n\}$  is said to be convergent to  $a$  iff  $|d(a_n, a)| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 8.2.2.** Suppose  $(A, d)$  be complex valued metric space and  $\{a_n\}$  be a sequence in  $A$ . Then the sequence  $\{a_n\}$  is said to be a Cauchy sequence iff  $|d(a_n, a_{n+m})| \rightarrow 0$  as  $n \rightarrow \infty$ .

Azam *et al.* [29] obtained the following extension of the Banach contraction principle in complex valued metric spaces:

**Theorem 8.2.3.** Suppose  $(A, d)$  be a complete complex valued metric space. Let the mappings  $S, T : A \rightarrow A$  satisfy:

$$d(Sa, Tb) \lesssim \lambda d(a, b) + \frac{\mu d(a, Sa)d(b, Tb)}{1 + d(a, b)}, \text{ for every } a, b \in A$$

where  $\lambda, \mu$  are nonnegative reals with  $\lambda + \mu < 1$ . Then  $S$  and  $T$  have a unique common fixed point.

**Proposition 8.2.4.** A self-mapping  $f$  on a complex valued metric space  $(A, d)$  is said to be continuous at a point  $a \in A$  if and only if  $\{f(a_n)\}$  is convergent to  $f(a)$  whenever  $\{a_n\}$  is convergent to  $a$ .

### 8.3 Fixed Points for $(\alpha, \phi)$ -contractive pair of mappings of rational type

We start this section by presenting the new concepts of  $(\alpha, \phi)$ -contractive mapping of rational type and  $\alpha$ -admissible pair of mappings.

Let  $\mathbb{C}^+ = \{z \in \mathbb{C} | \Re(z) \geq 0, \Im(z) \geq 0\}$ . Denote with  $\varphi$  the family of nondecreasing functions  $\phi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  such that  $\sum_{n=1}^{+\infty} |\phi^n(t)| < +\infty$  for each  $t \succ 0$ , where  $\phi^n$  is the  $n$ -th iterate of  $\phi$ .

**Remark 8.3.1.** Clearly, for every mapping  $\phi \in \varphi$ , we have  $|\phi(t)| = \phi(|t|)$  for every  $t \succ 0$ .

**Lemma 8.3.1.** For every function  $\phi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  the following holds:

if  $\phi$  is nondecreasing, then for each  $t \succ 0$ ,  $\lim_{n \rightarrow +\infty} |\phi^n(t)| = 0$  implies  $\phi(t) \prec t$ .

*Proof.* Suppose that  $\lim_{n \rightarrow +\infty} |\phi^n(t)| = 0$ . Let us assume that  $\phi(t) \succ t$  for some  $t \succ 0$ . Then  $\phi^n(t) \succ t$  for some  $t \succ 0$  for  $n = 1, 2, 3, \dots$ . That is,  $|\phi^n(t)| > |t|$  for some  $t \succ 0$  for  $n = 1, 2, 3, \dots$ . Thus,  $\lim_{n \rightarrow +\infty} |\phi^n(t)| \not\rightarrow 0$ , a contradiction. Also, if  $\phi(t) = t$  for some  $t \succ 0$ , then  $\lim_{n \rightarrow +\infty} |\phi^n(t)| \not\rightarrow 0$ . Hence, for all  $t \succ 0$ ,  $\phi(t) \prec t$ .  $\square$

**Definition 8.3.1.** Let  $\Delta$  be the family of functions  $\alpha : A \times A \rightarrow [0, \infty)$  satisfying  $\alpha(a, b) = \alpha(b, a)$  for all  $a, b \in A$ .

**Definition 8.3.2.** Let  $T : A \rightarrow A$  be a given mapping of a complex valued metric space  $(A, d)$ . The mapping  $T$  is said to be an  $(\alpha, \phi)$ -contractive mapping of rational type if there exists two functions  $\alpha \in \Delta$  and  $\phi \in \varphi$  satisfying

$$\alpha(a, b)d(Ta, Tb) \lesssim \phi(M(a, b)),$$

where  $M(a, b) = \max \left\{ d(a, b), \frac{d(a, Ta)d(b, Tb)}{1 + d(a, b)}, \frac{d(b, Ta)d(a, Tb)}{1 + d(a, b)} \right\}$ , for every  $a, b \in A$ .

