DUALITY IN MATHEMATICAL PROGRAMMING UNDER GENERALIZED CONVEXITY

A

Thesis

submitted in fulfilment of the requirements for the award of degree of

DOCTOR OF PHILOSOPHY

IN

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BY

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CANDIDATE'S DECLARATION

I hereby certify that the work being presented in the thesis entitled **"Duality In Mathematical Programming Under Generalized Convexity"** in fulfilment of the requirements for the award of degree of Doctor of Philosophy, submitted in the School of Mathematics & Computer Applications of Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. S. K. Gupta, Assistant Professor, Department of Mathematics, Indian Institute of Technology Patna and Dr. M. K. Sharma, Assistant Professor, School of Mathematics & Computer Applications, Thapar University, Patiala.

The matter embodied in this thesis has not been submitted by me to any other university or institute for the award of any other degree.

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TO

MY PARENTS

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Abstract

The work being presented in the present thesis is devoted to the study of duality results for some mathematical programming problems under generalized convexity assumptions. The chapterwise summary of the thesis is as follows:

Chapter 1 is introductory and consist of nonlinear and multiobjective programming problems, definitions, notations and prerequisites of the present work. A brief account of the related studies made by various authors in the field and a summary of the thesis are also presented.

In **Chapter 2**, we have considered Wolfe type second-order multiobjective symmetric dual programs involving nondifferentiable functions and appropriate duality theorems are established using the notion of second-order F-convexity assumptions. Moreover, an example has been given which is second-order F-convex but not convex. Further, these symmetric dual programs are generalized over arbitrary cones and usual duality results are obtained under second-order (F, α, ρ, d)-convexity assumptions. A non-trivial example which shows that second-order (F, α, ρ, d)-convex functions are generalization of second-order F-convex functions has also been exemplified.

In **Chapter 3**, a new pair of second-order multiobjective symmetric dual programs in which the objective function is optimized with respect to an arbitrary closed convex cone is formulated and appropriate duality relations are then obtained under K-η-bonvexity assumptions. We identify a function lying exclusively in the class of K-η-bonvex and not in class of invex function already existing in literature. Self duality for this pair is also obtained by assuming the functions involved to be skew-symmetric. Further, we have considered a pair of Mond-Weir type nondifferentiable multiobjective second-order symmetric dual programs over arbitrary cones, where each of the objective function contains a square root term with positive semidefinite matrix in $R^{n\times n}$. Weak, strong and converse duality results are then established under K-η-bonvexity/second-order K-F-convexity assumptions.

In **Chapter 4**, we have established duality relations for a pair of second-order mixed symmetric dual programs involving nondifferentiable functions under second-order F-convexity/pseudoconvexity assumptions. Next, we have considered a pair of mixed second-order symmetric dual programs over cones and obtained duality results under second-order (F,ρ) convexity/pseudoconvexity assumptions. This mixed formulation unifies two second-order symmetric dual formulations exist in the literature. Several known results [39, 55, 57, 70] are obtained as special cases.

In **Chapter 5**, we have formulated a pair of second-order multiobjective mixed symmetric dual programs over arbitrary cones and obtained appropriate duality results under second-order invexity/pseudoinvexity assumptions. Further, we construct a pair of multiobjective second-order mixed nondifferentiable symmetric dual programs involving the square root of a positive semidefinite quadratic function, $(x^T Bx)^{\frac{1}{2}}$. The usual duality results are then established using the notion of second-order F-convexity/pseudoconvexity assumptions.

Agarwal et al. [3] extended the results of Chen [43] over arbitrary cones and proved appropriate duality relations under higher-order K-F-convexity assumptions. Mond-Weir type duality has been discussed in both the papers. In **Chapter 6**, we have studied higher-order Wolfe type multiobjective symmetric dual programs over arbitrary cones and the duality results are then established under higher-order (F, α, ρ, d)-convexity/pseudo-convexity assumptions. We have also illustrated a nontrivial example of function lying in the class of higher-order K-(F, α, ρ, d)-convex but not in class of higher-order K-F-convex. Further, we consider the higher-order multiobjective symmetric nondifferentiable dual programs in which the objective function is optimized with respect to an arbitrary closed convex cone and proved duality theorems under higher-order-K-(F, α, ρ, d)-convexity assumptions.

In **Chapter 7**, motivated by Lai et al. [85], Lai and Lee [84] and Antczak [19], we have discussed sufficient optimality conditions and duality theorems for a nondifferentiable minimax fractional programming problem with B-(p, r)-invexity. An example which is B-$(1, 1)$-invex but not (p, r)-invex is exemplified. We also illustrate another example which is $(-1, 1)$-invex but not convex.

In **Chapter 8**, we have formulated a pair of multiobjective fractional variational symmetric dual problems for a class of nondifferentiable functions over arbitrary cones and achieved duality results under generalized (F, α, ρ, d)-convexity assumptions. A self duality theorem is also obtained by assuming the functions involved to be skew-symmetric.

At the last, an **Appendix A** has been given, in which we establish a strong duality theorem for a pair of multiobjective second-order symmetric dual programs. This removes an omission in an earlier result in Yang et al. [140].

List of Research Papers

1. “Higher-order (F, α, ρ, d)-convexity and symmetric duality in multiobjective programming ”, Computers and Mathematics with Applications, 60 (2010) 2373-2381.

2. “A note on multiobjective second-order symmetric duality ”, European Journal of Operational Research, 201 (2010) 649-651.

3. “Nondifferentiable multiobjective second-order symmetric duality ”, Optimization Letters, 5 (2011) 125-139.

4. “Generalized second-order mixed symmetric duality in nondifferentiable mathematical programming ” Abstract and Applied Analysis, volume 2011, Article ID 103597, 14 pages, doi:10.1155/2011/103597.

5. “Duality in nondifferentiable minimax fractional programming with B-(p, r)-invexity ”, Journal of Inequalities and Applications (Accepted).

6. “Multiobjective second-order nondifferentiable symmetric duality involving (F, α, ρ, d)-convex functions ”, Journal of Applied Mathematics and Informatics, 28 (2010) 1395-1408.

7. “Mond-Weir type nondifferentiable multiobjective second-order symmetric duality with cone constraints ”, International Journal of Mathematics in Operational Research, 3 (2011) 414-430.

List of Figures

Table of Contents

Chapter 1

INTRODUCTION

Optimization theory is one of the most lively and exciting branches of modern mathematics. It lies at the heart of applied mathematics. Optimality conditions and duality have played an important role in the development of mathematical programming. The necessary and sufficient optimality conditions allows one to devise efficient numerical methods for practical solution of a given optimization problem. Optimality conditions were first investigated by John [73]. These conditions with an added requirement on the lagrange multiplier for the objective function were independently derived by Karush [74], and Kuhn and Tucker [82]. Thus those conditions are known as the Karush-Kuhn-Tucker conditions (KKT conditions). KKT conditions [74, 82] have laid down foundations for many computational techniques in mathematical programming, and are also a great deal responsible for the development of duality theory. The idea of duality is to associate every programming problem being solved another mathematical program, which happens to be solved simultaneously. The new mathematical program satisfies some important properties and leads to rich economic interpretations related to the original programming problem, characterization of its optimality conditions and its solutions. The original problem is called the primal problem and this new program is known as dual program. It is well known that duality principles connect these two programs in such a way that the existence of an optimal solution to one of them guarantees an optimal solution to the other and optimal values of the two problems are equal. Duality has been used for many theoretical and computational developments in mathematical programming and in other diverse fields, including engineering, management science, control theory, business problems and economics. The existence of duality theory helps to develop numerical algorithms as it provides suitable stopping rules for primal and dual problems. A pair of dual problems is called symmetric if the dual of the dual is the primal problem.

Convexity plays an important role in nonlinear programming. In inequality constrained problems, KKT conditions are sufficient for optimality if the functions are convex. These conditions are used in many algorithms to solve nonlinear programming problems. However, the application of the KKT sufficient optimality conditions cannot be restricted to convex problems as many mathematical models used in decision sciences, economics, management sciences, applied mathematics and engineering involve non convex functions. So, it is a necessity to generalize the notion of convexity and to explore the extent of the validity of results to larger classes of optimization problems. Thus, the literature contains various generalizations of convex functions.

A mathematical programming problem with single objective function is called a scalar (or single objective) programming problem. However, just considering one criterion at a time usually does not represent real life problems well because in most of the cases, two or more objectives are associated with a problem. Such a mathematical optimization model with two or more objectives is called a multiobjective programming problem. Multiobjective programming also known as multi-criteria or multi-attribute optimization, is the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints. Multiobjective optimization problems can be found in various fields: product and process design, finance, aircraft design, the oil and gas industry, automobile design, or wherever optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives. Maximizing profit and minimizing the cost of a product; maximizing performance and minimizing fuel consumption of a vehicle; and minimizing weight while maximizing the strength of a particular component are examples of multiobjective optimization problems. The theory and algorithms for multiobjective problems is discussed by Miettinen [90].

The origin of the vector minimum problem can be traced to early developments in utility theory in economics. Pareto [109] began the study of multiobjective programming problems reducing them to a single objective one. However, the problem was first explicitly defined and studied by Kuhn and Tucker [82]. To eliminate certain anomalous efficient solutions they also proposed a slightly restricted definition of efficiency, called proper efficiency. Later, Geoffrion [52] modified this concept and called an efficient solution to be properly efficient if the ratio of gain (in every objective) to loss (in at least one other objective) is always finite. He also derived necessary and sufficient conditions for properly efficient solution of convex multiobjective programming problems. His work motivated many researchers in this field.

The present chapter is divided into three sections. The first section gives important preliminaries. The second section contains a review of various developments in single and multiobjective mathematical programming which are relevant to the thesis and the last one presents a summary of the thesis.

Sections, subsections, theorems, remarks, equations etc., are numbered consecutively along with the chapter number. For example, Section 2.1 means Section 1 of Chapter 2, Subsection 6.4.1 means Subsection 1 of Section 4 in Chapter 6 and Theorem 6.1 means Theorem 1 in Chapter 6.

1.1 Preliminaries

1.1.1 Notations and definitions

Unless stated otherwise throughout the thesis the following notations are used. R^n denotes the n-dimensional Euclidean space, $R^1 = \mathrm{R}$ the set of all real numbers, $R^n_+ = \{x \in R^n : x_j \geqq 0,\ j = 1, 2, \cdots, n\}$ the non-negative orthant of R^n and R_+ the set of nonnegative real numbers. All vectors will be considered as column vectors. We will use superscript T to denote transpose of a vector or a matrix. The index sets are $K = \{1, 2, \cdots, k\}$, $M = \{1, 2, \cdots, m\}$ and $Q = \{1, 2, \cdots, q\}$. For $r \in K$, the set $K_r = K - \{r\}$. In some chapters the symbol K has been used for a cone while in some others it denotes an index set. It will be clear from the context. Small letters are used to denote vectors or vector functions. A small letter with a subscript represents a component of a vector or a vector function. For $a,\ b \in R^n$,

$$a \geqq b \Leftrightarrow a_i \geqq b_i,\ i = 1, 2, ..., n,$$
$$a \geq b \Leftrightarrow a \geqq b \text{ and } a \neq b,$$
$$a > b \Leftrightarrow a_i > b_i,\ i = 1, 2, ..., n.$$
$$a \nleq b \text{ means negation of } a \leq b.$$

It may be noted that for scalars a and b, we shall use the inequalities $\geqq$ and $>$.

The vector $\nabla f(\bar{x})$ denotes the gradient of a scalar differentiable function $f : R^n \to R$ at $\bar{x}$, and is defined as

$$\nabla f(\bar{x}) = \left[\tfrac{\partial}{\partial x_1} f(\bar{x}),\ \tfrac{\partial}{\partial x_2} f(\bar{x}), \cdots, \tfrac{\partial}{\partial x_n} f(\bar{x})\right]^T$$

and for a vector valued differentiable function $f : R^n \to R^k$, the symbol $\nabla f(\bar{x})$ denotes $k \times n$ Jacobian matrix of f at $\bar{x}$, whose ith row is the vector $\nabla f_i(\bar{x})^T$. If the function $f : R^n \to R$ is twice differentiable at $\bar{x}$, in addition to the gradient vector there exists an $n \times n$ symmetric matrix $\nabla_{xx} f$ or $\nabla^2 f$, called the Hessian matrix of f at $\bar{x}$. The element in i^{th} row and j^{th} column of the Hessian matrix is the second-order partial derivative $\frac{\partial^2 f(\bar{x})}{\partial x_i \partial x_j}$. A vector valued function is differentiable if each of its components is differentiable and is twice differentiable if each of its components is twice differentiable.

Let $\phi : R^n \times R^m \to R$ be twice differentiable function, $\nabla_x \phi(\bar{x}, \bar{y})$ and $\nabla_y \phi(\bar{x}, \bar{y})$ denote the gradient (column) vectors with respect to x and y at $(\bar{x}, \bar{y})$ respectively, and $\nabla_{xx} \phi(\bar{x}, \bar{y})$ and $\nabla_{yx} \phi(\bar{x}, \bar{y})$ denote respectively the $n \times n$ and $n \times m$ matrices of second-order partial derivatives evaluated at $(\bar{x}, \bar{y})$.

In the subsequent chapters, we need the following definitions :

Definition 1.1 [22]. A convex set C of R^n is called a convex cone if for each $x \in C$ and $\lambda \geqq 0, \quad \lambda x \in C$.

Definition 1.2 [123]. The positive polar cone C^* of the cone C is defined by

$$C^* = \{z \in R^n : x^T z \geqq 0,\ \text{for all } x \in C\}.$$

Definition 1.3 [102, 113]. Let S be a compact convex set in R^n. The support function of S is defined by

$$s(x|S) = \max\ \{x^T y : y \in S\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that

$$s(y|S) \geqq s(x|S) + z^T(y - x)\ \ \text{for all } y \in S.$$

The subdifferential of $s(x|S)$ is given by

$$\partial s(x|S) = \{z \in S : z^T x = s(x|S)\}.$$

For any set $S \subset R^n$ the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0,\ \text{for all } z \in S\}.$$

It can be easily seen that for a compact convex set S, y is in $N_S(x)$ if and only if $s(y|S) = x^T y$, or equivalently, x is in $\partial s(y|S)$.

Definition 1.4 [3, 140]. A functional $F : X \times X \times R^n \to R$ (where $X \subseteq R^n$) is said to be sublinear in the third variable, if for any $x, \bar{x} \in X$

(A) $F(x, \bar{x}; a_1 + a_2) \leqq F(x, \bar{x}; a_1) + F(x, \bar{x}; a_2)$ for all $a_1,\ a_2 \in R^n$,

(B) $F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a)$ for all $\alpha \in R_+$ and $a \in R^n$.

1.1.2 General mathematical programming problem

The general mathematical programming problem can be stated as follows :

(NLP) Minimize $\theta(x)$

subject to $x \in P = \{x \in X : g(x) \leqq 0\}$,

where X is an open subset of R^n, the functions $\theta : X \to R$ and $g : X \to R^m$ are differentiable on X.

This problem is called a **scalar (or single objective)** mathematical programming problem. The function θ is known as the **objective function**, the components of g as the **constraint functions** and the corresponding inequalities as **constraints**.

The set P is called the **feasible set** and any point $\bar{x} \in P$ is called a **feasible point** or simply **feasible**.

If the objective function and all the constraints are linear in a mathematical programming problem, then the problem is called a linear programming problem and if either the objective function or at least one of the constraints are nonlinear functions then the problem is named as nonlinear programming problem.

If $\bar{x} \in P$ and $\theta(x) \geqq \theta(\bar{x})$ for each $x \in P$, then $\bar{x}$ is called **an optimal solution or a global optimal solution, or simply a solution** of (NLP). If $\bar{x} \in P$ and if there exists an δ- neighborhood $N_\delta(\bar{x})$ around $\bar{x}$ such that $\theta(x) \geqq \theta(\bar{x})$ for each $x \in P \bigcap N_\delta(\bar{x})$, then $\bar{x}$ is called **a local optimal solution**.

1.1.3 Convex functions and extensions

At $\bar{x} \in X$, θ is said to be

(i) Convex if for all $x \in X$

$$\theta[\lambda x + (1-\lambda)\bar{x}] \leqq \lambda\theta(x) + (1-\lambda)\theta(\bar{x}), \quad \text{for all} \quad \lambda, \quad 0 \leqq \lambda \leqq 1,$$

or equivalently, if

$$\theta(x) - \theta(\bar{x}) \geqq \nabla\theta(\bar{x})^T(x-\bar{x}) \quad \text{when} \quad \theta \text{ is differentiable at } \bar{x}.$$

The function θ is said to be strictly convex if the above conditions hold as strict inequalities for $x \neq \bar{x}$, $0 < \lambda < 1$.

(ii) Quasiconvex if for all $x \in X$

$$\theta(x) \leqq \theta(\bar{x}) \Rightarrow \theta[\lambda x + (1-\lambda)\bar{x}] \leqq \theta(\bar{x}), \quad \text{for all} \quad \lambda, \; 0 \leqq \lambda \leqq 1,$$

or equivalently, if

$$\theta(x) \leqq \theta(\bar{x}) \Rightarrow \nabla\theta(\bar{x})^T(x-\bar{x}) \leqq 0 \quad \text{when} \quad \theta \text{ is differentiable at } \bar{x}.$$

(iii) Pseudoconvex if θ is differentiable at $\bar{x}$ and for all $x \in X$,

$$\nabla\theta(\bar{x})^T(x-\bar{x}) \geqq 0 \Rightarrow \theta(x) \geqq \theta(\bar{x}),$$

or equivalently, if

$$\theta(x) < \theta(\bar{x}) \Rightarrow \nabla\theta(\bar{x})^T(x-\bar{x}) < 0.$$

(iv) Strictly Pseudoconvex if θ is differentiable at $\bar{x}$ and for all $x \in X$ $(x \neq \bar{x})$

$$\theta(x) \leqq \theta(\bar{x}) \Rightarrow \nabla\theta(\bar{x})^T(x-\bar{x}) < 0,$$

or equivalently, if

$$\nabla\theta(\bar{x})^T(x-\bar{x}) \geqq 0 \Rightarrow \theta(x) > \theta(\bar{x}).$$

In 1981, Hanson [67] introduced the concept of invexity replacing the difference vector $x-\bar{x}$ in the definition of a convex function by any vector function $\eta(x,\bar{x})$ and established KKT type sufficient optimality conditions for a nonlinear programming problem. Below, we define these functions which were named invex by Craven [47] and $\eta-$convex by Kaul and Kaur [75].

(v) θ is said to be **invex** if there exists a function $\eta : X \times X \mapsto R^n$ such that for all $x, \bar{x} \in X$

$$\theta(x) - \theta(\bar{x}) \geqq \eta(x,\bar{x})^T\nabla\theta(\bar{x}).$$

(vi) θ is said to be **pseudoinvex** if there exists a function $\eta : X \times X \mapsto R^n$ such that for all $x, \bar{x} \in X$

$$\eta(x,\bar{x})^T\nabla\theta(\bar{x}) \geqq 0 \implies \theta(x) \geqq \theta(\bar{x}).$$

The concept of $(F,\rho)-$convexity was introduced by Preda [111] as an extension of $F-$convexity [69] and $\rho-$convexity [128]. Gulati and Islam [61] obtained sufficient optimality conditions and duality theorems for multiobjective programming problems under generalized F-convexity. Later, the concepts of (F,α,ρ,d)-convex functions and higher-order (F,α,ρ,d)-type-I functions were introduced by Liang et al. [86] and by Suneja et al. [125], respectively. Let $F : X \times X \times R^n \to R$ be a sublinear functional with respect to third variable. Then θ is said to be

(vii) $F-$ **convex** at $\bar{x}$ if for all $x \in X$,

$$\theta(x) - \theta(\bar{x}) \geqq F(x,\bar{x};\nabla\theta(\bar{x})).$$

(viii) $F-$ **pseudoconvex** at $\bar{x}$ if for all $x \in X$,

$$F(x,\bar{x};\nabla\theta(\bar{x})) \geqq 0 \implies \theta(x) \geqq \theta(\bar{x}).$$

(ix) (F,ρ)**-convex** at $\bar{x} \in X$ if there exists $d : X \times X \mapsto R$ and $\rho \in R$ such that for all $x \in X$

$$F(x,\bar{x};\nabla\theta(\bar{x})) + \rho d^2(x,\bar{x}) \leqq \theta(x) - \theta(\bar{x}).$$

(x) (F, ρ)**-pseudoconvex** at $\bar{x} \in X$ if there exists $d : X \times X \mapsto R$ and $\rho \in R$ such that for all $x \in X$

$$F(x, \bar{x}; \nabla\theta(\bar{x})) \geqq -\rho d^2(x, \bar{x}) \Rightarrow \theta(x) \geqq \theta(\bar{x}),$$

or equivalently,

$$\theta(x) < \theta(\bar{x}) \Rightarrow F(x, \bar{x}; \nabla\theta(\bar{x})) < -\rho d^2(x, \bar{x}).$$

(xi) (F, α, ρ, d)**-convex** at $\bar{x} \in X$ if there exists a function $\alpha : X \times X \to R_+ \setminus \{0\}$, $d : X \times X \mapsto R$ and $\rho \in R$ such that for each $x \in X$

$$\theta(x) - \theta(\bar{x}) \geqq F(x, \bar{x}; \alpha(x, \bar{x})\nabla\theta(\bar{x})) + \rho d^2(x, \bar{x}).$$

(xii) (F, α, ρ, d)**-pseudoconvex** at $\bar{x} \in X$ if there exists a function $\alpha : X \times X \to R_+ \setminus \{0\}$, $d : X \times X \mapsto R$ and $\rho \in R$ such that for each $x \in X$

$$F(x, \bar{x}; \alpha(x, \bar{x})\nabla\theta(\bar{x})) \geqq -\rho d^2(x, \bar{x}) \Rightarrow \theta(x) \geqq \theta(\bar{x}).$$

A real valued differentiable function θ is **concave** or **pseudoconcave** iff $-\theta$ is **convex** or **pseudoconvex**. Other definitions follow similarly.

1.1.4 Further developments in convexity

Bector and Chandra [24] introduced the following concept of bonvex functions which were later on extended to $\eta-$ bonvex by Pandey [107]. The function θ is said to be

(xiii) Second-order Convex (Bonvex) if θ is twice differentiable at $\bar{x}$ and for all $x \in X$, $p \in R^n$

$$\theta(x) - \theta(\bar{x}) \geqq (\nabla\theta(\bar{x}) + \nabla^2\theta(\bar{x})p)^T(x - \bar{x}) - \tfrac{1}{2}p^T\nabla^2\theta(\bar{x})p.$$

(xiv) Second-order Pseudoconvex (Pseudobonvex) if θ is twice differentiable at $\bar{x}$ and for all $x \in X$, $p \in R^n$

$$(\nabla\theta(\bar{x}) + \nabla^2\theta(\bar{x})p)^T(x - \bar{x}) \geqq 0 \Rightarrow \theta(x) \geqq \theta(\bar{x}) - \tfrac{1}{2}p^T\nabla^2\theta(\bar{x})p.$$

(xv) η**-bonvex** if there exists a function $\eta : X \times X \to R^n$ such that for all $p \in R^n$ and $x \in X$,

$$\theta(x) - \theta(\bar{x}) \geqq \eta^T(x, \bar{x})[\nabla_x\theta(\bar{x}) + \nabla_{xx}\theta(\bar{x})p] - \tfrac{1}{2}p^T\nabla_{xx}\theta(\bar{x})p.$$

(xvi) η**-pseudobonvex** if there exists a function $\eta : X \times X \to R^n$ such that for all $p \in R^n$ and $x \in X$,

$$\eta^T(x,\bar{x})[\nabla_x\theta(\bar{x}) + \nabla_{xx}\theta(\bar{x})p] \geqq 0 \Rightarrow \theta(x) \geqq \theta(\bar{x}) - \tfrac{1}{2}p^T\nabla_{xx}\theta(\bar{x})p.$$

(xvii) second-order F-convex at $\bar{x} \in X$ if for all $p \in R^n$ and $x \in X$,

$$\theta(x) - \theta(\bar{x}) + \tfrac{1}{2}p^T\nabla_{xx}\theta(\bar{x})p \geqq F(x,\bar{x};\nabla_x\theta(\bar{x}) + \nabla_{xx}\theta(\bar{x})p).$$

(xviii) second-order F-pseudoconvex at $\bar{x} \in X$ if for all $p \in R^n$ and $x \in X$,

$$F(x,\bar{x};\nabla_x\theta(\bar{x}) + \nabla_{xx}\theta(\bar{x})p) \geqq 0$$
$$\Rightarrow \theta(x) \geqq \theta(\bar{x}) - \tfrac{1}{2}p^T\nabla_{xx}\theta(\bar{x})p.$$

Antczak [17] introduce the definition of a p-invex set with respect to η and the definition of a (p,r)-invex set with respect to η. The new class of nonconvex functions, called B-(p,r)-invex functions and many well-known classes of generalized invex functions as its subclasses and their properties are studied in [18].

Lemma 1.1 (Generalized Schwartz Inequality). Let A be a positive semidefinite symmetric matrix of order n. Then, for all $x, z \in R^n$,

$$x^TAz \leqq (x^TAx)^{\frac{1}{2}}(z^TAz)^{\frac{1}{2}}.$$

Equality holds if for some $\lambda \geqq 0$, $Ax = \lambda Az$.

1.1.5 Optimality conditions for single objective programming

The problem of optimizing a numerical function of one or more variables subject to constraints on the variables is called the mathematical programming, or constrained optimization problem. When either the objective function or atleast one of the constraints are nonlinear, the problem is called a nonlinear programming problem, a discipline playing an increasingly imperative role in diverse fields such as operations research and management science, engineering, economics, system analysis and computer science. Necessary optimality conditions for (NLP) were first investigated by John [73]. Fritz John conditions have not only shed new light on the notion of Lagrangian multipliers but also gave an easier approach to develop the Lagrangian multiplier rule for equality constrained optimization. The Fritz John necessary conditions are as follows:

Theorem 1.1 [73]. (Fritz John necessary conditions). If $\bar{x} \in P$ is an optimal solution of (NLP), then there exist $\bar{u} \in R$ and $\bar{v} \in R^m$ such that

$$\begin{aligned} &\bar{u}\nabla f(\bar{x}) + \nabla \bar{v}^T g(\bar{x}) = 0, \\ &\bar{v}^T g(\bar{x}) = 0, \\ &(\bar{u}, \bar{v}) \geqq 0, \quad (\bar{u}, \bar{v}) \neq 0. \end{aligned}$$

In the above conditions, the scalars $\bar{u}$ and $\bar{v}_i,\ i = 1, 2, \cdots, m$ are called Lagrangian multipliers. If the Lagrangian multiplier $\bar{u}$ is equal to zero, the Fritz John conditions do not make use of any information pertaining to the gradient of the objective function. In this case any function can replace f and there will be no change in the above necessary conditions. So the Fritz John conditions are of no practical value in locating an optimal point when $\bar{u} = 0$. In order to exclude such cases, some restrictions are imposed on the constraints. In the literature these restrictions are termed as constraint qualifications. Some of these constraint qualifications make use only of the convexity properties, while others make use mostly of the differentiability of the functions defining the feasible region P.

We state below the Kuhn-Tucker constraint qualifications [74, 82].

The Kuhn-Tucker Constraint Qualification.

The vector function g is said to satisfy the Kuhn-Tucker constraint qualification at $\bar{x} \in P$ if g is differentiable at $\bar{x}$ and if

$$\left.\begin{array}{l} y \in R^n \\ \nabla g_J(\bar{x}) y \leqq 0 \end{array}\right\} \implies \left\{\begin{array}{l} \text{There exists an } n\text{-dimensional vector} \\ \text{function } e \text{ on the interval } [0,\ 1] \text{ such that} \\ (a)\ e(0) = \bar{x} \\ \\ (b)\ e(t) \in P \text{ for } 0 \leqq t \leqq 1 \\ \\ (c)\ e \text{ is differentiable at } t = 0 \\ \\ \text{and } \frac{d}{dt}e(0) = \lambda y \text{ for some } \lambda > 0 \end{array}\right.$$

where $J = \{j \in M : g_j(\bar{x}) = 0\}$.

Assuming one or the other constraint qualification many authors have developed necessary optimality conditions for (NLP) that are precisely the Fritz John conditions with the added property that $\bar{u} > 0$.

Theorem 1.2 [74, 82] (Karush-Kuhn-Tucker Necessary Conditions). If $\bar{x} \in P$ is an optimal solution of (NLP) and g satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$, then there exists $\bar{v} \in R^m$ such that

$$\begin{aligned} &\nabla f(\bar{x}) + \bar{v}^T \nabla g(\bar{x}) = 0, \\ &\bar{v}^T g(\bar{x}) = 0, \\ &\bar{v} \geqq 0. \end{aligned}$$

The above necessary conditions hold under any constraint qualification [88]. Kuhn and Tucker [82] also proved that the above necessary conditions are sufficient for optimality under convexity assumptions.

1.1.6 Duality in mathematical programming

Duality has long been a central tool in the development of optimization theory. Neumann [106] introduced the concept of duality in linear programming. He formulated the following dual pair and proved usual duality relations.

Primal Problem (PP)

$$\begin{aligned} \text{Minimize } & z(x) = c^T x \\ \text{subject to } \quad & Ax \geqq b, \\ & x \geqq 0. \end{aligned}$$

Dual Problem (DP)

$$\begin{aligned} \text{Maximize } & w(y) = b^T y \\ \text{subject to } \quad & A^T y \leqq c, \\ & y \geqq 0. \end{aligned}$$

The above pair shows that if the primal problem is a minimization of a linear function over a set of linear constraints, then the dual is a maximization of another linear function over a set of linear constraints. Moreover, dual of the dual is again the primal problem.

Duality in nonlinear programming has also been developed extensively. It originated with the duality results of quadratic programming given by Dennis [49]. Wolfe [132] formulated the following dual to (NLP):

$$\begin{aligned} &\text{Maximize } \ \theta(y)+\mu^T g(y)\\ &\text{subject to } \ \ \nabla\theta(y)+\nabla\mu^T g(y)=0,\\ &\qquad\qquad\quad y\in X,\ \mu \geqq 0 \end{aligned}$$

and proved duality theorems assuming θ and g to be convex. Mangasarian [88] pointed out that these duality relations do not hold under weaker convexity assumptions. Mond and Weir [104] introduced the following dual to (NLP):

$$\begin{aligned} &\text{Maximize } \ \theta(y)\\ &\text{subject to } \ \ \nabla\theta(y)+\nabla\mu^T g(y)=0,\\ &\qquad\qquad\quad \mu^T g(y)\geqq 0,\\ &\qquad\qquad\quad y\in X,\ \mu \geqq 0 \end{aligned}$$

and proved duality relations by weakening the convexity assumptions of θ and g to pseudoconvexity of θ and quasiconvexity of $\mu^T g$. They also discussed duality results for the problems involving both equality and inequality constraints.

1.1.7 General multiobjective programming problem

The **general multiobjective programming problem** in n-dimensional Euclidean space can be stated as follows:

(P) $$\begin{aligned} &\text{Minimize } \ f(x)=\{f_1(x),f_2(x),...,f_k(x)\}\\ &\text{subject to } \ x\in X^\circ=\{x\in X: g(x)\leqq 0\}, \end{aligned}$$

where X is an open subset of R^n, the functions $f: X\to R^k$ and $g: X\to R^m$ are differentiable on X.

In such problems an optimal solution in the sense of one that minimizes all the objective functions simultaneously, (called an **ideal solution**) exists rarely. Often the several objectives are conflicting in nature. For example, it may be impossible to select an alternative to a problem which would maximize both profit and market share for a company. The existence of multiple objectives leads to many interesting questions, which do not arise in single objective models. It is difficult to obtain a unique solution since these problems rarely have feasible points that simultaneously minimize (or maximize) all the objectives. The concept of optimal solution in multiobjective optimization problems is clearly related to the preference attitude of the decision maker. A good decision is based on the principle that there is no other alternate that can be better in some aspect of consideration. Such a point is called an efficient point. An efficient solution is also known as noninferior or nondominated or Pareto optimal solution. Efficiency found its way into operations research in the pioneer work of Koopmans [81]. Later, the concept of weak efficiency has also been introduced. We define below these two concepts of optimality in multiobjective programming.

Definition 1.5 [122]. A point $\bar{x} \in X^{\circ}$ is said to be a weak efficient solution (weak minimum) of the vector minimum problem (P), if there exists no $x \in X^{\circ}$ such that

$$f(x) < f(\bar{x}).$$

Definition 1.6 [122]. A point $\bar{x} \in X^{\circ}$ is said to be an efficient (or nondominated or noninferior or Pareto optimal) solution of the vector minimum problem (P), if there exists no $x \in X^{\circ}$ such that

$$f(x) \leq f(\bar{x}).$$

This definition is based upon the intuitive conviction that the point $\bar{x}$ is chosen as the optimal solution if no criterion can be improved without worsening at least one other criterion.

It can be easily seen that the definitions of efficient/weak efficient solutions for multiobjective programming problems are extension of the definition of optimal solution for scalar programming problem.

The following definition of properly efficient solution is due to Geoffrion [52], who modified this concept earlier introduced by Kuhn and Tucker [82] :

Definition 1.7. An efficient solution $\bar{x}$ of the vector minimum problem (P) is said to be a properly efficient solution, if there exists a scalar $M > 0$ such that for each $r \in K = \{1, 2, ..., k\}$ and $x \in X^\circ$ satisfying $f_r(x) < f_r(\bar{x})$, we have

$$f_r(\bar{x}) - f_r(x) \leqq M[f_j(x) - f_j(\bar{x})]$$

for atleast one $j \in K \backslash \{r\}$ satisfying $f_j(\bar{x}) < f_j(x)$.

A k-dimensional vector function $f = (f_1, f_2, \ldots, f_k)$ is said to be convex at $\bar{x}$ (or on X) if for each $j \in K$, f_j is convex at $\bar{x}$ (or on X). The other definitions discussed in Subsection 1.1.3 and 1.1.4 follow similarly.

1.1.8 Optimality in multiobjective programming

In the classical work in 1951, Kuhn and Tucker [82] discussed some interesting results for (P). Since then, research in this field has made remarkable progress both theoretically and computationally. Some of the earliest attempts to obtain necessary and sufficient conditions for efficiency were carried out by Kuhn and Tucker [82] and Arrow et al. [20]. Geoffrion [52] introduced the following scalar parametric problem:

(EP) Minimize $\lambda^T f(x) = \sum\limits_{i \in K} \lambda_i f_i(x)$
subject to $x \in X^\circ$,

where $\lambda_i, (i \in K)$ are strictly positive parameters (often normalized according to $\sum\limits_{i \in K} \lambda_i = 1$) and related its optimal solution with a properly efficient solution of (P).

The comprehensive theorem in Geoffrion [52] includes the following necessary and sufficient conditions. Though Geoffrion [52] assumed the Kuhn-Tucker constraint qualification for Theorem 1.2, it holds under any constraint qualification.

Theorem 1.3 (Karush-Kuhn-Tucker type necessary conditions). Let $\bar{x} \in X^\circ$ be a properly efficient solution of (P) and let g satisfy a constraint qualification at $\bar{x}$. Then there exist $\bar{u} \in R^k$ and $\bar{v} \in R^m$, such that

$$\nabla \bar{u}^T f(\bar{x}) + \nabla \bar{v}^T g(\bar{x}) = 0,$$

$$\bar{v}^T g(\bar{x}) = 0,$$

$$\bar{u} > 0, \ \bar{v} \geqq 0, \ \sum_{i=1}^{k} \bar{u}_i = 1.$$

Theorem 1.4 (Karush-Kuhn-Tucker type sufficient conditions). Let f and g be convex at $\bar{x} \in X^\circ$. If there exist $\bar{u} \in R^k$ and $\bar{v} \in R^m$, such that

$$\nabla \bar{u}^T f(\bar{x}) + \nabla \bar{v}^T g(\bar{x}) = 0,$$
$$\bar{v}^T g(\bar{x}) = 0,$$
$$\bar{u} > 0,\ \bar{v} \geqq 0,\ \sum_{i=1}^{k} \bar{u}_i = 1,$$

then $\bar{x}$ is a properly efficient solution of (P).

Kaul et al. [76] established the following KKT type necessary conditions for an efficient solution of (P):

Theorem 1.5 (Karush-Kuhn-Tucker type necessary conditions). Assume that x^* is an efficient solution for (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist a nonzero vector $\bar{u} \in R^k$ and $\bar{v} \in R^m$ such that

$$\nabla \bar{u}^T f(x^*) + \nabla \bar{v}^T g(x^*) = 0,$$
$$\bar{v}^T g(x^*) = 0,$$
$$\bar{v} \geqq 0,\ \bar{u} \geqq 0,\ \sum_{i=1}^{k} \bar{u}_i = 1.$$

A survey of recent developments in multiobjective optimization has appeared in [44].

1.1.9 Duality in multiobjective programming

Duality plays an important role in mathematical programming has been extended to multiobjective optimization since late 1970's. Isermann [71] developed multiobjective duality in the linear case. Duality for linear vector maximum problems with matrix variables was discussed by Corley [45]. For the nonlinear cases duality has been developed by Bitran [34], Craven [47], Tanino and Sawaragi [127] etc. These studies differ in their approach as well as the sense in which 'Optimality' is defined for the multiobjective programming problem. Bitran's development associates a matrix, rather than a vector, to efficient points of the saddle point dual. Craven treats the problem from the strong vector minimization view point rather than the Pareto minimization. Tanino and Sawaragi [127] developed a duality theory for convex multiobjective problems using a vector valued Lagrangian function and exploring the properties of primal and dual point to set maps. Further, optimality conditions for multiobjective problems can be found in Wang [129], Miettinen [90] and Pardalos et al. [108].

Bector et al. [28] and Singh [116, 117] discussed Mond-Weir type duality in multiobjective programming using the constraint qualification [100] based on the idea of convergence vector under different generalized convexity assumptions. They considered the following dual problem of (P):

(D) Maximize $f(y)$

$$\begin{aligned} \text{subject to } \; & \lambda^T \nabla f(y) + \mu^T \nabla g(y) = 0, \\ & \mu^T g(y) \geqq 0, \quad \lambda > 0, \; \mu \geqq 0, \end{aligned}$$

and proved the duality theorems relating the efficient solutions of (P) and (D). Gulati and Talaat [63] obtained these duality relations without needing any constraint qualification under weaker convexity assumptions. They established the following duality results for the above Mond-Weir type dual :

Theorem 1.6 (Strong duality). Let $\bar{x}$ be a properly efficient solution for (P) and let g satisfy the Kuhn-Tucker constraint qualification at $\bar{x}$. Then there exist $(\bar{\lambda}, \bar{\mu})$, such that $(\bar{y} = \bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible solution for (D) and the objective values of (P) and (D) are equal. Also, if $\bar{\lambda}^T f$ is pseudoconvex and $\bar{\mu}^T g$ is quasiconvex at $\bar{y}$ for every dual feasible solution (y, λ, μ), then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a properly efficient solution for (D).

Theorem 1.7 (Converse duality). Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a weak efficient solution for (D), the $n \times n$ Hessian matrix $\nabla^2(\bar{\lambda}^T f(\bar{y}) + \bar{\mu}^T g(\bar{y}))$ be positive or negative definite and $\nabla f_i(\bar{y}),\; i = 1, 2, \cdots, k$ be linearly independent. If $\bar{\lambda}^T f$ is pseudoconvex and $\bar{\mu}^T g$ is quasiconvex at $\bar{y}$, then $\bar{y}$ is a properly efficient solution for (P).

Recently, Chinchuluun and Pardalos [44] have discussed optimality conditions and various applications for multiobjective problems. They have also given a new concept of epsilon Pareto optimal solution for such type of problems.

1.2 Review of the Related Work

1.2.1 Symmetric and self duality

In mathematical programming, a pair of primal and dual problems is called symmetric if the dual of the dual is the primal problem, that is, if the dual problem is expressed in the form of the primal problem, then its dual is the primal problem. Duality in linear programming is always symmetric. However, the majority of dual formulations in nonlinear programming do not possess this property. Dorn [50] introduced the concept of symmetric duality in quadratic programming. Dantzig et al. [48] formulated the following pair of symmetric dual programs and established weak and strong duality theorems :

(PS)
$$\begin{aligned} &\text{Minimize } \; F(x,y) = H(x,y) - y^T\nabla_y H(x,y)\\ &\text{subject to } \; \nabla_y H(x,y) \leqq 0,\\ &\qquad\qquad\qquad x, y \geqq 0. \end{aligned}$$

(DS)
$$\begin{aligned} &\text{Maximize } \; G(u,v) = H(u,v) - u^T\nabla_x H(u,v)\\ &\text{subject to } \; \nabla_x H(u,v) \geqq 0,\\ &\qquad\qquad\qquad u, v \geqq 0. \end{aligned}$$

where $H : R^n \times R^m \to R$ is a twice differentiable function.

For the weak duality theorem, Dantzig et al. [48] required $H(.,y)$ to be convex in x and $H(x,.)$ to be concave in y. To weaken the convexity-concavity assumption on $H(x,y)$ to pseudoconvexity-pseudoconcavity, Mond and Weir [103] considered the following pair of symmetric dual programs:

(PM)
$$\begin{aligned} &\text{Minimize } \; H(x,y)\\ &\text{subject to } \; \nabla_y H(x,y) \leqq 0,\\ &\qquad\qquad y^T\nabla_y H(x,y) \geqq 0,\\ &\qquad\qquad\qquad\qquad x \geqq 0. \end{aligned}$$

(DM)
$$\begin{aligned} &\text{Maximize } \; H(u,v)\\ &\text{subject to } \; \nabla_x H(u,v) \geqq 0, \end{aligned}$$

$$
\begin{aligned}
u^T\nabla_x H(u,v) &\leqq 0,\\
v &\geqq 0.
\end{aligned}
$$

and discussed duality theorems under pseudoconvexity-pseudoconcavity assumptions.

A program is said to be self dual [51] if the dual can be recast in the form of the primal, that is, primal and dual formulations are equivalent. Mond and Cottle [97] observed that the symmetric dual programs of Dantzig et al. [48] are self duals if $H(x,y)$ is skew symmetric, and gave self duality results.

Chandra and Husain [37] studied symmetric and self duality for the following nondifferentiable symmetric duals assuming the convexity/concavity of $H(x,y)$:

$$
\begin{aligned}
\text{Minimize} \quad & H(x,y) - y^T\nabla_y H(x,y) + (x^TBx)^{\frac{1}{2}}\\
\text{subject to} \quad & \nabla_y H(x,y) - Cw \leqq 0,\\
& w^TCw \leqq 1,\\
& x \geqq 0,\\
& y \geqq 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{Maximize} \quad & H(x,y) - x^T\nabla_x H(x,y) - (y^TCy)^{\frac{1}{2}}\\
\text{subject to} \quad & \nabla_x H(x,y) + Bz \geqq 0,\\
& z^TBz \leqq 1,\\
& x \geqq 0,\\
& y \geqq 0.
\end{aligned}
$$

Subsequently, Chandra et al. [35] formulated a pair of nondifferentiable symmetric dual programs in the spirit of Mond and Weir [103], and discussed duality results involving pseudoconvexity/pseudoconcavity assumptions. Later on, Mond and Schechter [102] studied Wolfe and Mond-Weir type nondifferentiable symmetric dual problems, in which the objective function contains a support function. They established duality results under convexity/concavity assumptions for Wolfe type model and pseudoconvexity/pseudoconcavity assumptions for Mond-Weir type model, respectively.

1.2.2 Symmetric duality with cone constraints

Bazaraa and Goode [21] generalized the formulation of symmetric duality introduced by Dantzig et al. [48] to include the case where the inequality constraints are defined via closed convex cones and their polars. Such a formulation enables one to consider infinitely many constraints of the inequality type. The new formulation retains the symmetric duality of the original programs. They studied the following Wolfe's type symmetric dual pair over arbitrary cones :

(PC) Minimize $F(x,y) = H(x,y) - y^T\nabla_y H(x,y)$

subject to $\nabla_y H(x,y) \in C_2^\star$,

$(x,y) \in C_1 \times C_2$,

(DC) Maximize $G(u,v) = H(u,v) - u^T\nabla_x H(u,v)$

subject to $-\nabla_x H(u,v) \in C_1^\star$,

$(u,v) \in C_1 \times C_2$,

where

- (i) C_1 and C_2 are closed convex cones with non-empty interiors in R^n and R^m, respectively.
- (ii) For $i = 1, 2$, $C_i^\star$ is the polar of C_i.
- (iii) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$ and $H : S_1 \times S_2 \mapsto R$ is a twice differentiable function.

Chandra and Kumar [41] formulated the following Mond-Weir type symmetric dual programs over arbitrary cones :

(PMC) Minimize $H(x,y)$

subject to $\nabla_y H(x,y) \in C_2^\star$,

$y^T\nabla_y H(x,y) \geqq 0$,

$x \in C_1$.

(DMC) Maximize $H(u,v)$

$$\begin{aligned} \text{subject to} \quad & -\nabla_x H(u,v) \in C_1^{\star}, \\ & u^T \nabla_x H(u,v) \leqq 0, \\ & v \in C_2. \end{aligned}$$

and proved usual duality theorems under pseudoinvexity type assumptions.

To review the work wherein the objective function is optimized with respect to a cone, consider the following problem :

(P1) K-minimize $f(x)$

subject to $-\, g(x) \in C,\ x \in X,$

where $X \subseteq R^n$ is open, $f : X \to R^k$, $g : X \to R^m$, C is a closed convex cone in R^m and K is a closed convex pointed cone in R^k with nonempty interior. Let int K denote interior of K.

Definition 1.8 [77, 123]. A point $\bar{x} \in X^o$ is said to be weak efficient solution of (P1) if there exists no $x \in X^o$ such that $f(\bar{x}) - f(x) \in \text{int } K$.

Definition 1.9 [77]. A point $\bar{x} \in X^\circ$ is said to be an efficient solution of (P1) if there exists no $x \in X^\circ$ such that $f(\bar{x}) - f(x) \in K \backslash \{0\}$.

Suneja et al. [123] formulated the following pair of multiobjective dual programs over cones and proved duality results under K-convexity assumptions:

(P2) K-minimize $f(x,y) - [y^T \nabla_y (\lambda^T f)(x,y)]e$

$$\begin{aligned} \text{subject to} \quad & -\, \nabla_y (\lambda^T f)(x,y) \in C_2^*,\ \lambda^T e = 1, \\ & \lambda \in K^*,\ (x,y) \in C_1 \times C_2, \end{aligned}$$

(D2) K-maximize $f(u,v) - [u^T \nabla_x (\lambda^T f)(u,v)]e$

$$\begin{aligned} \text{subject to} \quad & \nabla_x (\lambda^T f)(u,v) \in C_1^*,\ \lambda^T e = 1, \\ & \lambda \in K^*,\ (u,v) \in C_1 \times C_2, \end{aligned}$$

where $e \in \text{int } K$.

Later, Khurana [77] discussed the following Mond-Weir type symmetric dual problems :

(P3) K-minimize $f(x,y)$

subject to $-\, \nabla_y (\lambda^T f)(x,y) \in C_2^*,$

$$y^T\nabla_y(\lambda^T f)(x,y) \geqq 0,$$
$$\lambda \in K^*,\ x \in C_1,$$

(D3) $$\begin{aligned} &K\text{-maximize} \quad f(u,v) \\ &\text{subject to} \quad \nabla_x(\lambda^T f)(u,v) \in C_1^*, \\ &\qquad u^T\nabla_x(\lambda^T f)(u,v) \leqq 0, \\ &\qquad \lambda \in K^*,\ v \in C_2, \end{aligned}$$

and proved the duality theorems under K-pseudoinvexity/strongly K-pseudoinvexity assumptions. Kim and Kim [78] extended the above two results to nondifferentiable multiobjective symmetric dual programs containing support functions. Recently, Ahmad and Husain [12] formulated the mixed type symmetric dual problems over cones which unifies the above two dual formulations.

1.2.3 Second-order duality

The study of second-order duality is significant due to the computational advantage over the first-order duality as it provides tighter bounds for the value of the objective function, when approximations are used.

Mangasarian [89] first formulated the second-order dual for a nonlinear programming problem and established duality results under somewhat involved assumptions. Mond [94] considered the following second-order symmetric dual programs:

$$\begin{aligned} &\text{Minimize} \quad H(x,y) - y^T\nabla_y H(x,y) - y^T\nabla_{yy}H(x,y)p - \tfrac{1}{2}p^T\nabla_{yy}H(x,y)p \\ &\text{subject to} \quad \nabla_y H(x,y) + \nabla_{yy}H(x,y)p \leqq 0, \\ &\qquad x \geqq 0. \end{aligned}$$

$$\begin{aligned} &\text{Maximize} \quad H(x,y) - x^T\nabla_x H(x,y) - x^T\nabla_{xx}H(x,y)r - \tfrac{1}{2}r^T\nabla_{xx}H(x,y)r \\ &\text{subject to} \quad \nabla_x H(x,y) + \nabla_{xx}H(x,y)r \geqq 0, \\ &\qquad y \geqq 0, \end{aligned}$$

and proved second-order duality theorems under simpler assumptions.

Gulati et al. [55] formulated the pairs of second-order nonlinear symmetric dual programs and obtained usual duality results under η_1-convexity/η_2-concavity and η_1-pseudoconvexity/η_2-pseudoconcavity assumptions. Yang et al. [137] achieved duality relations for a pair of Wolfe type non-differentiable second order symmetric primal and dual problems under second-order F-convexity assumptions. Suneja et al. [124] studied the following pair of MondWeir type second-order multiobjective symmetric dual programs and proved the duality results under η-bonvexity/η-pseudobonvexity assumptions:

Primal (P4)

$$\begin{aligned}
\text{minimize} \quad & F(x,y,p) = \big(F_1(x,y,p), F_2(x,y,p), \ldots, F_k(x,y,p)\big) \\
\text{subject to} \quad & \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \leqq 0, \\
& y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \geqq 0, \\
& \lambda > 0.
\end{aligned}$$

Dual (D4)

$$\begin{aligned}
\text{maximize} \quad & G(u,v,q) = \big(G_1(u,v,q), G_2(u,v,q), \ldots, G_k(u,v,q)\big) \\
\text{subject to} \quad & \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \geqq 0, \\
& u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \leqq 0, \\
& \lambda > 0,
\end{aligned}$$

where

$$F_i(x,y,p) = f_i(x,y) - \tfrac{1}{2}p_i^T \nabla_{yy} f_i(x,y)p_i,$$

$$G_i(u,v,q) = f_i(u,v) - \tfrac{1}{2}q_i^T \nabla_{xx} f_i(u,v)q_i,$$

$$\lambda_i \in R,\ p_i \in R^m,\ q_i \in R^n, \quad \text{for } i = 1,2,\ldots,k.$$

Also $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T,\ p = (p_1, p_2, \ldots, p_k),\ q = (q_1, q_2, \ldots, q_k)$.

Yang et al. [139] extended the results of Suneja et al. [124] to the nondifferentiable case. Later, Gulati and Gupta [57] considered a pair of Wolfe type second-order nondifferentiable symmetric dual programs and proved weak, strong and converse duality theorems under η_1-bonvexity assumptions.

Srivastava and Bhatia [120] studied second-order symmetric duality for multiobjective programs using second-order (F, ρ)-convexity assumptions. Recently, Ahmad and Husain [13] formulated the following pair of Wolfe type multiobjective second order symmetric dual programs with cone constraints:

Primal Problem (WP)

minimize $\phi(x,y)-[y^T\nabla_y(\lambda^T\phi)(x,y)]e-[y^T\nabla_{yy}(\lambda^T\phi)(x,y)p]e-\frac{1}{2}[p^T\nabla_{yy}(\lambda^T\phi)(x,y)p]e$

subject to

$$\nabla_y(\lambda^T\phi)(x,y)+\nabla_{yy}(\lambda^T\phi)(x,y)p \in C_2^*,$$

$$x \in C_1.$$

$$\lambda > 0, \quad \lambda^T e = 1.$$

Dual Problem (WD)

maximize $\phi(u,v)-[u^T\nabla_x(\lambda^T\phi)(u,v)]e-[u^T\nabla_{xx}(\lambda^T\phi)(u,v)q]e-\frac{1}{2}[q^T\nabla_{xx}(\lambda^T\phi)(u,v)q]e$

subject to

$$-\nabla_x(\lambda^T\phi)(u,v)-\nabla_{xx}(\lambda^T\phi)(u,v)q \in C_1^*,$$

$$v \in C_2,$$

$$\lambda > 0, \quad \lambda^T e = 1,$$

where $\phi(x,y) : S_1 \times S_2 \to R^k (S_1 \subseteq R^n,\ S_2 \subseteq R^m), p \in R^m, q \in R^n, \lambda \in R^k$ and $e = (1,1,\ldots,1)^T \in R^k$, and usual duality results are established under second-order invexity assumptions. They also point out certain omissions and inconsistencies in the earlier work of Mishra [91] and Mishra and Wang [93]. Some papers [7, 9, 62, 64, 68, 124, 138, 140, 141] also deal with the study of second-order duality for nonlinear programs.

1.2.4 Higher-order duality

The concept of higher-order duality for the nonlinear programming problems has given by Mangasarian [89] by introducing twice differentiable functions $h : R^n \times R^n \mapsto R$ and $k : R^n \times R^n \mapsto R^m$. The higher-order dual pair studied in [89] is :

Primal problem (P5)

$$\begin{aligned} &\text{Minimize} \;\; f(x) \\ &\text{subject to} \quad g(x) \leqq 0. \end{aligned}$$

Dual problem (D5)

$$\begin{aligned} &\text{Maximize} \;\; f(x) + h(x,p) + u^T g(x) + u^T k(x,p) \\ &\text{subject to} \quad \nabla_p h(x,p) + \nabla_p u^T k(x,p) = 0, \\ &\qquad\qquad\qquad\qquad u \geqq 0, \end{aligned}$$

where $f : R^n \to R$ and $g : R^n \to R^m$.

Chen [43] studied the following higher-order multiobjective Mond-Weir type dual involving nondifferentiable functions and established duality results by introducing the definition of higher-order F-convexity:

(MP) minimize $(f_1(x,y) + s(x|C_1) - y^T z_1 + h_1(x,y,p_1) - p_1^T[\nabla_{p_1} h_1(x,y,p_1)], ...,$

$$f_k(x,y) + s(x|C_k) - y^T z_k + h_k(x,y,p_k) - p_k^T[\nabla_{p_k} h_1(x,y,p_k)])$$

subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x,y) - z_i + \nabla_{p_i} h_i(x,y,p_i)] \leqq 0,$$

$$y^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x,y) - z_i + \nabla_{p_i} h_i(x,y,p_i)] \geqq 0,$$

$$z_i \in D_i, \;\; i = 1,2,...,k, \lambda > 0, \;\; \lambda^T e = 1.$$

(MD) maximize $(f_1(u,v) - s(v|D_1) + u^T w_1 + g_1(u,v,r_1) - r_1^T[\nabla_{r_1} g_1(u,v,r_1)], ...,$

$$f_k(u,v) - s(v|D_k) + u^T w_1 + g_k(u,v,r_k) - r_k^T[\nabla_{r_k} g_k(u,v,r_k)])$$

subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u,v) + w_i + \nabla_{r_i} g_i(u,v,r_i)] \geqq 0,$$

$$u^T \sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + w_i + \nabla_{r_i} g_i(u,v,r_i)] \leqq 0,$$

$$w_i \in C_i,\ i = 1,2,...,k, \lambda > 0,\ \lambda^T e = 1,$$

where C_i and $D_i, i = 1,2,...,k$, are compact convex sets in R^n and R^m, respectively; $f_i : R^n \times R^m \rightarrow R$, $h_i : R^n \times R^m \times R^m \rightarrow R$ and $g_i : R^n \times R^m \times R^n \rightarrow R$ are differentiable functions for $i = 1,2,...,k$ and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)^T \in R^k$.

Later on, Gulati and Gupta [58] proved duality theorems for a pair of Wolfe type higher-order nondifferentiable symmetric dual programs under higher-order F-convexity assumptions. Ahmad et al. [14] introduced a new class of higher-order (F, α, ρ, d)-type I functions and studied various higher-order duality results for a pair of general Mond-Weir type higher-order nondifferentiable multiobjective dual programs.

1.2.5 Minimax Programming

Schmitendorf [115] considered the following minimax programming problem:

(NP) Minimize $\sup_{y \in Y} \phi(x,y)$

subject to $x \in S = \{x \in R^n : C(x) \leqq 0\}$,

where S is the set of feasible solutions of (NP), Y is a compact subset of R^m ; $\phi(.,.) : R^n \times R^m \rightarrow R$ is C^1 on $R^n \times R^m$ and $C(.) : R^n \rightarrow R^p$ is C^1 on R^n.

Schmitendorf [115] established the following necessary and sufficient optimality conditions for (NP) by defining the set

$$\hat{Y}(x) = \{y \in Y : \phi(x,y) = \sup_{z \in Y} \phi(x,z)\}.$$

Theorem 1.8 (Necessary Conditions). Let x^* be a solution to the problem (NP) and the vectors $\nabla_x C_i(x^*), i \in I(x^*)$, are linearly independent. Then there exist a positive integer α, scalars $\lambda_i \geqq 0, i = 1,2,\cdots,\alpha$, scalars $\mu_i \geqq 0, i = 1,2,\cdots,p$ and vectors $y_i \in \hat{Y}(x^*), i = 1,2,\cdots,\alpha$ such that

$$\sum_{i=1}^{\alpha} \lambda_i \nabla_x \phi(x^*, y_i) + \sum_{i=1}^{p} \mu_i \nabla_x C_i(x^*) = 0,$$

$$\mu_i C_i(x^*) = 0, i = 1, 2, \cdots, p,$$

$$\sum_{i=1}^{\alpha} \lambda_i \neq 0.$$

Also, if β is the number of nonzero μ_i, $1 \leqq \alpha + \beta \leqq n + 1$.

Theorem 1.9 (Sufficient Conditions). Let $x^* \in S$. Let $C(.)$ be a convex function of x and, for every $y \in Y$, let $\phi(., y)$ be a convex function of x. If there is a positive integer α, $1 \leqq \alpha \leqq n + 1$, if there are scalars $\lambda_i \geqq 0, i = 1, 2, \cdots, \alpha$, $\sum\limits_{i=1}^{\alpha} \lambda_i \neq 0$, and scalars $\mu_i \geqq 0, i = 1, 2, \cdots, p$, and if there are vectors $y_i \in \hat{Y}(x^*), i = 1, 2, \cdots, \alpha$, such that

$$\sum_{i=1}^{\alpha} \lambda_i \nabla_x \phi(x^*, y_i) + \sum_{i=1}^{p} \mu_i \nabla_x C_i(x^*) = 0,$$

$$\mu_i C_i(x^*) = 0, \ i = 1, 2, \cdots, p,$$

then x^* is a minimax solution.

Tanimoto [126] applied these optimality conditions to define the following dual problem:

$$\max_{(s,\lambda,\bar{y}) \in K} \quad \sup_{(x,\mu) \in H(s,\lambda,\bar{y})} f(x) + \sum_{j=1}^{p} \mu_j g_j(x),$$

where K is the set of triplets $(s, \lambda, \bar{y})$, where s ranges over the integers $1 \leqq s \leqq n + 1, \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)$ with $\lambda_i > 0$ $(i = 1, 2, \cdots, s)$, $\sum\limits_{i=1}^{s} \lambda_i = 1$ and $H(s, \lambda, \bar{y})$ denotes the set of all $(x, \mu) \in R^n \times R^p$ satisfying

$$\sum_{i=1}^{s} \lambda_i \bigtriangledown_x \phi(x, y_i) + \sum_{j=1}^{p} \mu_j \bigtriangledown g_j(x) = 0,$$

$$\{y_1, y_2, \cdots, y_s\} \subset Y(x)$$

$$\mu = (\mu_1, \mu_2, \cdots, \mu_p) \geqq 0,$$

$$(s, \lambda, \bar{y}) \in K,$$

and derived the duality theorems for convex minimax programming. Weir [131] relaxed the convexity assumptions in the sufficient optimality of [115] and employed the optimality conditions to construct several dual problems.

1.2.6 Duality in fractional programming problems

The mathematical programming problem in which the objective function is a ratio of two numerical functions is called a fractional programming problem.

Bector and Chandra [25] studied duality for the following fractional programming problems:

Primal Problem (P6)

$$\text{Minimize } \frac{f(x)}{h(x)}$$

$$\text{subject to } x \in X, \quad g(x) \leqq 0$$

Dual Problem (D6)

$$\text{Maximize } \quad \frac{f(x) + \mu^T g(y)}{h(y)}$$

$$\text{subject to } \nabla \left(\frac{f(x) + \mu^T g(y)}{h(y)} \right) = 0,$$

$$y \in X, \mu \geqq 0,$$

where X is an open subset of R^n and $f, \ h : X \mapsto R, \ \ g : X \mapsto R^m$, and for all $x, \ h(x) > 0$ and $f(x) \geqq 0$ (if h is nonlinear).

Mond [96] considered the following pair of nondifferentiable fractional programming problems :

Primal Problem (P7)

$$\text{Maximize } \frac{f(x) - (x^T Bx)^{\frac{1}{2}}}{g(x) + (x^T Dx)^{\frac{1}{2}}}$$

$$\text{subject to } h(x) \leqq 0.$$

Dual Problem (D7)

$$\text{Minimize } G(u, y, v, w, p) = p$$

$$\text{subject to } \nabla y^T h(u) + p \nabla g(u) + Bu + pDw = \nabla f(u),$$

$$- f(u) + u^T Bv + pg(u) + pu^T Dw + y^T h(u) \geqq 0,$$

$$y, p \geqq 0,$$

$$v^T Bv \leqq 1, w^T Dw \leqq 1.$$

and further necessary and sufficient conditions for optimality as well as appropriate duality theorems are also established. Lai et al. [85] obtained necessary and sufficient optimality conditions for nondifferentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct a parametric dual model and also discussed duality theorems. Later on, Yang et al. [136] studied symmetric duality for the class of nondifferentiable multiobjective fractional programming problems and proved duality results under invexity/incavity and pseudoinvexity/pseudoincavity assumptions.

Kim et al. [79] considered a class of nondifferentiable multiobjective fractional programs in which each component of the objective function contains a term involving the support function of a compact convex set and established necessary and sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems. Antczak [19] proved optimality conditions for a class of generalized fractional minimax programming problems involving B-(p,r)-invexity functions and established duality theorems for various duality models. Later on, Suneja et al. [125] introduced a new class of higher-order (F,ρ,σ)-type I functions for multiobjective programming problems and established higher-order duality results for higher-order Mond-Weir and Schaible type nondifferentiable multiobjective fractional programming dual programs where the objective functions and the constraints contain support functions of compact convex sets in R^n under higher-order (F,ρ,σ)-type I assumptions.

1.2.7 Duality in variational problems

Calculus of variation provides an excellent uniform analytical method to find that curve connecting two given points which either maximizes or minimizes some given integral. For example, to determine a curve which will generate the surface of revolution of smallest area when revolved about the x-axis. In general, we wish to find the curve $x = x(t)$ where $x(a) = \alpha_1$ and $x(b) = \alpha_2$ such that for some given known function $f(t, x(t), \dot{x}(t))$ of variables $t,\ x,\ \dot{x}$, the integral

$$\int_a^b f(t, x, \dot{x}) dt$$

is either a maximum or minimum (also called extremum values). The curve which satisfies this property is said to be an extremal. The problem of finding a piecewise smooth extremal $x = x(t)$ for the above program is known as a variational problem.

