FIXED POINT THEOREMS FOR MAPPINGS IN SOME ABSTRACT SPACES

A

THESIS

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REQUIREMENT FOR THE AWARD OF THE DEGREE

OF

DOCTOR OF PHILOSOPHY

(MATHEMATICS)

By

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March, 2014

DEDICATED TO
MY BELOVED
FAMILY

CANDIDATE'S DECLARATION

I hereby certify that the work being presented in the thesis entitled ***"Fixed Point Theorems for Mappings in Some Abstract Spaces"*** *in fulfillment of the requirements for the award of the degree of Doctor of Philosophy, submitted in the School of Mathematics and Computer Applications of Thapar University, Patiala, is an authentic record of my own work carried out during a period from August,* 2009 *to March,* 2014 *under the supervision of Dr. S.S. Bhatia and Dr. Sanjay Kumar.*

The matter embodied in the thesis has not been submitted by me for the award of any other degree.

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Date: **(*Saurabh Manro*)**

LIST OF SYMBOLS

$A = B,\ A \neq B$:	Equality and Inequality for sets
$f(x)$ or fx :	Image of x under f
iff :	if and only if
inf :	infimum (or greatest lower bound)
sup:	supremum (or least upper bound)
$\mathbf{N}$:	The set of all natural numbers
$\mathbf{R}$:	The set of all real numbers
$\mathbf{R}^{+}$:	The set of all non negative real numbers
$\mathbf{R}^{n}$:	n-dimensional Euclidean space
$\mathbf{Q}$:	The set of all rational numbers
$\lvert x \rvert$:	Absolute value of x
$I = I_X$:	The identity map
$\mathbf{U}$	Universal set

ABSTRACT

The present thesis entitled **"Fixed Point Theorems for Mappings in Some Abstract Spaces"** comprises certain investigations carried out by me at the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, under the supervision of Dr. S.S. Bhatia, Professor, SMCA, Thapar University, Patiala and Dr. Sanjay Kumar, Assistant Professor, Department of Mathematics, Deenbandhu Chhotu Ram University of Science & Technology, Murthal (Sonepat).

The aim of this work is to extend, generalize and unify various results in different abstract spaces for various mappings such as compatible mappings, R-weakly commuting mappings and their different variants.

The thesis consists of seven chapters. **Chapter-I** is introductory in nature, where we fix notations and terminology to be used in the subsequent chapters. In this chapter, some basic definitions, classical and recent results related to metric fixed point theory have been presented. Some of these results have been extended and generalized in the subsequent chapters. A brief summary of each chapter has been given towards the end of this chapter.

In Chapter-II, the concepts of compatible mappings and weakly compatible mappings of type (A) in G-metric spaces have been introduced. Some properties related to these mappings have also been proved in this chapter. The notion of the property ($E.A$) has been used to prove a common fixed point theorem for weakly compatible mappings. The results presented in this chapter have been applied to obtain the solution of an integral equation and the bounded solution of a functional equation arising in dynamic programming.

In Chapter-III, some new common fixed point theorems in G-metric spaces have been established by using the notions of compatibility, variants of R-weakly commutativity and weakly reciprocal continuity. Consequently, the results presented in

this chapter have improved and sharpened many related results present in the existing literature. Also, some examples in support of these results have been given.

The aim of **Chapter-IV** is to introduce a new property called “common limit in the range” for four self-mappings in fuzzy metric spaces and using this property, some common fixed point theorems have been established. The results presented in this chapter have generalized many known related results. Further, some common fixed theorems for R-weakly commuting mappings and some of their variants have been proved in fuzzy metric spaces.

The objective of **Chapter-V** is to prove a common fixed point theorem in intuitionistic fuzzy metric spaces by using pointwise R-weak commutativity and reciprocal continuity of mappings satisfying some contractive conditions. Towards the end of this chapter, some common fixed point theorems for weakly compatible mappings along with common ($E.A$) like property have been proved by using some implicit relations.

In **Chapter-VI,** some common fixed point theorems for pairs of compatible and subsequentially continuous mappings in intuitionistic fuzzy metric spaces via an implicit relation have been proved. The results proved in this chapter have improved many known related common fixed point theorems in these spaces. Further, some common fixed theorems for weakly compatible mappings in intuitionistic fuzzy metric spaces via an implicit relation and the common property ($E.A$) have been established.

Chapter-VII is devoted to the extension of the work done by Saadati *et al.* [112] in the frame work of modified intuitionistic fuzzy metric spaces. In this chapter, some common fixed point theorems have been established using weakly reciprocal continuous, non-compatible self-mappings via some implicit relations.

Finally, based on the present study, relevant topics for further research have been suggested.

List of Published / Accepted Research Papers

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3. Manro, S., Bhatia, S.S. and Kumar, S., Common fixed point theorems in intuitionistic fuzzy metric spaces using *R*- weakly commuting mappings, **Applied Mathematics (USA), 3(3)(2012), 225-230.**
4. Manro, S., Bhatia, S.S. and Kumar, S., Common fixed point theorems in fuzzy metric spaces, **Annals of Fuzzy Mathematics and Informatics (Korea), 3(1)(2012), 151- 158.**
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7. Manro, S., Kumar, S., Bhatia, S.S. and Tas, K., Common fixed point theorems in modified intuitionistic fuzzy metric spaces, **Journal of Applied mathematics (USA), Vol. 2013, Article ID 189321, 13 pages (SCI).**
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CHAPTER- I

INTRODUCTION

Let X be a non-empty set and $f : X \to X$ be a mapping. A point $x \in X$ is called a **fixed point** of f if $fx = x$, that is, x remains invariant under the mapping f. In graphical terms, a fixed point is the point of intersection of the graph of the curve $y = fx$ and the line $y = x$.

Fixed point theory is one of the most dynamic research area in nonlinear analysis. It has a wide range of applications in fields such as economics, computer science and many others. The most important and core result in this direction was given by Polish mathematician Stefan Banach [18] in 1922, popularly known as Banach contraction principle. This principle not only guarantees the existence and uniqueness of the fixed points of self-mappings but also gives a procedure to obtain the fixed points of these mappings. This technique is used as a powerful tool to solve various problems in different branches of Mathematics such as integral equations, ordinary differential equations, partial differential equations, dynamic programming, game theory etc. Motivated by the Banach contraction principle, several researchers (see, e.g., [22],[27], [32],[37],[52],[66],[104],[107],[123],[128],[141]) have introduced various other types of contractive conditions and have established number of related fixed point theorems. In 1977, Rhoades [108] investigated various contractive mappings that have appeared in the literature in the last twenty years and have analyzed the inter relationships among them.

The generalization of existing idea of Banach is a very important aspect for the advancement in the field of fixed point theory**.** In this thesis, fixed point results in some abstract spaces like G-metric spaces, fuzzy metric spaces, intuitionistic fuzzy metric spaces and modified intuitionistic fuzzy metric spaces have been presented and many existing results in the literature have been generalized.

To give sufficient background for subsequent chapters, we, in this introductory chapter present some basic definitions, notations and some classical and recent results related to this work. However, some of the basic definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

In Section 1.1, some basic definitions and preliminaries related to metric fixed point theory have been presented. In Section 1.2, various types of mappings in metric spaces have been presented. However, some of these mappings in context of other abstract spaces will be defined in their respective chapters. Section 1.3 deals with various abstract spaces used in the present work. Section 1.4 highlights the objectives formulated for the present study. The summary of the work presented in the subsequent chapters of this thesis are finally presented in last section.

1.1 PRELIMINARIES RELATED TO METRIC FIXED POINT THEORY

Metric spaces play a significant role in the study of functional analysis and topology. The term ‘metric’ is derived from the word ‘metor’ (measure). In 1906, Frechet [40] introduced the notion of metric space in his doctoral thesis submitted to the University of Paris. However, definition given by the German mathematician, Hausdroff [53] in 1914 is commonly used and is stated as:

Definition 1.1.1. Let X be any non-empty set. A **metric** on X is a real valued function $d : X \times X \to \mathbf{R}$ which satisfies the following conditions: for all $x, y, z \in X$,

(i)	$d(x, y) \geq 0$;	(Positivity)
(ii)	$d(x, y) = 0$ iff $x = y$;	(Identity)
(iii)	$d(x, y) = d(y, x)$;	(Symmetry)
(iv)	$d(x, z) \leq d(x, y) + d(y, z)$.	(Triangle inequality)

A **metric space** is a non-empty set X equipped with a metric d on X and is denoted by (X, d). Geometrically, $d(x, y)$ represents distance between two points x and y on the real line.

Example 1.1.1.

(i) The set $\mathbf{R}$ of real numbers with the distance function $d(x, y) = |x - y|$ is a metric space.

(ii) Every normed linear space is a metric space under the metric $d(x, y) = \|x - y\|$.

Definition 1.1.2. A sequence $\{x_n\}$ of a metric space (X, d) is said to be

(i) **convergent** to a point $x \in X$ if $\lim_{n \to \infty} d(x_n, x) = 0$.

(ii) **Cauchy** sequence if $\lim_{n,m\to\infty} d\left(x_n, x_m\right) = 0.$

Definition 1.1.3. A metric space (X, d)is said to be **complete** if every Cauchy sequence in (X, d) is convergent in X.

In a metric space, every convergent sequence is a Cauchy sequence but the converse is not true. For instance, Euclidean n-space with the Euclidean distance is complete metric space where as the set of rational numbers with metric $d(x,y)=|x-y|$ is not a complete metric space.

Definition 1.1.4.([49], p.1) Let X be a non-empty set and $f : X \to X$ be a mapping. A point $x \in X$ is called a **fixed point** of f if x remains invariant under f i.e., $fx = x$. In graphical terms, a fixed point means the point (x, fx) is on the line $y = x$, or in other words, the graph of f has a point in common with line $y = x$.

Example 1.1.2.

(i) A mapping $f:\mathbf{R}\to\mathbf{R}$ defined by $fx = 2x$, for all $x\in\mathbf{R}$, $x = 0$ is the unique fixed point.

(ii) A mapping $f:\mathbf{R}\to\mathbf{R}$ defined by $fx = x^2$, for all $x\in\mathbf{R}$, has two fixed points 0 and 1.

(iii) A mapping $f:\mathbf{R}\to\mathbf{R}$ defined by $fx = x$ (identity mapping), for all $x\in\mathbf{R}$, has infinitely many fixed points. i.e., every point of $\mathbf{R}$ is a fixed point of f.

(iv) A translation mapping $f:\mathbf{R}\to\mathbf{R}$ defined by $fx = x + 2$, for all $x\in\mathbf{R}$, has no fixed point.

These examples infer that a mapping may have a unique fixed point, more than one fixed point or infinitely many fixed points and even it may not have any fixed point.

Definition 1.1.5.([6], Vol.141,p.1) Let $\left(X,d\right)$ be a metric space. Then a mapping $f : X \to X$ is called **Lipschitzian mapping** if there exists a real number $M > 0$ such that

$$d\left(fx, fy\right) \le M\ d\left(x,y\right), \text{for all } x,y\in X.$$

Definition 1.1.6.([6], Vol.141,p.1) Let $\left(X,d\right)$ be a metric space. Then a mapping $f : X \to X$ is called **contraction mapping** if there exists a real number M, $0\le M<1$ such that

$$d\left(fx, fy\right) \le M\ d\left(x,y\right), \text{for all } x,y\in X.$$

Definition 1.1.7.([6], Vol.141,p.112) Let (X,d) be a metric space. Then a mapping $f: X \to X$ is called **non-expansive mapping** if

$$d(fx, fy) \leq d(x,y), \text{ for all } x, y \in X.$$

Definition 1.1.8.([6], Vol.141,p.112) Let (X,d) be a metric space. Then a mapping $f: X \to X$ is called **contractive mapping** if

$$d(fx, fy) < d(x,y), \text{ for all } x, y \in X \text{ and } x \neq y.$$

It is clear that contraction $\Rightarrow$ contractive $\Rightarrow$ non-expansive $\Rightarrow$ Lipschitzian.

However, converse may not be true in either case as:

(i) The identity mapping $I{:}X\to X$, where X is a metric space, is non-expansive but not contractive because for all $x, y \in X$, $d(Ix, Iy) = d(x,y)$.

(ii) Let $X = [1,\infty)$ be a complete metric space equipped with the metric of absolute value. Define $f: X\to X$ given by $fx = x + \frac{1}{x}$, for all $x \in X$. Then f is a contractive mapping, while f is not a contraction.

(iii) Let $X = [1,\infty)$ and define for all $x, y \in X$, $d(x, y) = |x - y|$. Define $f: X\to X$ given by $fx = 3x$. Then for $M = 3$, f is a Lipschitzian mapping, while f is not a contraction.

The study of fixed point theory was initiated by famous Dutch mathematician Brouwer [25] in 1912. Brouwer established the following result in this direction:

"*Let C be a closed unit ball in* $\mathbf{R}^n$ *and let f be a continuous self-mapping on C. Then f has a fixed point in C.*"

Later on, Schauder [118] extended Brouwer's result for infinite dimensional spaces. The credit of making fixed point theory useful and popular goes to Polish mathematician Stefan Banach. In 1922, Banach proved a fixed point theorem, which ensures the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach [18] is known as **Banach fixed point theorem or Banach contraction principle** stated as**:**

"*Let* (*X*, *d*) *be a complete metric space and f be a contraction mapping on X. Then f has a unique fixed point in X, i.e., every contraction mapping on a complete metric space has a unique fixed point.*"

Banach contraction principle has many applications but it suffers from one drawback, i.e., it requires the mapping to be continuous at all points of its domain. In 1968, Kannan [66] proved that there are mappings that have a discontinuity in their domain but still have fixed point, although such mappings are continuous at their fixed points. This theorem of Kannan is stated as:

"*Let* (*X*, *d*) *be a complete metric space and f is a self-mapping of X satisfying*

$$d\left(fx, fy\right) \leq k\ \left[d\left(fx, x\right) + d\left(fy, y\right)\right],$$

for all $x, y \in X$, *and k is any real number such that* $k \in \left[0, \frac{1}{2}\right)$. *Then f has a unique fixed point in X.*"

Following the appearance of Kannan's paper [66,67], many researchers started working along this direction and established related fixed point theorems.

In 2002, Branciari [24] obtained a fixed point result for a single self-mapping satisfying an analogue of Banach contraction principle for an integral-type inequality as follows:

"*Let* (*X*, *d*) *be a complete metric space,* $k \in [0,1)$ *and f is a self-mapping of X such that for all* $x, y \in X$, $\int_0^{d(fx, fy)} \phi(t)dt \leq k \int_0^{d(x,y)} \phi(t)dt$, *where* $\phi : [0, \infty) \to [0, \infty]$ *is a Lebesgue integrable mapping which is summable* (*i.e., with finite integral*) *on each compact subset of* $[0, \infty)$, *non-negative, and such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$, *then f has a unique fixed point in X.*"

Recently, many researchers (see, e.g., [7],[46],[55],[58],[73],[92-97],[110],[126]) have obtained common fixed point results on metric spaces.

1.2 VARIOUS TYPE OF MAPPINGS IN METRIC SPACES

Definition 1.2.1.[62] A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be **commuting** if $fgx = gfx$, for all $x \in X$.

Example 1.2.1. Let $X = [0, 1]$. Define self-mappings *f* and *g* on *X* by $fx = 2x$ and $gx = 3x$, for all $x \in X$. Then *f* and *g* are commuting mappings.

In 1982, Sessa [125] introduced the notion of weak commutativity, which is a weaker hypothesis than commutativity.

Definition 1.2.2. A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be **weakly commuting** if $d\left(fgx, gfx\right) \le d\left(fx, gx\right),$ for all $x \in X.$

Every pair of commuting self-mappings is weakly commuting but the converse is not true.

Example 1.2.2.[125] Let *X* = [0, 1] and *d* be the usual metric on *X*. Define the mappings $f, g : X \to X$ by $fx = \frac{x}{2}$ and $gx = \frac{x}{x+2},$ for all $x \in X.$ Then for any $x \in X,$

$$d(fgx, gfx) \le d(fx, gx).$$

This implies that *f* and *g* are weakly commuting on *X* but are not commuting on *X,* since $gfx \ne fgx$ for any non-zero $x \in X$.

In 1986, Jungck [63] defined the concept of compatible mappings which is more general than that of commuting and weakly commuting mappings.

Definition 1.2.3. A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be **compatible** mappings if $\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) = 0,$ whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X.$

Indeed, weakly commuting mappings are compatible, but the converse is not necessarily true.

Example 1.2.3.[63] Let $X = \left[0, \infty\right)$ be endowed with usual metric *d* and $f, g : X \to X$ such that $fx = x^3$ and $gx = 2x^3.$ Then $fgx \ne gfx$. So, *f* and *g* are not commuting on *X* and $\left|fgx - gfx\right| > \left|fx - gx\right|.$ Therefore, *f* and *g* are not weakly commuting on *X*. Also, for any sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X$ then

$$\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) = \lim_{n\to\infty}\left|fgx_n - gfx_n\right| = 0.$$

Therefore, *f* and *g* are compatible

Later, various authors have defined different variants of compatibility.

Definition 1.2.4. A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be:

(i) **compatible of type (*A*)[64]** if

$$\lim_{n\to\infty} d\left(fgx_n, ggx_n\right) = 0 \text{ and } \lim_{n\to\infty} d\left(gfx_n, ffx_n\right) = 0;$$

(ii) **compatible of type (*B*)[102]** if

$$\lim_{n\to\infty} d\left(fgx_n, ggx_n\right) \le \frac{1}{2}\left[\lim_{n\to\infty} d\left(fgx_n, fz\right) + \lim_{n\to\infty} d\left(fz, ffx_n\right)\right]$$

and

$$\lim_{n\to\infty} d\left(gfx_n, ffx_n\right) \le \frac{1}{2}\left[\lim_{n\to\infty} d\left(gfx_n, gz\right) + \lim_{n\to\infty} d\left(gz, ggx_n\right)\right];$$

(iii) **compatible of type (*C*)[101]** if

$$\lim_{n\to\infty} d\left(fgx_n, ggx_n\right) \le \frac{1}{3}\left[\lim_{n\to\infty} d\left(fgx_n, fz\right) + \lim_{n\to\infty} d\left(fz, ffx_n\right) + \lim_{n\to\infty} d\left(fz, ggx_n\right)\right]$$

and

$$\lim_{n\to\infty} d\left(gfx_n, ffx_n\right) \le \frac{1}{3}\left[\lim_{n\to\infty} d\left(gfx_n, gz\right) + \lim_{n\to\infty} d\left(gz, ggx_n\right) + \lim_{n\to\infty} d\left(gz, ffx_n\right)\right];$$

(iv) **compatible of type (*P*)[100]** if $\lim_{n\to\infty} d\left(ffx_n, ggx_n\right) = 0$;

whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$, in each of the above types.

In 1994, Pant [92] introduced the concept of *R*-weakly commuting mappings as follows:

Definition 1.2.5. A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be ***R*-weakly commuting** if there exists some real number *R* > 0 such that

$$d\left(fgx, gfx\right) \le R\, d\left(fx, gx\right), \text{ for all } x \in X.$$

Definition 1.2.6. A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be **pointwise *R*-weakly commuting** on *X* if for a given $x \in X$, there exists *R* > 0 such that

$$d\left(fgx, gfx\right) \le R\, d\left(fx, gx\right).$$

Remark 1.2.1.[92] (i) For $R = 1$, every R -weakly commuting pair is weakly commuting. (ii) Weak commutativity implies R-weak commutativity. However, R-weak commutativity implies weak commutativity only when $R \leq 1$.

Later on, Pathak *et al.* [99] introduced an improved notion of R-weak commutativity of mappings by defining R-weak commutativity of type (A_g) and (A_f).

Definition 1.2.7. A pair (f, g) of self-mappings of a metric space (X, d) is said to be:

(i) ***R*-weakly commuting of type (*A*$_g$)** if there exists some real number $R > 0$ such that

$$d\left(gfx, ffx\right) \leq R\, d\left(fx, gx\right), \text{for all } x \in X.$$

(ii) ***R*-weakly commuting of type (*A*$_f$)** if there exists some real number $R > 0$ such that

$$d\left(fgx, ggx\right) \leq R\, d\left(fx, gx\right), \text{ for all } x \in X.$$

Moreover, such mappings commute at their coincidence points.

Pant [93] in 1998, initiated the following notion of non-compatible mappings.

Definition 1.2.8. A pair (f, g) of self-mappings of a metric space (X, d) is said to be **non-compatible** if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$ but $\lim_{n\to\infty} d\left(fgx_n, gfx_n\right)$ is either non-zero or does not exist.

In 1998, Jungck and Rhoades [65] introduced the notion of weakly compatible mappings which is more general than that of compatibility as follows:

Definition 1.2.9. A pair (f, g) of self-mappings of a metric space (X, d) is said to be **weakly compatible mappings** if the mappings commute at all of their coincidence points, i.e., $fx = gx$ for some $x \in X$ implies $fgx = gfx$.

Also, compatible mappings are weakly compatible but the converse is not true.

Example 1.2.4. Let $X = [2, 20]$ and d be the usual metric on X. Define the mappings f, g: $X \to X$ by $fx = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 6, & 2 < x \leq 5 \end{cases}$ and $gx = \begin{cases} 2, & x = 2 \\ 12, & 2 < x \leq 5 \\ x - 3, & x > 5. \end{cases}$

Define a sequence $\{x_n\}$ in X by $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$. Clearly, mappings f and g are non-compatible and weakly compatible at the coincidence point $x = 2$.

In 2008, Al-Thagafi and Shahzad [14] introduced the concept of occasionally weakly compatible (*owc*) mappings which is a proper generalization of weakly compatible mappings.

Definition 1.2.10. A pair (f, g) of self-mappings of a metric space (X, d) is said to be **occasionally weakly compatible (*owc*)** if there exists a point $x \in X$ which is a coincidence point of f and g at which f and g commute.

The following example shows that the weakly compatible self-mappings form a proper subclass of *owc* self-mappings.

Example 1.2.5.[14] Let $X = [0, \infty)$ with usual metric. Define f, g: $X \to X$ by $fx = 2x$ and $gx = x^2$, for all $x \in X$. Then $fx = gx$ at $x = 0, 2$ but $fg(0) = gf(0)$ and $fg(2) \neq gf(2)$. Therefore, mappings f and g are *owc* but not weakly compatible.

In 2002, Aamri and Moutawakil [1] generalized the concept of non-compatible mappings by introducing a new property called property ($E.A$). This property is defined as:

Definition 1.2.11. A pair (f, g) of self-mappings of a metric space (X, d) is said to be satisfy **property ($E.A$),** if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \text{ for some } z \in X.$$

Example 1.2.6.[1] Let $X = [0, \infty)$. Define f, g : $X \to X$ by $fx = \frac{x}{4}$ and $gx = \frac{3x}{4}$, for all $x \in X$. Consider the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$. Clearly, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 \in X$. Then f and g satisfy property ($E.A$).

In 2005, Liu *et al.* [75] introduced the notion of common property ($E.A$) which contains the property ($E.A$) for pairs of self-mappings.

Definition 1.2.12. Two pairs (f, g) and (h, k) of self-mappings of a metric space (X, d) are said to satisfy the **common property ($E.A$)** if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hy_n = \lim_{n\to\infty} ky_n = z \text{ for some } z \in X.$$

Example 1.2.7. Define *f*, *g*, *h* and *k* on $X = [1,15)$ by

$$fx = \begin{cases} 1, & x \in \{1\} \cup (3,15) \\ 14, & x \in (1,3] \end{cases}, \quad gx = \begin{cases} 1, & x \in \{1\} \cup (3,15) \\ 5, & x \in (1,3] \end{cases}$$

$$hx = \begin{cases} 3, & x = 1 \\ 6, & x \in (1,3] \\ \dfrac{x+1}{4}, & x \in (3,15) \end{cases} \text{ and } kx = \begin{cases} 2, & x = 1 \\ 11, & x \in (1,3] \\ x-2, & x \in (3,15). \end{cases}.$$

Take $\{x_n\} = \left\{3 + \frac{1}{n}\right\}$ and $\{y_n\} = \left\{3 + \frac{1}{n}\right\}$, clearly

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hy_n = \lim_{n\to\infty} ky_n = 1 \in X.$$

Therefore, (*f*, *g*) and (*h*, *k*) satisfies the common property (*E.A*).

In 2011, Sintunavarat and Kumam [133] introduced a new property called as "common limit in the range of *g* property" (i.e., CLR_g property), defined as:

Definition 1.2.13. A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be satisfy the **common limit in the range of *g* property** if there exists a sequence $\{x_n\}$ in *X* such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gz \quad \text{for some } z \in X.$$

Example 1.2.8.[133] Let (*X*, *d*) be the usual metric space where $X = [0,\infty)$. Define *f*, *g*: *X*→*X* by $fx = x+1$ and $gx = 2x$, for all $x \in X$. Consider the sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = g(1) = 2$, therefore, *f* and *g* satisfy the (CLR_g) property.

In 2012, Imdad *et al.* [59] defined the common limit in the range of *g* and *k* (CLR_{gk}) property and thereby extending the notion of (CLR_g) property as under:

Definition 1.2.14. Let *X* be any non-empty set. Two pairs (*f*, *g*) and (*h*, *k*) of self-mappings on *X* are said to satisfy the **common limit in the range of *g* and *k* (CLR_{gk}) property** if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hy_n = \lim_{n\to\infty} ky_n = z \text{ where } z \in g(X) \cap k(X).$$

Chauhan *et al.* [29] defined the notion of "the joint common limit in the range ($JCLR_{gk}$) property" for two pairs of self-mappings as follows:

Definition 1.2.15. Let X be any non-empty set. Two pairs (f, g) and (h, k) of self-mappings on X are said to satisfy the **joint common limit in the range of *g* and *k* ($JCLR_{gk}$) property** if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hy_n = \lim_{n\to\infty} ky_n = gz = kz \text{ for some } z \in X.$$

Example 1.2.9. Let $X = [-1,1]$. Define self-mappings f, g, h and k on X by $fx = \frac{x}{3}, hx = -\frac{x}{3}, gx = x$ and $kx = -x,$ for all $x \in X.$ Then with sequences $\{x_n\} = \left\{\frac{1}{n}\right\}$ and $\{y_n\} = \left\{-\frac{1}{n}\right\}$ in X, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hy_n = \lim_{n\to\infty} ky_n = g0 = k0.$ This shows that the pairs (f, g) and (h, k) satisfy the ($JCLR_{gk}$) property. Since $0 \in g(X) \cap k(X)$, both pairs (f, g) and (h, k) satisfy the (CLR_{gk}) property.

In 1999, Pant [95] introduced a new continuity condition, known as reciprocal continuity as follows:

Definition 1.2.16. A pair (f, g) of self-mappings of a metric space (X, d) is said to be **reciprocally continuous** if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz,$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

It is to be noted that if f and g are both continuous, then f and g are reciprocally continuous but the converse is not true.

Example 1.2.10.[95] Let $X = [2, 20]$ and d be usual metric on X. Define mappings $f, g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \\ 3, & x > 2 \end{cases} \text{ and } gx = \begin{cases} 2, & x = 2 \\ 6, & x > 2. \end{cases}$$

Then f and g are reciprocally continuous mappings for the sequence $\{x_n\}$ = {2} but are not continuous at $x = 2$.

In 2011, Pant *et al.* [96], generalized the concept of reciprocal continuity by introducing the notion of weak reciprocal continuity.

Definition 1.2.17. A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be **weakly reciprocally continuous** if $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$, whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Example 1.2.11.[96] Let $X = [2, 20]$ and *d* be usual metric on *X*. Define mappings $f, g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 6, & 2 < x \leq 5 \end{cases} \text{ and } gx = \begin{cases} 2, & x = 2 \\ 12, & 2 < x \leq 5 \\ (x+1)/3 & x > 5. \end{cases}$$

Clearly, mappings *f* and *g* are weakly reciprocal continuous mappings but are not reciprocal continuous.

1.3. VARIOUS TYPE OF ABSRACT SPACES

We, now present various abstract spaces such as *G*-metric spaces, Menger spaces, Fuzzy metric spaces, Intuitionistic fuzzy metric spaces and Modified intuitionistic fuzzy metric spaces which will be used in the later chapters.

G- METRIC SPACE

In sixties, Gahler [41,42] introduced the concept of 2-metric space and proved that a 2-metric is a generalization of usual notion of a metric. But different authors proved that there is no relation between these two functions. For instance, Ha *et al.* [51] showed that a 2-metric need not be a continuous function of its variables, whereas an ordinary metric is continuous function of its variables. These considerations led Dhage [36], in 1992, to introduce a new class of generalized metric called as *D*-metric. Dhage claimed in his doctoral research work that *D*-metric provides a generalization of ordinary metric functions. Further, in 2004, Mustafa and Sims [87] demonstrated that most of the claims concerning the fundamental topological properties of *D*-metric spaces are incorrect and proved that a *D*-metric need not be a continuous function of its variables. These considerations led researchers to seek a more appropriate notion of generalized metric space. In 2006, Mustafa and Sims [88] introduced the concept of

G-metric spaces to overcome fundamental flaws in Dhage's theory of generalized metric spaces.

Definition 1.3.1. Let X be a non-empty set, and let G: $X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following axioms: for all $x, y, z, a \in X$,

(G1) $G(x,y,z)=0$ if $x=y=z$;

(G2) $G(x,x,y)>0$ with $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ with $z \neq y$;

(G4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=\ldots$;

(G5) $G(x,y,z) \leq G(x,a,a)+G(a,y,z)$,

then the function G is called a **Generalized metric** or more specifically a ***G*-metric** on X and the pair (X, G) is called a ***G*-metric space**.

Definition 1.3.2.[88] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ in X, is said to be ***G*-convergent** to a point $x \in X$ if $\lim\limits_{m,n\to\infty} G(x, x_n, x_m)=0,$ i.e., for each $\varepsilon > 0,$ there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon,$ for all $n, m \geq N.$

Definition 1.3.3.[88] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ in X, is said to be a ***G*-Cauchy sequence** if, $\lim\limits_{m,n,k\to\infty} G(x_n, x_m, x_k)=0,$ i.e., for each $\varepsilon > 0,$ there exists a positive integer N such that $G(x_n, x_m, x_k) < \varepsilon,$ for all $n, m, k \geq N.$

Definition 1.3.4.[88] A G-metric space (X, G) is said to be ***G*-complete** if every G-Cauchy sequence in (X, G) is G-convergent in X.

Definition 1.3.5.[88] A G – metric space (X, G) is called a **symmetric *G*-metric space** if

$$G(x, y, y) = G(x, x, y), \text{ for all } x, y \in X.$$

Proposition 1.3.1.[88] *Let* (X, G) *be a G-metric space. Then the following statements are equivalent*:

(i) $\{x_n\}$ *is G-convergent to x*;

(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$;

(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$;

(iv) $G(x_n, x_m, x) \to 0$ *as* $m, n \to \infty$.

Proposition 1.3.2.[88] *If* (*X*, *G*) *is a G-metric space. Then the following statements are equivalent*:

(i) *sequence* $\{x_n\}$ *is G – Cauchy*;

(ii) *for each* $\varepsilon > 0$, *there exists a positive integer N such that* $G(x_n, x_m, x_m) < \varepsilon$, *for all* $n, m \geq N$.

Proposition 1.3.3.[88] *Let* (*X*, *G*) *be a G-metric space. Then the function G*(*x*, *y*, *z*) *is jointly continuous in all three of its variables*.

Proposition 1.3.4.[88] *Every G-metric space* (*X*, *G*) *defines a metric space* (X, d_G), *where* $d_G(x, y) = G(x, y, y) + G(y, x, x)$, *for all* $x, y \in X$.

Proposition 1.3.5.[88] *A G-metric space* (*X*, *G*) *is G-complete if and only if* (X, d_G) *is a complete metric space.*

Proposition 1.3.6.[88] *Let* (*X*, *G*) *be a G-metric space. Then for any* $x, y, z, a \in X$, *it follows that*:

(i) *If* $G(x, y, z) = 0$, *then* $x = y = z$;

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;

(iii) $G(x, y, y) \leq 2G(y, x, x)$;

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;

(v) $G(x, y, z) \leq [G(x, a, a) + G(y, a, a) + G(z, a, a)]$.

In 2005, Mustafa [83] in his Ph.D. thesis characterized the well known Banach contraction principle in the context of *G*-metric spaces in the following way:

Theorem 1.3.1.[83] *Let* (*X*, *G*) *be a complete G-metric space and* $f : X \to X$ *be a mapping satisfying the following condition*:

$$G(fx, fy, fz) \leq k\, G(x, y, z), \text{ for all } x, y, z \in X \text{ and } 0 \leq k < 1.$$

Then f has a unique fixed point.

Motivated by the work of Mustafa and Sims [88], various researchers (see, e.g., [3],[4],[16],[78],[84-86],[89],[115],[117],[127]) have proved number of well known results in *G*-metric spaces.

PROBABILISTIC METRIC SPACES

In 1942, Menger [77] introduced the notion of a probabilistic metric space (*PM*-space). In order to define such spaces, Menger used the concept of distribution functions instead of non-negative real-valued metric. The notion of probabilistic metric space corresponds to situations when we do not know exactly the distance between two points i.e., we know only probabilities of possible values of this distance.

Definition 1.3.6.([120], Vol.16, p.43) A mapping $f:\mathbf{R}\to\mathbf{R}^{+}$ is called a **distribution function** if it is non-decreasing, left continuous with $\inf\{ft : t\in\mathbf{R}\}=0$ and $\sup\{ft : t\in\mathbf{R}\}=1$. Generally, the set of all distribution functions is denoted by *L*.

An example of distribution function is Heaviside function *H*(*t*), defined by

$$H(t)=\begin{cases}0, & t\le 0\\ 1, & t>0.\end{cases}$$

Definition 1.3.7.([120], Vol.16, p.124) The ordered pair (X,F) is said to be **probabilistic metric space** if X is a non-empty set and *F* is a mapping from $X\times X$ to *L* satisfying the following conditions: for all $x,y,z\in X$ and *t*, *s* > 0,

(i) $F(x,y\ ;t)=1$ if and only if *x* = *y*;

(ii) $F(x,y\ ;0)=0$;

(iii) $F(x,y\ ;t)=F(y,x\ ;t)$;

(iv) If $F(x,y\ ;t)=1$ and $F(y,z\ ;s)=1$, then $F(x,z\ ;t+s)=1$.

Note that $F(x,y\ ;t)$ may be interpreted as the probability that the distance between *x* and *y* is less than *t*.

Definition 1.3.8.([120], Vol.16, p.65) A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a ***t*-norm** if $*$ satisfies the following conditions: for all $a,b,c,d\in[0,1]$,

(i) $*$ is commutative and associative;

(ii) $a*1=a$;

(iii) $a*b\le c*d$ whenever $a\le c$ and $b\le d$.

The following two basic *t*-norms are frequently used in our work:

(i) The **minimum *t*-norm** $*_M$ is defined by

$$a *_M b = \min\{a, b\}.$$

(ii) The **product *t* -norm** $*_P$ is defined by

$$a *_P b = ab.$$

Definition 1.3.9.([120], Vol.16, p.125) A **Menger space** is a triplet $(X, F, *)$, where (X, F) is *PM*-space and $*$ is a $t-$norm with the condition:

$$F(x,z\ ;t+s) \geq F(x,y\ ;t) * F(y,z\ ;s) \quad \text{(Menger triangle inequality)}$$

holds for all $x, y, z \in X$ and $t, s > 0$.

Every metric space (X, d) can always be realized as *PM*-space by $F : X \times X \to L$ defined by $F(x, y\ ;t) = H(t - d(x, y))$, for all $x, y \in X$ and $t > 0$.

