Fixed Point Theorems for Different Mappings in Various Spaces

A Thesis

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by

Pooja Dhawan
Reg no: 951411004

under the supervison of

Dr. Jatinderdeep Kaur
Associate Professor
School of Mathematics
TIET, Patiala 147004 INDIA

Dr. Sanjeev Bakshi
Assistant Professor
School of Mathematics
TIET, Patiala 147004 INDIA

THAPAR INSTITUTE
OF ENGINEERING & TECHNOLOGY
(Deemed to be University)

TIET
PATIALA-147004, PUNJAB, INDIA
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Candidate's Declaration

I hereby certify that the work, which is being presented in the thesis, entitled **Fixed Point Theorems for Different Mappings in Various Spaces**, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work carried out during the period **26 July 2014** to **16 May, 2019** under the supervision of **Dr. Jatinderdeep Kaur**, Associate Professor, School of Mathematics and **Dr. Sanjeev Bakshi**, Assistant Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

Date: 11-02-2020

(Pooja Dhawan)
Candidate

It is certified that the above statement made by the candidate is correct to the best of our knowledge.

(Dr. Jatinderdeep Kaur)
Associate Professor
School of Mathematics
TIET, Patiala, 147004 INDIA

(Dr. Sanjeev Bakshi)
Assistant Professor
School of Mathematics
TIET, Patiala, 147004 INDIA

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(Pooja Dhawan)

List of Symbols

$U = V, U \neq V$:	Equality and Inequality for sets
$[u, v], [u, v)$ etc.:	Intervals on the real line
$f(u)$ *or* fu:	Image of u under f
iff:	if and only if
sup:	Supremum (or least upper bound)
inf:	Infimum (or greatest lower bound)
$\lim\ u_n = u$:	Limit of a sequence u_n
max:	Maximum
min:	Minimum
$\mathbb{N}$:	The set of all natural numbers
$\mathbb{R}$:	The set of all real numbers
$\mathbb{R}^+$:	The set of all non negative real numbers
$\mathbb{R}^n$:	n-dimensional Euclidean space
$\|u\|$:	Absolute value of u
$\rightarrow$:	Implication
$\Leftrightarrow$:	Logical equivalence
$F(\mathfrak{D})$:	The set of fixed points of $\mathfrak{D}$
$I = I_A$:	The identity mapping
2^A:	The collection of all subsets of A
$\hat{d}(u, v)$:	distance from one point to another
$\hat{d}(u, B)$:	distance between the point u and the set B
$B(a, r)$:	$\{b \in A : \hat{d}(a, b) < r\}$
$B(\bar{a}, r)$:	$\{b \in A : \hat{d}(a, b) \leq r\}$
ϕ:	empty set
$\in$:	belonging to or belong to
$\notin$:	does not belong to

Abstract

The present thesis entitled "**Fixed Point Theorems for Different Mappings in Various Spaces**" comprises certain investigations carried out by me at the School of Mathematics (SOM), Thapar Institute of Engineering and Technology, Patiala, under the supervision of **Dr. Jatinderdeep Kaur**, Associate Prof., SOM, TIET, Patiala and **Dr. Sanjeev Bakshi**, Assistant. Prof., SOM, TIET, Patiala.

Nonlinear analysis deals with solving nonlinear problems in many branches of mathematics, physics and in industry. Fixed-point theory is an important branch of nonlinear analysis. It is used to investigate the conditions under which single-valued or multivalued mappings have solutions. Numerous problems occuring in different branches of mathematics, such as differential equations, optimization theory and variational analysis can be modeled by the equation

$$u = Du$$

where D is a nonlinear operator defined on a metric space. The solutions of this equation are called fixed points of D. In 1906, the French mathematician Frechét [57] introduced metric spaces which helped in the development of fixed point theory.

The first fixed point theorem in metric spaces for contraction mappings was proved by a Polish mathematician Banach [20] in 1922. This theorem is known as Banach's fixed point theorem or the Banach contraction principle. By now, this result has become one of the most popular and effective tool in solving existence problems in many branches of mathematical analysis. Due to the simplicity and usefulness of this basic theorem, it has become a very popular tool for proving the existence and uniqueness theorems in various branches of mathematical analysis. In 1968, Kannan [84] introduced a contractive condition which possessed a unique fixed point like that of Banach. However, unlike the Banach condition, Kannan [84] proved that there are mappings that have a discontinuity in their domain but still have fixed point, although such mappings are continuous at their fixed point. This paper of Kannan [84] lead to lot of improvements and extensions of Banach contraction principle.

In the present thesis, several fixed point results in various abstract spaces such as Quasi partial metric spaces, b-metric-like spaces, partially ordered metric spaces and Partial Hausdorff metric spaces and for various types of contraction and ex-

pansion mappings have been discussed and thereby many existing results have been extended and generalized.

The present thesis consists of seven chapters. **Chapter 1** is introductory. In this chapter, apart from setting up the notations and terminologies to be used in the sequel, a brief review of the work done in the area of fixed point theory is presented. Further, a systematic plan of the results presented in the subsequent chapters is given towards the end of this chapter.

In **Chapter 2**, Some fixed point results in Hausdorff metric spaces using $\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$-multivalued contractive type mappings are investigated. Various theorems regarding this class of contractive pair of mappings have been studied in this chapter. The results presented in this chapter extend some well-known relevant results (such as Kikkawa [91], Nadler [111], Samet *et al.* [140]) existing in literature. Some illustrative examples are provided to demonstrate the main results. A result in homotopy theory is presented as an application of these results. In attempt to give extension to the results of Shahi *et al.* [147], some fixed point results in partially ordered metric spaces for (ξ, α)-expansive mappings are studied. Along with these results, an application to periodic boundary value problem is presented to show the usefulness of our main results.

Chapter 3 deals with some interesting fixed results in Quasi partial metric spaces. In this chapter, a new approach in the field of aggregation theory and metric aggregation is introduced. Firstly, the notion of expansion between quasi partial metric spaces through distance aggregation perspective along with suitable aggregation properties is defined. After that, with the help of aggregation functions, the concept of projective Ψ-expansion has been introduced and several fixed point results in Quasi partial metric spaces are obtained through this notion. Furthermore, sufficient conditions are also provided to characterize aggregation operator to ensure the existence and uniqueness of fixed point. The results derived in this chapter generalize the results presented by Borsik and Doboš [26], Martin *et al.* [101], Massanet and Valero [102], Mayor and Valero [105]. Some comparative examples are also given to check the efficacy of obtained results. Moreover, an application to asymptotic complexity analysis has also been presented in the end of this chapter.

In **Chapter 4**, the concept of contraction has been extended by introducing

$\mathfrak{D}$-Contraction defined on a family $\mathfrak{F}$ of bounded functions. Also, a new notion of fixed function for a metric space is introduced. Some fixed function theorems have been obtained by using different forms of $\mathfrak{D}$-Contraction along with demonstrative examples. In order to support the applicability of these results, an application to intensity modulated radiation therapy (IMRT) has also been presented following the FMO (Fluency map optimization) due to Shepard *et al.* [148] and Tian *et al.* [151]. This application is based on determining the best suitable treatment plan for tumor patients getting intensity modulated radiation therapy(IMRT). The fixed function obtained in this way represents the suitable doses for a number of tumor patients at the same time.

Chapter 5 deals with the generalization of the results of Wang *et al.* [153] and Jungck [78] by introducing a new notion of $\mathfrak{P}$-expansion defined on a family $\mathfrak{F}$ of bounded functions. Some fixed function theorems in complete metric spaces have been investigated by using various kinds of generalized expansive conditions. In addition to these results, some common fixed function theorems for a pair of weakly compatible mappings are also derived. Moreover, an application based on deciding the suitable doses of intensity values for the patients under Tomotherapy is also given in the end of the chapter.

In **Chapter 6**, the notion of $\mathcal{F}$-generalized multivalued contractive type mappings is introduced by using $\mathcal{C}$-class functions. Some common fixed point results for weakly isotone increasing set-valued mappings in the setting of ordered partial metric spaces are investigated for this new class. The results presented in this chapter generalize various relevant results from the current literature *e.g.* Ansari [13], Nashine [113], Nazari *et al.* [114], Wang *et al.* [153] and references therein.

Chapter 7 is devoted to some common and coincidence fixed point results for a sequence of functions in complete metric spaces. There exists vast literature showing the existence of fixed points using expansive mappings. But the existence of common and coincidence fixed points for a sequence of functions using expansive mappings is still not explored much. In this chapter, some results for a sequence of mappings satisfying generalized weakly expansive conditions in the setting of quasi partial metric spaces have been investigated. To demonstrate the usability of presented results, some useful deductions along with some examples are also provided. In the end of this chapter, some common coupled fixed point theorems are investigated in the framework of b-metric-like spaces in order to generalize the results of Bhaskar and Lakshmikantham [24], Alghamdi *et al.* [8]. Towards the end of this chapter, some relevant topics for further research have been suggested based on the present study.

The thesis is concluded by listing the Bibliography with various publications which are cited in this research work.

Research Publications

Published/Accepted Papers in Referred Journals

1. Pooja Dhawan and Jatinderdeep Kaur, Some common fixed point theorems in ordered partial metric spaces via $\mathcal{F}$-generalized contractive type mappings, Mathematics, 2019, 7(2), 193. (SCIE) (I.F. 1.109)
2. Pooja Dhawan, Jatinderdeep Kaur and Vishal Gupta, Novel results on fixed function and their application based on the best approximation of the treatment plan for tumor patients getting intensity modulated radiation therapy (IMRT), Proceedings of the Estonian Academy of Sciences, 2019, 68(3), 1-12. (SCIE) (I.F. 0.843)
3. Pooja Dhawan, Kapil Jain and Jatinderdeep Kaur, $\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$-multivalued contractive mappings and related results in complete metric spaces with an application, Mathematics, 2019, 7(1), 68. (SCIE) (I.F. 1.109)
4. Pooja Dhawan, Jatinderdeep Kaur and Vishal Gupta, Some new results on expansion mapping and their application, Journal of Intelligent and Fuzzy Systems, 2019, 37(6), 5611-5618. (SCIE) (I.F. 1.637)
5. Pooja Dhawan, Jatinderdeep Kaur and Sanjeev Bakshi, Fixed points of expansive mappings in quasi partial metric spaces, International journal of Advance Research in Science and Engineering, 5(11), 2016.
6. Pooja Dhawan and Jatinderdeep Kaur, Fixed Point Theorems for (ξ, α)-Expansive Mappings in Partially Ordered Sets, International Journal of Computer and Mathematical Sciences, 6(6), 2017.

Communicated Papers

1. Vishal Gupta, Pooja Dhawan and Jatinderdeep Kaur, On Ψ-projective expansion, Quasi partial metrics aggregation with an Application, Journal of Applied Analysis and Computation. (Accepted)
2. Pooja Dhawan and Jatinderdeep Kaur, Existence of coincidence and common fixed points for a sequence of mappings in Quasi partial metric spaces, Journal

of Advanced Research in Dynamical and Control Systems.

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Chapter 1

Introduction

1.1 Origin of Fixed Point Theory

The natural way to prove whether an equation possesses a solution or not is to simply put it up as a fixed point problem. So, we can say that the problem of finding the fixed points of a function $g(u)$ is same as the problem of finding the solution of the equation $G(u) = 0$, where $g(u) = G(u) + u$. In other words, the problem is to find such a function h for which $h(u) = u$. To clarify this fact, consider the quadratic equation $u^2 - 7u + 6 = 0$. The roots of this simple quadratic equation are $u = 1$ and $u = 6$. Let us rewrite this equation in the following way:

$$u = \frac{u^2 + 6}{7}.$$

By assuming $g(u) = \frac{u^2+6}{7}$, we obtain that $u = 1$ and $u = 6$ are fixed points of g. So, we can say that the problem of locating the fixed points of $g(u)$ is identical to the problem of figure out the solution of the equation $G(u) = 0$, where $g(u) = G(u)+u$. The results which are concerned with the existence of fixed points are called fixed point theorems. However, many fixed point results existing in literature are not much constructive as these results only guarantee the existence of fixed point and do not provide any help in finding the same. Generally, in mathematics, the problem of solving a system of equations can be transformed to the problem of finding fixed points of a self-map $\mathfrak{D}$ defined on a suitable space U.

In 1886, Poincare [123] firstly worked with the concept of fixed point. His fixed point result is known as "*Poincare's last geometric theorem*" which asserts the existence of at least two fixed points for an area preserving twist homeomorphism of an annulus. The first fixed point result for a topological space was given by Dutch mathematician Brouwer [30] in 1912 which states that "*A continuous self mapping defined on the closed unit ball in Euclidean space has at least one fixed point*". His result was applicable to finite dimensional spaces and it forms a base for many fixed point results. This result proved to be a key theorem due to its use in various areas of mathematics and economics. Later on, this theorem was extended for set-valued

functions by Kakutani [82] in 1941. Till today, there exists a large number of generalizations to *Brouwer's fixed point theorem.*
In the present thesis, the main emphasis has been given on Metric fixed point theory. This introductory chapter is primitive in nature. It contains some elementary concepts and related results which will be regularly utilized in the subsequent chapters of this thesis. Many useful and well known results and notations related to metric fixed point theory have not been mentioned in this introductory chapter due to space limitations. For more results, one can cite Aggarwal *et al.* [4], Goebel and Kirk [59], Istratescu [75], Takahashi [150].

Some definitions and results presented in this introductory chapter are further written in subsequent chapters for the sake of convenience.

1.2 Metric Fixed Point Theory

In 1906, Maurice Fréchet [57] introduced Metric Space in his work "*Sur quelques points du Calcul fonctionnel*". A metric space is a collection of those elements of a set for which distances between all the elements of that set are defined. Those distances, taken together, forms a metric on the set.

Definition 1.2.1 *[4] Let $(U, \hat{d})$ be a metric space, then a mapping $\mathfrak{D} : U \to U$ is said to be Lipschitz if we have,*

$$\hat{d}(\mathfrak{D}(u), \mathfrak{D}(v)) \leq a\ \hat{d}(u, v) \quad for \quad all \quad u, v \in U \text{ and } a \geq 0. \tag{1.2.1}$$

We notice that a Lipschitz map is evidently continuous. The smallest value of a such that (1.2.1) holds true, is called the Lipschitz constant for $\mathfrak{D}$ (denoted by L). If $L < 1$, the mapping $\mathfrak{D}$ is said to be a contraction mapping and if $L = 1$, the mapping $\mathfrak{D}$ is nonexpansive.
In 1922, the first fixed point result for a complete metric space came into light which was proved by Polish mathematician Stefan Banach [20] and his result is popularly known as "*Banach Contraction Principle*". This principle ensures that the application of a continuous self mapping on two points of a complete metric space contracts the distance between those two points. According to his result, "*A contraction self mapping defined on a complete metric space possesses a unique fixed point which can be obtained as the limit of an iteration scheme constructed*

by applying repeated images of the mapping (starting from an arbitrary point of space)".

Definition 1.2.2 *[20] For a metric space $(U, \hat{d})$, a mapping $D : U \to U$ is called a contraction mapping on U if for any real number λ with $0 \leq \lambda < 1$, the following inequality holds:*

$$\hat{d}(Du, Dv) \leq \lambda \ \hat{d}(u, v) \quad for \quad all \ u, v \in U.$$

Theorem 1.2.3 *[20] Let $(U, \hat{d})$ be a complete metric space and D be the contraction mapping defined on U. Then D possesses a unique fixed point u in U i.e. $Du = u$.*

This classic result is a natural ingredient in fixed point theory due to its utility and simplicity. This theorem is one of the most well known and important existence principles in Mathematics. Since the theorem and its many equivalent formulations or extensions are classical tools in showing the existence of solutions for many problems in pure and applied mathematics, a lot of work has been done in this direction. During last few decades, Banach's result has been generalized in different forms. For details, the articles of Ćirić [42], Kirk and Sims [92], Rhoades [131, 132], Rus [138] need special attention.
Banach contraction principle was the only result till 1968 to show the existence of a fixed point. But the drawback of this classic result was the requirement of a continuous mapping. In 1968, this problem was resolved by Kannan [84] by using modified contraction conditions. He proved these results for discontinuous mappings although these maps are continuous at their fixed points. Following his work, extensive research began in this direction and various contractive conditions are presented by the researchers in next two decades.

Kannan [84] used the following contraction condition for a discontinuous self-mapping $\mathfrak{D} : U \to U$ to prove his results:

$$\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \leq a[\hat{d}(u, \mathfrak{D}u) + \hat{d}(v, \mathfrak{D}v)],$$

$\forall \ u, v \in U$ where $a \in [0, \frac{1}{2})$.

After that, Chatterjea [34] and Reich [129] presented some results by using modified contractive conditions.

Theorem 1.2.4 *[34] Let $(U, \hat{d})$ be a complete metric space and D be the self mapping defined on U which satisfy the condition*

$$\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \leq a[\hat{d}(u, \mathfrak{D}v) + \hat{d}(v, \mathfrak{D}u)],$$

$\forall\ u, v \in U$ where $a \in [0, \frac{1}{2})$. Then D admits a unique fixed point in U.

Theorem 1.2.5 *[129] Let $(U, \hat{d})$ be a complete metric space and D be the self mapping defined on U which satisfy the condition*

$$\hat{d}(Du, Dv) \leq \alpha\ \hat{d}(u, Du) + \beta\ \hat{d}(v, Dv) + \gamma\ \hat{d}(u, v);$$

for all $u, v \in U$ and α, β, γ non negative with $\alpha + \beta + \gamma < 1$. Then D admits a unique fixed point in U.

In 1973, inspired from these results, Hardy and Rogers [65] established the following contraction condition to prove their result:

$$\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \leq a_1[\hat{d}(u, \mathfrak{D}u) + \hat{d}(v, \mathfrak{D}v)] + a_2[\hat{d}(v, \mathfrak{D}u) + \hat{d}(u, \mathfrak{D}v)] + a_3\ \hat{d}(u, v),$$

for all $u, v \in U$ where $a_1, a_2, a_3 \geq 0,\ 2a_1 + 2a_2 + a_3 < 1$.
Later on, in 1974, a more generalized contractive condition was obtained by Ćirić [42] to prove the uniqueness.

$$\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \leq a\ m(u, v),$$

$\forall\ u, v \in U$ and $a \in [0, 1)$ where

$$m(u, v) = max\{\hat{d}(u, v), \hat{d}(u, \mathfrak{D}u), \hat{d}(v, \mathfrak{D}v), \hat{d}(v, \mathfrak{D}u), \hat{d}(u, \mathfrak{D}v)\}.$$

Numerous authors such as Edelstein [54], Reich [130] have introduced a number of contractive conditions following the same way. In 1977, Rhoades [131] presented a study to compare these various contractive conditions.
In 1976, Jungck [78] began a research line in the field of commuting mappings and common fixed points. These results are the extensions of existing fixed point results.

Theorem 1.2.6 *[78] Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two commuting self maps of a complete metric space $(U, \hat{d})$ such that $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$ and $\mathfrak{D}_2$ is continuous. If there exists a real* *number* $\lambda \in (0,1)$ *satisfying*

$$\hat{d}(\mathfrak{D}_1 u, \mathfrak{D}_1 v) \leq \lambda\ \hat{d}(\mathfrak{D}_2 u, \mathfrak{D}_2 v), \ for\ all\ u, v \in U.$$

Then the maps $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *possess a unique common fixed point.*

Jungck's theorem has so many applications in various fields and many authors such as Chandok and Narang [35], Chugh and Kumar [40], Vats *et al.* [41], Dhage [50], Rhoades [133] generalized his results in numerous ways.
In 1982, Sessa [146] extended Jungck's result by introducing the concept of weakly commutative mappings:

Definition 1.2.7 *[146] Let* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *be two self maps of a complete metric space* $(U, \hat{d})$. *Then the pair* $(\mathfrak{D}_1, \mathfrak{D}_2)$ *is called weakly commuting if*

$$\hat{d}(\mathfrak{D}_1\mathfrak{D}_2 u, \mathfrak{D}_2\mathfrak{D}_1 u) \leq \hat{d}(\mathfrak{D}_1 u, \mathfrak{D}_2 u), \ for\ all\ u \in U.$$

Every pair of commuting maps is weakly commuting but the converse need not be true.

In 1988, Jungck [79] introduced Compatible maps and proved some coincidence fixed point results. His works have been the basis for a number of papers.
Later on, Bhaskar and Lakshmikantham [24] introduced the notion of coupled fixed points. Then, Lakshmikantham and Ćirić [96] generalized these results to prove common coupled fixed point and coupled coincidence theorems.

Definition 1.2.8 ***Coupled fixed point****:*
A point $(u, v) \in U \times U$ *is said to be a coupled fixed point for a mapping* $\mathfrak{D} : U \times U \to U$ *if* $\mathfrak{D}(u, v) = u$ *and* $\mathfrak{D}(v, u) = v$.

Definition 1.2.9 ***Coupled coincidence point****:*
A point $(u, v) \in U \times U$ *is said to be a coupled coincidence point for the mappings* $\mathfrak{D}, \hat{\mathfrak{D}} : U \times U \to U$ *if* $\mathfrak{D}(u, v) = \hat{\mathfrak{D}}(u, v)$ *and* $\mathfrak{D}(v, u) = \hat{\mathfrak{D}}(v, u)$.

Definition 1.2.10 ***Common coupled fixed point****:*
A point $(u, v) \in U \times U$ *is called a common coupled fixed point of the mappings* $\mathfrak{D}, \hat{\mathfrak{D}} : U \times U \to U$ *if* $u = \mathfrak{D}(u, v) = \hat{\mathfrak{D}}(u, v)$ *and* $v = \mathfrak{D}(v, u) = \hat{\mathfrak{D}}(v, u)$.

After that, many researchers such as Abbas *et al.* [1], Choudhary and Kundu [38], Samet [140] *etc.* gave generalizations to above results with different types of mappings.
In 1969, Nadler [111] introduced multi-valued contractive mappings and proved

some fixed point results in a complete metric space.

Definition 1.2.11 *[111] Let U and V be non-empty sets. Then a mapping $\mathfrak{D}$ from U to V is said to be a multi-valued mapping if $\mathfrak{D}$ is a function from U to the power set of V . We generally denote a multi-valued mapping by $\mathfrak{D} : U \to 2^V$.*

Definition 1.2.12 *[111] A point $u_0 \in U$ is called a fixed point of the multi-valued map $\mathfrak{D}$ if $u_0 \in \mathfrak{D}u_0$.*

Each single valued map can be seen as a multi-valued map. Let $\hat{\mathfrak{D}} : U \to V$ be a single valued map. Now define $\mathfrak{D} : U \to 2^V$ by $\mathfrak{D}u = \{\hat{\mathfrak{D}}(u)\}$.
It is to be noted that $\mathfrak{D}$ is multi-valued map *iff* for each $u \in U$, $\mathfrak{D}u \subseteq V$. Nadler's fixed point result for multivalued mappings is as follows:

Theorem 1.2.13 *[111] Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{D} : U \to \mathcal{P}_{cb}(U)$ be a multivalued mapping satisfying*

$$\mathcal{H}(\mathfrak{D}u, \mathfrak{D}v) \leq l \ \hat{d}(u, v),$$

$\forall \ u, v \in U$ and $l \in [0, 1)$ where
1. $\mathcal{P}_{cb}(U)$ is the family of all closed and bounded subsets of U.
2. $\mathcal{H}(U, V) = max\{\rho(U, V), \rho(V, U)\}$ for $U, V \in \mathcal{P}_{cb}(U)$
3. $\rho(U, V) = sup\{\hat{d}(u, V) : u \in U\}$, $\hat{d}(u, V) = inf\{\hat{d}(u, v) : v \in V\}$.
Then $\mathfrak{D}$ possesses a fixed point.

Many authors gave generalizations to Nadler's result in various ways (see [37], [76], [127], [128]). In particular, Kikkawa and Suzuki [91] presented the following version of Nadler's result by introducing a strictly decreasing function to modify the contraction.

Theorem 1.2.14 *[91] Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{D} : U \to \mathcal{P}_{cb}(U)$ be a multivalued mapping satisfying*

$$\eta(r) \ \hat{d}(u, \mathfrak{D}u) \leq \hat{d}(u, v) \ \Rightarrow \ \mathcal{H}(\mathfrak{D}u, \mathfrak{D}v) \leq r \ \hat{d}(u, v),$$

for all $u, v \in U$ and for some $r \in [0, 1)$ where $\eta : [0, 1) \to (\frac{1}{2}, 1]$ is a mapping defined by $\eta(r) = \frac{1}{1+r}$. Then $\mathfrak{D}$ possesses a fixed point.

The research for expansive type mappings for a metric space was initiated by Wang *et al.* [153]. Till now, there exists a large variety of results in the field of expansive mappings.

Theorem 1.2.15 *[153] Let $(U, \hat{d})$ be a complete metric space. If $\mathfrak{D} : U \to U$ is an onto mapping and there exists a constant $c > 1$ such that*

$$\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \geq c\ \hat{d}(u, v)\ for\ every\ u, v \in U.$$

Then, $\mathfrak{D}$ has a unique fixed point in U.

In 1992, Daffer and Kaneko [48] established some fixed point theorems using a pair of expansive type mappings.

Theorem 1.2.16 *[48] Let $(U, \hat{d})$ be a complete metric space. If $\mathfrak{D}_1 : U \to U$ is a surjective mapping and $\mathfrak{D}_2 : U \to U$ is an injective mapping. If there exists a constant $c > 1$ such that*

$$\hat{d}(\mathfrak{D}_1 u, \mathfrak{D}_1 v) \geq c\ \hat{d}(\mathfrak{D}_2 u, \mathfrak{D}_2 v)\ for\ every\ u, v \in U.$$

Then, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ possess a unique common fixed point in U.

Following this concept, several authors such as Ahmad *et al.* [5], Ahmed *et al.* [6], Dhawan *et al.* [52], Han and Xu [64], Shahi *et al.* [147] worked on various results for expansion mappings.

1.3 Various kinds of abstract spaces

1.3.1 Partial Metric Spaces

Matthews [104] presented the idea of partial metric spaces in 1994 by extending the notion of metric spaces using dataflow networks. Matthews [104] introduced a surprising property in this space that self distance of a point of the space may be non-zero (*i.e.* the self distance $p(u, u) = 0$ need not hold). The inspiration for the concept of non-zero self distance is explained by Bukatin *et al.* [33] in a presentation. In fact, the results presented by Matthews [104] play a vital role in the theory of computation. Various generalizations of this result have been used in constructing computation models in the field of computer science (for details, see [67], [116], [134], [135], [138]).
Following are the definitions and results presented by Matthews [104] in partial metric spaces:

Definition 1.3.1 *[104] Let U be a non-empty set. A function $p : U \times U \to \mathbb{R}^+$*

is said to be a partial metric on U if the following postulates hold true:

$(P1)$ $u = v \; if \; and \; only \; if \; p(u,u) = p(v,v) = p(u,v)$;
$(P2)$ $p(u,u) \leq p(u,v)$;
$(P3)$ $p(u,v) = p(v,u)$;
$(P4)$ $p(u,w) \leq p(u,v) + p(v,w) - p(v,v)$.

for all $u, v, w \in U$. The set U equipped with the metric p defined above is called a partial metric space and it is denoted by (U, p) (in short PMS).

Every partial metric p generates a T_0 topology τ_p on U with a base containing the collection of open balls $\{B_p(u, \epsilon), u \in U, \epsilon > 0\}$, where

$$B_p(u, \epsilon) = \{v \in U : p(u,v) < p(u,u) + \epsilon\},$$

$\forall \; u \in U$ and $\epsilon > 0$.

The mappings $d_p, d_m : U \times U \to \mathcal{R}^+$ given by

$$d_p(u,v) = 2p(u,v) - p(u,u) - p(v,v),$$

and

$$d_m(u,v) = max\{p(u,v) - p(u,u), p(u,v) - p(v,v)\}$$

are the metrics on U induced by partial metric p. It is evident that the metrics d_p and d_m are equivalent.

Definition 1.3.2 *[104] For a partial metric space (U, p), a sequence $\{u_n\}$ in U is said to be*
(i) convergent if there exists a point $u \in U$ such that $p(u,u) = \lim_{n\to\infty} p(u_n, u)$,
(ii) a Cauchy sequence if the $\lim_{m,n\to\infty} p(u_n, u_m)$ exists (and is finite).

Definition 1.3.3 *[104] A partial metric space (U, p) is said to be complete if every Cauchy sequence $\{u_n\}$ in U converges w.r.t. τ_p to a point $u \in U$ such that $p(u,u) = \lim_{n,m\to\infty} p(u_n, u_m)$.*

Theorem 1.3.4 *[104] Let (U, p) be a partial metric space. Then*
(i) $\{u_n\}$ is said to be a Cauchy sequence in (U, p) iff *it is a Cauchy sequence in the metric space (U, d_p),*
(ii) (U, p) is complete iff *the metric space (u, d_p) is complete. Also, $\lim_{n\to\infty} d_p(u_n, u) =$*

0 *iff* $p(u,u) = \lim_{n\to\infty} p(u_n,u) = \lim_{m,n\to\infty} p(u_n,u_m)$.

Lemma 1.3.5 *[104] Let (U,p) be a partial metric space and let $\{u_n\}$ be a sequence in U such that*

$$\lim_{n\to\infty} p(u_n,u_{n+1}) = 0.$$

If the sequence $\{u_{2n}\}$ is not a Cauchy sequence in (U,p), then there exist $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n(k) > m(k) > k$ such that the following four sequences

$$p(u_{2m(k)},u_{2n(k)+1}), p(u_{2m(k)},u_{2n(k)}), p(u_{2m(k)-1},u_{2n(k)+1}), p(u_{2m(k)-1},u_{2n(k)})$$

tend to $\epsilon > 0$ when $k \to \infty$.

Lemma 1.3.6 *[104] If the sequence $\{u_n\}$ with $\lim_{n\to\infty} d_p(u_{n+1},u_n) = 0$ is not a Cauchy sequence in (U,p), then for each $\epsilon > 0$, there exist two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n(k) > m(k) > k$ such that the following four sequences*

$$p(u_{m(k)},u_{n(k)+1}), p(u_{m(k)},u_{n(k)}), p(u_{m(k)-1},u_{n(k)+1}), p(u_{m(k)-1},u_{n(k)})$$

tend to $\epsilon > 0$ when $k \to \infty$.

Following these results, several authors obtained various results in this context. For details, the references ([11], [12], [17], [44], [87], [120], [121], [134]) can be cited.

1.3.2 Partial Ordered Metric Spaces

Definition 1.3.7 *(***Partial Order***) A partial order on a non-empty set U is a binary relation $\preceq$ satisfying the following postulates:*
(i) U is reflexive i.e. *$u \preceq u$ for every $u \in U$,*
(ii) U is antisymmetric i.e. *if $u \preceq v$ and $v \preceq u$, then $u = v$,*
(iii) U is transitive i.e. *if $u \preceq v$ and $v \preceq w$, then $u \preceq w$.*
for every $u, v, w \in U$. A set along with a partial order $\preceq$ defined on it, is called a partially ordered set. The collection of subsets of a given set along with partial order "$\subseteq$" and the set of natural numbers with partial order of "divisibility" are some common examples of partially ordered sets.

Definition 1.3.8 *(***Comparable elements***) Let $(U,\preceq)$ be a partially ordered set and $u, v \in U$. Then $u, v \in U$ are said to be comparable if either $u \preceq v$ or $v \preceq u$.*

Definition 1.3.9 *Let $(U, \preceq)$ be a non-empty set and $u, v \in U$. Then the space $(U, \hat{d}, \preceq)$ is said to be a partially ordered metric space if*
(i) $(U, \preceq)$ is a partially ordered set and
(ii) $(U, \hat{d})$ is a metric space.

Turinici [152] firstly proved some fixed point results for ordered metrizable uniform spaces in 1986. Later on, Ran and Reurings [126] obtained some applications of Turinici's results to solve matrix equations.

Theorem 1.3.10 *[126] Let U be a non-empty set equipped with a partial order such that each pair $u, v \in U$ has a lower or upper bound. Let $(U, \hat{d})$ be a metric space where $\hat{d}$ is a complete metric defined on U. Let $\mathfrak{D} : U \to U$ be a continuous monotone self mapping. Assume that*
(i) there exists $a \in (0,1)$ such that $\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \leq a\ \hat{d}(u, v)$ for every $u, v \in U$ and $u \succeq v$;
(ii) there exists $u_0 \in U$ such that $\mathfrak{D}u_0 \preceq u_0$.
Then $\mathfrak{D}$ possesses a unique fixed point $u^* \in U$ and for every $w \in U$, the iterative sequence $\{\mathfrak{D}^n(w)\}$ (where $n \in \mathbb{N}$) of $\mathfrak{D}$ starting from w converges to $u^* \in U$.

After that, Nieto and López [117, 118] used the results of Ran and Reurings [126] to solve first order differential equations. In [117], Nieto and López obtained some results by removing condition of continuity of the mapping $\mathfrak{D}$.

Theorem 1.3.11 *[117] Let U be a non-empty set equipped with a partial order such that each pair $u, v \in U$ has a lower or upper bound. Let $(U, \hat{d})$ be a metric space where $\hat{d}$ is a complete metric defined on U. Let $\mathfrak{D} : U \to U$ be an increasing self mapping. Assume that*
(i) there exists $a \in (0,1)$ such that $\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \leq a\ \hat{d}(u, v)$ for every $u, v \in U$ and $u \succeq v$;
(ii) there exists $u_0 \in U$ such that $u_0 \preceq \mathfrak{D}u_0$.
(iii) if $\{u_n\}$ is an increasing sequence converging to $u \in U$, then $u_n \preceq u$ for each $n \in \mathbb{N}$. Then $\mathfrak{D}$ possesses a unique fixed point $u^* \in U$ and for every $w \in U$, the iterative sequence $\{\mathfrak{D}^n(w)\}$ (where $n \in \mathbb{N}$) of $\mathfrak{D}$ starting from w converges to $u^* \in U$.

In 2008, Aggarwal *et al.* [3] extended these results by using modified contractive conditions.

Theorem 1.3.12 *[3] Let U be a non-empty set equipped with a partial order $\preceq$ and a complete metric $\hat{d}$. Let $\mathfrak{D} : U \to U$ be an increasing self mapping satisfying the following conditions:*

(i) there exists an increasing map $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$ and

$$\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \leq \varphi(max\{\hat{d}(u,v), \hat{d}(u,\mathfrak{D}u), \hat{d}(v,\mathfrak{D}v), \frac{1}{2}[\hat{d}(u,\mathfrak{D}v) + \hat{d}(v,\mathfrak{D}u)]\});$$

(ii) there exists $u_0 \in U$ such that $u_0 \preceq \mathfrak{D}u_0$;
(iii) either $\mathfrak{D}$ is continuous or if $\{u_n\}$ is an increasing sequence in U converging to $u \in U$, then $u_n \preceq u$ for all $n \in \mathbb{N}$;
for every $u, v \in U$ with $u \succeq v$. Then $\mathfrak{D}$ possesses at least one fixed point in U.

Then, Bhaskar and Lakshmikantham [96] initiated the research for coupled fixed points in partially ordered metric spaces. Following are the results presented by [96].

Theorem 1.3.13 *Let $(U, \hat{d})$ be a partially ordered metric space where $\hat{d}$ be a complete metric and $\preceq$ be the partial order defined on U. Let $\mathfrak{D} : U \times U \to U$ be a continuous mapping with mixed monotone property. Assume that there exists $c \in [0, 1)$ such that*

$$\hat{d}(\mathfrak{D}(l,m), \mathfrak{D}(u,v)) \leq \frac{c}{2}[\hat{d}(l,u) + d(m,v)],$$

for each $l \geq u, m \leq v$.
If there exist some $l_0, m_0 \in U$ which satisfy $l_0 \leq \mathfrak{D}(l_0, m_0)$ and $m_0 \geq \mathfrak{D}(m_0, l_0)$, then, there exist $l, m \in U$ such that $l = \mathfrak{D}(l,m)$ and $m = \mathfrak{D}(m,l)$.

Theorem 1.3.14 *In addition to the hypotheses of Theorem 1.3.13, assume that there exists $(u,v), (u^*, v^*) \in U \times U$, there exists a pair $(n_1, n_2) \in U \times U$ which is comparable to (u,v), then $\mathfrak{D}$ possesses a unique coupled fixed point.*

Later on, Lakshmikantham and Ćirić [96] presented the notion of g-monotone mapping and investigated some coupled coincidence and common coupled fixed point results.

Definition 1.3.15 *[96] Let $(U, \preceq)$ be a partially ordered set and $\mathfrak{D} : U \times U \to U$ and $g : U \to U$ be two mappings. Then the mapping $\mathfrak{D}$ is said to possess mixed g-monotone property if $\mathfrak{D}(u,v)$ is monotonically g-nondecreasing in u and is monotonically g-nonincreasing in v* i.e.

$$u_1, u_2 \in U \text{ and } gu_1 \preceq gu_2 \text{ implies } \mathfrak{D}(u_1, v) \preceq \mathfrak{D}(u_2, v),$$
$$v_1, v_2 \in U \text{ and } gv_1 \preceq gv_2 \text{ implies } \mathfrak{D}(u, v_2) \preceq \mathfrak{D}(u, v_1), \text{ for each } u, v \in U.$$

Definition 1.3.16 *[96] Let $(U, \preceq)$ be a partially ordered set. A pair $(u, v) \in U \times U$ is called a coupled coincidence point for the mappings $\mathfrak{D} : U \times U \to U$ and $g : U \to U$ if $\mathfrak{D}(u, v) = gu$ and $\mathfrak{D}(v, u) = gv$.*

Definition 1.3.17 *[96] Let $(U, \preceq)$ be a partially ordered set. The mappings $\mathfrak{D} : U \times U \to U$ and $g : U \to U$ are said to be commutative mappings if $g(\mathfrak{D}(u, v)) = \mathfrak{D}(gu, gv)$ for every $u, v \in U$.*

Theorem 1.3.18 *[96] Let $(U, \hat{d} \preceq)$ be a partially ordered set where $\hat{d}$ is a complete metric and $\mathfrak{D} : U \times U \to U$ and $g : U \to U$ be two mappings such that $\mathfrak{D}$ has mixed g-monotone property. Assume that there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ and $\lim_{s \to t^+} \phi(s) < t$ for every $t > 0$ with*

$$\hat{d}(\mathfrak{D}(l, m), \mathfrak{D}(u, v)) \leq \phi\left(\frac{\hat{d}(gl, gu) + \hat{d}(gm, gv)}{2}\right)$$

for each $l, m, u, v \in U$ with $gl \leq gu$ and $gm \geq gv$. Let $\mathfrak{D}(U \times U) \subseteq g(U)$, g commutes with $\mathfrak{D}$ and be continuous. Also, assume that either
(1) $\mathfrak{D}$ is continuous or;
(2) U has the following properties:
(a) if a nonincreasing sequence $\{u_n\}$ converges to u, then $u \leq u_n$ for all n;
(b) if a nondecreasing sequence $\{v_n\}$ converges to v, then $v_n \leq v$ for all n.
if there exist $u_0, v_0 \in U$ such that $gu_0 \leq \mathfrak{D}(u_0, v_0)$ and $gv_0 \geq \mathfrak{D}(v_0, u_0)$, then there exist $u, v \in U$ such that $gu = \mathfrak{D}(u, v)$ and $gv = \mathfrak{D}(v, u)$ i.e. *$\mathfrak{D}$ and g have a coupled coincidence point.*

Motivated by the above results, several authors obtained a number of common and coupled coincidence fixed point results in partially ordered metric spaces (For details, see [1], [16], [17], [38], [39]).

1.3.3 Quasi Partial Metric Spaces

In 2013, Karapinar *et al.* [86] introduced the notion of quasi partial metric spaces by removing the symmetric axiom from the properties of metric spaces. Quasi partial metric is a generalized form of quasi metric due to Bakhtin [19] and partial metric due to [104].

Definition 1.3.19 *[19] Let U be a non-empty set. A function $\hat{d} : U \times U \to \mathbb{R}^+$ is said to be a quasi metric if it satisfies the following axioms:*

(QM1) $\hat{d}(u,u) = 0$,
(QM2) $\hat{d}(u,v) \leq \hat{d}(u,w) + \hat{d}(w,v)$,
for all $u, v, w \in U$. *The pair* $(U, \hat{d})$ *is said to be a quasi-metric space.*

Definition 1.3.20 *[104] Let* U *be a non-empty set. A function* $p : U \times U \to \mathbb{R}^+$ *is said to be a partial metric on* U *if the following postulates hold true:*

$(P1)$ $u = v$ *if and only if* $p(u,u) = p(v,v) = p(u,v)$;
$(P2)$ $p(u,u) \leq p(u,v)$;
$(P3)$ $p(u,v) = p(v,u)$;
$(P4)$ $p(u,w) \leq p(u,v) + p(v,w) - p(v,v)$.

for all $u, v, w \in U$. *The set* U *equipped with the metric* p *defined above is called a partial metric space and it is denoted by* (U, p) *(in short PMS).*

Following are the definitions and results presented by Karapinar *et al.* [86] in quasi partial metric spaces:

Definition 1.3.21 *[86] A mapping* $q : U \times U \to \mathbb{R}^+$ *is said to be a quasi partial metric if the following conditions hold:*

$(q1)$ *if* $0 \leq q(u,u) = q(u,v) = q(v,v)$, *then* $u = v$;
$(q2)$ $q(u,u) \leq q(u,v)$;
$(q3)$ $q(u,u) \leq q(v,u)$;
$(q4)$ $q(u,v) \leq q(u,w) + q(w,v) - q(w,w)$;

for all $u, v \in U$. *Then the pair* (U, q) *is called a quasi partial metric space (QPMS).*

If $q(v,u) = q(u,v)$ for each $u, v \in U$, then (U, q) reduces to partial metric space (PMS). Also, for a quasi-partial metric q on U, the mapping $d_q : U \times U \to \mathbb{R}_+$ defined by

$$d_q(u,v) = q(u,v) + q(v,u) - q(u,u) - q(v,v)$$

is a (usual) metric on U.

Definition 1.3.22 *[86] Let* (U, q) *be a quasi partial metric space (QPMS). Then*

1. *a sequence* $\{u_n\} \subset U$ *converges to* $u \in U$ *if and only if* $q(u,u) = \lim_{n\to\infty} q(u, u_n) = \lim_{n\to\infty} q(u_n, u)$;
2. *a sequence* $\{u_n\} \subset U$ *is called a Cauchy sequence* iff $\lim_{m,n\to\infty} q(u_m, u_n)$ *and* $\lim_{n,m\to\infty} q(u_n, u_m)$ *exist and are finite;*

3. *the quasi partial metric space (U, q) is said to be complete if every Cauchy sequence $\{u_n\} \subset U$ converges to some $u \in U$ such that $q(u, u) = \lim_{m,n\to\infty} q(u_m, u_n) = \lim_{n,m\to\infty} q(u_n, u_m)$;*

4. *a mapping $\mathfrak{D} : U \to U$ is said to be continuous at $u_0 \in U$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{D}(B(u_0, \delta)) \subset B(\mathfrak{D}(u_0), \epsilon)$.*

Lemma 1.3.23 *[86] Let (U, q) be a quasi partial metric space (QPMS). Then a sequence $\{u_n\} \subset U$ converges to $u \in U$* iff *$q(u, u) = \lim_{n\to\infty} q(u, u_n) = \lim_{n\to\infty} q(u_n, u)$.*

Lemma 1.3.24 *[86] For a quasi-partial metric q on U,*

$$p_q(u, v) = \frac{1}{2}[q(u, v) + q(v, u)];\ \ u, v \in U$$

is a partial metric on U.

Lemma 1.3.25 *[86] Let (U, q) be a QPMS. Let (U, p_q) be the corresponding PMS and let (U, d_{p_q}) be the corresponding metric space. The following statements are equivalent:*

1. *The sequence $\{u_n\}$ is Cauchy in (U, q) .*
2. *The sequence $\{u_n\}$ is Cauchy in (U, p_q).*
3. *The sequence $\{u_n\}$ is Cauchy in (U, d_{p_q}).*

Lemma 1.3.26 *[86] Let (U, q) be a QPMS. Let (U, p_q) be the corresponding PMS and let (U, d_{p_q}) be the corresponding metric space. The following statements are equivalent:*

1. *(U, q) is complete.*
2. *(U, p_q) is complete.*
3. *(U, d_{p_q}) is complete.*

Moreover,

$$\begin{aligned} \lim_{n\to\infty} d_{p_q}(u, u_n) &= 0 \Leftrightarrow p_q(u, u) = \lim_{n\to\infty} p_q(u, u_n) = \lim_{n,m\to\infty} p_q(u_n, u_m) \\ &\Leftrightarrow q(u, u) = \lim_{n\to\infty} q(u, u_n) = \lim_{n,m\to\infty} q(u_n, u_m) \\ &= \lim_{n\to\infty} q(u_n, u) = \lim_{m,n\to\infty} q(u_m, u_n). \end{aligned}$$

Lemma 1.3.27 *[86] Let (U, q) be a quasi partial metric space. Then $q(u, v) = 0 \Rightarrow u = v$ and if $y \neq v$, then $q(u, v) > 0$ and $q(v, u) > 0$.*

Karapinar *et al.* [86] proved the following results:

Theorem 1.3.28 *[86] Let (U, q) be a quasi-partial metric space and $\mathfrak{D} : U \to U$ be a self mapping. Then the following conditions hold:*

1. *There exists $\varphi : U \to \mathbb{R}^+$ such that $q(u, \mathfrak{D}u) \leq \varphi(u) - \varphi(\mathfrak{D}u)$ for all $u \in U$ if and only if $\sum_{n=0}^{\infty} q(\mathfrak{D}^n u, \mathfrak{D}^{n+1} u)$ converges for all $u \in O(u)$.*

2. *There exists $\varphi : U \to \mathbb{R}^+$ such that $q(u, \mathfrak{D}u) \leq \varphi(u) - \varphi(\mathfrak{D}u)$ for all $u \in U$ if and only if $\sum_{n=0}^{\infty} q(\mathfrak{D}^n u, \mathfrak{D}^{n+1} u)$ converges for all $u \in O(u)$.*

where $O(u) = \{u, \mathfrak{D}u, \mathfrak{D}^2 u, ...\}$ is called the orbit of u.

Recently, some authors have extensively worked on Quasi Partial metric spaces. For details, one can refer to ([52], [56], [88], [94]).

1.3.4 Hausdorff Metric Spaces

In 1905, Pompeiu [124] introduced the concept of distance between two closed sets in his thesis under the guidance of Poincare [123]. According to his definition, "*If U and V are two closed and bounded sets, then the distance between the point $u \in U$ and the set V is defined as*

$$\hat{d}(u, V) = min\{\hat{d}(u, v) : \ v \in V\},$$

where $\hat{d}(u, v)$ denotes the usual distance between the points u and v."

After that, Pompeiu [124] defined the the notion of asymmetric distance between two sets U and V as

$$D(U, V) = max\{\hat{d}(u, V) : \ u \in U\}.$$

In 1914, by using Pompeiu's distance, German mathematician Felix Hausdorff [66] defined a new distance notion denoted by $\mathcal{H}(U, V)$ called Hausdorff metric where

$$\mathcal{H}(U, V) = max\{D(U, V), D(V, U)\}.$$

Hausdorff distance is the greatest of all the distances measured from a point in one set to some point in the other set.
In the past decades, the research in the field of multivalued mappings has been

developed in various ways. Nadler [111] initiated the work on multivalued mappings Hausdorff metric spaces and presented a multivalued version of Banach's result in a robust way.

Definition 1.3.29 *[111] Let U and V be non-empty sets. Then a mapping $\mathfrak{D}$ from U to V is said to be a multi-valued mapping if $\mathfrak{D}$ is a function from U to the power set of V. We generally denote a multi-valued mapping by $\mathfrak{D} : U \to 2^V$.*

Definition 1.3.30 *[111] A point $u_0 \in U$ is called a fixed point of the multi-valued map $\mathfrak{D}$ if $u_0 \in \mathfrak{D}u_0$.*

Theorem 1.3.31 *[111] Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{D} : U \to \mathcal{P}_{cb}(U)$ be a multivalued mapping satisfying*

$$\mathcal{H}(\mathfrak{D}u, \mathfrak{D}v) \leq l \ \hat{d}(u, v),$$

for all $u, v \in U$ and $l \in [0, 1)$ where
$\mathcal{P}_{cb}(U)$ is the collection of all non-empty closed and bounded subsets of U;
$\mathcal{H}(U, V) = max\{\rho(U, V), \rho(V, U)\}$ for $U, V \in \mathcal{P}_{cb}(U)$ and;
$\rho(U, V) = sup\{\hat{d}(u, V) : u \in U\}$, $\hat{d}(U, V) = inf\{\hat{d}(u, v) : v \in V\}$.
Then $\mathfrak{D}$ possesses a fixed point.

Afterwards, several extensions were obtained by making use of various types of contractive conditions (see [43], [45], [48], [145]).

