CONSTRUCTION OF NEW ITERATIVE METHOD TO SOLVE LINEAR SYSTEMS USING GENERALIZED INVERSE

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Mathematics and Computing

Submitted by

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DEDICATED

TO

GOD, MY PARENTS

AND

SUPERVISOR

CERTIFICATE

I hereby certify that the work which is being presented in the thesis "**Construction of new iterative method to solve linear systems using generalized inverse**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics, Thapar institute of engineering and technology, Patiala is an authentic record of my own work carried under the supervision of **Dr. Sanjeev Kumar.**

The matter presented here in this thesis has not been submitted for award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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(Chirag Chandel)

ABSTRACT

Numerical analysis constitutes the study, modification and analysis of methods explicitly which can't be solved analytically. Applied field of engineering require extensive knowledge and detailed analysis for solving linear system of equations. Hence our basic and fundamental task is to determine and generate methods which help us in finding the solutions. Thus, the thesis presented here gives comprehensive view of various techniques required in study linear systems. The thesis consists of four chapter as follows:

CHAPTER 1: This chapter constitutes basic definitions and concepts used in study of Linear algebra which are implemented in numerical techniques. Further, several techniques used for finding Moore-Penrose inverse are also included in light to solve real life engineering problems.

CHAPTER 2: In this chapter assessment of different iterative schemes used to determine the generalized inverse of matrices are defined. The schemes are organized in their increasing order of convergence and their survey is being done.

CHAPTER 3: The chapter contains a new iterative method for solving the system of linear equations. The concept has also been extended to Moore-Penrose inverse. The convergence criteria of proposed method along with detailed analysis is provided.

CHAPTER 4: In this chapter numerical comparisons of new method with methods of same as well as different orders are provided. Some engineering applications have been provided for this purpose. The results and various factors related to problems are obtained using *Mathematica*[11]

GLOSSARY OF SYMBOLS

$V(F)$	Vector Space V over Field F
$\| \|$	Norm
$*$	Conjugate Tranpose
	Moore-Penrose Inverse
;	Transpose
Z_0	Initial Guess
Z_{t+1}	Iteration at $(t+1)^{th}$ step
p	Order of convergence
σ	Singular values
δ	Zero of function
a_t	Sequence of approximation
κ	Condition number
X	$(x_1, x_2, \cdots, x_n)^{;}$ is a column vector
ϵ_{t+1}	Error at $(t+1)^{th}$ iteration

Contents

Chapter 1

Introduction and fundamental concepts

Numerical analysis is field of mathematics which deals with methods to find the approximate solutions to the problems. Several complex engineering problems and science applications cannot be solved analytically. Hence one has to depend on iterative techniques which helps devise algorithms and provide approximate solutions of the problems.

With the advent and easy availability of digital computers it has become more easy and efficient to get the numerical solutions of the problems. Earlier with limited growth of computation only simple analytical approach were used, but with the evolution in field of computers and greater accessibility, problem formulation and solution interpretation has become more efficient. The large and complicated geometries of science and engineering are modeled by using equations and precise solutions are obtained.

There are several spheres of mathematics that make rigorous use of numerical techniques to achieve results. Some of its domains are root finding problems, solving initial value and boundary value problem, approximation and interpolation of unknown functions, solution of integral using finite difference methods, problem of finding eigenvalues/eigenvectors and solution of systems etc.

The problem involving numerical computations can be solved using generally two approaches [1].

1. Direct approach
2. Indirect approach

Direct approach also known as analytical methods are those which are directly applied to problem. There are certain formulae's based on various conditions and when fulfilled on problems, the problems are solved by using these methods. These methods are generally easy to use, apply and examine. These methods can be used generally to solve problems with less dimensions. On the other hand the methods that cannot be solved directly by using any results or formulae are known as indirect methods or numerical methods.These methods consist of series of steps that are repeated till stopping criteria is achieved.

One of the principal field where numerical techniques are used is solving the system of equations which can be both linear as well as non-linear. However in linear systems the real world problems consists of huge number of equations which makes it an effective tool for problem solution. There are several direct methods such as Gauss elimination, Gauss Jordan etc in order to solve such systems, however incase of large sparse matrix or singular and rectangular matrices the inverse are evaluated using iterative methods and are called indirect methods. The iterative methods include initial guess and forms successive approximations which are obtained till tolerance is achieved. Examples include Gauss Jacobi, Power method, SOR methods etc which are also known as numerical methods.

1.1 Definition Review

1. **Matrix**:
   It is defined as arrangement of elements in particular order that satisfies certain set of rules for multiplication and addition. It can also be referred to as array of identical elements arranged in similar order. A matrix can be both rectangular or square depending upon the order of a matrix. It is classified as singular if its determinant is zero otherwise non-singular.

2. **Vector Space**:
   If (F,+,.) is field, known as field of scalars and V is an non-empty set known as set of vectors, then V is called vector space over field F denoted by $V(F)$ if there is defined internal composition + on V known as vector addition w.r.t to which V is an abelian group i.e. $(V, +)$ is an abelian.
   Also there is defined external composition in V over F known as scalar multiplication. Both the internal and external composition satisfy the following properties [1].

   (a) $(\gamma + \beta).a = a.\gamma + b.\beta \; \forall a \in F$ and $\gamma \in V$
   (b) $(a + b)\gamma = a.\gamma + b.\gamma \; \forall a, b \in F$ and $\gamma \in V$
   (c) $(ab)\gamma = a(b\gamma) \; \forall a, b \in F$ and $\gamma \in V$
   (d) $1.\gamma = \gamma \; \forall 1 \in F$ and $\gamma \in V$

3. **Error**:
   It is defined as difference between the actual value of solution and approximate value of the solution. The error of an element or solution depend upon the significant figures and also vary according to various error techniques. The error are of generally two types

   (a) Absolute error: Let z^* denotes the approximated value of exact value z. Then in such case the absolute error is defined as magnitude of difference between z and z^*. It is represented as

   $$F_a = |z - z^*| \tag{1.1.1}$$

   (b) Relative error: Let z^* denotes the approximated value of exact value z. Then relative error is defined as absolute error related to exact value z. It is represented as

   $$F_r = \frac{|z - z^*|}{|z|} \tag{1.1.2}$$

4. **Norm**:
Norm is a tool which is used to measure the size of objects such as vectors or matrices. It is basically a unit of measurement which relates how two vectors are close or far from each other. In matrices it tells us about measure of matrices which gives its value in form of real valued number.

Norm is mapping $f : P \times P \longrightarrow K$, where P can be either vector or matrix and K is real number. If P be any matrix then $\|P\|$ satisfies certain properties:

(a) $\|P\| \geq 0$ and $\|P\| = 0$ iff $P = 0$

(b) $\|aP\| = |a|\|P\|$

(c) $\|P + R\| \leq \|P\| + \|R\|$

(d) $\|P.R\| \leq \|P\|\|R\|$

The two basic norms are

- Euclidean norm: If P be any matrix of order $m \times n$ then Euclidean norm is defined as sum of square root of square of each element of matrix *i.e* $\|P\|_E = (\sum (a_{ij})^{\frac{1}{2}})^{\frac{1}{2}}$.
- Infinity norm: It P be any matrix of order $m \times n$ then Infinity norm is obtained by summation of absolute values of each row of matrix P and taking the maximum out of all the obtained values *i.e*
$\|P\|_\infty = max\,(\sum |a_{ij}|)$ for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$

1.2 Order of convergence

A sequence of approximations $a_0, a_1, \ldots a_t$ denoted by $\{a_t\}$ is said to have order of convergence p is there exists a finite non zero constant c such that

$$|\delta - a_{t+1}| = c\,|\delta - a_t|^p, \tag{1.2.1}$$

where c lies between $[0,1]$ and is called as asymptotic error constant. If the above condition is satisfied we say the sequence has order of convergence p. Also $\epsilon_{t+1} = \delta - a_{t+1}$ represents the error at $(t+1)^{th}$ iteration. Similarly the error at t^{th} iteration will be $\epsilon_t = \delta - a_t$. Hence the equation (1.2.1) can be represented as

$$|\epsilon_{t+1}| = c\,|\epsilon_t|^p \tag{1.2.2}$$

1.3 Condition number

It is defined as ratio of relative change in the output of function value to relative change in the input and it is denoted by symbol κ($kappa$).

