

**SOME METHODS FOR ANALYZING THE FUZZY  
CRITICAL PATH FOR A PROJECT NETWORK**

*Thesis submitted in partial fulfillment of the requirement for*

*The award of the degree of*

*Masters of Science*

*in*

**Mathematics and Computing**

*Submitted by*

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**JULY 2010**

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**INDIA**

**DEDICATED**

**TO**

**GOD, MY PARENTS AND MY SUPERVISOR**

## CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled “**Some methods for analyzing the fuzzy critical path for a project network**” in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Amit Kumar**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

  
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## ACKNOWLEDGEMENT

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First of all, I would like to thank the Almighty for granting perseverance. I would like to express my gratitude to **Dr. Amit Kumar, Assistant Professor, SMCA, Thapar University, Patiala**, for their patient guidance and support throughout this work. I was truly very fortunate to have the opportunity to work under him as a student. It was both an honor and a privilege to work with him. He also provides help in technical writing and presentation style and I found this guidance to be extremely valuable. I take this opportunity to express my sincere thanks to **Prof. S. S. Bhatia, Head SMCA, Thapar University, Patiala**, for their valuable support and help without which it would not have been possible for me to complete this work.

I am also thankful to **Mrs. Parampreet Kaur**, Research Scholar, SMCA and to all my friends who devoted their valuable time and helped me in all possible ways towards successful completion of this work. I do not find enough words with which I can express my feeling of thanks to the entire research scholars, faculty and staff of SMCA, Thapar University, Patiala, for their help, inspiration and moral support which went a long way in successful completion of my work. I thank all those who have contributed directly or indirectly to this work.

Lastly, and more importantly, I would like to thank my parents for their years of unyielding love and encouragement. They have always wanted the best for me and I admire my parent's determination and sacrifice to put me through college.

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# ABSTRACT

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In today's highly competitive business environment, project management's ability to schedule activities and monitor progress within strict cost, time and performance guidelines is becoming increasingly important to obtain competitive priorities such as on-time delivery and customization. When the activity times in the project are deterministic and known, critical path method (CPM) has been demonstrated to be a useful tool in managing projects in an efficient manner to meet this challenge. However, in practical situations this requirement is usually hard to fulfill, since many of the activities will be executed for the first time. So when project activity times cannot be specified with certainty due to the lack of duration information or poor definitions of the activity, then to deal with such real life situations, Zadeh (1965) introduced the concept of fuzzy set. Since there is always uncertainty about the time duration of activities in the network planning, so fuzzy critical path method (FCPM) was proposed since the late 1970s.

This thesis is devoted to critical path analysis under fuzzy environment.

The chapter-wise summary of the thesis is as follows:

**Chapter 1** is introductory in nature. This chapter includes basic concepts used throughout the work.

**Chapter 2** presents brief review of the work done in the area of finding the critical path under fuzzy environment.

In **Chapter 3**, a method based on the ranking value of a fuzzy number is presented to perform critical path analysis in a fuzzy environment. The trapezoidal fuzzy numbers, given by decision makers or characterized by historical data, are utilized to assess the activity times in a project network. To illustrate the presented

method a numerical example is solved. Presented method is applied to find fuzzy critical path of an airport's cargo ground operation system.

**Chapter 4,** In the previous chapter, a method is presented to find fuzzy critical path using ranking function. Although results obtained are mathematically correct, but the obtained results have no physical meaning, since there exists a negative part in calculated values of the latest fuzzy time and total slack fuzzy time, which represents that time may be negative. To overcome this shortcoming a new method is presented in this chapter.

**Chapter 5,** In the previous chapter, all the parameters are represented by trapezoidal fuzzy numbers, but in real life situations it is not always possible to represent all the parameters by same type of fuzzy numbers. In this chapter, a method is presented to find the fuzzy critical path of a given project network by representing the parameters by different types of fuzzy numbers.

**Chapter 6,** In the previous chapter, a method is presented to find fuzzy critical path of a given network by representing the parameters by different types of fuzzy numbers. In this chapter, an alternative method is represented to solve same type of problem. It is shown that the results of the presented method and the results obtained by using method in the previous chapter are identical while the method presented in this chapter is easy as compared to the method presented in previous chapter.

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## *Chapter 1*

### **INTRODUCTION**

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Government and industrial establishments always plan new projects for future development and expansion. A project is a collection of well defined tasks called activities and when all these activities are carried out, the project is said to be completed. The basic feature of all the projects is that these are non-repetitive and involves hundreds and thousands of activities. These projects are of big size and great complexity, e.g. construction projects like building a bridge, introduction of new project in market etc. Such projects involve large number of interrelated activities which must be completed in a specified time, in a specified sequence and require resources such as personnel, money, materials, facilities and space. All these projects require huge expenditure and time and management cannot be guided by past experience in their planning and control.

The main objective before starting a project is to schedule the required activities in an efficient manner so as to complete it on or before a specified time limit at a minimum cost of its completion. Hence, before starting any project, it is necessary to prepare a plan for scheduling and controlling the various activities involved with the given project. This will help in undertaking the project, possibly identifying bottlenecks and even discovering alternate work-plan for the project. An effective method of planning must provide a clear picture of the relationship between the activities or operations making up the project and show how delays at any particular stage will affect the remainder of the project. By this the resources can be reallocated well in advance and proper corrective action can also be taken to avoid last minute panic and delays.

When the activity times in the project are deterministic and known, critical path method (CPM) has been demonstrated to be a useful tool in managing projects in an efficient manner to meet this challenge. The purpose of CPM is to identify critical activities on the critical path so that resources may be concentrated on these activities in order to reduce the length of project period.

### **1.1 Critical path analysis**

CPM is an effective tool for scheduling and planning. A CPM is a route between two or more operations which minimize (or maximize) some measure of performance. This can also be defined as the sequence of activities which will require greatest normal time to accomplish i.e. in a critical path the sequence of activities with longest durations are singled out. It is called a critical path because any delay in activities lying on this path would cause a delay in the whole project. The objective of critical path analysis is to estimate the total project duration and to assign starting and finishing times to all activities involved in the project. This helps in checking actual progress against the scheduled duration of the project.

### **1.2 Significance of using critical path method**

In this section, significance of using CPM is discussed [30].

1. A network diagram helps translation of highly complex project into a set of simple and logical arranged activities and therefore,
  - help in the clarity of thoughts and actions.
  - help in clear and unambiguous communication developing from top to bottom and vice-versa among the people responsible for executing the project.

2. Detailed analysis of network help project incharge to peep into future because
  - difficulties and problems that can be reasonably expected to crop up during the course of execution, being foreseen ahead of actual execution.
  - delays and holdups during course of execution get minimized. Corrective action can also be taken well in time.
3. Isolates activities which control the project completion and therefore, results in expeditious completion of the project.
4. Helps in division of responsibilities and therefore, enhance effective coordination among different departments/agencies involved.
5. Helps in timely allocation of resources to various activities to achieve optimal utilization of resources.

### **1.3 Applications of critical path method**

CPM has been successfully used in many applications [29], including:

1. Constructing dam or canal system in a region.
2. Scheduling construction projects such as office building, and swimming pools.
3. Developing a countdown and “hold” procedure for the launching of space flights.
4. Installing a new computer system.
5. Designing and marketing a new product.
6. Completing a corporate merger.
7. Building a ship.
8. Developing supersonic planes.
9. Maintaining or Overhauling of aeroplanes or oil refinery.
10. Designing a prototype of a machine.

## 1.4 Basic definitions

In this section some basic definitions are reviewed [29, 30].

**Definition 1.1** A network is a graphic representation of a project's operations and is composed of activities and events that must be completed to reach the end objective of a project, showing the planning sequence of their accomplishments, their dependence and inter-relationships. The followings are some basic definitions used in network analysis.

**Definition 1.2** Any individual operation, which utilizes resources and has a beginning and an end, is called activity. An arrow is used to represent an activity with its head indicating the direction of progress in the project. These are usually classified into following four categories.

(a) **Predecessor activity:** An activity that must be completed immediately prior to the start of another activity is called predecessor activity.

(b) **Successor activity:** An activity which started immediately after one or more of other activities are completed is called successor activity.

(c) **Concurrent activity:** An activity that can be accomplished concurrently is known as concurrent activity. It may be noted that an activity can be predecessor or a successor to an event or it may be concurrent with one or more of the other activities.

(d) **Dummy activity:** An activity which does not consume either any resource and/or time is known as dummy activity.

A dummy activity in the network is added only to establish the given precedence relationship among activities of the project and is needed when (a) two or more parallel activities in a project have same head and tail events, or (b) two or more activities have

some (but not all) of their immediate predecessor activities in common. A dummy activity is depicted by dotted line in the network diagram.

**Definition 1.3** Events in the network diagram represent project milestones, such as the start or the completion of an activity or activities, and occur at a particular instant of time at which some specific part of the project has been or is to be achieved.

The events can be further classified into following two categories:

(a) **Merge Event:** An event which represents the joint completion of more than one activity is known as merge event.

(b) **Burst Event:** An event which represents the initiation of more than one activity is known as burst event.

Events in the network diagram are identified by numbers. Each event should be identified by a number higher than that allotted to its immediately preceding event to indicate progress of work.

**Definition 1.4** The earliest starting time for an activity  $(i, j)$  is the earliest time at which an activity can possibly start without delaying the project completion. It is denoted by  $ES_{ij}$ .

**Definition 1.5** The earliest finishing time for an activity  $(i, j)$  is the earliest time at which an activity can possibly finish without affecting the project completion. It is denoted by  $EF_{ij}$  and is calculated as:

$$EF_{ij} = ES_{ij} + t_{ij}, \text{ where } t_{ij} \text{ is the duration of an activity } (i, j).$$

**Definition 1.6** The earliest occurrence time of an event,  $j$  is the earliest time for an event to occur immediately after all the preceding activities have been completed without delaying the entire project. It is denoted by  $E_j$  and is calculated as:

$$E_j = \text{Maximum}_i \{ES_{ij} + t_{ij} \mid \text{for all immediate predecessor activities of node } j\}$$

where,  $ES_{ij} = E_i$ .

**Definition 1.7** The latest finishing time for an activity  $(i, j)$  is the latest time at which an activity must finish without delaying the project completion. It is denoted by  $LF_{ij}$ .

**Definition 1.8** The latest starting time for an activity  $(i, j)$  is the latest possible time at which an activity can start without delaying the project completion. It is denoted by  $LS_{ij}$  and is calculated as:

$$LS_{ij} = LF_{ij} - t_{ij}$$

**Definition 1.9** The latest allowable time of an event,  $i$  is the latest time at which an event can occur without causing a delay in already determined project's completion time. It is denoted by  $L_i$  and is calculated as:

$$L_i = \text{Minimum}_j \{LF_{ij} - t_{ij} \mid \text{for all immediate successor activities of node } i\}$$

where,  $LF_{ij} = L_j$ .

**Definition 1.10** The float is the length of time to which an activity can be delayed or extended without delaying the total project completion time. Mainly three types of floats are

are defined for each activity of the project.

**(a) Total Float:** The amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time. Mathematically, the total float of an activity  $(i, j)$  is the difference between the latest start time and earliest start time of that activity, i.e.,

$$TF_{ij} = LS_{ij} - ES_{ij} = LF_{ij} - EF_{ij}$$

**(b) Free Float:** The time by which the completion of an activity could be delayed beyond the earliest finish time without affecting the earliest start of a subsequent (succeeding) activity. Mathematically, free float of an activity  $(i, j)$  is calculated as:

$$FF_{ij} = (E_j - E_i) - t_{ij} = E_j - EF_{ij}$$

**(c) Independent Float:** The amount of time by which the start of an activity can be delayed without affecting the earliest start time of any immediately following activities, assuming that preceding activity has finished at its latest finish time. Mathematically, independent float of an activity  $(i, j)$  is calculated as:

$$IF_{ij} = E_j - L_i - t_{ij}$$

**Definition 1.11** The freedom for scheduling or to start the event is called slack. It is calculated by the difference between latest occurrence time and earliest occurrence time of that event.

$$\text{Slack} = L_i - E_i$$

**Definition 1.12** An activity is said to be critical if a delay in its start will cause a further delay in the completion of entire project.

**Definition 1.13** Critical path is the sequence of critical activities that form a continuous path between the start of a project and its completion.

**Definition 1.14** Length of the critical path is the sum of the individual times of all the critical activities lying on it and defines the longest time to complete the project.

## **1.5 Assumptions in critical path method**

The critical path method is based upon following assumptions [11].

- (a) The whole project can be divided in a set of independent and predictable activities but there are some practical constraints in starting and finishing nodes of some activities.
- (b) The precedence and succession relationship of various activities can be depicted by network diagram.
- (c) The duration of various activities can be estimated.
- (d) Duration is inversely proportional to cost of the resources allotted to the activity.

## **1.6 Methods to find critical path of a project network**

In the literature the following methods are used to determine critical path of a given project network.

1. Method based upon forward and backward pass calculations.
2. Method based upon linear programming.

### 1.6.1 Method based upon forward and backward pass calculations

In this method, the following steps are followed to find critical path of a given project network.

#### (a) Forward pass calculations (For Earliest event time)

Before starting computations, the occurrence time of initial network is fixed. Then, the forward pass computation yields the earliest start and earliest finish time for each activity  $(i, j)$ , and indirectly the earliest expected occurrence time for each event. This is mainly done in the following three steps:

**Step 1:** For easiness, the forward pass computations start by assuming the earliest starting time of zero for the initial project event

**Step 2:** Calculate  $E_j, j = 2, 3, \dots, n$  by using Definition 1.6, where  $n$  is the number of nodes

**Step 3:** Calculate  $EF_{ij}$  for each activity by using Definition 1.5

#### (b) Backward pass calculation (For Latest allowable time)

These can be computed by reversing the method of calculation used for earliest event times. This is done in the following steps:

**Step 1:** For ending node  $n$  assume  $L_n = E_n$

**Step 2:** Calculate  $L_j, j = n-1, n-2, \dots, 1$  by using Definition 1.9

### (c) Determination of critical path

Calculate total float for each activity. The activities having zero total float value are critical activities. Critical path is obtained by sequence of critical activities.

### 1.6.2 Method based upon linear programming

Although the previously described method for finding a critical path in a project network is easily programmed on a computer, linear programming can also be used to determine the length of the critical path. Define

$x_j$  = the time that the event corresponding to node  $j$  occurs.

For each activity  $(i, j)$ , we know that before node  $j$  occurs, node  $i$  must occur and activity  $(i, j)$  must be completed. This implies that for each arc  $(i, j)$  in the project network,  $x_j \geq x_i + t_{ij}$ . Let  $F$  be the node that represents completion of the project. Our goal is to minimize the time required to complete the project, so we can use an objective of  $z = x_F - x_1$ .

$$\begin{array}{ll} \text{Minimize} & x_F - x_1 \\ \text{subject to} & x_j \geq x_i + t_{ij} \quad \forall (i, j) \in A \\ & x_i \geq 0 \quad \forall i \in N \end{array}$$

where  $A$  is set of arcs and  $N$  is the set of nodes. Find the optimal solution by solving above linear programming problem. The critical activities correspond to the constraints that are satisfied as strict equations by the obtained optimal solution.

## 1.7 Fuzzy set theory

The successful implementation of critical path method requires the availability of a clear determined duration and cost for each activity, including normal durations, and normal costs. However, this requirement is sometimes difficult to fulfill in practical situations because activity durations are estimates of the actual time required, and there is liable to be a significant amount of uncertainty associated with the actual durations.

For example, it may be stated that activity A may take “about 7 days”. What does “about” really mean? What is the possible interval of values which would represent “about 7”? Could the range be from 5 to 7? or 1 to 14? The point is that the modifier “about” is fuzzy. So in these instances when project activity times cannot be specified with certainty due to the lack of duration information or poor definitions of the activity, then fuzzy sets or fuzzy numbers can be used to model the activities and their durations.

Consequently, the unknown or the vagueness about the time duration and the cost of activities in network planning has led to the development of fuzzy critical path (FCP) method and fuzzy time-cost tradeoff problem. Although the FCP method has been investigated over the past years, very few of studies take fuzzy time-cost tradeoff problem into account.

### 1.7.1 Basic definitions

In this section, some basic definitions related to fuzzy set theory are reviewed [16].

**Definition 1.15** A crisp set or a classical set  $A$  is defined as a collection of well defined objects. The objects are called elements of  $A$ .

**Definition 1.16** A crisp set  $A$ , defined on the universal set  $X$ , can also be represented by  $A = \{(x, \mu_A(x)) : x \in X\}$  where  $\mu_A(x) : X \rightarrow \{0,1\}$  is called characteristic function defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad (1.1)$$

**Definition 1.17** The characteristic function  $\mu_A$  of a crisp set  $A \subseteq X$  assigns a value either 0 or 1 to each member in  $X$ . This function can be generalized to a function  $\mu_{\tilde{A}}$  such that the value assigned to the element of the universal set  $X$  fall within a specified range  $[0,1]$  i.e.,  $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ . The assigned values indicate the membership grade of the element in the set  $A$ . The function  $\mu_{\tilde{A}}$  is called the membership function and the set  $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$  defined by  $\mu_{\tilde{A}}$  for each  $x \in X$  is called a fuzzy set.

**Definition 1.18** Let  $\tilde{A}$  be a fuzzy set and  $\alpha$  be a real number in the interval  $[0,1]$ . The crisp set  $A_\alpha$  defined by  $A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$  is called  $\alpha$ -cut of  $\tilde{A}$ . The crisp set  $A_{\alpha^+} = \{x \in X : \mu_{\tilde{A}}(x) > \alpha\}$  is called strong  $\alpha$ -cut of  $\tilde{A}$ .

**Definition 1.19** A fuzzy set  $\tilde{A}$ , defined on the universal set of real numbers  $\mathbb{R}$ , is said to be a fuzzy number if its membership function has the following characteristics:

1.  $\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0,1]$  is continuous.
2.  $\mu_{\tilde{A}}(x) = 0$  for all  $x \in (-\infty, a] \cup [d, \infty)$ .

3. is strictly increasing on  $[a, b]$  and strictly decreasing on  $[c, d]$ .
4.  $\mu_{\tilde{A}}(x) = 1$  for all  $x \in [b, c]$ .