Mond and Hanson [98] introduced the concept of duality in variational problems. They studied the following pair of dual problems under convexity assumptions :

(P8) Minimize $\int\limits_{t_0}^{t_1} f(t, x, \dot{x}) dt$

subject to $Q(t, x, \dot{x}) \geqq 0,\ x(t_0) = x_0,\ x(t_1) = x_1,$

(D8) Maximize $\int\limits_{t_0}^{t_1} [f(t, x, \dot{x}) - \lambda(t) Q(t, x, \dot{x})] dt$

subject to $f_x(t, x, \dot{x}) - \lambda(t) Q_x(t, x, \dot{x}) = \dfrac{d}{dt}[f_{\dot{x}}(t, x, \dot{x}) - \lambda(t) Q_{\dot{x}}(t, x, \dot{x})],$

$\lambda(t) \geqq 0,\ x(t_0) = x_0,\ x(t_1) = x_1,$

where $I = [a, b]$ is a real interval, $f : I \times R^n \times R^n \to R$ and $g : I \times R^n \times R^n \to R^m$; $x(t)$ is an n-dimensional piecewise smooth function of t and $\dot{x}(t)$ is the derivative of $x(t)$. Bector et al. [27] extended symmetric duality to variational problems, providing continuous analogous of the former results while Chandra and Husain [38] studied symmetric and self duality for fractional variational problems as dynamic generalizations.

Mond and Hanson [99] studied the following dual programs:

(P9) Minimize $\int_a^b [f(t, x, \dot{x}, y, \dot{y}) - y(t)^T \nabla_y f(t, x, \dot{x}, y, \dot{y}) + y(t)^T \frac{d}{dt} \nabla_{\dot{y}} f(t, x, \dot{x}, y, \dot{y})] dt$

subject to $x(a) = \alpha,\ x(b) = \beta$

$y(a) = \gamma,\ y(b) = \delta$

$\frac{d}{dt} \nabla_{\dot{y}} f(t, x, \dot{x}, y, \dot{y}) \geqq \nabla_y f(t, x, \dot{x}, y, \dot{y})$

$x(t) \geqq 0.$

(D9) Maximize $\int_a^b [f(t, u, \dot{u}, v, \dot{v}) - u(t)^T \nabla_x f(t, u, \dot{u}, v, \dot{v}) + u(t)^T \frac{d}{dt} \nabla_{\dot{x}} f(t, u, \dot{u}, v, \dot{v})] dt$

subject to $u(a) = \alpha,\ u(b) = \beta$

$v(a) = \gamma,\ v(b) = \delta$

$$\frac{d}{dt}\nabla_{\dot{x}} f(t,u,\dot{u},v,\dot{v}) \leqq \nabla_x f(t,u,\dot{u},v,\dot{v})$$
$$v(t) \geqq 0,$$

where $x : I \to R^n$, $y : I \to R^m$ and $f(t,x,\dot{x},y,\dot{y})$ is a continuously differentiable scalar function. They assumed f to be convex in x and $\dot{x}$ for each y and $\dot{y}$ and concave in y and $\dot{y}$ for each x and $\dot{x}$ to prove duality results.

If the constraints $x(t) \geqq 0$ and $v(t) \geqq 0$ are removed from the above primal and dual problems, we get the pair considered in Smart and Mond [119], wherein weak duality theorem is proved assuming the functional $\int_a^b f(t,x,\dot{x},y,\dot{y})dt$ to be invex in x and $\dot{x}$ and -$\int_a^b f(t,x,\dot{x},y,\dot{y})dt$ to be invex in y and $\dot{y}$. Later, Mond and Husain [101] obtained sufficient optimality conditions under weaker invexity assumptions and proved various duality results for a Mond-Weir type dual. Bector and Husain [29] generalized Wolfe and Mond-Weir type duals to their multiobjective analogue.

1.3 Summary of the thesis

The aim of the present thesis is to study the duality for some dual mathematical programming problems under generalized convexity assumptions. The results obtained are discussed in Chapter 2 to 8 and in Appendix A. Chapter 1 is introductory. Chapterwise summary is as follows:

In Chapter 2, we consider the following pair of second-order Wolfe type nondifferentiable multiobjective programming problem with k-objectives:

Primal problem (SP)

Minimize $L(x,y,\lambda,p) = f(x,y) + S(x \mid C)e_k - y^T\nabla_y(\lambda^T f)(x,y)e_k - y^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k$
$$- \tfrac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k,$$

subject to

$$\nabla_y(\lambda^T f)(x,y) - z + \nabla_{yy}(\lambda^T f)(x,y)p \leqq 0,$$

$$z \in D,$$

$$\lambda > 0, \quad \lambda^T e_k = 1.$$

Dual problem (SD)

Maximize $M(u,v,\lambda,r) = f(u,v) - S(v \mid D)e_k - u^T\nabla_x(\lambda^T f)(u,v)e_k - u^T(\nabla_{xx}(\lambda^T f)(u,v)r)e_k$
$$- \tfrac{1}{2}r^T(\nabla_{xx}(\lambda^T f)(u,v)r)e_k,$$

subject to

$$\nabla_x(\lambda^T f)(u,v) + w + \nabla_{xx}(\lambda^T f)(u,v)r \geqq 0,$$

$$w \in C,$$

$$\lambda > 0, \quad \lambda^T e_k = 1,$$

where

(i) f is a differentiable function from $R^n \times R^m \to R^k$, $e_k = (1,\ldots,1)^T \in R^k$,

(ii) r, w and p, z are vectors in R^n and R^m, respectively, $\lambda \in R^k$ and

(iii) C and D are compact convex sets in R^n and R^m, respectively.

The usual duality theorems are further established under second-order F-convexity assumptions. We illustrate a non trivial example which is second-order F-convex but not convex. Another example which is second-order (F,α,ρ,d)-convex but not second-order F-convex is also exemplified. Further, in Section 2.5, we studied a symmetric multiobjective dual programs over arbitrary cones and established appropriate duality results under second-order (F,α,ρ,d)-convexity assumptions.

In Chapter 3, we formulate the following pair of multiobjective second-order symmetric dual programs in which the objective function is optimized with respect to an arbitrary closed convex cone:

Primal problem (WP)

K-minimize

$$G(x,y,\lambda,p) = \Big(f_1(x,y) - y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy}f_i(x,y)p_i) - \tfrac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy}f_i(x,y)p_i),$$
$$\ldots, f_k(x,y) - y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy}f_i(x,y)p_i) - \tfrac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy}f_i(x,y)p_i)\Big)$$

subject to

$$-\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{yy}f_i(x,y)p_i) \in C_2^*,$$

$$\lambda^T e_k = 1,$$

$$\lambda \in \text{int } K^*,\ x \in C_1.$$

Dual problem (WD)

K-maximize

$$H(u,v,\lambda,q) = \Big(f_1(u,v) - u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) - \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i (q_i^T \nabla_{xx} f_i(u,v) q_i),$$
$$\ldots, f_k(u,v) - u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) - \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i (q_i^T \nabla_{xx} f_i(u,v) q_i)\Big)$$

subject to

$$\sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) \in C_1^*,$$

$$\lambda^T e_k = 1,$$

$$\lambda \in \text{int } K^*,\ v \in C_2\ ,$$

where

(*i*) $f_i : S_1 \times S_2 \to R,\ i = 1, 2, ..., k$ is a differentiable function of x and y, $e_k = (1, \ldots, 1)^T \in\ R^k$,

(*ii*) q_i and p_i are vectors in R^n and R^m, respectively, for $i = 1, 2, \ldots, k$ and $\lambda \in R^k$.

(*iii*) C_1 and C_2 be closed convex cones in R^n and R^m, respectively, with nonempty interiors.

(*iv*) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

For the above dual pair, usual duality theorems are established under K-η-bonvexity assumptions. We also illustrate an example which is K-η-bonvex but not invex. Furhter, self duality is obtained by assuming the functions involved to be skew-symmetric. In Section 3.5, a pair of Mond-Weir type nondifferentiable multiobjective second-order symmetric dual programs over arbitrary cones is formulated, where each of the objective function contains a square root term with positive semidefinite matrix in $R^{n\times n}$. Weak, strong and converse duality results are then proved under second-order K-F-convexity/K-η-bonvexity assumptions. Some special cases are discussed in the last of the chapter.

In Chapter 4, we have established duality relations for the following pair of second-order mixed symmetric dual programs involving nondifferentiable functions under second-order F-convexity/pseudoconvexity assumptions:

Primal Problem (SMNP)

minimize

$$G(x^1,y^1,x^2,y^2,z^2,p,r) = f(x^1,y^1) + S(x^1 \mid C_1) + g(x^2,y^2) + S(x^2 \mid C_2) - (y^2)^T z^2$$
$$-(y^1)^T[\nabla_{y^1} f(x^1,y^1)+\nabla_{y^1y^1} f(x^1,y^1)p]-\tfrac{1}{2}p^T\nabla_{y^1y^1} f(x^1,y^1)p-\tfrac{1}{2}r^T\nabla_{y^2y^2} g(x^2,y^2)r,$$

subject to

$$\nabla_{y^1} f(x^1,y^1) - z^1 + \nabla_{y^1y^1} f(x^1,y^1)p \leqq 0,$$

$$\nabla_{y^2} g(x^2,y^2) - z^2 + \nabla_{y^2y^2} g(x^2,y^2)r \leqq 0,$$

$$(y^2)^T\left[\nabla_{y^2} g(x^2,y^2) - z^2 + \nabla_{y^2y^2} g(x^2,y^2)r\right] \geqq 0,$$

$$z^1 \in D_1,\ z^2 \in D_2.$$

Dual Problem (SMND)

maximize

$$H(u^1,v^1,u^2,v^2,w^2,q,s) = f(u^1,v^1) - S(v^1 \mid D_1) + g(u^2,v^2) - S(v^2 \mid D_2) + (u^2)^T w^2$$
$$-(u^1)^T[\nabla_{x^1} f(u^1,v^1)+\nabla_{x^1x^1} f(u^1,v^1)q]-\tfrac{1}{2}q^T\nabla_{x^1x^1} f(u^1,v^1)q-\tfrac{1}{2}s^T\nabla_{x^2x^2} g(u^2,v^2)s,$$

subject to

$$\nabla_{x^1} f(u^1,v^1) + w^1 + \nabla_{x^1x^1} f(u^1,v^1)q \geqq 0,$$

$$\nabla_{x^2} g(u^2,v^2) + w^2 + \nabla_{x^2x^2} g(u^2,v^2)s \geqq 0,$$

$$(u^2)^T\left[\nabla_{x^2} g(u^2,v^2) + w^2 + \nabla_{x^2x^2} g(u^2,v^2)s\right] \leqq 0,$$

$$w^1 \in C_1,\ w^2 \in C_2,$$

where

(*i*) $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions, (*ii*) $C_1,\ C_2,\ D_1$ and D_2 are compact convex sets in $R^{|J_1|},\ R^{|J_2|},\ R^{|K_1|}$ and $R^{|K_2|}$, respectively, and

(*iii*) $p \in R^{|K_1|}, r \in R^{|K_2|}, q \in R^{|J_1|}$ and $s \in R^{|J_2|}$,

let $J_1 \subseteq N = \{1, 2, \ldots, n\}, K_1 \subseteq M = \{1, 2, \ldots, m\}$ and $J_2 = N \backslash J_1$ and $K_2 = M \backslash K_1$. Suppose $|J_1|$ denote the number of elements in J_1. The other symbols $|J_2|, |K_1|$ and $|K_2|$ are defined similarly.

Next, we have considered a pair of mixed second-order symmetric dual programs over arbitrary cones and obtained duality results under second-order (F, ρ) convexity/pseudoconvexity assumptions.

In Chapter 5, the duality results for a pair of second-order multiobjective mixed symmetric dual programs over arbitrary cones are obtained under second-order invexity/pseudoinvexity assumptions. In Section 5.3, we have formulated nondifferentiable multiobjective second-order mixed symmetric dual programs. Weak, strong and converse duality theorems are proved for aforementioned pair using the notion of second-order F-convexity/pseudoconvexity assumptions.

In Chapter 6, we have studied a pair of higher-order Wolfe type multiobjective symmetric dual programs over arbitrary cones and proved the usual duality results under higher-order (F, α, ρ, d)-convexity/pseudo-convexity assumptions. Mond-Weir type higher-order dual is also discussed. We have also illustrated a non-trivial example of function lying in the class of higher-order K-(F, α, ρ, d)-convex but not in class of higher-order K-F-convex. Further, we consider the higher-order multiobjective symmetric nondifferentiable dual programs in which the objective function is optimized with respect to an arbitrary closed convex cone and proved duality theorems under higher-order-K-(F, α, ρ, d)-convexity assumptions.

Antczak [19] proved optimality conditions for a class of generalized fractional minimax programming problems involving B-(p, r)-invexity functions and established duality theorems for various duality models. In Chapter 7, we have discussed sufficient optimality conditions and duality theorems for a nondifferentiable minimax fractional programming problem with B-(p,r)-invexity. An example which is B-$(1,1)$-invex but not (p,r)-invex is exemplified. We also illustrate another example which is $(-1,1)$-invex but not convex.

In Chapter 8, the duality theorems for the following pair of nondifferentiable multiobjective fractional variational symmetric dual problems over arbitrary cones are proved under generalized (F,α,ρ,d)-convexity assumptions:

(FVP)

Minimize $$p=\big(p_1,p_2,\dots,p_l\big)$$

subject to

$$x(a)=0=x(b),\quad y(a)=0=y(b),$$

$$\dot{x}(a)=0=\dot{x}(b),\quad \dot{y}(a)=0=\dot{y}(b),$$

$$\begin{aligned}&\int_a^b\Big\{f^i(t,x,\dot{x},y,\dot{y})+s(x\mid B_i)-y^Tz^i\Big\}dt\\&\quad-p_i\int_a^b\Big\{g^i(t,x,\dot{x},y,\dot{y})-s(x\mid E_i)+y^Tr^i\Big\}dt=0,\quad i\in L,\end{aligned}$$

$$-\sum_{i=1}^{l}\lambda_i\Big\{\big[f^i_y-Df^i_{\dot{y}}-z^i\big]-p_i\big[g^i_y-Dg^i_{\dot{y}}+r^i\big]\Big\}\in C_2^*,\quad t\in I,$$

$$y^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f^i_y-Df^i_{\dot{y}}-z^i\big]-p_i\big[g^i_y-Dg^i_{\dot{y}}+r^i\big]\Big\}\geqq 0,\quad t\in I,$$

$$\lambda>0,\quad x(t)\in C_1,\quad t\in I,$$

$$z^i\in D_i,\quad r^i\in H_i,\quad i=1,2,\dots,l.$$

(FVD)

Maximize $$q=\big(q_1,q_2,\dots,q_l\big)$$

subject to

$$u(a)=0=u(b),\quad v(a)=0=v(b),$$

$$\dot{u}(a)=0=\dot{u}(b),\quad \dot{u}(a)=0=\dot{u}(b),$$

$$\int_a^b \Big\{ f^i(t,u,\dot{u},v,\dot{v}) - s(v \mid D_i) + u^T w^i \Big\} dt$$

$$-q_i \int_a^b \Big\{ g^i(t,u,\dot{u},v,\dot{v}) + s(v \mid H_i) - u^T s^i \Big\} dt = 0, \quad i \in L,$$

$$\sum_{i=1}^{l} \lambda_i \Big\{ \big[f_x^i - Df_{\dot{x}}^i + w^i \big] - q_i \big[g_x^i - Dg_{\dot{x}}^i - s^i \big] \Big\} \in C_1^*, \quad t \in I,$$

$$u^T \sum_{i=1}^{l} \lambda_i \Big\{ \big[f_x^i - Df_{\dot{x}}^i + w^i \big] - q_i \big[g_x^i - Dg_{\dot{x}}^i - s^i \big] \Big\} \leqq 0, \quad t \in I,$$

$$\lambda > 0, \quad v(t) \in C_2, \quad t \in I,$$

$$w^i \in B_i, \quad s^i \in E_i, \quad i = 1, 2, \ldots, l.$$

where $f^i : I \times R^n \times R^n \times R^m \times R^m \to R$, $g^i : I \times R^n \times R^n \times R^m \times R^m \to R$, $i \in L = \{1, 2, \ldots, l\}$, are continuously differentiable functions and $I = [a, b]$ be a real interval and $C_1 \subset R^n$, $C_2 \subset R^m$, be closed convex cones with nonempty interiors having positive polars C_1^* and C_2^*, respectively and

$$p_i = \frac{F^i(x,y)}{G^i(x,y)} = \frac{\displaystyle\int_a^b \Big\{ f^i(t,x,\dot{x},y,\dot{y}) + s(x \mid B_i) - y^T z^i \Big\} dt}{\displaystyle\int_a^b \Big\{ g^i(t,x,\dot{x},y,\dot{y}) - s(x \mid E_i) + y^T r^i \Big\} dt}$$

and

$$q_i = \frac{M^i(u,v)}{L^i(u,v)} = \frac{\displaystyle\int_a^b \Big\{ f^i(t,u,\dot{u},v,\dot{v}) - s(v \mid D_i) + u^T w^i \Big\} dt}{\displaystyle\int_a^b \Big\{ g^i(t,u,\dot{u},v,\dot{v}) + s(v \mid H_i) - u^T s^i \Big\} dt}.$$

A self duality theorem is also obtained by assuming the functions involved to be skew-symmetric.

At the last, an **Appendix A** has been given, in which we establish a strong duality theorem for a pair of multiobjective second-order symmetric dual programs. This removes an omission in an earlier result in Yang et al. [140].

Chapter 2

NONDIFFERENTIABLE MULTIOBJECTIVE SECOND-ORDER SYMMETRIC DUAL PROGRAMS[1]

2.1 Introduction

Motivated by the concept of second-order duality in nonlinear problems, introduced by Mangasarian [89], several researchers [13, 56, 62, 64] have worked in this field. Zhang and Mond [141] extended the class of (F, ρ)-convex functions to second-order (F, ρ)-convex functions and obtained duality results for second-order Mangasarian type, Mond-Weir type and generalized Mond-Weir type multiobjective dual problems. Later, Ahmad and Husain [11] introduced second-order (F, α, ρ, d)-convex functions, their generalizations and developed weak, strong and strict converse duality theorems for second-order Mond-Weir type multiobjective dual programs.

Recently, Gulati and Geeta [56] formulated a pair of Mond-Weir type second-order multiobjective symmetric dual programs over arbitrary cones and established duality results under pseudoinvexity/K-F- convexity assumptions.

This chapter is organized as follows. Section 2.2 contains notations and definitions used in this chapter. In Section 2.3, we illustrate an example which is second-order F-convex but not convex. Another example which is second-order (F, α, ρ, d)-convex

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[1] A part of this chapter has appeared in *Optimization Letters* 5 (2011) 125-139 and the remaining part has appeared in *Journal of Applied Mathematics and Informatics* 28 (2010) 1395-1408.

but not second-order F-convex is also exemplified. In Section 2.4, we consider a pair of Wolfe type second-order multiobjective symmetric dual programs involving nondifferentiable functions and prove appropriate duality relations using the notion of second-order F-convexity assumptions. Further, in Section 2.5, we generalize the symmetric multiobjective dual programs over arbitrary cones and establish usual duality results under second-order (F, α, ρ, d)-convexity assumptions. The last Section contains some particular cases of the programs considered in this chapter.

2.2 Notations and definitions

Consider a function $f = (f_1, f_2, \ldots, f_k) : X \mapsto R^k$ differentiable at $u \in X$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in R^k$ and $d = (d_1, d_2, \ldots, d_k) \in R^k$.

Definition 2.1 [11]. A twice differentiable function f_i over X is said to be second-order (F, α, ρ_i, d_i)-convex at u on X, if $\forall x \in X$, there exist vector $q \in R^n$, a real valued function $\alpha : X \times X \to R_+ \backslash \{0\}$, a real valued function $d_i(\cdot, \cdot) : X \times X \to R$ and a real number ρ_i such that

$$f_i(x) - f_i(u) + \tfrac{1}{2} q^T \nabla_{xx} f_i(u) q \geqq F[x, u; \alpha(x,u)(\nabla_x f_i(u) + \nabla_{xx} f_i(u) q)] + \rho_i d_i^2(x, u).$$

A twice differentiable vector function $f : X \mapsto R^k$ is said to be second-order (F, α, ρ, d)-convex at u, if each of its components f_i is second-order (F, α, ρ_i, d_i)-convex at u.

Remarks

(i) If $k = 1$ and $q = 0$, the above definition become that of (F, α, ρ, d)-convex functions introduced by Liang et al. [87].

(ii) For single objective programming problem and $\alpha(x, u) = 1$, the definition of second-order (F, α, ρ_i, d_i)-convexity reduces to second-order (F, ρ)-convexity given by Srivastava and Bhatia [120].

2.3 Examples

2.1. An example of a non-convex function which is second-order F-convex.

Let $X = [2.5, 3.5] \subset R$. Let the function $\psi : X \to R$ be defined by

$$\psi(x) = e^{-x^2} + 8\sqrt{x} + \sin 2x$$

and the functional $F : X \times X \times R \to R$ be given by

$$F(x, u; a) = a(u^2 - 13u - 1).$$

Now,

$$\begin{aligned}
G &= \psi(x) - \psi(u) + \tfrac{1}{2} r^T \nabla_{xx}\psi(u) r - F(x, u; \nabla_x \psi(u) + \nabla_{xx}\psi(u) r) \\
&= e^{-x^2} - e^{-u^2} + 8(\sqrt{x} - \sqrt{u}) + \sin 2x - \sin 2u + r^2[(2u^2 - 1)e^{-u^2} \\
&\qquad -u^{-\frac{3}{2}} - 2\sin 2u] + (13u + 1 - u^2)\{-2ue^{-u^2} + \tfrac{4}{\sqrt{u}} + 2\cos 2u \\
&\qquad\qquad +r[2(2u^2 - 1)e^{-u^2} - 2u^{-\frac{3}{2}} - 4\sin 2u]\} \\
&= e^{-x^2} - e^{-u^2} + 8(\sqrt{x} - \sqrt{u}) + \sin 2x - \sin 2u + (13u + 1 - u^2) \\
&\qquad [-2ue^{-u^2} + \tfrac{4}{\sqrt{u}} + 2\cos 2u] + r^2[(2u^2 - 1)e^{-u^2} - u^{-\frac{3}{2}} - 2\sin 2u] \\
&\qquad\quad +(13u + 1 - u^2)[2r(2u^2 - 1)e^{-u^2} - 2ru^{-\frac{3}{2}} - 4r\sin 2u] \\
&= \psi_1 + \psi_2 \text{ (say)}
\end{aligned}$$

where

$$\begin{aligned}
\psi_1 &= e^{-x^2} - e^{-u^2} + 8(\sqrt{x} - \sqrt{u}) + \sin 2x - \sin 2u + (13u + 1 - u^2)[-2ue^{-u^2} \\
&\qquad + \tfrac{4}{\sqrt{u}} + 2\cos 2u] \geqq 0 \;\; \forall\, x, u \in X, \text{ as can be seen from Figure 2.1}
\end{aligned}$$

and

$$\begin{aligned}
\psi_2 &= r^2[(2u^2 - 1)e^{-u^2} - u^{-\frac{3}{2}} - 2\sin 2u] + (13u + 1 - u^2)[2r(2u^2 - 1)e^{-u^2} - 2ru^{-\frac{3}{2}} \\
&\quad -4r\sin 2u] \geqq 0 \;\forall u \in X \text{ and } r \in (-10^{18}, 10^{18}) \text{ as can be seen from Figure 2.2.}
\end{aligned}$$

Hence $G \geqq 0$. Therefore, ψ is second-order F-convex. But for $x = 2.6$ and $u = 3.4$, $\psi(x) - \psi(u) = -3.228$ and $(x - u)^T \nabla_x \psi(u) = -3.126$, which implies

$\psi(x) - \psi(u) \ngeqq (x - u)^T \nabla_x \psi(u)$.

Hence ψ is not convex.

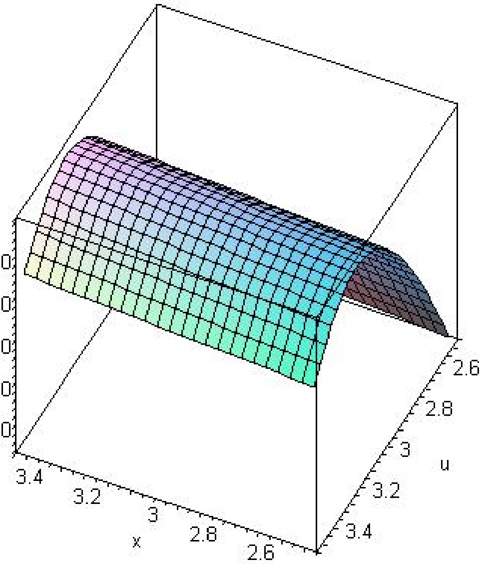

Figure 2.1: Graph of ψ_1 against x and u

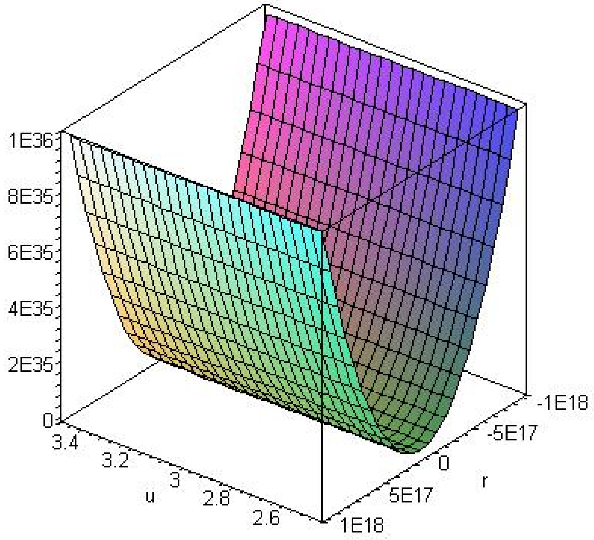

Figure 2.2: Graph of ψ_2 against u and r

2.2. An example of non trivial function which is second-order (F, α, ρ, d)-convex but not second-order F-convex.

Let $X = [0.7, 0.75] \subset R$. Let the function $f : X \to R$ be defined by $f(x) = \sin^2 x$ and $\alpha : X \times X \to R_+ \backslash \{0\}$ be identified by $\alpha(x, u) = (x + u + 1)$. Let the functional $F : X \times X \times R \to R$ be defined by $F(x, u; a) = \dfrac{a}{(x + u + 1)}$ and $d : X \times X \to R$ be given by $d(x, u) = \sqrt{x^2 + u^2}$.

For $\rho = -4$, we have

$L = f(x) - f(u) + \frac{1}{2}q^T \nabla_{xx} f(u)q - F[x, u; \alpha(x, u)(\nabla_x f(u) + \nabla_{xx} f(u)q)] - \rho d^2(x, u)$

$\quad = \sin^2 x - \sin^2 u + q^2 \cos 2u - F[x, u; (x+u+1)(\sin 2u + 2q \cos 2u)] - (-4)(\sqrt{x^2 + u^2})^2$

$\quad = \sin^2 x - \sin^2 u - \sin 2u + 4x^2 + q^2 \cos 2u - 2q \cos 2u + 4u^2$

$\quad = f_1 + f_2$ (say)

where

$f_1 = \sin^2 x - \sin^2 u - \sin 2u + 4x^2$

$\quad \geqq 0 \ \forall \ x, u \in X$ as can be seen from Figure 2.3

and

$f_2 = q^2 \cos 2u - 2q \cos 2u + 4u^2$

$\quad \geqq 0 \ \forall \ u \in X$ and $q \in (-10^{18}, 10^{18})$ as can be seen from Figure 2.4.

Hence $L \geqq 0$. Therefore f is second-order (F, α, ρ, d)-convex.

But f is not second-order F-convex since for $q = 1$, we have

$M = f(x) - f(u) + \frac{1}{2}q^T \nabla_{xx} f(u)q - F[x, u; \nabla_x f(u) + \nabla_{xx} f(u)q]$

$\quad = \sin^2 x - \sin^2 u + \cos 2u - F[x, u; \sin 2u + 2 \cos 2u]$

$\quad = \sin^2 x - \sin^2 u + \cos 2u - \frac{1}{(x+u+1)}(\sin 2u + 2 \cos 2u)$

$\quad < 0 \ \forall \ x, u \in X$ as be can seen from Figure 2.5.

Hence the function f is second-order (F, α, ρ, d)-convex but is not second-order F-convex.

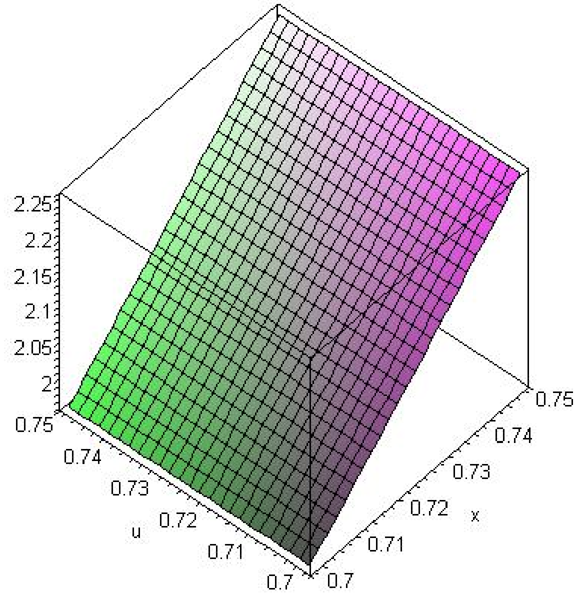

Figure 2.3: Graph of f_1 against u and x

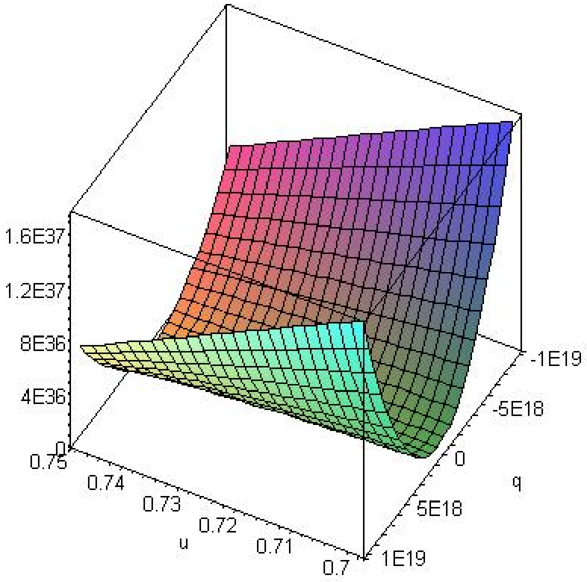

Figure 2.4: Graph of f_2 against u and q

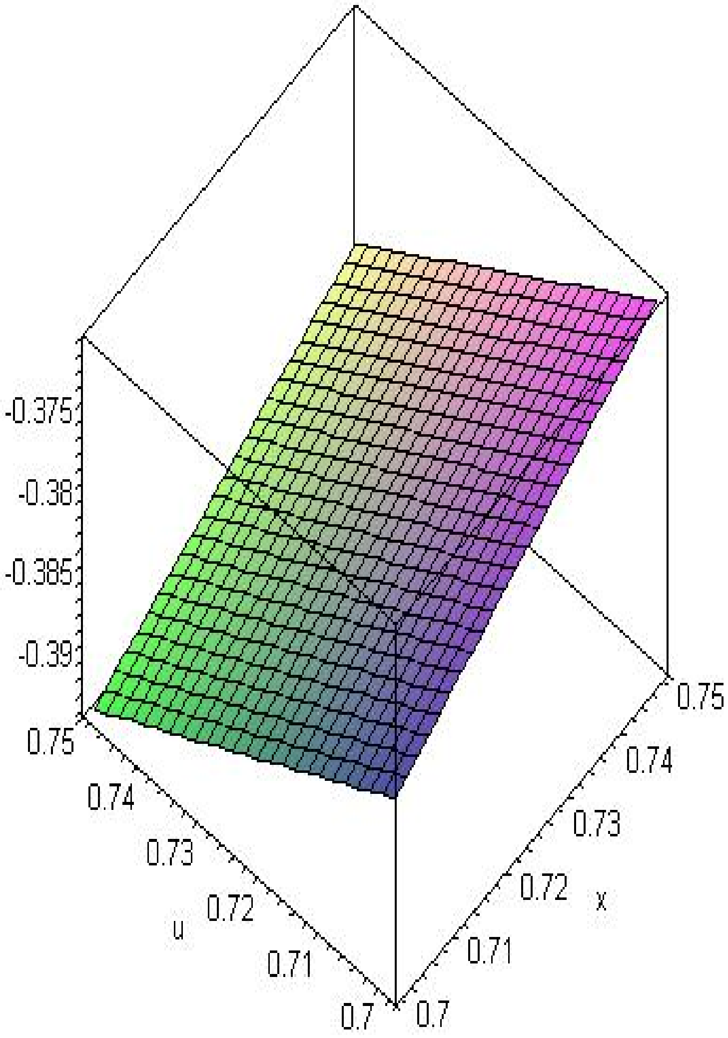

Figure 2.5: Graph of M against u and x

2.4 Wolfe type symmetric duality

Wolfe type second-order symmetric duality has been discussed by Ahmad and Husain [9], Yang et al. [137] and Gulati and Gupta [57] for single objective involving nondifferentiable functions and by Yang et al. [140] and Ahmad and Husain [13] for multiobjective programming problems. Later on, an ommission in the strong duality theorem in Yang et al. [140] has been rectified by Gupta and Kailey [64]. The work cited in [13, 64, 140] involved differentiable functions.

Consider the following pair of second-order Wolfe type nondifferentiable multiobjective programming problem with k-objectives:

Primal problem (SP)

Minimize $L(x,y,\lambda,p) = f(x,y)+S(x \mid C)e_k - y^T\nabla_y(\lambda^T f)(x,y)e_k - y^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k$

$$-\tfrac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k,$$

subject to

$$\nabla_y(\lambda^T f)(x,y) - z + \nabla_{yy}(\lambda^T f)(x,y)p \leqq 0, \tag{2.1}$$

$$z \in D, \tag{2.2}$$

$$\lambda > 0, \quad \lambda^T e_k = 1. \tag{2.3}$$

Dual problem (SD)

Maximize $M(u,v,\lambda,r) = f(u,v)-S(v \mid D)e_k - u^T\nabla_x(\lambda^T f)(u,v)e_k - u^T(\nabla_{xx}(\lambda^T f)(u,v)r)e_k$

$$-\tfrac{1}{2}r^T(\nabla_{xx}(\lambda^T f)(u,v)r)e_k,$$

subject to

$$\nabla_x(\lambda^T f)(u,v) + w + \nabla_{xx}(\lambda^T f)(u,v)r \geqq 0, \tag{2.4}$$

$$w \in C, \tag{2.5}$$

$$\lambda > 0, \quad \lambda^T e_k = 1, \tag{2.6}$$

where

(i) $f : R^n \times R^m \to R^k$ is a differentiable function, $e_k = (1,\ldots,1)^T \in R^k$,

(ii) r, w and p, z are vectors in R^n and R^m, respectively, $\lambda \in R^k$ and

(iii) C and D are compact convex sets in R^n and R^m, respectively.

We now prove the following duality results for the pair of problems (SP) and (SD).

Theorem 2.1 (Weak duality). Let (x, y, λ, z, p) be feasible for the primal problem (SP) and (u, v, λ, w, r) be feasible for the dual problem (SD). Let

(i) $f_i(., v) + (.)^T w \ \ (1 \leqq i \leqq k)$ be second-order F-convex at u,

(ii) $f_i(x, .) - (.)^T z \ \ (1 \leqq i \leqq k)$ be second-order G-concave at y,

where the sublinear functionals $F : R^n \times R^n \times R^n \mapsto R$ and $G : R^m \times R^m \times R^m \mapsto R$ satisfy the following conditions:

(iii) $F(x, u; a) + a^T u \geqq 0, \quad \text{for all } a \in R^n_+,$

(iv) $G(v, y; b) + b^T y \geqq 0, \quad \text{for all } b \in R^m_+.$

Then

$$L(x, y, \lambda, p) \nleq M(u, v, \lambda, r). \tag{2.7}$$

Proof. Suppose, to the contrary, that (2.7) is not true, that is,
$L(x, y, \lambda, p) \leq M(u, v, \lambda, r)$,
or

$$f(x,y)+S(x \mid C)e_k-y^T\nabla_y(\lambda^T f)(x,y)e_k-y^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k-\frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k$$
$$\leq f(u,v)-S(v \mid D)e_k-u^T\nabla_x(\lambda^T f)(u,v)e_k-u^T(\nabla_{xx}(\lambda^T f)(u,v)r)e_k-\frac{1}{2}r^T(\nabla_{xx}(\lambda^T f)(u,v)r)e_k.$$

Now, since $\lambda > 0$ and $\lambda^T e_k = 1$, we get

$$\begin{aligned}(\lambda^T f)(x,y) + S(x \mid C) - y^T\nabla_y(\lambda^T f)(x,y) - y^T(\nabla_{yy}(\lambda^T f)(x,y)p)\\ -\frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p) < (\lambda^T f)(u,v) - S(v \mid D) - u^T\nabla_x(\lambda^T f)(u,v)\\ -u^T(\nabla_{xx}(\lambda^T f)(u,v)r)-\frac{1}{2}r^T(\nabla_{xx}(\lambda^T f)(u,v)r).\end{aligned} \tag{2.8}$$

Since (x, y, λ, z, p) is feasible for the primal problem (SP) and (u, v, λ, w, r) is feasible for the dual problem (SD), by the dual constraint (2.4), the vector $\xi = \nabla_x(\lambda^T f)(u, v) + w + \nabla_{xx}(\lambda^T f)(u, v)r \in R^n_+$ and so from the hypothesis (iii), we obtain

$$F(x, u; \xi) + \xi^T u \geqq 0. \tag{2.9}$$

Similarly,

$$G(v, y; \gamma) + \gamma^T y \geqq 0, \tag{2.10}$$

for the vector $\gamma = -\{\nabla_y(\lambda^T f)(x, y) - z + \nabla_{yy}(\lambda^T f)(x, y)p\} \in R^m_+$.

From the second-order F-convexity of $f_i(., v) + (.)^T w$ $(1 \leqq i \leqq k)$, we have

$$f_i(x, v) + x^T w - f_i(u, v) - u^T w + \frac{1}{2} r^T (\nabla_{xx} f_i(u, v)r)$$
$$\geqq F(x, u; \nabla_x f_i(u, v) + w + \nabla_{xx} f_i(u, v)r).$$

It follows from $\lambda > 0$, $\lambda^T e_k = 1$ and sublinearity of F that

$$(\lambda^T f)(x, v) + x^T w - (\lambda^T f)(u, v) - u^T w$$
$$+ \frac{1}{2} r^T (\nabla_{xx}(\lambda^T f)(u, v)r) \geqq F(x, u; \nabla_x(\lambda^T f)(u, v) + w + \nabla_{xx}(\lambda^T f)(u, v)r),$$

or

$$(\lambda^T f)(x, v) + x^T w - (\lambda^T f)(u, v) - u^T w + \frac{1}{2} r^T (\nabla_{xx}(\lambda^T f)(u, v)r) \geqq F(x, u; \xi). \tag{2.11}$$

Using (2.9) in (2.11), we have

$$(\lambda^T f)(x, v) + x^T w - (\lambda^T f)(u, v) - u^T w + \frac{1}{2} r^T (\nabla_{xx}(\lambda^T f)(u, v)r) \geqq -u^T \xi. \tag{2.12}$$

From the second-order G-concavity of $f_i(x, .) - (.)^T z$ $(1 \leqq i \leqq k)$, we get

$$f_i(x, y) - y^T z - f_i(x, v) + v^T z - \frac{1}{2} p^T (\nabla_{yy} f_i(x, y)p)$$
$$\geqq G(v, y; -\nabla_y f_i(x, y) + z - \nabla_{yy} f_i(x, y)p).$$

It follows from $\lambda > 0$, $\lambda^T e_k = 1$ and sublinearity of G that

$$(\lambda^T f)(x, y) - y^T z - (\lambda^T f)(x, v) + v^T z$$
$$- \frac{1}{2} p^T (\nabla_{yy}(\lambda^T f)(x, y)p) \geqq G(v, y; -\nabla_y(\lambda^T f)(x, y) + z - \nabla_{yy}(\lambda^T f)(x, y)p),$$

or

$$(\lambda^T f)(x, y) - y^T z - (\lambda^T f)(x, v) + v^T z - \frac{1}{2} p^T (\nabla_{yy}(\lambda^T f)(x, y)p) \geqq G(v, y; \gamma). \tag{2.13}$$

From (2.10) and (2.13), we get

$$(\lambda^T f)(x,y) - y^T z - (\lambda^T f)(x,v) + v^T z - \frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p) \geqq -y^T\gamma. \tag{2.14}$$

Adding the inequalities (2.12) and (2.14), we obtain

$$(\lambda^T f)(x,y) + x^T w - y^T z + y^T\gamma - \frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)$$
$$\geqq (\lambda^T f)(u,v) + u^T w - v^T z - u^T\xi - \frac{1}{2}r^T(\nabla_{xx}(\lambda^T f)(u,v)r).$$

Substituting the values of ξ and γ, we have

$$(\lambda^T f)(x,y) + x^T w - y^T\nabla_y(\lambda^T f)(x,y) - y^T(\nabla_{yy}(\lambda^T f)(x,y)p) - \frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)$$
$$\geqq (\lambda^T f)(u,v) - v^T z - u^T\nabla_x(\lambda^T f)(u,v) - u^T(\nabla_{xx}(\lambda^T f)(u,v)r) - \frac{1}{2}r^T(\nabla_{xx}(\lambda^T f)(u,v)r).$$

Finally, since $x^T w \leqq S(x \mid C)$, $v^T z \leqq S(v \mid D)$ and $\lambda^T e_k = 1$, the last inequality yields

$$(\lambda^T f)(x,y) + S(x \mid C) - y^T\nabla_y(\lambda^T f)(x,y) - y^T(\nabla_{yy}(\lambda^T f)(x,y)p) - \frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)$$
$$\geqq (\lambda^T f)(u,v) - S(v \mid D) - u^T\nabla_x(\lambda^T f)(u,v) - u^T(\nabla_{xx}(\lambda^T f)(u,v)r) - \frac{1}{2}r^T(\nabla_{xx}(\lambda^T f)(u,v)r),$$

which contradicts (2.8). Hence the result.

Theorem 2.2 (Strong duality). Let $f : R^n \times R^m \to R^k$ be thrice differentiable function and let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ be a weak efficient solution of (SP). If

(*i*) the matrix $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is non singular,

(*ii*) the vectors $\{\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\}$ are linearly independent,

(*iii*) the vector $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} \notin \mathrm{span}\{\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\}\backslash\{0\}$ and

(*iv*) $\bar{p} \neq 0$ implies $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} \neq 0$,

then

(*I*) $\bar{p} = 0$, there exist $\bar{w} \in C$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ is feasible for (SD) and

(*II*) $L(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p}) = M(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$.

Also, if the hypotheses of Theorem 2.1 are satisfied for all feasible solutions of (SP) and (SD), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ is an efficient solution for (SD).

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ is a weak efficient solution of (SP), there exist $\alpha \in R^k$,

$\beta \in R^m$, $\gamma \in R^n$, $\delta \in R^k$ and $\mu \in R$, such that the following Fritz John optimality conditions [114] are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$:

$$\begin{aligned}\{\alpha^T(\nabla_x f(\bar{x},\bar{y}) + \gamma e_k) + \nabla_{xy}(\bar{\lambda}^T f)(\bar{x},\bar{y})(\beta - (\alpha^T e_k)\bar{y})\\ +\nabla_x[\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}](\beta - (\alpha^T e_k)(\bar{y} + \frac{1}{2}\bar{p}))\} = 0,\end{aligned} \tag{2.15}$$

$$\begin{aligned}[\nabla_y f(\bar{x},\bar{y})]^T[\alpha - (\alpha^T e_k)\bar{\lambda}] + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})(\beta - (\alpha^T e_k)(\bar{y} + \bar{p}))\\ +\nabla_y[\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}](\beta - (\alpha^T e_k)(\bar{y} + \frac{1}{2}\bar{p})) = 0,\end{aligned} \tag{2.16}$$

$$\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})(\beta - (\alpha^T e_k)(\bar{y}+\bar{p})) = 0, \tag{2.17}$$

$$\begin{aligned}\nabla_y f(\bar{x},\bar{y})(\beta - (\alpha^T e_k)\bar{y}) + \mu e_k - \delta + [(\beta - (\alpha^T e_k)(\bar{y} + \frac{1}{2}\bar{p}))^T\\ \times\nabla_{yy} f_1(\bar{x},\bar{y})\bar{p}, \ldots, (\beta - (\alpha^T e_k)(\bar{y} + \frac{1}{2}\bar{p}))^T \nabla_{yy} f_k(\bar{x},\bar{y})\bar{p}] = 0,\end{aligned} \tag{2.18}$$

$$\beta^T[\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y}) - \bar{z} + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}] = 0, \tag{2.19}$$

$$\delta^T \bar{\lambda} = 0, \tag{2.20}$$

$$\beta \in N_D(\bar{z}), \tag{2.21}$$

$$\gamma \in C, \quad \gamma^T \bar{x} = S(\bar{x} \mid C), \tag{2.22}$$

$$(\alpha, \beta, \delta) \geqq 0, \ (\alpha, \beta, \delta, \mu) \neq 0. \tag{2.23}$$

Since $\bar{\lambda} > 0$ and $\delta \geqq 0$, (2.20) yields $\delta = 0$.

By hypothesis (i), (2.17) gives

$$\beta = (\alpha^T e_k)(\bar{y}+\bar{p}). \tag{2.24}$$

Suppose $\alpha = 0$, then (2.24) implies $\beta = 0$ and (2.18) yields $\mu = 0$. Consequently $(\alpha, \beta, \delta, \mu) = 0$, contradicting (2.23). Hence, $\alpha \geq 0$ or

$$\alpha^T e_k > 0. \tag{2.25}$$

Now, we claim that $\bar{p} = 0$. Indeed, if $\bar{p} \neq 0$, then hypothesis (iv) implies $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} \neq 0$.

Using (2.24) and (2.25) in (2.16), we get

$$\nabla_y[\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p}]\bar{p} = -\frac{2}{\alpha^T e_k}\nabla_y f(\bar{x}, \bar{y})[\alpha - (\alpha^T e_k)\bar{\lambda}], \tag{2.26}$$

which contradicts the hypothesis (iii). Hence,

$$\bar{p} = 0. \tag{2.27}$$

Now, from (2.25)-(2.27), we have

$[\nabla_y f(\bar{x}, \bar{y})]^T[\alpha - (\alpha^T e_k)\bar{\lambda}] = 0$.

But the vectors $\{\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\}$ are linearly independent implies

$$\alpha = (\alpha^T e_k)\bar{\lambda}. \tag{2.28}$$

From (2.24) and (2.27), we get

$$\beta = (\alpha^T e_k)\bar{y}. \tag{2.29}$$

Using equations (2.25), (2.27)-(2.29) in (2.15), we have

$$\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \gamma = 0. \tag{2.30}$$

Now, taking $\bar{w} = \gamma \in C$ in (2.30), we find that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ satisfies the constraints from (2.4) to (2.6), of (SD), and is therefore a feasible solution for the dual problem (SD).

Further, using (2.25), (2.27), (2.29) in (2.19), we obtain

$$\bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = \bar{y}^T \bar{z}. \tag{2.31}$$

Moreover, since $\beta = (\alpha^T e_k)\bar{y}$ and $\alpha^T e_k > 0$, (2.21) implies $\bar{y} \in N_D(\bar{z})$, so that

$$\bar{y}^T \bar{z} = S(\bar{y} \mid D). \tag{2.32}$$

Therefore, using (2.22), (2.27) and (2.30)-(2.32), we get

$$f(\bar{x},\bar{y})+S(\bar{x}\mid C)e_k-\bar{y}^T\nabla_y(\bar{\lambda}^Tf)(\bar{x},\bar{y})e_k-\bar{y}^T(\nabla_{yy}(\bar{\lambda}^Tf)(\bar{x},\bar{y})\bar{p})e_k$$
$$-\tfrac{1}{2}\bar{p}^T(\nabla_{yy}(\bar{\lambda}^Tf)(\bar{x},\bar{y})\bar{p})e_k=f(\bar{x},\bar{y})-S(\bar{y}\mid D)e_k-\bar{x}^T\nabla_x(\bar{\lambda}^Tf)(\bar{x},\bar{y})e_k$$
$$-\bar{x}^T(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{x},\bar{y})\bar{r})e_k-\tfrac{1}{2}\bar{r}^T(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{x},\bar{y})\bar{r})e_k,$$

that is, the two objective function values are equal.

Now, let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r}=0)$ is not an efficient solution of (SD), then there exist $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{r}=0)$ feasible for (SD), such that,

$$f(\bar{x},\bar{y})-S(\bar{y}\mid D)e_k-\bar{x}^T[\nabla_x(\bar{\lambda}^Tf)(\bar{x},\bar{y})e_k+(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{x},\bar{y})\bar{r})e_k]$$
$$-\tfrac{1}{2}\bar{r}^T(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{x},\bar{y})\bar{r})e_k\leq f(\bar{u},\bar{v})-S(\bar{v}\mid D)e_k-\bar{u}^T[\nabla_x(\bar{\lambda}^Tf)(\bar{u},\bar{v})e_k$$
$$+(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{u},\bar{v})\bar{r})e_k]-\tfrac{1}{2}\bar{r}^T(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{u},\bar{v})\bar{r})e_k.$$

Since $\bar{x}^T\nabla_x(\bar{\lambda}^Tf)(\bar{x},\bar{y})=-S(\bar{x}\mid C)$ and $\bar{y}^T\nabla_y(\bar{\lambda}^Tf)(\bar{x},\bar{y})=S(\bar{y}\mid D)$ and $\bar{p}=0$,

$$\{f(\bar{x},\bar{y})+S(\bar{x}\mid C)e_k-\bar{y}^T[\nabla_y(\bar{\lambda}^Tf)(\bar{x},\bar{y})e_k+(\nabla_{yy}(\bar{\lambda}^Tf)(\bar{x},\bar{y})\bar{p})e_k]\}$$
$$-\tfrac{1}{2}\bar{p}^T(\nabla_{yy}(\bar{\lambda}^Tf)(\bar{x},\bar{y})\bar{p})e_k\leq\{f(\bar{u},\bar{v})-S(\bar{v}\mid D)e_k-\bar{u}^T[\nabla_x(\bar{\lambda}^Tf)(\bar{u},\bar{v})e_k$$
$$+(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{u},\bar{v})\bar{r})e_k]\}-\tfrac{1}{2}\bar{r}^T(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{u},\bar{v})\bar{r})e_k,$$

that is

$$L(\bar{x},\bar{y},\bar{\lambda},\bar{p})\leq M(\bar{u},\bar{v},\bar{\lambda},\bar{r}),$$

which contradicts weak duality theorem. Hence, $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r}=0)$ is an efficient solution of (SD).

Theorem 2.3 (Converse duality). Let $f:R^n\times R^m\rightarrow R^k$ be thrice differentiable function and let $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{r})$ be a weak efficient solution of (SD). If

- *(i)* the matrix $\nabla_{xx}(\bar{\lambda}^Tf)(\bar{u},\bar{v})$ is non singular,
- *(ii)* the vectors $\{\nabla_xf_1(\bar{u},\bar{v}),\ldots,\nabla_xf_k(\bar{u},\bar{v})\}$ are linearly independent,
- *(iii)* the vector $\nabla_x(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{u},\bar{v})\bar{r})\bar{r}\notin\text{span}\{\nabla_xf_1(\bar{u},\bar{v}),\ldots,\nabla_xf_k(\bar{u},\bar{v})\}\backslash\{0\}$ and
- *(iv)* $\bar{r}\neq0$ implies $\nabla_x(\nabla_{xx}(\bar{\lambda}^Tf)(\bar{u},\bar{v})\bar{r})\bar{r}\neq0$,

  then

- *(I)* $\bar{r}=0$, there exist $\bar{z}\in D$ such that $(\bar{u},\bar{v},\bar{\lambda},\bar{z},\bar{p}=0)$ is feasible for (SP) and

(II) $L(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p}) = M(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r})$.

Also, if the hypotheses of Theorem 2.1 are satisfied for all feasible solutions of (SP) and (SD), then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ is an efficient solution for (SP).

Proof. It follows on the lines of Theorem 2.2.

2.4.1 Remarks

(i) Let $n = m = 2$, $f_1(x, y) = x_1^2 + x_2^2 - y_1^2 - y_2^2$, $f_2(x, y) = x_1 + x_2^2 - y_2$, $C = \{(\xi_1, \xi_2) : 0 \leqq \xi_i \leqq 1, i = 1, 2\}$, $D = \{(\varsigma_1, \varsigma_2) : \varsigma_1 - \varsigma_2 = 0, 0 \leqq \varsigma_1 \leqq 1\}$.

Then

$S(x \mid C) = (x_1 + |x_1| + x_2 + |x_2|)/2$ and $S(y \mid D) = (y_1 + y_2 + |y_1 + y_2|)/2$.

Problem (SP) and (SD) become

Primal problem (ESP)

$$\text{Minimize} \quad L(x, y, \lambda, p) = \{G_1(x, y, \lambda, p), G_2(x, y, \lambda, p)\}$$

$$\text{subject to} \quad \{2\lambda_1(y_1 + p_1) + z_1, 2\lambda_1(y_2 + p_2) + z_2 + \lambda_2\} \geqq 0,$$

$$z_1 - z_2 = 0, \ 0 \leqq z_1 \leqq 1,$$

$$\lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1.$$

Dual problem (ESD)

$$\text{Maximize} \quad M(u, v, \lambda, r) = \{H_1(u, v, \lambda, r), H_2(u, v, \lambda, r)\}$$

$$\text{subject to} \quad \{2\lambda_1(u_1 + r_1) + \lambda_2 + w_1, 2(u_2 + r_2) + w_2\} \geqq 0,$$

$$0 \leqq w_i \leqq 1, \ i = 1, 2,$$

$$\lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1,$$

where

$$G_i(x, y, \lambda, p) = x_1^j + x_2^2 + (-1)^j \lambda_j y_2 + (-1)^i \lambda_j (y_1^2 + y_2^2)$$
$$+ \lambda_1[(y_1 + p_1)^2 + (y_2 + p_2)^2] + (x_1 + |x_1| + x_2 + |x_2|)/2,$$

$$H_i(u, v, \lambda, r) = -\{(j-1)v_1^2 + v_2^j + (v_1 + v_2 + |v_1 + v_2|)/2$$
$$+ (u_2 + r_2)^2 + \lambda_1(u_1 + r_1)^2 + (-1)^i \lambda_j u_1(u_1 - 1)\},$$

and $i + j = 3$ and $i = 1, 2$.

Therefore, our results also give the duality relations for (ESP) and (ESD), which cannot obtained from the work in [9, 48, 55, 57, 89, 137, 140], because the above dual models are a pair of second-order multiobjective symmetric problems involving nondifferentiable terms $S(x \mid C)$ and $S(y \mid D)$.

(ii) We can also construct a pair of Wolfe type second-order symmetric dual programs by taking $C = \{Ay : y^T Ay \leqq 1\}$ and $D = \{Bx : x^T Bx \leqq 1\}$ in our models (SP) and (SD), where A and B are positive semidefinite matrices. For C and D so defined, $(x^T Ax)^{1/2} = S(x \mid C)$ and $(y^T By)^{1/2} = S(y \mid D)$. Thus, duality results for such a dual pair are obtained.

2.5 Symmetric duality with cone constraints

In this section, we establish duality theorems under second-order (F, α, ρ, d)-convexity for the following pair of second-order Wolfe type nondifferentiable multiobjective programming problems with k-objectives over arbitrary cones:

Primal problem (PP)

Minimize $G(x, y, \lambda, p) = \{G_1(x, y, \lambda, p), G_2(x, y, \lambda, p), \cdots, G_k(x, y, \lambda, p)\}$

subject to

$$-\nabla_y(\lambda^T f)(x, y) + z - \nabla_{yy}(\lambda^T f)(x, y)p \in C_2^*, \tag{2.33}$$

$$z \in E, \tag{2.34}$$

$$\lambda^T e_k = 1, \tag{2.35}$$

$$\lambda > 0,\ x \in C_1. \tag{2.36}$$

Dual problem (DP)

Maximize $H(u, v, \lambda, q) = \{H_1(u, v, \lambda, q), H_2(u, v, \lambda, q), \cdots, H_k(u, v, \lambda, q)\}$

subject to

$$\nabla_x(\lambda^T f)(u, v) + w + \nabla_{xx}(\lambda^T f)(u, v)q \in C_1^*, \tag{2.37}$$

$$w \in D, \tag{2.38}$$

$$\lambda^T e_k = 1, \tag{2.39}$$

$$\lambda > 0,\ v \in C_2\ , \tag{2.40}$$

where for $i = 1, 2, \cdots, k$,

$$\begin{aligned} G_i(x, y, \lambda, p) = {} & f_i(x, y) + S(x \mid D) - y^T\nabla_y(\lambda^T f)(x, y) \\ & -y^T(\nabla_{yy}(\lambda^T f)(x, y)p) - \tfrac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x, y)p), \\ H_i(u, v, \lambda, q) = {} & f_i(u, v) - S(v \mid E) - u^T\nabla_x(\lambda^T f)(u, v) \\ & -u^T(\nabla_{xx}(\lambda^T f)(u, v)q) - \tfrac{1}{2}q^T(\nabla_{xx}(\lambda^T f)(u, v)q). \end{aligned}$$

and

(*i*) $f = \{f_1, f_2, \cdots, f_k\}$ is a differentiable function from $S_1 \times S_2 \to R^k$ ($S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets), $e_k = (1, \ldots, 1)^T \in R^k$,

(*ii*) q and p are vectors in R^n and R^m, respectively, $\lambda \in R^k$,

(*iii*) D and E are compact convex sets in R^n and R^m, respectively.

Theorem 2.4 (Weak duality). Let (x, y, λ, z, p) be feasible for the primal problem (PP) and (u, v, λ, w, q) be feasible for the dual problem (DP). Let the sublinear functionals $F : S_1 \times S_1 \times R^n \mapsto R$ and $G : S_2 \times S_2 \times R^m \mapsto R$ satisfy the following conditions:

$F(x, u; a) + \alpha_1^{-1}a^T u \geqq 0, \quad \text{for all } a \in C_1^*, \qquad \text{(A)}$

$G(v, y; b) + \alpha_2^{-1}b^T y \geqq 0, \quad \text{for all } b \in C_2^*. \qquad \text{(B)}$

Suppose that either (i) $\sum_{i=1}^{k} \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x, u))^2 + \rho_i^{(2)}(d_i^{(2)}(v, y))^2] \geqq 0$ or (ii) $\rho_i^{(1)} \geqq 0$ and $\rho_i^{(2)} \geqq 0$, for all i. Furthermore, assume that $f_i(., v) + (.)^T w$ $(1 \leqq i \leqq k)$ is second-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convex at u and $f_i(x, .) - (.)^T z$ $(1 \leqq i \leqq k)$ be second-order $(G, \alpha_2, \rho_i^{(2)}, d_i^{(2)})$-concave at y. Then

$$G(x, y, \lambda, p) \nleq H(u, v, \lambda, q). \tag{2.41}$$

Proof. Since $f_i(., v) + (.)^T w$ $(1 \leqq i \leqq k)$ is second-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convex, we have

$f_i(x, v) + x^T w - f_i(u, v) - u^T w + \tfrac{1}{2}q^T(\nabla_{xx}f_i(u, v)q)$

$$\geqq F[x,u;\alpha_1(x,u)(\nabla_x f_i(u,v)+w+\nabla_{xx} f_i(u,v)q)]+\rho_i^{(1)}(d_i^{(1)}(x,u))^2.$$

It follows from $\lambda > 0$, $\lambda^T e_k = 1$ and sublinearity of F that

$$\begin{aligned}(\lambda^T f)(x,v)+x^T w-(\lambda^T f)(u,v)-u^T w+\frac{1}{2}q^T(\nabla_{xx}(\lambda^T f)(u,v)q)\\ \geqq F[x,u;\alpha_1(x,u)(\nabla_x(\lambda^T f)(u,v)+w+\nabla_{xx}(\lambda^T f)(u,v)q)]\\ +\sum_{i=1}^{k}\lambda_i\rho_i^{(1)}(d_i^{(1)}(x,u))^2. \quad (2.42)\end{aligned}$$

As $f_i(x,.)-(.)^T z$ $(1 \leqq i \leqq k)$ is second-order $(G,\alpha_2,\rho_i^{(2)},d_i^{(2)})$-concave therefore we get

$$\begin{aligned}&f_i(x,y)-y^T z-f_i(x,v)+v^T z-\tfrac{1}{2}p^T(\nabla_{yy} f_i(x,y)p)\\ &\quad\geqq G[v,y;-\alpha_2(v,y)(\nabla_y f_i(x,y)-z+\nabla_{yy} f_i(x,y)p)]+\rho_i^{(2)}(d_i^{(2)}(v,y))^2.\end{aligned}$$

Using $\lambda > 0$, $\lambda^T e_k = 1$ and sublinearity of G it follows that

$$\begin{aligned}(\lambda^T f)(x,y)-y^T z-(\lambda^T f)(x,v)+v^T z-\frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)\\ \geqq G[v,y;-\alpha_2(v,y)(\nabla_y(\lambda^T f)(x,y)-z+\nabla_{yy}(\lambda^T f)(x,y)p)]\\ +\sum_{i=1}^{k}\lambda_i\rho_i^{(2)}(d_i^{(2)}(v,y))^2. \quad (2.43)\end{aligned}$$

Adding the inequalities (2.42) and (2.43), we obtain

$$\begin{aligned}(\lambda^T f)(x,y)-(\lambda^T f)(u,v)+x^T w-u^T w-y^T z+v^T z+\frac{1}{2}q^T(\nabla_{xx}(\lambda^T f)(u,v)q)\\ -\frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)\geqq F[x,u;\alpha_1(x,u)(\nabla_x(\lambda^T f)(u,v)+w\\ +\nabla_{xx}(\lambda^T f)(u,v)q)]+G[v,y;-\alpha_2(v,y)(\nabla_y(\lambda^T f)(x,y)-z\\ +\nabla_{yy}(\lambda^T f)(x,y)p)]+\sum_{i=1}^{k}\lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2+\rho_i^{(2)}(d_i^{(2)}(v,y))^2]. \quad (2.44)\end{aligned}$$

Since (x,y,λ,z,p) is feasible for the primal problem (PP) and (u,v,λ,w,q) is feasible for the dual problem (DP), $\alpha_1(x,u) > 0$, by the dual constraint (2.37), the vector $a=\alpha_1(x,u)(\nabla_x(\lambda^T f)(u,v)+w+\nabla_{xx}(\lambda^T f)(u,v)q)\in C_1^*$ and so from the hypothesis (A), we obtain

$$F(x,u;a)+\alpha_1^{-1}a^T u \geqq 0. \quad (2.45)$$

Similarly,

$$G(v,y;b)+\alpha_2^{-1}b^Ty \geqq 0, \tag{2.46}$$

for the vector $\quad b=-\alpha_2(v,y)\{\nabla_y(\lambda^Tf)(x,y)-z+\nabla_{yy}(\lambda^Tf)(x,y)p\}\in C_2^*$.