$$\begin{aligned} \kappa &= |f(z) - f(z^*)/f(z)/z - z^*/z| \\ &= |f(z) - f(z^*).z/f(z).z - z^*| \\ &\approx z.f'(z)/f(z). \end{aligned} \quad (1.3.1)$$

If this condition number is small $i.e.$ close to 1 we say that the function is well conditioned $i.e$ it is stable otherwise ill conditioned(unstable). Also in terms of matrix the condition number is calculated with the help of norm. Let

$$Px = b, \quad (1.3.2)$$

be any system of equation and let x^* be its approximate solution.Then conditions number for the given matrix P w.r.t $\|.\|$ is defined as

$$\kappa = \|P\|\|P^{-1}\|. \quad (1.3.3)$$

Also in terms of relative error it is defined as

$$\frac{\|x - x^*\|}{\|x\|} \leq \kappa \frac{\|f\|}{\|b\|}, \quad (1.3.4)$$

where f is residual vector of approximate solution x^*. Hence kappa gives relation between the accuracy of approximate solution and the residual vector r.

1.4 Gauss Elimination Method

Gauss elimination is direct method used to solve system of equation without actually finding the inverse of matrix. It consists of combination of two steps

1. Elimination
2. Backward substitution

Elimination is process of reducing the given matrix P of system of equation (1.3) into upper triangular matrix using elementary row transformation, whereas backward substitution uses substitution to find the values of variable x_i of vector x. Consider system of equation $Px = b$

where

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}, \tag{1.4.1}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \tag{1.4.2}$$

Thus after applying the elementary row transformation, the new upper triangular matrix say P^* is obtained as:

$$P^* = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & p_{22} & \cdots & p_{2m} \\ 0 & 0 & 0 & p_{nn} \end{pmatrix}. \tag{1.4.3}$$

Thus, rewriting the system of equation after transformation

$$\begin{aligned} p_{11}x_1 + p_{12}x_2 + \cdots + p_{1n}x_n &= a_{1,n+1}, \\ p_{22}x_2 + \cdots + p_{2n}x_n &= a_{2,n+1}, \\ \vdots &= \vdots \\ p_{nn}x_n &= p_{n,n+1}. \end{aligned} \tag{1.4.4}$$

Now performing the backward substitution for n^{th} step gives,

$$x_n = \frac{p_{n,n+1}}{p_{nn}}, \tag{1.4.5}$$

Continuing in this manner, the i^{th} vector of x is evaluated as follows:

$$x_i = \frac{p_{i,n+1} - \sum p_{ij}x_j}{p_{ii}}, \text{ for each } i = 1, 2, \ldots, n-1. \tag{1.4.6}$$

REMARK: The Gauss Elimination method will fail if any of diagonal elements after elimination is zero. However this can be corrected by swapping or interchanging rows based on various pivoting strategies.

1.5 Gauss Seidel and Jacobi's Method

The Gauss Jacobi's and Seidel methods are the iterative or indirect methods for solving linear system (1.3) which have higher dimensions. The reason for using iterative methods instead of direct methods (analytical) is small storage and less time consumed while computing the large sparse matrix. However, in case of smaller dimension these method fail to achieve accuracy in stipulated amount of time in comparison to directs method such as gauss elimination etc.

Let us consider the system $Px = b$ where the matrix P is of order $n \times n$. Then in gauss Jacobi's method we will solve each equation to obtain x_i corresponding to each i^{th} equation given by

$$x_i = \sum \left(-\frac{p_{ii}x_j}{p_{ii}} \right) + \frac{b_i}{p_{ii}} \text{ for } i = 1, 2, \ldots, n. \quad (1.5.1)$$

Now for each $k \geq 1$ we will find each component $(x_i)^k$ from the previous component $(x_i)^{k-1}$ we will continue in the similar manner and will stop when given tolerance is achieved.

Remark: In case, no initial guess is given start with $x_i's = 0$ for each i. The important thing that must be kept in mind is this method is feasible only if diagonal elements are non-zero.

The Gauss Seidel method is slight modification of Gauss Jacobi. It states that instead of deriving each component $(x_i)^k$ from the previous component $(x_i)^{k-1}$ for each i. We see that all components for $i > 1$, $(x_1)^k$, $(x_2)^k, \ldots (x_{i-1})^k$ has already been computed and are better approximations for each k. Hence it is rational to compute $(x_i)^k$ using these already computed improved results. Hence for each i we get solution as

$$(x_i)^k = \frac{1}{p_{ii}}[-\sum (p_{ij}(x_j)^k - \sum (p_{ij}(x_j)^{k-1}) + b_i]. \quad (1.5.2)$$

Continuing in the similar manner the process will stop when given tolerance is achieved. Hence the above modified method is called Gauss Seidel method

1.6 Inverse using Moore-Penrose properties

The main disadvantage of finding the solution of system was that inverse doesn't exists for singular as well as rectangular matrices. Thus to overcome this drawback Penrose in 1955 gave Penrose equations which laid the foundation for inverse of generalized matrices. By generalized inverse, it meant that the inverse exists for matrices of any order and retain some properties of usual inverse. This approach for calculating the inverse is entirely based on Moore-Penrose inverse properties. For that consider a matrix P of order $m \times n$. If matrix

P has full column rank *i.e* its rank equal to number of columns n . Then the inverse is denoted by

$$P^{\dagger} = (P'P)^{-1}P' \tag{1.6.1}$$

Similarly if the rank of the matrix if full row rank *i.e* its rank equal to number of rows m. Then the inverse is denoted by

$$P^{\dagger} = P'(P'P)^{-1} \tag{1.6.2}$$

1.7 Full Rank Factorization Method

Let us consider a matrix P of order $m \times n$ and rank r. Then the full rank factorization methods divides or factorizes the matrix P into product of two matrix say R and S which will be either full column rank or full row rank matrices. Also the rank of both matrices must be equal to r. For example let us assume a matrix given by

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}. \tag{1.7.1}$$

Then, by applying full rank factorization over it we will get corresponding full rank and full column matrices of order two given by R and S respectively and its product will be equal to P *i.e*

$$P = RS \tag{1.7.2}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 3 \end{pmatrix}. \tag{1.7.3}$$

Now by using the property method for each full column matrix (1.6.1) and full row matrix (1.6.2) the inverse is obtained and their product will give inverse of matrix *i.e* $P^{\dagger} = S^{\dagger}R^{\dagger}$.

1.8 Singular Value Decomposition

The singular value decomposition is the method which is used to find the generalized inverse of any matrix i.e. both rectangular or square. As the name suggests it consists of basic two steps. The first step constitutes finding the singular values of matrix and second step consists of transforming any given matrix say P into product of orthogonal or unitary matrices. This is an extended concept of diagonalization[2] which has applications is least square problems using decompositions.

Singular values: Let P be any matrix then the singular values of these matrices will be obtained by first finding out all the non-negative eigenvalues of either PP^* or P^*P and then taking the square root of these values. For example consider a matrix given by

$$P = \begin{pmatrix} 1.732 & 2 \\ 0 & 1.732 \end{pmatrix} \tag{1.8.1}$$

Then eigen values corresponding to matric PP^* are 1 and 9. Hence its singular values are 1 and 3.
Now if P be any matrix of order $m \times n$ then SVD method will transform it into the form $P = ADB^*$ where order of A and B are $m \times m$ and $n \times n$. If we suppose the rank of matrix P be r.

1. The columns of matrix A and B will consist of eigenvectors obtained after solving PP^* and P^*P respectively.

2. The matrix D of order $m \times n$ will consists of singular values of matrix P upto order r and rest of the entries will be zero.