**Definition 1.20** A fuzzy number  $\tilde{A} = (a, b, c)$ , shown in Fig. 1.1, is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \leq x \leq b \\ \frac{(c-x)}{(c-b)}, & b \leq x \leq c \end{cases} \quad \text{where } a, b, c \in \mathbb{R}. \quad (1.2)$$

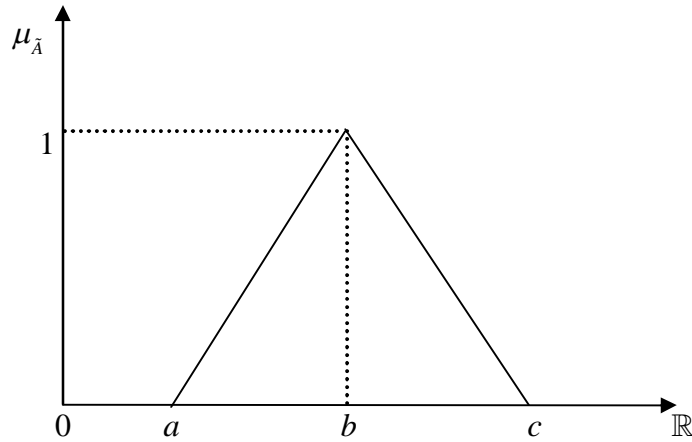


Fig. 1.1

A triangular fuzzy number

**Definition 1.21** A fuzzy number  $\tilde{A} = (a, b, c, d)$ , shown in Fig. 1.2, is said to be trapezoidal fuzzy number (TFN) if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{(d-x)}{(d-c)}, & c \leq x \leq d \\ 0, & \text{otherwise} \end{cases} \quad \text{where } a, b, c, d \in \mathbb{R} \quad (1.3)$$

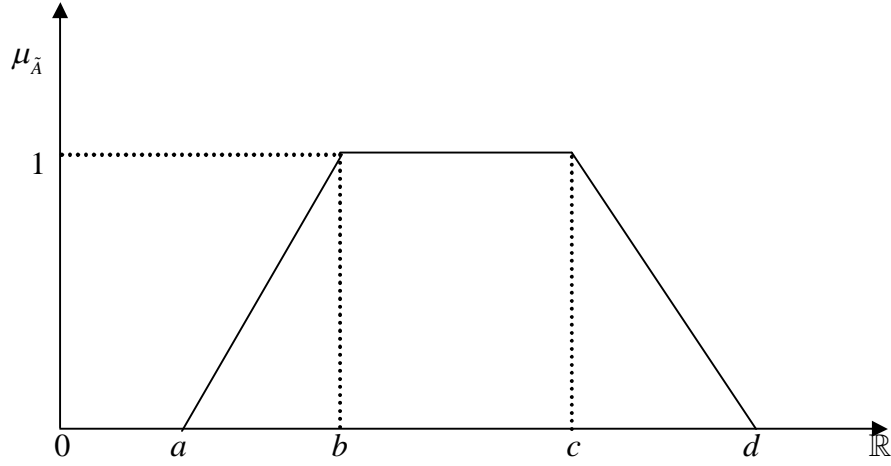


Fig. 1.2

A triangular fuzzy number

**Definition 1.22** Support of fuzzy number  $\tilde{A}$  is defined as:

$$S(\tilde{A}) = \{x \in X \mid \mu_{\tilde{A}}(x) > 0\}$$

**Definition 1.23** A fuzzy number  $\tilde{M}$  is said to be an  $L-R$  fuzzy number if

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \leq m, \alpha \geq 0 \\ R\left(\frac{x-m}{\beta}\right), & x \geq m, \beta \geq 0 \end{cases} \quad (1.4)$$

where  $m$  is the mean value of  $\tilde{M}$  and  $\alpha$  and  $\beta$  are left and right spreads, respectively,

and a function  $L(\cdot)$  the left shape function satisfying:

$$(1) L(x) = L(-x)$$

$$(2) L(0) = 1 \text{ and } L(1) = 0$$

$$(3) L(x) \text{ is non-increasing on } [0,1)$$

Naturally, a right shape function  $R(\cdot)$  is similarly defined as  $L(\cdot)$ . Using its mean value and left and right spreads, and shape functions, such a  $L-R$  fuzzy number  $\tilde{M}$  is symbolically written as:

$$\tilde{M} = (m, \alpha, \beta)_{L-R}$$

Clearly,  $\tilde{M} = (m, \alpha, \beta)_{L-R}$  is positive, if and only if,  $m - \alpha > 0$ , (note that  $L(1) = 0$ ).

Consider the following well-known functions, as examples of typical shape functions:

$$L(x) = \text{Maximum}(0, 1 - |x|^p), \quad p > 0$$

$$L(x) = e^{-|x|^p}, \quad p > 0$$

$$L(x) = e^{-px}, \quad p > 0$$

$$L(x) = \frac{1}{(1 - |x|^p)}, \quad p > 0$$

If  $L(x)$  and  $R(x)$  be linear functions, then the corresponding  $L-R$  number is said to be a triangular fuzzy number. Note that we use a fixed function  $L(\cdot)$  and a fixed function  $R(\cdot)$  for all fuzzy numbers in each problem.

### 1.7.2 Arithmetic operations

In this section, addition and subtraction operations between TFNs are reviewed [16].

Let  $\tilde{A} = (a_1, b_1, c_1, d_1)$  and  $\tilde{B} = (a_2, b_2, c_2, d_2)$  be any two TFNs, then:

Addition  $\oplus$ :

$$\tilde{A} \oplus \tilde{B} = (a_1, b_1, c_1, d_1) \oplus (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

Subtraction  $\ominus$ :

$$\tilde{A} \ominus \tilde{B} = (a_1, b_1, c_1, d_1) \ominus (a_2, b_2, c_2, d_2) = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$$

## *Chapter 2*

### **LITERATURE REVIEW**

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Gazdik [10] developed a fuzzy network of a priori unknown project to estimate the activity durations and used fuzzy algebraic operators to calculate the duration of the project and its critical path.

Kaufmann and Gupta [16] devoted a chapter of their book to the critical path method in which activity times are represented by triangular fuzzy numbers. A six step procedure is summarized for developing activity estimates, determining activity float times, and identifying the critical path.

Chang et al. [4] combined the composite and comparison methods of analyzing fuzzy numbers into an efficient procedure for solving project scheduling problems. The comparison method first eliminates activities that are not on highly critical paths. The composite method then determines the path with the highest degree of criticality. The fuzzy Delphi method [16] is used to determine the activity time estimates. The solution procedure is demonstrated in a 9 node, 14 activity project scheduling problem with activity times represented by triangular fuzzy numbers.

Nasution [26] argued that for a given  $\alpha$ -cut level of the slack, the availability of the fuzzy slack in critical path models provides sufficient information to determine the critical path. A fuzzy procedure utilizing interactive fuzzy subtraction is used to compute the latest allowable time and slack for activities. The procedure is demonstrated for a ten event network where activity times are represented by trapezoidal fuzzy numbers.

Hapke et al. [13] presented a fuzzy project scheduling decision support system. The fuzzy project scheduling system is used to allocate resources among dependent activities in a software project scheduling environment. The fuzzy project scheduling system used *L-R* type flat fuzzy numbers to model uncertain activity durations. Expected project completion time and maximum lateness are identified as the project performance measures and a sample problem is demonstrated for a software engineering project involving 53 activities. The fuzzy project scheduling system presented allows the estimation of project completion times and the ability to analyze the risk associated with overstepping the required project completion time.

Lorterapong [25] introduced a resource-constrained project scheduling method that addresses three performance objectives: (1) expected project completion time; (2) resource utilization; and (3) resource interruption. Fuzzy set theory is used to model the vagueness that is inherent with linguistic descriptions often used by people when describing activity durations. The analysis presented provides a framework for allocating resources in an uncertain project environment.

Chen et al. [7] incorporated time-window constraint and time schedule constraint into the traditional activity network. They developed a linear time algorithm for finding the critical path in an activity network with these time-constraints.

Yao and Lin [38] proposed a method for ranking fuzzy numbers without the need for any assumptions and used both positive and negative values to define ordering which then is applied to critical path method. They made a fuzzy activity network using triangular fuzzy numbers to represent the fuzzy data for the duration of each activity. Then they eliminated the fuzziness using the signed distance ranking for those fuzzy numbers to co-

construct the activity network, in the fuzzy sense. After that the fuzzy critical path can easily be obtained.

Chanas and Zielinski [2] introduced the concept of criticality in the network with fuzzy activity times. They also presented two methods of calculation of the path degree of criticality (according to the proposed concept of fuzzy criticality).

Lin [21] presented a new approach to a fuzzy critical path method for activity networks based on statistical confidence interval estimates and a ranking method for level  $(1-\alpha)$  fuzzy numbers. His focus was to introduce an approach that combined fuzzy set theory with statistics that includes the signed distance ranking of level  $(1-\alpha)$  fuzzy numbers derived from  $(1-\alpha)*100\%$ .

Kuchta [19] defined a fuzzy way of measuring the criticality of project activities and of the whole project. For this, he took into account the decision maker's attitude and project network structure. He argued that the criticality measure obtained may serve as a measure of risk or of the supervision effort needed and can help in making the decision whether to accept or to reject the project.

Chanas et al. [1] introduced and analyzed the notion of the necessary criticality (both with respect to path and to activity) of a network with imprecisely defined (by means of intervals or fuzzy intervals) activity duration times. They also proposed algorithms of calculating the degree of the necessary criticality of paths.

Lin [22] introduced a new approach to fuzzy critical path method that based on statistical confidence interval estimates which is the extension of crisp activity network. He also focused to introduce the approach to combine statistics with fuzzy mathematics, which includes the signed-distance ranking of level  $(1-\alpha)$  fuzzy numbers.

Chang et al. [3] proposed a method to systematically determine the critical path under the work flow model and give an overall example that shows how the method works.

Lin and Yao [24] proposed an approach that combines fuzzy mathematics with statistics to solve practical problems in unknown or vague situations. They introduced a fuzzy critical path method based on statistical confidence-interval estimates and a signed distance ranking for  $(1-\alpha)$  fuzzy number levels.

Dubois et al. [8] presented the basis for a correct calculation of latest starting dates, slack times and criticality degrees of tasks in task network with fuzzy processing times. They showed that this problem is rather easy to solve for particular network topologies namely series-parallel graphs.

Liang and Han [20] presented an algorithm to perform fuzzy critical path analysis for project network problem. By using this algorithm, they showed that the ambiguities involved in the assessment activity times in a project network can be effectively improved and thus a more convincing and effective project management decision-making can be obtained.

Zielinski [40] extended some results for interval numbers to the fuzzy case for determining the possibility distributions describing latest starting times for activities. He also proposed the time algorithms for computing the intervals of the possible values of the latest starting times of an activity in general networks with interval durations and extended the results to the networks with fuzzy durations.

Jassbi and Khanmohammadi [14] introduced a new approach based on beta membership functions for estimated durations and the delays of activities. Also they showed

that beta shape fuzzy values are used as the possibility of performing activities at estimated times.

Han et al. [12] used the trapezoidal fuzzy numbers to make the fuzzy measures of activity times characterized by linguistic value and proposed an algorithm for finding the fuzzy critical path of a project network. They also introduced the method that reduce the complexity of airport's ground operation model development and computations for solving problems, as well as incorporate the decision maker's risk attitude into the decision process. This method is utilized to perform critical path analysis for the Chiang Kai-shek airport's ground operation work.

Tian and Li [33] presented a method of modelling the well-define collaboration process based on workflow technology, probability theory and fuzzy-time high level Petri nets. They also offered theoretic basis and available means of time management, resource scheduling and conflict solving in collaborative work process.

Yakhchali et al. [34] focused on the full picture of necessarily critical path in the network with interval and fuzzy time lags. They generalized the results obtained for networks with interval activity duration times and time lags to the case of networks with fuzzy activity duration times. For this purpose, they implemented an algorithm for calculating the degree of necessary criticality of a path. They also proposed a linear programming approach to determine the necessity degree that a path is critical in the network with fuzzy time lags.

Yakhchali et al. [35] assumed time lags which are common practice in the different projects, are imprecise and they discussed the problems of possibly critical paths in the networks with interval-valued activity and time lag durations.

Chen [5] proposed an approach to critical path analysis for a project network with activity times being fuzzy numbers, in that membership function of fuzzy total duration time is constructed which is based on the extension principle and linear programming.

Koo et al. [18] presented a formal identification and resequencing process to support the correct and rapid development of sequencing alternatives in construction schedules.

Chen and Hsueh [6] presented a simple approach to solve the critical path method problem with fuzzy activity times (being fuzzy numbers) on the basis of the linear programming formulation and the fuzzy number ranking method that are more realistic than crisp ones. They also defined that the most critical path and the relative path degree of criticality which are easy to use in practice.

Yakhchali et al. [37] introduced the algorithms for computing of the interval value of the latest starting times and maximal floats of activities in the network with interval activity and time lag durations. They also compared the complexity of proposed algorithm with former algorithm. They also generalized the interval activity and time lag durations into fuzzy numbers.

Lin [23] presented a fuzzy approach based on statistical confidence interval estimates and a distance ranking method for  $(1-\alpha)$  fuzzy number levels. He also defined a theorem based on statistical confidence interval estimates through which the fuzzy critical path and fuzzy time-cost tradeoff problems can be solved efficiently.

Sharafi et al. [28] presented a new method based on fuzzy theory for solving fuzzy project scheduling in fuzzy environment. Also into this model, they assumed for first time that the relationship between activities are not crisp and are supposed fuzzy num-

bers. They have proposed a method that calculates all parameters of project such as earliest and latest start and finish time and slack times.

Eshtehardian et al. [9] introduced a new approach for scheduling problem, considering uncertainties in activities execution time by using fuzzy set theory. They showed that the project manager may apply his own risk acceptance level to obtain a critical path with new critical activities using  $\alpha$ -cut property. They also demonstrated that the project manager could reflect his/her degree of optimism with choosing different value of optimism index.

Yakhchali and Ghodsypour [36] introduced the problems of determining possible values of earliest and latest starting times of an activity in networks with minimal time lags and imprecise durations that are represented by means of interval or fuzzy numbers.

Sireesha and Shankar [31] presented a new method based on fuzzy theory for solving fuzzy project scheduling in fuzzy environment. Assuming that the duration of activities are triangular fuzzy numbers, they computed total float time of each activity and fuzzy critical path without computing forward and backward pass calculations.

Shankar et al. [27] presented an analytical method for measuring the criticality in a fuzzy project network, where the duration time of each activity is represented by a trapezoidal fuzzy number. He used a new defuzzification formula for trapezoidal fuzzy number and apply to the float time (slack time) for each activity in the fuzzy project network to find the critical path.

## Chapter 3

# A RANKING VALUE BASED METHOD FOR ANALYZING THE FUZZY CRITICAL PATH

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In this chapter a method based on the ranking value of a fuzzy number is presented to perform critical path analysis in a fuzzy environment. TFNs given by decision makers or characterized by historical data, are utilized to assess the activity times in a project network. By using this algorithm, the ambiguities involved in the assessment activity times in a project network can be effectively improved and thus a more convincing and effective project management decision-making can be obtained. To illustrate the presented method a numerical example is solved. Presented method is applied to find FCP of an airport's cargo ground operation system.

### 3.1 Ranking value of a fuzzy number

Let  $\tilde{A}_i = (a_i, b_i, c_i, d_i)$ ,  $i = 1, 2, \dots, n$ , be fuzzy numbers with membership functions  $\mu_{\tilde{A}_i}$  respectively. Let  $x_1 = \text{Infimum } D$ ,  $x_2 = \text{Supremum } D$ , and  $D_i = \{x \mid \mu_{\tilde{A}_i}(x) > 0\}$ ,  $i = 1, 2, \dots, n$

Then ranking value of fuzzy number  $\tilde{A}_i$ ,  $R(\tilde{A}_i)$ , is defined as

$$R(\tilde{A}_i) = \beta[(d_i - x_1)/(x_2 - x_1 - c_i + d_i)] + (1 - \beta)[1 - (x_2 - a_i)/(x_2 - x_1 + b_i - a_i)] \quad (3.1)$$

The value of  $\beta$  can be referred to as decision maker's risk index. If  $\beta < 0.5$ , it implies that the decision maker is a risk averter. If  $\beta = 0.5$ , it implies that the risk attitude of decision maker is neutral. If  $\beta > 0.5$ , it implies that the decision maker is risk lover. For a FCP

analysis problem, using the TFNs such as  $\tilde{FET}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$  to denote the fuzzy activity time of activity  $\tilde{A}_{ij}$ , the decision maker's risk attitude index  $\beta$  can be obtained by

$$\beta = \left[ \sum_i \sum_j \frac{b_{ij} - a_{ij}}{\left( (b_{ij} - a_{ij}) + (d_{ij} - c_{ij}) \right)} \right] / t, \quad \tilde{A}_{ij} \in ACT., \quad (3.2)$$

where  $ACT$  and  $t$  denote the set of all activities and the number of activities in a project network, respectively. For fuzzy number  $\tilde{A}_i$  with membership functions  $\mu_{\tilde{A}_i}(x)$  define

$$m_i = \text{minimum}\{x \mid \mu_{\tilde{A}_i}(x) = 1\} + \text{maximum}\{x \mid \mu_{\tilde{A}_i}(x) = 1\}$$

Now we rank the fuzzy numbers  $\tilde{A}_i$  and  $\tilde{A}_j$  according to the following rules:

$$\tilde{A}_i > \tilde{A}_j \Leftrightarrow R(\tilde{A}_i) > R(\tilde{A}_j),$$

$$\text{or, } R(\tilde{A}_i) = R(\tilde{A}_j) \text{ and } m_i > m_j,$$

$$\tilde{A}_i = \tilde{A}_j \Leftrightarrow R(\tilde{A}_i) = R(\tilde{A}_j) \text{ and } m_i = m_j$$

We can write  $x_1$  as  $x_1 = \text{Minimum}\{a_1, a_2, \dots, a_n\}$  and  $x_2$  as  $x_2 = \text{Maximum}\{d_1, d_2, \dots, d_n\}$ .