**Definition 8.3.3.** Let  $S, T$  be given self-mappings of a complex valued metric space  $(A, d)$ . The pair  $(S, T)$  is said to be an  $(\alpha, \phi)$ -contractive pair of mappings of rational type if there exists two functions  $\alpha \in \Delta$  and  $\phi \in \varphi$  such that

$$\alpha(a, b)d(Sa, Tb) \lesssim \phi(M(a, b)), \quad (8.3.1)$$

where  $M(a, b) = \max \left\{ d(a, b), \frac{d(a, Sa)d(b, Tb)}{1 + d(a, b)}, \frac{d(b, Sa)d(a, Tb)}{1 + d(a, b)} \right\}$ , for every  $a, b \in A$

**Definition 8.3.4.** Let  $S, T : A \rightarrow A$  and  $\alpha \in \Delta$ . The pair  $(S, T)$  is said to be an  $\alpha$ -admissible pair of mappings if

$$\alpha(a, b) \geq 1 \Rightarrow \alpha(Sa, Tb) \geq 1, \text{ for all } a, b \in A.$$

The following examples demonstrate the concept of  $\alpha$ -admissible pair of mappings:

**Example 8.3.5.** Let  $A = [0, +\infty)$ . Define  $S, T : A \rightarrow A$  by

$$S(a) = \begin{cases} 2a - \frac{3}{2} & a \geq 1, \\ \frac{a}{3} & a \in [0, 1), \\ 0 & a < 0. \end{cases} \text{ and } T(a) = \ln(1 - a).$$

Define the mapping  $\alpha : A \times A \rightarrow [0, +\infty)$  by

$$\alpha(a, b) = \begin{cases} 1/4 & \text{if } (a, b) \neq (0, 0), \\ 1 & \text{if } (a, b) = (0, 0). \end{cases}$$

It can easily be seen that  $\alpha(Sa, Tb) \geq 1$  and  $\alpha(a, b) = \alpha(b, a)$  for all  $a, b \in A$ , which proves that the pair  $(S, T)$  is an  $\alpha$ -admissible pair of mappings.

**Example 8.3.6.** Let  $A = [1, +\infty)$ . Define  $T, S : A \rightarrow A$  and  $\alpha : A \times A \rightarrow [0, +\infty)$  by

$$S(a) = a^2, T(a) = \sqrt{a} \text{ for every } a \in A \text{ and } \alpha(a, b) = \begin{cases} 2 & \text{if } a, b \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

It can easily be seen that  $\alpha(Sa, Tb) \geq 1$  and  $\alpha(a, b) = \alpha(b, a)$  for all  $a, b \in A$ , which clearly shows that the pair  $(S, T)$  is an  $\alpha$ -admissible pair of mappings.

The main results of this section are given as follows:

**Theorem 8.3.2.** Let the pair  $(S, T)$  of mappings is an  $(\alpha, \phi)$ -contractive pair of mappings of rational type in a complete complex valued metric space  $(A, d)$ . Also, suppose that:

- (i) the pair  $(S, T)$  is an  $\alpha$ -admissible pair of mappings,
- (ii) there exists  $a_0 \in A$  such that  $\alpha(a_0, Sa_0) \geq 1$ ,
- (iii) the mappings  $S$  and  $T$  are continuous.

Then,  $S$  and  $T$  have a common fixed point.

*Proof.* Let  $a_0 \in A$  be such that  $\alpha(a_0, Sa_0) \geq 1$ . Consider the sequence  $\{a_n\}$  in  $A$  defined by

$$\begin{aligned} a_{2k+1} &= Sa_{2k}, \\ a_{2k+2} &= Ta_{2k+1}, \quad k = 0, 1, 2, \dots \end{aligned}$$

Owing to the  $\alpha$ -admissible property of the pair of mappings, we have

$$\alpha(a_0, a_1) = \alpha(a_0, Sa_0) \geq 1 \Rightarrow \alpha(Sa_0, Ta_1) = \alpha(a_1, a_2) \geq 1.$$

By induction and using the symmetry property of the function  $\alpha$ , we get

$$\alpha(a_n, a_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}. \quad (8.3.2)$$

If  $a_{2k} = a_{2k+1}$ , for all  $k$ , then  $a_{2k}$  is a fixed point of  $S$ . Consider,

$$\begin{aligned} d(a_{2k+1}, a_{2k+2}) &= d(Sa_{2k}, Ta_{2k+1}) \\ &\lesssim \alpha(a_{2k}, a_{2k+1})d(Sa_{2k}, Ta_{2k+1}) \\ &\lesssim \phi(M(a_{2k}, a_{2k+1})). \end{aligned}$$

The continuity of  $\phi$  at  $t = 0$  implies that  $a_{2k+1} = a_{2k+2}$  thereby implying that  $a_{2k}$  is a common fixed point of  $S$  and  $T$ . For this, it is assumed that  $a_n \neq a_{n+1}$  for every  $n \in \mathbb{N}$ .