The concept of contraction mappings for *PM*-spaces was initiated by Sehgal [122] in his Ph.D thesis in the following way:

Definition 1.3.10. A self-mapping f on a *PM*-space (X, F) is said to be **probability contraction** if there exists $k \in (0,1)$ such that

$$F(fx, fy\ ;kt) \geq F(x, y\ ;t), \text{ for all } x, y \in X \text{ and } t > 0.$$

In 1972, Sehgal and Bharucha-Reid [124] obtained a generalization of Banach contraction principle on a complete Menger space which has turned out to be a milestone in developing fixed point theorems in Menger space.

Theorem 1.3.2. *Let* $(X, F, *)$ *be a complete Menger space where* $a * b = \min\{a, b\}$ *for all* $a, b \in [0,1]$ *and* $f : X \to X$ *be a mapping satisfying the following condition*:

$$F(fx, fy\ ;kt) \geq F(x, y\ ;t), \text{ for all } x, y \in X,\ t > 0 \text{ and } k \in (0,1).$$

Then f has a unique fixed point.

After this result, many researchers have obtained fixed point results on Menger spaces(see,e.g.,[13],[28],[39],[60],[68],[70],[74],[80],[106],[130-132]).

FUZZY METRIC SPACES

After the appearance of the notion of fuzzy set, originally introduced by Zadeh [140], the literature in this area has risen very quickly because of its interesting applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences, image processing, control theory, communication etc. One of the most interesting research topics in fuzzy topology is to find an appropriate definition of fuzzy metric space. Many authors have considered this problem and have introduced it in different ways. Inspired by Menger spaces, Kramosil and Michalek [69] generalized the concept of probabilistic metric space to the fuzzy framework.

Definition 1.3.11.[140] A **fuzzy set** *A* in a universe *X* is an object $A=\left\{\left(x,\mu_A\left(x\right)\right):x\in X\right\}$ where $\mu_A:X\rightarrow\left[0,1\right]$ is the membership function or grade of membership of the *A*.

Definition 1.3.12.[69] The 3-tuple $(X,M,*)$ is said to be a **fuzzy metric space** if *X* is an arbitrary set, $*$ is a continuous *t*-norm and *M* is a fuzzy set on $X^2\times\left[0,\infty\right)$ satisfying the following conditions: for all $x,y,z\in X$ and $t,s>0$,

(i) $M(x,y,0)=0$;

(ii) $M(x,y,t)=1$ if and only if $x=y$;

(iii) $M(x,y,t)=M(y,x,t)$;

(iv) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$;

(v) $M(x,y,\bullet):\left[0,\infty\right)\rightarrow\left[0,1\right]$ is left continuous;

In what follows, fuzzy metric spaces in the sense of Kramosil and Michalek [69] will be referred to as *KM*-fuzzy metric spaces. The function *M*(*x*, *y*, *t*) denote the degree of nearness between *x* and *y* w.r.t. *t* respectively.

In order to introduce a Hausdorff topology on the fuzzy metric spaces, George and Veeramani [43] modified the notion of *KM*-fuzzy metric space in a slight but in an appealing way as follows:

Definition 1.3.13. The 3-tuple $(X, M, *)$ is said to be a **fuzzy metric space** if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(i) $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ if and only if $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(v) $M(x, y, \bullet) : (0, \infty) \to [0,1]$ is continuous.

Fuzzy metric spaces introduced by George and Veeramani [43] will be referred to as GV-fuzzy metric space.

Remark 1.3.1.[48] The mapping $M(x, y, \bullet)$ is non-decreasing for all $x, y \in X$ in fuzzy metric space $(X, M, *)$.

Definition 1.3.14.[43] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be

(i) **convergent** to a point $x \in X$, if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$, for all $n \geq n_0$.

(ii) **Cauchy sequence** if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $m, n \geq n_0$.

Definition 1.3.15.[43] A fuzzy metric space $(X, M, *)$ is said to be **complete** if every Cauchy sequence is convergent.

Definition 1.3.16.[43] Let $(X, M, *)$ be a fuzzy metric space. The **open ball** $B(x, r, t)$ for $t > 0$ with center $x \in X$ and radius r, $0 < r < 1$, is defined as

$$B(x, r, t) = \{ y \in X : M(x, y, t) > 1 - r \}.$$

Definition 1.3.17.[43] Let $(X, M, *)$ be a fuzzy metric space. Define

$$\tau = \{A \subset X;\ x \in A \text{ iff there exists } r,\ 0 < r < 1 \text{ and } t > 0 \text{ such that } B(x, r, t) \subset A\},$$

then τ is a **topology** on X.

Example 1.3.1.[43](Induced fuzzy metric spaces)

Let (*X*, *d*) be a metric space. Denote $a*b=ab$ for all $a,b\in[0,1]$, where $*$ is a continuous *t*-norm and let M_d be fuzzy set on $X^2\times[0,\infty)$ defined as follows:

$$M_d(x,y,t)=\frac{ht^n}{ht^n+md(x,y)},$$

for all $h,m,n\in\mathbf{R}^+$ and $t>0$. Then $(X,M_d,*)$ is a fuzzy metric space.

Remark 1.3.2.[43] It should be noted that the above example holds even if the continuous *t*-norm $*$ is replaced by $a*b=\min\{a,b\}$, for all $a,b\in[0,1]$ and hence function *M* defined in above example is a fuzzy metric with respect to any continuous *t*-norm. In the above example, by taking *h* = *m* = *n* = 1, we get

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}, \text{ for all } x,y\in X \text{ and } t>0.$$

This fuzzy metric induced by a metric *d* is called the **standard fuzzy metric**.

Example 1.3.2.[43] Let $X=\mathbf{R}$ and *t*-norm $*$ defined by $a*b=a.\,b$, for all $a,b\in[0,1]$. Let *M* is the fuzzy set on $X^2\times[0,\infty)$, defined by

$$M(x,y,t)=\begin{cases}\left(\exp\left(\dfrac{|x-y|}{t}\right)\right)^{-1}, & t>0\\ 0, & t=0\end{cases} \quad \text{for all } x,y\in X.$$

Then $(X,M,*)$ be a fuzzy metric space.

In 1988, Grabiec [48] presented Banach contraction principle in the frame work of *KM*-fuzzy metric space as under:

Theorem 1.3.3. *Let* $(X,M,*)$ *be a complete fuzzy metric space such that* $\lim_{t\to\infty}M(x,y,t)=1$, *for all* $x,y\in X$. *Let f is a self-mapping of X satisfying*

$$M(fx,fy,kt)\geq M(x,y,t), \text{ for all } x,y\in X,\ t>0$$

where k is any real number such that $k\in(0,1)$. *Then f has a unique fixed point in X.*

Thereafter, many authors (see e.g.,[2],[30],[31],[38],[44],[45],[47],[54],[72],[79],[91],[134]) established fuzzy version of most of the classical metric common fixed point theorems.

INTUITIONISTIC FUZZY METRIC SPACES

As a generalization of fuzzy sets, Atanassov [15] introduced and studied the concept of intuitionistic fuzzy sets which was subsequently developed by many authors (see, e.g., [33],[98]). Intuitionistic fuzzy set can be utilized as a proper tool for representing hesitancy concerning both membership and non-membership of an element to a set. To be more precise, a basic assumption of fuzzy set theory is that if we specify the degree of membership of an element in a fuzzy set as a real number from [0,1], say *p*, then the degree of its non-membership is automatically determined as (1 - *p*), need not hold for intuitionistic fuzzy sets. In intuitionistic fuzzy set theory, it is assumed that non-membership should not be more than (1- *p*). Thus, by employing intuitionistic fuzzy sets in data bases, one can express a hesitation concerning examined objects. Intuitionistic fuzzy set has many applications in different areas such as in medical diagnosis, in decision making and in chemistry.

Definition 1.3.18.[15] An **intuitionistic fuzzy set** *A* in a universe *X* is an object $A=\left\{\left(x,\mu_A\left(x\right),\gamma_A(x)\right):x\in X\right\}$, where for all $x\in X$, $\mu_A(x)\left(\in[0,1]\right)$ is called the degree of membership of *x* in *A*, $\gamma_A(x)\left(\in[0,1]\right)$ is called the degree of non-membership of *x* in *A*, and $0\leq\mu_A\left(x\right)+\gamma_A\left(x\right)\leq 1$ for every $x\in X$.

In 2004, Park [98] defined the notion of intuitionistic fuzzy metric space with the help of continuous *t*-norm and continuous *t*-conorm as a generalization of *GV*-fuzzy metric space.

Definition 1.3.19.[119] A binary operation $\lozenge:[0,1]\times[0,1]\to[0,1]$ is called a **continuous *t*-conorm** if ◊ satisfies the following conditions: for all $a,b,c,d\in\left[0,1\right]$,

(i) ◊ is commutative and associative;

(ii) ◊ is continuous;

(iii) $a\lozenge 0=a$;

(iv) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$.

Definition 1.3.20. A 5-tuple $(X, M, N, *, \lozenge)$ is said to be an **intuitionistic fuzzy metric space** if X is an arbitrary set, $*$ is a continuous t-norm, $\lozenge$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(i) $M(x,y,t) + N(x,y,t) \leq 1$;

(ii) $M(x,y,t) > 0$;

(iii) $M(x,y,t) = 1$ if and only if $x = y$;

(iv) $M(x,y,t) = M(y,x,t)$;

(v) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;

(vi) $M(x,y,\bullet) : (0,\infty) \to (0,1]$ is continuous;

(vii) $N(x,y,t) > 0$;

(viii) $N(x,y,t) = 0$ if and only if $x = y$;

(ix) $N(x,y,t) = N(y,x,t)$;

(x) $N(x,y,t) \lozenge N(y,z,s) \geq N(x,z,t+s)$;

(xi) $N(x,y,\bullet) : (0,\infty) \to (0,1]$ is continuous.

Then (*M*, *N*) is called an intuitionistic fuzzy metric on *X*. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t, respectively.

Remark 1.3.3.[98] Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \lozenge)$ such that t-norm $*$ and t-conorm $\lozenge$ are associated as

$$x \lozenge y = 1 - \big((1-x) * (1-y)\big), \text{ for all } x, y \in X.$$

Remark 1.3.4.[98] The mappings $M(x,y,\bullet)$ and $N(x,y,\bullet)$ are respectively non-decreasing and non-increasing in an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ for all $x, y \in X$.

Example 1.3.3.[98] (Induced intuitionistic fuzzy metric space)

Let (*X*, *d*) be a metric space. Denote $a * b = a.b$ and $a \Diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. Let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)} \text{ and } N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)},$$

for all $h, k, m, n \in \mathbf{R}^+$. Then $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space.

Remark 1.3.5.[98] It should be noted that the above example holds even with the *t*-norm by $a * b = \min\{a, b\}$ and *t*-conorm $a \Diamond b = \max\{a, b\}$, for all $a, b \in [0,1]$ and hence functions (*M*, *N*) defined in above example is an intuitionistic fuzzy metric with respect to any continuous *t*-norm and continuous *t*-conorm. In the above example by taking *h* = *k* = *m* = *n* = 1, we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \text{ and } N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

This intuitionistic fuzzy metric induced by a metric *d* is called the **standard intuitionistic fuzzy metric.**

Definition 1.3.21.[98] Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space, and $r \in (0,1)$, $t > 0$ and $x \in X$. The set

$$B_{(M,N)}(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\}$$

is called the **open ball** with centre *x* and radius *r* with respect to *t*.

Definition 1.3.22.[98] Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space, and $r \in (0,1)$, $t > 0$ and $x \in X$. The set

$$B_{(M,N)}[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r, N(x, y, t) \leq r\}$$

is called the **closed ball** with centre *x* and radius *r* with respect to *t*.

Remark 1.3.6.[98] Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Define

$\tau_{(M,N)} = \{A \subset X : \text{for each } x \in A, \text{there exist } t > 0 \text{ and } r \in (0,1) \text{ such that } B_{(M,N)}(x, r, t) \subset A\}$.

Then $\tau_{(M,N)}$ is a topology on *X*.

Definition 1.3.23.[98] Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. A subset A of X is said to be **IF-bounded** if there exists $t > 0$ and $r \in (0,1)$ such that $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$, for all $x, y \in A$.

Definition 1.3.24.[98] Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be:

(i) **Cauchy sequence** if, for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$, such that

$$M(x_n, x_m, t) > 1 - \varepsilon \text{ and } N(x_n, x_m, t) < \varepsilon, \text{ for all } n, m \geq n_0.$$

(ii) **convergent** to a point $x \in X$ if, for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$, such that

$$M(x_n, x, t) > 1 - \varepsilon \text{ and } N(x_n, x, t) < \varepsilon, \text{ for all } n \geq n_0.$$

Definition 1.3.25.[98] An intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be **complete** if every Cauchy sequence in X is convergent with respect to $\tau_{(M,N)}$.

Alaca *et al.* [11], in 2006, redefined the notion of intuitionistic fuzzy metric space as a generalization of *KM*-fuzzy metric space. Alaca *et al.* [11], further proved well-known fixed point theorems of Banach [18] and Edelstein [37] in intuitionistic fuzzy metric spaces with the help of Grabiec [48].

Definition 1.3.26. A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an **intuitionistic fuzzy metric space** if X is a non-empty set, $*$ is a continuous *t*-norm, $\Diamond$ is a continuous *t*-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(i) $M(x, y, t) + N(x, y, t) \leq 1$;

(ii) $M(x, y, 0) = 0$;

(iii) $M(x, y, t) = 1$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$;

(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(vi) $M(x, y, \bullet) : [0, \infty) \to [0,1]$ is left continuous;

(vii) $\lim_{t \to \infty} M(x, y, t) = 1$;

(viii) $N(x,y,0)=1$;

(ix) $N(x,y,t)=0$ if and only if $x=y$;

(x) $N(x,y,t)=N(y,x,t)$;

(xi) $N(x,y,t) \lozenge N(y,z,s) \geq N(x,z,t+s)$;

(xii) $N(x,y,\bullet):[0,\infty)\to[0,1]$ is right continuous;

(xiii) $\lim_{t\to\infty} N(x,\ y,\ t) = 0$.

Then (*M*, *N*) is called an intuitionistic fuzzy metric on *X*. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t, respectively.

Definition 1.3.27.[11] Let $(X,M,N,*,\lozenge)$ be an intuitionistic fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be:

(*a*) **Cauchy sequence** if for all $t>0$ and $p>0$,

$$\lim_{n\to\infty} M(x_{n+p},\ x_n,\ t) = 1 \text{ and } \lim_{n\to\infty} N(x_{n+p},\ x_n,\ t) = 0.$$

(*b*) **convergent** to a point $x\in X$ if for all $t>0$,

$$\lim_{n\to\infty} M(x_n,\ x,\ t) = 1 \text{ and } \lim_{n\to\infty} N(x_n,\ x,\ t) = 0.$$

Remark 1.3.7.[11] Since $*$ and $\lozenge$ are continuous *t*-norm and *t*-conorm, the limit determined from (vii) and (xiii) is unique.

Example 1.3.4.[10] Let $X=[0,1]$ with the usual metric and let $*$ be the continuous *t*-norm and $\lozenge$ be the continuous *t*-conorm defined by $a*b=ab$ and $a\lozenge b=\max\{a,b\}$ respectively, for all $a,b\in[0,1]$. For each $t\in(0,\infty)$ and $x,y\in X$, define (*M*, *N*) by

$$M(x,y,t)=\begin{cases}\dfrac{t}{t+|x-y|}, & t>0\\ 0, & t=0\end{cases} \quad \text{and} \quad N(x,y,t)=\begin{cases}\dfrac{|x-y|}{t+|x-y|}, & t>0\\ 1, & t=0.\end{cases}$$

Clearly, $(X,M,N,*,\lozenge)$ is a complete intuitionistic fuzzy metric space.

Alaca *et al.* [11] characterized the Banach contraction principle in the context of intuitionistic fuzzy metric spaces in the following way:

Theorem 1.3.4. *Let* $(X, M, N, *, \Diamond)$ *be a complete intuitionistic fuzzy metric space. Let* $f : X \to X$ *be a mapping satisfying the following condition*:

$$M(fx, fy, kt) \geq M(x, y, t) \text{ and } N(fx, fy, kt) \leq N(x, y, t),$$

for all $x, y \in X, t > 0$ *and* $0 < k < 1$. *Then f has a unique fixed point.*

Based on the notion of intuitionistic fuzzy metric spaces, various authors (see, e.g., [9],[12],[56],[71],[82],[111],[136]) obtained common fixed point theorems for mappings satisfying different contractive conditions.

MODIFIED INTUITIONISTIC FUZZY METRIC SPACES

In 2006, Gregori *et al.* [50] showed that the topology induced by fuzzy metric coincides with topology induced by intuitionistic fuzzy metric. In view of this observation, Saadati *et al.* [112], in 2008, reframed the idea of intuitionistic fuzzy metric spaces and proposed a new notion under the name of modified intuitionistic fuzzy metric spaces with the help of the notion of continuous *t*-representable.

Now, we present following notions which have been used in the sequel:

Consider the set L^* and the operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice (see, Deschrijver and Kerre [35]). Units of L^* have been denoted by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

Definition 1.3.28.[34] A **triangular norm** (*t*-norm) on L^* is a mapping $\mathscr{T} : (L^*)^2 \to L^*$ satisfying the following conditions: for all $x, x', y, y', z \in L^*$,

(i) $\mathscr{T}(x, 1_{L^*}) = x$; (boundary condition)

(ii) $\mathscr{T}(x, y) = \mathscr{T}(y, x)$; (commutativity)

(iii) $\mathscr{T}(x, \mathscr{T}(y, z)) = \mathscr{T}(\mathscr{T}(x, y), z)$; (associativity)

(iv) If $x \leq_{L^*} x'$ and $y \leq_{L^*} y'$ implies $\mathscr{T}(x, y) \leq_{L^*} \mathscr{T}(x', y')$. (monotonicity)

Definition 1.3.29.[34] A continuous *t*-norm $\mathscr{T}$ on L^* is called **continuous *t* – representable** iff there exists a continuous *t*-norm * and a continuous *t*-conorm ◊ on [0, 1] such that for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathscr{T}(x, y) = (x_1 * y_1,\ x_2 \lozenge\ y_2).$$

Definition 1.3.30.[112] Let M, N are fuzzy sets from $X^2 \times (0,\infty) \to [0,1]$ such that $M(x,y,t) + N(x,y,t) \leq 1$, for all $x, y \in X$ and $t > 0$. The 3- tuple (X, $\zeta_{M,N}$, $\mathscr{T}$) is said to be a **modified intuitionistic fuzzy metric space,** if X is an arbitrary non-empty set, $\mathscr{T}$ is a continuous t- representable and $\zeta_{M,N}$ is an intuitionistic set from $X^2 \times (0,\infty) \to L^*$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(i) $\zeta_{M,N}(x,y,t) >_{L^*} 0_{L^*}$;

(ii) $\zeta_{M,N}(x,y,t) = 1_{L^*}$ iff $x = y$;

(iii) $\zeta_{M,N}(x,y,t) = \zeta_{M,N}(y,x,t)$;

(iv) $\zeta_{M,N}(x,y,t+s) \geq_{L^*} \mathscr{T}(\zeta_{M,N}(x,z,t), \zeta_{M,N}(z,y,s))$;

(v) $\zeta_{M,N}(x,y,\bullet) : (0,\infty) \to L^*$ is continuous.

In this case, $\zeta_{M,N}$ is called a modified intuitionistic fuzzy metric. Here,

$$\zeta_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)), \text{ for all } x, y \in X \text{ and } t > 0.$$

Remark 1.3.8.[112] In a modified intuitionistic fuzzy metric space $(X, \zeta_{M,N}, \mathscr{T})$, $\zeta_{M,N}(x,y,t)$ is non-decreasing with respect to t in (L^*, $\leq_{L^*}$) for all $x, y \in X$ and $t > 0$.

Definition 1.3.31.[112] A sequence $\{x_n\}$ in a modified intuitionistic fuzzy metric space (X, $\zeta_{M,N}$, $\mathscr{T}$) is said to be a **Cauchy sequence** if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $\zeta_{M,N}(x_n, x_m, t) >_{L^*} (1-\varepsilon, \varepsilon)$, for each $n, m \geq n_0$.

Definition 1.3.32.[112] A sequence $\{x_n\}$ in a modified intuitionistic fuzzy metric space (X, $\zeta_{M,N}$, $\mathscr{T}$) is said to be **convergent** to $x \in X$ if $\lim_{n\to\infty} \zeta_{M,N}(x_n, x, t) = 1_{L^*}$, for all $t > 0$.

Motivated by the work of Saadati *et al.* [112], various researchers (see, e.g., [5],[113],[114],[121]) have proved number of well-known results in modified intuitionistic fuzzy metric spaces.

1.4 OBJECTIVES OF THE STUDY

In specific terms, the objectives of this study are as follows:

(i) To extend and unify some of the results using integral contractive condition for fixed point theorem in some abstract spaces.

(ii) To generalize the results using compatible mappings, *R*-weakly commuting mappings and their variants in different spaces.

Keeping these objectives in view, some results have been established in various abstract spaces such as *G*-metric space, fuzzy metric space, intuitionistic fuzzy metric space and modified intuitionistic fuzzy metric space by using variants of compatible and *R*-weakly commuting mappings.

1.5 THESIS ORGANIZATION

The rest of this thesis is organized as follows:

In Chapter-II, the concepts of compatible mappings and weakly compatible mappings of type (*A*) in *G*-metric spaces have been introduced. Some properties related to these mappings have also been proved in this chapter. The notion of the property (*E.A*) has been used to prove a common fixed point theorem for weakly compatible mappings. The results presented in this chapter have been applied to obtain the solution of an integral equation and the bounded solution of a functional equation arising in dynamic programming.

In Chapter-III, some new common fixed point theorems in *G*-metric spaces have been established by utilizing the concepts of compatibility, variants of *R*-weakly commutativity and weakly reciprocal continuity. Consequently, the results presented in this chapter have improved and sharpened many related common fixed point theorems in the existing literature. Also, some examples in support of these results have been given.

The aim of **Chapter-IV** is to introduce a new property called “common limit in the range” for four self-mappings in fuzzy metric spaces. Also, some common fixed point results have been established by using this new property in fuzzy metric spaces. Further, some common fixed theorems using *R*-weakly commuting mappings and some of their variants have been proved in fuzzy metric spaces.

The objective of chapter **Chapter-V** is to prove a common fixed point theorem in intuitionistic fuzzy metric spaces by using pointwise *R*-weak commutativity and reciprocal continuity of mappings satisfying some contractive conditions. Towards the end of this chapter, some common fixed point theorems for weakly compatible mappings via an implicit relation have been proved in intuitionistic fuzzy metric spaces by using the common (*E.A*) like property.

In **Chapter-VI,** some common fixed point theorems for two pairs of compatible and sub-sequentially continuous mappings in intuitionistic fuzzy metric spaces via an implicit relation have been proved. The results proved in this chapter have improved many known related common fixed point theorems in these spaces. Further, some common fixed point theorems for weakly compatible mappings in intuitionistic fuzzy metric spaces via an implicit relation and the common property (*E.A*) have been established.

Chapter-VII is devoted to the extension of the work done by Saadati *et al.* [112] in the frame work of modified intuitionistic fuzzy metric spaces. In this chapter, some common fixed point theorems have been established using weakly reciprocal continuous, non-compatible self-mappings on a modified intuitionistic fuzzy metric space via some implicit relations.

Finally, based on the present study, relevant topics for further research have been suggested.

CHAPTER- II

COMMON FIXED POINT THEOREMS FOR VARIANTS OF COMPATIBLE MAPPINGS IN *G*-METRIC SPACES

In this chapter, some results for variants of compatible mappings along with property (*E.A*) in *G*- metric spaces have been proved. This chapter has been divided into various sections. In Section 2.1, the concepts of compatible mappings and weakly compatible mappings of type (*A*) in *G*-metric spaces have been introduced. Section 2.2 deals with some properties related to compatible mappings and weakly compatible mappings of type (*A*) in *G*-metric spaces have been proved. In Section 2.3, a common fixed point theorem for weakly compatible mappings has been proved by using the notion of the property (*E.A*). In Section 2.4, applications to the solution of integral equation and the bounded solution of a functional equation arising in dynamic programming have been given by making use of the result proved in the Section 2.3.

2.1 INTRODUCTION

In 2006, Mustafa and Sims [88] introduced the concept of *G*-metric spaces as follows:

Definition 2.1.1. Let X be a non-empty set, and let $G: X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following axioms: for all $x, y, z, a \in X$,

(G1) $G(x,y,z)=0$ if $x=y=z$;

(G2) $G(x,x,y)>0$ with $x \neq y$;

(G3) $G(x,x,y) \le G(x,y,z)$ with $z \neq y$;

(G4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=\ldots$;

(G5) $G(x,y,z) \le G(x,a,a)+G(a,y,z)$;

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then the function G is called a **Generalized metric** or more specifically a ***G*-metric** on X and the pair (X, G) is called a ***G*-metric space**.

Definition 2.1.2.[88] Let (X, G) and (X^0, G^0) be G-metric spaces and $f : (X, G) \to (X^0, G^0)$ be a function, then f is said to be ***G*-continuous** at a point $u \in X$ if and only if for a given $\varepsilon > 0,$ there exists $\delta > 0$ such that $G(u, x, y) < \delta$ implies $G^0\left(fu, fx, fy\right) < \varepsilon,$ for all $x, y \in X.$ A function f is G-continuous on X if and only if it is G-continuous at all $u \in X.$

In this chapter, the notion of compatible mappings, compatible mappings of type (A) and weakly compatible mappings of type (A) in G-metric spaces have been introduced as under:

Definition 2.1.3. A pair (f, g) of self-mappings of a G-metric space (X, G) is said to be **compatible mappings** if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ or $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X.$

Definition 2.1.4. A pair (f, g) of self-mappings of a G- metric space (X, G) is said to be **non-compatible** if there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X,$ but either

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) \neq 0, \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) \neq 0 \text{ or non-existent.}$$

Definition 2.1.5. A pair (f, g) of self-mappings of a G-metric space (X, G) is said to be **compatible mappings of type (*A*)** if

$$\lim_{n\to\infty} G(gfx_n, ffx_n, ffx_n) = 0 \text{ and } \lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X.$

Definition 2.1.6. A pair (f, g) of self-mappings of a G-metric space (X, G) is said to be **weakly compatible of type (*A*)** if

$$\lim_{n\to\infty} G(gfx_n, ffx_n, ffx_n) \leq \lim_{n\to\infty} G(fgx_n, ffx_n, ffx_n)$$

and

$$\lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) \leq \lim_{n\to\infty} G(gfx_n, ggx_n, ggx_n)$$

whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

2.2 COMPATIBLE MAPPINGS AND WEAKLY COMPATIBLE MAPPINGS OF TYPE (*A*)

The following proposition shows that under certain conditions, compatible mappings and compatible mappings of type (*A*) are equivalent:

Proposition 2.2.1. *Let f and g be G-continuous mappings from a G-metric space* (*X*, *G*) *into itself. Then f and g are compatible if and only if they are compatible of type* (*A*).

Proof. Suppose f and g are compatible mappings. Let $\{x_n\}$ be a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

By (*G*5), we have

$$G(gfx_n, ffx_n, ffx_n,) \leq G(gfx_n, fgx_n, fgx_n) + G(fgx_n, ffx_n, ffx_n).$$

Since f and g are compatible and f is *G*-continuous, we have

$$\lim_{n\to\infty} G(gfx_n, ffx_n, ffx_n) = 0.$$

Similarly, as g is *G*-continuous, we have

$$\lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) = 0.$$

Therefore, f and g are compatible of type (*A*).

Conversely, suppose that f and g are compatible of type (*A*).

Suppose that $\{x_n\}$ be a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Since g is *G*-continuous, therefore,

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gu.$$

By (*G*5),

$$G(fgx_n, gfx_n, gfx_n) \leq G(fgx_n, ggx_n, ggx_n) + G(ggx_n, gfx_n, gfx_n).$$

Since f and g are compatible mappings of type (*A*) and g is *G*-continuous, we have

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0.$$

Similarly, as f is *G*-continuous, we have

$$\lim_{n\to\infty} G\left(gfx_n,\, fgx_n,\, fgx_n\right) = 0.$$

Therefore, f and g are compatible. □

The following propositions show that under certain conditions, compatible mappings of type (A) and weakly compatible mappings of type (A) are equivalent:

Proposition 2.2.2. *Let f and g be two compatible mappings of type* (A) *from a G-metric space* (X, G) *into itself. Then f and g are weakly compatible mappings of type* (A).

Proof. As given, pair $\{f, g\}$ is compatible of type (A), then we have

$$0 = \lim_{n\to\infty} G(gfx_n,\, ffx_n,\, ffx_n) \le \lim_{n\to\infty} G(fgx_n,\, ffx_n,\, ffx_n)$$

and

$$0 = \lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) \le \lim_{n\to\infty} G(gfx_n, ggx_n, ggx_n),$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Therefore, f and g are weakly compatible of type (A). □

Proposition 2.2.3. *Let f and g be G-continuous mappings of a G-metric space* (X, G) *into itself. If f and g are weakly compatible of type* (A), *then they are compatible of type* (A).

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

As f and g are G-continuous mappings, therefore,

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = fu \text{ and } \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gu.$$

Also, f and g are weakly compatible of type (A), thus we have

$$\lim_{n\to\infty} G(gfx_n,\, ffx_n,\, ffx_n) \le \lim_{n\to\infty} G(fgx_n,\, ffx_n,\, ffx_n) = \mathrm{G}\left(fu,\, fu,\, fu\right) = 0$$

and

$$\lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) \le \lim_{n\to\infty} G(gfx_n, ggx_n, ggx_n) = G\left(gu, gu, gu\right) = 0.$$

This implies that,

$$\lim_{n\to\infty} G(gfx_n,\, ffx_n,\, ffx_n) = 0 \text{ and } \lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Hence, f and g are compatible of type (A). □

The following Proposition 2.2.4 is a direct consequence of Propositions 2.2.1, 2.2.2 and 2.2.3:

Proposition 2.2.4. *Let* f *and* g *be G-continuous mappings from a G-metric space* (X, G) *into itself. Then*

(i) f *and* g *are compatible of type* (A) *if and only if they are weakly compatible of type* (A).

(ii) f *and* g *are compatible if and only if they are weakly compatible of type* (A).

Proposition 2.2.5. *Let* f *and* g *be weakly compatible mappings of type* (A) *from a G-metric space* (X, G) *into itself. If* $fu = gu$ *for some* $u \in X$, *then* $fgu = ffu = ggu = gfu.$

Proof. Suppose that $\{x_n\}$ be a sequence in X defined by $x_n = u$, $n = 1, 2, 3, \ldots$ and $fu = gu.$ Then we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fu = gu.$ Since, f and g be weakly compatible mappings of type (A), we have

$$\begin{aligned} G(fgu, ggu, ggu) &= \lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) \\ &\le \lim_{n\to\infty} G(gfx_n, ggx_n, ggx_n) = G(gfu, ggu, ggu) \\ &= G(ggu, ggu, ggu) = 0. \end{aligned}$$

Hence, we have $fgu = ggu.$ Therefore, $fgu = ffu = ggu = gfu.$ □

Proposition 2.2.6. *Let* f *and* g *be weakly compatible mappings of type* (A) *from a G-metric space* (X, G) *into itself. Suppose* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ *for some* $u \in X$. *Then we have the following*:

(i) $\lim_{n\to\infty} gfx_n = fu$ *if* f *is G- continuous at* $u \in X$,

(ii) $\lim_{n\to\infty} fgx_n = gu$ *if* g *is G-continuous at* $u \in X$,

(iii) $fgu = ggu$ *and* $fu = gu$ *if* f *and* g *are G-continuous at* $u \in X$.

Proof. (i) Suppose $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$. Since, f is G-continuous, we have $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = fu.$

By (G5), we have

$$G\left(gfx_n,\, fu,\, fu\right) \le G\left(gfx_n,\, ffx_n,\, ffx_n\right) + G\left(\, ffx_n,\, fu,\, fu\right).$$

Taking limit $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} G\left(gfx_n,\, fu,\, fu\right) &\le \lim_{n\to\infty} G\left(gfx_n,\, ffx_n,\, ffx_n\right) + \lim_{n\to\infty} G\left(\, ffx_n,\, fu,\, fu\right)\\ &\le \lim_{n\to\infty} G\left(\, fgx_n,\, ffx_n,\, ffx_n\right) + G\left(\, fu,\, fu,\, fu\right)\\ &= G\left(\, fu,\, fu,\, fu\right) + G\left(\, fu,\, fu,\, fu\right) = 0 + 0 = 0.\end{aligned}$$

i.e., $\lim_{n\to\infty} G\left(gfx_n,\, fu,\, fu\right) = 0.$

This implies $\lim_{n\to\infty} gfx_n = fu.$ This completes the proof of part (i).

(ii) The proof of this part is similar to the proof of part (i).

(iii) Since g is G-continuous at u, we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u.$ This implies $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gu.$ Since f is G-continuous at u, then by the use of (i) $\lim_{n\to\infty} gfx_n = fu.$ Hence, by uniqueness of the limit, we have $fu = gu$ and so by Proposition 2.2.5, we get $fgu = gfu.$ □

2.3 COMMON FIXED POINT THEOREMS FORWEAKLY COMPATIBLE MAPPINGS USING PROPERTY (*E.A*)

In this section, the notion of the property (*E.A*) has been used to prove a common fixed point theorem for weakly compatible mappings.

Let Φ denote the set of all continuous functions $\phi : [0,\infty) \to [0,\infty)$ such that

(i) ϕ is non-decreasing;

(ii) $\lim_{n\to\infty} \phi^n(t) = 0,$ for all $t \in [0,\infty).$

If $\phi \in \Phi,$ then it is an easy to show that $\phi(0) = 0$ and $\phi(t) < t,$ for all $t \in \left(0,\infty\right)$ (see, Matkowski [76]).

The main result of this chapter is the following theorem:

Theorem 2.3.1. *Let* (X, G) *be a G-metric space and* A, B, S, $T : X \to X$ *be four mappings such that*:

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$;

(ii) *one of the pairs* (A, S) *or* (B, T) *satisfies property* (*E.A*);

(iii) *for all* $x, y \in X$, $\phi \in \Phi$,

$$G(Ax, By, By) \le \phi\left[\max\left\{G(Sx, Ty, Ty), G(Sx, By, By), G(Ty, By, By)\right\}\right];$$

(iv) *one of* $A(X)$, $B(X)$, $S(X)$ *or* $T(X)$ *is a complete subspace of* X.

Then the pairs (A, S) *and* (B, T) *have a coincidence point.*

Further, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point in* X.

Proof. Suppose the pair (B, T) satisfies the property (*E.A*). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p$ for some $p \in X$. Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sy_n = p$.

Now, we show that $\lim_{n\to\infty} Ay_n = p$.

From (iii), we have

$$G(Ay_n, Bx_n, Bx_n) \le \phi\left[\max\left\{G(Sy_n, Tx_n, Tx_n), G(Sy_n, Bx_n, Bx_n), G(Tx_n, Bx_n, Bx_n)\right\}\right].$$

Taking the limit as $n \to \infty$ and using the fact that $\phi(r)$ is continuous at $r = 0$, we get

$$\begin{aligned}\lim_{n\to\infty} G(Ay_n, p, p) &\le \phi\left[\max\left\{G(p, p, p), G(p, p, p), G(p, p, p)\right\}\right]\\ &= \phi\left[\max\{0, 0, 0\}\right] = \phi(0) = 0,\end{aligned}$$

and so, $\lim_{n\to\infty} Ay_n = p$.

Thus, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p$.

Suppose that $S(X)$ is a complete subspace of X. Then $p = Su$ for some $u \in X$. Subsequently, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p = Su$.

We now prove that $Au = Su = p$.