Taking  $\beta$  value calculated by equation (3.2) and by using equation (3.1), we can easily calculate the ranking values of the  $n$  TFNs. Then based on the ranking rules described above, the ranking of the  $n$  TFNs can be effectively determined.

## 3.2 Fuzzy critical path analysis

In this section, a method to analyze the FCP for a project network is presented.

### 3.2.1 Notations

The notations that will be used in the presented method are as follows:

$N$  : The set of all nodes in a project network.

$A_{ij}$  : The activity between nodes  $i$  and  $j$ .

$\widetilde{FET}_{ij}$  : The fuzzy activity time of  $A_{ij}$ .

$\widetilde{FES}_j$  : The earliest fuzzy time of node  $j$ .

$\widetilde{FLF}_j$  : The latest fuzzy time of node  $j$ .

$\widetilde{FTS}_{ij}$  : The total slack fuzzy time of  $A_{ij}$ .

$S(j)$  : The set of all successor activities of node  $j$ .

$NS(j)$  : The set of all nodes connected to all successor activities of node  $j$ , i.e.,

$$NS(j) = \{k \mid A_{jk} \in S(j), k \in N\}$$

$F(j)$  : The set of all predecessor activities of node  $j$ .

$NP(j)$  : The set of all nodes connected to all predecessor activities of node  $j$ , i.e.,

$$NP(j) = \{i \mid A_{ij} \in N\}$$

$P_i$  : The  $i^{\text{th}}$  path.

$P$  : The set of all paths in a project network.

$\widetilde{FCPM}(P_k)$  : The fuzzy completion time of path  $P_k$  in a project network.

### 3.2.2 Some important properties

In this section, some properties that will be used in this method for analyzing the FCP are presented. Set the initial node to be zero for starting i.e.  $\widetilde{FES}_1 = (0, 0, 0, 0)$ . Then, the following properties are true:

**Property 3.1:**  $\widetilde{FES}_j = \text{Maximum}\{\widetilde{FES}_i \oplus \widetilde{FET}_{ij} \mid i \in NP(j), j \neq 1, j \in N\}$

**Property 3.2:**  $\widetilde{FLF}_j = \text{Minimum}\{\widetilde{FLF}_k \ominus \widetilde{FET}_{jk} \mid k \in NS(j), j \neq n, j \in N\}$

**Property 3.3:**  $\widetilde{FTS}_{ij} = \widetilde{FLF}_j \ominus (\widetilde{FES}_i \oplus \widetilde{FET}_{ij}), 1 \leq i < j \leq n; i, j \in N$

**Property 3.4:**  $\widetilde{FCPM}(P_k) = \sum_{\substack{1 \leq i < j \leq n \\ i, j \in P_k}} \widetilde{FTS}_{ij}, P_k \in P$

**Definition 3.1** Assume that there exists a path  $P_c$  in a project network such that

$$\widetilde{FCPM}(P_c) = \text{Minimum}\{\widetilde{FCPM}(P_i) \mid P_i \in P\}$$

then the path  $P_c$  is a FCP.

### 3.2.3 Method for finding the fuzzy critical path

The FCP of a project network can be obtained by using the following steps:

1. Identify activities in a project
2. Establish precedence relationships of all activities
3. Estimate the fuzzy activity time with respect to each activity
4. Construct the project network
5. Let  $\widetilde{FES}_1 = (0, 0, 0, 0)$  and calculate  $\widetilde{FES}_j, j = 2, 3, \dots, n$ , by using property 3.1
6. Let  $\widetilde{FLF}_n = \widetilde{FES}_n$  and calculate  $\widetilde{FLF}_j, j = n-1, n-2, \dots, 2, 1$ , by using property 3.2
7. Calculate  $\widetilde{FTS}_{ij}$  with respect to each activity in a project network by using property 3.3
8. Find all the possible paths and calculate  $\widetilde{FCPM}(P_k)$  by using property 3.4
9. Find the FCP by using Definition 3.1
10. Find the grade of membership that the project can be completed at scheduled time

### 3.2.4 Numerical example

To illustrate the method described in the previous section a numerical example is solved.

**Example 3.1:** Suppose there is a project network, as shown in Fig. 3.1, with the set of node  $N = \{1, 2, 3, 4\}$ , the fuzzy activity time for each activity as shown in Table 3.1. All of the durations are in hours. Find the FCP for the given network.

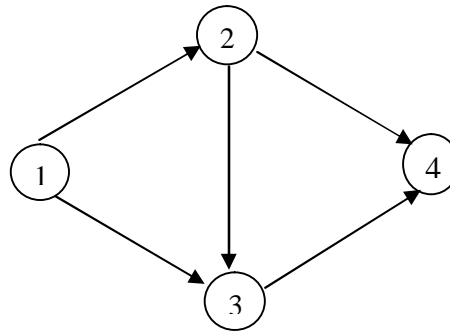


Fig. 3.1

A project network

Table 3.1 Fuzzy activity time for each activity

Activity $A_{ij}$	Fuzzy activity times (Hours) $\widetilde{FET}_{ij}$
$A_{12}$	Approximately 5 Hours (3, 5, 5, 7)
$A_{13}$	Approximately 10 Hours (5, 10, 10, 15)
$A_{23}$	Approximately between 3 and 4 Hours (1, 3, 4, 5)
$A_{24}$	Approximately between 4 and 5 Hours (2, 4, 5, 6)
$A_{34}$	Approximately between 8 and 10 Hours (6, 8, 10, 11)

By using equation (3.2),

$$\beta = \left( \left( \frac{(5-3)}{(5-3)+(7-5)} \right) + \left( \frac{(10-5)}{(10-5)+(15-10)} \right) + \left( \frac{(3-1)}{(3-1)+(5-4)} \right) \right. \\ \left. + \left( \frac{(4-2)}{(4-2)+(6-5)} \right) + \left( \frac{(8-6)}{(8-6)+(11-10)} \right) \right) / 5 = 0.6$$

The FCP of the given network, shown in Fig. 3.1, can be obtained by using the following steps:

**Step 1:** Set  $\tilde{FES}_1 = (0,0,0,0)$  and calculate  $\tilde{FES}_j, j = 2,3,4$ , by using property 3.1

$$\tilde{FES}_2 = \tilde{FES}_1 \oplus \tilde{FET}_{12} = (0,0,0,0) \oplus (3,5,5,7) = (3,5,5,7)$$

$$\tilde{FES}_3 = \text{Maximum}\{\tilde{FES}_1 \oplus \tilde{FET}_{13}, \tilde{FES}_2 \oplus \tilde{FET}_{23}\} = \text{Maximum}\{(5,10,10,15), (4,8,9,12)\}$$

Taking  $\beta = 0.6$ , the ranking value of  $(5,10,10,15)$  and  $(4,8,9,12)$  can be obtained:

$$x_1 = \text{Minimum}\{5,4\} = 4, \text{ and } x_2 = \text{Maximum}\{15,12\} = 15$$

$$R((5,10,10,15)) = 0.6[(15-4)/(15-4-10+15)] + 0.4[1-(15-5)/(15-4+10-5)] = 0.5625$$

$$R((4,8,9,12)) = 0.6[(12-4)/(15-4-9+12)] + 0.4[1-(15-4)/(15-4+8-4)] = 0.4495$$

Since  $R((5,10,10,15)) > R((4,8,9,12))$ ,  $\tilde{FES}_3 = (5,10,10,15)$

$$\tilde{FES}_4 = \text{Maximum}\{\tilde{FES}_2 \oplus \tilde{FET}_{24}, \tilde{FES}_3 \oplus \tilde{FET}_{34}\} = \text{Maximum}\{(5,9,10,13), (11,18,20,26)\}$$

By using equation (3.1) and taking  $\beta = 0.6$ ,  $\tilde{FES}_4 = (11,18,20,26)$

**Step 2:** Set  $\tilde{FLF}_4 = (11,18,20,26)$  and calculate  $\tilde{FLF}_j, j = 3,2,1$ , by using property 3.2

$$\tilde{FLF}_3 = \tilde{FLF}_4 \ominus \tilde{FET}_{34} = (11,18,20,26) \ominus (6,8,10,11) = (0,8,12,20)$$

$$\tilde{FLF}_2 = \text{Minimum}\{\tilde{FLF}_4 \ominus \tilde{FET}_{24}, \tilde{FLF}_3 \ominus \tilde{FET}_{23}\} = \text{Minimum}\{(5,13,16,24), (-5,4,9,19)\}$$

Taking  $\beta = 0.6$ ,  $\widetilde{FLF}_2 = (-5, 4, 9, 19)$

$$\widetilde{FLF}_1 = \text{Minimum}\{\widetilde{FLF}_3 \ominus \widetilde{FET}_{13}, \widetilde{FLF}_2 \ominus \widetilde{FET}_{12}\} = (-15, -2, 2, 15)$$

**Step 3:** Calculate  $\widetilde{FTS}_{ij}$  with respect to each activity by property 3.3

$$\widetilde{FTS}_{12} = \widetilde{FLF}_2 \ominus (\widetilde{FES}_1 \oplus \widetilde{FET}_{12}) = (-12, -1, 4, 16)$$

$$\widetilde{FTS}_{13} = \widetilde{FLF}_3 \ominus (\widetilde{FES}_1 \oplus \widetilde{FET}_{13}) = (-15, -2, 2, 15)$$

$$\widetilde{FTS}_{23} = \widetilde{FLF}_3 \ominus (\widetilde{FES}_2 \oplus \widetilde{FET}_{23}) = (-12, -1, 4, 16)$$

$$\widetilde{FTS}_{24} = \widetilde{FLF}_4 \ominus (\widetilde{FES}_2 \oplus \widetilde{FET}_{24}) = (-2, 8, 11, 21)$$

$$\widetilde{FTS}_{34} = \widetilde{FLF}_4 \ominus (\widetilde{FES}_3 \oplus \widetilde{FET}_{34}) = (-15, -2, 2, 15)$$

**Step 4:** Find all the possible paths and calculate  $\widetilde{FCPM}(P_k)$  by using property 3.4

$$P = \{(1, 2, 4), (1, 2, 3, 4), (1, 3, 4)\}$$

$$1. P_1 = (1, 2, 4) \quad \widetilde{FCPM}(P_1) = \widetilde{FTS}_{12} \oplus \widetilde{FTS}_{24} = (-14, 7, 15, 37)$$

$$2. P_2 = (1, 2, 3, 4) \quad \widetilde{FCPM}(P_2) = \widetilde{FTS}_{12} \oplus \widetilde{FTS}_{23} \oplus \widetilde{FTS}_{34} = (-39, -4, 10, 47)$$

$$3. P_3 = (1, 3, 4) \quad \widetilde{FCPM}(P_3) = \widetilde{FTS}_{13} \oplus \widetilde{FTS}_{34} = (-30, -4, 4, 30)$$

**Step 5:** Find the FCP by taking  $\beta = 0.6$ . The ranking value of  $\widetilde{FCPM}(P_i)$ ,  $i = 1, 2, 3$ , can be obtained.

$$R\left(\widetilde{FCPM}(P_1)\right) = 0.5942$$

$$R\left(\widetilde{FCPM}(P_2)\right) = 0.5352$$

$$R\left(\tilde{FCPM}(P_3)\right) = 0.4946$$

Since  $R\left(\tilde{FCPM}(P_3)\right) < R\left(\tilde{FCPM}(P_2)\right) < R\left(\tilde{FCPM}(P_1)\right)$  the FCP is  $P_3$  and the project completion time is approximately between 18 and 20 hours, i.e. (11,18,20,26) and the grade of membership at any scheduled time can be obtained. For example, the grade of membership that the project's completion time being within 15 hours is 0.5714.

### 3.3 Application of method to airport's cargo ground operation system

As the volume of cargo traffic has grown and the demand for cargo transport continues to rise, surface congestion has become an increasing problem, within an airport's cargo terminal. If an airport terminal's internal operations and service systems are inefficient, there will be a delay in ground operations. Therefore, cargo operations' time needs to be shortened and passengers' luggage must be processed before cargo goods in order to maintain customer satisfaction. To raise the profitability, and the efficiency of the airport cargo terminal, use of FCP analysis will help out. Fig. 3.2 shows an international airport cargo terminal's ground operation procedures network. With the set of node  $N = \{1,2,3,4,5\}$ , the fuzzy activity time for each activity is shown in Table 3.2.

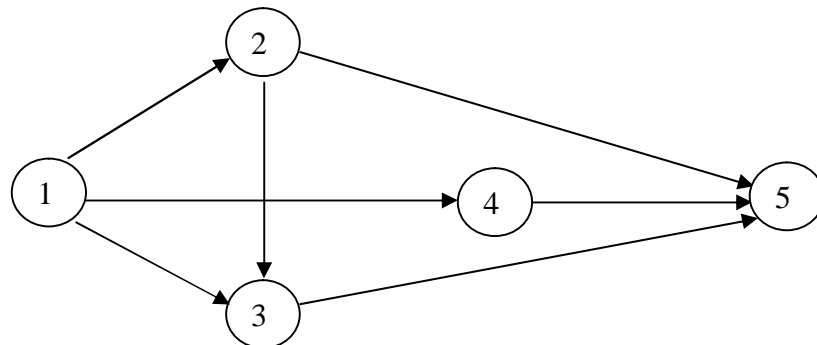


Fig. 3.2

International airport cargo terminal's ground operation procedures

Table 3.2 Fuzzy activity time for each activity

Activity $A_{ij}$	Description	Fuzzy activity time (Minutes) $\tilde{FET}_{ij}$
$A_{12}$	Customs office cargo clearance with document approval (Named $C_2$ )	Approximately 15 minutes (10,15,15,20)
$A_{13}$	Customs office cargo clearance with inspection (Named $C_3$ )	Approximately 40 minutes (30,40,40,50)
$A_{23}$	Customs office cargo clearance with document approval and inspection (Named $C_2, C_3$ )	Approximately between 30 and 60 minutes (30,40,50,60)
$A_{14}$	Customs office inspection-exempt cargo clearance (Named $C_1$ )	Approximately between 15 and 30 minutes (15,20,25,30)
$A_{25}$	Customs office after cargo clearance with document approval, releasing cargo and packing cargo waiting for loading	Approximately between 60 and 180 minutes (60,100,150,180)
$A_{35}$	Customs office after cargo clearance with inspection, releasing cargo and packing cargo waiting for loading	Approximately between 60 and 180 minutes (60,100,150,180)
$A_{45}$	Customs office after inspection-exempt cargo clearance, releasing cargo waiting for loading	Approximately between 60 and 180 minutes (60,100,150,180)

**Solution:** The FCP of the network, shown in Fig. 3.2, can be obtained as:

**Step 1:** Set  $\tilde{FES}_1 = (0,0,0,0)$  and calculate  $\tilde{FES}_j, j = 2,3,4,5$  by using property 3.1

$$\tilde{FES}_2 = \tilde{FES}_1 \oplus \tilde{FET}_{12} = (0,0,0,0) \oplus (10,25,15,20) = (10,15,15,20)$$

$$\begin{aligned} \tilde{FES}_3 &= \text{Maximum}\{\tilde{FES}_1 \oplus \tilde{FET}_{13}, \tilde{FES}_2 \oplus \tilde{FET}_{23}\} \\ &= \text{Maximum}\{(30,40,40,50), (40,55,65,80)\} \end{aligned}$$

Using equation (3.2), we get  $\beta = 0.5306$ , the ranking value of (30,40,40,50) and (40,55,65,80) can be obtained as:  $x_1 = \text{Minimum}\{30,40\} = 30, x_2 = \text{Maximum}\{50,80\} = 80$

$$\begin{aligned}
R((30,40,40,50)) &= 0.5306[(50-30)/(80-30-40+50)] \\
&\quad + 0.4694[1-(80-30)/(80-30+40-30)] \\
&= 0.2551
\end{aligned}$$

$$\begin{aligned}
R((40,55,65,80)) &= 0.5306[(80-30)/(80-30+65-80)] \\
&\quad + 0.4694[1-(80-40)/(80-30+55-40)] \\
&= 0.5887
\end{aligned}$$

Since  $R((40,55,65,80)) > R((30,40,40,50))$ ,  $\tilde{FES}_3 = (40,55,65,80)$

$$\tilde{FES}_4 = \tilde{FES}_1 \oplus \tilde{FET}_{14} = (0,0,0,0) \oplus (15,20,25,30) = (15,20,25,30)$$

$$\begin{aligned}
\tilde{FES}_5 &= \text{Maximum}\{\tilde{FES}_2 \oplus \tilde{FET}_{25}, \tilde{FES}_3 \oplus \tilde{FET}_{35}, \tilde{FES}_4 \oplus \tilde{FET}_{45}\} \\
&= (100,155,215,260)
\end{aligned}$$

**Step 2:** Set  $\tilde{FLF}_5 = (100,155,215,260)$  and calculate  $\tilde{FLF}_j$ ,  $j = 4,3,2,1$  by using property

3.2

$$\tilde{FLF}_4 = \tilde{FLF}_5 \ominus \tilde{FET}_{45} = (100,155,215,260) \ominus (60,100,150,180) = (-80,5,115,200)$$

$$\tilde{FLF}_3 = \tilde{FLF}_5 \ominus \tilde{FET}_{35} = (-80,5,115,200)$$

$$\tilde{FLF}_2 = \text{Minimum}\{\tilde{FLF}_5 \ominus \tilde{FET}_{25}, \tilde{FLF}_3 \ominus \tilde{FET}_{23}\}$$

$$\tilde{FLF}_2 = \text{Minimum}\{(-80,5,115,200), (-140,-45,75,170)\} = (-140,-45,75,170)$$

$$\tilde{FLF}_1 = (-160,-60,60,160)$$

**Step 3:** Calculate  $\tilde{FTS}_{ij}$  with respect to each activity by property 3.3

$$\tilde{FTS}_{12} = \tilde{FLF}_2 \ominus (\tilde{FES}_1 \oplus \tilde{FET}_{12})$$

$$\widetilde{FTS}_{12} = (-140, -45, 75, 170) \ominus ((0, 0, 0, 0) \oplus (10, 15, 15, 20)) = (-160, -60, 60, 160)$$

$$\widetilde{FTS}_{13} = \widetilde{FLF}_3 \ominus (\widetilde{FES}_1 \oplus \widetilde{FET}_{13}) = (-130, -35, 75, 170)$$

$$\widetilde{FTS}_{14} = \widetilde{FLF}_4 \ominus (\widetilde{FES}_1 \oplus \widetilde{FET}_{14}) = (-110, -20, 95, 185)$$

$$\widetilde{FTS}_{23} = \widetilde{FLF}_3 \ominus (\widetilde{FES}_2 \oplus \widetilde{FET}_{23}) = (-160, -60, 60, 160)$$

$$\widetilde{FTS}_{25} = \widetilde{FLF}_5 \ominus (\widetilde{FES}_2 \oplus \widetilde{FET}_{25}) = (-100, -10, 100, 190)$$

$$\widetilde{FTS}_{35} = \widetilde{FLF}_5 \ominus (\widetilde{FES}_3 \oplus \widetilde{FET}_{35}) = (-160, -60, 60, 160)$$

$$\widetilde{FTS}_{45} = \widetilde{FLF}_5 \ominus (\widetilde{FES}_4 \oplus \widetilde{FET}_{45}) = (-110, -20, 95, 185)$$

