EQUILIBRIUM STRUCTURE OF NON-SYNCHRONOUSLY ROTATING AND TIDALLY DISTORTED BINARY STARS

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Submitted by

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CERIFTICATE

This is to certify that the thesis entitled "Equilibrium Structure of Non-synchronously Rotating and Tidally distorted binary stars" submitted by Ms. Hanspreet Kaur of M.Sc.(Mathematics and computing), Thapar University, Patiala, was carried out by me under supervisions of Dr.A.K.Lal. She has not submitted this material for credit towards any other degree at Thapar University Patiala or any other University.

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Dedicated to my

Parents and God

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ABSTRACT

The problem of determining the structure of equilibrium structures of stars distorted by rotation tidal forces has a great importance in astrophysics. Such a problems will help in better understanding the nature of inner structure of rotating stars and stars in binary/ or multiple systems. Thus, there is need for in depth investigation the equilibrium structure of gaseous sphere.

Analytic study of determining the equilibrium structure of rotationally and tidally distorted stellar models is quite complex. Therefore, investigation attempted to solve such a problems in some approximate way. In one such attempt Mohan, Saxsena and Agarwal (62,63) used kippenhahn and Thomas averaging technique together with the results of Kopal(32) on Roche equipotential, to determine the effects of rotation and tidal forces on the equilibrium structures of eccentric binaries. They also implemented this approach in the case of the polytropic models of the stars as well as certain realistic main sequence stars. Several authors such as Lal(40), Sharma(84), Saini(80), Pathenia(72) used this method to study some astrophysical problems. However, the problems of determining the equilibrium structures of non-synchronously rotating binary is not satisfactory tacked so far.

The thesis consists of three chapters. Chapter one deals with the importance of studying equilibrium structure of rotating and tidally distorted gaseous spheres. In the present thesis an attempt has been made to modify the Roche equipotential that takes into account for non-synchronous rotation and eccentric orbits. Using average technique of Kippenhahn and Thomas(28) is conjunction with results of modified Roche equipotential. Kippenhahn and Thomas(28) averaging technique is also discussed briefly in this chapter. This chapter ends with the brief survey of literature available on this subject and summary of the work presented in the succeeding chapter.

In chapter II, we firstly discuss the approach of Sepinsky et al.(2007) for determining the Roche equipotential, which takes into account the motion of mass elements in non-synchronous, eccentric binary and planetary system. The results of such a

roche equipotential are next used in conjunctions with Kippenhahn and Thomas averaging technique to develop analytic expressions representing equilibrium structure of non-synchronously rotating and tidally distorted stellar models. The equilibrium structures equations marginally deviates from the equilibrium structure equations earlier discussed by Mohan, Saxena and Agarwal (62,63) for non-synchronously rotation.

The methodology developed in chapter – II has next been used in chapter-III to determine the equilibrium structure of rotationally and tidally distorted polytropic models. Certain results of physical parameter have been computed for various polytropic indices 1.5, 3.0 and 4.0. Conclusion based on the present study has finally been discussed.

CONTENTS

CHAPTER-I

INTRODUCTION

This chapter is introduction in nature. We first discuss the astrophysical significance of the theoretical study of the problem of determining the effect of rotation and /or tidal distortions on the equilibrium structures of gaseous spheres. A brief survey of the literature available on the subject on the subject is discussing in section 1.2. In section 1.4 we show how Kippenhahn and Thomas(30) averaging technique to derive the differential equations which determine the equilibrium structures of rotationally and tidally distorted gaseous spheres.

1.1 SIGNIFICANCE OF DETERMINING THE EQUILIBRIUM STRUCTURE OF ROTATIONALLY AND TIDALLY DISTORTED GASEOUS SPHERES

The theoretical model of a star is essentially a self gravitating gaseous sphere in hydrostatic and thermal equilibrium. Theoretical studies of the problems of the equilibrium structure of gaseous sphere have often been studied out to understand the nature of internal structures of the stars. Some stars are observed to be single and some of them are in groups. It is also observed that single stars are rotating about their axes of rotation. Stars with one companion known as binary star and more than one companion known as multiple star. Binary and Multiple stars are rotating about their axes as well as revolving around one another. If we assume the equilibrium model of a single non-rotating star as a gaseous sphere, the equilibrium model of a rotating stars will be rotationally distorted gaseous sphere. Similarly, the equilibrium model of the stars appearing in a binary or multiple system, then such a model will be a tidally distorted gaseous sphere if it is not rotating and a rotationally and tidally distorted gaseous sphere if the star is rotating as well.

In binary system of stars, the two stars normally rotate about their own axis as well as revolve about their common center of mass. In binary star system, one is primary star having more massive than the companion star called secondary star.

Keeping this in view an attempt has been made in the present thesis to investigate certain aspects of the problems of equilibrium structures of non-

synchronously rotationally and tidally distorted gaseous spheres which still need more investigations.

1.2 BRIEF SURVEY OF THE LITERATURE

Most of the theoretical studies about the equilibrium structures of the stars have been carried out in literature by assuming the star to be an undistorted spherical gaseous sphere. Several authors worked on the problems of determining the equilibrium structure of rotationally and tidally distorted stellar model. Extensive literature is now available on this subject (see for instance Chandrasekhar (9), Eddington (19), Mentzel et al. (52), Cox and Guili (14), Kippenhahn and Weigert (31), Clement (11), Kopal (33-35), Tassoul (90), Cox (13), Bohm-Vitense et al (3).

The theoretical investigation related to the problems of determining the equilibrium structures and stability of rotating and self gravitating objects, possibly begun with the work of Newton. He was the first to realize the importance of the law of gravitation for explaining the figures of celestial bodies. Later on Maclaurian, Clairaut, Laplace, Legendre, Jacobi, Poincare etc. contributed ideas, necessary for the development of the general theory of rotating bodies. Maclaurian, Jacobi, Kelvin and Jeans investigated in detail the problem of structure and stability of rotating liquid masses assuming uniform rotation.

In the year 1923, Edward Arthur Milne developed a technique for constructing the first detailed model for a slowly rotating star in pure radiative equilibrium. Later on, in the year 1933, the technique of Milne was generalized and applied to slightly distorted polytropes by Chandrasekhar. The effect of uniform rotation on slowly rotating Cowling star obeying simple Kramer's opacity has been studied by Sweet and Roy (89). Much of the work on the effect of rotation on stellar interiors is summarized in the review article of Authors such as Kruszewski (38), Limber (45), Roberts (74), James (29), Hurley and Roberts (27), Roxburgh and Durney (77), Martin (50), Sackmann and Anand (80), Linnell (46,47), Endal and Sofia (22), Smith (86), Lubow (49), Kopal (34), Durney (18), Deupree (16), Einsel and Spurzem (21), Maeder and Zahn (53), Reyniers and Smeyres

(73) and Sood and Singh (87) have also investigated the problems of equilibrium structures of rotating stars. Meynet and Meader (53) studied the effects of rotation on the equilibrium structure and evolution of massive stars. Mender et al (51) investigated the theoretical models of low mass pre-main sequence rotating stars.

Equilibrium structures of the stars which appear in binary and multiple systems are likely to be affected by both the rotational as well as the tidal effect of the companion stars. Attempts have been made in literature to determine the effects of rotation and tidal distortions on the equilibrium structure the stars in binary and multiple systems. In a series of papers Chandrasekhar(6,7,8) developed a first order analysis which he applied to the study of the rotational problem, the tidal problem and the binary star problem. The method, however, was found unsuitable when the separation between the components in only a few times the undisturbed radius of the primary. Monaghan (64) modified it to get more accurate results near the surface.