Using (2.45), (2.46) and hypothesis (i) or (ii) in (2.44), we have

$$\begin{aligned}&(\lambda^Tf)(x,y)-(\lambda^Tf)(u,v)+x^Tw-u^Tw-y^Tz+v^Tz\\ &\quad+\tfrac{1}{2}q^T(\nabla_{xx}(\lambda^Tf)(u,v)q)-\tfrac{1}{2}p^T(\nabla_{yy}(\lambda^Tf)(x,y)p)\geqq -\alpha_1^{-1}u^Ta-\alpha_2^{-1}y^Tb.\end{aligned}$$

Substituting the values of a and b, we have

$$\begin{aligned}&(\lambda^Tf)(x,y)+x^Tw-y^T\nabla_y(\lambda^Tf)(x,y)-y^T(\nabla_{yy}(\lambda^Tf)(x,y)p)\\ &\qquad-\tfrac{1}{2}p^T(\nabla_{yy}(\lambda^Tf)(x,y)p)\geqq(\lambda^Tf)(u,v)-v^Tz-u^T\nabla_x(\lambda^Tf)(u,v)\\ &\qquad\qquad-u^T(\nabla_{xx}(\lambda^Tf)(u,v)q)-\tfrac{1}{2}q^T(\nabla_{xx}(\lambda^Tf)(u,v)q).\end{aligned}$$

In view of the fact that $x^Tw\leqq S(x\mid D)$, $v^Tz\leqq S(v\mid E)$ and $\lambda^Te_k=1$, the last inequality yields

$$\begin{aligned}&(\lambda^Tf)(x,y)+S(x\mid D)-y^T\nabla_y(\lambda^Tf)(x,y)-y^T(\nabla_{yy}(\lambda^Tf)(x,y)p)\\ &\qquad-\frac{1}{2}p^T(\nabla_{yy}(\lambda^Tf)(x,y)p)\geqq(\lambda^Tf)(u,v)-S(v\mid E)-u^T\nabla_x(\lambda^Tf)(u,v)\\ &\qquad\qquad-u^T(\nabla_{xx}(\lambda^Tf)(u,v)q)-\frac{1}{2}q^T(\nabla_{xx}(\lambda^Tf)(u,v)q). \qquad (2.47)\end{aligned}$$

Now suppose contrary to the result that (2.41) not holds, that is ,

$$\begin{aligned}&\{G_1(x,y,\lambda,p),G_2(x,y,\lambda,p),\ldots,G_k(x,y,\lambda,p)\}\\ &\qquad\leq\{H_1(u,v,\lambda,q),H_2(u,v,\lambda,q),\ldots,H_k(u,v,\lambda,q)\}\end{aligned}$$

or

$$\begin{aligned}&f(x,y)+S(x\mid D)e_k-y^T\nabla_y(\lambda^Tf)(x,y)e_k-y^T(\nabla_{yy}(\lambda^Tf)(x,y)p)e_k\\ &\quad-\tfrac{1}{2}p^T(\nabla_{yy}(\lambda^Tf)(x,y)p)e_k\leq f(u,v)-S(v\mid E)e_k-u^T\nabla_x(\lambda^Tf)(u,v)e_k\\ &\qquad-u^T(\nabla_{xx}(\lambda^Tf)(u,v)q)e_k-\tfrac{1}{2}q^T(\nabla_{xx}(\lambda^Tf)(u,v)q)e_k.\end{aligned}$$

Since $\lambda>0$ and $\lambda^Te_k=1$, we obtain

$$\begin{aligned}&(\lambda^Tf)(x,y)+S(x\mid D)-y^T\nabla_y(\lambda^Tf)(x,y)-y^T(\nabla_{yy}(\lambda^Tf)(x,y)p)\\ &\quad-\tfrac{1}{2}p^T(\nabla_{yy}(\lambda^Tf)(x,y)p)<(\lambda^Tf)(u,v)-S(v\mid E)-u^T\nabla_x(\lambda^Tf)(u,v)\\ &\qquad-u^T(\nabla_{xx}(\lambda^Tf)(u,v)q)-\tfrac{1}{2}q^T(\nabla_{xx}(\lambda^Tf)(u,v)q),\end{aligned}$$

which contradicts (2.47). Hence (2.41) holds.

We now state a weak duality theorem under second-order F-convexity assumptions. Its proof follows on the lines of Theorem 2.4 on taking $\alpha_1(x,u) = 1,\ \alpha_2(v,y) = 1$, and $\rho_1 = 0,\ \rho_2 = 0$.

Theorem 2.5 (Weak duality). Let (x, y, λ, z, p) be feasible for the primal problem (PP) and (u, v, λ, w, q) be feasible for the dual problem (DP). Suppose that the sublinear functionals $F : S_1 \times S_1 \times R^n \mapsto R$ and $G : S_2 \times S_2 \times R^m \mapsto R$ satisfy the following conditions:

$$F(x,u;a) + a^T u \geqq 0, \quad \text{for all } a \in C_1^*, \qquad \text{(A)}$$

$$G(v,y;b) + b^T y \geqq 0, \quad \text{for all } b \in C_2^*. \qquad \text{(B)}$$

Furthermore, assume that $f_i(.,v) + (.)^T w$ $(1 \leqq i \leqq k)$ is second-order F-convex at u and $f_i(x,.) - (.)^T z$ $(1 \leqq i \leqq k)$ is second-order G-concave at y. Then $G(x,y,\lambda,p) \nleq H(u,v,\lambda,q)$.

Theorem 2.6 (Strong duality). Let $f : S_1 \times S_2 \to R^k$ be thrice differentiable function and let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ be a weak efficient solution of (PP). Suppose that

(i) the matrix $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})$ is non singular,

(ii) the vectors $\nabla_y f_1(\bar{x},\bar{y}), \ldots, \nabla_y f_k(\bar{x},\bar{y})$ are linearly independent,

(iii) the vector $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p})\bar{p} \notin \text{span}\{\nabla_y f_1(\bar{x},\bar{y}), \ldots, \nabla_y f_k(\bar{x},\bar{y})\}\backslash\{0\}$ and

(iv) $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p})\bar{p} = 0$ implies $\bar{p} = 0$

then

(I) $\bar{p} = 0$, there exist $\bar{w} \in D$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q} = 0)$ is feasible for (DP), and

(II) $G(\bar{x},\bar{y},\bar{\lambda},\bar{p}) = H(\bar{x},\bar{y},\bar{\lambda},\bar{q})$.

Also, if the hypotheses of Theorem 2.4 or 2.5 are satisfied for all feasible solutions of (PP) and (DP), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q} = 0)$ is an efficient solution for (DP).

Proof. Its proof follows on the lines of Theorem 2.2 of this chapter and Theorem 2 in Ahmad and Husain [13].

A converse duality theorem may be merely stated as its proof would run analogously to that of Theorem 2.6.

Theorem 2.7 (Converse duality). Let $f : S_1 \times S_2 \rightarrow R^k$ be thrice differentiable function and let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{q})$ be a weak efficient solution of (DP). Suppose that

- (i) the matrix $\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})$ is non singular,
- (ii) the vectors $\nabla_x f_1(\bar{u}, \bar{v}), \ldots, \nabla_x f_k(\bar{u}, \bar{v})$ are linearly independent,
- (iii) the vector $\nabla_x(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{q})\bar{q} \notin \text{span}\{\nabla_x f_1(\bar{u}, \bar{v}), \ldots, \nabla_x f_k(\bar{u}, \bar{v})\} \backslash \{0\}$ and
- (iv) $\nabla_x(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{q})\bar{q} = 0$ implies $\bar{q} = 0$,
  then
- (I) $\bar{q} = 0$, there exist $\bar{z} \in E$ such that $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ is feasible for (PP), and
- (II) $G(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p}) = H(\bar{u}, \bar{v}, \bar{\lambda}, \bar{q})$.

Also, if the hypotheses of Theorem 2.4 or 2.5 are satisfied for all feasible solutions of (PP) and (DP), then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ is an efficient solution for (PP).

2.6 Special cases

In this section, we consider some special cases of the problems studied in Section 2.4 and Section 2.5.

- (i) For $k = 1$, our programs (SP) and (SD) reduce to the problems (PP) and (DP) considered in Gulati and Gupta [57].
- (ii) For scalar programming problem, our problems (SP) and (SD) reduces to the single objective nondifferentiable symmetric dual programs considered in Yang et al. [137].

- (*iii*) If $k = 1$ and we take $D = \{Ay : y^T Ay \leqq 1\}$, $E = \{Bx : x^T Bx \leqq 1\}$, where A and B are positive semidefinite matrices, then $(x^T Ax)^{1/2} = S(x \mid D)$ and $(y^T By)^{1/2} = S(y \mid E)$. In this case (SP) and (SD) reduce to the problems considered in Ahmad and Husain [9].

- (*iv*) If $D = \{0\}$ and $E = \{0\}$, then (SP) and (SD) reduce to (MP) and (MD) considered in Yang et al. [140] along with the nonnegativity restrictions $x \geqq 0$ and $v \geqq 0$. However, taking $F(x, u; a) = (x - u)^T a$ and $G(v, y; b) = (v - y)^T b$ along with the hypotheses (A) and (B) of Theorem 1 in [140] gives $x \geqq 0$ and $v \geqq 0$.

- (*v*) If $D = \{0\}$ and $E = \{0\}$, then our problems (PP) and (DP) reduce to the programs studied in Ahmad and Husain [13].

Chapter 3

SYMMETRIC DUALITY IN MULTIOBJECTIVE PROGRAMMING INVOLVING GENERALIZED CONE-CONVEX FUNCTIONS [1]

3.1 Introduction

Ahmad and Husain [13] formulated a pair of Wolfe type multiobjective second order symmetric dual programs with cone constraints and established duality results under second-order invexity assumptions. They pointed out certain omissions and inconsistencies in the earlier work of Mishra [91] and Mishra and Wang [93].

This chapter is divided into five sections. Section 3.2 contains notations and definitions used in this chapter. In Section 3.3, we identify a function lying exclusively in the class of K-η-bonvex and not in class of invex function already existing in literature. In the next Section, we consider a new pair of second-order multiobjective symmetric dual programs in which the objective function is optimized with respect to an arbitrary closed convex cone and established appropriate duality theorems under K-η-bonvexity assumptions. Self duality is also obtained by assuming the functions

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[1] A part of this chapter has appeared in *International Journal of Mathematics in Operational Research*3 (2011) 414-430.

involved to be skew-symmetric. In Section 3.5, a pair of Mond-Weir type nondifferentiable multiobjective second-order symmetric dual programs over arbitrary cones is formulated, where each of the objective function contains a square root term with positive semidefinite matrix in $R^{n\times n}$. Weak, strong and converse duality results are then proved under second-order K-F-convexity/K-η-bonvexity assumptions. Some particular cases of the programs considered in this chapter are discussed in the last Section.

3.2 Notations and definitions

Let C_1 and C_2 be closed convex cones in R^n and R^m, respectively, with nonempty interiors. Suppose that $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$. Let $f_i : S_1 \times S_2 \to R,\ i = 1, 2, \ldots, k$, be differentiable functions of x and y.

Definition 3.1. A twice differentiable function $f : S_1 \times S_2 \to R^k$ is said to be K-η_1-bonvex in the first variable at $u \in S_1$ for fixed $v \in S_2$, if there exists a function $\eta_1 : S_1 \times S_1 \to R^n$ such that for $x \in S_1,\ q_i \in R^n, i = 1, 2, ..., k$,

$$(f_1(x,v) - f_1(u,v)+\frac{1}{2}q_1^T\nabla_{xx}f_1(u,v)q_1 - \eta_1^T(x,u)(\nabla_x f_1(u,v) + \nabla_{xx}f_1(u,v)q_1), ...,$$
$$f_k(x,v) - f_k(u,v)+\frac{1}{2}q_k^T\nabla_{xx}f_k(u,v)q_k - \eta_1^T(x,u)(\nabla_x f_k(u,v) + \nabla_{xx}f_k(u,v)q_k)) \in K$$

and $f(x,y)$ is said to be K-η_2-bonvex in the second variable at $v \in S_2$ for fixed $u \in S_1$, if there exists a function $\eta_2 : S_2 \times S_2 \to R^m$ such that for $y \in S_2,\ p_i \in R^m, i = 1, 2, ..., k$,

$$(f_1(u,y) - f_1(u,v)+\frac{1}{2}p_1^T\nabla_{yy}f_1(u,v)p_1 - \eta_2^T(y,v)(\nabla_y f_1(u,v) + \nabla_{yy}f_1(u,v)p_1), ...,$$
$$f_k(u,y) - f_k(u,v)+\frac{1}{2}p_k^T\nabla_{yy}f_k(u,v)p_k - \eta_2^T(y,v)(\nabla_y f_k(u,v) + \nabla_{yy}f_k(u,v)p_k)) \in K.$$

Definition 3.2. A twice differentiable function $f : S_1 \times S_2 \to R^k$ is said to be second-order K-F-convex in the first variable at $u \in S_1$ for fixed $v \in S_2$, if there exists a sublinear functional $F : S_1 \times S_1 \times R^n \to R$ such that for $x \in S_1,\ q_i \in R^n,\ i = 1, 2, ..., k$,

$$(f_1(x,v) - f_1(u,v)+\frac{1}{2}q_1^T\nabla_{xx}f_1(u,v)q_1 - F(x,u;\nabla_x f_1(u,v) + \nabla_{xx}f_1(u,v)q_1), ...,$$
$$f_k(x,v) - f_k(u,v)+\frac{1}{2}q_k^T\nabla_{xx}f_k(u,v)q_k - F(x,u;\nabla_x f_k(u,v) + \nabla_{xx}f_k(u,v)q_k)) \in K$$

and $f(x,y)$ is said to be second-order K-G-convex in the second variable at $v \in S_2$ for fixed $u \in S_1$, if there exists a sublinear functional $G : S_2 \times S_2 \times R^m \to R$ such that for $y \in S_2,\ p_i \in R^m, i = 1,2,...,k,$

$$(f_1(u,y) - f_1(u,v) + \frac{1}{2}p_1^T \nabla_{yy} f_1(u,v)p_1 - G(y,v;\nabla_y f_1(u,v) + \nabla_{yy} f_1(u,v)p_1), ...,$$
$$f_k(u,y) - f_k(u,v) + \frac{1}{2}p_k^T \nabla_{yy} f_k(u,v)p_k - G(y,v;\nabla_y f_k(u,v) + \nabla_{yy} f_k(u,v)p_k)) \in K.$$

Remarks

(*i*) If $K = R_+, p = 0$ and $q = 0$, then the above definition reduces to η-convexity/invexity in [75, 100].

(*ii*) If $K = R_+$, then the K-η-bonvexity becomes η-bonvexity [55, 124].

(*iii*) Eliminating the second-order terms and substituting $\eta_1(x,u) = (x-u)$ and $\eta_2(v,y) = (v-y)$, the definition reduces to K-convexity as in [123].

3.3 Example

An example of non trivial function which is K-η-bonvex but not invex.

Let $X = [2, 2.35] \subset R,\ n = m = 1$ and $k = 2$. Consider the function $f : X \to R^2$ be defined by $f(x) = (f_1(x), f_2(x))$, where $f_1(x) = x^3 \sin\frac{2}{x},\ f_2(x) = \sin^2 x$.

Let

$$K = \{(x,y) : x \geqq 4y, x \geqq 0\}$$

and $\eta : X \times X \to R$ be defined by

$$\eta(x,u) = 12(1-u).$$

To show that f is K-η-bonvex, we have to prove that

$$\big(f_1(x) - f_1(u) + \frac{1}{2}q_1^T(\nabla_{xx} f_1(u)q_1) - \eta^T(x,u)[\nabla_x f_1(u) + \nabla_{xx} f_1(u)q_1],$$
$$f_2(x) - f_2(u) + \frac{1}{2}q_2^T(\nabla_{xx} f_2(u)q_2) - \eta^T(x,u)[\nabla_x f_2(u) + \nabla_{xx} f_2(u)q_2]\big) \in K,$$

or

$$\big(x^3 \sin\tfrac{2}{x} - u^3 \sin\tfrac{2}{u} + q_1^2[3u \sin\tfrac{2}{u} - \tfrac{2}{u}\sin\tfrac{2}{u} - 4\cos\tfrac{2}{u}] - 12(1-u)$$

$$[-2u\cos\tfrac{2}{u}+3u^2\sin\tfrac{2}{u}+q_1(6u\sin\tfrac{2}{u}-\tfrac{4}{u}\sin\tfrac{2}{u}-8\cos\tfrac{2}{u})],\sin^2 x-\sin^2 u$$
$$+q_2^2\cos 2u-12(1-u)[\sin 2u+2q_2\cos 2u]\big)\in K.$$

Let

$$\phi = x^3\sin\tfrac{2}{x}-u^3\sin\tfrac{2}{u}+q_1^2[3u\sin\tfrac{2}{u}-\tfrac{2}{u}\sin\tfrac{2}{u}-4\cos\tfrac{2}{u}]$$
$$-12(1-u)[-2u\cos\tfrac{2}{u}+3u^2\sin\tfrac{2}{u}+q_1(6u\sin\tfrac{2}{u}-\tfrac{4}{u}\sin\tfrac{2}{u}-8\cos\tfrac{2}{u})]$$
$$=\phi_1+\phi_2 \text{ (say)}$$

and

$$\psi=\sin^2 x-\sin^2 u+q_2^2\cos 2u-12(1-u)[\sin 2u+2q_2\cos 2u].$$
$$=\psi_1+\psi_2 \text{ (say)}$$

where

$$\phi_1{=}x^3\sin\tfrac{2}{x}-u^3\sin\tfrac{2}{u}-12(1-u)[-2u\cos\tfrac{2}{u}+3u^2\sin\tfrac{2}{u}]$$

$\geqq 0 \ \ \forall\ x,u\in X$, as can be seen from Figure 3.1.

$$\phi_2{=}q_1^2[3u\sin\tfrac{2}{u}-\tfrac{2}{u}\sin\tfrac{2}{u}-4\cos\tfrac{2}{u}]-12(1-u)[q_1(6u\sin\tfrac{2}{u}-\tfrac{4}{u}\sin\tfrac{2}{u}-8\cos\tfrac{2}{u})]$$

$\geqq 0\ \forall\ u\in X$ and $q_1\in(-10^{18},10^{18})$ as can be seen from Figure 3.2.

$$\psi_1=\sin^2 x-\sin^2 u-12(1-u)\sin 2u$$

$<0 \ \ \forall\ x,u\in X$, as can be seen from Figure 3.3

and

$$\psi_2=q_2^2\cos 2u-24(1-u)q_2\cos 2u$$

$\leqq 0\ \forall\ u\in X$ and $q_2\in(-10^{18},10^{18})$ as can be seen from Figure 3.4.

Hence $\phi\geqq 0$ and $\psi<0$. This implies $\phi-4\psi\geqq 0$ and $\phi\geqq 0$. Thus $(\phi,\psi)\in K$ and hence f is K-η-bonvex function.

Next, to prove f is not invex, either

$f_1(x)-f_1(u)-\eta^T(x,u)\nabla_x f_1(u)\ngeqq 0$ or

$f_2(x)-f_2(u)-\eta^T(x,u)\nabla_x f_2(u)\ngeqq 0$.

Since $f_2(x)-f_2(u)-\eta^T(x,u)\nabla_x f_2(u)=\sin^2 x-\sin^2 u-12(1-u)\sin 2u$

$<0\ \forall\ x,u\in X$, as can be seen from Figure 3.3.

Therefore, f is not invex. Hence f is K-η-bonvex but not invex.

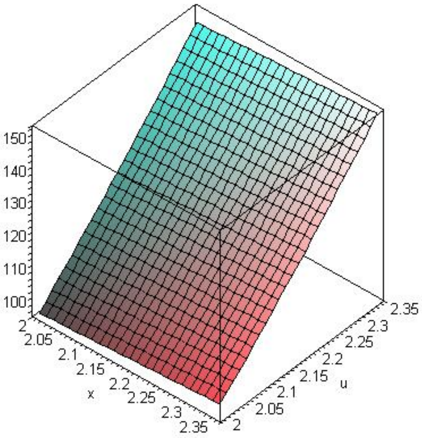

Figure 3.1: Graph of ϕ_1 against x and u

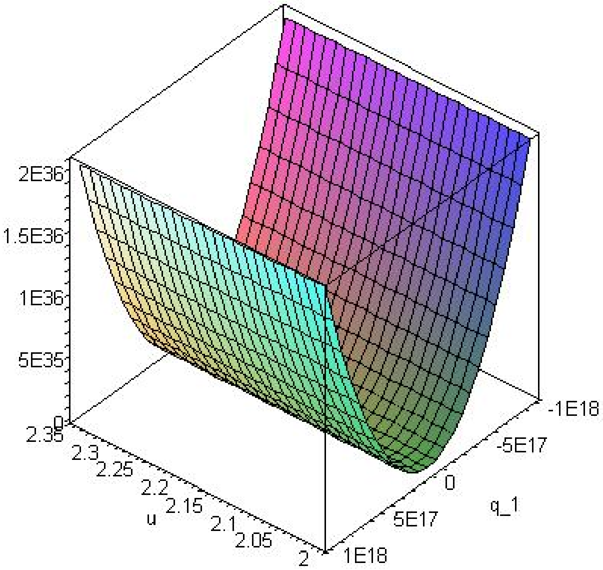

Figure 3.2: Graph of ϕ_2 against u and q_1

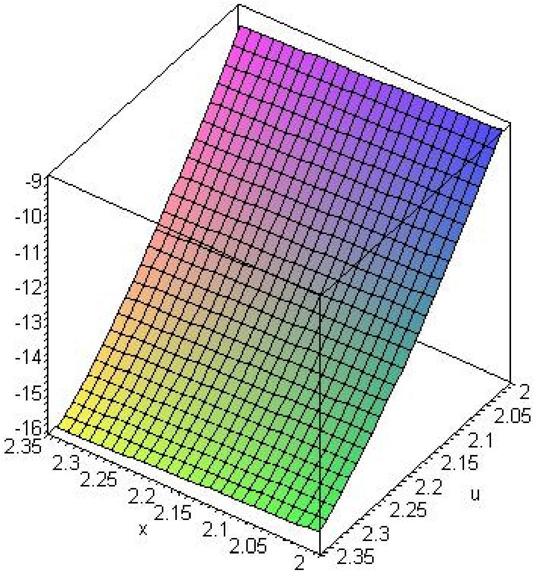

Figure 3.3: Graph of ψ_1 against x and u

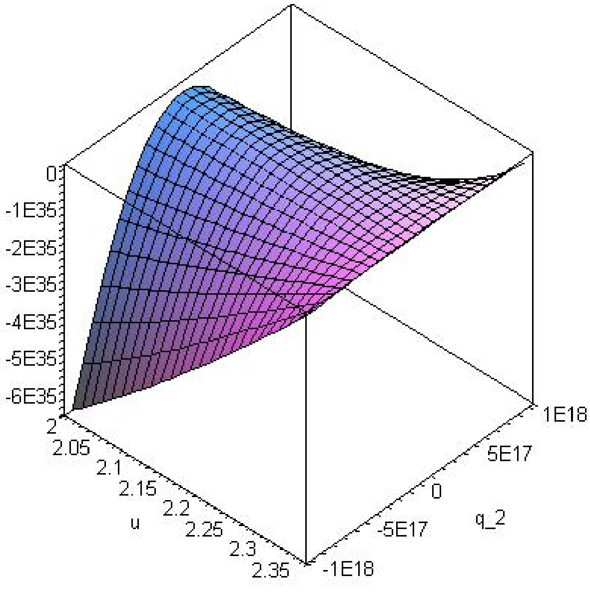

Figure 3.4: Graph of ψ_2 against u and q_2

3.4 Problem formulation and duality results

The formulations where the inequality constraints are defined via closed convex cones and their polars enable us to consider many constraints of the inequality type. Motivated by the concept of formulations with cones constraints, Suneja et al. [123] formulated a pair of symmetric dual multiobjective programs over arbitrary cones in which the objective function is optimized with respect to an arbitrary closed convex cone by assuming the function involved to be cone convex. Khurana [77] studied a pair of Mond-Weir type symmetric dual multiobjective programming problems and obtained duality results under cone-pseudoinvex assumptions. Mishra and Lai [92] extended the results of Khurana [77] to second-order and established appropriate duality relations under cone-second-order pseudoinvexity assumptions.

Recently, Ahmad and Husain [13] and Gulati et al. [62] proved duality results for second-order multiobjective symmetric dual programs with cone constraints under second-order invexity assumptions.

Now we consider the following pair of multiobjective second-order symmetric dual programs over arbitrary cones:

Primal problem (WP)

K-minimize

$$G(x,y,\lambda,p) = \Big(f_1(x,y) - y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - \frac{1}{2}\sum_{i=1}^{k} \lambda_i(p_i^T \nabla_{yy} f_i(x,y)p_i),$$

$$\ldots, f_k(x,y) - y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - \frac{1}{2}\sum_{i=1}^{k} \lambda_i(p_i^T \nabla_{yy} f_i(x,y)p_i)\Big)$$

subject to

$$-\sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \in C_2^*, \tag{3.1}$$

$$\lambda^T e_k = 1, \tag{3.2}$$

$$\lambda \in \text{int } K^*,\ x \in C_1. \tag{3.3}$$

Dual problem (WD)

K-maximize

$$H(u,v,\lambda,q)=(f_1(u,v)-u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i)-\frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx}f_i(u,v)q_i),$$

$$\ldots,f_k(u,v)-u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i)-\frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx}f_i(u,v)q_i))$$

subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i)\in C_1^*, \tag{3.4}$$

$$\lambda^T e_k=1, \tag{3.5}$$

$$\lambda\in \text{int } K^*,\ v\in C_2\ , \tag{3.6}$$

where

(*i*) $f_i : S_1\times S_2\to R,\ i=1,2,...,k$ is a differentiable function of x and y, $e_k=(1,\ldots,1)^T\in\ R^k$,

(*ii*) $p=(p_1,p_2,\ldots,p_k)$, $q=(q_1,q_2,\ldots,q_k)$, q_i and p_i are vectors in R^n and R^m, respectively, for $i=1,2,\ldots,k$ and $\lambda\in R^k$.

Theorem 3.1 (Weak duality). Let (x,y,λ,p) be feasible for (WP) and (u,v,λ,q) be feasible for (WD). Let

(*i*) $f(.,v)$ be K-η_1-bonvex in the first variable at u,

(*ii*) $-f(x,.)$ be K-η_2-bonvex in the second variable at y,

(*iii*) $\eta_1(x,u)+u\in C_1,\qquad$ for all $\ x\in C_1$,

(*iv*) $\eta_2(v,y)+y\in C_2,\qquad$ for all $\ v\in C_2$.

Then

$G(x,y,\lambda,p)-H(u,v,\lambda,q)\notin -K\backslash\{0\}$.

Proof. Suppose, to the contrary, that

$G(x,y,\lambda,p)-H(u,v,\lambda,q)\in -K\backslash\{0\}$.

that is,

$$\{[f_1(x,y) - y^T \sum_{i=1}^k \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - \frac{1}{2}\sum_{i=1}^k \lambda_i(p_i^T \nabla_{yy} f_i(x,y)p_i),$$

$$\ldots, f_k(x,y) - y^T \sum_{i=1}^k \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - \frac{1}{2}\sum_{i=1}^k \lambda_i(p_i^T \nabla_{yy} f_i(x,y)p_i)]$$

$$-[f_1(u,v) - u^T \sum_{i=1}^k \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - \frac{1}{2}\sum_{i=1}^k \lambda_i(q_i^T \nabla_{xx} f_i(u,v)q_i),$$

$$\ldots, f_k(u,v) - u^T \sum_{i=1}^k \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - \frac{1}{2}\sum_{i=1}^k \lambda_i(q_i^T \nabla_{xx} f_i(u,v)q_i)]\} \in -K\backslash\{0\}.$$

Since $\lambda \in \text{int } K^*$ and $\lambda \neq 0$, we obtain

$$\sum_{i=1}^k \lambda_i\{f_i(x,y) - y^T \sum_{i=1}^k \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - \frac{1}{2}\sum_{i=1}^k \lambda_i(p_i^T \nabla_{yy} f_i(x,y)p_i)$$

$$-(f_i(u,v) - u^T \sum_{i=1}^k \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - \frac{1}{2}\sum_{i=1}^k \lambda_i(q_i^T \nabla_{xx} f_i(u,v)q_i))\} < 0. \quad (3.7)$$

By K-η_1-bonvexity of $f(.,v)$, we have

$$\{f_1(x,v) - f_1(u,v) + \frac{1}{2}q_1^T \nabla_{xx} f_1(u,v)q_1 - \eta_1^T(x,u)[\nabla_x f_1(u,v) + \nabla_{xx} f_1(u,v)q_1], \ldots,$$

$$f_k(x,v) - f_k(u,v) + \frac{1}{2}q_k^T \nabla_{xx} f_k(u,v)q_k - \eta_1^T(x,u)[\nabla_x f_k(u,v) + \nabla_{xx} f_k(u,v)q_k]\} \in K.$$

Using $\lambda \in \text{int } K^*$, we get

$$\sum_{i=1}^k \lambda_i\{f_i(x,v) - f_i(u,v) + \frac{1}{2}q_i^T \nabla_{xx} f_i(u,v)q_i$$

$$-\eta_1^T(x,u)[\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i]\} \geqq 0. \quad (3.8)$$

Since (u,v,λ,q) is feasible for (WD), from the dual constraint (3.4) and hypothesis (*iii*), it follows that

$$[\eta_1(x,u) + u]^T \sum_{i=1}^k \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \geqq 0,$$

which implies

$$\eta_1^T(x,u) \sum_{i=1}^k \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i)$$

$$\geqq -u^T \sum_{i=1}^k \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i). \quad (3.9)$$

Using (3.8) and (3.9), we obtain

$$\sum_{i=1}^{k}\lambda_i(f_i(x,v)-f_i(u,v)+\frac{1}{2}q_i^T\nabla_{xx}f_i(u,v)q_i)$$
$$\geqq -u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i). \qquad (3.10)$$

Similarly, by K-η_2-bonvexity of $-f(x,.)$, from (3.1) and hypothesis (iv), we get,

$$\sum_{i=1}^{k}\lambda_i(f_i(x,y)-f_i(x,v)-\frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i)$$
$$\geqq y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i). \qquad (3.11)$$

It follows from (3.10), (3.11) and $\lambda^T e_k = 1$ that

$$\sum_{i=1}^{k}\lambda_i\{f_i(x,y)-y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i)-\frac{1}{2}\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{yy}f_i(x,y)p_i)$$
$$-(f_i(u,v)-u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx}f_i(u,v)q_i)-\frac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx}f_i(u,v)q_i))\}\geqq 0,$$

which contradicts (3.7). Hence the result.

If the variable λ in the problems (WP) and (WD) is fixed to be $\bar{\lambda}$, we shall denote these problems by $(WP)_{\bar{\lambda}}$ and $(WD)_{\bar{\lambda}}$.

Theorem 3.2 (Strong duality). Let $f : S_1\times S_2 \to R^k$ be thrice differentiable function and let $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ be a weak efficient solution of (WP). Suppose that

(i) the matrix $\nabla_{yy}f_i(\bar{x},\bar{y})$ is non singular for $i=1,2,\ldots,k$,

(ii) the vectors $\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})$ are linearly independent,

(iii) $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y(\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i))\bar{p}_i \notin \text{span}\{\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})\}\backslash\{0\}$,

(iv) $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y(\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i))\bar{p}_i = 0$ implies $\bar{p}_i = 0 \ \ \forall\, i$, and

(v) K is closed convex pointed cone with $R_+^k \subseteq K$.

Then $\bar{p} = 0$, $(\bar{x}, \bar{y}, \bar{q} = 0)$ is feasible for $(WD)_{\bar{\lambda}}$, and the objective function values of (WP) and $(WD)_{\bar{\lambda}}$ are equal. Furthermore, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (WP) and $(WD)_{\bar{\lambda}}$, then $(\bar{x}, \bar{y}, \bar{q} = 0)$ is an efficient solution for $(WD)_{\bar{\lambda}}$.

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is a weak efficient solution of (WP), there exist $\bar{\alpha} \in K^*$, $\bar{\beta} \in C_2$, $\bar{\eta} \in R$, such that the following by Fritz-John optimality conditions ([123], Lemma 1) are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$(for simplicity, we write $\nabla_x f_i, \nabla_{xy} f_i$ instead of $\nabla_x f_i(\bar{x}, \bar{y}), \nabla_{xy} f_i(\bar{x}, \bar{y})$ etc.):

$$(x - \bar{x})^T \{\sum_{i=1}^{k} \bar{\alpha}_i \nabla_x f_i + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{xy} f_i [\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}]$$

$$+ \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_x (\nabla_{yy} f_i \bar{p}_i)][\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p}_i)]\} \geqq 0, \quad \text{for all} \quad x \in C_1, \tag{3.12}$$

$$(y - \bar{y})^T \{\sum_{i=1}^{k} \bar{\alpha}_i \nabla_y f_i + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{yy} f_i [\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}] + \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_y (\nabla_{yy} f_i \bar{p}_i)][\bar{\beta}$$

$$-(\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p}_i)] - \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i](\bar{\alpha}^T e_k)\} \geqq 0, \quad \text{for all} \quad y \in R^m, \tag{3.13}$$

$$\nabla_y f(\bar{x}, \bar{y})(\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}) + \bar{\eta} e_k + \{(\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p}_1))^T \nabla_{yy} f_1(\bar{x}, \bar{y})\bar{p}_1,$$

$$\ldots, (\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p}_k))^T \nabla_{yy} f_k(\bar{x}, \bar{y})\bar{p}_k\} = 0, \tag{3.14}$$

$$[(\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p}_i))\bar{\lambda}_i]^T \nabla_{yy} f_i = 0, \ \ i = 1, 2, \ldots, k, \tag{3.15}$$

$$\bar{\beta}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) = 0, \tag{3.16}$$

$$\bar{\eta}^T [\bar{\lambda}^T e_k - 1] = 0, \tag{3.17}$$

$$(\bar{\alpha}, \bar{\beta}, \bar{\eta}) \neq 0. \tag{3.18}$$

Inequalities (3.13) and (3.14) are equivalent to

$$\sum_{i=1}^{k} \bar{\alpha}_i \nabla_y f_i + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{yy} f_i [\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}]$$

$$+ \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_y (\nabla_{yy} f_i \bar{p}_i)][\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p}_i)] - \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i](\bar{\alpha}^T e_k) = 0, \tag{3.19}$$

$$\nabla_y f_i[\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}]+\nabla_{yy} f_i\bar{p}_i[\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\frac{1}{2}\bar{p}_i)]+\bar{\eta}=0, \;\; i=1,2,\ldots,k. \tag{3.20}$$

Since $R^k_+ \subseteq K \Rightarrow K^* \subseteq R^k_+$ which implies $\mathrm{int}(K^*) \subseteq \mathrm{int}(R^k_+)$.

As $\bar{\lambda} \in \mathrm{int}\ (K^*)$, therefore $\bar{\lambda} > 0$.

As $\nabla_{yy} f_i$ is nonsingular and $\bar{\lambda}_i > 0$ for $i = 1, 2, \ldots, k$, (3.15) implies

$$\bar{\beta} = (\bar{\alpha}^T e_k)(\bar{y} + \bar{p}_i), \;\; i = 1, 2, \ldots, k. \tag{3.21}$$

If $\bar{\alpha} = 0$ then (3.21) yields $\bar{\beta} = 0$. Further, the equation (3.20) gives $\bar{\eta} = 0$. Consequently $(\bar{\alpha}, \bar{\beta}, \bar{\eta}) = 0$, contradicting (3.18). Hence $\bar{\alpha} \neq 0$. Further, $\bar{\alpha} \in K^* \subseteq R^k_+$ implies

$$\bar{\alpha}^T e_k > 0. \tag{3.22}$$

Now, we claim that $\bar{p}_i = 0$ for $i = 1, 2, \ldots, k$.

Using (3.21) and (3.22) in (3.19), we get

$$\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_y(\nabla_{yy} f_i\bar{p}_i))\bar{p}_i = \frac{-2}{\bar{\alpha}^T e_k}\sum_{i=1}^{k} \nabla_y f_i[\bar{\alpha}_i-(\bar{\alpha}^T e_k)\bar{\lambda}_i], \tag{3.23}$$

which by hypotheses (iii) and (iv) implies

$$\bar{p}_i = 0, \; \forall\, i = 1, 2, \ldots k, \tag{3.24}$$

and thus relation (3.21) gives

$$\bar{\beta} = (\bar{\alpha}^T e_k)\bar{y}. \tag{3.25}$$

Equations (3.22)-(3.24), yields

$$\sum_{i=1}^{k}(\bar{\alpha}_i - (\bar{\alpha}^T e_k)\bar{\lambda}_i)\nabla_y f_i = 0,$$

which on using hypothesis (ii) gives

$$\bar{\alpha}_i = (\bar{\alpha}^T e_k)\bar{\lambda}_i, \;\; i = 1, 2, \ldots, k. \tag{3.26}$$

Using (3.22), (3.24)-(3.26) in (3.12), we have

$$(x-\bar{x})^T\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x f_i \geqq 0, \quad \text{for all} \quad x\in C_1. \tag{3.27}$$

Let $x\in C_1$. Then $x+\bar{x}\in C_1$ and so (3.27) implies

$$x^T\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x f_i \geqq 0, \quad \text{for all} \quad x\in C_1.$$

Therefore,

$$\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x f_i \in C_1^*. \tag{3.28}$$

Also, from (3.25), we have

$$\bar{y}=\frac{\bar{\beta}}{\bar{\alpha}^T e_k}\in C_2.$$

Thus $(\bar{x},\bar{y},\bar{q}=0)$ satisfies the constraints of $(WD)_{\bar{\lambda}}$ and so it is a feasible solution for the dual problem $(WD)_{\bar{\lambda}}$.

Now, letting $x=0$ and $x=2\bar{x}$ in (3.27), we get

$$\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x f_i = 0. \tag{3.29}$$

Further, from (3.16), (3.22), (3.24) and (3.25), we obtain

$$\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y f_i = 0. \tag{3.30}$$

Therefore, using (3.24), (3.29) and (3.30), we get

$$\Big(f_1(\bar{x},\bar{y})-\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i+\nabla_{yy}f_i\bar{p}_i)-\tfrac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{p}_i^T\nabla_{yy}f_i\bar{p}_i),$$

$$\ldots, f_k(\bar{x},\bar{y})-\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i+\nabla_{yy}f_i\bar{p}_i)-\tfrac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{p}_i^T\nabla_{yy}f_i\bar{p}_i)\Big)$$

$$=\Big(f_1(\bar{x},\bar{y})-\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_x f_i+\nabla_{xx}f_i\bar{q}_i)-\tfrac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\bar{q}_i^T\nabla_{xx}f_i\bar{q}_i),$$

$$\dots, f_k(\bar{x},\bar{y}) - \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i + \nabla_{xx} f_i \bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{xx} f_i \bar{q}_i))$$

that is, the two objective function values are equal.

Now, let $(\bar{x},\bar{y},\bar{q}=0)$ is not an efficient solution of $(WD)_{\bar{\lambda}}$, then there exist $(\bar{u},\bar{v},\bar{q}=0)$ feasible for $(WD)_{\bar{\lambda}}$, such that,

$$\{f_1(\bar{x},\bar{y}) - \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \nabla_{xx} f_i(\bar{x},\bar{y})\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{xx} f_i(\bar{x},\bar{y})\bar{q}_i),$$
$$\dots, f_k(\bar{x},\bar{y}) - \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \nabla_{xx} f_i(\bar{x},\bar{y})\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{xx} f_i(\bar{x},\bar{y})\bar{q}_i)\}$$
$$-\{f_1(\bar{u},\bar{v}) - \bar{u}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i),$$
$$\dots, f_k(\bar{u},\bar{v}) - \bar{u}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i)\} \in -K\backslash\{0\}.$$

As $\quad \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i(\bar{x},\bar{y}) = \bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_y f_i(\bar{x},\bar{y}) \quad$ and $\quad \bar{p}_i = 0$, for $i = 1, 2, \dots, k$,

$$\{f_1(\bar{x},\bar{y}) - \bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) + \nabla_{yy} f_i(\bar{x},\bar{y})\bar{p}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{p}_i^T \nabla_{yy} f_i(\bar{x},\bar{y})\bar{p}_i),$$
$$\dots, f_k(\bar{x},\bar{y}) - \bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) + \nabla_{yy} f_i(\bar{x},\bar{y})\bar{p}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{p}_i^T \nabla_{yy} f_i(\bar{x},\bar{y})\bar{p}_i)\}$$
$$-\{f_1(\bar{u},\bar{v}) - \bar{u}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i),$$
$$\dots, f_k(\bar{u},\bar{v}) - \bar{u}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i) - \tfrac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i)\} \in -K\backslash\{0\},$$

which contradicts Theorem 3.1. Hence $(\bar{x},\bar{y},\bar{q}=0)$ is an efficient solution of $(WD)_{\bar{\lambda}}$.

Theorem 3.3 (Converse duality). Let $f : S_1 \times S_2 \to R^k$ be thrice differentiable function and let $(\bar{u},\bar{v},\bar{\lambda},\bar{q})$ be a weak efficient solution of (WD). Suppose that

- *(i)* the matrix $\nabla_{xx} f_i(\bar{u},\bar{v})$ is non singular for $i = 1, 2, \dots, k$,
- *(ii)* the vectors $\nabla_x f_1(\bar{u},\bar{v}), \dots, \nabla_x f_k(\bar{u},\bar{v})$ are linearly independent,
- *(iii)* $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x(\nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i))\bar{q}_i \notin \operatorname{span}\{\nabla_x f_1(\bar{u},\bar{v}), \dots, \nabla_x f_k(\bar{u},\bar{v})\}\backslash\{0\}$,
- *(iv)* $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x(\nabla_{xx} f_i(\bar{u},\bar{v})\bar{q}_i))\bar{q}_i = 0$ implies $\bar{q}_i = 0 \;\; \forall\, i$, and
- *(v)* K is closed convex pointed cone with $R^k_+ \subseteq K$.

Then $\bar{q} = 0$, $(\bar{u}, \bar{v}, \bar{p} = 0)$ is feasible for $(WP)_{\bar{\lambda}}$ and the objective function values of $(WP)_{\bar{\lambda}}$ and (WD) are equal. Furthermore, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of $(WP)_{\bar{\lambda}}$ and (WD), then $(\bar{u}, \bar{v}, \bar{p} = 0)$ is an efficient solution for $(WP)_{\bar{\lambda}}$.

Proof. It follows on the lines of Theorem 3.2.

Self duality

A mathematical programming problem is said to be self dual if it is formally identical with its dual, that is, if the dual is recast in the form of primal, the new problem so obtained is the same as the primal. In general (WP) and (WD) are not self dual without an added restriction on f. The vector function $f : R^n \times R^n \to R^k$ is said to be skew symmetric if for all $x, y \in R^n$,

$f_i(x, y) = -f_i(y, x), \quad i \in \{1, 2, \ldots, k\}$.

For the programs (WP) and (WD) self duality exists under the following assumptions: (i) $m = n$, (ii) $C_1 = C_2$, (iii) the vector function $f(x, y)$ to be skew symmetric, i.e., $f(x, y) = -f(y, x)$.

Now recasting the dual problem (WD) as a minimization problem:

$(WD)'$

$$K\text{-minimize} \quad \Big(-f_1(u,v)+u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx} f_i(u,v)q_i)+\tfrac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx} f_i(u,v)q_i),$$
$$\ldots, -f_k(u,v)+u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{xx} f_i(u,v)q_i)+\tfrac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{xx} f_i(u,v)q_i)\Big)$$

$$\text{subject to} \quad \sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \in C_1^*,$$
$$\lambda^T e_k = 1,$$
$$\lambda \in \text{int } K^*, \ v \in C_2 \ .$$

Since f is skew symmetric,

$\nabla_x f_i(u,v) = -\nabla_y f_i(v,u)$ and $\nabla_{xx} f_i(u,v) = -\nabla_{yy} f_i(v,u)$ for $i = 1, 2, \ldots, k$.

Therefore, the problem $(WD)'$ reduces to,

$$K\text{-minimize} \quad \Big(f_1(v,u)-u^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(v,u)+\nabla_{yy} f_i(v,u)q_i)-\tfrac{1}{2}\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{yy} f_i(v,u)q_i),$$

$$\ldots, f_k(v,u) - u^T \sum_{i=1}^k \lambda_i(\nabla_y f_i(v,u) + \nabla_{yy} f_i(v,u) q_i) - \tfrac{1}{2} \sum_{i=1}^k \lambda_i (q_i^T \nabla_{yy} f_i(v,u) q_i))$$

subject to

$$-\sum_{i=1}^k \lambda_i(\nabla_y f_i(v,u) + \nabla_{yy} f_i(v,u) q_i) \in C_2^*,$$

$$\lambda^T e_k = 1,$$

$$\lambda \in \text{int } K^*, \ v \in C_1 .$$

This shows that $(WD)'$ is formally identical to (WP), that is, the objective and constraint functions are identical. Hence (WP) is self dual. Consequently, the feasibility of (x, y, λ, p) for (WP) implies the feasibility of (y, x, λ, p) for (WD) and conversely.

3.5 Mond-Weir type second-order symmetric duality over arbitrary cones

Consider the following pair of Mond-Weir type second-order nondifferentiable multiobjective symmetric dual programming problems:

Primal (MP)

K-minimize $\quad L(x,y,w,p) = (L_1(x,y,w,p), L_2(x,y,w,p), ..., L_k(x,y,w,p))$

subject to

$$-\sum_{i=1}^k \lambda_i(\nabla_y f_i(x,y) - E_i w_i + \nabla_{yy} f_i(x,y) p_i) \in C_2^*, \tag{3.31}$$

$$y^T \sum_{i=1}^k \lambda_i(\nabla_y f_i(x,y) - E_i w_i + \nabla_{yy} f_i(x,y) p_i) \geqq 0, \tag{3.32}$$

$$w_i^T E_i w_i \leqq 1, \ i = 1, 2, ..., k, \tag{3.33}$$

$$\lambda \in \text{int } K^*, \quad x \in C_1. \tag{3.34}$$

Dual (MD)

K-maximize $\quad M(u,v,z,q) = (M_1(u,v,z,q), M_2(u,v,z,q), ..., M_k(u,v,z,q))$

subject to

$$\sum_{i=1}^k \lambda_i(\nabla_x f_i(u,v) + B_i z_i + \nabla_{xx} f_i(u,v) q_i) \in C_1^*, \tag{3.35}$$

$$u^T \sum_{i=1}^k \lambda_i(\nabla_x f_i(u,v) + B_i z_i + \nabla_{xx} f_i(u,v) q_i) \leqq 0, \tag{3.36}$$

$$z_i^T B_i z_i \leqq 1,\ i = 1, 2, ..., k, \tag{3.37}$$

$$\lambda \in \text{int } K^*, \quad v \in C_2, \tag{3.38}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)^T \in R^k$, $p = (p_1, p_2, \ldots, p_k)$, $q = (q_1, q_2, \ldots, q_k)$ and for $i = 1, 2, ..., k$,

$L_i(x, y, w, p) = f_i(x, y) + (x^T B_i x)^{\frac{1}{2}} - y^T E_i w_i - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i$,

$M_i(u, v, z, q) = f_i(u, v) - (v^T E_i v)^{\frac{1}{2}} + u^T B_i z_i - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i$ and

(*i*) $f_i : S_1 \times S_2 \rightarrow R$ is a thrice differentiable function of x and y,

(*ii*) B_i and E_i are positive semidefinite matrices in $R^{n \times n}$ and $R^{m \times m}$, respectively,

(*iii*) C_1^* and C_2^* are positive polar cones of C_1 and C_2,

(*iv*) p_i, w_i and q_i, z_i are vectors in R^m and R^n, respectively,

(*v*) K is closed convex pointed cone in R^k such that int $K \neq \phi$ and K^* is its positive polar cone.

Theorem 3.4 (Weak duality). Let (x, y, λ, w, p) be feasible for (MP) and (u, v, λ, z, q) be feasible for (MD). Let

(*i*) $\{f_1(., v) + (.)^T B_1 z_1, ..., f_k(., v) + (.)^T B_k z_k\}$ be second-order K-F-convex in the first variable at u,

(*ii*) $-\{f_1(x, .) - (.)^T E_1 w_1, ..., f_k(x, .) - (.)^T E_k w_k\}$ be second-order K-G-convex in the second variable at y,

(*iii*) $R_+^k \subseteq K$,

where the sublinear functionals $F : S_1 \times S_1 \times R^n \rightarrow R$ and $G : S_2 \times S_2 \times R^m \rightarrow R$ satisfy the following conditions:

(*iv*) $F(x, u; a) + a^T u \geqq 0$, for all $a \in C_1^*$ and

(*v*) $G(v, y; b) + b^T y \geqq 0$, for all $b \in C_2^*$.

Then

$L(x, y, w, p) - M(u, v, z, q) \notin -K\backslash\{0\}$.

Proof. Suppose, to the contrary, that

$L(x, y, w, p) - M(u, v, z, q) \in -K\backslash\{0\}$.

Since $\lambda \in \text{int } K^*$, we obtain

$$\sum_{i=1}^{k} \lambda_i(f_i(x, y) + (x^T B_i x)^{\frac{1}{2}} - y^T E_i w_i - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i)$$
$$< \sum_{i=1}^{k} \lambda_i(f_i(u, v) - (v^T E_i v)^{\frac{1}{2}} + u^T B_i z_i - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i). \quad (3.39)$$

By second-order K-F-convexity of $\{f_1(., v) + (.)^T B_1 z_1, ..., f_k(., v) + (.)^T B_k z_k\}$ at u, we have

$$(f_1(x, v) + x^T B_1 z_1 - f_1(u, v) - u^T B_1 z_1 + \tfrac{1}{2} q_1^T \nabla_{xx} f_1(u, v) q_1 - F(x, u; \nabla_x f_1(u, v) + B_1 z_1 + \nabla_{xx} f_1(u, v) q_1), ..., f_k(x, v) + x^T B_k z_k - f_k(u, v) - u^T B_k z_k + \tfrac{1}{2} q_k^T \nabla_{xx} f_k(u, v) q_k - F(x, u; \nabla_x f_k(u, v) + B_k z_k + \nabla_{xx} f_k(u, v) q_k)) \in K.$$

It follows from hypothesis (iii) that $K^* \subseteq R_+^k$ and hence $\lambda > 0$.

By using the sublinearity of F, we get

$$\sum_{i=1}^{k} \lambda_i(f_i(x, v) + x^T B_i z_i - f_i(u, v) - u^T B_i z_i + \tfrac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i)$$
$$\geqq F(x, u; \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + B_i z_i + \nabla_{xx} f_i(u, v) q_i))$$
$$\geqq -u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + B_i z_i + \nabla_{xx} f_i(u, v) q_i)$$

(by hypothesis (iv) and constraint (3.35), for $a = \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + B_i z_i + \nabla_{xx} f_i(u, v) q_i) \in C_1^*$)

$$\geqq 0 \ \text{(by constraint (3.36))}. \quad (3.40)$$

Similarly, using hypotheses (ii) and (v) along with primal constraints (3.31), (3.32) and (3.34) and the sublinearity of G, we get

$$\sum_{i=1}^{k} \lambda_i(f_i(x, y) - y^T E_i w_i - f_i(x, v) + v^T E_i w_i - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i) \geqq 0. \quad (3.41)$$

Combining inequalities (3.40) and (3.41), we get

$$\sum_{i=1}^{k}\lambda_i(f_i(x,y)+x^TB_iz_i-y^TE_iw_i-\tfrac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i)$$
$$\geqq\sum_{i=1}^{k}\lambda_i(f_i(u,v)-v^TE_iw_i+u^TB_iz_i-\tfrac{1}{2}q_i^T\nabla_{xx}f_i(u,v)q_i).$$

Applying the Schwartz inequality and using (3.33) and (3.37), we obtain

$$\sum_{i=1}^{k}\lambda_i(f_i(x,y)+(x^TB_ix)^{\frac{1}{2}}-y^TE_iw_i-\tfrac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i)$$
$$\geqq\sum_{i=1}^{k}\lambda_i(f_i(u,v)-(v^TE_iv)^{\frac{1}{2}}+u^TB_iz_i-\tfrac{1}{2}q_i^T\nabla_{xx}f_i(u,v)q_i),$$

which contradicts (3.39). Hence the result.

If the variable λ in the problems (MP) and (MD) is fixed to be $\bar{\lambda}$, we shall denote these problems by $(\text{MP})_{\bar{\lambda}}$ and $(\text{MD})_{\bar{\lambda}}$, respectively.

Theorem 3.5 (Strong duality). Let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ be an efficient solution for (MP). Suppose that

(i) $\nabla_{yy}f_i(\bar{x},\bar{y})$ is positive definite for $i=1,2,...,k$ and $\sum_{i=1}^{k}\bar{\lambda}_i\bar{p}_i^T(\nabla_y f_i(\bar{x},\bar{y})-E_i\bar{w}_i)\geqq 0$ or $\nabla_{yy}f_i(\bar{x},\bar{y})$ is negative definite for $i=1,2,...,k$ and $\sum_{i=1}^{k}\bar{\lambda}_i\bar{p}_i^T(\nabla_y f_i(\bar{x},\bar{y})-E_i\bar{w}_i)\leqq 0$,

(ii) the set $\{\nabla_y f_i(\bar{x},\bar{y})-E_i\bar{w}_i+\nabla_{yy}f_i(\bar{x},\bar{y})\bar{p}_i, i=1,2,\ldots,k\}$ is linearly independent,

(iii) K is closed convex pointed cone with $R_+^k\subseteq K$.

Then $\bar{p}=0$, there exist $\bar{z}$ such that $(\bar{x},\bar{y},\bar{z},\bar{q}=0)$ is feasible for $(\text{MD})_{\bar{\lambda}}$, and the objective function values of (MP) and $(\text{MD})_{\bar{\lambda}}$ are equal. Furthermore, if the hypotheses of Theorem 3.4 are satisfied for all feasible solutions of (MP) and $(\text{MD})_{\bar{\lambda}}$, then $(\bar{x},\bar{y},\bar{z},\bar{q})$ is an efficient solution for $(\text{MD})_{\bar{\lambda}}$.

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ is an efficient solution for (MP), by the Fritz John necessary optimality conditions (Suneja et al. [123]), there exist $\alpha\in K^*$, $\beta\in C_2$, $\gamma\in R_+$ and $\nu\in R^k$, such that the following conditions are satisfied at $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ (for simplicity, we write $\nabla_x f_i,\nabla_{xy}f_i$ instead of $\nabla_x f_i(\bar{x},\bar{y}),\nabla_{xy}f_i(\bar{x},\bar{y})$ etc.):

$$(x-\bar{x})^T[\sum_{i=1}^{k}\alpha_i(\nabla_x f_i+B_i\bar{z}_i-\frac{1}{2}\nabla_x(\nabla_{yy}f_i\bar{p}_i)\bar{p}_i)$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yx}f_i+\nabla_x(\nabla_{yy}f_i\bar{p}_i))(\beta-\gamma\bar{y})]\geqq 0\ \text{ for all } x\in C_1,\quad (3.42)$$

$$(y-\bar{y})^T[\sum_{i=1}^{k}\alpha_i(\nabla_y f_i - E_i\bar{w}_i - \frac{1}{2}\nabla_y(\nabla_{yy}f_i\bar{p}_i)\bar{p}_i) + \sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i + \nabla_y(\nabla_{yy}f_i\bar{p}_i))$$

$$\times(\beta-\gamma\bar{y}) - \gamma\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i)] \geqq 0 \text{ for all } y \in R^m, \tag{3.43}$$

$$[(\beta-\gamma\bar{y})^T(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i)](\lambda_i - \bar{\lambda}_i) \geqq 0, \forall\ i \text{ for all } \lambda \in \text{int } K^*, \tag{3.44}$$

$$\alpha_i E_i\bar{y} + (\beta-\gamma\bar{y})\bar{\lambda}_i E_i = 2\nu_i E_i\bar{w}_i,\ i=1,2,...,k, \tag{3.45}$$

$$[(\beta-\gamma\bar{y})\bar{\lambda}_i - \alpha_i\bar{p}_i]^T\nabla_{yy}f_i = 0,\ i=1,2,...,k, \tag{3.46}$$

$$\bar{x}^T B_i\bar{z}_i = (\bar{x}^T B_i\bar{x})^{\frac{1}{2}},\ i=1,2,...,k, \tag{3.47}$$

$$\beta^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i) = 0, \tag{3.48}$$

$$\gamma\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i) = 0, \tag{3.49}$$

$$\nu_i(\bar{w}_i^T E_i\bar{w}_i - 1) = 0,\ i=1,2,...,k, \tag{3.50}$$

$$\bar{z}_i^T B_i\bar{z}_i \leqq 1,\ i=1,2,...,k, \tag{3.51}$$

$$(\alpha,\beta,\gamma,\nu) \neq 0. \tag{3.52}$$

Inequalities (3.43) and (3.44) are equivalent to

$$\sum_{i=1}^{k}\alpha_i(\nabla_y f_i - E_i\bar{w}_i - \frac{1}{2}\nabla_y(\nabla_{yy}f_i\bar{p}_i)\bar{p}_i) + \sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i + \nabla_y(\nabla_{yy}f_i\bar{p}_i))$$

$$\times(\beta-\gamma\bar{y}) - \gamma\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i) = 0, \tag{3.53}$$

and

$$(\beta-\gamma\bar{y})^T(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i) = 0,\ i=1,2,...,k. \tag{3.54}$$

Since $\nabla_{yy}f_i$ is positive or negative definite, for $i=1,2,...,k$, equation (3.46) yield

$$(\beta-\gamma\bar{y})\bar{\lambda}_i = \alpha_i\bar{p}_i \qquad i=1,2,...,k. \tag{3.55}$$

On rearranging (3.53), we get

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)(\nabla_y f_i - E_i\bar{w}_i) + \sum_{i=1}^{k}\bar{\lambda}_i\nabla_{yy}f_i(\beta - \gamma\bar{y} - \gamma\bar{p}_i) + \sum_{i=1}^{k}\nabla_y(\nabla_{yy}f_i\bar{p}_i)((\beta-\gamma\bar{y})\bar{\lambda}_i - \tfrac{1}{2}\alpha_i\bar{p}_i) = 0,$$

which by equation (3.55) becomes

$$\sum_{i=1}^{k}(\alpha_i-\gamma\bar{\lambda}_i)(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i) + \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y(\nabla_{yy}f_i\bar{p}_i)(\beta-\gamma\bar{y}) = 0. \tag{3.56}$$

Now, let $\alpha_j = 0$ for some j. Since $R_+^k \subseteq K \Rightarrow K^* \subseteq R_+^k$ which further gives int $K^* \subseteq$ int R_+^k and therefore $\bar{\lambda} > 0$.

Then relation (3.55) yields

$$\beta = \gamma\bar{y}. \tag{3.57}$$

This reduces (3.56) to

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i) = 0,$$

which on using hypothesis (ii), gives

$$\alpha_i = \gamma\bar{\lambda}_i,\ i = 1,2,...,k. \tag{3.58}$$

Since $\bar{\lambda}_i > 0$ for $i = 1,2,...,k$ and $\alpha_j = 0$ for some j, therefore (3.58) implies $\gamma = 0$ and thus from (3.57) and (3.58) respectively, we get $\beta = 0$ and $\alpha_i = 0,\ i = 1,2,...,k$. Further, the equation (3.45) and (3.50) gives $\nu_i = 0,\ i = 1,2,...,k$. Thus $(\alpha,\beta,\gamma,\nu) = 0$, a contradiction to (3.52).

Hence $\alpha_i \neq 0$, for all i. As $\alpha \in K^* \subseteq R_+^k$ implies

$$\alpha_i > 0,\ i = 1,2,...,k. \tag{3.59}$$

Using (3.55), $\alpha > 0$ and $\bar{\lambda} > 0$ in (3.54), we get

$$\bar{\lambda}_i\bar{p}_i^T(\nabla_y f_i - E_i\bar{w}_i + \nabla_{yy}f_i\bar{p}_i) = 0, \quad i = 1,2,...,k,$$

which further gives

$$\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^T (\nabla_y f_i - E_i \bar{w}_i) + \sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^T \nabla_{yy} f_i \bar{p}_i = 0, \tag{3.60}$$

which contradicts hypothesis (i) unless

$$\bar{p}_i = 0, \ i = 1, 2, ..., k. \tag{3.61}$$

And thus from (3.55), we get

$$\beta = \gamma \bar{y}. \tag{3.62}$$

Equations (3.56), (3.61) and (3.62) yields

$$\sum_{i=1}^{k} (\alpha_i - \gamma \bar{\lambda}_i)(\nabla_y f_i - E_i \bar{w}_i) = 0,$$

which on using hypothesis (ii) gives

$$\alpha_i = \gamma \bar{\lambda}_i \ i = 1, 2, ..., k. \tag{3.63}$$

Since $\bar{\lambda}_i > 0$ and $\alpha_i > 0$ for $i = 1, 2, ..., k$, the above relation implies

$$\gamma > 0. \tag{3.64}$$

Therefore (3.62) yields $\bar{y} = \dfrac{\beta}{\gamma} \in C_2$.
Using (3.61) to (3.64) in (3.42), we get

$$(x - \bar{x})^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i + B_i \bar{z}_i) \geqq 0 \text{ for all } x \in C_1. \tag{3.65}$$

Since for each $x \in C_1$, $\bar{x} \in C_1$, $\bar{x} + x \in C_1$ as C_1 is a closed convex cone, inequality (3.65) gives

$$x^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i + B_i \bar{z}_i) \geqq 0 \qquad \text{for all } x \in C_1,$$

or

$\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i + B_i \bar{z}_i) \in C_1^*$.

Putting $x = 0$ in (3.65), we obtain

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i + B_i \bar{z}_i) \leqq 0.$$

Hence $(\bar{x}, \bar{y}, \bar{z}, \bar{q} = 0)$ satisfies the dual constraints in $(\text{MD})_{\bar{\lambda}}$ and so it is a feasible solution for the dual problem $(\text{MD})_{\bar{\lambda}}$.

Now, letting $\dfrac{2\nu_i}{\alpha_i} = a$. Then $a \geqq 0$ and from (3.45) and (3.62)

$$E_i \bar{y} = a E_i \bar{w}_i, \tag{3.66}$$

which is the condition for equality in the schwartz inequality. Therefore

$$\bar{y}^T E_i \bar{w}_i = (\bar{y}^T E_i \bar{y})^{\frac{1}{2}} (\bar{w}_i^T E_i \bar{w}_i)^{\frac{1}{2}}. \tag{3.67}$$

In case $\nu_i > 0$, (3.50) gives $\bar{w}_i^T E_i \bar{w}_i = 1$ and so $\bar{y}^T E_i \bar{w}_i = (\bar{y}^T E_i \bar{y})^{\frac{1}{2}}$.

In case $\nu_i = 0$, (3.66) gives $E_i \bar{y} = 0$ and so $\bar{y}^T E_i \bar{w}_i = (\bar{y}^T E_i \bar{y})^{\frac{1}{2}} = 0$.