**Step 4:** Find all possible paths and calculate  $\widetilde{FCPM}(P_k)$  by using property 3.4

$$P = \{(1, 2, 5), (1, 2, 3, 5), (1, 3, 5), (1, 4, 5)\}$$

$$P_1 = (1, 2, 5) \quad \widetilde{FCPM}(P_1) = \widetilde{FTS}_{12} \oplus \widetilde{FTS}_{25} = (-260, -70, 160, 350)$$

$$P_2 = (1, 2, 3, 5) \quad \widetilde{FCPM}(P_2) = \widetilde{FTS}_{12} \oplus \widetilde{FTS}_{23} \oplus \widetilde{FTS}_{35} = (-480, -180, 180, 480)$$

$$P_3 = (1, 3, 5) \quad \widetilde{FCPM}(P_3) = \widetilde{FTS}_{13} \oplus \widetilde{FTS}_{35} = (-290, -95, 135, 330)$$

$$P_4 = (1, 4, 5) \quad \widetilde{FCPM}(P_4) = \widetilde{FTS}_{14} \oplus \widetilde{FTS}_{45} = (-220, -40, 190, 370)$$

**Step 5:** Find the FCP, taking  $\beta = 0.5306$ , the ranking value of  $\widetilde{FCPM}(P_i), i = 1, 2, 3, 4$  can be obtained.

$$R(\widetilde{FCPM}(P_1)) = 0.5503$$

$$R(\widetilde{FCPM}(P_2)) = 0.516$$

$$R(\tilde{FCPM}(P_3)) = 0.5286$$

$$R(\tilde{FCPM}(P_4)) = 0.5768$$

Since  $R(\tilde{FCPM}(P_2)) < R(\tilde{FCPM}(P_3)) < R(\tilde{FCPM}(P_1)) < R(\tilde{FCPM}(P_4))$ , the FCP is  $P_2$  and airport's ground operation completion time is approximately between 155 and 255 minutes. As it can be seen from this analysis, the path where cargo ground service operations influence combination carriers is  $P_2$ . If carriers are assigned through this path for ground service operations to export by authorization, the average export operations time wasted will be at its longest.

### 3.4 Conclusion

In project evaluation analysis, it is very often that information available for making decision is vague and uncertain. Therefore, it is rather difficult to obtain exact activity assessment data. This chapter proposes an algorithm to tackle the problem in fuzzy project decision analysis. Thus by conducting fuzzy or non-fuzzy activity time assessments, the decision-makers can obtain the FCP automatically.

## Chapter 4

# A PROJECT SCHEDULING METHOD FOR ANALYZING THE FUZZY CRITICAL PATH

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In the last chapter, a method is presented to find FCP using ranking function. Although results obtained in last chapter are mathematically correct, but the obtained results have no physical meaning, since there exists a negative part in calculated values of the latest fuzzy time and total slack fuzzy time, which represents that time may be negative. To overcome this shortcoming, a new method is presented in this chapter.

The method presented in this chapter is based on fuzzy theory. Assuming that the duration of activities are TFN, in this method we compute several project characteristics such as earliest times, latest times, and, slack times in terms of TFN. In this method, an approach called modified backward pass (MBP) is presented which is based on a linear programming problem, removes negative and infeasible solutions which can be generated by other methods in the backward pass calculation. To illustrate the presented method a numerical example is solved.

### 4.1 Maximum and Minimum of trapezoidal fuzzy numbers

In this section, maximum and minimum of TFNs is defined.

Let  $\tilde{A} = (a_1, b_1, c_1, d_1)$  and  $\tilde{B} = (a_2, b_2, c_2, d_2)$  be any two TFNs, then

$$\begin{aligned} \tilde{\text{MAXIMUM}}(\tilde{A}, \tilde{B}) = & (\text{Maximum}(a_1, a_2), \text{Maximum}(b_1, b_2), \text{Maximum}(c_1, c_2) \\ & , \text{Maximum}(d_1, d_2)) \end{aligned} \quad (4.1)$$

$$\begin{aligned} \widetilde{\text{MINIMUM}}(\widetilde{A}, \widetilde{B}) = & (\text{Minimum}(a_1, a_2), \text{Minimum}(b_1, b_2), \text{Minimum}(c_1, c_2) \\ & , \text{Minimum}(d_1, d_2)) \end{aligned} \quad (4.2)$$

where  $\widetilde{\text{MAXIMUM}}$  and  $\widetilde{\text{MINIMUM}}$  are fuzzy maximum and minimum respectively.

## 4.2 Fuzzy project network

A network  $N = \langle V, A, \widetilde{T} \rangle$ , being a fuzzy project model is given.  $V$  is a set of events and  $A \subset V \times V$  is a set of activities. The network  $N$  is a directed and acyclic graph in the fuzzy environment. The set  $V = \{1, 2, \dots, n\}$  is labeled in such a way that the following condition holds:  $(i, j) \in A \Rightarrow i < j$ . In the fuzzy environment the duration of this activity ( $\widetilde{T}$ ) is a positive TFN:  $\widetilde{T}_{ij} = (t_{ij}^1, t_{ij}^2, t_{ij}^3, t_{ij}^4)$ . Let us denote by  $P(j) = \{i \in V \mid (i, j) \in A\}$  the set of predecessors of event  $j \in V$  and by  $S(i) = \{j \in V \mid (i, j) \in A\}$  the set of successors of event  $i \in V$ . Starting time of the fuzzy project model is a positive TFN:  $\widetilde{T}_s = (t_s^1, t_s^2, t_s^3, t_s^4)$ .

## 4.3 Fuzzy project scheduling

Fuzzy project scheduling consists of the forward pass and MBP calculations to obtain the substantial project characteristics. In this section, for the fuzzy project network, the characteristics such as earliest times, latest times, and slack times are obtained.

### 4.3.1 Fuzzy forward pass calculations

The earliest times and project completion time in a project network can be calculated by forward pass:

$$\widetilde{E}_j = (e_j^1, e_j^2, e_j^3, e_j^4) = \begin{cases} \widetilde{\text{MAXIMUM}}_{i \in P(j)} \{ \widetilde{E}_i \oplus \widetilde{T}_{ij} \} & , P(j) \neq \phi \\ \widetilde{T}_s = (t_s^1, t_s^2, t_s^3, t_s^4) & , P(j) = \phi \end{cases} \quad (4.3)$$

$$\widetilde{ES}_{ij} = (es_{ij}^1, es_{ij}^2, es_{ij}^3, es_{ij}^4) = \widetilde{E}_i \quad (4.4)$$

$$\widetilde{EF}_{ij} = (ef_{ij}^1, ef_{ij}^2, ef_{ij}^3, ef_{ij}^4) = \widetilde{ES}_{ij} \oplus \widetilde{T}_{ij} \quad (4.5)$$

$$\widetilde{T}_F = (t_F^1, t_F^2, t_F^3, t_F^4) = \text{MAXIMUM}_{i \in V}(\widetilde{E}_i) \quad (4.6)$$

where  $\widetilde{E}_j$  is the fuzzy earliest time of event  $j$ ;  $\widetilde{T}_s$  is the fuzzy time of starting the project;  $\widetilde{ES}_{ij}$  is the fuzzy earliest starting of activity  $(i, j)$ ;  $\widetilde{EF}_{ij}$  is the fuzzy earliest finishing of activity  $(i, j)$  and  $\widetilde{T}_F$  is the fuzzy time of project completion.

### 4.3.2 Fuzzy modified backward pass calculations

Backward pass calculations are employed to calculate the latest time in the project network. The fuzzy latest time of event  $i$  ( $\widetilde{L}_i$ ) can be written as:

$$\widetilde{L}_i = \begin{cases} \text{MINIMUM}\{\widetilde{L}_j \ominus \widetilde{T}_{ij}\} & , S(i) \neq \phi \\ \widetilde{T}_F & , S(i) = \phi \end{cases} \quad (4.7)$$

In a crisp environment, the equation of  $A + B - B = A$  is always correct but the addition and subtraction are not always inverse in the fuzzy theory. For example, for a typical data, in which  $S(2) = \{3\}$ ,  $\widetilde{L}_3 = (10, 20, 22, 30)$ , and  $\widetilde{T}_{23} = (5, 10, 15, 20)$ , using (4.7), it is found that:  $\widetilde{L}_2 = (-10, 5, 12, 25)$ . It is clear that  $\widetilde{L}_2$  is a TFN with a negative part. It depicts that the latest time of event 2 may happen in a negative time, but negative time is not feasible. To avoid this, we present a new approach which we call MBP. Therefore, according to the concept of  $\widetilde{L}_i$  and using (4.7), the fuzzy latest time of event  $i$  ( $\widetilde{L}_i$ ) can be defined as:

$$\tilde{L}_i = \begin{cases} \underset{j \in S(i)}{\widetilde{\text{MINIMUM}}}\{\tilde{X}_j \mid \tilde{X}_j \oplus \tilde{T}_{ij} = \tilde{L}_j\} & , S(i) \neq \phi \\ \tilde{T}_F & , S(i) = \phi \end{cases} \quad (4.8)$$

This relation leads to a positive TFN  $(l_i^1, l_i^2, l_i^3, l_i^4)$ . Therefore, the problem due to the appearance of the negative time is removed. But in some cases, the calculated  $\tilde{L}_i$  may not satisfy the definition of TFN. For example, for a typical project data, in which  $S(2) = \{3\}$ ,  $\tilde{L}_3 = (10, 20, 22, 30)$ , and  $\tilde{T}_{23} = (5, 10, 15, 20)$ , using (4.8), it is found that:  $\tilde{L}_2 = (5, 10, 7, 10)$ . It can be seen that,  $\tilde{L}_2$  is not a TFN. By adding the trapezoidal condition, following relation is obtained:

$$\tilde{L}_i = \begin{cases} \underset{j \in S(i)}{\widetilde{\text{MINIMUM}}}\left\{ \underset{\tilde{X}_j \text{ is a positive TFN}}{\widetilde{\text{MAXIMUM}}}\tilde{X}_j \mid \begin{array}{l} \tilde{X}_j \oplus \tilde{T}_{ij} \leq \tilde{L}_j \\ \tilde{X}_j \text{ is a positive TFN} \end{array} \right\} & , S(i) \neq \phi \\ \tilde{T}_F & , S(i) = \phi \end{cases} \quad (4.9)$$

In the above relation, we define the relation  $\leq$  for any two TFNs such as  $\tilde{A} = (a^1, a^2, a^3, a^4)$  and  $\tilde{B} = (b^1, b^2, b^3, b^4)$  as:  $\tilde{A} \leq \tilde{B} \approx a^1 \leq b^1, a^2 \leq b^2, a^3 \leq b^3, a^4 \leq b^4$  where  $S(i) \neq \phi$ . Relation (4.9) results in the following fuzzy mathematical programming problem:

$$\begin{aligned} \tilde{L}_i = \underset{j \in S(i)}{\widetilde{\text{MINIMIZE}}}\left\{ \underset{\tilde{X}_j = (x_j^1, x_j^2, x_j^3, x_j^4)}{\widetilde{\text{MAXIMUM}}}\tilde{X}_j \right\} \\ \text{subject to } \tilde{X}_j \oplus \tilde{T}_{ij} \leq \tilde{L}_j \quad , j \in S(i) \\ 0 \leq x_j^1 \leq x_j^2 \leq x_j^3 \leq x_j^4 \end{aligned} \quad (4.10)$$

This problem can be rearranged in a more convenient form as following:

$$\begin{aligned}
\tilde{L}_i &= \widetilde{\text{MINIMIZE}} \quad \tilde{Y}_i = (y_i^1, y_i^2, y_i^3, y_i^4) \\
\text{subject to} \quad & x_j^1 \leq y_i^1 && \forall j \in S(i) \\
& x_j^2 \leq y_i^2 && \forall j \in S(i) \\
& x_j^3 \leq y_i^3 && \forall j \in S(i) \\
& x_j^4 \leq y_i^4 && \forall j \in S(i) \\
& x_j^1 \leq l_j^1 - t_{ij}^1 && \forall j \in S(i) \\
& x_j^2 \leq l_j^2 - t_{ij}^2 && \forall j \in S(i) \\
& x_j^3 \leq l_j^3 - t_{ij}^3 && \forall j \in S(i) \\
& x_j^4 \leq l_j^4 - t_{ij}^4 && \forall j \in S(i) \\
& x_j^3 \leq x_j^4 \\
& x_j^2 \leq x_j^3 \\
& x_j^1 \leq x_j^2 \\
& 0 \leq x_j^1
\end{aligned} \tag{4.11}$$

It can be easily observed that the optimal solution of this problem,  $\tilde{L}_i$  is obtained as a positive TFN using a simple recursive relation:

$$\begin{aligned}
\tilde{L}_i &= (l_i^1, l_i^2, l_i^3, l_i^4) \\
l_i^4 &= \text{Maximum}(0, \text{Minimum}_{j \in S(i)}(l_j^4 - t_{ij}^4)) \\
l_i^3 &= \text{Maximum}(0, \text{Minimum}(l_i^4, \text{Minimum}_{j \in S(i)}(l_j^3 - t_{ij}^3))) \\
l_i^2 &= \text{Maximum}(0, \text{Minimum}(l_i^3, \text{Minimum}_{j \in S(i)}(l_j^2 - t_{ij}^2)))
\end{aligned} \tag{4.12}$$

$$l_i^1 = \text{Maximum}(0, \text{Minimum}(l_i^2, \text{Minimum}_{j \in S(i)}(l_j^1 - t_{ij}^1)))$$

In the MBP, the fuzzy latest finishing of activities, denoted by  $\tilde{LF}_{ij}$ , is calculated as follows:

$$\tilde{LF}_{ij} = (lf_{ij}^1, lf_{ij}^2, lf_{ij}^3, lf_{ij}^4) = \tilde{L}_j \quad (4.13)$$

and fuzzy latest starting of activities can be calculated using following relation.

$$\tilde{LS}_{ij} = \tilde{LF}_{ij} \ominus \tilde{T}_{ij} \quad (4.14)$$

$$\tilde{LS}_{ij} = (ls_{ij}^1, ls_{ij}^2, ls_{ij}^3, ls_{ij}^4)$$

$$ls_{ij}^4 = \text{Maximum}(0, (lf_{ij}^4 - t_{ij}^4))$$

$$ls_{ij}^3 = \text{Maximum}(0, \text{Minimum}(lf_{ij}^4, (lf_{ij}^3 - t_{ij}^3)))$$

$$ls_{ij}^2 = \text{Maximum}(0, \text{Minimum}(lf_{ij}^3, (lf_{ij}^2 - t_{ij}^2))) \quad (4.15)$$

$$ls_{ij}^1 = \text{Maximum}(0, \text{Minimum}(lf_{ij}^2, (lf_{ij}^1 - t_{ij}^1)))$$

### 4.3.3 Fuzzy slack times

There are three types of slacks for an activity, i.e. fuzzy total slack, fuzzy free slack and fuzzy independent slack, denoted by  $\tilde{TF}_{ij}$ ,  $\tilde{FF}_{ij}$  and  $\tilde{IF}_{ij}$  respectively. These are calculated using following relations:

$$\tilde{TF}_{ij} = \tilde{LF}_{ij} \ominus \tilde{EF}_{ij} \quad (4.16)$$

$$\tilde{FF}_{ij} = \tilde{E}_j \ominus \tilde{EF}_{ij} \quad (4.17)$$

$$\tilde{IF}_{ij} = \tilde{E}_j \ominus \tilde{L}_i \ominus \tilde{T}_{ij} \quad (4.18)$$

$$\tilde{TF}_{ij} = (tf_{ij}^1, tf_{ij}^2, tf_{ij}^3, tf_{ij}^4)$$

$$tf_{ij}^4 = \text{Maximum}(0, (lf_{ij}^4 - ef_{ij}^4))$$

$$tf_{ij}^3 = \text{Maximum}(0, \text{Minimum}(tf_{ij}^4, (lf_{ij}^3 - ef_{ij}^3))) \quad (4.19)$$

$$tf_{ij}^2 = \text{Maximum}(0, \text{Minimum}(tf_{ij}^3, (lf_{ij}^2 - ef_{ij}^2)))$$

$$tf_{ij}^1 = \text{Maximum}(0, \text{Minimum}(tf_{ij}^2, (lf_{ij}^1 - ef_{ij}^1)))$$

$$\tilde{FF}_{ij} = (ff_{ij}^1, ff_{ij}^2, ff_{ij}^3, ff_{ij}^4)$$

$$ff_{ij}^4 = \text{Maximum}(0, (e_j^4 - ef_{ij}^4))$$

$$ff_{ij}^3 = \text{Maximum}(0, \text{Minimum}(ff_{ij}^4, (e_j^3 - ef_{ij}^3)))$$

$$ff_{ij}^2 = \text{Maximum}(0, \text{Minimum}(ff_{ij}^3, (e_j^2 - ef_{ij}^2))) \quad (4.20)$$

$$ff_{ij}^1 = \text{Maximum}(0, \text{Minimum}(ff_{ij}^2, (e_j^1 - ef_{ij}^1)))$$

$$\tilde{IF}_{ij} = (if_{ij}^1, if_{ij}^2, if_{ij}^3, if_{ij}^4)$$

$$if_{ij}^4 = \text{Maximum}(0, (e_j^4 - l_i^4 - t_{ij}^4))$$

$$if_{ij}^3 = \text{Maximum}(0, \text{Minimum}(if_{ij}^4, (e_j^3 - l_i^3 - t_{ij}^3)))$$

$$if_{ij}^2 = \text{Maximum}(0, \text{Minimum}(if_{ij}^3, (e_j^2 - l_i^2 - t_{ij}^2))) \quad (4.21)$$

$$if_{ij}^1 = \text{Maximum}(0, \text{Minimum}(if_{ij}^2, (e_j^1 - l_i^1 - t_{ij}^1)))$$

#### 4.4 Numerical example

To illustrate the method described in the previous section, a numerical example is solved.