The method of Monaghan and Roxburgh (66) to study the structure of the primary component of a synchronous close binary was further extended by Naylor and Anand (67). Kippenhahn and Thomas (30) suggested a practical way of analyzing the effects of rotation and tidal distortions on the equilibrium structures of the stars by approximating the actual equipotentials surfaces of the star by roche equipotentials.

Chan and Chau (4) developed a method which allows an efficient and accurate investigation of the structure and evolution of a rotationally and tidally distorted star in close binary systems. Tassoul and Tassoul (92) considered the meridional circulation in rotating stars and mean study motions in rotationally and tidally distorted stars. Later Tassoul and Tassoul (91) extended the earlier work to study the reflection effects in close binaries when there is meridional circulation in rotating stars. Nelson et al (68) have discussed the evolution of rotationally and tidally distorted low-mass close binary systems. Lopezorti et al (48) analyzed the equilibrium configurations of close binary systems by expanding the auto gravitational, centrifugal and tidal potentials in Clairaut coordinates. Lal et al (41) have discussed the equilibrium structures of rotationally and tidally distorted primary component of binary stars taking in account the effect of mass variation inside the star.

Orlov (69) have generalized the roche model as is applied in the case of double star. In this model the point nuclei of the roche model has been substituted by polytropic gas nuclei of finite dimensions. Plavec (71) presented tables of roche model for the use of investigators in close binary systems. Avni and Schiller (2) studied the roche potential systems where the stellar rotation axis is not aligned with the orbital revolution axis. Eggleton (20) computed the effective radii of Roche lobes and compared the results with earlier results available in literature. Mochnacki (54) accurately integrated Roche model for close binary system in synchronous rotation to give volume, radii, surface area, mean gravities and mean inverse gravities in normalized form. Seidov (82) derived the exact analytical formula for the potential and mass ratio as a function of Lagrangian point's position in the classical Roche model of the close binary stars. Csataryova and Skopal (15) derive approximate analytical formulas for the basic parameters of the roche lobe, its radius and the position of the L_1 point for asynchronously rotating component in binary system. Lal et al (42) studied the validity of series expression being used for determining the position of a point on a Roche equipotential in case of rotating stars and stars in binary systems.

Most of the authors have studied the equilibrium structures of stars having solid body rotation. The influence of uniform rotation on a global structure of the white dwarf models has been considered by Chandrasekhar (10), Suda (88) and Lal et al (40). The most detailed models of uniformly rotating white dwarf are due to Anand (1), Roxburgh (76), Monaghan et al (54). Some of the authors such as Ostriker and Tassoul (70), Shapiro and Teukolsky (83) have noted the stability analysis of uniformly rotating white dwarf stars. Ostriker and Bodenheimer (69) , Smart and Monaghan (85), and Blinnikov (5) extensively analyzed the models of zero-temperature white dwarf in non-uniform rotation, Hachisu et al. (25) studied the fate of merging double white dwarf and presented a numerical method, Lal et al (40) presented a method for computing equilibrium structure of differentially rotating and tidally distorted white dwarf models of stars.

Mohan and Singh (60) have used the Kippenhahn and Thomas (30) averaging technique in conjunction with certain results of Kopal (32) on Roche equipotential to study the effect of rotation and tidal distortions on the small adiabatic oscillations of stars

in binary system. Sepenisky et al (85) investigated the existence and properties of equipotential surfaces and Lagrangian points in non-synchronous, eccentric binary star and planetary systems under the assumption of quasi-static equilibrium. Roche (75) studied effect of slow uniform rotation on the tidal effects in close binary system. Deupree and Karkas (17) studied the structure and evolution of close binary stars using the two-dimensional stellar structure algorithm. They have computed a series of solar composition for stellar evolution sequences of binary.

1.3 BASIC EQUATIONS DETERMINIG THE EQUILIBRIUM STRUCTURE OF GASEOUS SPHERE

The system of basic equations of the equilibrium structure of gaseous sphere in hydrostatic and thermal equilibrium, are well established in literature. These equations pertain to the problem of the equilibrium structure of stellar models. Let P and ρ denote the pressure and the density at a point, respectively, distant r from the center of the sphere.

1.3.1 Mass conservation

Let $M(r)$ be the mass contained within radius r, then the mass contained with the shell from r to $r+dr$ is then

$$M(r+dr)-M(r)=\rho(r)\,dV \tag{1.1a}$$

where $dV=4\pi r^2 dr$ is the volume of the shell. Now the left-hand-side can be rewritten as,

$$\frac{dM(r)}{dr}dr=\rho(r)4\pi r^2 dr \tag{1.1b}$$

Cancelling out the factor dr, we obtain

$$\frac{dM(r)}{dr} = \rho(r) 4\pi r^2 \tag{1.1c}$$

1.3.2 Hydrostatic equilibrium

For a star in hydrostatic equilibrium, gravity is balanced by pressure. Now, for a small element in a shell from r to $r + dr$. The area of this element is dA.

A coordinate system, where the radial component increases outwards, establishes.

Then the pressure force acting on the inner side of radius r is positive, while the pressure force acting on the outer side of radius $r + dr$ is negative.

The total pressure force is then

$$P(r)dA - P(r+dr)dA = [P(r) - P(r+dr)]dA = -\frac{dP}{dr} dr\, dA \tag{1.2a}$$

The gravity force, due to spherical symmetry, points toward the centre and therefore has a negative sign i.e.

$$-\frac{GM_r dm}{r^2} \tag{1.2b}$$

where dm is the mass of the element. Obviously

$$dm = \rho(r)\, dV = \rho(r)\, dA dr \tag{1.2c}$$

Therefore the gravity is given by

$$-\frac{GM_r \rho(r) dA Dr}{r^2} \tag{1.2d}$$

So from Newton's second law, we have

$$\rho(r)dAdr\frac{d^2r}{dt^2} = -\frac{dP}{dr}dr\,dA - \frac{GM_r\rho(r)dAdr}{r^2}$$

(1.2e)

We can cancel out the factors dr and dA, and arrive at:

$$\rho(r)\frac{d^2r}{dt^2} = -\frac{dP}{dr} - \frac{GM_r\rho(r)}{r^2}$$

(1.2f)

If the star is in hydrostatic equilibrium, then the sum of all forces must vanish, i.e.,

$$-\frac{dP}{dr} - \frac{GM_r\rho(r)}{r^2} = 0$$

(1.2g)

Moving the second term to the right hand side, we have

$$\frac{dP}{dr} = -\frac{GM_r\rho(r)}{r^2}$$

(1.2h)

1.3.3 Energy conservation

Stars lose energy via radiation. The radiation loss must be balanced by energy generated by nuclear reactions. The energy conservation equation express this in mathematical terms as:

Let $L(r)$ be the energy flow across the sphere with radius r, in units of W, then the net energy loss in the shell from r to $r+dr$ is

$$L(r+dr) - L(r) = \frac{dL(r)}{dr}dr$$

(1.3a)

If ε is the energy generation per kg, then the total energy generated in the shell is

$$dE = \varepsilon\rho(r)4\pi r^2 dr$$

(1.3b)

For the gas to be in thermal equilibrium, the radiation loss must be equal to the energy gain from nuclear burning. Therefore we have

$$\frac{dL(r)}{dr}dr = \varepsilon\rho(r)4\pi r^2 dr$$

(1.3c)

dr Cancels out, so we have

$$\frac{dL(r)}{dr} = \varepsilon\rho(r)4\pi r^2$$

(1.3d)

In problems, where the thermal properties of the model are either not to be investigated or are not important, the equilibrium structure of the gaseous sphere may be determined by solving equations (1.1c and 1.2h) using some suitable equations of state together with boundary conditions

At the centre $r = 0,\ M(r) = 0$

At the surface $r = 0,\ M(r) = 0, P = 0, or\ P_s, \rho = 0 or\ \rho_s$

A number of the theoretical as well as numerical studies regarding the equilibrium structure of gaseous spheres, particularly those which have particular reference to the problems of the equilibrium structures of the stars are available in literature (Chandrasekhar (8), Eddington (19), Meznal et al. (49), Cox and Giuli (13), Kippenhahn and Weigert (29)).