1.3.5 b-metric-like Spaces

In 1989, Bakhtin [19] established b-metric spaces as an extension of metric spaces by defining a b-metric constant ($\geq$1) in triangle inequality of metric axiom.

Definition 1.3.32 *[19] **(b-metric Space)** For a non-empty set U, the function $B : U \times U \to [0, +\infty)$ is called a b-metric space on the set U if it satisfies the following properties:*

- $(B1)$ $B(u, v) = 0 \Leftrightarrow u = v$;
- $(B2)$ $B(u, v) = B(v, u)$;
- $(B3)$ $B(u, v) \leq s(B(u, w) + B(w, v))$ *for all* $u, v, w \in U$ (*where* $s \geq 1$).

The pair (U, B) is called a b-metric space.

This idea gave researchers to think in a magnificent way for their fixed point results. After that, in 2013, Alghamdi [8] introduced an enlargement to b-metric space named b-metric-like spaces by considering non-zero self distance property.

Definition 1.3.33 *[8] **(b-metric-like Space)** For a non-empty set U, A b-metric-like on the set U is a mapping $B : U \times U \to [0, +\infty)$ if the following three axioms hold true for all $u, v, w \in U$:*

$(B1)$ $B(u, v) = 0 \Rightarrow u = v$;

$(B2)$ $B(u, v) = B(v, u)$;

$(B3)$ $B(u, v) \leq s(B(u, w) + B(w, v))$ *where* $s \geq 1$.

The set U equipped with a metric B defined on it, is called a b-metric like space and is denoted by (U, B).

Definition 1.3.34 *[8] Let $(U, \hat{d})$ be a b-metric-like space. For any sequence $\{u_n\}$ of points of a non-empty set U, a point $u \in U$ is said to be the limit of the sequence $\{u_n\}$ if for each $\epsilon > 0$, there exists some $n_\epsilon \in \mathbb{N}$, for which $\hat{d}(u_n, u) < \epsilon$ for all $n \geq n_\epsilon$ and we say that the sequence u_n is convergent to u.*

Definition 1.3.35 *[8] For a b-metric-like space $(U, \hat{d})$,*
(i) A sequence u_n in U is called Cauchy if for each $\epsilon > 0$, there exists some $n_\epsilon \in \mathbb{N}$, for which $\hat{d}(u_n, u_m) < \epsilon$ for all $n, m \geq n_\epsilon$;
(ii) A b-metric-like space $(U, \hat{d})$ is called a complete space iff *every Cauchy sequence u_n in U converges in U.*

Afterwards, a number of authors established the related results in various abstract spaces (see [46], [71], [157]).

1.3.6 Cone Metric Spaces

In 2007, Huang and Zhang [70] introduced the concept of cone metric space for an ordered real Banach space.

Definition 1.3.36 ***Cone***
A cone D is a subset of a real Banach space B with the following properties:

(1) D *is closed, non − empty and* $D \neq \{0\}$;

(2) *if* a, b *are any two nonnegative real numbers and* $u, v \in D$, *then* $au + bv \in D$;

(3) $D \cap \ (-D) = \{0\}$.

For a cone $D \subseteq B$, the partial ordering $\leq$ w.r.t. *D is defined by $u \leq v$ if and only if $v - u \in D$.*

Definition 1.3.37 ***Cone Metric Space***
Let U be any non-empty set and B be a real Banach space with partial ordering $\leq$ w.r.t. *the cone $D \subseteq B$. Then the mapping $\hat{d} : U \times U \to B$ is said to be a cone metric on U if it satisfies the following axioms:*

$(d1)$ $\quad 0 \leq \hat{d}(u, v) \quad for \ \ all \ \ u, v \in U;$

$(d2)$ $\quad \hat{d}(u, v) = 0 \ \ if \ \ and \ \ only \ \ if \ \ u = v;$

$(d3)$ $\quad \hat{d}(u, v) = \hat{d}(v, u) \quad for \ \ all \ \ u, v \in U;$

$(d4)$ $\quad \hat{d}(u, v) \leq \hat{d}(u, w) + \hat{d}(w, v) \quad for \ \ all \ \ u, v, w \in U.$

The set U equipped with the metric $\hat{d}$ defined as above is called a cone metric space and it is denoted by $(U, \hat{d})$.

1.4 Various kinds of Contraction/Expansion Mappings

1.4.1 $(\alpha - \psi)$ contractive mappings

In 2012, Samet *et al.* [142] introduced the idea of $(\alpha - \psi)$ contractive mappings and α-admissible mappings by establishing a new family of nondecreasing functions and obtained some fixed points results.

Definition 1.4.1 *[142] Let $(U, \hat{d})$ be a metric space and Ψ be the family of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:*

(1) $\displaystyle\sum_{n=1}^{+\infty} \psi^n(u) < +\infty \ for \ every \ u > 0, \ where \ \psi^n \ is \ \ n^{th} \ iterate \ of \ \psi;$

(2) ψ *is nondecreasing.*

Then a mapping $\mathfrak{D} : U \to U$ is called $(\alpha - \psi)$ contractive mapping if there exists two functions $\alpha : U \times U \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(u, v) \ \hat{d}(\mathfrak{D}u, \mathfrak{D}v) \leq \psi(\hat{d}(u, v)) \ for \ all \ u, v \in U.$$

Definition 1.4.2 *[142] Let $\mathfrak{D} : U \to U$ be a self mapping and $\alpha : U \times U \to [0, +\infty)$. The mapping $\mathfrak{D}$ is called α-admissible if*

$$\alpha(u, v) \geq 1 \Rightarrow \alpha(\mathfrak{D}u, \mathfrak{D}v) \geq 1.$$

for all $u, v \in U$.

Following are the results proved by Samet *et al.* [142]:

Theorem 1.4.3 *[142] Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{D} : U \to U$ be an $(\alpha - \psi)$-contractive mapping satisfying the following conditions:*

- *(i)* $\mathfrak{D}$ *is* $\alpha -$ *admissible*;
- *(ii)* *there exists* $u_0 \in U$ *such that* $\alpha(u_0, \mathfrak{D}u_0) \geq 1$;
- *(iii)* $\mathfrak{D}$ *is continuous.*

Then $\mathfrak{D}$ has a fixed point in U.

Theorem 1.4.4 *[142] Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{D} : U \to U$ be an $(\alpha - \psi)$-contractive mapping satisfying the following conditions:*

- *(i)* $\mathfrak{D}$ *is* $\alpha -$ *admissible*;
- *(ii)* *there exists* $u_0 \in U$ *such that* $\alpha(u_0, \mathfrak{D}u_0) \geq 1$;
- *(iii)* *if* $\{u_n\}$ *is a sequence in* U *such that* $\alpha(u_n, u_{n+1}) \geq 1$ *for all* n *and* $u_n \to u \in U$ *as* $n \to +\infty$, *then* $\alpha(u_n, u) \geq 1$ *for all* n.

Then $\mathfrak{D}$ has a fixed point in U.

By adding the following condition (H_1) to the hypotheses of above theorems, Samet *et al.* [140] showed the uniqueness of the fixed point:
(H_1): If there exists $w \in U$ such that for all $u, v \in U$,

$$\alpha(u, w) \geq 1 \text{ and } \alpha(v, w) \geq 1 \text{ where } u \geq w, \quad v \geq w.$$

Theorem 1.4.5 *[142] Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{D} : U \times U \to U$ be a given mapping. Assume that there exists $\psi \in \Psi$ and a function $\alpha : U^2 \times U^2 \to [0, +\infty)$ such that*

$$\alpha((l, m), (u, v)) \ \hat{d}(\mathfrak{D}(l, m), \mathfrak{D}(u, v)) \leq \frac{1}{2}\psi(d(l, u) + d(m, v)),$$

for all $(l,m),(u,v) \in U \times U$. *Also,*
(i) *For every pair* $(l,m),(u,v) \in U \times U$, *we have*

$$\alpha((l,m),(u,v)) \geq 1 \Rightarrow \alpha((\mathfrak{D}(l,m),\mathfrak{D}(m,l)),(\mathfrak{D}(u,v),\mathfrak{D}(v,u)) \geq 1;$$

(ii) *there exists* $(u_0,v_0) \in U \times U$ *such that*

$$\begin{aligned}
\alpha((u_0,v_0),(\mathfrak{D}(u_0,v_0),\mathfrak{D}(v_0,u_0))) &\geq 1 \text{ and} \\
\alpha((\mathfrak{D}(v_0,u_0),\mathfrak{D}(u_0,v_0)),(v_0,u_0)) &\geq 1;
\end{aligned}$$

(iii) $\mathfrak{D}$ *is continuous.*
Then $\mathfrak{D}$ *possesses a coupled fixed point.*

Theorem 1.4.6 *[142] Let* $(U,\hat{d})$ *be a complete metric space and* $\mathfrak{D} : U \times U \to U$ *be a given mapping. Assume that there exists* $\psi \in \Psi$ *and a function* $\alpha : U^2 \times U^2 \to [0,+\infty)$ *such that*

$$\alpha((l,m),(u,v))\ \hat{d}(\mathfrak{D}(l,m),\mathfrak{D}(u,v)) \leq \frac{1}{2}\psi(d(l,u)+d(m,v)),$$

for all $(l,m),(u,v) \in U \times U$. *Also,*
(i) *For every pair* $(l,m),(u,v) \in U \times U$, *we have*

$$\alpha((l,m),(u,v)) \geq 1 \Rightarrow \alpha((\mathfrak{D}(l,m),\mathfrak{D}(m,l)),(\mathfrak{D}(u,v),\mathfrak{D}(v,u)) \geq 1;$$

(ii) *there exists* $(u_0,v_0) \in U \times U$ *such that*

$$\begin{aligned}
\alpha((u_0,v_0),(\mathfrak{D}(u_0,v_0),\mathfrak{D}(v_0,u_0))) &\geq 1 \text{ and} \\
\alpha((\mathfrak{D}(v_0,u_0),\mathfrak{D}(u_0,v_0)),(v_0,u_0)) &\geq 1;
\end{aligned}$$

(iii) *if* $\{u_n\}$ *and* $\{v_n\}$ *are two sequences in* U *such that*

$$\alpha((u_n,v_n),(u_{n+1},v_{n+1})) \geq 1 \text{ and } \alpha((v_{n+1},u_{n+1}),(v_n,u_n)) \geq 1,$$

$u_n \to u \in U$ *and* $v_n \to v \in U$ *as* $n \to +\infty$, *then,*

$$\alpha((u_n,v_n),(u,v)) \geq 1 \text{ and } \alpha((v,u),(v_n,u_n)) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Then $\mathfrak{D}$ *possesses a coupled fixed point.*

These classic results gave a new direction to researchers. Various authors such as Asl *et al.* [14], Karapinar [90], Mursaleen [107] *etc.* gave extensions to these results by using more generalized contractive conditions.

1.4.2 ϕ-projective contraction mappings

A huge interest in the mathematical theory of aggregation is developing these days because of its wide range of applications. In numerous fields, there is different sets of data coming from different sources which needs to obtain a conclusion. It can be done in a compact way using numerical aggregation. Borsik and Doboš [26] investigated the problem of aggregation for a collection of metrics (which need not be finite) resulting in a single one. Borsik and Doboš [26] studied the properties of those functions that permit a collection of metrics to merge and a single metric is obtained in return.
By [26], a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a metric aggregation function provided that the function $d_\Phi : U \times U \to \mathbb{R}_+$ is a metric for every pair of metric spaces $(U_1, \hat{d}_1)$ and $(U_2, \hat{d}_2)$, where $U = U_1 \times U_2$ and

$$\hat{d}_\Phi((u,v),(z,w)) = \Phi(\hat{d}_1(u,z), \hat{d}_2(v,w))$$

for all $(u,v),\ (z,w) \in U$.

Borsik and Doboš [26] defined the monotonicity and sub-additivity of Φ as follows:
A function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is called monotone if $u \preceq v \Rightarrow \ \Phi(u) \leq \Phi(v)$ for all $u, v \in \mathbb{R}_+^n$ and sub-additive if $\Phi(u+v) \leq \Phi(u) + \Phi(v)\ \forall\ u, v \in \mathbb{R}_+^n$ where $\preceq$ stands for the following pointwise order relation on $\mathbb{R}_+^n$:

$$u \preceq v \ \Leftrightarrow \ u_i \leq v_i \ \ ; \ \ i = 1, ...n.$$

On the other hand, Herburt [68] defined homogeneity as follows:

Definition 1.4.7 *[68] A function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be homogeneous if $\Phi(\alpha u) = \alpha\ \Phi(u)$ for each $u \in \mathbb{R}_+^n$ and $\alpha \in \mathbb{R}_+$.*

Following the work of Borsik and Doboš [26], Massanet and Valero [102] studied the aggregation problem for partial metric spaces.

Definition 1.4.8 *[102] A function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be a partial metric aggregation function provided that the function $P_\Phi : U \times U \to [0, +\infty[$ is a partial metric for every arbitrary collection of partial metric spaces $\{(U_i, p_i)\}_{i=1}^n$, where $U = U_1 \times U_2 ... \times U_n$ and*

$$P_\Phi(u, v) = \Phi(p_1(u_1, v_1), ..., p_n(u_n, v_n))$$

for all $u = (u_1, ...u_n) \in U$, $v = (v_1, ...v_n) \in U$.

Recently, Alghamdi *et al.* [9] defined contraction between partial metric spaces through distance aggregation perspective along with suitable aggregation properties. With the help of aggregation functions, Alghamdi *et al.* [9] introduced projective Φ-contraction and proved some fixed point results through this notion.

Lemma 1.4.9 *[9] Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial metric aggregation function, then Φ is monotone.*

Theorem 1.4.10 *[9] $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is a partial metric aggregation function if and only if for each $u, v, z, w \in \mathbb{R}_+^n$, we have*

(1) $\Phi(u) + \Phi(v) \leq \Phi(z) + \Psi(w)$ *whenever* $u + v \preceq z + w, \quad v \preceq z, \quad v \preceq w.$

(2) $\Phi(u) = \Phi(v) = \Phi(z) \Rightarrow u = v = z$ *whenever* $v \preceq u, \quad z \preceq u.$

The notions of contraction and completeness are defined as follows:.

Remark 1.4.11 *[9] Let $\{U_i\}_{i=1}^n$ be a collection of non-empty sets and $U = \prod_{i=1}^n U_i$. Let $\mathfrak{D}$ be a self mapping defined on U with coordinate functions $\mathfrak{D}_\mathfrak{i} : U \to U_i$, $i = 1, ...n$ such that*

$$\mathfrak{D}(u) = (\mathfrak{D}_\mathfrak{1}(u), ..., \mathfrak{D}_\mathfrak{n}(u)) \quad for\ all\ u \in U.$$

Definition 1.4.12 *[9] Let $\{(U_i, p_i)\}_{i=1}^n$ be a family of arbitrarily chosen quasi partial metric spaces and $U = \prod_{i=1}^n U_i$. Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial metric aggregation function. Then the mapping $\mathfrak{D} : U \to U$ is called a projective Φ-contraction from (U, P_Φ) into itself, if there exists n constants $\lambda_1, ..., \lambda_n > 1$ such that*

$$p_i(\mathfrak{D}_i(u), \mathfrak{D}_i(u)) \leq \lambda_i \Phi(p_1(u_1, v_1), ...p_n(u_n, v_n))$$

for all $u, v \in U$, where P_Φ is the partial metric induced by aggregation of the collec-

tion of partial metric spaces $\{(U_i, p_i)\}_{i=1}^n$ *through aggregation function* Φ.

Theorem 1.4.13 *[9] Let* $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ *be a homogeneous partial metric aggregation function such that* $\Phi(1, ..., 1) = 1 = \Phi(1_i)$ *for all* $i = 1, ...n$. *Let* $\{(U_i, p_i)\}_{i=1}^n$ *be a family of partial metric spaces (chosen arbitrarily) and* $U = \prod_{i=1}^n U_i$. *Let us assume that the partial metric spaces* (U_i, p_i), $i = 1, ...n$ *are complete. Then the partial metric space* (U, P_Φ) *is complete, where* P_Φ *is partial metric aggregation induced through* Φ *by aggregation of family of partial metric spaces* $\{(U_i, p_i)\}_{i=1}^n$.

Theorem 1.4.14 *[9] Let* $\{(U_i, p_i)\}_{i=1}^n$ *be a family of arbitrarily chosen partial metric spaces with complete metrics* p_i; $i = 1, ..., n$ *and* $U = \prod_{i=1}^n U_i$. *Let* Φ *be a homogeneous partial metric aggregation function such that* $\Phi(1, ..., 1) = \Phi(1_i) = 1$; $i = 1, ..., n$ *and* $\mathfrak{D}$ *is a projective* Φ*-contraction.*
Then $\mathfrak{D}$ *has a unique fixed point* u^*.

Following the work of Massanet and Valero [102], some authors have worked on aggregation till now from different perspectives in order to generalize Banach's results (see [101], [105]).

1.4.3 (ξ, α)-expansive mappings

Shahi *et al.* [147] introduced (ξ, α)-expansive mappings in 2012 and investigated some fixed point results for this class in attempt to extend the work of Samet *et al.* [140]. Following are the definitions and results presented by Shahi *et al.* [147]:

Definition 1.4.15 *[147] Let* χ *be the family of nondecreasing functions* $\xi : [0, +\infty) \to [0, +\infty)$ *such that* $\sum_{n=1}^{+\infty} \xi^n(a) < +\infty$ *for every* $a > 0$, *where* ξ^n *is* n^{th} *iterate of* ξ *and* $\xi(u + v) = \xi(u) + \xi(u)$ *for all* $u, v \in [0, +\infty)$. *Also, for all* $a > 0, \xi(a) < a$.

Definition 1.4.16 *[147] If* $(U, \hat{d})$ *be a metric space and* $\mathfrak{D} : U \to U$ *is a given mapping. Then* $\mathfrak{D}$ *is called an* (ξ, α)*-expansive mapping if there exists two functions* $\xi \in \chi$ *and* $\alpha : U \times U \to [0, +\infty)$ *such that*

$$\xi(\hat{d}(\mathfrak{D}u, \mathfrak{D}v)) \geq \alpha(u, v)\ \hat{d}(u, v)\ \text{for all}\ \ u, v \in U.$$

Theorem 1.4.17 *[147] Let* $(U, \hat{d})$ *be a complete metric space and* $\mathfrak{D} : U \to U$ *be* *a bijective, (ξ, α)-expansive mapping satisfying the following conditions:*

(i) $\mathfrak{D}^{-1}$ *is* $\alpha -$ *admissible*;

(ii) *there exists* $u_0 \in U$ *such that* $\alpha(u_0, \mathfrak{D}^{-1}u_0) \geq 1$;

(iii) $\mathfrak{D}$ *is continuous.*

Then $\mathfrak{D}$ possesses a fixed point.

Then, Shahi *et al.* [147] proved another result by removing the continuity condition in above theorem:

Theorem 1.4.18 *[147] Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{D} : U \to U$ be a bijective, (ξ, α)-expansive mapping satisfying the following conditions:*

(i) $\mathfrak{D}^{-1}$ *is* $\alpha -$ *admissible*;

(ii) *there exists* $u_0 \in U$ *such that* $\alpha(u_0, \mathfrak{D}^{-1}u_0) \geq 1$;

(iii) *If* $\{u_n\}$ *is a sequence in* U *such that* $\alpha(u_n, u_{n+1}) \geq 1$ *for every* $n \in \mathbb{N}$ *and* $\lim_n u_n = u$, *then* $\alpha(\mathfrak{D}^{-1}u_n, \mathfrak{D}^{-1}u) \geq 1$ *for all* n.

Then $\mathfrak{D}$ possesses a fixed point.

To ensure the uniqueness, Shahi *et al.* [147] considered an additional hypothesis.
(H_1): If there exists $w \in U$ such that for all $u, v \in U$,

$$\alpha(u, w) \geq 1 \ \ and \ \ \alpha(v, w) \geq 1.$$

The next result shows the existence of coupled fixed points in complete metric spaces due to Bhaskar and Lakshmikantham [24] and Shahi *et al.* [147]:

Theorem 1.4.19 *[147] Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{D} : U \times U \to U$ be a bijective mapping. Assume that there exists $\xi \in \chi$ and a function $\alpha : U^2 \times U^2 \to [0, +\infty)$ such that*

$$\xi(\hat{d}(\mathfrak{D}(l, m), \mathfrak{D}(u, v))) \geq \frac{1}{2}\ \alpha((l, m), (u, v))[\hat{d}(l, u) + d(m, v)]$$

for all $(l, m), (u, v) \in U \times U$. Also,
(i) For every pair $(l, m), (u, v) \in U \times U$, we have

$$\alpha((l, m), (u, v)) \geq 1 \Rightarrow \alpha(\mathfrak{D}^{-1}l, \mathfrak{D}^{-1}u) \geq 1;$$

(ii) there exists $(u_0, v_0) \in U \times U$ *such that*

$$\begin{aligned}\alpha((u_0, v_0), (a, b)) &\geq 1; \quad (a, b) \leq (u_0, v_0) \quad \text{and} \\ \alpha((b, a), (v_0, u_0)) &\geq 1; \quad (v_0, u_0) \leq (b, a)\end{aligned}$$

where $\mathfrak{D}^{-1}u_0 = (a, b)$ *and* $\mathfrak{D}^{-1}v_0 = (b, a)$*;*
(iii) $\mathfrak{D}$ *is continuous.*
Then $\mathfrak{D}$ *possesses a coupled fixed point.*

The next result shows that Theorem 1.4.19 holds true even if the condition of continuity of the mapping $\mathfrak{D}$ is removed. To prove this result, the continuity condition was replaced by a new hypotheses.

Theorem 1.4.20 *[147] Let* $(U, \hat{d})$ *be a complete metric space and* $\mathfrak{D} : U \times U \to U$ *be a bijective mapping. Assume that there exists* $\xi \in \chi$ *and a function* $\alpha : U^2 \times U^2 \to [0, +\infty)$ *such that*

$$\xi(\hat{d}(\mathfrak{D}(l, m), \mathfrak{D}(u, v))) \geq \frac{1}{2}\ \alpha((l, m), (u, v))[\hat{d}(l, u) + d(m, v)]$$

for all $(l, m), (u, v) \in U \times U$*. Also,*
(i) For every pair $(l, m), (u, v) \in U \times U$*, we have*

$$\alpha((l, m), (u, v)) \geq 1 \Rightarrow \alpha(\mathfrak{D}^{-1}l, \mathfrak{D}^{-1}u) \geq 1;$$

(ii) there exists $(u_0, v_0) \in U \times U$ *such that*

$$\begin{aligned}\alpha((u_0, v_0), (a, b)) &\geq 1; \quad (a, b) \leq (u_0, v_0) \quad \text{and} \\ \alpha((b, a), (v_0, u_0)) &\geq 1; \quad (v_0, u_0) \leq (b, a)\end{aligned}$$

where $\mathfrak{D}^{-1}u_0 = (a, b)$ *and* $\mathfrak{D}^{-1}v_0 = (b, a)$*;*
(iii) if $\{u_n\}$ *and* $\{v_n\}$ *are two sequences in* U *such that*

$$\alpha((u_n, v_n), (u_{n+1}, v_{n+1})) \geq 1 \text{ and } \alpha((v_{n+1}, u_{n+1}), (v_n, u_n)) \geq 1,$$

$u_n \to u \in U$ *and* $v_n \to v \in U$ *as* $n \to +\infty$*, then,*

$$\alpha(\mathfrak{D}^{-1}(u_n, v_n), \mathfrak{D}^{-1}(u, v)) \geq 1 \text{ and } \alpha(\mathfrak{D}^{-1}(v, u), \mathfrak{D}^{-1}(v_n, u_n)) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Then $\mathfrak{D}$ *possesses a coupled fixed point.*

From past few years, there exists considerable research in this direction. Various authors have proved fixed points theorems using more generalized expansive conditions and using various abstract spaces for this class of mappings (see [10], [83], [89], [95]).

1.4.4 Objectives of the study

The objectives of the presented study are as follows:

(i) To study the existence and uniqueness of fixed points/functions in various abstract spaces such as Quasi partial metric, Partially ordered metric, Hausdorff metric, b-metric-like *etc.* and for various types of generalized contractions along with their applications.

(ii) To study the fixed point/function results for various types of contraction or expansion mappings with applications.

(iii) To study the coupled, common coupled and coupled coincidence fixed point/function theorems using various mappings in various abstract spaces such as b-metric like spaces, Partially ordered metric spaces, Partially ordered b-metric like spaces *etc.* along with their applications.

In view of aforementioned objectives, some results have been derived in various abstract spaces such as Quasi partial metric spaces, Partially ordered metric spaces, Hausdorff metric spaces, b-metric-like spaces with suitable applications. Some results are obtained for various types of contraction/expansion mappings such as $\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$-multivalued contractive mappings, Ψ-projective expansion through metric aggregation functions, $\mathfrak{D}$-contraction for complete metric spaces, $\mathfrak{P}$-expansion along with interesting applications. Also, there are some coupled, common coupled and coupled coincidence fixed point results for a sequence of mappings and for generalized set-valued mappings.

1.4.5 Thesis Organization

The presented work consists of seven chapters. Each chapter has been divided in various sections. The numeric expressions like 3.5.2 indicates Theorem (Lemma/

Corollary/Proposition/Definition/Remark) 2 of Section 5 of Chapter 3. The conditions, hypotheses or properties of these Chapters are denoted by (i), (ii),(iii),... . As a standard notation, the numerals written in square brackets refer to the references from bibliography. A brief review of the research presented in this thesis is as follows:

In **Chapter 2**, Some fixed point results in Hausdorff metric spaces using $\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$-multivalued contractive type mappings are investigated. Various theorems regarding this class of contractive pair of mappings have been studied in this chapter. The results presented in this chapter extend some well-known relevant results (such as Kikkawa [91], Nadler [111], Samet *et al.* [140]) existing in literature. Some illustrative examples are provided to demonstrate the usefulness of main results. A result in homotopy theory is presented as an application of these results. In attempt to give extension to the results of Shahi *et al.* [147], some fixed point results in partially ordered metric spaces for (ξ, α)-expansive mappings are studied. Along with these results, an application to periodic boundary value problem is presented to show the usefulness of our main results.

Chapter 3 deals with some interesting fixed results in Quasi partial metric spaces. In this chapter, a new approach in the field of aggregation theory and metric aggregation is introduced. Firstly, the notion of expansion between quasi partial metric spaces through distance aggregation perspective along with suitable aggregation properties is defined. After that, with the help of aggregation functions, the concept of projective Ψ-expansion has been introduced and several fixed point results in Quasi partial metric spaces are obtained through this notion. Furthermore, sufficient conditions are also provided to characterize aggregation operator to ensure the existence and uniqueness of fixed point. The results derived in this chapter generalize the results presented by Borsik and Doboš [26], Martin *et al.* [101], Massanet and Valero [102], Mayor and Valero [105]. Some comparative examples are also given to check the efficacy of obtained results. Moreover, an application to asymptotic complexity analysis has also been presented in the end of this chapter.

In **Chapter 4**, the concept of contraction has been extended by defining a new notion of $\mathfrak{D}$-Contraction on a family $\mathfrak{F}$ of bounded functions. Also, a new concept of fixed function for a metric space is introduced in this chapter. Various fixed function theorems have been obtained by using different forms of $\mathfrak{D}$-Contraction along with demonstrative examples. In order to support the applicability of these results, an application to intensity modulated radiation therapy (IMRT) has also been presented following the FMO (Fluency map optimization) due to Shepard *et al.* [148] and Tian *et al.* [151]. This application is based on the treatment plan for tumor patients getting IMRT. The fixed function obtained in this way represents the suitable doses for a number of tumor patients at the same time.

Chapter 5 deals with the generalization of the results of Wang *et al.* [153] and Jungck [78] by introducing a new notion of $\mathfrak{P}$-expansion defined on a collection $\mathfrak{F}$ of bounded functions. Some fixed function theorems are investigated by using various kinds of generalized expansive conditions. In addition to these results, some common fixed function theorems have also been derived for weakly compatible mappings. Moreover, an application based on suitable plan of intensity values for the patients under Tomotherapy is also given in the end of the chapter.

In **Chapter 6**, a new notion of $\mathcal{F}$-generalized multivalued contractive type mappings is introduced through $\mathfrak{C}$-class due to Ansari [13]. Using this class, some common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces are investigated. The results presented in this chapter generalize various relevant results from the current literature *e.g.* Ansari [13], Nashine [113], Nazari *et al.* [114], Wang *et al.* [153] and references therein.

Chapter 7 is devoted to some common coupled and coincidence fixed point results for various abstract spaces. There exists vast literature showing the existence of fixed points using expansive mappings. But the existence of common and coincidence fixed points for a sequence of functions using expansive mappings is still not explored much. In this chapter, some results for a sequence of mappings with generalized expansive conditions in quasi partial metric spaces have been investigated. To demonstrate the usability of presented results, some useful deductions along with an example are also provided.
After that, some common coupled fixed point theorems are investigated in b-metric-like spaces which generalize the results of Bhaskar and Lakshmikantham [24], Alghamdi *et al.* [8].
The thesis is concluded by listing the Bibliography with various publications which are cited in this research work.

Chapter 2

Fixed Point Theorems for Various Abstract Spaces

2.1 Introduction

Metric fixed point theory has played a keyrole in the development of nonlinear functional analysis. It has a wide variety of applications in numerous field such as computer science (see [103], [136], [144]) and economics (see [25], [119]) *etc.* However, the credit of placing the above idea into an abstract framework suitable for broad applications goes to Polish Mathematician Banach [20]. In 1922, he established a fixed point theorem which has been affirmed the base for the rest of fixed point theory. According to this principle: "*Each contraction defined on a complete metric space U possesses a unique fixed point*".

In 2013, Rahimi *et al.* [125] presented a study on various types of abstract spaces and investigated the relationships among them. In 1914, German mathematician Hausdorff [66] defined a new distance notion denoted by $\mathcal{H}(\mathcal{U}, \mathcal{V})$ and is known as Hausdorff metric where

$$\mathcal{H}(\mathcal{U}, \mathcal{V}) = max\{\rho(\mathcal{U}, \mathcal{V}), \rho(\mathcal{V}, \mathcal{U})\} \ for \ \mathcal{U}, \mathcal{V} \in \mathcal{P}_{cb}(\mathcal{U}),$$

$\mathcal{P}_{cb}(\mathcal{U})$: Collection of all non-empty closed and bounded subsets of $\mathcal{U}$ and
$\rho(\mathcal{U}, \mathcal{V}) : sup\{\hat{d}(u, \mathcal{V}) : u \in \mathcal{U}\}, \ \hat{d}(u, \mathcal{V}) = inf\{\hat{d}(u, v) : v \in \mathcal{V}\}$.

The first fixed point theorem in this context was given by Markin [100] in 1968. After that, this research list continued with the results of Nadler [111] and Covits and Nadler [45] in the subsequent years. These results were essentially based on Hausdorff metric for their contraction conditions.

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The result presented by Nadler [111] is as follows:

Theorem 2.1.1 *[111] Let $(\mathcal{U}, \hat{d})$ be a complete metric space and $\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ be a multivalued mapping satisfying*

$$\mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq l\ \hat{d}(u, v),$$

for all $u, v \in \mathcal{U}$ and for some $l \in [0, 1)$. Then $\mathcal{D}$ possesses a fixed point.

Inspired by this result, various fixed point theorems concerning multivalued contractions appeared in the last few decades. For instance, in 1983, Reich [127] formulated the following problem in Hausdorff metric spaces:

Theorem 2.1.2 *[127] Let $(\mathcal{U}, \hat{d})$ be a complete metric space and $\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ be a multivalued mapping satisfying*

$$\mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq l\ \alpha(\hat{d}(u, v))\hat{d}(u, v)$$

for all $u, v \in \mathcal{U}$, where $\alpha : (0, \infty) \to [0, 1)$ fulfills

$$\limsup_{s \to t^+} \alpha(s) < 1$$

for all $t \in (0, \infty)$ and $l \in [0, 1)$. Then $\mathcal{D}$ possesses a fixed point.

After that, in 2008, Kikkawa and Suzuki [91] presented the following version of Nadler's result by introducing a strictly decreasing function to modify the contractive condition in Hausdorff metric spaces:

Theorem 2.1.3 *[91] Let $(\mathcal{U}, \hat{d})$ be a complete metric space and $\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ be a multivalued mapping satisfying*

$$\eta(r)\ \hat{d}(u, \mathcal{D}u) \leq \hat{d}(u, v) \Rightarrow \mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq r\ \hat{d}(u, v),$$

for all $u, v \in \mathcal{U}$ and for some $r \in [0, 1)$ where $\eta : [0, 1) \to (\frac{1}{2}, 1]$ is a mapping defined by $\eta(r) = \frac{1}{(1+r)}$.
Then $\mathcal{D}$ possesses a fixed point.

Till now, several authors generalized these results and presented different kinds of extended versions of Hausdorff metric spaces (see [15], [60], [155]). In a survey paper, Berinde and Pacurar [23] highlighted the role of Pompeiu-Hausdorff metric in fixed point theory in numerous fields.

On the other hand, Turinici [152] introduced the concept of ordering to prove the existence of fixed points for ordered metrizable uniform spaces. Later on, Ran and Reurings [126] presented some applications of Turinici's results for partially ordered metric spaces to solve matrix equations. A partial order on a non-empty set $\mathcal{U}$ is a binary relation such that $\mathcal{U}$ is reflexive, antisymmetric and transitive. A set along with a partial order defined on it, is called a partially ordered set. Ran and Reurings [126] proved the following partial ordered version of of Banach contraction principle:

Theorem 2.1.4 *[126] Let $\mathcal{U}$ be a non-empty set equipped with a partial order such that each pair $u, v \in \mathcal{U}$ has a lower or upper bound. Let $(\mathcal{U}, \hat{d})$ be a metric space where $\hat{d}$ is a complete metric defined on $\mathcal{U}$. Let $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ be a continuous monotone self mapping. Assume that*

1. *there exists $a \in (0,1)$ such that $\hat{d}(u, \mathcal{D}v) \leq a\ \hat{d}(u, v)$ for every $u, v \in \mathcal{U}$ and $u \succeq v$;*
2. *there exists $u_0 \in \mathcal{U}$ such that $\mathcal{D}u_0 \preceq u_0$.*

Then $\mathcal{D}$ possesses a unique fixed point $u^* \in \mathcal{U}$ and for every $w \in \mathcal{U}$, the iterative sequence $\{\mathcal{D}^n(w)\}$ (where $n \in \mathbb{N}$) of $\mathcal{D}$ starting from w converges to $u^* \in \mathcal{U}$.

Recently, Shahi *et al.* [147] presented the notion of (ξ, α)-expansive mappings in complete metric spaces and investigated the existence of a fixed points for this class by following the idea of expansive mappings presented by Wang *et al.* in [153]. The result of Wang *et al.* [153] is as follows:

Theorem 2.1.5 *[153] Let $(\mathcal{U}, \hat{d})$ be a complete metric space. If $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ be an onto mapping and there exists a constant $q > 1$ such that*

$$\hat{d}(\mathcal{D}u, \mathcal{D}v) \geq q\ \hat{d}(u, v) \quad for\ every\ u, v \in \mathcal{U}.$$

Then $\mathcal{D}$ has a unique fixed point in $\mathcal{U}$.

In this chapter, a new class of contractive pair of mappings called $\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$-multivalued contractive type mappings in Hausdorff metric spaces is introduced. Various fixed point theorems regarding this class of contractive pair of mappings have been studied in this chapter. The results presented in this chapter extend and generalize various well-known relevant results (such as Kikkawa [91], Nadler [111], Samet *et al.* [142]) existing in literature. A result in homotopy theory is presented as an application of these results. In attempt to extend the results of Shahi *et al.* [147], some fixed point results for (ξ, α)-expansive mappings in partially ordered metric spaces are studied. Also, some coupled fixed point theorems have also been derived for this class by following the idea of Bhaskar and Lakshmikantham [24]. Along with these results, an application to periodic boundary value problem is also presented to illustrate the usefulness of our main results.

The contents of this chapter are divided into different sections. Section 2.2 deals with some preliminaries related to this chapter. In section 2.3, various fixed point theorems have been proved for ($\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$)-multivalued contractive type mappings in Hausdorff metric spaces along with an application to homotopy theory. Section 2.4 is concerned with the generalization of (ξ, α)-expansive mappings in partially ordered metric spaces with an application to periodic boundary value problem.

2.2 Preliminaries

This section presents some elementary definitions, notations and results that are to be used in later sections along with some new terminologies.

Pompeiu [124] defined the distance notion between two closed sets as follows:

Definition 2.2.1 *[124] If $\mathcal{U}$ and $\mathcal{V}$ are two closed and bounded sets, then the distance between the point $u \in \mathcal{U}$ and the set $\mathcal{V}$ is defined as*

$$\hat{d}(u, \mathcal{V}) = min\{\hat{d}(u, v) : \ v \in \mathcal{V}\},$$

where $\hat{d}(u, v)$ is the (Euclidean) distance between the points u and v.

Using the above notion, Hausdorff [66] defined Hausdorff distance as below:

Definition 2.2.2 *[66] If $\mathcal{U}$ and $\mathcal{V}$ are two closed and bounded sets, then the hausdorff distance between these two sets is defined as*

$$\mathcal{H}(\mathcal{U}, \mathcal{V}) = max\{\rho(\mathcal{U}, \mathcal{V}), \rho(\mathcal{V}, \mathcal{U})\} \ where$$

$$\rho(\mathcal{U}, \mathcal{V}) = sup\{\hat{d}(u, \mathcal{V}) : u \in \mathcal{U}\}, \ \hat{d}(u, \mathcal{V}) = inf\{\hat{d}(u, v) : v \in \mathcal{V}\}.$$

The results and definitions given by Samet *et al.* [142] are given below:

Definition 2.2.3 *[142] Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:*

(i) $\sum_{n=1}^{+\infty}\psi^n(s) < +\infty$ *for every* $s > 0$*, where* ψ^n *is* n^{th} *iterate of* ψ*;*
(ii) ψ *is nondecreasing.*

Lemma 2.2.4 *[142] If* $\psi : [0, \infty) \to [0, \infty)$ *is a nondecreasing function, then* $\psi(s) < s$ *for all* $s > 0$*.*

Lemma 2.2.5 *[142] If* $\psi \in \Psi$*, then the function* ψ *is continuous at* 0*.*

Definition 2.2.6 *[142] If* $(\mathcal{U}, \hat{d})$ *is a metric space and* $\mathcal{D}$ *is a self mapping defined on* $\mathcal{U}$*, then* $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ *is called an* $(\alpha$*-*$\psi)$*-contractive mapping if there exists* $\psi \in \Psi$ *and* $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ *such that*

$$\alpha(u, v)\ \hat{d}(\mathcal{D}u, \mathcal{D}v) \leq \psi(\hat{d}(u, v)) \quad for \quad all \quad u, v \in \mathcal{U}.$$

Definition 2.2.7 *[142] Let* $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ *and* $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty)$*. Then the mapping* $\mathcal{D}$ *is said to be* α*-admissible if*

$$\alpha(u, v) \geq 1 \Rightarrow \alpha(\mathcal{D}u, \mathcal{D}v) \geq 1.$$

Theorem 2.2.8 *[142] Let* $(\mathcal{U}, \hat{d})$ *be a complete metric space and* $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ *be an* $(\alpha - \psi)$*-contractive mapping satisfying the following conditions:*

$$\begin{aligned} (i) &\quad \mathcal{D}\ is\ \ \alpha - admissible; \\ (ii) &\quad there\ exists\ \ u_0 \in \mathcal{U}\ \ such\ that\ \ \alpha(u_0, \mathcal{D}u_0) \geq 1; \\ (iii) &\quad \mathcal{D}\ is\ continuous. \end{aligned}$$

Then $\mathcal{D}$ *has a fixed point in* $\mathcal{U}$*.*

Other related results due to Samet *et al.* [142] have already been presented in chapter 1.
Now, the new notion of $(\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}})$-multivalued contractive type mappings in the framework of Hausdorff metric spaces is presented.

Definition 2.2.9 *Let* $\Psi_{\mathcal{H}}$ *be the family of nondecreasing functions* $\psi_{\mathcal{H}} : [0, +\infty) \to [0, +\infty)$ *such that*

$$\begin{aligned} (i) &\quad \sum_{n=1}^{+\infty} n\ \psi_{\mathcal{H}}^n(s) < +\infty\ for\ every\ s > 0,\ where\ \psi_{\mathcal{H}}^n\ is\ n^{th}\ iterate\ of\ \psi_{\mathcal{H}}; \\ (ii) &\quad \psi_{\mathcal{H}}(s) < s\ for\ all\ s > 0; \\ (iii) &\quad \psi_{\mathcal{H}}(s) + \psi_{\mathcal{H}}(t) = \psi_{\mathcal{H}}(s + t)\ for\ all\ s, t \in [0, +\infty). \end{aligned}$$

Definition 2.2.10 *Let $(\mathcal{U}, \hat{d})$ be a metric space and $\mathcal{D}$ be a self mapping defined on $\mathcal{U}$, then $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ is called an ($\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$)-contractive mapping if there exists $\psi_{\mathcal{H}} \in \Psi_{\mathcal{H}}$ and $\alpha_{\mathcal{H}} : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ such that*

$$\alpha_{\mathcal{H}}(u, v)\ \mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq \psi_{\mathcal{H}}(\hat{d}(u, v)) \quad for \quad all \quad u, v \in \mathcal{U}. \tag{2.2.1}$$

Definition 2.2.11 *The mapping $\mathcal{D}$ is called $\alpha_{\mathcal{H}}$-admissible if*

$$\alpha_{\mathcal{H}}(u, v) \geq 1 \Rightarrow \alpha_{\mathcal{H}}(s, t) \geq 1\ for\ every\ s \in \mathcal{D}u\ and\ t \in \mathcal{D}v \quad for \quad s \in \mathcal{D}u \quad and \quad t \in \mathcal{D}v.$$

Remark 2.2.12 *If $\mathcal{D}$ is a contraction mapping defined on $\mathcal{U}$, then $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ is called an ($\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$)-contractive mapping where $\alpha_{\mathcal{H}}(u, v) = 1$ for all $u, v \in \mathcal{U}$ and $\psi_{\mathcal{H}}(s) = ks$ for all $s \geq 0$ and some $k \in [0, 1)$.*

Recently, Shahi *et al.* [147] introduced the notion of (ξ, α)-expansive mappings in complete metric spaces by following the idea of Samet *et al.* [142] as follows:

Definition 2.2.13 *[147] Let χ be the family of nondecreasing functions $\xi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:*

(i) $\sum_{n=1}^{+\infty} \xi^n(s) < +\infty$ *for every* $s > 0$, *where* ξ^n *is* n^{th} *iterate of* ξ;

(ii) $\xi(s + t) = \xi(s) + \xi(t)$ *for all* $s, t \in [0, +\infty)$;

(iii) $\xi(s) < s$ *for all* $s > 0$.

Definition 2.2.14 *[147] Let $(\mathcal{U}, \hat{d})$ be a metric space and $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ be a given mapping. Then $\mathcal{D}$ is called an (ξ, α)-expansive mapping if there exists two functions $\xi \in \chi$ and $\alpha : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ such that*

$$\xi(\hat{d}(\mathcal{D}u, \mathcal{D}v)) \geq \alpha(u, v)\ \hat{d}(u, v);$$

and α-admissible if $\alpha(u, v) \geq 1 \Rightarrow \alpha(\mathcal{D}u, \mathcal{D}v) \geq 1$ for all $u, v \in \mathcal{U}$.

Definition 2.2.15 *[147] Let $(\mathcal{U}, \preceq)$ be a partially ordered space equipped with a complete metric d and $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ be a given mapping. $\mathcal{D}$ is said to be an (ξ, α)-expansive mapping if there exist two functions $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ and $\xi \in \chi$ such that with $u \geq v$,*

$$\xi(\hat{d}(\mathcal{D}u, \mathcal{D}v)) \geq \alpha(u, v)\ \hat{d}(u, v) \quad for \quad all \quad u, v \in \mathcal{U}. \tag{2.2.2}$$

2.3 Fixed point theorems in Hausdorff metric spaces

2.3.1 Main Results

Theorem 2.3.1 *Let* $(\mathcal{U}, \hat{d})$ *be a complete metric space and* $\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ *be a multivalued* ($\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$)*-contractive mapping such that*

(i) $\mathcal{D}$ *is* $\alpha_{\mathcal{H}} -$ *admissible;*

(ii) *there exists* $u_0 \in \mathcal{U}$ *such that* $\alpha_{\mathcal{H}}(u_0, s) \geq 1$ *for every* $s \in \mathcal{D}u_0$*;*

(iii) *if* $\{u_n\}$ *is a sequence in* $\mathcal{U}$ *such that* $\alpha_{\mathcal{H}}(u_n, u_{n+1}) \geq 1$ *for all* n *and* $\{u_n\} \to u \in \mathcal{U}$ *as* $n \to \infty$*, then* $\alpha_{\mathcal{H}}(u_n, u) \geq 1 \ \forall \ n \in \mathbb{N}$.

Then $\mathcal{D}$ *has a fixed point.*

Proof Let $u_0 \in \mathcal{U}$. Let $u_1 \in \mathcal{D}u_0$ with $u_0 \neq u_1$. Then, by assumption (ii), $\alpha_{\mathcal{H}}(u_0, u_1) \geq 1$. Let us suppose that there exists $u_2 \in \mathcal{D}u_1$ such that

$$\hat{d}(u_1, u_2) \leq \mathcal{H}(\mathcal{D}u_0, \mathcal{D}u_1) + \psi_{\mathcal{H}}(\hat{d}(u_0, u_1)).$$

Similarly, there exists $u_3 \in \mathcal{D}u_2$ such that

$$\hat{d}(u_2, u_3) \leq \mathcal{H}(\mathcal{D}u_1, \mathcal{D}u_2) + \psi^2_{\mathcal{H}}(\hat{d}(u_0, u_1)).$$

Following the same pattern, we get a sequence $\{u_n\}$ in $\mathcal{U}$ such that there exists $u_{n+1} \in \mathcal{D}u_n$ and

$$\hat{d}(u_n, u_{n+1}) \leq \mathcal{H}(\mathcal{D}u_{n-1}, \mathcal{D}u_n) + \psi^n_{\mathcal{H}}(\hat{d}(u_0, u_1)).$$

Since $\mathcal{D}$ is $\alpha_{\mathcal{H}}$-admissible, therefore

$$\alpha_{\mathcal{H}}(u_0, u_1) \geq 1 \Rightarrow \alpha_{\mathcal{H}}(u_1, u_2) \geq 1.$$

By mathematical induction, we obtain

$$\alpha_{\mathcal{H}}(u_n, u_{n+1}) \geq 1 \ \ \forall \ \ n \in \mathbb{N}. \tag{2.3.1}$$

Now, using (2.2.1) and (2.3.1)

$$
\begin{aligned}
\hat{d}(u_n, u_{n+1}) &\leq \mathcal{H}(\mathcal{D}u_{n-1}, \mathcal{D}u_n) + \psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) \\
&\leq \alpha_{\mathcal{H}}(u_{n-1}, u_n)\mathcal{H}(\mathcal{D}u_{n-1}, \mathcal{D}u_n) + \psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) \\
&\leq \psi_{\mathcal{H}}(\hat{d}(u_{n-1}, u_n)) + \psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) \\
&= \psi_{\mathcal{H}}[\hat{d}(u_{n-1}, u_n)) + \psi_{\mathcal{H}}^{n-1}(\hat{d}(u_0, u_1))] \\
&\leq \psi_{\mathcal{H}}[\psi_{\mathcal{H}}(\hat{d}(u_{n-2}, u_{n-1})) + \psi_{\mathcal{H}}^{n-1}(\hat{d}(u_0, u_1)) + \psi_{\mathcal{H}}^{n-1}(\hat{d}(u_0, u_1))] \\
&= \psi_{\mathcal{H}}[\psi_{\mathcal{H}}(\hat{d}(u_{n-2}, u_{n-1})) + 2\psi_{\mathcal{H}}^{n-1}(\hat{d}(u_0, u_1))] \\
&= \psi_{\mathcal{H}}^2(\hat{d}(u_{n-2}, u_{n-1})) + 2\psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) + n\psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) \\
&= (n+1)\psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)).
\end{aligned}
$$

"Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$
\sum_{n \geq n_0} (n+1)\psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) < \epsilon.
$$

For $m, n \in \mathbb{N}$ with $m > n > n_0$, we have

$$
\begin{aligned}
\hat{d}(u_n, u_m) &\leq \sum_{k=n}^{m-1} \hat{d}(u_k, u_{k+1}) \\
&\leq \sum_{k=n}^{m-1} (k+1)\psi_{\mathcal{H}}^k(\hat{d}(u_0, u_1)) \\
&\leq \sum_{k \geq n} (k+1)\psi_{\mathcal{H}}^k(\hat{d}(u_0, u_1)) \\
&\leq \sum_{k \geq n_0} (k+1)\psi_{\mathcal{H}}^k(\hat{d}(u_0, u_1)) \\
&< \epsilon.
\end{aligned}
$$

Thus, $\{u_n\}$ is a Cauchy sequence in $\mathcal{U}$. But $(\mathcal{U}, \hat{d})$ is a complete metric space. Therefore, there exists $u \in \mathcal{U}$ such that $u_n \to u$ as $n \to \infty$.
Now it is shown that u is a fixed point of $\mathcal{D}$.

$$
\begin{aligned}
\hat{d}(u,\mathcal{D}u) &\le \hat{d}(u,u_n) + \hat{d}(u_n,\mathcal{D}u)\\
&\le \hat{d}(u,u_n) + \mathcal{H}(\mathcal{D}u_{n-1},\mathcal{D}u)\\
&\le \hat{d}(u,u_n) + \alpha_{\mathcal{H}}(u_{n-1},u)\mathcal{H}(\mathcal{D}u_{n-1},\mathcal{D}u)\\
&\le \hat{d}(u,u_n) + \psi_{\mathcal{H}}(\hat{d}(u_{n-1},u))\\
&< \hat{d}(u,u_n) + \hat{d}(u_{n-1},u)\\
&\to 0 \;\; as \;\; n\to\infty.
\end{aligned}
$$

Thus, $\hat{d}(u,\mathcal{D}u)=0$. As $\mathcal{D}u$ is closed, therefore, $u\in\mathcal{D}u$.