**Example 4.1:** Suppose there is a project network, as shown in Fig. 4.1, the fuzzy activity time for each activity is given in Table 4.1.

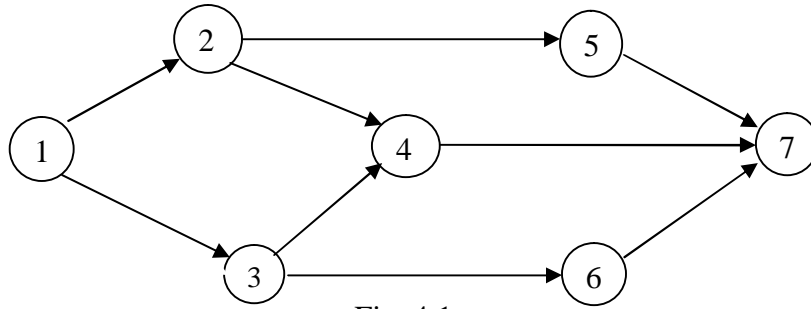


Fig. 4.1

A project network

Table 4.1 Fuzzy activity time for each activity

Activity ( $i, j$ )	Duration ( $\tilde{T}_{ij}$ )
(1,2)	(25,28,32,35)
(1,3)	(40,55,65,70)
(2,4)	(32,37,43,48)
(3,4)	(20,25,35,40)
(2,5)	(35,38,42,45)
(3,6)	(42,45,55,60)
(4,7)	(60,65,75,85)
(5,7)	(65,75,85,90)
(6,7)	(15,18,22,26)

**Fuzzy earliest time of events**

$$P(1) = \phi \Rightarrow \tilde{E}_1 = (e_1^1, e_1^2, e_1^3, e_1^4) = \tilde{T}_s = (0,0,0,0)$$

$$P(2) = \{1\} \Rightarrow \tilde{E}_2 = (e_2^1, e_2^2, e_2^3, e_2^4) = \tilde{E}_1 \oplus \tilde{T}_{12} = (25, 28, 32, 35)$$

$$P(4) = \{2,3\} \Rightarrow \tilde{E}_4 = \text{Maximum} \{ \tilde{E}_2 \oplus \tilde{T}_{24}, \tilde{E}_3 \oplus \tilde{T}_{34} \}$$

$$\tilde{E}_4 = \text{Maximum} \{(57, 65, 75, 83), (60, 80, 100, 110)\} = (60, 80, 100, 110)$$

Similarly using (4.3),  $\tilde{E}_5$ ,  $\tilde{E}_6$  and  $\tilde{E}_7$  can be calculated.

### Fuzzy latest time of events

$$\tilde{L}_7 = \tilde{E}_7 = (125, 145, 175, 195)$$

$$S(6) = \{7\} \Rightarrow \tilde{L}_6 = (l_6^1, l_6^2, l_6^3, l_6^4)$$

$$l_6^4 = \text{Maximum}(0, (l_7^4 - t_{67}^4)) = \text{Maximum}(0, 169) = 169$$

$$l_6^3 = \text{Maximum}(0, \text{Minimum}(l_7^4, (l_7^3 - t_{67}^3))) = \text{Maximum}(0, \text{Minimum}(169, 153)) = 153$$

$$l_6^2 = \text{Maximum}(0, \text{Minimum}(l_7^3, (l_7^2 - t_{67}^2))) = \text{Maximum}(0, \text{Minimum}(153, 127)) = 127$$

$$l_6^1 = \text{Maximum}(0, \text{Minimum}(l_7^2, (l_7^1 - t_{67}^1))) = \text{Maximum}(0, \text{Minimum}(127, 110)) = 110$$

$$\tilde{L}_6 = (110, 127, 153, 169)$$

$$S(2) = \{4, 5\} \Rightarrow \tilde{L}_2 = (l_2^1, l_2^2, l_2^3, l_2^4)$$

$$l_2^4 = \text{Maximum}(0, \text{Minimum}((l_5^4 - t_{25}^4), (l_4^4 - t_{24}^4))) = \text{Maximum}(0, \text{Minimum}(60, 62)) = 60$$

$$l_2^3 = \text{Maximum}(0, \text{Minimum}(l_2^4, \text{Minimum}((l_5^3 - t_{25}^3), (l_4^3 - t_{24}^3))))$$

$$= \text{Maximum}(0, \text{Minimum}(60, \text{Minimum}(48, 57))) = 48$$

$$l_2^2 = \text{Maximum}(0, \text{Minimum}(l_2^3, \text{Minimum}((l_5^2 - t_{25}^2), (l_4^2 - t_{24}^2))))$$

$$= \text{Maximum}(0, \text{Minimum}(48, \text{Minimum}(32, 43))) = 32$$

$$l_2^1 = \text{Maximum}(0, \text{Minimum}(l_2^2, \text{Minimum}((l_5^1 - t_{25}^1), (l_4^1 - t_{24}^1))))$$

$$= \text{Maximum}(0, \text{Minimum}(32, \text{Minimum}(25, 33))) = 25$$

$$\tilde{L}_2 = (25, 32, 48, 60)$$

The fuzzy time of project completion,  $\tilde{T}_F$ , is calculated using (4.6) as: (125, 145, 175, 195).

Table 4.2 represents fuzzy earliest time and latest times of events by using (4.3) and (4.12).

Table 4.2 Values of fuzzy earliest time and fuzzy latest time for each event

Event ( <i>i</i> )	$\tilde{E}_i$	$\tilde{L}_i$
1	(0,0,0,0)	(0,0,0,0)
2	(25,28,32,35)	(25,32,48,60)
3	(40,55,65,70)	(45,55,65,70)
4	(60,80,100,110)	(65,80,100,110)
5	(60,66,74,80)	(60,70,90,105)
6	(82,100,120,130)	(110,127,153,169)
7	(125,145,175,195)	(125,145,175,195)

$\tilde{ES}_{ij}$ ,  $\tilde{EF}_{ij}$ ,  $\tilde{LF}_{ij}$  and  $\tilde{LS}_{ij}$  are obtained using (4.4), (4.5), (4.13) and (4.14) respectively.

$$\tilde{ES}_{12} = \tilde{E}_1 = (0,0,0,0)$$

$$\tilde{ES}_{24} = \tilde{E}_2 = (25,28,32,35)$$

$$\tilde{EF}_{24} = \tilde{ES}_{24} \oplus \tilde{T}_{24} = (57,65,75,83)$$

$$\tilde{LF}_{24} = \tilde{L}_4 = (65,80,100,110)$$

$$\tilde{LS}_{24} = (ls_{24}^1, ls_{24}^2, ls_{24}^3, ls_{24}^4)$$

$$ls_{24}^4 = \text{Maximum}(0, \text{Minimum}(lf_{24}^4 - t_{24}^4)) = \text{Maximum}(0, 62) = 62$$

$$ls_{24}^3 = \text{Maximum}(0, \text{Minimum}(lf_{24}^4, (lf_{24}^3 - t_{24}^3))) = \text{Maximum}(0, \text{Minimum}(62, 57)) = 57$$

$$ls_{24}^2 = \text{Maximum}(0, \text{Minimum}(lf_{24}^3, (lf_{24}^2 - t_{24}^2))) = \text{Maximum}(0, \text{Minimum}(57, 43)) = 43$$

$$ls_{24}^1 = \text{Maximum}(0, \text{Minimum}(lf_{24}^2, (lf_{24}^1 - t_{24}^1))) = \text{Maximum}(0, \text{Minimum}(43, 33)) = 33$$

Calculated values of  $\tilde{ES}_{ij}$ ,  $\tilde{EF}_{ij}$ ,  $\tilde{LS}_{ij}$  and  $\tilde{LF}_{ij}$  have been presented in the Table 4.3.

Table 4.3 Values of  $\tilde{ES}_{ij}$ ,  $\tilde{EF}_{ij}$ ,  $\tilde{LS}_{ij}$  and  $\tilde{LF}_{ij}$  for each activity

$(i, j)$	$\tilde{ES}_{ij}$	$\tilde{EF}_{ij}$	$\tilde{LS}_{ij}$	$\tilde{LF}_{ij}$
(1, 2)	(0, 0, 0, 0)	(25, 28, 32, 35)	(0, 4, 16, 25)	(45, 55, 65, 70)
(1, 3)	(0, 0, 0, 0)	(40, 55, 65, 70)	(0, 0, 0, 0)	(65, 80, 100, 110)
(2, 4)	(25, 28, 32, 35)	(57, 65, 75, 83)	(33, 43, 57, 62)	(65, 80, 100, 110)
(3, 4)	(40, 55, 65, 70)	(60, 80, 100, 110)	(45, 55, 65, 70)	(65, 80, 100, 110)
(2, 5)	(25, 28, 32, 35)	(60, 66, 74, 80)	(25, 32, 48, 60)	(60, 70, 90, 105)
(3, 6)	(40, 55, 65, 70)	(82, 100, 120, 130)	(68, 82, 98, 109)	(110, 127, 253, 169)
(4, 7)	(60, 80, 100, 110)	(120, 145, 175, 195)	(65, 80, 100, 110)	(125, 145, 175, 195)
(5, 7)	(60, 66, 74, 80)	(125, 141, 159, 170)	(60, 70, 90, 105)	(125, 145, 175, 195)
(6, 7)	(82, 100, 120, 130)	(97, 118, 142, 156)	(110, 127, 153, 169)	(125, 145, 175, 195)

Fuzzy slack times are calculated using (4.19), (4.20) and (4.21).

$$\tilde{TF}_{12} = \tilde{LF}_{12} \ominus \tilde{EF}_{12} = (tf_{12}^1, tf_{12}^2, tf_{12}^3, tf_{12}^4)$$

$$tf_{12}^4 = \text{Maximum}(0, (lf_{12}^4 - ef_{12}^4)) = \text{Maximum}(0, 25) = 25$$

$$tf_{12}^3 = \text{Maximum}(0, \text{Minimum}(tf_{12}^4, (lf_{12}^4 - ef_{12}^4))) = \text{Maximum}(0, \text{Minimum}(25, 16)) = 16$$

$$tf_{12}^2 = \text{Maximum}(0, \text{Minimum}(tf_{12}^3, (lf_{12}^2 - ef_{12}^2))) = \text{Maximum}(0, \text{Minimum}(16, 4)) = 4$$

$$tf_{12}^1 = \text{Maximum}(0, \text{Minimum}(tf_{12}^2, (lf_{12}^1 - ef_{12}^1))) = \text{Maximum}(0, \text{Minimum}(4, 0)) = 0$$

$$\tilde{TF}_{12} = (0, 4, 16, 25)$$

$$\tilde{FF}_{12} = \tilde{E}_2 \ominus \tilde{EF}_{12}$$

$$ff_{12}^4 = \text{Maximum}(0, (e_2^4 - ef_{12}^4)) = \text{Maximum}(0, (0)) = 0$$

$$ff_{12}^3 = \text{Maximum}(0, \text{Minimum}(ff_{12}^4, (e_2^3 - ef_{12}^3))) = \text{Maximum}(0, \text{Minimum}(0, 0)) = 0$$

$$ff_{12}^2 = \text{Maximum}(0, \text{Minimum}(ff_{12}^3, (e_2^2 - ef_{12}^2))) = \text{Maximum}(0, \text{Minimum}(0, 0)) = 0$$

$$ff_{12}^1 = \text{Maximum}(0, \text{Minimum}(ff_{12}^2, (e_2^1 - ef_{12}^1))) = \text{Maximum}(0, \text{Minimum}(0, 0)) = 0$$

$$\tilde{FF}_{12} = (0, 0, 0, 0)$$

$$if_{24}^4 = \text{Maximum}(0, (e_4^4 - l_2^4 - d_{24}^4)) = \text{Maximum}(0, 2) = 2$$

$$if_{24}^3 = \text{Maximum}(0, \text{Minimum}(e_4^4, (e_4^3 - l_2^3 - d_{13}^3))) = \text{Maximum}(0, \text{Minimum}(2, 9)) = 2$$

$$if_{24}^2 = \text{Maximum}(0, \text{Minimum}(e_4^3, (e_4^2 - l_2^2 - d_{13}^2))) = \text{Maximum}(0, \text{Minimum}(2, 11)) = 2$$

$$if_{24}^1 = \text{Maximum}(0, \text{Minimum}(e_4^2, (e_4^1 - l_2^1 - d_{13}^1))) = \text{Maximum}(0, \text{Minimum}(2, 3)) = 2$$

$$\tilde{IF}_{24} = (2, 2, 2, 2)$$

Calculated values of  $\tilde{TF}_{ij}$ ,  $\tilde{FF}_{ij}$ ,  $\tilde{IF}_{ij}$  and  $\tilde{IF}_{ij}$  are presented in Table 4.4.

Table 4.4 Values of  $\tilde{TF}_{ij}$ ,  $\tilde{FF}_{ij}$  and  $\tilde{IF}_{ij}$  for each activity

$(i, j)$	$\tilde{TF}_{ij}$	$\tilde{FF}_{ij}$	$\tilde{IF}_{ij}$
(1, 2)	(0, 4, 16, 25)	(0, 0, 0, 0)	(0, 0, 0, 0)
(1, 3)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(2, 4)	(8, 15, 25, 27)	(3, 15, 25, 27)	(2, 2, 2, 2)
(3, 4)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(2, 5)	(0, 4, 16, 25)	(0, 0, 0, 0)	(0, 0, 0, 0)
(3, 6)	(27, 27, 33, 39)	(0, 0, 0, 0)	(0, 0, 0, 0)
(4, 7)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(5, 7)	(0, 4, 16, 25)	(0, 4, 16, 25)	(0, 0, 0, 0)
(6, 7)	(27, 27, 33, 39)	(27, 27, 33, 39)	(0, 0, 0, 0)

## **4.5 Conclusion**

Previous works on network scheduling using fuzzy set theory have a drawback that they do not support the backward pass calculations in direct manner similar to that used in the forward pass. In this chapter a method based on the fuzzy theory is presented to solve the project scheduling problem under the fuzzy environment. A major advantage of this method is to employ direct arithmetic fuzzy operations in obtaining meaningful computable results. In this method, we presented a new approach called MBP. This approach, based on a linear programming (LP) problem, removes negative and infeasible solutions which can be generated by other methods in the backward pass calculation.

## Chapter 5

# LINEAR PROGRAMMING BASED METHOD FOR ANALYZING THE FUZZY CRITICAL PATH

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In the previous chapter, all the parameters are represented by TFNs, but in real life situations it is not always possible to represent all the parameters by same type of fuzzy numbers. In this chapter, a method is presented to find the FCP of a given project network by representing the parameters by different types of fuzzy numbers.

The basic idea is based on the extension principle and linear programming formulation. A pair of linear programs parameterized by possibility level  $\alpha$  is formulated to calculate the lower and upper bounds of the fuzzy total duration time at  $\alpha$ . By enumerating different values of  $\alpha$ , the membership function of the fuzzy total duration time is constructed, and the FCPs are identified at the same time. Moreover, by applying the Yager ranking method, definitions of the most critical path and the relative degree of criticality of paths are developed. Since the total duration time is completely expressed by a membership function rather than by a crisp value, the fuzziness of activity times is conserved completely, and more information is provided for critical path analysis.

### 5.1 Linear programming formulation of fuzzy critical path problems

Consider a project network  $S = \langle V, A, t \rangle$ , consisting of a finite set  $V$  of nodes (events) and a set  $A \subset V \times V$  of arcs with crisp activity times, which are determined by a function  $t: A \rightarrow \mathbb{R}^+$  and attached to the arcs. Denote  $t_{ij}$  as the time period of activity  $(i, j) \in A$ . Two basic results provided by CPM are the total duration time needed to com-

plete the project and the critical path. One of the procedures for finding a critical path is to first determine the earliest occurrence times via a forward pass and the latest occurrence times via a backward pass for each activity, and then we can identify a critical activity defined as an activity with the earliest occurrence time and the latest occurrence time being equal.

An alternative way to determine the total duration and find critical paths is by using linear programming technique [32]. The idea is based on the concept that a CPM problem can be thought of as the opposite of the shortest path problem, to determine a critical path in the project network it is sufficient to find the longest path from start to finish. The linear programming formulation assumes that a unit flow enters the project network at the start node and leaves at the finish node. Let  $x_{ij}$  be the decision variable denoting the amount of flow in activity  $(i, j) \in A$ . Since only one unit of flow could be in any arc at any one time, the variable  $x_{ij}$  must assume binary values (0 or 1) only. The CPM problem with  $n$  nodes is formulated as:

$$\begin{aligned}
 D = \text{Maximize} \quad & \sum_{i=1}^n \sum_{j=1}^n t_{ij} x_{ij} \\
 \text{subject to} \quad & \sum_{j=1}^n x_{1j} = 1, \\
 & \sum_{j=1}^n x_{ij} = \sum_{k=1}^n x_{ki}, \quad i = 2, \dots, n-1 \\
 & \sum_{k=1}^n x_{kn} = 1, \\
 & x_{ij} = 0 \text{ or } 1, (i, j) \in A.
 \end{aligned} \tag{5.1}$$

The objective is to maximize the total duration time of the project network from node 1

to node  $n$ . The constraints are called the flow conservation equations and indicate that the flow may be neither created nor destroyed in the project network. The critical path for this project network consists of a set of activities  $(i, j) \in A$  from the start to the finish in which each activity in the path corresponds to the optimal decision variable  $x_{ij}^* = 1$  in the optimal solution to model (5.1).