1.4 AVERAGING TECHNIQUE OF KIPPENHAHN AND THOMAS

In order to study the effects of rotation and tidal distortions on the equilibrium structure of gaseous spheres, Kippenhahn and Thomas (28) developed the concept of topologically equivalent surfaces corresponding to actual equipotential surfaces of a rotationally and tidally distorted model. They define on these equivalent spherical surfaces, quantities such as $\bar{f}, \bar{g}$ etc. which denote certain averages of the quantities f, g, respectively on the actual equipotential surfaces. If Ψ denotes the total potential due to gravitation, rotation and tidal forces of a rotationally and tidally distorted model at an arbitrary point $P(x, y, z)$ then $\Psi(x, y, z)$ = constant, is a equipotential surface. Let V_Ψ be the volume enclosed by the equipotential surface. Ψ = constant and S_Ψ the surface area of this equipotential surface. For any function $f(x, y, z)$ they define $\bar{f}$ as its mean value over the equipotential surface Ψ = constant by the relation

$$\bar{f} = \frac{1}{S_\Psi} \int_{\Psi=const} f d\sigma \tag{1.4}$$

where $d\sigma$ denotes the surface element of the equipotential surface Ψ = constant. Clearly $\bar{f}$ is a function of equipotential surface Ψ = constant only and can be obtained as equation(1.4) for each equipotential surface Ψ = constant. Kippenhahn and Thomas (1970) also define a variable r_Ψ in analogy with the radius of sphere by the relation

$$V_\Psi = \frac{4}{3} \pi r_\Psi^{\,3} \tag{1.5}$$

Also by definition

$$S_\Psi = \int_{\Psi=const} d\sigma \tag{1.6}$$

Obviously, in general, S_Ψ is not equal to $4\pi r_\Psi^{\,2}$. Kippenhahn and Thomas (28) define a function $g(x, y, z)$ by the relation

$$g = \frac{d\Psi}{dn} \tag{1.7}$$

This g corresponds to the force of gravity of a sphere. The distance dn between two neighboring surfaces Ψ = constant and $\Psi + d\Psi$ = constant is, general, not constant (i.e. not same at all points of the surface). They used (1.7) to compute the mean values $\bar{g}$ and $\bar{g}^{-1}$ with the help of relations

$$\bar{g} = \frac{1}{S_\Psi} \int_{\Psi=const} \frac{d\Psi}{dn} d\sigma \tag{1.8}$$

$$\bar{g}^{-1} = \frac{1}{S_\Psi} \int_{\Psi=const} \left(\frac{d\Psi}{dn} \right)^{-1} d\sigma \tag{1.9}$$

Both $\bar{g}$ and $\bar{g}^{-1}$ are functions of Ψ alone and represent the value of g and $\bar{g}^{-1}$ respectively over the topologically equivalent spherical surface. The volume dV_Ψ between the surface Ψ = constant and $\Psi + d\Psi$ = constant is given by

$$dV_\Psi = \int_{\Psi=const} dn d\sigma = \int_{\Psi=const} \left(\frac{d\Psi}{dn} \right)^{-1} dn = S_\Psi \bar{g}^{-1} d\Psi \tag{1.10}$$

Kippenhahn and Thomas() also defined non-dimensional parameters u, v and w as

$$u = \frac{S_\Psi}{4\pi r_\Psi{}^2}, \quad v = \frac{\bar{g} r_\Psi{}^2}{GM_\Psi}, \quad w = \frac{\bar{g}^{-1} GM_\Psi}{r_\Psi{}^2} \tag{1.11}$$

where M_Ψ is the mass enclosed by equipotential surface Ψ = constant.

We may thus regard the equipotential surface Ψ = constant to be topologically equivalent to a sphere of radius r_Ψ for which various functions are defined by the above relations. It may be noticed that if Ψ is the gravitational potential of a sphere then the surface Ψ = constant is spherical surface with $r_\Psi = r$ for which u=1 and $g = GM_\Psi / r_\Psi{}^2$ is constant on these spheres and therefore v and w are constants and equal to 1.

Equations (1.4) to (1.11) are purely mathematical definitions, which have been applied by Kippenhahn and Thomas (28) to gravitational fields of gaseous spheres distorted by rotational and tidal forces. In hydrostatic equilibrium the equipotential surfaces are also surface of equipressure and equidensity. Therefore, on an equipotential surface the pressure P_Ψ and the density ρ_Ψ are also constant. Using these concepts of Kippenhahn and Thomas (28) obtained the equations governing the equilibrium structure of a rotationally and tidally distorted stellar model in the following manner

From equation (1.5) the mass dM_Ψ between the equipotential surface $\Psi =$ constant and $\Psi + d\Psi =$ constant is given by

$$dM_\Psi = dV_\Psi \rho_\Psi = 4\pi r_\Psi^{\,2} \rho_\Psi dr_\Psi \tag{1.12}$$

Thus, we get

$$\frac{dM_\Psi}{dr_\Psi} = 4\pi r_\Psi^{\,2} \rho_\Psi \tag{1.13}$$

From equation (1.9) and (1.12) we have

$$d\Psi = \frac{d\Psi}{dV_\Psi} dV_\Psi = \left(\frac{dV_\Psi}{d\Psi} \right)^{-1} \frac{dM_\Psi}{\rho_\Psi} = \frac{dM_\Psi}{S_\Psi \bar{g}^{-1} \rho_\Psi} \tag{1.14}$$

Using relations (1.10), we get

$$d\Psi = \frac{GM_\Psi dM_\Psi}{4\pi r_\Psi^{\,4} \rho_\Psi uw} \tag{1.15}$$

The conditions for hydrostatic equilibrium, $dP_\Psi / d\Psi = -\rho_\Psi$, can now be written with equation (1.10) in the form

$$\frac{dP_\Psi}{dM_\Psi} = -\frac{GM_\Psi}{4\pi r_\Psi^{\,4}} f_p \tag{1.16}$$

where

$$f_p = \frac{1}{uw} = \frac{4\pi r_\Psi^{\,4}}{GM_\Psi} \frac{1}{S_\Psi \bar{g}^{-1}}$$

The factor f_p is a function of Ψ only. If Ψ is known the equipotential surface can be determined, and then consequently values of $S_\Psi, r_\Psi, \bar{g}, \bar{g}^{-1}$ for equipotential surface can be obtained simply from the geometry of the equipotentials. The mass M_Ψ which depends on the density distribution ρ_Ψ can be determined by integrating the equation (1.13). similarly the other structure equations derived by Kippenhahn and Thomas (30), which includes the effect of rotation and tidal distortions on the equilibrium structure of gaseous sphere are as follows.