3. The Product of all these matrices in given order will give the initial matrix P.

For example say P be matrix of order 2×3 given by

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{1.8.2}$$

Then singular values related to the matrix P are 1.414 and 1.732 which makes the diagonal elements of matrix D. Thus finding the eigenvectors corresponding to PP^* and P^*P we get matrices A, B given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.8.3}$$

and

$$B = \begin{pmatrix} 0.70 & 0.57 & 0.40 \\ 0 & 0.57 & -.81 \\ 0.70 & 0.57 & 0.40 \end{pmatrix}. \tag{1.8.4}$$

Now in order to find the generalized inverse, if P be a matrix and A, B be unitary or orthogonal matrices then inverse is given as:

$$P^{\dagger} = B^{*}D^{\dagger}A^{*} \tag{1.8.5}$$

1.9 Decell's Method

Decell's method derives the inverse of matrix using Cayley Hamilton theorem [3]. If P be any complex matrix of order $m \times n$ and $f(\beta)=(-1)^n\ [\beta^n + a_1\beta^{n-1} + a_2\beta^{n-2} + \ldots + a_{n-1}\beta + a_n]$ represent the characteristics polynomial of PP^{*}.
Then, if there exists a largest nonzero integer given by $a_k \neq 0$ then its inverse is calculated by

$$P^{\dagger}=-a_k^{-1}P^{*}[(PP^{*})^{k-1} + a_1(PP^{*})^{k-1} + \cdots + a_{k-1}].$$

However if $k = 0$ where $a_k \neq 0$ then its inverse is zero.

1.10 Penrose Equations

Let us take P be any matrix and X be its generalized inverse, then various Penrose equations are defined as:

$$PXP = P, \tag{1.10.1}$$
$$XPX = X, \tag{1.10.2}$$
$$(PX)^{*} = PX, \tag{1.10.3}$$
$$(XP)^{*} = XP, \tag{1.10.4}$$

Based on the above four conditions, different types of generalized inverse are categorized:

1. X is said to be 1-inverse if conditions (1.10.1) is satisfied. It is denoted as P^{-}
2. X is said to be 1,2- inverse if conditions (1.10.1), (1.10.2) are satisfied . It is denoted as $(P_r)^{-}$
3. X is said to be minimum norm solution if conditions (1.10.4), (1.10.1) are satisfied. It is denoted as $(P_m)^{-}$

4. X is said to be least square inverse if conditions (1.10.1), (1.10.3) are satisfied. It is denoted as $(P_l)^-$
5. X is said to be Moore-Penrose inverse if conditions (1.10.1), (1.10.2), (1.10.3), (1.10.4) are satisfied. It is denoted as $P^\dagger$

Remark: The last defined inverses are not unique except Moore-Penrose inverse which is always unique for matrix.

The Penrose equations are modified and inverse is defined as

$$P^\dagger = XPY, \tag{1.10.5}$$

where

$$X^* = P(PP^*)^{-1}, \tag{1.10.6}$$

and

$$Y = A^*(AA^*)^{-1}. \tag{1.10.7}$$

Consider a matrix of order 3 × 6 as follows:

$$P = \begin{bmatrix} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 & -1 & 1 \end{bmatrix}. \tag{1.10.8}$$

Using equation (1.10.6) and (1.10.7)

$$X = \begin{bmatrix} 0.21 & 0.15 & 0 \\ 0.15 & 0.36 & 0 \\ -0.05 & 0.21 & 0 \\ 0.26 & -0.05 & 0 \\ -0.21 & -0.05 & 0 \\ 0.15 & 0.36 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} -0.33 & -0.33 & 0.33 \\ 0.66 & 0.66 & 0.33 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.10.9}$$

Thus, by using (1.10.5) inverse of matrix P is:-

$$P^\dagger = \begin{bmatrix} 0.08 & 0.03 & 0.12 \\ -0.01 & 0.19 & 0.17 \\ -0.10 & -0.15 & 0.05 \\ 0.19 & -0.12 & 0.07 \\ -0.08 & -0.03 & 0.12 \\ -0.01 & 0.19 & 0.17 \end{bmatrix}. \tag{1.10.10}$$

1.11 Drazin Pseudoinverse

This method is used widely and specifically for finding the inverse of singular matrices of order $n \times n$ and is also known as Drazin inverse. It is denoted by P^d, P_d, P^D. It is also used for finding the eigenvalue and eigenvector inverse of matrix. Let P be matrix of order $n \times n$ then deriving equations for Drazin inverse are given by

$$\begin{aligned} PX &= XP \\ P^{k+1}X &= P^k, \end{aligned}$$

where k is any positive integer and is called index of matrix which leads to inverse of matrix. In order to derive the inverse using Drazin method following steps must be computed:

1. Begin with determining the index of a matrix P using (1.11.1) and let it be k.
2. Characteristics polynomial and minimal polynomial corresponding to given matrix are calculated. Let us denote minimal polynomial by $m(\beta)$ where β represents the corresponding eigenvalue of matrix.
3. Obtain a polynomial say $r(\beta)$ from minimal polynomial using the result $m(\beta) = c\beta^k(1 - \beta r(\beta))$.
4. Finally Drazin inverse is obtained by using the result

   $P^d = P^{l}(r(P))^{l+1}$, where $l \geq k$.

Chapter 2

Review of iterative methods of various orders for finding the generalized inverse

This chapter contains review of various iterative methods for finding the inverse of matrices of any order. The applicability of each method is tested on non-singular matrices and their theoretical analysis is done. Along with this, by aid of Penrose properties each method can be extended to evaluate the pseudoinverse of singular as well as rectangular matrices

2.1 Brief History

The beginning of inverse stared with idea of general reciprocal that was started by Moore [4]. The concept was evolved over the time defining certain characteristics which was applied to solve system of linear equations. As he started progressing with his work, he gave concept of inverse for solving a square matrix which was used in estimations in least square methods. In similar course of time several mathematician [5] worked on various applications and found various conditions used to find inverse. Since the work of Moore [6] started getting depleted Penrose [7] renewed it and presented it in new way. Simultaneously, Rao [8] a great mathematicians originated a method to find inverse of square matrices and reflected it in various fields. Hence contributions of all these expertise made the whole concept more cumbersome. Thus the work was assembled, organized and new abbreviations was given which was to be used to avoid ambiguity.

Several Methods such as Gauss Elimination, Gauss Jordan etc. were available to solve the inverse of non-singular matrices. These were called as direct methods which were effective in the case of matrices of lesser order. However for matrices that were non-singular and of larger order there was need to find inverse. Hence the term Generalized inverse were coined. Now there were many pre existing conditions but four most important conditions was given by Penrose [7] whose combination provided several types of inverse depending on requirement.

As mentioned earlier the direct methods are already available however when matrix is of higher dimension or system of equation is quite large these methods become inefficient and use large storage capacity which reduces the computational ability. Thus several iterative methods were deduced with different order of convergence and matrix by matrix multiplication which enhanced the efficiency.

2.2 Iterative method of Order two

The first iterative technique used to find the inverse was developed using Newton's method and is known as Schulz's iterative method [9]. This technique gives effective results when applied generally to square matrices. We know classical Newton Raphson's method is given by

$$Z_{t+1} = Z_t - \frac{g(Z_t)}{g'(Z_t)} \text{ where } t \geq 0 \tag{2.2.1}$$

Provided $g'(Z_t) \neq 0$. Let us assume P be any matrix whose inverse is given by Z.

$$g(Z) = P - \frac{1}{Z}, \tag{2.2.2}$$

$$\Rightarrow g'(Z) = \frac{1}{Z^2}, \tag{2.2.3}$$

substituting value of (2.2.2) and (2.2.3) in (2.2.1) and after simplification, one get

$$Z_{t+1} = Z_t - PZ^2_t + Z_t, \tag{2.2.4}$$

$$Z_{t+1} = Z_t(2I - PZ_t). \tag{2.2.5}$$

Where I is the identity matrix of order same as P.
Thus the iterative method given by (2.2.5) is known as Schulz's method of order two.