Thus in either case

$$\bar{y}^T E_i \bar{w}_i = (\bar{y}^T E_i \bar{y})^{\frac{1}{2}}. \tag{3.68}$$

Hence
$$\begin{aligned} L_i(\bar{x}, \bar{y}, \bar{w}, \bar{p} = 0) &= f_i(\bar{x}, \bar{y}) + (\bar{x}^T B_i \bar{x})^{\frac{1}{2}} - \bar{y}^T E_i \bar{w}_i \\ &= f_i(\bar{x}, \bar{y}) + \bar{x}^T B_i \bar{z}_i - (\bar{y}^T E_i \bar{y})^{\frac{1}{2}} \\ &= M_i(\bar{x}, \bar{y}, \bar{z}, \bar{q} = 0) \ \{\text{using (3.47) and (3.68)}\}, \end{aligned}$$

that is, the two objective function values are equal.

Now, let $(\bar{x}, \bar{y}, \bar{z}, \bar{q})$ is not an efficient solution for $(\text{MD})_{\bar{\lambda}}$, then there exists $(\bar{u}, \bar{v}, \bar{z}, \bar{q})$ feasible for $(\text{MD})_{\bar{\lambda}}$, such that

$$\begin{aligned} &\{f_1(\bar{x}, \bar{y}) - (\bar{y}^T E_1 \bar{y})^{\frac{1}{2}} + \bar{x}^T B_1 \bar{z}_1 - \tfrac{1}{2}\bar{q}_1^T \nabla_{xx} f_1(\bar{x}, \bar{y}) \bar{q}_1, ..., \\ &\qquad f_k(\bar{x}, \bar{y}) - (\bar{y}^T E_k \bar{y})^{\frac{1}{2}} + \bar{x}^T B_k \bar{z}_k - \tfrac{1}{2}\bar{q}_k^T \nabla_{xx} f_k(\bar{x}, \bar{y}) \bar{q}_k\} - \\ &\{f_1(\bar{u}, \bar{v}) - (\bar{v}^T E_1 \bar{v})^{\frac{1}{2}} + \bar{u}^T B_1 \bar{z}_1 - \tfrac{1}{2}\bar{q}_1^T \nabla_{xx} f_1(\bar{u}, \bar{v}) \bar{q}_1, ..., \\ &\qquad f_k(\bar{u}, \bar{v}) - (\bar{v}^T E_k \bar{v})^{\frac{1}{2}} + \bar{u}^T B_k \bar{z}_k - \tfrac{1}{2}\bar{q}_k^T \nabla_{xx} f_k(\bar{u}, \bar{v}) \bar{q}_k\} \in -K \backslash \{0\}. \end{aligned}$$

Since $\bar{p} = 0,\ \bar{q} = 0,\ \bar{x}^T B_i \bar{z}_i = (\bar{x}^T B_i \bar{x})^{\frac{1}{2}}$ and $\bar{y}^T E_i \bar{w}_i = (\bar{y}^T E_i \bar{y})^{\frac{1}{2}},\ i = 1, 2, ..., k,$ we have

$$\begin{aligned}&\{f_1(\bar{x}, \bar{y}) + (\bar{x}^T B_1 \bar{x})^{\frac{1}{2}} - \bar{y}^T E_1 \bar{w}_1 - \tfrac{1}{2}\bar{p}_1^T \nabla_{yy} f_1(\bar{x}, \bar{y})\bar{p}_1, ...,\\ &\qquad f_k(\bar{x}, \bar{y}) + (\bar{x}^T B_k \bar{x})^{\frac{1}{2}} - \bar{y}^T E_k \bar{w}_k - \tfrac{1}{2}\bar{p}_k^T \nabla_{yy} f_k(\bar{x}, \bar{y})\bar{p}_k\} -\\ &\{f_1(\bar{u}, \bar{v}) - (\bar{v}^T E_1 \bar{v})^{\frac{1}{2}} + \bar{u}^T B_1 \bar{z}_1 - \tfrac{1}{2}\bar{q}_1^T \nabla_{xx} f_1(\bar{u}, \bar{v})\bar{q}_1, ...,\\ &\qquad f_k(\bar{u}, \bar{v}) - (\bar{v}^T E_k \bar{v})^{\frac{1}{2}} + \bar{u}^T B_k \bar{z}_k - \tfrac{1}{2}\bar{q}_k^T \nabla_{xx} f_k(\bar{u}, \bar{v})\bar{q}_k\} \in -K\backslash\{0\},\end{aligned}$$

which contradicts Theorem 3.4. Hence $(\bar{x}, \bar{y}, \bar{z}, \bar{q})$ is an efficient solution for $(\text{MD})_{\bar{\lambda}}$.

Theorem 3.6 (Converse duality). Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{q})$ be an efficient solution for (MD). Suppose that

(i) $\nabla_{xx} f_i(\bar{u}, \bar{v})$ is positive definite for $i = 1, 2, ..., k$ and $\sum_{i=1}^{k} \bar{\lambda}_i \bar{q}_i^T(\nabla_x f_i(\bar{u}, \bar{v}) + B_i \bar{z}_i) \geqq 0$ or $\nabla_{xx} f_i(\bar{u}, \bar{v})$ is negative definite for $i = 1, 2, ..., k$ and $\sum_{i=1}^{k} \bar{\lambda}_i \bar{q}_i^T(\nabla_x f_i(\bar{u}, \bar{v}) + B_i \bar{z}_i) \leqq 0,$

(ii) the set $\{\nabla_x f_i(\bar{u}, \bar{v}) + B_i \bar{z}_i + \nabla_{xx} f_i(\bar{u}, \bar{v})\bar{q}_i, i = 1, 2, \ldots, k\}$ is linearly independent,

(iii) K is closed convex pointed cone with $R_+^k \subseteq K$.

Then $\bar{q} = 0$, there exist $\bar{w}$ such that $(\bar{u}, \bar{v}, \bar{w}, \bar{p} = 0)$ is feasible for $(\text{MP})_{\bar{\lambda}}$, and the objective function values of $(\text{MP})_{\bar{\lambda}}$ and (MD) are equal. Furthermore, if the hypotheses of Theorem 3.4 are satisfied for all feasible solutions of $(\text{MP})_{\bar{\lambda}}$ and (MD), then $(\bar{u}, \bar{v}, \bar{w}, \bar{p} = 0)$ is an efficient solution for $(\text{MP})_{\bar{\lambda}}$.

Proof. Follows on the lines of Theorem 3.5.

3.6 Special cases

In this section, we consider some of the special cases of our problems studied in Section 3.4 and 3.5.

(i) If $K = R_+^k$, $p_i = p$ and $q_i = q, \forall\ i$, then our programs (WP) and (WD) gives the duality results for the dual pair studied in [13, 62].

(ii) If $B_i = E_i = \{0\}$, for $i = 1, 2, ..., k$, $p = q = 0$, then (MP) and (MD) are reduced to the programs studied in Khurana [77]. This shows that our programs are generalization of programs of Khurana [77] as they involves nondifferentiable and second-order terms. In addition, if we take $k = 1$, $K = R_+$, $\lambda = 1 \in \text{int } R_+$, in (MP) and (MD), then we get the programs considered in Chandra and Kumar [41].

Further for all cases, $K = R^k_+$, $C_1 = R^n_+$ and $C_2 = R^m_+$.

(*iii*) If $k = 1$, our problems (WP) and (WD) reduce to programs studied in Gulati et al. [55] which in turn gives the problems considered in [89, 95].

(*iv*) If $p_i = p$ and $q_i = q$, $i = 1, 2, ..., k$ in (WP) and (WD), then the programs of Yang et al. [140] are obtained with $x \geqq 0$ and $v \geqq 0$. However, taking $F(x, u; a) = (x - u)^T a$ and $G(v, y; b) = (v - y)^T b$ in hypothesis (A) and (B) of Theorem 1 in [140] gives $x \geqq 0$ and $v \geqq 0$.

(*v*) Our problems (MP) and (MD) reduce to the programs studied in Ahmad and Husain [10]. This explains that (MP) and (MD) are generalization of programs of Ahmad and Husain [10] as these programs are discussed over cones.

Chapter 4

MIXED TYPE SECOND-ORDER SYMMETRIC DUALITY WITH GENERALIZED F-CONVEXITY[1]

4.1 Introduction

Mixed duality is a fruitful theory which plays an important role in mathematical programming. Mixed programming problems are in general more difficult than linear programming problems. The concept of mixed duality is interesting and useful both from theoretical as well as from algorithmic point of view. Xu [133] formulated two mixed type duals in multiobjective fractional programming and proved usual duality theorems. Chandra et al. [39] and Yang et al. [135] discussed a mixed symmetric dual formulation for a nonlinear progrmming problem and for a class of nondifferentiable nonlinear programming problems, respectively. Later on, Ahmad [6] formulated mixed type symmetric dual in multiobjective programming problems ignoring nonnegativity restrictions of Bector et al. [26].

This chapter is organized as follows. In Section 4.2, we give some notations and definitions used in the chapter. The next Section deals with the study of a pair of second-order mixed symmetric dual programs for a class of nondifferentiable nonlinear programming problems. Weak, strong and converse duality theorems are proved using

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the notion of second-order F-convexity/pseudoconvexity assumptions. In Section 4.4, we formulate a pair of mixed second-order symmetric dual programs over cones and proved duality results under second-order (F, ρ) convexity/pseudoconvexity assumptions. The last section contains some particular cases of the programs considered in this chapter.

4.2 Notations and definitions

For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, let $J_1 \subseteq N, K_1 \subseteq M$ and $J_2 = N \backslash J_1$ and $K_2 = M \backslash K_1$. Let $|J_1|$ denote the number of elements in J_1. The other symbols $|J_2|, |K_1|$ and $|K_2|$ are defined similarly. Let $x^1 \in R^{|J_1|}, x^2 \in R^{|J_2|}$. Then any $x \in R^n$ can be written as (x^1, x^2). Similarly for $y^1 \in R^{|K_1|}, y^2 \in R^{|K_2|}, y \in R^m$ can be written as (y^1, y^2). It may be noted here that if $J_1 = \emptyset$, then $|J_1| = 0, J_2 = N$ and therefore $|J_2| = n$. In this case, $R^{|J_1|}, R^{|J_2|}$ and $R^{|J_1|} \times R^{|K_1|}$ will be zero-dimensional, n-dimensional and $|K_1|$-dimensional Euclidean spaces, respectively. Similarly, we can describe the cases $J_2 = \emptyset, K_1 = \emptyset$ or $K_2 = \emptyset$. Let C_1, C_2, C_3 and C_4 be closed convex cones with nonempty interiors in $R^{|J_1|}$, $R^{|J_2|}$, $R^{|K_1|}$ and $R^{|K_2|}$ respectively.

Now we consider a sublinear functional $F : X \times X \times R^n \to R$ (where $X \subseteq R^n$), and a real valued twice differentiable function $\psi : X \to R$.

Definition 4.1. ψ is said to be second-order (F, ρ) convex at $u \in X$, if $\forall\ q \in R^n$ and $x \in X$, there exist a real valued function $d : X \times X \to R$ and a real number ρ, such that

$$\psi(x) - \psi(u) + \frac{1}{2} q^T \nabla_{xx}\psi(u) q \geqq F\left(x, u; \nabla_x \psi(u) + \nabla_{xx}\psi(u) q\right) + \rho d^2(x, u).$$

Definition 4.2. ψ is said to be second-order (F, ρ) pseudoconvex at $u \in X$, if $\forall\ q \in R^n$ and $x \in X$, if there exist a real valued function $d : X \times X \to R$ and a real number ρ, such that

$$F\left(x, u; \nabla_x \psi(u) + \nabla_{xx}\psi(u) q\right) \geqq 0$$
$$\Rightarrow \psi(x) \geqq \psi(u) - \frac{1}{2} q^T \nabla_{xx}\psi(u) q + \rho d^2(x, u).$$

4.3 Second-order mixed nondifferentiable symmetric dual programs

Consider the following pair of mixed nondifferentiable second-order symmetric dual programs:

Primal Problem (SMNP)

minimize

$$G(x^1,y^1,x^2,y^2,z^2,p,r) = f(x^1,y^1) + S(x^1 \mid C_1) + g(x^2,y^2) + S(x^2 \mid C_2) - (y^2)^T z^2$$
$$-(y^1)^T[\nabla_{y^1} f(x^1,y^1) + \nabla_{y^1y^1} f(x^1,y^1)p] - \frac{1}{2}p^T \nabla_{y^1y^1} f(x^1,y^1)p - \frac{1}{2}r^T \nabla_{y^2y^2} g(x^2,y^2)r,$$

subject to

$$\nabla_{y^1} f(x^1,y^1) - z^1 + \nabla_{y^1y^1} f(x^1,y^1)p \leqq 0, \tag{4.1}$$

$$\nabla_{y^2} g(x^2,y^2) - z^2 + \nabla_{y^2y^2} g(x^2,y^2)r \leqq 0, \tag{4.2}$$

$$(y^2)^T \left[\nabla_{y^2} g(x^2,y^2) - z^2 + \nabla_{y^2y^2} g(x^2,y^2)r\right] \geqq 0, \tag{4.3}$$

$$z^1 \in D_1,\ z^2 \in D_2. \tag{4.4}$$

Dual Problem (SMND)

maximize

$$H(u^1,v^1,u^2,v^2,w^2,q,s) = f(u^1,v^1) - S(v^1 \mid D_1) + g(u^2,v^2) - S(v^2 \mid D_2) + (u^2)^T w^2$$
$$-(u^1)^T[\nabla_{x^1} f(u^1,v^1) + \nabla_{x^1x^1} f(u^1,v^1)q] - \frac{1}{2}q^T \nabla_{x^1x^1} f(u^1,v^1)q - \frac{1}{2}s^T \nabla_{x^2x^2} g(u^2,v^2)s,$$

subject to

$$\nabla_{x^1} f(u^1,v^1) + w^1 + \nabla_{x^1x^1} f(u^1,v^1)q \geqq 0, \tag{4.5}$$

$$\nabla_{x^2} g(u^2,v^2) + w^2 + \nabla_{x^2x^2} g(u^2,v^2)s \geqq 0, \tag{4.6}$$

$$(u^2)^T \left[\nabla_{x^2} g(u^2,v^2) + w^2 + \nabla_{x^2x^2} g(u^2,v^2)s\right] \leqq 0, \tag{4.7}$$

$$w^1 \in C_1,\ w^2 \in C_2, \tag{4.8}$$

where

(*i*) $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions, (*ii*) $C_1,\ C_2,\ D_1$ and D_2 are compact convex sets in $R^{|J_1|},\ R^{|J_2|},\ R^{|K_1|}$ and $R^{|K_2|}$, respectively, and

(*iii*) $p \in R^{|K_1|}, r \in R^{|K_2|}, q \in R^{|J_1|}$ and $s \in R^{|J_2|}$.

Theorem 4.1 (Weak duality). Let $(x^1, y^1, x^2, y^2, z^1, z^2, p, r)$ be feasible for (SMNP) and $(u^1, v^1, u^2, v^2, w^1, w^2, q, s)$ be feasible for (SMND). Let the sublinear functionals $F_1 : R^{|J_1|} \times R^{|J_1|} \times R^{|J_1|} \mapsto R$, $F_2 : R^{|K_1|} \times R^{|K_1|} \times R^{|K_1|} \mapsto R$, $G_1 : R^{|J_2|} \times R^{|J_2|} \times R^{|J_2|} \mapsto R$ and $G_2 : R^{|K_2|} \times R^{|K_2|} \times R^{|K_2|} \mapsto R$ satisfy the following conditions:

$$F_1(x^1, u^1; a^1) + (a^1)^T u^1 \geqq 0, \quad \text{for all } a^1 \in R_+^{|J_1|}, \tag{A}$$

$$F_2(v^1, y^1; a^2) + (a^2)^T y^1 \geqq 0, \quad \text{for all } a^2 \in R_+^{|K_1|}, \tag{B}$$

$$G_1(x^2, u^2; b^1) + (b^1)^T u^2 \geqq 0, \quad \text{for all } b^1 \in R_+^{|J_2|}, \tag{C}$$

$$G_2(v^2, y^2; b^2) + (b^2)^T y^2 \geqq 0, \quad \text{for all } b^2 \in R_+^{|K_2|}. \tag{D}$$

Suppose that

(*i*) $f(., v^1) + (.)^T w^1$ be second-order F_1-convex at u^1, and $f(x^1, .) - (.)^T z^1$ be second-order F_2-concave at y^1,

(*ii*) $g(., v^2) + (.)^T w^2$ be second-order G_1-pseudoconvex at u^2, and $g(x^2, .) - (.)^T z^2$ be second-order G_2-pseudoconcave at y^2.

Then

$G(x^1, y^1, x^2, y^2, z^2, p, r) \geqq H(u^1, v^1, u^2, v^2, w^2, q, s)$.

Proof. By the second-order F_1-convexity of $f(., v^1) + (.)^T w^1$ at u^1 and the second-order F_2 concavity of $f(x^1, .) - (.)^T z^1$ at y^1, we have

$$\begin{aligned} &f(x^1, v^1) + (x^1)^T w^1 - f(u^1, v^1) - (u^1)^T w^1 + \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1, v^1) q \\ &\qquad \geqq F_1\left(x^1, u^1; \nabla_{x^1} f(u^1, v^1) + w^1 + \nabla_{x^1 x^1} f(u^1, v^1) q\right) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &f(x^1, y^1) - (y^1)^T z_1 - f(x^1, v^1) + (v^1)^T z_1 - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1, y^1) p \\ &\qquad \geqq F_2\left(v^1, y^1; -\left(\nabla_{y^1} f(x^1, y^1) - z^1 + \nabla_{y^1 y^1} f(x^1, y^1) p\right)\right). \end{aligned} \tag{4.10}$$

Since $(x^1, y^1, x^2, y^2, z^1, z^2, p, r)$ is feasible for primal problem (SMNP) and $(u^1, v^1, u^2, v^2, w^1, w^2, q, s)$ is feasible for dual problem (SMND), by the dual constraint (4.5), the vector $a^1 = \nabla_{x^1} f(u^1, v^1) + w^1 + \nabla_{x^1 x^1} f(u^1, v^1)q \in R_+^{|J_1|}$, and so from the hypothesis (A), we obtain

$$F_1(x^1, u^1; a^1) + (a^1)^T u^1 \geqq 0. \tag{4.11}$$

Similarly,

$$F_2(v^1, y^1; a^2) + (a^2)^T y^1 \geqq 0, \tag{4.12}$$

for the vector $a^2 = -\left[\nabla_{y^1} f(x^1, y^1) - z^1 + \nabla_{y^1 y^1} f(x^1, y^1)p\right] \in R_+^{|K_1|}$.
Using (4.11) in (4.9) and (4.12) in (4.10), we have

$$f(x^1, v^1) + (x^1)^T w^1 - f(u^1, v^1) - (u^1)^T w^1 + \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1, v^1)q \geqq -(u^1)^T a^1,$$

and

$$f(x^1, y^1) - (y^1)^T z_1 - f(x^1, v^1) + (v^1)^T z_1 - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1, y^1)p \geqq -(y^1)^T a^2.$$

Adding the above inequalities, we obtain

$$\begin{aligned} f(x^1, y^1) + (x^1)^T w^1 - (y^1)^T z_1 + (y^1)^T a^2 - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1, y^1)p& \\ \geqq f(u^1, v^1) + (u^1)^T w^1 - (v^1)^T z_1 - (u^1)^T a^1 - \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1, v^1)q.& \end{aligned} \tag{4.13}$$

Substituting the values of a^1 and a^2 in (4.13), we get

$$\begin{aligned} f(x^1, y^1) + (x^1)^T w^1 - (y^1)^T [\nabla_{y^1} f(x^1, y^1) + \nabla_{y^1 y^1} f(x^1, y^1)p] - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1, y^1)p& \\ \geqq f(u^1, v^1) - (v^1)^T z_1 - (u^1)^T [\nabla_{x^1} f(u^1, v^1) + \nabla_{x^1 x^1} f(u^1, v^1)q] - \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1, v^1)q.& \end{aligned} \tag{4.14}$$

Using $(x^1)^T w^1 \leqq S(x^1 \mid C_1)$ and $(v^1)^T z^1 \leqq S(v^1 \mid D_1)$, we have

$$\begin{aligned} f(x^1, y^1) + S(x^1 \mid C_1) - (y^1)^T [\nabla_{y^1} f(x^1, y^1) + \nabla_{y^1 y^1} f(x^1, y^1)p] - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1, y^1)p& \\ \geqq f(u^1, v^1) - S(v^1 \mid D_1) - (u^1)^T [\nabla_{x^1} f(u^1, v^1) + \nabla_{x^1 x^1} f(u^1, v^1)q] - \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1, v^1)q.& \end{aligned} \tag{4.15}$$

By hypothesis (C) and the dual constraint (4.6), we obtain

$$G_1(x^2, u^2; \nabla_{x^2} g(u^2, v^2) + w^2 + \nabla_{x^2 x^2} g(u^2, v^2)s) \geqq -(u^2)^T \left[\nabla_{x^2} g(u^2, v^2) + w^2 + \nabla_{x^2 x^2} g(u^2, v^2)s\right],$$

which on using the dual constraint (4.7) yields

$G_1(x^2, u^2; \nabla_{x^2} g(u^2, v^2) + w^2 + \nabla_{x^2x^2} g(u^2, v^2)s) \geqq 0.$

Since $g(., v^2) + (.)^T w^2$ be second-order G_1-pseudoconvex at u^2, we have

$$g(x^2, v^2) + (x^2)^T w^2 \geqq g(u^2, v^2) + (u^2)^T w^2 - \frac{1}{2} s^T \nabla_{x^2x^2} g(u^2, v^2)s. \tag{4.16}$$

Similarly, from (4.2) and (4.3) and hypothesis (D) along with second-order G_2-pseudoconcavity of $g(x^2, .) - (.)^T z^2$ at y^2, we get

$$g(x^2, y^2) - (y^2)^T z^2 \geqq g(x^2, v^2) - (v^2)^T z^2 + \frac{1}{2} r^T \nabla_{y^2y^2} g(x^2, y^2)r. \tag{4.17}$$

Adding equations (4.16) and (4.17), we obtain

$$\begin{aligned} &g(x^2, y^2) + (x^2)^T w^2 - (y^2)^T z^2 - \frac{1}{2} r^T \nabla_{y^2y^2} g(x^2, y^2)r \\ &\qquad \geqq g(u^2, v^2) + (u^2)^T w^2 - (v^2)^T z^2 - \frac{1}{2} s^T \nabla_{x^2x^2} g(u^2, v^2)s. \end{aligned}$$

Using $(x^2)^T w^2 \leqq S(x^2 \mid C_2)$ and $(v^2)^T z^2 \leqq S(v^2 \mid D_2)$, we have

$$\begin{aligned} &g(x^2, y^2) + S(x^2 \mid C_2) - (y^2)^T z^2 - \frac{1}{2} r^T \nabla_{y^2y^2} g(x^2, y^2)r \\ &\qquad \geqq g(u^2, v^2) + (u^2)^T w^2 - S(v^2 \mid D_2) - \frac{1}{2} s^T \nabla_{x^2x^2} g(u^2, v^2)s. \end{aligned} \tag{4.18}$$

Inequalities (4.15) and (4.18) together yield

$$\begin{aligned} &f(x^1, y^1) + S(x^1 \mid C_1) + g(x^2, y^2) + S(x^2 \mid C_2) - (y^2)^T z^2 \\ &\qquad -(y^1)^T[\nabla_{y^1} f(x^1, y^1) + \nabla_{y^1y^1} f(x^1, y^1)p] - \frac{1}{2} p^T \nabla_{y^1y^1} f(x^1, y^1)p - \frac{1}{2} r^T \nabla_{y^2y^2} g(x^2, y^2)r \\ &\geqq f(u^1, v^1) - S(v^1 \mid D_1) + g(u^2, v^2) - S(v^2 \mid D_2) + (u^2)^T w^2 \\ &\qquad -(u^1)^T[\nabla_{x^1} f(u^1, v^1) + \nabla_{x^1x^1} f(u^1, v^1)q] - \frac{1}{2} q^T \nabla_{x^1x^1} f(u^1, v^1)q - \frac{1}{2} s^T \nabla_{x^2x^2} g(u^2, v^2)s, \end{aligned}$$

that is,

$G(x^1, y^1, x^2, y^2, z^2, p, r) \geqq H(u^1, v^1, u^2, v^2, w^2, q, s).$

Theorem 4.2 (Strong duality). Let $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ be differentiable functions and let $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{p}, \bar{r})$ be a local optimal solution of (SMNP). Suppose that

(i) the matrix $\nabla_{y^1y^1} f(\bar{x}^1, \bar{y}^1)$ is non singular,

(ii) $\nabla_{y^2y^2} g(\bar{x}^2, \bar{y}^2)$ is positive definite and $\bar{r}^T (\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - \bar{z}^2) \geqq 0$ or $\nabla_{y^2y^2} g(\bar{x}^2, \bar{y}^2)$

is negative definite and $\bar{r}^T\left(\nabla_{y^2} g\left(\bar{x}^2, \bar{y}^2\right) - \bar{z}^2\right) \leqq 0$,

(iii) $\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - \bar{z}^2 + \nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\bar{r} \neq 0$, and

(iv) one of the matrices $\frac{\partial}{\partial y_i^1}\left(\nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1)\right),\ i = 1, 2, \ldots, |K_1|$, is positive or negative definite.

Then $\bar{p} = 0$, $\bar{r} = 0$, there exist $\bar{w}^1 \in C_1$ and $\bar{w}^2 \in C_2$ such that $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{w}^1, \bar{w}^2, \bar{q} = 0, \bar{s} = 0)$ is feasible for (SMND) and the objective function values of (SMNP) and (SMND) are equal. Furthermore, if the assumptions of Theorem 4.1 are satisfied for all feasible solutions of (SMNP) and (SMND), then $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{p}, \bar{r})$ and $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{w}^1, \bar{w}^2, \bar{q}, \bar{s})$ are global optimal solutions for (SMNP) and (SMND), respectively.

Proof. Since $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{p}, \bar{r})$ is a local optimal solution of (SMNP), there exist $\alpha \in R$, $\beta \in R^{|K_1|}$, $\gamma \in R^{|K_2|}$, $\delta \in R$, $\eta_1 \in R^{|J_1|}$ and $\eta_2 \in R^{|J_2|}$ such that the following by Fritz John optimality conditions [114] are satisfied at $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{p}, \bar{r})$:

$$\alpha^T(\nabla_{x^1} f(\bar{x}^1, \bar{y}^1) + \eta_1) + \nabla_{y^1 x^1} f(\bar{x}^1, \bar{y}^1)\left[\beta - \alpha\bar{y}^1\right] + \nabla_{x^1}\left(\nabla_{y^1 y^1} f\left(\bar{x}^1, \bar{y}^1\right)\bar{p}\right)\left[\beta - \alpha\left(\bar{y}^1 + \frac{1}{2}\bar{p}\right)\right] = 0, \quad (4.19)$$

$$\alpha^T(\nabla_{x^2} g(\bar{x}^2, \bar{y}^2) + \eta_2) + \nabla_{y^2 x^2} g(\bar{x}^2, \bar{y}^2)\left[\gamma - \delta\bar{y}^2\right] + \nabla_{x^2}\left(\nabla_{y^2 y^2} g\left(\bar{x}^2, \bar{y}^2\right)\bar{r}\right)\left[\gamma - \delta\bar{y}^2 - \frac{1}{2}\alpha\bar{r}\right] = 0, \quad (4.20)$$

$$\nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1)\left[\beta - \alpha\left(\bar{y}^1 + \bar{p}\right)\right] + \nabla_{y^1}\left(\nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1)\bar{p}\right)\left[\beta - \alpha\left(\bar{y}^1 + \frac{1}{2}\bar{p}\right)\right] = 0, \quad (4.21)$$

$$\left[\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - \bar{z}^2\right]\left[\alpha - \delta\right] + \nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\left[\gamma - \delta\left(\bar{y}^2 + \bar{r}\right)\right] + \nabla_{y^2}\left(\nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\bar{r}\right)\left[\gamma - \delta\bar{y}^2 - \frac{1}{2}\alpha\bar{r}\right] = 0, \quad (4.22)$$

$$\nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1)\left[\beta - \alpha\left(\bar{y}^1 + \bar{p}\right)\right] = 0, \quad (4.23)$$

$$\nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\left[\gamma - \delta\bar{y}^2 - \alpha\bar{r}\right] = 0, \quad (4.24)$$

$$\beta^T\left[\nabla_{y^1}f(\bar{x}^1,\bar{y}^1)-\bar{z}^1+\nabla_{y^1y^1}f(\bar{x}^1,\bar{y}^1)\bar{p}\right]=0, \tag{4.25}$$

$$\gamma^T\left[\nabla_{y^2}g(\bar{x}^2,\bar{y}^2)-\bar{z}^2+\nabla_{y^2y^2}g(\bar{x}^2,\bar{y}^2)\bar{r}\right]=0, \tag{4.26}$$

$$\delta(\bar{y}^2)^T\left[\nabla_{y^2}g(\bar{x}^2,\bar{y}^2)-\bar{z}^2+\nabla_{y^2y^2}g(\bar{x}^2,\bar{y}^2)\bar{r}\right]=0, \tag{4.27}$$

$$\beta\in N_{D_1}(\bar{z}^1), \tag{4.28}$$

$$(\alpha-\delta)\bar{y}^2+\gamma\in N_{D_2}(\bar{z}^2), \tag{4.29}$$

$$\eta_1\in C_1,\quad \eta_1^T\bar{x}^1=S(\bar{x}^1\mid C_1), \tag{4.30}$$

$$\eta_2\in C_2,\quad \eta_2^T\bar{x}^2=S(\bar{x}^2\mid C_2), \tag{4.31}$$

$$(\alpha,\beta,\gamma,\delta)\geq 0,\quad (\alpha,\beta,\gamma,\delta)\neq 0. \tag{4.32}$$

By hypothesis (i), (4.23) gives

$$\beta=\alpha\left(\bar{y}^1+\bar{p}\right). \tag{4.33}$$

Since $\nabla_{y^2y^2}g(\bar{x}^2,\bar{y}^2)$ is positive or negative definite, (4.24) yields

$$\gamma=\delta\bar{y}^2+\alpha\bar{r}. \tag{4.34}$$

Suppose $\alpha=0$, then (4.34) implies
$\gamma=\delta\bar{y}^2$.
Using (4.34) in (4.22), we get
$(\alpha-\delta)\left[\nabla_{y^2}g(\bar{x}^2,\bar{y}^2)-\bar{z}^2+\nabla_{y^2y^2}g(\bar{x}^2,\bar{y}^2)\bar{r}\right]+\frac{1}{2}\nabla_{y^2}\left(\nabla_{y^2y^2}g(\bar{x}^2,\bar{y}^2)\bar{r}\right)\left[\gamma-\delta\bar{y}^2\right]=0,$
which on using hypothesis (iii) and $\gamma=\delta\bar{y}^2$ yields
$\alpha=\delta$.
As $\alpha=0$, therefore the equations $\alpha=\delta$ and $\gamma=\delta\bar{y}^2$ give $\delta=0$ and $\gamma=0$, respectively. Further, (4.33) implies $\beta=0$. Consequently, $(\alpha,\beta,\gamma,\delta)=0$, contradicting (4.32). Hence we have,

$$\alpha>0. \tag{4.35}$$

Subtracting (4.27) from (4.26) yields

$$\left[\gamma - \delta(\bar{y}^2)\right]^T \left[\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - \bar{z}^2 + \nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\bar{r}\right] = 0.$$

Using (4.34) and (4.35) in above equation, we get

$$\bar{r}^T \left(\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - \bar{z}^2\right) + \bar{r}^T \nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\bar{r} = 0, \tag{4.36}$$

which contradicts hypothesis (ii) unless

$$\bar{r} = 0. \tag{4.37}$$

Equation (4.34) yields

$$\gamma = \delta \bar{y}^2. \tag{4.38}$$

Using (4.37) and (4.38) in (4.22), we obtain

$$(\alpha - \delta)\left(\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - \bar{z}^2\right) = 0,$$

which on using hypothesis (iii) and (4.37) gives

$$\alpha = \delta. \tag{4.39}$$

Since $\alpha > 0$, therefore

$$\delta > 0. \tag{4.40}$$

Now, using (4.33) and (4.35) in (4.21), we get

$$(\nabla_{y^1}(\nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1)\bar{p}))\bar{p} = 0,$$

which by hypothesis (iv) implies

$$\bar{p} = 0. \tag{4.41}$$

By equation (4.33) and (4.41), we have

$$\beta = \alpha \bar{y}^1. \tag{4.42}$$

Using (4.35), (4.41) and (4.42) in (4.19), we get

$$\nabla_{x^1} f(\bar{x}^1, \bar{y}^1) + \eta_1 = 0. \tag{4.43}$$

Equations (4.20), (4.35), (4.37) and (4.38) give

$$\nabla_{x^2} g(\bar{x}^2, \bar{y}^2) + \eta_2 = 0, \tag{4.44}$$

and hence, we also have

$$(\bar{x}^2)^T \left(\nabla_{x^2} g(\bar{x}^2, \bar{y}^2) + \eta_2\right) = 0. \tag{4.45}$$

Thus $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{w}^1 = \eta_1, \bar{w}^2 = \eta_2, \bar{q} = 0, \bar{s} = 0)$ satisfies the dual constraints from (4.5) to (4.8) and so it is a feasible solution for the dual problem (SMND).
Further, using (4.35), (4.41), (4.42) in (4.25), we obtain

$$(\bar{y}^1)^T \nabla_{y^1} f(\bar{x}^1, \bar{y}^1) = (\bar{y}^1)^T \bar{z}^1. \tag{4.46}$$

Moreover, since $\beta = \alpha \bar{y}^1$ and $\alpha > 0$, (4.28) implies $\bar{y}^1 \in N_{D_1}(\bar{z}^1)$, so that

$$(\bar{y}^1)^T \bar{z}^1 = S(\bar{y}^1 \mid D_1). \tag{4.47}$$

From (4.29) and (4.38)-(4.40), we get
$\bar{y}^2 \in N_{D_2}(\bar{z}^2)$.
Since D_2 is a compact convex set in $R^{|K_2|}$,

$$(\bar{y}^2)^T \bar{z}^2 = S(\bar{y}^2 \mid D_2). \tag{4.48}$$

Therefore, using (4.30), (4.31), (4.37), (4.41), (4.43) and (4.46)-(4.48), we obtain

$$\begin{aligned}
&f(\bar{x}^1, \bar{y}^1) + S(\bar{x}^1 \mid C_1) + g(\bar{x}^2, \bar{y}^2) + S(\bar{x}^2 \mid C_2) - (\bar{y}^2)^T \bar{z}^2 - (\bar{y}^1)^T [\nabla_{y^1} f(\bar{x}^1, \bar{y}^1) \\
&\qquad + \nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1) \bar{p}] - \tfrac{1}{2} \bar{p}^T \nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1) \bar{p} - \tfrac{1}{2} \bar{r}^T \nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2) \bar{r} \\
&= f(\bar{x}^1, \bar{y}^1) - S(\bar{y}^1 \mid D_1) + g(\bar{x}^2, \bar{y}^2) - S(\bar{y}^2 \mid D_2) + (\bar{x}^2)^T \bar{w}^2 - (\bar{x}^1)^T [\nabla_{x^1} f(\bar{x}^1, \bar{y}^1) \\
&\qquad + \nabla_{x^1 x^1} f(\bar{x}^1, \bar{y}^1) \bar{q}] - \tfrac{1}{2} \bar{q}^T \nabla_{x^1 x^1} f(\bar{x}^1, \bar{y}^1) \bar{q} - \tfrac{1}{2} \bar{s}^T \nabla_{x^2 x^2} g(\bar{x}^2, \bar{y}^2) \bar{s},
\end{aligned}$$

that is, the two objective function values are equal.
Finally, from Theorem 4.1, we get that $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{p}, \bar{r})$ and $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{w}^1, \bar{w}^2, \bar{q}, \bar{s})$ are global optimal solutions for (SMNP) and (SMND), respectively.

Theorem 4.3 (Converse duality). Let $f : R^{|J_1|} \times R^{|K_1|} \rightarrow R$ and $g : R^{|J_2|} \times R^{|K_2|} \rightarrow R$ be differentiable functions and let $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{w}^1, \bar{w}^2, \bar{q}, \bar{s})$ be a local optimal solution of (SMND). Suppose that

(i) the matrix $\nabla_{x^1x^1} f(\bar{u}^1, \bar{v}^1)$ is non singular,

(ii) $\nabla_{x^2x^2} g\left(\bar{u}^2, \bar{v}^2\right)$ is positive definite and $\bar{s}^T\left(\nabla_{x^2} g\left(\bar{u}^2, \bar{v}^2\right) + \bar{w}^2\right) \geqq 0$ or $\nabla_{x^2x^2} g\left(\bar{u}^2, \bar{v}^2\right)$ is negative definite and $\bar{s}^T\left(\nabla_{x^2} g\left(\bar{u}^2, \bar{v}^2\right) + \bar{w}^2\right) \leqq 0$,

(iii) $\nabla_{x^2} g(\bar{u}^2, \bar{v}^2) + \bar{w}^2 + \nabla_{x^2x^2} g(\bar{u}^2, \bar{v}^2)\bar{s} \neq 0$, and

(iv) one of the matrices $\frac{\partial}{\partial x_i^1}\left(\nabla_{x^1x^1} f(\bar{u}^1, \bar{v}^1)\right),\ i = 1, 2, \ldots, |J_1|$, is positive or negative definite.

Then $\bar{q} = 0$, $\bar{s} = 0$, there exist $\bar{z}^1 \in D_1$ and $\bar{z}^2 \in D_2$ such that $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{z}^1, \bar{z}^2, \bar{p} = 0, \bar{r} = 0)$ is feasible for (SMNP) and the objective function values of (SMNP) and (SMND) are equal. Furthermore, if the assumptions of Theorem 4.1 are satisfied for all feasible solutions of (SMNP) and (SMND), then $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{w}^1, \bar{w}^2, \bar{q}, \bar{s})$ and $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{z}^1, \bar{z}^2, \bar{p}, \bar{r})$ are global optimal solutions for (SMND) and (SMNP), respectively.

Proof. It follows on the lines of Theorem 4.2.

4.4 Second-order mixed symmetric dual programs with cone constraints

Now we consider the following pair of mixed second-order symmetric dual programs:

Primal Problem (SMP)

$$\text{minimize } L(x^1, y^1, x^2, y^2, p, r) = f(x^1, y^1) + g(x^2, y^2) - (y^1)^T[\nabla_{y^1} f(x^1, y^1) + \nabla_{y^1y^1} f(x^1, y^1)p] - \frac{1}{2}p^T \nabla_{y^1y^1} f(x^1, y^1)p - \frac{1}{2}r^T \nabla_{y^2y^2} g(x^2, y^2)r,$$

subject to

$$-\left[\nabla_{y^1} f(x^1, y^1) + \nabla_{y^1y^1} f(x^1, y^1)p\right] \in C_3^*, \tag{4.49}$$

$$-\left[\nabla_{y^2} g(x^2, y^2) + \nabla_{y^2y^2} g(x^2, y^2)r\right] \in C_4^*, \tag{4.50}$$

$$(y^2)^T\left[\nabla_{y^2} g(x^2, y^2) + \nabla_{y^2y^2} g(x^2, y^2)r\right] \geqq 0, \tag{4.51}$$

$$x^1 \in C_1,\ x^2 \in C_2. \tag{4.52}$$

Dual Problem (SMD)

maximize $M(u^1, v^1, u^2, v^2, q, s) = f(u^1, v^1) + g(u^2, v^2) - (u^1)^T[\nabla_{x^1} f(u^1, v^1)$

$$+\nabla_{x^1x^1} f(u^1, v^1)q] - \frac{1}{2}q^T \nabla_{x^1x^1} f(u^1, v^1)q - \frac{1}{2}s^T \nabla_{x^2x^2} g(u^2, v^2)s,$$

subject to

$$\nabla_{x^1} f(u^1, v^1) + \nabla_{x^1x^1} f(u^1, v^1)q \in C_1^*, \tag{4.53}$$

$$\nabla_{x^2} g(u^2, v^2) + \nabla_{x^2x^2} g(u^2, v^2)s \in C_2^*, \tag{4.54}$$

$$(u^2)^T \left[\nabla_{x^2} g(u^2, v^2) + \nabla_{x^2x^2} g(u^2, v^2)s\right] \leqq 0, \tag{4.55}$$

$$v^1 \in C_3,\ v^2 \in C_4. \tag{4.56}$$

(*i*) $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions,

(*ii*) $p \in R^{|K_1|}, r \in R^{|K_2|}, q \in R^{|J_1|}$ and $s \in R^{|J_2|}$.

Theorem 4.4 (Weak duality). Let $(x^1, y^1, x^2, y^2, p, r)$ be feasible for (SMP) and $(u^1, v^1, u^2, v^2, q, s)$ be feasible for (SMD). Let the sublinear functionals $F_1 : R^{|J_1|} \times R^{|J_1|} \times R^{|J_1|} \mapsto R$, $F_2 : R^{|K_1|} \times R^{|K_1|} \times R^{|K_1|} \mapsto R$, $G_1 : R^{|J_2|} \times R^{|J_2|} \times R^{|J_2|} \mapsto R$ and $G_2 : R^{|K_2|} \times R^{|K_2|} \times R^{|K_2|} \mapsto R$ satisfy the following conditions:

$$F_1(x^1, u^1; a^1) + (a^1)^T u^1 \geqq 0, \quad \text{for all } a^1 \in C_1^*, \tag{A}$$

$$F_2(v^1, y^1; a^2) + (a^2)^T y^1 \geqq 0, \quad \text{for all } a^2 \in C_3^*, \tag{B}$$

$$G_1(x^2, u^2; b^1) + (b^1)^T u^2 \geqq 0, \quad \text{for all } b^1 \in C_2^*, \tag{C}$$

$$G_2(v^2, y^2; b^2) + (b^2)^T y^2 \geqq 0, \quad \text{for all } b^2 \in C_4^*. \tag{D}$$

Suppose that

(*i*) $f(., v^1)$ be second-order (F_1, ρ_1) convex at u^1, and $-f(x^1, .)$ be second-order (F_2, ρ_2) convex at y^1,

(*ii*) $g(., v^2)$ be second-order (G_1, σ_1) pseudoconvex at u^2, and $-g(x^2, .)$ be second-order (G_2, σ_2) pseudoconvex at y^2,

(*iii*) either $\rho_1 d_1^2(x^1, u^1) + \rho_2 d_2^2(v^1, y^1) \geqq 0$ or $\rho_1, \rho_2 \geqq 0$, and

(iv) either $\sigma_1 d_3^2(x^2,u^2)+\sigma_2 d_4^2(v^2,y^2) \geqq 0$ or $\sigma_1,\sigma_2 \geqq 0$.

Then
$L(x^1,y^1,x^2,y^2,p,r) \geqq M(u^1,v^1,u^2,v^2,q,s)$.

Proof. By the second-order (F_1,ρ_1) convexity of $f(.,v^1)$ at u^1 and the second-order (F_2,ρ_2) convexity of $-f(x^1,.)$ at y^1, we have

$$\begin{aligned} f(x^1,v^1)-f(u^1,v^1)+\frac{1}{2}q^T\nabla_{x^1x^1}f(u^1,v^1)q \\ \geqq F_1\left(x^1,u^1;\nabla_{x^1}f(u^1,v^1)+\nabla_{x^1x^1}f(u^1,v^1)q\right)+\rho_1 d_1^2(x^1,u^1) \end{aligned} \tag{4.57}$$

and

$$\begin{aligned} f(x^1,y^1)-f(x^1,v^1)-\frac{1}{2}p^T\nabla_{y^1y^1}f(x^1,y^1)p \\ \geqq F_2\left(v^1,y^1;-\left(\nabla_{y^1}f(x^1,y^1)+\nabla_{y^1y^1}f(x^1,y^1)p\right)\right)+\rho_2 d_2^2(v^1,y^1). \end{aligned} \tag{4.58}$$

Adding the inequalities (4.57) and (4.58), we obtain

$$\begin{aligned} f(x^1,y^1)-f(u^1,v^1)+\frac{1}{2}q^T\nabla_{x^1x^1}f(u^1,v^1)q-\frac{1}{2}p^T\nabla_{y^1y^1}f(x^1,y^1)p \geqq F_1(x^1,u^1;\nabla_{x^1}f(u^1,v^1) \\ +\nabla_{x^1x^1}f(u^1,v^1)q)+F_2\left(v^1,y^1;-\left(\nabla_{y^1}f(x^1,y^1)+\nabla_{y^1y^1}f(x^1,y^1)p\right)\right) \\ +\rho_1 d_1^2(x^1,u^1)+\rho_2 d_2^2(v^1,y^1). \end{aligned} \tag{4.59}$$

Since (x^1,y^1,x^2,y^2,p,r) is feasible for primal problem (SMP) and (u^1,v^1,u^2,v^2,q,s) is feasible for dual problem (SMD), by the dual constraint (4.53), the vector $a^1 = \nabla_{x^1}f(u^1,v^1)+\nabla_{x^1x^1}f(u^1,v^1)q \in C_1^*$, and so from the hypothesis (A), we obtain

$$F_1(x^1,u^1;a^1)+(a^1)^Tu^1 \geqq 0. \tag{4.60}$$

Similarly,

$$F_2(v^1,y^1;a^2)+(a^2)^Ty^1 \geqq 0, \tag{4.61}$$

for the vector $a^2 = -\left[\nabla_{y^1}f(x^1,y^1)+\nabla_{y^1y^1}f(x^1,y^1)p\right] \in C_3^*$.
Using (4.60) and (4.61) and hypothesis (iii) in (4.59), we have

$f(x^1, y^1) - f(u^1, v^1) + \frac{1}{2}q^T \nabla_{x^1x^1} f(u^1, v^1)q - \frac{1}{2}p^T \nabla_{y^1y^1} f(x^1, y^1)p \geqq -(u^1)^T a^1 - (y^1)^T a^2.$
Substituting the values of a^1 and a^2, we get

$$f(x^1, y^1) - (y^1)^T[\nabla_{y^1} f(x^1, y^1) + \nabla_{y^1y^1} f(x^1, y^1)p] - \frac{1}{2}p^T \nabla_{y^1y^1} f(x^1, y^1)p \geqq f(u^1, v^1) - (u^1)^T[\nabla_{x^1} f(u^1, v^1) + \nabla_{x^1x^1} f(u^1, v^1)q] - \frac{1}{2}q^T \nabla_{x^1x^1} f(u^1, v^1)q. \quad (4.62)$$

By hypothesis (C) and the dual constraint (4.54), we obtain
$G_1(x^2, u^2; \nabla_{x^2} g(u^2, v^2) + \nabla_{x^2x^2} g(u^2, v^2)s) \geqq -(u^2)^T \left[\nabla_{x^2} g(u^2, v^2) + \nabla_{x^2x^2} g(u^2, v^2)s\right],$
which on using the dual constraint (4.55) yields
$G_1(x^2, u^2; \nabla_{x^2} g(u^2, v^2) + \nabla_{x^2x^2} g(u^2, v^2)s) \geqq 0.$
Since $g(., v^2)$ be second-order (G_1, σ_1) pseudoconvex at u^2, we have

$$g(x^2, v^2) \geqq g(u^2, v^2) - \frac{1}{2}s^T \nabla_{x^2x^2} g(u^2, v^2)s + \sigma_1 d_3^2(x^2, u^2). \quad (4.63)$$

Similarly, from (4.50) and (4.51) and hypothesis (D) along with second-order (G_2, σ_2) pseudoconvexity of $-g(x^2, .)$ at y^2, we get

$$g(x^2, y^2) \geqq g(x^2, v^2) + \frac{1}{2}r^T \nabla_{y^2y^2} g(x^2, y^2)r + \sigma_2 d_4^2(v^2, y^2). \quad (4.64)$$

Adding equations (4.63) and (4.64) and using hypothesis (iv), we obtain

$$g(x^2, y^2) - \frac{1}{2}r^T \nabla_{y^2y^2} g(x^2, y^2)r \geqq g(u^2, v^2) - \frac{1}{2}s^T \nabla_{x^2x^2} g(u^2, v^2)s. \quad (4.65)$$

Equations (4.62) and (4.65) together yield

$$f(x^1, y^1) + g(x^2, y^2) - (y^1)^T[\nabla_{y^1} f(x^1, y^1) + \nabla_{y^1y^1} f(x^1, y^1)p] - \frac{1}{2}p^T \nabla_{y^1y^1} f(x^1, y^1)p - \frac{1}{2}r^T \nabla_{y^2y^2} g(x^2, y^2)r \geqq f(u^1, v^1) + g(u^2, v^2) - (u^1)^T[\nabla_{x^1} f(u^1, v^1) + \nabla_{x^1x^1} f(u^1, v^1)q] - \frac{1}{2}q^T \nabla_{x^1x^1} f(u^1, v^1)q - \frac{1}{2}s^T \nabla_{x^2x^2} g(u^2, v^2)s,$$

that is,
$L(x^1, y^1, x^2, y^2, p, r) \geqq M(u^1, v^1, u^2, v^2, q, s).$

Theorem 4.5 (Strong duality). Let $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ be differentiable functions and let $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{p}, \bar{r})$ be a local optimal solution of (SMP). Suppose that

(i) the matrix $\nabla_{y^1y^1} f(\bar{x}^1, \bar{y}^1)$ is non singular,

(ii) $\nabla_{y^2y^2} g\,(\bar{x}^2, \bar{y}^2)$ is positive definite and $\nabla_{y^2} g\,(\bar{x}^2, \bar{y}^2)\,\bar{r} \geqq 0$ or $\nabla_{y^2y^2} g\,(\bar{x}^2, \bar{y}^2)$ is negative definite and $\nabla_{y^2} g\,(\bar{x}^2, \bar{y}^2)\,\bar{r} \leqq 0$,

(iii) $\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) + \nabla_{y^2y^2} g(\bar{x}^2, \bar{y}^2)\bar{r} \neq 0$, and

(iv) one of the matrices $\frac{\partial}{\partial y_i^1}\left(\nabla_{y^1y^1} f(\bar{x}^1, \bar{y}^1)\right),\ i = 1, 2, \ldots, |K_1|$, is positive or negative definite.

Then $\bar{p} = 0$, $\bar{r} = 0$, $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{q} = 0, \bar{s} = 0)$ is feasible for (SMD) and the objective function values of (SMP) and (SMD) are equal. Furthermore, if the assumptions of Theorem 4.4 are satisfied for all feasible solutions of (SMP) and (SMD), then $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{p}, \bar{r})$ and $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{q}, \bar{s})$ are global optimal solutions for (SMP) and (SMD), respectively.

Proof. Its proof follows on the lines of Theorem 4.2 of this chapter, taking $\eta_1 = 0$, $\eta_2 = 0$ and Theorem 3.2 in [12].

A converse duality theorem may be merely stated as its proof would run analogously to that of Theorem 4.5.

Theorem 4.6 (Converse duality). Let $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ be differentiable functions and let $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{q}, \bar{s})$ be a local optimal solution of (SMD). Suppose that

(i) the matrix $\nabla_{x^1x^1} f(\bar{u}^1, \bar{v}^1)$ is non singular,

(ii) $\nabla_{x^2x^2} g\,(\bar{u}^2, \bar{v}^2)$ is positive definite and $\nabla_{x^2} g\,(\bar{u}^2, \bar{v}^2)\,\bar{s} \geqq 0$ or $\nabla_{x^2x^2} g\,(\bar{u}^2, \bar{v}^2)$ is negative definite and $\nabla_{x^2} g\,(\bar{u}^2, \bar{v}^2)\,\bar{s} \leqq 0$,

(iii) $\nabla_{x^2} g(\bar{u}^2, \bar{v}^2) + \nabla_{x^2x^2} g(\bar{u}^2, \bar{v}^2)\bar{s} \neq 0$, and

(iv) one of the matrices $\frac{\partial}{\partial x_i^1}\left(\nabla_{x^1x^1} f(\bar{u}^1, \bar{v}^1)\right),\ i = 1, 2, \ldots, |J_1|$, is positive or negative definite.

Then $\bar{q} = 0$, $\bar{s} = 0$, $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{p} = 0, \bar{r} = 0)$ is feasible for (SMP) and the objective function values of (SMP) and (SMD) are equal. Furthermore, if the assumptions of Theorem 4.4 are satisfied for all feasible solutions of (SMP) and (SMD), then $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{q}, \bar{s})$ and $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{p}, \bar{r})$ are global optimal solutions for (SMD) and

(SMP), respectively.

4.5 Special cases

In this section, we will consider some of the special cases of the problems considered in Section 4.3 and Section 4.4 by choosing particular forms of J_1, J_2, K_1 and K_2 and closed convex cones C_1, C_2, C_3 and C_4 and compact convex sets.

(*i*) If $J_2 = \emptyset$ and $K_2 = \emptyset$, then our problems (SMNP) and (SMND) reduce to the programs (PP) and (DP) studied in Gulati and Gupta [57].

(*ii*) If $J_2 = \emptyset$, $K_2 = \emptyset$, $C_1 = \{0\}$ and $D_1 = \{0\}$, then (SMNP) and (SMND) are reduced to the programs (SP) and (SD) studied in Gulati et al. [55] with the omission of nonnegativity constraints from (SP) and (SD).

(*iii*) If $J_1 = \emptyset$ and $K_1 = \emptyset$ in (SMNP) and (SMND), then the programs studied in [70] are obtained.

Further for all cases, $C_1 = R_+^{|J_1|}, C_2 = R_+^{|J_2|}, C_3 = R_+^{|K_1|}$ and $C_4 = R_+^{|K_2|}$.

(*iv*) If $J_2 = \emptyset$ and $K_2 = \emptyset$, then our problems (SMP) and (SMD) reduce to the programs (SP) and (SD) studied in Gulati et al. [55] and if $J_1 = \emptyset$ and $K_1 = \emptyset$ in (SMP) and (SMD), then the programs (SP1) and (SD1) of [55] are obtained.

(*v*) By eliminating the second-order terms, our problems (SMP) and (SMD) reduce to the mixed symmetric dual programs studied by Chandra et al. [39].

Chapter 5

SECOND-ORDER MULTIOBJECTIVE MIXED SYMMETRIC DUAL PROGRAMS [1]

5.1 Introduction

Bector et al. [26] and Yang et al. [135] obtained duality results for mixed symmetric multiobjective differentiable and single objective nondifferentiable programming problems. Aghezzaf [4] formulated second-order mixed type dual for multiobjective programming problems and achieved duality theorems under generalized second-order (F, ρ)-convexity assumptions. Ahmad [6] studied invexity/generalized invexity for mixed type symmetric dual in multiobjective programming problems ignoring nonnegativity constraints of Bector et al. [26] but adjoining an additional constraint on invexity/generalized invexity. Recently, Ahmad and Husain [12] discussed a pair of multiobjective mixed symmetric dual programs over arbitrary cones and established duality results under K-preinvexity/K-pseudoinvexity assumptions.

In the present chapter, we formulate a pair of second-order multiobjective mixed symmetric dual programs over arbitrary cones. Weak, strong and converse duality theorems are proved for these programs under second-order invexity/pseudoinvexity

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[1] A part of this chapter has appeared in *Nonlinear Analysis: Real World Applications*12 (2011) 3373-3383.

assumptions. For notations, refer Section 4.2 of Chapter 4. In mathematical programming there are a large number of papers discussing duality theory for a problem involving the square root of a positive semidefinite quadratic function, $(x^TBx)^{\frac{1}{2}}$ [5, 10, 35, 37, 110, 118]. These types of problems are important due to the fact that, even though the objective function and/or constraint functions are nonsmooth, a simple representation for the dual problem may be found. Using such a function we construct a pair of multiobjective second-order mixed symmetric dual programs in Section 5.3. The usual duality results are then established using the notion of second-order F-convexity/pseudoconvexity assumptions. Special cases are discussed to show that our study extends some of the known results in [10, 13, 55, 62, 124, 140].

5.2 Mixed type second-order multiobjective symmetric duality with cone constraints

Now we consider the following pair of multiobjective mixed second-order symmetric dual programs:

Primal Problem (MSP)

minimize

$$G(x^1,y^1,x^2,y^2,\lambda,p,r^1,\ldots,r^l) = \Big\{G_1\left(x^1,y^1,x^2,y^2,\lambda,p,r^1,\ldots,r^l\right), \\ G_2\left(x^1,y^1,x^2,y^2,\lambda,p,r^1,\ldots,r^l\right),\ldots,G_l\left(x^1,y^1,x^2,y^2,\lambda,p,r^1,\ldots,r^l\right)\Big\}$$

subject to

$$-\sum_{i=1}^{l}\lambda_i[\nabla_{y^1}f_i(x^1,y^1)+\nabla_{y^1y^1}f_i(x^1,y^1)p]\in C_3^*, \tag{5.1}$$

$$-\sum_{i=1}^{l}\lambda_i[\nabla_{y^2}g_i(x^2,y^2)+\nabla_{y^2y^2}g_i(x^2,y^2)r^i]\in C_4^*, \tag{5.2}$$

$$(y^2)^T\sum_{i=1}^{l}\lambda_i\left(\nabla_{y^2}g_i(x^2,y^2)+\nabla_{y^2y^2}g_i(x^2,y^2)r^i\right)\geqq 0, \tag{5.3}$$

$$\lambda^Te_l=1, \tag{5.4}$$

$$\lambda>0,\ x^1\in C_1,\ x^2\in C_2. \tag{5.5}$$

Dual Problem (MSD)

maximize

$$H(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l) = \Big\{ H_1\left(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l\right),$$
$$H_2\left(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l\right), \ldots, H_l\left(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l\right) \Big\}$$

subject to

$$\sum_{i=1}^{l} \lambda_i[\nabla_{x^1} f_i(u^1, v^1) + \nabla_{x^1x^1} f_i(u^1, v^1)q] \in C_1^*, \tag{5.6}$$

$$\sum_{i=1}^{l} \lambda_i[\nabla_{x^2} g_i(u^2, v^2) + \nabla_{x^2x^2} g_i(u^2, v^2)s^i] \in C_2^*, \tag{5.7}$$

$$(u^2)^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{x^2} g_i(u^2, v^2) + \nabla_{x^2x^2} g_i(u^2, v^2)s^i\right) \leqq 0, \tag{5.8}$$

$$\lambda^T e_l = 1, \tag{5.9}$$

$$\lambda > 0, \;\; v^1 \in C_3, \;\; v^2 \in C_4. \tag{5.10}$$

where for $i = 1, 2, \ldots, l$

$$\begin{aligned} G_i(x^1, y^1, x^2, y^2, \lambda, p, r^1, \ldots, r^l) &= f_i(x^1, y^1) + g_i(x^2, y^2) \\ &- (y^1)^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{y^1} f_i(x^1, y^1) + \nabla_{y^1y^1} f_i(x^1, y^1)p\right) \\ &- \tfrac{1}{2} p^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{y^1y^1} f_i(x^1, y^1)p\right) - \tfrac{1}{2}(r^i)^T \nabla_{y^2y^2} g_i(x^2, y^2) r^i, \end{aligned}$$

$$\begin{aligned} H_i(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l) &= f_i(u^1, v^1) + g_i(u^2, v^2) \\ &- (u^1)^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{x^1} f_i(u^1, v^1) + \nabla_{x^1x^1} f_i(u^1, v^1)q\right) \\ &- \tfrac{1}{2} q^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{x^1x^1} f_i(u^1, v^1)q\right) - \tfrac{1}{2}(s^i)^T \nabla_{x^2x^2} g_i(u^2, v^2) s^i, \end{aligned}$$

and

(*i*) $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ and $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions

(*ii*) C_1, C_2, C_3 and C_4 are closed convex cones with nonempty interiors in $R^{|J_1|}$, $R^{|J_2|}$, $R^{|K_1|}$ and $R^{|K_2|}$ respectively, (*iii*) $p \in R^{|K_1|}$, $q \in R^{|J_1|}$, $r^i \in R^{|K_2|}$, $s^i \in R^{|J_2|}$, $\lambda \in R^l$ and $e_l = (1, \ldots, 1)^T \in R^l$.

Theorem 5.1 (Weak duality). Let $(x^1, y^1, x^2, y^2, \lambda, p, r^1, \ldots, r^l)$ be feasible for (MSP) and $(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l)$ be feasible for (MSD). Let

(*i*) $\sum\limits_{i=1}^{l} \lambda_i f_i(., v^1)$ be second-order invex with respect to η_1 for fixed v^1, and $-\sum\limits_{i=1}^{l} \lambda_i f_i(x^1, .)$ be second-order invex with respect to η_2 for fixed x^1 with $\eta_1(x^1, u^1) + u^1 \in C_1$ and $\eta_2(v^1, y^1) + y^1 \in C_3$ and

(*ii*) $\sum\limits_{i=1}^{l} \lambda_i g_i(., v^2)$ be second-order pseudoinvex with respect to η_3 for fixed v^2, and $-\sum\limits_{i=1}^{l} \lambda_i g_i(x^2, .)$ be second-order pseudoinvex with respect to η_4 for fixed x^2 with $\eta_3(x^2, u^2) + u^2 \in C_2$ and $\eta_4(v^2, y^2) + y^2 \in C_4$.

Then

$G(x^1, y^1, x^2, y^2, \lambda, p, r^1, \ldots, r^l) \nleq H(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l)$.