For chemically homogenous spheres, nuclear energy generation rate ε depends only upon density ρ_Ψ and temperature T_Ψ and are, therefore, constant on equipotential surface. Thus, if L_Ψ is the energy passes per second through the equipotential surface Ψ = constant, then

$$\frac{dL_\Psi}{dM_\Psi} = \varepsilon \tag{1.17}$$

Using equation (1.12), it can be written as

$$\frac{dL_\Psi}{dr_\Psi} = 4\pi r_\Psi^{\,2} \rho_\Psi \varepsilon \tag{1.18}$$

If the energy is transported by radiation, then the energy transport equation is

$$F_\Psi = -\frac{4acT_\Psi^{\,3}}{3\kappa} \frac{d\Psi}{dn} \frac{dT_\Psi}{dM_\Psi} \frac{4\pi r_\Psi^{\,4} uw}{GM_\Psi} \tag{1.19}$$

where F_Ψ is the radioactive flux on the equipotential surface Ψ = constant. By integrating F_Ψ over the equipotential surface Ψ = constant , we get

$$L_\Psi = \int_{\Psi=const} F_\Psi d\sigma$$

$$= -\frac{4acT_\Psi{}^3}{3\kappa}\frac{dT_\Psi}{dM_\Psi}uw\frac{4\pi r_\Psi{}^4}{GM_\Psi}\int_{\Psi=const}\left(\frac{d\Psi}{dn}\right)d\sigma$$

$$= \frac{64\pi^2 acT_\Psi{}^3 r_\Psi{}^4}{3\kappa}u^2 vw\frac{dT_\Psi}{dM_\Psi}$$

(1.20)

So that

$$\frac{dT_\Psi}{dM_\Psi} = -\frac{3\kappa L_\Psi}{64\pi^2 acT_\Psi{}^3 r_\Psi{}^4} f_T \tag{1.21}$$

Using equation (1.12), this equation can be expressed as

$$\frac{dT_\Psi}{dr_\Psi} = -\frac{3\kappa L_\Psi \rho_\Psi}{16\pi acT_\Psi{}^3 r_\Psi{}^2} f_T \tag{1.22}$$

where $$f_T = \frac{1}{u^2 vw}$$

Equations (1.13),(1.16),(1.17) and(1.21) which are the four basic equations governing the equilibrium structure of a gaseous sphere distorted by rotational and tidal forces.

These reduce to the normal equations used for determining the equilibrium structures of spherical models of stars when distortion parameters *u, v, w* are set one each. The boundary conditions which the above equations have to satisfy are:

$M_\Psi = 0$, $L_\Psi = 0$, at the center $r_\Psi = 0$ (1.23a)

$M_\Psi = M_0$, $L_\Psi = L_{\Psi S}$, $P_\Psi = 0$, $T_\Psi = 0$

Or $P_\Psi = P_{\Psi S}$, $T_\Psi = T_{\Psi S}$ at the free surface $r_\Psi = R_\Psi$ (1.23b)

Where M_0 is the total mass of the model and $L_{\Psi S}$, $P_{\Psi S}$, $T_{\Psi S}$ are the values of L_Ψ, P_Ψ, T_Ψ respectively, on the outermost equipotential surface.

1.5 THE PRESENT WORK

The problem of determining the structure of equilibrium structures of stars distorted by rotation tidal forces has a great importance in astrophysics. Such a problems will help in better understanding the nature of inner structure of rotating stars and stars in binary/ or multiple systems. Thus, there is need for in depth investigation the equilibrium structure of gaseous sphere.

Analytic study of determining the equilibrium structure of rotationally and tidally distorted stellar models is quite complex. Therefore, investigation attempted to solve such a problems in some approximate way. In one such attempt Mohan, Saxsena and Agarwal (62,63) used kippenhahn and Thomas averaging technique together with the results of Kopal (32) on Roche equipotential, to determine the effects of rotation and tidal forces on the equilibrium structures of eccentric binaries. They also implemented this approach in the case of the polytropic models of the stars as well as certain realistic main sequence stars. Several authors such as Lal (40), Sharma (84), Saini (80), Pathenia (72) used this method to study some astrophysical problems. However, the problems of determining the equilibrium structures of non-synchronously rotating binary is not satisfactory tacked so far.

The thesis consists of three chapters. Chapter one deals with the importance of studying equilibrium structure of rotating and tidally distorted gaseous spheres. In the present thesis an attempt has been made to modify the Roche equipotential that takes into account for non-synchronous rotation and eccentric orbits. Using average technique of Kippenhahn and Thomas (28) is conjunction with results of modified Roche equipotential. Kippenhahn and Thomas (28) averaging technique is also discussed briefly in this chapter. This chapter ends with the brief survey of literature available on this subject and summary of the work presented in the succeeding chapter.

In chapter II, we firstly discuss the approach of Sepinsky et al.(2007) for determining the Roche equipotential, which takes into account the motion of mass elements in non-synchronous, eccentric binary and planetary system. The results of such a roche equipotential are next used in conjunctions with Kippenhahn and Thomas

averaging technique to develop analytic expressions representing equilibrium structure of non-synchronously rotating and tidally distorted stellar models. The equilibrium structures equations marginally deviates from the equilibrium structure equations earlier discussed by Mohan, Saxena and Agarwal (62, 63) for non-synchronously rotation.

The methodology developed in chapter – II has next been used in chapter-III to determine the equilibrium structure of rotationally and tidally distorted polytropic models. Certain results of physical parameter have been computed for various polytropic indices 1.5, 3.0 and 4.0. Conclusion based on the present study has finally been discussed.

Chapter - II

EQUILIBRIUM STRUCTURES OF NONSYNCRONOUSLY ROTATIONLLY AND TIDALLY DISTORTED STELLAR MODELS

This chapter is organized as follows. In section 2.1, we find the Roche equipotential of eccentric binaries. In section 2.2,we'll apply the approach of Mohan, Saxena and Agarwal for computing the equilibrium structure of rotationally and tidally distorted gaseous sphere.

2.1 ROCHE EQUIPOTENTIAL OF DISTORTED STARS

To introduce the concept of Roche equipotential, we have two components of a close binary system called as primary and secondary star. The primary component is assumed very much massive than secondary, so we assumed it to be point mass. Both the components of binary systems are supposed to be rotating about their axes as well as revolving about their common center of mass in a fixed Keplerian orbit with period P_{orb}, semi major axis a, and eccentricity e. The distance D between the mass centers of the two stars and the Keplerian orbital angular velocity Ω_k of the stars are, at any time given by

$$D(t) = \frac{a(1-e^2)}{1+e\cos\nu} \tag{2.1}$$

$$\Omega_\kappa(t) = \frac{2\pi}{P_{orb}} \frac{(1+e\cos\nu)^2}{(1-e^2)^{3/2}} \tag{2.2}$$

where ν is the true anomaly.

Following Kopal, Mohan, and Saxena 's results on Roche equipotential which are showing below Suppose that m_1 and m_2 are the masses of two components of close binary system separated by distance D. Now assumed that the primary component of this system of mass m_1 is much massive than its secondary companion star of mass m_2 ($m_1 >> m_2$) which is assumed to be point mass. We assume that the primary component rotates uniformly about axis perpendicular to the orbital plane with a fixed rotational angular velocity Ω_1 parallel as in the same direction as the orbital angular velocity. Following Sepinsky et al (2007), total potential Ψ due to the gravitational, centrifugal forces and orbital motion acting at the point mass P is given as

$$\Psi = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} + \frac{1}{2}\Omega_1(x^2 + y^2) - \frac{Gm_2 x}{D^2} \tag{2.3}$$

Where $r_1^2 = x^2 + y^2 + z^2$ and $r_2^2 = (D-x)^2 + y^2 + z^2$ represent the distances of point P from the center of gravity of two components. Ω denotes the keplerian orbital angular velocity of rotation of the system about axis z-axis and passing through center of gravity of the system.

The first term on the right hand side of the equation (2.3) is potential arises due to mass m_1 of primary component, second term is the potential due to mass m_2 of secondary component of mass m_1,third term due to centrifugal force and last term is due to orbital motion. Equation (2.3) holds outside the components of binary system.