2.3 Iterative method of order three

One of the methods for solving the linear systems apart from using direct methods are preconditioners. Let us consider a system of equation given by $Px = b$ then here we consider a matrix say Q which is invertible and coined as preconditioning matrix. This preconditoner will use iterations to solve either left or right conditioner system and here $b = Q^{-1}x$. The other important point that must be kept in mind while solving these is that matrix PQ must approximates well to identity matrix.

However the drawback of such systems are that it is tedious to find matrix Q. Hence we look for method which gives results with faster convergence. Thus we obtained a iterative technique which has order of convergence *three* and is referred to as Chebyshev's iterative method [10].

It is defined as

$$Z_{t+1} = Z_t - \frac{g(Z_t)}{g'(Z_t)} - \frac{[g(Z_t)]^2 . g''(Z_t)}{2.[g'(Z_t)]^2} \tag{2.3.1}$$

Let us assume P be any matrix whose inverse is given by Z then
$g(Z) = P - \frac{1}{Z}$,
$g'(Z) = \frac{1}{Z^2}$
$g''(Z) = -2Z^3$.
Substituting values of $g(Z)$, $g'(Z)$, $g''(Z)$ in equation (2.3.1), one gets

$$Z_{t+1} = (2I - PZ_t)z_t + [(PZ_t)^2 + I - 2PZ_t]Z_t,$$
$$Z_{t+1} = Z_t[3I - 3PZ_t + (PZ_t)^2], \text{ where } t \geq 0. \tag{2.3.2}$$

2.4 Iterative method of order four and five

The specific and detailed solution of inverse in case of complex rectangular matrices can be found using SVD method but due to its high operational and computational cost it fails to achieve the optimum objective.

Hence iterative technique consisting of series was proposed by Li et al.[11] which is used to solve system of order *four* and *five* and is presented below.

$$Z_{t+1} = Z_t[(D^1)_k I - (D^2)_k P Z_t + \cdots + (-1)^{k-1}(D^k)_k P (Z_t)^{k-1}] \text{ for each } k = 2, 3, \ldots\ldots$$

Here k represents the order of method. Substituting different values of k will give iterative methods of different order. This iterative method also holds for computing the Moore-Penrose inverse with higher order of convergence [12].

2.5 Iterative method of order six

The iterative method of order *six* for finding the inverse was computed by Soleymani et al.[13] in 2013. The concept was also extended for finding the weighted Moore-Penrose inverse of matrix. We know the matrix equation is given by

$$g(Z) = P - \frac{1}{Z}. \tag{2.5.1}$$

Using the combination of the following iterative functions

$$\begin{cases} Q_t = Z_t - g'(Z_t)g(Z_t), \\ R_t = Q_t - [(Q_t - Z_t)^{-1}(g(Q_t) - g(Z_t))]^{-1}g(Q_t), \\ S_t = R_t - [(R_t - Q_t)^{-1}(g(R_t) - g(Q_t))]^{-1} g(R_t), \\ Z_{t+1} = S_t - [(Z_t - S_t)^{-1}(g(Z_t) - g(W_t))]^{-1}g(S_t). \end{cases} \tag{2.5.2}$$

and applying the iterative functions on (2.5.1), the obtained iterative method is defined as

$$Z_{t+1} = Z_t[6I - 15PZ_t + 20(PZ_t)^2 - 15(PZ_t)^3 + 6(PZ_t)^4 - (PZ_t)^5] \tag{2.5.3}$$

The above final equation gives the iterative method of order *six*. Now we must try to simplify the above given iterative procedure in order to achieve less matrix by matrix multiplication.

$$\begin{cases} M = PZ_t, \\ N = M(-I + M), \\ Z_{t+1} = Z_t(2I - M)(3I - 2M + N)(I + N). \end{cases} \tag{2.5.4}$$

Hence the above equation given *sixth* order iterative method with *five* matrix by matrix multiplications for solving the system of Linear equation

2.6 Iterative method of order eight

In year 2013, F.Solyemani computed the iterative method of order eight[14] for finding the inverse of matrix. The Matrix equation for inverse is given by

$$g(Z) = P - \frac{1}{Z}. \tag{2.6.1}$$

Using the following non linear solver on (2.6.1)

$$\begin{cases} Q_t = Z_t - g'(Z_t)g(Z_t), \\ R_t = Z_t - (2^{-1}g(Z_t))(g'(Z_t)^{-1} + g'(Q_t)^{-1}), \\ S_t = R_t - [(R_t - Q_t)^{-1}(g(R_t) - g(Q_t))]^{-1} g(R_t), \end{cases} \tag{2.6.2}$$

$$Z_{t+1} = S_t - [(R_t - S_t)^{-1}(g(R_t) -$$

The derived iterative method is given as:

$$Z_{t+1} = Z_t(6I - 15PZ_t + 20(PZ_t)^2 - 15(PZ_t)^3 + 6(PZ_t)^4 - (PZ_t)^5), \qquad (2.6.3)$$

which is the required iterative method of eight order. Now in order to increase the computational efficiency the method must be made more comprehensible by reducing number of matrix by matrix multiplications.

$$\begin{cases} L_t = PZ_t, \\ S_t = 9I + L_t[-16I + L_t(14I + L_t(-6I + L_t))], \\ Z_{t+1} = \frac{-1}{4} Z_t S_t(-4I + L_t S_t). \end{cases} \qquad (2.6.4)$$

Hence we obtained the *eight* order method and iterative sequence Z_t converges to P^{-1} under given conditions.

2.7 Iterative method of order nine

Important aspect of iterative methods are that the method is said to be more efficient if it contains higher order of convergence with less matrix multiplications. Hence several methods with higher convergence were found. However the other precautions kept in mind was that, the iterative method must converge and have less number of iteration. Thus by applying the following nonlinear solver on matrix equation (2.6.1) the ninth order iterative method developed by Sharifi et al. [15] is obtained as.

$$\begin{cases} Q_t = Z_t - g'(Z_t)^{-1}g(Z_t) - 2^{-1} \times \\ (g'(Z_t)^{-1}g''(Z_t))^2, \; R_t = Q_t - 2^{-1}g'(Q_t)^{-1}g(Q_t), \\ Z_{t+1} = Q_t - g(R_t)^{-1}g(Q_t). \end{cases} \qquad (2.7.1)$$

After solving the above given equations, one obtains the iterative method as

$$Z_{t+1} = \frac{1}{4}(3I + PZ_t(-3I + PZ_t))(4I - (-I + PZ_t))^3(-3I + PZ_t(-3I + PZ_t(-3I + PZ_t)))^2, \qquad (2.7.2)$$

Streamlining the above method for lesser matrix multiplications, we get

$$\begin{cases} M = PZ_t, \\ N = 3I + M(-3I + M), \\ O = MN, \\ Z_{t+1} = \frac{-1}{4} Z_t N[-13I + O(15I + O(-7I + O))]. \end{cases}$$

Hence the *ninth* order method with *seven* matrix by matrix multiplications is achieved.

2.8 Initial Guesses

It is well known fact that in order to preserve or achieve convergence of any iterative methods it is required to have correct choice for initial value Z_0. There are several initial choices available depending upon the type of matrices we deal with. Usually for Newton method we take starting values as

$$Z_0 = \frac{2P^*}{\sigma_l^2 + \sigma_r^2}, \tag{2.8.1}$$

where σ_l and σ_r represents the largest and smallest singular values of matrix P. This initial guess was given by Pan and Schreiber [16] and is used for square non-singular matrices.

The other options of initial guess for square matrices are given by Ben-Israel and Greville [17] as

$$Z_0 = \alpha P^* \tag{2.8.2}$$

,

where $0 < \alpha < \frac{2}{(\|P\|_2)^2}$ or $Z_0 = \alpha I$ where I represent the identity matrix and must satisfy the result $\|I - \alpha P\| < 1$.