From (iii), we have

$$G(Au, Bx_n, Bx_n) \le \phi\left[\max\left\{G(Su, Tx_n, Tx_n), G(Su, Bx_n, Bx_n), G(Tx_n, Bx_n, Bx_n)\right\}\right].$$

Taking the limit as $n \to \infty$, and using the property of ϕ, we get

$$\begin{aligned} G(Au, Su, Su) &\le \phi\left[\max\left\{G(p,p,p), G(p,p,p), G(p,p,p)\right\}\right] \\ &= \phi\left[\max\{0,0,0\}\right] = \phi(0) = 0, \end{aligned}$$

which implies *Au* = *Su*. Therefore, *u* is a coincidence point for the pair (*A*, *S*).

The weak compatibility of *A* and *S* implies that *ASu* = *SAu* and hence *AAu* = *ASu* = *SAu* = *SSu*. Since, $A(X) \subseteq T(X)$, there exists $v \in X$ such that *Au* = *Tv*. We claim that *Tv* = *Bv* or *Au* = *Bv*.

Suppose not, from (iii) and using the fact that $\phi(r) < r$, we have

$$\begin{aligned} G(Au, Bv, Bv) &\le \phi\left[\max\left\{G(Su, Tv, Tv), G(Su, Bv, Bv), G(Tv, Bv, Bv)\right\}\right] \\ &= \phi\left[\max\left\{0, G(Au, Bv, Bv), G(Au, Bv, Bv)\right\}\right] \\ &= \phi\left[G(Au, Bv, Bv)\right] < G(Au, Bv, Bv), \end{aligned}$$

which is a contradiction to the supposition that $Au \neq Bv$. Hence *Au* = *Bv* = *Tv*. It follows therefore that the pair (*B*, *T*) has also a coincidence point. The weak compatibility of *B* and *T* implies that *BTv* = *TBv* = *TTv* = *BBv*.

Finally, we show that *Au* is a common fixed point of *A* and *S*.

Suppose not, then from (iii) and by using the property of ϕ, we get

$$\begin{aligned} G(AAu, Au, Au) &= G(AAu, Bv, Bv) \\ &\le \phi\left[\max\left\{G(SAu, Tv, Tv), G(SAu, Bv, Bv), G(Tv, Bv, Bv)\right\}\right] \\ &= \phi\left[\max\left\{G(AAu, Au, Au), G(AAu, Au, Au), G(Au, Au, Au)\right\}\right] \\ &= \phi\left[\max\left\{G(AAu, Au, Au), G(AAu, Au, Au), 0\right\}\right] \\ &= \phi\left[G(AAu, Au, Au)\right] < G(AAu, Au, Au), \end{aligned}$$

which gives a contradiction. Therefore, *Au* = *AAu* = *SAu* is a common fixed point of *A* and *S*.

Similarly, one can prove that *Bv* is a common fixed point of *B* and *T*. Since *Au* = *Bv*, we deduce that *Au* is a common fixed point of *A*, *B*, *S* and *T*.

For uniqueness, suppose x_0 and y_0, with $x_0 \neq y_0$ are two common fixed points of *A*, *B*, *S* and *T*. Then, from (iii), and using the property of ϕ, we have

$$\begin{aligned} G(x_0, y_0, y_0) &= G(Ax_0, By_0, By_0) \\ &\leq \phi\left[\max\left\{G(Sx_0, Ty_0, Ty_0), G(Sx_0, By_0, By_0), G(Ty_0, By_0, By_0)\right\}\right] \\ &= \phi\left[\max\left\{G(x_0, y_0, y_0), G(x_0, y_0, y_0), G(y_0, y_0, y_0)\right\}\right] \\ &= \phi\left[G(x_0, y_0, y_0)\right] \\ &< G(x_0, y_0, y_0), \end{aligned}$$

which is a contradiction and so there must be $x_0 = y_0$. Therefore *A*, *B*, *S* and *T* have a unique common fixed point. On the similar lines, one can obtain the same conclusion in case one of *A*(*X*), *B*(*X*) or *T*(*X*) is a complete subspace of *X* instead of *S*(*X*), and in case the pair (*A*, *S*) satisfies the property (*E.A*) in place of pair (*B*, *T*). This completes the proof of the theorem. □

By choosing *S* = *T* in Theorem 2.3.1, the following result involving three self-mappings is deduced:

Corollary 2.3.1. *Let* (*X*, *G*) *be a G-metric space and A, B, S* : *X* $\rightarrow$ *X be three mappings such that*:

(i) $A(X) \subseteq S(X)$ *and* $B(X) \subseteq S(X)$;

(ii) *one of the pairs* (*A*, *S*) *and* (*B*, *S*) *satisfies the property* (*E.A*);

(iii) *for all* $x, y \in X$, $\phi \in \Phi$,

$$G(Ax, By, By) \leq \phi\left[\max\left\{G(Sx, Sy, Sy), G(Sx, By, By), G(Sy, By, By)\right\}\right];$$

(iv) *one of A*(*X*), *B*(*X*)*or S*(*X*) *is a complete subspace of X.*

Then the pairs (*A*, *S*) *and* (*B*, *S*) *have a coincidence point.*

Further, if (*A*, *S*) *and* (*B*, *S*) *are weakly compatible, then A, B and S have a unique common fixed point in X.*

The following example illustrates the validity of Corollary 2.3.1:

Example 2.3.1. Let $X = [0, 2]$ and $G : X \times X \times X \rightarrow [0, \infty)$ be defined by

$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}, \text{ for all } x, y, z \in X.$$

Define also $A, B, S : X \rightarrow X$ by $Ax = 1$, $Bx = 2 - x$ and $Sx = x$, for all $x \in X$ and $\phi : [0, \infty) \rightarrow [0, \infty)$ by $\phi(t) = \frac{t}{2}$, for all $t \geq 0$. Clearly, the hypotheses (i) and (iv) of the

Corollary 2.3.1 hold trivially. Moreover, the pair (*A*, *S*) satisfies the property (*E.A*). Here we show only that the hypothesis (iii) holds. In fact, for all $x, y \in X$, we have

$$\begin{aligned} &G\left(Ax, By, By\right) = G\left(1, 2-y, 2-y\right) = \left|1-y\right|, \\ &G\left(Sx, Sy, Sy\right) = G\left(x, y, y\right) = \left|x-y\right|, \\ &G\left(Sx, By, By\right) = G\left(x, 2-y, 2-y\right) = \left|2-x-y\right|, \\ &G\left(Sy, By, By\right) = G\left(y, 2-y, 2-y\right) = 2\left|1-y\right| \end{aligned}$$

and consequently

$$\begin{aligned} G\left(Ax, By, By\right) &= \left|1-y\right| \\ &= \frac{1}{2} 2\left|1-y\right| = \phi\left(G(Sy, By, By)\right) \\ &\le \phi\left[\max\left\{G(Sx, Sy, Sy), G(Sx, By, By), G(Sy, By, By)\right\}\right]. \end{aligned}$$

Then, by the Corollary 2.3.1, the pairs (*A*, *S*) and (*B*, *S*) have a coincidence point, that is, $x = 1$. Moreover, if (*A*, *S*) and (*B*, *S*) are weakly compatible, then $x = 1$ is the unique common fixed point of *A*, *B* and *S* in *X*.

2.4 APPLICATIONS TO INTEGRAL AND FUNCTIONAL EQUATIONS

In this section, two useful applications of Corollary 2.3.1 are presented. Firstly, the solution of an integral equation is obtained by applying the Corollary 2.3.1.

Let $\Omega = \left[0,1\right]$ and $C\left(\Omega\right)$ be the space of all real valued continuous functions defined on Ω. Clearly, this space endowed with the *G*-metric given by

$$G(x, y, z) = \sup_{t\in\Omega}\left|x(t) - y(t)\right| + \sup_{t\in\Omega}\left|y(t) - z(t)\right| + \sup_{t\in\Omega}\left|z(t) - x(t)\right|, \text{ for all } x, y, z \in C\left(\Omega\right),$$

is a *G*-complete metric space.

Let $p : \Omega \times \mathbf{R} \to \mathbf{R}$ and $q : \Omega \times \Omega \times \mathbf{R} \to \mathbf{R}$ be two continuous functions. Consider an integral equation of the following type:

$$p\left(t, x(t)\right) = \int_{\Omega} q\left(t, s, x(s)\right) ds, \text{ for all } s, t \in \Omega . \tag{2.4.1}$$

Theorem 2.4.1. *Suppose there exists* $H : \Omega \times \mathbf{R} \to \left[0, \infty\right)$ *such that*

(i) $H\left(s, v(t)\right) \le \int_{\Omega} q\left(t, s, u(s)\right) ds \le p\left(s, v(t)\right)$, *for all* $s, t \in \Omega$;

(ii) $p\left(s, v(t)\right) - H\left(s, v(t)\right) \le k\left|p\left(s, v(t)\right) - v(t)\right|$, *where* $k \in \left(0,1\right)$.

Then the integral equation (2.4.1) *has a unique solution in* $C(\Omega)$.

Proof. Define $(Ax)(t)=\int_\Omega q(t,s,x(s))ds$ and $(Bx)(t)=p(t,x(t))$. Now

$$\begin{aligned}G(Ax,By,By)&=2\sup_{t\in\Omega}\left|(Ax)(t)-(By)(t)\right|\\&=2\sup_{t\in\Omega}\left|p(t,y(t))-\int_\Omega q(t,s,x(s))ds\right|\\&\le 2\sup_{t\in\Omega}\left|p(t,y(t))-H(t,y(t))\right|\\&\le 2k\sup_{t\in\Omega}\left|p(t,y(t))-y(t)\right|=k\,G(y,By,By).\end{aligned}$$

Thus, all the hypotheses of the Corollary 2.3.1 are satisfied by choosing $S=I_{C(\Omega)}$ (the identity mapping on Ω), and $\phi(r)=kr,$ for all $r\ge 0,$ $k\in(0,1)$. Therefore, there is a unique solution of the integral equation (2.4.1) in $C(\Omega)$. □

Further, by using the Corollary 2.3.1, the existence and uniqueness of the bounded solution of a functional equation arising in dynamic programming has been presented.

Here, we assume that *U* and *V* are Banach spaces, $W\subseteq U$ is a state space and $D\subseteq V$ is a decision space.

It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well (see, e.g., [19-21]). In particular, the problem of dynamic programming related to multistage process reduces to the problem of solving the functional equation

$$Q(x)=\sup_{y\in D}\left\{f(x,y)+K\left(x,y,Q\left(\tau(x,y)\right)\right)\right\},\ x\in W, \tag{2.4.2}$$

where $\tau:W\times D\to W,\ f:W\times D\to \mathbf{R}, K:W\times D\times\mathbf{R}\to\mathbf{R}.$

Let *B*(*W*) denote the space of all bounded real-valued functions on *W*. Clearly, this space endowed with the *G*-metric given by

$$G(h,k,p)=\max\left\{\sup_{x\in W}\left|hx-kx\right|,\sup_{x\in W}\left|kx-px\right|,\sup_{x\in W}\left|px-hx\right|\right\},\ \text{for all } h,k,p\in B(W),$$

is a *G*-complete metric space.

Theorem 2.4.2. *Let* $K:W\times D\times\mathbf{R}\to\mathbf{R}$ *and* $f:W\times D\to\mathbf{R}$ *be two bounded functions and let* $A:B(W)\to B(W)$ *be defined by*

$$A(hx)=\sup_{y\in D}\left\{ f\left(x,y\right)+K\left(x,y,h\left(\tau\left(x,y\right)\right)\right)\right\}, \tag{2.4.3}$$

for all $h\in B(W)$ *and* $x\in W$. *Assume that the following condition holds*:

$$\left|K\left(x,y,h_1\left(\tau\left(x,y\right)\right)\right)-K\left(x,y,h_2\left(\tau\left(x,y\right)\right)\right)\right|\le \phi\left(\left|h_1x-h_2x\right|\right),$$

where $x\in W, h_1,h_2\in B(W), y\in D$ *and* $\phi\in\Phi$. *Then functional equation* (2.4.2) *has a unique bounded solution.*

Proof. Note that $\left(B(W),G\right)$ is a complete G-metric space. Let ε be an arbitrary positive number, $x\in W$ and $h_1,h_2\in B(W)$, then there exist $y_1,y_2\in D$ such that

$$A(h_1x)< f\left(x,y_1\right)+K\left(x,y_1,h_1\left(\tau\left(x,y_1\right)\right)\right)+\varepsilon, \tag{2.4.4}$$

$$A(h_2x)< f\left(x,y_2\right)+K\left(x,y_2,h_2\left(\tau\left(x,y_2\right)\right)\right)+\varepsilon, \tag{2.4.5}$$

$$A(h_1x)\ge f\left(x,y_2\right)+K\left(x,y_2,h_1\left(\tau\left(x,y_2\right)\right)\right), \tag{2.4.6}$$

$$A(h_2x)\ge f\left(x,y_1\right)+K\left(x,y_1,h_2\left(\tau\left(x,y_1\right)\right)\right). \tag{2.4.7}$$

Then, from (2.4.4) and (2.4.7), it follows easily that

$$\begin{aligned}A(h_1x)-A(h_2x)&<K\left(x,y_1,h_1\left(\tau\left(x,y_1\right)\right)\right)-K\left(x,y_1,h_2\left(\tau\left(x,y_1\right)\right)\right)+\varepsilon\\&\le\left|K\left(x,y_1,h_1\left(\tau\left(x,y_1\right)\right)\right)-K\left(x,y_1,h_2\left(\tau\left(x,y_1\right)\right)\right)\right|+\varepsilon\\&\le\phi\left(\left|h_1x-h_2x\right|\right)+\varepsilon.\end{aligned}$$

Hence, we get

$$A(h_1x)-A(h_2x)<\phi\left(\left|h_1x-h_2x\right|\right)+\varepsilon. \tag{2.4.8}$$

Similarly, from (2.4.5) and (2.4.6), we obtain

$$A(h_2x)-A(h_1x)<\phi\left(\left|h_1x-h_2x\right|\right)+\varepsilon. \tag{2.4.9}$$

Therefore, from (2.4.8) and (2.4.9), we have

$$\left|A(h_1x)-A(h_2x)\right|<\phi\left(\left|h_1x-h_2x\right|\right)+\varepsilon, \tag{2.4.10}$$

which implies,

$$G\left(A(h_1x),A(h_2x),A(h_2x)\right)<\phi\left(G\left(h_1x,h_2x,h_2x\right)\right)+\varepsilon.$$

The above inequality is true for any $x \in W$. Since $\varepsilon > 0$ is arbitrary, we get

$$G\left(A(h_1 x), A(h_2 x), A(h_2 x)\right) \leq \phi\left(G\left(h_1 x, h_2 x, h_2 x\right)\right).$$

Thus, all the hypothesis of the Corollary 2.3.1 are satisfied with A = B and $S = I_{B(W)}$ (the identity mapping on B(W)). Therefore, the functional equation (2.4.2) has a unique bounded solution. □

CHAPTER-III

COMMON FIXED POINT THEOREMS FOR VARIANTS OF *R*-WEAKLY COMMUTING MAPPINGS IN *G*-METRIC SPACES

In this chapter, some common fixed point theorems for variants of *R*-weakly commuting mappings in *G*-metric spaces have been proved. This chapter has been divided into various sections. In Section 3.1, the concept of *R*-weakly commuting mapping and its variants in *G*-metric spaces have been introduced. In Section 3.2, a common fixed point theorem has been established by using *R*-weakly commuting mappings in *G*-metric spaces. Section 3.3 deals with some new common fixed point theorems in *G*-metric spaces for some variants of *R*-weakly commuting mappings by utilizing the concept of weak reciprocal continuity. The results presented in this chapter have consequently improved and sharpened many known common fixed point theorems in the existing literature. Also, some examples in support of these results have been given.

3.1 INTRODUCTION

In 1994, Pant [92] introduced the notion of *R*-weakly commuting mappings in metric spaces as follows:

A pair (f, g) of self-mappings of a metric space (X, d) is said to be *R*-weakly commuting if there exists some real number $R > 0$ such that

$$d\left(fgx, gfx\right) \leq R\, d\left(fx, gx\right), \text{ for all } x \in X.$$

Now, we introduce the concept of *R*-weakly commuting mappings and its variants in *G*-metric spaces as follows:

Definition 3.1.1. A pair (f, g) of self-mappings of a *G*-metric space (X, G) is said to be ***R*-weakly commuting at a point** $x \in X$ if

$$G\left(fgx, gfx, gfx\right) \leq R\, G\left(fx, gx, gx\right), \text{ for some } R > 0.$$

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Definition 3.1.2. A pair (*f*, *g*) of self-mappings of a *G*-metric space (*X*, *G*) is said to be **pointwise *R*-weakly commuting** on *X* if given $x \in X$, there exists $R > 0$ such that

$$G\left(fgx, gfx, gfx\right) \le R\, G\left(fx, gx, gx\right).$$

Definition 3.1.3. A pair (*f*, *g*) of self-mappings of a *G*-metric space (*X*, *G*) is said to be

(i) ***R*-weakly commuting of type (A_g)** if there exists some $R > 0$ such that

$$G\left(ffx, gfx, gfx\right) \le R\, G\left(fx, gx, gx\right), \text{for all } x \in X.$$

(ii) ***R*-weakly commuting of type (A_f)** if there exists some $R > 0$ such that

$$G\left(fgx, ggx, ggx\right) \le R\, G\left(fx, gx, gx\right), \text{for all } x \in X.$$

Definition 3.1.4. A pair (*f*, *g*) of self-mappings of a *G*-metric space (*X*, *G*) is said to be ***R*-weakly commuting of type (*P*)** if there exists some $R > 0$ such that

$$G\left(ffx, ggx, ggx\right) \le R\, G\left(fx, gx, gx\right), \text{ for all } x \in X.$$

3.2 A COMMON FIXED POINT THEOREM FOR *R*-WEAKLY COMMUTING MAPPINGS

In this section, the following result on complete *G*-metric space has been established:

Theorem 3.2.1. *Let* (*X*, *G*) *be a complete G-metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) *f or g is continuous*;

(iii) $G(fx, fy, fz) \le q\, G(gx, gy, gz)$, *for every* $x, y, z \in X$ *and* $0 \le q < 1$.

Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in *X*. By (i), one can choose a point $x_1 \in X$ such that $fx_0 = gx_1$. In general, choose x_{n+1} such that $y_n = fx_n = gx_{n+1}$.

Now, we show that $\{y_n\}$ is a *G*-Cauchy sequence in *X*. For proving this, we take $x = x_n, y = x_{n+1}, z = x_{n+1}$ in (iii), we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le q\, G(gx_n, gx_{n+1}, gx_{n+1}) = q\, G(fx_{n-1}, fx_n, fx_n).$$

Continuing in the same way, we get

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le q^n\, G(fx_0, fx_1, fx_1).$$

This implies,

$$G(y_n, y_{n+1}, y_{n+1}) \le q^n\, G(y_0, y_1, y_1).$$

Therefore, for all $n, m \in \mathbf{N}$, $n < m$,

$$\begin{aligned} G(y_n, y_m, y_m) &\le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \ldots + G(y_{m-1}, y_m, y_m) \\ &\le \left(q^n + q^{n+1} + \ldots + q^{m-1}\right) G(y_0, y_1, y_1) \\ &\le \left(q^n + q^{n+1} + \ldots\right) G(y_0, y_1, y_1) \\ &= \frac{q^n}{(1-q)} G(y_0, y_1, y_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $\{y_n\}$ is a *G*-Cauchy sequence in *X*. Since (*X*, *G*) is complete *G*-metric space, therefore, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z.$ Let us suppose that the mapping f is continuous. Therefore, $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ffx_n = fz.$

Since *f* and *g* are *R*-weakly commuting,

$$G\left(fgx_n, gfx_n, gfx_n\right) \le R\, G\left(fx_n, gx_n, gx_n\right), \text{where } R > 0.$$

On letting $n \to \infty$, we get $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = fz.$

We now prove that *z* = *fz*. Suppose $z \ne fz$, then *G*(*z*, *fz*, *fz*) > 0.

On setting $x = x_n$, $y = fx_n$, $z = fx_n$ in (iii), we have

$$G\left(fx_n, ffx_n, ffx_n\right) \le q\, G\left(gx_n, gfx_n, gfx_n\right).$$

Letting limit as $n \to \infty$, we get

$$G(z, fz, fz) \le q\, G(z, fz, fz) < G(z, fz, fz),$$

which is a contradiction.

Therefore, *z* = *fz*. Since *f*(*X*) $\subseteq$ *g*(*X*), we can find $z_1 \in X$ such that $z = fz = gz_1$.

Now put $x = fx_n$, $y = z = z_1$ in (iii), we have

$$G(ffx_n, fz_1, fz_1) \le q\, G(gfx_n, gz_1, gz_1).$$

Taking limit as $n \to \infty$, we get

$$G(fz, fz_1, fz_1) \leq q\, G(fz, gz_1, gz_1) = q\, G(fz, fz, fz) = 0,$$

which implies that $fz = fz_1$, i.e., $z = fz = fz_1 = gz_1$. Also, by using definition of *R*-weakly commutativity,

$$G(fz, gz, gz) = G(fgz_1, gfz_1, gfz_1) \leq R\, G(fz_1, gz_1, gz_1) = 0,$$

implies that *fz* = *gz* = *z*. Thus, *z* is a common fixed point of *f* and *g*.

Uniqueness: Assume that *w* ($\neq$z) be another common fixed point of *f* and *g*.

Then, $G(z, w, w) > 0$ and

$$\begin{aligned} G(z, w, w) &= G(fz, fw, fw) \leq q\, G(gz, gw, gw) \\ &= q\, G(z, w, w) < G(z, w, w), \end{aligned}$$

a contradiction. Therefore, *z* = *w*. Hence uniqueness follows. □

The following example demonstrates the validity of Theorem 3.2.1:

Example 3.2.1. Let *X* = [-1, 1] and let $G: X \times X \times X \to \mathbf{R}^+$ be the *G*-metric defined by $G(x, y, z) = \left(|x - y| + |y - z| + |z - x|\right)$, for all $x, y, z \in X$. Then (*X*, *G*) is a complete *G*-metric space. Define self-mappings *f* and *g* on *X* by *fx* = *x* and *gx* = 2*x* -1, for all $x \in X$. Here we note that,

(i) $f(X) \subseteq g(X)$;

(ii) *f* is continuous on *X*;

(iii) $G(fx, fy, fz) \leq q\, G(fx, gy, gz)$, holds for all $x, y, z \in X$, $\frac{1}{2} \leq q < 1$.

Further, the mappings *f* and *g* are *R*-weakly commuting. Thus, all the conditions of the Theorem 3.2.1 are satisfied and *x* = 1 is the unique common fixed point of *f* and *g*.

3.3 COMMON FIXED POINT THEOREMS FOR VARIANTS OF *R*-WEAKLY COMMUTING MAPPINGS BY USING WEAK RECIPROCAL CONTINUITY

In this section, we prove the following theorems by using variants of *R*-weakly commuting mappings:

Theorem 3.3.1. *Let f and g be weakly reciprocally continuous self-mappings of a complete G- metric space* (X, G) *satisfying the following conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) $G(fx, fy, fz) \leq q\, G(gx, gy, gz)$, *for every* $x, y, z \in X$ *and* $0 \leq q < 1$.

If f and g are either compatible or R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f) *or R-weakly commuting of type* (P), *then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in *X*. As $f(X) \subseteq g(X)$, therefore there exists a sequence of points $\{x_n\}$ such that $fx_n = gx_{n+1}$.

Define a sequence $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$. (3.3.1)

Sequence $\{y_n\}$ is a *G* – Cauchy sequence in *X* (the proof follows the same lines as in Theorem 3.2.1). Now, since (*X*, *G*) is complete *G*-metric space, therefore there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Hence, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$.

Suppose that *f* and *g* are compatible mappings. Weak reciprocal continuity of *f* and *g* implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$. Let $\lim_{n\to\infty} gfx_n = gz$, then the compatibility of *f* and *g* gives, $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, i.e., $G\left(\lim_{n\to\infty} fgx_n, gz, gz\right) = 0$.

Hence, $\lim_{n\to\infty} fgx_n = gz$.

By (3.3.1), we have $\lim_{n\to\infty} fgx_{n+1} = \lim_{n\to\infty} ffx_n = gz$.

Therefore, by the use of (ii), we get

$$G(fz,\ ffx_n,\ ffx_n) \leq q\ G(gz,\ gfx_n,\ gfx_n).$$

Letting $n \to \infty$ on both sides, we have

$$G(fz, gz, gz) \leq q\, G\left(gz,\ gz,\ gz\right) = 0.$$

This gives, $fz = gz$. Again, compatibility of f and g implies commutativity at a coincidence point. Hence $gfz = fgz = ffz = ggz$. Now, we claim that $fz = ffz$. Suppose $fz \neq ffz,$ then by using (ii) we obtain,

$$\begin{aligned} G\left(fz,\ ffz,\ ffz\right) &\leq q\, G\left(gz,\ gfz,\ gfz\right) \\ G\left(fz,\ ffz,\ ffz\right) &\leq q\, G\left(fz,\ ffz,\ ffz\right), \end{aligned}$$

which is a contraction since $q \in [0,1)$. Hence, $fz = ffz = gfz$ and fz is a common fixed point of f and g.

Next suppose that $\lim_{n\to\infty} fgx_n = fz.$ Then $f(X) \subseteq g(X)$ implies that $fz = gu$ for some $u \in X$ and therefore, $\lim_{n\to\infty} fgx_n = gu.$

Compatibility of f and g implies, $\lim_{n\to\infty} gfx_n = gu.$

By virtue of (3.3.1), we have $\lim_{n\to\infty} fgx_{n+1} = \lim_{n\to\infty} ffx_n = gu.$

Using (ii), we get

$$G(fu,\ ffx_n,\ ffx_n) \leq q\, G(gu, gfx_n, gfx_n).$$

Letting $n \to \infty$ on both sides, we have

$$G(fu, gu, gu) \leq q\, G(gu, gu, gu) = 0,$$

this gives, $fu = gu$. Compatibility of f and g yields $fgu = ggu = ffu = gfu$. Finally, we claim that $fu = ffu$. Suppose that $fu \neq ffu$, then by using (ii) we obtain,

$$\begin{aligned} G(fu,\ ffu,\ ffu) &\leq q\, G(gu, gfu, gfu) \\ G(fu,\ ffu,\ ffu) &\leq q\, G(fu,\ ffu,\ ffu), \end{aligned}$$

which again gives a contradiction, since $q \in [0,1)$. Hence $fu = ffu = gfu$. Therefore, fu is a common fixed point of f and g.

Now, suppose that f and g are R-weakly commuting of type (A_g). Now, weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz.$ Let us first assume that $\lim_{n\to\infty} gfx_n = gz.$ Then R-weak commutativity of type (A_g) of f and g yields

$$G\left(ffx_n, gfx_n, gfx_n\right) \leq R\, G\left(fx_n, gx_n, gx_n\right) \text{ where } R > 0.$$

Letting $n \to \infty$ on both sides, we have

$$G\left(\lim_{n\to\infty} ffx_n, gz, gz\right) \leq R\, G(z, z, z) = 0 .$$

This gives, $\lim_{n\to\infty} ffx_n = gz$. Also, using (ii) we get

$$G(fz, ffx_n, ffx_n) \leq q\, G(gz, gfx_n, gfx_n) .$$

Letting $n \to \infty$ on both sides, we have

$$G(fz, gz, gz) \leq q\, G(gz, gz, gz) = 0.$$

Hence, we get *fz* = *gz*. Again, by using *R*-weak commutativity of type (A_g),

$$G(ffz, gfz, gfz) \leq R\, G(fz, gz, gz) = 0.$$

This yields *ffz* = *gfz*. Therefore, *ffz* = *fgz* = *gfz* = *ggz*. We claim that *fz* = *ffz*. Suppose $fz \neq ffz,$ using (ii), we get

$$G(fz, ffz, ffz) \leq q\, G(gz, gfz, gfz)$$
$$G(fz, ffz, ffz) \leq q\, G(fz, ffz, ffz),$$

a contradiction. Hence *fz* = *ffz* = *gfz* and *fz* is a common fixed point of *f* and *g*.
Similarly, we can prove if $\lim_{n\to\infty} fgx_n = fz.$

On the other hand, if *f* and *g* are *R*-weakly commuting mappings of type (A_f), then by following the similar steps as presented above, it can easily be proved that *fz* is a common fixed point of *f* and *g*.

Finally now, suppose that *f* and *g* are *R*-weakly commuting of type (*P*). Weak reciprocal continuity of *f* and *g* implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz.$ Let us assume that $\lim_{n\to\infty} gfx_n = gz.$ Since pair (*f*, *g*) is *R*-weak commuting of type (*P*), we have

$$G(ffx_n, ggx_n, ggx_n) \leq R\, G(fx_n, gx_n, gx_n) \text{ where } R > 0.$$

Letting $n \to \infty$ on both sides, we get

$$G\left(\lim_{n\to\infty} ffx_n, \lim_{n\to\infty} ggx_n, \lim_{n\to\infty} ggx_n\right) \leq R\, G(z, z, z) = 0.$$

This gives, $G\left(\lim_{n\to\infty} ffx_n, \lim_{n\to\infty} ggx_n, \lim_{n\to\infty} ggx_n\right) = 0.$

Using (i) and (3.3.1), we have, $gfx_{n-1} = ggx_n \to gz$ as $n \to \infty$, this gives, $ffx_n \to gz$ as $n \to \infty$. Also, by using (ii) we have

$$G(fz, ffx_n, ffx_n) \le q\, G(gz, gfx_n, gfx_n).$$

Letting $n \to \infty$ on both sides, we get

$$G(fz, gz, gz) \le q\, G(gz, gz, gz) = 0.$$

This implies that $fz = gz$. Again, by using R-weak commutativity of type (P),

$$G\left(ffz, ggz, ggz\right) \le R\, G\left(fz, gz, gz\right) = 0 \text{ where } R > 0.$$

This yields $ffz = ggz$. Therefore, $ffz = fgz = gfz = ggz$.

Lastly, we claim that $fz = ffz$. Suppose that $fz \ne ffz$.

Using (ii), we have

$$G(fz, ffz, ffz) \le q\, G(gz, gfz, gfz)$$
$$G(fz, ffz, ffz) \le q\, G(fz, ffz, ffz),$$

which is a contradiction. Therefore, $fz = ffz$.

Hence $fz = ffz = gfz$ and fz is a common fixed point of f and g.

This result holds good even if $\lim_{n\to\infty} fgx_n = fz$ is considered instead of $\lim_{n\to\infty} gfx_n = gz$.

Uniqueness of the common fixed point in each of the three types of mappings can easily be obtained by using (ii). □

The following example illustrates the Theorem 3.3.1:

Example 3.3.1. Let (X, G) be G- metric space, where $X = [2, 20]$ and for all $x, y, z \in X$, $G(x, y, z) = \left(|x - y| + |y - z| + |z - x|\right)$. Define $f,\ g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 6, & 2 < x \le 5 \end{cases} \text{ and } gx = \begin{cases} 2, & x = 2 \\ 12, & 2 < x \le 5 \\ (x+1)/3, & x > 5. \end{cases}$$

It can be easily verified that

(i) $f\left(X\right) \subseteq g\left(X\right)$;

(ii) f and g satisfies condition (ii) of Theorem 3.3.1;

(iii) f and g are R-weakly commuting of type(A_g);

(iv) f and g are weakly reciprocally continuous for sequences $\{x_n\} = \{2\}$ or $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$ for each in X.

Thus, f and g satisfy all the conditions of Theorem 3.3.1 and have a unique common fixed point at $x = 2$.

We now establish the following common fixed point theorems for a non-compatible pair of self- mappings in a G-metric space:

Theorem 3.3.2. *Let f and g be weakly reciprocally continuous non-compatible self-mappings of G- metric space* (X, G) *satisfying the following conditions*:

(i) $f(X) \subseteq g(X)$;

(ii) $G(fx, fy, fz) \leq k\, G(gx, gy, gz), \; \text{for all } k \geq 0, \; x, y, z \in X$;

(iii) $G(fx, ffx, ffx) < \max \begin{Bmatrix} G(gx, gfx, gfx), G(fx, gx, gx), G(ffx, gfx, gfx), \\ G(fx, gfx, gfx), G(gx, ffx, ffx) \end{Bmatrix}$,

for all $x \in X$, *provided the right hand side is non-zero.*

If f and g are R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f), *then f and g have common fixed point.*

Proof. Given that f and g are non-compatible mappings, therefore there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$, but either

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) \neq 0, \; \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) \neq 0 \text{ or non-existent.}$$

By (i), for each x_n, there exists y_n in X such that $fx_n = gy_n$.

Thus, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} gy_n = z.$

By using (ii), we obtain

$$\begin{aligned} &G(fx_n, fy_n, fy_n) \leq k\, G(gx_n, gy_n, gy_n) \\ &G(z, \lim_{n\to\infty} fy_n, \lim_{n\to\infty} fy_n) \leq k\, G(z, z, z) = 0. \end{aligned}$$

This gives, $\lim_{n\to\infty} fy_n = z.$ Therefore, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} fy_n = z.$

Now, suppose that f and g are R-weakly commuting of type (A_g). Then weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz.$ Similarly, $\lim_{n\to\infty} fgy_n = fz$ or $\lim_{n\to\infty} gfy_n = gz.$ Let us first assume that $\lim_{n\to\infty} gfy_n = gz.$

Then R-weak commutativity of type (A_g) of f and g yields that

$$G(ffy_n, gfy_n, gfy_n) \le R\, G(fy_n, gy_n, gy_n)$$
$$G\left(\lim_{n\to\infty} ffy_n, gz, gz\right) \le R\, G(z, z, z) = 0,$$

where $R > 0$.

This gives, $\lim_{n\to\infty} ffy_n = gz$. Using (ii), we get

$$G(ffy_n, fz, fz) \le k\, G(gfy_n, gz, gz).$$

Letting $n \to \infty$ on both sides, we have

$$G(gz, fz, fz) \le k\, G(gz, gz, gz) = 0.$$

This implies that $fz = gz$. Again, by virtue of R-weak commutativity of type (A_g),

$$G(ffz, gfz, gfz) \le R\, G(fz, gz, gz) = 0.$$

This yields $ffz = gfz$ and $ffz = fgz = gfz = ggz$. If $fz \ne ffz$, then by the use of (iii), we get

$$\begin{aligned} G(fz, ffz, ffz) &< \max\begin{Bmatrix} G(gz, gfz, gfz), G(fz, gz, gz), G(ffz, gfz, gfz), \\ G(fz, gfz, gfz), G(gz, ffz, ffz) \end{Bmatrix} \\ &= \max\begin{Bmatrix} G(fz, ffz, ffz), G(fz, fz, fz), G(ffz, ffz, ffz), \\ G(fz, ffz, ffz), G(fz, ffz, ffz) \end{Bmatrix} \\ G(fz, ffz, ffz) &< G(fz, ffz, ffz), \end{aligned}$$

which is a contradiction. Hence $fz = ffz = gfz$ and fz is a common fixed point of f and g.

This result holds good even if $\lim_{n\to\infty} fgy_n = fz$ is considered instead of $\lim_{n\to\infty} gfy_n = gz$.

On the other hand, if f and g are R-weakly commuting of type (A_f), then the proof follows the similar lines. □

The following example illustrates the validity of Theorem 3.3.2:

Example 3.3.2. Let (X, G) be G- metric space, where $X = [2, 20]$ and

$$G(x, y, z) = \left(|x-y| + |y-z| + |z-x|\right), \text{ for all } x, y, z \in X.$$

Define $f, g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 6, & 2 < x \le 5 \end{cases} \text{ and } gx = \begin{cases} 2, & x = 2 \\ 11, & 2 < x \le 5 \\ (x+1)/3, & x > 5. \end{cases}$$

Let $\{x_n\}$ be a sequence in *X* such that either $\{x_n\}=\{2\}$ or $\{x_n\}=\left\{5+\frac{1}{n}\right\}$ for each *n*. Then clearly, f and g satisfy all the conditions of Theorem 3.3.2 and *x* = 2 is a common fixed point.