Example 2.3.1 *Let $\mathcal{U}=\mathbb{R}$ equipped with standard metric $\hat{d}$ and multivalued metric $\mathcal{H}$. Let the mapping $\mathcal{D}:\mathcal{U}\to\mathcal{P}_{cb}(\mathcal{U})$ be defined by*

$$
\mathcal{D}u=\begin{cases}\{2u-\frac{3}{2},3\}, & if \;\; u>1,\\ \{\frac{u}{2},\frac{u}{4}\}, & if \;\; 0\le u\le 1,\\ \{0\}, & if \;\; u<0.\end{cases}
$$

Now, $\mathcal{H}(\mathcal{D}1,\mathcal{D}2)=\frac{11}{4}>1=\hat{d}(1,2)$.
Therefore, Theorem 2.1.1 is not applicable.
Define $\alpha_{\mathcal{H}}:\mathcal{U}\times\mathcal{U}\to[0,\infty)$ by

$$
\alpha_{\mathcal{H}}(u,v)=\begin{cases}1, & if \;\; u,v\in[0,1],\\ 0, & otherwise.\end{cases}
$$

Let $\psi_{\mathcal{H}}(t)=\frac{t}{2}\;\;\forall\;\; t\ge 0$.
Then, it can be easily checked that

$$
\alpha_{\mathcal{H}}(u,v)\;\mathcal{H}(\mathcal{D}u,\mathcal{D}v)\le\psi_{\mathcal{H}}(\hat{d}(u,v)).
$$

Thus, $\mathcal{D}$ is an ($\alpha_{\mathcal{H}}$-$\psi_{\mathcal{H}}$)-contractive mapping.
Let $u_0=1$, then $\mathcal{D}u_0=\{\frac{1}{2},\frac{1}{4}\}$. Therefore, $\alpha_{\mathcal{H}}(u_0,s)\ge 1$ for every $s\in\mathcal{D}u_0$. Let $u,v\in\mathcal{U}$ be such that $\alpha_{\mathcal{H}}(u,v)\ge 1$, then, $u,v\in[0,1]$ and therefore, $\mathcal{D}u=\{\frac{u}{2},\frac{u}{4}\}$, $\mathcal{D}v=\{\frac{v}{2},\frac{v}{4}\}$ which implies $\alpha_{\mathcal{H}}(s,t)\ge 1$ for every $s\in\mathcal{D}u,\;\; t\in\mathcal{D}v$. Thus, $\mathcal{D}$ is $\alpha_{\mathcal{H}}$-admissible.
Let $\{u_n\}$ be a sequence in $\mathcal{U}$ such that $\alpha_{\mathcal{H}}(u_n,u_{n+1})\ge 1\;\;\forall\;\; n\in\mathbb{N}$ and $u_n\to u\in\mathcal{U}$. Then, $u_n\in[0,1]$ by definition of $\alpha_{\mathcal{H}}$. Therefore, $u\in[0,1]$ and hence $\alpha_{\mathcal{H}}(u_n,u)\ge 1$. Thus, all the hypotheses of Theorem 2.3.1 are satisfied. Therefore, $\mathcal{D}$ has a fixed

point namely 0.

Theorem 2.3.2 *Let $(\mathcal{U}, \hat{d})$ be a complete metric space and $\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ be a multivalued mapping such that*

(i) *there exists a continuous function $\psi_{\mathcal{H}} \in \Psi_{\mathcal{H}}$ and $\alpha_{\mathcal{H}} : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ such that for all $u, v \in \mathcal{U}$,*
$\hat{d}(u, \mathcal{D}u) - \psi_{\mathcal{H}}(\hat{d}(u, v)) \leq \hat{d}(u, v) \quad \Rightarrow \quad \alpha_{\mathcal{H}}(u, v)\mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq \psi_{\mathcal{H}}(\hat{d}(u, v))$;

(ii) *$\mathcal{D}$ is $\alpha_{\mathcal{H}} -$ admissible;*

(iii) *there exists $u_0 \in \mathcal{U}$ such that $\alpha_{\mathcal{H}}(u_0, s) \geq 1$ for every $s \in \mathcal{D}u_0$;*

(iv) *if $\{u_n\}$ is a sequence in $\mathcal{U}$ such that $\alpha_{\mathcal{H}}(u_n, u_{n+1}) \geq 1$ for all n and $\{u_n\} \to z$, then $\alpha_{\mathcal{H}}(u_n, u) \geq 1 \;\; \forall u \in \{u_k : k \in N\} \cup \{z\}$.*

Then $\mathcal{D}$ has a fixed point.

Proof Let $u_0 \in \mathcal{U}$. Let $u_1 \in \mathcal{D}u_0$ with $u_0 \neq u_1$. Then, by assumption (ii), $\alpha_{\mathcal{H}}(u_0, u_1) \geq 1$.
Now,

$$\hat{d}(u_0, \mathcal{D}u_0) - \psi_{\mathcal{H}}(\hat{d}(u_0, u_1)) \leq \hat{d}(u_0, \mathcal{D}u_0) \leq \hat{d}(u_0, u_1).$$

By assumption (i), we get

$$\alpha_{\mathcal{H}}(u_0, u_1)\ \mathcal{H}(\mathcal{D}u_0, \mathcal{D}u_1) \leq \psi_{\mathcal{H}}(\hat{d}(u_0, u_1)).$$

This implies

$$\hat{d}(u_1, \mathcal{D}u_1) \leq \alpha_{\mathcal{H}}(u_0, u_1)\ \mathcal{H}(\mathcal{D}u_0, \mathcal{D}u_1) \leq \psi_{\mathcal{H}}(\hat{d}(u_0, u_1)).$$

Let $u_2 \in \mathcal{D}u_1$ be such that

$$\hat{d}(u_1, u_2) < \psi_{\mathcal{H}}(\hat{d}(u_0, u_1)) + \psi_{\mathcal{H}}(\hat{d}(u_0, u_1)).$$

Again, we have

$$\hat{d}(u_1, \mathcal{D}u_1) - \psi_{\mathcal{H}}(\hat{d}(u_1, u_2)) \leq \hat{d}(u_1, \mathcal{D}u_1) \leq \hat{d}(u_1, u_2).$$

By assumption, we have

$$\alpha_{\mathcal{H}}(u_1, u_2)\ \mathcal{H}(\mathcal{D}u_1, \mathcal{D}u_2) \leq \psi_{\mathcal{H}}(\hat{d}(u_1, u_2));$$

which implies

$$\hat{d}(u_2, \mathcal{D}u_2) \leq \alpha_{\mathcal{H}}(u_1, u_2)\ \mathcal{H}(\mathcal{D}u_1, \mathcal{D}u_2) \leq \psi_{\mathcal{H}}(\hat{d}(u_1, u_2)).$$

Let $u_3 \in \mathcal{D}u_2$ be such that

$$\hat{d}(u_2, u_3) < \psi_{\mathcal{H}}(\hat{d}(u_1, u_2)) + \psi_{\mathcal{H}}^2(\hat{d}(u_0, u_1)).$$

Thus, we get a sequence $u_{n+1} \in \mathcal{D}u_n$ such that

$$\begin{aligned}
\hat{d}(u_n, u_{n+1}) &< \psi_{\mathcal{H}}(\hat{d}(u_{n-1}, u_n)) + \psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) \\
&\leq \psi_{\mathcal{H}}[(\hat{d}(u_{n-1}, u_n)) + \psi_{\mathcal{H}}^{n-1}(\hat{d}(u_0, u_1))] \\
&\leq \psi_{\mathcal{H}}[\psi_{\mathcal{H}}(\hat{d}(u_{n-2}, u_{n-1})) + \psi_{\mathcal{H}}^{n-1}(\hat{d}(u_0, u_1)) + \psi_{\mathcal{H}}^{n-1}(\hat{d}(u_0, u_1))] \\
&= \psi_{\mathcal{H}}^2(\hat{d}(u_{n-2}, u_{n-1})) + 2\psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) \\
&\ . \\
&\ . \\
&\ . \\
&\leq \psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) + n\psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) \\
&= (n+1)\psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)).
\end{aligned}$$

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n \geq n_0} (n+1)\psi_{\mathcal{H}}^n(\hat{d}(u_0, u_1)) < \epsilon.$$

For $m, n \in \mathbb{N}$ with $m > n > n_0$, we have

$$\begin{aligned}
\hat{d}(u_n, u_m) &\leq \sum_{k=n}^{m-1} \hat{d}(u_k, u_{k+1}) \\
&\leq \sum_{k=n}^{m-1} (k+1)\psi_{\mathcal{H}}^k(\hat{d}(u_0, u_1)) \\
&\leq \sum_{k \geq n} (k+1)\psi_{\mathcal{H}}^k(\hat{d}(u_0, u_1)) \\
&\leq \sum_{k \geq n_0} (k+1)\psi_{\mathcal{H}}^k(\hat{d}(u_0, u_1)) \\
&< \epsilon.
\end{aligned}$$

Thus, $\{u_n\}$ is a Cauchy sequence in $\mathcal{U}$. But $(\mathcal{U}, \hat{d})$ is a complete metric space.

Therefore, there exists $z \in \mathcal{U}$ such that $u_n \to z$ as $n \to \infty$.
Next, it is shown that $\hat{d}(z, \mathcal{D}u) \leq \psi_{\mathcal{H}}(\hat{d}(z, u)) \;\; \forall \; u \in \{u_k : k \in N\} - \{z\}$.
Since $u_n \to z$, there exists $n_0 \in \mathbb{N}$ such that $\hat{d}(z, u_n) \leq \frac{1}{3}\hat{d}(z, u)$ for all $n \geq n_0$.
Then,

$$\begin{aligned} \hat{d}(u_n, \mathcal{D}u_n) - \psi_{\mathcal{H}}(\hat{d}(u, u_n)) &\leq \hat{d}(u_n, \mathcal{D}u_n) \\ &\leq \hat{d}(u_n, u_{n+1}) \\ &\leq \hat{d}(u_n, z) + \hat{d}(z, u_{n+1}) \\ &\leq \frac{2}{3}\hat{d}(u, z) \\ &\leq \hat{d}(u, z) - \hat{d}(z, u_n) \\ &\leq \hat{d}(u_n, u). \end{aligned}$$

Therefore, by assumption,

$$\begin{aligned} \alpha_{\mathcal{H}}(u_n, u) \; \mathcal{H}(\mathcal{D}u_n, \mathcal{D}u) &\leq \psi_{\mathcal{H}}(\hat{d}(u_n, u)) \\ \Rightarrow \quad \mathcal{H}(\mathcal{D}u_n, \mathcal{D}u) &\leq \psi_{\mathcal{H}}(\hat{d}(u_n, u)) \\ \Rightarrow \quad \hat{d}(u_{n+1}, \mathcal{D}u) &\leq \psi_{\mathcal{H}}(\hat{d}(u_n, u)) \quad for \quad n \geq n_0. \end{aligned}$$

Letting $n \to \infty$,

$$\hat{d}(z, \mathcal{D}u) \leq \psi_{\mathcal{H}}(\hat{d}(z, u)).$$

Next, it is proved that

$$\alpha_{\mathcal{H}}(u, z) \; \mathcal{H}(\mathcal{D}u, \mathcal{D}z) \leq \psi_{\mathcal{H}}(\hat{d}(z, u)) \;\; \forall \;\; u \in \{u_k : k \in N\}.$$

If $u = z$, then it is obvious. So, let $u \neq z$. Then for every $n \in \mathbb{N}$, $\exists \;\; v_n \in \mathcal{D}u$ such that

$$\hat{d}(z, v_n) \leq \hat{d}(z, \mathcal{D}u) + \frac{1}{n}\hat{d}(u, z).$$

Now,

$$\begin{aligned} \hat{d}(u, \mathcal{D}u) &\leq \hat{d}(u, v_n) \\ &\leq \hat{d}(u, z) + \hat{d}(z, v_n) \\ &\leq \hat{d}(u, z) + \hat{d}(z, \mathcal{D}u) + \frac{1}{n}\hat{d}(u, z) \\ &\leq \hat{d}(u, z) + \psi_{\mathcal{H}}(\hat{d}(z, u)) + \frac{1}{n}\hat{d}(u, z). \end{aligned}$$

By taking $n \to \infty$, we get

$$\hat{d}(u, \mathcal{D}u) - \psi_{\mathcal{H}}(\hat{d}(z, u)) \leq \hat{d}(u, z).$$

Thus, by assumption, $\alpha_{\mathcal{H}}(u, z)\ \mathcal{H}(\mathcal{D}u, \mathcal{D}z) \leq \psi_{\mathcal{H}}(\hat{d}(z, u)) \quad \forall\ u \in \{u_k : k \in N\}$. Finally,

$$\begin{aligned} \hat{d}(z, \mathcal{D}z) &= \lim_{n\to\infty} \hat{d}(u_{n+1}, \mathcal{D}z) \\ &\leq \limsup_{n\to\infty} \mathcal{H}(\mathcal{D}u_n, \mathcal{D}z) \\ &\leq \limsup_{n\to\infty} \frac{\psi_{\mathcal{H}}(\hat{d}(u_n, z))}{\alpha_{\mathcal{H}}(u_n, z)} \\ &= 0. \end{aligned}$$

and $\mathcal{D}z$ is closed, therefore, $z \in \mathcal{D}z$.

Example 2.3.2 *Let $\mathcal{U} = \{a, b, c, d, e\}$. Define $\hat{d} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ as follows:*
$\hat{d}(a, b) = \hat{d}(a, c) = 3.5, \quad \hat{d}(a, d) = \hat{d}(a, c) = 8, \quad \hat{d}(b, c) = 7, \quad \hat{d}(b, d) = \hat{d}(b, e) = \hat{d}(c, d) = \hat{d}(c, e) = 4.5, \quad \hat{d}(d, e) = 2$ *and* $\hat{d}(u, u) = 0, \quad \hat{d}(u, v) = \hat{d}(v, u) \quad \forall u, v \in \mathcal{U}$.
Then $(\mathcal{U}, \hat{d})$ is a complete metric space. Define $\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ as follows:
$\mathcal{D}(a) = \mathcal{D}(b) = \mathcal{D}(c) = \{a\}, \quad \mathcal{D}(d) = \{a, b\}, \quad \mathcal{D}(e) = \{c\}$.
Define $\psi_{\mathcal{H}} : [0, \infty) \to [0, \infty)$ as $\psi_{\mathcal{H}}(t) = \frac{4}{5}t$ and $\alpha_{\mathcal{H}} : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ as

$$\alpha_{\mathcal{H}}(u, v) = \begin{cases} 0, & if \quad (u = e \quad and \quad v \neq d) \quad or \quad (u \neq d \quad and \quad v = e), \\ \frac{1}{2}, & if \quad (u = e \quad and \quad v = d) \quad or \quad (u = d \quad and \quad v = e), \\ 1, & if \quad otherwise. \end{cases}$$

Now three cases arise:
Case I: If $u, v \in \{a, b, c\}$, then $\mathcal{H}(\mathcal{D}u, \mathcal{D}v) = 0$. Therefore,

$$\alpha_{\mathcal{H}}(u, v)\ \mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq \psi_{\mathcal{H}}(\hat{d}(u, v)).$$

Case II: If $u \in \{a, b, c\}, \quad v \in \{d, e\}$, then $\mathcal{H}(\mathcal{D}u, \mathcal{D}v) = 3.5$. Therefore,

$$\alpha_{\mathcal{H}}(u, v)\ \mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq \psi_{\mathcal{H}}(\hat{d}(u, v)).$$

Case III: If $u = d, \quad v = e$ or $u = e, \quad v = d$ then $\mathcal{H}(\mathcal{D}u, \mathcal{D}v) = 7$ and $\hat{d}(u, \mathcal{D}u) = 4.5$. But in this case, $\alpha_{\mathcal{H}}(u, v)\ \mathcal{H}(\mathcal{D}u, \mathcal{D}v) = \frac{1}{2} \times 7 \nleq \frac{8}{5} = \psi_{\mathcal{H}}(\hat{d}(u, v))$. Also, in this case, $\hat{d}(u, \mathcal{D}u) - \psi_{\mathcal{H}}(\hat{d}(u, v)) = 4.5 - \frac{8}{5} = 2.9 \nleq 2 = \hat{d}(u, v)$. Thus, combining all cases, *we have*

$$\hat{d}(u, \mathcal{D}u) - \psi_{\mathcal{H}}(\hat{d}(u,v)) \leq \hat{d}(u,v) \;\; \Rightarrow \;\; \alpha_{\mathcal{H}}(u,v)\,\mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq \psi_{\mathcal{H}}(\hat{d}(u,v)) \;\; \forall\, u, v \in \mathcal{U}.$$

Also, $\alpha_{\mathcal{H}}(u,v) \geq 1 \;\; \Rightarrow \;\; u, v \in \{a, b, c, d\}$*, therefore, for every* $u \in \mathcal{D}u$ *and* $v \in \mathcal{D}v$*, we have* $\alpha_{\mathcal{H}}(u,v) \geq 1$*. Thus,* $\mathcal{D}$ *is* $\alpha_{\mathcal{H}}$*-admissible. Now, take* $u_0 \in \{a, b, c, d\}$*, then, we have,* $\alpha_{\mathcal{H}}(u_0, s) \geq 1$ *for every* $s \in \mathcal{D}u_0$*. Also, if* $\{u_n\}$ *is a sequence in* $\mathcal{U}$ *such that* $\alpha_{\mathcal{H}}(u_n, u_{n+1}) \geq 1$ *for all* n *and* $u_n \to z$*, then* $\alpha_{\mathcal{H}}(u_n, u) \geq 1$ *for all* $u \in \{u_n\} \cup \{z\}$*. Hence, all the conditions of Theorem 2.3.2 are satisfied. Therefore,* $\mathcal{D}$ *has a fixed point. In this example,* a *is a fixed point.*

Theorem 2.3.3 *Let* $(\mathcal{U}, \hat{d})$ *be a complete metric space and* $\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ *be a multivalued (*$\alpha_{\mathcal{H}}$*-*$\psi_{\mathcal{H}}$*)-contractive mapping such that*

(*i*) *there exists a continuous function* $\psi_{\mathcal{H}} \in \Psi_{\mathcal{H}}$ *and* $\alpha_{\mathcal{H}} : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ *such that for all* $u, v \in \mathcal{U}$,
$\hat{d}(u, \mathcal{D}u) - \psi_{\mathcal{H}}(\hat{d}(u,v)) \leq \hat{d}(u,v) \;\; \Rightarrow \;\; \alpha_{\mathcal{H}}(u,v)\;\mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq \psi_{\mathcal{H}}(M(u,v))$
where $M(u,v) = max\{\hat{d}(u, \mathcal{D}v), \hat{d}(v, \mathcal{D}u)\}$;

(*ii*) $\mathcal{D}$ *is* $\alpha_{\mathcal{H}} - admissible$;

(*iii*) *there exists* $u_0 \in \mathcal{U}$ *such that* $\alpha_{\mathcal{H}}(u_0, s) \geq 1$ *for every* $s \in \mathcal{D}u_0$;

(*iv*) *if* $\{u_n\}$ *is a sequence in* $\mathcal{U}$ *such that* $\alpha_{\mathcal{H}}(u_n, u_{n+1}) \geq 1$ *for all* n *and* $\{u_n\} \to z$*, then* $\alpha_{\mathcal{H}}(u_n, u) \geq 1 \;\; \forall u \in \{u_k : k \in N\} \cup \{z\}$.

Then $\mathcal{D}$ *has a fixed point.*

Proof Let $u_0 \in \mathcal{U}$. Let $u_1 \in \mathcal{D}u_0$ with $u_0 \neq u_1$. Then, by assumption (*ii*), $\alpha_{\mathcal{H}}(u_0, u_1) \geq 1$. Now,

$$\hat{d}(u_0, \mathcal{D}u_0) - \psi_{\mathcal{H}}(\hat{d}(u_0, u_1)) \leq \hat{d}(u_0, \mathcal{D}u_0) \leq \hat{d}(u_0, u_1).$$

By assumption (*i*), we have

$$\alpha_{\mathcal{H}}(u_0, u_1)\;\mathcal{H}(\mathcal{D}u_0, \mathcal{D}u_1) \leq \psi_{\mathcal{H}}(M(u_0, u_1)).$$

This implies

$$\hat{d}(u_1, \mathcal{D}u_1) \leq \alpha_{\mathcal{H}}(u_0, u_1)\;\mathcal{H}(\mathcal{D}u_0, \mathcal{D}u_1) \leq \psi_{\mathcal{H}}(M(u_0, u_1)).$$

Let $u_2 \in \mathcal{D}u_1$ be such that

$$\hat{d}(u_1, u_2) < \psi_{\mathcal{H}}(M(u_0, u_1)) + \psi_{\mathcal{H}}(M(u_0, u_1)).$$

Again, we have

$$\hat{d}(u_1, \mathcal{D}u_1) - \psi_{\mathcal{H}}(\hat{d}(u_1, u_2)) \leq \hat{d}(u_1, \mathcal{D}u_1) \leq \hat{d}(u_1, u_2).$$

By assumption, we have

$$\alpha_{\mathcal{H}}(u_1, u_2)\ \mathcal{H}(\mathcal{D}u_1, \mathcal{D}u_2) \leq \psi_{\mathcal{H}}(M(u_1, u_2)).$$

which implies

$$\hat{d}(u_2, \mathcal{D}u_2) \leq \alpha_{\mathcal{H}}(u_1, u_2)\mathcal{H}(\mathcal{D}u_1, \mathcal{D}u_2) \leq \psi_{\mathcal{H}}(M(u_1, u_2)).$$

Let $u_3 \in \mathcal{D}u_2$ be such that

$$\hat{d}(u_2, u_3) < \psi_{\mathcal{H}}(M(u_1, u_2)) + \psi^2_{\mathcal{H}}(M(u_0, u_1)).$$

Thus, we get a sequence $u_{n+1} \in \mathcal{D}u_n$ such that

$$\begin{aligned}
\hat{d}(u_n, u_{n+1}) &< \psi_{\mathcal{H}}(M(u_{n-1}, u_n)) + \psi^n_{\mathcal{H}}(M(u_0, u_1)) \\
&\leq \psi_{\mathcal{H}}[(M(u_{n-1}, u_n)) + \psi^{n-1}_{\mathcal{H}}(M(u_0, u_1))] \\
&\leq \psi_{\mathcal{H}}[\psi_{\mathcal{H}}(M(u_{n-2}, u_{n-1})) + \psi^{n-1}_{\mathcal{H}}(M(u_0, u_1)) + \psi^{n-1}_{\mathcal{H}}(M(u_0, u_1))] \\
&= \psi^2_{\mathcal{H}}(M(u_{n-2}, u_{n-1})) + 2\psi^n_{\mathcal{H}}(M(u_0, u_1)) \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&\leq \psi^n_{\mathcal{H}}(M(u_0, u_1)) + n\psi^n_{\mathcal{H}}(M(u_0, u_1)) \\
&= (n+1)\psi^n_{\mathcal{H}}(M(u_0, u_1)).
\end{aligned}$$

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n \geq n_0} (n+1)\psi^n_{\mathcal{H}}(M(u_0, u_1)) < \epsilon.$$

For $m, n \in \mathbb{N}$ with $m > n > n_0$, we have

$$\begin{aligned}
\hat{d}(u_n, u_m) &\leq \sum_{k=n}^{m-1} \hat{d}(u_k, u_{k+1}) \\
&\leq \sum_{k=n}^{m-1} (k+1)\psi_{\mathcal{H}}^k(M(u_0, u_1)) \\
&\leq \sum_{k \geq n} (k+1)\psi_{\mathcal{H}}^k(M(u_0, u_1)) \\
&\leq \sum_{k \geq n_0} (k+1)\psi_{\mathcal{H}}^k(M(u_0, u_1)) \\
&< \epsilon.
\end{aligned}$$

Thus, $\{u_n\}$ is a Cauchy sequence in $\mathcal{U}$. But $(\mathcal{U}, \hat{d})$ is a complete metric space. Therefore, there exists $z \in \mathcal{U}$ such that $u_n \to z$ as $n \to \infty$.
Next, it is shown that $\hat{d}(z, \mathcal{D}u) \leq \psi_{\mathcal{H}}(\hat{d}(z, u)) \;\; \forall \;\; u \in \{u_k : k \in N\} - \{z\}$.
Since $u_n \to z$, there exists $n_0 \in \mathbb{N}$ such that $\hat{d}(z, u_n) \leq \frac{1}{3}\hat{d}(z, u)$ for all $n \geq n_0$.
Then,

$$\begin{aligned}
\hat{d}(u_n, \mathcal{D}u_n) - \psi_{\mathcal{H}}(\hat{d}(u, u_n)) &\leq \hat{d}(u_n, \mathcal{D}u_n) \\
&\leq \hat{d}(u_n, u_{n+1}) \\
&\leq \hat{d}(u_n, z) + \hat{d}(z, u_{n+1}) \\
&\leq \frac{2}{3}\hat{d}(u, z) \\
&= \hat{d}(u, z) - \frac{1}{3}\hat{d}(u, z) \\
&\leq \hat{d}(u, z) - \hat{d}(z, u_n) \\
&\leq \hat{d}(u_n, u).
\end{aligned}$$

Therefore, by assumption,

$$\begin{aligned}
&\alpha_{\mathcal{H}}(u_n, u)\; \mathcal{H}(\mathcal{D}u_n, \mathcal{D}u) \leq \psi_{\mathcal{H}}(M(u_n, u)) \\
&\quad \Rightarrow \;\; \mathcal{H}(\mathcal{D}u_n, \mathcal{D}u) \leq \psi_{\mathcal{H}}(M(u_n, u)) \\
&\quad \Rightarrow \;\; \hat{d}(u_{n+1}, \mathcal{D}u) \leq \psi_{\mathcal{H}}(M(u_n, u)) \;\; for \;\; n \geq n_0. \qquad (2.3.2)
\end{aligned}$$

Now, $M(u_n, u) = max\{\hat{d}(u_n, \mathcal{D}u), \hat{d}(u, \mathcal{D}u_n)\}$. If $M(u_n, u) = \hat{d}(u, \mathcal{D}u_n)$ for infinitely many n, then by (2.3.2),

$$\hat{d}(u_{n+1}, \mathcal{D}u) \leq \psi_{\mathcal{H}}(\hat{d}(u, \mathcal{D}u_n)) \leq \psi_{\mathcal{H}}(\hat{d}(u, u_{n+1})) \;\; for \;\; infinitely \;\; many \;\; n.$$

Letting $n \to \infty$,

$$\hat{d}(z, \mathcal{D}u) \leq \psi_{\mathcal{H}}(\hat{d}(u, z)).$$

If $M(u_n, u) = \hat{d}(u_n, \mathcal{D}u)$ for infinitely many n, then by (2.3.2),

$$\hat{d}(u_{n+1}, \mathcal{D}u) \leq \psi_{\mathcal{H}}(\hat{d}(u_n, \mathcal{D}u)) \;\; for \;\; infinitely \;\; many \;\; n.$$

Letting $n \to \infty$,

$$\hat{d}(z, \mathcal{D}u) \leq \psi_{\mathcal{H}}(\hat{d}(z, \mathcal{D}u)). \tag{2.3.3}$$

Suppose that $\hat{d}(z, \mathcal{D}u) > 0$, then

$$\psi_{\mathcal{H}}(\hat{d}(z, \mathcal{D}u)) < \hat{d}(z, \mathcal{D}u);$$

which is contradictory to (2.3.3). Therefore, $\hat{d}(z, \mathcal{D}u) = 0$. Hence $\hat{d}(z, \mathcal{D}u) \leq \psi_{\mathcal{H}}(\hat{d}(u, z))$.
Further, it is proved that

$$\alpha_{\mathcal{H}}(u, z) \; \mathcal{H}(\mathcal{D}u, \mathcal{D}z) \leq \psi_{\mathcal{H}}(M(z, u)) \;\; \forall \;\; u \in \{u_k : k \in N\}.$$

If $u = z$, then it is obvious. So, let $u \neq z$. Then for every $n \in \mathbb{N}$, $\exists \;\; v_n \in \mathcal{D}u$ such that

$$\hat{d}(z, v_n) \leq \hat{d}(z, \mathcal{D}u) + \frac{1}{n}\hat{d}(u, z).$$

Now,

$$\begin{aligned}
\hat{d}(u, \mathcal{D}u) &\leq \hat{d}(u, v_n) \\
&\leq \hat{d}(u, z) + \hat{d}(z, v_n) \\
&\leq \hat{d}(u, z) + \hat{d}(z, \mathcal{D}u) + \frac{1}{n}\hat{d}(u, z) \\
&\leq \hat{d}(u, z) + \psi_{\mathcal{H}}(\hat{d}(z, u)) + \frac{1}{n}\hat{d}(u, z).
\end{aligned}$$

By taking $n \to \infty$, we get

$$\hat{d}(u, \mathcal{D}u) - \psi_{\mathcal{H}}(\hat{d}(z, u)) \leq \hat{d}(u, z).$$

Thus, by assumption, $\alpha_{\mathcal{H}}(u, z) \; \mathcal{H}(\mathcal{D}u, \mathcal{D}z) \leq \psi_{\mathcal{H}}(M(z, u)) \;\; \forall \; u \in \{u_k : k \in N\}$.
Finally, it has been shown that $\hat{d}(z, \mathcal{D}z) = 0$. Let us suppose the contrary to this

statement *i.e.* $\hat{d}(z, \mathcal{D}z) > 0$.

$$\begin{aligned}
\hat{d}(z, \mathcal{D}z) &= \lim_{n\to\infty} \hat{d}(u_{n+1}, \mathcal{D}z) \\
&\leq \limsup_{n\to\infty} \mathcal{H}(\mathcal{D}u_n, \mathcal{D}z) \\
&\leq \limsup_{n\to\infty} \frac{\psi_{\mathcal{H}}(M(u_n, z))}{\alpha_{\mathcal{H}}(u_n, z)} \\
&= \limsup_{n\to\infty} \frac{\psi_{\mathcal{H}}(max\{\hat{d}(u_n, \mathcal{D}z), \hat{d}(z, \mathcal{D}u_n)\})}{\alpha_{\mathcal{H}}(u_n, z)} \\
&\leq \limsup_{n\to\infty} \frac{\psi_{\mathcal{H}}(max\{\hat{d}(u_n, \mathcal{D}z), \hat{d}(z, u_{n+1})\})}{\alpha_{\mathcal{H}}(u_n, z)} \\
\Rightarrow \hat{d}(z, \mathcal{D}z) &\leq \frac{\psi_{\mathcal{H}}(\hat{d}(z, \mathcal{D}z))}{\liminf_{n\to\infty} \alpha_{\mathcal{H}}(u_n, z)} \\
&< \frac{\hat{d}(z, \mathcal{D}z)}{\liminf_{n\to\infty} \alpha_{\mathcal{H}}(u_n, z)}.
\end{aligned}$$

which is not possible as $\liminf_{n\to\infty} \alpha_{\mathcal{H}}(u_n, z) \geq 1$. Thus, $\hat{d}(z, \mathcal{D}z) = 0$. Since $\mathcal{D}z$ is closed, therefore, $z \in \mathcal{D}z$.

2.3.2 Consequences

Nadler's fixed point theorem

Theorem 2.3.4 *(Nadler [111]) Let $(\mathcal{U}, \hat{d})$ be a complete metric space and $\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ be a multivalued mapping satisfying*

$$\mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq l\ \hat{d}(u, v),$$

for all $u, v \in \mathcal{U}$ and $l \in [0, 1)$. Then $\mathcal{D}$ possesses a fixed point.

Proof Let $\alpha_{\mathcal{H}} : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ be the mapping defined by $\alpha_{\mathcal{H}}(u, v) = 1$ for all $u, v \in \mathcal{U}$ and $\psi_{\mathcal{H}} : [0, +\infty) \to [0, +\infty)$ defined by $\psi_{\mathcal{H}}(u) = lu$ where $l \in [0, 1)$. It is easy to show that all the hypotheses of Theorem 2.3.1 are satisfied. Consequently, $\mathcal{D}$ has a fixed point.

Suzuki-Kikkawa's fixed point theorem

Theorem 2.3.5 *(Suzuki, Kikkawa [91]) Let $(\mathcal{U}, \hat{d})$ be a complete metric space and*

$\mathcal{D} : \mathcal{U} \to \mathcal{P}_{cb}(\mathcal{U})$ *be a multivalued mapping satisfying*

$$\eta(r)\ \hat{d}(u, \mathcal{D}u) \leq \hat{d}(u, v) \ \Rightarrow\ \mathcal{H}(\mathcal{D}u, \mathcal{D}v) \leq r\ \hat{d}(u, v),$$

for all $u, v \in \mathcal{U}$ *and* $r \in [0, 1)$ *where* $\eta : [0, 1) \to (\frac{1}{2}, 1]$ *is a mapping defined by* $\eta(r) = \frac{1}{1+r}$. *Then* $\mathcal{D}$ *possesses a fixed point.*

Proof Let $\alpha_{\mathcal{H}} : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ be the mapping defined by $\alpha_{\mathcal{H}}(u, v) = 1$ for all $u, v \in \mathcal{U}$ and $\psi_{\mathcal{H}} : [0, +\infty) \to [0, +\infty)$ defined by $\psi_{\mathcal{H}}(u) = ru$ where $r \in [0, 1)$. It is easy to show that all the hypotheses of Theorem 2.3.2 are satisfied. Consequently, $\mathcal{D}$ has a fixed point.

2.3.3 Application to Homotopy Theory

In this section, a result in homotopy theory in the context of fixed points is presented through our functions used in Theorem 2.3.1.

Theorem 2.3.6 *Let* $(\mathcal{U}, \hat{d})$ *be a complete metric space,* $\mathcal{M}$ *and* $\mathcal{N}$ *are open and closed subsets of* $\mathcal{U}$ *respectively such that* $\mathcal{M} \subset \mathcal{N}$. *For* $a, b \in \mathbb{R}$, *let* $\mathcal{D} : \mathcal{N} \times [a, b] \to \mathcal{P}_{cb}(\mathcal{U})$ *be an operator which satisfies the conditions given below:*
(i) $v \notin \mathcal{D}(v, t)$ *for every* $v \in \mathcal{N}/\mathcal{M}$ *and* $t \in [a, b]$;
(ii) there exists $\psi_{\mathcal{H}} \in \Psi_{\mathcal{H}}$ *and* $\alpha_{\mathcal{H}} : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ *such that*

$$\alpha_{\mathcal{H}}(u, v)\ \mathcal{H}(\mathcal{D}(u, t), \mathcal{D}(v, t)) \leq \psi_{\mathcal{H}}(\hat{d}(u, v))$$

for each pair $(u, v) \in \mathcal{N} \times \mathcal{N}$ *and* $t \in [a, b]$;
(iii) there exists a continuous function $\varsigma : [a, b] \to \mathbb{R}$ *such that for every* $s, t \in [a, b]$ *and* $u \in \mathcal{N}$, *we have*

$$\mathcal{H}(\mathcal{D}(u, s), \mathcal{D}(u, t)) \leq \psi_{\mathcal{H}}(|\varsigma(s) - \varsigma(t)|);$$

(iv) if $u^* \in \mathcal{D}(u^*, t)$, *then* $\mathcal{D}(u^*, t) = \{u^*\}$;
(v) there exists $u_0 \in \mathcal{U}$ *such that* $u_0 \in \mathcal{D}(u_0, t)$;
(vi) the function $\xi_{\mathcal{H}} : [0, \infty) \to [0, \infty)$ *is continuous and strictly nondecreasing defined by* $\xi_{\mathcal{H}}(u) = u - \psi_{\mathcal{H}}(u)$.
If $\mathcal{D}(. , t^*)$ *has a fixed point in* $\mathcal{N}$ *for some* $t^* \in [a, b]$, *then* $\mathcal{D}(. , t)$ *has a fixed point in* $\mathcal{M}$ *for all* $t \in [a, b]$. *Furthermore, for fixed* $t \in [a, b]$, *this fixed point is unique if* $\psi_{\mathcal{H}}(t) = \frac{t}{2}$.

Proof Define a mapping $\alpha_{\mathcal{H}} : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ by

$$\alpha_{\mathcal{H}}(u, v) = \begin{cases} 3 & u \in \mathcal{D}(u, t),\ v \in \mathcal{D}(v, t), \\ 0 & otherwise \end{cases}$$

for $t \in [a, b]$. Let $\alpha_{\mathcal{H}}(u, v) \geq 1$ which implies that $u \in \mathcal{D}(u, t)$ and $v \in \mathcal{D}(v, t)\ \ \forall\, t \in [a, b]$. By assumption (iv), $\mathcal{D}(u, t) = \{u\}$ and $\mathcal{D}(v, t) = \{v\}$. It follows that $\mathcal{D}$ is $\alpha_{\mathcal{H}}$-admissible.
By assumption (v), there exists $u_0 \in \mathcal{U}$ such that $u_0 \in \mathcal{D}(u_0, t)$ for all t *i.e.* $\alpha_{\mathcal{H}}(u_0, u_0) \geq 1$. Assume that $\alpha_{\mathcal{H}}(u_n, u_{n+1}) \geq 1$ for all n and $u_n \to z$ as $n \to \infty$ for a sequence $\{u_n\}$ in $\mathcal{U}$, then $u_n \in \mathcal{D}(u_n, t)$ and $u_{n+1} \in \mathcal{D}(u_{n+1}, t)$ for all n and $t \in [a, b]$. This implies that $z \in \mathcal{D}(z, t)$ and thus $\alpha_{\mathcal{H}}(u_n, z) \geq 1$. Let

$$\mathcal{V} = \{t \in [a, b] \mid u \in \mathcal{D}(u, t) \ \ for \ \ u \in \mathcal{M}\}.$$

As $\mathcal{D}(.\ , t^*)$ has a fixed point in $\mathcal{N}$ for some $t^* \in [a, b]$, therefore, there exists $u \in \mathcal{N}$ such that $u \in \mathcal{D}(u, t^*)$ and by assumption (i), $u \in \mathcal{D}(u, t^*)$ for $t^* \in [a, b]$ and $u \in \mathcal{M}$. So, $\mathcal{V}$ is a non-empty set. Now, it is proved that $\mathcal{V}$ is open as well as closed in $[a, b]$. Let $u_0 \in \mathcal{D}(u_0, t_0)$ for $u_0 \in \mathcal{M}$ and $t_0 \in \mathcal{V}$. The set $\mathcal{M}$ being open in $(\mathcal{U}, \hat{d})$ implies the existence of $r > 0$ such that $\mathcal{B}_{\hat{d}}(u_0, r) \subseteq \mathcal{M}$. Let $\epsilon = r - \psi_{\mathcal{H}}(r) > 0$.
As ς is continuous on $t_0 \in [a, b]$, there exists $\alpha_\epsilon > 0$ such that $|\varsigma(t) - \varsigma(t_0)| < \epsilon$ for all $t \in (t_0 - \alpha_\epsilon, t_0 + \alpha_\epsilon)$.
Let $t \in (t_0 - \alpha_\epsilon, t_0 + \alpha_\epsilon)$ for $u \in \mathcal{B}_{\hat{d}}(u_0, r) = \{u \in \mathcal{U} \mid \hat{d}(u_0, u) < r\}$ and $u \in \mathcal{D}(u, t)$, we have

$$\begin{aligned} \hat{d}(u, u_0) &= \hat{d}(\mathcal{D}(u, t), u_0) \\ &\leq \mathcal{H}(\mathcal{D}(u, t), \mathcal{D}(u_0, t_0)) + \mathcal{H}(\mathcal{D}(u, t_0), \mathcal{D}(u_0, t_0)). \end{aligned}$$

Since $u_0 \in \mathcal{D}(u_0, t_0)$ and $u \in \mathcal{B}_{\hat{d}}(u_0, r) \subseteq \mathcal{M} \subset \mathcal{N}$ with $t_0 \in [a, b]$, therefore, $\alpha_{\mathcal{H}}(u, u_0) \geq 1$. So, by assumption (ii) and (iii), we obtain

$$\begin{aligned} \hat{d}(u, u_0) &\leq \psi_{\mathcal{H}}(|\varsigma(t) - \varsigma(t_0)|) + \alpha_{\mathcal{H}}(u, u_0)\ \mathcal{H}(\mathcal{D}(u, t_0), \mathcal{D}(u_0, t_0)) \\ &\leq \psi_{\mathcal{H}}(\varsigma(t) - \varsigma(t_0)) + \psi_{\mathcal{H}}(\hat{d}(u, u_0)) \\ &\leq \psi_{\mathcal{H}}(\epsilon) + \psi_{\mathcal{H}}(\hat{d}(u, u_0)) \\ &< \psi_{\mathcal{H}}(\epsilon) + \psi_{\mathcal{H}}(r) \\ &= \psi_{\mathcal{H}}(r - \psi_{\mathcal{H}}(r)) + \psi_{\mathcal{H}}(r) \end{aligned}$$

$$\begin{aligned} &< r - \psi_{\mathcal{H}}(r) + \psi_{\mathcal{H}}(r) \\ &= r. \end{aligned}$$

Thus, for every fixed $t \in (t_0 - \alpha_\epsilon, t_0 + \alpha_\epsilon)$, $\mathcal{D}(.\ ,t) : \overline{\mathcal{B}_{\hat{d}}(u_0,r)} \rightarrow \mathcal{P}_{cb}(\overline{\mathcal{B}_{\hat{d}}(u_0,r)})$ fulfills all the hypotheses of Theorem 2.3.1 and therefore, $\mathcal{D}(.\ ,t)$ has a fixed point in $\overline{\mathcal{B}_{\hat{d}}(u_0,r)} \subset \mathcal{N}$. But, by assumption (i), this fixed point belongs to $\mathcal{M}$. Therefore, $(t_0 - \alpha_\epsilon, t_0 + \alpha_\epsilon) \subseteq \mathcal{V}$ and thus $\mathcal{V}$ is open in $[a,b]$.
Next, it is shown that the set $\mathcal{V}$ is closed. Let $\{t_n\}$ be a sequence in $\mathcal{V}$ with $t_n \rightarrow t' \in [a,b]$ as $n \rightarrow \infty$. Now, it will be proved that $t' \in \mathcal{V}$. By definition of $\mathcal{V}$, there exists $u_n \in \mathcal{M}$ such that $u_n \in \mathcal{D}(u_n, t_n)$ for all n. For $n, m \in \mathbb{N}$ $(m > n)$,

$$\begin{aligned} \hat{d}(u_n, u_m) &\leq \mathcal{H}(\mathcal{D}(u_n,t_n), \mathcal{D}(u_m,t_m)) \\ &\leq \mathcal{H}(\mathcal{D}(u_n,t_n), \mathcal{D}(u_n,t_m)) + \mathcal{H}(\mathcal{D}(u_n,t_m), \mathcal{D}(u_m,t_m)) \\ &\leq \psi_{\mathcal{H}}(|\varsigma(t_n) - \varsigma(t_m)|) + \alpha_{\mathcal{H}}(u_n,u_m)\mathcal{H}(\mathcal{D}(u_n,t_m), \mathcal{D}(u_m,t_m)) \\ &\leq \psi_{\mathcal{H}}(|\varsigma(t_n) - \varsigma(t_m)|) + \psi_{\mathcal{H}}(\hat{d}(u_n,u_m)) \\ \Rightarrow \hat{d}(u_n,u_m) - \psi_{\mathcal{H}}(\hat{d}(u_n,u_m)) &\leq \psi_{\mathcal{H}}(|\varsigma(t_n) - \varsigma(t_m)|) \\ \Rightarrow \xi_{\mathcal{H}}(\hat{d}(u_n,u_m)) &\leq \psi_{\mathcal{H}}(|\varsigma(t_n) - \varsigma(t_m)|) \\ &< |\varsigma(t_n) - \varsigma(t_m)| \\ \Rightarrow \hat{d}(u_n,u_m) &< \xi_{\mathcal{H}}^{-1}(|\varsigma(t_n) - \varsigma(t_m)|). \end{aligned}$$

Continuity of functions $\xi_{\mathcal{H}}^{-1}$, ς and convergence of the sequence $\{t_n\}$ implies that

$$\lim_{m,n\rightarrow\infty} \hat{d}(u_n,u_m) = 0.$$

i.e. $\{u_n\}$ is a Cauchy sequence in $\mathcal{U}$ and since $(\mathcal{U}, \hat{d})$ is complete, therefore, there exists $u' \in \mathcal{N}$ such that $\lim_{n\rightarrow\infty} u_n = u'$.
Now,

$$\begin{aligned} \hat{d}(u_n, \mathcal{D}(u',t')) &\leq \mathcal{H}(\mathcal{D}(u_n,t_n), \mathcal{D}(u',t')) \\ &\leq \mathcal{H}(\mathcal{D}(u_n,t_n), \mathcal{D}(u_n,t')) + \mathcal{H}(\mathcal{D}(u_n,t'), \mathcal{D}(u',t')) \\ &\leq \psi_{\mathcal{H}}(|\varsigma(t_n) - \varsigma(t')|) + \alpha_{\mathcal{H}}(u_n,u')\mathcal{H}(\mathcal{D}(u_n,t'), \mathcal{D}(u',t')) \\ \Rightarrow \hat{d}(u_n, \mathcal{D}(u',t')) &\leq \psi_{\mathcal{H}}(|\varsigma(t_n) - \varsigma(t')|) + \psi_{\mathcal{H}}(\hat{d}(u_n,u')). \end{aligned}$$

Taking limit $n \rightarrow \infty$, we get $\lim_{n\rightarrow\infty} \hat{d}(u_n, \mathcal{D}(u',t')) = 0$ and therefore, $\hat{d}(u', \mathcal{D}(u',t')) = \lim_{n\rightarrow\infty} \hat{d}(u_n, \mathcal{D}(u',t')) = 0$. This implies that $u' \in \mathcal{D}(u',t')$ and by assumption (i), $u' \in \mathcal{M}$. So, $t' \in \mathcal{V}$ and hence $\mathcal{V}$ is closed in $[a,b]$.

Thus, $\mathcal{V} = [a, b]$ and $\mathcal{D}(. \ , t)$ has a fixed point in $\mathcal{M}$ for all t in $[a, b]$.
For uniqueness of this fixed point, let us consider another fixed point $v' \in \mathcal{D}(v', t)$ for fixed $t \in [a, b]$.

$$\begin{aligned} \hat{d}(u', v') &\leq \mathcal{H}(\mathcal{D}(u', t), \mathcal{D}(v', t)) \\ &\leq \alpha_{\mathcal{H}}(u', v') \ \mathcal{H}(\mathcal{D}(u', t), \mathcal{D}(v', t)) \\ &\leq \psi_{\mathcal{H}}(\hat{d}(u', v')). \end{aligned}$$

If $\psi_{\mathcal{H}}(t) = \frac{t}{2} \ \ \forall \ t \geq 0$, the uniqueness follows.

2.4 Fixed point theorems in Partially Ordered metric spaces

2.4.1 Main Results

Theorem 2.4.1 *Let $(\mathcal{U}, \preceq, \hat{d})$ be a partially ordered metric space where $\hat{d}$ is a complete metric. Let $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ be a bijective, (ξ, α)-expansive mapping satisfying the following conditions:*

- (i) $\mathcal{D}^{-1}$ *is nondecreasing*;
- (ii) $\mathcal{D}^{-1}$ *is ordered* $\alpha - admissible$;
- (iii) *there exists some* $u_0 \in \mathcal{U}$ *such that* $\alpha(u_0, \mathcal{D}^{-1}u_0) \geq 1$ *where* $\mathcal{D}^{-1}u_0 \leq u_0$;
- (iv) $\mathcal{D}^{-1}$ *is continuous.*

Then $\mathcal{D}$ possesses a fixed point in $\mathcal{U}$.