Intuitively, if any of the activity duration time  $t_{ij}$ , the coefficient in the objective function of model (5.1), is fuzzy, the total duration time  $D$  becomes fuzzy as well. The conventional CPM problem defined in (5.1) is then modified into the CPM problem with fuzzy parameters in the objective function. Consequently, it cannot be maximized directly. Consider a project network  $S_f = \langle V, A, \tilde{T} \rangle$  with fuzzy activity times.  $V$  and  $A$  are the same as in the crisp case except that the activity times are approximately known and defined by function  $\tilde{T}: A \rightarrow FN(\mathbb{R}^+)$ , where  $FN(\mathbb{R}^+)$  is the set of non-negative fuzzy numbers. Denote fuzzy number  $\tilde{T}_{ij}$  as the fuzzy duration time of activity  $(i, j) \in A$ , and its membership function is  $\mu_{\tilde{T}_{ij}}(t_{ij})$ . We then have

$$\tilde{T}_{ij} = \{(t_{ij}, \mu_{\tilde{T}_{ij}}(t_{ij})) \mid t_{ij} \in S(\tilde{T}_{ij})\}, \quad (i, j) \in A,$$

where  $S(\tilde{T}_{ij})$  is the support of  $\tilde{T}_{ij}$ , which denotes the universe set of activity time of activity  $(i, j) \in A$ . Consequently, the fuzzy CPM problem is of following form:

$$\begin{aligned} \tilde{D} = \text{Maximize} \quad & \sum_{i=1}^n \sum_{j=1}^n \tilde{T}_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^n x_{1j} = 1, \end{aligned}$$

$$\sum_{j=1}^n x_{ij} = \sum_{k=1}^n x_{ki}, \quad i = 2, 3, \dots, n-1, \quad (5.2)$$

$$\sum_{k=1}^n x_{kn} = 1,$$

$$x_{ij} = 0 \text{ or } 1, \quad (i, j) \in A.$$

Similar to model (5.1), the objective is to maximize the total duration time of the project network from node 1 to node  $n$  except that the total duration times are fuzzy, i.e., the optimal solution to model (5.2) can be used to identify the critical path with longest path length in the set of all paths in  $S_f = \langle V, A, \tilde{T} \rangle$  denoted as  $P_f$ . Model (5.2) is essentially a linear program with fuzzy coefficients in the objective function; one straightforward approach is to apply the existing fuzzy linear programming techniques to the fuzzy CPM problem. Note that the maximal objective value in model (5.2) is not a real number but a fuzzy number. But, most of the existing techniques provide only crisp solutions. If the obtained objective value is a crisp value, then some helpful information for project management may be lost. To conserve the fuzziness of the activity times, we must derive the membership function of the total duration time  $\mu_{\tilde{D}}(d)$ .

## 5.2 Method to find the fuzzy critical path

The presented approach is based on a combination of the concept of  $\alpha$ -cut, Zadeh's extension principle, two-level mathematical programming, and parametric programming. One approach to construct the membership function  $\mu_{\tilde{D}}(d)$  is to derive the  $\alpha$ -cuts of  $\mu_{\tilde{D}}(d)$ . The  $\alpha$ -cut of  $\tilde{T}_{ij}$  is defined as follows:

$$(T_{ij})_{\alpha} = \{t_{ij} \in S(\tilde{T}_{ij}) \mid \mu_{\tilde{T}_{ij}} \geq \alpha\} \quad (5.3)$$

$$\begin{aligned}
(T_{ij})_\alpha &= \left[ \underset{t_{ij}}{\text{Infimum}}\{t_{ij} \in S(\tilde{T}_{ij}) \mid \mu_{\tilde{T}_{ij}}(t_{ij}) \geq \alpha\}, \underset{t_{ij}}{\text{Supremum}}\{t_{ij} \in S(\tilde{T}_{ij}) \mid \mu_{\tilde{T}_{ij}}(t_{ij}) \geq \alpha\} \right] \\
&= [(T_{ij})_\alpha^L, (T_{ij})_\alpha^U]
\end{aligned} \tag{5.4}$$

### 5.2.1 Finding the fuzzy total duration time

On the basis of Zadeh's extension principle, the membership function  $\mu_{\tilde{D}}(d)$  is defined as:

$$\mu_{\tilde{D}}(d) = \underset{t_{ij} \in \mathbb{R}^+, (i,j) \in A}{\text{Supremum}} \underset{(i,j) \in A}{\text{Minimum}} \{ \mu_{\tilde{T}_{ij}}(t_{ij}) \mid d = D(t) \}, \tag{5.5}$$

where  $D(t)$  is defined in model (5.1) and  $t$  is the vector of activity times. We can use its  $\alpha$ -cut to construct its membership function. In (5.5),  $\mu_{\tilde{D}}(d)$  is the minimum of  $\mu_{\tilde{T}_{ij}}(t_{ij}) \forall (i, j) \in A$ . To deal with the membership value, we need  $\mu_{\tilde{T}_{ij}}(t_{ij}) \geq \alpha \forall (i, j) \in A$  and at least one  $\mu_{\tilde{T}_{ij}}(t_{ij})$  equal to  $\alpha$  such that  $d = D(t)$  to satisfy  $\mu_{\tilde{D}}(d) = \alpha$ . To find the membership function  $\mu_{\tilde{D}}(d)$ , it suffices to find the left shape function and the right shape function of  $\mu_{\tilde{D}}(d)$ . This is equivalent to find the lower bound  $D_\alpha^L$  and the upper bound  $D_\alpha^U$  of the  $\alpha$ -cut of  $\mu_{\tilde{D}}(d)$ :

$$D_\alpha^L = \text{Minimum}\{D(t) \mid (T_{ij})_\alpha^L \leq t_{ij} \leq (T_{ij})_\alpha^U \forall (i, j) \in A\}, \tag{5.6}$$

$$D_\alpha^U = \text{Maximum}\{D(t) \mid (T_{ij})_\alpha^L \leq t_{ij} \leq (T_{ij})_\alpha^U \forall (i, j) \in A\}, \tag{5.7}$$

which can be reformulated as a pair of two-level mathematical programs:

$$D_\alpha^L = \text{Minimum}_{(T_{ij})_\alpha^L \leq t_{ij} \leq (T_{ij})_\alpha^U} \left\{ \begin{array}{l} \text{Maximize} \quad \sum_{i=1}^n \sum_{j=1}^n t_{ij} x_{ij} \\ \text{subject to} \quad \sum_{j=1}^n x_{1j} = 1, \\ \sum_{j=1}^n x_{ij} = \sum_{k=1}^n x_{ki}, \quad i = 2, 3, \dots, n-1, \\ \sum_{k=1}^n x_{kn} = 1, \\ x_{ij} \geq 0, \quad (i, j) \in A \end{array} \right. \quad (5.8)$$

$$D_\alpha^U = \text{Maximum}_{(T_{ij})_\alpha^L \leq t_{ij} \leq (T_{ij})_\alpha^U} \left\{ \begin{array}{l} \text{Maximize} \quad \sum_{i=1}^n \sum_{j=1}^n t_{ij} x_{ij} \\ \text{subject to} \quad \sum_{j=1}^n x_{1j} = 1, \\ \sum_{j=1}^n x_{ij} = \sum_{k=1}^n x_{ki}, \quad i = 2, 3, \dots, n-1, \\ \sum_{k=1}^n x_{kn} = 1, \\ x_{ij} \geq 0, \quad (i, j) \in A \end{array} \right. \quad (5.9)$$

At least one  $t_{ij}$ ,  $(i, j) \in A$  must hit the boundary of their  $\alpha$ -cuts to satisfy  $\mu_{\bar{d}}(d) = \alpha$ . To solve model (5.8), we formulate the dual problem of second level problem to become a minimization problem that is consistent with minimization problem of first level. Let  $y_j$  denote the dual variable corresponding to the  $j^{\text{th}}$  general constraint of the second level problem in model (5.8). So, we get

$$D_\alpha^L = \text{Minimum}_{\substack{(T_{ij})_\alpha^L \leq t_{ij} \leq (T_{ij})_\alpha^U \\ \forall (i, j) \in A}} \left\{ \begin{array}{l} \text{Minimize} \quad (y_n - y_1) \\ \text{subject to} \quad y_j \geq y_i + t_{ij}, \quad (i, j) \in A \\ y_i, y_j \quad \text{unrestricted in sign} \quad \forall (i, j) \in A \end{array} \right. \quad (5.10)$$

where,  $y_j$  represents the occurrence time of the node  $j$ , and the value of  $(y_n - y_1)$  represents the duration of the project. The objective of the dual problem is to find the required minimal duration time such that all precedence relationships among the different activities are satisfied. Each constraint specifies the precedence relationships among activities; for example, the constraint  $y_j \geq y_i + t_{ij}$  shows that  $y_j$ , the occurrence time of node  $j$ , cannot be any earlier than time  $y_i + t_{ij}$ . In model (5.10), since  $(T_{ij})_\alpha^L \leq t_{ij} \leq (T_{ij})_\alpha^U \quad \forall (i, j) \in A$ , one can derive the lower bound of the objective value by setting  $t_{ij}$  to its lower bound  $(T_{ij})_\alpha^L \quad \forall (i, j) \in A$ . Thus, model (5.10) can be reformulated as:

$$D_\alpha^L = \text{Minimum} \begin{cases} \text{Minimize} & y_n - y_1 \\ \text{subject to} & y_j \geq y_i + (T_{ij})_\alpha^L, \quad (i, j) \in A, \\ & y_i, y_j \text{ unrestricted in sign} \quad \forall (i, j) \in A \end{cases} \quad (5.11)$$

Model (5.11) can be rewritten as the following conventional one-level linear program:

$$\begin{aligned} D_\alpha^L = & \text{Minimize} \quad y_n - y_1 \\ & \text{subject to} \quad y_j \geq y_i + (T_{ij})_\alpha^L, \quad (i, j) \in A \\ & y_i, y_j \text{ unrestricted in sign} \quad \forall (i, j) \in A \end{aligned} \quad (5.12)$$

In model (5.12), since all  $t_{ij}$  are set to the lower bounds of their  $\alpha$ -cuts, this assures  $\mu_D(d) = \alpha$ , as required by (5.5). The minimal objective value parameterized by  $\alpha$  represents the lower bound of the total duration time of this project network at possibility level  $\alpha$ .

$$D_{\alpha}^U = \text{Maximum} \left\{ \begin{array}{l} \text{Maximize } \sum_{i=1}^n \sum_{j=1}^n (T_{ij})_{\alpha}^U x_{ij} \\ \text{subject to } \sum_{j=1}^n x_{1j} = 1, \\ \sum_{j=1}^n x_{ij} = \sum_{k=1}^n x_{ki}, \quad i = 2, 3, \dots, n-1, \\ \sum_{k=1}^n x_{kn} = 1 \\ x_{ij} \geq 0, \quad (i, j) \in A. \end{array} \right. \quad (5.13)$$

Model (5.13) can be rewritten as the following conventional one-level linear program:

$$D_{\alpha}^U = \text{Maximize } \sum_{i=1}^n \sum_{j=1}^n (T_{ij})_{\alpha}^U x_{ij}$$

$$\text{subject to } \sum_{j=1}^n x_{1j} = 1,$$

$$\sum_{j=1}^n x_{ij} = \sum_{k=1}^n x_{ki}, \quad i = 2, 3, \dots, n-1,$$

$$\sum_{k=1}^n x_{kn} = 1$$

$$x_{ij} \geq 0, \quad (i, j) \in A. \quad (5.14)$$

The maximal objective value parameterized by  $\alpha$  represents the upper bound of the total duration time of this project network at possibility level  $\alpha$ . For two possibility levels  $\alpha_1$  and  $\alpha_2$  such that  $0 < \alpha_2 < \alpha_1 \leq 1$ , we have  $[(T_{ij})_{\alpha_1}^L, (T_{ij})_{\alpha_1}^U] \subseteq [(T_{ij})_{\alpha_2}^L, (T_{ij})_{\alpha_2}^U]$ . Therefore, the feasible region defined by  $\alpha_2$  in model (5.12) is smaller than the one defined by  $\alpha_1$ , thus  $D_{\alpha_1}^L \geq D_{\alpha_2}^L$ ; and the objective value defined by  $\alpha_2$  in model (5.14) is larger than the one defined by  $\alpha_1$ , therefore,  $D_{\alpha_1}^U \leq D_{\alpha_2}^U$ , i.e., the left shape function is non-decreasing and the right shape function is non-increasing. This assures the convexity of  $\tilde{D}$ . Note that these

optimal solutions are parameterized by  $\alpha$ . If both  $D_\alpha^L$  and  $D_\alpha^U$  are invertible with respect to  $\alpha$ , then a left shape function  $LS(d) = (D_\alpha^L)^{-1}$  and a right shape function  $RS(d) = (D_\alpha^U)^{-1}$  can be obtained, from which the membership function  $\mu_{\tilde{D}}(d)$  is constructed as follows:

$$\mu_{\tilde{D}}(d) = \begin{cases} LS(d), & a_1 \leq d \leq a_2, \\ 1, & a_2 \leq d \leq a_3, \\ RS(d), & a_3 \leq d \leq a_4, \end{cases} \quad (5.15)$$

the numerical solutions for  $D_\alpha^L$  and  $D_\alpha^U$  at different possibility level  $\alpha$  can be obtained to approximate the shapes of  $LS(z)$  and  $RS(z)$ .

### 5.2.2 Finding the fuzzy critical paths

Critical paths at each possibility level  $\alpha$  could be identified from the optimal solution of models (5.12) and (5.14). For model (5.12), the critical activities correspond to the constraints that are satisfied as strict equations by the obtained optimal solution. For model (5.14), the critical path for this project network consists of an activity  $(i, j) \in A$  from the start to the finish in which each activity in the path corresponds to the optimal decision variable  $x_{ij}^* = 1$ . Consequently, at each possibility level  $\alpha$ , at least one critical path can be found for the lower bound of the fuzzy total duration time and at least one for the upper bound of the fuzzy total duration time.

### 5.2.3 Crisp transformation of critical paths

According to the above, we can find at least two critical paths at a certain possibility level  $\alpha$ . However, they may be different, even for the same possibility level, i.e. for example, for two distinct possibility levels  $\alpha_1$  and  $\alpha_2$ , we may find four distinct critical paths,

namely  $(FCP)_{\alpha_1}^L$ ,  $(FCP)_{\alpha_1}^U$ ,  $(FCP)_{\alpha_2}^L$  and  $(FCP)_{\alpha_2}^U$ . Although this completely conserves all the fuzziness of activity times, but from a practical point of view, we cannot clearly know which is the most critical path that contains the most critical activities that must be controlled. To resolve this problem, we first transform the fuzzy activity times of critical activities on a fuzzy critical path into crisp times, and then summing these crisp activity times to find the crisp length of this FCP. Finally, these crisp total duration times are compared to find the largest one. We defuzzyfy the fuzzy activity times of critical activities on the fuzzy critical path into crisp ones by using Yager ranking index.

#### 5.2.4 Yager ranking index

Yager proposed a procedure for ordering fuzzy sets, in which a ranking index  $I(\tilde{t})$  is calculated for the convex fuzzy number  $\tilde{t}$  from its  $\alpha$ -cut  ${}^\alpha t = [t_\alpha^L, t_\alpha^U]$  according to the following formula:

$$I(\tilde{t}) = \int_0^1 \frac{1}{2} (t_\alpha^L + t_\alpha^U) d\alpha, \quad (5.16)$$

which is the center of the mean value of  $\tilde{t}$ . Consider two fuzzy numbers  $\tilde{D}_1$  and  $\tilde{D}_2$ , the case of  $I(\tilde{D}_1) \geq I(\tilde{D}_2)$  implies that  $\tilde{D}_1 \geq \tilde{D}_2$ , and then  $\text{maximum}\{\tilde{D}_1, \tilde{D}_2\} = \tilde{D}_1$ . Yager's ranking method possesses linearity and additivity properties.

**Definition 5.1** The length  $L_{p_f}$  of any path  $p_f \in P_f$  in a project network  $S_f = \langle V, A, \tilde{T} \rangle$  with activity times being fuzzy numbers is defined as the sum of the Yager ranking indices of all fuzzy activity times on this path, i.e.  $L_{p_f} = \sum_{\forall (i,j) \in p_f} I(\tilde{T}_{ij})$ .

A path is critical in crisp environment if and only if it is the longest path in the network project.

**Definition 5.2** The most critical path (mcp) in a project network  $S_f = \langle V, A, \tilde{T} \rangle$  with activity times being fuzzy numbers is defined as the FCP whose path length having largest Yager ranking index.

Consider that there are  $m$  critical paths  $FCP_k, k = 1, 2, \dots, m$  with sets of critical activities  $FCA_k, k = 1, 2, \dots, m$ , respectively. Then the length of the most critical path is defined as

$$L_{mcp}^{\max} = \text{Maximum}_k \left\{ \sum_{\forall (i,j) \in FCA_k, k=1,2,\dots,m} I(\tilde{T}_{ij}) \right\} \quad (5.17)$$

### 5.2.5 Relative path degree of criticality

Consider a project network having  $G$  paths. Setting the path degree of criticality of its most critical path as 1.0, the relative degree of criticality of a path can be defined as:

**Definition 5.3** The relative degree of criticality of a path  $p_{fg} \in P_{fg}, g \in \{1, 2, \dots, G\}$  is defined as the ratio of the Yager ranking index of this path to the one of the most critical path:

$$R \deg(p_{fg}) = \frac{\sum_{\forall (i,j) \in p_{fg}} I(\tilde{T}_{ij})}{\text{Maximum}_k \left\{ \sum_{\forall (i,j) \in FCA_k, k=1,2,\dots,m} I(\tilde{T}_{ij}) \right\}} = \frac{\sum_{\forall (i,j) \in p_{fg}} I(\tilde{T}_{ij})}{L_{mcp}^{\max}} = \frac{I(p_{fg})}{L_{mcp}^{\max}} \quad (5.18)$$

### 5.3 Numerical example

**Example 5.1:** Consider a project whose corresponding network is given in Fig. 5.3. The activity times are fuzzy numbers of  $L_{ij} - R_{ij}$  type,  $(i, j) \in A$ , as shown in Table 5.1.