Use of the inequalities (8.3.1) and (8.3.2) gives,

$$\begin{aligned} d(a_{2k}, a_{2k+1}) &= d(Sa_{2k}, Ta_{2k-1}) \\ &\lesssim \alpha(a_{2k}, a_{2k-1})d(Sa_{2k}, Ta_{2k-1}) \\ &\lesssim \phi(M(a_{2k}, a_{2k-1})), \end{aligned}$$

and thus

$$|d(a_{2k}, a_{2k+1})| \leq |\phi(M(a_{2k}, a_{2k-1}))| = \phi(|M(a_{2k}, a_{2k-1})|). \quad (8.3.3)$$

On the other hand,

$$\begin{aligned} M(a_{2k}, a_{2k-1}) &= \max \left\{ d(a_{2k}, a_{2k-1}), \frac{d(a_{2k}, Sa_{2k})d(a_{2k-1}, Ta_{2k-1})}{1 + d(a_{2k}, a_{2k-1})}, \frac{d(a_{2k-1}, Sa_{2k})d(a_{2k}, Ta_{2k-1})}{1 + d(a_{2k}, a_{2k-1})} \right\} \\ &= \max \left\{ d(a_{2k}, a_{2k-1}), \frac{d(a_{2k}, a_{2k+1})d(a_{2k-1}, a_{2k})}{1 + d(a_{2k}, a_{2k-1})}, \frac{d(a_{2k-1}, a_{2k+1})d(a_{2k}, a_{2k})}{1 + d(a_{2k}, a_{2k-1})} \right\} \\ &= \max \left\{ d(a_{2k}, a_{2k-1}), \frac{d(a_{2k}, a_{2k+1})d(a_{2k-1}, a_{2k})}{1 + d(a_{2k}, a_{2k-1})} \right\}, \end{aligned}$$

that is,

$$\begin{aligned} |M(a_{2k}, a_{2k-1})| &= \max \left\{ |d(a_{2k}, a_{2k-1})|, \left| \frac{d(a_{2k}, a_{2k+1})d(a_{2k-1}, a_{2k})}{1 + d(a_{2k}, a_{2k-1})} \right| \right\} \\ &= \max \left\{ |d(a_{2k}, a_{2k-1})|, \frac{|d(a_{2k}, a_{2k+1})||d(a_{2k-1}, a_{2k})|}{|1 + d(a_{2k}, a_{2k-1})|} \right\}. \end{aligned}$$

Since  $|1 + d(a_{2k}, a_{2k-1})| > |d(a_{2k}, a_{2k-1})|$ , therefore

$$|M(a_{2k}, a_{2k-1})| < \max \{ |d(a_{2k}, a_{2k-1})|, |d(a_{2k}, a_{2k+1})| \}. \quad (8.3.4)$$

Owing to monotonicity of the function  $\phi$  and in view of (8.3.3) and (8.3.4),

$$\begin{aligned} |d(a_{2k}, a_{2k+1})| &\leq \phi(|M(a_{2k}, a_{2k-1})|) \\ &\leq \phi(\max \{ |d(a_{2k}, a_{2k-1})|, |d(a_{2k}, a_{2k+1})| \}). \end{aligned} \quad (8.3.5)$$

If for some  $k$ , we have  $|d(a_{2k}, a_{2k-1})| \leq |d(a_{2k}, a_{2k+1})|$ , which in view of (8.3.5) gives

$$0 < |d(a_{2k}, a_{2k+1})| \leq \phi(|d(a_{2k}, a_{2k+1})|) < |d(a_{2k}, a_{2k+1})|, \quad (8.3.6)$$

which is a contradiction. Thus, for all  $k$ ,

$$\max \{|d(a_{2k}, a_{2k-1})|, |d(a_{2k}, a_{2k+1})|\} = |d(a_{2k}, a_{2k-1})|. \quad (8.3.7)$$

In view of (8.3.5) and (8.3.7), we infer that

$$|d(a_{2k}, a_{2k+1})| \leq \phi(|d(a_{2k}, a_{2k-1})|), \text{ for all } k. \quad (8.3.8)$$

Continuing this process inductively, we obtain

$$|d(a_{2k}, a_{2k+1})| \leq \phi^{2k}(|d(a_1, a_0)|), \text{ for all } k. \quad (8.3.9)$$

Therefore, for any  $m > n$ ,

$$\begin{aligned} |d(a_n, a_m)| &\leq |d(a_n, a_{n+1})| + |d(a_{n+1}, a_{n+2})| + \dots + |d(a_{m-1}, a_m)| \\ &\leq \phi^n(|d(a_1, a_0)|) + \phi^{n+1}(|d(a_1, a_0)|) + \dots + \phi^{m-1}(|d(a_1, a_0)|). \end{aligned} \quad (8.3.10)$$