If say the matrix P is diagonally dominant then starting values will be

$$Z_0 = diag(1/p_{11}, 1/p_{22}, \cdots 1/p_{mm}).$$

For symmetric positive definite matrix we will take $Z_0 = (\frac{1}{\|P\|_F})$ as initial approximations where $\|\cdot\|_F$ is Forbenius norm.
In case of singular or rectangular matrices, the initial guess is taken as

$$Z_0 = \frac{P^*}{\|P\|_1 \|P\|_\infty} \tag{2.8.2}$$

Chapter 3

Construction of new iterative method for finding generalized inverse

3.1 Main Idea

As discussed in earlier chapters the main aim of solving the inverse by using iterative methods are to avoid high storage and more computational time drawbacks. In case of matrices with maximum elements as zero i.e. sparse matrix, the cost becomes exorbitant which causes hurdle to effectiveness to method. These were the purposes which enabled to come up with iterative techniques as invention of new digital computers came into effect. The direct methods were highly inefficient in terms of operational costs and could not be implemented on large systems.

The other important thing that is required in the iterative methods are initial guess. Various methods are subject to change in number of iterations and accuracy depending upon the initial guess. There are several initial guess that must be taken into account before starting the iterative procedure.
As we know that most iterative methods usually contains Schulz's formulae and basic matrix function defined as $g(Z) = P - \frac{1}{Z}$. Using Schulz's formuale and some more functions we get an iterative method as defined in equations below given as

$$
\begin{cases}
Q_t = Z_t - \dfrac{g(Z_t)}{g'(Z_t)}, \\
R_t = Q_t - \left[\dfrac{g(Q_t)}{Q_t} \cdot \dfrac{g(Z_t) + g(Q_t)}{g(Z_t) - g(Q_t)}\right], \\
S_t = R_t - \dfrac{g(R_t)}{2g'(R_t)}\left[2 + \dfrac{g(R_t) \cdot g(R_t)}{g'(R_t) \cdot g'(R_t)}\right].
\end{cases}
\tag{3.1.1}
$$

Now we know that the method is more potentially accurate if it has higher order of convergence and in comparison to it less matrix by matrix multiplication are achieved. This method here achieves order of convergence *nine* and its error equation is defined as

$$8(P_2)^6(-2P_2^2 + P_3)(\epsilon)^9 + O(\epsilon)^{10} \quad (3.1.2)$$

Hence one can write the matrix inverse of the above method as

$$\begin{aligned} Z_{t+1} = {} & Z_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2.PZ_t))(3 + PZ_t(-2 + PZ_t)^2 \\ & (2 + PZ_t(-3 + 2PZ_t)))(-3 + PZ_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t))), \end{aligned} \quad (3.1.3)$$

$$\begin{aligned} Z_{t+1} = {} & (Z_t)[24.I - 252(PZ_t) + 1538(PZ_t)^2 - 6129(PZ_t)^3 + 16998(PZ_t)^4 \\ & - 31400(PZ_t)^5 + 50652(PZ_t)^6 - 56403(PZ_t)^7 + 47236(PZ_t)^8 - 29580(PZ_t)^9 \\ & + 13626(PZ_t)^{10} - 4475(PZ_t)^{11} + 990(PZ_t)^{12} - 132(PZ_t)^{13} + 8(PZ_t)^{14}], \end{aligned} \quad (3.1.4)$$

In order to achieve high level of accuracy, one must simplify it in order to get less matrix by matrix multiplication. Thus the final new iterative method is given as

$$\begin{cases} Q = PZ_t, \\ R = (-2I + Q)(-2I + Q)(2I + Q(-3 + 2Q)), \\ S = QR, \\ Z_{t+1} = Z_t R(3I + S(-3I + S)). \end{cases} \quad (3.1.5)$$

which has order of convergence *nine* and matrix by matrix multiplication *eight*.

3.2 Convergence analysis of new method

Theorem 3.2.1 *Let P be any matrix denoted by $P = [p_{ij}]$ of order $n \times n$ and say Z_0 be initial approximation satisfying $\|I - PZ_0\| < 1$. then, iterative method (3.1.5) converges with ninth order to P^{-1}.*

Proof Let us assume (3.2.1) holds and the residual matrix be F_t such that $F_t = I - PZ_t$. Also for initial approximation *i.e* t=0, $F_0 = I - PZ_0$

$$\begin{aligned} F_{t+1} &= I - PZ_{t+1} \\ &= I - PZ_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t))(3 + PZ_t(-2 + PZ_t)^2 \\ &\quad (2 + PZ_t(-3 + 2PZ_t)))(-3 + PZ_t(-2 + PZ_t)^2 \\ &\quad (2 + PZ_t(-3 + 2PZ_t))), \end{aligned} \quad (3.2.1)$$

$$
\begin{aligned}
&= I - P(Z_t)[24.I - 252(PZ_t) + 1538(PZ_t)^2 - 6129(PZ_t)^3 + 16998(PZ_t)^4 \\
&- 31400(PZ_t)^5 + 50652(PZ_t)^6 - 56403(PZ_t)^7 + 47236(PZ_t)^8 \\
&- 29580(PZ_t)^9 + 13626(PZ_t)^{10} - 4475(PZ_t)^{1} + 990(PZ_t)^{12} \\
&- 132(PZ_t)^{13} + 8(PZ_t)^{14}],
\end{aligned}
$$

$$
\begin{aligned}
&= -(-I + PZ_t)^9(I + PZ_t(-5 + 2PZ_t))^3, \\
&= (I - PZ_t)^9(I + PZ_t(-5 + 2PZ_t))^3, \\
&= (F_t)^9(I - 5PZ_t + 2P^2Z_t^2)^3, \\
&= (F_t)^9(I - PZ_t - 4PZ_t + 2P^2Z_t^2).^3, \\
&= (F_t)^9(F_t + 2(-2PZ_t + 2P^2Z_t^2)).
\end{aligned} \tag{3.2.2}
$$

Let us now consider

$$PZ_t = \varphi. \tag{3.2.3}$$

substituting the value of φ in (3.2.2), one gets

$$
\begin{aligned}
&= (F_t)^9(F_t + 2(-2.\varphi + (\varphi)^2)), \\
&= (F_t)^9(F_t - 2((\varphi)^2 + I - I - 2.\varphi)), \\
&= (F_t)^9(F_t - 2((F_t)^2 + I)), \\
&= (F_t)^9(F_t - 2((F_t)^2 + I)).
\end{aligned} \tag{3.2.4}
$$

Now applying norm on both sides of (3.2.4), one deduces

$$\|F_{t+1}\| \leq \|(F_t)\|^9(\|F_t\| - 2(\|F_t\| + I)). \tag{3.2.5}$$

Also $\|F_0\| \leq 1$. So by using the concept of induction and (3.2.1), (3.2.5) , we deduce

$$\|F_{t+1}\| \leq \|(F_t)^9\|(\|F_t\| - 2(\|F_t\| + I)) \leq 1. \tag{3.2.6}$$

Assuming the fact $\|F_t\| \leq 1$, one obtain

$$\|F_{t+1}\| \leq \|(F_t)^9\|(\|F_t\| - 2(\|F_t\| + I)) \leq \|F_t\|^9. \tag{3.2.7}$$

Again by given result, one can get

$$\|F_{t+1}\| \leq \|F_t\|^9 \leq \cdots \leq (\|F_0\|^9)^{t+1}. \tag{3.2.8}$$

Thus $I - PZ_t \rightarrow 0$ *as* $n \rightarrow \infty \implies Z_t \rightarrow P^{-1}$.
Hence we have obtained that the method converges. Now we will show that order of convergence of sequence is nine.
Let us consider an error matrix given by

$$
\begin{aligned}
\varepsilon_t &= P^{-1} - Z_t, \\
P\varepsilon_t &= I - PZ_t = F_t, \\
P\varepsilon_{t+1} &= F_{t+1} = (P\varepsilon_t)^9(P\varepsilon_t - 2((P\varepsilon_t)^2 + I)), \\
\|P\varepsilon_{t+1}\| &\leq \|\varepsilon_t\|^9(P\|\varepsilon_t\| - 2((\|\varepsilon_t\|)^2 + I)), \\
&= \beta_t\|\varepsilon_t\|^9.
\end{aligned}
\tag{3.2.9}
$$

where $\beta_t = (P\|\varepsilon_t\| - 2((\|\varepsilon_t\|)^2 + I))$.