Proof As $\mathcal{D}^{-1}$ is nondecreasing and $\mathcal{D}^{-1}u_0 \leq u_0$, we obtain

$$u_0 \geq \mathcal{D}^{-1}u_0 \geq \mathcal{D}^{-2}u_0 \geq ... \geq \mathcal{D}^{-n}u_0; \tag{2.4.1}$$

and by (ii), $\mathcal{D}^{-1}$ is ordered α-admissible, therefore we have

$$\alpha(u_0, \mathcal{D}^{-1}u_0) \geq 1 \Rightarrow \alpha(\mathcal{D}^{-1}u_0, \mathcal{D}^{-2}u_0) \geq 1 \Rightarrow ... \Rightarrow \alpha(\mathcal{D}^{-n}u_0, \mathcal{D}^{-(n+1)}u_0) \geq 1. \tag{2.4.2}$$

Now using (2.2.2) and (2.4.2), we get

$$\begin{aligned}\hat{d}(u_0, \mathcal{D}^{-1}u_0) &\leq \alpha(u_0, \mathcal{D}^{-1}u_0)\ \hat{d}(u_0, \mathcal{D}^{-1}u_0)\\ &\leq \xi(\hat{d}(\mathcal{D}u_0, u_0)).\end{aligned}$$

Continuing in this manner, we get

$$\hat{d}(\mathcal{D}^{-n}u_0, \mathcal{D}^{-(n+1)}u_0) \leq \xi^n(\hat{d}(\mathcal{D}u_0, u_0)).$$

Note that $\hat{d}(\mathcal{D}^{-n}u_0, \mathcal{D}^{-(n+1)}u_0) \to 0$ as $n \to +\infty$. Next it is proved that the sequence $\{\mathcal{D}^{-n}u_0\}$ is a Cauchy sequence in $\mathcal{U}$. For given $\epsilon > 0$, let us suppose that there exists $n_\epsilon \in \mathbb{N}$ such that

$$\sum_{n \geq n_\epsilon} \xi^n(\hat{d}(\mathcal{D}u_0, u_0)) < \epsilon.$$

For $n, m \geq 0$ with $m > n > n_\epsilon$, we have by using triangular inequality,

$$\begin{aligned}\hat{d}(\mathcal{D}^{-n}u_0, \mathcal{D}^{-m}u_0) &\leq \hat{d}(\mathcal{D}^{-n}u_0, \mathcal{D}^{-(n+1)}u_0) + \hat{d}(\mathcal{D}^{-(n+1)}u_0, \mathcal{D}^{-(n+2)}u_0) + ...\\ &\quad + \hat{d}(\mathcal{D}^{-(m+1)}u_0, \mathcal{D}^{-m}u_0)\\ &\leq \xi^n(\hat{d}(\mathcal{D}u_0, u_0)) + \xi^{n+1}(\hat{d}(\mathcal{D}u_0, u_0)) + ... + \xi^m(\hat{d}(\mathcal{D}u_0, u_0))\\ &= \sum_{l=n}^{m} \xi^l(\hat{d}(\mathcal{D}u_0, u_0))\\ &\leq \sum_{n \geq n_\epsilon} \xi^n(\hat{d}(\mathcal{D}u_0, u_0)) < \epsilon.\end{aligned}$$

By completeness of partially ordered metric space $(\mathcal{U}, \preceq, \hat{d})$, the sequence $\{\mathcal{D}^{-n}u_0\}$ is convergent being a Cauchy sequence. Let $u \in \mathcal{U}$ be the limit of this sequence. Let $\epsilon > 0$ be given. By continuity of the mapping $\mathcal{D}^{-1}$, there exists $\delta > 0$ such that $\hat{d}(\hat{u}, u) < \delta$ implies $\hat{d}(\mathcal{D}^{-1}\hat{u}, \mathcal{D}^{-1}u) < \frac{\epsilon}{2}$ for each $\hat{u} \in \mathcal{U}$. Assume that $\varsigma = min\{\frac{\epsilon}{2}, \delta\}$ is given. Since $\{\mathcal{D}^{-n}u_0\}$ is convergent to u, therefore there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0 (n \in \mathbb{N})$, $\hat{d}(\mathcal{D}^{-n}u_0, u) < \varsigma$.
Now, for $n \geq n_0 (n \in \mathbb{N})$, we obtain

$$\begin{aligned}\hat{d}(\mathcal{D}^{-1}u, u) &\leq \hat{d}(\mathcal{D}^{-1}u, \mathcal{D}^{-(n+1)}u_0) + \hat{d}(\mathcal{D}^{-(n+1)}u_0, u)\\ &< \frac{\epsilon}{2} + \varsigma \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;\end{aligned}$$

which implies that $\hat{d}(\mathcal{D}^{-1}u, u) = 0$ *i.e.* $\mathcal{D}^{-1}u = u$ and $\mathcal{D}u = \mathcal{D}(\mathcal{D}^{-1}u) = (\mathcal{D}\mathcal{D}^{-1})u = u$. This completes the proof.

Example 2.4.1 *Let $\mathcal{U} = \mathbb{R}^+$ and $\preceq$ be the partial order defined on $\mathcal{U}$. Let $\hat{d}$ be the standard metric $\hat{d}(u, v) = |u - v|$ for all $u, v \in \mathbb{R}$. Define the mappings $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ and $\alpha : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ by*

$$\mathcal{D}(u) = \begin{cases} 2u^2 & u \geq 0, \\ 0 & u < 0 \ ; \end{cases}$$

and

$$\alpha(u, v) = \begin{cases} 1 & u, v \geq 0, \\ 0 & otherwise. \end{cases}$$

Note that $(\mathbb{R}^+, \hat{d})$ is a complete metric space and the mapping $\mathcal{D}$ is nondecreasing and continuous. Let $\xi(s) = \frac{s}{2}$ for all $s \geq 0$. Clearly, $\mathcal{D}$ is an (ξ, α)-expansive mapping. For all $u, v \in \mathbb{R}^+$ with $u \geq v$, if $\alpha(u, v) \geq 1$, then we get

$$\hat{d}(\mathcal{D}u, \mathcal{D}v) = |\mathcal{D}u - \mathcal{D}v| = |2u^2 - 2v^2|;$$

and

$$\begin{aligned} \xi(\hat{d}(\mathcal{D}u, \mathcal{D}v)) &= \xi(|2u^2 - 2v^2|) \\ &= |u^2 - v^2| \geq \alpha(u, v) \ \hat{d}(u, v). \end{aligned}$$

Also, $\mathcal{D}^{-1}$ is ordered α-admissible. For this, let $u, v \in \mathbb{R}^+$ such that $\alpha(u, v) \geq 1$ where $u \geq v$ and by definition of $\mathcal{D}^{-1}$, we have

$$\begin{aligned} \mathcal{D}^{-1}u &= \sqrt{u} \geq 0, \quad \mathcal{D}^{-1}v = \sqrt{v} \geq 0 \\ \Rightarrow \alpha(\mathcal{D}^{-1}u, \mathcal{D}^{-1}v) &\geq 1. \end{aligned}$$

Also, there exists $u_0 \in \mathcal{U}$ such that $\alpha(u_0, \mathcal{D}^{-1}u_0) \geq 1$. For $u_0 = 1$, we have

$$\alpha(1, \mathcal{D}^{-1}1) = \alpha(1, 1) \geq 1 \ \text{ and } \ \mathcal{D}^{-1}u_0 = \mathcal{D}^{-1}1 = 1 \leq 1 = u_0.$$

Thus, all the required conditions of Theorem 2.4.1 are fulfilled. Consequently, $\mathcal{D}$ has a fixed point. In this example, 0 is a fixed point of $\mathcal{D}$.

The next theorem shows the existence of a fixed point for the mapping $\mathcal{D}$ where $\mathcal{D}$ need not be necessarily continuous. The following hypothesis helps us to prove the result:

(H_1): If $\{u_n\}$ is a nondecreasing sequence in $\mathcal{U}$ converging to $u \in \mathcal{U}$, then $u_n \preceq \mathcal{U}$

for all $n \in \mathbb{N}$.
(H_2): If $\{u_n\}$ is a sequence in $\mathcal{U}$ such that $\alpha(u_n, u_{n+1}) \geq 1$ for every $n \in \mathbb{N}$ and $\lim_n u_n = u$, then $\alpha(\mathcal{D}^{-1}u_n,\ \mathcal{D}^{-1}u) \geq 1$ for all n.

Theorem 2.4.2 *If the condition of continuity of $\mathcal{D}$ in Theorem 2.4.1 is replaced by the conditions (H_1) and (H_2), then $\mathcal{D}$ has a fixed point in $\mathcal{U}$.*

Proof Following the proof of Theorem 2.4.1, there exists $u \in \mathcal{U}$ such that $\lim_n \mathcal{D}^{-1}u_0 = u$. Next it is proved that u is a fixed point for $\mathcal{D}$. For given $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that for $n \geq n_0,\ \ n \in \mathbb{N}$, we have by convergence of sequence $\{\mathcal{D}^{-1}u_0\}$,

$$\hat{d}(\mathcal{D}^{-n}u_0, u) < \frac{\epsilon}{2}.$$

Then by hypothesis (H_1), we obtain

$$\mathcal{D}^{-n}u_0 \leq u.$$

Now using (2.2.2) and triangular inequality, we have

$$\begin{aligned} \hat{d}(\mathcal{D}^{-1}u, u) &\leq \hat{d}(\mathcal{D}^{-1}u, \mathcal{D}^{-(n+1)}u_0) + \hat{d}(\mathcal{D}^{-(n+1)}u_0, u) \\ &= \hat{d}(\mathcal{D}^{-1}u, \mathcal{D}^{-1}.\mathcal{D}^{-n}u_0) + \hat{d}(\mathcal{D}^{-(n+1)}u_0, u) \end{aligned}$$

$$\begin{aligned} &\leq \alpha(\mathcal{D}^{-(n+1)}u_0, \mathcal{D}^{-1}u_0)\ \hat{d}(\mathcal{D}^{-1}.\mathcal{D}^{-n}u_0, \mathcal{D}^{-1}u) + \hat{d}(\mathcal{D}^{-(n+1)}u_0, u) \\ &\leq \xi(\hat{d}(\mathcal{D}^{-n}u_0, u)) + \hat{d}(\mathcal{D}^{-(n+1)}u_0, u) \\ &< \hat{d}(\mathcal{D}^{-n}u_0, u) + \hat{d}(\mathcal{D}^{-(n+1)}u_0, u) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This gives an end to our proof.

Example 2.4.2 *Let $\mathcal{U} = \mathbb{R}^+$ and $\preceq$ be the partial order defined on $\mathcal{U}$. Let $\hat{d}$ be the standard metric $\hat{d}(u,v) = |u - v|$ for all $u, v \in \mathbb{R}$. Define the mappings $\mathcal{D} : \mathcal{U} \to \mathcal{U}$ and $\alpha : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ by*

$$\mathcal{D}(u) = \begin{cases} 2u - 1 & u \geq \frac{1}{2}, \\ \frac{u}{2} & u \in \left[0, \frac{1}{2}\right) \ ; \end{cases}$$

and

$$\alpha(u,v) = \begin{cases} 0 & u,v \in \left[0, \frac{1}{2}\right), \\ 1 & otherwise. \end{cases}$$

Clearly, $\mathcal{D}^{-1}$ is nondecreasing and discontinuous mapping. Moreover, $\mathcal{D}$ is an (ξ,α)-expansive mapping with $\xi(s) = \frac{s}{2}$ for all $s \geq 0$. For all $u, v \in \mathbb{R}$ with $u \geq v$, if $\alpha(u,v) \geq 1$, then we get

$$\hat{d}(\mathcal{D}u, \mathcal{D}v) = \left|2u - \frac{1}{2} - 2v + \frac{1}{2}\right| = 2|u-v|;$$

and

$$\begin{aligned} \xi(\hat{d}(\mathcal{D}u, \mathcal{D}v)) &= \xi(2|u-v|) \\ &= |u-v| \geq \alpha(u,v)\hat{d}(u,v). \end{aligned}$$

Also, $\mathcal{D}^{-1}$ is ordered α-admissible. For this, let $u, v \in \mathcal{U}$ such that $\alpha(u,v) \geq 1$ where $u \geq v \Rightarrow u \geq \frac{1}{2}$, $v \geq \frac{1}{2}$ and by definition of $\mathcal{D}^{-1}$, we have

$$\mathcal{D}^{-1}u = \frac{u}{2} + \frac{1}{2} \geq \frac{1}{2}, \quad \mathcal{D}^{-1}v = \frac{v}{2} + \frac{1}{2} \geq \frac{1}{2}$$

$$\Rightarrow \alpha(\mathcal{D}^{-1}u, \mathcal{D}^{-1}v) \geq 1.$$

Thus, $\mathcal{D}^{-1}$ is ordered α-admissible.

Also, there exists $u_0 \in \mathcal{U}$ such that $\alpha(u_0, \mathcal{D}^{-1}u_0) \geq 1$. For $u_0 = 1$, we have

$$\alpha(1, \mathcal{D}^{-1}1) = \alpha(1,1) \geq 1 \ \text{ and } \ \mathcal{D}^{-1}u_0 = 1 \leq 1 = u_0.$$

Finally, let $\{u_n\}$ be a nondecreasing sequence in $\mathbb{R}$ such that $\alpha(u_n, u_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $\lim_n u_n = u \in \mathcal{U}$. Since $\alpha(u_n, u_{n+1}) \geq 1$, by definition of α, we have $u_n \geq \frac{1}{2}$ for each n and consequently $u \geq \frac{1}{2}$ and $\alpha(\mathcal{D}^{-1}u_n, \mathcal{D}^{-1}u) \geq 1$. As $\{u_n\}$ is nondecreasing, therefore, $u_n \leq u$. Thus, all the required hypothesis of Theorem 2.4.2 are satisfied, so $\mathcal{D}$ has a fixed point. In this example, 0 and 1 are two fixed points of $\mathcal{D}$.

In the following result, an additional hypothesis is considered to ensure the uniqueness of the fixed point.

Theorem 2.4.3 *By adding the condition (H) to the hypotheses of Theorem 2.4.1 and 2.4.2, the uniqueness of the fixed point of $\mathcal{D}$ is obtained.*

(H): If there exists $w \in \mathcal{U}$ *such that for all* $u, v \in \mathcal{U}$,

$$\alpha(u,w) \geq 1 \text{ and } \alpha(v,w) \geq 1 \text{ where } u \geq w,\ v \geq w.$$

Proof Let us assume that u and v are two fixed points of $\mathcal{D}$ *i.e.* $\mathcal{D}(u) = u$ and $\mathcal{D}(v) = v$. By condition (H), there exists $w \in \mathcal{U}$ such that

$$\alpha(u,w) \geq 1 \quad \text{and} \quad \alpha(v,w) \geq 1 \quad \text{where} \quad u \geq w, \quad v \geq w. \tag{2.4.3}$$

Using (2.4.3) and ordered α-admissibility of $\mathcal{D}^{-1}$, we obtain

$$\alpha(u, \mathcal{D}^{-1}w) \geq 1 \quad \text{and} \quad \alpha(v, \mathcal{D}^{-1}w) \geq 1 \quad \text{where} \quad u \geq \mathcal{D}^{-1}w, \quad v \geq \mathcal{D}^{-1}w.$$

Therefore, by repeating this process, we have

$$\alpha(u, \mathcal{D}^{-n}w) \geq 1 \quad \text{and} \quad \alpha(v, \mathcal{D}^{-n}w) \geq 1 \quad \text{where} \quad u \geq \mathcal{D}^{-n}w, \quad v \geq \mathcal{D}^{-n}w; \tag{2.4.4}$$

for all $n \in \mathbb{N}$. Now using (2.2.2) and (2.4.4), we get

$$\begin{aligned}
\hat{d}(u, \mathcal{D}^{-n}w) &= \hat{d}(\mathcal{D}^{-1}u, \mathcal{D}^{-1}(\mathcal{D}^{-(n-1)}w)) \\
&\leq \alpha(u, \mathcal{D}^{-n}w)\ \hat{d}(\mathcal{D}^{-1}u, \mathcal{D}^{-1}(\mathcal{D}^{-(n-1)}w)) \\
&\leq \xi(\hat{d}(u, (\mathcal{D}^{-(n-1)}w)) \\
&\leq \xi(\xi(\hat{d}(u, (\mathcal{D}^{-(n-2)}w))) \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&\leq \xi^n(\hat{d}(u,w));
\end{aligned}$$

for all $n \in \mathbb{N}$. Now $\lim_n \mathcal{D}^{-n}w = u$ as $n \to \infty$. Using similar steps, we have $\lim_n \mathcal{D}^{-n}w = v$. The uniqueness of the limit of $\mathcal{D}^{-n}w$ implies $u = v$ *i.e.* $\mathcal{D}$ has a unique fixed point.

2.4.2 Application to Periodic Boundary Value Problem

Consider the periodic boundary value problem

$$\begin{cases} u'(t) = h(t, u(t)), & t \in I = [0, T] \\ u(0) = u(T), & otherwise; \end{cases} \tag{2.4.5}$$

where $h : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $T > 0$.
A function $u \in C^1(I, \mathbb{R})$ satisfying the above conditions is a solution to problem (2.4.5). A lower solution of problem (2.4.5) is a function $u \in C^1(I, \mathbb{R})$ such that

$$\begin{cases} u'(t) \leq h(t, u(t)), & t \in I = [0, T] \\ u(0) \leq u(T), & otherwise; \end{cases}$$

and upper solution for problem (2.4.5) satisfies the reversed conditions.
This application presents the existence of a unique solution for given problem using (ξ, α)-expansive mappings with suitable conditions.

Theorem 2.4.4 *Consider the given periodic boundary value problem where $h : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following conditions:*

1. *there exists $\lambda > 0$ such that $h(t, v) + \lambda v - h(t, u) - \lambda u \geq \lambda(v - u)$ for all $u, v \in \mathbb{R}$ with $v \geq u$;*

2. *there exists a function $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\zeta(\mathcal{B}^{-1}w(t), w(t)) \geq 0$ with $\mathcal{B}^{-1}w(t) \leq w(t)$ for all $t \in I$; $\zeta(u, v) \geq 0$; $u, v \in \mathbb{R}$ and $w \in C(I, \mathbb{R})$ is a lower solution to problem (3.1) and $\mathcal{B} : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ is a bijective mapping defined as*

$$\mathcal{B}(u(t)) = 2\int_0^T G(t, s)[h(s, u(s)) + \lambda u(s)]ds = w(t);$$

   for $t \in I$, $u, w \in C(I, \mathbb{R})$ where $\mathcal{B}^{-1}w(t) = u(t)$;

3. *for all $t \in I$ and $u, v \in C(I, \mathbb{R})$; $\zeta(u(t), v(t)) \geq 0 \Rightarrow \zeta(\mathcal{B}^{-1}u(t), \mathcal{B}^{-1}v(t)) \geq 0$;*

4. *If $u_n \to u \in C(I, \mathbb{R})$ and $\zeta(u_n, u_{n+1}) \geq 0$, then $\zeta(u_n, u) \geq 0$ for all $n \in \mathbb{N}$.*

Then, existence of a lower solution of given problem ensures the existence of a unique solution for the same.

Proof The given problem (2.4.5) can be rewritten as

$$\begin{cases} u'(t) + \lambda u(t) = h(t, u(t)) + \lambda u(t), & t \in I = [0, T] \\ u(0) = u(T), & otherwise; \end{cases}$$

which is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds;$$

where $G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s < t \leq T \\ \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t < s \leq T. \end{cases}$

If $u \in C(I,\mathbb{R})$ is a fixed point of $\mathcal{B}$, then $u \in C^1(I,\mathbb{R})$ is a solution of the given problem. Let us assume that $C(I,\mathbb{R}) = U$. Let us define a partial order relation on U as

$$u \preceq v \Leftrightarrow u(t) \leq v(t), \quad u,v \in U, \quad t \in I.$$

Thus, U is a partially ordered set. Now define the metric d on U as

$$\hat{d}(u,v) = \sup_{t\in I} \ |u(t) - v(t)|, \quad u,v \in U, \quad t \in I.$$

Note that $(U,\hat{d})$ is a complete metric space. Consider a nondecreasing sequence $\{u_n\}$ in $C(I,\mathbb{R})$ converging to $u \in C(I,\mathbb{R})$. Then, for all $t \in I$, we have

$$u_1(t) \leq u_2(t) \leq u_3(t) \leq ... \leq u_n(t) \leq ...$$

This implies that $u_n(t) \leq u(t) \quad \forall \ t \in I, \quad n \in \mathbb{N}$. Thus, $u_n \leq u$ for all $n \in \mathbb{N}$. Also, it can be easily verified that $\mathcal{B}$ is a nondecreasing mapping as $G(t,s) > 0$ for each $(t,s) \in I \times I$ and for $u \geq v$, we have, $u(t) \geq v(t); \quad u,v \in U$.
Define the mappings $\xi \in \chi$ by $\xi(u) = \frac{u}{2}$ for all $u \geq 0$ and

$$\alpha(u,v) = \begin{cases} 1, & \zeta(u(t),v(t)) \geq 0, \quad t \in I \\ 0, & otherwise. \end{cases}$$

By condition 1 of Theorem 2.4.4 and definition of G, we obtain for $u \succeq v$,

$$\begin{aligned} \xi(\hat{d}(\mathcal{B}(u),\mathcal{B}(v))) &= \frac{1}{2}\hat{d}(\mathcal{B}(u),\mathcal{B}(v)) \\ &= \frac{1}{2}\sup_{t\in I} \ |\mathcal{B}(u(t)) - \mathcal{B}(v(t))| \\ &= \frac{1}{2}\sup_{t,s\in I} \ \left|2\int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds - 2\int_0^T G(t,s)[h(s,v(s)) + \lambda v(s)]ds\right| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{t,s\in I} \left| \int_0^T G(t,s)[h(s,u(s)) + \lambda u(s) - h(s,v(s)) - \lambda v(s)]ds \right| \\
&\geq \sup_{t,s\in I} \left| \int_0^T G(t,s)[\lambda(u(s) - v(s)]ds \right|.
\end{aligned}
$$

Since $G(t,s) > 0$, $\lambda > 0$ and $u(s) \geq v(s)$, therefore the above inequality becomes

$$
\begin{aligned}
\xi(\hat{d}(\mathcal{B}(u), \mathcal{B}(v))) &\geq \int_0^T \sup_{t,s\in I} G(t,s).\ \lambda \sup_{t,s\in I} |(u(s) - v(s)|ds \\
&= \int_0^T \sup_{t,s\in I} G(t,s).\ \ \lambda\ \hat{d}(u,v)ds \\
&= \lambda\ \hat{d}(u,v) \int_0^T \sup_{t,s\in I} G(t,s)ds \\
&= \lambda\ \hat{d}(u,v) \sup_{t,s\in I} \frac{1}{e^{\lambda t} - 1} \left(\frac{1}{\lambda} e^{\lambda(T+s-t)}|_0^t + \frac{1}{\lambda} e^{\lambda(s-t)}|_t^T \right) \\
&= \lambda\ \hat{d}(u,v) \times \frac{1}{\lambda} = \hat{d}(u,v) \\
\Rightarrow \xi(\hat{d}(\mathcal{B}(u), \mathcal{B}(v))) &\geq \ \hat{d}(u,v).
\end{aligned}
$$

By definition of α, we get

$$
\Rightarrow \xi(\hat{d}(\mathcal{B}(u), \mathcal{B}(v))) \geq \alpha(u,v)\ \hat{d}(u,v) \quad for\ all\ u,v \in U\ with\ u \succeq v.
$$

This implies that $\mathcal{B}$ is an (ξ,α)-expansive mapping. Now, by assumption 3 of Theorem 2.4.4, for $u \succeq v$, $\ u,v \in U$, we have

$$
\begin{aligned}
\alpha(u,v) \geq 1 \Rightarrow \zeta(u(t), v(t)) \geq 0 \ &\Rightarrow\ \zeta(\mathcal{B}^{-1}u(t), \mathcal{B}^{-1}v(t)) \geq 0 \\
&\Rightarrow\ \alpha(\mathcal{B}^{-1}u, \mathcal{B}^{-1}v) \geq 1.
\end{aligned}
$$

Therefore, $\mathcal{B}^{-1}$ is ordered α-admissible. Let ρ be a lower solution of (2.4.5), then by assumption 2,

$$
\zeta(\mathcal{B}^{-1}\rho(t), \rho(t)) \geq 0 \Rightarrow \alpha(\mathcal{B}^{-1}\rho,\ \ \rho) \geq 1 \ \ and\ \ \mathcal{B}^{-1}\rho \leq \rho.
$$

Also, if $u_n \to u \in U$ for all $n \in \mathbb{N}$, we have

$$
\zeta(u_n, u_{n+1}) \geq 0 \quad \Rightarrow \quad \zeta(u_n, u) \geq 0.
$$

This implies $\alpha(u_n, u_{n+1}) \geq 1 \quad \Rightarrow \quad \alpha(u_n, u) \geq 1$.

Thus all the required conditions for Theorem 2.4.2 are fulfilled. Consequently, $\mathcal{B}$ has a fixed point and thus given problem (2.4.5) possesses a solution. The uniqueness follows from (2.4.3).

Remark 2.4.5 *Theorem 2.4.4 also hold true if the existence of the lower solution to given problem is replaced by the upper solution.*

Chapter 3

Fixed Point Theorems for Quasi Partial Metric Spaces through Aggregation

3.1 Introduction

A huge interest in the theory of aggregation is developing these days because it has a wide variety of applications. In many fields, there are different sets of data through different sources which are needed to be concluded. It can be easily done using numerical aggregation. Borsik and Doboš [26] investigated the problem of aggregation for a collection of metrics (which need not be finite) resulting in a single one. They studied the properties of those functions that permit a collection of metrics to merge and a single metric is obtained in return.

Definition 3.1.1 *[26] A function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a metric aggregation function provided that the function $\hat{d}_\Phi : U \times U \to \mathbb{R}_+$ is a metric for every pair of metric spaces $(U_1, \hat{d}_1)$ and $(U_2, \hat{d}_2)$, where $U = U_1 \times U_2$ and*

$$\hat{d}_\Phi((u,v),(z,w)) = \Phi(\hat{d}_1(u,z), \hat{d}_2(v,w));$$

for all (u,v), $(z,w) \in U$.

In [68], Herburt defined homogeneity as follows:

Definition 3.1.2 *[68] A function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be homogeneous if $\Phi(\alpha u) = \alpha \ \Phi(u)$ for each $u \in \mathbb{R}_+^n$ and $\alpha \in \mathbb{R}_+$.*

In 1994, Matthews [104] introduced partial metric spaces with an application in denotational semantics. Many authors worked on this notion afterwards e.g.

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Heckmann [67] defined weak partial metric following Matthews' [104] notion and Romaguera and Valero [136] introduced a quantitative computational model for partial metric spaces with formal balls. Motivated by these generalizations presented by above authors; Massanet and Valero [102] studied the aggregation problem in the setting of partial metric spaces. Recently, Karapinar *et al.* [86] introduced quasi partial metric spaces by removing the symmetric axiom from the definition of partial metric spaces. All the related definitions and results are stated in preliminaries section. On the other hand, Wang *et al.* [153] introduced the notion of expansive mappings for a metric space which led to many useful results afterwards.
The main focus of this chapter is to present the notion of expansive mappings in quasi partial metric spaces and the involvement of aggregation functions in such a way that the previous results existing in literature can be retrieved as a particular case of our new ones.

The chapter is organized as follows: Section 3.2 consists of some relevant preliminaries. In section 3.3, some fixed point results in quasi partial metric *via* expansive mappings are presented. In section 3.4, the notion of quasi partial metric aggregation along with the properties and conditions required to characterize distance aggregation operators is introduced. In section 3.5, the concept of Projective Ψ-expansion is introduced via these distance operators and some fixed point results are obtained through it. Section 3.6 deals with an application to asymptotic complexity analysis.

3.2 Preliminaries

Karapinar *et al.* [86] defined the quasi partial metric spaces as follows:

Definition 3.2.1 *[86] A mapping $q : U \times U \to \mathbb{R}^+$ is said to be a quasi partial metric if the following conditions hold:*

(q1) $if\ 0 \leq q(u,u) = q(u,v) = q(v,v),\ then\ u = v;$

(q2) $q(u,u) \leq q(u,v);$

(q3) $q(u,u) \leq q(v,u);$

(q4) $q(u,w) \leq q(u,v) + q(v,w) - q(v,v);$

for all $u, v, w \in U$. Then the pair (U, q) is called a quasi partial metric space (QPMS).

If $q(v, u) = q(u, v)$ for each $u, v \in U$, then (U, q) reduces to partial metric space (PMS). Also, for a quasi-partial metric q on U, the mapping $\hat{d}_q : U \times U \to \mathbb{R}_+$ defined by

$$\hat{d}_q(u, v) = q(u, v) + q(v, u) - q(u, u) - q(v, v);$$

is a (usual) metric on U.

Lemma 3.2.2 *[86] Let (U, q) be a QPMS. Let (U, p_q) be the corresponding partial metric space and let (U, d_{p_q}) be the corresponding metric space. The following statements are equivalent:*

1. *The sequence $\{u_n\}$ is Cauchy in (U, q) .*
2. *The sequence $\{u_n\}$ is Cauchy in (U, p_q).*
3. *The sequence $\{u_n\}$ is Cauchy in (U, d_{p_q}).*

Lemma 3.2.3 *[86] Let (U, q) be a QPMS. Let (U, p_q) be the corresponding PMS and let (U, d_{p_q}) be the corresponding metric space. The following statements are equivalent:*

1. *(U, q) is complete.*
2. *(U, p_q) is complete.*
3. *(U, d_{p_q}) is complete.*

Moreover,

$$\begin{aligned}
\lim_{n\to\infty} d_{p_q}(u, u_n) &= 0 \Leftrightarrow p_q(u, u) = \lim_{n\to\infty} p_q(u, u_n) = \lim_{n,m\to\infty} p_q(u_n, u_m) \\
&\Leftrightarrow q(u, u) = \lim_{n\to\infty} q(u, u_n) = \lim_{n,m\to\infty} q(u_n, u_m) \\
&= \lim_{n\to\infty} q(u_n, u) = \lim_{m,n\to\infty} q(u_m, u_n).
\end{aligned}$$

Borsik and Doboš [26] defined the monotonicity and sub-additivity of Φ as follows:

Definition 3.2.4 *[26] A function $\Phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is said to be monotone if $u \preceq v \Rightarrow \Phi(u) \leq \Phi(v)$ for all $u, v \in \mathbb{R}^n_+$ and sub-additive if $\Phi(u + v) \leq \Phi(u) + \Phi(v)$ for all $u, v \in \mathbb{R}^n_+$ where $\preceq$ stands for the following pointwise order relation on $\mathbb{R}^n_+$:*

$$u \preceq v \quad \Leftrightarrow \quad u_i \leq v_i \quad ; \quad i = 1, ...n.$$

The result given by Wang *et al.* [153] for expansive mappings is stated below:

Theorem 3.2.5 *[153] Let $\mathfrak{D} : U \to U$ be an onto mapping defined on a complete metric space $(U, \hat{d})$ satisfying the condition*

$$\hat{d}(\mathfrak{D}u, \mathfrak{D}v) \geq c\ \hat{d}(u, v) \quad \forall\ u, v \in U;$$

where $c > 1$. Then $\mathfrak{D}$ has a unique fixed point in U.

3.3 Quasi partial metric and expansive mappings

The following lemma will be helpful in proving our main results.

Lemma 3.3.1 *Let (U, q) be a quasi partial metric space and $\{u_n\}$ be a sequence of points of U. If there exists a number $k \in (0, 1)$ such that*

$$q(u_{n+1}, u_n) \leq k\ q(u_n, u_{n-1}) \quad ;\ n = 1, 2, ... \tag{3.3.1}$$

Then $\{u_n\}$ is a Cauchy sequence in U.

Proof By considering successively the condition (3.3.1), we obtain

$$q(u_{n+1}, u_n) \leq kq(u_n, u_{n-1}) \leq k^2 q(u_{n-1}, u_{n-2}) \leq \quad ... \quad \leq k^n q(u_1, u_0).$$

Also,

$$max\{q(u_n, u_n), q(u_{n+1}, u_{n+1})\} \leq q(u_{n+1}, u_n) \leq k^n q(u_1, u_0).$$

Therefore,

$$\begin{aligned}
d_q(u_n, u_{n+1}) &= q(u_n, u_{n+1}) + q(u_{n+1}, u_n) - q(u_n, u_n) - q(u_{n+1}, u_{n+1}) \\
&\leq q(u_n, u_{n+1}) + q(u_{n+1}, u_n) + q(u_n, u_n) + q(u_{n+1}, u_{n+1}) \\
&\leq k^n q(u_0, u_1) + k^n q(u_1, u_0) + k^n q(u_1, u_0) + k^n q(u_1, u_0) \\
&= 3k^n q(u_1, u_0) + k^n q(u_0, u_1) \quad where \quad k < 1 \\
\Rightarrow \lim_{n\to\infty} d_q(u_n, u_{n+1}) &= 0.
\end{aligned}$$

Similarly, it can be shown that

$$\lim_{n\to\infty} d_q(u_{n+1}, u_n) = 0.$$

Further,

$$\begin{aligned} d_q(u_n, u_m) &= d_q(u_n, u_{n+1}) + d_q(u_{n+1}, u_{n+2}) + ... + d_q(u_{m-1}, u_m) \\ &\le 3k^n q(u_1, u_0) + k^n q(u_0, u_1) + 3k^{n+1} q(u_1, u_0) + k^{n+1} q(u_0, u_1) \\ &\quad +... + 3k^{m-1} q(u_1, u_0) + k^{m-1} q(u_0, u_1) \\ &= 3k^n q(u_1, u_0)[1 + k + ... + k^{m-1}] + k^n q(u_0, u_1)[1 + k + ... + k^{m-1}] \\ &\le \frac{3k^n}{1-k} q(u_1, u_0) + \frac{k^n}{1-k} q(u_0, u_1). \end{aligned}$$

This shows that $\{u_n\}$ is a Cauchy sequence in U *w.r.t.* metric d_q. From Lemma 3.2.2, $\{u_n\}$ is Cauchy in quasi partial metric space (U, q).

Theorem 3.3.2 *Let (U, q) be a complete quasi partial metric space and $\mathfrak{D} : U \to U$ be a bijective mapping. Suppose that there exists $c_1, c_2, c_3 \ge 0$ such that $c_1 + c_2 + c_3 > 1$ and*

$$q(\mathfrak{D}u, \mathfrak{D}v) \ge c_1\ q(u, v) + c_2\ q(u, \mathfrak{D}u) + c_3\ q(v, \mathfrak{D}v) \quad \forall\ u, v \in U. \tag{3.3.2}$$

Then $\mathfrak{D}$ has a fixed point in U.

Proof Let $u_0 \in U$. Since $\mathfrak{D}$ is bijective, there exists $u_1 \in U$ such that $\mathfrak{D}u_1 = u_0$. Define a sequence $\{u_n\}$ in U such that $u_{n-1} = \mathfrak{D}u_n; \quad n = 1, 2, ...$ If $u_{n-1} = u_n$ for some n, then the result is trivial. Therefore, assume that $u_{n-1} \ne u_n$ for all n.
By (3.3.2),

$$\begin{aligned} q(u_n, u_{n-1}) &= q(\mathfrak{D}u_{n+1}, \mathfrak{D}u_n) \\ &\ge c_1\ q(u_{n+1}, u_n) + c_2\ q(u_{n+1}, \mathfrak{D}u_{n+1}) + c_3\ q(u_n, \mathfrak{D}u_n) \\ &= c_1\ q(u_{n+1}, u_n) + c_2\ q(u_{n+1}, u_n) + c_3\ q(u_n, u_{n-1}) \\ \Rightarrow (1 - c_3) q(u_n, u_{n-1}) &\ge (c_1 + c_2) q(u_{n+1}, u_n) \\ \Rightarrow q(u_{n+1}, u_n) &\le \left(\frac{1 - c_3}{c_1 + c_2}\right) q(u_n, u_{n-1}). \end{aligned}$$

Since $c_1 + c_2 \ne 0$ and $(1 - c_3) > 0$, therefore

$$q(u_{n+1}, u_n) \le \lambda q(u_n, u_{n-1}) \ where \ \ \lambda = \left(\frac{1 - c_3}{c_1 + c_2}\right) < 1.$$

Thus, by Lemma 3.3.1, $\{u_n\}$ is a Cauchy sequence in U and since (U, q) is complete, therefore, (U, d_q) is complete where d_q is the usual metric induced by quasi metric q. Therefore, $\{u_n\}$ is convergent in U *w.r.t.* metric d_q. Let $u^* \in U$ be such that

$\underset{n\to\infty}{lim}\, d_q(u^*, u_n) = 0$. By Lemma 3.2.3, we have

$$\begin{aligned} q(u^*, u^*) &= \lim_{n\to\infty} q(u^*, u_n) = \lim_{n,m\to\infty} q(u_n, u_m) \\ &= \lim_{n\to\infty} q(u_n, u^*) = \lim_{m,n\to\infty} q(u_m, u_n). \end{aligned} \tag{3.3.3}$$

Again, by Lemma 3.3.1, the sequence $\{u_n\}$ is Cauchy in (U, d_q) *i.e.*

$$\lim_{m,n\to\infty} d_q(u_m, u_n) = 0.$$

Also,

$$\begin{aligned} max\{q(u_n, u_n), q(u_{n+1}, u_{n+1})\} &\leq q(u_{n+1}, u_n) \\ &\leq \lambda q(u_n, u_{n-1}) \\ &\leq ... \leq \lambda^n q(u_1, u_0). \end{aligned}$$

Therefore, $\underset{n\to\infty}{lim}\, q(u_n, u_n) = 0$. By definition of metric d_q, we have, $\underset{m,n\to\infty}{lim}\, q(u_m, u_n) = 0$ and $\underset{n,m\to\infty}{lim}\, q(u_n, u_m) = 0$. Thus, by (3.3.3), we obtain

$$q(u^*, u^*) = \lim_{n\to\infty} q(u^*, u_n) = \lim_{n\to\infty} q(u_n, u^*) = 0;$$

which shows that $\{u_n\}$ is convergent in quasi partial metric space (U, q). Let $u \in U$ be such that $u^* = \mathfrak{D}u$. Next, it is proved that $u^* = u$. Let us suppose that $u^* \neq u$. Then

$$\begin{aligned} q(u_n, u^*) &= q(\mathfrak{D}u_{n+1}, \mathfrak{D}u) \\ &\geq c_1\ q(u_{n+1}, u) + c_2\ q(u_{n+1}, \mathfrak{D}u_{n+1}) + c_3\ q(u, \mathfrak{D}u). \end{aligned}$$

As $n \to \infty$, above inequality becomes

$$\begin{aligned} 0 = q(u^*, u^*) &\geq c_1\ q(u^*, u) + c_3\ q(u, u^*) \\ \Rightarrow c_1\ q(u^*, u) &+ c_3\ q(u, u^*) \leq 0; \end{aligned}$$

but as $u^* \neq u$, we have, $q(u^*, u) > 0$ and $q(u, u^*) > 0$ with $c_1, c_3 \geq 0$. Thus, we arrive at a contradiction. Therefore, $u^* = u$ and hence $u^* = \mathfrak{D}u = u$.
This completes the proof.

Corollary 3.3.3 *Let (U, q) be a complete quasi partial metric space and $\mathfrak{D} : U \to$*

U be a bijective mapping. Suppose that there exists a constant $c > 1$ such that

$$q(\mathfrak{D}u, \mathfrak{D}v) \geq c\ q(u, v) \quad \forall\ \ u, v \in U.$$

Then $\mathfrak{D}$ has a unique fixed point in U.

Proof From above Theorem, it follows that $\mathfrak{D}$ has a fixed point u^* in U by putting $c_1 = c$ and $c_2, c_3 = 0$ in inequality (3.3.3).
For uniqueness, let z be another fixed point of $\mathfrak{D}$. Then,

$$q(u^*, z) = q(\mathfrak{D}u^*, \mathfrak{D}z) \geq c\ q(u^*, z) \quad \text{where} \ \ c > 1.$$

which is a contradiction. Therefore, $u^* = z$.

Corollary 3.3.4 *Let (U, q) be a quasi partial metric space with q as complete metric and $\mathfrak{D} : U \to U$ be a bijective mapping. Suppose that there exists a positive integer n and a constant $c > 1$ such that*

$$q(\mathfrak{D}^n u, \mathfrak{D}^n v) \geq c\ q(u, v) \quad \forall\ u, v \in U.$$

Then $\mathfrak{D}$ has a unique fixed point in U.

Proof From Corollary 3.3.3, $\mathfrak{D}^n$ has a unique fixed point u^* in U. Also, $\mathfrak{D}^n(\mathfrak{D}u^*) = \mathfrak{D}(\mathfrak{D}^n u^*) = \mathfrak{D}u^*$ which shows that $\mathfrak{D}u^*$ is also a fixed point of $\mathfrak{D}^n$. By uniqueness of fixed point in $\mathfrak{D}^n$, we have $\mathfrak{D}u^* = u^*$ and thus $\mathfrak{D}^n$ and $\mathfrak{D}$ both have a unique fixed point u^*.

Example 3.3.1 *Let $U = \mathbb{R}^+$ and $q(u, v) = max\{u - v, v - u\} + u$. Then (U, q) is a complete quasi partial metric space. Let $\mathfrak{D}u = 3u^2$ for all $u \in U$.*

Note that $\mathfrak{D}$ is a bijective mapping. Also, for all $u \preceq v$, we obtain

$$\begin{aligned} q(\mathfrak{D}u, \mathfrak{D}v) &= q(3u^2, 3v^2) \\ &= 3v^2 \\ &\geq c\ q(u, v) \ \ \text{where} \ \ c = 3 > 1. \end{aligned}$$

Thus, the condition of expansion is also satisfied for $\mathfrak{D}$.
Hence, all the conditions of Corollary 3.3.3 are fulfilled. Therefore, there exists a unique fixed point of $\mathfrak{D}$. Here, 0 is the unique fixed point.

Theorem 3.3.5 *Let (U, q) be a quasi partial metric space with q as complete met-*

ric and $\mathfrak{D} : U \to U$ be a continuous bijective mapping. Suppose that there exists a positive real number $c > 1$ such that

$$q(\mathfrak{D}u, \mathfrak{D}v) \geq c\ \rho \quad where \quad \rho \in \{q(u, \mathfrak{D}u), q(v, \mathfrak{D}v)\}. \tag{3.3.4}$$

Then $\mathfrak{D}$ possesses a fixed point in U.

Proof Following Theorem 3.3.2, we can construct a sequence $\{u_n\}$ of points of U such that $u_{n-1} = \mathfrak{D}u_n$. Let us assume that $u_{n-1} \neq u_n \quad \forall \quad n = 1, 2, ...$(otherwise, $u_p = u_{p-1}$ for some $p \in n$ and thus u_p is a fixed point of $\mathfrak{D}$).
By (3.3.4), it follows that

$$q(u_n, u_{n-1}) = q(\mathfrak{D}u_{n+1}, \mathfrak{D}u_n) \geq c\rho_n \quad where \quad \rho_n \in \{q(u_n, u_{n-1}), q(u_{n+1}, u_n)\}.$$

Then two cases arises:
Case I:If $\rho_n = q(u_n, u_{n-1})$, then

$$q(u_n, u_{n-1}) \geq c\ q(u_n, u_{n-1}) > q(u_n, u_{n-1});$$

which is a contradiction.
Case II:If $\rho_n = q(u_{n+1}, u_n)$, then

$$\begin{aligned} q(u_n, u_{n-1}) &\geq c\ q(u_{n+1}, u_n) \\ \Rightarrow k\ q(u_n, u_{n-1}) &\geq q(u_{n+1}, u_n) \quad where \quad k \in (0, 1). \end{aligned}$$

Thus, by Lemma 3.3.1, the sequence $\{u_n\}$ is a Cauchy sequence in U and as U is complete, this sequence converges to some point u^* in U. By continuity of $\mathfrak{D}$, it follows that u^* is a fixed point of $\mathfrak{D}$.

3.4 Quasi partial metric aggregation

This section presents the notion of quasi partial metric aggregation functions and some of its properties required to prove the main results.

Definition 3.4.1 *A function $\Psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is said to be a quasi partial metric aggregation function provided that the function $Q_\Psi : U \times U \to [0, +\infty[$ is a quasi partial metric for every collection of Quasi partial metric spaces $\{(U_i, q_i)\}_{i=1}^n$, where*

$U = U_1 \times U_2 ... \times U_n$ *and*

$$Q_\Psi(u,v) = \Psi(q_1(u_1,v_1),...,q_n(u_n,v_n));$$

for all $u = (u_1, ...u_n) \in U,\ v = (v_1, ...v_n) \in U$.

Next results characterize quasi partial metric aggregation function.

Proposition 3.4.2 *Let* $\Psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ *be a quasi partial metric aggregation function, then* Ψ *is monotone.*

Proof Consider the quasi partial metric $q : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined as $q(u,v) = max\{(u-v),(v-u)\}+u\ \ \forall\, u,v \in \mathbb{R}_+$. Since Ψ is a quasi partial metric aggregation function, therefore the function $Q_\Psi : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ defined by

$$Q_\Psi(u,v) = \Psi(q(u_1,v_1),...,q(u_n,v_n));$$

for all $u = (u_1, ...u_n),\ v = (v_1, ...v_n) \in \mathbb{R}^n_+$ is a quasi partial metric.
Consider $u, v \in \mathbb{R}^n_+$ where $u \preceq v$. Then

$$\begin{aligned}
\Psi(u) &= \Psi(u_1,...,u_n)\\
&= \Psi(q(u_1,u_1),...,q(u_n,u_n))\\
&= Q_\Psi(u,u)\\
&\leq Q_\Psi(u,v)\\
&= \Psi(q(u_1,v_1),...,q(u_n,v_n))\\
&= \Psi(v_1,...,v_n)\\
&= \Psi(v)\\
\Rightarrow \Psi(u) &\leq \Psi(v).
\end{aligned}$$

Proposition 3.4.3 *Let* $\Psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ *be a quasi partial metric aggregation function, If* $\Psi(u) = 0$ *for some* $u \in \mathbb{R}^n_+$, *then* $u = \bar{0}$ *where* $\bar{0} = (0,...,0) \in \mathbb{R}^n_+$.

Proof Let us assume that $\Psi(u) = 0$ for some $u \in \mathbb{R}^n_+$. Since Ψ is a quasi partial metric aggregation function, therefore, by Proposition 3.4.2, Ψ is monotone and thus $\Psi\left(\frac{u}{3}\right) \leq \Psi(u)$ which implies that $\Psi\left(\frac{u}{3}\right) = 0$.
Consider the quasi partial metric $Q_\Psi : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ introduced in Proposition 3.4.2. Now,

$$Q_\Psi\left(\frac{u}{3},u\right) \;=\; \Psi\left(q\left(\frac{u_1}{3},u_1\right),...,q\left(\frac{u_n}{3},u_n\right)\right)$$

$$
\begin{aligned}
&= \Psi(u_1, ..., u_n) \\
&= \Psi(u) = 0 \textit{ and} \\
Q_\Psi(u,u) &= \Psi(q(u_1,u_1), ..., q(u_n,u_n)) \\
&= \Psi(u) = 0. \\
\textit{Also, } Q_\Psi\left(\frac{u}{3}, \frac{u}{3}\right) &= \Psi\left(q\left(\frac{u_1}{3}, \frac{u_1}{3}\right), ..., q\left(\frac{u_n}{3}, \frac{u_n}{3}\right)\right) \\
&= \Psi\left(\frac{u}{3}\right) = 0.
\end{aligned}
$$

Thus, by definition of quasi partial metric, $\frac{u}{3} = u$ and therefore, $u = \bar{0}$.