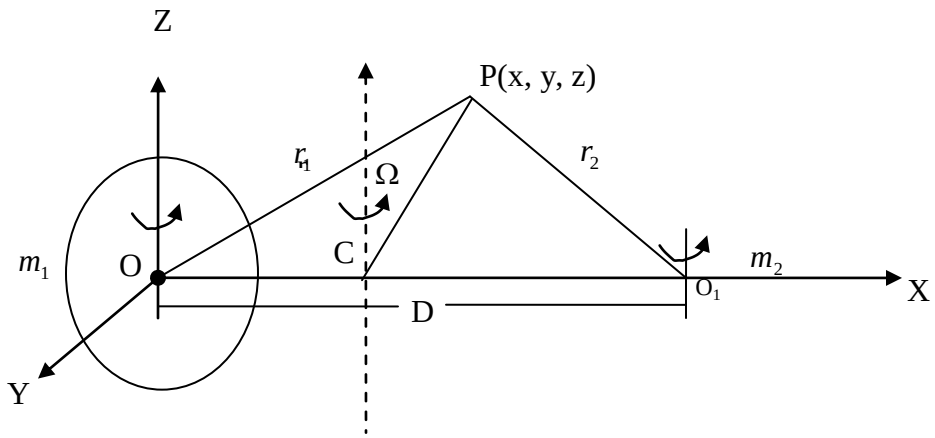

Fig 1.1: Axis of for a binary system

Equation (2.3) can be expressed in nondimensional form as….

$$\Psi^* = \frac{1}{r^*} + q\left[\frac{1}{\sqrt{1-2\lambda r^* + r^{*2}}} - \lambda r^*\right] + nr^{*2}(1-\nu^2) - \frac{q^2}{2(1+q)} \tag{2.4}$$

Where $\Psi^* = \frac{D\Psi}{Gm_1} - \frac{m_2^{\,2}}{2m_1(m_1+m_2)}$ is the non-dimensional form of total potential Ψ and $r^* = \frac{r}{D}$ is non-dimensional form of r. Also λ = sinθcosϕ, μ = sinθsinϕ and ν = cosθ(r,θ,ϕ are the spherical coordinate of the point P). Moreover,

$$q = \frac{m_2}{m_1} \tag{2.5}$$

is non-dimensional parameter representing the ratio of mass of the secondary over primary and 2n represent the square of the normalized angular velocity Ω. In the equation (2.3) when q=0, it become potential of a purely rotating spherical model and for n=0, it become potential of a non-rotating spherical model.

The surface Ψ = constant on the left hand side of equation (2.4) are known as roche equipotentials. Roche equipotential in non-dimensional form shows as Ψ^* = constant .

Now we define

$$r_0 = \frac{1}{\Psi^* - q} \tag{2.6}$$

Kopal has also shown that on the surface of roche equipotentials, (r,θ,ϕ) are connected through the relation…

$$r^* = r_0\left[1 + C_2 r_0 + C_3 r_0^{\,3} + C_4 r_0^{\,4} + C_5 r_0^{\,5} + C_6 r_0^{\,6} + C_7 r_0^{\,7} + C_8 r_0^{\,8} + C_9 r_0^{\,9} + C_{10} r_0^{\,10} + \ldots\right] \tag{2.7}$$

Where

$$C_2 = -\frac{q^2}{2},\ C_3 = a_0 = qP_2 + n(1-\nu^2),\ C_4 = qP_3,$$
$$C_5 = qP_4, C_6 = qP_5 + 3C_3^{\ 2},\ C_7 = qP_6 + 7qP_3C_3,$$
$$C_8 = qP_7 + 8C_3C_5 + 4q^2P_3^{\ 2},$$
$$C_9 = qP_8 + 9qP_5C_3 + 9q^2P_3P_4,$$
$$C_{10} = qP_9 + 5q^2P_4^{\ 2} + 10q^2P_3P_5 + 10qP_6C_3$$

This expression agrees with the expression earlier developed by Kopal (32) except the coefficient of C_2.

Here, $P_J = P_J(\lambda)$ are Legendre polynomials and terms of higher powers of n and q are neglected. This relation helps us to finding the shapes of equipotetials Ψ^*=constant.

The volume enclosed by the equipotential surface Ψ^* = constant is as follow:

$$V_\Psi = \frac{2}{3}\int_{-1}^{1}\int_{-\sqrt{1-\nu^2}}^{\sqrt{1-\nu^2}} \frac{r^3 d\lambda d\nu}{\mu}$$

(2.8)

Kopal(32) has shown the expression of V_Ψ in terms of r_0 defined by equation(2.7), can be expressed as:

$$V_\Psi = \frac{4\pi D^3 r_0^{\ 3}}{3}\left[1 - \frac{3}{2}q^2 r_0 + 2n r_0^{\ 3} + \left(\frac{12}{5}q^2 + \frac{8}{5}nq + \frac{32}{5}n^2\right)r_0^{\ 6} + \frac{15}{7}q^2 r_0^{\ 8} + 2q^2 r_0^{\ 10} + \ldots\right]$$

(2.9)

where terms are upto 2nd order of smallness in powers of n and q are retained.

Following the approach of Kopal (32), Mohan and Singh (59) obtained the expression of surface area S_Ψ and the values of averages or parameters r_Ψ, $\overline{g}$, $\overline{g}^{-1}$ on the roche equipotential Ψ^* = constant are obtained. These are…

$$S_\Psi = 4\pi {r_0}^2 D^2 \left[1 - q^2 r_0 + \frac{4}{3} n {r_0}^3 + \left(\frac{7}{5} q^2 + \frac{14}{15} nq + \frac{56}{15} n^2 \right) {r_0}^6 + \frac{9}{7} q^2 {r_0}^8 + \frac{11}{7} q^2 {r_0}^{10} + \ldots \right]$$

(2.10)

$$r_\Psi = \left(\frac{3V_\Psi}{4\pi} \right)^{\frac{1}{3}}$$

$$= D r_0 \left[1 - \frac{1}{2} q^2 r_0 + \frac{2}{3} n {r_0}^3 + \left(\frac{4}{5} q^2 + \frac{8}{15} nq + \frac{76}{45} n^2 \right) {r_0}^6 + \frac{5}{7} q^2 {r_0}^8 + \frac{2}{3} q^2 {r_0}^{10} + \ldots \right]$$

(2.11)

$$\bar{g} = \frac{2}{S_\Psi} \int_{-1}^{1} \int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \left(\frac{d\Psi}{dn} \right) \frac{r^2 d\lambda d\nu}{\mu}$$

$$= \frac{GM_\Psi}{D^2} \frac{1}{{r_0}^2} \left[1 + q^2 r_0 - \frac{8}{3} n {r_0}^3 - \left(2q^2 + \frac{4}{3} nq + \frac{28}{9} n^2 \right) {r_0}^6 - \frac{15}{7} q^2 {r_0}^8 - \frac{7}{3} q^2 {r_0}^{10} \right]$$

(2.12)

$$g^{-1} = \frac{2}{S_\Psi} \int_{-1}^{1} \int_{-\sqrt{1-\nu^2}}^{\sqrt{1-\nu^2}} \left(\frac{d\Psi}{dn} \right)^{-1} \frac{r^2 d\lambda d\nu}{\mu}$$

$$= \frac{{r_0}^2}{D^2} \left[1 + q^2 r_0 + \frac{8}{3} n {r_0}^3 + \left(\frac{26}{5} q^2 + \frac{52}{15} nq + \frac{524}{45} n^2 \right) {r_0}^6 + \frac{40}{7} q^2 {r_0}^8 - \frac{11}{7} q^2 {r_0}^{10} + \ldots \right]$$

(2.13)

where

$$\left(\frac{d\Psi}{dn} \right)^2 = {\Psi_r}^2 + \frac{1-\lambda^2}{r^2} {\Psi_\lambda}^2 + \frac{1-\nu^2}{r^2} {\Psi_\nu}^2 - \frac{2\lambda\nu}{r^2} \Psi_\lambda \Psi_\nu$$

(2.14)

2.2 MOHAN, SAXENA AND AGARWAL'S APPROACH FOR COMPUTING THE EQUILIBRIUM STRUCTURES OF ROTATIONALLY AND TIDALLY DISTORTED GASEOUS SPHERES

Mohan, Saxena and agarwal (62, 63) used concept of Roche equipotentials proposed by Kopal in conjunction with Kippenhahn and Thomas's averaging approach to explicitly obtain equations governing the equilibrium structures and periods of radial and non-radical oscillations of rotationally and/or tidally distorted stars and applied these to analyze the problems of rotating stars and stars in binary system.