3.3 Extending the concept for finding the Moore-Penrose Inverse

Consider a matrix P of any order the we know $P^\dagger$ represents the Moore Penrose inverse of matrix P if the four given following properties are satisfies

$$
\begin{aligned}
PXP &= P, \\
XPX &= X, \\
(PX)^* &= PX, \\
(XP)^* &= XP,
\end{aligned}
$$

Here X represents the inverse of matrix. The Moore-Penrose holds true for any matrix whether it is rectangular or singular.

Lemma 3.3.1 *Consider a rectangular matrix P of order $k \times l$ then the sequence $\{Z_t\}$ satisfies the following properties for any $t \geq 0$.*

1. $(PZ_t)^* = PZ_t$.
2. $(Z_tP)^* = Z_tP$
3. $Z_tPP^\dagger = Z_t$
4. $P^\dagger PZ_t = Z_t$

Proof We will prove the following results using mathematical induction .
Consider Case 1, for $t = 0$ and let $Z_0 = \alpha P^*$ be initial approximation. Hence we get,

$$\begin{aligned}(PZ_0)^* &= (P\alpha P^*)^*,\\ &= (\alpha P^*)^* P^*,\\ &= (P^*)^*\alpha P^*,\\ &= P\alpha P^*,\\ &= PZ_0.\end{aligned}$$

$$\begin{aligned}(Z_0P)^* &= (Z_0.P^*P)^*,\\ &= P^*(Z_0P^*)^*,\\ &= P^*Z_0P,\\ &= Z_0P.\end{aligned}$$

Using the Properties of Moore - Penrose inverse, we know $PP^\dagger = (PP^\dagger)^*$ and $P^\dagger P = (P^\dagger P)^*$.
Thus,

$$\begin{aligned}Z_0PP^\dagger &= (\alpha P^*)PP^\dagger,\\ &= \alpha P^*(P^\dagger)^*P^*,\\ &= \alpha(PP^\dagger P)^*,\\ &= \alpha P^*,\\ &= Z_0.\end{aligned}$$

and

$$\begin{aligned}P^\dagger PZ_0 &= \alpha P^\dagger PP^*,\\ &= \alpha P^*(P^\dagger)^*P^*,\\ &= \alpha(PP^\dagger P)^*,\\ &= (\alpha P^*),\\ &= Z_0.\end{aligned}$$

Now let the result holds for some $t > 0$ *i.e* $(PZ_t) = PZ_t$. We will show the result also holds true for $(t + 1)$ elements

$$(PZ_{t+1})^* = [PZ_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t))(3 + PZ_t(-2 + PZ_t)^2 (2 + PZ_t(-3 + 2PZ_t)))(-3 + PZ_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t)))]^*,$$

$$= [P(Z_t)(24I - 252(PZ_t) + 1538(PZ_t)^2 - 6129(PZ_t)^3 + 16998(PZ_t)^4 - 31400(PZ_t)^5 + 50652(PZ_t)^6 - 56403(PZ_t)^7 + 47236(PZ_t)^8 - 29580(PZ_t)^9 + 13626(PZ_t)^{10} - 4475(PZ_t)^{11} + 990(PZ_t)^{12} - 132(PZ_t)^{13} + 8(PZ_t)^{14})]^*,$$

$$= [24(PZ_t) - 252(PZ_t)^2 + 1538(PZ_t)^3 - 6129(PZ_t)^4 + 16998(PZ_t)^5 - 31400(PZ_t)^6 + 50652(PZ_t)^7 - 56403(PZ_t)^8 + 47236(PZ_t)^9 - 29580(PZ_t)^{10} + 13626(PZ_t)^{11} - 4475(PZ_t)^{12} + 990(PZ_t)^{13} - 132(PZ_t)^{14} + 8(PZ_t)^{15}]^*.$$

Let us assume $J_t = (PZ_t)$

$$= [24(J_t)^* - 252(J_t^*)^2 + 1538(J_t^*)^3 - 6129(J_t^*)^4 + 16998(J_t^*)^5 - 31400(J_t^*)^6 + 50652(J_t^*)^7 - 56403(J_t^*)^8 + 47236(J_t^*)^9 - 29580(J_t^*)^{10} + 13626(J_t^*)^{11} - 4475(J_t^*)^{12} + 990(J_t^*)^{13} - 132(J_t^*)^{14} + 8(J_t^*)^{15}]^*,$$

$$= PZ_t[24I - 252(PZ_t) + 1538(PZ_t)^2 - 6129(PZ_t)^3 + 16998(PZ_t)^4 - 31400(PZ_t)^5 + 50652(PZ_t)^6 - 56403(PZ_t)^7 + 47236(PZ_t)^8 - 29580(PZ_t)^9 + 13626(PZ_t)^{10} - 4475(PZ_t)^{11} + 990(PZ_t)^{12} - 132(PZ_t)^{13} + 8(PZ_t)^{14}],$$

$$= P[Z_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t)) (3 + PZ_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t))) (-3 + PZ_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t)))],$$

$$= PZ_{t+1}.$$

Hence the result holds true for $(t + 1)$. Continuing in similar manner result also holds for $(Z_tP)^* = Z_tP$

For the third inequality let the result holds true for t elements *i.e* $Z_tPP^{\dagger} = Z_t$, By using induction we will show the result also holds true for $(t+1)$.

Hence we will show $Z_{t+1}PP^{+} = Z_{t+1}$

$$\begin{aligned} Z_{t+1}PP^{+} &= [Z_t(24I - 252(PZ_t) + 1538(PZ_t)^2 - 6129(PZ_t)^3 \\ &+ 16998(PZ_t)^4 - 31400(PZ_t)^5 + 50652(PZ_t)^6 - 56403(PZ_t)^7 \\ &+ 47236(PZ_t)^8 - 29580(PZ_t)^9 + 13626(PZ_t)^{10} - 4475(PZ_t)^{11} \\ &+ 990(PZ_t)^{12} - 132(PZ_t)^{13} + 8(PZ_t)^{14})]PP^{+}, \end{aligned}$$

$$\begin{aligned} &= [Z_t(24I - 252(PZ_t)PP^{\dagger} + 1538(PZ_t)^2.PP^{\dagger} - 6129(PZ_t)^3PP^{\dagger} \\ &+ 16998(PZ_t)^4PP^{\dagger} - 31400(PZ_t)^5PP^{\dagger} + 50652(PZ_t)^6PP^{\dagger} - 56403(PZ_t)^7PP^{\dagger} \\ &+ 47236(PZ_t)^8PP^{\dagger} - 29580(PZ_t)^9PP^{\dagger} + 13626(PZ_t)^{10}PP^{\dagger} - 4475(PZ_t)^{11}PP^{\dagger} \\ &+ 990(PZ_t)^{12}PP^{\dagger} - 132(PZ_t)^{13}PP^{\dagger} + 8(PZ_t)^{14}PP^{\dagger})], \end{aligned}$$

Let us assume $J_t = (PZ_t)$

$$\begin{aligned} &= [24(Z_t)PP^{+} - 252(J_t)PPP^{\dagger} + 1538(J_t)^2Z_tPP^{\dagger} - 6129(J_t)^3Z_tPP^{\dagger} \\ &+ 16998(J_t)^4Z_tPP^{\dagger} - 31400(J_t)^5Z_tPP^{\dagger} + 50652(J_t)^6Z_tPP^{\dagger} \\ &- 56403(J_t)^7Z_tPP^{\dagger} + 47236(J_t)^8Z_tPP^{\dagger} - 29580(J_t)^9Z_tPP^{\dagger} + 13626(J_t)^{10}Z_tPP^{\dagger} \\ &- 4475(J_t)^{11}Z_tPP^{\dagger} + 990(J_t)^{12}Z_tPP^{\dagger} - 132(J_t)^{13}Z_tPP^{\dagger} + 8(J_t)^{14}Z_tPP^{\dagger}], \end{aligned}$$

$$\begin{aligned} &= z_t[24I - 252(PZ_t) + 1538(PZ_t)^2 - 6129(PZ_t)^3 \\ &+ 16998(PZ_t)^4 - 31400(PZ_t)^5 + 50652(PZ_t)^6 - 56403(PZ_t)^7 \\ &+ 47236(PZ_t)^8 - 29580(PZ_t)^9 + 13626(PZ_t)^{10} - 4475(PZ_t)^{11} \\ &+ 990(PZ_t)^{12} - 132(PZ_t)^{13} + 8(PZ_t)^{14}], \end{aligned}$$

$$\begin{aligned} &= [Z_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t)) \\ &(3 + PZ_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t))) \\ &(-3 + PZ_t(-2 + PZ_t)^2(2 + PZ_t(-3 + 2PZ_t)))], \\ &= Z_{t+1}. \end{aligned}$$