Remark 3.3.1. The conclusions of Theorem 3.3.2 remain true even if condition (iii) of Theorem 3.3.2 is replaced by following condition:

$$G(fx,\, ffx,\, ffx) < G(gx, ggx, ggx), \text{ whenever } gx \neq ggx \text{ for all } x \in X.$$

Remark 3.3.2. Theorem 3.3.1 and 3.3.2 and generalize the results of Pant *et al.*[96] for a pair of mappings in *G* - metric space.

A natural way to unify and prove several fixed point theorems in a simple manner is to consider an implicit contractive type relation instead of the usual contractive conditions. Popa [103], initiated a study of implicit contractive type conditions for proving several classical fixed point theorems.

Now, we establish some fixed point theorems by making use of implicit relations.

Let Θ denote the class of continuous functions $\theta:\left(\mathbf{R}^{+}\right)^{5}\to\mathbf{R}^{+}$ such that $\theta(x,0,0,x,x)=x,$ for all $x\in\mathbf{R}^{+}.$

The following functions θ_1 and θ_2 defined by

(i) $\theta_1(x_1,x_2,x_3,x_4,x_5)=\max\{x_1,x_2,x_3,x_4,x_5\}$;

(iii) $\theta_2(x_1,x_2,x_3,x_4,x_5)=\sqrt[3]{x_1x_4x_5}$,

belong to the class Θ.

Theorem 3.3.3. *Let f and g be weakly reciprocally continuous non-compatible self-mappings of G- metric space* (*X, G*) *satisfying the following conditions*:

(i) $f(X)\subseteq g(X)$;

(ii) $\int_0^{G(fx,fy,fz)}\psi(s)ds \le \int_0^{G(gx,gy,gz)}\psi(s)ds,$ *for all* $x,y,z\in X$;

(iii) $\int_0^{G(fx,ffx,ffx)}\psi(s)ds < \int_0^{\omega}\psi(s)ds$

where for some $\theta\in\Theta,\ x\in X,$

$\omega=\theta\{G(gx,gfx,gfx),G(fx,gx,gx),G(ffx,gfx,gfx),G(fx,gfx,gfx),G(gx,ffx,ffx)\},$

provided right-hand side is non-zero and $\psi:\mathbf{R}^{+}\to\mathbf{R}$ *is a Lebesgue integrable non-negative summable mapping such that* $\int_0^{\varepsilon}\psi(s)ds>0$ *for each* $\varepsilon>0$.

If f and g are R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f), *then f and g have common fixed point.*

Proof. Let f and g are non-compatible mappings. Therefore, there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=z\in X$, but either $\lim_{n\to\infty} G(fgx_n,gfx_n,gfx_n)\neq 0$, $\lim_{n\to\infty} G(gfx_n,fgx_n,fgx_n)\neq 0$ or non-existent. Since $f(X)\subseteq g(X)$, for each x_n, there exists y_n in X such that $fx_n=gy_n$. Thus $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=\lim_{n\to\infty} gy_n=z$. By virtue of this and using (ii), we obtain

$$\int_0^{G(fx_n,fy_n,fy_n)}\psi(s)ds\le\int_0^{G(gx_n,gy_n,gy_n)}\psi(s)ds.$$

Letting $n\to\infty$ at both sides, we have

$$\int_0^{G(z,\lim_{n\to\infty} fy_n,\lim_{n\to\infty} fy_n)}\psi(s)ds\le\int_0^{G(z,z,z)}\psi(s)ds,$$

which implies that, $\lim_{n\to\infty} fy_n=z$. Therefore, we have

$$\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=\lim_{n\to\infty} gy_n=\lim_{n\to\infty} fy_n=z.$$

Suppose that f and g are R-weakly commuting of type (A_g). Then weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fgx_n=fz$ or $\lim_{n\to\infty} gfx_n=gz$. Similarly, $\lim_{n\to\infty} fgy_n=fz$ or $\lim_{n\to\infty} gfy_n=gz$. Let us first assume that $\lim_{n\to\infty} gfy_n=gz$. Then, R-weak commutativity of type (A_g) of f and g yields

$$G(ffy_n,gfy_n,gfy_n)\le R\,G(fy_n,gy_n,gy_n)$$

$$G\left(\lim_{n\to\infty} ffy_n,gz,gz\right)\le R\,G(z,z,z)=0$$

where $R>0$.

This gives, $\lim_{n\to\infty} ffy_n=gz$. Using (ii), we get

$$\int_0^{G(ffy_n,fz,fz)}\psi(s)ds\le\int_0^{G(gfy_n,gz,gz)}\psi(s)ds.$$

Letting $n\to\infty$ on both sides,

$$\int_0^{G(gz,fz,fz)} \psi(s)ds \le \int_0^{G(gz,gz,gz)} \psi(s)ds,$$

we have $fz = gz$. Again, by virtue of R-weak commutativity of type (A_g), we have

$$G\big(ffz, gfz, gfz\big) \le k\, G\big(fz, gz, gz\big) = 0\,.$$

This yields that $ffz = gfz$ and $ffz = fgz = gfz = ggz$. If $fz \ne ffz$ then by using (iii), we get

$$\int_0^{G(fz,ffz,ffz)} \psi(s)ds < \int_0^{\omega} \psi(s)ds$$

where

$$\begin{aligned}\omega &= \theta\{G(gz,gfz,gfz), G(fz,gz,gz), G(ffz,gfz,gfz), G(fz,gfz,gfz), G(gz,ffz,ffz)\}\\ &= \theta\{G(fz,ffz,ffz), G(fz,fz,fz), G(ffz,ffz,ffz), G(fz,ffz,ffz), G(fz,ffz,ffz)\}\\ &= \theta\{G(fz,ffz,ffz), 0, 0, G(fz,ffz,ffz), G(fz,ffz,ffz)\}\\ &= G(fz,ffz,ffz),\end{aligned}$$

which gives

$$\int_0^{G(fz,ffz,ffz)} \psi(s)ds < \int_0^{G(fz,ffz,ffz)} \psi(s)ds,$$

a contradiction. Hence $fz = ffz = gfz$ and fz is a common fixed point of f and g. Similarly, we can prove, if $\lim_{n\to\infty} fgy_n = fz,$ then fz is a common fixed point of f and g.

If f and g are R-weakly commuting of type (A_f), then the proof follows the similar lines. Hence the proof of the theorem is established. □

If we take θ as θ_1 and θ_2, then we get the following corollaries respectively:

Corollary 3.3.1. *Let f and g be weakly reciprocally continuous non-compatible self-mappings of G- metric space* (X, G) *satisfying the following conditions*:

(i) $f(X) \subseteq g(X)$;

(ii) $\int_0^{G(fx,fy,fz)} \psi(s)ds \le \int_0^{G(gx,gy,gz)} \psi(s)ds, \textit{for all } x, y, z \in X$;

(iii) $\int_0^{G(fx,ffx,ffx)} \psi(s)ds < \int_0^{\upsilon} \psi(s)ds, \textit{where}$

$$\upsilon = \max\begin{Bmatrix} G(gx,gfx,gfx), G(fx,gx,gx), G(ffx,gfx,gfx), \\ G(fx,gfx,gfx), G(gx,ffx,ffx) \end{Bmatrix},$$

for all $x \in X$, *provided right-hand side is non-zero and* $\psi : \mathbf{R}^+ \to \mathbf{R}$ *is a Lebesgue integrable non-negative summable mapping such that* $\int_0^{\varepsilon} \psi(s)ds > 0$ *for each* $\varepsilon > 0$.

If f and g are R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f), *then f and g have common fixed point.*

Corollary 3.3.2.*Let f and g be weakly reciprocally continuous non-compatible self-mappings of G- metric space* (*X, G*) *satisfying the following conditions*:

(i) $f(X) \subseteq g(X)$;

(ii) $\int_0^{G(fx,fy,fz)} \psi(s)ds \le \int_0^{G(gx,gy,gz)} \psi(s)ds, \textit{for all } x, y, z \in X$;

(iii) $\int_0^{G(fx,ffx,ffx)} \psi(s)ds < \int_0^{\sigma} \psi(s)ds, \textit{where}$

$$\sigma = \sqrt[3]{G(gx, gfx, gfx).G(fx, gfx, gfx).G(gx, ffx, ffx)},$$

for all $x \in X$, *provided right-hand side is non-zero and* $\psi : \mathbf{R}^+ \to \mathbf{R}$ *is a Lebesgue integrable non-negative summable mapping such that* $\int_0^{\varepsilon} \psi(s)ds > 0$ *for each* $\varepsilon > 0$.

If f and g are R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f), *then f and g have common fixed point.*

CHAPTER- IV

COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES

In this chapter, common fixed point theorems for weakly compatible mappings on fuzzy metric spaces have been proved by making use of some implicit relations. This chapter has been divided into various sections. In Section 4.1, a new property called “common limit in the range” property for four self-mappings in fuzzy metric spaces has been introduced. Section 4.2 presents the lemmas required to prove the main results of this chapter. In Section 4.3, some new common fixed point theorems for weakly compatible mappings in fuzzy metric spaces by using common limit in the range property have been presented. Section 4.4 deals with some common fixed theorems for R-weakly commuting mappings and some of their variants in fuzzy metric spaces.

4.1 INTRODUCTION

Kramosil and Michalek [69] introduced fuzzy metric spaces by generalizing the concept of probabilistic metric spaces to fuzzy situations using continuous t-norm in the following way:

Definition 4.1.1.([120], Vol.16, p.65) A binary operation $*$: $[0,1]\times[0,1] \rightarrow [0,1]$ is **continuous *t*-norm** if $*$ satisfies the following conditions: for all $a,b,c,d \in [0,1]$,

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$;

(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$.

The results obtained in this chapter have been published in **Annals of Fuzzy Mathematics and Informatics, 3(1)(2012), 151- 158** and **Journal of Advanced research in Applied Mathematics, 6(1)(2014), 27-41.**

Definition 4.1.2. The 3-tuple $(X, M, *)$ is said to be a **fuzzy metric space** if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(i) $M(x, y, 0) = 0$;

(ii) $M(x, y, t) = 1$ if and only if $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t+s)$;

(v) $M(x, y, \bullet) : [0, \infty) \to [0, 1]$ is left continuous;

In 1994, Mishra *et al.* [81] introduced the notion of compatible mappings in fuzzy metric spaces akin to the notion of compatible mappings in metric spaces as follows:

Definition 4.1.3. A pair (f , g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be **compatible** if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be **non-compatible** if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$ but $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \ne 1$ or non-existent for at least one $t > 0$.

In 1994, Pant [92] introduced the notion of R-weakly commuting mappings in metric spaces. Later, Vasuki [139] introduced the notion of R-weakly commuting mappings in fuzzy metric spaces as:

Definition 4.1.4. A pair (f , g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be ***R*- weakly commuting** if there exists some $R > 0$ such that

$$M(fgx, gfx, t) \ge M\left(fx, gx, \frac{t}{R}\right), \text{ for all } x \in X \text{ and } t > 0.$$

In 1997, Pathak *et al.* [99] improved the notion of R-weakly commuting mappings in metric spaces by introducing the notions of R-weakly commutativity of type (A_g) and R-weakly commutativity of type (A_f). Imdad and Ali [57] in 2008, generalized the notion of R-weakly commutativity of type (A_g) and R-weakly commutativity of type (A_f) in fuzzy metric with inspiration from Pathak *et al.* [99]. Further, they have also introduced the notion of *R*-weakly commuting mappings of type (*P*) in fuzzy metric spaces.

Definition 4.1.5. A pair (f , g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be ***R*- weakly commuting of type (A_f)** if there exists some $R > 0$ such that

$$M(ggx, fgx, t) \geq M\left(gx, fx, \frac{t}{R}\right), \text{ for all } x \in X \text{ and } t > 0.$$

Definition 4.1.6. A pair (f , g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be ***R*- weakly commuting of type (A_g)** if there exists some $R > 0$ such that

$$M(ffx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right), \text{ for all } x \in X \text{ and } t > 0.$$

Definition 4.1.7. A pair (f , g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be ***R*- weakly commuting of type (*P*)** if there exists some $R > 0$ such that

$$M(ffx, ggx, t) \geq M\left(fx, gx, \frac{t}{R}\right), \text{ for all } x \in X \text{ and } t > 0.$$

Definition 4.1.8.[8] Two finite families of self-mappings $\{A_i\}_{i=1}^{m}$ and $\{B_j\}_{j=1}^{n}$ on a set X are said to be **pairwise commuting** if

(i) $A_i A_j = A_j A_i$, $i, j \in \{1, 2, 3, ...m\}$,

(ii) $B_i B_j = B_j B_i$, $i, j \in \{1, 2, 3, ...n\}$,

(iii) $A_i B_j = B_j A_i$, $i \in \{1, 2, 3, ...m\}$, $j \in \{1, 2, 3, ...n\}$.

We, now introduce a new property called as “common limit in the range property” for four self-mappings, with an inspiration from Sintunavarat and Kumam [133] .

Definition 4.1.9. The pairs (A, S) and (B, T) on a fuzzy metric space $(X, M, *)$ is said to share the ***common limit in the range of S property*** if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz$ for some $z \in X$.

Example 4.1.1. Let $(X, M, *)$ be a fuzzy metric space with $X = [-1, 1]$ and

$$M(x, y, t) = \begin{cases} e^{\frac{-|x-y|}{t}}, & t > 0 \\ 0, & t = 0 \end{cases} \text{ for all } x, y \in X.$$

Define self-mappings *A*, *B*, *S* and *T* on *X* by $Ax = \frac{x}{3}$, $Bx = -\frac{x}{3}$, $Sx = x$, $Tx = -x$ for all $x \in X$. Then, with the sequences $\{x_n\} = \left\{\frac{1}{n}\right\}$ and $\{y_n\} = \left\{-\frac{1}{n}\right\}$ in *X*, one can easily verify that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = S(0).$$

This shows that the pairs (*A*, *S*) and (*B*, *T*) share the *common limit in the range of S property.*

4.2 LEMMAS

The proof of our main results is based upon the following lemmas, out of which first two are due to Mishra *et al.* [81].

Lemma 4.2.1. *Let* $\{y_n\}$ *is a sequence in a fuzzy metric space* $(X, M, *)$. *If there exists a constant* $k \in (0,1)$ *such that* $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ *for all* $n = 0,1,2,\ldots$, $t > 0$, *then* $\{y_n\}$ *is a Cauchy sequence in* X.

Lemma 4.2.2. *Let* $(X, M, *)$ *be a fuzzy metric space. If there exists a constant* $k \in (0,1)$ *such that* $M(x, y, kt) \geq M(x, y, t)$ *for all* $x, y \in X$, $t > 0$, *then* $x = y$.

Popa [103] introduced the idea of implicit relations in metric spaces which are fruitful in unifying new contraction conditions and proved some fixed point results by using these relations. Singh and Jain [129] further extended the results of Popa [103] in fuzzy metric spaces.

We have used the following implicit relation for the proof of our main result:

Implicit relation. Let Ψ_4 be the set of all continuous functions $F : [0,1]^4 \to \mathbf{R}$ satisfying the following conditions:

(F_1) for every $u > 0, v \geq 0$ with $F(u,v,u,v) \geq 0$ or $F(u,v,v,u) \geq 0$, we have $u > v$;

(F_2) $F(u,u,1,1) < 0$ for all $u \in (0,1)$.

Example 4.2.1. Define $F:[0,1]^4 \to \mathbf{R}$ as $F(t_1,t_2,t_3,t_4)=t_1-\varphi(\min\{t_2,t_3,t_4\})$, where $\varphi:[0,1]\to[0,1]$ is increasing and continuous function such that $\varphi(t)>t$ for all $t\in(0,1)$. Clearly, $F\in\Psi_4$.

Example 4.2.2. Define $F:[0,1]^4 \to \mathbf{R}$ as $F(t_1,t_2,t_3,t_4)=\int_0^{t_1}\phi(t)dt-\varphi\left(\int_0^{\min\{t_2,t_3,t_4\}}\phi(t)dt\right)$, where $\varphi:[0,1]\to[0,1]$ is continuous and increasing function such that $\varphi(t)>t$ for all $t\in(0,1)$ and $\phi:\mathbf{R}^+\to\mathbf{R}^+$ is a Lebesgue integrable function which is summable and satisfies $0<\int_0^{\varepsilon}\phi(t)dt<1$ for all $0<\varepsilon<1$ and $\int_0^1\phi(t)dt=1$. Clearly, $F\in\Psi_4$.

Lemma 4.2.3. *Let A, B, S and T be four self-mappings of a fuzzy metric space* $(X,M,*)$ *satisfying the following conditions*:

(i) *the pair* (*A*, *S*) *satisfies the common limit in the range of S property* (*or* (*B*, *T*) *satisfies the common limit in the range of T property*);

(ii) *for any* $x,y\in X$, $F\in\Psi_4$ *and* $t>0$ *such that*

$$F\left(M(Ax,By,t),M(Sx,Ty,t),M(Sx,Ax,t),M(Ty,By,t)\right)\geq 0;$$

(iii) $A(X)\subseteq T(X)$ (*or* $B(X)\subseteq S(X)$).

Then the pairs (*A*, *S*) *and* (*B*, *T*) *share the common limit in the range of S property* (*or T property*).

Proof. Suppose that the pair (*A*, *S*) satisfies the common limit in the range of *S* property, then there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=Sz$ for some point $z\in X$. Then as $A(X)\subseteq T(X)$, hence, for each x_n, there exist y_n in *X* such that $Ax_n=Ty_n$. Therefore, $\lim_{n\to\infty}Ty_n=Sz$.

Thus in all, we have $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Ty_n=Sz.$

Now, we assert that $\lim_{n\to\infty}By_n=Sz$. Suppose not, then applying inequality (ii), we get

$$F\left(M(Ax_n,By_n,t),M(Sx_n,Ty_n,t),M(Sx_n,Ax_n,t),M(Ty_n,By_n,t)\right)\geq 0.$$

Taking $n\to\infty$, we obtain

$$F\left(M(Sz, \lim_{n\to\infty} By_n, t), M(Sz, Sz, t), M(Sz, Sz, t), M(Sz, \lim_{n\to\infty} By_n, t)\right) \geq 0$$

$$F\left(M(Sz, \lim_{n\to\infty} By_n, t), 1, 1, M(Sz, \lim_{n\to\infty} By_n, t)\right) \geq 0.$$

By (F_2), we get $M(Sz, \lim_{n\to\infty} By_n, t) > 1$, which is contradiction and therefore, $\lim_{n\to\infty} By_n = Sz$. Thus the pairs (*A*, *S*) and (*B*, *T*) share the common limit in the range of *S* property. Similarly, it can proved that if (*B*, *T*) satisfies common limit in the range of *T* property, then the pairs (*A*, *S*) and (*B*, *T*) share the common limit in the range of *T* property. □

4.3 COMMON FIXED POINT THEOREMS USING "COMMON LIMIT IN THE RANGE PROPERTY"

Now we establish the following theorems concerning weakly compatible mappings by using common limit in the range property.

Theorem 4.3.1. *Let A, B, S and T be self-mappings of a fuzzy metric space* $(X, M, *)$ *satisfying following conditions*:

(i) *the pair* (*A*, *S*) *satisfies the common limit in the range of S property* (*or* (*B*, *T*) *satisfies the common limit in the range of T property*);

(ii) *for any* $x, y \in X$, $F \in \Psi_4$ *and t* > 0,

$$F\left(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\right) \geq 0;$$

(iii) $A(X) \subseteq T(X)$ (*or* $B(X) \subseteq S(X)$).

Then the pairs (*A*, *S*) *and* (*B*, *T*) *have a point of coincidence each. Moreover*, *A*, *B*, *S and T have a unique common fixed point provided that both the pairs* (*A*, *S*) *and* (*B*, *T*) *are weakly compatible.*

Proof. In view of Lemma 4.2.3, the pairs (*A*, *S*) and (*B*, *T*) share the common limit in the range of *S* property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz, \text{ for some } z \in X.$$

We, first assert that *Az* = *Sz*. Suppose not, then by (ii), we have

$$F\left(M(Az, By_n, t), M(Sz, Ty_n, t), M(Sz, Az, t), M(Ty_n, By_n, t)\right) \geq 0.$$

Taking $n \to \infty$, we obtain

$$F\left(M(Az,Sz,t),M(Sz,Sz,t),M(Sz,Az,t),M(Sz,Sz,t)\right)\geq 0$$
$$F\left(M(Az,Sz,t),1,M(Sz,Az,t),1\right)\geq 0.$$

By (F_1), we get $M(Sz,Az,t)>1,$ which is a contradiction and therefore, *Az* = *Sz*.

Moreover, since $A(X)\subseteq T(X),$ therefore, there exists a $v\in X$ such that *Az* = *Tv*.

Now, we assert that *Bv* = *Tv*. Suppose not, then by the use of (ii), we have

$$F\left(M(Az,Bv,t),M(Sz,Tv,t),M(Sz,Az,t),M(Tv,Bv,t)\right)\geq 0$$
$$F\left(M(Tv,Bv,t),M(Tv,Tv,t),M(Tv,Tv,t),M(Tv,Bv,t)\right)\geq 0$$
$$F\left(M(Tv,Bv,t),1,1,M(Tv,Bv,t)\right)\geq 0.$$

Again, by using (F_1) we get $M(Tv,Bv,t)>1$, a contradiction and therefore,

$$Tv = Bv = Az = Sz.$$

Since the pairs (*A*, *S*) and (*B*, *T*) are weakly compatible and *Az* = *Sz* and *Tv* = *Bv*, therefore, *ASz* = *SAz* = *AAz* = *SSz*, *BTv* = *TBv* = *TTv* = *BB*v.

Finally, we assert that *AAz* = *Az*. Suppose not, then by (ii), we have

$$F\left(M(AAz,Bv,t),M(SAz,Tv,t),M(SAz,AAz,t),M(Tv,Bv,t)\right)\geq 0$$
$$F\left(M(AAz,Bv,t),M(AAz,Bv,t),M(AAz,AAz,t),M(Bv,Bv,t)\right)\geq 0$$
$$F\left(M(AAz,Az,t),M(AAz,Az,t),1,1\right)\geq 0$$

which is contradiction to (F_2) and therefore, *AAz* = *Az* = *SAz*. Hence, *Az* is common fixed point of *A* and *S*. Similarly, one can easily prove that *BBv* = *Bv* = *TBv*, i.e., *Bv* is common fixed point of *B* and *T*. Since *Az* = *Bv*, therefore, *Az* is common fixed point of *A*, *S*, *B* and *T*. The uniqueness of common fixed point is an easy consequence of inequality (ii). □

The following example illustrates the validity of Theorem 4.3.1:

Example 4.3.1. Let $(X,M,*)$ be a fuzzy metric space, where *X* = [0, 2) with a *t*- norm defined by $a * b = \min\{a, b\}$ for all $a,b\in[0,1]$ and

$$M(x,y,t)=\begin{cases}\dfrac{t}{t+|x-y|}, & t>0\\ 0 & , t=0\end{cases} \text{ for all } x,y\in X.$$

Define $F:[0,1]^4\to\mathbf{R}$ as $F\left(t_1,t_2,t_3,t_4\right)=t_1-\varphi\left(\min\left\{t_2,t_3,t_4\right\}\right),$ where $\varphi:[0,1]\to[0,1]$ is continuous and increasing function such that $\varphi(t)>t$ for all $t\in(0,1)$ and define self-mappings *A*, *B*, *S*, *T* on *X* by

$$Ax = Bx = 1, \quad Sx = \begin{cases} 1, & x \in \mathbf{Q} \\ \frac{2}{3}, & x \notin \mathbf{Q} \end{cases}, \quad Tx = \begin{cases} 1, & x \in \mathbf{Q} \\ \frac{1}{3}, & x \notin \mathbf{Q}. \end{cases}$$

Clearly, the pairs (*A*, *S*) and (*B*, *T*) satisfy all conditions of Theorem 4.3.1 and shares the common limit in the range of *S* property. Hence, mappings *A*, *B*, *S* and *T* have a unique common fixed point $x = 1$.

By choosing *A*, *B*, *S* and *T* suitably, one can derive following corollaries:

Corollary 4.3.1. *Let A and S be two self-mappings of a fuzzy metric space* $(X, M, *)$ *satisfying*:

(i) *the pair* (*A*, *S*) *satisfies the common limit in the range of S property*;

(ii) $A(X) \subseteq T(X)$;

(iii) *for any* $x, y \in X$, $F \in \Psi_4$ *and* $t > 0$ *such that*

$$F\left(M(Ax, Ay, t), M(Sx, Sy, t), M(Sx, Ax, t), M(Sy, Ay, t)\right) \geq 0.$$

Then A and S have a point of coincidence. Moreover, A and S have a unique common fixed point provided that A and S are weakly compatible.

Proof. Taking $B = A$ and $T = S$ in Theorem 4.3.1, the proof follows. □

Corollary 4.3.2. *Let A, B, S and T be four self-mappings of a fuzzy metric space* $(X, M, *)$ *satisfying following conditions*:

(i) *for any* $x, y \in X$, $F \in \Psi_4$ *and* $t > 0$ *such that*

$$F\left(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\right) \geq 0;$$

(ii) $A(X) \subseteq T(X)$ (*or* $B(X) \subseteq S(X)$);

(iii) *the pairs* (*A*, *S*) *and* (*B*, *T*) *share the common limit in the range of S property* (*or T property*).

Then the pairs (*A*, *S*) *and* (*B*, *T*) *have a point of coincidence each. Moreover*, *A*, *B*, *S and T have a unique common fixed point provided that both the pairs* (*A*, *S*) *and* (*B*, *T*) *are weakly compatible.*

Proof. Proof follows on the same lines as of Theorem 4.3.1 by using Lemma 4.2.3. □

Corollary 4.3.3. *The conclusions of Theorem* 4.3.1 *remain true even if condition* (ii) *of Theorem* 4.3.1. *is replaced by one of the following conditions*: *for any* $x, y \in X$ *and* $t > 0$,

(i) $M(Ax, By, t) \geq \varphi\left(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\}\right)$,

where $\varphi : [0,1] \to [0,1]$ *is increasing and continuous function such that* $\varphi(t) > t$ *for all* $t \in (0,1)$.

(ii) $M(Ax, By, t) \geq k\left(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\}\right)$ *where* $k > 1$.

(iii) $M(Ax, By, t) \geq k\, M(Sx, Ty, t) + \min\{M(Sx, Ax, t), M(Ty, By, t)\}$ *where* $k > 0$.

(iv) $M(Ax, By, t) \geq a\, M(Sx, Ty, t) + b\, M(Sx, Ax, t) + c\, M(Ty, By, t)$

where $a > 1$, $b, c \geq 0$ *and* $b, c \neq 1$.

(v) $M(Ax, By, t) \geq a\, M(Sx, Ty, t) + b\left[M(Sx, Ax, t) + M(Ty, By, t)\right]$

where $a > 1$ *and* $0 \leq b < 1$.

(vi) $M(Ax, By, t) \geq k\, M(Sx, Ty, t) M(Sx, Ax, t) M(Ty, By, t)$ *where* $k > 1$.

Theorem 4.3.2. *Let A, B, S and T be four self-mappings of a fuzzy metric space* $(X, M, *)$ *satisfying following conditions*:

(i) *for any* $x, y \in X, F \in \Psi_4$ *and* $t > 0$ *such that*

$$F\left(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\right) \geq 0;$$

(ii) $A(X) \subseteq T(X)$ (*or* $B(X) \subseteq S(X)$);

(iii) *the pair* (*A*, *S*) *satisfies property* (*E.A*) *and S*(*X*) *is a closed subspace of X* (*or* (*B*, *T*) *satisfies property* (*E.A*) *and T*(*X*) *is a closed subspace of X*).

Then the pairs (*A*, *S*) *and* (*B*, *T*) *have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs* (*A*, *S*) *and* (*B*, *T*) *are weakly compatible.*

Proof. Suppose that the pair (*A*, *S*) satisfy property (*E.A*). Then there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some $p \in X$. Since *S*(*X*) is a closed subspace of *X*. Therefore, *p* = *Sz*, for some $z \in X$. Hence, the pair (*A*, *S*) satisfies the common limit in the range of *S* property. Thus, by the use of Corollary 4.3.2 and Lemma 4.2.3, mappings *A*, *B*, *S* and *T* have a unique common fixed point. □

Corollary 4.3.4. *Let A, B, S and T be self-mappings of a fuzzy metric space* $(X, M, *)$ *satisfying following conditions*:

(i) *for any* $x, y \in X$, $F \in \Psi_4$ *and* $t > 0$ *such that*

$$F\left(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\right) \geq 0;$$

(ii) $A(X) \subseteq T(X)$ (*or* $B(X) \subseteq S(X)$);

(iii) *the pairs* (*A*, *S*) *and* (*B*, *T*) *satisfies common property* (*E.A*) *and* *S*(*X*) *is a closed subspace of X.*

Then the pairs (*A*, *S*) *and* (*B*, *T*) *have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs* (*A*, *S*) *and* (*B*, *T*) *are weakly compatible.*

Proof. Since the pairs (*A*, *S*) and (*B*, *T*) satisfies common property (*E.A*), therefore, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = p \text{ for some } p \in X.$$

It follows from *S*(*X*) is a closed subspace of *X* that *p* = *Sz* for some $z \in X$ and thus the pairs (*A*, *S*) and (*B*, *T*) share the common limit in the range of *S* property. Therefore, by Theorem 4.3.1, mappings *A*, *B*, *S* and *T* have a unique common fixed point. □

The result of Theorem 4.3.2 holds true even if the condition of property (*E.A*) is replaced by noncompatibility condition as shown below:

Corollary 4.3.5. *Let A, B, S and T be self-mappings of a fuzzy metric space* $(X, M, *)$ *satisfying following conditions*:

(i) *for any* $x, y \in X$, $F \in \Psi_4$ *and* $t > 0$ *such that*

$$F\left(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\right) \geq 0;$$

(ii) $A(X) \subseteq T(X)$ (*or* $B(X) \subseteq S(X)$);

(iii) *the pair* (*A*, *S*) (*or* (*B*, *T*)) *is non-compatible mappings and* *S*(*X*) *is a closed subspace of X.*

Then the pairs (*A*, *S*) *and* (*B*, *T*) *have a point of coincidence each. Moreover*, *A*, *B*, *S and T have a unique common fixed point provided that both the pairs* (*A*, *S*) *and* (*B*, *T*) *are weakly compatible.*

Proof. Since the pair (*A*, *S*) is non-compatible mappings, we get *A* and *S* satisfy property (*E.A*). Therefore, by Theorem 4.3.2, we get *A*, *B*, *S* and *T* have a unique common fixed point in *X*. □

Although, property (*E.A*) (common property (*E.A*)) is an essential tool to claim the existence of common fixed points of some mappings. But, this property requires the condition of closedness of *S*(*X*). In Theorem 4.3.1, the condition of closedness of subspace *S*(*X*) has been weakened. Therefore, it is interesting to use common limit in the range of *S* property as another auxiliary tool to claim the existence of a common fixed point.

As an application of Theorem 4.3.1, a common fixed point theorem for four finite families of mappings on fuzzy metric spaces has been established.

Theorem 4.3.3. *Let* $\{A_1, A_2, ..., A_m\}$, $\{B_1, B_2, ..., B_n\}$, $\{S_1, S_2, ..., S_p\}$ *and* $\{T_1, T_2, ..., T_q\}$ *be four finite families of self-mappings of a fuzzy metric space* $(X, M, *)$ *such that* $A = A_1.A_2.....A_m$, $B = B_1.B_2.....B_n$, $S = S_1.S_2.....S_p$ *and* $T = T_1.T_2.....T_q$ *and satisfying following conditions*:

(i) *for any* $x, y \in X$, $F \in \Psi_4$ *and* $t > 0$ *such that*

$$F\left(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\right) \geq 0\,;$$

(ii) $A(X) \subseteq T(X)$ (*or* $B(X) \subseteq S(X)$);

(iii) *the pair* (*A*, *S*) (*or* (*B*, *T*)) *satisfies the common limit in the range of S property. Then*

(*a*) *the pairs* (A, S) *and* (B, T) *have a point of coincidence each.*

(*b*) A_i, S_k, B_r *and* T_t *have a unique common fixed point provided that the pairs of families* $(\{A_i\}, \{S_k\})$ *and* $(\{B_r\}, \{T_t\})$ *commute pairwise for all i* = 1, 2, . . . ,*m*, *k* = 1, 2, . . . , *p*, *r* = 1, 2, . . . , *n*, *t* = 1, 2, . . . , *q*.

Proof. As the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_t\})$ commute pairwise, therefore,

$$\begin{aligned} AS &= \left(A_1A_2\ldots A_m\right)\left(S_1S_2\ldots S_p\right) = \left(A_1A_2\ldots A_{m-1}\right)\left(A_mS_1S_2\ldots S_p\right) \\ &= \left(A_1A_2\ldots A_{m-1}\right)\left(S_1S_2\ldots S_pA_m\right) = \left(A_1A_2\ldots A_{m-2}\right)\left(A_{m-1}S_1S_2\ldots S_pA_m\right) \\ &= \left(A_1A_2\ldots A_{m-2}\right)\left(S_1S_2\ldots S_pA_{m-1}A_m\right) = \ldots = A_1\left(S_1S_2\ldots S_pA_2\ldots A_{m-1}A_m\right) \\ &= \left(S_1S_2\ldots S_p\right)\left(A_1A_2\ldots A_m\right) = SA. \end{aligned}$$

Similarly, one can prove that $BT = TB$. Hence, the pair (A, S) and (B, T) are weakly compatible. Now, using Theorem 4.3.1, we conclude that A, S, B and T have a unique common fixed point in X, say z.

Now, we prove that z remains the fixed point of all the component mappings. For this consider

$$\begin{aligned} A\left(A_iz\right) &= \left(A_1A_2\ldots A_m\right)A_iz = \left(A_1A_2\ldots A_{m-1}\right)\left(A_mA_i\right)z \\ &= \left(A_1A_2\ldots A_{m-1}\right)\left(A_iA_m\right)z = \left(A_1A_2\ldots A_{m-2}\right)\left(A_{m-1}A_iA_m\right)z \\ &= \left(A_1A_2\ldots A_{m-2}\right)\left(A_iA_{m-1}A_m\right)z = \ldots = A_1\left(A_iA_2\ldots A_{m-1}A_m\right)z \\ &= \left(A_1A_i\right)\left(A_2A_3\ldots A_m\right)z = \left(A_iA_1\right)\left(A_2A_3\ldots A_m\right)z \\ &= A_i\left(A_1A_2\ldots A_m\right)z = A_iAz = A_iz. \end{aligned}$$

Similarly, one can prove that

$$\begin{aligned} &A\left(S_kz\right) = S_k\left(Az\right) = S_kz,\; S\left(S_kz\right) = S_k\left(Sz\right) = S_kz, \\ &S\left(A_iz\right) = A_i\left(Sz\right) = A_iz,\; B\left(B_rz\right) = B_r\left(Bz\right) = B_rz, \\ &B\left(T_tz\right) = T_t\left(Bz\right) = T_tz,\; T\left(T_tz\right) = T_t\left(Tz\right) = T_tz \end{aligned}$$

and $T\left(B_rz\right) = B_r\left(Tz\right) = B_rz,$

which show that (for all i, r, k and t) A_iz and S_kz are other fixed point of the pair (A, S) whereas B_rz and T_tz are other fixed points of the pair (B, T). Since A, B, S and T have a unique common fixed point, so, we get

$$z = A_iz = S_kz = B_rz = T_tz,$$

for all $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, p$, $r = 1, 2, \ldots, n$, $t = 1, 2, \ldots, q$.