Lemma 3.4.4 *For every $u, v, w, t \in \mathbb{R}_+$ such that $u \leq w + t - v$ where $v \leq w$ and $v \leq t$, there exists $l, m, n \in \mathbb{R}^2_+$ for which $\tilde{q}(l,m) = w + t - v$, $\tilde{q}(l,n) = w$, $\tilde{q}(n,m) = t$ and $\tilde{q}(n,n) = v$ where $\tilde{q} : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}_+$ is the quasi partial metric defined by $\tilde{q}(l,m) = max\{l_1, m_1\} + max\{l_2, m_2\} + max\{(m_1 - l_1), 0\}$ for every $l = (l_1, l_2)$, $m = (m_1, m_2) \in \mathbb{R}^2_+$.*

Proof It can be easily seen that $\tilde{q}$ is the quasi partial metric defined on $\mathbb{R}^2_+$. Furthermore, the following points of $\mathbb{R}^2_+$ satisfy the required conditions:

$$
l = \left(w - \frac{v}{2}, \frac{v}{2}\right), \; m = \left(\frac{v}{2}, t - \frac{v}{2}\right), \; n = \left(\frac{v}{2}, \frac{v}{2}\right).
$$

Lemma 3.4.5 *For every $u, v, w \in \mathbb{R}_+$ such that $u \geq v$ and $u \geq w$, there exists $l, m \in I(\mathbb{R})$ for which $\hat{q}(l,m) = u$, $\hat{q}(l,l) = v$ and $\hat{q}(m,m) = w$ where $\hat{q} : I(\mathbb{R}) \times I(\mathbb{R}) \to \mathbb{R}_+$ is the quasi partial metric defined as $\hat{q}(l,m) = max\{l_2, m_2\} - min\{l_1, m_1\} + max\{(m_1 - l_1), 0\}$ for every $l = [l_1, l_2]$, $m = [m_1, m_2] \in I(\mathbb{R})$ where $I(\mathbb{R})$ denotes an interval in $\mathbb{R}_+$.*

Proof It can be easily seen that $\hat{q}$ is the quasi partial metric defined on $I(\mathbb{R})$. Furthermore, the following elements of $I(\mathbb{R})$ fulfill the required conditions:

$$
l = [-v, 0], \; m = [-u, -u + w].
$$

Theorem 3.4.6 *$\Psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a quasi partial metric aggregation function if and only if for each $u, v, z, w \in \mathbb{R}^n_+$, we have*

(1) $\Psi(u) + \Psi(v) \leq \Psi(z) + \Psi(w)$ *whenever* $u + v \preceq z + w$, $v \preceq z$, $v \preceq w$.

(2) $\Psi(u) = \Psi(v) = \Psi(z) \Rightarrow u = v = z$ *whenever* $v \preceq u$, $z \preceq u$.

Proof Suppose that Ψ is a quasi partial metric aggregation function. Let $u, v, z, w \in \mathbb{R}^n_+$ where $u + v \preceq z + w$, $v \preceq z$, $v \preceq w$. Then by Lemma 3.4.4, there exists

$\hat{u_i}, \hat{v_i}, \hat{z_i} \in \mathbb{R}^2_+$ such that $\tilde{q}(\hat{u_i}, \hat{v_i}) = z_i + w_i - v_i$, $\tilde{q}(\hat{u_i}, \hat{z_i}) = z_i$, $\tilde{q}(\hat{z_i}, \hat{v_i}) = w_i$ and $\tilde{q}(\hat{z_i}, \hat{z_i}) = v_i$ for all $i = 1, ..., n$.
Let $U = \prod_{i=1}^{n} \mathbb{R}^2_+$ and $\hat{u} = (\hat{u_1}, ..., \hat{u_n})$, $\hat{v} = (\hat{v_1}, ..., \hat{v_n})$, $\hat{z} = (\hat{z_1}, ..., \hat{z_n})$. Then $\hat{u}, \hat{v}, \hat{z} \in U$. Consider the quasi partial metric Q_Ψ defined on U by

$$Q_\Psi(\hat{u}, \hat{v}) = \Psi(\tilde{q}(\hat{u_1}, \hat{v_1}), ..., \tilde{q}(\hat{u_n}, \hat{v_n}));$$

for every $\hat{u}, \hat{v} \in U$. Then

$$\begin{aligned} \Psi(z + w - v) = Q_\Psi(\hat{u}, \hat{v}) &\leq Q_\Psi(\hat{u}, \hat{z}) + Q_\Psi(\hat{z}, \hat{v}) - Q_\Psi(\hat{z}, \hat{z}) \\ &= \Psi(z) + \Psi(w) - \Psi(v) \end{aligned}$$

$$\Rightarrow \Psi(z + w - v) \leq \Psi(z) + \Psi(w) - \Psi(v).$$

Also, monotonicity of Ψ implies that

$$\begin{aligned} \Psi(u) \leq \Psi(z + w - v) &\leq \Psi(z) + \Psi(w) - \Psi(v) \\ \Rightarrow \Psi(u) + \Psi(v) &\leq \Psi(z) + \Psi(w). \end{aligned}$$

Thus, condition (1) is proved. Now, let $u, v, z \in \mathbb{R}^n_+$ where $v \preceq u$, $z \preceq u$. Then by Lemma 3.4.5, there exists $\hat{u}, \hat{v} \in I(\mathbb{R})$ such that $\hat{q}(\hat{u_i}, \hat{u_i}) = v_i$, $\hat{q}(\hat{v_i}, \hat{v_i}) = z_i$, and $\hat{q}(\hat{u_i}, \hat{v_i}) = u_i$ for each $i = 1, ..., n$. Let $U = \prod_{i=1}^{n} I(\mathbb{R})$ and $\hat{u} = (\hat{u_1}, ..., \hat{u_n}), \hat{v} = (\hat{v_1}, ..., \hat{v_n}), \hat{z} = (\hat{z_1}, ..., \hat{z_n})$. Then $\hat{u}, \hat{v}, \hat{z} \in U$.
Consider the quasi partial metric Q_Ψ on U defined by

$$Q_\Psi(\hat{u}, \hat{v}) = \Psi(\tilde{q}(\hat{u_1}, \hat{v_1}), ..., \tilde{q}(\hat{u_n}, \hat{v_n}));$$

for every $\hat{u}, \hat{v} \in U$. Let $\Psi(\hat{u}) = \Psi(\hat{v}) = \Psi(\hat{z})$ for $\hat{u}, \hat{v}, \hat{z} \in \mathbb{R}^n_+$. It is easy to see that $Q_\Psi(\hat{u}, \hat{v}) = Q_\Psi(\hat{u}, \hat{u}) = Q_\Psi(\hat{v}, \hat{v})$ and therefore, by definition of quasi partial metric, $\hat{u} = \hat{v}$. By Lemma 3.4.5, $\hat{u} = [-v, 0]$ and $\hat{v} = [-u, -u + z]$ and Consequently, $u = v = z$. Hence, condition (2) is proved.
Conversely, let us assume that conditions (1) and (2) hold. Now, it will be shown that Ψ is a quasi partial metric aggregation function. Let $\{(U_i, q_i)\}_{i=1}^{n}$ be a family of arbitrarily chosen quasi partial metric spaces and $U = \prod_{i=1}^{n} U_i$. Let $u, v \in U$ and $Q_\Psi(u, v) = Q_\Psi(u, v) = Q_\Psi(v, v)$. Then, by condition (2), we have $q_i(u_i, v_i) = q_i(u_i, u_i) = q_i(v_i, v_i)$ $\forall\, i = 1, ..., n$. It follows by definition that $u_i = v_i$ $\forall\, i = 1, ..., n$ and thus, $u = v$. Now, set $v = w = \bar{0}$ in condition (1). Since $q_i(u_i, u_i) \leq q_i(u_i, v_i)$

and $q_i(u_i, u_i) \leq q_i(v_i, u_i) \ \ \forall \ i = 1, ..., n$, we obtain

$$\begin{aligned} Q_\Psi(u, u) &= \Psi(q_1(u_1, u_1), ..., q_n(u_n, u_n)) \\ &\leq \Psi(q_1(u_1, v_1), ..., q_n(u_n, v_n)) \\ &= Q_\Psi(u, v); \end{aligned}$$

as Ψ is monotone. Similarly, $Q_\Psi(u, u) \leq Q_\Psi(v, u)$. Also, by (1), for each $u, v, z \in U$,

$$\begin{aligned} Q_\Psi(u, z) &= \Psi(q_i(u_i, z_i)) \\ &\leq \Psi(q_i(u_i, v_i)) + \Psi(q_i(v_i, z_i)) - \Psi(q_i(v_i, v_i)) \\ &= Q_\Psi(u, v) + Q_\Psi(v, z) - Q_\Psi(v, v). \end{aligned}$$

Thus, all the axioms are satisfied for quasi partial metric Q_Ψ induced through aggregation of quasi partial metrics q_i where $i = 1, ..., n$.

3.5 Projective expansion and quasi partial metric aggregation

Remark 3.5.1 *Let $\{U_i\}_{i=1}^n$ be a collection of non-empty sets and $U = \prod_{i=1}^n U_i$. Let $\mathfrak{D}$ be a self mapping defined on U with coordinate functions $\mathfrak{D}_\mathfrak{i} : U \to U_i, \ i = 1, ...n$ such that*

$$\mathfrak{D}(u) = (\mathfrak{D}_\mathfrak{1}(u), ..., \mathfrak{D}_\mathfrak{n}(u)) \ \ for \ all \ \ u \in U.$$

Definition 3.5.2 *Let $\{(U_i, q_i)\}_{i=1}^n$ be a family of arbitrarily chosen quasi partial metric spaces and $U = \prod_{i=1}^n U_i$. Let $\Psi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a quasi partial metric aggregation function. Then the mapping $\mathfrak{D} : U \to U$ is called a projective Ψ-expansion from (U, Q_Ψ) into itself, if there exists n constants $\lambda_1, ..., \lambda_n > 1$ such that*

$$q_i(\mathfrak{D}_i(u), \mathfrak{D}_i(v)) \geq \lambda_i \Psi(q_1(u_1, v_1), ...q_n(u_n, v_n));$$

for all $u, v \in U$, where Q_Ψ is the quasi partial metric induced by aggregation of the collection of quasi partial metric spaces $\{(U_i, q_i)\}_{i=1}^n$ through aggregation function Ψ.

Note that if we put $n = 1$ and Ψ an identity function in Definition 3.5.2, then, the result given by Wang *et al.* [153] becomes a particular case of Ψ-projective expansion (see Theorem 3.2.5).

Example 3.5.1 *Let* $U_i = [0,1];\quad i = 1,2$ *and* q *be the quasi partial metric defined as* $q(u,v) = max\{(u-v),(v-u)\} + u$ *for all* $(u,v) \in [0,1] \times [0,1]$*. Let* $\{(U_i, q_i)\}_{i=1}^2$ *be the complete quasi partial metric spaces where* $q_1 = q_2 = q$ *and* $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *be the function defined as* $\Psi((u_1,u_2)) = \frac{u_1+u_2}{2} + \frac{1}{2}$ *for all* $u \in \mathbb{R}_+^2$*.*

It can be easily verified that conditions (1) and (2) hold in the statement of Theorem 3.4.6 and therefore, it is a quasi partial metric aggregation function. Define the mapping $\mathfrak{D} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ *by* $\mathfrak{D}(u) = (2,2)$ *for each* $u = (u_1,u_2) \in \mathbb{R}_+^2$*. It is not hard to see that* $\mathfrak{D}$ *is a projective* Ψ*- expansion as for* $u = (1,0),\ v = (0,0)$*, we obtain by definition of* q*,*

$$\begin{aligned} q(\mathfrak{D}_i(u), \mathfrak{D}_i(v)) &= q(2,2) \\ &= 2 \geq \frac{4}{3}\Psi(q_1(1,0), q_2(0,0)); \end{aligned}$$

for all $u, v \in \mathbb{R}_+^2$*.*

Example 3.5.2 *Let* $U_i = \mathbb{R}_+;\quad i = 1,2$ *and* q *be the quasi partial metric defined as* $q(u,v) = max\{(u-v),(v-u)\} + u$ *for all* $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$*. Let* $\{(U_i, q_i)\}_{i=1}^2$ *be the complete quasi partial metric spaces where* $q_1 = q_2 = q$ *and* $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *be the function defined as* $\Psi((u_1,u_2)) = (u_1 + u_2)$ *for all* $u \in \mathbb{R}_+^2$*.*

It can be easily verified that conditions (1) and (2) hold in the statement of Theorem 3.4.6 and therefore, it is a quasi partial metric aggregation function. Define the mapping $\mathfrak{D} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ *by* $\mathfrak{D}(u) = (4(u_1+u_2), 4(u_1+u_2))$ *for each* $u = (u_1,u_2) \in \mathbb{R}_+^2$*. It is not hard to see that* $\mathfrak{D}$ *is a projective* Ψ*- expansion as*

Case I*: For* $u \succeq v$*, we obtain by definition of* q*,*

$$\begin{aligned} q(\mathfrak{D}_i(u), \mathfrak{D}_i(v)) &= q(4(u_1+u_2), 4(v_1+v_2)) \\ &= 8(u_1+u_2) - 4(v_1+v_2) \\ &\geq 3[2(u_1+u_2) - (v_1+v_2)] \\ &= \lambda\Psi(q_1(u_1,v_1)q_2(u_2,v_2)); \end{aligned}$$

for all $u, v \in \mathbb{R}_+^2$ *where* $\lambda = 3 > 1$*.*

Case II*: For* $u \preceq v$*, we obtain by definition of* q*,*

$$q(\mathfrak{D}_i(u), \mathfrak{D}_i(v)) = q(4(u_1+u_2), 4(v_1+v_2))$$

$$
\begin{aligned}
&= 4(v_1 + v_2) \\
&\geq 3(v_1 + v_2) \\
&= \lambda\Psi(q_1(u_1, v_1)q_2(u_2, v_2));
\end{aligned}
$$

for all $u, v \in \mathbb{R}^2_+$ *where* $\lambda = 3 > 1$.

The next result has been presented in order to prove the existence and uniqueness of fixed point in quasi partial metric space considered *via* aggregation.

Lemma 3.5.3 *Let* $\Psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ *be a homogeneous quasi partial metric aggregation function such that* $\Psi(1, ..., 1) = 1 = \Psi(1_i)$ *for all* $i = 1, ...n$*. Let* $\{(U_i, q_i)\}_{i=1}^n$ *be a family of quasi partial metric spaces (chosen arbitrarily) and* $U = \prod_{i=1}^n U_i$*. Let us assume that the quasi partial metric spaces* (U_i, q_i), $\;i = 1, ...n$ *are complete. Then the quasi partial metric space* (U, Q_Ψ) *is complete where* Q_Ψ *is quasi partial metric aggregation induced through* Ψ *by aggregation of family of quasi partial metric spaces* $\{(U_i, q_i)\}_{i=1}^n$.

Proof Let $\{u^p\}_{p\in\mathbb{N}}$ be a Cauchy sequence in (U, Q_Ψ). Then there exists $l \in \mathbb{R}_+$ such that $\lim_{p,r} Q_\Psi(u^p, u^r) = l$ and for given $\epsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that $Q_\Psi(u^p, u^r) < \epsilon + l$ for all $p, r \geq p_0$. This implies that

$$\Psi(q_1(u_1^p, u_1^r), ..., q_1(u_n^p, u_n^r)) < \epsilon + l.$$

Since Ψ is a quasi partial metric aggregation, therefore Ψ is monotone and thus we have

$$\Psi(q_i(u_i^p, u_i^r).1_i) \leq \Psi(q_i(u_i^p, u_i^r)) < \epsilon + l \quad for \quad all \quad i = 1, ...n;$$

and as Ψ is homogeneous, it follows that

$$q_i(u_i^p, u_i^r) = q_i(u_i^p, u_i^r)\Psi(1_i) = \Psi(q_i(u_i^p, u_i^r).1_i) < \epsilon + l;$$

for all $i = 1, ...n$ and for all $p, r \geq p_0$.
This shows that there exists $u_i \in U_i$ such that $\lim_p u_i^p = u_i$ and $\lim_{p,r} q_i(u_i^p, u_i^r) = q_i(u_i, u_i) = \lim_p q_i(u_i, u_i^p) = \lim_p q_i(u_i^p, u_i) = l$ for all $i = 1, ...n$. Also, since (U_i, q_i) is complete quasi partial metric space for all $i = 1, ...n$; therefore for given $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $q_i(u_j, u_j^m) - q_i(u_j, u_j) < \frac{\epsilon}{3}$ for all $m \geq m_0$ and for all $i = 1, ..., n$.

Again by Theorem 3.4.6, we obtain

$$\begin{aligned} Q_\Psi(u,u^m) - Q_\Psi(u,u) &= \Psi(q_1(u_1,u_1^m),...,q_1(u_n,u_n^m)) - \Psi(q_1(u_1,u_1),...,q_1(u_n,u_n)) \\ &\leq \Psi\left(\frac{\epsilon}{3},...,\frac{\epsilon}{3}\right) \\ &= \frac{\epsilon}{3}\Psi(1,...,1) \\ &< \epsilon. \end{aligned}$$

as Ψ is homogeneous and $\Psi(1,...1) = 1$. Thus, $Q_\Psi(u,u^m) - Q_\Psi(u,u) < \epsilon$ for all $m \geq m_0$ and $\lim_m Q_\Psi(u,u^m) = Q_\Psi(u,u)$. Similarly, it can be shown that $\lim_m Q_\Psi(u^m,u) = Q_\Psi(u,u)$. Also, $Q_\Psi(u,u) = \Psi(q_1(u_1,u_1),...,q_1(u_n,u_n)) = \Psi(l,...l) = l\Psi(1,...1) = l$. Hence the quasi partial metric space (U,Q_Ψ) is complete.

In the next theorem, it is proved that every projective Ψ-expansion satisfying the condition $\Psi(1,...,1) \geq 1$, is an expansion.

Theorem 3.5.4 *Let $\{(U_i,q_i)\}_{i=1}^n$ be a family of quasi partial metric spaces (chosen arbitrarily) with $U = \prod_{i=1}^n U_i$. Let Ψ be a homogeneous quasi partial metric aggregation function such that $\Psi(1,...,1) \geq 1$ and $\mathfrak{D}$ is a projective Ψ-expansion. Then, $\mathfrak{D}$ is an expansion from the quasi partial metric space (U,Q_Ψ) to itself where Q_Ψ is the quasi partial metric (defined earlier).*

Proof It follows from Proposition 3.4.2 that Ψ is monotone. For $u,v \in U$, monotonicity of Ψ and nature of mapping $\mathfrak{D}$ implies

$$\begin{aligned} Q_\Psi(\mathfrak{D}(u),\mathfrak{D}(v)) &= \Psi(q_1(\mathfrak{D}_1(u),\mathfrak{D}_1(v)),...,q_n(\mathfrak{D}_n(u),\mathfrak{D}_n(v))) \\ &\geq \Psi(\lambda_1\Psi(q_1(u_1,v_1),...,q_n(u_n,v_n)),...\lambda_n\Psi(q_1(u_1,v_1),...,q_n(u_n,v_n))) \\ &\geq \Psi(\lambda\Psi(q_1(u_1,v_1),...,q_n(u_n,v_n)),...\lambda\Psi(q_1(u_1,v_1),...,q_n(u_n,v_n))); \end{aligned}$$

where $\lambda = min\{\lambda_1,...,\lambda_n\}$. Homogeneity of Ψ yields

$$\begin{aligned} &\Psi(\lambda\Psi(q_1(u_1,v_1),...,q_n(u_n,v_n)),...\lambda\Psi(q_1(u_1,v_1),...,q_n(u_n,v_n))) \\ &\quad = \lambda\Psi(1,...,1)\Psi(q_1(u_1,v_1),...,q_n(u_n,v_n)). \end{aligned}$$

Thus, above inequality becomes

$$\begin{aligned} Q_\Psi(\mathfrak{D}(u),\mathfrak{D}(v)) &\geq \lambda\Psi(1,...,1)Q_\Psi(u,v) \\ &\geq \lambda Q_\Psi(u,v). \end{aligned}$$

Hence, the mapping $\mathfrak{D}$ is an expansion from the quasi partial metric space (U,Q_Ψ)

to itself.

Next result shows the existence and uniqueness of fixed point.

Corollary 3.5.5 *Let* $\{(U_i, q_i)\}_{i=1}^n$ *be a family of arbitrarily chosen quasi partial metric spaces with complete metrics* q_i; $i = 1, ..., n$ *and* $U = \prod_{i=1}^n U_i$. *Let* Ψ *be a homogeneous quasi partial metric aggregation function such that* $\Psi(1, ..., 1) = \Psi(1_i) = 1$; $i = 1, ..., n$ *and* $\mathfrak{D}$ *is an onto projective* Ψ*-expansion. Then* $\mathfrak{D}$ *has a unique fixed point* u^*.

Proof By Lemma 3.5.3, it follows that the quasi partial metric space (U, Q_Ψ) is complete and by Theorem 3.5.4 shows that $\mathfrak{D}$ is an expansion from the quasi partial metric space (U, Q_Ψ) to itself. By Corollary 3.3.3, we see that $\mathfrak{D}$ has a unique fixed point u^* in U.

According to these results, every projective Ψ-expansion is an expansion but does the converse hold? The next example is an answer to this query *i.e.* every expansive mapping need not be a Ψ-projective expansion.

Example 3.5.3 *Let* $U_i = \mathbb{R}_+$; $i = 1, 2$ *and* q *be the quasi partial metric defined as* $q(u, v) = max\{(u - v), (v - u)\} + u$ *for all* $(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+$. *Let* $\{(U_i, q_i)\}_{i=1}^2$ *be the collection of complete quasi partial metric spaces where* $q_1 = q_2 = q$ *and* $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *be the function defined as* $\Psi((u_1, u_2)) = (u_1 + u_2)$ *for all* $u \in \mathbb{R}_+^2$.
It can be easily verified that conditions (1) and (2) hold in the statement of Theorem 3.4.6 and therefore, it is a quasi partial metric aggregation function. Moreover, $\Psi(1, 1) = 2 \geq 1$. *Define the mapping* $\mathfrak{D} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ *by* $\mathfrak{D}(u) = (2(u_1+u_2), 2(u_1+u_2))$ *for each* $u = (u_1, u_2) \in \mathbb{R}_+^2$. *It is not hard to see that* $\mathfrak{D}$ *is an expansion as*

$$\begin{aligned} Q_\Psi(\mathfrak{D}(u), \mathfrak{D}(v)) &= Q_\Psi((2(u_1 + u_2), 2(u_1 + u_2)), (2(v_1 + v_2), 2(v_1 + v_2))) \\ &= \Psi(q_1(2(u_1 + u_2), 2(v_1 + v_2)), q_2(2(u_1 + u_2), 2(v_1 + v_2))). \end{aligned}$$

Case I*: For* $u \succeq v$, *we obtain by definition of* q,

$$\begin{aligned} Q_\Psi(\mathfrak{D}(u), \mathfrak{D}(v)) &= 8(u_1 + u_2) - 4(v_1 + v_2) \\ &\geq 2[2(u_1 + u_2) - (v_1 + v_2)] \\ &= 2\Psi(2u_1 - v_1, 2u_2 - v_2) \\ &= 2\Psi(q_1(u_1, v_1), q_2(u_2, v_2)) \\ \Rightarrow Q_\Psi(\mathfrak{D}(u), \mathfrak{D}(v)) &\geq \lambda Q_\Psi(u, v); \end{aligned}$$

for all $u, v \in \mathbb{R}_+^2$ *where* $\lambda = 2 > 1$.

Case II: *For $u \preceq v$, we obtain by definition of q,*

$$\begin{aligned} Q_\Psi(\mathfrak{D}(u), \mathfrak{D}(v)) &= 4(v_1 + v_2) \\ &\geq 2(v_1 + v_2) \\ &= 2\ \Psi(q_1(u_1, v_1), q_2(u_2, v_2)) \\ \Rightarrow Q_\Psi(\mathfrak{D}(u), \mathfrak{D}(v)) &\geq \lambda Q_\Psi(u, v); \end{aligned}$$

for all $u, v \in \mathbb{R}^2_+$ where $\lambda = 2 > 1$. It follows that $\mathfrak{D}$ is an expansion from quasi partial metric space $(\mathbb{R}^2_+, Q_\Psi)$ into itself. Next, it is shown that $\mathfrak{D}$ is not a projective Ψ-expansion.
Consider $u, v \in \mathbb{R}^2_+$ with $u = (0, 1)$ and $v = (1, 0)$. Then

$$q(\mathfrak{D}_i(u), \mathfrak{D}_i(v)) = q(2, 2) = max\{0, 0\} + 2 = 2;$$

and

$$\lambda\Psi(q_1(u_1, v_1), q_2(u_2, v_2)) = \lambda\Psi(q_1(0, 1)q_2(1, 0)) = \lambda\Psi(1, 2) = 3\lambda.$$

Thus, $q(\mathfrak{D}_i(u), \mathfrak{D}_i(v)) \not\geq \lambda\Psi(q_1(u_1, v_1)q_2(u_2, v_2))$ as $\lambda > 1$.

The next example shows that the assumption 'Ψ is homogeneous' cannot be omitted in the statement of Theorem 3.5.4.

Example 3.5.4 *Let $([0, 1], q)$ be the complete quasi partial metric space such that q denotes the restriction of the quasi partial metric introduced in Proposition 3.4.2 to $[0, 1]$. Consider the family of complete quasi partial metric spaces $\{([0, 1], q_i)\}_{i=1,2}$ such that $q_1 = q_2 = q$. Define the function $\Psi : \mathbb{R}^2_+ \to \mathbb{R}_+$ by $\Psi(u) = u_1 + u_2 + \frac{1}{4}$ for all $u \in \mathbb{R}^2_+$. It is easy to see that for the function Ψ assertions (1) and (2) hold in the statement of Theorem 3.4.6 and, thus, it is a quasi partial metric aggregation function. Moreover, it is clear that $\Psi(1, 1) \geq 1$. However, Ψ is not homogeneous. Indeed,*

$$\Psi(2, 2) = \frac{17}{4} \neq \frac{18}{4} = 2\Psi(1, 1).$$

Next, consider the mapping $\mathfrak{D} : [0, 1]^2 \to [0, 1]^2$ defined by $\mathfrak{D}(u) = (0, 0)$ for all $u \in [0, 1]^2$. It is clear that $\mathfrak{D}$ is a projective Ψ-expansion. Nevertheless, $\mathfrak{D}$ is not an expansion from $([0, 1]^2, Q_\Psi)$ into itself, where Q_Ψ is the quasi partial metric induced by aggregation of the family of quasi partial metric spaces $\{([0, 1], q_i)\}_{i=1,2}$, through

Ψ. Indeed,

$$Q_\Psi(\mathfrak{D}(0,0),\mathfrak{D}(0,0)) = Q_\Psi((0,0),(0,0)) = \Psi(0,0) = \frac{1}{4}.$$

Therefore, there does not exist $\lambda > 1$ such that

$$Q_\Psi(\mathfrak{D}(0,0),\mathfrak{D}(0,0)) \geq \lambda Q_\Psi((0,0),(0,0)).$$

The next example shows that the assumption $\Psi(1,...,1) \geq 1$ cannot be omitted in the statement of Theorem 3.5.4 in order to guarantee that a projective Ψ-expansion is also an expansion from (U, Q_Ψ) into itself.

Example 3.5.5 *Let $\{([0,1], q_i)\}_i = 1, 2.$ be the family of complete quasi partial metric spaces such that $q_1 = q_2 = q$. Define the function $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $\Psi(u) = \frac{u_1+u_2}{3}$ for all $u \in \mathbb{R}_+^2$. It is easy to see that Ψ is a homogeneous quasi partial metric aggregation function. Nevertheless, $\Psi(1,1) = \frac{2}{3} < 1$.*
Next, consider the mapping $\mathfrak{D} : [0,1]^2 \to [0,1]^2$ defined by $\mathfrak{D}(u) = (2(u_1+u_2), 2(u_1+u_2))$ for all $u \in [0,1]^2$. Then we have for $u \succeq v$,

$$\begin{aligned}
q(\mathfrak{D}_i(u),\mathfrak{D}_i(v)) &= q(2(u_1+u_2), 2(v_1+v_2)) \\
&= 4(u_1+u_2) - 2(v_1+v_2) \\
&\geq 3\left(\frac{2}{3}(u_1+u_2) - \frac{1}{3}(v_1+v_2)\right) \\
&= \lambda\Psi(q(u_1,v_1), q(u_2,v_2)) \text{ with } \lambda = 3 > 1;
\end{aligned}$$

for all $u, v \in [0,1]^2$ and for $i = 1, 2$. Similar is the case for $u \preceq v$. So, $\mathfrak{D}$ is a projective Ψ-expansion. However, $\mathfrak{D}$ is not an expansion from the quasi partial metric space $([0,1]^2, Q_\Psi)$ into itself where Q_Ψ is the quasi partial metric induced by aggregation of the family of quasi partial metric spaces $\{([0,1], q_i)\}_{i=1,2}$, through Ψ. Take $u, v \in [0,1]^2$ given by $u = (0,0)$ and $v = (0,1)$. Then there does not exist $\lambda > 1$ such that

$$Q_\Psi(\mathfrak{D}(0,0),\mathfrak{D}(0,1)) \geq \lambda Q_\Psi((0,0),(2,2)).$$

Since

$$\begin{aligned}
Q_\Psi(\mathfrak{D}(0,0),\mathfrak{D}(0,1)) &= Q_\Psi((0,0),(2,2)) \\
&= \Psi(2,2) \\
&= \frac{4}{3}.
\end{aligned}$$

3.6 Application

In the field of computer science, the complexity analysis of an algorithm using minimum resources is an arduous task. This analysis is based on determining the quantity of existing resources such that running time, memory space, distribution of data *etc.* The objective of complexity analysis is to assess which of the algorithm is most suitable or in other words, the algorithm which takes minimum running time with minimum space even with large inputs and other suitable resources. This is usually done by means of asymptotic analysis where the running time of an algorithm A is denoted by a mapping $\mathfrak{D}_A : \mathbb{N} \to (0, \infty)$. The time or space taken by an algorithm to solve the problem under consideration is denoted by $\mathfrak{D}_A(n)$ where $n \in \mathbb{N}$ represents the size of input data to be processed. Let $S(\mathfrak{D}_A)$ denotes the set of all functions from $\mathbb{N}$ to $(0, \infty)$.

When the complexity analysis of an algorithm has to be determined, one approaches to asymptotic complexity analysis rather than exact analysis. So, they try to find such an algorithm that takes "approximately" minimum running time, minimum space even with large inputs and other suitable resources.

Let $f \in S(\mathfrak{D}_A)$ denote the running time or space taken by an algorithm. Then, we can define an asymptotic upper bound for f in the following way:

If there exists $n_0 \in \mathbb{N}$, $k \in \mathbb{R}^+$ and a function $g \in S(\mathfrak{D}_A)$ such that $f(n) \leq kg(n)$ for all $n \in \mathbb{N}$ such that $n_0 \leq n$. Then, g gives an asymptotic upper bound of f, and represents an "approximate" information of the algorithm. We write it as $f \in \mathcal{U}(g)$. Similarly, we can also define an asymptotic lower bound for the algorithm. The notation $f \in \mathcal{L}(g)$ means that there exists $n_0 \in \mathbb{N}$, $k \in \mathbb{R}^+$ and a function $g \in S(\mathfrak{D}_A)$ such that $kg(n) \leq f(n)$ for all $n \in \mathbb{N}$ such that $n_0 \leq n$. The best situation is the case when we can find such a function f which satisfy the condition $f \in \mho(g)$ where $\mho(g) = \mathcal{U}(g) \cap \mathcal{L}(g)$. In this case, the function f represents a 'tight' asymptotic bound of algorithm *i.e.* it represents the total asymptotic information about the most suitable resources to solve the problem under consideration.

Let the pair $(\mathcal{C}, \hat{d}^c)$ represents the complexity space, where

$$\mathcal{C} = \left\{ f \in S(\mathfrak{D}_A) : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\};$$

and $\hat{d}^c$ is the complete quasi partial metric on $\mathcal{C}$ defined by

$$\hat{d}^c(f, g) = \sum_{n=1}^{\infty} 2^{-n} max \left\{ \frac{1}{f(n)} - \frac{1}{g(n)}, \ \frac{1}{g(n)} - \frac{1}{f(n)} \right\} + \frac{1}{f(n)}.$$

The members of $\mathcal{C}$ are called complexity functions and $\hat{d}^c(f,g)$ represents the complexity distance from f to g. Then $\hat{d}^c(f,g) = 0$ means 'f is as efficient as g'.
The problem has been solved by using Divide and Conquer method given by [137]. In this procedure, we split the problem into subproblems (depending upon different resources) and solve them separately using same algorithm to find the suitable solution. After obtaining the solutions of the subproblems, all subproblems will be aggregated to obtain a global solution to the original problem which will represent an algorithm with all approximately suitable resources. The next result explores the significance of above theory.

Proposition 3.6.1 *Let $\mathfrak{D}$ be an onto self mapping defined on $\mathcal{C}$ with coordinate functions $\mathfrak{D}_\mathfrak{i} : \mathcal{C} \to \mathcal{C}_i,\ i = 1, ...n$ such that*

$$\mathfrak{D}(f)(n) = (\mathfrak{D}_\mathfrak{1}(f)(n), ..., \mathfrak{D}_\mathfrak{n}(f)(n)) \quad for \quad each \quad f \in \mathcal{C} \quad and\ n \in \mathbb{N};$$

satisfying the expansive inequality

$$\hat{d}^c_i(\mathfrak{D}_i(f)(n), \mathfrak{D}_i(g)(n)) \geq \lambda_i \Psi(\hat{d}^c_1(f_1, g_1), ...\hat{d}^c_n(f_n, g_n));$$

for all $f_i, g_i \in \mathcal{C}$, $i = 1, 2, ..., n$ and $\lambda_1, ..., \lambda_n > 1$. Then $\mathfrak{D} \in \mho(g)$.

Proof The members $\mathcal{C}_i,\ \ i = 1, 2, ...n;\ n \in \mathbb{N}$ of complexity class $\mathcal{C}$ are constructed in such a way that they will be based on different resources such as time, space, data *etc* and $\mathcal{C} = \prod_{i=1}^n \mathcal{C}_i$. It is easy to see that $(\mathcal{C}_i, \hat{d}^c_i)$ is a collection of complete quasi partial metric spaces. Let $\Psi : \mathbb{R}^n_+ \to \mathbb{R}_+$ be the function used to aggregate these members and it is defined in such a way that $\Psi(1, 1, ..., 1) \geq 1$. Let $\mathfrak{D}$ be an onto self mapping defined on $\mathcal{C}$ with coordinate functions $\mathfrak{D}_\mathfrak{i} : \mathcal{C} \to \mathcal{C}_i,\ i = 1, ...n$ such that

$$\mathfrak{D}(f)(n) = (\mathfrak{D}_\mathfrak{1}(f)(n), ..., \mathfrak{D}_\mathfrak{n}(f)(n)) \quad for \quad each \quad f \in \mathcal{C} \quad and\ n \in \mathbb{N};$$

satisfying the expansive inequality

$$\hat{d}^c_i(\mathfrak{D}_i(f)(n), \mathfrak{D}_i(g)(n)) \geq \lambda_i \Psi(\hat{d}^c_1(f_1, g_1), ...\hat{d}^c_n(f_n, g_n));$$

for all $f_i, g_i \in \mathcal{C}$, $i = 1, 2, ..., n$ and $\lambda_1, ..., \lambda_n > 1$.
Thus, all the conditions of Corollary 3.5.5 are fulfilled and therefore, $\mathfrak{D}$ has a fixed point f^* *i.e.* $\mathfrak{D} \in \mathcal{L}(g) \cap \mathcal{U}(g)$. It follows that $\mathfrak{D} \in \mho(g)$.

Chapter 4

$\mathfrak{D}$-Contraction and Related Fixed Point Theorems

4.1 Introduction

The concept of *"Contraction"* for a metric space was firstly introduced by polish mathematician Banach [20] to prove the existence and uniqueness of a fixed point. His principle known as "*Banach Contraction"* ensures that the application of a continuous self mapping on two points of a complete metric space contracts the distance between those two points. According to his result, "*A contraction self mapping defined on a complete metric space possesses a unique fixed point which can be obtained as the limit of an iteration scheme constructed by applying repeated images of the mapping (starting from an arbitrary point of space)"*. After that, many authors including Chatterjea [34], Ćirić [42], Kannan [84] gave extensions to this result by presenting more robust contractive conditions.
By now, there exists considerable literature on all these generalizations in various spaces which are applicable in numerous fields. For more details, references [4], [26], [35], [84], [131], [147], [61], [62] can be cited.

This chapter deals with a unique approach in the field of contraction mappings introduced with a family of bounded functions. The contents of this chapter have been divided into four sections. Section 4.2 is concerned with some basic definitions and results related to this paper. In section 4.3, main results have been presented with some illustrative examples whereas section 4.4 deals with an application to Intensity modulation radiation therapy.

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4.2 Preliminaries

Following are the results due to Banach [20]:

Definition 4.2.1 *[20] For a metric space $(U, \hat{d})$, a mapping $D : U \to U$ is called a contraction mapping on U if for any real number λ with $0 \leq \lambda < 1$, the following inequality holds:*

$$\hat{d}(Du, Dv) \leq \lambda \hat{d}(u, v) \quad for \quad all \ u, v \in U.$$

Remark 4.2.2 *It can be easily seen that the distance between the images of any two points of a given set is contracting by a uniform factor $\lambda < 1$.*

Example 4.2.1 *[20] Let $U = \mathbb{R}^2$ be a set equipped with standard metric $\hat{d}$*

$$\hat{d}((u_1, v_1), (u_2, v_2)) = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2} \ for \ all \quad u_1, u_2, v_1, v_2 \in U;$$

and $D : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined as $Du = \frac{3}{8}u$ for all $u \in \mathbb{R}^2$ where $u = (u_1, u_2)$. Then D is a contraction on U as $\hat{d}(Du, Dv) = \frac{3}{8}\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} = \frac{3}{8}\hat{d}(u, v)$.

Theorem 4.2.3 *[20] Let $(U, \hat{d})$ be a complete metric space and D be the contraction mapping defined on U. Then D possesses a unique fixed point u in U i.e. $Du = u$.*

After this well known result, Reich [129] presented the following theorem:

Theorem 4.2.4 *[129] Let $(U, \hat{d})$ be a complete metric space and D be the self mapping defined on U which satisfy the condition*

$$\hat{d}(Du, Dv) \leq \alpha \ \hat{d}(u, Du) + \beta \ \hat{d}(v, Dv) + \gamma \ \hat{d}(u, v);$$

for all $u, v \in U$ and α, β, γ non negative with $\alpha + \beta + \gamma < 1$. Then D admits a unique fixed point in U.

In 2012, Samet *et. al.* [142] obtained some fixed point results by defining $(\alpha - \psi)$-contractive mapping as follows:

Definition 4.2.5 *[142] Let Ψ be the family of all functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying the following properties:*

(1) $\sum_{n=1}^{+\infty} \psi^n(s) < +\infty$ *for every* $s > 0$, *where* ψ^n *is* n^{th} *iterate of* ψ ;

(2) ψ *is nondecreasing.*

Definition 4.2.6 *[142] Let $(U,\hat{d})$ be a metric space and D be a self mapping defined on U. The mapping D is said to be an $(\alpha-\psi)$-contractive mapping if there exists two functions $\alpha : U \times U \to [0,+\infty)$ and $\psi \in \Psi$ satisfying*

$$\alpha(u,v)\ \hat{d}(Du,Dv) \le \psi(\hat{d}(u,v)) \quad for \quad all \quad u,v \in U;$$

and α-admissible if

$$\alpha(u,v) \ge 1 \Rightarrow \alpha(Du,Dv) \ge 1 \ for \quad every\ u,v \in U.$$

4.3 Fixed function and related theorems

This section presents some fixed function theorems using the notions of fixed function and $\mathfrak{D}$-Contraction.

Definition 4.3.1 ***Fixed function**: Let $\mathfrak{D}$ be any self mapping defined on a family of functions $\mathfrak{F}$, then $f \in \mathfrak{F}$ is said to be fixed function of $\mathfrak{D}$ if $\mathfrak{D}f = f$.*

Example 4.3.1 *Let $U = [1,2]$ and the mapping $\mathfrak{D}$ be defined as $\mathfrak{D}f(u) = f^2(u) - 2f(u) + 2$ for all $f \in \mathfrak{F}$ and $u \in U$. Then $f(u) = 2$ for all $u \in U$ and $f(u) = 1$ for all $u \in U$ are two fixed functions of $\mathfrak{D}$.*

Example 4.3.2 *Let $U = \mathbb{R}^+$ and $\mathfrak{D}$ be the self mapping on $\mathfrak{F}$. Let $f \in \mathfrak{F}$ be a function defined on U as*

$$f(u) = \begin{cases} -1 & 0 \le u \le 1 \\ 0 & otherwise. \end{cases}$$

Then f^3 is a fixed function of $\mathfrak{D}$.

Definition 4.3.2 *Let $(U,\hat{d})$ be a complete metric space and let $\mathfrak{F}$ be the collection of all bounded functions defined on U. Let $\mathfrak{D}$ be any self mapping on $\mathfrak{F}$. Then the given mapping is called $\mathfrak{D}$-contraction mapping on $\mathfrak{F}$, if for any real number*

$\lambda \in [0,1)$, *we have*

$$d^*(\mathfrak{D}f, \mathfrak{D}g) \leq \lambda\ d^*(f,g) \quad for \quad all \quad f,g \in \mathfrak{F};$$

where

$$d^*(f,g) = sup\{\hat{d}(f(u), g(v))|\ u,v \in U\}. \tag{4.3.1}$$

Remark 4.3.3 *Clearly,* d^* *is a metric on* $\mathfrak{F}$ *as* $d^*(f,g) = 0 \Leftrightarrow f \sim g$ *for all* $f,g \in \mathfrak{F}$*. Also, for all* $u,v \in U$ *and* $f,g,h \in \mathfrak{F}$,

$$\begin{aligned} |f(u) - g(v)| &\leq |f(u) - h(w)| + |h(w) - g(v)| \\ &\leq \sup\{|f(u) - h(w)|\ |u,w \in U\} \\ &\quad + \sup\{|h(w) - g(v)|\ |w,v \in U\} \\ \Rightarrow \sup\{|f(u) - g(v)|\ |u,v \in U\} &\leq \sup\{|f(u) - h(w)|\ |u,w \in U\} \\ &\quad + \sup\{|h(w) - g(v)|\ |w,v \in U\} \\ \Rightarrow d^*(f,g) &\leq d^*(f,h) + d^*(h,g). \end{aligned}$$

Theorem 4.3.4 *Let* $(U,\hat{d})$ *be a complete metric space with metric* $\hat{d}$ *defined as* $\hat{d}(u,v) = |u - v|$ *for all* $u,v \in U$*. Let* $\mathfrak{F}$ *be the collection of all bounded functions* f *defined on* U *with metric* d^* *(as defined in (4.3.1)). Also, let* $\mathfrak{D}$ *be the* $\mathfrak{D}$*-contraction mapping defined on* $\mathfrak{F}$*. Then there exists a unique fixed function* $f \in \mathfrak{F}$ *i.e. there exists some* $f \in \mathfrak{F}$ *such that* $\mathfrak{D}f = f$.

Proof Let f,g be any two functions from the family $\mathfrak{F}$. Since $\mathfrak{D}$ is the $\mathfrak{D}$-contraction mapping on $\mathfrak{F}$, therefore, there exists a real number $\lambda \in [0,1)$ such that

$$d^*(\mathfrak{D}f, \mathfrak{D}g) \leq \lambda\ d^*(f,g) \quad for \quad all \quad f,g \in \mathfrak{F};$$

where

$$d^*(f,g) = sup\{\hat{d}(f(u), g(v))|\ u,v \in U\}.$$

This further implies that

$$\begin{aligned} d^*(\mathfrak{D}^2 f, \mathfrak{D}^2 g) &\leq \lambda d^*(\mathfrak{D}f, \mathfrak{D}g) \\ &\leq \lambda^2 d^*(f,g) \quad for \quad all \quad f,g \in \mathfrak{F}. \end{aligned}$$

Continuing in the same manner, we get

$$d^*(\mathfrak{D}^n f, \mathfrak{D}^n g) \leq \lambda^n d^*(f,g) \ for \ \ all \ \ f,g \in \mathfrak{F}. \tag{4.3.2}$$

Step I: *Existence of Cauchy sequence.*

Let f_0 be any function in $\mathfrak{F}$. Let us define the sequence $\{f_n\}_{(n\in\mathbb{N})}$ by setting

$$f_1 = \mathfrak{D}(f_0),$$

$$f_2 = \mathfrak{D}(f_1) = \mathfrak{D}^2(f_0),$$

$$. \quad . \quad .$$

$$. \quad . \quad .$$

$$f_n = \mathfrak{D}(f_{n-1}) = \mathfrak{D}^2(f_{n-2}) = ... = \mathfrak{D}^n(f_0).$$

Let $p, q \in N$ be some positive integers with $p > q$. Let $p = q + t$ where $t \geq 1$. Now, by using (4.3.2), we get

$$\begin{aligned}
d^*(f_q, f_p) &= d^*(f_q, f_{q+t}) \\
&\leq d^*(f_q, f_{q+1}) + d^*(f_{q+1}, f_{q+2}) + ... + d^*(f_{q+t-1}, f_{q+t}) \\
&= d^*(\mathfrak{D}^q f_0, \mathfrak{D}^q f_1) + d^*(\mathfrak{D}^{q+1} f_0, \mathfrak{D}^{q+1} f_1) + ... \\
&\quad + d^*(\mathfrak{D}^{q+t-1} f_0, \mathfrak{D}^{q+t-1} f_1) \\
&\leq \lambda^q d^*(f_0, f_1) + \lambda^{q+1} d^*(f_0, f_1) + ... \\
&\quad + \lambda^{q+t-1} d^*(f_0, f_1) \\
&= \lambda^q d^*(f_0, f_1).[1 + \lambda + \lambda^2 + ... + \lambda^{t-1}] \\
&\leq \frac{\lambda^q}{1-\lambda} d^*(f_0, f_1) \ \ where \ \ \lambda < 1.
\end{aligned}$$

Since $\mathfrak{F}$ is a family of bounded functions, therefore, $d^*(f_q, f_p) \to 0 \ as \ p \ , \ q \to \infty$. Hence, $\{f_n\}_{(n\in\mathbb{N})}$ is a Cauchy sequence in $\mathfrak{F}$.

Step II: *Existence of fixed function.*

As $\mathfrak{F}$ is the family of bounded functions defined on complete metric space $(U, \hat{d})$, therefore, $(\mathfrak{F}, d^*)$ is a complete metric space and thus the sequence $\{f_n\}_{(n\in\mathbb{N})}$ is convergent in $\mathfrak{F}$.

Let $f \in \mathfrak{F}$ be the limit of $\{f_n\}_{(n\in\mathbb{N})}$ *i.e* $\lim_{n\to\infty} \ \ f_n = f$. By the continuity of $\mathfrak{D}$, we get

$$\lim_{n\to\infty} \mathfrak{D} f_n = \mathfrak{D} f.$$

Also, $\mathfrak{D} f_n = f_{n+1} \to f \ as \ n \to \infty$.
Thus, uniqueness of limit implies that $\mathfrak{D} f = f$. This shows that f is a fixed function of $\mathfrak{D}$.

Step III: *Uniqueness of fixed function.*
Let g be another fixed function of $\mathfrak{D}$ $i.e$ $\mathfrak{D} g = g$ and $f \nsim g$. Now,

$$\begin{aligned} 0 \leq d^*(f,g) &= d^*(\mathfrak{D} f, \mathfrak{D} g) \\ &\leq \lambda d^*(f,g) \\ &< d^*(f,g). \end{aligned}$$

Thus, we arrive at a contradiction. Hence, f is a unique fixed function of $\mathfrak{D}$.

Example 4.3.3 *Let $U = \mathbb{R}$ and $\hat{d}$ be the metric defined on $\mathbb{R}$. Clearly, $(U, \hat{d})$ is a complete metric space. Let $\mathfrak{F}$ be the family of bounded functions defined on U and d^* be the metric on $\mathfrak{F}$ defined as*

$$d^*(f,g) = sup\{\hat{d}(f(u), g(v))|\ u, v \in U\}.$$

It can be easily seen that $(\mathfrak{F}, d^*)$ is a complete metric space being the family of bounded functions defined on complete metric space $(U, \hat{d})$. Let

$$f(u) = \begin{cases} 1 & u \ is \ \ rational \\ 0 & u \ \ is \ \ irrational \ ; \end{cases}$$

and

$$g(u) = \begin{cases} -1 & u \ is \ \ rational \\ \ \ 0 & u \ \ is \ \ irrational. \end{cases}$$

Let the mapping $\mathfrak{D}$ be defined as $\mathfrak{D} f = f^2$ for all $f \in \mathfrak{F}$. Then, it remains to show that the mapping $\mathfrak{D}$ is a $\mathfrak{D}$-contraction mapping. For this, we have

$$\begin{aligned} d^*(\mathfrak{D} f, \mathfrak{D} g) &= sup\{\ \hat{d}(\mathfrak{D} f(u), \mathfrak{D} g(v))\ |u, v \in U\} \\ &= sup\{|f^2(u) - g^2(v)|\ |u, v \in U\} \end{aligned}$$

$$\leq \lambda \; sup\{|f(u)-g(v)| \; |u,v \in U\} \;\; where \;\; 0 \leq \lambda < 1$$
$$\Rightarrow d^*(\mathfrak{D}f, \mathfrak{D}g) \leq \lambda d^*(f,g).$$

Since all the conditions required for Theorem 4.3.4 are fulfilled, therefore, there exists a unique fixed function of $\mathfrak{D}$. *In this example,* f^2, f^4, f^6 etc. *yield same fixed function of* $\mathfrak{D}$.