In order to determine the inner structure of a rotationally and tidally distorted gaseous sphere, the system of equations (1.17) has to be integrating numerically subject to the boundary conditions (1.18) specified therein. Therefore, the evaluation of the actual equipotential surface of a rotationally and tidally distorted gaseous sphere is complicated. Kippenhahn and Thomas (48) proposed that the evaluation of the distortion parameter u, v, w ,f_p and f_T and appearing in the stellar structure equations are given by

$$u = \frac{S_\Psi}{4\pi r_\Psi{}^2}$$

$$= \left[1 - \left(\frac{1}{5}q^2 + \frac{2}{15}nq + \frac{4}{45}n^2\right) r_0{}^6 - \frac{1}{7}q^2 r_0{}^8 - \frac{1}{9}q^2 r_0{}^{10}\right] \qquad (2.15)$$

$$v = \frac{\bar{g} r_\Psi{}^2}{GM_\Psi}$$

$$= \left[1 - \frac{4}{3} n r_0^{\,3} - \left(\frac{2}{5} q^2 + \frac{4}{15} nq + \frac{128}{45} n^2 \right) r_0^{\,6} - \frac{5}{7} q^2 r_0^{\,8} - q^2 r_0^{\,10} \right] \tag{2.16}$$

$$w = \frac{g^{-1}}{r_\Psi^{\,2}} GM_\Psi$$

$$= \left[1 + 2q^2 r_0 + \frac{4}{3} n r_0^{\,3} + \left(\frac{18}{5} q^2 + \frac{12}{5} nq + \frac{272}{45} n^2 \right) r_0^{\,6} + \frac{30}{7} q^2 r_0^{\,8} - \frac{23}{9} q^2 r_0^{\,10} + \ldots \right]$$

(2.17)

$$f_p = \frac{1}{uw}$$

$$= \left[\, 1 - 2q^2 r_0 - \frac{4}{3} n r_0^{\,3} - \left(\frac{17}{5} q^2 + \frac{34}{15} nq + \frac{188}{45} n^2 \right) r_0^{\,6} - \frac{29}{7} q^2 r_0^{\,8} + \frac{8}{3} q^2 r_0^{\,10} - \ldots \right]$$

(2.18)

$$f_T = \frac{1}{u^2 vw} \qquad = \left[1 - 2q^2 r_0 - \left(\frac{14}{5} q^2 + \frac{28}{15} nq + \frac{56}{45} n^2 \right) r_0^{\,6} - \frac{23}{7} q^2 r_0^{\,8} + \frac{34}{9} q^2 r_0^{\,10} - \ldots \right]$$

(2.19)

where $r_\Psi^{*} = r_\Psi/\mathrm{D}$ is the non-dimensional form of r_Ψ and the terms up to second order of smallness in n and q are retained.

The values of M_Ψ, P_Ψ, L_Ψ etc. on the various equipotential surfaces of a rotationally and tidally distorted gaseous spheres may now be obtained by solving the system of differentially equations(1.19) with boundary conditions and using the values of distortion parameters f_p and f_T as given in equation (2.18) and (2.19).

It may be noted that while approximating the equipotential surfaces of a rotationally and tidally distorted model by Roche equipotentials, the structure of the star is not approximated by the structure of a Roche model. In the case of no distortion (n=q=0), equation (2.15 to 2.19) gives $u=v=w=1$ and the system of differential equation (1.23) reduce to the equations governing the equilibrium structure of the original undistorted star and not of the Roche model. Usual methods for stellar structure equations such as Henyey et al (48) method can be now used to integrate the system of differential equation (1.23) governing the equilibrium structure of a rotationally and tidally distorted gaseous sphere. At every step, the values of the distortion parameters, u, v, w, f_p, f_T have to be computed using equation (2.22). in the case of thermal properties are not considered important and only hydrostatic equilibrium of a rotationally and tidally distorted gaseous spheres is to be investigated then we need only to integrate equation (1.9) and (1.12) subject to the boundary conditions

At the center $r_\Psi = 0$, $M_\Psi = 0$

and at the free surface $r_\Psi = R_\Psi$ (2.22a)

$$M_\Psi = M_0, P_\Psi = 0, \rho_\Psi = 0 \; or \; P_\Psi = P_{\Psi S}, \rho_\Psi = \rho_{\Psi S} \qquad (2.22b)$$

In case the star is being distorted forces alone (or tidal forces alone) we may set q=0 (n=0) and still use the above approach to determine the equilibrium structure of corresponding purely rotationally distorted or purely tidally distorted model.

Mohan and Saxena (57,58) found it more convenient to work with r_0 in place of M_ψ or r_ψ as independent variable by using equation (2.9) which is connected with the variable r_ψ through relation (2.20) and (2.21). Saxena (80) expressed the system of differential equations governing the equilibrium structure of a rotationally and tidally distorted stellar model as

$$\frac{dM_\Psi}{dr_0} = 4\pi D^3 \rho_\Psi {r_0}^2 f_1 \tag{2.23}$$

$$\frac{dP_\Psi}{dr_0} = -\frac{GM_\Psi}{D{r_0}^2}\rho_\Psi f_2 \tag{2.24}$$

$$\frac{dL_\Psi}{dr_0} = 4\pi \varepsilon D^3 \rho_\Psi {r_0}^2 f_1 \tag{2.25}$$

here f_1, f_2, f_3 are functions of n, q and r_0 incorporating the effects of rotation and tidal distortions on the equilibrium structure equations of a stellae model. The explicit expressions for these distortion parameters as given by Saxena (80) are

$$f_1 = \left[1 - 2q^2 r_0 + 4n{r_0}^3 + \left(\frac{36}{5}q^2 + \frac{24}{5}nq + \frac{96}{5}n^2\right){r_0}^6 + \frac{55}{7}q^2{r_0}^8 + \frac{26}{3}q^2{r_0}^{10} + \ldots\right] \tag{2.26}$$

$$f_2 = \left[1 - 2q^2 r_0 + \left(\frac{3}{5}q^2 + \frac{2}{5}nq + \frac{4}{15}n^2\right){r_0}^6 + \frac{6}{7}q^2{r_0}^8 + \frac{26}{3}q^2{r_0}^{10} + \ldots\right] \tag{2..27}$$

$$f_3 = \left[1 - 2q^2 r_0 + \frac{4}{3}n{r_0}^3 + \left(\frac{6}{5}q^2 + \frac{4}{5}nq + \frac{224}{45}n^2\right){r_0}^6 + \frac{12}{7}q^2{r_0}^8 + \frac{88}{9}q^2{r_0}^{10} + \ldots\right] \tag{2.28}$$

In these above expressions terms up to second order of smallness in n,q r_0^{10} and r_0 upto in are retained. The boundary conditions governing the system of differential equation (2.23 -2.25) are:

At the center

$$r_0 = 0,\ M_\Psi = 0\ ,\ L_\Psi = 0 \tag{2.29}$$

And at the free surfaces, $r_{0=}\, r_{0S}$

$$M_\Psi = M_0, L_\Psi = L_{\Psi S}$$

$$P_\Psi = 0,\ \rho_\Psi = 0, T_\Psi = 0 \text{ or } P_\Psi = P_{\Psi S,}\ \rho_\Psi = \rho_{\Psi S,}\ T_\Psi = T_{\Psi S} \tag{2.30a}$$

where r_{0S} being the value of r_0 at the free surfaces.