Since  $\sum_{n=1}^{+\infty} \phi^n(t) < +\infty$  for every  $t > 0$ , letting  $m, n \rightarrow +\infty$  gives

$$|d(a_n, a_m)| \rightarrow 0 \text{ as } m, n \rightarrow +\infty.$$

Thus,  $\{a_n\}$  is a Cauchy sequence in the complex valued metric space  $(A, d)$  by Lemma 8.2.2. The completeness of the space  $A$  implies the existence of a point  $a^* \in A$  such that  $a_n \rightarrow a^*$  as  $n \rightarrow +\infty$ . Due to the continuity of  $S$ , we have by Proposition 8.2.4 that  $a_{2k+1} = Sa_{2k} \rightarrow Sa^*$  as  $k \rightarrow +\infty$ . By the uniqueness of the limit, we get  $Sa^* = a^*$ . Similarly, by the continuity of  $T$ , we have  $Ta^* = a^*$ . Thus, the mappings  $S$  and  $T$  have a common fixed point  $a^*$ .  $\square$

The following example demonstrates the validity of the above theorem.

**Example 8.3.7.** Consider the set  $A = [0, +\infty)$  equipped with the complex valued metric  $d(a, b) = i|a - b|$ . Define the mappings  $S, T : A \rightarrow A$  and  $\alpha \in \Delta$  by

$$S(a) = \begin{cases} 3a - \frac{8}{3} & a \geq 1, \\ \frac{a}{3} & a \in [0, 1). \end{cases}$$

$$T(a) = \begin{cases} 5a - \frac{14}{3} & a \geq 1, \\ \frac{a}{3} & a \in [0, 1). \end{cases}$$

and

$$\alpha(a, b) = \begin{cases} 1 & a, b \in [0, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the pair  $(S, T)$  is an  $(\alpha, \phi)$ -contractive pair of mappings of rational type with  $\phi(a) = a/2$  for all  $a \geq 0$ . In fact,

$$\alpha(a, b)d(Sa, Tb) \lesssim \phi(M(a, b)), \text{ for all } a, b \in A.$$

Also, for  $a_0 = 0$ , we have

$$\alpha(0, S0) = 1.$$

Thus, there exists  $a_0 \in A$  such that  $\alpha(a_0, Sa_0) \geq 1$ . Obviously,  $S$  and  $T$  are continuous and so it remains to show that the pair  $(S, T)$  is an  $\alpha$ -admissible pair of mappings. For this, suppose that  $\alpha(a, b) \geq 1$  for some  $a, b \in A$  thereby implying that  $a, b \in [0, 1)$ . So, in view of the definitions of  $S, T$  and  $\alpha$ , we have

$$Sa = \frac{a}{3} \in [0, 1), Tb = \frac{b}{3} \in [0, 1) \text{ and } \alpha(Sa, Tb) = 1.$$

Clearly,  $\alpha(a, b) = \alpha(b, a)$  for all  $a, b \in A$ . Thus, the pair  $(S, T)$  of mappings is an  $\alpha$ -admissible pair of mappings. All the conditions of Theorem 8.3.2 are fulfilled thereby

implying that  $S$  and  $T$  have a common fixed point. In this example, 0 is the common fixed point of  $S$  and  $T$ .

In the next theorem, we show that Theorem 8.3.2 holds true even if the continuity requirement of  $S$  and  $T$  is replaced with a new condition as mentioned below:

**Theorem 8.3.3.** Let the pair  $(S, T)$  of mappings is an  $(\alpha, \phi)$ -contractive pair of mappings of rational type in a complete complex valued metric space  $(A, d)$ . Also, suppose that:

- (i) the pair  $(S, T)$  of mappings is  $\alpha$ -admissible,
- (ii) there exists a point  $a_0 \in A$  satisfying  $\alpha(a_0, Sa_0) \geq 1$ ,
- (iii) if  $\{a_n\}$  is a sequence in  $A$  such that  $\alpha(a_n, a_{n+1}) \geq 1$  for all  $n$  and  $a_n \rightarrow a \in A$  as  $n \rightarrow +\infty$ , then  $\alpha(a_n, a) \geq 1$  for all  $n$ . Then,  $S$  and  $T$  have a common fixed point.