Theorem 3.3.2 *Consider a matrix P of order $k \times l$ which is complex rectangular and singular and iterative sequence Z_t for $t = 0, 1, \cdots$. Then for any $t \geq 0$ the sequence converges to $P^{\dagger}$ which has order of convergence nine and initial approximation $Z_0 = \alpha P^*$.*

Proof Using Lemma (3.3.1), we consider $F_t = Z_t - P^\dagger$ as error matrix for finding the Moore - Penrose inverse. Now let

$$\begin{aligned} PF_{t+1} &= PZ_{t+1} - PP^\dagger, \\ &= PZ_{t+1} - I + I - PP^\dagger, \\ &= -F_{t+1} + I - PP^\dagger, \\ &= -[(F_t)^9(F_t - 2((F_t)^2 + I))] + I - PP^\dagger, \\ &= (PF_t)^9(PF_t - 2((PF_t)^2 + I)). \end{aligned} \tag{3.3.1}$$

By using identity given by

$$(I - PP^\dagger)^k = (I - PP^\dagger) \;\forall\; k. \tag{3.3.2}$$

Denoting $M = PP^\dagger$ and $N = I - PZ_0$, we get

$$\begin{aligned} M.M &= PP^\dagger PP^\dagger \\ \text{and } M.N &= PP^\dagger(I - PZ_0) \\ &= PP^\dagger - PPP^\dagger Z_0 \\ &= PP^\dagger - PZ_0 \\ &= PP^\dagger - PZ_0PP^\dagger \\ &= (I - PZ_0)PP^\dagger \\ &= NM \end{aligned} \tag{3.3.3}$$

Also we know by Stanimirovic and Ctevokvic illic [18] that if there exists two complex matrices M and N of order $n \times n$ respectively, then $\rho(MN) = \rho(N)$. Using the result we get

$$\rho(PP^\dagger - PZ_0) \leq \rho(I - PZ_0)$$

Now by using initial approximations $Z_0 = \alpha P^*$, we get

$$\rho(P(P^\dagger - \alpha_0 P^*)) \leq \rho(I - P\alpha_0 P^*) = max|1 - \lambda_i(\alpha PP^*)| \text{ for } 1 \leq i \leq s \tag{3.3.4}$$

where s represents the number of singular values. Also we know that max $|1-\lambda_i(\alpha PP^*)| < 1$ for $1 \leq i \leq s$.

And that there exists ϵ, $\|.\|$ such that

$$\|P(Z_0 - P^{\dagger})\| \leq \rho(P(Z_0 - P^{\dagger})) + \epsilon < 1 \quad i.e \; \|PF_0\| \leq 1 \tag{3.3.5}$$

Using the same concept, we obtain

$$\|PF_{t+1}\| \leq \|PF_t\| \; (^{9}\|PF_t\|) \tag{3.3.6}$$

Hence finding Moore-Penrose inverse of new iterative technique we get

$$\begin{aligned} \|Z_{t+1} - P^{\dagger}\| &= \|PP^{\dagger}Z_{t+1} - P^{\dagger}PP^{\dagger}\| \\ &\leq \|P^{\dagger}\|\|PZ_{t+1} - PP^{\dagger}\| \\ &= \|P^{\dagger}\|\|PF_{t+1}\| \end{aligned} \tag{3.3.7}$$

By using results (3.3.6) and (3.3.7), we get

$$\|F_{t+1}\| \leq \|P\|\|P^{\dagger}\|\|F_t\|^{9} \tag{3.3.8}$$

Hence we get iterative sequence which converges to Moore-Penrose $P^{\dagger}$ when $n \rightarrow +\infty$.

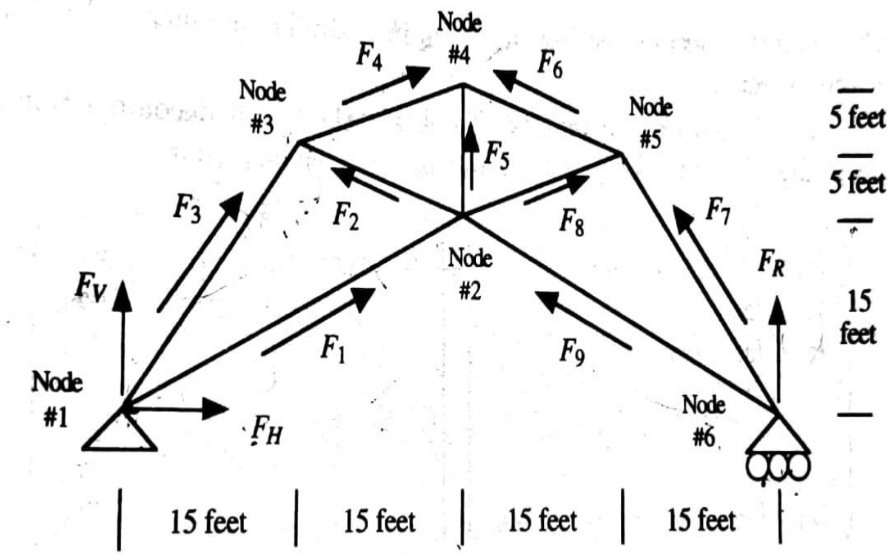
Chapter 4

Numerical reports

For checking how productive the recommended iterative method is, four problems are being tested from several fields of engineering and science. The order two, three, eight and nine are taken into account while checking how fruitful the results are and is compared with the new method. The most important factor while analyzing the results are time used for computations and residual factors, hence for that commands such as *Timeused*[] and $N\| Z_{t+1} - Z_t \|$ are used.

In all the methods used for comparing numerical results the initial guess is taken as $\dfrac{P^*}{\|P\|_1 \|P\|_\infty}$ where $\|P\|_1$ =1-norm and $\|P\|_\infty$ = infinity norm and required tolerance is set to $\frac{1}{2} \times 10^{-250}$. The programs are implemented in *Mathematica*11.1.1. The detailing for system used for programming is *Microsoft windows 8.1 intel(R), Pentium(i5) CPU, 2.60 GHz, 4 GB RAM, 64 Bit operating system.* The most important aspect that must be kept which evaluating is that the higher the order of convergence and less the number of iteration, the more effective is the method.

Problem 1 (Civil Engineering Problem)
Given a truss which is subjected to load as shown in the figure given below. We need to find reaction and member forces at within the plane truss. The load configuration of truss is given as below. At node 3 and 5, 500 pounds of force are directed vertically downwards whereas at node 4 it is 1000 pounds.