In fact

$$r_{0S} = \frac{1}{\Psi_S^{\ *} - q}$$

(2.30b)

where is the non dimensional form of the total potential ψ on the outermost equipotential surface of $\psi_s^{\ *}$ a rotationally and tidally distorted stellar model. In case of no distortion f_P = f_T =1 and the above equations reduce to the usual equations governing the equilibrium structure of an undistorted gaseous sphere.

CHAPTER-III

EQUILIBRIUM STRUCTURES OF NON-SYNCHRONOUSLY ROTAING AND TIDALLY DISTORTED POLYTROPIC MODEL OF THE STARS

In section 3.1 we briefly discuss the internal structure of the polytropic stars. The methodology developed in chapter –II is next used in section 3.2 to determine the equilibrium structure of rotationally and tidally distorted polytropic models. Numerical computations have been performed to obtain the inner structure, shapes and values of other physical parameters of rotationally and tidally distorted polytropic models of stars are obtained in section 3.3. Analysis of the results with conclusion is finally discussed in section 3.4.

3.1 POLYTROPIC MODELS OF STARS

Polytropic models have frequently been used in literature to explain the inner structures of realistic stars. A polytropic model is a model in which the quantity of heat supplied (dq) is directly proportional to the instantaneous change of temperature (dt), so that dq/dt is constant. In a polytropic model the pressure P and the density ρ at an arbitrary point in the model are given by the relation

$$P = P_c\theta^{N+1} \text{ and } \rho = \rho_c\theta^N \tag{3.1}$$

where P_c and ρ_c are, respectively the pressure and density at the center of the model and θ $(0 \le \theta \le 1)$ is a parameter depending upon the distance of the chosen point from the center. The index N used in relation (3.1) is known as polytropic index of the model. For cases of practical interest to the problems of stellar structure N lies between zero to five. Index N is a measure of the central condensation of the model (larger the value of N greater is the central condensation). Whereas a polytropic model of index zero has a homogenous structure in which the density is uniform throughout the model, a polytropic model of index five is very highly centrally condensed model whose radius extends to infinity.

The equilibrium structure of polytropic model of index N is determined by the solution of the nonlinear differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^N \tag{3.2}$$

subject to the boundary conditions $\theta=1, \frac{d\theta}{d\xi}=0$ at the center $\xi=0$ (3.3a)

and θ=0 at the surface (3.3b)

Equation (3.2) is known as the Lane-Emden equation. Analytical solutions of this equation are only possible for N=0, 1, and 5. Chandrasekhar (8) and several other investigators have obtained numerical solutions of Lane-Emden equation (3.2) which satisfy condition (3.3) for various values of the polytropic index N lying between 0 and 5.

Once the solution of Lane-Emden equation (3.2) has been obtained which satisfies condition (3.3), the values of various physical parameters of the star having inner structure as a polytrope of index N can be determined.

3.2 EQUILIBRIUM STRUCTURES OF ROTATIONALLY AND TIDALLY DISTORTED POLYTROPIC MODELS OF STARS

Suppose a polytropic model is subject to rotation and tidal distortion then its structure becomes rotationally and tidally distorted polytropic model. Following the approach of section (2.2) we may approximate the equipotential surfaces of such a distorted model by modified Roche equipotential surfaces which incorporate the effect of eccentricity besides centrifugal force.

Let P_Ψ denote the pressure and ρ_Ψ density on the equipotential surface Ψ^*=constant of the distorted model. Then the value of the density and pressure on the equivalent surface of the corresponding topologically equivalent spherical model will also be ρ_Ψ and P_Ψ, respectively. We assume that the distorted model behaves like a polytropic model so that in its case also ρ_Ψ and P_Ψ are connected through the polytropic relations of the type

$$P_\Psi = P_{C\Psi}\theta_\Psi{}^{N+1} \text{ and } \rho_\Psi = \rho_{C\Psi}\theta_\Psi{}^{N} \tag{3.3}$$

where N is polytropic index and $P_{C\Psi}, \rho_{C\Psi}$ the values of P_Ψ and ρ_ψ respectively at the center. Here θ_Ψ represents some average of the value of polytropic parameter θ at various point on the equipotential surface Ψ^* = constant. In the case of polytropic models, the equations (2.23) and (2.24) which govern the hydrostatic equilibrium structure of rotationally and tidally distorted gaseous sphere can be combined together with the equation (3.3) to yield

$$\frac{1}{r_0{}^2}\frac{d}{dr_0}\left(\frac{r_0{}^2}{f_2}\frac{d\theta_\Psi}{dr_0}\right) = -\frac{R^2}{l^2} f_1\theta_\Psi{}^{N} \tag{3.4}$$

where

$$l^2 = \frac{(N+1)P_{C\Psi}}{4\pi G\rho_{C\Psi}{}^2}$$

and f_1, f_2 are the distortion parameters as given in equation (2.26) and (2.27).

As regards the boundary conditions, P_Ψ and ρ_Ψ must be maximum at the center and zero at the free surface. These require θ_Ψ to be maximum at center and zero at the free surface. This leads to the conditions: θ_Ψ =1 and $\frac{d\theta_\Psi}{dr_0} = 0$ at the center and θ_Ψ =0 at the free surface. The boundary conditions which satisfy equation (3.4) are

$$\theta_\Psi = 1,\ \frac{d\theta_\Psi}{dr_0} = 0, \text{ at the center } r_0 = 0 \tag{3.5a}$$

and $$\theta_\Psi = 0, \text{ at the surface } r_0 = r_{0S} \tag{3.5b}$$

where r_{0S} is the value of r_0 at surface.

In the absence of any distortions ($f_1 = f_2 = 1$), equation (3.4) reduces to Lane-Emden equation governing the equilibrium structure of an undistorted polytropic model of index N in non-dimensional form.

The quantity *l* as defined in equation (3.4) is of the dimension of length. If we set $r_\Psi = l\xi$, then ξ is a nondimensional variable defined for the topologically equivalent spherical model. It corresponds to Emden variable ξ of Lane-Emden equation for an undistorted spherical polytropic model.

If we set $R = l\xi_u$ where ξ_u is the value of ξ at the outermost surface of the undistorted polytropic model in equation (3.4), the differential equation governing the equilibrium structure of a rotationally and tidally distorted polytropic model may be written in nondimensional form as

$$\frac{d}{dr_0}\left[A(r_0,n,q)\frac{d\theta_\Psi}{dr_0}\right] = -\frac{\xi_u^{\,2}}{K^2}\theta_\Psi^{\,N} r_0^{\,2} B(r_0,n,q) \tag{3.6}$$

where

$$A(r_0,n,q) = \frac{r_0^{\,2}}{f_2} = r_0^{\,2}\left[1 + 2q^2 r_0 - \left(\frac{3}{5}q^2 + \frac{2}{5}nq + \frac{4}{15}n^2\right)r_0^{\,6} - \frac{6}{7}q^2 r_0^{\,8} - \frac{26}{3}q^2 r_0^{\,10} - \ldots\right]$$

$$B(r_0,n,q) = f_1 = \left[1 - 2q^2 r_0 + 4n r_0^{\,3} + \left(\frac{36}{5}q^2 + \frac{24}{5}nq + \frac{96}{5}n^2\right)r_0^{\,6} + \frac{55}{7}q^2 r_0^{\,8} + \frac{26}{3}q^2 r_0^{\,10} + \ldots\right]$$

with $r_0 = \dfrac{1}{\Psi^* - q}$ and in these expression terms up to second order of smallness in n,q and up to $r_0^{\,10}$ in r_0 are retained. The dimensionless *K* in equation (3.6) is the ratio of the undistorted radius R_Ψ of the primary to the separation *D* between the centers of primary and secondary star. In fact

$$\frac{D}{l} = \frac{D\xi_u}{l\xi_u} = \frac{D}{R_\Psi}\xi_u = \frac{1}{K}\xi_u \tag{3.7}$$

where ξ_u is the value of ξ at the outermost surface of the undistorted polytropic model.