*Proof.* The proof of Theorem 8.3.2 shows that the sequence  $\{a_n\}$  is a Cauchy sequence in the complete complex valued metric space  $(A, d)$ . This implies the existence of  $u \in A$  satisfying  $a_n \rightarrow u$  as  $n \rightarrow +\infty$ . The use of inequality (8.3.2) and the condition (iii) gives

$$\alpha(a_n, u) \geq 1 \quad \forall n \in \mathbb{N} \quad (8.3.11)$$

In view of (8.3.1) and (8.3.11), we have

$$\begin{aligned} d(a_{2n(k)+1}, Tu) &= d(Sa_{2n(k)}, Tu) \\ &\lesssim \alpha(a_{2n(k)}, u)d(Sa_{2n(k)}, Tu) \\ &\lesssim \phi(M(a_{2n(k)}, u)) \end{aligned}$$

that is,

$$|d(a_{2n(k)+1}, Tu)| \leq |\phi(M(a_{2n(k)}, u))| \quad (8.3.12)$$

Moreover,

$$\phi(M(a_{2n(k)}, u)) = \max \left\{ \phi(d(a_{2n(k)}, u)), \phi \left( \frac{d(a_{2n(k)}, a_{2n(k)+1}) \cdot d(u, Tu)}{1 + d(a_{2n(k)}, u)} \right), \phi \left( \frac{d(u, a_{2n(k)+1}) \cdot d(a_{2n(k)}, Tu)}{1 + d(a_{2n(k)}, u)} \right) \right\}$$

Also,

$$|\phi(M(a_{2n(k)}, u))| = \max \left\{ |\phi(d(a_{2n(k)}, u))|, \left| \phi \left( \frac{d(a_{2n(k)}, a_{2n(k)+1}) \cdot d(u, Tu)}{1 + d(a_{2n(k)}, u)} \right) \right|, \left| \phi \left( \frac{d(u, a_{2n(k)+1}) \cdot d(a_{2n(k)}, Tu)}{1 + d(a_{2n(k)}, u)} \right) \right| \right\}$$

Letting  $k \rightarrow \infty$  in the above equality gives

$$\lim_{k \rightarrow \infty} |\phi(M(a_{2n(k)}, u))| = 0. \quad (8.3.13)$$

Now, on making  $k \rightarrow \infty$  in inequality (8.3.12) and using (8.3.13),

$$|d(u, Tu)| = 0 \quad (8.3.14)$$

which is a contradiction yielding thereby  $u = Tu$ . Similarly, it can easily be shown that  $u = Su$ . Hence proved.  $\square$

The following example proves the validity of the above theorem:

**Example 8.3.8.** Consider the set  $A = [0, +\infty)$  with the complex valued metric  $d(a, b) = i|a - b|$  for all  $a, b \in A$ . Define the mappings  $S, T : A \rightarrow A$  and  $\alpha \in \Delta$  by

$$Sa = \begin{cases} 2a - \frac{3}{2} & a \geq 1, \\ \frac{a}{4} & 0 \leq a < 1. \end{cases}$$

$$Ta = \begin{cases} 3a - 3 & a \geq 1, \\ \frac{a}{4} & 0 \leq a < 1. \end{cases}$$

and

$$\alpha(a, b) = \begin{cases} 1 & a, b \in [0, \frac{1}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 8.3.2 is not applicable in this case due to the reason that  $S$  and  $T$  are discontinuous at 1. Clearly, the pair  $(S, T)$  is an  $(\alpha, \phi)$ -contractive pair of mappings of rational type with  $\phi(a) = \frac{a}{3}$  for all  $a \succsim 0$ . In fact,

$$\alpha(a, b)d(Sa, Tb) \preceq \phi(M(a, b)), \text{ for all } a, b \in A,$$

Also, for  $a_0 = 0$ , we have

$$\alpha(0, S0) = 1.$$

Thus, there exists  $a_0 \in A$  such that  $\alpha(a_0, Sa_0) \geq 1$ . Clearly,  $\alpha(a, b) = \alpha(b, a)$  for all  $a, b \in A$ . Now, we proceed to show the  $\alpha$ -admissible property of the pair  $(S, T)$  of mappings. For this, let  $\alpha(a, b) \geq 1$  for some  $a, b \in A$ . This implies that  $a, b \in [0, \frac{1}{2}]$  and by the definition of  $S, T$  and  $\alpha$ , we have

$$Sa = \frac{a}{4} \in [0, \frac{1}{2}], Tb = \frac{b}{4} \in [0, \frac{1}{2}] \text{ and } \alpha(Sa, Tb) = 1$$

that is, the pair  $(S, T)$  is an  $\alpha$ -admissible pair of mappings.

Finally, assume that

$$\alpha(a_n, a_{n+1}) \geq 1,$$

where  $\{a_n\}$  is a sequence in  $A$  and  $\{a_n\} \rightarrow a \in A$  as  $n \rightarrow +\infty$ . Since  $\alpha(a_n, a_{n+1}) \geq 1$  for each  $n$ , by the definition of  $\alpha$ , we have  $a_n \in [0, \frac{1}{2}]$  for all  $n$  and  $a \in [0, \frac{1}{2}]$ . Then,  $\alpha(a_n, a) = 1$ .