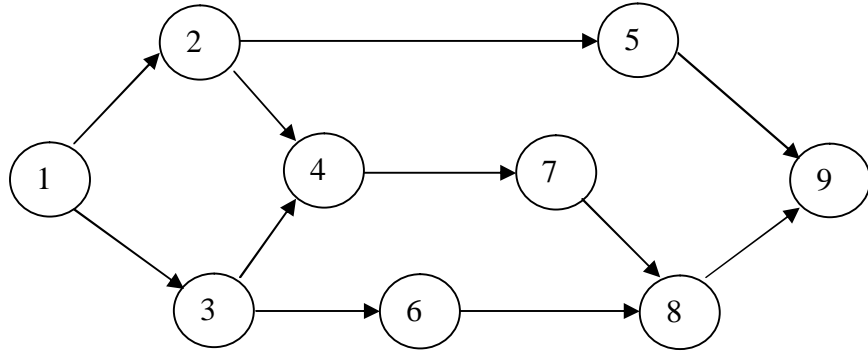


Fig. 5.1  
A project network

Table 5.1 Fuzzy activity time and left and right shape functions for each activity

$\tilde{T}_{ij}$	$L_{ij}(x)$	$R_{ij}(x)$
$\tilde{T}_{12} = (1, 1.5, 1, 1)_{L_{12}-R_{12}}$	$L_{12}(x) = \text{Maximum}(1 - x^2, 0)$	$R_{12}(x) = \text{Maximum}(0, 1 - x)$
$\tilde{T}_{13} = (2, 3, 0, 2)_{L_{13}-R_{13}}$	$L_{13}(x) = e^{-x}$	$R_{13}(x) = \text{Maximum}(0, 1 - x)$
$\tilde{T}_{24} = (0, 0, 0, 0)_{L_{24}-R_{24}}$	$L_{24}(x) = \text{Maximum}(0, 1 - x)$	$R_{24}(x) = \text{Maximum}(0, 1 - x)$
$\tilde{T}_{25} = (2, 3, 1, 2)_{L_{25}-R_{25}}$	$L_{25}(x) = \text{Maximum}(0, 1 - x^4)$	$R_{25}(x) = e^{-x}$
$\tilde{T}_{34} = (0, 0, 0, 0)_{L_{34}-R_{34}}$	$L_{34}(x) = \text{Maximum}(0, 1 - x)$	$R_{34}(x) = \text{Maximum}(1 - x^2, 0)$
$\tilde{T}_{36} = (6, 7, 0, 2)_{L_{36}-R_{36}}$	$L_{36}(x) = e^{-x^2}$	$R_{36}(x) = \text{Maximum}(1 - x^2, 0)$
$\tilde{T}_{46} = (5, 5, 1, 1)_{L_{46}-R_{46}}$	$L_{46}(x) = \text{Maximum}(0, 1 - x)$	$R_{46}(x) = \text{Maximum}(0, 1 - x^4)$
$\tilde{T}_{47} = (9, 9, 1, 1)_{L_{47}-R_{47}}$	$L_{47}(x) = \text{Maximum}(0, 1 - x^4)$	$R_{47}(x) = e^{-x}$
$\tilde{T}_{59} = (8, 9, 2, 4)_{L_{59}-R_{59}}$	$L_{59}(x) = \text{Maximum}(0, 1 - x^4)$	$R_{59}(x) = \text{Maximum}(0, 1 - x^2)$
$\tilde{T}_{68} = (4, 4, 2, 2)_{L_{68}-R_{68}}$	$L_{68}(x) = \text{Maximum}(0, 1 - x^2)$	$R_{68}(x) = \text{Maximum}(0, 1 - x^4)$
$\tilde{T}_{78} = (3, 4, 2, 0)_{L_{78}-R_{78}}$	$L_{78}(x) = \text{Maximum}(0, 1 - x)$	$R_{78}(x) = \text{Maximum}(0, 1 - x^4)$
$\tilde{T}_{89} = (6, 9, 2, 3)_{L_{89}-R_{89}}$	$L_{89}(x) = \text{Maximum}(0, 1 - x^2)$	$R_{89}(x) = e^{-x^2}$

According to model (5.12), problem can be formulated as:

$$\tilde{D} = \text{Maximize} \left( \tilde{T}_{12}x_{12} + \tilde{T}_{13}x_{13} + \tilde{T}_{24}x_{24} + \tilde{T}_{25}x_{25} + \tilde{T}_{34}x_{34} + \tilde{T}_{36}x_{36} + \tilde{T}_{46}x_{46} + \tilde{T}_{47}x_{47} \right. \\ \left. + \tilde{T}_{59}x_{59} + \tilde{T}_{68}x_{68} + \tilde{T}_{78}x_{78} + \tilde{T}_{89}x_{89} \right)$$

$$\text{subject to } x_{12} + x_{13} = 1,$$

$$x_{12} = x_{24} + x_{25},$$

$$x_{13} = x_{34} + x_{36},$$

$$x_{24} + x_{34} = x_{47} + x_{46},$$

$$x_{25} = x_{59},$$

$$x_{46} + x_{36} = x_{68},$$

$$x_{47} = x_{78},$$

$$x_{78} + x_{68} = x_{89},$$

$$x_{59} + x_{89} = 1,$$

$$x_{12}, x_{13}, x_{24}, x_{25}, x_{34}, x_{36}, x_{46}, x_{47}, x_{59}, x_{68}, x_{78}, x_{89} \geq 0$$

The  $\alpha$ -cuts of all fuzzy activity times  $\tilde{T}_{ij} \quad \forall (i, j) \in A$  can be obtained using (1.4) and

Definition (1.18) and are summarized in Table 5.2.

$$L_{13} \left( \frac{2 - (\tilde{T}_{13})_{\alpha}^L}{0} \right) = \alpha \quad \Rightarrow \quad e^{-\left( \frac{2 - (\tilde{T}_{13})_{\alpha}^L}{0} \right)} = \alpha$$

$$\Rightarrow \quad -\ln \alpha = \left( \frac{2 - (\tilde{T}_{13})_{\alpha}^L}{0} \right) \quad \Rightarrow \quad (\tilde{T}_{13})_{\alpha}^L = 2$$

$$R_{13} \left( \frac{(\tilde{T}_{13})_{\alpha}^U - 3}{2} \right) = \alpha \quad \Rightarrow \quad 1 - \left( \frac{(\tilde{T}_{13})_{\alpha}^U - 3}{2} \right) = \alpha$$

$$\Rightarrow 2 - 2\alpha = (\tilde{T}_{13})_{\alpha}^U - 3 \Rightarrow (\tilde{T}_{13})_{\alpha}^U = 2 - 5\alpha$$

Table 5.2 The  $\alpha$ -cuts of fuzzy activity times

$\tilde{T}_{ij}$	$(T_{ij})_{\alpha}^L$	$(T_{ij})_{\alpha}^U$
$\tilde{T}_{12} = (1, 1.5, 1, 1)_{L_{12}-R_{12}}$	$(T_{12})_{\alpha}^L = 1 - \sqrt{1 - \alpha}$	$(T_{12})_{\alpha}^U = 2.5 - \alpha$
$\tilde{T}_{13} = (2, 3, 0, 2)_{L_{13}-R_{13}}$	$(T_{13})_{\alpha}^L = 2$	$(T_{13})_{\alpha}^U = 5 - 2\alpha$
$\tilde{T}_{24} = (0, 0, 0, 0)_{L_{24}-R_{24}}$	$(T_{24})_{\alpha}^L = 0$	$(T_{24})_{\alpha}^U = 0$
$\tilde{T}_{25} = (2, 3, 1, 2)_{L_{25}-R_{25}}$	$(T_{25})_{\alpha}^L = 2 - \sqrt[4]{1 - \alpha}$	$(T_{25})_{\alpha}^U = 3 - 2 \ln \alpha$
$\tilde{T}_{34} = (0, 0, 0, 0)_{L_{34}-R_{34}}$	$(T_{34})_{\alpha}^L = 0$	$(T_{25})_{\alpha}^U = 0$
$\tilde{T}_{36} = (6, 7, 0, 2)_{L_{36}-R_{36}}$	$(T_{36})_{\alpha}^L = 6$	$(T_{25})_{\alpha}^U = 7 + 2\sqrt{1 - \alpha}$
$\tilde{T}_{46} = (5, 5, 1, 1)_{L_{46}-R_{46}}$	$(T_{46})_{\alpha}^L = 4 + \alpha$	$(T_{46})_{\alpha}^U = 5 + \sqrt[4]{1 - \alpha}$
$\tilde{T}_{47} = (9, 9, 1, 1)_{L_{47}-R_{47}}$	$(T_{47})_{\alpha}^L = 9 - \sqrt[4]{1 - \alpha}$	$(T_{47})_{\alpha}^U = 9 - \ln \alpha$
$\tilde{T}_{59} = (8, 9, 2, 4)_{L_{59}-R_{59}}$	$(T_{59})_{\alpha}^L = 8 - 2\sqrt[4]{1 - \alpha}$	$(T_{59})_{\alpha}^U = 9 + 4\sqrt{1 - \alpha}$
$\tilde{T}_{68} = (4, 4, 2, 2)_{L_{68}-R_{68}}$	$(T_{68})_{\alpha}^L = 4 - 2\sqrt{1 - \alpha}$	$(T_{68})_{\alpha}^U = 4 + 2\sqrt[4]{1 - \alpha}$
$\tilde{T}_{78} = (3, 4, 2, 0)_{L_{78}-R_{78}}$	$(T_{78})_{\alpha}^L = 1 + 2\alpha$	$(T_{78})_{\alpha}^U = 4$
$\tilde{T}_{89} = (6, 9, 2, 3)_{L_{89}-R_{89}}$	$(T_{89})_{\alpha}^L = 6 - 2\sqrt{1 - \alpha}$	$(T_{89})_{\alpha}^U = 9 + 3\sqrt{-\ln \alpha}$

According to model (5.12) and (5.14), the lower and upper bounds of  $\tilde{D}$  at possibility level  $\alpha$ ,  $\alpha \in (0, 1]$ , can be solved as:

$$D_{\alpha}^L = \text{Minimize } (y_9 - y_1)$$

$$\text{subject to } y_2 \geq y_1 + (1 - \sqrt{1 - \alpha}),$$

$$y_3 \geq y_1 + 2,$$

$$y_4 \geq y_2,$$

$$y_5 \geq y_2 + (2 - \sqrt[4]{1 - \alpha}),$$

$$y_6 \geq y_3 + 6,$$

$$y_6 \geq y_4 + (4 + \alpha),$$

$$y_7 \geq y_4 + (9 - \sqrt[4]{1 - \alpha}),$$

$$y_8 \geq y_6 + (4 - 2\sqrt{1 - \alpha}),$$

$$y_8 \geq y_7 + (1 + 2\alpha),$$

$$y_9 \geq y_5 + (8 + 2\sqrt[4]{1 - \alpha}),$$

$$y_9 \geq y_8 + (6 - 2\sqrt{1 - \alpha}),$$

$$D_\alpha^U = \text{Maximize } (2.5 - \alpha)x_{12} + (5 - 2\alpha)x_{13} + (3 - 2\ln \alpha)x_{25} + (7 + 2\sqrt{1 - \alpha})x_{36}$$

$$+ (5 + \sqrt[4]{1 - \alpha})x_{46} + (9 - \ln \alpha)x_{47} + (9 + 4\sqrt{1 - \alpha})x_{59} + (4 + 2\sqrt[4]{1 - \alpha})x_{68} + 4x_{78}$$

$$+ (9 + 3\sqrt{-\ln \alpha})x_{89}$$

$$\text{subject to } x_{12} + x_{13} = 1,$$

$$x_{12} = x_{24} + x_{25},$$

$$x_{13} = x_{34} + x_{36},$$

$$x_{24} + x_{34} = x_{46} + x_{47},$$

$$x_{25} = x_{59},$$

$$x_{46} + x_{36} = x_{68},$$

$$x_{47} = x_{78},$$

$$x_{78} + x_{68} = x_{89},$$

$$x_{59} + x_{89} = 1,$$

$$x_{12}, x_{13}, x_{24}, x_{25}, x_{34}, x_{36}, x_{46}, x_{47}, x_{59}, x_{68}, x_{78}, x_{89} \geq 0.$$

A mathematical programming software Lingo is used to solve the above linear programs.

Table 5.3 lists the total duration time and the fuzzy critical paths at twelve distinct  $\alpha$  values.

Table 5.3 The  $\alpha$  – cuts of total duration and the corresponding critical paths

$\alpha$	Lower Bound		Upper Bound	
	$D_\alpha^L$	Critical Path	$D_\alpha^U$	Critical Path
1.0	20.0000	1-3-4-7-8-9	25.0000	1-3-4-7-8-9
0.9	18.6052	1-3-4-7-8-9	26.2791	1-3-4-7-8-9
0.8	18.0368	1-3-4-7-8-9	27.0491	1-3-6-8-9
0.7	17.5645	1-3-4-7-8-9	27.9673	1-3-6-8-9
0.6	17.1398	1-3-4-7-8-9	28.7996	1-3-6-8-9
0.5	16.7449	1-3-4-7-8-9	29.5937	1-3-6-8-9
0.4	16.3707	1-3-4-7-8-9	30.3811	1-3-6-8-9
0.3	16.0120	1-3-4-7-8-9	31.1945	1-3-6-8-9
0.2	15.6654	1-3-4-7-8-9	32.0863	1-3-6-8-9
0.1	15.3286	1-3-4-7-8-9	33.6549	1-3-4-7-8-9
0.01	15.0325	1-3-4-7-8-9	38.0231	1-3-4-7-8-9
0.001	15.0033	1-3-4-7-8-9	41.7905	1-3-4-7-8-9

The approximate membership function of the fuzzy total duration time is shown in Fig. 5.2.

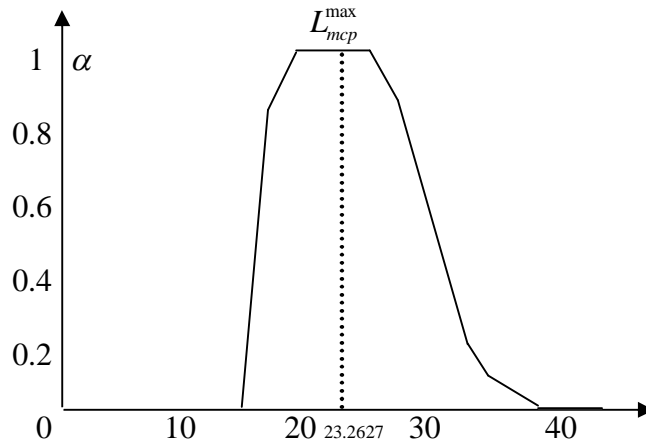


Fig. 5.2

Fuzzy total duration time

### 5.3.1 Discussion

In above example, the case of  $\alpha = 0$  is not discussed since several shape functions are not defined on  $\alpha = 0$ . The  $\alpha = 1.0$  cut shows the total duration time that is most likely, and the  $\alpha$ -cut that approaches 0 shows the range in which the total duration time could appear. In this example, although the total duration time is fuzzy, its most likely value falls between 20 and 25, and it is almost impossible for its value to fall outside the range of 15.0033 and 41.7905. Note that in Table 5.3, two distinct fuzzy critical paths  $FCP_1 = \{1 \rightarrow 3 \rightarrow 6 \rightarrow 8 \rightarrow 9\}$  and  $FCP_2 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 8 \rightarrow 9\}$  have been found, Notably, for the lower bounds of the total duration time at all possibility levels  $\alpha$ , the critical path is identified as  $FCP_2 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 8 \rightarrow 9\}$  and no results are given on lower bound for  $FCP_1 = \{1 \rightarrow 3 \rightarrow 6 \rightarrow 8 \rightarrow 9\}$ , indicating that it is impossible that the path  $FCP_1$  becomes the critical path at all  $\alpha$  possibility levels. However, for the upper bounds

the total duration time, the critical path is identified as  $FCP_1$  at  $\alpha = 0.2, 0.3, \dots, 0.8$ , and is  $FCP_2$  at other possibility levels. Following this, it seems that the possibility of  $FCP_2$  being the most critical path is larger than that of  $FCP_1$ . This can be verified as follows. By applying Yager's method shown in (5.16), the Yager ranking indices for fuzzy activity times  $\tilde{T}_{ij}$  are calculated as:  $I(\tilde{T}_{13}) = 3, I(\tilde{T}_{36}) \cong 7.1667, I(\tilde{T}_{34}) = 0, I(\tilde{T}_{47}) = 9.1, I(\tilde{T}_{68}) = 4.1333, I(\tilde{T}_{78}) = 3,$  and  $I(\tilde{T}_{89}) \cong 8.1627. I(FCP_1) \cong 22.4627, I(FCP_2) = 23.2627.$  Of these  $I(FCP_2)$  is the largest, indicates that  $FCP_2$  is the most critical path. According to Definition 5.3, the relative path degree of criticality of any path can be calculated via (5.18).

$$R \deg(FCP_1) = \frac{I(FCP_1)}{\max\{I(FCP_1), I(FCP_2)\}} \cong 0.9656$$

## 5.4 Conclusion

Several researchers have investigated the critical path analysis in the project network with fuzzy activity times. Clearly, when the activity times are fuzzy, the total duration time is also fuzzy. This chapter develops an approach to find the membership function of the fuzzy total duration time when the activity times are fuzzy numbers. The underlying idea is based on linear programming formulation and Zadeh's extension principle to transform the fuzzy CPM problem to a pair of parametric linear programs. By enumerating different values of the possibility level  $\alpha$ , the lower and upper bound of the  $\alpha$ -cuts of the fuzzy total duration time are calculated to approximate the membership function. Clearly, if the obtained total duration time is a crisp value, then it may lose some useful information. In this chapter, the fuzzy total duration time is expressed by a membership function that completely conserves the fuzziness of activity times.

## *Chapter 6*

# **MODIFIED LINEAR PROGRAMMING METHOD FOR ANALYZING THE FUZZY CRITICAL PATH**

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In the previous chapter, a method is presented to find FCP of a given network by representing the parameters by different types of fuzzy numbers. In this chapter, an alternative method is represented to solve same type of problem and by solving same numerical problem, it is shown that the results of the presented method and the results obtained by using method in the previous chapter are identical while the method presented in this chapter is easy as compared to the method presented in previous chapter.