This shows that z is a unique common fixed point of $\{A_i\}_{i=1}^{m}$, $\{S_k\}_{k=1}^{p}$, $\{B_r\}_{r=1}^{n}$ and $\{T_t\}_{t=1}^{q}$.

Hence, the proof of the theorem is established. □

Next result involves a lower semi continuous function $\psi:[0,1]\to[0,1]$ such that $\psi(t)>t$ for all $t\in(0,1)$ along with $\psi(0)=0$ and $\psi(1)=1$.

Theorem 4.3.4. *Let A, B, S and T be four self-mappings on a fuzzy metric space* $(X,M,*)$ *satisfying following conditions*:

(i) *the pair* (*A*, *S*) *satisfies common limit in the range of S property* (*or pair* (*B*, *T*) *satisfies common limit in the range of T property*);

(ii) $A(X)\subseteq T(X)$ (*or* $B(X)\subseteq S(X)$);

(iii) *for all* $x,y\in X$, $t>0$,

$$\int_0^{M(Ax,By,t)}\phi(s)ds\geq\psi\left(\int_0^{\sigma}\phi(s)ds\right),$$

where $\sigma=\min\{M(Sx,Ty,t),M(Sx,Ax,t),M(Ty,By,t)\}$ *and* $\phi:\mathbf{R}^+\to\mathbf{R}^+$ *is a Lebesgue summable integrable function satisfies* $0<\int_0^{\varepsilon}\phi(s)ds<1$ *for all* $0<\varepsilon<1$, *and* $\int_0^{1}\phi(s)ds=1$.

Then the pairs (*A*, *S*) *and* (*B*, *T*) *have a point of coincidence each.*

Moreover, the *mappings A, B, S and T have a unique common fixed point provided that both the pairs* (*A*, *S*) *and* (*B*, *T*) *are weakly compatible.*

Proof. Suppose that the pair (*A*, *S*) satisfies the common limit in the range of *S* property, then there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=Sz$ for some point $z\in X$. Since $A(X)\subseteq T(X)$, hence, for each x_n, there exist y_n in *X* such that $Ax_n=Ty_n$. Therefore, $\lim_{n\to\infty}Ty_n=Sz$. Thus in all, we have $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Ty_n=Sz$. Now, we assert that $\lim_{n\to\infty}By_n=Sz$. Suppose not, then applying inequality (iii), we get

$$\int_0^{M(Ax_n,By_n,t)}\phi(s)ds\geq\psi\left(\int_0^{\sigma}\phi(s)ds\right),$$

where,

$$\sigma=\min\{M(Sx_n,Ty_n,t),M(Sx_n,Ax_n,t),M(Ty_n,By_n,t)\}.$$

Taking $n\to\infty$, we obtain

$$\int_0^{M(Sz,\lim_{n\to\infty}By_n,t)}\phi(s)ds\geq\psi\left(\int_0^{\sigma}\phi(s)ds\right),$$

where,

$$\begin{aligned}\sigma &= \min\left\{M(Sz,Sz,t), M(Sz,Sz,t), M(Sz,\lim_{n\to\infty} By_n,t)\right\}\\ &= \min\left\{1,1,M(Sz,\lim_{n\to\infty} By_n,t)\right\}\\ &= M(Sz,\lim_{n\to\infty} By_n,t).\end{aligned}$$

This gives,

$$\int_0^{M(Sz,\lim_{n\to\infty} By_n,t)} \phi(s)ds \geq \psi\left(\int_0^{M(Sz,\lim_{n\to\infty} By_n,t)} \phi(s)ds\right) > \int_0^{M(Sz,\lim_{n\to\infty} By_n,t)} \phi(s)ds,$$

which is a contradiction, therefore, $\lim_{n\to\infty} By_n = Sz.$ Thus, the pairs (*A*, *S*) and (*B*, *T*) share the common limit in the range of *S* property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz \text{ for some } z \in X.$$

Now, in order to prove *A*, *B*, *S* and *T* have fixed point, we first assert that $Az = Sz.$ Suppose not, then by (iii), we have

$$\int_0^{M(Az,By_n,t)} \phi(s)ds \geq \psi\left(\int_0^{\sigma} \phi(s)ds\right),$$

where,

$$\sigma = \min\{M(Sz,Ty_n,t), M(Sz,Az,t), M(Ty_n,By_n,t)\}.$$

Taking $n \to \infty$, we get

$$\int_0^{M(Az,Sz,t)} \phi(s)ds \geq \psi\left(\int_0^{\sigma} \phi(s)ds\right),$$

where,

$$\begin{aligned}\sigma &= \min\{M(Sz,Sz,t), M(Sz,Az,t), M(Sz,Sz,t)\}\\ &= \min\{1, M(Sz,Az,t), 1\}\\ &= M(Sz,Az,t).\end{aligned}$$

This gives,

$$\int_0^{M(Az,Sz,t)} \phi(s)ds \geq \psi\left(\int_0^{M(Sz,Az,t)} \phi(s)ds\right) > \int_0^{M(Sz,Az,t)} \phi(s)ds,$$

which is a contradiction and therefore, $Az = Sz.$

Since, $A(X) \subseteq T(X)$, there exists a $v \in X$ such that $Az = Tv.$

Now, we assert that $Tv = Bv.$ Suppose not, then by (iii), we get

$$\int_0^{M(Az,Bv,t)} \phi(s)ds \geq \psi\left(\int_0^{\sigma} \phi(s)ds\right)$$

or

$$\int_0^{M(Tv,Bv,t)} \phi(s)ds \geq \psi\left(\int_0^{\sigma} \phi(s)ds\right),$$

where,

$$\begin{aligned} \sigma &= \min\{M(Sz,Tv,t), M(Sz,Az,t), M(Tv,Bv,t)\} \\ &= \min\{M(Tv,Tv,t), M(Tv,Tv,t), M(Tv,Bv,t)\} \\ &= \min\{1,1,M(Tv,Bv,t)\} \\ &= M(Tv,Bv,t). \end{aligned}$$

This gives,

$$\int_0^{M(Tv,Bv,t)} \phi(s)ds \geq \psi\left(\int_0^{M(Tv,Bv,t)} \phi(s)ds\right) > \int_0^{M(Tv,Bv,t)} \phi(s)ds,$$

a contradiction and therefore, $Tv = Bv = Az = Sz.$

Since the pairs (*A*, *S*) and (*B*, *T*) are weakly compatible and $Az = Sz$ and $Tv = Bv,$ therefore,

$$ASz = SAz = AAz = SSz, \quad BTv = TBv = TTv = BBv.$$

Finally, we assert that $AAz = Az.$ Suppose not, then again by (iii), we have

$$\int_0^{M(AAz,Bv,t)} \phi(s)ds \geq \psi\left(\int_0^{\sigma} \phi(s)ds\right),$$

where,

$$\begin{aligned} \sigma &= \min\{M(SAz,Tv,t), M(SAz,AAz,t), M(Tv,Bv,t)\} \\ &= \min\{M(AAz,Bv,t), M(AAz,AAz,t), M(Bv,Bv,t)\} \\ &= \min\{M(AAz,Az,t),1,1\} \\ &= M(AAz,Az,t). \end{aligned}$$

This gives,

$$\int_0^{M(AAz,Bv,t)} \phi(s)ds \geq \psi\left(\int_0^{M(AAz,Az,t)} \phi(s)ds\right) > \int_0^{M(AAz,Az,t)} \phi(s)ds,$$

which is a contradiction and therefore, $AAz = Az = SAz,$ which gives Az is common fixed point of *A* and *S*. Similarly, one can easily prove that $BBv = Bv = TBv,$ i.e., Bv is common fixed point of *B* and *T*. Since $Az = Bv$, therefore Az is common fixed point of *A*, *S*, *B* and *T*. The uniqueness of common fixed point is an easy consequence of inequality (iii). □

Remark 4.3.1. Results similar to Theorems 4.3.2, 4.3.3 and Corollaries 4.3.1, 4.3.2, 4.3.3 can also be outlined in respect to Theorem 4.3.4.

4.4 COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS

The following theorems concerning weakly compatible mappings have been established in this section:

Theorem 4.4.1. *Let* $(X, M, *)$ *be a complete fuzzy metric space. Let f and g be weakly compatible self-mappings of X satisfying*:

(i) $M(gx, gy, kt) \geq M(fx, fy, t)$ *for all* $x, y \in X$, $t > 0$ *and* $0 < k < 1$;

(ii) $g(X) \subseteq f(X)$.

If one of g(*X*) *or f*(*X*) *is complete, then f and g have a unique common fixed point.*

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$, choose $x_1 \in X$ such that $gx_0 = fx_1$. In general, choose $x_{n+1} \in X$ such that $y_n = fx_{n+1} = gx_n$.

Then by (i), we have

$$M(fx_n, fx_{n+1}, t) = M(gx_{n-1}, gx_n, t)$$
$$\geq M\left(fx_{n-1}, fx_n, \frac{t}{k}\right) = M\left(gx_{n-2}, gx_{n-1}, \frac{t}{k}\right) \geq \ldots \geq M\left(fx_0, fx_1, \frac{t}{k^n}\right).$$

Therefore, for any *p*,

$$M(fx_n, fx_{n+p}, t) \geq M\left(fx_n, fx_{n+1}, \frac{t}{p}\right) \geq .^{(p-times)}.. \geq M\left(fx_{n+p-1}, fx_{n+p}, \frac{t}{p}\right)$$
$$\geq M\left(fx_0, fx_1, \frac{t}{pk^n}\right) \geq .^{(p-times)}.. \geq M\left(fx_0, fx_1, \frac{t}{pk^{n+p-1}}\right).$$

As $n \to \infty$, $\{fx_n\} = \{y_n\}$ is Cauchy sequence and so, by completeness of *X*, $\{fx_n\}$ is convergent sequence. Therefore, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$.

Since, *f*(*X*) is complete, so there exist a point $p \in X$ such that *fp* = *z*.

Now, from (i), we have

$$M(gp, gx_n, kt) \geq M(fp, fx_n, t).$$

Taking the limit as $n \to \infty,$ we get

$$M(gp, z, kt) \geq M(fp, z, t) = M(z, z, t) = 1.$$

This gives, *gp* = *z* = *fp*.

As *f* and *g* are weakly compatible, therefore, *fgp* = *gfp* i.e., *fz* = *gz*.

We now, show that *z* is fixed point of *f* and *g*.

From (i),

$$M(gz, gx_n, kt) \geq M(fz, fx_n, t).$$

Taking the limit as $n \to \infty,$ we get

$$M(gz, z, kt) \geq M(fz, z, t)$$
$$M(gz, z, kt) \geq M(gz, z, t).$$

Therefore, by Lemma 4.2.2, we have $gz = z = fz.$

Hence, *z* is a common fixed point of *f* and *g*.

For uniqueness, let *w* be another fixed point of *f* and *g*, then by (i),

$$M(gz, gw, kt) \geq M(fz, fw, t)$$

or

$$M(z, w, kt) \geq M(z, w, t).$$

By using Lemma 4.2.2, we get $z = w.$

Therefore, *z* is unique common fixed point of *f* and *g*. □

Now, we furnish the following example to establish the validity of the results proved in Theorem 4.4.1.

Example 4.4.1. Let *X* = [0, 1]. Define

$$M(x, y, t) = \begin{cases} \dfrac{t}{t + |x - y|}, & t > 0 \\ 0, & t = 0 \end{cases} \quad \text{for all } x, y \in X.$$

Clearly, $(X, M, *)$ is a complete fuzzy metric space. Define self-mappings *f* and *g* on *X* by $fx = \frac{x}{2},\ gx = \frac{x}{6}.$ Then $g(X) \subseteq f(X)$ and for $\frac{1}{3} \leq k < 1,$ condition (i) satisfied. However, mappings are weakly compatible at *x* = 0. Hence, all conditions of

Theorem 4.4.1 are satisfied and $x = 0$ is a unique common fixed point of *f* and *g*.

The results to the above theorem hold true even if we replace complete fuzzy metric space with fuzzy metric space.

Theorem 4.4.2. *Let* $(X, M, *)$ *be a fuzzy metric space. Let f and g be weakly compatible self- mappings of X satisfying following condition*:

(i) $M(gx, gy, kt) \geq M(fx, fy, t)$ *for all* $x, y \in X$, $t > 0$ *and* $0 < k < 1$;

(ii) $g(X) \subseteq f(X)$.

If one of g(X) or f(X) is complete, then f and g have a unique common fixed point.

Proof. From the proof of Theorem 4.4.1, we conclude that $\{fx_n\} = \{y_n\}$ is Cauchy sequence in *X*. Now, suppose that *f*(*X*) is a complete subspace of *X*, then the subsequence of $\{y_n\}$ must get a limit in *f*(*X*). Let it be *u* and $u \in f^{-1}v$ then $fv = u$ for some $v \in X$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore, the sequence $\{y_n\}$ also converges. Now, from (i),

$$M(gv, gx_n, kt) \geq M(fv, fx_n, t).$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} M(gv, u, kt) &\geq M(fv, u, t) \\ &= M(u, u, t) = 1. \end{aligned}$$

This gives, $gv = u = fv$ which shows that pair (*f*, *g*) has a point of coincidence.

Since, *f* and *g* are weakly compatible, therefore, $fgv = gfv$, i.e., $fu = gu$.

Now, we show that *u* is a fixed point of *f* and *g*. From (i), we have

$$M(gu, gx_n, kt) \geq M(fu, fx_n, t).$$

Taking the limit as $n \to \infty$, we get

$$M(gu, u, kt) \geq M(fu, u, t)$$
$$M(gu, u, kt) \geq M(gu, u, t),$$

by Lemma 4.2.2, we get $gu = u = fu$.

Hence, *u* is a fixed point of *f* and *g*.

For uniqueness, let *w* be another fixed point of *f* and *g*, then by (i),

$$M(gz, gw, kt) \geq M(fz, fw, t)$$
$$M(z, w, kt) \geq M(z, w, t),$$

by Lemma 4.2.2, we get $z = w$.

Therefore, *z* is unique common fixed point of *f* and *g*. □

Theorem 4.4.3. *Theorems* 4.4.1 *and* 4.4.2 *remain true if weakly compatible self-mappings are replaced by any one of the following self-mappings*:

(i) *R-weakly commuting*;

(ii) *R-weakly commuting of type* (A_f);

(iii) *R-weakly commuting of type* (A_g);

(iv) *R-weakly commuting of type* (P);

(v) *weakly commuting property.*

Proof. (i) Using the hypothesis of Theorems 4.4.1 and 4.4.2, the proof for the existence of coincidence point follows the same lines. So, let *x* be any arbitrary coincidence point for the pair (*f*, *g*), then by using *R*-weak commutativity, one gets

$$M(fgx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right) = M\left(fx, fx, \frac{t}{R}\right) = 1 \text{ where } R > 0.$$

This gives, *fgx* = *gfx*. Thus, the pair (*f*, *g*) is weakly compatible.

(ii) In case (*f*, *g*) is a *R*-weakly commuting pair of type (A_f), then

$$M(ggx, fgx, t) \geq M\left(gx, fx, \frac{t}{R}\right) = M\left(fx, fx, \frac{t}{R}\right) = 1$$

which gives, $ggx = fgx$.

Also,

$$M(fgx, gfx, t) \geq M\left(fgx, ggx, \frac{t}{2}\right) * M\left(ggx, gfx, \frac{t}{2}\right) = 1 * 1 = 1$$

which gives, $fgx = gfx$. Thus, the pair (*f*, *g*) is weakly compatible.

(iii) In case (*f*, *g*) is an *R*-weakly commuting pair of type (A_g), then

$$M(ffx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right) = M\left(fx, fx, \frac{t}{R}\right) = 1$$

which gives $gfx = ffx$.

Also,

$$M(fgx, gfx, t) \geq M\left(fgx,\ ffx, \frac{t}{2}\right) * M\left(ffx, gfx, \frac{t}{2}\right) = 1 * 1 = 1$$

which gives, $fgx = gfx.$ Thus, the pair (*f*, *g*) is weakly compatible.

Similarly, if pair (*f*, *g*) is *R*-weakly commuting of type (P) or weakly commuting property, then (*f*, *g*) also commutes at their point of coincidence.

Hence, in view of Theorem 4.4.2, *f* and *g* have a unique common fixed point in all the five cases. This proves the assertion. □

CHAPTER- V

COMMON FIXED POINT THEOREMS FOR RECIPROCAL CONTINUOUS AND WEAKLY COMPATIBLE MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACES

In this chapter, some common fixed point theorems have been obtained for reciprocal continuous mappings and weakly compatible mappings by using common (*E.A*) like property and an implicit relation in intuitionistic fuzzy metric spaces. This chapter has been divided into various sections. Section 5.1 deals with the preliminaries related to intuitionistic fuzzy metric spaces. In Section 5.2, the lemmas required to prove the main results have been presented. In Section 5.3, some common fixed point theorems have been established by using reciprocal continuity of mappings satisfying contractive conditions in an intuitionistic fuzzy metric space. Section 5.4 deals with common fixed point theorems for weakly compatible mappings by using an implicit relation in intuitionistic fuzzy metric spaces along with the common (*E.A*) like property.

5.1 INTRODUCTION

Inspired by the idea of intuitionistic fuzzy sets, Alaca *et al.* [11] defined the notion of intuitionistic fuzzy metric spaces with the help of continuous *t*-norms and continuous *t*-conorms as a generalization of fuzzy metric space due to Kramosil and Michálek [69].

Definition 5.1.1.([120], Vol.16, p.65) A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is **continuous *t*-norm** if it satisfies the following conditions: for all $a,b,c,d \in \left[0,1\right]$,

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a*1=a$;

(iv) $a*b \le c*d$, whenever $a \le c$ and $b \le d$.

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Definition 5.1.2.[119] A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is **continuous *t*-conorm** if $\Diamond$ satisfies the following conditions: for all $a, b, c, d \in [0,1]$,

(i) $\Diamond$ is commutative and associative;

(ii) $\Diamond$ is continuous;

(iii) $a \Diamond 0 = a$;

(iv) $a \Diamond b \le c \Diamond d$, whenever $a \le c$ and $b \le d$.

Definition 5.1.3. A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an **intuitionistic fuzzy metric space** if X is an arbitrary set, $*$ is a continuous *t*-norm, $\Diamond$ is a continuous *t*-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the conditions: for all $x, y, z \in X$ and $s, t > 0$,

(i) $M(x, y, t) + N(x, y, t) \le 1$;

(ii) $M(x, y, 0) = 0$;

(iii) $M(x, y, t) = 1$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$;

(v) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;

(vi) $M(x, y, \bullet) : [0, \infty) \to [0,1]$ is left continuous;

(vii) $\lim_{t \to \infty} M(x, y, t) = 1$;

(viii) $N(x, y, 0) = 1$;

(ix) $N(x, y, t) = 0$ if and only if $x = y$;

(x) $N(x, y, t) = N(y, x, t)$;

(xi) $N(x, y, t) \Diamond N(y, z, s) \ge N(x, z, t + s)$;

(xii) $N(x, y, \bullet) : [0, \infty) \to [0,1]$ is right continuous;

(xiii) $\lim_{t \to \infty} N(x, y, t) = 0$.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t respectively.

Turkoglu *et al.* [137] extended the notion of compatible mappings to intuitionistic fuzzy metric spaces as follows:

Definition 5.1.4. A pair (*A*, *S*) of self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be **compatible** if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0 \text{ for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

In 2006, Turkoglu *et al.* [136] defined the notion of *R*-weakly commuting mappings in intuitionistic fuzzy metric spaces as follows:

Definition 5.1.5. A pair (*A*, *S*) of self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be **weakly commuting** if

$$M(ASx, SAx, t) \geq M(Ax, Sx, t) \text{ and } N(ASx, SAx, t) \leq N(Ax, Sx, t),$$

for all $x \in X$ and $t > 0$.

Definition 5.1.6. A pair (*A*, *S*) of self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be **pointwise** R**-weakly commuting**, if given $x \in X$, there exists $R > 0$ such that for all $t > 0$

$$M(ASx, SAx, t) \geq M\left(Ax, Sx, \frac{t}{R}\right) \text{ and } N(ASx, SAx, t) \leq N\left(Ax, Sx, \frac{t}{R}\right).$$

Inspired by Wadhwa *et al*. [138], we adopt the following notions in intuitionistic fuzzy metric spaces:

Definition 5.1.7. A pair (*A*, *S*) of self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to satisfy the ***E.A* like property** if there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in A(X)$ or $z \in S(X)$.

Definition 5.1.8. Two pairs (*A*, *S*) and (*B, T*) of self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ are said to satisfy the **common (*E.A*) like property** if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in S(X) \cap T(X)$ or $z \in A(X) \cap B(X)$.

5.2 LEMMAS

The proofs of our main results are based upon the following lemmas, of which first two lemmas are due to Alaca *et al.* [10].

Lemma 5.2.1. *Let* $(X, M, N, *, \Diamond)$ *be an intuitionistic fuzzy metric space and* $\{y_n\}$ *be a sequence in X. If there exists a real number* $k \in (0,1)$ *such that*

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t) \text{ and } N(y_n, y_{n+1}, kt) \leq N(y_{n-1}, y_n, t),$$

for all $t > 0$, $n = 0,1,2,\ldots$, *then* $\{y_n\}$ *is a Cauchy sequence in* X.

Lemma 5.2.2. *Let* $(X, M, N, *, \Diamond)$ *be an intuitionistic fuzzy metric space. If there exists a real number* $k \in (0,1)$ *such that*

$$M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t) \text{ for all } x, y \in X,\ t > 0,$$

then x = y.

Lemma 5.2.3. *Let* $(X, M, N, *, \Diamond)$ *be an intuitionistic fuzzy metric space with continuous t-norm* $*$ *and continuous t-conorm* $\Diamond$ *defined by* $t * t \geq t$ *and* $(1-t)\Diamond(1-t) \leq (1-t)$ *for all* $t \in [0,1]$ *respectively. Further, pairs* (A, S) *and* (B, T) *satisfying the following conditions*:

(i) $A(X) \subseteq T(X),\ B(X) \subseteq S(X)$;

(ii) *there exists a constant* $k \in (0,1)$ *such that*

$$M(Ax, By, kt) \geq \begin{pmatrix} M(Ty, By, t) * M(Sx, Ax, t) * M(Sx, By, \alpha t) * \\ M(Ty, Ax, (2-\alpha)t) * M(Ty, Sx, t) \end{pmatrix}$$

and

$$N(Ax, By, kt) \leq \begin{pmatrix} N(Ty, By, t) \Diamond N(Sx, Ax, t) \Diamond N(Sx, By, \alpha t) \Diamond \\ N(Ty, Ax, (2-\alpha)t) \Diamond N(Ty, Sx, t) \end{pmatrix},$$

for all $x, y \in X$, $t > 0$ *and* $\alpha \in (0,2)$.

Then the continuity of one of the mappings in compatible pair (A, S) *or* (B, T) *on* $(X, M, N, *, \Diamond)$ *implies their reciprocal continuity.*

Proof. First, assume that (A, S) is a compatible pair and S is continuous. We have to show that pair (A, S) is reciprocally continuous. Let $\{u_n\}$ be a sequence in X such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Su_n = z$ for some $z \in X$.

Since S is continuous, we have $\lim_{n\to\infty} SAu_n = \lim_{n\to\infty} SSu_n = Sz.$ Compatibility of pair (A, S) implies

$$\lim_{n\to\infty} M(ASu_n, SAu_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ASu_n, SAu_n, t) = 0.$$

This gives,

$$\lim_{n\to\infty} M(ASu_n, Sz, t) = 1 \text{ and } \lim_{n\to\infty} N(ASu_n, Sz, t) = 0.$$

That is, $\lim_{n\to\infty} ASu_n = Sz.$ By (i), for each n, there exists $v_n \in X$ such that $ASu_n = Tv_n.$ Thus, we have $\lim_{n\to\infty} ASu_n = \lim_{n\to\infty} SSu_n = \lim_{n\to\infty} SAu_n = \lim_{n\to\infty} Tv_n = Sz.$

Now, we claim that $\lim_{n\to\infty} Bv_n = Sz.$

Take $\alpha = 1$ in (ii), we have

$$M(ASu_n, Bv_n, kt) \geq \begin{pmatrix} M(Tv_n, Bv_n, t) * M(SSu_n, ASu_n, t) * M(SSu_n, Bv_n, t) * \\ M(Tv_n, ASu_n, t) * M(Tv_n, SSu_n, t) \end{pmatrix}$$

and

$$N(ASu_n, Bv_n, kt) \leq \begin{pmatrix} N(Tv_n, Bv_n, t) \Diamond N(SSu_n, ASu_n, t) \Diamond N(SSu_n, Bv_n, t) \Diamond \\ N(Tv_n, ASu_n, t) \Diamond N(Tv_n, SSu_n, t) \end{pmatrix}.$$

Taking $n \to \infty,$ we get

$$M\left(Sz, \lim_{n\to\infty} Bv_n, kt\right) \geq \begin{pmatrix} M\left(Sz, \lim_{n\to\infty} Bv_n, t\right) * M(Sz, Sz, t) * M\left(Sz, \lim_{n\to\infty} Bv_n, t\right) * \\ M(Sz, Sz, t) * M(Sz, Sz, t) \end{pmatrix}$$

and

$$N\left(Sz, \lim_{n\to\infty} Bv_n, kt\right) \leq \begin{pmatrix} N\left(Sz, \lim_{n\to\infty} Bv_n, t\right) \Diamond N(Sz, Sz, t) \Diamond N\left(Sz, \lim_{n\to\infty} Bv_n, t\right) \Diamond \\ N(Sz, Sz, t) \Diamond N(Sz, Sz, t) \end{pmatrix}.$$

This gives,

$$M\left(Sz, \lim_{n\to\infty} Bv_n, kt\right) \geq M\left(Sz, \lim_{n\to\infty} Bv_n, t\right)$$

and

$$N\left(Sz, \lim_{n\to\infty} Bv_n, kt\right) \leq N\left(Sz, \lim_{n\to\infty} Bv_n, t\right).$$

Therefore, from Lemma 5.2.2**,** we have $\lim_{n\to\infty} Bv_n = Sz.$

Next, we show that $Az = Sz.$ Take $\alpha = 1$ in (ii), we have

$$M(Az, Bv_n, kt) \geq \begin{pmatrix} M(Tv_n, Bv_n, t) * M(Sz, Az, t) * M(Sz, Bv_n, t) \\ *M(Tv_n, Az, t) * M(Tv_n, Sz, t) \end{pmatrix}$$

and

$$N(Az, Bv_n, kt) \leq \begin{pmatrix} N(Tv_n, Bv_n, t) \diamond N(Sz, Az, t) \diamond N(Sz, Bv_n, t) \\ \diamond N(Tv_n, Az, t) \diamond N(Tv_n, Sz, t) \end{pmatrix}.$$

Proceeding limit as $n \to \infty,$ we get

$$M(Az, Sz, kt) \geq \begin{pmatrix} M(Sz, Sz, t) * M(Sz, Az, t) * M(Sz, Sz, t) \\ *M(Sz, Az, t) * M(Sz, Sz, t) \end{pmatrix}$$

and

$$N(Az, Sz, kt) \leq \begin{pmatrix} N(Sz, Sz, t) \diamond N(Sz, Az, t) \diamond N(Sz, Sz, t) \\ \diamond N(Sz, Az, t) \diamond N(Sz, Sz, t) \end{pmatrix}.$$

That is,

$$M(Az, Sz, kt) \geq M(Sz, Az, t) \text{ and } N(Az, Sz, kt) \leq N(Sz, Az, t),$$

therefore, by use of Lemma 5.2.2, we have $Az = Sz.$

Hence, $\lim_{n\to\infty} SAu_n = \lim_{n\to\infty} ASu_n = Sz = Az$.

This proves that A and S are reciprocally continuous on X . Similarly, it can be proved that pair (B,T) is reciprocally continuous if the pair (B,T) is assumed to be compatible and T is continuous. □

5.3 COMMON FIXED POINT THEOREMS FOR RECIPROCAL CONTINUOUS MAPPINGS

We establish the following theorem in this section:

Theorem 5.3.1. *Let* $(X, M, N, *, \diamond)$ *be a complete intuitionistic fuzzy metric space with continuous t-norm* $*$ *and continuous t-conorm* $\diamond$ *defined by* $t * t \geq t$ *and* $(1-t) \diamond (1-t) \leq (1-t)$ *for all* $t \in [0,1]$ *respectively. Further, let* (A,S) *and* (B,T) *be pointwise* R *- weakly commuting pairs of self- mappings of* X *satisfying*:

(i) $A(X) \subseteq T(X),\ B(X) \subseteq S(X)$;

(ii) *there exists a constant* $k \in (0,1)$ *such that*

$$M(Ax,By,kt) \geq \begin{pmatrix} M(Ty,By,t) * M(Sx,Ax,t) * M(Sx,By,\alpha t) * \\ M(Ty,Ax,(2-\alpha)t) * M(Ty,Sx,t) \end{pmatrix}$$

and

$$N(Ax,By,kt) \leq \begin{pmatrix} N(Ty,By,t) \Diamond N(Sx,Ax,t) \Diamond N(Sx,By,\alpha t) \Diamond \\ N(Ty,Ax,(2-\alpha)t) \Diamond N(Ty,Sx,t) \end{pmatrix},$$

for all $x, y \in X$, $t > 0$ *and* $\alpha \in (0,2)$;

(iii) *one of the mappings in compatible pair* (A,S) *or* (B,T) *is continuous.*

Then A, B, S *and* T *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in *X*. By (i), we define the sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for all $n = 1, 2, 3, \ldots$. We show that $\{y_n\}$ is a Cauchy sequence in X. Take $\alpha = (1-\beta)$, where $\beta \in (0,1)$ in (ii), we have

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) &= M(Bx_{2n+1}, Ax_{2n+2}, kt) \\ &= M(Ax_{2n+2}, Bx_{2n+1}, kt) \\ &\geq \begin{pmatrix} M(Tx_{2n+1}, Bx_{2n+1}, t) * M(Sx_{2n+2}, Ax_{2n+2}, t) * M(Sx_{2n+2}, Bx_{2n+1}, (1-\beta)t) \\ * M(Tx_{2n+1}, Ax_{2n+2}, (1+\beta)t) * M(Tx_{2n+1}, Sx_{2n+2}, t) \end{pmatrix} \\ &= \begin{pmatrix} M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+1}, (1-\beta)t) \\ * M(y_{2n}, y_{2n+2}, (1+\beta)t) * M(y_{2n}, y_{2n+1}, t) \end{pmatrix} \\ &\geq \begin{pmatrix} M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * 1 * M(y_{2n}, y_{2n+1}, t) \\ * M(y_{2n+1}, y_{2n+2}, \beta t) * M(y_{2n}, y_{2n+1}, t) \end{pmatrix} \\ &\geq \left(M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, \beta t) \right). \end{aligned}$$

Taking $\beta \to 1$, we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, t)$$
$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n}, y_{2n+1}, t)$$
$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t).$$

Similarly, we can show that

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

Also,

$$\begin{aligned}
N(y_{2n+1},y_{2n+2},kt) &= N(Bx_{2n+1},Ax_{2n+2},kt)\\
&= N(Ax_{2n+2},Bx_{2n+1},kt)\\
&\le \begin{pmatrix} N(Tx_{2n+1},Bx_{2n+1},t)\Diamond N(Sx_{2n+2},Ax_{2n+2},t)\Diamond N(Sx_{2n+2},Bx_{2n+1},(1-\beta)t)\\ \Diamond N(Tx_{2n+1},Ax_{2n+2},(1+\beta)t)\Diamond N(Tx_{2n+1},Sx_{2n+2},t) \end{pmatrix}\\
&= \begin{pmatrix} N(y_{2n},y_{2n+1},t)\Diamond N(y_{2n+1},y_{2n+2},t)\Diamond N(y_{2n+1},y_{2n+1},(1-\beta)t)\\ \Diamond N(y_{2n},y_{2n+2},(1+\beta)t)\Diamond N(y_{2n},y_{2n+1},t) \end{pmatrix}\\
&\le \begin{pmatrix} N(y_{2n},y_{2n+1},t)\Diamond N(y_{2n+1},y_{2n+2},t)\Diamond 0\Diamond N(y_{2n},y_{2n+1},t)\\ \Diamond N(y_{2n+1},y_{2n+2},\beta t)\Diamond N(y_{2n},y_{2n+1},t) \end{pmatrix}\\
&\le \left(N(y_{2n},y_{2n+1},t)\Diamond N(y_{2n+1},y_{2n+2},t)\Diamond N(y_{2n+1},y_{2n+2},\beta t)\right).
\end{aligned}$$

Taking $\beta \to 1$, we get

$$\begin{aligned}
&N(y_{2n+1},y_{2n+2},kt) \le N(y_{2n},y_{2n+1},t)\Diamond N(y_{2n+1},y_{2n+2},t)\Diamond N(y_{2n+1},y_{2n+2},t)\\
&N(y_{2n+1},y_{2n+2},kt) \le N(y_{2n},y_{2n+1},t)\Diamond N(y_{2n+1},y_{2n+2},t) \le N(y_{2n},y_{2n+1},t)\\
&N(y_{2n+1},y_{2n+2},kt) \le N(y_{2n},y_{2n+1},t).
\end{aligned}$$

Similarly, it can be shown that

$$N(y_{2n+2},y_{2n+3},kt) \le N(y_{2n+1},y_{2n+2},t).$$

Therefore, for $n = 1,2,3,\ldots$ and $t > 0$, we have

$$M(y_n,y_{n+1},kt) \ge M(y_{n-1},y_n,t) \text{ and } N(y_n,y_{n+1},kt) \le N(y_{n-1},y_n,t).$$

Hence, by Lemma 5.2.1, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, so $\{y_n\}$ converges to $z \in X$. Its subsequences $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Bx_{2n+1}\}$ and $\{Sx_{2n+2}\}$ also converge to z.