Therefore, all the conditions of Theorem 8.3.3 are satisfied thereby implying that  $S$  and  $T$  have common fixed points. Here, 0 and  $\frac{3}{2}$  are two common fixed points of  $S$  and  $T$ .

For the uniqueness of the common fixed point, the following condition has been considered:

(A) For every  $a, b \in A$ , there exists  $c \in A$  satisfying  $\alpha(a, c) \geq 1$  and  $\alpha(b, c) \geq 1$ .

**Theorem 8.3.4.** If the condition (A) along with the hypotheses of Theorems 8.3.2 and 8.3.3 holds, then there exists a unique fixed point of  $T$ .

*Proof.* Suppose that the mappings  $S$  and  $T$  have two common fixed points  $u$  and  $v$ . The condition (A) implies the existence of a point  $w \in A$  such that

$$\alpha(u, w) \geq 1 \text{ and } \alpha(v, w) \geq 1. \quad (8.3.15)$$

Applying the  $\alpha$ -admissible property of the pair  $(S, T)$  of mappings, we get for all  $n \in \mathbb{N}$

$$\alpha(u, T^n w) \geq 1 \text{ and } \alpha(v, T^n w) \geq 1. \quad (8.3.16)$$

Let us define the sequence  $\{w_n\}$  in  $A$  by  $w_{n+1} = Tw_n$  for every  $n \geq 0$  and  $w_0 = w$ . In view of (8.3.16), we have for all  $n$ ,

$$d(u, w_{2n+1}) = d(Su, Tw_{2n}) \lesssim \alpha(u, w_{2n})d(Su, Tw_{2n}) \lesssim \phi(M(u, w_{2n})), \quad (8.3.17)$$

that is,

$$|d(u, w_{2n+1})| \leq |\phi(M(u, w_{2n}))| = \phi(|M(u, w_{2n})|). \quad (8.3.18)$$

On the other hand,

$$M(u, w_{2n}) = \max \left\{ d(u, w_{2n}), \frac{d(u, Su) \cdot d(w_{2n}, Tw_{2n})}{1 + d(u, w_{2n})}, \frac{d(w_{2n}, Su) \cdot d(u, Tw_{2n})}{1 + d(u, w_{2n})} \right\}.$$

Also,

$$\begin{aligned} |M(u, w_{2n})| &= \max \left\{ |d(u, w_{2n})|, \frac{|d(u, Su)| |d(w_{2n}, Tw_{2n})|}{|1 + d(u, w_{2n})|}, \frac{|d(w_{2n}, Su)| |d(u, Tw_{2n})|}{|1 + d(u, w_{2n})|} \right\} \\ &= \max \left\{ |d(u, w_{2n})|, \frac{|d(w_{2n}, u)| |d(u, w_{2n+1})|}{|1 + d(u, w_{2n})|} \right\}. \end{aligned}$$

Since  $|1 + d(u, w_{2n})| > |d(u, w_{2n})|$ , therefore

$$|M(u, w_{2n})| < \max \{|d(u, w_{2n})|, |d(u, w_{2n+1})|\}. \quad (8.3.19)$$

By making use of inequality (8.3.19) and the monotone property of  $\psi$ , we have from (8.3.18)

$$|d(u, w_{2n+1})| \leq \phi(\max \{|d(u, w_{2n})|, |d(u, w_{2n+1})|\}), \quad (8.3.20)$$

for all  $n$ . Now, without loss to the generality, suppose that  $d(u, w_{2n}) \succ 0$  for all  $n$ . Assume that  $\max \{|d(u, w_{2n})|, |d(u, w_{2n+1})|\} = |d(u, w_{2n+1})|$ , so from (8.3.20),

$$|d(u, w_{2n+1})| \leq \phi(|d(u, w_{2n+1})|) < |d(u, w_{2n+1})|, \quad (8.3.21)$$

which is a contradiction. Hence,

$$\max \{|d(u, w_{2n})|, |d(u, w_{2n+1})|\} = |d(u, w_{2n})| \quad (8.3.22)$$

and

$$|d(u, w_{2n+1})| \leq \phi(|d(u, w_{2n})|). \quad (8.3.23)$$

for all  $n$ . Continuing this process, we obtain

$$|d(u, w_n)| \leq \phi^n(|d(u, w_0)|) \forall n \geq 1. \quad (8.3.24)$$

Letting  $n \rightarrow \infty$  in the above inequality implies that

$$\lim_{n \rightarrow \infty} |d(u, w_n)| = 0. \quad (8.3.25)$$

Similarly,

$$\lim_{n \rightarrow \infty} |d(v, w_n)| = 0. \quad (8.3.26)$$

The equations (8.3.25) and (8.3.26) implies that  $u = v$ . Thus, there exists a unique common fixed point  $u$  of mappings  $S$  and  $T$ .  $\square$