### **6.1 Modified method to find the fuzzy critical path**

In this method, model (5.2) is transformed into crisp one via defuzzifying the fuzzy activity times in the objective function into crisp ones by using a fuzzy ranking method. To find the critical path in fuzzy environments, it suffices to solve the crisp linear programming model which is transformed from model (5.2), where the fuzzy objective value of model (5.2) is defuzzified to a crisp one based on a fuzzy number ranking method. For defuzzification, it is required to select such fuzzy number ranking method which is simple and could still be applied to identify the critical path by using the streamlined network simplex method [32]. One popular approach that meets these requirements is the Yager's ranking method. According to (5.17), since it is calculated for the convex fuzzy number  $\tilde{t}$  from the extreme values of its  $\alpha$ -cut,  $t_\alpha^L$  and  $t_\alpha^U$ , rather than its membership function, it is not require knowing the explicit form of the membership functions of the fuzzy numbers to be ranked.

### 6.1.1 Crisp transformation

Consider the critical path problem formulated as model (5.2) with  $m$  paths. Let  $x_{ij}^k, (i, j) \in A$  be the  $k^{th}$  basic feasible solution which corresponds to the  $k^{th}$  path

$p_k, k = 1, 2, \dots, m$ . Then,  $\tilde{D}^{(k)} = \sum_{i=1}^n \sum_{j=1}^n \tilde{T}_{ij} x_{ij}^k$  is the fuzzy total duration time of the  $k^{th}$  path.

Of these  $m$  paths, the one with the largest total duration time

$\tilde{D}^* = \text{maximum}\{\tilde{D}^{(k)}, k = 1, 2, \dots, m\}$  could be identified as a FCP, called the most critical

path. By applying this method to find the  $\tilde{D}^*$  it suffices to find the largest Yager ranking

index  $I(\tilde{D}^*) = \text{maximum}\{I(\tilde{D}^{(k)}), k = 1, 2, \dots, m\}$ . Furthermore, since the Yager's method

possesses the properties of linearity and additivity, we have

$$I(\tilde{D}^{(k)}) = I\left(\sum_{i=1}^n \sum_{j=1}^n \tilde{T}_{ij} x_{ij}^{(k)}\right) = \sum_{i=1}^n \sum_{j=1}^n I(\tilde{T}_{ij}) x_{ij}^{(k)}$$

i.e., the maximum fuzzy objective value  $\tilde{D}^*$  corresponds to the maximum ranking index

$I(\tilde{D}^*) = \text{Maximum}_k \left\{ \sum_{i=1}^n \sum_{j=1}^n I(\tilde{T}_{ij}) x_{ij}^{(k)} \right\}$ . Consequently, the critical path problem with

fuzzy activity times can be formulated as follows:

$$\begin{aligned} I(\tilde{D}^*) = \quad & \text{Maximize} \quad \sum_{i=1}^n \sum_{j=1}^n I(\tilde{T}_{ij}) x_{ij} \\ & \text{subject to} \quad \sum_{j=1}^n x_{1j} = 1, \\ & \quad \quad \quad \sum_{j=1}^n x_{ij} = \sum_{k=1}^n x_{ki}, \quad i = 2, \dots, n-1, \\ & \quad \quad \quad \sum_{k=1}^n x_{kn} = 1, \\ & \quad \quad \quad x_{ij} \geq 0, \quad (i, j) \in A. \end{aligned} \tag{6.1}$$

From the obtained optimal basic feasible solution  $x_{ij}^*, (i, j) \in A$  of model (6.1), the most critical path  $p^*$  can be identified, and the fuzzy total duration time can be calculated as

$$\tilde{D}^* = \sum_{i=1}^n \sum_{j=1}^n \tilde{T}_{ij} x_{ij}^* .$$

Since this crisp transformation is based upon the Yager's method which possesses sound properties of linearity, and additivity, model (6.1) can be proved to be correct. Furthermore, next section the dual formulation of the fuzzy CPM problem is also provided for the proof of the correctness of this crisp transformation.

### 6.1.2 Dual verification

According to the duality theorem of linear programming, the primal and the dual models have the same objective value [32]. Thus, one way to show the validity of the above crisp transformation is to formulate the dual of model (5.2):

$$\begin{aligned} & \text{Minimize} && y_n - y_1 \\ & \text{subject to} && y_j - y_i \geq T_{ij}, \quad (i, j) \in A, \\ & && y_i, y_j \text{ unrestricted in sign}, \quad \forall (i, j) \in A \end{aligned} \tag{6.2}$$

where  $y_i$  and  $y_j$  denotes the occurrence time of nodes  $i$  and  $j$ , respectively. Each constraint specifies the precedence relationships among the different activities. The objective is to find the shortest time span such that all precedence relationships are satisfied. When activity times are fuzzy numbers, Model (6.2) becomes

$$\begin{aligned} & \text{Minimize} && y_n - y_1 \\ & \text{subject to} && y_j - y_i \geq \tilde{T}_{ij}, \quad (i, j) \in A, \\ & && y_i, y_j \text{ unrestricted in sign}, \quad \forall (i, j) \in A \end{aligned} \tag{6.3}$$

which is a linear program with right-handed values of constraints being fuzzy numbers. One way to deal with model (6.3) is to apply the crisp transformation stated above, and then we have

$$\begin{aligned}
& \text{Minimize} && y_n - y_1 \\
& \text{subject to} && y_j - y_i \geq I(\tilde{T}_{ij}), \quad (i, j) \in A, \\
& && y_i, y_j \text{ unrestricted in sign}, \quad \forall (i, j) \in A
\end{aligned} \tag{6.4}$$

The dual of model (6.4) is exactly the same as model (6.1), which proves the correctness of the crisp transformation stated in the preceding subsection.

### 6.1.3 Relative path degree of criticality

Consider the basic feasible solution to model (6.1). As stated in section 6.1.1, any basic feasible solution is corresponding to one path in the project network. Setting the path degree of criticality of the most critical path as 1.0, denoted as  $\text{deg}_{\text{Cr}}^R(p^*) = 1$ , the relative path degree of criticality of the  $k^{\text{th}}$  path  $p_k, k = 1, 2, \dots, m$ , can be defined as follows:

$$\text{deg}_{\text{Cr}}^R(p_k) = \frac{\sum_{i=1}^n \sum_{j=1}^n I(\tilde{T}_{ij}) x_{ij}^{(k)}}{\sum_{i=1}^n \sum_{j=1}^n I(\tilde{T}_{ij}) x_{ij}^*} = \frac{I(\tilde{D}^{(k)})}{I(\tilde{D}^*)}, \quad k = 1, 2, \dots, m. \tag{6.5}$$

## 6.2 Numerical example

**Example 6.1:** Consider a project whose corresponding network is given in Fig 6.1. The activity times fuzzy numbers of  $L_{ij} - R_{ij}$  type,  $(i, j) \in A$ , as shown in Table 6.1.

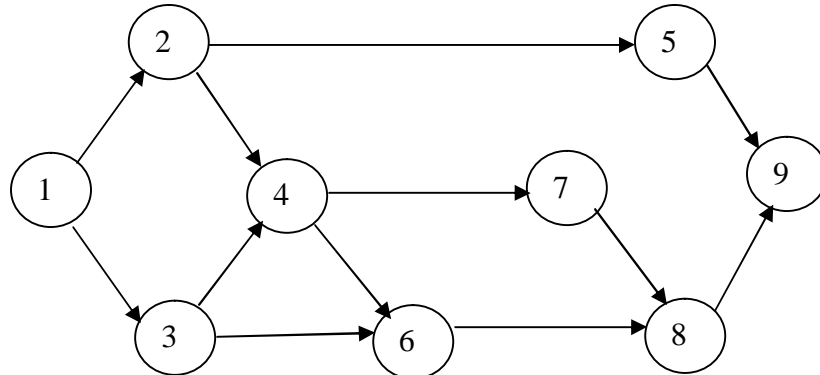


Fig. 6.1

A project network

Table 6.1 Fuzzy activity time and left and right shape functions for each activity

$\tilde{T}_{ij}$	$L_{ij}(x)$	$R_{ij}(x)$
$\tilde{T}_{12} = (1, 1.5, 1, 1)_{L_{12}-R_{12}}$	$L_{12}(x) = \text{Maximum}(1 - x^2, 0)$	$R_{12}(x) = \text{Maximum}(0, 1 - x)$
$\tilde{T}_{13} = (2, 3, 0, 2)_{L_{13}-R_{13}}$	$L_{13}(x) = e^{-x}$	$R_{13}(x) = \text{Maximum}(0, 1 - x)$
$\tilde{T}_{24} = (0, 0, 0, 0)_{L_{24}-R_{24}}$	$L_{24}(x) = \text{Maximum}(0, 1 - x)$	$R_{24}(x) = \text{Maximum}(0, 1 - x)$
$\tilde{T}_{25} = (2, 3, 1, 2)_{L_{25}-R_{25}}$	$L_{25}(x) = \text{Maximum}(0, 1 - x^4)$	$R_{25}(x) = e^{-x}$
$\tilde{T}_{34} = (0, 0, 0, 0)_{L_{34}-R_{34}}$	$L_{34}(x) = \text{Maximum}(0, 1 - x)$	$R_{34}(x) = \text{Maximum}(1 - x^2, 0)$
$\tilde{T}_{36} = (6, 7, 0, 2)_{L_{36}-R_{36}}$	$L_{36}(x) = e^{-x^2}$	$R_{36}(x) = \text{Maximum}(1 - x^2, 0)$
$\tilde{T}_{46} = (5, 5, 1, 1)_{L_{46}-R_{46}}$	$L_{46}(x) = \text{Maximum}(0, 1 - x)$	$R_{46}(x) = \text{Maximum}(0, 1 - x^4)$
$\tilde{T}_{47} = (9, 9, 1, 1)_{L_{47}-R_{47}}$	$L_{47}(x) = \text{Maximum}(0, 1 - x^4)$	$R_{47}(x) = e^{-x}$
$\tilde{T}_{59} = (8, 9, 2, 4)_{L_{59}-R_{59}}$	$L_{59}(x) = \text{Maximum}(0, 1 - x^4)$	$R_{59}(x) = \text{Maximum}(0, 1 - x^2)$
$\tilde{T}_{68} = (4, 4, 2, 2)_{L_{68}-R_{68}}$	$L_{68}(x) = \text{Maximum}(0, 1 - x^2)$	$R_{68}(x) = \text{Maximum}(0, 1 - x^4)$
$\tilde{T}_{78} = (3, 4, 2, 0)_{L_{78}-R_{78}}$	$L_{78}(x) = \text{Maximum}(0, 1 - x)$	$R_{78}(x) = \text{Maximum}(0, 1 - x^4)$
$\tilde{T}_{89} = (6, 9, 2, 3)_{L_{89}-R_{89}}$	$L_{89}(x) = \text{Maximum}(0, 1 - x^2)$	$R_{89}(x) = e^{-x^2}$

**Solution:** According to model (5.2), this problem can be formulated as follows:

$$\begin{aligned} \text{Maximize } & (\tilde{T}_{12}x_{12} + \tilde{T}_{13}x_{13} + \tilde{T}_{24}x_{24} + \tilde{T}_{25}x_{25} + \tilde{T}_{34}x_{34} + \tilde{T}_{36}x_{36} + \tilde{T}_{46}x_{46} + \tilde{T}_{47}x_{47} + \tilde{T}_{59}x_{59} \\ & + \tilde{T}_{68}x_{68} + \tilde{T}_{78}x_{78} + \tilde{T}_{89}x_{89}) \end{aligned}$$

$$\text{subject to } x_{12} + x_{13} = 1,$$

$$x_{12} = x_{24} + x_{25},$$

$$x_{13} = x_{34} + x_{36},$$

$$x_{24} + x_{34} = x_{46} + x_{47},$$

$$x_{25} = x_{59},$$

$$x_{36} + x_{46} = x_{68},$$

$$x_{47} = x_{78},$$

$$x_{68} + x_{78} = x_{89},$$

$$x_{59} + x_{89} = 1,$$

$$x_{12}, x_{13}, x_{24}, x_{25}, x_{34}, x_{36}, x_{46}, x_{47}, x_{59}, x_{68}, x_{78}, x_{89} \geq 0.$$

The associated mathematical programming problem based on model (6.1) is:

$$\begin{aligned} \text{Maximize } & I(\tilde{T}_{12})x_{12} + I(\tilde{T}_{13})x_{13} + I(\tilde{T}_{24})x_{24} + I(\tilde{T}_{25})x_{25} + I(\tilde{T}_{34})x_{34} + I(\tilde{T}_{36})x_{36} + I(\tilde{T}_{46})x_{46} \\ & + I(\tilde{T}_{47})x_{47} + I(\tilde{T}_{59})x_{59} + I(\tilde{T}_{68})x_{68} + I(\tilde{T}_{78})x_{78} + I(\tilde{T}_{89})x_{89} \end{aligned}$$

$$\text{subject to } x_{12} + x_{13} = 1,$$

$$x_{12} = x_{24} + x_{25},$$

$$x_{13} = x_{34} + x_{36},$$

$$x_{24} + x_{34} = x_{46} + x_{47},$$

$$x_{25} = x_{59},$$

$$x_{36} + x_{46} = x_{68}, \tag{6.6}$$

$$x_{47} = x_{78},$$

$$x_{68} + x_{78} = x_{89},$$

$$x_{59} + x_{89} = 1,$$

$$x_{12}, x_{13}, x_{24}, x_{25}, x_{34}, x_{36}, x_{46}, x_{47}, x_{59}, x_{68}, x_{78}, x_{89} \geq 0.$$

The Yager's ranking indices for  $\tilde{T}_{ij}$  are calculated as:

$$(t_{12})_{\alpha}^L = 1 - \sqrt{1 - \alpha} \quad (t_{12})_{\alpha}^U = (2.5 - \alpha)$$

$$I(\tilde{T}_{12}) = \int_0^1 \frac{1}{2} \left( (1 - \sqrt{1 - \alpha}) + (2.5 - \alpha) \right) d\alpha = 1.16667$$

Similarly values can be calculated for others activities.

$$I(\tilde{T}_{36}) = 7.16667, I(\tilde{T}_{13}) = I(\tilde{T}_{78}) = 3, I(\tilde{T}_{24}) = I(\tilde{T}_{34}) = 0, I(\tilde{T}_{25}) = 3.1, I(\tilde{T}_{46}) = 5.15$$

$$I(\tilde{T}_{47}) = 9.1, I(\tilde{T}_{59}) = 9.03333, I(\tilde{T}_{68}) = 4.13333, I(\tilde{T}_{89}) = 8.16267,$$

Substituting these values into model (6.6) results in a conventional critical path problem that can be solved using a mathematical programming solver e.g. Lingo. An optimal solution of  $x_{13}^* = x_{34}^* = x_{47}^* = x_{78}^* = x_{89}^* = 1$ , and  $x_{12}^* = x_{24}^* = x_{25}^* = x_{36}^* = x_{46}^* = x_{59}^* = x_{68}^* = 0$  with  $I(\tilde{D}^*) = 23.26267$  is obtained. That is, the most critical path is  $p^* = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 8 \rightarrow 9\}$ . To verify this path is the critical one, we find all paths for comparing their Yager's ranking indices. There are only six paths starting from Node 1 to Node 9.

$$p_1 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 8 \rightarrow 9\}, \quad p_2 = \{1 \rightarrow 3 \rightarrow 6 \rightarrow 8 \rightarrow 9\},$$

$$p_3 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 6 \rightarrow 8 \rightarrow 9\}, \quad p_4 = \{1 \rightarrow 2 \rightarrow 5 \rightarrow 9\},$$

$$p_5 = \{1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 8 \rightarrow 9\}, \quad p_6 = \{1 \rightarrow 2 \rightarrow 4 \rightarrow 7 \rightarrow 8 \rightarrow 9\}.$$

The ranking indices  $I(\tilde{D}_{p_i}), i = 1, 2, \dots, 6$ , are calculated by using the Yager's approach.

$$I(\tilde{D}_{p_1}) = 3 + 0 + 9.1 + 3 + 8.16267 = 23.26267$$

$$I(\tilde{D}_{p_2}) = 3 + 7.16667 + 4.13333 + 8.16267 = 22.46267$$

$$I(\tilde{D}_{p_3}) = 20.446, \quad I(\tilde{D}_{p_4}) = 13.3, \quad I(\tilde{D}_{p_5}) = 18.61267, \quad I(\tilde{D}_{p_6}) = 21.42934$$

The maximum of these six ranking indices is  $I(\tilde{D}_{p_1})=23.26267$ , indicating  $p_1$  is really the critical path. Setting the relative path degree of criticality of the most critical path  $p_1$  to be 1, the relative path degrees of criticality of other paths can be calculated using (6.5). The results are listed in Table 6.2.

Table 6.2 Relative path degree of criticality

Path	$\text{deg}_{\text{Cr}}^R(p_i)$
$p_1 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 8 \rightarrow 9\}$	1
$p_2 = \{1 \rightarrow 3 \rightarrow 6 \rightarrow 8 \rightarrow 9\}$	0.9574
$p_6 = \{1 \rightarrow 2 \rightarrow 4 \rightarrow 7 \rightarrow 8 \rightarrow 9\}$	0.9212
$p_3 = \{1 \rightarrow 3 \rightarrow 4 \rightarrow 6 \rightarrow 8 \rightarrow 9\}$	0.8789
$p_5 = \{1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 8 \rightarrow 9\}$	0.8001
$p_4 = \{1 \rightarrow 2 \rightarrow 5 \rightarrow 9\}$	0.5717

### 6.3 Conclusion

This chapter presents a simple approach to solve the CPM problem with fuzzy activity times that are more realistic than crisp ones. On the basis of Yager's ranking method, the fuzzy CPM problem is transformed to a crisp one which can be solved by using the conventional streamlined linear programming approaches. The obtained FCP is assured to be the most critical one from the viewpoint of Yager. We then calculated critical path and the relative path degree of criticality of a path. Clearly, the presented approach is not confined to the fuzzy activity times of  $L-R$  type. Other types such as  $L-L$  type, triangular type, and trapezoidal type are also applicable. Moreover, it is not require the knowledge of the explicit form of membership functions of the fuzzy activity times.

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