Example 4.3.4 *Let* $U = [0,1]$ *and* $\hat{d}$ *be the metric defined on* U. *Let* $\mathfrak{F} = C[0,1]$ *(i.e set of all real valued continuous functions defined on [0,1]) and the mapping* $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ *be defined as*

$$\mathfrak{D}f(u) = \frac{2}{3}f(u) \quad for \quad all \quad f \in \mathfrak{F} \;\; and \;\; u \in [0,1].$$

Here, $(U, \hat{d})$ *is a complete metric space and* $\mathfrak{F} = C[0,1]$ *is the collection of all real valued continuous (and hence bounded) functions defined on* $U = [0,1]$. *Let* $f_n(u) = \frac{u^n}{n}$ *for all* $u \in [0,1]$. *Then* $\{f_n(u)\}_{(u \in [0,1])}$ *is a uniformly convergent sequence in* $\mathfrak{F}$ *and therefore is a Cauchy sequence. Also, the given mapping is a* $\mathfrak{D}$*-contraction mapping as*

$$\begin{aligned} d^*(\mathfrak{D}f, \mathfrak{D}g) &= d^*\left(\frac{2}{3}f, \frac{2}{3}g\right) \\ &= sup\left\{\hat{d}\left(\frac{2}{3}f(u), \frac{2}{3}g(v)\right) \; |u,v \in U\right\} \\ &= \frac{2}{3}sup\{\; \hat{d}(f(u),g(v)) \; |u,v \in U\} \\ &< \lambda d^*(f,g) \;\; for \;\; \frac{2}{3} < \lambda < 1. \end{aligned}$$

Since all the conditions required for Theorem 4.3.4 are fulfilled, therefore, there exists a unique fixed function of $\mathfrak{D}$. *In this example, null function is a unique fixed function.*

Theorem 4.3.5 *Let* $(U, \hat{d})$ *be a complete metric space (where* $\hat{d}$ *is the metric as defined earlier) and* $\mathfrak{F}$ *be the collection of all bounded functions* f *defined on* U *with metric* d^* *(as defined in (4.3.1)). Also, let* $\mathfrak{D}$ *be the modified* $\mathfrak{D}$*-contraction mapping on* $\mathfrak{F}$ *satisfying*

$$d^*(\mathfrak{D}f, \mathfrak{D}g) \leq \alpha \; d^*(f, \mathfrak{D}f) + \beta \; d^*(g, \mathfrak{D}g) + \gamma \; d^*(f,g);$$

for all $f, g \in \mathfrak{F}$; α, β, γ *non negative with* $\alpha + \beta + \gamma < 1$. *Then* $\mathfrak{D}$ *has a unique fixed function.*

Proof Let $f_0 \in \mathfrak{F}$ be any arbitrary function and $f_n = \mathfrak{D} f_{n-1} = \mathfrak{D}^n f_0$.

Step I: *Existence of Cauchy sequence in $\mathfrak{F}$. For this, consider*

$$\begin{aligned}
d^*(f_1, f_2) &= d^*(\mathfrak{D} f_0, \mathfrak{D} f_1) \\
&\leq \alpha\, d^*(f_0, \mathfrak{D} f_0) + \beta\, d^*(f_1, \mathfrak{D} f_1) + \gamma\, d^*(f_0, f_1) \\
&= \alpha\, d^*(f_0, f_1) + \beta\, d^*(f_1, f_2) + \gamma\, d^*(f_0, f_1) \\
&= (\alpha + \gamma) d^*(f_0, f_1) + \beta\, d^*(f_1, f_2) \\
\Rightarrow (1-\beta) d^*(f_1, f_2) &\leq (\alpha + \gamma) d^*(f_0, f_1) \\
\Rightarrow d^*(f_1, f_2) &\leq \left(\frac{\alpha+\gamma}{1-\beta}\right) d^*(f_0, f_1) \qquad (where \quad \beta < 1).
\end{aligned}$$

Similarly

$$\begin{aligned}
d^*(f_2, f_3) &\leq \left(\frac{\alpha+\gamma}{1-\beta}\right) d^*(f_1, f_2) \\
&\leq \left(\frac{\alpha+\gamma}{1-\beta}\right)^2 d^*(f_0, f_1);
\end{aligned}$$

and so on.
As $\left(\frac{\alpha+\gamma}{1-\beta}\right) < 1$ and $f_0, f_1 \in \mathfrak{F}$ are bounded, therefore, $\{f_n\}_{(n \in \mathbb{N})}$ is a Cauchy sequence in $\mathfrak{F}$. Since $\mathfrak{F}$ is complete being the family of bounded functions defined on complete metric space $(U, \hat{d})$, therefore, the sequence $\{f_n\}_{(n \in \mathbb{N})}$ is convergent in $\mathfrak{F}$ (say it converges to $f \in \mathfrak{F}$).

Step II: *Existence of fixed function.*
Now it will be shown that f is a fixed function of $\mathfrak{D}$. Let s be any arbitrary positive integer. Now,

$$\begin{aligned}
d^*(f, \mathfrak{D} f) &\leq d^*(f, f_s) + d^*(f_s, \mathfrak{D} f) \\
&= d^*(f, f_s) + d^*(\mathfrak{D} f_{s-1}, \mathfrak{D} f) \\
&= d^*(f, f_s) + d^*(\mathfrak{D} f, \mathfrak{D} f_{s-1}) \\
\Rightarrow d^*(f, \mathfrak{D} f) &\leq d^*(f, f_s) + \alpha\, d^*(f, \mathfrak{D} f) + \beta\, d^*(f_{s-1}, \mathfrak{D} f_{s-1}) + \gamma\, d^*(f, f_{s-1}) \\
\Rightarrow (1-\alpha) d^*(f, \mathfrak{D} f) &\leq d^*(f, f_s) + \beta\, d^*(f_{s-1}, \mathfrak{D} f_{s-1}) + \gamma\, d^*(f, f_{s-1}).
\end{aligned}$$

The right side expression can be made arbitrarily small enough by taking s sufficiently large. Thus $0 \leq d^*(f, \mathfrak{D} f) < \epsilon$. This implies that $d^*(f, \mathfrak{D} f) = 0$ *i.e.* f is a fixed function of $\mathfrak{D}$.

Step III: *Uniqueness of fixed function.*
Suppose $g \in \mathfrak{F}$ be another fixed function of $\mathfrak{D}$ *i.e* $\mathfrak{D}g = g$ and $g \nsim f$. Then,

$$\begin{aligned} d^*(f,g) &= d^*(\mathfrak{D}f, \mathfrak{D}g) \\ &\leq \alpha\ d^*(f,\mathfrak{D}f) + \beta\ d^*(g,\mathfrak{D}g) + \gamma\ d^*(f,g) \\ \Rightarrow (1-\gamma)d^*(f,g) &\leq 0 \qquad (where\ \ \gamma < 1) \\ \Rightarrow d^*(f,g) &\leq 0; \end{aligned}$$

which is a contradiction to our assumption. This implies that f is unique.

Definition 4.3.6 *The mapping $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ is said to be an $(\alpha - \psi)$-contractive mapping if there exists two functions $\alpha : U \times U \to [0, +\infty)$ and $\psi \in \Psi$ satisfying*

$$\alpha(f(u), g(v))\ d^*(\mathfrak{D}f, \mathfrak{D}g) \leq \psi(d^*(f,g)); \tag{4.3.3}$$

for all $f, g \in \mathfrak{F}$ and $u, v \in U$.

Definition 4.3.7 *Let $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ and $\alpha : U \times U \to [0, +\infty)$. The mapping $\mathfrak{D}$ is called an α-admissible mapping if*

$$\alpha(f(u), g(v)) \geq 1 \Rightarrow \alpha(\mathfrak{D}f(u), \mathfrak{D}g(v)) \geq 1;$$

for every $f, g \in \mathfrak{F}$ and $u, v \in U$.

Theorem 4.3.8 *Let $(U, \hat{d})$ be a complete metric space and $\mathfrak{F}$ be the collection of all bounded functions f (defined on U) with metric d^* (as defined in (4.3.1)). Let $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ be an $(\alpha - \psi)$-contractive mapping. Also, suppose that*

(i) $\mathfrak{D}\ \ is\ \ \alpha - admissible;$

(ii) $there\ is\ some\ f_0 \in \mathfrak{F}\ for\ which\ \ \alpha(f_0(u), \mathfrak{D}f_0(v)) \geq 1\ for\ all\ u, v \in U;$

(iii) $\mathfrak{D}\ \ is\ \ continuous.$

Then $\mathfrak{D}$ possesses a fixed function in $\mathfrak{F}$.

Proof Let $f_0 \in \mathfrak{F}$ be a function such that

$$\alpha(f_0(u), \mathfrak{D}f_0(v)) \geq 1\ \ for\ \ all\ \ u, v \in U.$$

Define the sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathfrak{F}$ by $f_{n+1} = \mathfrak{D}f_n$ for every $n \in \mathbb{N}$. If $f_n = f_{n+1}$ for some $n \in \mathbb{N}$, then f_n is a fixed function of $\mathfrak{D}$. Let us assume that $f_n \neq f_{n+1}$ for every $n \in \mathbb{N}$.
As by condition (i), $\mathfrak{D}$ is α-admissible, therefore for all $u, v \in U$, we have

$$\begin{aligned} \alpha(f_0(u), f_1(v)) &= \alpha(f_0(u), \mathfrak{D}f_0(v)) \geq 1 \\ \Rightarrow \alpha(\mathfrak{D}f_0(u), \mathfrak{D}f_1(v)) &= \alpha(f_1(u), f_2(v)) \geq 1. \end{aligned}$$

By mathematical induction, we get

$$\alpha(f_n(u), f_{n+1}(v)) \geq 1 \quad for \quad all \quad n \in \mathbb{N} \quad and \quad u, v \in U. \tag{4.3.4}$$

Using (4.3.3) and (4.3.4),

$$\begin{aligned} d^*(f_n, f_{n+1}) &= d^*(\mathfrak{D}f_{n-1}, \mathfrak{D}f_n) \\ &\leq \alpha(f_{n-1}(u), f_n(v))d^*(\mathfrak{D}f_{n-1}, \mathfrak{D}f_n) \\ &\leq \psi(d^*(f_{n-1}, f_n)). \end{aligned}$$

Repetition of above process implies

$$d^*(f_n, f_{n+1}) \leq \psi^n(d^*(f_0, f_1)) \quad for \quad all \quad n \in \mathbb{N}.$$

Let $n > m \geq N$ for $N \in \mathbb{N}$. Using triangular inequality, we have

$$\begin{aligned} d^*(f_m, f_n) &\leq d^*(f_m, f_{m+1}) + d^*(f_{m+1}, f_{m+2}) + d^*(f_{m+2}, f_{m+3}) + \\ &\quad ... + d^*(f_{n-1}, f_n) \\ &\leq \psi^m(d^*(f_0, f_1)) + \psi^{m+1}(d^*(f_0, f_1)) + ... + \psi^{n-1}(d^*(f_0, f_1)) \\ &= \sum_{k=m}^{n-1} \psi^k(d^*(f_0, f_1)). \end{aligned}$$

As $\sum_{n=1}^{+\infty}\psi^n(u) < +\infty$ for each $u > 0$, so $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{F}$ and since $\mathfrak{F}$ is a collection of bounded functions defined on complete metric space $(U, \hat{d})$, therefore $(\mathfrak{F}, d^*)$ is itself a complete metric space. Therefore, there exists a function $f \in \mathfrak{F}$ such that $f_n \to f$ as $n \to +\infty$.
As $\mathfrak{D}$ is a continuous mapping, therefore, we have

$$\mathfrak{D}f_n \to \mathfrak{D}f \quad as \quad n \to +\infty \quad \Rightarrow f_{n+1} \to \mathfrak{D}f \quad as \quad n \to +\infty.$$

Since limit of a convergent sequence is always unique, therefore, we have $f = \mathfrak{D}f$ *i.e.* f is a fixed function of $\mathfrak{D}$. This completes the proof.

Example 4.3.5 *Let* $U = [0, 2]$ *and* $\hat{d}(u, v) = |u - v|$. *Let* $\mathfrak{F}$ *be the family of bounded functions on* $[0, 2]$ *and* $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ *be defined as* $\mathfrak{D}f = f^2$ *and* d^* *be the metric defined on* $\mathfrak{F}$ *as*

$$d^*(f, g) = \int_0^2 |f(u) - g(u)| du.$$

Let

$$f(u) = \begin{cases} 1 & u \in [0, 1] \\ 0 & otherwise \ ; \end{cases}$$

$$g(u) = \begin{cases} -1 & u \in [0, 1] \\ 0 & otherwise \ ; \end{cases}$$

and

$$\alpha(f(u), g(v)) = \begin{cases} 2 & u \in [0, 1] \\ 0 & otherwise \ . \end{cases}$$

Clearly, $(U, \hat{d})$ *is a complete metric space and* $\mathfrak{D}$ *is a continuous mapping. Moreover,* f *and* g *are bounded functions and* $\mathfrak{D}$ *is* α*-admissible as*

$$\alpha(f(u), g(v)) \geq 1 \quad \Rightarrow \quad u \in [0, 1];$$

and for $u \in [0, 1]$, *we have*

$$\alpha(\mathfrak{D}f(u), \mathfrak{D}g(v)) = \alpha(f^2(u), g^2(v)) \geq 1.$$

Now, it is proved that $\mathfrak{D}$ *is an* $(\alpha - \psi)$*-contractive mapping. To prove this, Let* $\psi \in \Psi$ *be a function defined as* $\psi(u) = \frac{u}{2}$.
Case I: *When* $u \in [0, 1]$, *then*

$$\begin{aligned} \alpha(f(u), g(v)) \ d^*(\mathfrak{D}f, \mathfrak{D}g) &= 2 \int_0^2 |f^2(u) - g^2(u)| du \\ &= 2 \int_0^1 |f^2(u) - g^2(u)| du + 2 \int_1^2 |f^2(u) - g^2(u)| du \end{aligned}$$

$$\begin{aligned} &= 2(0)+2(0)=0 \\ &\leq u=\psi(d^*(f,g)). \end{aligned}$$

Case II: *When* $u \in (1,2]$, *then*

$$\alpha(f(u),g(v))\ d^*(\mathfrak{D}f,\mathfrak{D}g)=0 \leq \psi(d^*(f,g)).$$

Thus, all the conditions needed for Theorem 4.3.8 are fulfilled, so, there must exists a fixed function in $\mathfrak{F}$. *In this example,* f *is a fixed function of* $\mathfrak{D}$.

Uniqueness: By considering the following hypothesis, the uniqueness of fixed function in Theorem 4.3.8 will be assured.
(H): for all $f,g \in \mathfrak{F}$, there exists $h \in \mathfrak{F}$ such that

$$\alpha(f(u),h(v)) \geq 1 \ \ and \ \ \alpha(g(u),h(v)) \geq 1.$$

Theorem 4.3.9 *Adding condition (H) to the hypothesis of Theorem 4.3.8, we obtain the uniqueness of fixed function of* $\mathfrak{D}$.

Proof Let us suppose that f^* and g^* be two fixed functions of $\mathfrak{D}$. From (H), there exists some $h^* \in \mathfrak{F}$ such that

$$\alpha(f^*(u),h^*(v)) \geq 1 \ \ and \ \ \alpha(g^*(u),h^*(v)) \geq 1. \tag{4.3.5}$$

Since $\mathfrak{D}$ is α-admissible, by (4.3.5), we have

$$\alpha(f^*(u),\mathfrak{D}^n h^*(v)) \geq 1 \ \ and \ \ \alpha(g^*(u),\mathfrak{D}^n h^*(v)) \geq 1 \ \ for \ \ all \ \ n \in \mathbb{N}. \tag{4.3.6}$$

Using (4.3.6) and $(\alpha-\psi)$-contractive condition

$$\begin{aligned} d^*(f^*,\mathfrak{D}^n h^*) &= d^*(\mathfrak{D}f^*,\mathfrak{D}(\mathfrak{D}^{n-1}h^*)) \\ &\leq \alpha(f^*(u),\mathfrak{D}^{n-1}h^*(v))\ d^*(\mathfrak{D}f^*,\mathfrak{D}(\mathfrak{D}^{n-1}h^*)) \\ &\leq \psi(d^*(f^*,\mathfrak{D}^{n-1}h^*)); \end{aligned}$$

which implies that $d^*(f^*,\mathfrak{D}^n h^*) \leq \psi^n(d^*(f^*,h^*))$ for all $n \in \mathbb{N}$. Taking limit $n \to +\infty$, we get

$$\mathfrak{D}^n h^* \to f^*.$$

Similarly,

$$\mathfrak{D}^n h^* \to g^*.$$

Uniqueness of limit gives $f^* = g^*$. This proves the theorem.

4.4 Application to Intensity modulated radiation therapy

The application in this section is based on determining the best suitable treatment plan for tumor patients getting intensity modulated radiation therapy (IMRT). Bortfeld [27] and Shepard *et al.* [148] presented some useful techniques to develop algorithms for the problems encountered in tomotherapy. In these techniques, a dose deposition coefficient (DDC) matrix is often computed to decide the dose distribution to each voxel in required volume of interest from every beamlet with unit intensity. But we usually get a large set of data during calculation that requires a huge computer memory and computational efficiency. As a result, small values from DDC matrix are usually truncated that affects the quality of treatment plan. Fixed point method is very efficient and effective technique to solve this problem. In this technique, a proper DDC matrix truncation has been used that significantly improves accuracy of results. Following Tian *et al.* [151]'s FMO model, the DDC matrix was divided into two components $\mathfrak{D}_1$ and $\mathfrak{D}_2$ on the basis of a threshold value. The matrix $\mathfrak{D}_1$ (major component) consists those values of DDC matrix which are higher than the threshold whereas the minor component $\mathfrak{D}_2$ consists remaining values. In fact, $\mathfrak{D}_1$ represents those doses which correspond to tumor area voxels (specifically) while $\mathfrak{D}_2$ represents scatter doses passing at large distances. Following this concept, the treatment plan for more than a patient at a time, is presented through our results in a more effective way. The results proposed in this paper provide a very efficient and easy technique for estimation of suitable treatment plan.

In the present case, two tumor patients have been considered with different tumor levels. Let U denotes the set of all threshold intensity values (with unit Gy) to be given on particular days and in particular sessions. A patient is getting the therapy two times a day. Days and sessions are denoted by D and S respectively.

$$U = \begin{cases} (1, D_1S_1), (\frac{1}{2}, D_1S_2), (1, D_2S_1), (\frac{1}{2}, D_2S_2) & Patient - I, \\ (1, D_1S_1), (2, D_1S_2), (1, D_2S_1), (2, D_2S_2) & Patient - II. \end{cases}$$

Note that U is complete being a closed and bounded subset of $\mathbb{R}^2$. Let $\mathfrak{F} = \{f_1, f_2\}$ be the family of dose functions and each function represents different dose distributions (to tumor locations) of different tumor patients during IMRT.

$$f_1(u) = \begin{cases} 2u & Patient - I, \\ u & Patient - II. \end{cases} \quad and \quad f_2(u) = \begin{cases} \frac{u}{3} & Patient - I, \\ \frac{2u}{3} & Patient - II. \end{cases}$$

It is to be noted that $\mathfrak{F}$ is the family of bounded functions. Let $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ be the mapping defined as $\mathfrak{D}f = f^2 - 2f + 2 \ \ \forall \ f \in \mathfrak{F}$. It is required to prove that $\mathfrak{D}$ is a $\mathfrak{D}$-contraction mapping. For $u, v \in U$, we have the following cases:
For Patient-I

Case I- When $u = v = 1$. Then

$$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{5}{9} \ \ and \ \ |f_1 - f_2| = \frac{5}{3}.$$

Case II- When $u = v = \frac{1}{2}$. Then

$$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{25}{36} \ \ and \ \ |f_1 - f_2| = \frac{5}{6}.$$

Case III- When $u = 1, \ \ v = \frac{1}{2}$. Then

$$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{11}{36} \ \ and \ \ |f_1 - f_2| = \frac{11}{6}.$$

Case IV- When $u = \frac{1}{2}, \ \ v = 1$. Then

$$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{4}{9} \ \ and \ \ |f_1 - f_2| = \frac{2}{3}.$$

For Patient-II

Case I- When $u = v = 1$. Then

$$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{1}{9} \ \ and \ \ |f_1 - f_2| = \frac{1}{3}.$$

Case II- When $u = v = 2$. Then

$$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{8}{9} \ \ and \ \ |f_1 - f_2| = \frac{2}{3}.$$

Case III- When $u = 1, \ \ v = 2$. Then

$$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{1}{9} \ \ and \ \ |f_1 - f_2| = \frac{1}{3}.$$

Case IV- When $u = 2, \ \ v = 1$. Then

$$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{8}{9} \ \ and \ \ |f_1 - f_2| = \frac{4}{3}.$$

From all above cases, for Patient-I

$$\begin{aligned} d^*(\mathfrak{D}f_1, \mathfrak{D}f_2) &= sup\{|\mathfrak{D}f_1 - \mathfrak{D}f_2| \ u, v \in U\} \\ &= \frac{25}{36} \leq \frac{2}{3} \times \frac{11}{6} \\ &= \lambda \ d^*(f_1, f_2); \end{aligned}$$

and for Patient-II

$$\begin{aligned} d^*(\mathfrak{D}f_1, \mathfrak{D}f_2) &= sup\{|\mathfrak{D}f_1 - \mathfrak{D}f_2| \ u, v \in U\} \\ &= \frac{8}{9} \leq \frac{2}{3} \times \frac{4}{3} \\ &= \lambda \ d^*(f_1, f_2); \end{aligned}$$

where $\lambda = \frac{2}{3} < 1$.

Thus, all the conditions required for Theorem 4.3.4 are fulfilled. Therefore, there exists a unique fixed function f_1 of $\mathfrak{D}$ that yields suitable doses for two patients at the same time.

Chapter 5

Fixed Function Theorems in Complete Metric Spaces for Expansive Mappings

5.1 Introduction and Preliminaries

The study of fixed points was initiated by Poincare [123] in 1886. His result is known as "*Poincare's last geometric theorem*" which ensures the existence of at least two fixed points for an area preserving twist homeomorphism of an annulus. On the other hand, in 1984, Wang *et al.* [153] began a research line in the field of expansive mappings and presented some fixed point results in metric spaces.

Theorem 5.1.1 *[153] Let $(U, \hat{d})$ be a complete metric space. If $D : U \to U$ is an onto mapping and there exists a constant $a > 1$ such that*

$$\hat{d}(Du_1, Du_2) \geq a\ \hat{d}(u_1, u_2);$$

for each $u_1, u_2 \in U$. Then D has a unique fixed point in U.

In [78], Jungck defined interdependence between commuting mappings and fixed points.

Definition 5.1.2 *[78] Two self mappings D_1 and D_2 of a metric space $(U, \hat{d})$ are said to be commuting if $D_1D_2u = D_2D_1u$ for all $u \in U$.*

Theorem 5.1.3 *[78] Let D_1 be a continuous self mapping defined on a complete metric space $(U, \hat{d})$. Then D_1 has a fixed point in U iff there exists $a \in (0, 1)$ and a mapping $D_2 : U \to U$ which commutes with D_1 and satisfies $D_2(U) \subset D_1(U)$ and $\hat{d}(D_2(u), D_2(v)) \leq a\ \hat{d}(D_1(u), D_1(v))$ for all $u, v \in U$. The mappings D_1 and D_2 have a unique common fixed point if above inequality holds.*

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After that, Jungck [79] defined weakly compatible mappings.

Definition 5.1.4 *[79] Let D_1 and D_2 be two self mappings defined on a set U. If $w = D_1u = D_2u$ for some u in U, then u is called a coincidence point of D_1 and D_2, and w is called a point of coincidence of D_1 and D_2. Self mappings D_1 and D_2 are said to be weakly compatible if $D_1D_2u = D_2D_1u$ for some u in U such that $D_1u = D_2u$.*
The concept of weakly compatible mappings is more general as compared to commuting mappings.

In this chapter, following chapter 4, the concept of fixed function is used to prove some fixed function theorems *via* expansive mappings in complete metric spaces. A new notion of $\mathfrak{P}$-expansion is introduced in this chapter and various fixed function theorems are derived using different forms of this class of expansive mapping. Also, some common fixed function theorems for a pair of weakly compatible mappings are derived and the credibility of obtained results have been verified through examples and a nice application to Tomotherapy.

This chapter is organized as follows: Section 5.2 consists of some fixed function results in complete metric spaces. In section 5.3, some common fixed function theorems for a pair of weakly compatible mappings are obtained. Section 5.4 deals with an application to Tomotherapy.

5.2 Fixed function theorems for expansive mappings

Definition 5.2.1 *Let $(U, \hat{d})$ be a complete metric space where metric $\hat{d}$ is defined as $\hat{d}(u, v) = |u - v| \ \forall \ u, v \in U$ and $\mathfrak{F}$ be the collection of all bounded functions defined on U equipped with metric d^* where $d^*(f, g) = sup\{\hat{d}(f(u), g(v))|\ u, v \in U\}$ ($\hat{d}$ is the usual metric). Let $\mathfrak{P} : \mathfrak{F} \to \mathfrak{F}$ be a self mapping. Then $\mathfrak{P}$ is called an expansive mapping if for all $f, g \in \mathfrak{F}$, there exists a number $\alpha > 1$ such that*

$$d^*(\mathfrak{P}f, \mathfrak{P}g) \geq \alpha\ d^*(f, g).$$

Example 5.2.1 *Let U be the set of positive real numbers and the mapping $\mathfrak{P}$ is* *defined as* $\mathfrak{P}f = 5f + 2 \ \ \forall \ f \in \mathfrak{F}$. *Then* $\mathfrak{P}$ *is an expansive mapping as for all* $u, v \in U; \ \ f, g \in \mathfrak{F}$, *we have*

$$\begin{aligned} d^*(\mathfrak{P}f, \mathfrak{P}g) &= sup\{|5f(u) + 2 - 5g(v) - 2| \ |u, v \in U\} \\ &= sup\{5|f(u) - g(v)| \ |u, v \in U\} \\ &> sup\{|f(u) - g(v)| \ |u, v \in U\} \\ &= \alpha \ d^*(f, g) \quad \ where \quad \alpha = 2. \end{aligned}$$

Lemma 5.2.2 *Let* $(U, \hat{d})$ *be a complete metric space and* $\mathfrak{F}$ *be the collection of all bounded functions defined on* U *equipped with metric* d^*. *Let* $\{f_k\}_{(k\in\mathbb{N})}$ *be a sequence of functions in* $\mathfrak{F}$. *If there exists a number* $\alpha \in (0, 1)$ *such that*

$$d^*(f_{k+1}, f_k) \le \alpha \ d^*(f_k, f_{k-1}), \quad k = 1, 2, ... \tag{5.2.1}$$

Then $\{f_k\}_{(k\in\mathbb{N})}$ *is a Cauchy sequence in* $\mathfrak{F}$.

Proof By induction, we get from (5.2.1),

$$\begin{aligned} d^*(f_{k+1}, f_k) &\le \alpha \ d^*(f_k, f_{k-1}) \le \alpha^2 d^*(f_{k-1}, f_{k-2}) \\ &\le ... \le \alpha^k d^*(f_1, f_0). \end{aligned}$$

Let $p, q \in \mathbb{N}$ be some positive integers with $p > q$. Let $p = q + t$ where $t \ge 1$. Then,

$$\begin{aligned} d^*(f_q, f_p) &= d^*(f_q, f_{q+t}) \\ &\le d^*(f_q, f_{q+1}) + d^*(f_{q+1}, f_{q+2}) + ... \\ &\quad + d^*(f_{q+t-1}, f_{q+t}) \\ &\le \alpha^q d^*(f_1, f_0) + \alpha^{q+1} d^*(f_1, f_0) + ... \\ &\quad + \alpha^{q+t-1} d^*(f_1, f_0) \\ &\le \frac{\alpha^q}{1-\alpha} d^*(f_1, f_0) \ \ where \ \ \alpha < 1. \end{aligned}$$

Taking limit $q \to \infty$, we get

$$d^*(f_q, f_p) < \epsilon.$$

This shows that $\{f_k\}_{(k\in\mathbb{N})}$ is a Cauchy sequence in $\mathfrak{F}$.

Theorem 5.2.3 *Let* $(U, \hat{d})$ *be a complete metric space with metric* $\hat{d}$ *defined as* $\hat{d}(u, v) = |u - v|$ *for all* $u, v \in U$. *Let* $\mathfrak{F}$ *be the collection of all bounded functions defined on* U *with metric* d^*. *Also, let* $\mathfrak{P}$ *be an onto expansive self mapping defined* *on $\mathfrak{F}$. Suppose that there exists $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma > 1$ such that*

$$d^*(\mathfrak{P}f, \mathfrak{P}g) \geq \alpha\ d^*(f, g) + \beta\ d^*(f, \mathfrak{P}f) + \gamma\ d^*(g, \mathfrak{P}g);$$

for all $f, g \in \mathfrak{F}$ (where $f \nsim g$). Then $\mathfrak{P}$ has a fixed function in $\mathfrak{F}$.

Proof Let f_0 be any arbitrary function in $\mathfrak{F}$. Since $\mathfrak{P}$ is an onto mapping, there exists $f_1 \in \mathfrak{F}$ such that $\mathfrak{P}f_1 = f_0$. Proceeding in the same way, we can define a sequence $\{f_k\}_{(k \in \mathbb{N})}$ in $\mathfrak{F}$ such that $f_{k-1} = \mathfrak{P}f_k, \quad k = 1, 2, ...$
Assume that $f_k \nsim f_{k-1} \ \ \forall \ \ k = 1, 2, ...$ (because otherwise f_k is a fixed function of $\mathfrak{P}$). Then,

$$\begin{aligned}
d^*(f_{k-1}, f_k) &= d^*(\mathfrak{P}f_k, \mathfrak{P}f_{k+1}) \\
&\geq \alpha\ d^*(f_k, f_{k+1}) + \beta\ d^*(f_k, \mathfrak{P}f_k) + \gamma\ d^*(f_{k+1}, \mathfrak{P}f_{k+1}) \\
&= \alpha\ d^*(f_k, f_{k+1}) + \beta\ d^*(f_k, f_{k-1}) + \gamma\ d^*(f_{k+1}, f_k) \\
\Rightarrow (1-\beta) d^*(f_{k-1}, f_k) &\geq (\alpha + \gamma) d^*(f_k, f_{k+1}) \\
\Rightarrow d^*(f_k, f_{k+1}) &\leq \left(\frac{1-\beta}{\alpha+\gamma} \right) d^*(f_{k-1}, f_k) \\
&= s\ d^*(f_{k-1}, f_k) \quad where \quad s < 1.
\end{aligned}$$

By Lemma 5.2.2, $\{f_k\}_{(k \in \mathbb{N})}$ is a Cauchy sequence in $\mathfrak{F}$. Since $(\mathfrak{F}, d^*)$ is complete being a collection of functions defined on a complete metric space (U, d), the sequence $\{f_k\}_{(k \in \mathbb{N})}$ is convergent in $\mathfrak{F}$.
Let $f^* \in \mathfrak{F}$ be the limit of this sequence. Accordingly, there exists $f \in \mathfrak{F}$ such that $\mathfrak{P}f = f^*$. Now,

$$\begin{aligned}
d^*(f_k, f^*) &= d^*(\mathfrak{P}f_{k+1}, \mathfrak{P}f) \\
&\geq \alpha\ d^*(f_{k+1}, f) + \beta\ d^*(f_{k+1}, \mathfrak{P}f_{k+1}) + \gamma\ d^*(f, \mathfrak{P}f).
\end{aligned}$$

As $k \to \infty$, the above inequality becomes

$$\begin{aligned}
0 = d^*(f^*, f^*) &\geq \alpha\ d^*(f^*, f) + \beta(0) + \gamma\ d^*(f, f^*) \\
&\Rightarrow 0 \geq (\alpha + \gamma) d^*(f, f^*);
\end{aligned}$$

which implies that $d^*(f, f^*) = 0$ *i.e.* $f \sim f^*$. Thus, f^* is a fixed function of D.
This completes the proof.

Remark 5.2.4 *By setting $\beta = \gamma = 0$ in Theorem 5.2.3, the following result is obtained:*

Corollary 5.2.5 *Let $(U, \hat{d})$ be a complete metric space with metric $\hat{d}$ defined as $\hat{d}(u, v) = |u - v|$ for all $u, v \in U$. Let $\mathfrak{F}$ be the collection of all bounded functions defined on U with metric d^*. Also, let $\mathfrak{P}$ be an onto expansive self mapping defined on $\mathfrak{F}$. Suppose that there exists $\alpha > 1$ such that*

$$d^*(\mathfrak{P}f, \mathfrak{P}g) \geq \alpha\ d^*(f, g);$$

for all $f, g \in \mathfrak{F}$ (where $f \nsim g$). Then $\mathfrak{P}$ has a unique fixed function in $\mathfrak{F}$.

Proof For uniqueness, let g^* be another fixed function of $\mathfrak{P}$. Then

$$\begin{aligned} d^*(f^*, g^*) &= d^*(\mathfrak{P}f^*, \mathfrak{P}g^*) \\ &\geq \alpha\ d^*(f^*, g^*); \end{aligned}$$

which is not possible as $\alpha > 1$. Therefore, the fixed function f^* is unique.
This completes the proof.

Corollary 5.2.6 *Let $(U, \hat{d})$ be a complete metric space with metric $\hat{d}$ defined as $\hat{d}(u, v) = |u - v|$ for all $u, v \in U$ and $\mathfrak{F}$ be the collection of all bounded functions defined on U along with metric d^*. Also, let $\mathfrak{P}$ be an onto expansive self mapping defined on $\mathfrak{F}$. Suppose that there exists a +ve integer n and $\alpha > 1$ such that*

$$d^*(\mathfrak{P}^n f, \mathfrak{P}^n g) \geq \alpha\ d^*(f, g);$$

for all $f, g \in \mathfrak{F}$(where $f \nsim g$). Then $\mathfrak{P}$ has a unique fixed function in $\mathfrak{F}$.

Proof By Corollary 5.2.5,

$$\begin{aligned} d^*(\mathfrak{P}^n f, \mathfrak{P}^n g) &\geq \alpha\ d^*(\mathfrak{P}^{n-1} f, \mathfrak{P}^{n-1} g) \\ &\geq \alpha^2 d^*(\mathfrak{P}^{n-2} f, \mathfrak{P}^{n-2} g) \\ ... &\geq \alpha^n d^*(f, g); \end{aligned}$$

and thus $\mathfrak{P}^n$ has a unique fixed function f^*. But $\mathfrak{P}^n(\mathfrak{P}f^*) = \mathfrak{P}(\mathfrak{P}^n f^*) = \mathfrak{P}f^*$, therefore, $\mathfrak{P}f^*$ is also a fixed function of $\mathfrak{P}^n$ but the uniqueness implies $\mathfrak{P}f^* = f^*$. Thus, $\mathfrak{P}$ has a unique fixed function in $\mathfrak{F}$.

Example 5.2.2 *Let $U = \mathbb{R}^+$ and $\hat{d}$ be the usual metric defined on U. Let $\mathfrak{F}$ be the family of bounded functions defined on U and d^* be the metric defined on $\mathfrak{F}$ as*

$$d^*(f, g) = sup\{\hat{d}(f(u), g(v))|\ u, v \in U\}.$$

Note that $(U, \hat{d})$ is a complete metric space and thus $(\mathfrak{F}, d^*)$ is also complete being collection of bounded functions defined on U. Let

$$f(u) = \begin{cases} -2 & u \text{ is rational}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$g(u) = \begin{cases} 1 & u \text{ is rational}, \\ 2 & \text{otherwise}. \end{cases}$$

be two arbitrary functions from the family $\mathfrak{F}$. Let the mapping $\mathfrak{P}$ be defined as $\mathfrak{P}f = f^2 + 3f \quad \forall\ f \in \mathfrak{F}$. It is easy to check that $\mathfrak{P}$ is an expansive mapping as for $u, v \in U;\ f, g \in \mathfrak{F}$, we have the following cases for $\alpha = \frac{6}{5}$:

***Case-I**: If u, v are rational numbers. Then*

$$\begin{aligned} d^*(\mathfrak{P}f, \mathfrak{P}g) &= sup\{d(\mathfrak{P}f(u), \mathfrak{P}g(v))|\ u, v \in U\} \\ &= sup\{|\mathfrak{P}f(u) - \mathfrak{P}g(v)|\ u, v \in U\} \\ &= sup\{|f^2(u) + 3f(u) - g^2(v) - 3g(v)|\ u, v \in U\} \\ &= 6 \geq \frac{6}{5} \times 3 \\ &= \alpha\ sup\{|f(u) - g(v)|\ u, v \in U\} \\ &= \alpha\ d^*(f, g). \end{aligned}$$

***Case-II**: If u, v are not rational numbers. Then*

$$\begin{aligned} d^*(\mathfrak{P}f, \mathfrak{P}g) &= sup\{d(\mathfrak{P}f(u), \mathfrak{P}g(v))|\ u, v \in U\} \\ &= sup\{|\mathfrak{P}f(u) - \mathfrak{P}g(v)|\ u, v \in U\} \\ &= sup\{|f^2(u) + 3f(u) - g^2(v) - 3g(v)|\ u, v \in U\} \\ &= 10 \geq \frac{6}{5} \times 2 \\ &= \alpha\ sup\{|f(u) - g(v)|\ u, v \in U\} \\ &= \alpha\ d^*(f, g). \end{aligned}$$

***Case-III**: If u is a rational number and v is not. Then*

$$\begin{aligned} d^*(\mathfrak{P}f, \mathfrak{P}g) &= sup\{d(\mathfrak{P}f(u), \mathfrak{P}g(v))|\ u, v \in U\} \\ &= sup\{|\mathfrak{P}f(u) - \mathfrak{P}g(v)|\ u, v \in U\} \end{aligned}$$

$$
\begin{aligned}
&= sup\{|f^2(u) + 3f(u) - g^2(v) - 3g(v)|\ u, v \in U\} \\
&= 12 \geq \frac{6}{5} \times 4 \\
&= \alpha\ sup\{|f(u) - g(v)|\ u, v \in U\} \\
&= \alpha\ d^*(f, g).
\end{aligned}
$$

Case-IV: *If v is a rational number and u is not. Then*

$$
\begin{aligned}
d^*(\mathfrak{P}f, \mathfrak{P}g) &= sup\{d(\mathfrak{P}f(u), \mathfrak{P}g(v))|\ u, v \in U\} \\
&= sup\{|\mathfrak{P}f(u) - \mathfrak{P}g(v)|\ u, v \in U\} \\
&= sup\{|f^2(u) + 3f(u) - g^2(v) - 3g(v)|\ u, v \in U\} \\
&= 4 \geq \frac{6}{5} \times 1 \\
&= \alpha\ sup\{|f(u) - g(v)|\ u, v \in U\} \\
&= \alpha\ d^*(f, g).
\end{aligned}
$$

Thus, all the conditions of Corollary 5.2.5 are satisfied. So, there exists a unique fixed function. In this example, f is the unique fixed function.

Theorem 5.2.7 *Let $(U, \hat{d})$ be a complete metric space with metric space equipped with distance metric $\hat{d}$ and $(\mathfrak{F}, d^*)$ is a metric space where $\mathfrak{F}$ is the family of all bounded functions defined on U. Also, let $\mathfrak{P} : \mathfrak{F} \to \mathfrak{F}$ be a continuous surjective mapping. Assume that there exists a constant $\alpha > 1$ such that for all $f, g \in \mathfrak{F}$ (where $f \nsim g$),*

$$d^*(\mathfrak{P}f, \mathfrak{P}g) \geq \alpha\ M(f, g);$$

where

$$M(f, g) = max\{d^*(f, g), d^*(f, \mathfrak{P}f), d^*(g, \mathfrak{P}g)\}.$$

Then $\mathfrak{P}$ has a fixed function in $\mathfrak{F}$.

Proof Following Theorem 5.2.3, construct a sequence $\{f_k\}_{k \in \mathbb{N}}$ such that $f_{k-1} = \mathfrak{P}f_k \ \forall \ k \geq 1$ where $f_{k-1} \neq f_k$ for all k. By given condition,

$$d^*(f_{k-1}, f_k) = d^*(\mathfrak{P}f_k, \mathfrak{P}f_{k+1}) \geq \alpha\ M(f_k, f_{k+1}); \tag{5.2.2}$$

where

$$
\begin{aligned}
M(f_k, f_{k+1}) &= max\{d^*(f_k, f_{k+1}), d^*(f_k, \mathfrak{P}f_k), d^*(f_{k+1}, \mathfrak{P}f_{k+1})\} \\
&= max\{d^*(f_k, f_{k+1}), d^*(f_k, f_{k-1})\}.
\end{aligned}
$$

If $M(f_k, f_{k+1}) = d^*(f_k, f_{k-1})$, then by (5.2.2)

$$d^*(f_{k-1}, f_k) \geq \alpha\ d^*(f_k, f_{k-1}) \ \ where \ \ \alpha > 1;$$

which implies that $d^*(f_k, f_{k-1}) = 0$ *i.e.* $f_k = f_{k-1}$ which is a contradiction. If $M(f_k, f_{k+1}) = d^*(f_k, f_{k+1})$, then we have

$$d^*(f_{k-1}, f_k) \geq \alpha\ d^*(f_k, f_{k+1});$$

and thus by Lemma 5.2.2, $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{F}$.
Since $(\mathfrak{F}, d^*)$ is complete being a collection of bounded functions defined on a complete metric space $(U, \hat{d})$, the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges in $\mathfrak{F}$. Accordingly, there must exist some $f \in \mathfrak{F}$ such that $\lim\limits_{k\to\infty} d^*(f_k, f) = 0$.
As $\mathfrak{P}$ is continuous, we have

$$\begin{aligned} \lim_{k\to\infty} d^*(\mathfrak{P}f_k, \mathfrak{P}f) &= 0 \\ \Rightarrow \lim_{k\to\infty} d^*(f_{k-1}, \mathfrak{P}f) &= 0 \\ \Rightarrow d^*(f, \mathfrak{P}f) &= 0; \end{aligned}$$

which shows that f is a fixed function of $\mathfrak{P}$.

5.3 Common fixed function theorems for various expansive mappings

Theorem 5.3.1 *Let $(U, \hat{d})$ be a complete metric space equipped with distance metric $\hat{d}$ and $(\mathfrak{F}, d^*)$ is a metric space where $\mathfrak{F}$ is the family of all bounded functions defined on U and d^* where $d^*(f, g) = sup\{\hat{d}(f(u), g(v))|\ u, v \in U\}$. Also, let $\mathfrak{P}_1 : \mathfrak{F} \to \mathfrak{F}$ and $\mathfrak{P}_2 : \mathfrak{F} \to \mathfrak{F}$ be two weakly compatible self maps such that $\mathfrak{P}_2(\mathfrak{F}) \subseteq \mathfrak{P}_1(\mathfrak{F})$. Assume that there exists a constant $\alpha > 1$ such that for all $f, g \in \mathfrak{F}$ (where $f \nsim g$),*

$$d^*(\mathfrak{P}_1 f, \mathfrak{P}_1 g) \geq \alpha\ d^*(\mathfrak{P}_2 f, \mathfrak{P}_2 g). \tag{5.3.1}$$

If one of the subspaces $\mathfrak{P}_1(\mathfrak{F})$ or $\mathfrak{P}_2(\mathfrak{F})$ is complete, then $\mathfrak{P}_1$ and $\mathfrak{P}_2$ have a unique common fixed function in $\mathfrak{F}$.

Proof Let $f_0 \in \mathfrak{F}$ be any arbitrary function. Since $\mathfrak{P}_2(\mathfrak{F}) \subseteq \mathfrak{P}_1(\mathfrak{F})$, therefore there exists some $f_1 \in \mathfrak{F}$ such that $\mathfrak{P}_1 f_1 = \mathfrak{P}_2 f_0$. In general, we have some $f_{n+1} \in \mathfrak{F}$ such that

$$g_n = \mathfrak{P}_1 f_{n+1} = \mathfrak{P}_2 f_n.$$

By (5.3.1),

$$\begin{aligned} d^*(\mathfrak{P}_2 f_n, \mathfrak{P}_2 f_{n+1}) &\leq \frac{1}{\alpha}\ d^*(\mathfrak{P}_1 f_n, \mathfrak{P}_1 f_{n+1}) \\ &= \frac{1}{\alpha}\ d^*(\mathfrak{P}_2 f_{n-1}, \mathfrak{P}_2 f_n). \end{aligned}$$

By Lemma 5.2.2, above inequality shows that $\{\mathfrak{P}_2 f_n\}$ is a Cauchy sequence in $\mathfrak{F}$. Since $(\mathfrak{F}, d^*)$ is complete being the collection of bounded functions defined on complete metric space $(U, \hat{d})$, therefore, $\{\mathfrak{P}_2 f_n\}$ is convergent in $\mathfrak{F}$. Let the limit of this sequence be f^* *i.e.*

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \mathfrak{P}_1 f_n = \lim_{n\to\infty} \mathfrak{P}_2 f_n = f^*.$$

As $\mathfrak{P}_1(\mathfrak{F})$ is a complete subspace of $\mathfrak{F}$, therefore, there exists a function $\hat{f} \in \mathfrak{F}$ such that $\mathfrak{P}_1 \hat{f} = f^*$. By (5.3.1), we obtain

$$d^*(\mathfrak{P}_2 \hat{f}, \mathfrak{P}_2 f_n) \leq \frac{1}{\alpha}\ d^*(\mathfrak{P}_1 \hat{f}, \mathfrak{P}_1 f_n).$$

Taking limit $n \to \infty$,

$$\begin{aligned} d^*(\mathfrak{P}_2 \hat{f}, \mathfrak{P}_2 f_n f^*) &\leq \frac{1}{\alpha}\ d^*(\mathfrak{P}_1 \hat{f}, f^*) \\ &= \frac{1}{\alpha}\ d^*(f^*, f^*). \end{aligned}$$

This implies that $\mathfrak{P}_2 \hat{f} = f^*$. Thus, $\mathfrak{P}_1 \hat{f} = \mathfrak{P}_2 \hat{f} = f^*$. As $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are weakly compatible mappings, therefore, by definition,

$$\begin{aligned} \mathfrak{P}_1 \mathfrak{P}_2 \hat{f} &= \mathfrak{P}_2 \mathfrak{P}_1 \hat{f} \\ \mathfrak{P}_1 f^* &= \mathfrak{P}_2 f^*. \end{aligned} \tag{5.3.2}$$

Now, it remains to show that f^* is a fixed point of $\mathfrak{P}_1$ and $\mathfrak{P}_2$. By (5.3.1),

$$d^*(\mathfrak{P}_1 f^*, \mathfrak{P}_1 f_n) \geq \alpha\ d^*(\mathfrak{P}_2 f^*, \mathfrak{P}_2 f_n).$$

Taking limit $n \to \infty$,

$$d^*(\mathfrak{P}_1 f^*, f^*) \geq \alpha\ d^*(\mathfrak{P}_2 f^*, f^*).$$

By (5.3.2), we have, $\mathfrak{P}_1 f^* = f^*$ which shows that f^* is a fixed function of $\mathfrak{P}_1$ and $\mathfrak{P}_2$. For uniqueness, let g^* be another fixed function of $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Then,

$$\begin{aligned} d^*(f^*, g^*) = d^*(\mathfrak{P}_1 f^*, \mathfrak{P}_1 g^*) &\geq \alpha\ d^*(\mathfrak{P}_2 f^*, \mathfrak{P}_2 g^*) \\ &= \alpha\ d^*(f^*, g^*); \end{aligned}$$

which is a contradiction. This completes the proof.

Example 5.3.1 *Let $U = [0, 2]$ be any set equipped with usual metric $\hat{d}$. Let $\mathfrak{F}$ be the family of bounded functions defined on U and d^* be the metric defined on $\mathfrak{F}$ as*

$$d^*(f, g) = sup\{\hat{d}(f(u), g(v))|\ u, v \in U\}.$$

Let the mappings $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be defined as $\mathfrak{P}_1 f = f$ and $\mathfrak{P}_2 f = \frac{f}{2}$ for all $f \in \mathfrak{F}$. Note that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are weakly compatible for all $f \in \mathfrak{F}$ and $\mathfrak{P}_2(\mathfrak{F}) \subseteq \mathfrak{P}_1(\mathfrak{F})$. It remains to show that the inequality (5.3.1) in Theorem 5.3.1 is satisfied.

$$\begin{aligned} d^*(\mathfrak{P}_1 f, \mathfrak{P}_1 g) &= d^*(f, g) \\ &= sup\{|f(u) - g(v)|\ |\ u, v \in U\} \\ &\geq \alpha\ sup\left\{\frac{|f(u) - g(v)|}{2}\ |\ \ u, v \in U\right\} \\ &= \alpha\ d^*(\mathfrak{P}_2 f, \mathfrak{P}_2 g) \quad where \quad \alpha = 2 > 1. \end{aligned}$$

Thus, all the conditions required for Theorem 5.3.1 are fulfilled, therefore, there exists a unique fixed function. In this example, Null function is the unique fixed function.