## 8.4 Consequences

By substituting  $\alpha(a, b) = 1$  for every  $a, b \in A$  in Theorem 8.3.4, the following fixed point theorem has been obtained:

**Corollary 8.4.1.** Let  $S, T$  be given self-mappings of a complete complex valued metric space  $(A, d)$ . Let there exists a function  $\phi \in \varphi$  such that

$$d(Sa, Tb) \lesssim \phi(M(a, b)), \text{ for every } a, b \in A,$$

where  $M(a, b) = \max \left\{ d(a, b), \frac{d(a, Sa)d(b, Tb)}{1 + d(a, b)}, \frac{d(b, Sa)d(a, Tb)}{1 + d(a, b)} \right\}$ . Then  $S$  and  $T$  have a unique common fixed point.

The following fixed point results can be easily derived from Corollary 8.4.1.

**Corollary 8.4.2.** (see Rouzkard and Imdad [156]) Let  $S, T : A \rightarrow A$  be mappings of a complete complex valued metric space  $(A, d)$  satisfying for all  $a, b \in A$ ,

$$d(Sa, Tb) \lesssim \lambda d(a, b) + \frac{\mu d(a, Sa)d(b, Tb) + \gamma d(b, Sa)d(a, Tb)}{1 + d(a, b)}, \quad (8.4.1)$$

where  $\lambda, \mu, \gamma$  are nonnegative real numbers such that  $\lambda + \mu + \gamma < 1$ . Then,  $S$  and  $T$  have a unique common fixed point.

**Corollary 8.4.3.** (see Azam *et al.* [29]) Let  $S, T : A \rightarrow A$  be mappings of a complete complex valued metric space  $(A, d)$  satisfying for all  $a, b \in A$ ,

$$d(Sa, Tb) \lesssim \lambda d(a, b) + \frac{\mu d(a, Sa)d(b, Tb)}{1 + d(a, b)}, \quad (8.4.2)$$

where  $\lambda, \mu$  are nonnegative real numbers such that  $\lambda + \mu < 1$ . Then,  $S$  and  $T$  have a unique common fixed point.

**Corollary 8.4.4.** (see Rouzkard and Imdad [156]) Let  $T : A \rightarrow A$  be a mapping of a complete complex valued metric space  $(A, d)$  satisfying for all  $a, b \in A$ ,

$$d(Ta, Tb) \lesssim \lambda d(a, b) + \frac{\mu d(a, Ta)d(b, Tb) + \gamma d(b, Ta)d(a, Tb)}{1 + d(a, b)}, \quad (8.4.3)$$

where  $\lambda, \mu, \gamma$  are nonnegative real numbers such that  $\lambda + \mu + \gamma < 1$ . Then  $T$  has a unique fixed point.

**Corollary 8.4.5.** Let  $S, T : A \rightarrow A$  be mappings of a complete complex valued metric space  $(A, d)$ . Suppose that there exists a function  $\phi \in \varphi$  such that

$$d(Sa, Tb) \lesssim \phi(d(a, b)), \quad (8.4.4)$$

for all  $a, b \in A$ . Then  $S$  and  $T$  have a unique common fixed point.

**Corollary 8.4.6.** Let  $S, T : A \rightarrow A$  be mappings of a complete complex valued metric space  $(A, d)$ . Suppose that there exists a constant  $\lambda \in [0, 1)$  such that

$$d(Sa, Tb) \lesssim \lambda \max \left\{ d(a, b), \frac{d(a, Sa)d(b, Tb)}{1 + d(a, b)}, \frac{d(b, Sa)d(a, Tb)}{1 + d(a, b)} \right\}, \quad (8.4.5)$$

for all  $a, b \in A$ . Then  $S$  and  $T$  have a unique common fixed point.

**Corollary 8.4.7.** Let  $S, T : A \rightarrow A$  be mappings of a complete complex valued metric space  $(A, d)$ . Suppose that there exist constants  $A, B, C \geq 0$  with  $(A + 2B + 2C) \in [0, 1)$  such that

$$d(Sa, Tb) \lesssim Ad(a, b) + B \left[ \frac{d(a, Sa)d(b, Tb)}{1 + d(a, b)} \right] + C \left[ \frac{d(b, Sa)d(a, Tb)}{1 + d(a, b)} \right], \quad (8.4.6)$$

for all  $a, b \in A$ . Then  $S$  and  $T$  have a unique common fixed point.