Proof. Suppose, to the contrary that

$G(x^1, y^1, x^2, y^2, \lambda, p, r^1, \ldots, r^l) \leq H(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l)$,

that is,

$$\Big\{G_1\left(x^1, y^1, x^2, y^2, \lambda, p, r^1, \ldots, r^l\right), G_2\left(x^1, y^1, x^2, y^2, \lambda, p, r^1, \ldots, r^l\right), \ldots, G_l\left(x^1, y^1, x^2, y^2, \lambda, p, r^1, \ldots, r^l\right)\Big\} \leq \Big\{H_1\left(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l\right), H_2\left(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l\right), \ldots, H_l\left(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l\right)\Big\}.$$

Then, since $\lambda > 0$, we have

$$\sum_{i=1}^{l} \lambda_i \Big\{ f_i(x^1, y^1) + g_i(x^2, y^2) - (y^1)^T \sum_{i=1}^{l} \lambda_i (\nabla_{y^1} f_i(x^1, y^1) + \nabla_{y^1 y^1} f_i(x^1, y^1) p) - \frac{1}{2} p^T \sum_{i=1}^{l} \lambda_i (\nabla_{y^1 y^1} f_i(x^1, y^1) p) - \frac{1}{2} (r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i \Big\}$$

$$< \sum_{i=1}^{l} \lambda_i \Big\{ f_i(u^1, v^1) + g_i(u^2, v^2) - (u^1)^T \sum_{i=1}^{l} \lambda_i (\nabla_{x^1} f_i(u^1, v^1) + \nabla_{x^1 x^1} f_i(u^1, v^1) q) - \frac{1}{2} q^T \sum_{i=1}^{l} \lambda_i (\nabla_{x^1 x^1} f_i(u^1, v^1) q) - \frac{1}{2} (s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i \Big\}. \tag{5.11}$$

By second-order invexity of $\sum\limits_{i=1}^{l} \lambda_i f_i(.,v^1)$ and $-\sum\limits_{i=1}^{l} \lambda_i f_i(x^1,.)$ with respect to η_1 and η_2 for fixed v^1 and x^1, respectively, we have

$$\sum_{i=1}^{l} \lambda_i \{f_i(x^1,v^1) - f_i(u^1,v^1) + \frac{1}{2} q^T \nabla_{x^1 x^1} f_i(u^1,v^1) q\}$$
$$\geqq \eta_1^T(x^1,u^1) \sum_{i=1}^{l} \lambda_i [\nabla_{x^1} f_i(u^1,v^1) + \nabla_{x^1 x^1} f_i(u^1,v^1) q], \qquad (5.12)$$

and

$$\sum_{i=1}^{l} \lambda_i \{f_i(x^1,y^1) - f_i(x^1,v^1) - \frac{1}{2} p^T \nabla_{y^1 y^1} f_i(x^1,y^1) p\}$$
$$\geqq -\eta_2^T(v^1,y^1) \sum_{i=1}^{l} \lambda_i [\nabla_{y^1} f_i(x^1,y^1) + \nabla_{y^1 y^1} f_i(x^1,y^1) p]. \qquad (5.13)$$

Since $(u^1, v^1, u^2, v^2, \lambda, q, s^1, \ldots, s^l)$ is feasible for (MSD), from the dual constraint (5.6) and $\eta_1(x^1,u^1) + u^1 \in C_1$, it follows that
$[\eta_1(x^1,u^1) + u^1]^T \sum\limits_{i=1}^{l} \lambda_i [\nabla_{x^1} f_i(u^1,v^1) + \nabla_{x^1 x^1} f_i(u^1,v^1) q] \geqq 0$,
which implies

$$\eta_1^T(x^1,u^1) \sum_{i=1}^{l} \lambda_i [\nabla_{x^1} f_i(u^1,v^1) + \nabla_{x^1 x^1} f_i(u^1,v^1) q]$$
$$\geqq -(u^1)^T \sum_{i=1}^{l} \lambda_i [\nabla_{x^1} f_i(u^1,v^1) + \nabla_{x^1 x^1} f_i(u^1,v^1) q]. \qquad (5.14)$$

Similarly, from the primal constraint (5.1) and $\eta_2(v^1,y^1) + y^1 \in C_3$, we have

$$-\eta_2^T(v^1,y^1) \sum_{i=1}^{l} \lambda_i [\nabla_{y^1} f_i(x^1,y^1) + \nabla_{y^1 y^1} f_i(x^1,y^1) p]$$
$$\geqq (y^1)^T \sum_{i=1}^{l} \lambda_i [\nabla_{y^1} f_i(x^1,y^1) + \nabla_{y^1 y^1} f_i(x^1,y^1) p]. \qquad (5.15)$$

Using (5.14) in (5.12) and (5.15) in (5.13), we get
$$\sum_{i=1}^{l} \lambda_i \{f_i(x^1,v^1) - f_i(u^1,v^1) + \tfrac{1}{2} q^T \nabla_{x^1 x^1} f_i(u^1,v^1) q\}$$
$$\geqq -(u^1)^T \sum_{i=1}^{l} \lambda_i [\nabla_{x^1} f_i(u^1,v^1) + \nabla_{x^1 x^1} f_i(u^1,v^1) q],$$

and

$$\sum_{i=1}^{l} \lambda_i\{f_i(x^1,y^1) - f_i(x^1,v^1) - \tfrac{1}{2}p^T\nabla_{y^1y^1}f_i(x^1,y^1)p\}$$

$$\geqq (y^1)^T \sum_{i=1}^{l} \lambda_i[\nabla_{y^1}f_i(x^1,y^1) + \nabla_{y^1y^1}f_i(x^1,y^1)p].$$

It follows from the above inequalities and $\lambda^T e_l = 1$ that

$$\sum_{i=1}^{l} \lambda_i\bigg\{f_i(x^1,y^1) - (y^1)^T \sum_{i=1}^{l} \lambda_i[\nabla_{y^1}f_i(x^1,y^1) + \nabla_{y^1y^1}f_i(x^1,y^1)p]$$

$$-\frac{1}{2}p^T \sum_{i=1}^{l} \lambda_i(\nabla_{y^1y^1}f_i(x^1,y^1)p)\bigg\} \geqq \sum_{i=1}^{l} \lambda_i\bigg\{f_i(u^1,v^1) - (u^1)^T \sum_{i=1}^{l} \lambda_i[\nabla_{x^1}f_i(u^1,v^1)$$

$$+\nabla_{x^1x^1}f_i(u^1,v^1)q] - \frac{1}{2}q^T \sum_{i=1}^{l} \lambda_i(\nabla_{x^1x^1}f_i(u^1,v^1)q)\bigg\}. \qquad (5.16)$$

By $\eta_3(x^2,u^2) + u^2 \in C_2$ and the dual constraint (5.7), we get

$[\eta_3(x^2,u^2) + u^2]^T \sum_{i=1}^{l} \lambda_i[\nabla_{x^2}g_i(u^2,v^2) + \nabla_{x^2x^2}g_i(u^2,v^2)s^i] \geqq 0,$

which on using the dual constraint (5.8) yields

$\eta_3^T(x^2,u^2) \sum_{i=1}^{l} \lambda_i[\nabla_{x^2}g_i(u^2,v^2) + \nabla_{x^2x^2}g_i(u^2,v^2)s^i] \geqq 0.$

Since $\sum_{i=1}^{l} \lambda_i g_i(.,v^2)$ is second-order pseudoinvex with respect to η_3 for fixed v^2, we have

$$\sum_{i=1}^{l} \lambda_i[g_i(x^2,v^2)] \geqq \sum_{i=1}^{l} \lambda_i[g_i(u^2,v^2) - \frac{1}{2}(s^i)^T\nabla_{x^2x^2}g_i(u^2,v^2)s^i]. \qquad (5.17)$$

Similarly, from (5.2) and (5.3) and $\eta_4(v^2,y^2) + y^2 \in C_4$ along with second-order pseudoinvexity of $-\sum_{i=1}^{l} \lambda_i g_i(x^2,.)$ with respect to η_4 for fixed x^2, we have

$$\sum_{i=1}^{l} \lambda_i[g_i(x^2,y^2)] \geqq \sum_{i=1}^{l} \lambda_i[g_i(x^2,v^2) + \frac{1}{2}(r^i)^T\nabla_{y^2y^2}g_i(x^2,y^2)r^i]. \qquad (5.18)$$

Combining (5.17) and (5.18), we have

$$\sum_{i=1}^{l} \lambda_i\left[g_i(x^2,y^2) - \frac{1}{2}(r^i)^T\nabla_{y^2y^2}g_i(x^2,y^2)r^i\right]$$

$$\geqq \sum_{i=1}^{l} \lambda_i\left[g_i(u^2,v^2) - \frac{1}{2}(s^i)^T\nabla_{x^2x^2}g_i(u^2,v^2)s^i\right]. \qquad (5.19)$$

Equations (5.16) and (5.19) together yield

$$\sum_{i=1}^{l}\lambda_i\Bigg\{f_i(x^1,y^1)+g_i(x^2,y^2)-(y^1)^T\sum_{i=1}^{l}\lambda_i(\nabla_{y^1}f_i(x^1,y^1)+\nabla_{y^1y^1}f_i(x^1,y^1)p)$$
$$-\tfrac{1}{2}p^T\sum_{i=1}^{l}\lambda_i(\nabla_{y^1y^1}f_i(x^1,y^1)p)-\tfrac{1}{2}(r^i)^T\nabla_{y^2y^2}g_i(x^2,y^2)r^i\Bigg\}$$
$$\geqq\sum_{i=1}^{l}\lambda_i\Bigg\{f_i(u^1,v^1)+g_i(u^2,v^2)-(u^1)^T\sum_{i=1}^{l}\lambda_i(\nabla_{x^1}f_i(u^1,v^1)+\nabla_{x^1x^1}f_i(u^1,v^1)q)$$
$$-\tfrac{1}{2}q^T\sum_{i=1}^{l}\lambda_i(\nabla_{x^1x^1}f_i(u^1,v^1)q)-\tfrac{1}{2}(s^i)^T\nabla_{x^2x^2}g_i(u^2,v^2)s^i\Bigg\},$$

which contradicts (5.11). Hence the results.

Theorem 5.2 (Strong duality). Let $f : R^{|J_1|}\times R^{|K_1|}\to R^l$ and $g : R^{|J_2|}\times R^{|K_2|}\to R^l$ be differentiable functions and let $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}^1,\ldots,\bar{r}^l)$ be a weak efficient solution of (MSP). Suppose that

(*i*) the matrix $\nabla_{y^1y^1}(\bar{\lambda}^Tf)(\bar{x}^1,\bar{y}^1)$ is non singular,

(*ii*) the matrices $\nabla_{y^2y^2}g_i\,(\bar{x}^2,\bar{y}^2)$ for $i=1,2,\ldots,l$ are non singular,

(*iii*) the set $\Big\{\nabla_{y^2}g_1(\bar{x}^2,\bar{y}^2)+\nabla_{y^2y^2}g_1(\bar{x}^2,\bar{y}^2)\bar{r}^1,\ldots,\nabla_{y^2}g_l(\bar{x}^2,\bar{y}^2)+\nabla_{y^2y^2}g_l(\bar{x}^2,\bar{y}^2)\bar{r}^l\Big\}$ is linearly independent,

(*iv*) for some $\varrho\in R^l_+$ $(\varrho>0)$ and $\bar{r}^i\in R^{|K_2|}$, $\bar{r}^i\neq 0$ $(i=1,2,\ldots,l)$ implies that $\sum_{i=1}^{l}\varrho_i(\bar{r}^i)^T\Big[\nabla_{y^2}g_i(\bar{x}^2,\bar{y}^2)+\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i\Big]\neq 0,$

(*v*) $\sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{y^1}(\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p}))\bar{p}\notin \operatorname{span}\{\nabla_{y^1}f_1(\bar{x}^1,\bar{y}^1),\ldots,\nabla_{y^1}f_l(\bar{x}^1,\bar{y}^1)\}\backslash\{0\}$, and

(*vi*) $\sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{y^1}(\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p}))\bar{p}=0$ implies $\bar{p}=0$.

Then $\bar{p}=0$, $\bar{r}^i=0$, for $i=1,2,\ldots,l$, $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{q}=0,\bar{s}^1=\ldots=\bar{s}^l=0)$ is feasible for (MSD) and the objective function values of (MSP) and (MSD) are equal. Furthermore, if the assumptions of Theorem 5.1 are satisfied for all feasible solutions of (MSP) and (MSD), then $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{q}=0,\bar{s}^1=\ldots=\bar{s}^l=0)$ is an efficient solution for (MSD).

Proof. Since $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}^1,\ldots,\bar{r}^l)$ is a weak efficient solution of (MSP), there exist $\alpha \in R^l_+$, $\beta \in C_3$, $\nu \in C_4$, $\xi \in R_+$, $\eta \in R$ and $\delta \in R^l_+$ such that the following by Fritz John optimality conditions ([123], Lemma 1) are satisfied at $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}^1, \ldots, \bar{r}^l)$:

$$\begin{aligned}\Bigg\{&\sum_{i=1}^{l}\alpha_i\nabla_{x^1}f_i\left(\bar{x}^1,\bar{y}^1\right)+\sum_{i=1}^{l}\bar{\lambda}_i\nabla_{y^1x^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\left[\beta-(\alpha^Te_l)\bar{y}^1\right]\\ &+\sum_{i=1}^{l}\bar{\lambda}_i\left(\nabla_{x^1}\left(\nabla_{y^1y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\bar{p}\right)\right)\left[\beta-(\alpha^Te_l)(\bar{y}^1+\frac{1}{2}\bar{p})\right]\Bigg\}(x^1-\bar{x}^1)\geqq 0,\ \forall x^1\in C_1,\end{aligned}\tag{5.20}$$

$$\begin{aligned}\Bigg\{&\sum_{i=1}^{l}\alpha_i\left[\nabla_{x^2}g_i(\bar{x}^2,\bar{y}^2)-\frac{1}{2}\left(\nabla_{x^2}\left(\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\right)\bar{r}^i\right]+\sum_{i=1}^{l}\bar{\lambda}_i\Bigg[\nabla_{y^2x^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\\ &\qquad+\nabla_{x^2}\left(\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\Bigg][\nu-\xi\bar{y}^2]\Bigg\}(x^2-\bar{x}^2)\geqq 0,\ \forall x^2\in C_2,\end{aligned}\tag{5.21}$$

$$\begin{aligned}&\sum_{i=1}^{l}\alpha_i\nabla_{y^1}f_i(\bar{x}^1,\bar{y}^1)+\sum_{i=1}^{l}\bar{\lambda}_i\nabla_{y^1y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\left[\beta-\left(\alpha^Te_l\right)\bar{y}^1\right]\\ &\quad+\sum_{i=1}^{l}\bar{\lambda}_i\left[\nabla_{y^1}\left(\nabla_{y^1y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\bar{p}\right)\right]\left[\beta-\left(\alpha^Te_l\right)\left(\bar{y}^1+\frac{1}{2}\bar{p}\right)\right]\\ &\qquad-\sum_{i=1}^{l}\bar{\lambda}_i\left[\nabla_{y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)+\nabla_{y^1y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\bar{p}\right]\left(\alpha^Te_l\right)=0,\end{aligned}\tag{5.22}$$

$$\begin{aligned}&\sum_{i=1}^{l}\alpha_i\left[\nabla_{y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)-\frac{1}{2}\left(\nabla_{y^2}\left(\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\bar{r}^i\right)\right]+\sum_{i=1}^{l}\bar{\lambda}_i\Bigg[\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\\ &+\nabla_{y^2}\left(\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\Bigg]\left[\nu-\xi\bar{y}^2\right]-\xi\sum_{i=1}^{l}\bar{\lambda}_i\left(\nabla_{y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)+\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)=0,\end{aligned}\tag{5.23}$$

$$\begin{aligned}&\left[\beta-\left(\alpha^Te_l\right)\bar{y}^1\right]\nabla_{y^1}f\left(\bar{x}^1,\bar{y}^1\right)+\left[\nu-\xi\bar{y}^2\right]\nabla_{y^2}g(\bar{x}^2,\bar{y}^2)-\delta+\eta e_l+\Bigg\{\Bigg[\beta-\left(\alpha^Te_l\right)\\ &\quad\left(\bar{y}^1+\frac{1}{2}\bar{p}\right)\Bigg]^T\nabla_{y^1y^1}f_1\left(\bar{x}^1,\bar{y}^1\right)\bar{p},\ldots,\left[\beta-\left(\alpha^Te_l\right)\left(\bar{y}^1+\frac{1}{2}\bar{p}\right)\right]^T\nabla_{y^1y^1}f_l\left(\bar{x}^1,\bar{y}^1\right)\bar{p}\Bigg\}\\ &\qquad+\left\{\left[\nu-\xi\bar{y}^2\right]^T\nabla_{y^2y^2}g_1(\bar{x}^2,\bar{y}^2)\bar{r}^1,\ldots,\left[\nu-\xi\bar{y}^2\right]^T\nabla_{y^2y^2}g_l\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^l\right\}=0,\end{aligned}\tag{5.24}$$

$$\nabla_{y^1y^1}(\bar{\lambda}^Tf)(\bar{x}^1,\bar{y}^1)\left[\beta-\left(\alpha^Te_l\right)\left(\bar{y}^1+\bar{p}\right)\right]=0,\tag{5.25}$$

$$\left[\left(\nu - \xi\bar{y}^2\right)\bar{\lambda}_i - \alpha_i\bar{r}^i\right]^T \nabla_{y^2y^2} g_i(\bar{x}^2,\bar{y}^2) = 0, \qquad i = 1,2,\dots,l, \tag{5.26}$$

$$\beta^T \sum_{i=1}^{l} \bar{\lambda}_i[\nabla_{y^1} f_i(\bar{x}^1,\bar{y}^1) + \nabla_{y^1y^1} f_i(\bar{x}^1,\bar{y}^1)\bar{p}] = 0, \tag{5.27}$$

$$\nu^T \sum_{i=1}^{l} \bar{\lambda}_i[\nabla_{y^2} g_i(\bar{x}^2,\bar{y}^2) + \nabla_{y^2y^2} g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i] = 0, \tag{5.28}$$

$$\xi(\bar{y}^2)^T \sum_{i=1}^{l} \bar{\lambda}_i[\nabla_{y^2} g_i(\bar{x}^2,\bar{y}^2) + \nabla_{y^2y^2} g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i] = 0, \tag{5.29}$$

$$\eta[\bar{\lambda}^T e_l - 1] = 0, \tag{5.30}$$

$$\delta^T\bar{\lambda} = 0, \tag{5.31}$$

$$(\alpha,\beta,\nu,\xi,\eta,\delta) \neq 0. \tag{5.32}$$

Since $\bar{\lambda} > 0$ and $\delta \geqq 0$, (5.31) implies $\delta = 0$.

As $\nabla_{y^1y^1}(\bar{\lambda}^T f)(\bar{x}^1,\bar{y}^1)$ is nonsingular, by (5.25), we get

$$\beta = (\alpha^T e_l)(\bar{y}^1 + \bar{p}). \tag{5.33}$$

From (5.26) and hypothesis (ii), we have

$$(\nu - \xi\bar{y}^2)\bar{\lambda}_i = \alpha_i\bar{r}^i, \quad i = 1,2,\dots,l. \tag{5.34}$$

Now, we claim that $\alpha_i \neq 0,\ i = 1,2,\dots,l$. Otherwise, if for some l_0, $\alpha_{l_0} = 0$, then it follows from $\lambda_{l_0} > 0$, and (5.34) that

$\nu = \xi\bar{y}^2$.

From (5.23), we get

$$\sum_{i=1}^{l}\left(\alpha_i - \xi\bar{\lambda}_i\right)\nabla_{y^2} g_i\left(\bar{x}^2,\bar{y}^2\right) + \sum_{i=1}^{l}\bar{\lambda}_i\nabla_{y^2y^2} g_i\left(\bar{x}^2,\bar{y}^2\right)\left[\nu - \xi\bar{y}^2 - \xi\bar{r}^i\right] + \sum_{i=1}^{l}\nabla_{y^2}\left(\nabla_{y^2y^2} g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\left[\left(\nu - \xi\bar{y}^2\right)\bar{\lambda}_i - \tfrac{1}{2}\alpha_i\bar{r}^i\right] = 0,$$

By using (5.34), it follows that

$$\sum_{i=1}^{l}\left(\alpha_i - \xi\bar{\lambda}_i\right)\left[\nabla_{y^2} g_i\left(\bar{x}^2,\bar{y}^2\right) + \nabla_{y^2y^2} g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right] + \tfrac{1}{2}\sum_{i=1}^{l}\bar{\lambda}_i\left[\nabla_{y^2}\left(\nabla_{y^2y^2} g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\right]\left[\nu - \xi\bar{y}^2\right] = 0,$$

which on using hypothesis (iii) and $\nu = \xi \bar{y}^2$ yields $\alpha_i = \xi \bar{\lambda}_i, \quad i = 1, 2, \ldots, l$.
As $\bar{\lambda}_i > 0, \quad i = 1, 2, \ldots, l$ and $\alpha_{l_0} = 0$, for some l_0, the above equation shows $\xi = 0$.
Now, using $\xi = 0$, the equations $\nu = \xi \bar{y}^2$ and $\alpha_i = \xi \bar{\lambda}_i, \ i = 1, 2, \ldots, l$ give $\nu = 0$ and $\alpha_i = 0, \ \forall \ i$. Further, (5.33) and (5.24) implies $\beta = 0$ and $\eta = 0$, respectively.
Consequently, $(\alpha, \beta, \nu, \xi, \eta, \delta) = 0$, contradicting (5.32). Hence $\alpha_i > 0, \ \forall \ i$.
Subtracting (5.29) from (5.28) yields

$$\left[\nu - \xi \bar{y}^2\right]^T \sum_{i=1}^{l} \bar{\lambda}_i \left[\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) + \nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2) \bar{r}^i\right] = 0,$$

Using (5.34), we get

$$\sum_{i=1}^{l} \alpha_i (\bar{r}^i)^T \left[\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) + \nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2) \bar{r}^i\right] = 0.$$

By the hypothesis (iv) with $\alpha_i > 0, \ i = 1, 2, \ldots, l$, we have

$$\bar{r}^i = 0, \quad i = 1, 2, \ldots, l. \tag{5.35}$$

As $\bar{\lambda}_i > 0, \quad i = 1, 2, \ldots, l$, (5.34) yields

$$\nu = \xi \bar{y}^2. \tag{5.36}$$

Using (5.35) and (5.36) in (5.23), we get
$\sum_{i=1}^{l} \left(\alpha_i - \xi \bar{\lambda}_i\right) \left[\nabla_{y^2} g_i \left(\bar{x}^2, \bar{y}^2\right)\right] = 0,$
which on using hypothesis (iii) and (5.35) gives

$$\alpha_i = \xi \bar{\lambda}_i, \quad i = 1, 2, \ldots, l. \tag{5.37}$$

From (5.37) and $\bar{\lambda}^T e_l = 1$, it is clear that $\alpha^T e_l = \xi(\bar{\lambda}^T e_l) = \xi$. since $\alpha_i > 0, \ i = 1, 2, \ldots, l$, therefore

$$\xi > 0. \tag{5.38}$$

Now using (5.33) and $\alpha_i > 0, \ \forall \ i$ in (5.22), we get

$$\sum_{i=1}^{l}\bar{\lambda}_i\left[\nabla_{y^1}(\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p})\right]\bar{p}=\tfrac{-2}{\alpha^Te_l}\sum_{i=1}^{l}\nabla_{y^1}f_i(\bar{x}^1,\bar{y}^1)[\alpha_i-(\alpha^Te_l)\bar{\lambda}_i]$$

which by hypothesis (v) and (vi) implies

$$\bar{p}=0. \tag{5.39}$$

From (5.39) and (5.33), we obtain

$$\beta=(\alpha^Te_l)\bar{y}^1. \tag{5.40}$$

Using (5.37)-(5.40) in (5.20), we have

$$(x^1-\bar{x}^1)^T\sum_{i=1}^{l}\bar{\lambda}_i\nabla_{x^1}f_i(\bar{x}^1,\bar{y}^1)\geqq 0,\ \text{ for all } x^1\in C_1. \tag{5.41}$$

Let $x^1\in C_1$. Then $x^1+\bar{x}^1\in C_1$, as C_1 is a closed convex cone, and so (5.41) implies

$$(x^1)^T\sum_{i=1}^{l}\bar{\lambda}_i\nabla_{x^1}f_i(\bar{x}^1,\bar{y}^1)\geqq 0,\quad \text{for all } x^1\in C_1.$$

Therefore,

$$\sum_{i=1}^{l}\bar{\lambda}_i\nabla_{x^1}f_i(\bar{x}^1,\bar{y}^1)\in C_1^*. \tag{5.42}$$

Also from $\alpha_i>0,\ i=1,2,\ldots,l$ and (5.40), we have
$\bar{y}^1=\dfrac{\beta}{\alpha^Te_l}\in C_3$.
Moreover, equations (5.21) and (5.35)-(5.38) give

$$(x^2-\bar{x}^2)^T\sum_{i=1}^{l}\bar{\lambda}_i\nabla_{x^2}g_i(\bar{x}^2,\bar{y}^2)\geqq 0,\ \text{ for all } x^2\in C_2. \tag{5.43}$$

Let $x^2\in C_2$. Then $x^2+\bar{x}^2\in C_2$, as C_2 is a closed convex cone, and so (5.43) implies

$$(x^2)^T\sum_{i=1}^{l}\bar{\lambda}_i\nabla_{x^2}g_i(\bar{x}^2,\bar{y}^2)\geqq 0,\quad \text{for all } x^2\in C_2.$$

Therefore,

$$\sum_{i=1}^{l} \bar{\lambda}_i \nabla_{x^2} g_i(\bar{x}^2, \bar{y}^2) \in C_2^*. \tag{5.44}$$

Also from (5.36) and (5.38), we have
$\bar{y}^2 = \dfrac{\nu}{\xi} \in C_4.$
Now, letting $x^2 = 0$ and $x^2 = 2\bar{x}^2$ in (5.43), we get

$$(\bar{x}^2)^T \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{x^2} g_i(\bar{x}^2, \bar{y}^2) = 0. \tag{5.45}$$

Thus $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{\lambda}, \bar{q} = 0, \bar{s}^1 = \ldots = \bar{s}^l = 0)$ satisfies the dual constraints from (5.6) to (5.10) and so it is a feasible solution for the dual problem (MSD).
Similarly, by putting $x^1 = 0$ and $x^1 = 2\bar{x}^1$ in (5.41), we obtain

$$(\bar{x}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) = 0. \tag{5.46}$$

Further, using $\alpha_i > 0, \ \forall \ i$, (5.39) and (5.40) in (5.27), we get

$$(\bar{y}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1) = 0. \tag{5.47}$$

Therefore, using (5.35), (5.39), (5.46) and (5.47), we get

$$\begin{aligned}
&\Bigg\{ f_1(\bar{x}^1, \bar{y}^1) + g_1(\bar{x}^2, \bar{y}^2) - (\bar{y}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1) + \nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}) - \tfrac{1}{2}\bar{p}^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}) \\
&- \tfrac{1}{2}(\bar{r}^1)^T \nabla_{y^2 y^2} g_1(\bar{x}^2, \bar{y}^2)\bar{r}^1, \ldots, f_l(\bar{x}^1, \bar{y}^1) + g_l(\bar{x}^2, \bar{y}^2) - (\bar{y}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1) + \nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}) \\
&- \tfrac{1}{2}\bar{p}^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p}) - \tfrac{1}{2}(\bar{r}^l)^T \nabla_{y^2 y^2} g_l(\bar{x}^2, \bar{y}^2)\bar{r}^l \Bigg\} \\
&= \Bigg\{ f_1(\bar{x}^1, \bar{y}^1) + g_1(\bar{x}^2, \bar{y}^2) - (\bar{x}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) + \nabla_{x^1 x^1} f_i(\bar{x}^1, \bar{y}^1)\bar{q}) - \tfrac{1}{2}\bar{q}^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^1 x^1} f_i(\bar{x}^1, \bar{y}^1)\bar{q}) \\
&- \tfrac{1}{2}(\bar{s}^1)^T \nabla_{x^2 x^2} g_1(\bar{x}^2, \bar{y}^2)\bar{s}^1, \ldots, f_l(\bar{x}^1, \bar{y}^1) + g_l(\bar{x}^2, \bar{y}^2) - (\bar{x}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) + \nabla_{x^1 x^1} f_i(\bar{x}^1, \bar{y}^1)\bar{q}) \\
&- \tfrac{1}{2}\bar{q}^T \sum_{i=1}^{l} \bar{\lambda}_i (\nabla_{x^1 x^1} f_i(\bar{x}^1, \bar{y}^1)\bar{q}) - \tfrac{1}{2}(\bar{s}^l)^T \nabla_{x^2 x^2} g_l(\bar{x}^2, \bar{y}^2)\bar{s}^l \Bigg\}
\end{aligned}$$

that is, the two objective function values are equal.

Now, let $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{q}=0,\bar{s}^1=\ldots=\bar{s}^l=0)$ is not an efficient solution of (MSD), then there exist $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{\lambda},\bar{q}=0,\bar{s}^1=\ldots=\bar{s}^l=0)$ feasible for (MSD), such that,

$$\Big\{G_1\left(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}^1,\ldots,\bar{r}^l\right),G_2\left(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}^1,\ldots,\bar{r}^l\right),$$
$$\ldots,G_l\left(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}^1,\ldots,\bar{r}^l\right)\Big\}\leq\Big\{H_1\left(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{\lambda},\bar{q},\bar{s}^1,\ldots,\bar{s}^l\right),$$
$$H_2\left(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{\lambda},\bar{q},\bar{s}^1,\ldots,\bar{s}^l\right),\ldots,H_l\left(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{\lambda},\bar{q},\bar{s}^1,\ldots,\bar{s}^l\right)\Big\},$$

which contradicts weak duality theorem. Hence $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{\lambda},\bar{q}=0,\bar{s}^1=\ldots=\bar{s}^l=0)$ is an efficient solution of (MSD).

Theorem 5.3 (Converse duality). Let $f:R^{|J_1|}\times R^{|K_1|}\rightarrow R^l$ and $g:R^{|J_2|}\times R^{|K_2|}\rightarrow R^l$ be differentiable functions and let $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{\lambda},\bar{q},\bar{s}^1,\ldots,\bar{s}^l)$ be a weak efficient solution of (MSD). Suppose that

(i) the matrix $\nabla_{x^1x^1}(\bar{\lambda}^Tf)(\bar{u}^1,\bar{v}^1)$ is non singular,

(ii) the matrices $\nabla_{x^2x^2}g_i\left(\bar{u}^2,\bar{v}^2\right)$ for $i=1,2,\ldots,l$ are non singular,

(iii) the set $\Big\{\nabla_{x^2}g_1(\bar{u}^2,\bar{v}^2)+\nabla_{x^2x^2}g_1(\bar{u}^2,\bar{v}^2)\bar{s}^1,\ldots,\nabla_{x^2}g_l(\bar{u}^2,\bar{v}^2)+\nabla_{x^2x^2}g_l(\bar{u}^2,\bar{v}^2)\bar{s}^l\Big\}$ is linearly independent,

(iv) for some $\varrho\in R^l_+$ $(\varrho>0)$ and $\bar{s}^i\in R^{|J_2|}$, $\bar{s}^i\neq 0$ $(i=1,2,\ldots,l)$ implies that $\sum\limits_{i=1}^{l}\varrho_i(\bar{s}^i)^T\Big[\nabla_{x^2}g_i(\bar{u}^2,\bar{v}^2)+\nabla_{x^2x^2}g_i(\bar{u}^2,\bar{v}^2)\bar{s}^i\Big]\neq 0$,

(v) $\sum\limits_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1}(\nabla_{x^1x^1}f_i(\bar{u}^1,\bar{v}^1)\bar{q}))\bar{q}\notin \operatorname{span}\{\nabla_{x^1}f_1(\bar{u}^1,\bar{v}^1),\ldots,\nabla_{x^1}f_l(\bar{u}^1,\bar{v}^1)\}\backslash\{0\}$, and

(vi) $\sum\limits_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1}(\nabla_{x^1x^1}f_i(\bar{u}^1,\bar{v}^1)\bar{q}))\bar{q}=0$ implies $\bar{q}=0$.

Then $\bar{q}=0$, $\bar{s}^i=0$, for $i=1,2,\ldots,l$, $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{\lambda},\bar{p}=0,\bar{r}^1=\ldots=\bar{r}^l=0)$ is feasible for (MSP) and the objective function values of (MSP) and (MSD) are equal. Furthermore, if the assumptions of Theorem 5.1 are satisfied for all feasible solutions of (MSP) and (MSD), then $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{\lambda},\bar{p}=0,\bar{r}^1=\ldots=\bar{r}^l=0)$ is an efficient solution for (MSP).

Proof. It follows on the lines of Theorem 5.2.

5.3 Multiobjective nondifferentiable second-order symmetric dual programs

Now we consider the following pair of multiobjective mixed second-order nondifferentiable symmetric dual programs and discuss their duality results:

Primal Problem (NKP)

minimize

$N(x^1, y^1, x^2, y^2, w^2, \lambda, p, r) = \left\{ N_1\big(x^1, y^1, x^2, y^2, w_1^2, \lambda, p, r^1\big), \dots, N_l\left(x^1, y^1, x^2, y^2, w_l^2, \lambda, p, r^l\right) \right\}$

subject to

$$\sum_{i=1}^{l} \lambda_i [\nabla_{y^1} f_i(x^1, y^1) - Ew^1 + \nabla_{y^1 y^1} f_i(x^1, y^1) p] \leqq 0, \tag{5.48}$$

$$\sum_{i=1}^{l} \lambda_i [\nabla_{y^2} g_i(x^2, y^2) - C_i w_i^2 + \nabla_{y^2 y^2} g_i(x^2, y^2) r^i] \leqq 0, \tag{5.49}$$

$$(y^2)^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{y^2} g_i(x^2, y^2) - C_i w_i^2 + \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\right) \geqq 0, \tag{5.50}$$

$$(w^1)^T E w^1 \leqq 1, \tag{5.51}$$

$$(w_i^2)^T C_i w_i^2 \leqq 1, \ i = 1, 2, \dots, l, \tag{5.52}$$

$$\lambda^T e_l = 1, \tag{5.53}$$

$$\lambda > 0. \tag{5.54}$$

Dual Problem (NKD)

maximize

$K(u^1, v^1, u^2, v^2, z^2, \lambda, q, s) = \left\{ K_1\big(u^1, v^1, u^2, v^2, z_1^2, \lambda, q, s^1\big), \dots, K_l\left(u^1, v^1, u^2, v^2, z_l^2, \lambda, q, s^l\right) \right\}$

subject to

$$\sum_{i=1}^{l} \lambda_i \big[\nabla_{x^1} f_i(u^1, v^1) + Dz^1 + \nabla_{x^1 x^1} f_i(u^1, v^1) q\big] \geqq 0, \tag{5.55}$$

$$\sum_{i=1}^{l}\lambda_i[\nabla_{x^2}g_i(u^2,v^2)+B_iz_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i]\geqq 0, \tag{5.56}$$

$$(u^2)^T\sum_{i=1}^{l}\lambda_i\left(\nabla_{x^2}g_i(u^2,v^2)+B_iz_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i\right)\leqq 0, \tag{5.57}$$

$$(z^1)^TDz^1\leqq 1, \tag{5.58}$$

$$(z_i^2)^TC_iz_i^2\leqq 1,\ i=1,2,\ldots,l, \tag{5.59}$$

$$\lambda^Te_l=1, \tag{5.60}$$

$$\lambda>0. \tag{5.61}$$

where $\lambda\in R^l$, $e_l=(1,\ldots,1)^T\in\ R^l$, $p,w^1\in R^{|K_1|}$ and $q,z^1\in R^{|J_1|}$, D and E are positive semidefinite matrices in $R^{|J_1|}\times R^{|J_1|}$ and $R^{|K_1|}\times R^{|K_1|}$, respectively and for $i=1,2,\ldots,l$

$$\begin{aligned}N_i(x^1,y^1,x^2,y^2,w_i^2,\lambda,p,r^i)&=f_i(x^1,y^1)+\left((x^1)^TDx^1\right)^{\frac{1}{2}}\\&\quad+g_i(x^2,y^2)+\left((x^2)^TB_ix^2\right)^{\frac{1}{2}}-(y^2)^TC_iw_i^2-(y^1)^T\sum_{i=1}^{l}\lambda_i\big(\nabla_{y^1}f_i(x^1,y^1)\\&\qquad+\nabla_{y^1y^1}f_i(x^1,y^1)p\big)-\tfrac{1}{2}p^T\sum_{i=1}^{l}\lambda_i\left(\nabla_{y^1y^1}f_i(x^1,y^1)p\right)-\tfrac{1}{2}(r^i)^T\nabla_{y^2y^2}g_i(x^2,y^2)r^i,\end{aligned}$$

$$\begin{aligned}K_i(u^1,v^1,u^2,v^2,z_i^2,\lambda,q,s^i)&=f_i(u^1,v^1)-\left((v^1)^TEv^1\right)^{\frac{1}{2}}\\&\quad+g_i(u^2,v^2)-\left((v^2)^TC_iv^2\right)^{\frac{1}{2}}+(u^2)^TB_iz_i^2-(u^1)^T\sum_{i=1}^{l}\lambda_i\big(\nabla_{x^1}f_i(u^1,v^1)\\&\qquad+\nabla_{x^1x^1}f_i(u^1,v^1)q\big)-\tfrac{1}{2}q^T\sum_{i=1}^{l}\lambda_i\left(\nabla_{x^1x^1}f_i(u^1,v^1)q\right)-\tfrac{1}{2}(s^i)^T\nabla_{x^2x^2}g_i(u^2,v^2)s^i,\end{aligned}$$

and

- *(i)* $f_i:R^{|J_1|}\times R^{|K_1|}\to R$ and $g_i:R^{|J_2|}\times R^{|K_2|}\to R$ are differentiable functions
- *(ii)* B_i and C_i are positive semidefinite matrices in $R^{|J_2|}\times R^{|J_2|}$ and $R^{|K_2|}\times R^{|K_2|}$, respectively,
- *(iii)* $r^i,w_i^2\in R^{|K_2|}$ and $s^i,z_i^2\in R^{|J_2|}$.

Theorem 5.4 (Weak duality). Let $(x^1,y^1,x^2,y^2,w^1,w^2,\lambda,p,r)$ be feasible for (NKP) and $(u^1,v^1,u^2,v^2,z^1,z^2,\lambda,q,s)$ be feasible for (NKD). Let the sublinear functionals

$F_1 : R^{|J_1|} \times R^{|J_1|} \times R^{|J_1|} \mapsto R$, $F_2 : R^{|K_1|} \times R^{|K_1|} \times R^{|K_1|} \mapsto R$, $G_1 : R^{|J_2|} \times R^{|J_2|} \times R^{|J_2|} \mapsto R$ and $G_2 : R^{|K_2|} \times R^{|K_2|} \times R^{|K_2|} \mapsto R$ satisfy the following conditions:

$$F_1(x^1, u^1; a^1) + (a^1)^T u^1 \geqq 0, \quad \text{for all } a^1 \in R_+^{|J_1|}, \tag{A}$$

$$F_2(v^1, y^1; a^2) + (a^2)^T y^1 \geqq 0, \quad \text{for all } a^2 \in R_+^{|K_1|}, \tag{B}$$

$$G_1(x^2, u^2; b^1) + (b^1)^T u^2 \geqq 0, \quad \text{for all } b^1 \in R_+^{|J_2|}, \tag{C}$$

$$G_2(v^2, y^2; b^2) + (b^2)^T y^2 \geqq 0, \quad \text{for all } b^2 \in R_+^{|K_2|}. \tag{D}$$

Suppose that

(*i*) $\sum_{i=1}^{l} \lambda_i\big(f_i(., v^1) + (.)^T D z^1\big)$ be second-order F_1-convex at u^1, and $\sum_{i=1}^{l} \lambda_i\big(f_i(x^1, .) - (.)^T E w^1\big)$ be second-order F_2-concave at y^1,

(*ii*) $\sum_{i=1}^{l} \lambda_i\big(g_i(., v^2) + (.)^T B_i z_i^2\big)$ be second-order G_1-pseudoconvex at u^2, and $\sum_{i=1}^{l} \lambda_i\big(g_i(x^2, .) - (.)^T C_i w_i^2\big)$ be second-order G_2-pseudoconcave at y^2.

Then

$N(x^1, y^1, x^2, y^2, w^2, \lambda, p, r) \nleq K(u^1, v^1, u^2, v^2, z^2, \lambda, q, s)$.

Proof. Suppose, to the contrary, that

$N(x^1, y^1, x^2, y^2, w^2, \lambda, p, r) \leq K(u^1, v^1, u^2, v^2, z^2, \lambda, q, s)$,

that is,

$$\Big\{ N_1\big(x^1, y^1, x^2, y^2, w_1^2, \lambda, p, r^1\big), \dots, N_l\left(x^1, y^1, x^2, y^2, w_l^2, \lambda, p, r^l\right) \Big\}$$
$$\leq \Big\{ K_1\big(u^1, v^1, u^2, v^2, z_1^2, \lambda, q, s^1\big), \dots, K_l\left(u^1, v^1, u^2, v^2, z_l^2, \lambda, q, s^l\right) \Big\}.$$

Then, since $\lambda > 0$, we have

$$\begin{aligned}
&\sum_{i=1}^{l} \lambda_i \Big\{ f_i(x^1, y^1) + \big((x^1)^T D x^1\big)^{\frac{1}{2}} + g_i(x^2, y^2) + \big((x^2)^T B_i x^2\big)^{\frac{1}{2}} - (y^2)^T C_i w_i^2 - (y^1)^T \sum_{i=1}^{l} \lambda_i \big(\nabla_{y^1} f_i(x^1, y^1) \\
&\quad + \nabla_{y^1 y^1} f_i(x^1, y^1) p\big) - \frac{1}{2} p^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{y^1 y^1} f_i(x^1, y^1) p\right) - \frac{1}{2} (r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i \Big\} < \sum_{i=1}^{l} \lambda_i \Big\{ f_i(u^1, v^1) \\
&\qquad - \big((v^1)^T E v^1\big)^{\frac{1}{2}} + g_i(u^2, v^2) - \big((v^2)^T C_i v^2\big)^{\frac{1}{2}} + (u^2)^T B_i z_i^2 - (u^1)^T \sum_{i=1}^{l} \lambda_i \big(\nabla_{x^1} f_i(u^1, v^1) \\
&\qquad\quad + \nabla_{x^1 x^1} f_i(u^1, v^1) q\big) - \frac{1}{2} q^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{x^1 x^1} f_i(u^1, v^1) q\right) - \frac{1}{2} (s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i \Big\}.
\end{aligned} \tag{5.62}$$

By the second-order F_1-convexity of $\sum\limits_{i=1}^{l}\lambda_i\big(f_i(.,v^1)+(.)^TDz^1\big)$ at u^1 and the second-order F_2 concavity of $\sum\limits_{i=1}^{l}\lambda_i\big(f_i(x^1,.)-(.)^TEw^1\big)$ at y^1, we have

$$\sum_{i=1}^{l}\lambda_i\bigg\{f_i(x^1,v^1)+(x^1)^TDz^1-f_i(u^1,v^1)-(u^1)^TDz^1+\frac{1}{2}q^T\nabla_{x^1x^1}f_i(u^1,v^1)q\bigg\}$$
$$\geqq F_1\left(x^1,u^1;\sum_{i=1}^{l}\lambda_i\big[\nabla_{x^1}f_i(u^1,v^1)+Dz^1+\nabla_{x^1x^1}f_i(u^1,v^1)q\big]\right) \quad (5.63)$$

and

$$\sum_{i=1}^{l}\lambda_i\bigg\{f_i(x^1,y^1)-(y^1)^TEw^1-f_i(x^1,v^1)+(v^1)^TEw^1-\frac{1}{2}p^T\nabla_{y^1y^1}f_i(x^1,y^1)p\bigg\}$$
$$\geqq F_2\left(v^1,y^1;-\sum_{i=1}^{l}\lambda_i[\nabla_{y^1}f_i(x^1,y^1)-Ew^1+\nabla_{y^1y^1}f_i(x^1,y^1)p]\right). \quad (5.64)$$

Since $(x^1,y^1,x^2,y^2,w^1,w^2,\lambda,p,r)$ is feasible for primal problem (NKP) and $(u^1,v^1,u^2,v^2,z^1,z^2,\lambda,q,s)$ is feasible for dual problem (NKD), by the dual constraint (5.55), the vector $a^1=\sum\limits_{i=1}^{l}\lambda_i\big[\nabla_{x^1}f_i(u^1,v^1)+Dz^1+\nabla_{x^1x^1}f_i(u^1,v^1)q\big]\in R_+^{|J_1|}$, and so from the hypothesis (A), we obtain

$$F_1(x^1,u^1;a^1)+(a^1)^Tu^1\geqq 0. \quad (5.65)$$

Similarly,

$$F_2(v^1,y^1;a^2)+(a^2)^Ty^1\geqq 0, \quad (5.66)$$

for the vector $a^2=-\sum\limits_{i=1}^{l}\lambda_i[\nabla_{y^1}f_i(x^1,y^1)-Ew^1+\nabla_{y^1y^1}f_i(x^1,y^1)p]\in R_+^{|K_1|}$.
Using (5.66) in (5.64) and (5.65) in (5.63), we have

$$\sum_{i=1}^{l}\lambda_i\bigg\{f_i(x^1,v^1)+(x^1)^TDz^1-f_i(u^1,v^1)-(u^1)^TDz^1+\tfrac{1}{2}q^T\nabla_{x^1x^1}f_i(u^1,v^1)q\bigg\}\geqq-(u^1)^Ta^1,$$

and

$$\sum_{i=1}^{l}\lambda_i\bigg\{f_i(x^1,y^1)-(y^1)^TEw^1-f_i(x^1,v^1)+(v^1)^TEw^1-\tfrac{1}{2}p^T\nabla_{y^1y^1}f_i(x^1,y^1)p\bigg\}\geqq-(y^1)^Ta^2.$$

Adding the above inequalities, we obtain

$$\sum_{i=1}^{l}\lambda_i\Big\{f_i(x^1,y^1)+(x^1)^TDz^1-(y^1)^TEw^1+(y^1)^Ta^2-\frac{1}{2}p^T\nabla_{y^1y^1}f_i(x^1,y^1)p\Big\}$$
$$\geqq\sum_{i=1}^{l}\lambda_i\Big\{f_i(u^1,v^1)-(v^1)^TEw^1+(u^1)^TDz^1-(u^1)^Ta^1-\frac{1}{2}q^T\nabla_{x^1x^1}f_i(u^1,v^1)q\Big\}. \tag{5.67}$$

Substituting the values of a^1 and a^2 in (5.67), we get

$$\sum_{i=1}^{l}\lambda_i\Big\{f_i(x^1,y^1)+(x^1)^TDz^1-(y^1)^T\sum_{i=1}^{l}\lambda_i\big(\nabla_{y^1}f_i(x^1,y^1)+\nabla_{y^1y^1}f_i(x^1,y^1)p\big)$$
$$-\tfrac{1}{2}p^T\sum_{i=1}^{l}\lambda_i\left(\nabla_{y^1y^1}f_i(x^1,y^1)p\right)\Big\}\geqq\sum_{i=1}^{l}\lambda_i\Big\{f_i(u^1,v^1)-(v^1)^TEw^1-(u^1)^T\sum_{i=1}^{l}\lambda_i\big(\nabla_{x^1}f_i(u^1,v^1)$$
$$+\nabla_{x^1x^1}f_i(u^1,v^1)q\big)-\tfrac{1}{2}q^T\sum_{i=1}^{l}\lambda_i\left(\nabla_{x^1x^1}f_i(u^1,v^1)q\right)-\tfrac{1}{2}(s^i)^T\nabla_{x^2x^2}g_i(u^2,v^2)s^i\Big\}.$$

Applying the Schwartz inequality and using (5.51) and (5.58), we have

$$\sum_{i=1}^{l}\lambda_i\Big\{f_i(x^1,y^1)+\big((x^1)^TDx^1\big)^{\frac{1}{2}}-(y^1)^T\sum_{i=1}^{l}\lambda_i\big(\nabla_{y^1}f_i(x^1,y^1)+\nabla_{y^1y^1}f_i(x^1,y^1)p\big)$$
$$-\frac{1}{2}p^T\sum_{i=1}^{l}\lambda_i\left(\nabla_{y^1y^1}f_i(x^1,y^1)p\right)\Big\}\geqq\sum_{i=1}^{l}\lambda_i\Big\{f_i(u^1,v^1)-\big((v^1)^TEv^1\big)^{\frac{1}{2}}$$
$$-(u^1)^T\sum_{i=1}^{l}\lambda_i\big(\nabla_{x^1}f_i(u^1,v^1)+\nabla_{x^1x^1}f_i(u^1,v^1)q\big)-\frac{1}{2}q^T\sum_{i=1}^{l}\lambda_i\left(\nabla_{x^1x^1}f_i(u^1,v^1)q\right)\Big\}. \tag{5.68}$$

By hypothesis (C) and the dual constraint (5.56), we obtain

$$G_1\Big(x^2,u^2;\sum_{i=1}^{l}\lambda_i[\nabla_{x^2}g_i(u^2,v^2)+B_iz_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i]\Big)$$
$$\geqq-(u^2)^T\left[\sum_{i=1}^{l}\lambda_i[\nabla_{x^2}g_i(u^2,v^2)+B_iz_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i]\right],$$

which on using the dual constraint (5.54) yields

$$G_1\Big(x^2,u^2;\sum_{i=1}^{l}\lambda_i[\nabla_{x^2}g_i(u^2,v^2)+B_iz_i^2+\nabla_{x^2x^2}g_i(u^2,v^2)s^i]\Big)\geqq 0.$$

Since $\sum_{i=1}^{l}\lambda_i\big(g_i(.,v^2)+(.)^TB_iz_i^2\big)$ be second-order G_1-pseudoconvex at u^2, we have

$$\sum_{i=1}^{l}\lambda_i\left[g_i(x^2,v^2)+(x^2)^TB_iz_i^2\right]$$
$$\geqq\sum_{i=1}^{l}\lambda_i\Big[g_i(u^2,v^2)+(u^2)^TB_iz_i^2-\frac{1}{2}(s^i)^T\nabla_{x^2x^2}g_i(u^2,v^2)s^i\Big]. \tag{5.69}$$

Similarly, from (5.49) and (5.50) and hypothesis (D) along with second-order G_2-pseudoconcavity of $\sum_{i=1}^{l} \lambda_i \big(g_i(x^2, .) - (.)^T C_i w_i^2\big)$ at y^2, we get

$$\sum_{i=1}^{l} \lambda_i \left[g_i(x^2, y^2) - (y^2)^T C_i w_i^2\right] \geqq \sum_{i=1}^{l} \lambda_i \left[g_i(x^2, v^2) - (v^2)^T C_i w_i^2 + \frac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\right]. \tag{5.70}$$

Adding equations (5.69) and (5.70), we obtain

$$\sum_{i=1}^{l} \lambda_i \left[g_i(x^2, y^2) + (x^2)^T B_i z_i^2 - (y^2)^T C_i w_i^2 - \tfrac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\right] \geqq \sum_{i=1}^{l} \lambda_i \left[g_i(u^2, v^2) + (u^2)^T B_i z_i^2 - (v^2)^T C_i w_i^2 - \tfrac{1}{2}(s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i\right].$$

Applying the Schwartz inequality, (5.52) and (5.59), we obtain,

$$\sum_{i=1}^{l} \lambda_i \left[g_i(x^2, y^2) + \big((x^2)^T B_i x^2\big)^{\frac{1}{2}} - (y^2)^T C_i w_i^2 - \frac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i\right] \geqq \sum_{i=1}^{l} \lambda_i \left[g_i(u^2, v^2) - \big((v^2)^T C_i v^2\big)^{\frac{1}{2}} + (u^2)^T B_i z_i^2 - \frac{1}{2}(s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i\right]. \tag{5.71}$$

Equations (5.68) and (5.71) together yield

$$\begin{aligned}
&\sum_{i=1}^{l} \lambda_i \Big\{ f_i(x^1, y^1) + \big((x^1)^T D x^1\big)^{\frac{1}{2}} + g_i(x^2, y^2) + \big((x^2)^T B_i x^2\big)^{\frac{1}{2}} - (y^2)^T C_i w_i^2 \\
&\quad - (y^1)^T \sum_{i=1}^{l} \lambda_i \big(\nabla_{y^1} f_i(x^1, y^1) + \nabla_{y^1 y^1} f_i(x^1, y^1) p\big) - \tfrac{1}{2} p^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{y^1 y^1} f_i(x^1, y^1) p\right) \\
&\qquad - \tfrac{1}{2}(r^i)^T \nabla_{y^2 y^2} g_i(x^2, y^2) r^i \Big\} \\
&\geqq \sum_{i=1}^{l} \lambda_i \Big\{ f_i(u^1, v^1) - \big((v^1)^T E v^1\big)^{\frac{1}{2}} + g_i(u^2, v^2) - \big((v^2)^T C_i v^2\big)^{\frac{1}{2}} + (u^2)^T B_i z_i^2 \\
&\quad - (u^1)^T \sum_{i=1}^{l} \lambda_i \big(\nabla_{x^1} f_i(u^1, v^1) + \nabla_{x^1 x^1} f_i(u^1, v^1) q\big) - \tfrac{1}{2} q^T \sum_{i=1}^{l} \lambda_i \left(\nabla_{x^1 x^1} f_i(u^1, v^1) q\right) \\
&\qquad - \tfrac{1}{2}(s^i)^T \nabla_{x^2 x^2} g_i(u^2, v^2) s^i \Big\}
\end{aligned}$$

which contradicts (5.62). Hence the results.

Theorem 5.5 (Strong duality). Let $f : R^{|J_1|} \times R^{|K_1|} \to R^l$ and $g : R^{|J_2|} \times R^{|K_2|} \to R^l$ be differentiable functions and let $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{w}^1, \bar{w}^2, \bar{\lambda}, \bar{p}, \bar{r})$ be a weak efficient solution of (NKP). Suppose that (*i*) the matrix $\nabla_{y^1y^1}(\bar\lambda^T f)(\bar x^1,\bar y^1)$ is non singular,

(*ii*) the matrices $\nabla_{y^2y^2} g_i\,(\bar x^2,\bar y^2)$ for $i=1,2,\ldots,l$ are non singular,

(*iii*) the set $\Big\{\nabla_{y^2} g_1(\bar x^2,\bar y^2) - C_1\bar w^1 + \nabla_{y^2y^2} g_1(\bar x^2,\bar y^2)\bar r^1,\ldots,\nabla_{y^2} g_l(\bar x^2,\bar y^2) - C_l\bar w^l + \nabla_{y^2y^2} g_l(\bar x^2,\bar y^2)\bar r^l\Big\}$ is linearly independent,

(*iv*) for some $\varrho\in R^l_+$ $(\varrho>0)$ and $\bar r^i\in R^{|K_2|}$, $\bar r^i\neq 0$ $(i=1,2,\ldots,l)$ implies that $\sum\limits_{i=1}^{l}\varrho_i(\bar r^i)^T\Big[\nabla_{y^2} g_i(\bar x^2,\bar y^2) - C_i\bar w_i^2 + \nabla_{y^2y^2} g_i(\bar x^2,\bar y^2)\bar r^i\Big]\neq 0$,

(*v*) $\sum\limits_{i=1}^{l}\bar\lambda_i(\nabla_{y^1}(\nabla_{y^1y^1} f_i(\bar x^1,\bar y^1)\bar p))\bar p\notin \operatorname{span}\{\nabla_{y^1} f_1(\bar x^1,\bar y^1),\ldots,\nabla_{y^1} f_l(\bar x^1,\bar y^1)\}\backslash\{0\}$, and

(*vi*) $\sum\limits_{i=1}^{l}\bar\lambda_i(\nabla_{y^1}(\nabla_{y^1y^1} f_i(\bar x^1,\bar y^1)\bar p))\bar p = 0$ implies $\bar p = 0$.

Then $\bar p = 0$, $\bar r = 0$, there exist $\bar z^1\in R^{|J_1|}$ and $\bar z_i^2\in R^{|J_2|}$, $i=1,2,\ldots,l$ such that $(\bar x^1,\bar y^1,\bar x^2,\bar y^2,\bar z^1,\bar z^2,\bar\lambda,\bar q=0,\bar s=0)$ is feasible for (NKD) and the objective function values of (NKP) and (NKD) are equal. Furthermore, if the assumptions of Theorem 5.4 are satisfied for all feasible solutions of (NKP) and (NKD), then $(\bar x^1,\bar y^1,\bar x^2,\bar y^2,\bar z^1,\bar z^2,\bar\lambda,\bar q=0,\bar s=0)$ is an efficient solution for (NKD).

Proof. Since $(\bar x^1,\bar y^1,\bar x^2,\bar y^2,\bar w^1,\bar w^2,\bar\lambda,\bar p,\bar r)$ is a weak efficient solution of (NKP), there exist $\alpha\in R^l_+$, $\beta\in R^{|K_1|}$, $\gamma\in R^{|K_2|}$, $\delta\in R_+$, $\mu\in R_+$, $\nu\in R^l_+$, $\eta\in R$ and $\psi\in R^l_+$ such that the following by Fritz John optimality conditions [114] are satisfied at $(\bar x^1,\bar y^1,\bar x^2,\bar y^2,\bar w^1,\bar w^2,\bar\lambda,\bar p,\bar r)$:

$$\sum_{i=1}^{l}\alpha_i\big(\nabla_{x^1} f_i\left(\bar x^1,\bar y^1\right) + D\bar z^1\big) + \sum_{i=1}^{l}\bar\lambda_i\nabla_{y^1x^1} f_i\left(\bar x^1,\bar y^1\right)\left[\beta - (\alpha^T e_l)\bar y^1\right]$$
$$+\sum_{i=1}^{l}\bar\lambda_i\left(\nabla_{x^1}\left(\nabla_{y^1y^1} f_i\left(\bar x^1,\bar y^1\right)\bar p\right)\right)\left[\beta - (\alpha^T e_l)(\bar y^1 + \frac{1}{2}\bar p)\right] = 0, \tag{5.72}$$

$$\sum_{i=1}^{l}\alpha_i\left[\nabla_{x^2} g_i(\bar x^2,\bar y^2) + B_i\bar z_i^2 - \frac{1}{2}\left(\nabla_{x^2}\left(\nabla_{y^2y^2} g_i\left(\bar x^2,\bar y^2\right)\bar r^i\right)\right)\bar r^i\right]$$
$$+\sum_{i=1}^{l}\bar\lambda_i\Big[\nabla_{y^2x^2} g_i\left(\bar x^2,\bar y^2\right) + \nabla_{x^2}\left(\nabla_{y^2y^2} g_i\left(\bar x^2,\bar y^2\right)\bar r^i\right)\Big][\gamma - \delta\bar y^2] = 0, \tag{5.73}$$

$$\begin{aligned}
&\sum_{i=1}^{l}\alpha_i\nabla_{y^1}f_i(\bar{x}^1,\bar{y}^1)+\sum_{i=1}^{l}\bar{\lambda}_i\nabla_{y^1y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\left[\beta-\left(\alpha^Te_l\right)\bar{y}^1\right]\\
&+\sum_{i=1}^{l}\bar{\lambda}_i\left[\nabla_{y^1}\left(\nabla_{y^1y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\bar{p}\right)\right]\left[\beta-\left(\alpha^Te_l\right)\left(\bar{y}^1+\frac{1}{2}\bar{p}\right)\right]-\sum_{i=1}^{l}\bar{\lambda}_i\Big[\nabla_{y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\\
&\qquad+\nabla_{y^1y^1}f_i\left(\bar{x}^1,\bar{y}^1\right)\bar{p}\Big]\left(\alpha^Te_l\right)=0,
\end{aligned}\tag{5.74}$$

$$\begin{aligned}
&\sum_{i=1}^{l}\alpha_i\left[\nabla_{y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)-C_i\bar{w}_i^2-\frac{1}{2}\left(\nabla_{y^2}\left(\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\bar{r}^i\right)\right]\\
&\qquad+\sum_{i=1}^{l}\bar{\lambda}_i\left[\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)+\nabla_{y^2}\left(\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\right]\left[\gamma-\delta\bar{y}^2\right]\\
&\qquad\qquad-\delta\sum_{i=1}^{l}\bar{\lambda}_i\left(\nabla_{y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)-C_i\bar{w}_i^2+\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)=0,
\end{aligned}\tag{5.75}$$

$$\left(-\beta E+\mu E\bar{w}^1\right)=0,\tag{5.76}$$

$$\alpha_iC_i\bar{y}^2+\bar{\lambda}_iC_i\left(\gamma-\delta\bar{y}^2\right)=2\nu_iC_i\bar{w}_i^2,\qquad i=1,2,\ldots,l,\tag{5.77}$$

$$\begin{aligned}
&\left[\beta-\left(\alpha^Te_l\right)\bar{y}^1\right]\nabla_{y^1}f\left(\bar{x}^1,\bar{y}^1\right)+\left[\gamma-\delta\bar{y}^2\right]\nabla_{y^2}g(\bar{x}^2,\bar{y}^2)-\xi+\eta e_l\\
&+\left\{\left[\beta-\left(\alpha^Te_l\right)\left(\bar{y}^1+\frac{1}{2}\bar{p}\right)\right]^T\nabla_{y^1y^1}f_1\left(\bar{x}^1,\bar{y}^1\right)\bar{p},\ldots,\left[\beta-\left(\alpha^Te_l\right)\left(\bar{y}^1+\frac{1}{2}\bar{p}\right)\right]^T\right.\\
&\qquad\left.\nabla_{y^1y^1}f_l\left(\bar{x}^1,\bar{y}^1\right)\bar{p}\right\}+\left\{\left[\gamma-\delta\bar{y}^2\right]^T\left(\nabla_{y^2y^2}g_1(\bar{x}^2,\bar{y}^2)\bar{r}^1-C_1\bar{w}_1^2\right),\right.\\
&\qquad\qquad\left.\ldots,\left[\gamma-\delta\bar{y}^2\right]^T\left(\nabla_{y^2y^2}g_l\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^l-C_l\bar{w}_l^2\right)\right\}=0,
\end{aligned}\tag{5.78}$$

$$\nabla_{y^1y^1}(\bar{\lambda}^Tf)(\bar{x}^1,\bar{y}^1)\left[\beta-\left(\alpha^Te_l\right)\left(\bar{y}^1+\bar{p}\right)\right]=0,\tag{5.79}$$

$$\left[\left(\gamma-\delta\bar{y}^2\right)\bar{\lambda}_i-\alpha_i\bar{r}^i\right]^T\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)=0,\qquad i=1,2,\ldots,l,\tag{5.80}$$

$$\beta^T\sum_{i=1}^{l}\bar{\lambda}_i[\nabla_{y^1}f_i(\bar{x}^1,\bar{y}^1)-E\bar{w}^1+\nabla_{y^1y^1}f_i(\bar{x}^1,\bar{y}^1)\bar{p}]=0,\tag{5.81}$$

$$\gamma^T\sum_{i=1}^{l}\bar{\lambda}_i[\nabla_{y^2}g_i(\bar{x}^2,\bar{y}^2)-C_i\bar{w}_i^2+\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i]=0,\tag{5.82}$$

$$\delta(\bar{y}^2)^T \sum_{i=1}^{l} \bar{\lambda}_i \left(\nabla_{y^2} g_i(\bar{x}^2, \bar{y}^2) - C_i \bar{w}_i^2 + \nabla_{y^2 y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}^i\right) = 0, \tag{5.83}$$

$$(\bar{x}^1)^T D \bar{z}^1 = \left((\bar{x}^1)^T D \bar{x}^1\right)^{\frac{1}{2}}, \tag{5.84}$$

$$(\bar{x}^2)^T B_i \bar{z}_i^2 = \left((\bar{x}^2)^T B_i \bar{x}^2\right)^{\frac{1}{2}}, \qquad i = 1, 2, \ldots, l, \tag{5.85}$$

$$(\bar{z}^1)^T D \bar{z}^1 \leqq 1, \tag{5.86}$$

$$(\bar{z}_i^2)^T B_i \bar{z}_i^2 \leqq 1, \qquad i = 1, 2, \ldots, l, \tag{5.87}$$

$$\mu\left((\bar{w}^1)^T E \bar{w}^1 - 1\right) = 0, \tag{5.88}$$

$$\nu_i\left((\bar{w}_i^2)^T C_i \bar{w}_i^2 - 1\right) = 0, \qquad i = 1, 2, \ldots, l, \tag{5.89}$$

$$\eta\left(\bar{\lambda}^T e_l - 1\right) = 0, \tag{5.90}$$

$$\xi^T \bar{\lambda} = 0, \tag{5.91}$$

$$(\alpha, \beta, \gamma, \delta, \mu, \nu, \xi) \geqq 0, \quad (\alpha, \beta, \gamma, \delta, \mu, \nu, \eta, \xi) \neq 0. \tag{5.92}$$

Since $\bar{\lambda} > 0$ and $\xi \geqq 0$, (5.91) implies $\xi = 0$.

By hypothesis (i), (5.79) gives

$$\beta = (\alpha^T e_l)(\bar{y}^1 + \bar{p}). \tag{5.93}$$

Since the matrices $\nabla_{y^2 y^2} g_i\,(\bar{x}^2, \bar{y}^2)$ for $i = 1, 2, \ldots, l$ are non singular, (5.80) yields

$$(\gamma - \delta\bar{y}^2)\bar{\lambda}_i = \alpha_i \bar{r}^i, \quad i = 1, 2, \ldots, l. \tag{5.94}$$

Now, we claim that $\alpha_i \neq 0,\ i = 1, 2, \ldots, l$. Otherwise, if for some k_0, $\alpha_{k_0} = 0$, then it follows from $\lambda_{k_0} > 0$, and (5.94) that

$\gamma = \delta\bar{y}^2$.