In structural engineering, analysis of structure basically involves calculation of forces and reactions.
In the given problem the structure is statically determinate truss (i.e the equations of equilibrium are sufficient to find the unknowns) and our objective is to calculate the reaction force (at supports) and member forces (compression and tension) generated due to given loading.
The system of equations after solving can be represented in the form of matrix as $PF = B$ which is of the order 12×12 where

$$P = \begin{pmatrix} 1 & 0 & A & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & B & 0 & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -B & E & 0 & 0 & 1 & 0 & 0 & E & -B & 0 \\ 0 & 0 & -A & -F & 0 & 0 & 0 & 0 & 0 & F & -A & 0 \\ 0 & 0 & 0 & -E & -D & E & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F & -C & F & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -E & -1 & -E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -F & 0 & F & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & -D & -E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -F & C & -F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & 0 & G & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C & 0 & H & 0 \end{pmatrix}.$$

Where $A = \cos(26.56), B = \sin(26.56), C = \cos(53.12), D = \sin(53.12), E = \sin(18.43), F = \cos(18.43)$, $G = \sin(28.56)$, $H = \cos(28.56)$

$$F = \begin{bmatrix} F_h \\ F_y \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_r \end{bmatrix}$$

where F_h and F_y are horizontal and vertical reaction forces respectively at static support and F_r is vertical reaction at dynamic support. $F_1, F_2, \cdots, F_9$ are internal/member forces of truss which can be either be compression or tension. The right hand side of equation is represented by vector given as

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -500 \\ 0 \\ -1000 \\ 0 \\ -500 \\ 0 \\ 0 \end{bmatrix}$$

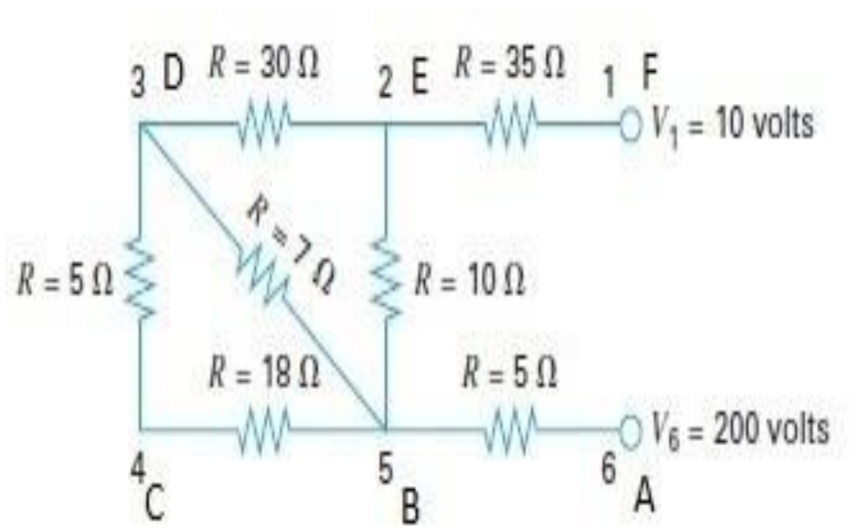
The tolerance used is $\frac{1}{2} \times 10^{-250}$ and initial guess is taken as $\frac{P^*}{\|P\|_1 \|P\|_\infty}$. The results are compared in table (4.1) with the methods of order two, three, eight and nine.

Method	Order	Iterations	Residue norm	Average Time (in seconds)
Schulz	2.000	34	1.15374×10^{-16}	4.095
Chebyshev's	3.000	23	1.17986×10^{-16}	3.11
F.Solyemani [14]	8.000	12	3.74366×10^{-16}	2.204
Sharifi et al. [15]	9.0001	12	1.77326×10^{-63}	1.999
Proposed method	9.0001	10	2.02551×10^{-197}	1.906

Table 4.1

Problem 2 (Electrical Engineering Problem)

In given circuit the resistance across each branch and voltage across initial point A and final point B is given below[19]. Let us denote Current by *I* and resistance by *R*. Our aim is to find current along each branch of circuit.



Now applying the kirchoff's voltage law in loop *i.e,* $\sum V = 0$ or $\sum IR = 0$ in circuit ABCDEF, the equations after applying the kirhoff's law is given as:-

$$
\begin{aligned}
200 - 5i_1 - 18i_2 - 5i_3 - 30i_4 - 35i_5 &= 10,\\
200 - 5i_1 - 7i_6 - 30i_4 - 35i_5 &= 10,\\
200 - 10i_7 - 35i_5 &= 10,\\
-18i_2 - 5i_3 + 7i_6 &= 0,\\
-30i_4 + 10i_7 - 7i_6 &= 0,\\
-18i_2 - 5i_3 - 30i_4 + 10i_7 &= 0.
\end{aligned}
$$

Here i_1, i_2, i_3, i_4, i_5, i_6 and i_7 represents the current along different branches of circuit. Using the above equations, system of linear equations in seven unknowns are given by

$$\begin{bmatrix} -5 & -18 & -5 & -30 & -35 & 0 & 0 \\ -5 & 0 & 0 & -30 & -35 & -7 & 0 \\ 0 & 0 & 0 & 0 & -35 & 0 & -10 \\ 0 & -18 & -5 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & -30 & 0 & -7 & 10 \\ 0 & -18 & -5 & -30 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = b,$$

where

$$b = \begin{bmatrix} -190 \\ -190 \\ -190 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now comparing the various factors for methods for several order in table (4.2), we get

Method	Order	Iterations	Residue norm	Average time(in seconds)
Schulz	2.000	15	4.04827×10^{-17}	2.704
Chebyshev's	3.0048	9	8.65239×10^{-11}	53
F.Solyemani [14]	8.000	7	1.08936×10^{-204}	1.655
Sharifi et al. [15]	9.0021	6	2.08677×10^{-38}	1.484
Proposed method	9.0001	6	8.6433×10^{-207}	1.453

Table 4.2

Problem 3 (Chemical Engineering Problem)

The variables of interest in an absorption column are steady state composition of solute in liquid, x_i and steady state composition of solute in the gas on each plate is y_i. We have been given six plate absorption column with inert composition x_0 =0.05 kg liquid and y_7 = 0.3 kg inert gas. And the liquid and gas flow rates are L=40.8 kg/min and $G = 66.7 kg/min$ respectively. Assuming that the equilibrium holds by $y_i = 0.72x_i$,We find the system $Px = b$ which represents the material balance around an arbitrary balance and is given as:

$$\begin{bmatrix} -88.824 & 48.024 & 0 & 0 & 0 & 0 \\ 40.8 & -88.824 & 48.024 & 0 & 0 & 0 \\ 0 & 40.8 & -88.824 & 48.024 & 0 & 0 \\ 0 & 0 & 40.8 & -88.824 & 48.024 & 0 \\ 0 & 0 & 0 & 40.8 & -88.824 & 48.024 \\ 0 & 0 & 0 & 0 & 40.8 & -88.824 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2.04 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20.01 \end{bmatrix}.$$

Solving the above system of equations and providing comparisons with iterative methods in table (4.3), the obtained values are:

Method	Order	Iterations	Residue norm	Average time(in seconds)
Schulz	2.000	17	9.11683×10^{-20}	2.375
Chebyshev's	3.000	9	2.07816×10^{-13}	4.718
F.Solyemani [14]	8.0211	6	2.97459×10^{-57}	1.625
Sharifi et al. [15]	9.0000	6	4.19485×10^{-84}	3.577
Proposed method	8.9569	5	5.92895×10^{-50}	1.9596

Table 4.3

Problem 4 (Academic Problem)

Let us consider a general matrix P of order 4×5. The system of equation is of the type $Px = B$. Thus in order to solve such system, inverse is obtained using iterative method. Also comparison with methods of various orders is provided in table (4.4)

$$P = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 \\ 3 & 0 & -3 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{bmatrix}$$

Method	Order	Iterations	Residue norm	Average time(in seconds)
Schulz	2.000	14	4.1534×10^{-95}	2.656
Chebyshev's	3.000	8	1.97737×10^{-38}	1.782
F.Solyemani [14]	8.0171	5	9.4283×10^{-34}	1.609
Sharifi et al. [15]	9.008	5	1.05616×10^{-45}	1.422
Proposed method	9.001	5	6.4227×10^{-250}	1.516

Table 4.4

Chapter 5

Conclusion

In this thesis we have presented a new iterative method for finding the inverse of matrices. The developed method has order of convergence nine. It has also been shown that method has eight matrix by matrix multiplications which reflects the method is computationally efficient. Further the concept of new iterative method has been extended to Moore-Penrose inverse. Also the proposed method has been compared with existing methods of various order. Further Numerical illustrations have showed that the proposed method takes less computational time and iterations then the existing methods which itself testifies the fact that method is more efficient.

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