Now, suppose that (A,S) is a compatible pair and S is continuous. Then by Lemma 5.2.3, the pair (A , S) is reciprocally continuous. This implies that $\lim_{n\to\infty} SAx_n = Sz$ and $\lim_{n\to\infty} ASx_n = Az.$

Since, (A,S) is a compatible pair, therefore,

$$\lim_{n\to\infty} M(ASx_n,SAx_n,t) = 1 \text{ and } \lim_{n\to\infty} N(ASx_n,SAx_n,t) = 0$$

which implies that

$$M(Az,Sz,t) = 1 \text{ and } N(Az,Sz,t) = 0.$$

Hence, $Sz = Az.$ Since $A(X) \subseteq T(X),$ therefore, there exists a point $p \in X$ such that $Sz = Az = Tp.$

Now, taking $\alpha = 1$ in (ii), we have

$$M(Az, Bp, kt) \geq \left(M(Tp, Bp, t) * M(Sz, Az, t) * M(Sz, Bp, t) * M(Tp, Az, t) * M(Tp, Sz, t) \right)$$
$$M(Az, Bp, kt) \geq \left(M(Az, Bp, t) * M(Az, Az, t) * M(Az, Bp, t) * M(Az, Az, t) * M(Az, Az, t) \right)$$

and

$$N(Az, Bp, kt) \leq \left(N(Tp, Bp, t) \lozenge N(Sz, Az, t) \lozenge N(Sz, Bp, t) \lozenge N(Tp, Az, t) \lozenge N(Tp, Sz, t) \right)$$
$$N(Az, Bp, kt) \leq \left(N(Az, Bp, t) \lozenge N(Az, Az, t) \lozenge N(Az, Bp, t) \lozenge N(Az, Az, t) \lozenge N(Az, Az, t) \right).$$

This gives,

$$M(Az, Bp, kt) \geq M(Az, Bp, t) \text{ and } N(Az, Bp, kt) \leq N(Az, Bp, t).$$

Thus, by Lemma 5.2.2, we have $Az = Bp.$

Since, A and S are pointwise R- weakly commuting mappings, therefore, there exists $R > 0,$ such that

$$M(ASz, SAz, t) \geq M\left(Az, Sz, \frac{t}{R} \right) = 1 \text{ and } N(ASz, SAz, t) \leq N\left(Az, Sz, \frac{t}{R} \right) = 0.$$

Hence, $ASz = SAz$ and $ASz = SAz = AAz = SSz.$

Similarly, B and T are pointwise R-weakly commuting mappings, we have $BBp = BTp = TBp = TTp.$

Again, taking $\alpha = 1$ in (ii), we have

$$M(AAz, Bp, kt) \geq \begin{pmatrix} M(Tp, Bp, t) * M(SAz, AAz, t) * M(SAz, Bp, t) \\ *M(Tp, AAz, t) * M(Tp, SAz, t) \end{pmatrix}$$
$$M(AAz, Az, kt) \geq \begin{pmatrix} M(Tp, Tp, t) * M(AAz, AAz, t) * M(AAz, Az, t) \\ *M(Az, AAz, t) * M(Az, AAz, t) \end{pmatrix}$$

and

$$N(AAz, Bp, kt) \leq \begin{pmatrix} N(Tp, Bp, t) \lozenge N(SAz, AAz, t) \lozenge N(SAz, Bp, t) \\ \lozenge N(Tp, AAz, t) \lozenge N(Tp, SAz, t) \end{pmatrix}$$
$$N(AAz, Az, kt) \leq \begin{pmatrix} N(Tp, Tp, t) \lozenge N(AAz, AAz, t) \lozenge N(AAz, Az, t) \\ \lozenge N(Az, AAz, t) \lozenge N(Az, AAz, t) \end{pmatrix}.$$

This gives,

$$M(AAz, Az, kt) \geq M(AAz, Az, t) \text{ and } N(AAz, Az, kt) \leq N(AAz, Az, t).$$

By Lemma 5.2.2, we have $SAz = AAz = Az.$ Hence Az is common fixed point of A and S. Similarly by (ii), $Bp = Az$ is a common fixed point of B and T. Hence, Az is a common fixed point of A, B, S and T.

Uniqueness: Suppose that Ap is another common fixed point of A, B, S and T.

Taking $\alpha = 1$ in (ii), we have

$$M(AAz, BAp, kt) \geq \begin{pmatrix} M(TAp, BAp, t) * M(SAz, AAz, t) * M(SAz, BAp, t) \\ *M(TAp, AAz, t) * M(TAp, SAz, t) \end{pmatrix}$$

$$M(Az, Ap, kt) \geq \begin{pmatrix} M(Ap, Ap, t) * M(Az, Az, t) * M(Az, Ap, t) \\ *M(Ap, Az, t) * M(Ap, Az, t) \end{pmatrix}$$

and

$$N(AAz, BAp, kt) \leq \begin{pmatrix} N(TAp, BAp, t) \Diamond N(SAz, AAz, t) \Diamond N(SAz, BAp, t) \\ \Diamond N(TAp, AAz, t) \Diamond N(TAp, SAz, t) \end{pmatrix}$$

$$N(Az, Ap, kt) \leq \begin{pmatrix} N(Ap, Ap, t) \Diamond N(Az, Az, t) \Diamond N(Az, Ap, t) \\ \Diamond N(Ap, Az, t) \Diamond N(Ap, Az, t) \end{pmatrix}.$$

This gives,

$M(Az, Ap, kt) \geq M(Az, Ap, t)$ and $N(Az, Ap, kt) \leq N(Az, Ap, t).$

By Lemma 5.2.2, we have $Ap = Az.$

Thus, uniqueness follows. □

Taking $S = T = I_X$ in above theorem, we get following result:

Corollary 5.3.1. *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space with continuous t-norm $*$ and continuous t-conorm $\Diamond$ defined by $t * t \geq t$ and $(1-t) \Diamond (1-t) \leq (1-t)$ for all $t \in [0,1]$. Further, let A and B are reciprocally continuous mappings on X satisfying*

$$M(Ax, By, kt) \geq \begin{pmatrix} M(y, By, t) * M(x, Ax, t) * M(x, By, \alpha t) \\ *M(y, Ax, (2-\alpha)t) * M(y, x, t) \end{pmatrix}$$

and

$$N(Ax, By, kt) \leq \begin{pmatrix} N(y, By, t) \Diamond N(x, Ax, t) \Diamond N(x, By, \alpha t) \\ \Diamond N(y, Ax, (2-\alpha)t) \Diamond N(y, x, t) \end{pmatrix}$$

for all $x, y \in X$, $t > 0$ *and* $\alpha \in (0,2)$.

Then A *and* B *have a unique common fixed point.*

The following example illustrates Theorem 5.3.1:

Example 5.3.1. Let $X = [2, 20]$ and pair (*M*, *N*) be the intuitionistic fuzzy set on $X^2 \times [0, \infty)$ defined by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0 \\ 0, & t=0 \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0 \\ 1, & t=0. \end{cases}$$

Then $(X, M, N, *, \Diamond)$ be complete intuitionistic fuzzy metric space where $*$ and $\Diamond$ are continuous *t*-norm and continuous *t*-conorm defined by $a * b = \min\{a,b\}$ and $a \Diamond b = \max\{a,b\}$ for all $a, b \in [0,1]$, respectively.

Now we define self-mappings A, B, S and T on X as

$$Ax = \begin{cases} 2, & x = 2 \\ 3, & x > 2, \end{cases} \qquad Bx = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 6, & 2 < x \le 5, \end{cases}$$

$$Sx = \begin{cases} 2, & x = 2 \\ 6, & x > 2 \end{cases} \quad \text{and } Tx = \begin{cases} 2, & x = 2 \\ 12, & 2 < x \le 5 \\ x-3, & x > 5. \end{cases}$$

Then A, B, S and T satisfy all the conditions of Theorem 5.3.1 with $k \in (0,1), \alpha \in (0,2)$ and $x = 2$ is a unique common fixed point.

5.4 COMMON FIXED POINT THEOREMS USING COMMON (*E.A*) LIKE PROPERTY

In this section, we prove the following fixed point theorems by using *E.A* like property in intuitionistic fuzzy metric spaces.

Theorem 5.4.1. *Let A and S be self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$ *satisfying the conditions*:

(i) *for all* $x, y \in X, t > 0,$ *there exists* $q \in (0,1)$ *such that,*

$$M(Ax, Ay, qt) \ge \min\{M(Ax, Sy, t), M(Sx, Ay, t), M(Sx, Sy, t)\}$$

and

$$N(Ax, Ay, qt) \leq \max\{N(Ax, Sy, t), N(Sx, Ay, t), N(Sx, Sy, t)\};$$

(ii) *A and S satisfy the E.A like property;*

(iii) *A and S are weakly compatible mappings.*

Then A and S have a unique common fixed point in X.

Proof. Since, *A* and *S* satisfy the *E.A* like property, therefore, there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in A(X)$ or $z \in S(X)$.

Suppose that $z \in S(X)$. Therefore, *z* = *Su* for some $u \in X$. First, we show that *Au* = *Su*. Using (i), we have

$$M(Au, Ax_n, qt) \geq \min\{M(Au, Sx_n, t), M(Su, Ax_n, t), M(Su, Sx_n, t)\}$$

and

$$N(Au, Ax_n, qt) \leq \max\{N(Au, Sx_n, t), N(Su, Ax_n, t), N(Su, Sx_n, t)\}.$$

Taking the limit as $n \to \infty$, we get

$$M(Au, z, qt) \geq \min\{M(Au, z, t), M(z, z, t), M(z, z, t)\}$$
$$M(Au, z, qt) \geq M(Au, z, t)$$

and

$$N(Au, z, qt) \leq \max\{N(Au, z, t), N(z, z, t), N(z, z, t)\}$$
$$N(Au, z, qt) \leq N(Au, z, t).$$

Therefore, by Lemma 5.2.2, we have *Au* = *z* = *Su*. Hence, *u* is a point of coincidence of *A* and *S*. Since, *A* and *S* are weakly compatible mappings, therefore, *Az* = *ASu* = *AAu* = *SSu* = *SAu* = *Sz*.

Next, we show that *Az* = *z*.

By (i), we have

$$M(Az, Ax_n, qt) \geq \min\{M(Az, Sx_n, t), M(Sz, Ax_n, t), M(Sz, Sx_n, t)\}$$

and

$$N(Az, Ax_n, qt) \leq \max\{N(Az, Sx_n, t), N(Sz, Ax_n, t), N(Sz, Sx_n, t)\}.$$

Taking the limit as $n \to \infty$, we have

$$M(Az, z, qt) \geq \min\{M(Az, z, t), M(Az, z, t), M(Az, z, t)\}$$
$$M(Az, z, qt) \geq M(Az, z, t)$$

and

$$N(Az, z, qt) \le \max\{N(Az, z, t), N(Az, z, t), N(Az, z, t)\}$$
$$N(Az, z, qt) \le N(Az, z, t).$$

Therefore, by Lemma 5.2.2, we have $Az = z = Sz$. Hence, z is a common fixed point of A and S. For uniqueness, let w be another fixed point of A and S, then by (i), we have

$$M(Az, Aw, qt) \ge \min\{M(Az, Sw, t), M(Sz, Aw, t), M(Sz, Sw, t)\}$$
$$M(z, w, qt) \ge \min\{M(z, w, t), M(z, w, t), M(z, w, t)\}$$
$$M(z, w, qt) \ge M(z, w, t)$$

and

$$N(Az, Aw, qt) \le \max\{N(Az, Sw, t), N(Sz, Aw, t), N(Sz, Sw, t)\}$$
$$N(z, w, qt) \le \max\{N(z, w, t), N(z, w, t), N(z, w, t)\}$$
$$N(z, w, qt) \le N(z, w, t).$$

Hence, by Lemma 5.2.2, we have $z = w$.

This proves the main result. □

We, now use the concept of implicit relation for establishing various common fixed point results in intuitionistic fuzzy metric spaces.

Let M_4 be the set of all real continuous functions ϕ and $\psi : [0,1]^4 \to \mathbf{R}$, which are non-decreasing in the first argument and are satisfying the following conditions:

(A) $\phi(u,1,u,1) \ge 0 \Rightarrow u \ge 1;$

(B) $\phi(u,1,1,u) \ge 0 \Rightarrow u \ge 1;$

(C) $\phi(u,u,1,1) \ge 0 \Rightarrow u \ge 1;$

(D) $\psi(u,0,u,0) \le 0 \Rightarrow u \le 0;$

(E) $\psi(u,0,0,u) \le 0 \Rightarrow u \le 0;$

(F) $\psi(u,u,0,0) \le 0 \Rightarrow u \le 0$ for all $u \ge 0$.

Example 5.4.1. Define $\phi, \psi : [0,1]^4 \to \mathbf{R}$ as $\phi(t_1,t_2,t_3,t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4$ and $\psi(t_1,t_2,t_3,t_4) = 12t_1 - 9t_2 + 8t_3 - 11t_4$. Clearly, ϕ and ψ satisfies all conditions (A), (B), (C), (D), (E) and (F). Therefore, $\phi,\ \psi \in M_4$.

Theorem 5.4.2. *Let A, B, S and T be self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$ *satisfying the following conditions that* :

(i) *the pair* (*A*, *S*) *and* (*B*, *T*) *share the common* (*E.A*) *like property*;

(ii) *for any* $x, y \in X$, $\phi, \psi \in M_4$ *and for all* $t > 0$, *there exists* $k \in (0,1)$ *such that,*

$$\phi(M(Ax, By, kt), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)) \geq 0$$

and

$$\psi(N(Ax, By, kt), N(Sx, Ty, t), N(Sx, Ax, t), N(Ty, By, t)) \leq 0.$$

Then the pairs (*A*, *S*) *and* (*B*, *T*) *have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided the pairs* (*A*, *S*) *and* (*B*, *T*) *are weakly compatible.*

Proof. In view of (i), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in S(X) \cap T(X)$ or $z \in A(X) \cap B(X)$.

Suppose that $z \in S(X) \cap T(X)$, then $z \in S(X)$, therefore, there exists a point $u \in X$ such that $z = Su$.

We claim that $Au = z$. Taking $x = u$, $y = y_n$ in (ii), we have

$$\phi(M(Au, By_n, kt), M(Su, Ty_n, t), M(Su, Au, t), M(Ty_n, By_n, t)) \geq 0$$

and

$$\psi(N(Au, By_n, kt), N(Su, Ty_n, t), N(Su, Au, t), N(Ty_n, By_n, t)) \leq 0 .$$

Letting the limit as $n \to \infty$, we get

$$\phi(M(Au, z, kt), M(z, z, t), M(z, Au, t), M(z, z, t)) \geq 0$$
$$\phi(M(Au, z, kt), 1, M(Au, z, t), 1) \geq 0$$

and

$$\psi(N(Au, z, kt), N(z, z, t), N(z, Au, t), N(z, z, t)) \leq 0$$
$$\psi(N(Au, z, kt), 0, N(Au, z, t), 0) \leq 0.$$

As ϕ and ψ is non-decreasing in the first argument, we have

$$\phi(M(Au, z, t), 1, M(Au, z, t), 1) \geq 0$$

and

$$\psi(N(Au, z, t), 0, N(Au, z, t), 0) \leq 0.$$

Using implicit relations (*A*) and (*D*), we get

$$M(Au, z, t) \geq 1 \text{ and } N(Au, z, t) \leq 0 .$$

Hence $M(Au,z,t)=1$ and $N(Au,z,t)=0$. Therefore, $Au = z = Su$, which shows that u is a coincidence point of the pair (A, S).

Since $z \in T(X)$ implies that there exists $v \in X$ such that $Tv = z = Au = Su$. Now, we show that $Bv = z$. Putting $x = u$, $y = v$ in (ii), we have

$$\phi(M(Au,Bv,kt),M(Su,Tv,t),M(Su,Au,t),M(Tv,Bv,t)) \geq 0$$
$$\phi(M(z,Bv,kt),1,1,M(z,Bv,t)) \geq 0$$

and

$$\psi(N(Au,Bv,kt),N(Su,Tv,t),N(Su,Au,t),N(Tv,Bv,t)) \leq 0$$
$$\psi(N(z,Bv,kt),0,0,N(z,Bv,t)) \leq 0.$$

Since ϕ and ψ is non-decreasing in the first argument, we have

$$\phi(M(z,Bv,t),1,1,M(z,Bv,t)) \geq 0$$

and

$$\psi(N(z,Bv,t),0,0,N(z,Bv,t)) \leq 0.$$

Using implicit relations (B) and (E), we get $M(z,Bv,t) \geq 1$ and $N(z,Bv,t) \leq 0$. Hence $M(z,Bv,t)=1$ and $N(z,Bv,t)=0$. Therefore, $Bv = z = Tv$, which shows that v is a coincidence point of the pair (B, T).

Since the pairs (A, S) and (B, T) are weakly compatible and $Au = Su$, $Bv = Tv$, therefore, $Az = ASu = SAu = Sz$, $Bz = BTv = TBv = Tz$.

Now, we show that $Az = z$. From (ii), we have

$$\phi(M(Az,Bv,kt),M(Sz,Tv,t),M(Sz,Az,t),M(Tv,Bv,t)) \geq 0$$
$$\phi(M(Az,z,kt),M(Az,z,t),M(Az,Az,t),M(Bv,Bv,t)) \geq 0$$
$$\phi(M(Az,z,kt),M(Az,z,t),1,1) \geq 0$$

and

$$\psi(N(Az,Bv,kt),N(Sz,Tv,t),N(Sz,Az,t),N(Tv,Bv,t)) \leq 0$$
$$\psi(N(Az,z,kt),N(Az,z,t),N(Az,Az,t),N(Bv,Bv,t)) \leq 0$$
$$\psi(N(Az,z,kt),N(Az,z,t),0,0) \leq 0.$$

Since ϕ and ψ is non-decreasing in the first argument, we have

$$\phi(M(Az,z,t),M(Az,z,t),1,1) \geq 0$$

and

$$\psi(N(Az,z,t),N(Az,z,t),0,0) \leq 0.$$

Using implicit relations (*C*) and (*F*), we get $M(Az,z,t) \geq 1$ and $N(Az,z,t) \leq 0.$ Hence $M(Az,z,t) = 1$ and $N(Az,z,t) = 0.$ Therefore, *Az* = *z* = *Sz*.

Similarly, one can prove that *Bz* = *Tz* = *z*. Hence, *Az* = *Bz* = *Sz* = *Tz*, and *z* is common fixed point of *A*, *B*, *S* and *T*.

Uniqueness: Let *z* and *w* be two common fixed points of *A*, *B*, *S* and *T*, then by using (ii), we have

$$\phi(M(Az,Bw,kt), M(Sz,Tw,t), M(Sz,Az,t), M(Tw,Bw,t)) \geq 0$$
$$\phi(M(z,w,kt), M(z,w,t), M(z,z,t), M(w,w,t)) \geq 0$$
$$\phi(M(z,w,kt), M(z,w,t),1,1) \geq 0$$
$$\phi(M(z,w,t), M(z,w,t),1,1) \geq 0$$

and

$$\psi(N(Az,Bw,kt), N(Sz,Tw,t), N(Sz,Az,t), N(Tw,Bw,t)) \leq 0$$
$$\psi(N(z,w,kt), N(z,w,t), N(z,z,t), N(w,w,t)) \leq 0$$
$$\psi(N(z,w,kt), N(z,w,t), 0, 0) \leq 0$$
$$\psi(N(z,w,t), N(z,w,t), 0, 0) \leq 0.$$

Using implicit relations (*C*) and (*F*), we have $M(z,w,t) \geq 1$ and $N(z,w,t) \leq 0.$

Hence, $M(z,w,t) = 1$ and $N(z,w,t) = 0.$

Therefore, *z* = *w*. $\square$

The case *B* = *A* and *T* = *S* in the Theorem 5.4.2 yields following corollary:

Corollary 5.4.1. *Let A and S be self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$ *satisfying the following*:

(i) *the pair* (*A*, *S*) *satisfies the E.A like property*;

(ii) *for any* $x, y \in X$, $\phi, \psi \in M_4$ *and for all* $t > 0,$ *there exists* $k \in (0,1)$ *such that*

$$\phi(M(Ax,Ay,kt), M(Sx,Sy,t), M(Sx,Ax,t), M(Sy,Ay,t)) \geq 0$$

and

$$\psi(N(Ax,Ay,kt), N(Sx,Sy,t), N(Sx,Ax,t), N(Sy,Ay,t)) \leq 0.$$

Then A and S have a point of coincidence each. Moreover, if the pair (*A*, *S*) *is weakly compatible, then A and S have a unique common fixed point.*

Remark 5.4.1. Theorem 5.4.2 remains valid if condition (ii) is replaced by the following:

$$M(Ax,By,kt)\ge\min\left\{\begin{array}{l}M(Sx,Ty,t),M(By,Sx,t),M(Ax,Ty,t),\\ M(Sx,Ax,t),\dfrac{aM(Ax,By,t)+bM(Ax,Ty,t)}{aM(By,Ty,t)+b},\\ \dfrac{cM(Sx,By,t)+dM(Sx,Ty,t)}{cM(By,Ty,t)+d},\dfrac{eM(Ax,Ty,t)+fM(Sx,Ax,t)}{eM(Sx,Ty,t)+f}\end{array}\right\}$$

and

$$N(Ax,By,kt)\le\max\left\{\begin{array}{l}N(Sx,Ty,t),N(By,Sx,t),N(Ax,Ty,t),\\ N(Sx,Ax,t),\dfrac{aN(Ax,By,t)+bN(Ax,Ty,t)}{aN(By,Ty,t)+b},\\ \dfrac{cN(Sx,By,t)+dN(Sx,Ty,t)}{cN(By,Ty,t)+d},\dfrac{eN(Ax,Ty,t)+fN(Sx,Ax,t)}{eN(Sx,Ty,t)+f}\end{array}\right\}$$

for all $x,y\in X$, $t>0$, where $k\in(0,1)$ and $a, b, c, d, e, f > 0$ with the pairs (a, b), (c, d) and (e, f) cannot be simultaneously zero.

The following example illustrates Theorem 5.4.2.

Example 5.4.2. Let $X=[0,2)$ and pair (M, N) be the intuitionistic fuzzy set on $X^2\times[0,\infty)$ defined by

$$M(x,y,t)=\begin{cases}\dfrac{t}{t+|x-y|}, & t>0\\ 0, & t=0\end{cases}\quad\text{and}\quad N(x,y,t)=\begin{cases}\dfrac{|x-y|}{t+|x-y|}, & t>0\\ 1, & t=0.\end{cases}$$

Then $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space where $*$ and $\Diamond$ are continuous t-norm and continuous t-conorm defined by $a*b=\min\{a,b\}$ and $a\Diamond b=\max\{a,b\}$ for all $a,b\in[0,1]$, respectively.

Define $\phi,\psi:[0,1]^4\to\mathbf{R}$ as $\phi(t_1,t_2,t_3,t_4)=14t_1-12t_2+6t_3-8t_4$

and $\psi(t_1,t_2,t_3,t_4)=12t_1-9t_2+8t_3-11t_4$. Clearly, ϕ and ψ satisfies all conditions (A), (B), (C), (D), (E) and (F). Therefore, $\phi,\ \psi\in M_4$. Now we define self-mappings A,B,S and $T:X\to X$ by

$$Ax = Bx = 1, \quad Sx = \begin{cases} 1, & x \in \mathbf{Q} \\ \frac{2}{3}, & x \notin \mathbf{Q}, \end{cases} \qquad Tx = \begin{cases} 1, & x \in \mathbf{Q} \\ \frac{1}{3}, & x \notin \mathbf{Q}. \end{cases}$$

Consider the sequences $\{x_n\} = \left\{\frac{1}{n}\right\}$ and $\{y_n\} = \left\{\frac{-1}{n}\right\}$ in X, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in S(X) \cap T(X)$$

which shows that pairs (A, S) and (B, T) share the common ($E.A$) like property. By a routine calculation, one can easily verify the condition (ii). Thus, all the conditions of Theorem 5.4.2 are satisfied and therefore, $x = 1$ is the unique common fixed point of A, B, S and T.

CHAPTER- VI

COMMON FIXED POINT THEOREMS FOR VARIANTS OF COMPATIBLE MAPPINGS IN INTUIONISTIC FUZZY METRIC SPACES

In this chapter, common fixed point theorems for variants of compatible mappings along with common property (*E.A*) using implicit relation in intuitionistic fuzzy metric spaces have been established. This chapter has been divided into various sections. Section 6.1 deals with some definitions (not covered in Chapter V) related to intuitionistic fuzzy metric spaces. In Section 6.2, common fixed theorems have been proved for weakly compatible mappings with the help of an implicit relation in intuitionistic fuzzy metric spaces by employing the common property (*E.A*). Section 6.3 focuses on proving some common fixed point theorems for two pairs of compatible and subsequentially continuous mappings satisfying an implicit relation in intuitionistic fuzzy metric spaces.

6.1 INTRODUCTION

In this section, the notion of subcompatibility and subsequential continuity in an intuitionistic fuzzy metric space have been introduced in the similar way as introduced by H. Bouhadjera and Godet-Thobie [23] for metric space.

Definition 6.1.1. A pair (*A*, *S*) of self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be **subcompatible** if there exists a sequence $\{x_n\}$ in *X* with $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and for all $t > 0$,

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0.$$

Definition 6.1.2. A pair (*A*, *S*) of self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be **subsequentially continuous** if there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$, $\lim_{n\to\infty} SAx_n = Sz$.

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Definition 6.1.3.[8] Two families of self-mappings $\{A_i\}_{i=1}^{m}$ and $\{B_j\}_{j=1}^{n}$ are said to be **pairwise commuting** if

(i) $A_i A_j = A_j A_i$, $i, j \in \{1,2,3,...m\}$,

(ii) $B_i B_j = B_j B_i$, $i, j \in \{1,2,3,...n\}$,

(iii) $A_i B_j = B_j A_i$, $i \in \{1,2,3,...n\}$, $j \in \{1,2,3,...n\}$.

6.2 COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS USING COMMON PROPERTY (*E.A*)

In this section, the concept of implicit relation has been used for establishing various common fixed point results in intuitionistic fuzzy metric spaces. This concept plays a vital rule in the proof of the main results.

Let M_4 be the set of all real-valued continuous functions ϕ and $\psi : [0,1]^4 \to \mathbf{R}$, non-decreasing in the first argument and satisfying the following conditions:

(A) $\phi(u,1,u,1) \geq 0 \Rightarrow u \geq 1$;

(B) $\phi(u,1,1,u) \geq 0 \Rightarrow u \geq 1$;

(C) $\phi(u,u,1,1) \geq 0 \Rightarrow u \geq 1$;

(D) $\psi(u,0,u,0) \leq 0 \Rightarrow u \leq 0$;

(E) $\psi(u,0,0,u) \leq 0 \Rightarrow u \leq 0$;

(F) $\psi(u,u,0,0) \leq 0 \Rightarrow u \leq 0$ for all $u \geq 0$.

Example 6.2.1. Define $\phi, \psi : [0,1]^4 \to \mathbf{R}$ as $\phi(t_1,t_2,t_3,t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4$ and $\psi(t_1,t_2,t_3,t_4) = 12t_1 - 9t_2 + 8t_3 - 11t_4$. Clearly, ϕ and ψ satisfies all conditions (A), (B), (C), (D), (E) and (F). Therefore, $\phi, \psi \in M_4$.

We, now establish the following results:

Theorem 6.2.1. *Let A, B, S and T be self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$ *satisfying the following conditions that*:

(i) *the pair* (A, S) (*or* (B, T)) *satisfies the property* ($E.A$);

(ii) *for any* $x, y \in X$, $\phi, \psi \in M_4$ *and for all* $t > 0$, *there exists* $k \in (0,1)$ *such that*

$$\phi(M(Ax,By,kt),M(Sx,Ty,t),M(Sx,Ax,t),M(Ty,By,t))\geq 0$$
$$\psi(N(Ax,By,kt),N(Sx,Ty,t),N(Sx,Ax,t),N(Ty,By,t))\leq 0;$$

(ii) $A(X)\subseteq T(X)$ (*or* $B(X)\subseteq S(X)$).

Then the pairs (*A*, *S*) *and* (*B*, *T*) *share the common property* (*E.A*).

Proof. Suppose that the pair (*A*, *S*) satisfies property (*E.A*), then there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n=\lim\limits_{n\to\infty} Sx_n=z$ for some $z\in X$. Since $A(X)\subseteq T(X)$, therefore, for each x_n, there exist y_n in X such that $Ax_n=Ty_n$. This gives, $\lim\limits_{n\to\infty} Ax_n=\lim\limits_{n\to\infty} Sx_n=\lim\limits_{n\to\infty} Ty_n=z.$ Now, we claim that $\lim\limits_{n\to\infty} By_n=z.$

Applying inequality (ii), we obtain

$$\phi(M(Ax_n,By_n,kt),M(Sx_n,Ty_n,t),M(Sx_n,Ax_n,t),M(Ty_n,By_n,t))\geq 0$$

and

$$\psi(N(Ax_n,By_n,kt),N(Sx_n,Ty_n,t),N(Sx_n,Ax_n,t),N(Ty_n,By_n,t))\leq 0\,.$$

Taking limit as $n\to\infty$,

$$\phi(M(z,\lim_{n\to\infty} By_n,kt),M(z,z,t),M(z,z,t),M(z,\lim_{n\to\infty} By_n,t))\geq 0$$
$$\phi(M(z,\lim_{n\to\infty} By_n,kt),1,1,M(z,\lim_{n\to\infty} By_n,t))\geq 0$$

and

$$\psi(N(z,\lim_{n\to\infty} By_n,kt),N(z,z,t),N(z,z,t),N(z,\lim_{n\to\infty} By_n,t))\leq 0$$
$$\psi(N(z,\lim_{n\to\infty} By_n,kt),0,0,N(z,\lim_{n\to\infty} By_n,t))\leq 0.$$

Since ϕ and ψ is non-decreasing in the first argument, we have

$$\phi(M(z,\lim_{n\to\infty} By_n,t),1,1,M(z,\lim_{n\to\infty} By_n,t))\geq 0$$

and

$$\psi(N(z,\lim_{n\to\infty} By_n,t),0,0,N(z,\lim_{n\to\infty} By_n,t))\leq 0.$$

Using (*B*) and (*E*), we get

$$M(z,\lim_{n\to\infty} By_n,t)\geq 1 \text{ and } N(z,\lim_{n\to\infty} By_n,t)\leq 0\,.$$

Hence $M(z,\lim\limits_{n\to\infty} By_n,t)=1$ and $N(z,\lim\limits_{n\to\infty} By_n,t)=0.$

Therefore, $\lim\limits_{n\to\infty} By_n=z.$ Hence, the pairs (*A, S*) and (*B, T*) share the common property (*E.A*).

Similarly, if pair (B, T) satisfies property ($E.A$) and $B(X) \subseteq S(X)$, then pairs (A, S) and (B, T) share the common property ($E.A$). □

Theorem 6.2.2. *Let A, B, S and T be four self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$ *satisfying the following conditions*:

(i) *for any* $x, y \in X$, $\phi, \psi \in M_4$ *and for all* $t > 0$, *there exists* $k \in (0,1)$ *such that*

$$\phi(M(Ax, By, kt), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)) \geq 0$$
$$\psi(N(Ax, By, kt), N(Sx, Ty, t), N(Sx, Ax, t), N(Ty, By, t)) \leq 0;$$

(ii) *the pair* (A, S) *and* (B, T) *share the common property* ($E.A$);

(iii) $S(X)$ *and* $T(X)$ *are closed subsets of X.*

Then each of the pairs (A, S) *and* (B, T) *have a point of coincidence. Moreover, A, B, S and T have a unique common fixed point provided both the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Proof. In view of (ii), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z \text{ for some } z \in X.$$

Since $S(X)$ is a closed subset of X, therefore, there exists a point $u \in X$ such that $z = Su$. We, now claim that $Au = z$. By (i), we have

$$\phi(M(Au, By_n, kt), M(Su, Ty_n, t), M(Su, Au, t), M(Ty_n, By_n, t)) \geq 0$$

and

$$\psi(N(Au, By_n, kt), N(Su, Ty_n, t), N(Su, Au, t), N(Ty_n, By_n, t)) \leq 0.$$

Taking limit as $n \to \infty$,

$$\phi(M(Au, z, kt), M(z, z, t), M(z, Au, t), M(z, z, t)) \geq 0$$
$$\phi(M(Au, z, kt), 1, M(z, Au, t), 1) \geq 0$$

and

$$\psi(N(Au, z, kt), N(z, z, t), N(z, Au, t), N(z, z, t)) \leq 0$$
$$\psi(N(Au, z, kt), 0, N(z, Au, t), 0) \leq 0.$$

As ϕ and ψ are non-decreasing in the first argument, we have

$$\phi(M(Au, z, t), 1, M(z, Au, t), 1) \geq 0$$

and

$$\psi(N(Au, z, t), 0, N(z, Au, t), 0) \leq 0.$$

Using implicit relations (A) and (D), we have

$$M(Au,z,t) \geq 1 \text{ and } N(Au,z,t) \leq 0.$$

Hence $M(Au,z,t) = 1$ and $N(Au,z,t) = 0$.

Therefore, $Au = z = Su$ which shows that u is a coincidence point of the pair (A, S).

Since $T(X)$ is also a closed subset of X, therefore, $\lim_{n\to\infty} Ty_n = z$ in $T(X)$ and hence there exists $v \in X$ such that $Tv = z = Au = Su$. Now, we show that $Bv = z$.

By using inequality (i), we have

$$\phi(M(Au,Bv,kt), M(Su,Tv,t), M(Su,Au,t), M(Tv,Bv,t)) \geq 0$$
$$\phi(M(z,Bv,kt),1,1,M(z,Bv,t)) \geq 0$$

and

$$\psi(N(Au,Bv,kt), N(Su,Tv,t), N(Su,Au,t), N(Tv,Bv,t)) \leq 0$$
$$\psi(N(z,Bv,kt),0,0,N(z,Bv,t)) \leq 0.$$

As ϕ and ψ are non-decreasing in the first argument, we have

$$\phi(M(z,Bv,t),1,1,M(z,Bv,t)) \geq 0$$

and

$$\psi(N(z,Bv,t),0,0,N(z,Bv,t)) \leq 0.$$

Using implicit relations (*B*) and (*E*), we get

$$M(z,Bv,t) \geq 1 \text{ and } N(z,Bv,t) \leq 0.$$

Hence $M(z,Bv,t) = 1$ and $N(z,Bv,t) = 0$.

Therefore, $Bv = z = Tv$, which shows that v is a coincidence point of the pair (B, T).

Moreover, since the pairs (A, S) and (B, T) are weakly compatible and $Au = Su$, $Bv = Tv$, therefore, $Az = ASu = SAu = Sz$, $Bz = BTv = TBv = Tz$.

Next, we claim that $Az = z$ for showing the existence of a fixed point of A. By using inequality (i), we have

$$\phi(M(Az,Bv,kt), M(Sz,Tv,t), M(Sz,Az,t), M(Tv,Bv,t)) \geq 0$$
$$\phi(M(Az,z,kt), M(Az,z,t), M(Az,Az,t), M(Bv,Bv,t)) \geq 0$$
$$\phi(M(Az,z,kt), M(Az,z,t),1,1) \geq 0$$

and

$$\psi(N(Az,Bv,kt), N(Sz,Tv,t), N(Sz,Az,t), N(Tv,Bv,t)) \leq 0$$
$$\psi(N(Az,z,kt), N(Az,z,t), N(Az,Az,t), N(Bv,Bv,t)) \leq 0$$
$$\psi(N(Az,z,kt), N(Az,z,t),0,0) \leq 0.$$

Since ϕ and ψ are non-decreasing in the first argument, we have

$$\phi(M(Az,z,t),M(Az,z,t),1,1)\geq 0$$

and

$$\psi(N(Az,z,t),N(Az,z,t),0,0)\leq 0.$$

On using implicit relations (C) and (F), we get

$$M(Az,z,t)\geq 1 \text{ and } N(Az,z,t)\leq 0.$$

Hence, $M(Az,z,t)=1$ and $N(Az,z,t)=0.$ Therefore, $Az = z = Sz$.

Similarly, we can prove that $Bz = Tz = z$. Hence, $Az = Bz = Sz = Tz = z$, which implies that z is a common fixed point of A, B, S and T.