From (5.75), we get

$$\sum_{i=1}^{l} \left(\alpha_i - \delta\bar{\lambda}_i\right)\left(\nabla_{y^2} g_i\,(\bar{x}^2, \bar{y}^2) - C_i \bar{w}_i^2\right) + \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{y^2 y^2} g_i\,(\bar{x}^2, \bar{y}^2)\left[\gamma - \delta\bar{y}^2 - \delta\bar{r}^i\right]$$
$$+ \sum_{i=1}^{l} \nabla_{y^2}\left(\nabla_{y^2 y^2} g_i\,(\bar{x}^2, \bar{y}^2)\,\bar{r}^i\right)\left[\left(\gamma - \delta\bar{y}^2\right)\bar{\lambda}_i - \tfrac{1}{2}\alpha_i \bar{r}^i\right] = 0,$$

By using (5.94), it follows that

$$\sum_{i=1}^{l}\left(\alpha_i-\delta\bar{\lambda}_i\right)\left[\nabla_{y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)-C_i\bar{w}_i^2+\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right]$$
$$+\frac{1}{2}\sum_{i=1}^{l}\bar{\lambda}_i\left[\nabla_{y^2}\left(\nabla_{y^2y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)\bar{r}^i\right)\right]\left[\gamma-\delta\bar{y}^2\right]=0,$$

which on using hypothesis (iii) and $\gamma=\delta\bar{y}^2$ yields

$\alpha_i=\delta\bar{\lambda}_i,\quad i=1,2,\ldots,l.$

As $\bar{\lambda}_i>0,\quad i=1,2,\ldots,l$ and $\alpha_{k_0}=0$, for some k_0, the above equation shows $\delta=0$. Now, using $\delta=0$, the equations $\gamma=\delta\bar{y}^2$ and $\alpha_i=\delta\bar{\lambda}_i,\ i=1,2,\ldots,l$ give $\gamma=0$ and $\alpha_i=0,\ \forall\ i$. Further, (5.93) and (5.78) implies $\beta=0$ and $\eta=0$, respectively.

Also from equation (5.76) and (5.88), we have

$\bar{\mu}=\bar{\mu}\big((\bar{w}^1)^TE\bar{w}^1\big)=(\bar{w}^1)^T(\bar{\mu}E\bar{w}^1)=(\bar{w}^1)^T(E\beta)=0.$

From (5.77) and (5.89), we get

$\nu_i=0,\qquad i=1,2,\ldots,l.$

Consequently, $(\alpha,\beta,\gamma,\delta,\mu,\nu,\eta,\xi)=0$, contradicting (5.92). Hence $\alpha_i>0,\ \forall\ i$.

Subtracting (5.83) from (5.82) yields

$$\left[\gamma-\delta\bar{y}^2\right]^T\sum_{i=1}^{l}\bar{\lambda}_i\left[\nabla_{y^2}g_i(\bar{x}^2,\bar{y}^2)-C_i\bar{w}_i^2+\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i\right]=0,$$

Using (5.94), we get

$$\sum_{i=1}^{l}\alpha_i(\bar{r}^i)^T\left[\nabla_{y^2}g_i(\bar{x}^2,\bar{y}^2)-C_i\bar{w}_i^2+\nabla_{y^2y^2}g_i(\bar{x}^2,\bar{y}^2)\bar{r}^i\right]=0.$$

By the hypothesis (iv) with $\alpha_i>0,\ i=1,2,\ldots,l$, we have

$$\bar{r}^i=0,\ \ i=1,2,\ldots,l. \tag{5.95}$$

As $\bar{\lambda}_i>0,\quad i=1,2,\ldots,l$, (5.94) yields

$$\gamma=\delta\bar{y}^2. \tag{5.96}$$

Using (5.95) and (5.96) in (5.75), we get

$$\sum_{i=1}^{l}\left(\alpha_i-\delta\bar{\lambda}_i\right)\left[\nabla_{y^2}g_i\left(\bar{x}^2,\bar{y}^2\right)-C_i\bar{w}_i^2\right]=0,$$

which on using hypothesis (iii) and (5.95) gives

$$\alpha_i=\delta\bar{\lambda}_i,\ \ i=1,2,\ldots,l. \tag{5.97}$$

From (5.97) and $\bar{\lambda}^T e_l = 1$, it is clear that $\alpha^T e_l = \delta(\bar{\lambda}^T e_l) = \delta$. Since $\alpha_i > 0$, $i = 1, 2, \ldots, l$, therefore

$$\delta > 0. \tag{5.98}$$

Now using (5.93) and $\alpha_i > 0$, $\forall\ i$ in (5.74), we get

$$\sum_{i=1}^{l} \bar{\lambda}_i \Big[\nabla_{y^1}(\nabla_{y^1 y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p})\Big]\bar{p} = \frac{-2}{\alpha^T e_l} \sum_{i=1}^{l} \nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1)[\alpha_i - (\alpha^T e_l)\bar{\lambda}_i]$$

which by hypothesis (v) and (vi) implies

$$\bar{p} = 0. \tag{5.99}$$

From (5.99) and (5.93), we obtain

$$\beta = (\alpha^T e_l)\bar{y}^1. \tag{5.100}$$

Using (5.97)-(5.100) in (5.72), we have

$$\sum_{i=1}^{l} \bar{\lambda}_i \big(\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) + D\bar{z}^1\big) = 0. \tag{5.101}$$

Moreover, equations (5.73) and (5.95)-(5.98) give

$$\sum_{i=1}^{l} \bar{\lambda}_i \big(\nabla_{x^2} g_i(\bar{x}^2, \bar{y}^2) + B_i \bar{z}_i^2\big) = 0. \tag{5.102}$$

and hence, we also have

$$(\bar{x}^2)^T \sum_{i=1}^{l} \bar{\lambda}_i \big(\nabla_{x^2} g_i(\bar{x}^2, \bar{y}^2) + B_i \bar{z}_i^2\big) = 0. \tag{5.103}$$

Thus $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{\lambda}, \bar{q} = 0, \bar{s} = 0)$ satisfies the dual constraints from (5.55) to (5.61) and so it is a feasible solution for the dual problem (NKD). Now let $\frac{2\nu_i}{\alpha_i} = a$. Then $a \geqq 0$ and from (5.77) and (5.96)

$$C_i \bar{y}^2 = a C_i \bar{w}_i^2 \tag{5.104}$$

which is condition for equality in the Schwartz inequality. Therefore
$(\bar{y}^2)^T C_i \bar{w}_i^2 = \big((\bar{y}^2)^T C_i \bar{y}^2\big)^{\frac{1}{2}} \big((\bar{w}_i^2)^T C_i \bar{w}_i^2\big)^{\frac{1}{2}}$.
In case $\nu_i > 0$, (5.89) gives $(\bar{w}_i^2)^T C_i \bar{w}_i^2 = 1$ and so $(\bar{y}^2)^T C_i \bar{w}_i^2 = \big((\bar{y}^2)^T C_i \bar{y}^2\big)^{\frac{1}{2}}$. In case $\nu_i = 0$, (5.104) gives $C_i \bar{y}^2 = 0$ and so $(\bar{y}^2)^T C_i \bar{w}_i^2 = \big((\bar{y}^2)^T C_i \bar{y}^2\big)^{\frac{1}{2}} = 0$. Thus in either case

$$(\bar{y}^2)^T C_i \bar{w}_i^2 = \big((\bar{y}^2)^T C_i \bar{y}^2\big)^{\frac{1}{2}}. \tag{5.105}$$

Also, (5.101) yields

$$(\bar{x}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i \big(\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1) + D\bar{z}^1\big) = 0.$$

From (5.84), we have

$$(\bar{x}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i \big(\nabla_{x^1} f_i(\bar{x}^1, \bar{y}^1)\big) = -(\bar{x}^1)^T D\bar{z}^1 = -\big((\bar{x}^1)^T D\bar{x}^1\big)^{\frac{1}{2}}. \tag{5.106}$$

Further, using $\alpha_i > 0,\ i = 1, 2, \ldots, l$, (5.99) and (5.100) in (5.81), we get

$$(\bar{y}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i \big(\nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1)\big) = (\bar{y}^1)^T E\bar{w}^1. \tag{5.107}$$

Equation (5.76), implies
$E\beta = \mu E\bar{w}^1$.
Using (5.100) in the above equation

$$E\bar{y}^1 = \frac{\mu}{\alpha^T e_l} E\bar{w}^1. \tag{5.108}$$

Since equation (5.108) is the condition for the Schwartz inequality to hold as equality, so
$(\bar{y}^1)^T E\bar{w}^1 = \big((\bar{y}^1)^T E\bar{y}^1\big)^{\frac{1}{2}} \big((\bar{w}^1)^T E\bar{w}^1\big)^{\frac{1}{2}}$.
In case $\mu > 0$, equation (5.88) implies $(\bar{w}^1)^T E\bar{w}^1 = 1$ and so $(\bar{y}^1)^T E\bar{w}^1 = \big((\bar{y}^1)^T E\bar{y}^1\big)^{\frac{1}{2}}$.
In case $\mu = 0$, equation (5.108) gives $E\bar{y}^1 = 0$ and so $(\bar{y}^1)^T E\bar{w}^1 = \big((\bar{y}^1)^T E\bar{y}^1\big)^{\frac{1}{2}} = 0$.

Thus in either case $(\bar{y}^1)^T E\bar{w}^1 = \big((\bar{y}^1)^T E\bar{y}^1\big)^{\frac{1}{2}}$.

Now equation (5.107) becomes

$$(\bar{y}^1)^T \sum_{i=1}^{l} \bar{\lambda}_i\big(\nabla_{y^1} f_i(\bar{x}^1, \bar{y}^1)\big) = (\bar{y}^1)^T E\bar{w}^1 = \big((\bar{y}^1)^T E\bar{y}^1\big)^{\frac{1}{2}}. \tag{5.109}$$

Therefore, using (5.85), (5.95), (5.99), (5.105), (5.106) and (5.109), we get

$$\begin{aligned}
&\Big\{ f_1(\bar{x}^1,\bar{y}^1)+\big((\bar{x}^1)^T D\bar{x}^1\big)^{\frac{1}{2}}+g_1(\bar{x}^2,\bar{y}^2)+\big((\bar{x}^2)^T B_1\bar{x}^2\big)^{\frac{1}{2}}-(\bar{y}^2)^T C_1\bar{w}_1^2-(\bar{y}^1)^T \sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{y^1} f_i(\bar{x}^1,\bar{y}^1)\\
&\quad +\nabla_{y^1y^1} f_i(\bar{x}^1,\bar{y}^1)\bar{p})-\tfrac{1}{2}\bar{p}^T \sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{y^1y^1} f_i(\bar{x}^1,\bar{y}^1)\bar{p})-\tfrac{1}{2}(\bar{r}^1)^T\nabla_{y^2y^2} g_1(\bar{x}^2,\bar{y}^2)\bar{r}^1,\ldots, f_l(\bar{x}^1,\bar{y}^1)\\
&\qquad +\big((\bar{x}^1)^T D\bar{x}^1\big)^{\frac{1}{2}}+g_l(\bar{x}^2,\bar{y}^2)+\big((\bar{x}^2)^T B_l\bar{x}^2\big)^{\frac{1}{2}}-(\bar{y}^2)^T C_l\bar{w}_l^2-(\bar{y}^1)^T \sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{y^1} f_i(\bar{x}^1,\bar{y}^1)\\
&\qquad\quad +\nabla_{y^1y^1} f_i(\bar{x}^1,\bar{y}^1)\bar{p})-\tfrac{1}{2}\bar{p}^T \sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{y^1y^1} f_i(\bar{x}^1,\bar{y}^1)\bar{p})-\tfrac{1}{2}(\bar{r}^l)^T\nabla_{y^2y^2} g_l(\bar{x}^2,\bar{y}^2)\bar{r}^l \Big\}\\
&= \Big\{ f_1(\bar{x}^1,\bar{y}^1)-\big((\bar{y}^1)^T E\bar{y}^1\big)^{\frac{1}{2}}+g_1(\bar{x}^2,\bar{y}^2)-\big((\bar{y}^2)^T C_1\bar{y}^2\big)^{\frac{1}{2}}+(\bar{x}^2)^T B_1\bar{z}_1^2-(\bar{x}^1)^T \sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1} f_i(\bar{x}^1,\bar{y}^1)\\
&\quad +\nabla_{x^1x^1} f_i(\bar{x}^1,\bar{y}^1)\bar{q})-\tfrac{1}{2}\bar{q}^T \sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1x^1} f_i(\bar{x}^1,\bar{y}^1)\bar{q})-\tfrac{1}{2}(\bar{s}^1)^T\nabla_{x^2x^2} g_1(\bar{x}^2,\bar{y}^2)\bar{s}^1,\ldots, f_l(\bar{x}^1,\bar{y}^1)\\
&\qquad -\big((\bar{y}^1)^T E\bar{y}^1\big)^{\frac{1}{2}}+g_l(\bar{x}^2,\bar{y}^2)-\big((\bar{y}^2)^T C_l\bar{y}^2\big)^{\frac{1}{2}}+(\bar{x}^2)^T B_l\bar{z}_l^2-(\bar{x}^1)^T \sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1} f_i(\bar{x}^1,\bar{y}^1)\\
&\qquad\quad +\nabla_{x^1x^1} f_i(\bar{x}^1,\bar{y}^1)\bar{q})-\tfrac{1}{2}\bar{q}^T \sum_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1x^1} f_i(\bar{x}^1,\bar{y}^1)\bar{q})-\tfrac{1}{2}(\bar{s}^l)^T\nabla_{x^2x^2} g_l(\bar{x}^2,\bar{y}^2)\bar{s}^l \Big\}
\end{aligned}$$

that is, the two objective function values are equal.

Now, let $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}^2, \bar{\lambda}, \bar{q} = 0, \bar{s} = 0)$ is not an efficient solution of (NKD), then there exist $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{z}^1, \bar{z}^2, \bar{\lambda}, \bar{q} = 0, \bar{s} = 0)$ feasible for (NKD), such that,

$$\begin{aligned}
&\Big\{ N_1\big(x^1, y^1, x^2, y^2, w_1^2, \lambda, p, r^1\big), \ldots, N_l\left(x^1, y^1, x^2, y^2, w_l^2, \lambda, p, r^l\right) \Big\}\\
&\qquad\qquad \leq \Big\{ K_1\big(u^1, v^1, u^2, v^2, z_1^2, \lambda, q, s^1\big), \ldots, K_l\left(u^1, v^1, u^2, v^2, z_l^2, \lambda, q, s^l\right) \Big\},
\end{aligned}$$

which contradicts Theorem 5.4. Hence $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{z}^1, \bar{z}_1^2, \ldots, \bar{z}_l^2, \bar{\lambda}_1, \ldots, \bar{\lambda}_l, \bar{q} = 0, \bar{s}^1 = 0, \ldots, \bar{s}^l = 0)$ is an efficient solution of (NKD).

Theorem 5.6 (Converse duality). Let $f : R^{|J_1|} \times R^{|K_1|} \to R^l$ and $g : R^{|J_2|} \times R^{|K_2|} \to R^l$ be differentiable functions and let $(\bar{u}^1, \bar{v}^1, \bar{u}^2, \bar{v}^2, \bar{z}^1, \bar{z}^2, \bar{\lambda}, \bar{q}, \bar{s})$ be a weak efficient solution of (NKD). Suppose that

(*i*) the matrix $\nabla_{x^1x^1}(\bar{\lambda}^T f)(\bar{u}^1, \bar{v}^1)$ is non singular, (*ii*) the matrices $\nabla_{x^2x^2}g_i\,(\bar{u}^2,\bar{v}^2)$ for $i=1,2,\ldots,l$ are non singular,

(*iii*) the set $\Big\{\nabla_{x^2}g_1(\bar{u}^2,\bar{v}^2)+B_1\bar{z}^1+\nabla_{x^2x^2}g_1(\bar{u}^2,\bar{v}^2)\bar{s}^1,\ldots,\nabla_{x^2}g_l(\bar{u}^2,\bar{v}^2)+B_l\bar{z}^l+\nabla_{x^2x^2}g_l(\bar{u}^2,\bar{v}^2)\bar{s}^l\Big\}$ is linearly independent,

(*iv*) for some $\varrho\in R^l_+$ $(\varrho>0)$ and $\bar{s}^i\in R^{|J_2|}$, $\bar{s}^i\neq 0$ $(i=1,2,\ldots,l)$ implies that $\sum\limits_{i=1}^{l}\varrho_i(\bar{s}^i)^T\Big[\nabla_{x^2}g_i(\bar{u}^2,\bar{v}^2)+B_i\bar{z}_i^2+\nabla_{x^2x^2}g_i(\bar{u}^2,\bar{v}^2)\bar{s}^i\Big]\neq 0$,

(*v*) $\sum\limits_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1}(\nabla_{x^1x^1}f_i(\bar{u}^1,\bar{v}^1)\bar{q}))\bar{q}\notin \operatorname{span}\{\nabla_{x^1}f_1(\bar{u}^1,\bar{v}^1),\ldots,\nabla_{x^1}f_l(\bar{u}^1,\bar{v}^1)\}\backslash\{0\}$, and

(*vi*) $\sum\limits_{i=1}^{l}\bar{\lambda}_i(\nabla_{x^1}(\nabla_{x^1x^1}f_i(\bar{u}^1,\bar{v}^1)\bar{q}))\bar{q}=0$ implies $\bar{q}=0$.

Then $\bar{q}=0$, $\bar{s}=0$, there exist $\bar{w}^1\in R^{|K_1|}$ and $\bar{w}_i^2\in R^{|K_2|}$, $i=1,2,\ldots,l$ such that $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{w}^1,\bar{w}^2,\bar{\lambda},\bar{p}=0,\bar{r}=0)$ is feasible for (NKP) and the objective function values of (NKP) and (NKD) are equal. Furthermore, if the assumptions of Theorem 5.4 are satisfied for all feasible solutions of (NKP) and (NKD), then $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{w}^1,\bar{w}^2,\bar{\lambda},\bar{p}=0,\bar{r}=0)$ is an efficient solution for (NKP).

Proof. It follows on the lines of Theorem 5.5.

5.4 Special cases

In this section, we consider some of the special cases of the problems studied in this chapter in Section 5.2 and Section 5.3.

(*i*) If $J_2=\emptyset$ and $K_2=\emptyset$, then our programs (MSP) and (MSD) gives the duality results studied in [13, 62].

(*ii*) If $J_2=\emptyset$, $K_2=\emptyset$, $D=\{0\}$ and $E=\{0\}$ in (NKP) and (NKD), then the programs (MP) and (MD) of [140] are obtained.

(*iii*) If $J_1=\emptyset$, $K_1=\emptyset$, $B_i=\{0\}$ and $C_i=\{0\}$, $i=1,2,\ldots,l$, then (NKP) and (NKD) become the programs studied in [124].

(*iv*) If $J_1 = \emptyset$ and $K_1 = \emptyset$ in (NKP) and (NKD), then the programs (MP) and (MD) of [10] are obtained with nonnegativity constraints.

For $C_1 = R_+^{|J_1|}, C_2 = R_+^{|J_2|}, C_3 = R_+^{|K_1|}$ and $C_4 = R_+^{|K_2|}$, in (MSP) and (MSD), we obtain the following special cases :

(*v*) If we take $J_2 = \emptyset$, $K_2 = \emptyset$ and $l = 1$, then our problems (MSP) and (MSD) reduce to the programs (SP) and (SD) studied in Gulati et al. [55] and if $J_1 = \emptyset$, $K_1 = \emptyset$ and $l = 1$ in (MSP) and (MSD), then the programs (SP1) and (SD1) of [55] are obtained.

(*vi*) If $J_2 = \emptyset$ and $K_2 = \emptyset$ in (MSP) and (MSD), then the programs (MP) and (MD) of [140] are obtained with $x^1 \geqq 0$ and $v^1 \geqq 0$.

Chapter 6

HIGHER-ORDER MULTIOBJECTIVE SYMMETRIC DUALITY OVER CONES[1]

6.1 Introduction

Higher-order duality in nonlinear programming has been studied in last few years by many researchers [3, 14, 43, 53, 89, 105]. One practical advantage of higher-order duality is that it provides tighter bounds for the value of objective function of the primal problem when approximations are used because there are more parameters involved. Mangasarian [89] first formulated a class of higher-order dual problems for nonlinear programming problems. Mond and Zhang [105] obtained duality results for various higher-order dual problems under higher-order invexity assumptions.

Yang et al. [138] formulated several second order duals for scalar programming problem and proved duality results involving generalized F-convex functions. Zhang and Mond [141] extended the class of (F, ρ)-convex functions to second order (F, ρ)-convex functions and obtained duality results for multiobjective dual problems. Motivated by various concepts of generalized convexity, Liang et al. [86, 87] introduced a unified formulation of generalized convexity, called (F, α, ρ, d)- convexity and obtained some optimality conditions and duality results for the single objective fractional and multiobjective problems. Ahmad and Husain [11] discussed duality

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theorems for second-order Mond-Weir type multiobjective dual model under second-order (F, α, ρ, d)-convexity/pseudo–convexity assumptions. Later on, Ahmad et al. [14] formulated a general Mond-Weir type higher-order dual for nondifferentiable multiobjective programming problem and established higher-order duality theorems.

Chen [43] studied Mond-Weir type higher-order symmetric duality for multiobjective nondifferentiable programms by introducing higher-order F-convexity. Recently, Agarwal et al. [3] extended the results of Chen [43] over arbitrary cones and proved appropriate duality relations under higher-order K-F-convexity assumptions. Mond-Weir type duality has been discussed in both the papers.

This chapter is divided into five sections. Section 6.2 contains notations and definitions used in this chapter. In Section 6.3, we give a non-trivial example of function lying in the class of higher-order K-(F, α, ρ, d)-convex but not in class of higher-order K-F-convex. In the next Section, we have studied higher-order Wolfe type and Mond-Weir type multiobjective symmetric dual programs over arbitrary cones. Weak strong and converse duality theorems are then proved under higher-order (F, α, ρ, d)-convexity/pseudo-convexity assumptions. In Section 6.5, a pair of higher-order multiobjective nondifferentiable symmetric dual programs over arbitrary cones is formulated and duality theorems are then proved under higher-order-K-(F, α, ρ, d)-convexity assumptions. Finally, in the last Section, we have discussed some special cases of the programs studied in this chapter to show that this study extends some known results of the literature.

6.2 Notations and definitions

Let $F : S \times S \times R^n \to R$ (where $S \subseteq R^n$) be a sublinear functional. Now we consider a function $\phi = (\phi_1, \phi_2, \ldots, \phi_k) : S \mapsto R^k$ differentiable at $u \in S$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in R^k$ and $d = (d_1, d_2, \ldots, d_k) \in R^k$.

Definition 6.1. A twice differentiable function ϕ_i over S is said to be higher-order (F, α, ρ_i, d_i)-convex at u on S with respect to $\zeta_i : S \times R^n \mapsto R$, if for all $x \in S$ and

$q \in R^n$, there exist a real valued function $\alpha : S \times S \to R_+ \backslash \{0\}$, a real valued function $d_i(\cdot,\cdot) : S \times S \to R$ and a real number ρ_i such that

$$\phi_i(x) - \phi_i(u) - \zeta_i(u,q) + q^T \nabla_q \zeta_i(u,q) \geqq F[x,u;\alpha(x,u)(\nabla_x \phi_i(u) + \nabla_q \zeta_i(u,q))] + \rho_i d_i^2(x,u).$$

Definition 6.2. A twice differentiable function ϕ_i over S is said to be higher-order (F,α,ρ_i,d_i)-pseudoconvex at u on S with respect to $\zeta_i : S \times R^n \mapsto R$, if for all $x \in S$ and $q \in R^n$, there exist a real valued function $\alpha : S \times S \to R_+ \backslash \{0\}$, a real valued function $d_i(\cdot,\cdot) : S \times S \to R$ and a real number ρ_i such that

$$F[x,u;\alpha(x,u)(\nabla_x \phi_i(u) + \nabla_q \zeta_i(u,q))] + \rho_i d_i^2(x,u) \geqq 0$$
$$\Rightarrow \phi_i(x) - \phi_i(u) - \zeta_i(u,q) + q^T \nabla_q \zeta_i(u,q) \geqq 0.$$

A twice differentiable vector function $\phi : S \mapsto R^k$ is said to be higher-order (F,α,ρ,d)-convex/pseudoconvex at u, if each of its components ϕ_i is higher-order (F,α,ρ_i,d_i)-convex/pseudoconvex at u.

Definition 6.3. A twice differentiable function $\phi : S \to R^k$ is said to be higher-order-K-F-convex at u on S with respect to $\zeta : S \times R^n \mapsto R^k$, if for all $x \in S$ and $q \in R^n$ such that

$$\Big\{\phi_1(x) - \phi_1(u) - \zeta_1(u,q) + q^T \nabla_q \zeta_1(u,q) - F[x,u;\nabla_x \phi_1(u) + \nabla_q \zeta_1(u,q)], \ldots,$$
$$\phi_k(x) - \phi_k(u) - \zeta_k(u,q) + q^T \nabla_q \zeta_k(u,q) - F[x,u;\nabla_x \phi_k(u) + \nabla_q \zeta_k(u,q)]\Big\} \in K.$$

Definition 6.4. A twice differentiable function $\phi : S \to R^k$ is said to be higher-order-K-(F,α,ρ,d)-convex at u on S with respect to $\zeta : S \times R^n \mapsto R^k$, if for all $x \in S$ and $q \in R^n$, there exist vector $\rho \in R^k$, a real valued function $\alpha : S \times S \to R_+ \backslash \{0\}$ and $d : S \times S \to R^k$ such that

$$\Big\{\phi_1(x) - \phi_1(u) - \zeta_1(u,q) + q^T \nabla_q \zeta_1(u,q) - F[x,u;\alpha(x,u)(\nabla_x \phi_1(u) + \nabla_q \zeta_1(u,q))]$$
$$-\rho_1 d_1^2(x,u), \ldots, \phi_k(x) - \phi_k(u) - \zeta_k(u,q) + q^T \nabla_q \zeta_k(u,q) - F[x,u;\alpha(x,u)(\nabla_x \phi_k(u)$$
$$+\nabla_q \zeta_k(u,q))] - \rho_k d_k^2(x,u)\Big\} \in K.$$

Remarks

(i) If $k = 1$ and $\zeta(u,q) = 0$, then the definition of higher-order (F,α,ρ_i,d_i)-convexity become that of (F,α,ρ,d)-convex functions introduced by Liang et al. [87].

(*ii*) If $K = R_+$, $\alpha(x,u) = 1$ and $\zeta(u,q) = \frac{1}{2}q^T\nabla_{xx}\phi(u)q$, the definition of higher-order-K-(F,α,ρ,d)-convexity reduce to second-order (F,ρ)-convexity given by Srivastava and Bhatia [120].

(*iii*) If $K = R_+$, $\alpha(x,u) = 1$ and $\rho = 0$, then higher-order-K-(F,α,ρ,d)-convexity reduce to higher-order F-convexity (see [43, 58]).

(*iv*) If we take $K = R_+$, $\alpha(x,u) = 1$, $\rho = 0$, $\zeta(u,q) = \frac{1}{2}q^T\nabla_{xx}\phi(u)q$ and $F_{x,u}(a) = \eta(x,u)^Ta$, where η is a function from $S \times S$ to R^n, the definition of higher-order-K-(F,α,ρ,d)-convexity becomes that of second-order η-convexity given in [124].

6.3 Example

An example of non trivial function which is higher-order K-(F,α,ρ,d)-convex but not higher-order K-F-convex.

Let $X = [-2.5, -0.5] \subset R$, $n = m = 1$, $k = 2$, $K = \{(x,y) : x \geqq 0, y \geqq 0\}$. Consider the function $\psi : X \to R^2$ be defined by $\psi(x) = (\psi_1(x), \psi_2(x))$, where

$\psi_1(x) = x^3 \sin\dfrac{2}{x}$, $\psi_2(x) = x^3$,

and $\alpha : X \times X \to R_+\backslash\{0\}$ be identified by $\alpha(x,u) = (u^2+1)$. Let the functional $F : X \times X \times R \to R$ be defined by $F(x,u;a) = 12(1-u)a$. Suppose $d : X \times X \to R^2$ be given by $d(x,u) = (d_1(x,u), d_2(x,u))$, where

$d_1(x,u) = (x^4+u^2)^{\frac{3}{2}}$, $d_2(x,u) = (x^2+u^2)^{\frac{5}{2}}$

and $\zeta : X \times R \to R^2$ be defined by $\zeta(u,q) = (\zeta_1(u,q), \zeta_2(u,q))$, where

$\zeta_1(u,q) = 15u + 12q$, $\zeta_2(u,q) = \sin^2 u + q^2$.

For $\rho_1 = -4$ and $\rho_2 = -28$, we have

$$L = \Big\{\psi_1(x) - \psi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q) - F[x,u;\alpha(x,u)(\nabla_x\psi_1(u) + \nabla_q\zeta_1(u,q))]$$
$$-\rho_1 d_1^2(x,u), \psi_2(x) - \psi_2(u) - \zeta_2(u,q) + q^T\nabla_q\zeta_2(u,q) - F[x,u;\alpha(x,u)(\nabla_x\psi_2(u)$$

$$
\begin{aligned}
&\qquad\qquad\left.+\nabla_q\zeta_2(u,q))] - \rho_2 d_2^2(x,u)\right\} \in K.\\
&= \Big\{x^3\sin\tfrac{2}{x} - u^3\sin\tfrac{2}{u} - 15u - 12(1-u)\left[(u^2+1)(-2u\cos\tfrac{2}{u} + 3u^2\sin\tfrac{2}{u} + 12)\right]\\
&\qquad + 4(x^4+u^2)^3, x^3 - u^3 - \sin^2 u + q^2 - 12(1-u)\left[(u^2+1)(3u^2+2q)\right]\\
&\qquad\qquad \left.+ 28(x^2+u^2)^5\right\} \in K,\\
&= \{L_1, L_2\} \in K,
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 &= x^3\sin\tfrac{2}{x} - u^3\sin\tfrac{2}{u} - 15u - 12(1-u)\left[(u^2+1)(-2u\cos\tfrac{2}{u} + 3u^2\sin\tfrac{2}{u} + 12)\right]\\
&\qquad + 4(x^4+u^2)^3\\
&\geqq 0 \ \forall\ x, u \in X \text{ as can be seen from Figure 6.1}
\end{aligned}
$$

and

$$
\begin{aligned}
L_2 &= x^3 - u^3 - \sin^2 u + q^2 - 12(1-u)\left[(u^2+1)(3u^2+2q)\right] + 28(x^2+u^2)^5\\
&= L_{21} + L_{22}.
\end{aligned}
$$

Now $L_{21} = x^3 - u^3 - \sin^2 u - 12(1-u)\left[3u^2(u^2+1)\right] + 28(x^2+u^2)^5$

$$\geqq 0 \ \forall\ x, u \in X \text{ as can be seen from Figure 6.2}$$

and

$$
\begin{aligned}
L_{22} &= q^2 - 12(1-u)\left[2q(u^2+1)\right]\\
&\geqq 0 \ \forall\ u \in X \text{ and } q \in (-10^{18}, 10^{18}) \text{ as can be seen from Figure 6.3.}
\end{aligned}
$$

Thus $L_2 \geqq 0$. Therefore ψ is higher-order K- (F, α, ρ, d)-convex with respect to ζ.

Next, we will show that ψ is not higher-order K-F-convex with respect to ζ. To prove it, we will show that

$$
\begin{aligned}
M = \Big\{&\psi_1(x) - \psi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q) - F[x,u;\nabla_x\psi_1(u) + \nabla_q\zeta_1(u,q)], \psi_2(x)\\
&\left.-\psi_2(u) - \zeta_2(u,q) + q^T\nabla_q\zeta_2(u,q) - F[x,u;\nabla_x\psi_2(u) + \nabla_q\zeta_2(u,q)]\right\} \notin K
\end{aligned}
$$

i.e., either

$$\psi_1(x) - \psi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q) - F[x,u;\nabla_x\psi_1(u) + \nabla_q\zeta_1(u,q)] \ngeqq 0$$

or $\quad \psi_2(x) - \psi_2(u) - \zeta_2(u,q) + q^T\nabla_q\zeta_2(u,q) - F[x,u;\nabla_x\psi_2(u) + \nabla_q\zeta_2(u,q)] \ngeqq 0.$

Since

$$N = \psi_1(x) - \psi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q) - F[x,u;\nabla_x\psi_1(u) + \nabla_q\zeta_1(u,q)]$$

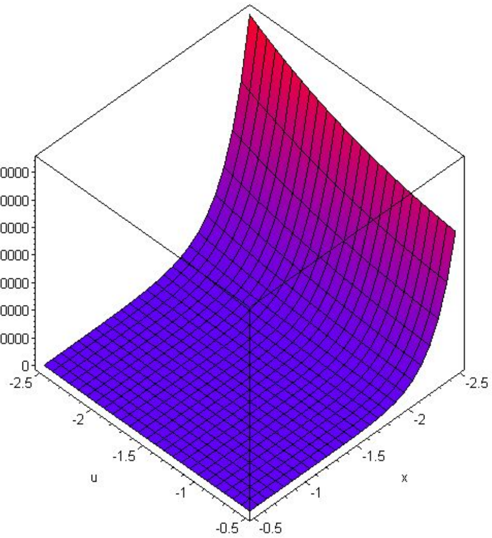

Figure 6.1: Graph of L_1 against u and x

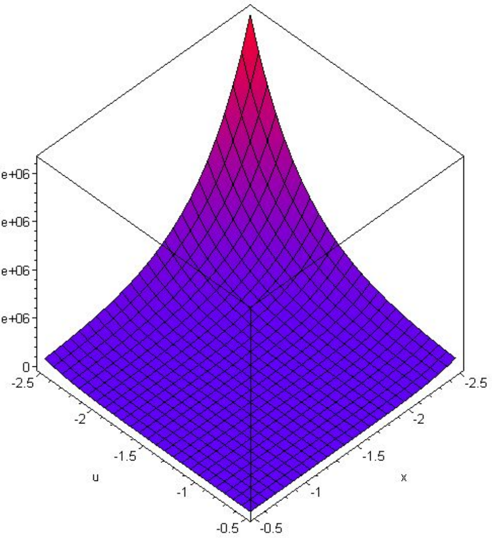

Figure 6.2: Graph of L_{21} against u and x

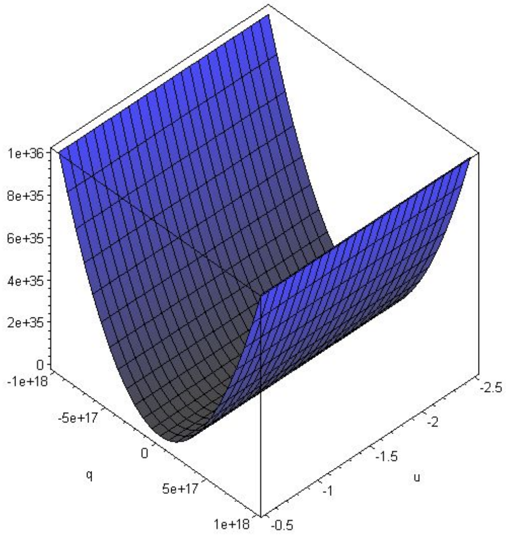

Figure 6.3: Graph of L_{22} against q and u

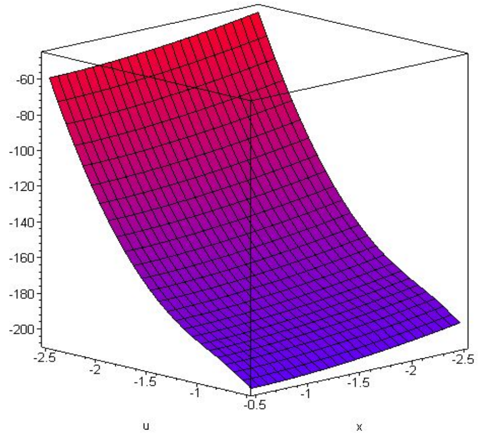

Figure 6.4: Graph of N against u and x

$$= x^3 \sin \tfrac{2}{x} - u^3 \sin \tfrac{2}{u} - 15u - 12(1-u) \left[-2u \cos \tfrac{2}{u} + 3u^2 \sin \tfrac{2}{u} + 12\right]$$

$< 0 \; \forall \; x, u \in X$ as be can seen from Figure 6.4.

Therefore, ψ is not higher-order K-F-convex with respect to ζ.

6.4 Higher-order symmetric duality in multiobjective programming

6.4.1 Wolfe type higher-order symmetric duality

In this section, we consider the following Wolfe type multiobjective higher-order symmetric dual programs:

Primal Problem (HWP)

minimize $\quad L(x, y, \lambda, p) = f(x, y) + (\lambda^T h)(x, y, p)e_k - p^T \nabla_p (\lambda^T h)(x, y, p)e_k$
$\qquad -y^T \nabla_y (\lambda^T f)(x, y)e_k - y^T \nabla_p (\lambda^T h)(x, y, p)e_k$

subject to

$$-\{\nabla_y (\lambda^T f)(x, y) + \nabla_p (\lambda^T h)(x, y, p)\} \in C_2^*, \tag{6.1}$$

$$\lambda^T e_k = 1, \tag{6.2}$$

$$\lambda > 0, \;\; x \in C_1. \tag{6.3}$$

Dual Problem (HWD)

maximize $\quad M(u, v, \lambda, r) = f(u, v) + (\lambda^T g)(u, v, r)e_k - r^T \nabla_r (\lambda^T g)(u, v, r)e_k$
$\qquad -u^T \nabla_x (\lambda^T f)(u, v)e_k - u^T \nabla_r (\lambda^T g)(u, v, r)e_k$

subject to

$$\nabla_x (\lambda^T f)(u, v) + \nabla_r (\lambda^T g)(u, v, r) \in C_1^*, \tag{6.4}$$

$$\lambda^T e_k = 1, \tag{6.5}$$

$$\lambda > 0, \;\; v \in C_2, \tag{6.6}$$

where

(i) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$, (*ii*) $f : S_1 \times S_2 \to R^k$, $h : S_1 \times S_2 \times R^m \to R^k$ and $g : S_1 \times S_2 \times R^n \to R^k$ are differentiable functions, $e_k = (1, \ldots, 1)^T \in R^k$, $\lambda \in R^k$ and

(*iii*) r and p are vectors in R^n and R^m, respectively.

Theorem 6.1 (Weak duality). Let (x, y, λ, p) be feasible for the primal problem (HWP) and (u, v, λ, r) be feasible for the dual problem (HWD). Let for $i = 1, 2, \ldots, k$

(*i*) $f_i(., v)$ be higher-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convex at u with respect to $g_i(u, v, r)$,

(*ii*) $-f_i(x, .)$ be higher-order $(G, \alpha_2, \rho_i^{(2)}, d_i^{(2)})$-convex at y with respect to $-h_i(x, y, p)$,

(*iii*) either (a) $\sum\limits_{i=1}^{k} \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x, u))^2 + \rho_i^{(2)}(d_i^{(2)}(v, y))^2] \geqq 0$ or (b) $\rho_i^{(1)} \geqq 0$ and $\rho_i^{(2)} \geqq 0$, for all i,

where the sublinear functionals $F : R^n \times R^n \times R^n \mapsto R$ and $G : R^m \times R^m \times R^m \mapsto R$ satisfy the following conditions:

(*iv*) $F(x, u; a) + \alpha_1^{-1} a^T u \geqq 0$, for all $a \in C_1^*$ and

(*v*) $G(v, y; b) + \alpha_2^{-1} b^T y \geqq 0$, for all $b \in C_2^*$.

Then

$$L(x, y, \lambda, p) \nleq M(u, v, \lambda, r). \tag{6.7}$$

Proof. Assume by contradiction that (6.7) is not true, that is ,
$L(x, y, \lambda, p) \leq M(u, v, \lambda, r)$, or
$f(x, y) + (\lambda^T h)(x, y, p)e_k - p^T \nabla_p(\lambda^T h)(x, y, p)e_k - y^T \nabla_y(\lambda^T f)(x, y)e_k - y^T \nabla_p(\lambda^T h)(x, y, p)e_k$
$\leq f(u, v) + (\lambda^T g)(u, v, r)e_k - r^T \nabla_r(\lambda^T g)(u, v, r)e_k - u^T \nabla_x(\lambda^T f)(u, v)e_k - u^T \nabla_r(\lambda^T g)(u, v, r)e_k.$
Since $\lambda > 0$ and $\lambda^T e_k = 1$, we obtain

$$\begin{aligned}(\lambda^T f)(x, y) &+ (\lambda^T h)(x, y, p) - p^T \nabla_p(\lambda^T h)(x, y, p) \\ &- y^T \nabla_y(\lambda^T f)(x, y) - y^T \nabla_p(\lambda^T h)(x, y, p) < (\lambda^T f)(u, v) + (\lambda^T g)(u, v, r) \\ &- r^T \nabla_r(\lambda^T g)(u, v, r) - u^T \nabla_x(\lambda^T f)(u, v) - u^T \nabla_r(\lambda^T g)(u, v, r).\end{aligned} \tag{6.8}$$

Since (x, y, λ, p) is feasible for the primal problem (HWP) and (u, v, λ, r) is feasible for the dual problem (HWD), $\alpha_1(x, u) > 0$, by the dual constraint (6.4), the vector $a = \alpha_1(x, u)[\nabla_x(\lambda^T f)(u, v) + \nabla_r(\lambda^T g)(u, v, r)] \in C_1^*$ and so from the hypothesis (iv), we obtain

$$F(x, u; a) + \alpha_1^{-1} a^T u \geqq 0. \tag{6.9}$$

Similarly,

$$G(v, y; b) + \alpha_2^{-1} b^T y \geqq 0, \tag{6.10}$$

for the vector $b = -\alpha_2(v, y)[\nabla_y(\lambda^T f)(x, y) + \nabla_p(\lambda^T h)(x, y, p)] \in C_2^*$.

By higher-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convexity of $f_i(., v)(1 \leqq i \leqq k)$ with respect to $g_i(u, v, r)$, we have

$$f_i(x, v) - f_i(u, v) - g_i(u, v, r) + r^T \nabla_r g_i(u, v, r)$$
$$\geqq F[x, u; \alpha_1(x, u)(\nabla_x f_i(u, v) + \nabla_r g_i(u, v, r))] + \rho_i^{(1)}(d_i^{(1)}(x, u))^2.$$

It follows from $\lambda > 0$ and sublinearity of F that

$$(\lambda^T f)(x, v) - (\lambda^T f)(u, v) - (\lambda^T g)(u, v, r) + r^T \nabla_r(\lambda^T g)(u, v, r)$$
$$\geqq F[x, u; \alpha_1(x, u)(\nabla_x(\lambda^T f)(u, v) + \nabla_r(\lambda^T g)(u, v, r))] + \sum_{i=1}^{k} \lambda_i \rho_i^{(1)}(d_i^{(1)}(x, u))^2,$$

or

$$(\lambda^T f)(x, v) - (\lambda^T f)(u, v) - (\lambda^T g)(u, v, r)$$
$$+ r^T \nabla_r(\lambda^T g)(u, v, r) - \sum_{i=1}^{k} \lambda_i \rho_i^{(1)}(d_i^{(1)}(x, u))^2 \geqq F(x, u; a). \tag{6.11}$$

Using (6.9) in (6.11), we have

$$(\lambda^T f)(x, v) - (\lambda^T f)(u, v) - (\lambda^T g)(u, v, r)$$
$$+ r^T \nabla_r(\lambda^T g)(u, v, r) - \sum_{i=1}^{k} \lambda_i \rho_i^{(1)}(d_i^{(1)}(x, u))^2 \geqq -\alpha_1^{-1} u^T a. \tag{6.12}$$

Similarly, using hypotheses (ii) and (v) along with primal constraint (6.1) and inequality (6.10), $\lambda > 0$, $\alpha_2(v, y) > 0$ and sublinearity of G, we get

$$(\lambda^T f)(x, y) - (\lambda^T f)(x, v) + (\lambda^T h)(x, y, p) \\ -p^T \nabla_p (\lambda^T h)(x, y, p) - \sum_{i=1}^{k} \lambda_i \rho_i^{(2)} (d_i^{(2)}(v, y))^2 \geqq -\alpha_2^{-1} y^T b. \tag{6.13}$$

Adding the inequalities (6.12) and (6.13), we obtain

$$(\lambda^T f)(x,y)+(\lambda^T h)(x,y,p)-p^T\nabla_p(\lambda^T h)(x,y,p)+\alpha_2^{-1}y^T b-(\lambda^T f)(u,v)-(\lambda^T g)(u,v,r) \\ +r^T\nabla_r(\lambda^T g)(u,v,r)+\alpha_1^{-1}u^T a \geqq \sum_{i=1}^{k} \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2+\rho_i^{(2)}(d_i^{(2)}(v,y))^2].$$

Using hypothesis (iii) in the above inequality, we get

$$(\lambda^T f)(x, y) + (\lambda^T h)(x, y, p) - p^T \nabla_p (\lambda^T h)(x, y, p) + \alpha_2^{-1} y^T b \\ \geqq (\lambda^T f)(u, v) + (\lambda^T g)(u, v, r) - r^T \nabla_r (\lambda^T g)(u, v, r) - \alpha_1^{-1} u^T a.$$

Finally, substituting the values of a and b, we have

$$(\lambda^T f)(x,y)+(\lambda^T h)(x,y,p)-p^T\nabla_p(\lambda^T h)(x,y,p)-y^T\nabla_y(\lambda^T f)(x,y)-y^T\nabla_p(\lambda^T h)(x,y,p) \\ \geqq (\lambda^T f)(u,v)+(\lambda^T g)(u,v,r)-r^T\nabla_r(\lambda^T g)(u,v,r)-u^T\nabla_x(\lambda^T f)(u,v)-u^T\nabla_r(\lambda^T g)(u,v,r),$$

which contradicts (6.8). Hence the result.

If the variable λ in the problems (HWP) and (HWD) is fixed to be $\bar{\lambda}$, we shall denote these problems by $(HWP)_{\bar{\lambda}}$ and $(HWD)_{\bar{\lambda}}$.

Theorem 6.2 (Strong duality). Let $f : S_1 \times S_2 \to R^k$ be twice differentiable function and let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ be a weak efficient solution of (HWP). Suppose that

(i) the matrix $\nabla_{pp}(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})$ is non singular,

(ii) the vectors $\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})$ are linearly independent,

(iii) the vector $\{\nabla_y(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p}\}$ $\notin \text{span}\{\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\}\backslash\{0\}$,

(iv) $\nabla_y(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p} = 0$ implies $\bar{p} = 0$ and

(v) $(\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = (\bar{\lambda}^T g)(\bar{x}, \bar{y}, 0)$, $\nabla_x(\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = \nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, 0)$, $\nabla_y(\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = 0$ and $\nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = 0$.

Then $\bar{p}=0$, $(\bar{x},\bar{y},\bar{r}=0)$ is feasible for $(HWD)_{\bar{\lambda}}$, and the objective values of (HWP) and $(HWD)_{\bar{\lambda}}$ are equal. Furthermore, if the hypotheses of Theorem 6.1 are satisfied for all feasible solutions of (HWP) and $(HWD)_{\bar{\lambda}}$, then $(\bar{x},\bar{y},\bar{r}=0)$ is an efficient solution for $(HWD)_{\bar{\lambda}}$.

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ is a weak efficient solution of (HWP), by the Fritz John necessary optimality conditions [123], there exist $\bar{\alpha}\in R^k_+$, $\bar{\beta}\in C_2$, $\bar{\mu}\in R^k_+$, $\bar{\eta}\in R$, such that the following conditions are satisfied at $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$:

$$\begin{aligned}&\{\bar{\alpha}^T\nabla_x f(\bar{x},\bar{y})+\nabla_x(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})(\bar{\alpha}^T e_k)+\nabla_{xy}(\bar{\lambda}^T f)(\bar{x},\bar{y})[\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}]\\&\quad+\nabla_{px}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})[\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\bar{p})]\}(x-\bar{x})\geqq 0,\quad \text{for all}\ \ x\in C_1,\end{aligned}\tag{6.14}$$

$$\begin{aligned}&\nabla_y f(\bar{x},\bar{y})[\bar{\alpha}-(\bar{\alpha}^T e_k)\bar{\lambda}]+[\nabla_y(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})-\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})](\bar{\alpha}^T e_k)\\&\quad+\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})[\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}]+\nabla_{py}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})[\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\bar{p})]=0,\end{aligned}\tag{6.15}$$

$$\nabla_{pp}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})[\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\bar{p})]=0,\tag{6.16}$$

$$\begin{aligned}&\nabla_y f(\bar{x},\bar{y})[\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}]+h(\bar{x},\bar{y},\bar{p})(\bar{\alpha}^T e_k)-\bar{\mu}+\bar{\eta}e_k\\&\qquad\qquad+\nabla_p h(\bar{x},\bar{y},\bar{p})[\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\bar{p})]=0,\end{aligned}\tag{6.17}$$

$$\bar{\beta}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y})+\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})]=0,\tag{6.18}$$

$$\bar{\mu}^T\bar{\lambda}=0,\tag{6.19}$$

$$\bar{\eta}^T[\bar{\lambda}^T e_k-1]=0,\tag{6.20}$$

$$(\bar{\alpha},\bar{\beta},\bar{\mu},\bar{\eta})\neq 0.\tag{6.21}$$

Since $\bar{\lambda}>0$ and $\bar{\mu}\geqq 0$, (6.19) yields $\bar{\mu}=0$.

From (6.16) and nonsingularity of $\nabla_{pp}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})$, we have

$$\bar{\beta}=(\bar{\alpha}^T e_k)(\bar{y}+\bar{p}).\tag{6.22}$$

If $\bar{\alpha}=0$, then (6.17) and (6.22) yields $\bar{\eta}=0$ and $\bar{\beta}=0$, respectively. Consequently $(\bar{\alpha},\bar{\beta},\bar{\mu},\bar{\eta})=0$, contradicting (6.21). Hence, $\bar{\alpha}\geq 0$ or

$$\bar{\alpha}^T e_k>0.\tag{6.23}$$

Now, using (6.22) and (6.23) in (6.15), we get

$$\begin{aligned}&\nabla_y(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p}) - \nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})\\ &\qquad +\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p} = -\frac{1}{\bar{\alpha}^T e_k}\nabla_y f(\bar{x},\bar{y})[\bar{\alpha}-(\bar{\alpha}^T e_k)\bar{\lambda}], \qquad (6.24)\end{aligned}$$

which by hypotheses (iii) and (iv) implies

$$\bar{p} = 0. \qquad (6.25)$$

Now using hypothesis (iii) in (6.24), we obtain
$\nabla_y f(\bar{x},\bar{y})[\bar{\alpha}-(\bar{\alpha}^T e_k)\bar{\lambda}] = 0.$
Since the vectors $\{\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})\}$ are linearly independent, therefore the above equation yields

$$\bar{\alpha} = (\bar{\alpha}^T e_k)\bar{\lambda}. \qquad (6.26)$$

From (6.25) in (6.22), we get

$$\bar{\beta} = (\bar{\alpha}^T e_k)\bar{y}. \qquad (6.27)$$

Using (6.23) and (6.25)-(6.27) in (6.14), we have
$\{\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_x(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})\}(x-\bar{x}) \geqq 0, \quad \text{for all} \quad x \in C_1.$
From hypothesis (v), for $\bar{r} = 0$, the above inequality yields

$$\{\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})\}(x-\bar{x}) \geqq 0. \qquad (6.28)$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$ and so (6.28) implies
$\{\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})\}x \geqq 0 \quad \text{for all} \quad x \in C_1.$
Therefore,

$$\{\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})\} \in C_1^*. \qquad (6.29)$$

Also, from (6.27), we have
$\bar{y} = \dfrac{\bar{\beta}}{\bar{\alpha}^T e_k} \in C_2.$

Thus $(\bar{x}, \bar{y}, \bar{r} = 0)$ satisfies the constraints from (6.4) to (6.6) in $(HWD)_{\bar{\lambda}}$ and so it is a feasible solution for the dual problem $(HWD)_{\bar{\lambda}}$.

Now, letting $x = 0$ and $x = 2\bar{x}$ in (6.28), we get

$$\bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})] = 0. \tag{6.30}$$

Further, from (6.18), (6.23) and (6.27), we obtain

$$\bar{y}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})] = 0. \tag{6.31}$$

Therefore, using (6.25), (6.30), (6.31) and hypothesis (v), for $\bar{r} = 0$, we get

$$f(\bar{x}, \bar{y}) + (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})e_k - \bar{p}^T\nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})e_k - \bar{y}^T\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y})e_k - \bar{y}^T\nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})e_k$$
$$= f(\bar{x}, \bar{y}) + (\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})e_k - \bar{r}^T\nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})e_k - \bar{x}^T\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y})e_k - \bar{x}^T\nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})e_k,$$

that is, the two objective values are equal.

Now, let $(\bar{x}, \bar{y}, \bar{r} = 0)$ is not an efficient solution of $(HWD)_{\bar{\lambda}}$, then there exist $(\bar{u}, \bar{v}, \bar{r} = 0)$ feasible for $(HWD)_{\bar{\lambda}}$ such that,

$$f(\bar{x}, \bar{y}) + (\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})e_k - \bar{r}^T\nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})e_k - \bar{x}^T\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y})e_k - \bar{x}^T\nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})e_k$$
$$\leq f(\bar{u}, \bar{v}) + (\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r})e_k - \bar{r}^T\nabla_r(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r})e_k - \bar{u}^T\nabla_x(\bar{\lambda}^T f)(\bar{u}, \bar{v})e_k - \bar{u}^T\nabla_r(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r})e_k.$$

As $\bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_r(\lambda^T g)(\bar{x}, \bar{y}, \bar{r})] = 0 = \bar{y}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})]$ and from hypothesis (v), for $\bar{r} = 0$, we obtain

$$f(\bar{x}, \bar{y}) + (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})e_k - \bar{p}^T\nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})e_k - \bar{y}^T\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y})e_k - \bar{y}^T\nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})e_k$$
$$\leq f(\bar{u}, \bar{v}) + (\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r})e_k - \bar{r}^T\nabla_r(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r})e_k - \bar{u}^T\nabla_x(\bar{\lambda}^T f)(\bar{u}, \bar{v})e_k - \bar{u}^T\nabla_r(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r})e_k,$$

which contradicts weak duality theorem. Hence $(\bar{x}, \bar{y}, \bar{r} = 0)$ is an efficient solution of $(HWD)_{\bar{\lambda}}$.

Theorem 6.3 (Converse duality). Let $f : S_1 \times S_2 \rightarrow R^k$ be twice differentiable function and let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r})$ be a weak efficient solution of (HWD). Suppose that

(*i*) the matrix $\nabla_{rr}(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r})$ is non singular,

(*ii*) the vectors $\nabla_x f_1(\bar{u}, \bar{v}), \ldots, \nabla_x f_k(\bar{u}, \bar{v})$ are linearly independent,

(*iii*) the vector $\{\nabla_x(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r}) - \nabla_r(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r}) + \nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{r}\}$
$\notin \operatorname{span}\{\nabla_x f_1(\bar{u}, \bar{v}), \ldots, \nabla_x f_k(\bar{u}, \bar{v})\}\backslash\{0\},$

(iv) $\nabla_x(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r}) - \nabla_r(\bar{\lambda}^T g)(\bar{u}, \bar{v}, \bar{r}) + \nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{r} = 0$ implies $\bar{r} = 0$ and

(v) $(\bar{\lambda}^T g)(\bar{u}, \bar{v}, 0) = (\bar{\lambda}^T h)(\bar{u}, \bar{v}, 0)$, $\nabla_y(\bar{\lambda}^T g)(\bar{u}, \bar{v}, 0) = \nabla_p(\bar{\lambda}^T h)(\bar{u}, \bar{v}, 0)$, $\nabla_x(\bar{\lambda}^T g)(\bar{u}, \bar{v}, 0) = 0$ and $\nabla_r(\bar{\lambda}^T g)(\bar{u}, \bar{v}, 0) = 0$.

Then $\bar{r} = 0$, $(\bar{u}, \bar{v}, \bar{p} = 0)$ is feasible for $(HWP)_{\bar{\lambda}}$, and the objective values of $(HWP)_{\bar{\lambda}}$ and (HWD) are equal. Furthermore, if the hypotheses of Theorem 6.1 are satisfied for all feasible solutions of $(HWP)_{\bar{\lambda}}$ and (HWD), then $(\bar{u}, \bar{v}, \bar{p} = 0)$ is an efficient solution for $(HWP)_{\bar{\lambda}}$.

Proof. It follows on the lines of Theorem 6.2.

6.4.2 Mond-Weir type higher-order symmetric duality

We now formulate following pair of Mond-Weir type higher-order multiobjective symmetric dual programs over cones:

Primal Problem (HMP)

minimize $\quad F(x, y, p_1, p_2, \ldots, p_k) = (F_1(x, y, p_1), F_2(x, y, p_2), \ldots, F_k(x, y, p_k))$

subject to

$$-\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)] \in C_2^*, \tag{6.32}$$

$$y^T \sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)] \geqq 0, \tag{6.33}$$

$$\lambda > 0, \quad x \in C_1. \tag{6.34}$$

Dual Problem (HMD)

maximize $\quad H(u, v, r_1, r_2, \ldots, r_k) = (H_1(u, v, r_1), H_2(u, v, r_2), \ldots, H_k(u, v, r_k))$

subject to

$$\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i)] \in C_1^*, \tag{6.35}$$

$$u^T \sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i)] \leqq 0, \tag{6.36}$$

$$\lambda > 0, \quad v \in C_2, \tag{6.37}$$

where for $i = 1, 2, ..., k$, $F_i(x, y, p_i) = f_i(x, y) + h_i(x, y, p_i) - p_i^T[\nabla_{p_i} h_i(x, y, p_i)]$, $H_i(u, v, r_i) = f_i(u, v) + g_i(u, v, r_i) - r_i^T[\nabla_{r_i} g_i(u, v, r_i)]$, $f_i : S_1 \times S_2 \to R$, $h_i : S_1 \times S_2 \times R^m \to R$ and $g_i : S_1 \times S_2 \times R^n \to R$ are differentiable functions, $p_i \in R^m$ and $r_i \in R^n$.

Theorem 6.4 (Weak duality). Let $(x, y, \lambda, p_1, p_2, \ldots, p_k)$ be feasible for the primal problem (HMP) and $(u, v, \lambda, r_1, r_2, \ldots, r_k)$ be feasible for the dual problem (HMD). Let for $i = 1, 2, \ldots, k$

(i) $f_i(., v)$ be higher-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convex at u with respect to $g_i(u, v, r_i)$,

(ii) $-f_i(x, .)$ be higher-order $(G, \alpha_2, \rho_i^{(2)}, d_i^{(2)})$-convex at y with respect to $-h_i(x, y, p_i)$,

(iii) either (a) $\sum\limits_{i=1}^{k} \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x, u))^2 + \rho_i^{(2)}(d_i^{(2)}(v, y))^2] \geqq 0$ or (b) $\rho_i^{(1)} \geqq 0$ and $\rho_i^{(2)} \geqq 0$, for all i,

where the sublinear functionals $F : R^n \times R^n \times R^n \mapsto R$ and $G : R^m \times R^m \times R^m \mapsto R$ satisfy the following conditions:

(iv) $F(x, u; a) + \alpha_1^{-1} a^T u \geqq 0$, for all $a \in C_1^*$ and

(v) $G(v, y; b) + \alpha_2^{-1} b^T y \geqq 0$, for all $b \in C_2^*$.

Then

$F(x, y, p_1, p_2, \ldots, p_k) \nleq H(u, v, r_1, r_2, \ldots, r_k)$.

Proof. Suppose, to the contrary, that

$F(x, y, p_1, p_2, \ldots, p_k) \leq H(u, v, r_1, r_2, \ldots, r_k)$.

Since $\lambda > 0$, we obtain

$$\sum_{i=1}^{k} \lambda_i[f_i(x, y) + h_i(x, y, p_i) - p_i^T(\nabla_{p_i} h_i(x, y, p_i))]$$
$$< \sum_{i=1}^{k} \lambda_i[f_i(u, v) + g_i(u, v, r_i) - r_i^T(\nabla_{r_i} g_i(u, v, r_i))]. \quad (6.38)$$

Since $(x, y, \lambda, p_1, p_2, \ldots, p_k)$ is feasible for the primal problem (HMP) and $(u, v, \lambda, r_1, r_2, \ldots, r_k)$ is feasible for the dual problem (HMD), $\alpha_1(x, u) > 0$, by the dual constraint (6.35), the vector $a = \alpha_1(x,u)\{\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_{r_i} g_i(u,v,r_i)]\} \in C_1^*$ and so from the hypothesis (iv), we obtain

$F(x,u;a) \geqq -\alpha_1^{-1} u^T a.$

Substituting value of a in above inequality, we obtain

$$F(x,u;\alpha_1(x,u)\{\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v)+\nabla_{r_i} g_i(u,v,r_i)]\}) \geqq -u^T \sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v)+\nabla_{r_i} g_i(u,v,r_i)]$$

$$\geqq 0 \text{ [by dual constraint (6.36)].} \tag{6.39}$$

By higher-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convexity of $f_i(.,v)(1 \leqq i \leqq k)$ with respect to $g_i(u,v,r_i)$, we have

$$f_i(x,v) - f_i(u,v) - g_i(u,v,r_i) + r_i^T \nabla_{r_i} g_i(u,v,r_i)$$
$$\geqq F[x,u;\alpha_1(x,u)(\nabla_x f_i(u,v) + \nabla_{r_i} g_i(u,v,r_i))] + \rho_i^{(1)}(d_i^{(1)}(x,u))^2.$$

Using sublinearity of function about the third variable, and multiplying each inequality by λ_i and summing over i, we obtain

$$\sum_{i=1}^{k} \lambda_i[f_i(x,v) - f_i(u,v) - g_i(u,v,r_i) + r_i^T(\nabla_{r_i} g_i(u,v,r_i)) - \rho_i^{(1)}(d_i^{(1)}(x,u))^2]$$
$$\geqq F\left(x,u;\alpha_1(x,u)\left\{\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_{r_i} g_i(u,v,r_i)]\right\}\right).$$
$$\geqq 0 \text{ [by (6.39)].} \tag{6.40}$$

Similarly, using hypotheses (ii) and (v) along with primal constraint (6.32) and (6.33), $\alpha_2(v,y) > 0$, sublinearity of G, we get

$$\sum_{i=1}^{k} \lambda_i[f_i(x,y) - f_i(x,v) + h_i(x,y,p_i)$$
$$-p_i^T(\nabla_{p_i} h_i(x,y,p_i)) - \rho_i^{(2)}(d_i^{(2)}(x,u))^2] \geqq 0. \tag{6.41}$$

Adding the inequalities (6.40) and (6.41), we obtain

$$\sum_{i=1}^{k} \lambda_i[f_i(x,y) + h_i(x,y,p_i) - p_i^T(\nabla_{p_i} h_i(x,y,p_i)) - f_i(u,v) - g_i(u,v,r_i)$$
$$+r_i^T(\nabla_{r_i} g_i(u,v,r_i))] \geqq \sum_{i=1}^{k} \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2 + \rho_i^{(2)}(d_i^{(2)}(v,y))^2]. \tag{6.42}$$

Using hypothesis (iii) in (6.42), we get

$$\sum_{i=1}^{k} \lambda_i[f_i(x,y) + h_i(x,y,p_i) - p_i^T(\nabla_{p_i} h_i(x,y,p_i))]$$
$$\geqq \sum_{i=1}^{k} \lambda_i[f_i(u,v) + g_i(u,v,r_i) - r_i^T(\nabla_{r_i} g_i(u,v,r_i))],$$

which contradicts (6.38). Hence the result.