The following corollary presents the well known Banach contraction principle in complex valued metric spaces:

**Corollary 8.4.8.** Let  $T$  be a self-mapping of a complete complex valued metric space  $(A, d)$ . Suppose that there exists a constant  $\lambda \in [0, 1)$  such that

$$d(Ta, Tb) \preceq \lambda d(a, b), \quad (8.4.7)$$

for all  $a, b \in A$ . Then  $T$  has a unique fixed point.

## 8.5 Application

In this section, an application of Theorem 8.3.4 has been presented:

**Theorem 8.5.1.** Consider the set  $A = C([a, b], \mathbb{R}^n)$ , where  $[a, b] \subseteq \mathbb{R}^+$  and the metric  $d : A \times A \rightarrow \mathbb{C}$  is defined by

$$d(u, v) = \max_{t \in [a, b]} |u(t) - v(t)| \sqrt{1 + a^2} e^{i \tan^{-1} a}.$$

Consider the Urysohn integral equations

$$\begin{aligned} u(t) &= \int_a^b K_1(t, s, u(s)) ds + g(t), & t \in [a, b], \\ u(t) &= \int_a^b K_2(t, s, u(s)) ds + h(t), & t \in [a, b] \end{aligned} \quad (8.5.1)$$

where  $t \in [a, b] \subset \mathbb{R}$ ,  $u, g, h \in A$ .

Define the mappings  $F_u, G_u \in A$  for each  $u \in A$  by

$$F_u(t) = \int_a^b K_1(t, s, u(s)) ds, \text{ for all } t \in [a, b],$$

and

$$G_u(t) = \int_a^b K_2(t, s, u(s)) ds, \text{ for all } t \in [a, b],$$

where  $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

If there exists two mappings  $\alpha \in \Delta$  and  $\phi \in \varphi$  fulfilling the following conditions for each  $u, v \in A$ ,

(i)  $\alpha(u, v) \|F_u(t) - G_v(t) + g(t) - h(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a} \lesssim \phi(\max\{A(u, v)(t), B(u, v)(t), C(u, v)(t)\})$ ,

where

$$A(u, v)(t) = \|u(t) - v(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a},$$

$$B(u, v)(t) = \frac{\|F_u(t) + g(t) - u(t)\|_\infty \|G_v(t) + h(t) - v(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}}{1 + \|u(t) - v(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}},$$

and

$$C(u, v)(t) = \frac{\|F_u(t) + g(t) - v(t)\|_\infty \|G_v(t) + h(t) - u(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}}{1 + \|u(t) - v(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}},$$

(ii) there exists  $u_0 \in A$  satisfying  $\alpha(u_0, F_{u_0}(t) + g(t)) \geq 1$ ,

(iii)  $\alpha(u, v) \geq 1 \Rightarrow \alpha(F_u(t) + g(t), G_v(t) + h(t)) \geq 1$ ,

(iv) there exists a  $z \in A$  satisfying  $\alpha(u, z) \geq 1$  and  $\alpha(v, z) \geq 1$ , then the system of integral equations (8.5.1) have a unique common solution.

*Proof.* Clearly,  $(A, d)$  is a complete complex valued metric space. Define the mappings  $T, S : A \rightarrow A$  by

$$Tu = G_u + h, Su = F_u + g.$$

Then,

$$d(Su, Tv) = \|F_u(t) - G_v(t) + g(t) - h(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}$$

$$d(u, Su) = \|F_u(t) + g(t) - u(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}$$

$$d(v, Tv) = \|G_v(t) + h(t) - v(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}$$

and

$$d(v, Su) = \|F_u(t) + g(t) - v(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}$$

$$d(u, Tv) = \|G_v(t) + h(t) - u(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}$$

It can be easily seen from (i) that

$$\alpha(u, v)d(Su, Tv) \lesssim \phi(M(u, v)),$$

where  $M(u, v) = \max \left\{ d(u, v), \frac{d(u, Su) \cdot d(v, Tv)}{1 + d(u, v)}, \frac{d(v, Su) \cdot d(u, Tv)}{1 + d(u, v)} \right\}$  for every  $u, v \in A$ . From condition (ii), we obtain that there exists  $u_0 \in A$  such that  $\alpha(u_0, Su_0) \geq 1$ . And, from condition (iii), we obtain that the pair  $(S, T)$  is an  $\alpha$ -admissible pair of mappings. Obviously,  $S$  and  $T$  are continuous. All the required conditions of Theorem 8.3.4 are true and so there exists a unique common solution of the system of integral equations (8.5.1).  $\square$

## 8.6 Scope for future work

Based on present research study, it is suggested that many fixed point results can be obtained in various other abstract spaces such as convex metric spaces, symmetric spaces, complex valued metric spaces, metric-like spaces etc by making use of various other contractive conditions.



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