Uniqueness: Let w be another common fixed points of A, B, S and T. Then by using (i), we have

$$\begin{aligned}&\phi(M(Az,Bw,kt),M(Sz,Tw,t),M(Sz,Az,t),M(Tw,Bw,t))\geq 0\\&\phi(M(z,w,kt),M(z,w,t),M(z,z,t),M(w,w,t))\geq 0\\&\phi(M(z,w,kt),M(z,w,t),1,1)\geq 0\\&\phi(M(z,w,t),M(z,w,t),1,1)\geq 0\end{aligned}$$

and

$$\begin{aligned}&\psi(N(Az,Bw,kt),N(Sz,Tw,t),N(Sz,Az,t),N(Tw,Bw,t))\leq 0\\&\psi(N(z,w,kt),N(z,w,t),N(z,z,t),N(w,w,t))\leq 0\\&\psi(N(z,w,kt),N(z,w,t),0,0)\leq 0\\&\psi(N(z,w,t),N(z,w,t),0,0)\leq 0.\end{aligned}$$

Using implicit relations (C) and (F), we have

$$M(z,w,t)\geq 1 \text{ and } N(z,w,t)\leq 0.$$

Hence, $M(z,w,t)=1$ and $N(z,w,t)=0.$

Therefore, $z = w$, i.e., mappings A, B, *S and T have a unique common fixed point*. □

Taking $B = A$ and $T = S$ in the Theorem 6.2.2 yields following corollary:

Corollary 6.2.1. *Let A and S be self-mappings of an intuitionistic fuzzy metric space* $(X,M,N,*,\lozenge)$ *satisfying the following conditions that*

(i) *the pair* (A, S) *satisfies the property* ($E.A$);

(ii) *for any* $x,y\in X$, $\phi,\psi\in M_4$ *and for all* $t>0,$ *there exists* $k\in(0,1)$ *such that*

$$\begin{aligned}&\phi(M(Ax,Ay,kt),M(Sx,Sy,t),M(Sx,Ax,t),M(Sy,Ay,t))\geq 0\\&\psi(N(Ax,Ay,kt),N(Sx,Sy,t),N(Sx,Ax,t),N(Sy,Ay,t))\leq 0;\end{aligned}$$

(iii) *$S(X)$ is a closed subset of X.*

Then A and S each have a point of coincidence. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

The following example demonstrates the validity of the Theorem 6.2.2:

Example 6.2.2. Let $X = [0,2)$ and pair (M, N) be the intuitionistic fuzzy set on $X^2 \times [0,\infty)$ defined by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0 \\ 0, & t=0 \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0 \\ 1, & t=0. \end{cases}$$

Then $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space where $*$ and $\Diamond$ are continuous t-norm and continuous t-conorm defined by $a*b = \min\{a,b\}$ and $a \Diamond b = \max\{a,b\}$ for all $a,b \in [0,1]$, respectively. Define $\phi, \psi : [0,1]^4 \to \mathbf{R}$ as $\phi(t_1,t_2,t_3,t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4$ and $\psi(t_1,t_2,t_3,t_4) = 12t_1 - 9t_2 + 8t_3 - 11t_4$.

Clearly, ϕ and ψ satisfies all conditions (*A*), (*B*), (*C*), (*D*), (*E*) and (*F*). Therefore, $\phi, \psi \in M_4$. Now we define self-mappings A, B, S and $T : X \to X$ by

$$Ax = Bx = 1, \quad Sx = \begin{cases} 1, & x \in \mathbf{Q} \\ \dfrac{2}{3}, & x \notin \mathbf{Q}, \end{cases} \qquad Tx = \begin{cases} 1, & x \in \mathbf{Q} \\ \dfrac{1}{3}, & x \notin \mathbf{Q}. \end{cases}$$

Choosing the sequences $\left\{x_n = \frac{1}{n}\right\}$ and $\left\{y_n = \frac{1}{n}\right\}$ in X, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in X$$

which shows that pairs (*A*, *S*) and (*B*, *T*) share the common property (*E.A*). By a routine calculation, one can verify the condition (i) of Theorem 6.2.2. Thus, all the conditions of Theorem 6.2.2 are satisfied and thereby proving that $x = 1$ is the unique common fixed point of *A*, *B*, *S* and *T*.

6.3 COMMON FIXED POINT THEOREMS FOR COMPATIBLE AND SUBSEQUENTIAL CONTINUOUS MAPPINGS

In this section, the following implicit relation has been utilized for establishing various common fixed point results in intuitionistic fuzzy metric spaces for compatible and subsequential continuous mappings.

Let M_6 denotes the set of all real-valued continuous functions ϕ and $\psi : [0,1]^6 \to \mathbf{R}$ satisfying the following conditions:

(A) $\phi(u,1,u,1,u,u) < 0,$ for all $u \in (0,1);$

(B) $\psi(v,0,v,0,v,v) > 0,$ for all $v \in (0,1).$

Example 6.3.1. Define $\phi,\ \psi : [0,1]^6 \to \mathbf{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \phi_1(\min\{t_2,t_3,t_4,t_5,t_6\}),$$

where $\phi_1 : [0,1] \to [0,1]$ is continuous increasing function such that $\phi_1(s) > s$ for all $s \in (0,1),$ and

$$\psi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \psi_1(\max\{t_2,t_3,t_4,t_5,t_6\}),$$

where $\psi_1 : [0,1] \to [0,1]$ is continuous increasing function such that $\psi_1(k) < k$ for all $k \in (0,1).$ Clearly, ϕ and ψ satisfy the conditions (A) and (B). Therefore, $\phi, \psi \in M_6.$

Now, we prove the following results:

Theorem 6.3.1. *Let A, B, S and T be four self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$. *If the pairs* (*A*, *S*) *and* (*B*, *T*) *are compatible and subsequentially continuous mappings, then*

(i) *the pair* (*A*, *S*) *has a coincidence point*;

(ii) *the pair* (*B*, *T*) *has a coincidence point.*

Further, A, B, S and T have a unique common fixed point provided A, B, S and T satisfy the following:

(iii) *for any* $x, y \in X$, $\phi, \psi \in M_6$ *and for all* $t > 0$,

$$\phi\big(M(Ax,By,t), M(Sx,Ax,t), M(Sx,Ty,t), M(Ty,By,t), M(Sx,By,t), M(Ty,Ax,t)\big) \geq 0$$
$$\psi\big(N(Ax,By,t), N(Sx,Ax,t), N(Sx,Ty,t), N(Ty,By,t), N(Sx,By,t), N(Ty,Ax,t)\big) \leq 0.$$

Proof. Since the pairs (*A*, *S*) and (*B*, *T*) are compatible and subsequentially continuous mappings, therefore, there exist sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,\ \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w \text{ for some } z, w \in X$$

and

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = M(Az, Sz, t) = 1,\ \lim_{n\to\infty} N(ASx_n, SAx_n, t) = N(Az, Sz, t) = 0$$

$$\lim_{n\to\infty} M(BTy_n, TBy_n, t) = M(Bw, Tw, t) = 1,\ \lim_{n\to\infty} N(BTy_n, TBy_n, t) = N(Bw, Tw, t) = 0$$

this gives *Az* = *Sz* and *Bw* = *Tw*. Hence, *z* is a coincidence point of *A* and *S*, and *w* is a coincidence point of *B* and *T*. This proves the assertion (i) and (ii).

Now, we prove that *z* = *w*. Let if possible $z \neq w$. Then by using (iii), we have

$$\phi\begin{pmatrix} M(Ax_n, By_n, t), M(Sx_n, Ax_n, t), M(Sx_n, Ty_n, t), \\ M(Ty_n, By_n, t), M(Sx_n, By_n, t), M(Ty_n, Ax_n, t) \end{pmatrix} \geq 0$$

and

$$\psi\begin{pmatrix} N(Ax_n, By_n, t), N(Sx_n, Ax_n, t), N(Sx_n, Ty_n, t), \\ N(Ty_n, By_n, t), N(Sx_n, By_n, t), N(Ty_n, Ax_n, t) \end{pmatrix} \leq 0.$$

On letting $n \to \infty,$ we have

$$\phi\big(M(z,w,t), M(z,z,t), M(z,w,t), M(w,w,t), M(z,w,t), M(w,z,t)\big) \geq 0$$
$$\phi\big(M(z,w,t), 1, M(z,w,t), 1, M(z,w,t), M(z,w,t)\big) \geq 0$$

and

$$\psi\big(N(z,w,t), N(z,z,t), N(z,w,t), N(w,w,t), N(z,w,t), N(w,z,t)\big) \leq 0$$
$$\psi\big(N(z,w,t), 0, N(z,w,t), 0, N(z,w,t), N(z,w,t)\big) \leq 0,$$

which is a contradiction in view of the implicit relations (*A*) and (*B*). Therefore, *z* = *w*.

Now, we assert that *Az* = *z*. If not, then by using (iii), we have

$$\phi\begin{pmatrix} M(Az, By_n, t), M(Sz, Az, t), M(Sz, Ty_n, t), \\ M(Ty_n, By_n, t), M(Sz, By_n, t), M(Ty_n, Az, t) \end{pmatrix} \geq 0$$

and

$$\psi\begin{pmatrix} N(Az, By_n, t), N(Sz, Az, t), N(Sz, Ty_n, t), \\ N(Ty_n, By_n, t), N(Sz, By_n, t), N(Ty_n, Az, t) \end{pmatrix} \leq 0.$$

Taking limit as $n \to \infty$, we have

$$\phi\big(M(Az,z,t), M(Az,Az,t), M(Az,z,t), M(z,z,t), M(Az,z,t), M(z,Az,t)\big) \geq 0$$

$$\phi\big(M(Az,z,t), 1, M(Az,z,t), 1, M(Az,z,t), M(Az,z,t)\big) \geq 0$$

and

$$\psi\big(N(Az,z,t), N(Az,Az,t), N(Az,z,t), N(z,z,t), N(Az,z,t), N(z,Az,t)\big) \leq 0$$

$$\psi\big(N(Az,z,t), 0, N(Az,z,t), 0, N(Az,z,t), N(Az,z,t)\big) \leq 0$$

which again is a contradiction in view of implicit relations (*A*) and (*B*). Therefore, *Az* = *z* = *Sz*.

Similarly, it can be proved that *Bz* = *z* = *Tz*. Therefore, we have, *z* = *Az* = *Bz* = *Sz* = *Tz*. i.e., *z* is common fixed point of *A*, *B*, *S* and *T*. The uniqueness of common fixed point follows easily from (iii). This completes the proof of the theorem. □

Now, we prove common fixed point theorem using reciprocal continuity together with subcompatibility as follows:

Theorem 6.3.2. *Let A, B, S and T be four self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$. *If the pairs* (*A*, *S*) *and* (*B*, *T*) *are subcompatible and reciprocally continuous mappings, then*

(i) *the pair* (*A*, *S*) *has a coincidence point,*

(ii) *the pair* (*B*, *T*) *has a coincidence point.*

Further, A, B, S and T have a unique common fixed point provided A, B, S and T satisfy the following condition:

(iii) *for any* $x, y \in X$, $\phi, \psi \in M_6$ *and for all t* > 0,

$$\phi\begin{pmatrix} M(Ax,By,t), M(Sx,Ax,t), M(Sx,Ty,t), \\ M(Ty,By,t), M(Sx,By,t), M(Ty,Ax,t) \end{pmatrix} \geq 0$$

$$\psi\begin{pmatrix} N(Ax,By,t), N(Sx,Ax,t), N(Sx,Ty,t), \\ N(Ty,By,t), N(Sx,By,t), N(Ty,Ax,t) \end{pmatrix} \leq 0.$$

Proof. By using the definition of reciprocal continuous and subcompatible mappings, the proof of this theorem follows on the same lines on that of Theorem 6.3.1. □

Corollary 6. 3.1. *The conclusions of Theorem* 6.3.1 *and Theorem* 6.3.2 *remain true if we replace the inequality* (iii) *of Theorem* 6.3.1 *or Theorem* 6.3.2 *by any one of the following conditions* (*iv*) *or* (*v*):

$(iv)\quad M(Ax,By,t) \ge F\left(\min\left\{\begin{matrix} M(Sx,Ty,t), M(Sx,Ax,t), M(Ty,By,t), \\ M(Sx,By,t), M(Ty,Ax,t) \end{matrix}\right\}\right)$

$$N(Ax,By,t) \le G\left(\max\left\{\begin{matrix} N(Sx,Ty,t), N(Sx,Ax,t), N(Ty,By,t), \\ N(Sx,By,t), N(Ty,Ax,t) \end{matrix}\right\}\right)$$

for all $x, y \in X$, *where* $F, G : [0,1] \to [0,1]$ *is increasing and continuous function such that* $F(t) > t$ *and* $G(t) < t$ *for all* $t \in (0,1)$.

$(v)\quad \int_0^{M(Ax,By,t)} \varphi(t)dt \ge F\left(\int_0^{\min\{M(Sx,Ty,t),M(Sx,Ax,t),M(Ty,By,t),M(Sx,By,t),M(Ty,Ax,t)\}} \varphi(t)dt\right)$

$$\int_0^{N(Ax,By,t)} \varphi(t)dt \le G\left(\int_0^{\max\{N(Sx,Ty,t),N(Sx,Ax,t),N(Ty,By,t),N(Sx,By,t),N(Ty,Ax,t)\}} \varphi(t)dt\right)$$

where $F, G : [0,1] \to [0,1]$ *is increasing and continuous function such that* $F(t) > t$ *and* $G(t) < t$ *for all* $t \in (0,1)$ *and* $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ *is a summable Lebesgue integrable function satisfies* $0 < \int_0^{\varepsilon} \varphi(s)ds < 1$, *for all* $0 < \varepsilon < 1$.

By setting *A* = *B* in Theorems 6.3.1, 6.3.2, we derive the following two corollaries for three mappings.

Corollary 6.3.2. *Let A, S and T be three self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$. *If the pairs* (*A*, *S*) *and* (*A*, *T*) *are compatible and subsequentially continuous mappings, then*

(i) *the pair* (*A*, *S*) *has a coincidence point*;

(ii) *the pair* (*A*, *T*) *has a coincidence point*.

Further, A, S and T have a unique common fixed point provided A, S and T satisfies the following:

(iii) *for any* $x, y \in X$, $\phi, \psi \in M_6$ *and for all t* > 0,

$$\phi\big(M(Ax,Ay,t), M(Sx,Ax,t), M(Sx,Ty,t), M(Ty,Ay,t), M(Sx,Ay,t), M(Ty,Ax,t)\big) \ge 0$$
$$\psi\big(N(Ax,Ay,t), N(Sx,Ax,t), N(Sx,Ty,t), N(Ty,Ay,t), N(Sx,Ay,t), N(Ty,Ax,t)\big) \le 0.$$

Corollary 6.3.3. *Let A, S and T be three self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$. *If the pairs* (*A*, *S*) *and* (*A*, *T*) *are subcompatible and reciprocally continuous mappings, then*

(i) *the pair* (*A*, *S*) *has a coincidence point*;

(ii) *the pair* (*A*, *T*) *has a coincidence point.*

Further, A, S and T have a unique common fixed point provided A, S and T satisfies the following:

(iii) *for any* $x, y \in X$, $\phi, \psi \in M_6$ *and for all* $t > 0$,

$$\phi\begin{pmatrix} M(Ax, Ay, t), M(Sx, Ax, t), M(Sx, Ty, t), \\ M(Ty, Ay, t), M(Sx, Ay, t), M(Ty, Ax, t) \end{pmatrix} \geq 0$$

$$\psi\begin{pmatrix} N(Ax, Ay, t), N(Sx, Ax, t), N(Sx, Ty, t), \\ N(Ty, Ay, t), N(Sx, Ay, t), N(Ty, Ax, t) \end{pmatrix} \leq 0.$$

Finally, by setting $A = B$ and $S = T$ in Theorems 6.3.1 and 6.3.2, we derive the following two corollaries:

Corollary 6.3.4. *Let A and S be two self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$. *If the pair* (A, S) *is compatible and subsequentially continuous mappings, then*

(i) *the pair* (A, S) *has a coincidence point.*

Further, A and S have a unique common fixed point provided A and S satisfy the following:

(ii) *for any* $x, y \in X$, $\phi, \psi \in M_6$ *and for all* $t > 0$, *we have*

$$\phi(M(Ax, Ay, t), M(Sx, Ax, t), M(Sx, Sy, t), M(Sy, Ay, t), M(Sx, Ay, t), M(Sy, Ax, t)) \geq 0$$
$$\psi(N(Ax, Ay, t), N(Sx, Ax, t), N(Sx, Sy, t), N(Sy, Ay, t), N(Sx, Ay, t), N(Sy, Ax, t)) \leq 0.$$

Corollary 6.3.5. *Let A and S be pair of self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$. *If the pair* (A, S) *is subcompatible and reciprocally continuous mappings, then*

(i) *the pair* (A, S) *has a coincidence point.*

Further, A and S have a unique common fixed point provided A and S satisfy the following:

(ii) *for any* $x, y \in X$, $\phi, \psi \in M_6$ *and for all t* > 0, *we have*

$$\phi\left(M(Ax, Ay, t), M(Sx, Ax, t), M(Sx, Sy, t), M(Sy, Ay, t), M(Sx, Ay, t), M(Sy, Ax, t)\right) \geq 0$$
$$\psi\left(N(Ax, Ay, t), N(Sx, Ax, t), N(Sx, Sy, t), N(Sy, Ay, t), N(Sx, Ay, t), N(Sy, Ax, t)\right) \leq 0.$$

Following two examples demonstrate the validity of Theorems 6.3.1 and 6.3.2.

Example 6.3.2. Let $X = (-3, \infty)$ and pair (*M*, *N*) be the intuitionistic fuzzy set on $X^2 \times [0, \infty)$ defined by

$$M(x, y, t) = \begin{cases} \dfrac{t}{t + |x - y|}, & t > 0 \\ 0, & t = 0 \end{cases} \quad \text{and} \quad N(x, y, t) = \begin{cases} \dfrac{|x - y|}{t + |x - y|}, & t > 0 \\ 1, & t = 0. \end{cases}$$

Then $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space where $*$ and $\Diamond$ are continuous *t*-norm and continuous *t*-conorm. Now we define self-mappings $A, B, S, T : X \to X$ as

$$Ax = Bx = \begin{cases} 0, & x \in (-3, 0) \\ \dfrac{x}{3}, & x \in [0, 1] \\ 2x - 1, & x \in (1, \infty) \end{cases}, \quad Sx = Tx = \begin{cases} \dfrac{x}{2}, & x \in (-3, 1] \\ 3x - 2, & x \in (1, \infty) \end{cases}$$

and $\phi, \psi : [0,1]^6 \to \mathbf{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi_1(\min\{t_2, t_3, t_4, t_5, t_6\}),$$

where $\phi_1 : [0,1] \to [0,1]$ is increasing and continuous function $\phi_1(s) = \sqrt{s}$ such that $\phi_1(s) > s$ for all $s \in (0,1)$, and

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi_1(\max\{t_2, t_3, t_4, t_5, t_6\}),$$

where $\psi_1 : [0,1] \to [0,1]$ is increasing and continuous function $\psi_1(k) = k^2$ such that $\psi_1(k) < k$ for all $k \in (0,1)$. Clearly, for the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$, (*A*, *S*) and (*B*, *T*) are compatible as well as subsequentially continuous. Thus, all the conditions of Theorem 6.3.1 are satisfied. Evidently, *x* = 0 is a coincidence as well as unique common fixed point of *A*, *B*, *S* and *T*.

Example 6.3.3. Let $X = \mathbf{R}$ and pair (*M*, *N*) be the intuitionistic fuzzy set on $X^2 \times [0,\infty)$ defined by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0 \\ 0, & t=0 \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0 \\ 1, & t=0. \end{cases}$$

Then $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space where $*$ and $\Diamond$ are continuous *t*-norm and continuous *t*-conorm. Now we define self-mappings $A, B, S, \ T$: $X \to X$ as

$$Ax = Bx = \begin{cases} x+1, & x \in (-\infty,1) \\ 2x-1, & x \in [1,\infty) \end{cases}, \quad Sx = Tx = \begin{cases} \dfrac{x}{2}, & x \in (-\infty,1) \\ 3x-2, & x \in [1,\infty) \end{cases}$$

and $\phi, \psi : [0,1]^6 \to \mathbf{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \phi_1(\min\{t_2,t_3,t_4,t_5,t_6\}),$$

where $\phi_1 : [0,1] \to [0,1]$ is increasing and continuous function $\phi_1(s) = \sqrt{s}$ such that $\phi_1(s) > s$ for all $s \in (0,1),$ and

$$\psi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \psi_1(\max\{t_2,t_3,t_4,t_5,t_6\}),$$

where $\psi_1 : [0,1] \to [0,1]$ is increasing and continuous function $\psi_1(k) = k^2$ such that $\psi_1(k) < k$ for all $k \in (0,1).$ Clearly, for the sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$, (*A*, *S*) and (*B*, *T*) are subcompatible as well as reciprocally continuous. Therefore, all the conditions of Theorem 6.3.2 are satisfied. Evidently, *x* = 1 is a coincidence as well as unique common fixed point of *A*, *B*, *S* and *T*.

As an application of Theorems 6.3.1 and 6.3.2, following common fixed point theorems have been proved for four finite families of mappings on intuitionistic fuzzy metric spaces:

Theorem 6.3.3. *Let* $\{A_1, A_2, ..., A_m\}$, $\{B_1, B_2, ..., B_n\}$, $\{S_1, S_2, ..., S_p\}$ *and* $\{T_1, T_2, ..., T_q\}$ *be four finite families of self-mappings of a intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$ *such that* $A = A_1.A_2.....A_m$, $B = B_1.B_2.....B_n$, $S = S_1.S_2.....S_p$ *and* $T = T_1.T_2.....T_q$ *satisfying the following conditions*:

(i) *for any* $x, y \in X$, $\phi, \psi \in M_6$ *and for all* $t > 0$,

$$\phi\begin{pmatrix} M(Ax, By, t), M(Sx, Ax, t), M(Sx, Ty, t), \\ M(Ty, By, t), M(Sx, By, t), M(Ty, Ax, t) \end{pmatrix} \geq 0$$

$$\psi\begin{pmatrix} N(Ax, By, t), N(Sx, Ax, t), N(Sx, Ty, t), \\ N(Ty, By, t), N(Sx, By, t), N(Ty, Ax, t) \end{pmatrix} \leq 0;$$

(ii) (A, S) *and* (B, T) *are subsequentially continuous mappings and*

(iii) *the pairs of families* $(\{A_i\}, \{S_k\})$ *and* $(\{B_r\}, \{T_t\})$ *commute pairwise.*

Then the pairs (A, S) *and* (B, T) *have a point of coincidence each. Moreover,* $\{A_i\}_{i=1}^{m}, \{S_k\}_{k=1}^{p}$, $\{B_r\}_{r=1}^{n}$ *and* $\{T_t\}_{t=1}^{q}$ *also have a unique common fixed point.*

Proof. By using (iii), we first show that *AS* = *SA* as

$$\begin{aligned} AS &= (A_1A_2...A_m)(S_1S_2...S_p) = (A_1A_2...A_{m-1})(A_mS_1S_2...S_p) \\ &= (A_1A_2...A_{m-1})(S_1S_2...S_pA_m) = (A_1A_2...A_{m-2})(A_{m-1}S_1S_2...S_pA_m) \\ &= (A_1A_2...A_{m-2})(S_1S_2...S_pA_{m-1}A_m) = ... = A_1(S_1S_2...S_pA_2...A_{m-1}A_m) \\ &= (S_1S_2...S_p)(A_1A_2...A_m) = SA. \end{aligned}$$

Similarly, one can prove that *BT* = *TB*. Hence, the pairs (*A*, *S*) and (*B*, *T*) are compatible. Now, using Theorem 6.3.1, we conclude that *A*, *B*, *S* and *T* have a unique common fixed point in *X*, say *z*.

Now, we prove that *z* remains the fixed point of all the component mappings. For this consider

$$\begin{aligned} A(A_iz) &= (A_1A_2...A_m)A_iz = (A_1A_2...A_{m-1})(A_mA_i)z \\ &= (A_1A_2...A_{m-1})(A_iA_m)z = (A_1A_2...A_{m-2})(A_{m-1}A_iA_m)z \\ &= (A_1A_2...A_{m-2})(A_iA_{m-1}A_m)z = ... = A_1(A_iA_2...A_{m-1}A_m)z \\ &= (A_1A_i)(A_2A_3...A_m)z = (A_iA_1)(A_2A_3...A_m)z \\ &= A_i(A_1A_2...A_m)z = A_iAz = A_iz. \end{aligned}$$

Similarly, one can prove that

$$\begin{aligned}&A(S_k z) = S_k(Az) = S_k z,\ S(S_k z) = S_k(Sz) = S_k z,\\ &S(A_i z) = A_i(Sz) = A_i z,\ B(B_r z) = B_r(Bz) = B_r z,\\ &B(T_t z) = T_t(Bz) = T_t z,\ T(T_t z) = T_t(Tz) = T_t z\end{aligned}$$

and $$T(B_r z) = B_r(Tz) = B_r z,$$

which show that (for all *i*, *r*, *k* and *t*) $A_i z$ and $S_k z$ are other fixed point of the pair (*A*, *S*) whereas $B_r z$ and $T_t z$ are other fixed points of the pair (*B*, *T*). Since *A*, *B*, *S* and *T* have a unique common fixed point, so, we get

$$z = A_i z = S_k z = B_r z = T_t z,$$

for all *i* = 1, 2, . . . ,*m*, *k* = 1, 2, . . . , *p*, *r* = 1, 2, . . . ,*n*, *t* = 1, 2, . . . , *q*, which shows that *z* is a unique common fixed point of $\{A_i\}_{i=1}^{m}, \{S_k\}_{k=1}^{p}, \{B_r\}_{r=1}^{n}$ and $\{T_t\}_{t=1}^{q}$. □

Theorem 6.3.4. *Let* $\{A_1, A_2, ..., A_m\}$, $\{B_1, B_2, ..., B_n\}$, $\{S_1, S_2, ..., S_p\}$ *and* $\{T_1, T_2, ..., T_q\}$ *be four finite families of self-mappings of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$ *such that* $A = A_1.A_2.....A_m$, $B = B_1.B_2.....B_n$, $S = S_1.S_2.....S_p$, $T = T_1.T_2.....T_q$ *satisfying the following conditions*:

(i) *for any* $x, y \in X$, $\phi, \psi \in M_6$ *and for all* *t* > 0,

$$\phi\begin{pmatrix} M(Ax, By, t), M(Sx, Ax, t), M(Sx, Ty, t), \\ M(Ty, By, t), M(Sx, By, t), M(Ty, Ax, t) \end{pmatrix} \geq 0$$

$$\psi\begin{pmatrix} N(Ax, By, t), N(Sx, Ax, t), N(Sx, Ty, t), \\ N(Ty, By, t), N(Sx, By, t), N(Ty, Ax, t) \end{pmatrix} \leq 0;$$

(ii) *the pairs of families* $(\{A_i\}, \{S_k\})$ *and* $(\{B_r\}, \{T_t\})$ *commute pairwise*;

(iii) *the pairs* (A, S) *and* (B, T) *are subcompatible and reciprocally continuous mappings.*

Then the pairs (A, S) *and* (B, T) *have a point of coincidence each. Moreover,* $\{A_i\}_{i=1}^{m}, \{S_k\}_{k=1}^{p}, \{B_r\}_{r=1}^{n}$ *and* $\{T_t\}_{t=1}^{q}$ *have a unique common fixed point.*

Proof. By using the definition of reciprocal continuous and subcompatible mappings, the proof of this theorem follows on the same lines as that of Theorem 6.3.3. □

CHAPTER- VII

COMMON FIXED POINT THEOREMS IN MODIFIED INTUIONISTIC FUZZY METRIC SPACES

This chapter aims to prove some common fixed point theorems in a modified intuitionistic fuzzy metric space by using variants of *R*-weakly commuting mappings and that of compatible continuous mappings. This chapter has been divided into various sections. Section 7.1 provides the necessary information about modified intuitionistic fuzzy metric spaces. In Section 7.2, the lemmas required to prove the main results have been presented. In Section 7.3, some common fixed point theorems have been established for non-compatible self-mappings and variants of *R*-weakly commuting mappings on a modified intuitionistic fuzzy metric space by using the concept of weakly reciprocal continuity.

7.1 INTRODUCTION

Now, we recall the following notions that will be used in the sequel.

Definition 7.1.1.[34] A **triangular norm (*t*-norm)** on L^* is a mapping $\mathscr{T}: (L^*)^2 \to L^*$ satisfying the following conditions: for all $x, x', y, y', z \in L^*$,

(i) $\mathscr{T}(x, 1_{L^*}) = x$; (boundary condition)

(ii) $\mathscr{T}(x, y) = \mathscr{T}(y, x)$; (commutativity)

(iii) $\mathscr{T}(x, \mathscr{T}(y, z)) = \mathscr{T}(\mathscr{T}(x, y), z)$; (associativity)

(iv) If, $x \leq_{L^*} x'$ and $y \leq_{L^*} y'$ implies $\mathscr{T}(x, y) \leq_{L^*} \mathscr{T}(x', y')$. (monotonicity)

where $L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}$.

Definition 7.1.2.[34, 35] A continuous *t*-norm $\mathscr{T}$ on L^* is called **continuous *t* – representable** iff there exists a continuous *t*- norm * and a continuous *t* – conorm ◊ on [0, 1] such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathscr{T}(x, y) = (x_1 * y_1,\ x_2 \lozenge y_2).$$

The results obtained in this chapter have been published in **Journal of Applied mathematics, Volume 2013, Article ID 189321, 13 pages (SCI).**

Definition 7.1.3.[112] Let M, N are fuzzy sets from $X^2 \times (0,\infty) \to [0,1]$ such that $M(x,y,t) + N(x,y,t) \le 1$, for all $x, y \in X$ and $t > 0$. The 3- tuple (X, $\zeta_{M,N}$, $\mathscr{T}$) is said to be a **modified intuitionistic fuzzy metric space,** if X is an arbitrary non-empty set, $\mathscr{T}$ is a continuous t- representable and $\zeta_{M,N}$ is an intuitionistic set from $X^2 \times (0,\infty) \to L^*$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(i) $\zeta_{M,N}(x,y,t) >_{L^*} 0_{L^*}$;

(ii) $\zeta_{M,N}(x,y,t) = 1_{L^*}$ iff $x = y$;

(iii) $\zeta_{M,N}(x,y,t) = \zeta_{M,N}(y,x,t)$;

(iv) $\zeta_{M,N}(x,y,t+s) \ge_{L^*} \mathscr{T}\left(\zeta_{M,N}(x,z,t), \zeta_{M,N}(z,y,s)\right)$;

(v) $\zeta_{M,N}(x,y,\bullet): (0,\infty) \to L^*$ is continuous.

In this case, $\zeta_{M,N}$ is called a modified intuitionistic fuzzy metric. Here,

$$\zeta_{M,N}(x,y,t) = \left(M(x,y,t), N(x,y,t)\right), \text{ for all } x, y \in X \text{ and } t > 0.$$

Definition 7.1.4.[112] A pair (f, g) of self-mappings of a modified intuitionistic fuzzy metric space (X, $\zeta_{M,N}$, $\mathscr{T}$) is said to be **compatible** if $\lim_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) = 1_{L^*}$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$.

A pair (f, g) of self-mappings of a modified intuitionistic fuzzy metric space (X, $\zeta_{M,N}$, $\mathscr{T}$) is said to be **non-compatible** if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$ but $\lim_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) \ne 1_{L^*}$ or non-existent for at least one $t > 0$.

Definition 7.1.5.[135] A pair (f, g) of self-mappings of a modified intuitionistic fuzzy metric space (X, $\zeta_{M,N}$, $\mathscr{T}$) is said to be ***R*-weakly commuting** at a point $x \in X$ if

$$\zeta_{M,N}(fgx, gfx, t) \ge_{L^*} \zeta_{M,N}\left(fx, gx, \frac{t}{R}\right), \text{ for some } R > 0 \text{ and } t > 0.$$

Definition 7.1.6.[135] A pair (f, g) of self-mappings of a modified intuitionistic fuzzy metric space (X, $\zeta_{M,N}$, $\mathscr{T}$) is said to be **pointwise *R*-weakly commuting** on X if given $x \in X$, there exists $R > 0$ such that

$$\zeta_{M,N}\left(fgx, gfx, t\right) \geq_{L^*} \zeta_{M,N}\left(fx, gx, \frac{t}{R}\right), \text{ for all } t > 0.$$

Definition 7.1.7.[135] A pair (*f*, *g*) of self-mappings of a modified intuitionistic fuzzy metric space (*X*, $\zeta_{M,N}$, $\mathscr{T}$) is said to be

(i) ***R*-weakly commuting of type (A_g)** if there exists some *R* > 0 such that

$$\zeta_{M,N}(ffx, gfx, t) \geq_{L^*} \zeta_{M,N}\left(fx, gx, \frac{t}{R}\right), \text{ for all } x \in X \text{ and } t > 0.$$

(ii) ***R*-weakly commuting of type (A_f)** if there exists some *R* > 0 such that

$$\zeta_{M,N}\left(fgx, ggx, t\right) \geq_{L^*} \zeta_{M,N}\left(fx, gx, \frac{t}{R}\right), \text{ for all } x \in X \text{ and } t > 0.$$

Definition 7.1.8.[61] A pair (*f*, *g*) of self-mappings of a modified intuitionistic fuzzy metric space (*X*, $\zeta_{M,N}$, $\mathscr{T}$) is said to be ***R*-weakly commuting of type (*P*)** if there exists some *R* > 0 such that

$$\zeta_{M,N}\left(ffx, ggx, t\right) \geq_{L^*} \zeta_{M,N}\left(fx, gx, \frac{t}{R}\right), \text{ for all } x \in X \text{ and } t > 0.$$

7.2 LEMMAS

The proof of our main results is based upon the following lemmas:

Lemma 7.2.1.[61] *Let (X,* $\zeta_{M,N}$ *,* $\mathscr{T}$*) be a modified intuitionistic fuzzy metric space. If there exists a constant* $k \in (0,1)$ *such that*

$$\zeta_{M,N}(x, y, kt) \geq_{L^*} \zeta_{M,N}(x, y, t),$$

for all $x, y \in X$, $t > 0$ *and then* $x = y$.

Lemma 7.2.2.[61] *Let (X,* $\zeta_{M,N}$ *,* $\mathscr{T}$*) be a modified intuitionistic fuzzy metric space and* $\{y_n\}$ *be a sequence in X. If there exists a constant* $k \in (0,1)$ *such that*

$$\zeta_{M,N}(y_n, y_{n+1}, kt) \geq_{L^*} \zeta_{M,N}(y_{n-1}, y_n, t),$$

for all $t > 0$ *and n* = 1, 2,3, . . ., *then* $\{y_n\}$ *is a Cauchy sequence in X.*

7.3 COMMON FIXED POINT THEOREMS FOR VARIANTS OF *R*-WEAKLY COMMUTING MAPPINGS

In the following section, some common fixed point theorems for variants of *R*-weakly commuting mappings by using weak reciprocal continuous mappings have been presented.

Theorem 7.3.1. *Let f and g be weakly reciprocally continuous self-mappings of a complete modified intuitionistic fuzzy metric space* $(X, \zeta_{M,N}, \mathscr{T})$ *satisfying the following conditions*:

(i) $f(X) \subseteq g(X)$;

(ii) *for any* $x, y \in X$, $t > 0$, $k \in (0,1)$ *such that*:

$$\zeta_{M,N}(fx, fy, kt) \geq_{L^*} \min\begin{Bmatrix} \zeta_{M,N}(gx, gy, t), \zeta_{M,N}(gx, fy, 2t), \\ \zeta_{M,N}(fx, gx, t), \zeta_{M,N}(fx, gy, t), \zeta_{M,N}(fy, gy, t) \end{Bmatrix}.$$

If f and g are either compatible or R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f) *or R-weakly commuting of type* (P), *then f and g have a unique common fixed point.*

Proof. Let x_0 be any point in *X*. Since $f(X) \subseteq g(X)$, then there exists a sequence of points $\{x_n\}$ such that $fx_n = gx_{n+1}$, for all *n* = 0, 1, 2,3, ...