If the variable λ in the problems (HMP) and (HMD) is fixed to be $\bar{\lambda}$, we shall denote these problems by $(HMP)_{\bar{\lambda}}$ and $(HMD)_{\bar{\lambda}}$.

Theorem 6.5 (Strong duality). Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_k)$ be an efficient solution of (HMP), $f_i : S_1 \times S_2 \to R$ is thrice differentiable function at $(\bar{x}, \bar{y})$, $h_i : S_1 \times S_2 \times R^m \to R$ is a twice differentiable function at $(\bar{x}, \bar{y}, \bar{p}_i)$, $g_i : S_1 \times S_2 \times R^n \to R$ is differentiable at $(\bar{x}, \bar{y}, \bar{r}_i)$, $i = 1, 2, \ldots, k$. If the following conditions hold:

(i) $h_i(\bar{x}, \bar{y}, 0) = 0,\ g_i(\bar{x}, \bar{y}, 0) = 0,\ \nabla_x h_i(\bar{x}, \bar{y}, 0) = \nabla_{r_i} g_i(\bar{x}, \bar{y}, 0),\ \nabla_{p_i} h_i(\bar{x}, \bar{y}, 0) = 0, \nabla_y h_i(\bar{x}, \bar{y}, 0) = 0, \quad i = 1, 2, \ldots, k,$

(ii) for all $i = 1, 2, \ldots, k$, the Hessian matrix $\nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)$ is positive or negative definite,

(iii) the set of vectors $\{\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\}_{i=1}^{k}$ is linearly independent,

(iv) the set of vectors $\{\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i), \nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)\}_{i=1}^{k}$ is linearly independent,

(v) for some $\alpha \in R^k$ $(\alpha > 0)$ and $p_i \in R^n,\ p_i \neq 0 \quad (i = 1, 2, \ldots, k)$ implies that $\sum_{i=1}^{k} \alpha_i p_i^T [\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] \neq 0.$

Then $\bar{p}_i = 0$ $(i = 1, 2, \ldots, k)$, $\ (\bar{x}, \bar{y}, \bar{r}_1 = \bar{r}_2 = \ldots = \bar{r}_k = 0)$ is feasible for $(\text{HMD})_{\bar{\lambda}}$, and the two objective values are equal. Furthermore, if the hypotheses of Theorem 6.4 are satisfied, then $(\bar{x}, \bar{y}, \bar{r}_1 = \bar{r}_2 = \ldots = \bar{r}_k = 0)$ is an efficient solution for $(\text{HMD})_{\bar{\lambda}}$.

Proof. Its proof follows on the lines of Theorem 3.2 in [3] on taking $K = R_+^k$ and omitting the nondifferentiable terms.

A converse duality theorem may be merely stated as its proof would run analogously to that of Theorem 6.5.

Theorem 6.6 (Converse duality). Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r}_1, \bar{r}_2, \ldots, \bar{r}_k)$ be an efficient solution of (HMD), $f_i : S_1 \times S_2 \to R$ is a thrice differentiable function at $(\bar{u}, \bar{v})$, $h_i : S_1 \times S_2 \times R^m \to R$ is a twice differentiable function at $(\bar{u}, \bar{v}, \bar{p}_i)$ and $g_i : S_1 \times S_2 \times R^n \to R$ is differentiable at $(\bar{u}, \bar{v}, \bar{r}_i)$, $i = 1, 2, \ldots, k$. If the following conditions hold:

(i) $h_i(\bar{u}, \bar{v}, 0) = 0,\ g_i(\bar{u}, \bar{v}, 0) = 0, \nabla_x g_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{r_i} g_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{p_i} h_i(\bar{u}, \bar{v}, 0) = \nabla_y g_i(\bar{u}, \bar{v}, 0),\quad i = 1, 2, \ldots, k,$

(ii) for all $i = 1, 2, \ldots, k$, the Hessian matrix $\nabla_{r_i r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i)$ is positive or negative definite,

(iii) the set of vectors $\{\nabla_x f_i(\bar{u}, \bar{v}) + \nabla_{r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i)\}_{i=1}^{k}$ is linearly independent,

(iv) the set of vectors $\{\nabla_x f_i(\bar{u}, \bar{v}) + \nabla_x g_i(\bar{u}, \bar{v}, \bar{r}_i), \nabla_x f_i(\bar{u}, \bar{v}) + \nabla_{r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i)\}_{i=1}^{k}$ is linearly independent,

(v) for some $\alpha \in R^k$ $(\alpha > 0)$ and $r_i \in R^n,\ r_i \neq 0\quad (i = 1, 2, \ldots, k)$ implies that $\sum\limits_{i=1}^{k} \alpha_i r_i^T [\nabla_x f_i(\bar{u}, \bar{v}) + \nabla_{r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i)] \neq 0.$

Then $\bar{r}_i = 0$ $(i = 1, 2, \ldots, k)$, $(\bar{u}, \bar{v}, \bar{p}_1 = \bar{p}_2 = \ldots = \bar{p}_k = 0)$ is feasible for $(\text{HMP})_{\bar{\lambda}}$, and the two objective values of are equal. Furthermore, if the hypotheses of Theorem 6.4 are satisfied, then $(\bar{u}, \bar{v}, \bar{p}_1 = \bar{p}_2 = \ldots = \bar{p}_k = 0)$ is an efficient solution for $(\text{HMP})_{\bar{\lambda}}$.

6.5 Nondifferentiable multiobjective higher-order symmetric duality

In this section, we consider the following Wolfe type nondifferentiable multiobjective higher-order symmetric dual programs:

Primal Problem (HNWP)

K-minimize $G(x, y, \lambda, p) = f(x, y) + S(x \mid D)e_k + (\lambda^T h)(x, y, p)e_k - p^T \nabla_p (\lambda^T h)(x, y, p)e_k$

$$-y^T\nabla_y(\lambda^T f)(x,y)e_k - y^T\nabla_p(\lambda^T h)(x,y,p)e_k$$

subject to

$$-\{\nabla_y(\lambda^T f)(x,y) - z + \nabla_p(\lambda^T h)(x,y,p)\} \in C_2^*, \tag{6.43}$$

$$z \in E, \tag{6.44}$$

$$\lambda^T e_k = 1, \tag{6.45}$$

$$\lambda \in \text{int}K^*,\ x \in C_1. \tag{6.46}$$

Dual Problem (HNWD)

K-maximize $H(u,v,\lambda,r) = f(u,v) - S(v \mid E)e_k + (\lambda^T g)(u,v,r)e_k - r^T\nabla_r(\lambda^T g)(u,v,r)e_k$

$$-u^T\nabla_x(\lambda^T f)(u,v)e_k - u^T\nabla_r(\lambda^T g)(u,v,r)e_k$$

subject to

$$\nabla_x(\lambda^T f)(u,v) + w + \nabla_r(\lambda^T g)(u,v,r) \in C_1^*, \tag{6.47}$$

$$w \in D, \tag{6.48}$$

$$\lambda^T e_k = 1, \tag{6.49}$$

$$\lambda \in \text{int}K^*,\ v \in C_2, \tag{6.50}$$

where

(*i*) C_1 and C_2 be closed convex cones with non-empty interiors in R^n and R^m, respectively,

(*ii*) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$,

(*iii*) $f : S_1 \times S_2 \to R^k$, $h : S_1 \times S_2 \times R^m \to R^k$ and $g : S_1 \times S_2 \times R^n \to R^k$ are differentiable functions, $e_k = (1,\ldots,1)^T \in R^k$, $\lambda \in R^k$,

(*iv*) r and p are vectors in R^n and R^m, respectively,

(*v*) D and E are compact convex sets in R^n and R^m, respectively and

(*vi*) $S(x \mid D)$ and $S(v \mid E)$ are the support functions of D and E, respectively.

We now prove the following duality results for the pair of problems (HNWP) and (HNWD).

Theorem 6.7 (Weak duality). Let (x, y, λ, z, p) be feasible for the primal problem (HNWP) and (u, v, λ, w, r) be feasible for the dual problem (HNWD). Let the sublinear functionals $F : R^n \times R^n \times R^n \mapsto R$ and $G : R^m \times R^m \times R^m \mapsto R$ satisfy the following conditions:

$F(x, u; a) + \alpha_1^{-1} a^T u \geqq 0$, for all $a \in C_1^*$, $\qquad (A)$

$G(v, y; b) + \alpha_2^{-1} b^T y \geqq 0$, for all $b \in C_2^*$. $\qquad (B)$

Suppose that either (i) $\sum\limits_{i=1}^{k} \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2 + \rho_i^{(2)}(d_i^{(2)}(v,y))^2] \geqq 0$ or (ii) $\rho^{(1)} \geqq 0$ and $\rho^{(2)} \geqq 0$. Furthermore, assume that $f(.,v) + (.)^T w e_k$ is higher-order-K-$(F, \alpha_1, \rho^{(1)}, d^{(1)})$-convex at u with respect to $g(u, v, r)$, $-\{f(x, .) - (.)^T z e_k\}$ is higher-order-K-$(G, \alpha_2, \rho^{(2)}, d^{(2)})$-convex at y with respect to $-h(x, y, p)$ and $R_+^k \subseteq K$. Then

$$G(x, y, \lambda, p) - H(u, v, \lambda, r) \notin -K\backslash\{0\}. \tag{6.51}$$

Proof. Since $f(.,v) + (.)^T w e_k$ is higher-order-K-$(F, \alpha_1, \rho^{(1)}, d^{(1)})$-convex with respect to $g(u, v, r)$, we have

$$\{f_1(x,v)+x^Tw-f_1(u,v)-u^Tw-g_1(u,v,r)+r^T\nabla_r g_1(u,v,r)-F[x,u;\alpha_1(x,u)(\nabla_x f_1(u,v) +w + \nabla_r g_1(u,v,r))] - \rho_1^{(1)}(d_1^{(1)}(x,u))^2, \ldots, f_k(x,v) + x^Tw - f_k(u,v) - u^Tw -g_k(u,v,r) + r^T\nabla_r g_k(u,v,r) - F[x,u;\alpha_1(x,u)(\nabla_x f_k(u,v) + w +\nabla_r g_k(u,v,r))] - \rho_k^{(1)}(d_k^{(1)}(x,u))^2\} \in K.$$

Since $R_+^k \subseteq K \Rightarrow K^* \subseteq R_+^k$ and hence $\lambda > 0$.

Further using sublinearity of F and $\lambda^T e_k = 1$, we get

$$(\lambda^T f)(x,v) + x^Tw - (\lambda^T f)(u,v) - u^Tw - (\lambda^T g)(u,v,r) + r^T\nabla_r(\lambda^T g)(u,v,r) -F[x,u;\alpha_1(x,u)(\nabla_x(\lambda^T f)(u,v) + w + \nabla_r(\lambda^T g)(u,v,r))] - \sum_{i=1}^{k} \lambda_i \rho_i^{(1)}(d_i^{(1)}(x,u))^2 \geqq 0. \tag{6.52}$$

As $-\{f(x,.) - (.)^T z e_k\}$ is higher-order-K-$(G, \alpha_2, \rho^{(2)}, d^{(2)})$-convex with respect to $-h(x, y, p)$, therefore we get

$$\{f_1(x,y)-y^Tz-f_1(x,v)+v^Tz+h_1(x,y,p)-p^T\nabla_p h_1(x,y,p)-G[v,y;-\alpha_2(v,y)(\nabla_y f_1(x,y)$$

$$-z+\nabla_p h_1(x,y,p))]-\rho_1^{(2)}(d_1^{(2)}(v,y))^2,\ldots,f_k(x,y)-y^Tz-f_k(x,v)+v^Tz$$
$$+h_k(x,y,p)-p^T\nabla_p h_k(x,y,p)-G[v,y;-\alpha_2(v,y)(\nabla_y f_k(x,y)-z$$
$$+\nabla_p h_k(x,y,p))]-\rho_k^{(2)}(d_k^{(2)}(v,y))^2\}\in K.$$

Using $\lambda\in \text{int}K^*$, $\lambda^Te_k=1$, and sublinearity of G, we obtain

$$(\lambda^Tf)(x,y)-y^Tz-(\lambda^Tf)(x,v)+v^Tz+(\lambda^Th)(x,y,p)-p^T\nabla_p(\lambda^Th)(x,y,p)-G[v,y;$$
$$-\alpha_2(v,y)(\nabla_y(\lambda^Tf)(x,y)-z+\nabla_p(\lambda^Th)(x,y,p))]-\sum_{i=1}^k\lambda_i\rho_i^{(2)}(d_i^{(2)}(v,y))^2\geqq 0. \tag{6.53}$$

Adding the inequalities (6.52) and (6.53), we have

$$(\lambda^Tf)(x,y)-(\lambda^Tf)(u,v)+x^Tw-u^Tw-y^Tz+v^Tz+(\lambda^Th)(x,y,p)-p^T\nabla_p(\lambda^Th)(x,y,p)$$
$$-(\lambda^Tg)(u,v,r)+r^T\nabla_r(\lambda^Tg)(u,v,r)\geqq F[x,u;\alpha_1(x,u)(\nabla_x(\lambda^Tf)(u,v)+w$$
$$+\nabla_r(\lambda^Tg)(u,v,r))]+G[v,y;-\alpha_2(v,y)(\nabla_y(\lambda^Tf)(x,y)-z+\nabla_p(\lambda^Th)(x,y,p))]$$
$$+\sum_{i=1}^k\lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2+\rho_i^{(2)}(d_i^{(2)}(v,y))^2]. \tag{6.54}$$

Now, since (x,y,λ,z,p) is feasible for the primal problem (HNWP) and (u,v,λ,w,r) is feasible for the dual problem (HNWD), $\alpha_1(x,u)>0$, by the dual constraint (6.47) and from the hypothesis (A), we obtain

$$F\Bigg(x,u;\alpha_1(x,u)[\nabla_x(\lambda^Tf)(u,v)+w+\nabla_r(\lambda^Tg)(u,v,r)]\Bigg)$$
$$+\Bigg[\nabla_x(\lambda^Tf)(u,v)+w+\nabla_r(\lambda^Tg)(u,v,r)\Bigg]^Tu\geqq 0. \tag{6.55}$$

Also, from hypothesis (B), $\alpha_2(v,y)>0$ and (6.43), we get

$$G\Bigg(v,y;-\alpha_2(v,y)[\nabla_y(\lambda^Tf)(x,y)-z+\nabla_p(\lambda^Th)(x,y,p)]\Bigg)$$
$$-\Bigg[\nabla_y(\lambda^Tf)(x,y)-z+\nabla_p(\lambda^Th)(x,y,p)\Bigg]^Ty\geqq 0. \tag{6.56}$$

Using (6.55), (6.56) and hypothesis (i) or (ii) in (6.54), we have

$$(\lambda^Tf)(x,y)+x^Tw+(\lambda^Th)(x,y,p)-p^T\nabla_p(\lambda^Th)(x,y,p)-y^T\nabla_y(\lambda^Tf)(x,y)-y^T\nabla_p(\lambda^Th)(x,y,p)$$
$$\geqq(\lambda^Tf)(u,v)-v^Tz+(\lambda^Tg)(u,v,r)-r^T\nabla_r(\lambda^Tg)(u,v,r)-u^T\nabla_x(\lambda^Tf)(u,v)-u^T\nabla_r(\lambda^Tg)(u,v,r).$$

In view of the fact that $x^T w \leqq S(x \mid D)$, $v^T z \leqq S(v \mid E)$ and $\lambda^T e_k = 1$, the last inequality yields

$$\begin{aligned}(\lambda^T f)(x,y) + S(x \mid D) + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) - y^T \nabla_y (\lambda^T f)(x,y) \\ -y^T \nabla_p (\lambda^T h)(x,y,p) \geqq (\lambda^T f)(u,v) - S(v \mid E) + (\lambda^T g)(u,v,r) \\ -r^T \nabla_r (\lambda^T g)(u,v,r) - u^T \nabla_x (\lambda^T f)(u,v) - u^T \nabla_r (\lambda^T g)(u,v,r).\end{aligned} \tag{6.57}$$

Now suppose contrary to the result that (6.51) not holds, that is ,
$G(x,y,\lambda,p) - H(u,v,\lambda,r) \in -K\backslash\{0\}$, or

$$\begin{aligned}\{f(x,y) + S(x \mid D)e_k + (\lambda^T h)(x,y,p)e_k - p^T \nabla_p (\lambda^T h)(x,y,p)e_k - y^T \nabla_y (\lambda^T f)(x,y)e_k \\ -y^T \nabla_p (\lambda^T h)(x,y,p)e_k\} - \{f(u,v) - S(v \mid E)e_k + (\lambda^T g)(u,v,r)e_k - r^T \nabla_r (\lambda^T g)(u,v,r)e_k \\ -u^T \nabla_x (\lambda^T f)(u,v)e_k - u^T \nabla_r (\lambda^T g)(u,v,r)e_k\} \in -K\backslash\{0\}.\end{aligned}$$

It follows from $\lambda \in \mathrm{int} K^*$ and $\lambda^T e_k = 1$ that

$$\begin{aligned}(\lambda^T f)(x,y) + S(x \mid D) + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) - y^T \nabla_y (\lambda^T f)(x,y) \\ -y^T \nabla_p (\lambda^T h)(x,y,p) < (\lambda^T f)(u,v) - S(v \mid E) + (\lambda^T g)(u,v,r) - r^T \nabla_r (\lambda^T g)(u,v,r) \\ -u^T \nabla_x (\lambda^T f)(u,v) - u^T \nabla_r (\lambda^T g)(u,v,r),\end{aligned}$$

which contradicts (6.57). Hence the result.

Theorem 6.8 (Strong duality). Let $f : S_1 \times S_2 \rightarrow R^k$ be twice differentiable function and let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ be a weak efficient solution of (HNWP). Suppose that

(i) the matrix $\nabla_{pp}(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})$ is non singular,

(ii) the vectors $\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})$ are linearly independent,

(iii) the vector $\{\nabla_y (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p}\}$
$\notin \mathrm{span}\{\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\}\backslash\{0\}$,

(iv) $\nabla_y (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p} = 0$ implies $\bar{p} = 0$,

(v) $(\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = (\bar{\lambda}^T g)(\bar{x}, \bar{y}, 0)$, $\nabla_y (\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = 0$, $\nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = 0$, $\nabla_x (\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = \nabla_r (\bar{\lambda}^T g)(\bar{x}, \bar{y}, 0)$ and

(vi) K is closed convex pointed cone with $R^k_+ \subseteq K$.

Then

(I) $\bar{p} = 0$, there exist $\bar{w} \in D$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ is feasible for (HNWD) and

(II) $G(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p}) = H(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$.

Also, if the hypotheses of Theorem 6.7 are satisfied for all feasible solutions of (HNWP) and (HNWD), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ is an efficient solution for (HNWD).

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ is a weak efficient solution of (HNWP), by the Fritz John necessary optimality conditions [123], there exist $\bar{\alpha} \in K^*$, $\bar{\beta} \in C_2$, $\bar{\eta} \in R$, such that the following conditions are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$:

$$\begin{aligned}&\{\bar{\alpha}^T(\nabla_x f(\bar{x}, \bar{y}) + \bar{\gamma} e_k) + \nabla_x(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})(\bar{\alpha}^T e_k) + \nabla_{xy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})[\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}] \\ &\qquad + \nabla_{px}(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})]\}(x - \bar{x}) \geqq 0, \quad \text{for all} \;\; x \in C_1, \end{aligned} \tag{6.58}$$

$$\begin{aligned}&\nabla_y f(\bar{x}, \bar{y})[\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}] + [\nabla_y(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})](\bar{\alpha}^T e_k) \\ &\qquad + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})[\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}] + \nabla_{py}(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})] = 0, \end{aligned} \tag{6.59}$$

$$\nabla_{pp}(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})] = 0, \tag{6.60}$$

$$\begin{aligned}&\{\nabla_y f(\bar{x}, \bar{y})[\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}] + h(\bar{x}, \bar{y}, \bar{p})(\bar{\alpha}^T e_k) + \bar{\eta} e_k \\ &\qquad + \nabla_p h(\bar{x}, \bar{y}, \bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})]\}(\lambda - \bar{\lambda}) \geqq 0, \quad \text{for all } \lambda \in \text{int } K^*, \end{aligned} \tag{6.61}$$

$$\bar{\beta}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) - \bar{z} + \nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})] = 0, \tag{6.62}$$

$$\bar{\eta}^T[\bar{\lambda}^T e_k - 1] = 0, \tag{6.63}$$

$$\bar{\beta} \in N_E(\bar{z}), \tag{6.64}$$

$$\bar{\gamma} \in D, \quad \bar{\gamma}^T \bar{x} = S(\bar{x} \mid D), \tag{6.65}$$

$$(\bar{\alpha}, \bar{\beta}, \bar{\eta}) \neq 0. \tag{6.66}$$

From (6.60) and nonsingularity of $\nabla_{pp}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})$, we have

$$\bar{\beta} = (\bar{\alpha}^T e_k)(\bar{y}+\bar{p}). \tag{6.67}$$

Also, (6.61) is equivalent to

$\nabla_y f(\bar{x},\bar{y})[\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}] + h(\bar{x},\bar{y},\bar{p})(\bar{\alpha}^T e_k) + \bar{\eta} e_k + \nabla_p h(\bar{x},\bar{y},\bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y}+\bar{p})] = 0.$

Now, we claim that $\bar{\alpha} \neq 0$. Since if $\bar{\alpha} = 0$ then (6.67) yields $\bar{\beta} = 0$. Further, the above equality gives $\bar{\eta} e_k = 0$ or $\bar{\eta} = 0$. Consequently $(\bar{\alpha},\bar{\beta},\bar{\eta}) = 0$, contradicting (6.66). Hence $\bar{\alpha} \neq 0$.

Since $R^k_+ \subseteq K \Rightarrow K^* \subseteq R^k_+$, $\bar{\alpha} \in K^* \Rightarrow \bar{\alpha} \geqq 0$.

But $\bar{\alpha} \neq 0 \Rightarrow \bar{\alpha} \geq 0$ or

$$\bar{\alpha}^T e_k > 0. \tag{6.68}$$

The relation (6.59), (6.67) and (6.68) give

$$\begin{aligned} &\nabla_y(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p}) - \nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p}) \\ &\qquad + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p} = -\frac{1}{\bar{\alpha}^T e_k}\nabla_y f(\bar{x},\bar{y})[\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}], \end{aligned} \tag{6.69}$$

which on using hypothesis (iii) and (iv) yield

$$\bar{p} = 0. \tag{6.70}$$

Now using hypothesis (iii) in (6.69), we obtain

$\nabla_y f(\bar{x},\bar{y})[\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}] = 0.$

From the condition (ii), we have

$$\bar{\alpha} = (\bar{\alpha}^T e_k)\bar{\lambda}. \tag{6.71}$$

Equations (6.67) and (6.70) give

$$\bar{\beta} = (\bar{\alpha}^T e_k)\bar{y}. \tag{6.72}$$

Using (6.68) and (6.70)-(6.72) in (6.58), we have

$\{\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \bar{\gamma} + \nabla_x(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})\}(x - \bar{x}) \geqq 0, \quad \text{for all} \quad x \in C_1.$

From hypothesis (v), for $\bar{r}=0$, the above inequality yields

$$\{\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})+\bar{\gamma}+\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})\}(x-\bar{x}) \geqq 0. \tag{6.73}$$

Let $x \in C_1$. Then $x+\bar{x} \in C_1$, as C_1 is closed convex cone, and so (6.73) shows that for every $x \in C_1$
$\{\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})+\bar{\gamma}+\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})\}^T x \geqq 0$,
which implies,

$$\{\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})+\bar{\gamma}+\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})\} \in C_1^*. \tag{6.74}$$

Also, from (6.68), (6.72) and $\bar{\beta} \in C_2$, we obtain $\bar{y} \in C_2$. Thus $(\bar{x},\bar{y},\bar{\lambda},\bar{w}=\bar{\gamma},\bar{r}=0)$ satisfies the dual constraints from (6.47) to (6.50) in (HNWD) and so it is a feasible solution for the dual problem (HNWD). Also, by letting $x=0$ and $x=2\bar{x}$ simultaneously in (6.73), we get
$\bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})+\bar{\gamma}+\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})]=0$,
or

$$\bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})+\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})]=-\bar{x}^T\bar{\gamma}=-S(\bar{x}\mid D). \tag{6.75}$$

From (6.64) and (6.72), $(\bar{\alpha}^T e_k)\bar{y} \in N_E(\bar{z})$. Since $\bar{\alpha}^T e_k>0$, $\bar{y} \in N_E(\bar{z})$. Also, as E is a compact convex set in R^m, $\bar{y}^T\bar{z}=S(\bar{y}\mid E)$.
Further, from (6.62), (6.68) and (6.72) and the above relation, we obtain

$$\bar{y}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y})+\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})]=\bar{y}^T\bar{z}=S(\bar{y}\mid E). \tag{6.76}$$

Therefore, using (6.70), (6.75), (6.76) and hypothesis (v), for $\bar{r}=0$, we get
$f(\bar{x},\bar{y})+S(\bar{x}\mid D)e_k+(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k-\bar{p}^T\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k-\bar{y}^T\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y})e_k$
$-\bar{y}^T\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k=f(\bar{x},\bar{y})-S(\bar{y}\mid E)e_k+(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k-\bar{r}^T\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k$
$-\bar{x}^T\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})e_k-\bar{x}^T\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k$,
that is, the two objective values are equal.
Now, let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r}=0)$ is not an efficient solution of (HNWD), then there exist $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{r}=0)$ feasible for (HNWD), such that,

$$f(\bar{x},\bar{y})-S(\bar{y}\mid E)e_k+(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k-\bar{r}^T\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k-\bar{x}^T\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})e_k$$
$$-\bar{x}^T\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k-f(\bar{u},\bar{v})+S(\bar{v}\mid E)e_k-(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k+\bar{r}^T\nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k$$
$$+\bar{u}^T\nabla_x(\bar{\lambda}^T f)(\bar{u},\bar{v})e_k+\bar{u}^T\nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k\in -K\backslash\{0\}.$$

As $\bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})+\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})]=-S(\bar{x}\mid D)$, $\bar{y}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y})+\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})]=S(\bar{y}\mid E)$ and from hypothesis (v), for $\bar{r}=0$, we obtain

$$\{f(\bar{x},\bar{y})+S(\bar{x}\mid D)e_k+(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k-\bar{p}^T\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k-\bar{y}^T\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y})e_k$$
$$-\bar{y}^T\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k\}-\{f(\bar{u},\bar{v})-S(\bar{v}\mid E)e_k+(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k-\bar{r}^T\nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k$$
$$-\bar{u}^T\nabla_x(\bar{\lambda}^T f)(\bar{u},\bar{v})e_k-\bar{u}^T\nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k\}\in -K\backslash\{0\},$$

which contradicts Theorem 6.7. Hence $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r}=0)$ is an efficient solution of (HNWD).

Theorem 6.9 (Converse duality). Let $f: S_1\times S_2\to R^k$ be twice differentiable function and let $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{r})$ be a weak efficient solution of (HNWD). Suppose that

(i) the matrix $\nabla_{rr}(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})$ is non singular,

(ii) the vectors $\nabla_x f_1(\bar{u},\bar{v}),\ldots,\nabla_x f_k(\bar{u},\bar{v})$ are linearly independent,

(iii) the vector $\{\nabla_x(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})-\nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})+\nabla_{xx}(\bar{\lambda}^T f)(\bar{u},\bar{v})\bar{r}\}$
$\notin \text{span}\{\nabla_x f_1(\bar{u},\bar{v}),\ldots,\nabla_x f_k(\bar{u},\bar{v})\}\backslash\{0\}$,

(iv) $\nabla_x(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})-\nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})+\nabla_{xx}(\bar{\lambda}^T f)(\bar{u},\bar{v})\bar{r}=0$ implies $\bar{r}=0$,

(v) $(\bar{\lambda}^T g)(\bar{u},\bar{v},0)=(\bar{\lambda}^T h)(\bar{u},\bar{v},0)$, $\nabla_x(\bar{\lambda}^T g)(\bar{u},\bar{v},0)=0$, $\nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},0)=0$,
$\nabla_y(\bar{\lambda}^T g)(\bar{u},\bar{v},0)=\nabla_p(\bar{\lambda}^T h)(\bar{u},\bar{v},0)$ and

(vi) K is closed convex pointed cone with $R^k_+\subseteq K$.

Then

(I) $\bar{r}=0$, there exist $\bar{z}\in E$ such that $(\bar{u},\bar{v},\bar{\lambda},\bar{z},\bar{p}=0)$ is feasible for (HNWP), and

(II) $G(\bar{u},\bar{v},\bar{\lambda},\bar{p})=H(\bar{u},\bar{v},\bar{\lambda},\bar{r})$.

Also, if the hypotheses of Theorem 6.7 are satisfied for all feasible solutions of (HNWP) and (HNWD), then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ is an efficient solution for (HNWP).

Proof. It follows on the lines of Theorem 6.8.

6.6 Special cases

In this section, we consider some of the special cases of the problems studied in Section 6.4. In all these cases, $K = R_+^k$, $C_1 = R_+^n$, $C_2 = R_+^m$, $(\lambda^T h)(x, y, p) = \frac{1}{2} p^T \nabla_{yy} (\lambda^T f)(x, y) p$ and $(\lambda^T g)(u, v, r) = \frac{1}{2} r^T \nabla_{xx} (\lambda^T f)(u, v) r$.

- *(i)* For $k = 1$, $D = \{Ay : y^T Ay \leqq 1\}$, $E = \{Bx : x^T Bx \leqq 1\}$, where A and B are positive semidefinite matrices, $(x^T Ax)^{1/2} = S(x \mid D)$ and $(y^T By)^{1/2} = S(y \mid E)$. In this case (HNWP) and (HNWD) reduce to the problems considered in Ahmad and Husain [9].

- *(ii)* Let $k = 1$, $D = \{0\}$ and $E = \{0\}$, then our problems (HNWP) and (HNWD) become the problems studied in Gulati et al. [55].

- *(iii)* If $k = 1$, then our problems (HNWP) and (HNWD) reduce to the programs studied in Yang et al. [137].

- *(iv)* Let $D = \{0\}$ and $E = \{0\}$, our problems (HNWP) and (HNWD) reduce to (MP) and (MD) considered in Yang et al. [140] along with nonnegativity restrictions $x \geqq 0$ and $v \geqq 0$. However, taking $F(x, u; a) = (x - u)^T a$ and $G(v, y; b) = (v - y)^T b$ along with the hypothesis (A) and (B) of Theorem 1 in [140] gives $x \geqq 0$ and $v \geqq 0$.

Chapter 7

Duality In Nondifferentiable Minimax Fractional Programming[1]

7.1 Introduction

The mathematical programming problem in which the objective function is a ratio of two numerical functions is called a fractional programming problem. Fractional programming is used in various fields of study. Most extensively it is used in business and economic situations, mainly in the situations of deficit of financial resources. Fractional programming problems have arisen in multiobjective programming [54, 130], game theory [36] and goal programming [42]. Problems of these type have been the subject of immense interest in the past few years.

The necessary and sufficient conditions for generalized minimax programming were first developed by Schmitendorf [115]. Tanimoto [126] applied these optimality conditions to define a dual problem and derived duality theorems. Bector and Bhatia [23] relaxed the convexity assumptions in the sufficient optimality condition in [115] and also employed the optimality conditions to construct several dual models which involve pseudo-convex and quasi-convex functions, and derived duality theorems. Yadav and Mukhrjee [134] established the optimality conditions to construct the two

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[1] The result of this chapter will appear in *Journal of Inequalities and Applications*.

dual problems and derived duality theorems for differentiable fractional minimax programming. Chandra and Kumar [40] pointed out that the formulation of Yadav and Mukhrjee [134] has some omissions and inconsistencies and they constructed two modified dual problems and proved duality theorems for differentiable fractional minimax programming.

Lai et al. [85] established necessary and sufficient optimality conditions for nondifferentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct a parametric dual model and also discussed duality theorems. Lai and Lee [84] obtained duality theorems for two parameter-free dual models of nondifferentiable minimax fractional problem involving generalized convexity assumptions. Recently, Antczak [19] proved optimality conditions for a class of generalized fractional minimax programming problems involving B-(p, r)-invexity functions and established duality theorems for various duality models.

In this chapter, motivated by Lai et al. [85], Lai and Lee [84] and Antczak [19], we discuss sufficient optimality conditions and duality theorems for a nondifferentiable minimax fractional programming problem with B-(p, r)-invexity. In Section 7.2, we give some preliminaries. An example which is B-$(1, 1)$-invex but not (p, r)-invex is exemplified. We also illustrate another example which is $(-1, 1)$-invex but not convex. In Section 7.4, we establish the sufficient optimality conditions. Duality results are presented in Section 7.5.

7.2 Notations and preliminaries

Definition 7.1. Let $f : X \to R$ (where $X \subseteq R^n$) be differentiable function and let $p,\ r$ be arbitrary real numbers. Then f is said to be (p, r)-invex (strictly (p, r)-invex) with respect to η at $u \in X$ on X if there exist a function $\eta : X \times X \to R^n$ such that, for all $x \in X$, the inequalities

$$\frac{1}{r} e^{rf(x)} \geqq \frac{1}{r} e^{rf(u)} \left[1 + \frac{r}{p} \nabla f(u) (e^{p\eta(x,u)} - \mathbf{1}) \right] \ (> \text{ if } x \neq u) \text{ for } p \neq 0,\ r \neq 0,$$

$$\frac{1}{r}e^{rf(x)} \geqq \frac{1}{r}e^{rf(u)}\left[1 + r\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1})\right] \ (> \text{if } x \neq u) \text{ for } p = 0,\ r \neq 0,$$

$$f(x) - f(u) \geqq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) \ (> \text{if } x \neq u) \text{ for } p \neq 0,\ r = 0,$$

$$f(x) - f(u) \geqq \nabla f(u)\eta(x,u) \ (> \text{if } x \neq u) \text{ for } p = 0,\ r = 0,$$

hold.

Definition 7.2 [19]. The differentiable function $f : X \to R$ (where $X \subseteq R^n$) is said to be (strictly) B-(p,r)-invex with respect to η and b at $u \in X$ on X if there exist a function $\eta : X \times X \to R^n$ and a function $b : X \times X \to R_+$ such that, for all $x \in X$, the inequalities

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) \geqq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) \ (> \text{if } x \neq u) \text{ for } p \neq 0,\ r \neq 0,$$

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))} - 1) \geqq \nabla f(u)\eta(x,u) \ (> \text{if } x \neq u) \text{ for } p = 0,\ r \neq 0,$$

$$b(x,u)(f(x) - f(u)) \geqq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) \ (> \text{if } x \neq u) \text{ for } p \neq 0,\ r = 0,$$

$$b(x,u)(f(x) - f(u)) \geqq \nabla f(u)\eta(x,u) \ (> \text{if } x \neq u) \text{ for } p = 0,\ r = 0,$$

hold.

f is said to be (strictly) B-(p,r)-invex with respect to η and b on X if it is B-(p,r)-invex with respect to same η and b at each $u \in X$ on X.

In this chapter, we consider the following nondifferentiable minimax fractional programming problem:

(FP)

$$\min_{x \in R^n} \sup_{y \in Y} \frac{l(x,y) + (x^T Dx)^{1/2}}{m(x,y) - (x^T Ex)^{1/2}}$$

$$\text{subject to } g(x) \leqq 0,\ x \in X$$

where Y is a compact subset of R^m, $l(.,.) : R^n \times R^m \to R,\ m(.,.) : R^n \times R^m \to R$, are C^1 functions on $R^n \times R^m$ and $g(.) : R^n \to R^p$ is C^1 function on R^n. D and E are $n \times n$ positive semidefinite matrices.

Let $S = \{x \in X : g(x) \leqq 0\}$ be the set of all feasible solutions of (FP). Any point

$x \in S$ is called the feasible point of (FP). For each $(x, y) \in R^n \times R^m$, we define

$$\phi(x, y) = \frac{l(x, y) + (x^T Dx)^{1/2}}{m(x, y) - (x^T Ex)^{1/2}},$$

such that for each $(x, y) \in S \times Y$,

$l(x, y) + (x^T Dx)^{1/2} \geqq 0$ and $m(x, y) - (x^T Ex)^{1/2} > 0$.

For each $x \in S$, we define

$H(x) = \{h \in H : g_h(x) = 0\}$,

where

$$H = \{1, 2, \cdots, p\},$$

$$Y(x) = \left\{ y \in Y : \frac{l(x, y) + (x^T Dx)^{1/2}}{m(x, y) - (x^T Ex)^{1/2}} = \sup_{z \in Y} \frac{l(x, z) + (x^T Dx)^{1/2}}{m(x, z) - (x^T Ex)^{1/2}} \right\}.$$

$$K(x) = \Big\{ (s, t, \tilde{y}) \in N \times R_+^s \times R^{ms} : 1 \leqq s \leqq n + 1,\ t = (t_1, t_2, \cdots, t_s) \in R_+^s \text{ with } \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_s) \text{ with } \bar{y}_i \in Y(x) (i = 1, 2, \cdots, s) \Big\}.$$

Since l and m are continuously differentiable and Y is compact in R^m, it follows that for each $x^* \in S, Y(x^*) \neq \emptyset$, and for any $\bar{y}_i \in Y(x^*)$, we have a positive constant

$$k_\circ = \phi(x^*, \bar{y}_i) = \frac{l(x^*, \bar{y}_i) + (x^{*T} Dx^*)^{1/2}}{m(x^*, \bar{y}_i) - (x^{*T} Ex^*)^{1/2}}.$$

If the functions l, g and m in problem (FP) are continuously differentiable with respect to $x \in R^n$, then Lai et al. [85] derived the following necessary conditions for optimality of (FP).

Theorem 7.1 (Necessary conditions). If x^* is a solution of (FP) satisfying $x^{*T} Dx^* > 0$, $x^{*T} Ex^* > 0$, and $\nabla g_h(x^*), h \in H(x^*)$ are linearly independent, then there exist $(s, t^*, \bar{y}) \in K(x^*), k_\circ \in R_+, w, v \in R^n$, and $\mu^* \in R_+^p$ such that

$$\sum_{i=1}^{s} t_i^* \Big\{ \nabla l(x^*, \bar{y}_i) + Dw - k_\circ (\nabla m(x^*, \bar{y}_i) - Ev) \Big\} + \nabla \sum_{h=1}^{p} \mu_h^* g_h(x^*) = 0, \tag{7.1}$$

$$l(x^*, \bar{y}_i) + (x^{*T} Dx^*)^{\frac{1}{2}} - k_\circ \left(m(x^*, \bar{y}_i) - (x^{*T} Ex^*)^{\frac{1}{2}} \right) = 0,\ i = 1, 2, \cdots, s, \tag{7.2}$$

$$\sum_{h=1}^{p} \mu_h^* g_h(x^*) = 0, \tag{7.3}$$

$$t_i^* \geqq 0 \ (i = 1, 2, \cdots, s), \sum_{i=1}^{s} t_i^* = 1, \tag{7.4}$$

$$\begin{cases} w^T D w \leqq 1, \ v^T E v \leqq 1, \\ \\ (x^{*T} D x^*)^{1/2} = x^{*T} D w, \\ \\ (x^{*T} E x^*)^{1/2} = x^{*T} E v. \end{cases} \tag{7.5}$$

Remark 7.1 [19]. It should be pointed out that the exponentials appearing on the right-hand sides of the inequalities above are understood to be taken componentwise and $\mathbf{1} = (1, 1, \ldots, 1) \in R^n$.

Remark 7.2. All the theorems in this chapter will be proved only in the case when $p \neq 0$, $r \neq 0$. The proofs in the other cases are easier than in this one. It follows from the form of inequalities which are given in Definition 7.2. Moreover, without limiting the generality considerations we shall assume that $r > 0$.

7.3 Examples

7.1. An example of non trivial function which is $(-1, 1)$-invex but not convex.

Let $X = [8.75, 9.15] \subset R$. Consider the function $f : X \to R$ defined by $f(x) = \log(\sin^2 x)$. Let $\eta : X \times X \to R$ be given by

$$\eta(x, u) = 12(1 + u).$$

To prove that f is $(-1, 1)$-invex, we have to show that

$$\frac{1}{r} e^{rf(x)} - \frac{1}{r} e^{rf(u)} \left[1 + \frac{r}{p} \nabla f(u) \left(e^{p\eta(x,u)} - \mathbf{1} \right) \right] \geqq 0, \text{ for } p = -1 \text{ and } r = 1.$$

Now, consider

$$\begin{aligned} \varphi &= e^{f(x)} - e^{f(u)} \left[1 - \nabla f(u) \left(e^{-\eta(x,u)} - \mathbf{1} \right) \right] \\ &= \sin^2 x + \sin 2u \left(e^{-12(1+u)} - 1 \right) - \sin^2 u \\ &\geqq 0 \ \ \forall \ x, u \in X, \text{ as can be seen from Figure 7.1.} \end{aligned}$$

Hence, f is $(-1, 1)$-invex.

Further, for $x = 8.8$ and $u = 9.1$, we have

$$\vartheta = f(x) - f(u) - (x-u)^T \nabla f(u)$$

$$= 2\log\left(\frac{\sin x}{\sin u}\right) - \frac{(x-u)\sin 2u}{\sin^2 u}$$

$$= -0.5701 < 0$$

Thus f is not convex function on X.

7.2. An example of non trivial function which is B-$(1,1)$-invex but not (p,r)-invex.

Let $X = [0.25, 0.45] \subset R$. Consider the function $f : X \to R$ defined by

$$f(x) = -x^2 + \log(8\sqrt{x}).$$

Let $\eta : X \times X \to R$ be given by

$$\eta(x,u) = \log(1 + 2u^2)$$

and $b : X \times X \to R_+$ be defined by

$$b(x,u) = 4\sin^2 x + \sin^2 u.$$

The function f defined above is B-$(1,1)$-invex as

$$\phi = b(x,u)(e^{(f(x)-f(u))} - 1) - \nabla f(u)(e^{\eta(x,u)} - \mathbf{1})$$

$$= \left[4\sin^2 x + \sin^2 u\right]\left[e^{(u^2-x^2)}\sqrt{\tfrac{x}{u}} - 1\right] - \left[u - 4u^3\right]$$

$$\geqq 0 \;\; \forall\, x, u \in X, \text{ as can be seen from Figure 7.2.}$$

But it is not (p,r) invex for all $p, r \in (-10^{17}, 10^{17})$ as

$$\psi = \tfrac{1}{r}e^{rf(x)} - \tfrac{1}{r}e^{rf(u)}\left[1 + \tfrac{r}{p}\nabla f(u)\left(e^{p\eta(x,u)} - 1\right)\right]$$

$$= \tfrac{1}{r}e^{1.4613\times r} - \tfrac{1}{r}e^{1.469\times r}\left[1 + 0.45 \times \tfrac{r}{p}\left(e^{0.3022\times p} - 1\right)\right]$$

(for $x = 0.4$ and $u = 0.42$)

< 0 as can be seen from Figure 7.3.

Hence f is B-$(1,1)$-invex but not (p,r)-invex.

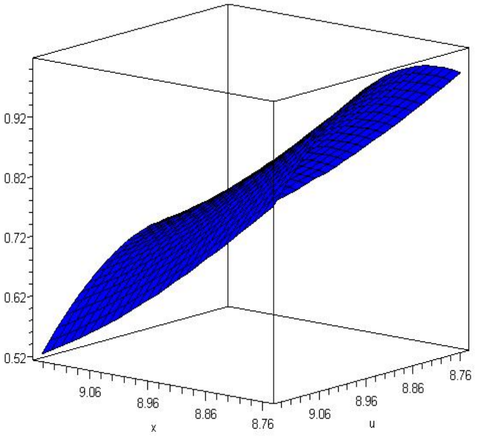

Figure 7.1: Graph of φ against x and u

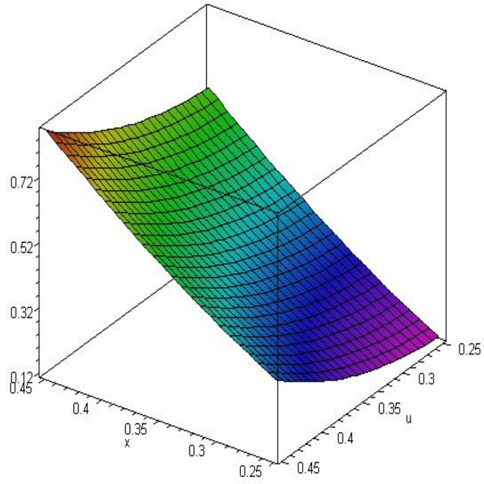

Figure 7.2: Graph of ϕ against x and u

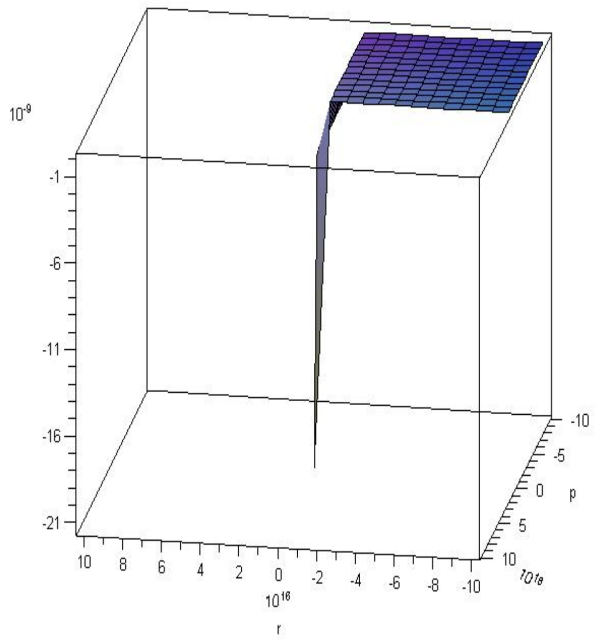

Figure 7.3: Graph of ψ against r and p

7.4 Sufficient conditions

Now, we prove the sufficient condition for optimality of (FP) under B-(p,r)-invexity assumptions.

Theorem 7.2 (Sufficient condition). Let x^* be a feasible solution of (FP) and there exist a positive integer $s, 1 \leqq s \leqq n+1,\ t^* \in R_+^s, \bar{y}_i \in Y(x^*)(i = 1,2,\cdots s), k_\circ \in R_+, w, v \in R^n$ and $\mu^* \in R_+^p$ satisfying the relations (7.1)-(7.5). Assume that

(*i*) $\sum\limits_{i=1}^{s} t_i^* \left(l(.,\bar{y}_i) + (.)^T Dw - k_\circ \left(m(.,\bar{y}_i) - (.)^T Ev \right) \right)$ is B-(p,r)-invex at x^* on S with respect to η and b satisfying $b(x,x^*) > 0$ for all $x \in S$,

(*ii*) $\sum\limits_{h=1}^{p} \mu_h^* g_h(.)$ is B_g-(p,r)-invex at x^* on S with respect to the same function η, and with respect to the function b_g, not necessarily, equal to b.

Then x^* is an optimal solution of (FP).

Proof. Suppose to the contrary that x^* is not an optimal solution of (FP). Then there exists an $\bar{x} \in S$ such that

$$\sup_{y \in Y} \frac{l(\bar{x},y) + (\bar{x}^T D \bar{x})^{1/2}}{m(\bar{x},y) - (\bar{x}^T E \bar{x})^{1/2}} < \sup_{y \in Y} \frac{l(x^*,y) + ({x^*}^T D x^*)^{1/2}}{m(x^*,y) - ({x^*}^T E x^*)^{1/2}}.$$

We note that

$$\sup_{y \in Y} \frac{l(x^*,y) + ({x^*}^T D x^*)^{1/2}}{m(x^*,y) - ({x^*}^T E x^*)^{1/2}} = \frac{l(x^*,\bar{y}_i) + ({x^*}^T D x^*)^{1/2}}{m(x^*,\bar{y}_i) - ({x^*}^T E x^*)^{1/2}} = k_\circ,$$

for $\bar{y}_i \in Y(x^*), i = 1,2,\cdots,s$ and

$$\frac{l(\bar{x},\bar{y}_i) + (\bar{x}^T D \bar{x})^{1/2}}{m(\bar{x},\bar{y}_i) - (\bar{x}^T E \bar{x})^{1/2}} \leqq \sup_{y \in Y} \frac{l(\bar{x},y) + (\bar{x}^T D \bar{x})^{1/2}}{m(\bar{x},y) - (\bar{x}^T E \bar{x})^{1/2}}.$$

Thus, we have

$$\frac{l(\bar{x},\bar{y}_i)+(\bar{x}^T D\bar{x})^{1/2}}{m(\bar{x},\bar{y}_i)-(\bar{x}^T E\bar{x})^{1/2}} < k_\circ,\ \text{for } i=1,2,\cdots,s.$$

It follows that

$$l(\bar{x},\bar{y}_i)+(\bar{x}^T D\bar{x})^{1/2}-k_\circ(m(\bar{x},\bar{y}_i)-(\bar{x}^T E\bar{x})^{1/2}) < 0,\ \text{for } i=1,2,\cdots,s. \tag{7.6}$$

From (7.2), (7.4)-(7.6) and Schwartz Inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^{s} t_i^*\{l(\bar{x},\bar{y}_i)+\bar{x}^T Dw-k_\circ(m(\bar{x},\bar{y}_i)-\bar{x}^T Ev)\}\\
&\qquad\leqq \sum_{i=1}^{s} t_i^*\{l(\bar{x},\bar{y}_i)+(\bar{x}^T D\bar{x})^{\frac{1}{2}}-k_\circ(m(\bar{x},\bar{y}_i)-(\bar{x}^T E\bar{x})^{\frac{1}{2}})\}\\
&\qquad< 0=\sum_{i=1}^{s} t_i^*\{l(x^*,\bar{y}_i)+(x^{*T} Dx^*)^{\frac{1}{2}}-k_\circ(m(x^*,\bar{y}_i)-(x^{*T} Ex^*)^{\frac{1}{2}})\}\\
&\qquad=\sum_{i=1}^{s} t_i^*\{l(x^*,\bar{y}_i)+x^{*T} Dw-k_\circ(m(x^*,\bar{y}_i)-x^{*T} Ev)\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{s} t_i^*\{l(\bar{x},\bar{y}_i)+\bar{x}^T Dw-k_\circ(m(\bar{x},\bar{y}_i)-\bar{x}^T Ev)\}\\
&\qquad< \sum_{i=1}^{s} t_i^*\{l(x^*,\bar{y}_i)+x^{*T} Dw-k_\circ(m(x^*,\bar{y}_i)-x^{*T} Ev)\}.
\end{aligned} \tag{7.7}$$

As $\sum_{i=1}^{s} t_i^*\left(l(.,\bar{y}_i)+(.)^T Dw-k_\circ\left(m(.,\bar{y}_i)-(.)^T Ev\right)\right)$ is B-(p,r)-invex at x^* on S with respect to η and b, we have

$$\begin{aligned}
&\frac{1}{r}b(x,x^*)\left\{e^{r\left[\sum_{i=1}^{s} t_i^*\left(l(x,\bar{y}_i)+x^T Dw-k_\circ\left(m(x,\bar{y}_i)-x^T Ev\right)\right)-\sum_{i=1}^{s} t_i^*\left(l(x^*,\bar{y}_i)+x^{*T} Dw-k_\circ\left(m(x^*,\bar{y}_i)-x^{*T} Ev\right)\right)\right]}-1\right\}\\
&\quad\geqq \frac{1}{p}\left\{\sum_{i=1}^{s} t_i^*\left(\nabla l(x^*,\bar{y}_i)+Dw-k_\circ\left(\nabla m(x^*,\bar{y}_i)-Ev\right)\right)\right\}\left\{e^{p\eta(x,x^*)}-\mathbf{1}\right\}
\end{aligned}$$

holds for all $x\in S$, and so for $x=\bar{x}$. Using (7.7) and $b(\bar{x},x^*)>0$ together with the inequality above, we get

$$\frac{1}{p}\left\{\sum_{i=1}^{s} t_i^*\left(\nabla l(x^*,\bar{y}_i)+Dw-k_\circ\left(\nabla m(x^*,\bar{y}_i)-Ev\right)\right)\right\}\left\{e^{p\eta(\bar{x},x^*)}-\mathbf{1}\right\}<0. \tag{7.8}$$

From the feasibility of $\bar{x}$ together with $\mu_h^*\geqq 0,\ h\in H$, we have

$$\sum_{h=1}^{p}\mu_h^* g_h(\bar{x})\leqq 0. \tag{7.9}$$

By B_g-(p,r)-invexity of $\sum\limits_{h=1}^{p}\mu_h^* g_h(.)$ at x^* on S with respect to the same function η, and with respect to the function b_g, we have

$$\tfrac{1}{r} b_g(\bar{x},x^*)\left\{e^{r\left[\sum\limits_{h=1}^{p}\mu_h^* g_h(\bar{x})-\sum\limits_{h=1}^{p}\mu_h^* g_h(x^*)\right]}-1\right\} \geqq \tfrac{1}{p}\sum_{h=1}^{p}\nabla\mu_h^* g_h(x^*)\left\{e^{p\eta(\bar{x},x^*)}-\mathbf{1}\right\}.$$

Since $b_g(x,x^*) \geq 0$ for all $x \in S$ then by (7.3) and (7.9), we obtain

$$\frac{1}{p}\sum_{h=1}^{p}\nabla\mu_h^* g_h(x^*)\left\{e^{p\eta(\bar{x},x^*)}-\mathbf{1}\right\} \leqq 0. \tag{7.10}$$

By adding the inequalities (7.8) and (7.10), we have

$$\tfrac{1}{p}\left\{\sum_{i=1}^{s} t_i^*\left(\nabla l(x^*,\bar{y}_i)+Dw-k_\circ\left(\nabla m(x^*,\bar{y}_i)-Ev\right)\right)+\sum_{h=1}^{p}\nabla\mu_h^* g_h(x^*)\right\}$$
$$\left\{e^{p\eta(\bar{x},x^*)}-\mathbf{1}\right\}<0,$$

which contradicts (7.1). Hence the result.

7.5 Duality model

Consider the following dual to (FP):

(FD) $$\max_{(s,t,\bar{y})\in K(a)} \quad \sup_{(a,\mu,k,v,w)\in H_1(s,t,\bar{y})} \quad k,$$

where $H_1(s,t,\bar{y})$ denotes the set of all $(a,\mu,k,v,w) \in R^n \times R_+^p \times R_+ \times R^n \times R^n$ satisfying

$$\sum_{i=1}^{s} t_i\{\nabla l(a,\bar{y}_i)+Dw-k(\nabla m(a,\bar{y}_i)-Ev)\}+\nabla\sum_{h=1}^{p}\mu_h g_h(a)=0, \tag{7.11}$$

$$\sum_{i=1}^{s} t_i\{l(a,\bar{y}_i)+a^TDw-k(m(a,\bar{y}_i)-a^TEv)\} \geqq 0, \tag{7.12}$$

$$\sum_{h=1}^{p}\mu_h g_h(a) \geqq 0, \tag{7.13}$$

$$(s,t,\bar{y}) \in K(a), \tag{7.14}$$

$$w^TDw \leqq 1,\ v^TEv \leqq 1. \tag{7.15}$$

If, for a triplet $(s,t,\bar{y}) \in K(a)$, the set $H_1(s,t,\bar{y})=\emptyset$, then we define the supremum over it to be $-\infty$. For convenience, we let

$$\psi_1(.) = \sum_{i=1}^{s} t_i\{l(., \bar{y}_i) + (.)^T Dw - k(m(., \bar{y}_i) - (.)^T Ev)\}.$$

Let S_{FD} denote a set of all feasible solutions for problem (FD). Moreover, let S_1 denote $S_1 = \{a \in R^n : (a, \mu, k, v, w, s, t, \bar{y}) \in S_{FD}\}$.

Theorem 7.3 (Weak duality). Let x be a feasible solution of (FP) and $(a, \mu, k, v, w, s, t, \bar{y})$ be a feasible of (FD). Let

(i) $\sum\limits_{i=1}^{s} t_i \left(l(., \bar{y}_i) + (.)^T Dw - k\left(m(., \bar{y}_i) - (.)^T Ev\right)\right)$ is B-(p, r)-invex at a on $S \cup S_1$ with respect to η and b satisfying $b(x, a) > 0$,

(ii) $\sum\limits_{h=1}^{p} \mu_h g_h(.)$ is B_g-(p, r)-invex at a on $S \cup S_1$ with respect to the same function η and with respect to the function b_g, not necessarily, equal to b.

Then

$$\sup_{y \in Y} \frac{l(x, y) + (x^T Dx)^{1/2}}{m(x, y) - (x^T Ex)^{1/2}} \geq k. \tag{7.16}$$

Proof. Suppose to the contrary that

$$\sup_{y \in Y} \frac{l(x, y) + (x^T Dx)^{1/2}}{m(x, y) - (x^T Ex)^{1/2}} < k.$$

Then, we have

$$l(x, \bar{y}_i) + (x^T Dx)^{1/2} - k(m(x, \bar{y}_i) - (x^T Ex)^{1/2}) < 0, \text{ for all } \bar{y}_i \in Y.$$

It follows from (7.4) that

$$t_i\{l(x, \bar{y}_i) + (x^T Dx)^{1/2} - k(m(x, \bar{y}_i) - (x^T Ex)^{1/2}) \leq 0, \tag{7.17}$$

with at least one strict inequality, since $t = (t_1, t_2, \cdots, t_s) \neq 0$.

From (7.12), (7.15) and (7.17), we have

$$\begin{aligned}
\psi_1(x) &= \sum_{i=1}^{s} t_i\{l(x,\bar{y}_i) + x^T Dw - k(m(x,\bar{y}_i) - x^T Ev)\} \\
&\leqq \sum_{i=1}^{s} t_i\{l(x,\bar{y}_i) + (x^T Dx)^{\frac{1}{2}} - k(m(x,\bar{y}_i) - (x^T Ex)^{\frac{1}{2}})\} \\
&< 0 \leqq \sum_{i=1}^{s} t_i\{l(a,\bar{y}_i) + a^T Dw - k(m(a,\bar{y}_i) - a^T Ev)\} \\
&= \psi_1(a).
\end{aligned}$$

Hence

$$\psi_1(x) < \psi_1(a). \tag{7.18}$$

Since $\sum\limits_{i=1}^{s} t_i \left(l(.,\bar{y}_i) + (.)^T Dw - k\left(m(.,\bar{y}_i) - (.)^T Ev\right)\right)$ is B-(p,r)-invex at a on $S \cup S_1$ with respect to η and b, we have

$$\tfrac{1}{r} b(x,a) \left\{ e^{r\left[\sum\limits_{i=1}^{s} t_i\left(l(x,\bar{y}_i)+x^T Dw-k\left(m(x,\bar{y}_i)-x^T Ev\right)\right)-\sum\limits_{i=1}^{s} t_i\left(l(a,\bar{y}_i)+a^T Dw-k\left(m(a,\bar{y}_i)-a^T Ev\right)\right)\right]} - 1 \right\}$$
$$\geqq \tfrac{1}{p} \left\{ \sum_{i=1}^{s} t_i \left(\nabla l(a,\bar{y}_i) + Dw - k\left(\nabla m(a,\bar{y}_i) - Ev\right)\right) \right\} \left\{ e^{p\eta(x,a)} - \mathbf{1} \right\}.$$

From (7.18) and $b(x,a) > 0$ together with the inequality above, we get

$$\frac{1}{p} \left\{ \sum_{i=1}^{s} t_i \left(\nabla l(a,\bar{y}_i) + Dw - k\left(\nabla m(a,\bar{y}_i) - Ev\right)\right) \right\} \left\{ e^{p\eta(x,a)} - \mathbf{1} \right\} < 0. \tag{7.19}$$

Using the feasibility of x together with $\mu_h \geqq 0,\ h \in H$, we obtain

$$\sum_{h=1}^{p} \mu_h g_h(x) \leqq 0. \tag{7.20}$$

From hypothesis (ii), we have

$$\tfrac{1}{r} b_g(x,a) \left\{ e^{r\left[\sum\limits_{h=1}^{p} \mu_h g_h(x) - \sum\limits_{h=1}^{p} \mu_h g_h(a)\right]} - 1 \right\} \geqq \tfrac{1}{p} \sum_{h=1}^{p} \nabla \mu_h g_h(a) \left\{ e^{p\eta(x,a)} - \mathbf{1} \right\}.$$

As $b_g(x,a) \geqq 0$ then by (7.13) and (7.20), we obtain

$$\frac{1}{p} \sum_{h=1}^{p} \nabla \mu_h g_h(a) \left\{ e^{p\eta(x,a)} - \mathbf{1} \right\} \leqq 0. \tag{7.21}$$

Thus, by (7.19) and (7.21), we obtain the inequality

$$\tfrac{1}{p} \left\{ \sum_{i=1}^{s} t_i \left(\nabla l(a,\bar{y}_i) + Dw - k\left(\nabla m(a,\bar{y}_i) - Ev\right)\right) + \sum_{h=1}^{p} \nabla \mu_h g_h(a) \right\}$$

$$\left\{e^{p\eta(x,a)} - \mathbf{1}\right\} < 0,$$

which contradicts (7.11). Hence (7.16) holds.

Theorem 7.4 (Strong duality). Let x^* be an optimal solution of (FP) and $\nabla g_h(x^*), h \in H(x^*)$ is linearly independent. Then there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}) \in H_1(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is a feasible solution of (FD). Further, if the hypotheses of Theorem 7.3 are satisfied for all feasible solutions $(a, \mu, k, v, w, s, t, \bar{y})$ of (FD), then $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is an optimal solution of (FD) and the two objectives have the same optimal values.

Proof. If x^* be an optimal solution of (FP) and $\nabla g_h(x^*), h \in H(x^*)$ is linearly independent, then by Theorem 7.1, there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}) \in H_1(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (FD) and problems (FP) and (FD) have the same objective values and

$$\bar{k} = \frac{l(x^*, \bar{y}_i^*) + (x^{*T} D x^*)^{1/2}}{m(x^*, \bar{y}_i^*) - (x^{*T} E x^*)^{1/2}}.$$

The optimality of this feasible solution for (FD) thus follows from Theorem 7.3.