5.4 Application to Tomotherapy

In this application, the treatment plan for more than a patient at a time, is presented through our results in a more effective way.

Let us consider the case of two lung cancer patients (with different tumor levels) getting Tomotherapy. Let U denote the set of all possible intensity values to be given on particular days and in particular sessions. Each patient is getting therapy in two sessions per day. Days and sessions are denoted by D_i and S_j $(i, j = 1, 2.)$ respectively.

$$U = \begin{cases} (1, D_1S_1), (\frac{1}{2}, D_1S_2), (1, D_2S_1), (\frac{1}{2}, D_2S_2) & Patient - I, \\ (2, D_1S_1), (1, D_1S_2), (2, D_2S_1), (1, D_2S_2) & Patient - II. \end{cases}$$

Note that U is complete being a closed and bounded subset of $\mathbb{R}^2$. Let $\mathfrak{F} = \{f, g\}$ be the family of dose functions and each function represents different doses (to tumor locations) of different tumor patients during Tomotherapy.

$$f(u) = \begin{cases} u & Patient - I, \\ \frac{u}{2} & Patient - II. \end{cases} \quad and \quad g(u) = \begin{cases} 2u & Patient - I, \\ \frac{3}{2}u & Patient - II. \end{cases}$$

Here, $\mathfrak{F}$ is the family of bounded functions. Let $\mathfrak{P} : \mathfrak{F} \to \mathfrak{F}$ be the mapping defined as $\mathfrak{P}f = f^2 - \frac{f}{2} + \frac{1}{2}$ for all $f \in \mathfrak{F}$. Now, it remains to prove that $\mathfrak{P}$ is an expansion mapping. For $u, v \in U$, we have the following cases:

For Patient-I

Case I- When $u = v = 1$. Then

$$|\mathfrak{P}f - \mathfrak{P}g| = \frac{5}{2} \ \ and \ \ |f - g| = 1.$$

Case II- When $u = 1, \quad v = \frac{1}{2}$. Then

$$|\mathfrak{P}f - \mathfrak{P}g| = 0 \ \ and \ \ |f - g| = 0.$$

Case III- When $u = \frac{1}{2}, \quad v = 1$. Then

$$|\mathfrak{P}f - \mathfrak{P}g| = 3 \ \ and \ \ |f - g| = \frac{3}{2}.$$

Case IV- When $u = v = \frac{1}{2}$. Then

$$|\mathfrak{P}f - \mathfrak{P}g| = \frac{1}{2} \ \ and \ \ |f - g| = \frac{1}{2}.$$

For Patient-II

Case I- When $u = v = 2$. Then

$$|\mathfrak{P}f - \mathfrak{P}g| = 7 \ \ and \ \ |f - g| = 2.$$

Case II- When $u = 2, \quad v = 1$. Then

$$|\mathfrak{P}f - \mathfrak{P}g| = 1 \ \ and \ \ |f - g| = \frac{1}{2}.$$

Case III- When $u = 1, \quad v = 2$. Then

$$|\mathfrak{P}f - \mathfrak{P}g| = \frac{15}{2} \ \ and \ \ |f - g| = \frac{5}{2}.$$

Case IV- When $u = v = 1$. Then

$$|\mathfrak{P}f - \mathfrak{P}g| = \frac{3}{2} \ \ and \ \ |f - g| = 1.$$

From all above cases, for Patient-I

$$\begin{aligned} d^*(\mathfrak{P}f, \mathfrak{P}g) &= sup\{|\mathfrak{P}f - \mathfrak{P}g| \ u, v \in U\} \\ &= 3 \\ &\geq 2 \times \frac{3}{2} \\ &= \alpha \ d^*(f, g). \end{aligned}$$

and for Patient-II

$$\begin{aligned} d^*(\mathfrak{P}f, \mathfrak{P}g) &= sup\{|\mathfrak{P}f - \mathfrak{P}g| \ u, v \in U\} \\ &= \frac{15}{2} \\ &\geq 2 \times \frac{5}{2} \\ &= \alpha \ d^*(f, g); \end{aligned}$$

where $\alpha = 2 > 1$.

Thus, all the conditions required for Corollary 5.2.5 are fulfilled. Therefore, there exists a unique fixed function f of $\mathfrak{P}$ that yields suitable doses for two patients at the same time.

Chapter 6

Common Fixed Point Theorems in Ordered Partial Metric Spaces

6.1 Introduction and Preliminaries

The study of common fixed points was initiated by Jungck [79] in 1986 and this concept attracted many researchers to prove the existence of fixed points by using various metrical contractions. On the other hand, the notion of partial metric spaces was presented by Matthews [104] which has been considered as one of the most interesting and outstanding generalizations of metric spaces. Many authors generalized this notion in different ways (See [21], [72, 73], [81], [112], [120]).
In 2010, Hong [69] defined the concept of approximative values to prove the existence of common fixed points for multivalued operators in the framework of ordered metric spaces. After that, Erduran [55] extended this concept and studied some fixed point results for multivalued mappings in partial metric spaces. In 2014, Ansari [13] introduced $\mathcal{C}$-class functions defined on $\mathbb{R}$.
In this chapter, the notion of $\mathcal{F}$-generalized contractive type mappings is introduced and some common fixed point theorems for multivalued mappings in ordered partial metric spaces by using $\mathcal{C}$-class functions and $\mathcal{F}$-generalized contractive type mappings are obtained through this new class of mappings.
Matthews [104] defined the partial metric space as follows:

Definition 6.1.1 *[104] Let U be a non empty set. A function $\hat{p} : U \times U \to \mathbb{R}^+$ is said to be a partial metric on U if the following postulates hold true for all $u, v, w \in U$:*

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($\hat{p}1$) $u = v \; if \; and \; only \; if \; \hat{p}(u,u) = \hat{p}(v,v) = \hat{p}(u,v)$;

($\hat{p}2$) $\hat{p}(u,u) \leq \hat{p}(u,v) \quad (small \; self - distance \; axiom)$;

($\hat{p}3$) $\hat{p}(u,v) = \hat{p}(v,u) \quad (symmetry)$;

($\hat{p}4$) $\hat{p}(u,w) \leq \hat{p}(u,v) + \hat{p}(v,w) - \hat{p}(v,v) \quad (modified \; triangle \; inequality)$;

for any $u, v, w \in U$. The pair $(U, \hat{p})$ is then called a partial metric space (in short PMS).

Each partial metric $\hat{p}$ on U generates a T_0 topology $\tau_{\hat{p}}$ on U which has a base, the family of open $\hat{p}$-balls $\{B_{\hat{p}}(u,\epsilon), u \in U, \epsilon > 0\}$, where

$$B_{\hat{p}}(u,\epsilon) = \{v \in U : \hat{p}(u,v) < \hat{p}(u,u) + \epsilon\};$$

for all $u \in U$ and $\epsilon > 0$.

If $\hat{p}$ is a partial metric defined on U, then the mapping $d_{\hat{p}} : U \times U \to \mathbb{R}^+$ given by

$$d_{\hat{p}}(u,v) = 2\hat{p}(u,v) - \hat{p}(u,u) - \hat{p}(v,v);$$

is a metric on U.

Following are the results due to Matthews [104]:

Definition 6.1.2 *[104] For a partial metric space $(U, \hat{p})$, a sequence $\{u_n\}$ in U is said to be*

1. *convergent if there exists a point $u \in U$ such that $\hat{p}(u,u) = \lim_{n\to\infty} \hat{p}(u_n, u)$,*
2. *a Cauchy sequence if the limit $\lim_{m,n\to\infty} \hat{p}(u_n, u_m)$ exists (and is finite).*

Definition 6.1.3 *[104] A partial metric space $(U, \hat{p})$ is said to be complete if every Cauchy sequence $\{u_n\}$ in U converges* w.r.t. *$\tau_{\hat{p}}$ to a point $u \in U$ such that $\hat{p}(u,u) = \lim_{n,m\to\infty} \hat{p}(u_n, u_m)$.*

Lemma 6.1.4 *[104] Let $(U, \hat{p})$ be a partial metric space. Then*

1. *$\{u_n\}$ is said to be a Cauchy sequence in $(U, \hat{p})$ iff it is a Cauchy sequence in the metric space $(U, d_{\hat{p}})$,*
2. *$(U, \hat{p})$ is complete iff the metric space $(u, d_{\hat{p}})$ is complete. Also, $\lim_{n\to\infty} d_{\hat{p}}(u_n, u) = 0$ iff $\hat{p}(u,u) = \lim_{n\to\infty} \hat{p}(u_n, u) = \lim_{m,n\to\infty} \hat{p}(u_n, u_m)$.*

Chen and Zhu [36] proved the following result:

Lemma 6.1.5 *[36] Let $(U,\hat{p})$ be a partial metric space and let $\{u_n\}$ be a sequence in U such that*

$$\lim_{n\to\infty} \hat{p}(u_n, u_{n+1}) = 0.$$

If the sequence $\{u_{2n}\}$ is not a Cauchy sequence in $(U,\hat{p})$, then there exist $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n(k) > m(k) > k$ such that the following four sequences

$$\hat{p}(u_{2m(k)}, u_{2n(k)+1}), \hat{p}(u_{2m(k)}, u_{2n(k)}), \hat{p}(u_{2m(k)-1}, u_{2n(k)+1}), \hat{p}(u_{2m(k)-1}, u_{2n(k)});$$

tend to $\epsilon > 0$ when $k \to \infty$.

Saluja *et al.* [139] prove the result given below for mappings involving rational type expressions in partial metric spaces:

Lemma 6.1.6 *[139] If the sequence $\{u_n\}$ with $\lim_{n\to\infty} d_{\hat{p}}(u_{n+1}, u_n) = 0$ is not a Cauchy sequence in $(U,\hat{p})$, then for each $\epsilon > 0$, there exist two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n(k) > m(k) > k$ such that the following four sequences*

$$\hat{p}(u_{m(k)}, u_{n(k)+1}), \hat{p}(u_{m(k)}, u_{n(k)}), \hat{p}(u_{m(k)-1}, u_{n(k)+1}), \hat{p}(u_{m(k)-1}, u_{n(k)});$$

tend to $\epsilon > 0$ when $k \to \infty$.

Let $CB^{\hat{p}}(U)$ be a family of all non-empty, closed and bounded subsets of the partial metric space $(U,\hat{p})$. Note that the notion of a closed set is obvious as $\tau_{\hat{p}}$ is the topology induced by $\hat{p}$ and boundedness in its standard form is given as follows: A_1 is a bounded subset in $(U,\hat{p})$ if there exists $M \geq 0$ and $u_0 \in U$ such that for each $a_1 \in A_1$, we have $a_1 \in B_{\hat{p}}(u_0, M)$ *i.e.* $\hat{p}(u_0, a_1) < \hat{p}(a_1, a_1) + M$.
For all $A_1, A_2 \in CB^{\hat{p}}(U)$ and $u \in U$,

$$\hat{p}(u, A_1) = \inf\{\hat{p}(u, v) : v \in A_1\},$$
$$\delta_{\hat{p}}(A_1, A_2) = \sup\{\hat{p}(a_1, A_2) : a_1 \in A_1\},$$
$$\delta_{\hat{p}}(A_2, A_1) = \sup\{\hat{p}(A_1, a_2) : a_2 \in A_2\},$$

and

$$H_p(A_1, A_2) = \max\{\delta_{\hat{p}}(A_1, A_2), \delta_{\hat{p}}(A_2, A_1)\}.$$

Note that $\hat{p}(u, A_1) = 0$ implies $d_{\hat{p}}(u, A_1) = 0$ where

$$d_{\hat{p}}(u, A_1) = \inf\{d_{\hat{p}}(u, a_1) : a_1 \in A_1\}.$$

The result of Altun *et al.* [12] for generalized contractions in partial metric spaces is stated below:

Theorem 6.1.7 *[12] Let $(U, \hat{p})$ be a partial metric space and let A_1 be any non-empty set in $(U, \hat{p})$, then $a_1 \in \overline{A_1}$ iff $\hat{p}(a_1, A_1) = \hat{p}(a_1, a_1)$ where $\overline{A_1}$ denotes the closure of A_1* w.r.t. *the partial metric $\hat{p}$. we say that A_1 is closed in $(U, \hat{p})$* iff *$A_1 = \overline{A_1}$.*

Aydi *et al.* [17] proved the result below for cyclic contractions in partial hausdorff metric spaces:

Proposition 6.1.8 *[17] Let $(U, \hat{p})$ be a partial metric space. For all $A_1, A_2, A_3 \in CB^{\hat{p}}(U)$, we have*

(h_1) $H_p(A_1, A_1) \leq H_p(A_1, A_2)$,

(h_2) $H_p(A_1, A_2) = H_p(A_2, A_1)$,

(h_3) $H_p(A_1, A_2) \leq H_p(A_1, A_3) + H_p(A_3, A_2) - \inf_{a_3 \in A_3} \hat{p}(a_3, a_3)$,

(h_4) $H_p(A_1, A_2) = 0 \Rightarrow A_1 = A_2$.

The mapping $H_p : CB^{\hat{p}}(U) \times CB^{\hat{p}}(U) \to [0, +\infty)$ is called the Partial Hausdorff metric induced by $\hat{p}$. Every Hausdorff metric is a Partial Hausdorff metric but the converse need not be true (*Example 2.6*, [17]).
Ran and Reurings [126] defined partially ordered metric spaces and comparability as follows:

Definition 6.1.9 *[126] For a non-empty set U, the space $(U, \hat{p}, \preceq)$ is called an ordered partial metric space if $(U, \hat{p})$ is a partial metric space and $(U, \preceq)$ is a partially ordered set.*
Let $(U, \preceq)$ be a partially ordered set. Then $u, v \in U$ are called comparable if $u \preceq v$ or $v \preceq u$.

Hong [69] defined partial order on members of ordered metric spaces in the following way:

Definition 6.1.10 *[69] Let A_1 and A_2 be any two non-empty subsets of an ordered set $(U, \preceq)$. The relation $\preceq_2$ between A_1 and A_2 is defined as follows:*

$$A_1 \preceq_2 A_2 \text{ if } a_1 \preceq a_2 \text{ for each } a_1 \in A_1 \text{ and } a_2 \in A_2.$$

Dhage *et al.* [51] defined the concept of weak compatibility for condensed mappings in 2003.

Definition 6.1.11 *[51] Let $(U, \preceq)$ be a partially ordered set. Two maps $\mathcal{D}, \mathcal{D}_2 : U \to 2^U$ are said to be weakly isotone increasing if for any $u \in U$, we have $\mathcal{D}u \preceq_2 \mathcal{D}_2 v$ for all $v \in \mathcal{D}u$ and $\mathcal{D}_2 u \preceq_2 \mathcal{D}v$ for all $v \in \mathcal{D}_2 u$.*
In particular, the mappings $\mathcal{D}, \mathcal{D}_2 : U \to U$ are called weakly isotone increasing if $\mathcal{D}u \preceq \mathcal{D}_2\mathcal{D}u$ and $\mathcal{D}_2 u \preceq \mathcal{D}\mathcal{D}_2 u$ hold for each $u \in U$.

Recently, Erduran [55] introduced the notion of g-approximative multivalued mapping in ordered partial metric spaces.

Definition 6.1.12 *[55] An ordered partial metric space is said to have a sequential limit comparison property if for every nonincreasing sequence (or nondecreasing sequence) $\{u_n\}$ in U, we have $u_n \to u$ implies $u \leq u_n$ (or $u_n \leq u$) respectively.*

Definition 6.1.13 *[55] A subset A of set U is said to be approximative if the set $P_A(u) = \{v \in A : \hat{p}(u, v) = \hat{p}(A, u)\}$ $\forall$ $u \in U$ is non-empty.*

A set-valued mapping $\mathcal{D}$ is said to have approximate values in U if $\mathcal{D}u$ is approximative for each $u \in U$.

Nazari and Mohitazar [115] defined the following class to prove some fixed point results in ordered partial metric spaces:

Definition 6.1.14 *[115] Denote by Υ the set of all functions $\xi : [0, +\infty)^4 \to [0, +\infty)$ with the following properties:*

1. *ξ is nondecreasing in third and fourth variables,*
2. *$\xi(s_1, s_2, s_3, s_4) = 0 \Leftrightarrow s_1 s_2 s_3 s_4 = 0$,*
3. *ξ is continuous.*

The following functions belong to Υ:

1. $\xi(s_1, s_2, s_3, s_4) = L \min\{s_1, s_2, s_3, s_4\}$ where $L > 0$,
2. $\xi(s_1, s_2, s_3, s_4) = s_1 s_2 s_3 s_4$,
3. $\xi(s_1, s_2, s_3, s_4) = \ln(1 + s_1 s_2 s_3 s_4)$,
4. $\xi(s_1, s_2, s_3, s_4) = \exp(s_1 s_2 s_3 s_4) - 1$.

For two mappings $\mathcal{D}, \mathcal{D}_2 : U \to 2^U$, we define

$$M(u,v) = \max\{\hat{p}(u,v), \hat{p}(u,\mathcal{D}u), \hat{p}(v,\mathcal{D}_2v), \frac{1}{2}[\hat{p}(v,\mathcal{D}u) + \hat{p}(u,\mathcal{D}_2v)]\}.$$

In 2014, the concept of C-class functions was introduced by Ansari [13]. By using this concept, many fixed point theorems in the literature can be generalized.

Definition 6.1.15 *[13] A mapping $\mathcal{F} : [0,\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following axioms:*

1. $\mathcal{F}(t_1, t_2) \leq t_1$,
2. $\mathcal{F}(t_1, t_2) = t_1$ *implies that either* $t_1 = 0$ *or* $t_2 = 0$ *for all* $t_1, t_2 \in [0,\infty)$.

We denote C-class functions by $\mathcal{C}$.

Example 6.1.1 *[13] Following are some members of class $\mathcal{C}$ for all $t_1, t_2 \in [0,\infty)$:*

1. $\mathcal{F}(t_1,t_2) = t_1 - t_2$, $\mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_2 = 0$;
2. $\mathcal{F}(t_1,t_2) = mt_1$, $0<m<1$, $\mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_1 = 0$;
3. $\mathcal{F}(t_1,t_2) = \frac{t_1}{(1+t_2)^r}$, $r \in (0,\infty)$, $\mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_1 = 0$ *or* $t_2 = 0$;
4. $\mathcal{F}(t_1,t_2) = \log(t_2 + a^{t_1})/(1+t_2)$, $a > 1$, $\mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_1 = 0$ *or* $t_2 = 0$;
5. $\mathcal{F}(t_1,t_2) = \ln(1 + a^{t_1})/2$, $a > e$, $\mathcal{F}(t_1,1) = t_1 \Rightarrow t_1 = 0$;
6. $\mathcal{F}(t_1,t_2) = (t_1 + l)^{(1/(1+t_2)^r)} - l$, $l > 1, r \in (0,\infty)$, $\mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_2 = 0$;
7. $\mathcal{F}(t_1,t_2) = t_1 \log_{t_2+a} a$, $a > 1$, $\mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_1 = 0$ *or* $t_2 = 0$;
8. $\mathcal{F}(t_1,t_2) = t_1 - (\frac{1+t_1}{2+t_1})(\frac{t_2}{1+t_2})$, $\mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_2 = 0$;
9. $\mathcal{F}(t_1,t_2) = t_1\beta(t_1)$ *where* $\beta : [0,\infty) \to [0,1)$ *is continuous,* $\mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_1 = 0$;

10. $\mathcal{F}(t_1,t_2) = t_1 - \frac{t_2}{k+t_2}, \mathcal{F}(t_1,t_2) = t_1 \Rightarrow t_2 = 0$*;*

Let Ψ be the family of continuous and monotone nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(t) = 0$ *iff* $t = 0$ and Φ_1 the family of continuous functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) = 0$ *iff* $t = 0$.
Let Φ_u be the family of continuous functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) \geq 0$. Note that $\Phi_1 \subset \Phi_u$.

In the next section, the notion of $\mathcal{F}$-generalized (ψ,φ,ξ)-contractive type mappings is defined and some common fixed point theorems for this new class have been proved.

6.2 Fixed point theorems for generalized (ψ,φ,ξ)-contractive type mappings

Definition 6.2.1 *Let U be an ordered partial metric space. Two mappings $\mathcal{D}, \mathcal{D}_2 : U \to 2^U$ are said to be $\mathcal{F}$-generalized (ψ,φ,ξ)-contractive type mappings if,*

$$\begin{aligned}\psi(H_{\hat{p}}(\mathcal{D}u, \mathcal{D}_2 v)) \;\leq\; & \mathcal{F}(\psi(M(u,v)), \phi(\psi(M(u,v))) + \xi(\hat{p}(u,\mathcal{D}u), \hat{p}(v,\mathcal{D}_2 v), \\ & \hat{p}(v,\mathcal{D}u) - \hat{p}(v,v), \hat{p}(u,\mathcal{D}_2 v) - \hat{p}(u,u)),\end{aligned}$$

for all $u, v \in U$ with u and v comparable and $\psi \in \Psi, \phi \in \Phi_u, \xi \in \Upsilon$ and $\mathcal{F} \in \mathcal{C}$.

Definition 6.2.2 ***Limit comparison property****: A non-empty set U is said to hold limit comparison property if for a sequence $\{u_n\} \in U$ such that $u_n \to u$, u_n is comparable to u for all $n \in \mathbb{N}$.*

Theorem 6.2.3 *Let U be a complete ordered partial metric space with the limit comparison property. Assume that $\mathcal{D}_1, \mathcal{D}_2 : U \to 2^U$ are weakly isotone increasing $\mathcal{F}$-generalized (ψ,φ,ξ)-contractive type mappings and satisfy approximative property. Suppose that there exists $u_0 \in U$ such that $\{u_0\} \preceq_2 \mathcal{D}_1 u_0$. Then $\mathcal{D}_1$, $\mathcal{D}_2$ have a common fixed point $u \in U$ such that $\hat{p}(u,u) = 0$.*

Proof Firstly, it is proved that if u is a fixed point of $\mathcal{D}_1$ such that $\hat{p}(u,u) = 0$, then it is a common fixed point of $\mathcal{D}_1$ and $\mathcal{D}_2$. By using given contractive condition and property 2) of ξ,

$$\psi(\hat{p}(u,\mathcal{D}_2 u)) \leq \psi(H_{\hat{p}}(\mathcal{D}_1 u, \mathcal{D}_2 u))$$

$$
\begin{aligned}
\psi(\hat{p}(u,\mathcal{D}_2u)) &\leq \mathcal{F}(\psi(M(u,u),\varphi(\psi(M(u,u)))) \\
&\quad + \xi(\hat{p}(u,\mathcal{D}_1u),\hat{p}(u,\mathcal{D}_2u),\hat{p}(u,\mathcal{D}_1u)-\hat{p}(u,u),\hat{p}(u,\mathcal{D}_2u)-\hat{p}(u,u)) \\
&= \mathcal{F}(\psi(M(u,u),\varphi(\psi(M(u,u)))) + \xi(0,\hat{p}(u,\mathcal{D}_2u),0,\hat{p}(u,\mathcal{D}_2u)-0) \\
&= \mathcal{F}(\psi(M(u,u),\varphi(\psi(M(u,u))));
\end{aligned} \tag{6.2.1}
$$

where

$$
\begin{aligned}
M(u,u) &= Max\left\{\hat{p}(u,u),\hat{p}(u,\mathcal{D}_1u),\hat{p}(u,\mathcal{D}_2u),\frac{\hat{p}(u,\mathcal{D}_2u)+\hat{p}(u,\mathcal{D}_1u)}{2}\right\} \\
&\leq Max\left\{\hat{p}(u,u),\hat{p}(u,u),\hat{p}(u,\mathcal{D}_2u),\frac{\hat{p}(u,\mathcal{D}_2u)+\hat{p}(u,u)}{2}\right\}.
\end{aligned}
$$

Thus by (6.2.1),

$$
\begin{aligned}
\psi(\hat{p}(u,\mathcal{D}_2u)) &\leq \mathcal{F}(\psi(\hat{p}(u,\mathcal{D}_2u),\varphi(\psi(\hat{p}(\mathcal{D}_2u,u)))) \\
&= \mathcal{F}(\psi(\hat{p}(u,\mathcal{D}_2u)),\varphi(\psi(\hat{p}(u,\mathcal{D}_2u)))).
\end{aligned}
$$

This implies that, $\psi(\hat{p}(u,\mathcal{D}_2u)) = 0$ or $\varphi(\psi(\hat{p}(u,\mathcal{D}_2u))) = 0$, therefore $\hat{p}(\mathcal{D}_2u,u) = 0$. Since $\mathcal{D}_2u$ satisfy approximative property, therefore there exist $v \in P_{\mathcal{D}_2}(u)$ such that $\hat{p}(v,u) = \hat{p}(u,\mathcal{D}_2u) = 0$ *i.e*, $v = u$. Thus $u \in \mathcal{D}_2u$.
Let $u_0 \in U$, if $u_0 \in \mathcal{D}_1u_0$, the proof is complete. Otherwise, from the fact that $\mathcal{D}_1u_0$ has approximative property, it follows that there exists $u_1 \in \mathcal{D}_1u_0$, with $u_1 \neq u_0$ such that

$$
\hat{p}(u_0,u_1) = \inf_{u\in\mathcal{D}_1u_0} \hat{p}(u,u_0) = \hat{p}(\mathcal{D}_1u_0,u_0).
$$

Again if $u_1 \in \mathcal{D}_2u_1$, the proof is complete. Otherwise, since $\mathcal{D}_2u_1$ has approximative property, it follows there exist $u_2 \in \mathcal{D}_2u_1$ with $u_2 \neq u_1$ such that

$$
\hat{p}(u_1,u_2) = \inf_{u\in\mathcal{D}_2u_1} \hat{p}(u,u_1) = \hat{p}(\mathcal{D}_2u_1,u_1).
$$

By repeating this process, we can find a sequence $\{u_n\}$ in U, such that $u_{2n+1} \in \mathcal{D}_1u_{2n}$ and

$$
\hat{p}(u_{2n+1},u_{2n}) = \hat{p}(\mathcal{D}_1u_{2n},u_{2n});
$$

and $u_{2n+2} \in \mathcal{D}_2u_{2n+1}$, with

$$
\hat{p}(u_{2n+2},u_{2n+1}) = \hat{p}(\mathcal{D}_2u_{2n+1},u_{2n+1}).
$$

On the other hand

$$\hat{p}(\mathcal{D}_1 u_{2n}, u_{2n}) \leq \underset{u\in\mathcal{D}_2 u_{2n-1}}{sup} \hat{p}(\mathcal{D}_1 u_{2n}, u)$$
$$\leq H_{\hat{p}}(\mathcal{D}_1 u_{2n}, \mathcal{D}_2 u_{2n-1}).$$

Therefore

$$\hat{p}(u_{2n+1}, u_{2n}) \leq H_{\hat{p}}(\mathcal{D}_1 u_{2n}, \mathcal{D}_2 u_{2n-1}); \tag{6.2.2}$$

and similarly

$$\hat{p}(u_{2n+2}, u_{2n+1}) \leq H_{\hat{p}}(\mathcal{D}_2 u_{2n+1}, \mathcal{D}_1 u_{2n}). \tag{6.2.3}$$

Since $u_0 \preceq_2 \mathcal{D}_1 u_0$ and $u_1 \in \mathcal{D}_1 u_0 \Rightarrow u_0 \preceq_2 u_1$. Also, since $\mathcal{D}_1$ and $\mathcal{D}_2$ are isotone increasing, therefore, $\mathcal{D}_1 u_0 \preceq_2 \mathcal{D}_2 v$ for all $v \in \mathcal{D}_1 u_0$ and thus $\mathcal{D}_1 u_0 \preceq_2 \mathcal{D}_2 u_1$. In particular, $u_1 \preceq_2 u_2$. Continuing this process, we obtain

$$u_1 \preceq u_2 \preceq ... \preceq u_n \preceq u_{n+1} \preceq ....$$

Now it is required to show that $\underset{n\to\infty}{lim} \hat{p}(u_{n+1}, u_n) = 0$.
Using (6.2.2) and the fact that $\mathcal{D}_1$ and $\mathcal{D}_2$ are $\mathcal{F}$-generalized (ψ, φ, ξ)-contractive mappings, we get

$$\begin{aligned}
\psi(\hat{p}(u_{2n+1}, u_{2n})) &\leq \psi(H_{\hat{p}}(\mathcal{D}_1 u_{2n}, \mathcal{D}_2 u_{2n-1}))\\
&\leq \mathcal{F}(\psi(M(u_{2n}, u_{2n-1}), \varphi(\psi(M(u_{2n}, u_{2n-1}))))\\
&+ \xi(\hat{p}(u_{2n}, \mathcal{D}_1 u_{2n}), \hat{p}(u_{2n-1}, \mathcal{D}_2 u_{2n-1}), \hat{p}(u_{2n}, \mathcal{D}_2 u_{2n-1}) - \hat{p}(u_{2n}, u_{2n})\\
&, \hat{p}(u_{2n+1}, \mathcal{D}_1 u_{2n}) - \hat{p}(u_{2n}, u_{2n})) \qquad (6.2.4)\\
&= \mathcal{F}(\psi(M(u_{2n}, u_{2n-1}), \varphi(\psi(M(u_{2n}, u_{2n-1}))))\\
&+ \xi(\hat{p}(u_{2n+1}, u_{2n}), \hat{p}(u_{2n-1}, u_{2n}), \hat{p}(u_{2n}, u_{2n}) - \hat{p}(u_{2n}, u_{2n})\\
&, \hat{p}(u_{2n+1}, u_{2n+1}) - \hat{p}(u_{2n-1}, u_{2n-1})\\
&\leq \mathcal{F}(\psi(M(u_{2n}, u_{2n-1}), \varphi(\psi(M(u_{2n}, u_{2n-1}))));
\end{aligned}$$

where

$$\begin{aligned}
M(u_{2n}, u_{2n-1}) &= Max\left\{\hat{p}(u_{2n}, u_{2n-1}), \hat{p}(u_{2n}, \mathcal{D}_1 u_{2n}), \hat{p}(u_{2n-1}, \mathcal{D}_2 u_{2n-1}), \frac{\hat{p}(u_{2n-1}, \mathcal{D}_1 u_{2n}) + \hat{p}(u_{2n}, \mathcal{D}_2 u_{2n-1})}{2}\right\}\\
&\leq Max\left\{\hat{p}(u_{2n}, u_{2n-1}), \hat{p}(u_{2n}, u_{2n+1}), \hat{p}(u_{2n-1}, u_{2n}), \frac{\hat{p}(u_{2n-1}, u_{2n+1}) + \hat{p}(u_{2n}, u_{2n})}{2}\right\}\\
&\leq Max\left\{\hat{p}(u_{2n}, u_{2n-1}), \hat{p}(u_{2n}, u_{2n+1}), \hat{p}(u_{2n-1}, u_{2n}), \frac{\hat{p}(u_{2n-1}, u_{2n1}) + \hat{p}(u_{2n}, u_{2n+1})}{2}\right\}\\
&= Max\{\hat{p}(u_{2n-1}, u_{2n}), \hat{p}(u_{2n}, u_{2n+1})\}.
\end{aligned}$$

If $Max\{\hat{p}(u_{2n-1}, u_{2n}), \hat{p}(u_{2n}, u_{2n+1})\} = \hat{p}(u_{2n}, u_{2n+1})$, then by (6.2.4),

$$\psi(\hat{p}(u_{2n+1}, u_{2n})) \leq \mathcal{F}(\psi(\hat{p}(u_{2n}, u_{2n-1})), \varphi(\psi(\hat{p}(u_{2n+1}, u_{2n}))));$$

which implies that $\psi(\hat{p}(u_{2n+1}, u_{2n})) = 0$ or $\varphi(\psi(\hat{p}(u_{2n+1}, u_{2n}))) = 0$. Therefore $\hat{p}(u_{2n+1}, u_{2n}) = 0$ which is a contradiction.

Thus, $\hat{p}(u_{2n}, u_{2n-1}) \leq M(u_{2n}, u_{2n-1}) \leq \hat{p}(u_{2n}, u_{2n-1})$ and so $M(u_{2n}, u_{2n-1}) = \hat{p}(u_{2n}, u_{2n-1})$. Also, by using (6.2.4), we get

$$\psi(\hat{p}(u_{2n+1}, u_{2n})) \leq \mathcal{F}(\psi(\hat{p}(u_{2n}, u_{2n-1})), \varphi(\psi(\hat{p}(u_{2n}, u_{2n-1})))) \leq \psi(\hat{p}(u_{2n}, u_{2n-1})). \qquad (6.2.5)$$

Proceeding as above,

$$\psi(\hat{p}(u_{2n+1}, u_{2n+2})) \leq \mathcal{F}(\psi(\hat{p}(u_{2n}, u_{2n+1})), \varphi(\psi(\hat{p}(u_{2n}, u_{2n+1})))) \leq \psi(\hat{p}(u_{2n}, u_{2n+1})). \qquad (6.2.6)$$

By (6.2.5) and (6.2.6),

$$\hat{p}(u_{n+1}, u_n) \leq \hat{p}(u_n, u_{n-1});$$

for each $n \in \mathbb{N}$.

Therefore, the sequence $\{\hat{p}(u_n, u_{n+1})\}$ is a nonnegative and non-increasing sequence and thus there exists $r > 0$ such that

$$\lim_{n\to\infty} \hat{p}(u_{n+1}, u_n) = r.$$

Now since φ is lower semicontinuous,

$$\varphi(\psi(r)) \leq \lim_{n\to\infty} inf \varphi(\psi(\hat{p}(u_n, u_{n-1}))).$$

Therefore, by (6.2.5), we obtain

$$\psi(r) \leq \mathcal{F}(\psi(r), \varphi(\psi(r))).$$

This implies $\psi(r) = 0$ or $\varphi(\psi(r)) = 0$. Hence $r = 0$.

Next, it remains to show that $\{u_n\}$ is a Cauchy sequence in U, *i.e.* to prove that

$$\lim_{n,m\to\infty} \hat{p}(u_n, u_m) = 0.$$

Assume that the sequence $\{u_{2n}\}$ is not a Cauchy sequence in $(U, \hat{p})$, then by Lemma 6.1.5, there exist $\epsilon > 0$ and two sequences $\{u_{m(k)}\}$ and $\{u_{n(k)}\}$ of $\{u_n\}$ with $n(k) > m(k) > k$ such that the sequences in Lemma 6.1.5 tend to ϵ as $k \to \infty$.

Using the given contractive condition,

$$\psi(\hat{p}(u_{2m(k)}, u_{2n(k)+1})) \leq \psi(H_p(\mathcal{D}_1 u_{2m(k)-1}, \mathcal{D}_2 u_{2n(k)}))$$
$$\leq \mathcal{F}(\psi(M(u_{2m(k)-1}, u_{2n(k)}))), \varphi(\psi(M(2u_{m(k)-1}, u_{2n(k)})))); \tag{6.2.7}$$

where

$$M(u_{2m(k)-1}, u_{2n(k)})) = Max \left\{ \begin{array}{r} \hat{p}(u_{2m(k)-1}, u_{2n(k)}), \hat{p}(u_{2m(k)-1}, \mathcal{D}_1 u_{2m(k)-1}), \hat{p}(u_{2n(k)}, \mathcal{D}_2 u_{2n(k)}), \\ \dfrac{\hat{p}(u_{2n(k)}, \mathcal{D}_1 u_{2m(k)-1}) + \hat{p}(u_{2m(k)-1}, \mathcal{D}_2 u_{2n(k)})}{2} \end{array} \right\}$$
$$= Max \left\{ \begin{array}{r} \hat{p}(u_{2m(k)-1}, u_{2n(k)}), \hat{p}(u_{2m(k)-1}, u_{2m(k)}), \hat{p}(u_{2n(k)}, u_{2n(k)+1}), \\ \dfrac{\hat{p}(u_{2n(k)}, u_{2m(k)}) + \hat{p}(u_{2m(k)-1}, u_{2n(k)+1})}{2} \end{array} \right\}$$
$$\to Max\{\epsilon, 0, 0, \epsilon\}$$
$$= \epsilon \quad as \quad k \to \infty.$$

Thus by (6.2.7) and for any $k \to \infty$,

$$\psi(\epsilon) \leq \mathcal{F}(\psi(\epsilon), \varphi(\psi(\epsilon))).$$

This implies that $\psi(\epsilon) = 0$ or $\varphi(\psi(\epsilon)) = 0$ and thus $\epsilon = 0$ which is a contradiction. Therefore the sequence $\{u_n\}$ is a Cauchy sequence. As $(U, \hat{p})$ is complete, the space $(U, d_{\hat{p}})$ is complete. Therefore, $\lim\limits_{n\to\infty} d_{\hat{p}}(u_n, u) = 0$ for some $u \in U$. Now by Lemma 6.1.4,

$$\hat{p}(u, u) = \lim_{n\to\infty} \hat{p}(u_n, u) = \lim_{m,n\to\infty} \hat{p}(u_n, u_m) = 0.$$

Since U has limit comparison property, therefore for $n \in \mathbb{N}$, u_n is comparable to u, therefore,

$$\hat{p}(u_{2n+2}, \mathcal{D}_1 u) \leq \sup_{u \in \mathcal{D}_2 u_{2n+1}} \hat{p}(u, \mathcal{D}_1 u) \leq H_{\hat{p}}(\mathcal{D}_2 u_{2n+1}, \mathcal{D}_1 u).$$

Thus,

$$\psi(\hat{p}(u_{2n+2}, \mathcal{D}_1 u)) \leq \psi(H_{\hat{p}}(\mathcal{D}_2 u_{2n+1}, \mathcal{D}_1 u))$$
$$\leq \mathcal{F}(\psi(M(u_{2n+1}, u)), \varphi(\psi(M(u_{2n+1}, u))))$$
$$+ \xi(\hat{p}(u_{2n+1}, \mathcal{D}_2 u_{2n+1}), \hat{p}(u, \mathcal{D}_1 u), \hat{p}(u_{2n+1}, \mathcal{D}_1 u) - \hat{p}(u_{2n+1}, u_{2n+1})$$
$$, \hat{p}(u, \mathcal{D}_2 u_{2n+1}) - \hat{p}(u, u)) \tag{6.2.8}$$

$$\leq \mathcal{F}(\psi(M(u_{2n+1},u)),\varphi(\psi(M(u_{2n+1},u))))$$
$$+\,\xi(\hat{p}(u_{2n+1},u_{2n+2}),\hat{p}(u,\mathcal{D}_1u),\hat{p}(u_{2n+1},\mathcal{D}_1u)-\hat{p}(u_{2n+1},u_{2n+1})$$
$$,\hat{p}(u,u_{2n+2})-\hat{p}(u,u));$$

where

$$\hat{p}(u,\mathcal{D}_1u)\leq M(u_{2n+1},u)$$

$$= Max\left\{\hat{p}(u_{2n+1},u),\hat{p}(u_{2n+1},\mathcal{D}_2u_{2n+1}),\hat{p}(u,\mathcal{D}_1u),\frac{\hat{p}(u_{2n+1},\mathcal{D}_1u)+\hat{p}(u,\mathcal{D}_2u_{2n+1})}{2}\right\}$$
$$\leq Max\left\{\hat{p}(u_{2n+1},u),\hat{p}(u_{2n+1},u_{2n+2}),\hat{p}(u,\mathcal{D}_1u),\frac{\hat{p}(u_{2n+1},\mathcal{D}_1u)+\hat{p}(u,u_{2n+2})}{2}\right\}.$$

Taking limit $n\to\infty$, we get $\lim_{n\to\infty} M(u_{2n+1},u) = d(u,\mathcal{D}_1u)$. Since φ is lower semicontinuous, taking limit $n\to\infty$ in (6.2.8) implies

$$\psi(\hat{p}(u,\mathcal{D}_1u))\leq \mathcal{F}(\psi(\hat{p}(u,\mathcal{D}_1u)),\varphi(\psi(\hat{p}(u,\mathcal{D}_1u))));$$

which further implies that $\psi(\hat{p}(u,\mathcal{D}_1u))=0$ or $\varphi(\psi(\hat{p}(u,\mathcal{D}_1u)))=0$.
Thus, $\hat{p}(u,\mathcal{D}_1u)=0$. Since $\mathcal{D}_1u$ has approximative property, there exist $v\in P_{\mathcal{D}_1u}$ such that $\hat{p}(v,u)=0$ *i.e* $v=u$, therefore $u\in\mathcal{D}_1u$. Thus u is a fixed point of $\mathcal{D}_1$. This completes the proof.

By putting $\mathcal{F}(t_1,t_2)=t_1-t_2$, The following result holds:

Corollary 6.2.4 *Let U be a complete ordered partial metric space satisfying limit comparison property. Let $\mathcal{D}_1,\mathcal{D}_2 : U\to 2^U$ be two weakly isotone increasing mappings holding approximative property such that*

$$H_p(\mathcal{D}_1u,\mathcal{D}_2v) \leq \psi(M(u,v))-\varphi(\psi(M(u,v))+\xi(\hat{p}(u,\mathcal{D}_1u),\hat{p}(v,\mathcal{D}_2v),$$
$$\hat{p}(v,\mathcal{D}_1u)-\hat{p}(v,v),\hat{p}(u,\mathcal{D}_2v)-\hat{p}(u,u));$$

for all $u,v\in U$ with u and v comparable and $\psi\in\Psi$, $\varphi\in\Phi$ and $\xi\in\Upsilon$. Suppose that there exists $u_0\in U$ such that $\{u_0\}\preceq_2\mathcal{D}_1u_0$. Then $\mathcal{D}_1$, $\mathcal{D}_2$ have a common fixed point $u\in U$, such that $\hat{p}(u,u)=0$.

On putting $\mathcal{F}(t_1,t_2)=mt_1$ and $\psi(t)=t$, the following result is obtained:

Corollary 6.2.5 *Let U be a complete ordered partial metric space satisfying limit comparison property. Suppose that $\mathcal{D}_1,\mathcal{D}_2 : U\to 2^U$ are two weakly isotone in-*

creasing mappings holding approximative property and there exists $k \in [0,1)$ such that

$$\begin{aligned} H_p(\mathcal{D}_1 u, \mathcal{D}_2 v) \;\; &\leq \;\; m\ \psi(M(u,v)) + \xi(\hat{p}(u, \mathcal{D}_1 u), \hat{p}(v, \mathcal{D}_2 v), \\ &\qquad \hat{p}(v, \mathcal{D}_1 u) - \hat{p}(v,v), \hat{p}(u, \mathcal{D}_2 v) - \hat{p}(u,u)); \end{aligned}$$

for all $u, v \in U$ with u and v comparable and $\psi \in \Psi$ and $\xi \in \Upsilon$. Suppose that there exists $u_0 \in U$ such that $\{u_0\} \preceq_2 \mathcal{D}_1 u_0$. Then $\mathcal{D}_1$, $\mathcal{D}_2$ have a common fixed point $u \in U$ such that $\hat{p}(u,u) = 0$.

By putting $\mathcal{D}_2 = \mathcal{D}_1$ in Theorem 6.2.3, the following Corollary holds:

Corollary 6.2.6 *Let U be a complete ordered partial metric space satisfying the limit comparison property. Suppose that $\mathcal{D}_1 : U \to 2^U$ is a $\mathcal{F}$-generalized (ψ, φ, ξ)-contractive type mapping holding approximative property. Let $\mathcal{D}_1$ be weakly isotone increasing and there exists $u_0 \in U$ such that $\{u_0\} \preceq_2 \mathcal{D}_1 u_0$. Then $\mathcal{D}_1$ has a fixed point $u \in U$ such that $\hat{p}(u,u) = 0$.*

6.3 Example

Example 6.3.1 *Let $U = [0,1]$ equipped with partial metric $\hat{p}$ defined by $\hat{p}(u,v) = max\{u,v\}$, for each $u, v \in U$. Define the partial order on U by*

$$u \preceq v \Leftrightarrow \hat{p}(u,u) = \hat{p}(u,v) \Leftrightarrow u = max\{u,v\} \Leftrightarrow v \leq u.$$

It is easy to check that $(U, \preceq)$ is a totally ordered set and $(U, \hat{p})$ is a complete partial metric space. Also, the mappings $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined as

$$\mathcal{D}_1 u = \begin{cases} \{0\} & if \quad u \in \{0, \frac{1}{2}\}, \\ \{0, \frac{1}{2}\} & otherwise, \end{cases} \quad and \quad \mathcal{D}_2 u = \begin{cases} \{0\} & if \quad u \in \{0, \frac{1}{2}\}, \\ \{\frac{1}{2}\} & otherwise. \end{cases}$$

Note that $\mathcal{D}_1$ and $\mathcal{D}_2$ are weakly isotone increasing as for $v, z \in \mathcal{D}_2 u$; $w \in \mathcal{D}_1 v \Rightarrow w = 0$. Thus, $w \leq z \Rightarrow z \preceq w$. Hence, for each $u \in U$; $\mathcal{D}_2 u \preceq_2 \mathcal{D}_1 v$ for each $v \in \mathcal{D}_2 u$. Similarly, for each $u \in U$, it can be easily shown that $\mathcal{D}_1 u \preceq_2 \mathcal{D}_2 v$ for all $v \in \mathcal{D}_1 u$.

Let $\psi(t) = 2t$ and $\varphi(t) = t/2$, $\mathcal{F}(t_1, t_2) = \frac{1}{2} t_1$ and $\xi(s_1, s_2, s_3, s_4) = s_1 s_2 s_3 s_4$. Next it is proved that the mappings $\mathcal{D}_1$ and $\mathcal{D}_2$ are $\mathcal{F}$-generalized (ψ, φ, ξ)- contractive mappings. Following cases arises:

Case I: *If* $u, v \in \{0, \frac{1}{2}\}$. *Then,*

$$\begin{aligned}\psi(H_{\hat{p}}(\mathcal{D}_1 u, \mathcal{D}_2 v)) &= \psi(H_{\hat{p}}(\{0\}, \{0\}))\\ &= \psi(0)\\ &= 0\\ &\leq \mathcal{F}(\psi(M(u,v)), \phi(\psi(M(u,v)))) + \xi(\hat{p}(u, \mathcal{D}_1 u), \hat{p}(v, \mathcal{D}_2 v),\\ &\quad \hat{p}(v, \mathcal{D}_1 u) - \hat{p}(v,v), \hat{p}(u, \mathcal{D}_2 v) - \hat{p}(u,u)).\end{aligned}$$

Case II: *If* $u = v = 1$. *Then,*

$$\begin{aligned}\psi(H_{\hat{p}}(\mathcal{D}_1 u, \mathcal{D}_2 v)) &= \psi\left(H_{\hat{p}}\left(\left\{0, \frac{1}{2}\right\}, \left\{\frac{1}{2}\right\}\right)\right)\\ &= \psi\left(\frac{1}{2}\right)\\ &= 1.\end{aligned}$$

Now,

$$\begin{aligned}M(u,v) &= Max\left\{\hat{p}(u,v), \hat{p}(u, \mathcal{D}_1 u), \hat{p}(v, \mathcal{D}_2 v), \frac{\hat{p}(u, \mathcal{D}_2 v) + \hat{p}(v, \mathcal{D}_1 u)}{2}\right\}\\ &\leq Max\left\{\hat{p}(1,1), \hat{p}(1, \{0, \frac{1}{2}\}), \hat{p}(1, \frac{1}{2}), \frac{\hat{p}(1, \{0, \frac{1}{2}\}) + \hat{p}(1, \frac{1}{2})}{2}\right\}\\ &= Max\{1, 1, 1, \frac{1}{2}(1+1)\}\\ &= 1.\end{aligned}$$

So,

$$\begin{aligned}\mathcal{F}(\psi(M(u,v)), \phi(\psi(M(u,v)))) + \xi(\hat{p}(u, \mathcal{D}_1 u), \hat{p}(v, \mathcal{D}_2 v), \hat{p}(v, \mathcal{D}_1 u) - \hat{p}(v,v), \hat{p}(u, \mathcal{D}_2 v) - \hat{p}(u,u))\\ = \frac{1}{2}\psi(1) + \xi(1, 1, 1-1, 1-1)\\ = 1.\end{aligned}$$

Thus, the contractive condition is proved. Similarly, the remaining cases can be discussed and proved.
Hence, all the hypotheses of Theorem 6.2.3 are fulfilled. Therefore, $\mathcal{D}_1$, $\mathcal{D}_2$ *have a common fixed point* $u = 0$.