In this section we present explicit expressions for computing the volume, surface area and the shape of a rotationally and tidally distorted polytropic model. Following the approach of Mohan and Saxena (54) the total volume enclosed by rotationally and tidally distorted polytropic model is given by

$$V_\Psi = \frac{4\pi}{3}\left(\frac{l\xi_u}{K}\right)^3 r_0{}^3\left[1-\frac{3}{2}q^2 r_{0S} + 2n r_{0S}{}^3 + \left(\frac{12}{5}q^2 + \frac{8}{5}nq + \frac{32}{5}n^2\right) r_{0S}{}^6 + \frac{15}{7}q^2 r_{0S}{}^8 + 2q^2 r_{0S}{}^{10} + \ldots\right] \quad (3.8)$$

$$S_\Psi = 4\pi r_{0S}{}^2\left(\frac{l\xi_u}{K}\right)^2\left[1-q^2 r_{0S} + \frac{4}{3}n r_{0S}{}^3 + \left(\frac{7}{5}q^2 + \frac{14}{15}nq + \frac{56}{15}n^2\right) r_{0S}{}^6 + \frac{9}{7}q^2 r_{0S}{}^8 + \frac{11}{7}q^2 r_{0S}{}^{10} + \ldots\right] \quad (3.9)$$

3.3 ANALYSIS AND CONCLUSION OF RESULTS

To obtain the inner structure, the shape, the volume and the surface area of a rotationally and tidally distorted polytropic model, equation (3.6) has be integrated numerically subject to the boundary condition (3.5) for specified values of the parameters N, ξ_u ,n,q and K which denote the polytropic index, the radius of the undistorted polytropic model, the non-dimensional measure of angular velocity of rotation , the ratio of the undistorted radius of the primary to the distance between the centers of the primary and secondary. For a polytropic model distorted by rotational forces alone *K*=1.

For obtaining the numerical solutions, equation (3.6) has been integrated numerically using fourth-order Runge-Kutta method for the specified values of input parameters. A series solution similar to the one variable for undistorted polytropic models (see Chandrasekhar (8) page 95) was developed to start integration at points near the center. The series solution is given by

$$\theta_\Psi = 1 - \frac{\xi_u^{\,2}}{6K^2} r_0^{\,2} + \frac{7q^2\xi_u^{\,3}}{18K^3} r_0^{\,3} + \frac{N\xi_u^{\,4}}{120K^4} r_0^{\,4} - \left(\frac{2n\xi_u^{\,2}}{15K^2} + \frac{7Nq^2\xi_u^{\,4}}{540K^4} + \frac{N\xi_u^{\,2}}{75K^2} \right) r_0^{\,5} - \left(\frac{(8N^2 - 5n)\xi_u^{\,6}}{15120K^6} \right) r_0^{\,6} + \ldots$$

(3.10)

Taking starting values from this series solution at r_0 = 0.005, numerical integration of equation (3.6) was carried forward using Runge-Kutta method of order four. Using the step length of 0.005, numerical integration was continued till θ_Ψ first became zero. Once outermost radius r_{0s} is known, the volume, surface area and shape of the distorted polytropic models can be obtained as equations (3.8 and 3.9)

Numerical results obtained from different parameters are tabulated in Table (3.1). The value of the parameter *K* has been taken as one for the rotationally distorted and tidally distorted models. The values of θ_Ψ, P_Ψ *and* ρ_Ψ for various types of distorted polytropic models of indices 1.5 ,3.0 and 4.0 are given in table 3.1.

The results for the Volume and Surface area given by table 3.2 shows that for values of different polytropic indices 1.5, 3.0 and 4.0, .on comparing the results of the volume and surface area of rotationally and tidally distorted model with the corresponding results of Mohan and Saxena( ), our results increased marginall by 2% and 1% respectively. The values of θ_Ψ remains approximately same as obtained by Mohan and Saxena ( ).

Table 3.1 Effect of inclusion of orbital motion on the equilibrium structures of rotating stars and stars in binary systems having polytropic structure

X	Uniformly Rotating			Tidally distorted			Non Synchronous binary		
	n=0.1, q=0.0, k=1.0			n=0.0 , q=0.1, k=0.5			n=0.1 , q=0.1, k=0.5		
	θ_Ψ	P_Ψ	ρ_Ψ	θ_Ψ	P_Ψ	ρ_Ψ	θ_Ψ	P_Ψ	ρ_Ψ
POLYTROPIC INDEX N = 1.5									
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	0.9189	0.8095	0.8809	0.9144	0.7996	0.8744	0.9151	0.8010	0.8754
0.4	0.7081	0.4220	0.5959	0.6948	0.4025	0.5792	0.6972	0.4059	0.5822
0.6	0.4413	0.1293	0.2931	0.4249	0.1176	0.2769	0.4289	0.1205	0.2809
0.8	0.1902	0.0157	0.0829	0.1811	0.0139	0.0770	0.1855	0.1483	0.0799
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
POLYTROPIC INDEX N=3.0									
0.0	1.0000	1.0000	1.0000	1.000	1.000	1.000	1.0000	1.0000	1.0000
0.2	0.7592	0.3323	0.4377	0.7506	0.3175	0.4230	0.7511	0.3183	0.4238
0.4	0.4137	0.0293	0.7084	0.4028	0.2633	0.0653	0.4034	0.2648	0.0656
0.6	0.1979	0.0015	0.0077	0.1913	0.0013	0.0070	0.1915	0.0013	0.0070
0.8	0.0748	0.0000	0.0004	0.7247	0.0000	0.0003	0.0724	0.0000	0.0003
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
POLYTROPIC INDEX N=4.0									
0.0	1.0000	0.7841	0.8068	1.0000	0.7640	0.7998	1.0000	0.7405	0.8002
0.2	0.4436	0.1719	0.0387	0.4354	0.0580	0.1025	0.4358	0.0157	0.0360
0.4	0.1804	0.0001	0.0010	0.1758	0.0001	0.0009	0.1760	0.0001	0.0053
0.6	0.0806	0.0000	0.0000	0.07834	0.0000	0.0000	0.0784	0.00002	000001

0.8	0.0302	0.0	0.0000	0.0292	0.0	0.0	0.0292	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table:3.2 Volumes and Surface areas of rotating stars and stars in binary systems having polytropic structures

Type of Model	N	q	K	V_Ψ x10^{-2}	S_Ψ x10^{-2}	V_Ψ x10^{-3}	S_Ψ x10^{-2}	V_Ψ x10^{-3}	S_Ψ x10^{-3}
				Polytropic Index N=1.5		Polytropic Index N=3.0		Polytropic Index N=4.0	
Uniformly Rotating	0.1	0.0	1.0	2.3222	1.8289	1.6345	6.7182	17.3364	3.2435
Tidally Distorted	0.0	0.1	0.5	2.0686	1.6914	1.4008	6.0542	14.5174	2.8778
Non-synchronous	0.1	0.1	0.5	2.1006	1.7076	1.4307	6.1354	14.8941	2.9251