Also, define a sequence $\{y_n\}$ in *X* as

$$y_n = fx_n = gx_{n+1}. \tag{7.3.1}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in *X*. For proving this, by the use of (ii), we have

$$\begin{aligned}\zeta_{M,N}(y_n, y_{n+1}, kt) &= \zeta_{M,N}(fx_n, fx_{n+1}, kt)\\ &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(gx_n, gx_{n+1}, t), \zeta_{M,N}(gx_n, fx_{n+1}, 2t), \zeta_{M,N}(fx_n, gx_n, t),\\ \zeta_{M,N}(fx_n, gx_{n+1}, t), \zeta_{M,N}(fx_{n+1}, gx_{n+1}, t)\end{matrix}\right\}\\ &= \min\left\{\begin{matrix}\zeta_{M,N}(y_{n-1}, y_n, t), \zeta_{M,N}(y_{n-1}, y_{n+1}, 2t), \zeta_{M,N}(y_n, y_{n-1}, t),\\ \zeta_{M,N}(y_n, y_n, t), \zeta_{M,N}(y_{n+1}, y_n, t)\end{matrix}\right\}\\ &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(y_{n-1}, y_n, t), \zeta_{M,N}(y_{n-1}, y_n, t), \zeta_{M,N}(y_n, y_{n+1}, t),\\ \zeta_{M,N}(y_n, y_{n-1}, t), 1_{L^*}, \zeta_{M,N}(y_{n+1}, y_n, t)\end{matrix}\right\}\\ &= \min\left\{\zeta_{M,N}(y_{n-1}, y_n, t), \zeta_{M,N}(y_n, y_{n+1}, t)\right\}\\ \zeta_{M,N}(y_n, y_{n+1}, kt) &\geq_{L^*} \zeta_{M,N}(y_{n-1}, y_n, t).\end{aligned}$$

Then, by Lemma 7.2.2, $\{y_n\}$ is a Cauchy sequence in *X*.

Since *X* is complete, therefore there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z.$ Hence, by (7.3.1), $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z.$

Suppose that *f* and *g* are compatible mappings. By weak reciprocal continuity of *f* and *g*, it implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$. Let $\lim_{n\to\infty} gfx_n = gz$, then the compatibility of *f* and *g* gives

$$\lim_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) = 1_{L^*}.$$

This gives, $\lim_{n\to\infty} fgx_n = gz.$ By (7.3.1), we get $\lim_{n\to\infty} fgx_{n+1} = \lim_{n\to\infty} ffx_n = gz.$

Therefore, by the use of (ii), we get

$$\zeta_{M,N}(fz, ffx_n, kt) \geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(gz, gfx_n, t), \zeta_{M,N}(gz, ffx_n, 2t), \zeta_{M,N}(fz, gz, t),\\ \zeta_{M,N}(fz, gfx_n, t), \zeta_{M,N}(ffx_n, gfx_n, t)\end{matrix}\right\}.$$

Taking $n \to \infty$, we get

$$\begin{aligned}\zeta_{M,N}(fz, gz, kt) &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(gz, gz, t), \zeta_{M,N}(gz, gz, 2t), \zeta_{M,N}(fz, gz, t),\\ \zeta_{M,N}(fz, gz, t), \zeta_{M,N}(gz, gz, t)\end{matrix}\right\}\\ &= \zeta_{M,N}(fz, gz, t).\end{aligned}$$

By using Lemma 7.2.1, we get *fz* = *gz*. Again, compatibility of *f* and *g* implies commutativity at a coincidence point. Hence *gfz* = *fgz* = *ffz* = *ggz*.

By using (ii), we obtain

$$\begin{aligned}\zeta_{M,N}(fz, ffz, kt) &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(gz, gfz, t), \zeta_{M,N}(gz, ffz, 2t), \zeta_{M,N}(fz, gz, t), \\ \zeta_{M,N}(fz, gfz, t), \zeta_{M,N}(ffz, gfz, t)\end{matrix}\right\} \\ &= \min\left\{\begin{matrix}\zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, ffz, 2t), 1_{L^*}, \\ \zeta_{M,N}(fz, ffz, t), 1_{L^*}\end{matrix}\right\} \\ &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, ffz, t), 1_{L^*}, \\ \zeta_{M,N}(fz, ffz, t), 1_{L^*}\end{matrix}\right\} \\ &= \zeta_{M,N}(fz, ffz, t).\end{aligned}$$

That is, $fz = ffz$. Therefore, $fz = ffz = gfz$ and fz is a common fixed point of f and g.

Next, suppose that $\lim_{n\to\infty} fgx_n = fz$. Then, $f(X) \subseteq g(X)$ implies that $fz = gu$ for some $u \in X$ and, therefore, $\lim_{n\to\infty} fgx_n = gu.$

The compatibility of f and g implies that $\lim_{n\to\infty} gfx_n = gu$. By the use of (7.3.1), we get

$$\lim_{n\to\infty} fgx_{n+1} = \lim_{n\to\infty} ffx_n = gu.$$

Using (ii), we have

$$\zeta_{M,N}(fu, ffx_n, kt) \geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(gu, gfx_n, t), \zeta_{M,N}(gu, ffx_n, 2t), \zeta_{M,N}(fu, gu, t), \\ \zeta_{M,N}(fu, gfx_n, t), \zeta_{M,N}(ffx_n, gfx_n, t)\end{matrix}\right\}.$$

Taking $n \to \infty,$ we get

$$\begin{aligned}\zeta_{M,N}(fu, gu, kt) &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(gu, gu, t), \zeta_{M,N}(gu, gu, 2t), \zeta_{M,N}(fu, gu, t), \\ \zeta_{M,N}(fu, gu, t), \zeta_{M,N}(gu, gu, t)\end{matrix}\right\} \\ &= \zeta_{M,N}(fu, gu, t).\end{aligned}$$

Therefore, by the use of Lemma 7.2.1, we get $fu = gu$.

Compatibility of f and g implies $fgu = ggu = ffu = gfu$. Finally, using (ii), we obtain

$$\begin{aligned}\zeta_{M,N}(fu, ffu, kt) &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(gu, gfu, t), \zeta_{M,N}(gu, ffu, 2t), \zeta_{M,N}(fu, gu, t), \\ \zeta_{M,N}(fu, gfu, t), \zeta_{M,N}(ffu, gfu, t)\end{matrix}\right\} \\ &= \min\left\{\begin{matrix}\zeta_{M,N}(fu, ffu, t), \zeta_{M,N}(fu, ffu, 2t), 1_{L^*}, \\ \zeta_{M,N}(fu, ffu, t), 1_{L^*}\end{matrix}\right\} \\ &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(fu, ffu, t), \zeta_{M,N}(fu, ffu, t), 1_{L^*}, \\ \zeta_{M,N}(fu, ffu, t), 1_{L^*}\end{matrix}\right\} \\ &= \zeta_{M,N}(fu, ffu, t).\end{aligned}$$

That is, $fu = ffu$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Now, suppose that f and g are R-weakly commuting of type (A_g). Weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$. Let us first assume that $\lim_{n\to\infty} gfx_n = gz$. Then, R-weak commutativity of type (A_g) of f and g yields

$$\zeta_{M,N}\left(ffx_n, gfx_n, t\right) \geq_{L^*} \zeta_{M,N}\left(fx_n, gx_n, \frac{t}{R}\right)$$

$$\lim_{n\to\infty} \zeta_{M,N}\left(ffx_n, gz, t\right) \geq_{L^*} \zeta_{M,N}\left(z, z, \frac{t}{R}\right) = 1_{L^*}.$$

This gives, $\lim_{n\to\infty} ffx_n = gz$. Also, using (ii), we get

$$\zeta_{M,N}(fz, ffx_n, kt) \geq_{L^*} \min\begin{Bmatrix} \zeta_{M,N}(gz, gfx_n, t), \zeta_{M,N}(gz, ffx_n, 2t), \zeta_{M,N}(fz, gz, t), \\ \zeta_{M,N}(fz, gfx_n, t), \zeta_{M,N}(ffx_n, gfx_n, t) \end{Bmatrix}.$$

Taking $n \to \infty$, we have

$$\begin{aligned} \zeta_{M,N}(fz, gz, kt) &\geq_{L^*} \min\begin{Bmatrix} \zeta_{M,N}(gz, gz, t), \zeta_{M,N}(gz, gz, 2t), \zeta_{M,N}(fz, gz, t), \\ \zeta_{M,N}(fz, gz, t), \zeta_{M,N}(gz, gz, t) \end{Bmatrix} \\ &= \zeta_{M,N}(fz, gz, t). \end{aligned}$$

Hence, by Lemma 7.2.1, we get $fz = gz$. Again, by using R-weak commutativity of type (A_g),

$$\zeta_{M,N}\left(ffz, gfz, t\right) \geq_{L^*} \zeta_{M,N}\left(fz, gz, \frac{t}{R}\right) = 1_{L^*}.$$

This yields $ffz = gfz$. Therefore, $ffz = fgz = gfz = ggz$. Using (ii), we get

$$\begin{aligned} \zeta_{M,N}(fz, ffz, kt) &\geq_{L^*} \min\begin{Bmatrix} \zeta_{M,N}(gz, gfz, t), \zeta_{M,N}(gz, ffz, 2t), \zeta_{M,N}(fz, gz, t), \\ \zeta_{M,N}(fz, gfz, t), \zeta_{M,N}(ffz, gfz, t) \end{Bmatrix} \\ &= \min\begin{Bmatrix} \zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, ffz, 2t), 1_{L^*}, \\ \zeta_{M,N}(fz, ffz, t), 1_{L^*} \end{Bmatrix} \\ &\geq_{L^*} \min\begin{Bmatrix} \zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, ffz, t), 1_{L^*}, \\ \zeta_{M,N}(fz, ffz, t), 1_{L^*} \end{Bmatrix} \\ &= \zeta_{M,N}(fz, ffz, t). \end{aligned}$$

That is, $fz = ffz$. Hence, $fz = ffz = gfz$ and fz is a common fixed point of f and g.

Similarly, we prove if $\lim_{n\to\infty} fgx_n = fz$.

Suppose that *f* and *g* are *R*-weakly commuting of type (A_f). Again, as done above, we can easily prove that *fz* is a common fixed point of *f* and *g*.

Finally, suppose that *f* and *g* are *R*-weakly commuting of type (*P*). Weak reciprocal continuity of *f* and *g* implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$. Let us assume that $\lim_{n\to\infty} gfx_n = gz$. Then, *R*-weak commutativity of type (*P*) of *f* and *g* yields

$$\zeta_{M,N}\left(ffx_n, ggx_n, t\right) \geq_{L^*} \zeta_{M,N}\left(fx_n, gx_n, \frac{t}{R}\right).$$

Taking limit as $n \to \infty$,

$$\lim_{n\to\infty} \zeta_{M,N}\left(ffx_n, ggx_n, t\right) \geq_{L^*} \zeta_{M,N}\left(z, z, \frac{t}{R}\right) = 1_{L^*}.$$

This gives $\lim_{n\to\infty} \zeta_{M,N}\left(ffx_n, ggx_n, t\right) = 1_{L^*}$.

Using (7.3.1), $\lim_{n\to\infty} gfx_{n-1} = \lim_{n\to\infty} ggx_n = gz.$

This gives $\lim_{n\to\infty} ffx_n = gz.$ Also, using (ii) we get

$$\zeta_{M,N}(fz, ffx_n, kt) \geq_{L^*} \min\left\{\begin{array}{l}\zeta_{M,N}(gz, gfx_n, t), \zeta_{M,N}(gz, ffx_n, 2t), \zeta_{M,N}(fz, gz, t), \\ \zeta_{M,N}(fz, gfx_n, t), \zeta_{M,N}(ffx_n, gfx_n, t)\end{array}\right\}.$$

Taking $n \to \infty,$ we have

$$\begin{aligned}\zeta_{M,N}(fz, gz, kt) &\geq_{L^*} \min\left\{\begin{array}{l}\zeta_{M,N}(gz, gz, t), \zeta_{M,N}(gz, gz, 2t), \zeta_{M,N}(fz, gz, t), \\ \zeta_{M,N}(fz, gz, t), \zeta_{M,N}(gz, gz, t)\end{array}\right\} \\ &= \min\left\{1_{L^*}, 1_{L^*}, \zeta_{M,N}(fz, gz, t), \zeta_{M,N}(fz, gz, t), 1_{L^*}\right\} \\ &= \zeta_{M,N}(fz, gz, t).\end{aligned}$$

By Lemma 7.2.1, we get *fz* = *gz*. Again, by using *R*-weak commutativity of type (*P*),

$$\zeta_{M,N}\left(ffz, ggz, t\right) \geq_{L^*} \zeta_{M,N}\left(fz, gz, \frac{t}{R}\right) = 1_{L^*}.$$

This yields *ffz* = *ggz.*

Therefore, *ffz* = *fgz* = *gfz* = *ggz*. Using (ii), we get

$$
\begin{aligned}
\zeta_{M,N}(fz, ffz, kt) &\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(gz, gfz, t), \zeta_{M,N}(gz, ffz, 2t), \zeta_{M,N}(fz, gz, t), \\ \zeta_{M,N}(fz, gfz, t), \zeta_{M,N}(ffz, gfz, t)\end{matrix}\right\} \\
&= \min\left\{\begin{matrix}\zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, ffz, 2t), \zeta_{M,N}(fz, fz, t), \\ \zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(ffz, ffz, t)\end{matrix}\right\} \\
&\geq_{L^*} \min\left\{\begin{matrix}\zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, fz, t), \\ \zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(ffz, ffz, t)\end{matrix}\right\} \\
&= \zeta_{M,N}(fz, ffz, t).
\end{aligned}
$$

By Lemma 7.2.1, we get $fz = ffz$. Hence, $fz = ffz = gfz$ and fz is a common fixed point of f and g.

Similarly, we prove that if $\lim_{n\to\infty} fgx_n = fz$.

Uniqueness of the common fixed point follows easily in each of the three cases by using (ii). □

The following example illustrates Theorem 7.3.1.

Example 7.3.1. Let $X = [2, 20]$ and for each $t > 0$, define

$$\zeta_{M,N}(x, y, t) = \left(\frac{t}{t + |x-y|}, \frac{|x-y|}{t+|x-y|}\right), \text{ for all } x, y \in X.$$

Then $(X, \zeta_{M,N}, \mathscr{T})$ is a complete modified intuitionistic fuzzy metric space.

Define $f, g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 6, & 2 < x \leq 5 \end{cases} \quad \text{and} \quad gx = \begin{cases} 2, & x = 2 \\ 12, & 2 < x \leq 5 \\ (x+1)/3, & x > 5. \end{cases}$$

Let $\{x_n\}$ be a sequence in X such that either $\{x_n\} = \{2\}$ or $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$ for each n.

Clearly, f and g satisfy all the conditions of Theorem 7.3.1 and have a unique common fixed point at $x = 2$.

Theorem 7.3.2. *Let f and g be weakly reciprocally continuous non-compatible self-mappings of a modified intuitionistic fuzzy metric space* $(X, \zeta_{M,N}, \mathscr{T})$ *satisfying*:

(i) $f(X) \subseteq g(X)$;

(ii) $\zeta_{M,N}(fx, fy, kt) \geq_{L^*} \zeta_{M,N}(gx, gy, t),\ \ for\ all\ k \geq 0,\ x, y \in X\ and\ t > 0$;

(iii) $\zeta_{M,N}(fx, ffx, t) >_{L^*} \max\left\{\begin{matrix}\zeta_{M,N}(gx, gfx, t), \zeta_{M,N}(fx, gx, t), \zeta_{M,N}(ffx, gfx, t), \\ \zeta_{M,N}(fx, gfx, t), \zeta_{M,N}(gx, ffx, t)\end{matrix}\right\}$,

whenever $fx \neq ffx$ *for all* $x \in X$ *and* $t > 0$.

If f and g are R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f) *or R-weakly commuting of type* (*P*), *then f and g have common fixed point.*

Proof. Since *f* and *g* are non-compatible mappings, there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$ but either $\lim_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) \neq 1_{L^*}$ or the limit does not exist. Since $f(X) \subseteq g(X)$, for each $\{x_n\}$, there exists $\{y_n\}$ in *X* such that $fx_n = gy_n$. Thus $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} gy_n = z$. By virtue of this and using (ii), we obtain

$$\begin{aligned}&\zeta_{M,N}(fx_n, fy_n, kt) \geq_{L^*} \zeta_{M,N}(gx_n, gy_n, t)\\ &\lim_{n\to\infty} \zeta_{M,N}(z, fy_n, kt) \geq_{L^*} \zeta_{M,N}(z, z, t) = 1_{L^*}.\end{aligned}$$

This gives, $\lim_{n\to\infty} fy_n = z$. Therefore, we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} fy_n = z$.

Suppose that *f* and *g* are *R*-weakly commuting of type (A_g). Then, weak reciprocal continuity of *f* and *g* implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$. Similarly, $\lim_{n\to\infty} fgy_n = fz$ or $\lim_{n\to\infty} gfy_n = gz$. Let us first assume that $\lim_{n\to\infty} gfy_n = gz$. Then *R*-weak commutativity of type (A_g) of *f* and *g* yields

$$\begin{aligned}&\zeta_{M,N}\left(ffy_n, gfy_n, t\right) \geq_{L^*} \zeta_{M,N}\left(fy_n, gy_n, \frac{t}{R}\right)\\ &\lim_{n\to\infty} \zeta_{M,N}\left(ffy_n, gz, t\right) \geq_{L^*} \zeta_{M,N}\left(z, z, \frac{t}{R}\right) = 1_{L^*}.\end{aligned}$$

This gives, $\lim_{n\to\infty} ffy_n = gz$. Using (ii), we get

$$\zeta_{M,N}(ffy_n, fz, kt) \geq_{L^*} \zeta_{M,N}(gfy_n, gz, t).$$

Taking the limit as $n \to \infty$, we have

$$\zeta_{M,N}(gz, fz, kt) \geq_{L^*} \zeta_{M,N}(gz, gz, t) = 1_{L^*}.$$

This implies that *fz* = *gz*. Again, by virtue of *R*-weak commutativity of type (A_g),

$$\zeta_{M,N}\left(ffz, gfz, t\right) \geq_{L^*} \zeta_{M,N}\left(fz, gz, \frac{t}{R}\right) = 1_{L^*}.$$

This yields $ffz = gfz$ and $ffz = fgz = gfz = ggz$.

If $fz \neq ffz$, then by using (iii), we get

$$\begin{aligned}\zeta_{M,N}(fz, ffz, t) >_{L^*} \max &\begin{Bmatrix}\zeta_{M,N}(gz, gfz, t), \zeta_{M,N}(fz, gz, t), \zeta_{M,N}(ffz, gfz, t), \\ \zeta_{M,N}(fz, gfz, t), \zeta_{M,N}(gz, ffz, t)\end{Bmatrix} \\ = \max &\begin{Bmatrix}\zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, gz, t), \zeta_{M,N}(ffz, gfz, t), \\ \zeta_{M,N}(fz, ffz, t), \zeta_{M,N}(fz, ffz, t)\end{Bmatrix}\end{aligned}$$

$$\zeta_{M,N}(fz, ffz, t) >_{L^*} 1_{L^*},$$

which is a contradiction. Hence, $fz = ffz = gfz$ and fz is a common fixed point of f and g.

Similarly, we can prove, if $\lim_{n\to\infty} fgy_n = fz$, then again fz is a common fixed point of f and g. Proof is similar if f and g are R-weakly commuting of type (A_f) or (P). □

The following example illustrates Theorem 7.3.2.

Example 7.3.2. Let $(X, \zeta_{M,N}, \mathscr{T})$ be a modified intuitionistic fuzzy metric space as defined in Example 7.3.1. Define $f, g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 6, & 2 < x \leq 5 \end{cases} \quad \text{and} \quad gx = \begin{cases} 2, & x = 2 \\ 11, & 2 < x \leq 5 \\ (x+1)/3, & x > 5. \end{cases}$$

Let $\{x_n\}$ be a sequence in X such that either $\{x_n\} = \{2\}$ or $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$ for each n. Then clearly, f and g satisfy all the conditions of Theorem 7.3.2 and have a common fixed point at $x = 2$.

Theorem 7.3.3. *Let f and g be weakly reciprocally continuous non-compatible self-mappings of a modified intuitionistic fuzzy metric space* $(X, \zeta_{M,N}, \mathscr{T})$ *satisfying the conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) $\zeta_{M,N}(fx, fy, kt) \geq_{L^*} \zeta_{M,N}(gx, gy, t)$, *for all* $k \geq 0$, $x, y \in X$ *and* $t > 0$;

(iii) $\zeta_{M,N}(fx, ffx, t) >_{L^*} \zeta_{M,N}(gx, ggx, t)$,

whenever $fx \neq ffx$ *for all* $x \in X$ *and* $t > 0$.

If f and g are R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f) *or R-weakly commuting of type* (P)*, then f and g have common fixed point.*

Proof. Since f and g are non-compatible mappings, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$, but $\lim_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) \neq 1_{L^*}$ or non-existent for at least one $t > 0$. Since $f(X) \subseteq g(X)$, for each x_n, there exists y_n in X such that $fx_n = gy_n$. Thus, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} gy_n = z$. By virtue of this and (ii), we obtain $\lim_{n\to\infty} fy_n = z$. Therefore, we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} fy_n = z$.

Suppose that f and g are R-weakly commuting of type (A_g). Then, weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$. Similarly, $\lim_{n\to\infty} fgy_n = fz$ or $\lim_{n\to\infty} gfy_n = gz$. Let us first assume that $\lim_{n\to\infty} gfy_n = gz$. Then, R-weak commutativity of type (A_g) of f and g yields

$$\zeta_{M,N}(ffy_n, gfy_n, t) \geq_{L^*} \zeta_{M,N}\left(fy_n, gy_n, \frac{t}{R}\right)$$

$$\lim_{n\to\infty} \zeta_{M,N}(ffy_n, gz, t) \geq_{L^*} \zeta_{M,N}\left(z, z, \frac{t}{R}\right) = 1_{L^*}.$$

This gives, $\lim_{n\to\infty} ffy_n = gz$. Using (ii), we get

$$\zeta_{M,N}(ffy_n, fz, kt) \geq_{L^*} \zeta_{M,N}(gfy_n, gz, t).$$

Taking $n \to \infty$, we have

$$\zeta_{M,N}(gz, fz, kt) \geq_{L^*} \zeta_{M,N}(gz, gz, t) = 1_{L^*}.$$

This implies that $fz = gz$. Again, by virtue of R-weak commutativity of type (A_g),

$$\zeta_{M,N}(ffz, gfz, t) \geq_{L^*} \zeta_{M,N}\left(fz, gz, \frac{t}{R}\right) = 1_{L^*}.$$

This yields $ffz = gfz$ and $ffz = fgz = gfz = ggz$. If $fz \neq ffz$, then by using (iii), we get

$$\zeta_{M,N}(fz, ffz, t) >_{L^*} \zeta_{M,N}(gz, ggz, t) = \zeta_{M,N}(fz, ffz, t),$$

which is a contradiction. Hence $fz = ffz = gfz$ and fz is a common fixed point of f and g.

Similarly, we can prove, if $\lim_{n\to\infty} fgy_n = fz$, then again fz is a common fixed point of f and g. Proof is similar if f and g are R-weakly commuting of type (A_f) or R-weakly commuting of type (P). □

The following example illustrates Theorem 7.3.3.

Example 7.3.3. Let (X, $\zeta_{M,N}$, $\mathscr{T}$) be a modified intuitionistic fuzzy metric space as defined in Example 7.3.1. Define $f, g : X \to X$ by

$$fx = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 4, & 2 < x \le 5 \end{cases} \quad \text{and} \quad gx = \begin{cases} 2, & x = 2 \\ 4, & 2 < x \le 5 \\ (x+1)/3, & x > 5. \end{cases}$$

Let $\{x_n\}$ be a sequence in X such that either $\{x_n\} = \{2\}$ or $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$ for each n. Then, f and g satisfy all the conditions of Theorem 7.3.3 and have two common fixed points at $x = 2$ and $x = 4$.

Now, we prove our results using a new implicit relation defined as:

Let Θ denote the class of those functions $\theta : (L^*)^5 \to L^*$ such that θ is continuous and $\theta(x, 1_{L^*}, 1_{L^*}, x, x) = x$. Some examples of $\theta \in \Theta$ are:

(i) $\theta_1(x_1, x_2, x_3, x_4, x_5) = \min\{x_1, x_2, x_3, x_4, x_5\}$;

(ii) $\theta_2(x_1, x_2, x_3, x_4, x_5) = \sqrt[3]{x_1 x_4 x_5}$.

Theorem 7.3.4. *Let f and g be weakly reciprocally continuous non-compatible self-mappings of a modified intuitionistic fuzzy metric space* (*X*, $\zeta_{M,N}$, $\mathscr{T}$) *satisfying conditions*:

(i) $f(X) \subseteq g(X)$;

(ii) $\int_0^{\zeta_{M,N}(fx, fy, t)} \psi(s)ds \ge_{L^*} \int_0^{\zeta_{M,N}(gx, gy, t)} \psi(s)ds$, *for all* $x, y \in X$, $t > 0$;

(iii) $\int_0^{\zeta_{M,N}(fx, ffx, t)} \psi(s)ds >_{L^*} \int_0^{\omega} \psi(s)ds$,

$$\text{where} \quad \omega = \theta \left\{ \begin{matrix} \zeta_{M,N}(gx, gfx, t), \zeta_{M,N}(fx, gx, t), \zeta_{M,N}(ffx, gfx, t), \\ \zeta_{M,N}(fx, gfx, t), \zeta_{M,N}(gx, ffx, t) \end{matrix} \right\}$$

whenever $fx \ne ffx$ *for all* $x \in X$, $t > 0$ *and for some* $\theta \in \Theta$, *where* $\psi : \mathbf{R}^+ \to \mathbf{R}$ *is a Lebesgue integrable mapping which is summable non-negative and such that* $\int_0^{\varepsilon} \psi(s)ds > 0$ *for each* $\varepsilon > 0$.

If f and g are R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f) *or R-weakly commuting of type* (P), *then f and g have common fixed point.*

Proof. Since f and g are non-compatible mappings, there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = z \in X$, but $\lim\limits_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) \neq 1_{L^*}$ or non-existent for at least one $t > 0$. Since $f(X) \subseteq g(X)$, for each x_n, there exists y_n in X such that $fx_n = gy_n$. Thus, $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} gy_n = z$. By virtue of this and using (ii), we obtain

$$\int_0^{\zeta_{M,N}(fx_n, fy_n, t)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(gx_n, gy_n, t)} \psi(s)ds.$$

Taking $n \to \infty$, we have

$$\int_0^{\zeta_{M,N}(z, \lim\limits_{n\to\infty} fy_n, t)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(z,z,t)} \psi(s)ds,$$

which implies that, $\lim\limits_{n\to\infty} fy_n = z$.

Therefore, $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} gy_n = \lim\limits_{n\to\infty} fy_n = z$.

Suppose that f and g are R-weakly commuting of type (A_g). Then, weak reciprocal continuity of f and g implies that $\lim\limits_{n\to\infty} fgx_n = fz$ or $\lim\limits_{n\to\infty} gfx_n = gz$. Similarly, $\lim\limits_{n\to\infty} fgy_n = fz$ or $\lim\limits_{n\to\infty} gfy_n = gz$. Let us first assume that $\lim\limits_{n\to\infty} gfy_n = gz$. Then R-weak commutativity of type (A_g) of f and g yields

$$\zeta_{M,N}(ffy_n, gfy_n, t) \geq_{L^*} \zeta_{M,N}\left(fy_n, gy_n, \frac{t}{R}\right)$$

$$\lim_{n\to\infty} \zeta_{M,N}(ffy_n, gz, t) \geq_{L^*} \zeta_{M,N}\left(z, z, \frac{t}{R}\right) = 1_{L^*}.$$

This gives, $\lim\limits_{n\to\infty} ffy_n = gz$. Using (ii), we get

$$\int_0^{\zeta_{M,N}(ffy_n, fz, kt)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(gfy_n, gz, t)} \psi(s)ds.$$

Taking $n \to \infty$, we have

$$\int_0^{\zeta_{M,N}(gz, fz, kt)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(gz, gz, t)} \psi(s)ds.$$

This implies that $fz = gz$. Again, by virtue of R-weak commutativity of type (A_g),

$$\zeta_{M,N}(ffz, gfz, t) \geq_{L^*} \zeta_{M,N}\left(fz, gz, \frac{t}{R}\right) = 1_{L^*}.$$

This yields $ffz = gfz$ and $ffz = fgz = gfz = ggz$. If $fz \neq ffz$, then by using (iii), we get

$$\int_0^{\zeta_{M,N}(fz,ffz,t)} \psi(s)ds >_{L^*} \int_0^{\omega} \psi(s)ds,$$

Where,

$$\omega = \theta\left\{\begin{matrix}\zeta_{M,N}(gz,gfz,t),\zeta_{M,N}(fz,gz,t),\zeta_{M,N}(ffz,gfz,t),\\ \zeta_{M,N}(fz,gfz,t),\zeta_{M,N}(gz,ffz,t)\end{matrix}\right\}$$

$$=\theta\left\{\begin{matrix}\zeta_{M,N}(fz,ffz,t),\zeta_{M,N}(fz,fz,t),\zeta_{M,N}(ffz,ffz,t),\\ \zeta_{M,N}(fz,ffz,t),\zeta_{M,N}(fz,ffz,t)\end{matrix}\right\}$$

$$=\theta\left\{\zeta_{M,N}(fz,ffz,t),1_{L^*},1_{L^*},\zeta_{M,N}(fz,ffz,t),\zeta_{M,N}(fz,ffz,t)\right\}$$

$$\omega = \zeta_{M,N}(fz,ffz,t).$$

This gives,

$$\int_0^{\zeta_{M,N}(fz,ffz,t)} \psi(s)ds >_{L^*} \int_0^{\zeta_{M,N}(fz,ffz,t)} \psi(s)ds,$$

which is a contradiction. Hence, $fz = ffz = gfz$ and fz is a common fixed point of f and g.

Similarly, we can prove, if $\lim_{n\to\infty} fgy_n = fz$, then, again fz is a common fixed point of f and g. Proof is similar if f and g are R-weakly commuting of type (A_f) or (P). □

Let Δ denote the class of those functions $\delta : (L^*)^4 \to L^*$ such that δ is continuous and $\delta(x,1_{L^*},x,1_{L^*}) = x$.

Examples of $\delta \in \Delta$ are:

(i) $\delta_1(x_1,x_2,x_3,x_4) = \min\{x_1,x_2,x_3,x_4\}$;

(ii) $\delta_2(x_1,x_2,x_3,x_4) = \sqrt{x_1x_3}$.

Now, we establish following result for the aforesaid implicit relation:

Theorem 7.3.5. *Let f and g be weakly reciprocally continuous non-compatible self-mappings of a modified intuitionistic fuzzy metric space* (X, $\zeta_{M,N}$, $\mathscr{T}$) *satisfying the conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) $\int_0^{\zeta_{M,N}(fx,fy,t)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(gx,gy,t)} \psi(s)ds$, *for all* $x,y \in X$, $t > 0$;

(iii) $\int_0^{\zeta_{M,N}(fx,ffx,t)} \psi(s)ds >_{L^*} \int_0^{\Omega} \psi(s)ds,$, *where* $\Omega = \delta\left(\zeta_{M,N}(gx, gfx, t), \zeta_{M,N}(fx, gx, t), \zeta_{M,N}(fx, gfx, t), \zeta_{M,N}(ffx, gfx, t)\right)$,

whenever $fx \neq ffx$ *for all* $x \in X$, $t > 0$ *and for some* $\delta \in \Delta$ *where* $\psi : \mathbf{R}^+ \to \mathbf{R}$ *is a Lebesgue integrable mapping which is summable non-negative and such that* $\int_0^{\varepsilon} \psi(s)ds > 0$ *for each* $\varepsilon > 0$.

If f and g are R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f) *or R-weakly commuting of type* (*P*), *then f and g have common fixed point.*

Proof. Since *f* and *g* are non-compatible mappings, there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$, but $\lim_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) \neq 1_{L^*}$ or non-existent for at least one *t* > 0. Since $f(X) \subseteq g(X)$, for each x_n, there exists y_n in *X* such that $fx_n = gy_n$. Thus $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} gy_n = z.$ By virtue of this and using (ii), we obtain

$$\int_0^{\zeta_{M,N}(fx_n, fy_n, t)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(gx_n, gy_n, t)} \psi(s)ds.$$

Taking $n \to \infty$, we have

$$\int_0^{\zeta_{M,N}\left(z, \lim_{n\to\infty} fy_n, t\right)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(z,z,t)} \psi(s)ds,$$

which implies that, $\lim_{n\to\infty} fy_n = z.$

Therefore, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} fy_n = z.$

Suppose that *f* and *g* are *R*-weakly commuting of type (A_g). Then, weak reciprocal continuity of *f* and *g* implies that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz.$ Similarly, $\lim_{n\to\infty} fgy_n = fz$ or $\lim_{n\to\infty} gfy_n = gz$. Let us first assume that $\lim_{n\to\infty} gfy_n = gz.$ Then *R*-weak commutativity of type (A_g) of *f* and *g* yields

$$\zeta_{M,N}(ffy_n, gfy_n, t) \geq_{L^*} \zeta_{M,N}\left(fy_n, gy_n, \frac{t}{R}\right)$$

$$\lim_{n\to\infty} \zeta_{M,N}(ffy_n, gz, t) \geq_{L^*} \zeta_{M,N}\left(z, z, \frac{t}{R}\right) = 1_{L^*}.$$

This gives $\lim_{n\to\infty} ffy_n = gz.$ Using (ii), we get

$$\int_0^{\zeta_{M,N}(ffy_n, fz, kt)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(gfy_n, gz, t)} \psi(s)ds.$$

Taking $n \to \infty,$ we have

$$\int_0^{\zeta_{M,N}(gz,\, fz,kt)} \psi(s)ds \geq_{L^*} \int_0^{\zeta_{M,N}(gz,gz,t)} \psi(s)ds.$$

This implies that $fz = gz$. Again, by virtue of R-weak commutativity of type (A_g),

$$\zeta_{M,N}\left(ffz, gfz, t\right) \geq_{L^*} \zeta_{M,N}\left(fz, gz, \frac{t}{R}\right) = 1_{L^*} \, .$$

This yields $ffz = gfz$ and $ffz = fgz = gfz = ggz$. If $fz \neq ffz,$ then by using (iii), we get

$$\int_0^{\zeta_{M,N}(\, fz,\, ffz,t)} \psi(s)ds >_{L^*} \int_0^{\Omega} \psi(s)ds,$$

where

$$\begin{aligned}\Omega &= \delta\Big(\zeta_{M,N}(gz, gfz, t), \zeta_{M,N}(\, fz, gz, t), \zeta_{M,N}(\, fz, gfz, t), \zeta_{M,N}(\, ffz, gfz, t)\Big)\\ &= \delta\Big(\zeta_{M,N}(\, fz,\, ffz, t), \zeta_{M,N}(\, fz,\, fz, t), \zeta_{M,N}(\, fz,\, ffz, t), \zeta_{M,N}(\, ffz,\, ffz, t)\Big)\\ &= \delta\Big(\zeta_{M,N}(\, fz,\, ffz, t), 1_{L^*}, \zeta_{M,N}(\, fz,\, ffz, t), 1_{L^*}\Big)\\ &= \zeta_{M,N}(\, fz,\, ffz, t).\end{aligned}$$

This gives,

$$\int_0^{\zeta_{M,N}(\, fz,\, ffz,t)} \psi(s)ds >_{L^*} \int_0^{\zeta_{M,N}(\, fz,\, ffz,t)} \psi(s)ds,$$

which is a contradiction. Hence, $fz = ffz = gfz$ and fz is a common fixed point of f and g.

Similarly, we can prove, if $\lim\limits_{n\to\infty} fgy_n = fz,$ then, again, fz is a common fixed point of f and g. Proof is similar if f and g are R-weakly commuting of type (A_f) or (P). □

SCOPE FOR FUTURE WORK

Based on present research study, it is suggested that common fixed point theorems for various other abstract spaces like complex valued metric spaces [17], convex metric spaces [26, 90], partial metric spaces [116] and Boolean-valued metric spaces [105] can be studied by making use of various other contractive conditions.

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