Chapter 7

Coupled, Common Coupled and Coincidence Fixed Point Theorems for Various Abstract Spaces

7.1 Introduction

In 2006, Bhaskar and Lakshmikantham [24] gave the concept of coupled fixed point and proved some fixed point results in partially ordered metric spaces. Later on, Lakshmikantham and Ćirić [96] generalized these results and proved coupled coincidence and common coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces.
In 1993, the idea of b-metric spaces was introduced and used by Bakhtin [19] and Czerwik [47] and after that, Alghamdi *et al.* [8] presented the notion of b-metric-like spaces which is a beautiful combination of b-metric and metric-like spaces.

Definition 7.1.1 *[8] For a non empty set U, b-metric-like on the set U is a mapping $B : U \times U \to [0, +\infty)$ if the following three axioms hold true for all $u, v, w \in U$:*

$(B1)$ $B(u, v) = 0 \Rightarrow u = v$;

$(B2)$ $B(u, v) = B(v, u)$;

$(B3)$ $B(u, v) \leq s(B(u, w) + B(w, v))$ *where* $s \geq 1$.

The set U equipped with a metric B defined on it, is called a b-metric like space and is denoted by (U, B).

Till today, numerous papers dealt with fixed point theory for single valued and multivalued operators in b-metric like spaces. The study of common, common coupled and coincidence fixed points of nonlinear mappings with different contractive conditions has become the center of intensive research activity from few decades. For more details, we refer the reader to ([6], [16], [17], [21], [38], [77], [85], [93]).

Lakshmikantham and Ćirić [96] defined coupled coincidence and common coupled fixed points as below:

Definition 7.1.2 *[96] A point $(u, v) \in U \times U$ is said to be a coupled coincidence point for the mappings $D_1, D_2 : U \times U \to U$ if $D_1(u, v) = D_2(u, v)$ and $D_1(v, u) = D_2(v, u)$.*

Definition 7.1.3 *[96] A point $(u, v) \in U \times U$ is called a common coupled fixed point of the mappings $D_1, D_2 : U \times U \to U$ if $u = D_1(u, v) = D_2(u, v)$ and $v = D_1(v, u) = D_2(v, u)$.*

In 2013, Karapinar [86] defined quasi partial metric spaces in his work by combining the properties of quasi metric space and partial metric space. Let us recall that for a non-empty set U, a mapping $q : U \times U \to \mathbb{R}^+$ is said to be a quasi partial metric if the following conditions hold:

$(q1)$ $0 \leq q(u, u) = q(u, v) = q(v, v),$ *then* $u = v$;

$(q2)$ $q(u, u) \leq q(u, v)$;

$(q3)$ $q(u, u) \leq q(v, u)$;

$(q4)$ $q(u, z) \leq q(u, v) + q(v, z) - q(v, v)$;

for all $u, v \in U$. Then the pair (U, q) is called a quasi partial metric space(QPMS).

The chapter is organized as follows: In section 7.2, related preliminaries are presented. Section 7.3 consists of some coupled fixed point theorems in partially ordered metric spaces. In section 7.4 and 7.5, some common and coincidence fixed point theorems in the setting of quasi partial metric spaces are investigated respectively. Section 7.6 deals with some common coupled fixed point theorems in b-metric-like spaces.

7.2 Preliminaries

In 2008, Abbas and Jungck [2] proved the following result in cone metric spaces:

Proposition 7.2.1 *[2] Let D_1 and D_2 be weakly compatible self mappings defined on a non-empty set U. If D_1 and D_2 have a unique point of coincidence $v = D_1 u = D_2 u$, then v is the unique common fixed point of D_1 and D_2.*

We shall require the following lemma due to Samet *et al.* [142] in order to prove some results in partially ordered metric spaces:

Lemma 7.2.2 *[142] Let $D_1 : \mathcal{U} \to \mathcal{U}$ be a given mapping. Define the mapping $D_2 : \mathcal{U} \times \mathcal{U} \to \mathcal{U} \times \mathcal{U}$ by $D_2(u, v) = (D_1(u, v), D_1(v, u))$ for all $(u, v), (v, u) \in \mathcal{U} \times \mathcal{U}$. Then (u, v) is a coupled fixed point of D_1 if and only if (u, v) is a fixed point of D_2.*

Recently, Dhawan *et al.* ([52], Chapter 3) proved some fixed point theorems for expansive mappings in quasi partial metric spaces by proving the following lemma:

Lemma 7.2.3 *[52] Let (U, q) be a quasi partial metric space and $\{u_n\}$ be a sequence of points of U. If there exists a number $k \in (0, 1)$ such that*

$$q(u_{n+1}, u_n) \leq k\ q(u_n, u_{n-1}); \quad n = 1, 2, ... \tag{7.2.1}$$

Then $\{u_n\}$ is a Cauchy sequence in U.

Following is the result due to Karapinar *et al.* [86]:

Lemma 7.2.4 *[86] Let (U, q) be a quasi partial metric space. Let (U, p_q) be the corresponding partial metric space and let (U, d_{p_q}) be the corresponding metric space. The following statements are equivalent:*

1. *(U, q) is complete.*
2. *(U, p_q) is complete.*
3. *(U, d_{p_q}) is complete.*

Moreover,

$$\begin{aligned}
\lim_{n\to\infty} d_{p_q}(u, u_n) &= 0 \Leftrightarrow p_q(u, u) = \lim_{n\to\infty} p_q(u, u_n) = \lim_{n,m\to\infty} p_q(u_n, u_m) \\
&\Leftrightarrow q(u, u) = \lim_{n\to\infty} q(u, u_n) = \lim_{n,m\to\infty} q(u_n, u_m) \\
&= \lim_{n\to\infty} q(u_n, u) = \lim_{m,n\to\infty} q(u_m, u_n).
\end{aligned}$$

Alghamdi *et al.* [8] defined the convergence and completeness in b-metric-like spaces as below:

Definition 7.2.5 *[8] Let $(U, \hat{d})$ be a b-metric-like space. For any sequence $\{a_n\}$ of points of non empty set U, the point $a \in U$ is said to be the limit of $\{a_n\}$ if for each $\epsilon > 0$, there exists some $n(\epsilon) \in \mathbb{N}$, for which $D(a_n, a) < \epsilon$, for all $n \in n(\epsilon)$ and we say that the sequence $\{a_n\}$ is convergent to a.*

Definition 7.2.6 *[8] For a b-metric-like space $(U, \hat{d})$:*

1. *A sequence $\{a_n\}$ is called Cauchy if for each $\epsilon > 0$, there exists some $n(\epsilon) \in \mathbb{N}$, for which $D(a_n, a_m) < \epsilon$ for all $n, m \geq n(\epsilon)$.*

2. *A b-metric-like space $(U, \hat{d})$ is called a complete space iff every Cauchy sequence $\{a_n\}$ in U converges in U.*

7.3 Coupled fixed point theorems in Partially Ordered metric spaces

Theorem 7.3.1 *Let $(\mathcal{U}, \hat{d})$ be a partially ordered metric space where $\hat{d}$ is a complete metric and $\preceq$ be the partial order defined on $\mathcal{U}$. Let $\mathfrak{D} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ be a nondecreasing and bijective mapping. Assume that there exists $\xi \in \chi$ and a function $\alpha : \mathcal{U}^2 \times \mathcal{U}^2 \to [0, +\infty)$ such that*

$$\xi(\hat{d}(\mathcal{D}(l, m), \mathcal{D}(u, v))) \geq \frac{1}{2}\alpha((l, m), (u, v))[\hat{d}(l, u) + \hat{d}(m, v)]; \tag{7.3.1}$$

for all $(l, m), (u, v) \in \mathcal{U} \times \mathcal{U}$. Also
(i) For every pair $(l, m), (u, v) \in \mathcal{U} \times \mathcal{U}$, we have

$$\alpha((l, m), (u, v)) \geq 1 \Rightarrow \alpha(\mathcal{D}^{-1}l, \mathcal{D}^{-1}u) \geq 1;$$

(ii) there exists $(l_0, m_0) \in \mathcal{U} \times \mathcal{U}$ such that

$$\begin{aligned} \alpha((l_0, m_0), (a, b)) &\geq 1; \quad (a, b) \leq (l_0, m_0) \quad \text{and} \\ \alpha((b, a), (m_0, l_0)) &\geq 1; \quad (m_0, l_0) \leq (b, a); \end{aligned}$$

where $\mathcal{D}^{-1}l_0 = (a, b)$ and $\mathcal{D}^{-1}m_0 = (b, a)$;
(iii) $\mathcal{D}$ is continuous.
Then $\mathcal{D}$ has a coupled fixed point.

Proof Let us define a mapping $\mathcal{D}$ given in [24] as a bijective mapping such that

$$\mathcal{D}^{-1}(u, v) = \mathcal{D}^{-1}(u) \quad \text{and} \quad \mathcal{D}^{-1}(v, u) = \mathcal{D}^{-1}(v).$$

The proof of this theorem consists in transporting this problem to partially ordered metric space (V, σ) where $V = \mathcal{U} \times \mathcal{U}$ and $\sigma((l, m), (u, v)) = \hat{d}(l, u) + \hat{d}(m, v)$ for all

$(l,m),(u,v) \in \mathcal{U} \times \mathcal{U}$ with $l \geq m$. Using (7.3.1), we have,

$$\xi(\hat{d}(\mathcal{D}(l,m),\mathcal{D}(u,v))) \geq \frac{1}{2}\alpha((l,m),(u,v))[\hat{d}(l,u)+\hat{d}(m,v)] \tag{7.3.2}$$

and

$$\xi(\hat{d}(\mathcal{D}(v,u),\mathcal{D}(m,l))) \geq \frac{1}{2}\alpha((v,u),(m,l))[\hat{d}(v,m)+\hat{d}(u,l)]. \tag{7.3.3}$$

Define the mapping $\zeta : V \times V \to [0,+\infty)$ by

$$\zeta(a_1,a_2),(b_1,b_2)) = min\{\alpha((a_1,a_2),(b_1,b_2)),\alpha((b_2,b_1),(a_2,a_1))\}; \tag{7.3.4}$$

for every pair $(a_1,a_2) = a, \ \ (b_1,b_2) = b \in V$. Adding the inequalities (7.3.2), (7.3.3) and using (7.3.4), we obtain

$$\xi(\hat{d}(\mathcal{D}(l,m),\mathcal{D}(u,v))) + \xi(\hat{d}(\mathcal{D}(v,u),\mathcal{D}(m,l))) \geq \zeta((l,m),(u,v))\sigma((u,v),(l,m));$$

for every $(l,m),(u,v) \in \mathcal{U} \times \mathcal{U} = V$. Using additive property $\xi(a+b) = \xi(a)+\xi(b)$, we get

$$\begin{aligned} \xi((\hat{d}(\mathcal{D}(l,m),\mathcal{D}(u,v))) + \hat{d}(\mathcal{D}(v,u),\mathcal{D}(m,l))) &\geq \zeta((l,m),(u,v))\sigma((u,v),(l,m)) \\ \Rightarrow \xi(\sigma(\mathcal{D}(l,m),\mathcal{D}(u,v))) &\geq \zeta((l,m),(u,v))\sigma((u,v),(l,m)); \end{aligned}$$

for all $(l,m),(u,v) \in \mathcal{U}\times\mathcal{U} = V$. More precisely, we can rewrite the above expression as

$$\xi(\sigma(\mathcal{D}\mu,\mathcal{D}\nu)) \geq \zeta(\mu,\nu)\sigma(\mu,\nu);$$

where $\mu = (l,m)$ and $\nu = (u,v) \in Y$.

Clearly, $\mathcal{D}$ is a continuous, nondecreasing and (ξ,ζ)-expansive mapping. Also, for $\mu,\nu \in Y$ with $\mu \geq \nu$, let $\zeta(\mu,\nu) \geq 1$ which implies $\zeta(\mathcal{D}^{-1}\mu,\mathcal{D}^{-1}\nu) \geq 1$. This shows that $\mathcal{D}^{-1}$ is ordered ζ-admissible. Moreover, by condition (ii), there exists $(l_0,m_0) \in V$ such that

$$\begin{aligned} \alpha((l_0,m_0),(a,b)) &\geq 1; \ \ (a,b) \leq (l_0,m_0) \quad and \\ \alpha((b,a),(m_0,l_0)) &\geq 1; \ \ (v_0,u_0) \leq (b,a) \\ \Rightarrow \alpha((l_0,m_0),\mathcal{D}^{-1}(l_0)) &\geq 1; \ \ \mathcal{D}^{-1}(l_0) \leq (l_0,m_0) \quad and \\ \alpha(\mathcal{D}^{-1}(m_0),(m_0,l_0)) &\geq 1; \ \ (m_0,l_0) \leq \mathcal{D}^{-1}(m_0) \end{aligned}$$

$$\Rightarrow \zeta((l_0,m_0),\mathcal{D}^{-1}(l_0,m_0)) \geq 1; \ \ \mathcal{D}^{-1}(l_0,m_0) \leq (l_0,m_0).$$

This fulfills condition (ii) of Theorem 2.4.1. Thus, the given problem has been transformed to problem of Theorem 2.4.1 with partially ordered metric space (V, σ) where $V = \mathcal{U} \times \mathcal{U}$. Therefore, $\mathcal{D}$ as well as $\mathcal{D}^{-1}$ possesses a fixed point and Lemma 7.2.2 ensures the existence of a coupled fixed point of $\mathcal{D}$.

7.4 Common fixed point results in Quasi partial metric spaces

Theorem 7.4.1 *Let (U, q) be a quasi partial metric space. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two weakly compatible mappings defined on U such that $\mathcal{D}_2(U) \subseteq \mathcal{D}_1(U)$. Suppose that there exists a positive real number $c > 1$ such that*

$$q(\mathcal{D}_1 u, \mathcal{D}_1 v) \geq c\ q(\mathcal{D}_2 u, \mathcal{D}_2 v); \tag{7.4.1}$$

for each pair $u, v \in U$.
If one of the two subspaces $\mathcal{D}_1(U)$ or $\mathcal{D}_2(U)$ is complete, then $\mathcal{D}_1$ and $\mathcal{D}_2$ have a unique common fixed point in U.

Proof Let u_0 be any arbitrary element of U. It is given that $\mathcal{D}_2(U) \subseteq \mathcal{D}_1(U)$, therefore, we can choose u_1 such that $v_1 = \mathcal{D}_1 u_1 = \mathcal{D}_2 u_0$. Proceeding in the same manner, we can obtain u_{n+1} such that $v_{n+1} = \mathcal{D}_1 u_{n+1} = \mathcal{D}_2 u_n$. Then, by (7.4.1), it follows that

$$\begin{aligned} q(v_{n+1}, v_{n+2}) &= q(\mathcal{D}_2 u_n, \mathcal{D}_2 u_{n+1}) \\ &\leq \frac{1}{c}\ q(\mathcal{D}_1 u_n, \mathcal{D}_1 u_{n+1}) \\ &= \frac{1}{c}\ q(\mathcal{D}_2 u_{n-1}, \mathcal{D}_2 u_n) \\ \Rightarrow q(v_{n+1}, v_{n+2}) &= \frac{1}{c}\ q(v_n, v_{n+1}); \end{aligned} \tag{7.4.2}$$

Again, by Lemma 7.2.3, the sequence $\{v_n\}$ is a Cauchy sequence in U. Moreover, $\mathcal{D}_2(U) \subseteq \mathcal{D}_1(U)$ and either $\mathcal{D}_1(U)$ or $\mathcal{D}_2(U)$ is complete. Therefore, following Lemma 7.2.4, the space $(\mathcal{D}_1(U), d_q)$ is complete and thus $v_n = \mathcal{D}_2 u_{n-1}$ converges in $(\mathcal{D}_1(U), d_q)$. In other words, there exists some $u^* \in \mathcal{D}_1(U)$ such that $\lim_{n\to\infty} d_q(u^*, v_n) = 0$.
Let us suppose that there exists some $z \in U$ such that $\mathcal{D}_1 z = u^*$. By Lemma 7.2.4,

we obtain

$$
\begin{aligned}
q(\mathcal{D}_1 z, u^*) = q(u^*, u^*) &= \lim_{n\to\infty} q(u^*, v_n) = \lim_{n,m\to\infty} q(v_n, v_m) \\
&= \lim_{n\to\infty} q(v_n, u^*) = \lim_{m,n\to\infty} q(v_m, v_n). \qquad (7.4.3)
\end{aligned}
$$

Since $\{v_n\}$ is Cauchy in the metric space $(\mathcal{D}_1(U), d_q)$, therefore, we get

$$
\lim_{m,n\to\infty} d_q(v_m, v_n) = 0. \qquad (7.4.4)
$$

Moreover, since $max\{q(v_{n+1}, v_{n+1}), q(v_n, v_n)\} \leq q(v_{n+1}, v_n)$. So, by mathematical induction and using (7.4.2), we get

$$
\begin{aligned}
max\{q(v_{n+1}, v_{n+1}), q(v_n, v_n)\} &\leq q(v_{n+1}, v_n) \\
&\leq \left(\frac{1}{c}\right)^n q(v_1, v_0);
\end{aligned}
$$

and thus,

$$
\lim_{n\to\infty} q(v_n, v_n) = 0. \qquad (7.4.5)
$$

Hence, by definition of d_q, we obtain from (7.4.4) and (7.4.5)

$$
\lim_{m,n\to\infty} q(v_m, v_n) = 0.
$$

Thus, by (7.4.3), we get

$$
\begin{aligned}
q(\mathcal{D}_1 z, u^*) = q(u^*, u^*) &= \lim_{n\to\infty} q(u^*, v_n) = \lim_{n,m\to\infty} q(v_n, v_m) \\
&= \lim_{n\to\infty} q(v_n, u^*) = \lim_{m,n\to\infty} q(v_m, v_n) = 0.
\end{aligned}
$$

Now, by (7.4.1)

$$
q(\mathcal{D}_2 z, \mathcal{D}_2 u_n) \leq \frac{1}{c} q(\mathcal{D}_1 z, \mathcal{D}_1 u_n).
$$

Taking limit $n \to \infty$, the above inequality becomes

$$
\begin{aligned}
q(\mathcal{D}_2 z, u^*) &\leq \frac{1}{c} q(\mathcal{D}_1 z, u^*) = 0 \\
\Rightarrow q(\mathcal{D}_2 z, u^*) &= 0 \ \Rightarrow \mathcal{D}_2 z = u^*.
\end{aligned}
$$

Hence, $\mathcal{D}_1 z = \mathcal{D}_2 z = u^*$ and as $\mathcal{D}_1$ and $\mathcal{D}_2$ are weakly compatible *i.e.* $\mathcal{D}_1\mathcal{D}_2 z = \mathcal{D}_2\mathcal{D}_1 z \ \ \forall\ z \in U$, therefore, we obtain $\mathcal{D}_1 u^* = \mathcal{D}_2 u^*$.

Next, it is proved that u^* is a common fixed point of $\mathcal{D}_1$ and $\mathcal{D}_2$. By (7.4.1), we have

$$q(\mathcal{D}_1u^*, \mathcal{D}_1u_n) \geq c\ q(\mathcal{D}_2u^*, \mathcal{D}_2u_n).$$

Taking limit $n \to \infty$, we get, $q(\mathcal{D}_1u^*, u^*) \geq c\ q(\mathcal{D}_2u^*, u^*) = c\ q(\mathcal{D}_1u^*, u^*)$.

$$\Rightarrow q(\mathcal{D}_1u^*, u^*) = 0 \quad i.e. \quad \mathcal{D}_1u^* = u^*.$$

Hence, $\mathcal{D}_1u^* = \mathcal{D}_2u^* = u^*$.

For uniqueness, let t be another common fixed point of $\mathcal{D}_1$ and $\mathcal{D}_2$. Then, $q(u^*, t) = q(\mathcal{D}_1u^*, \mathcal{D}_1t) \geq c\ q(\mathcal{D}_2u^*, \mathcal{D}_2t) = c\ q(u^*, t)$ which implies $q(u^*, t) = 0$ $i.e.$ $u^* = t$.

Example 7.4.1 *Let $U = [0,1]$ and $q(u,v) = max\{u-v, v-u\} + u \ \ \forall \ \ u,v \in U$. Then, (U,q) is a complete quasi partial metric space. Let $\mathcal{D}_1u = \frac{u}{3}$ and $\mathcal{D}_2u = \frac{u}{9}$. Then $\mathcal{D}_2(U) \subseteq \mathcal{D}_1(U)$ and $\mathcal{D}_1(U)$ is complete. Also, for $u \preceq v$, we get*

$$\begin{aligned} q(\mathcal{D}_1u, \mathcal{D}_1v) &= max\left\{\frac{u-v}{3}, \frac{v-u}{3}\right\} + \frac{u}{3} \\ &= \frac{v}{3} \geq c\ q(\mathcal{D}_2u, \mathcal{D}_2v); \end{aligned}$$

for all $u,v \in [0,1]$ and $1 < c < 3$.

Moreover, the mappings $\mathcal{D}_1$ and $\mathcal{D}_2$ are weakly compatible at $u = 0$. Thus, all the conditions are satisfied for Theorem 7.4.1. Here, 0 is the unique common fixed point.

Next example shows that the condition of weak compatibility cannot be omitted in the statement of Theorem 7.4.1.

Example 7.4.2 *Let $U = [0,2]$ and $q(u,v) = max\{u-v, v-u\} + u \ \ \forall \ \ u,v \in U$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the mappings defined as $\mathcal{D}_1u = 2-u$ and $\mathcal{D}_2u = \frac{2}{3} - \frac{u}{3} \ \ \forall \ \ u \in U$. Then $\mathcal{D}_2(U) \subseteq \mathcal{D}_1(U)$ and $\mathcal{D}_1(U)$ is complete.*
Also, for $u \preceq v$, we obtain

$$\begin{aligned} q(\mathcal{D}_1u, \mathcal{D}_1v) &= q(2-u, 2-v) \\ &= 2-v \geq \frac{c}{3}(2-v) = c\ q(\mathcal{D}_2u, \mathcal{D}_2v); \end{aligned}$$

for all $u,v \in [0,1]$ and $1 < c < 3$.
Thus, the inequality given in Theorem 7.4.1 is satisfied. But $\mathcal{D}_12 = \mathcal{D}_22 = 0$ and $\mathcal{D}_1\mathcal{D}_22 = 2$, $\mathcal{D}_2\mathcal{D}_12 = \frac{2}{3}$. Therefore, $\mathcal{D}_1$ and $\mathcal{D}_2$ are not weakly compatible. In this example, $\mathcal{D}_1$ and $\mathcal{D}_2$ do not have a common fixed point.

This shows that the condition of weak compatibility is essential for the existence of common fixed point.

7.5 Coincidence fixed point theorems in Quasi partial metric spaces

Theorem 7.5.1 *Let (U, q) be a complete quasi partial metric space. Let $D : U \to U$ be a self mapping and $\{h_i\}_{i=1}^{\infty}$ be a sequence of functions defined on U such that*

$$q(h_i u, h_j v) \geq a\ q(Du, Dv) + b\ q(h_i u, Du) + c\ q(h_j v, Dv); \tag{7.5.1}$$

for all $u, v \in U$; $i, j \in \mathbb{N}$; a, b, c real numbers with $a + b + c > 1$. Also, suppose that D and $\{h_i\}_{i=1}^{\infty}$ satisfy the following conditions:

1. *$Du \subseteq h_i u$ for each $i \in \mathbb{N}$;*
2. *$h_i(U)$ or $D(U)$ is complete; $i = 1, 2, ...$*

Then D and $\{h_i\}_{i=1}^{\infty}$ possess a point of coincidence $v \in U$ such that i.e. $v = h_1 u = h_2 u = ... = Du$ for $u \in U$. Moreover, if $a > 1$, then this point of coincidence is unique. Moreover, if D and $\{h_i\}_{i=1}^{\infty}$ are weakly compatible and $a > 1$, then D and $\{h_i\}_{i=1}^{\infty}$ have a unique common fixed point v in U.

Proof Let $u_0 \in U$ be arbitrary. As $Du \subseteq h_i u$, there exists $u_1, u_2 \in U$ such that $Du_0 = h_1 u_1$ and $Du_1 = h_2 u_2$. By continuing this process, there exists sequences $\{u_k\}$ and $\{v_k\}$ such that

$$v_k = Du_k = h_{k+1} u_{k+1} \quad \forall k \in \mathbb{N} \cup \{0\}.$$

If $u_k = u_{k+1}$ for some k, then u_k is the coincidence point of D and $\{h_i\}_{i=1}^{\infty}$. Therefore, without loss of generality, suppose that $u_k \neq u_{k+1}$ for all k.
By (7.5.1),

$$\begin{aligned} q(u_{k-1}, Du_k)) &= q(h_k u_k, h_{k+1} u_{k+1})) \\ &\geq a\ q(Du_k, Du_{k+1}) + b\ q(h_k u_k, Du_k) + c\ q(h_{k+1} u_{k+1}, Du_{k+1}) \\ &= a\ q(Du_k, Du_{k+1}) + b\ q(Du_{k-1}, Du_k) + c\ q(Du_k, Du_{k+1}) \\ \Rightarrow q(Du_k, Du_{k+1}) &\leq \left(\frac{1-b}{a+c}\right) q(Du_{k-1}, Du_k) \end{aligned}$$

$$= \lambda\, q(Du_{k-1}, Du_k) \quad \forall \;\; k \in \mathbb{N} \;\; where \;\; \lambda < 1.$$

By mathematical induction, we get

$$\begin{aligned} & q(Du_k, Du_{k+1}) \leq \lambda^k q(Du_0, Du_1) \quad \forall \;\; k \geq 0 \\ & or \quad q(v_k, v_{k+1}) \leq \lambda^k q(v_0, v_1) \quad \forall \;\; k \geq 0. \end{aligned} \tag{7.5.2}$$

For $n, m \in \mathbb{N}$ with $m > n$, repetition of (7.5.2) gives

$$\begin{aligned} q(v_n, v_m) &\leq q(v_n, v_{n+1}) + q(v_{n+1}, v_{n+2}) + ... + q(v_{m-1}, v_m) \\ &\leq \lambda^n q(v_0, v_1) + \lambda^{n+1} q(v_0, v_1) + ... + \lambda^{m-1} q(v_0, v_1) \\ &\leq \frac{\lambda^n}{1-\lambda} q(v_0, v_1). \end{aligned} \tag{7.5.3}$$

This implies that $\lim\limits_{n,m\to\infty} q(Du_n, Du_m) = \lim\limits_{n,m\to\infty} q(v_n, v_m) = 0$. Similarly, it can be proved that $\lim\limits_{m,n\to\infty} q(Du_m, Du_n) = 0$. Thus, $\{v_n\} = \{Du_n\}$ is a Cauchy sequence in $D(U)$. Suppose that $D(U)$ is complete. Then there exists $v \in D(U) \subseteq h_i(U)$ such that $v_n = Du_n \to v$ as $n \to \infty$. This implies $h_n u_n \to v$ for $n \in \mathbb{N}$. If $h_i(U)$ is complete, then $v \in h_i(U)$ for all $i \in \mathbb{N}$. This indicates that there exists $u \in U$ such that $h_i u = v \;\; \forall \;\; i \in \mathbb{N}$.
Now,

$$\begin{aligned} q(Du_{k-1}, h_k u) &= q(h_k u_k, h_k u) \\ &\geq a\; q(Du_k, Du) + b\; q(h_k u_k, Du_k) + c\; q(h_k u, Du) \\ &\geq a\; q(Du_k, Du) \\ \Rightarrow q(Du_k, Du) &\leq \frac{1}{a}\; q(Du_{k-1}, h_k u). \;\; (a \neq 0) \end{aligned} \tag{7.5.4}$$

Therefore, by (7.5.4),

$$\begin{aligned} q(v, Du) &\leq q(v, Du_n) + q(Du_n, Du) \\ &\leq q(v, Du_n) + \frac{1}{a}\; q(Du_{n-1}, h_n u) \\ &= q(v, Du_n) + \frac{1}{a}\; q(h_n u_n, h_n u). \end{aligned}$$

Taking limit $n \to \infty$, we have

$$q(v, Du) \leq q(v, v) + \frac{1}{a}\; q(v, v). \tag{7.5.5}$$

Since (U, q) is complete, therefore (U, d_q) is complete. Thus,

$$\begin{aligned}\lim_{n\to\infty} d_q(Du_n, v) &= 0 \\ \Leftrightarrow q(v,v) &= \lim_{n\to\infty} q(v, Du_n) = \lim_{n,m\to\infty} q(Du_n, Du_m) \\ &= \lim_{n\to\infty} q(Du_n, v) = \lim_{m,n\to\infty} q(Du_m, Du_n).\end{aligned}$$

Further, by (7.5.3)

$$\begin{aligned}\lim_{n,m\to\infty} q(Du_n, Du_m) &= 0 \\ \Leftrightarrow q(v,v) &= 0.\end{aligned}$$

So, (7.5.5) reduces to

$$q(v, Du) = 0 \quad i.e. \quad v = Du;$$

and hence $h_i u = Du = v$; $i = 1, 2, \ldots$. Thus, v is a point of coincidence of D and $\{h_i\}_{i=1}^{\infty}$.

Let $a > 1$ and $v^* \in U$ be another point of coincidence of D and $\{h_i\}_{i=1}^{\infty}$ such that $Du^* = h_n u^* = v^*$. Then,

$$\begin{aligned} q(v, v^*) = q(Du, Du^*) &= q(h_n u, h_n u^*) \\ &\geq a\ q(Du, Du^*) + b\ q(h_n u, Du) + c\ q(h_n u^*, Du^*) \\ &= a\ q(Du, Du^*) + b\ q(v,v) + c\ q(v^*, v^*) \\ &\geq a\ q(Du, Du^*) \\ &= a\ q(v, v^*) \\ \Rightarrow q(v, v^*) &\geq a\ q(v, v^*) \quad where \quad a > 1 \\ \Rightarrow q(v, v^*) &= 0 \quad \Rightarrow \quad v = v^*. \end{aligned}$$

Thus, D and $\{h_i\}_{i=1}^{\infty}$ have a unique point of coincidence in U.

Following Proposition 7.2.1, If D and $\{h_i\}_{i=1}^{\infty}$ are weakly compatible, then D and $\{h_i\}_{i=1}^{\infty}$ have a unique common fixed point in U.

The corollary given below is obtained by setting $D = I_U$ (identity map on U).

Corollary 7.5.2 *Let (U, q) be a complete quasi partial metric space. Let $\{h_i\}_{i=1}^{\infty}$*

be a sequence of functions defined on U such that

$$q(h_iu, h_jv) \geq a\ q(u, v) + b\ q(h_iu, u) + c\ q(h_jv, v);$$

for all $u, v \in U$; $i, j \in \mathbb{N}$; a, b, c real numbers with $a+b+c > 1$. If $h_i(U)$ is complete for all $i = 1, 2, ...,$ then $\{h_i\}_{i=1}^{\infty}$ possess a point of coincidence $v \in U$ such that i.e. $v = h_1u = h_2u = ...$ for $u \in U$. Moreover, if $a > 1$, then this point of coincidence is unique. If D and $\{h_i\}_{i=1}^{\infty}$ are weakly compatible with $a > 1$, then D and $\{h_i\}_{i=1}^{\infty}$ have a unique common fixed point v in U.

Corollary 7.5.3 *Let (U, q) be a complete quasi partial metric space. Let $D : U \to U$ be a self mapping and $\{h_i\}_{i=1}^{\infty}$ be a sequence of functions defined on U such that*

$$q(h_iu, h_jv) \geq a\ q(Du, Dv);$$

for all $u, v \in U$; $i, j \in \mathbb{N}$; with $a > 1$. Also, suppose that D and $\{h_i\}_{i=1}^{\infty}$ satisfy the following conditions:

1. *$Du \subseteq h_iu$ for each $i \in \mathbb{N}$;*
2. *$h_i(U)$ or $D(U)$ is complete; $i = 1, 2, ...$*

Then D and $\{h_i\}_{i=1}^{\infty}$ possess a unique point of coincidence $v \in U$ such that i.e. $v = h_1u = h_2u = ... = Du$ for $u \in U$. Moreover, If D and $\{h_i\}_{i=1}^{\infty}$ are weakly compatible, then D and $\{h_i\}_{i=1}^{\infty}$ have a unique common fixed point v in U.

Proof By putting $b = c = 0$ in Theorem 7.5.1, the required result is obtained.

The following corollary presents Wang's result [153]:

Corollary 7.5.4 *Let (U, q) be a complete quasi partial metric space. Let $\mathfrak{D}$ be a self mapping defined on U such that*

$$q(\mathfrak{D}u, \mathfrak{D}v) \geq a\ q(u, v);$$

for all $u, v \in U$ with $a > 1$. Then $\mathfrak{D}$ possesses a unique fixed point u in U.

Proof As every quasi partial metric space is a metric space, therefore, Wang's result follows by taking $h_iu = \mathfrak{D}u$; $i = 1$ and $Du = I_u$ and $b = c = 0$ in Theorem 7.5.1.

The corollary given below presents the result due to Banach [20]:

Corollary 7.5.5 *Let (U, q) be a complete quasi partial metric space. Let D be a* *self mapping defined on U such that*

$$q(Du, Dv) \leq \frac{1}{a}\ q(u, v);$$

for all $u, v \in U$ with $a > 1$. Then D possesses a unique fixed point u in U.

Proof As every quasi partial metric space is a metric space, the Banach's result follows by taking $h_i u = u \ \ \forall \ \ i$ in Corollary 7.5.3.

Example 7.5.1 *Let $U = [0, +\infty)$ equipped with the quasi partial metric $q : U \times U \to \mathbb{R}^+$ defined by $q(u, v) = \begin{cases} 0 & u = v, \\ v & u \neq v. \end{cases} \ \ \forall\ u, v \in U$.*
Note that (U, q) is a complete quasi partial metric space. Define $h_i : U \to U$ as follows:

$$h_i(u) \ \ = \ \ \begin{cases} u^2 & u \geq i, \\ 2u & 0 \leq u < i\ ; \end{cases}$$

for each $i \in \mathbb{N}$. Define the mapping $D : U \to U$ by $D(u) = \frac{u}{2}$. It remains to show that the mappings $\{h_i\}_{i=1}^{\infty}$ satisfy condition (7.5.1). Three cases arises:

***Case I**: If $u < i, v < j$, then $h_i(u) = 2u,\ \ h_j(v) = 2v$ and $q(h_i u, h_j v) = 2v$. Also,*

$$\begin{aligned} & a\ q(Du, Dv) + b\ q(h_i u, Du) + c\ q(h_j v, Dv) \\ &= a\ q(\frac{u}{2}, \frac{v}{2}) + b\ q(h_i u, \frac{u}{2}) + c\ q(h_j v, \frac{v}{2}) \\ &= a\ q(\frac{u}{2}, \frac{v}{2}) + b\ q(2u, \frac{u}{2}) + c\ q(2v, \frac{v}{2}) \\ &= a\ \frac{v}{2} + b\ \frac{u}{2} + c\ \frac{v}{2} \\ &= 2\ v - \frac{1}{2}\ v \quad \ where\ \ a = 4, \ \ b = 0, \ \ c = -1 \\ &= \frac{3}{2}\ v \leq 2\ v = q(h_i u, h_j v). \end{aligned}$$

***Case II**: If $u \geq i, v < j$, then $h_i(u) = u^2,\ \ h_j(v) = 2v$ and $q(h_i u, h_j v) = 2v$. Also,*

$$\begin{aligned} & a\ q(Du, Dv) + b\ q(h_i u, Du) + c\ q(h_j v, Dv) \\ &= a\ q(\frac{u}{2}, \frac{v}{2}) + b\ q(h_i u, \frac{u}{2}) + c\ q(h_j v, \frac{v}{2}) \end{aligned}$$

$$= 4\ \frac{v}{2} + 0 - \frac{v}{2}$$
$$= \frac{3}{2}\ v \leq 2\ v = q(h_i u, h_j v).$$

Case III: *If $u \geq i,\ v \geq j$, then $h_i(u) = u^2,\ h_j(v) = v^2$ and $q(h_i u, h_j v) = v^2$. Also,*

$$a\ q(Du, Dv) + b\ q(h_i u, Du) + c\ q(h_j v, Dv)$$
$$= a\ q(\frac{u}{2}, \frac{v}{2}) + b\ q(h_i u, \frac{u}{2}) + c\ q(h_j v, \frac{v}{2})$$
$$= 4\ \frac{v}{2} - \frac{v}{2}$$
$$= \frac{3}{2}\ v \leq v^2 = q(h_i u, h_j v).$$

Thus, in each case, the expansive inequality is satisfied. Therefore, by Theorem 7.5.1, there exists a unique point of coincidence of D and $\{h_i\}_{i=1}^{\infty}$. Here, 0 is the unique point of coincidence.

7.6 Common coupled fixed point theorems in b-metric-like spaces

Theorem 7.6.1 *Let (U, D) be a complete b-metric-like space having a parameter $K \geq 1$ and let the mappings $F_1, F_2 : U \times U \to U$ satisfy*

$$D(F_1(a,b), F_2(u,v)) \leq \alpha\frac{(D(a,u) + D(b,v))}{2} + \beta\frac{D(a, F_1(a,b))D(u, F_2(u,v))}{(1 + D(a,u) + D(b,v))} + \gamma\frac{D(u, F_1(a,b))D(a, F_2(u,v))}{(1 + D(a,u) + D(b,v))}; \quad (7.6.1)$$

for all $a, b, u, v \in U$ and $\alpha, \beta \geq 0$ with $K\alpha + \beta < 1$ and $\alpha + \gamma < 1$. Then F_1 and F_2 have a unique common coupled fixed point in U.

Proof Step 1: Firstly, it is shown that $\{a_n\}$, $\{b_n\}$ are Cauchy sequences in U. Let $a_0, b_0 \in U$ be arbitrary points. Define $a_{2k+1} = F_1(a_{2k}, b_{2k}),\ b_{2k+1} = F_1(b_{2k}, a_{2k})$ and $a_{2k+2} = F_2(a_{2k+1}, b_{2k+1}),\ b_{2k+2} = F_2(b_{2k+1}, a_{2k+1})$ for $k = 0, 1, 2, ...$
Now,

$$D(a_{2k+1}, a_{2k+2}) = D(F_1(a_{2k}, b_{2k}), F_2(a_{2k+1}, b_{2k+1}))$$
$$\leq \alpha\ \frac{D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1})}{2} + \beta\ \frac{D(a_{2k}, F_1(a_{2k}, b_{2k}))D(a_{2k+1}, F_2(a_{2k+1}b_{2k+1}))}{(1 + D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1}))}$$
$$+ \gamma\ \frac{D(a_{2k+1}, F_1(a_{2k}, b_{2k}))D(a_{2k}, F_2(a_{2k+1}, b_{2k+1}))}{(1 + D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1}))}$$

$$
\begin{aligned}
D(a_{2k+1}, a_{2k+2}) &= \alpha\ \frac{D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1})}{2} + \beta\ \frac{D(a_{2k}, a_{2k+1})D(a_{2k+1}, a_{2k+2})}{(1 + D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1}))} \\
&+ \gamma\ \frac{D(a_{2k+1}, a_{2k+1})D(a_{2k}, a_{2k+2})}{(1 + D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1}))} \\
&\leq \alpha\ \frac{D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1})}{2} + \beta\ \frac{D(a_{2k}, a_{2k+1})D(a_{2k+1}, a_{2k+2})}{(1 + D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1}))} \\
&+ \gamma\ [2D(a_{2k+1}, a_{2k+2})] \\
&\leq \alpha\ \frac{D(a_{2k}, a_{2k+1})}{2} + \alpha\ \frac{D(b_{2k}, b_{2k+1})}{2} + \beta\ D(a_{2k+1}, a_{2k+2}) \\
&+ \gamma\ [2D(a_{2k+1}, a_{2k+2})] \\
\Rightarrow (1 - \beta - 2\gamma)\ D(a_{2k+1}, a_{2k+2}) &\leq \alpha\ \frac{D(a_{2k}, a_{2k+1})}{2} + \alpha\ \frac{D(b_{2k}, b_{2k+1})}{2} \\
D(a_{2k+1}, a_{2k+2}) &\leq \alpha\ \frac{D(a_{2k}, a_{2k+1})}{2(1 - \beta - 2\gamma)} + \alpha\ \frac{D(b_{2k}, b_{2k+1})}{2(1 - \beta - 2\gamma)} \\
\Rightarrow D(a_{2k+1}, a_{2k+2}) &\leq \alpha\ \frac{D(a_{2k}, a_{2k+1})}{2(1 - \beta)} + \alpha\ \frac{D(b_{2k}, b_{2k+1})}{2(1 - \beta)}. \qquad (7.6.2)
\end{aligned}
$$

Continuing in the same manner, we have

$$
D(b_{2k+1}, b_{2k+2}) \leq \alpha\ \frac{D(b_{2k}, b_{2k+1})}{2(1 - \beta)} + \alpha\ \frac{D(a_{2k}, a_{2k+1})}{2(1 - \beta)}. \qquad (7.6.3)
$$

Adding (7.6.2) and (7.6.3), we get

$$
\begin{aligned}
[D(a_{2k+1}, a_{2k+2}) + D(b_{2k+1}, b_{2k+2})] &\leq \frac{\alpha}{1 - \beta}\ [D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1})] \\
&= h\ [D(a_{2k}, a_{2k+1}) + D(b_{2k}, b_{2k+1})];
\end{aligned}
$$

where $0 < h = \frac{\alpha}{1-\beta} < 1$.

Similarly,

$$
D(a_{2k+2}, a_{2k+3}) \leq \alpha\ \frac{D(a_{2k+1}, a_{2k+2})}{2(1 - \beta)} + \alpha\ \frac{D(b_{2k+1}, b_{2k+2})}{2(1 - \beta)}; \qquad (7.6.4)
$$

and

$$
D(b_{2k+2}, b_{2k+3}) \leq \alpha\ \frac{D(b_{2k+1}, b_{2k+2})}{2(1 - \beta)} + \alpha\ \frac{D(a_{2k+1}, a_{2k+2})}{2(1 - \beta)}. \qquad (7.6.5)
$$

Adding inequalities (7.6.4) and (7.6.5), we have

$$
\begin{aligned}
[D(a_{2k+2}, a_{2k+3}) + D(b_{2k+2}, b_{2k+3})] &\leq \frac{\alpha}{1 - \beta}\ [D(a_{2k+1}, a_{2k+2}) + D(b_{2k+1}, b_{2k+2})] \\
&= h\ [D(a_{2k+1}, a_{2k+2}) + D(b_{2k+1}, b_{2k+2})].
\end{aligned}
$$

Continuing in this way,

$$[D(a_n, a_{n+1})+D(b_n, b_{n+1})] \leq h\,[D(a_{n-1}, a_n)+D(b_{n-1}, b_n)] \leq ...h^n\,[D(a_0, a_1)+D(b_0, b_1)].$$

Now, if we put $[D(a_n, a_{n+1}) + D(b_n, b_{n+1})] = \delta_n$, then $\delta_n \leq h\ \delta_{n-1} \leq h^n\ \delta_0$.

For $m > n$, we have

$$\begin{aligned}
[D(a_n, a_m) + D(b_n, b_m)] &\leq K[D(a_n, a_{n+1}) + D(b_n, b_{n+1})] + ... + K^{m-n}[D(a_{m-1}, a_m) + D(b_{m-1}, b_m)] \\
&\leq K\ h^n \delta_0 + K^2\ hn + 1\delta_0 + ... + K^{m-n}\ h^{m-1}\delta_0 \\
&\leq K\ h^n[1 + (Kh) + (Kh)^2 + ...]\ \delta_0 \\
&\leq \frac{Kh^n}{1 - Kh}\ \delta_0\ as\ n \to \infty.
\end{aligned}$$

This shows that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in U. Since U is a complete b-metric-like space, there exists $a, b \in U$ such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$.

Step 2: Now, We show that (a, b) is common coupled fixed point of F_1 and F_2 *i.e.* $a = F_1(a, b)$ and $b = F_1(b, a)$.
We suppose on the contrary that $a \neq F_1(a, b)$ and $b \neq F_1(b, a)$ so that

$$D(a, F_1(a, b)) = l_1 > 0\ and\ D(b, F_1(b, a)) = l_2 > 0.$$

Consider

$$\begin{aligned}
l_1 = D(a, F_1(a, b)) &\leq K[D(a, a_{2k+2}) + D(a_{2K+2}, F_1(a, b)] \\
&= KD(a, a_{2k+2}) + KD(F_2(a_{2k+1}, b_{2k+1}), F_1(a, b)) \\
&\leq KD(a, a_{2k+2}) + K\alpha\ \frac{(D(a_{2k+1}, a) + D(b_{2k+1}, b))}{2} \\
&+ K\beta\ \frac{D(a, F_1(a, b))D(a_{2k+1}, F_2(a_{2k+1}, b_{2k+1}))}{1 + D(a_{2k+1,a} + D(b_{2k+1}, b))} + K\gamma\ \frac{D(a_{2k+1}, F_1(a, b))D(a, a_{2k+2})}{1 + D(a_{2k+1}, a) + D(b_{2k+1}, b)}.
\end{aligned}$$

By taking $k \to \infty$, we get $l_1 \leq 0$ which is a contradiction.
Therefore, $D(a, F_1(a, b)) = 0$. This implies $a = F_1(a, b)$. Similarly, we can prove that $b = F_1(b, a)$. In the same way, it can be easily proved that $a = F_2(a, b)$ and $b = F_2(b, a)$. Hence (a, b) is a common coupled fixed point of F_1 and F_2.

Step 3: Now it will be proved that F_1 and F_2 have a unique common coupled fixed point.

Let $(a^*, b^*) \in U \times U$ be another common coupled fixed point of F_1 and F_2. Then,

$$\begin{aligned} D(a, a^*) &= D(F_1(a, b), F_2(a^*, b^*)) \\ &\leq \alpha\ \frac{(D(a, a^*) + D(b, b^*))}{2} + \beta\ \frac{D(a, F_1(a, b))D(a^*, F_2(a^*, b^*))}{(1 + D(a, a^*) + D(b, b^*))} \\ &+ \gamma\ \frac{D(a^*, F_1(a, b))D(a, F_2(a^*, b^*))}{(1 + D(a, a^*) + D(b, b^*))} \\ &= \alpha\ \frac{(D(a, a^*) + D(b, b^*))}{2} + \beta\ \frac{D(a, a)D(a^*, a^*)}{(1 + D(a, a^*) + D(b, b^*))} \\ &+ \gamma\ \frac{D(a^*, a)D(a, a^*)}{(1 + D(a, a^*) + D(b, b^*))} \\ \Rightarrow D(a, a*) &\leq \alpha\ \frac{D(a, a^*)}{2} + \alpha\ \frac{D(b, b^*)}{2} + 4\beta\ D(a, a^*) + \gamma\ D(a^*, a) \\ &\leq \frac{2}{(2 - \alpha - 8\beta - 2\gamma)}\ D(b, b^*) \\ &\leq \frac{2}{(2 - \alpha - 2\beta)}\ D(b, b^*). \end{aligned} \tag{7.6.6}$$

Similarly, we can easily prove that

$$D(b, b^*) \leq \frac{2}{(2 - \alpha - 2\beta)} D(a, a^*). \tag{7.6.7}$$

Adding (7.6.6) and (7.6.7), we get

$$\begin{aligned} D(a, a^*) + D(b, b^*) &\leq \frac{2}{(2 - \alpha - 2\beta)} [D(a, a^*) + D(b, b^*)] \\ (2 - 2\alpha - 2\gamma)[D(a, a^*) + D(b, b^*)] &\leq 0 \\ \Rightarrow D(a, a^*) + D(b, b^*) &= 0. \end{aligned}$$

This implies, $a = a^*$ and $b = b^*$.

Corollary 7.6.2 *Let (U, D) be a complete b-metric-like space having a parameter $K \geq 1$ and let the mapping $F_2 : U \times U \to U$ satisfy*

$$\begin{aligned} D(F_1(a, b), F_1(u, v)) \ \leq\ & \alpha\ \frac{(D(a, u) + D(b, v))}{2} + \beta\ \frac{(D(a, F_1(a, b))D(u, F_1(u, v)))}{(1 + D(a, u) + D(b, v))} \\ & + \gamma\ \frac{(D(u, F_1(a, b))D(a, F_1(u, v)))}{(1 + D(a, u) + D(b, v))}; \end{aligned}$$

for all $a, b, u, v \in U$ and $\alpha, \beta \geq 0$ with $K\alpha + \beta < 1$ and $\alpha + \gamma < 1$. Then F_1 has a unique coupled fixed point in U.

Proof The above result can be easily proved by assuming $F_1 = F_2$ in Theorem 7.6.1.

7.7 Future scope of the work

The new notion of fixed function, various contraction and expansion maps presented in this work, can be applied in numerous abstract spaces such as Modular metric spaces, G_b-metric spaces, Partial Hausdorff metric spaces etc. as well as for different kinds of contraction and expansion mappings like weakly commuting mappings, compatible mappings etc. and hence can be generalized in many ways. Moreover, these new concepts can provide a wide range for applications in numerous areas such as Chemistry, Health Sciences, Economics etc. Also, coupled, common coupled and coupled coincidence fixed function results can also be studied for several spaces and maps existing in literature.

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