Theorem 7.5 (Strict converse duality). Let x^* and $(\bar{a}, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ be the optimal solutions of (FP) and (FD), respectively and $\nabla g_h(x^*), h \in H(x^*)$ is linearly independent. Suppose that $\sum_{i=1}^{s} t_i \left(l(., \bar{y}_i) + (.)^T D w - \bar{k}\left(m(., \bar{y}_i) - (.)^T E v\right)\right)$ is strictly B-(p, r)-invex at a on $S \cup S_1$ with respect to η and b satisfying $b(x, a) > 0$ for all $x \in S$. Furthermore assume that $\sum_{h=1}^{p} \mu_h g_h(.)$ is B_g-(p, r)-invex at a on $S \cup S_1$ with respect to the same function η and with respect to the function b_g, not necessarily, equal to the function b. Then $x^* = \bar{a}$, that is, $\bar{a}$ is an optimal point in (FP) and

$$\sup_{y \in Y} \frac{l(\bar{a}, \bar{y}^*) + (\bar{a}^T D \bar{a})^{1/2}}{m(\bar{a}, \bar{y}^*) - (\bar{a}^T E \bar{a})^{1/2}} = \bar{k}.$$

Proof. We shall assume that $x^* \neq \bar{a}$ and reach a contradiction. From the strong duality theorem (Theorem 7.4), it follows that

$$\sup_{y \in Y} \frac{l(x^*, \bar{y}^*) + (x^{*T} D x^*)^{1/2}}{m(x^*, \bar{y}^*) - (x^{*T} E x^*)^{1/2}} = \bar{k}. \tag{7.22}$$

By feasibility of x^* together with $\mu_h \geqq 0,\ h \in H$, we obtain

$$\sum_{h=1}^{p} \mu_h g_h(x^*) \leqq 0. \tag{7.23}$$

By assumption, $\sum\limits_{h=1}^{p} \mu_h g_h(.)$ is B_g-(p,r)-invex at a on $S \cup S_1$ with respect to η and with respect to the b_g. Then, by Definition 7.2, there exists a function b_g such that $b_g(x,a) \geqq 0$ for all $x \in S$ and $a \in S_1$. Hence by (7.13) and (7.23),
$\frac{1}{r} b_g(x^*, \bar{a}) \left\{ e^{r\left[\sum\limits_{h=1}^{p} \mu_h g_h(x^*) - \sum\limits_{h=1}^{p} \mu_h g_h(\bar{a})\right]} - 1 \right\} \leqq 0.$
Then, from Definition 7.2, we get

$$\frac{1}{p} \sum_{h=1}^{p} \nabla \mu_h g_h(\bar{a}) \left\{ e^{p\eta(x^*,\bar{a})} - \mathbf{1} \right\} \leqq 0. \tag{7.24}$$

Therefore, by (7.24), we obtain the inequality
$\frac{1}{p} \left\{ \sum\limits_{i=1}^{s} t_i \left(\nabla l(\bar{a}, \bar{y}_i) + Dw - \bar{k} \left(\nabla m(\bar{a}, \bar{y}_i) - Ev \right) \right) \right\} \left\{ e^{p\eta(x^*,\bar{a})} - \mathbf{1} \right\} \geqq 0.$
As $\sum\limits_{i=1}^{s} t_i \left(l(., \bar{y}_i) + (.)^T Dw - \bar{k} \left(m(., \bar{y}_i) - (.)^T Ev \right) \right)$ is strictly B-(p,r)-invex with respect to η and b at $\bar{a}$ on $S \cup S_1$. Then, by the Definition of strictly B-(p,r)-invexity and from above inequality, it follows that
$\frac{1}{r} b(x^*, \bar{a}) \times$
$\left\{ e^{r\left[\sum\limits_{i=1}^{s} t_i \left(l(x^*,\bar{y}_i) + x^{*T} Dw - \bar{k}\left(m(x^*,\bar{y}_i) - x^{*T} Ev \right) \right) - \sum\limits_{i=1}^{s} t_i \left(l(\bar{a},\bar{y}_i) + \bar{a}^T Dw - \bar{k}\left(m(\bar{a},\bar{y}_i) - \bar{a}^T Ev \right) \right) \right]} - 1 \right\} > 0.$
From the hypothesis $b(x^*, \bar{a}) > 0$, and the above inequality, we get
$$\sum_{i=1}^{s} t_i \left(l(x^*, \bar{y}_i) + x^{*T} Dw - \bar{k} \left(m(x^*, \bar{y}_i) - x^{*T} Ev \right) \right)$$
$$- \sum_{i=1}^{s} t_i \left(l(\bar{a}, \bar{y}_i) + \bar{a}^T Dw - \bar{k} \left(m(\bar{a}, \bar{y}_i) - \bar{a}^T Ev \right) \right) > 0.$$
Therefore, by (7.12),
$\sum\limits_{i=1}^{s} t_i \left(l(x^*, \bar{y}_i) + x^{*T} Dw - \bar{k} \left(m(x^*, \bar{y}_i) - x^{*T} Ev \right) \right) > 0.$
Since $t_i \geqq 0,\ i = 1, 2, \ldots, s$, therefore there exists i^* such that

$l(x^*, \bar{y}_i^*) + x^{*T} Dw - \bar{k} \left(m(x^*, \bar{y}_i^*) - x^{*T} Ev \right) > 0.$

Hence, we obtain the following inequality

$$\frac{l(x^*, \bar{y}_i^*) + (x^{*T} D x^*)^{1/2}}{m(x^*, \bar{y}_i^*) - (x^{*T} E x^*)^{1/2}} > \bar{k},$$

which contradicts (7.22). Hence the result.

Chapter 8

NONDIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL VARIATIONAL PROGRAMMING PROBLEMS

8.1 Introduction

Hanson [66] surveyed and generalized the relationship between mathematical programming and classical calculus of variation. Optimality conditions and duality results were obtained for scalar valued variational problems by Mond and Hanson [98] under convexity. Bector et al. [27] extended symmetric duality to variational programming, giving continuous analogous of the results of Dantzig et al. [48] and Mond and Weir [103], respectively.

Ahmad and Husain [11] introduced second-order (F, α, ρ, d)-convex functions and their generalizations and developed weak, strong and strict converse duality theorems for second-order Mond-Weir type multiobjective dual. Hachimi and Aghezzaf [65] extended the concepts of (F, α, ρ, d) type-I functions and generalized type-I functions to the continuous case and established various sufficient optimality conditions and mixed duality results for multiobjective variational problems. Recently, Ahmad and Sharma [15] formulated a pair of multiobjective fractional variational symmetric dual problems over arbitrary cones and obtained usual duality results under generalized F-convexity assumptions.

This chapter is organized as follows: In Section 8.2, we give some definitions and preliminaries related with the chapter. In Section 8.3, we formulate a pair of multiobjective fractional variational symmetric dual problems for a class of nondifferentiable functions over arbitrary cones and prove weak, strong and converse duality theorems under (F, α, ρ, d)-convexity/pseudoconvexity assumptions. Self duality for this pair is also discussed in Section 8.4.

8.2 Notations and definitions

Let $I = [a, b]$ be a real interval and $C_1 \subset R^n$, $C_2 \subset R^m$, be closed convex cones with nonempty interiors having positive polars C_1^* and C_2^*, respectively. Let for each $i \in L = \{1, 2, \ldots, l\}$, $f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ and $g^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$, where $x : I \to R^n$ and $y : I \to R^m$, with derivatives $\dot{x}$ and $\dot{y}$, are twice continuously differentiable functions. For $i \in L$, the symbols $f_x^i, f_{\dot{x}}^i, f_y^i$ and $f_{\dot{y}}^i$ denote gradient vectors of the scalar function $f^i(t, x, \dot{x}, y, \dot{y})$ with respect to $x, \dot{x}, y$ and $\dot{y}$, respectively. We have

$$f_x^i = (\frac{\partial f^i}{\partial x^1}, \ldots, \frac{\partial f^i}{\partial x^n})^T,\ f_{\dot{x}}^i = (\frac{\partial f^i}{\partial \dot{x}^1}, \ldots, \frac{\partial f^i}{\partial \dot{x}^n})^T.$$

Similarly, $g_x^i, g_{\dot{x}}^i, g_y^i$ and $g_{\dot{y}}^i$ denote the gradient vectors of $g^i(t, x, \dot{x}, y, \dot{y})$ with respect to x, $\dot{x}$, y and $\dot{y}$, respectively.

The following observations are used for proving duality results:

$Df_{\dot{y}}^i = f_{\dot{y}t}^i + f_{\dot{y}y}^i \dot{y} + f_{\dot{y}\dot{y}}^i \ddot{y} + f_{\dot{y}x}^i \dot{x} + f_{\dot{y}\dot{x}}^i \ddot{x}$.

Consequently,

$$\frac{\partial}{\partial y} Df_{\dot{y}}^i = Df_{\dot{y}y}^i,\quad \frac{\partial}{\partial \dot{y}} Df_{\dot{y}}^i = Df_{\dot{y}\dot{y}}^i + f_{\dot{y}y}^i,\quad \frac{\partial}{\partial \ddot{y}} Df_{\dot{y}}^i = f_{\dot{y}\dot{y}}^i,$$

$$\frac{\partial}{\partial x} Df_{\dot{y}}^i = Df_{\dot{y}x}^i,\quad \frac{\partial}{\partial \dot{x}} Df_{\dot{y}}^i = Df_{\dot{y}\dot{x}}^i + f_{\dot{y}x}^i,\quad \frac{\partial}{\partial \ddot{x}} Df_{\dot{y}}^i = f_{\dot{y}\dot{x}}^i,\quad i \in L.$$

Similarly, $Dg_{\dot{y}}^i$ can be defined.

Let $X(I, R^n)$ denotes the space of piecewise smooth functions x with norm

$\| x \| = \| x \|_\infty + \| Dx \|_\infty$,

where the differentiation operator D is given by

$u = Dx \Leftrightarrow x(t) = \alpha_1 + \int_a^t u(s)ds$,
where α_1 is a given boundary value. Therefore, $\frac{d}{dt} \equiv D$ except at discontinuities. Denote by $Y(I, R^m)$, the space of piecewise smooth functions $y : I \to R^m$ with the norm as that of space $X(I, R^n)$.
Consider the following multiobjective variational problem:
(NFP) Minimize

$$\int_a^b \phi(t, x(t), \dot{x}(t))dt = \left(\int_a^b \phi^1(t, x(t), \dot{x}(t))dt, \ldots, \int_a^b \phi^l(t, x(t), \dot{x}(t))dt \right)$$

$$\begin{aligned} \text{subject to} \quad x(a) &= 0 = x(b), \\ \dot{x}(a) &= 0 = \dot{x}(b), \\ h(t, x(t), \dot{x}(t)) &\leqq 0, \quad t \in I, \end{aligned}$$

where $\phi : I \times R^n \times R^n \to R^l$ and $h : I \times R^n \times R^n \to R^m$ are differentiable functions. Let S be the set of all feasible solutions of (NFP), i.e.,

$$S = \{x \in X(I, R^n) \mid \; x(a) = 0 = x(b), \; \dot{x}(a) = 0 = \dot{x}(b), \; h(t, x(t), \dot{x}(t)) \leqq 0, \; t \in I\}.$$

Definition 8.1. A point $\bar{x} \in S$ is an efficient (or pareto optimal) solution of (NFP) if there exist no other $x \in S$ such that

$$\int_a^b \phi(t, x(t), \dot{x}(t))dt \leq \int_a^b \phi(t, \bar{x}(t), \dot{\bar{x}}(t))dt.$$

Definition 8.2. A point $\bar{x} \in S$ is a weak efficient solution of (NFP) if there exist no other $x \in S$ such that

$$\int_a^b \phi(t, x(t), \dot{x}(t))dt < \int_a^b \phi(t, \bar{x}(t), \dot{\bar{x}}(t))dt.$$

Let F and G be sublinear functional with respect to third variable and $d = (d^1, d^2)$, where $d^1 = (d^1_1, d^1_2, \ldots, d^1_l) : I \times R^n \times R^n \times R^n \times R^n \to R^l$, $d^2 = (d^2_1, d^2_2, \ldots, d^2_l) : I \times R^m \times R^m \times R^m \times R^m \to R^l$. Let $f = (f^1, f^2, \ldots, f^l) : I \times R^n \times R^n \times R^m \times R^m \to R^l$ be differentiable function, $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_1 : R^n \times R^n \to R_+ \backslash \{0\}$, $\alpha_2 : R^m \times R^m \to$

$R_+ \backslash \{0\}$ and $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho^1_1, \rho^1_2, \ldots, \rho^1_l), \rho^2 = (\rho^2_1, \rho^2_2, \ldots, \rho^2_l) \in R^l$.

Definition 8.3. For each $i \in L$, $\int_a^b f^i(t, x, \dot{x}, y, \dot{y})$ is said to be $(F, \alpha_1, \rho^1_i, d^1_i)$-convex in x and $\dot{x}$ for fixed y and $\dot{y}$, if

$$\int_a^b f^i(t, x, \dot{x}, y, \dot{y})dt - \int_a^b f^i(t, u, \dot{u}, y, \dot{y})dt$$
$$\geqq \int_a^b F(t, x, \dot{x}, y, \dot{y}; \alpha_1(x, u)(f^i_x(t, u, \dot{u}, y, \dot{y})$$
$$- Df^i_{\dot{x}}(t, u, \dot{u}, y, \dot{y})))dt + \rho^1_i \int_a^b \left(d^1_i(t, x, \dot{x}, u, \dot{u})\right)^2 dt$$

for all $x,\ u : I \to R^n$ and for some arbitrary sublinear functional F.

Definition 8.4. For each $i \in L$, $\int_a^b f^i(t, x, \dot{x}, y, \dot{y})$ is said to be $(F, \alpha_1, \rho^1_i, d^1_i)$-pseudoconvex in x and $\dot{x}$ for fixed y and $\dot{y}$, if

$$\int_a^b F(t, x, \dot{x}, y, \dot{y}; \alpha_1(x, u)(f^i_x(t, u, \dot{u}, y, \dot{y}) - Df^i_{\dot{x}}(t, u, \dot{u}, y, \dot{y})))dt \geqq 0$$
$$\Rightarrow \int_a^b f^i(t, x, \dot{x}, y, \dot{y})dt \geqq \int_a^b f^i(t, u, \dot{u}, y, \dot{y})dt + \rho^1_i \int_a^b \left(d^1_i(t, x, \dot{x}, u, \dot{u})\right)^2 dt,$$

for all $x,\ u : I \to R^n$ and for some arbitrary sublinear functional F.

In the sequel, we will write $F(t, x, u; \xi)$ for $F(t, x, \dot{x}, u, \dot{u}; \xi)$ and $G(t, v, y; \eta)$ for $G(t, v, \dot{v}, y, \dot{y}; \eta)$.

8.3 Problem formulation and duality results

Kim and Lee [80] formulated a pair of symmetric dual variational problems in the spirit of Mond and Weir [103] using pseudo-invexity. Recently, Hachimi and Aghezzaf [65] proved mixed duality results for multiobjective variational programming problems using the concept of generalized (F, α, ρ, d)-type I functions. Very recently, Ahmad and Sharma [15] formulated multiobjective fractional variational symmetric dual problems over arbitrary cones and established usual duality theorems under generalized F-convexity assumptions.

Now, we consider the problem of finding functions $x : I \to R^n$ and $y : I \to R^m$, where $(\dot{x}, \dot{y})$ is piecewise smooth on I, to solve the following pair of symmetric dual multiobjective nondifferentiable fractional variational problems over arbitrary cones:

Primal Problem (VP)

minimize

$$\frac{\displaystyle\int_a^b \{f(t,x,\dot{x},y,\dot{y})+s(x\mid B)-y^Tz\}dt}{\displaystyle\int_a^b \{g(t,x,\dot{x},y,\dot{y})-s(x\mid E)+y^Tr\}dt}$$

$$=\Bigg(\frac{\displaystyle\int_a^b \{f^1(t,x,\dot{x},y,\dot{y})+s(x\mid B_1)-y^Tz^1\}dt}{\displaystyle\int_a^b \{g^1(t,x,\dot{x},y,\dot{y})-s(x\mid E_1)+y^Tr^1\}dt},\dots,\frac{\displaystyle\int_a^b \{f^l(t,x,\dot{x},y,\dot{y})+s(x\mid B_l)-y^Tz^l\}dt}{\displaystyle\int_a^b \{g^l(t,x,\dot{x},y,\dot{y})-s(x\mid E_l)+y^Tr^l\}dt}\Bigg)$$

subject to

$$x(a)=0=x(b),\quad y(a)=0=y(b),$$

$$\dot{x}(a)=0=\dot{x}(b),\quad \dot{y}(a)=0=\dot{y}(b),$$

$$-\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot{y}}^i-z^i\big]-\big[g_y^i-Dg_{\dot{y}}^i+r^i\big]\frac{F^i(x,y)}{G^i(x,y)}\Big\}\in C_2^*,\quad t\in I,$$

$$y^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot{y}}^i-z^i\big]-\big[g_y^i-Dg_{\dot{y}}^i+r^i\big]\frac{F^i(x,y)}{G^i(x,y)}\Big\}\geqq 0,\quad t\in I,$$

$$\lambda>0,\quad x(t)\in C_1,\quad t\in I,$$

$$z^i\in D_i,\quad r^i\in H_i,\quad i=1,2,\dots,l.$$

Dual Problem (VD)

maximize

$$\frac{\displaystyle\int_a^b \{f(t,u,\dot{u},v,\dot{v})-s(v\mid D)+u^Tw\}dt}{\displaystyle\int_a^b \{g(t,u,\dot{u},v,\dot{v})+s(v\mid H)-u^Ts\}dt}$$

$$=\Bigg(\frac{\displaystyle\int_a^b \{f^1(t,u,\dot{u},v,\dot{v})-s(v\mid D_1)+u^Tw^1\}dt}{\displaystyle\int_a^b \{g^1(t,u,\dot{u},v,\dot{v})+s(v\mid H_1)-u^Ts^1\}dt},\dots,\frac{\displaystyle\int_a^b \{f^l(t,u,\dot{u},v,\dot{v})-s(v\mid D_l)+u^Tw^l\}dt}{\displaystyle\int_a^b \{g^l(t,u,\dot{u},v,\dot{v})+s(v\mid H_l)-u^Ts^l\}dt}\Bigg)$$

subject to

$$u(a)=0=u(b),\quad v(a)=0=v(b),$$

$$\dot{u}(a)=0=\dot{u}(b),\quad \dot{v}(a)=0=\dot{v}(b),$$

$$\sum_{i=1}^{l}\lambda_i\Big\{\big[f_x^i-Df_{\dot{x}}^i+w^i\big]-\big[g_x^i-Dg_{\dot{x}}^i-s^i\big]\frac{M^i(u,v)}{L^i(u,v)}\Big\}\in C_1^*,\quad t\in I,$$

$$u^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f_x^i-Df_{\dot{x}}^i+w^i\big]-\big[g_x^i-Dg_{\dot{x}}^i-s^i\big]\frac{M^i(u,v)}{L^i(u,v)}\Big\}\leqq 0,\quad t\in I,$$

$$\lambda > 0, \quad v(t) \in C_2, \quad t \in I,$$
$$w^i \in B_i, \quad s^i \in E_i, \quad i = 1, 2, \ldots, l,$$

where $f^i : I \times R^n \times R^n \times R^m \times R^m \to R$ and $g^i : I \times R^n \times R^n \times R^m \times R^m \to R, \;\; i \in L,$ are continuously differentiable functions and

$$F^i(x,y) = \int_a^b \left\{ f^i(t, x, \dot{x}, y, \dot{y}) + s(x \mid B_i) - y^T z^i \right\} dt;$$

$$G^i(x,y) = \int_a^b \left\{ g^i(t, x, \dot{x}, y, \dot{y}) - s(x \mid E_i) + y^T r^i \right\} dt;$$

$$M^i(u,v) = \int_a^b \left\{ f^i(t, u, \dot{u}, v, \dot{v}) - s(v \mid D_i) + u^T w^i \right\} dt;$$

and

$$L^i(u,v) = \int_a^b \left\{ g^i(t, u, \dot{u}, v, \dot{v}) + s(v \mid H_i) - u^T s^i \right\} dt.$$

Assume that

$G^i(x,y) > 0, \; L^i(u,v) > 0, \; F^i(x,y) \geqq 0, \; M^i(u,v) \geqq 0, \; \forall \; i.$

In order to simplify the notations, let

$$p_i = \frac{F^i(x,y)}{G^i(x,y)} = \frac{\displaystyle\int_a^b \left\{ f^i(t, x, \dot{x}, y, \dot{y}) + s(x \mid B_i) - y^T z^i \right\} dt}{\displaystyle\int_a^b \left\{ g^i(t, x, \dot{x}, y, \dot{y}) - s(x \mid E_i) + y^T r^i \right\} dt}$$

and

$$q_i = \frac{M^i(u,v)}{L^i(u,v)} = \frac{\displaystyle\int_a^b \left\{ f^i(t, u, \dot{u}, v, \dot{v}) - s(v \mid D_i) + u^T w^i \right\} dt}{\displaystyle\int_a^b \left\{ g^i(t, u, \dot{u}, v, \dot{v}) + s(v \mid H_i) - u^T s^i \right\} dt}$$

and express problems (VP) and (VD) equivalently as follows:

Primal Problem (FVP)

Minimize $\quad p = \big(p_1, p_2, \ldots, p_l\big)$

subject to

$$x(a) = 0 = x(b), \quad y(a) = 0 = y(b), \tag{8.1}$$

$$\dot{x}(a) = 0 = \dot{x}(b), \quad \dot{y}(a) = 0 = \dot{y}(b), \tag{8.2}$$

$$\begin{aligned} &\int_a^b \left\{ f^i(t, x, \dot{x}, y, \dot{y}) + s(x \mid B_i) - y^T z^i \right\} dt \\ &\quad - p_i \int_a^b \left\{ g^i(t, x, \dot{x}, y, \dot{y}) - s(x \mid E_i) + y^T r^i \right\} dt = 0, \quad i \in L, \end{aligned} \tag{8.3}$$

$$-\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i - Df_{\dot{y}}^i - z^i\big] - p_i\big[g_y^i - Dg_{\dot{y}}^i + r^i\big]\Big\} \in C_2^*, \quad t \in I, \tag{8.4}$$

$$y^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i - Df_{\dot{y}}^i - z^i\big] - p_i\big[g_y^i - Dg_{\dot{y}}^i + r^i\big]\Big\} \geqq 0, \quad t \in I, \tag{8.5}$$

$$\lambda > 0, \quad x(t) \in C_1, \quad t \in I, \tag{8.6}$$

$$z^i \in D_i, \quad r^i \in H_i, \quad i = 1, 2, \ldots, l. \tag{8.7}$$

Dual Problem (FVD)

Maximize $\qquad q = \big(q_1, q_2, \ldots, q_l\big)$

subject to

$$u(a) = 0 = u(b), \quad v(a) = 0 = v(b), \tag{8.8}$$

$$\dot{u}(a) = 0 = \dot{u}(b), \quad \dot{u}(a) = 0 = \dot{u}(b), \tag{8.9}$$

$$\begin{aligned}&\int_a^b \Big\{f^i(t, u, \dot{u}, v, \dot{v}) - s(v \mid D_i) + u^T w^i\Big\}dt \\ &\quad -q_i\int_a^b \Big\{g^i(t, u, \dot{u}, v, \dot{v}) + s(v \mid H_i) - u^T s^i\Big\}dt = 0, \quad i \in L,\end{aligned} \tag{8.10}$$

$$\sum_{i=1}^{l}\lambda_i\Big\{\big[f_x^i - Df_{\dot{x}}^i + w^i\big] - q_i\big[g_x^i - Dg_{\dot{x}}^i - s^i\big]\Big\} \in C_1^*, \quad t \in I, \tag{8.11}$$

$$u^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f_x^i - Df_{\dot{x}}^i + w^i\big] - q_i\big[g_x^i - Dg_{\dot{x}}^i - s^i\big]\Big\} \leqq 0, \quad t \in I, \tag{8.12}$$

$$\lambda > 0, \quad v(t) \in C_2, \quad t \in I, \tag{8.13}$$

$$w^i \in B_i, \quad s^i \in E_i, \quad i = 1, 2, \ldots, l. \tag{8.14}$$

In the above problems (FVP) and (FVD), it is to be noted that p and q are nonnegative. Let P and Q denote the sets of feasible solutions of (FVP) and (FVD), respectively. In the subsequent analysis, usual duality theorems are discussed in terms of (FVP) and (FVD), but equally apply to (VP) and (VD).

Now we establish weak, strong, converse and self duality theorems for the problems (FVP) and (FVD).

Theorem 8.1 (Weak duality). Let $(x,y,p,\lambda,z,r)\in P$ and $(u,v,q,\lambda,w,s)\in Q$. Let the sublinear functionals $F : I\times R^n\times R^n\times R^n \mapsto R$ and $G : I\times R^m\times R^m\times R^m\mapsto R$ satisfy the following conditions:

$F\big(t,x,u;a\big)+\alpha_1^{-1}a^Tu \geqq 0,\quad \text{for all } a\in C_1^*,\quad t\in I \qquad\qquad \text{(A)}$

$G\big(t,v,y;b\big)+\alpha_2^{-1}b^Ty \geqq 0,\quad \text{for all } b\in C_2^*,\quad t\in I. \qquad\qquad \text{(B)}$

Suppose that either

$(i)\quad \sum\limits_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot{x},u,\dot{u})\Big]^2dt+\sum\limits_{i=1}^{l}\lambda_i\rho_i^2\int_a^b\Big[d_i^2(t,v,\dot{v},y,\dot{y})\Big]^2dt\geqq 0$

or $\rho_i^1\geqq 0$ and $\rho_i^2\geqq 0$, $\;i\in L$.

Furthermore, assume that $\sum\limits_{i=1}^{l}\lambda_i\int_a^b\Big\{\big(f^i(t,\cdot,\cdot,v,\dot{v})+(\cdot)^Tw^i\big)-q_i\big(g^i(t,\cdot,\cdot,v,\dot{v})-(\cdot)^Ts^i\big)\Big\}dt$ is $(F,\alpha_1,\rho_i^1,d_i^1)$-pseudoconvex in x and $\dot{x}$, $\sum\limits_{i=1}^{l}\lambda_i\int_a^b\Big\{\big(f^i(t,x,\dot{x},\cdot,\cdot)-(\cdot)^Tz^i\big)-p_i\big(g^i(t,x,\dot{x},\cdot,\cdot)+(\cdot)^Tr^i\big)\Big\}dt$ is $(G,\alpha_2,\rho_i^2,d_i^2)$-pseudoconcave in y and $\dot{y}$.

Then $p\nleqq q$.

Proof. As (x,y,p,λ,z,r) is feasible for the primal problem (FVP) and (u,v,q,λ,w,s) is feasible for the dual problem (FVD), $\alpha_1(x,u)>0$, by the dual constraint (8.11), the vector $a=\alpha_1(x,u)\sum\limits_{i=1}^{l}\lambda_i\Big\{\big[f_x^i-Df_{\dot{x}}^i+w^i\big]-q_i\big[g_x^i-Dg_{\dot{x}}^i-s^i\big]\Big\}\in C_1^*,\quad t\in I$, and so from hypothesis (A), we obtain

$$\begin{aligned}F\bigg(t,x,u;\alpha_1(x,u)\sum_{i=1}^{l}\lambda_i\Big\{\big[f_x^i-Df_{\dot{x}}^i+w^i\big]-q_i\big[g_x^i-Dg_{\dot{x}}^i-s^i\big]\Big\}\bigg)&\\ \geqq -u^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f_x^i-Df_{\dot{x}}^i+w^i\big]-q_i\big[g_x^i-Dg_{\dot{x}}^i-s^i\big]\Big\}&\\ \geqq\ 0 \text{ (by dual constraint (8.12)).}&\end{aligned}$$

By using sublinearity of F, we get

$$\sum_{i=1}^{l}\lambda_iF\bigg(t,x,u;\alpha_1(x,u)\Big\{\big[f_x^i-Df_{\dot{x}}^i+w^i\big]-q_i\big[g_x^i-Dg_{\dot{x}}^i-s^i\big]\Big\}\bigg)\geqq 0,$$

which implies

$$\sum_{i=1}^{l}\lambda_i\int_a^bF\bigg(t,x,u;\alpha_1(x,u)\Big\{\big[f_x^i-Df_{\dot{x}}^i+w^i\big]-q_i\big[g_x^i-Dg_{\dot{x}}^i-s^i\big]\Big\}\bigg)dt\geqq 0.\quad (8.15)$$

This, in view of $(F,\alpha_1,\rho_i^1,d_i^1)$-pseudoconvexity of $\sum\limits_{i=1}^{l}\lambda_i\displaystyle\int_a^b\{\big(f^i(t,\cdot,\cdot,v,\dot v)+(\cdot)^T w^i\big)-q_i\big(g^i(t,\cdot,\cdot,v,\dot v)-(\cdot)^T s^i\big)\}dt$ in x and $\dot x$, we have

$$\begin{aligned}\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,x,\dot x,v,\dot v)+x^Tw^i\Big)&\\ -q_i\Big(g^i(t,x,\dot x,v,\dot v)+s(v\mid H_i)-x^Ts^i\Big)\bigg\}dt&\\ \geqq\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,u,\dot u,v,\dot v)+u^Tw^i\Big)-q_i\Big(g^i(t,u,\dot u,v,\dot v)+s(v\mid H_i)&\\ -u^Ts^i\Big)\bigg\}dt+\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot x,u,\dot u)\Big]^2dt.&\end{aligned}\tag{8.16}$$

Since $x^Tw^i\leqq s(x\mid B_i),\ w^i\in B_i,\ i=1,2,\ldots,l$, (8.16) can be written as

$$\begin{aligned}\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,x,\dot x,v,\dot v)+s(x\mid B_i)\Big)&\\ -q_i\Big(g^i(t,x,\dot x,v,\dot v)+s(v\mid H_i)-x^Ts^i\Big)\bigg\}dt&\\ \geqq\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,u,\dot u,v,\dot v)+u^Tw^i\Big)-q_i\Big(g^i(t,u,\dot u,v,\dot v)+s(v\mid H_i)&\\ -u^Ts^i\Big)\bigg\}dt+\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot x,u,\dot u)\Big]^2dt.&\end{aligned}$$

From $x^Ts^i\leqq s(x\mid E_i),\ s^i\in E_i,\ v^Tr^i\leqq s(v\mid H_i),\ r^i\in H_i,\ q_i\geqq 0,\ i=1,2,\ldots,l$, by above inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,x,\dot x,v,\dot v)+s(x\mid B_i)-s(v\mid D_i)\Big)&\\ -q_i\Big(g^i(t,x,\dot x,v,\dot v)-s(x\mid E_i)+v^Tr^i\Big)\bigg\}dt\geqq\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot x,u,\dot u)\Big]^2dt.&\end{aligned}\tag{8.17}$$

By the primal constraint (8.4), $\alpha_2(v,y)>0$, taking vector $b=-\alpha_2(v,y)\sum\limits_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot y}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot y}^i+r^i\big]\Big\}\in C_2^*,\quad t\in I$, and using hypothesis (B), we obtain

$$\begin{aligned}G\bigg(t,v,y;-\alpha_2(v,y)\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot y}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot y}^i+r^i\big]\Big\}\bigg)&\\ \geqq y^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot y}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot y}^i+r^i\big]\Big\}&\end{aligned}$$

$$\geqq \ 0 \text{ (by primal constraint (8.5)).}$$

Using sublinearity of G, we get

$$\sum_{i=1}^{l} \lambda_i G\biggl(t, v, y; -\alpha_2(v, y)\biggl\{\bigl[f_y^i - Df_{\dot{y}}^i - z^i\bigr] - p_i\bigl[g_y^i - Dg_{\dot{y}}^i + r^i\bigr]\biggr\}\biggr) \geqq 0,$$

which implies

$$\sum_{i=1}^{l} \lambda_i \int_a^b G\biggl(t, v, y; -\alpha_2(v, y)\biggl\{\bigl[f_y^i - Df_{\dot{y}}^i - z^i\bigr] - p_i\bigl[g_y^i - Dg_{\dot{y}}^i + r^i\bigr]\biggr\}\biggr) dt \geqq 0. \qquad (8.18)$$

The $(G, \alpha_2, \rho_i^2, d_i^2)$-pseudoconcavity of $\sum_{i=1}^{l} \lambda_i \int_a^b \Bigl\{\bigl(f^i(t, x, \dot{x}, \cdot, \cdot) - (\cdot)^T z^i\bigr) - p_i\bigl(g^i(t, x, \dot{x}, \cdot, \cdot) + (\cdot)^T r^i\bigr)\Bigr\} dt$ in y and $\dot{y}$ yields

$$\begin{aligned}
&\sum_{i=1}^{l} \lambda_i \int_a^b \biggl\{\Bigl(f^i(t, x, \dot{x}, v, \dot{v}) - v^T z^i\Bigr) \\
&\qquad\qquad -p_i\Bigl(g^i(t, x, \dot{x}, v, \dot{v}) - s(x \mid E_i) + v^T r^i\Bigr)\biggr\} dt \\
&\leqq \sum_{i=1}^{l} \lambda_i \int_a^b \biggl\{\Bigl(f^i(t, x, \dot{x}, y, \dot{y}) - y^T z^i\Bigr) - p_i\Bigl(g^i(t, x, \dot{x}, y, \dot{y}) - s(x \mid E_i) \\
&\qquad\qquad +y^T r^i\Bigr)\biggr\} dt - \sum_{i=1}^{l} \lambda_i \rho_i^2 \int_a^b \Bigl[d_i^2(t, v, \dot{v}, y, \dot{y})\Bigr]^2 dt. \qquad (8.19)
\end{aligned}$$

Since $v^T z^i \leqq s(v \mid D_i),\ z^i \in D_i,\ i = 1, 2, \ldots, l$, (8.19) becomes

$$\begin{aligned}
&\sum_{i=1}^{l} \lambda_i \int_a^b \biggl\{\Bigl(f^i(t, x, \dot{x}, v, \dot{v}) + s(x \mid B_i) - s(v \mid D_i)\Bigr) \\
&\qquad -p_i\Bigl(g^i(t, x, \dot{x}, v, \dot{v}) - s(x \mid E_i) + v^T r^i\Bigr)\biggr\} dt \leqq -\sum_{i=1}^{l} \lambda_i \rho_i^2 \int_a^b \Bigl[d_i^2(t, v, \dot{v}, y, \dot{y})\Bigr]^2 dt. \quad (8.20)
\end{aligned}$$

From (8.17) and (8.20), we get

$$\sum_{i=1}^{l}\lambda_i\int_a^b\Bigg\{\Bigg(f^i(t,x,\dot{x},v,\dot{v})+s(x\mid B_i)-s(v\mid D_i)\Bigg)-q_i\Bigg(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)$$

$$+v^Tr^i\Bigg)\Bigg\}dt-\sum_{i=1}^{l}\lambda_i\int_a^b\Bigg\{\Bigg(f^i(t,x,\dot{x},v,\dot{v})+s(x\mid B_i)-s(v\mid D_i)\Bigg)-p_i\Bigg(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)$$

$$+v^Tr^i\Bigg)\Bigg\}dt\geqq\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot{x},u,\dot{u})\Big]^2dt+\sum_{i=1}^{l}\lambda_i\rho_i^2\int_a^b\Big[d_i^2(t,v,\dot{v},y,\dot{y})\Big]^2dt.\quad(8.$$

Using hypothesis (i) in (8.21), we obtain

$$\sum_{i=1}^{l}\lambda_i\big(p_i-q_i\big)\int_a^b\Bigg(g^i(t,x(t),\dot{x}(t),v(t),\dot{v}(t))-s(x\mid E_i)+v^Tr^i\Bigg)dt\geqq 0.\qquad(8.22)$$

If, for some $i\in L$, $p_i<q_i$ and $p_i\leqq q_j$, for all $j\neq i$, then from $\lambda>0$ and $\displaystyle\int_a^b\Bigg(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)+v^Tr^i\Bigg)dt>0,\quad i=1,2,\ldots,l$, implies that

$$\sum_{i=1}^{l}\lambda_i\big(p_i-q_i\big)\int_a^b\Bigg(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)+v^Tr^i\Bigg)dt<0,$$

which contradicts (8.22). Hence $p\nleq q$.

Theorem 8.2 (Weak duality). Let $(x,y,p,\lambda,z,r)\in P$ and $(u,v,q,\lambda,w,s)\in Q$. Let the sublinear functionals $F:I\times R^n\times R^n\times R^n\mapsto R$ and $G:I\times R^m\times R^m\times R^m\mapsto R$ satisfy the following conditions:

$F\big(t,x,u;a\big)+\alpha_1^{-1}a^Tu\geqq 0$, for all $a\in C_1^*$, $t\in I$ (A)

$G\big(t,v,y;b\big)+\alpha_2^{-1}b^Ty\geqq 0$, for all $b\in C_2^*$, $t\in I$. (B)

Suppose that either

(i) $\displaystyle\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot{x},u,\dot{u})\Big]^2dt+\sum_{i=1}^{l}\lambda_i\rho_i^2\int_a^b\Big[d_i^2(t,v,\dot{v},y,\dot{y})\Big]^2dt\geqq 0$

or $\rho_i^1\geqq 0$ and $\rho_i^2\geqq 0$, $i\in L$.

Furthermore, assume that $\displaystyle\int_a^b\Big\{\big(f^i(t,\cdot,\cdot,v,\dot{v})+(\cdot)^Tw^i\big)-q_i\big(g^i(t,\cdot,\cdot,v,\dot{v})-(\cdot)^Ts^i\big)\Big\}dt$ is $(F,\alpha_1,\rho_i^1,d_i^1)$-convex in x and $\dot{x}$, $\displaystyle\int_a^b\Big\{\big(f^i(t,x,\dot{x},\cdot,\cdot)-(\cdot)^Tz^i\big)-p_i\big(g^i(t,x,\dot{x},\cdot,\cdot)+(\cdot)^Tr^i\big)\Big\}dt$ is $(G,\alpha_2,\rho_i^2,d_i^2)$-concave in y and $\dot{y}$.

Then

$$p\nleq q.\qquad(8.23)$$

Proof. Assume by contradiction that (8.23) is not true, that is, $p \leq q$, i.e., $p_i < q_i$, for some $i \in L$ and $p_i \leqq q_j$, for all $j \neq i$, then from $\lambda > 0$ and $\int_a^b \Big(g^i(t, x, \dot{x}, v, \dot{v}) - s(x \mid E_i) + v^T r^i \Big) dt > 0, \quad i = 1, 2, \ldots, l$, we have

$$\sum_{i=1}^{l} \lambda_i \big(p_i - q_i\big) \int_a^b \Big(g^i(t, x, \dot{x}, v, \dot{v}) - s(x \mid E_i) + v^T r^i \Big) dt < 0. \tag{8.24}$$

Since (x, y, p, λ, z, r) is feasible for the primal problem (FVP) and (u, v, q, λ, w, s) is feasible for the dual problem (FVD), $\alpha_1(x, u) > 0$, by the dual constraint (8.11), the vector $a = \alpha_1(x, u) \sum_{i=1}^{l} \lambda_i \Big\{ \big[f_x^i - D f_{\dot{x}}^i + w^i\big] - q_i \big[g_x^i - D g_{\dot{x}}^i - s^i\big] \Big\} \in C_1^*, \quad t \in I$, and so from hypothesis (A), we obtain

$$\begin{aligned} F\Big(t, x, u; \alpha_1(x, u) \sum_{i=1}^{l} \lambda_i \Big\{ \big[f_x^i - D f_{\dot{x}}^i + w^i\big] - q_i \big[g_x^i - D g_{\dot{x}}^i - s^i\big] \Big\}\Big) \\ \geqq -u^T \sum_{i=1}^{l} \lambda_i \Big\{ \big[f_x^i - D f_{\dot{x}}^i + w^i\big] - q_i \big[g_x^i - D g_{\dot{x}}^i - s^i\big] \Big\} \\ \geqq \ 0 \text{ (by dual constraint (8.12)).} \end{aligned}$$

By using sublinearity of F, we get

$$\sum_{i=1}^{l} \lambda_i F\Big(t, x, u; \alpha_1(x, u) \Big\{ \big[f_x^i - D f_{\dot{x}}^i + w^i\big] - q_i \big[g_x^i - D g_{\dot{x}}^i - s^i\big] \Big\} \Big) \geqq 0,$$

which further implies

$$\sum_{i=1}^{l} \lambda_i \int_a^b F\Big(t, x, u; \alpha_1(x, u) \Big\{ \big[f_x^i - D f_{\dot{x}}^i + w^i\big] - q_i \big[g_x^i - D g_{\dot{x}}^i - s^i\big] \Big\} \Big) dt \geqq 0. \tag{8.25}$$

Since $\int_a^b \{ \big(f^i(t, \cdot, \cdot, v, \dot{v}) + (\cdot)^T w^i\big) - q_i \big(g^i(t, \cdot, \cdot, v, \dot{v}) - (\cdot)^T s^i\big) \} dt$ is $(F, \alpha_1, \rho_i^1, d_i^1)$-convex in x and $\dot{x}$, we have

$$\begin{aligned} &\int_a^b \Big\{ \Big(f^i(t, x, \dot{x}, v, \dot{v}) + x^T w^i \Big) - q_i \Big(g^i(t, x, \dot{x}, v, \dot{v}) + s(v \mid H_i) - x^T s^i \Big) \Big\} dt \\ &\quad - \int_a^b \Big\{ \Big(f^i(t, u, \dot{u}, v, \dot{v}) + u^T w^i \Big) - q_i \Big(g^i(t, u, \dot{u}, v, \dot{v}) + s(v \mid H_i) - u^T s^i \Big) \Big\} dt \\ &\qquad \geqq \int_a^b F\Big(t, x, u; \alpha_1(x, u) \Big\{ \big[f_x^i - D f_{\dot{x}}^i + w^i\big] - q_i \big[g_x^i - D g_{\dot{x}}^i - s^i\big] \Big\} \Big) dt \\ &\qquad\quad + \rho_i^1 \int_a^b \Big[d_i^1(t, x, \dot{x}, u, \dot{u}) \Big]^2 dt. \end{aligned} \tag{8.26}$$

Using (8.13) and (8.25) in (8.26), we obtain

$$\begin{aligned}&\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,x,\dot{x},v,\dot{v})+x^Tw^i\Big)-q_i\Big(g^i(t,x,\dot{x},v,\dot{v})+s(v\mid H_i)-x^Ts^i\Big)\bigg\}dt\\&\quad-\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,u,\dot{u},v,\dot{v})+u^Tw^i\Big)-q_i\Big(g^i(t,u,\dot{u},v,\dot{v})+s(v\mid H_i)-u^Ts^i\Big)\bigg\}dt\\&\qquad\qquad\geqq\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot{x},u,\dot{u})\Big]^2dt.\end{aligned}\tag{8.27}$$

Since $x^Tw^i\leqq s(x\mid B_i),\ w^i\in B_i,\ i=1,2,\dots,l,$ (8.27) can be written as

$$\begin{aligned}&\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,x,\dot{x},v,\dot{v})+s(x\mid B_i)-s(v\mid D_i)\Big)\\&\quad-q_i\Big(g^i(t,x,\dot{x},v,\dot{v})+s(v\mid H_i)-x^Ts^i\Big)\bigg\}dt\geqq\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot{x},u,\dot{u})\Big]^2dt.\end{aligned}$$

From $x^Ts^i\leqq s(x\mid E_i),\ s^i\in E_i,\ v^Tr^i\leqq s(v\mid H_i),\ r^i\in H_i,\ q_i\geqq 0,\ i=1,2,\dots,l,$ by above inequality, we obtain

$$\begin{aligned}&\sum_{i=1}^{l}\lambda_i\int_a^b\bigg\{\Big(f^i(t,x,\dot{x},v,\dot{v})+s(x\mid B_i)-s(v\mid D_i)\Big)\\&\quad-q_i\Big(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)+v^Tr^i\Big)\bigg\}dt\geqq\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b\Big[d_i^1(t,x,\dot{x},u,\dot{u})\Big]^2dt.\end{aligned}\tag{8.28}$$

By the primal constraint (8.4), $\alpha_2(v,y)>0$, taking vector $b=-\alpha_2(v,y)\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot{y}}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot{y}}^i+r^i\big]\Big\}\in C_2^*,\quad t\in I$, and using hypothesis (B), we obtain

$$\begin{aligned}&G\Big(t,v,y;-\alpha_2(v,y)\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot{y}}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot{y}}^i+r^i\big]\Big\}\Big)\\&\qquad\geqq y^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot{y}}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot{y}}^i+r^i\big]\Big\}\\&\qquad\geqq\ 0\ \text{(by primal constraint (8.5))}.\end{aligned}$$

Using sublinearity of G, we get

$$\sum_{i=1}^{l}\lambda_iG\Big(t,v,y;-\alpha_2(v,y)\Big\{\big[f_y^i-Df_{\dot{y}}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot{y}}^i+r^i\big]\Big\}\Big)\geqq 0,$$

which shows that

$$\sum_{i=1}^{l}\lambda_i\int_a^bG\Big(t,v,y;-\alpha_2(v,y)\Big\{\big[f_y^i-Df_{\dot{y}}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot{y}}^i+r^i\big]\Big\}\Big)dt\geqq 0.\tag{8.29}$$

The $(G,\alpha_2,\rho_i^2,d_i^2)$-concavity of $\int_a^b \{\big(f^i(t,x,\dot{x},\cdot,\cdot)-(\cdot)^T z^i\big)-p_i\big(g^i(t,x,\dot{x},\cdot,\cdot)+(\cdot)^T r^i\big)\}dt$ in y and $\dot{y}$ yields

$$\begin{aligned}
&\int_a^b \bigg\{\Big(f^i(t,x,\dot{x},v,\dot{v})-v^T z^i\Big)-p_i\Big(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)+v^T r^i\Big)\bigg\}dt+\int_a^b G\Big(t,v,y;\\
&\quad -\alpha_2(v,y)\Big\{\big[f_y^i-Df_{\dot{y}}^i-z^i\big]-p_i\big[g_y^i-Dg_{\dot{y}}^i+r^i\big]\Big\}\Big)dt\leqq\int_a^b \bigg\{\Big(f^i(t,x,\dot{x},y,\dot{y})\\
&\qquad -y^T z^i\Big)-p_i\Big(g^i(t,x,\dot{x},y,\dot{y})-s(x\mid E_i)+y^T r^i\Big)\bigg\}dt-\rho_i^2\int_a^b \Big[d_i^2(t,v,\dot{v},y,\dot{y})\Big]^2 dt. \quad (8.30)
\end{aligned}$$

From (8.6), (8.29) and (8.30), we have

$$\begin{aligned}
&\sum_{i=1}^{l}\lambda_i\int_a^b \bigg\{\Big(f^i(t,x,\dot{x},v,\dot{v})-v^T z^i\Big)-p_i\Big(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)+v^T r^i\Big)\bigg\}dt\\
&\quad\leqq\sum_{i=1}^{l}\lambda_i\int_a^b \bigg\{\Big(f^i(t,x,\dot{x},y,\dot{y})-y^T z^i\Big)-p_i\Big(g^i(t,x,\dot{x},y,\dot{y})-s(x\mid E_i)+y^T r^i\Big)\bigg\}dt\\
&\qquad -\sum_{i=1}^{l}\lambda_i\rho_i^2\int_a^b \Big[d_i^2(t,v,\dot{v},y,\dot{y})\Big]^2 dt. \quad (8.31)
\end{aligned}$$

Since $v^T z^i\leqq s(v\mid D_i),\ z^i\in D_i,\ i=1,2,\dots,l,$ (8.31) becomes

$$\begin{aligned}
&\sum_{i=1}^{l}\lambda_i\int_a^b \bigg\{\Big(f^i(t,x,\dot{x},v,\dot{v})+s(x\mid B_i)-s(v\mid D_i)\Big)\\
&\qquad -p_i\Big(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)+v^T r^i\Big)\bigg\}dt\leqq-\sum_{i=1}^{l}\lambda_i\rho_i^2\int_a^b \Big[d_i^2(t,v,\dot{v},y,\dot{y})\Big]^2 dt. \quad (8.32)
\end{aligned}$$

From (8.28) and (8.32), we get

$$\begin{aligned}
&\sum_{i=1}^{l}\lambda_i\int_a^b \bigg\{\Big(f^i(t,x,\dot{x},v,\dot{v})+s(x\mid B_i)-s(v\mid D_i)\Big)\\
&-q_i\Big(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)+v^T r^i\Big)\bigg\}dt\\
&\quad -\sum_{i=1}^{l}\lambda_i\int_a^b \bigg\{\Big(f^i(t,x,\dot{x},v,\dot{v})+s(x\mid B_i)-s(v\mid D_i)\Big)\\
&\qquad -p_i\Big(g^i(t,x,\dot{x},v,\dot{v})-s(x\mid E_i)+v^T r^i\Big)\bigg\}dt\\
&\qquad\quad \geqq\sum_{i=1}^{l}\lambda_i\rho_i^1\int_a^b \Big[d_i^1(t,x,\dot{x},u,\dot{u})\Big]^2 dt+\sum_{i=1}^{l}\lambda_i\rho_i^2\int_a^b \Big[d_i^2(t,v,\dot{v},y,\dot{y})\Big]^2 dt. \quad (8.33)
\end{aligned}$$

Using hypothesis (i) in (8.33), we obtain

$$\begin{aligned}&\sum_{i=1}^{l}\lambda_i\int_a^b\Big\{\Big(f^i(t,x,\dot x,v,\dot v)+s(x\mid B_i)-s(v\mid D_i)\Big)\\&\qquad\qquad\qquad -q_i\Big(g^i(t,x,\dot x,v,\dot v)-s(x\mid E_i)+v^Tr^i\Big)\Big\}dt\\&\geqq\sum_{i=1}^{l}\lambda_i\int_a^b\Big\{\Big(f^i(t,x,\dot x,v,\dot v)+s(x\mid B_i)-s(v\mid D_i)\Big)\\&\qquad\qquad\qquad -p_i\Big(g^i(t,x,\dot x,v,\dot v)-s(x\mid E_i)+v^Tr^i\Big)\Big\}dt.\end{aligned}$$

that is,

$$\sum_{i=1}^{l}\lambda_i\big(p_i-q_i\big)\int_a^b\Big(g^i(t,x(t),\dot x(t),v(t),\dot v(t))-s(x\mid E_i)+v^Tr^i\Big)dt\geqq 0, \tag{8.34}$$

which contradicts (8.24). Hence the result.

Any problem, say (FVD), in which λ is fixed to be $\bar\lambda$ will be denoted by $(FVD)_{\bar\lambda}$.

Theorem 8.3 (Strong duality). Let $(\bar x,\bar y,\bar p,\bar\lambda,\bar z,\bar r)$ be a weakly efficient solution for (FVP). Assume that

(*i*) $\Big[\sigma(t)^T\Big\{\sum_{i=1}^{l}\bar\lambda_i\big[\big(f^i_{yy}-\bar p_i g^i_{yy}\big)-D\big(f^i_{\dot y y}-\bar p_i g^i_{\dot y y}\big)\big]\Big\}+D\Big[\sigma(t)^T\sum_{i=1}^{l}\bar\lambda_i\big\{D(f^i_{\dot y\dot y}-\bar p_i g^i_{\dot y\dot y})\big\}\Big]+$ $D^2\big\{-\sigma(t)^T\sum_{i=1}^{l}\bar\lambda_i\big(f^i_{\dot y\dot y}-\bar p_i g^i_{\dot y\dot y}\big)\big\}\Big]\sigma(t)=0$ implies $\sigma(t)=0,\ \forall\ t\in I$, and

(*ii*) the set of vectors $\Big\{\big(f^i_y-\bar z^i\big)-\bar p_i\big(g^i_y+\bar r^i\big)-D\big(f^i_{\dot y}-\bar p_i g^i_{\dot y}\big)\Big\}_{i=1}^{l}$ is linearly independent.

Then there exist $\bar w^i\in R^n,\ \bar s^i\in R^n,\ i=1,2,\dots,l$, such that $(\bar x,\bar y,\bar p,\bar w,\bar s)$ is feasible for $(FVD)_{\bar\lambda}$, and the objective values of (FVP) and $(FVD)_{\bar\lambda}$ are equal. Furthermore, if the hypotheses of a weak duality theorem are satisfied for all feasible solutions of $(FVP)_{\bar\lambda}$ and $(FVD)_{\bar\lambda}$, then $(\bar x,\bar y,\bar p,\bar w,\bar s)$ is an efficient solution of $(FVD)_{\bar\lambda}$.

Proof. Since $(\bar x,\bar y,\bar p,\bar\lambda,\bar z,\bar r)$ be a weakly efficient solution of (FVP), by Fritz John optimality conditions [123], there exist $\alpha\in R^l,\ \beta\in R^l$, piecewise smooth $\gamma(t):I\to$

$C_2,\ \xi(t) : I \to R$ and $\delta \in R^l$ such that

$$\tilde{M} = \sum_{i=1}^{l} \alpha_i p_i + \sum_{i=1}^{l} \beta_i \Big[\big(f^i + s(\bar{x} \mid B_i) - \bar{y}^T \bar{z}^i\big) - \bar{p}_i\big(g^i - s(\bar{x} \mid E_i) + \bar{y}^T \bar{r}^i\big)\Big]$$
$$+ \big[\gamma - \xi \bar{y}^T\big]\Big[\sum_{i=1}^{l} \lambda_i \Big\{ \big(f_y^i - \bar{z}^i\big) - \bar{p}_i\big(g_y^i + \bar{r}^i\big) - D\big(f_{\dot{y}}^i - \bar{p}_i g_{\dot{y}}^i\big)\Big\}\Big] - \delta^T \bar{\lambda}$$

satisfies the following conditions at $(\bar{x}, \bar{y}, \bar{p}, \bar{\lambda}, \bar{z}, \bar{r})$:

$$\Big[\tilde{M}_x - D\tilde{M}_{\dot{x}} + D^2 \tilde{M}_{\ddot{x}}\Big]\Big\{x(t) - \bar{x}(t)\Big\} \geqq 0,\ \forall\ x(t) \in C_1,\ t \in I, \tag{8.35}$$

$$\tilde{M}_y - D\tilde{M}_{\dot{y}} + D^2 \tilde{M}_{\ddot{y}} = 0, \quad t \in I, \tag{8.36}$$

$$\tilde{M}_\lambda = 0, \quad t \in I, \tag{8.37}$$

$$\tilde{M}_p = 0, \quad t \in I, \tag{8.38}$$

$$\int_a^b \beta_i \Big[\big(f^i + s(\bar{x} \mid B_i) - \bar{y}^T \bar{z}^i\big)$$
$$-\bar{p}_i(g^i - s(\bar{x} \mid E_i) + \bar{y}^T \bar{r}^i)\Big] dt = 0,\ i \in L, \quad t \in I, \tag{8.39}$$

$$\gamma^T \sum_{i=1}^{l} \bar{\lambda}_i \Big\{ \big(f_y^i - \bar{z}^i\big) - \bar{p}_i\big(g_y^i + \bar{r}^i\big) - D\big(f_{\dot{y}}^i - \bar{p}_i g_{\dot{y}}^i\big)\Big\} = 0, \quad t \in I, \tag{8.40}$$

$$\xi \bar{y}^T \sum_{i=1}^{l} \lambda_i \Big\{ \big(f_y^i - \bar{z}^i\big) - \bar{p}_i\big(g_y^i + \bar{r}^i\big) - D\big(f_{\dot{y}}^i - \bar{p}_i g_{\dot{y}}^i\big)\Big\} = 0, \quad t \in I, \tag{8.41}$$

$$\delta^T \bar{\lambda} = 0, \tag{8.42}$$

$$s(\bar{x} \mid B_i) = \bar{x}^T \eta_i,\ \eta_i \in B_i,\ \ i = 1, 2, \ldots, l, \tag{8.43}$$

$$s(\bar{x} \mid E_i) = \bar{x}^T \theta_i,\ \theta_i \in E_i,\ \ i = 1, 2, \ldots, l, \tag{8.44}$$

$$\beta_i \bar{y}^T + \big[\gamma - \xi \bar{y}\big]\bar{\lambda}_i \in N_{D_i}(\bar{z}^i), \tag{8.45}$$

$$\bar{p}_i \Big[\beta_i \bar{y} + \big(\gamma - \xi \bar{y}\big)\bar{\lambda}_i\Big] \in N_{H_i}(\bar{r}^i), \tag{8.46}$$

$$(\alpha, \beta, \gamma, \xi, \delta) \neq 0, \quad t \in I. \tag{8.47}$$

The above relation hold throughout the interval I, except at the corners of $(\bar{x}, \bar{y}, \bar{p}, \bar{\lambda}, \bar{z}, \bar{r})$, where (8.35) and (8.36) hold for unique right and left hand limits. The piecewise smooth functions γ and ξ are continuously differentiable except possibly of the corners of $(\bar{x}, \bar{y}, \bar{p}, \bar{\lambda}, \bar{z}, \bar{r})$.

Using the observations on $Df^i_{\dot{y}}$ and $Dg^i_{\dot{y}},\ i = 1, 2, \ldots, l$, from Section 8.2, equations (8.35)-(8.38) become

$$\begin{aligned}
&\Bigg[\sum_{i=1}^{l} \beta_i \Big\{ \big[\big(f^i_x + \eta_i\big) - \bar{p}_i\big(g^i_x - \theta_i\big)\big] - D\big(f^i_{\dot{x}} - \bar{p}_i g^i_{\dot{x}}\big) \Big\} \\
&\qquad\qquad + \big[\gamma - \xi\bar{y}\big]^T \Bigg[\sum_{i=1}^{l} \bar{\lambda}_i \big\{ \big(f^i_{yx} - \bar{p}_i g^i_{yx}\big) - D\big(f^i_{\dot{y}x} - \bar{p}_i g^i_{\dot{y}x}\big) \big\} \Bigg] \\
&\qquad\qquad\qquad\qquad - D\Bigg[\big(\gamma - \xi\bar{y}\big)^T \sum_{i=1}^{l} \bar{\lambda}_i \big\{ \big(f^i_{y\dot{x}} - \bar{p}_i g^i_{y\dot{x}}\big) - D\big(f^i_{\dot{y}\dot{x}} - \bar{p}_i g^i_{\dot{y}\dot{x}}\big) \\
&\qquad\qquad\qquad\qquad\qquad - \big(f^i_{\dot{y}x} - \bar{p}_i g^i_{\dot{y}x}\big) \big\} \Bigg] + D^2 \Big\{ - \big[\gamma - \xi\bar{y}\big]^T \sum_{i=1}^{l} \big(f^i_{\dot{y}\dot{x}} - \bar{p}_i g^i_{\dot{y}\dot{x}}\big) \Big\} \Bigg] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\{ x(t) - \bar{x}(t) \Big\} \geqq 0,\ \forall\ x(t) \in C_1,\ t \in I,
\end{aligned} \tag{8.48}$$

$$\begin{aligned}
&\sum_{i=1}^{l} \big[\beta_i - \xi\bar{\lambda}_i\big] \Big\{ \big(f^i_y - \bar{z}^i\big) - \bar{p}_i\big(g^i_y + \bar{r}^i\big) - D\big(f^i_{\dot{y}} - \bar{p}_i g^i_{\dot{y}}\big) \Big\} \\
&\qquad\qquad + \big[\gamma - \xi\bar{y}\big]^T \Bigg[\sum_{i=1}^{l} \bar{\lambda}_i \big\{ \big(f^i_{yy} - \bar{p}_i g^i_{yy}\big) - D\big(f^i_{\dot{y}y} - \bar{p}_i g^i_{\dot{y}y}\big) \big\} \Bigg] \\
&\qquad\qquad\qquad\qquad + D\Bigg[\big(\gamma - \xi\bar{y}\big)^T \sum_{i=1}^{l} \bar{\lambda}_i \big\{ D\big(f^i_{\dot{y}\dot{y}} - \bar{p}_i g^i_{\dot{y}\dot{y}}\big) \big\} \Bigg] \\
&\qquad\qquad\qquad\qquad\qquad + D^2 \Big\{ - \big(\gamma - \xi\bar{y}\big)^T \sum_{i=1}^{l} \bar{\lambda}_i \big(f^i_{\dot{y}\dot{y}} - \bar{p}_i g^i_{\dot{y}\dot{y}}\big) \Big\} = 0, \quad t \in I,
\end{aligned} \tag{8.49}$$

$$\begin{aligned}
&\Big\{ \big(f^i_y - \bar{z}^i\big) - \bar{p}_i\big(g^i_y + \bar{r}^i\big) \\
&\qquad\qquad - D\big(f^i_{\dot{y}} - \bar{p}_i g^i_{\dot{y}}\big) \Big\} \Big[\gamma - \xi\bar{y} \Big]^T - \delta_i = 0, \quad i = 1, 2, \ldots, l, \quad t \in I,
\end{aligned} \tag{8.50}$$

and

$$\alpha_i - \beta_i\big(g^i - \theta_i + \bar{y}^T\bar{r}^i\big) - \big[\gamma - \xi\bar{y}\big]^T \big[g^i_y + \bar{r}^i - Dg^i_{\dot{y}}\big] = 0,\ \ i = 1, 2, \ldots, l,\ \ t \in I. \tag{8.51}$$

Since $\delta \geqq 0$ and $\bar{\lambda} > 0$, (8.42) yields $\delta = 0$. Consequently (8.50) becomes

$$\Big\{\big(f^i_y - \bar{z}^i\big) - \bar{p}_i\big(g^i_y + \bar{r}^i\big) - D\big(f^i_{\dot{y}} - \bar{p}_i g^i_{\dot{y}}\big)\Big\}\Big[\gamma - \xi\bar{y}\Big]^T = 0,\ i = 1, 2, \dots, l,\ t \in I. \tag{8.52}$$

Multiplying (8.49) by $\big[\gamma - \xi\bar{y}\big]$ and using (8.52), we have

$$\begin{aligned}\Big[\gamma - \xi\bar{y}\Big]^T \sum_{i=1}^{l} \bar{\lambda}_i \Big\{\big(f^i_{yy} - \bar{p}_i g^i_{yy}\big) - D\big(f^i_{\dot{y}y} - \bar{p}_i g^i_{\dot{y}y}\big)\Big\}\Big[\gamma - \xi\bar{y}\Big] \\ + D\Big[\big(\gamma - \xi\bar{y}\big)^T \sum_{i=1}^{l} \bar{\lambda}_i \big\{D\big(f^i_{\dot{y}\dot{y}} - \bar{p}_i g^i_{\dot{y}\dot{y}}\big)\big\}\Big]\Big[\gamma - \xi\bar{y}\Big] \\ + D^2\Big\{-\big(\gamma - \xi\bar{y}\big)^T \sum_{i=1}^{l} \bar{\lambda}_i \big(f^i_{\dot{y}\dot{y}} - \bar{p}_i g^i_{\dot{y}\dot{y}}\big)\Big\}\Big[\gamma - \xi\bar{y}\Big] = 0, \quad t \in I,\end{aligned} \tag{8.53}$$

This, in view of hypothesis (i), yields

$$\gamma - \xi\bar{y} = 0, \quad t \in I. \tag{8.54}$$

From (8.49), we have
$\sum\limits_{i=1}^{l} \big[\beta_i - \xi\bar{\lambda}_i\big]\Big\{\big(f^i_y - \bar{z}^i\big) - \bar{p}_i\big(g^i_y + \bar{r}^i\big) - D\big(f^i_{\dot{y}} - \bar{p}_i g^i_{\dot{y}}\big)\Big\} = 0,$
which, because of hypothesis (ii), implies

$\beta_i - \xi\bar{\lambda}_i = 0, \quad i = 1, 2, \dots, l,$

or

$$\beta = \xi\bar{\lambda}. \tag{8.55}$$

If, for $t \in I$, $\xi = 0$, then from (8.54) and (8.55), we get $\gamma = 0$ and $\beta = 0$, respectively. Also, (8.51) gives $\alpha = 0$. Hence $(\alpha, \beta, \gamma, \xi, \delta) = 0$, contradicting (8.47). Thus $\xi > 0, \quad t \in I$.

From (8.54), we have

$$\bar{y} = \frac{\gamma}{\xi} \in C_2, \quad t \in I. \tag{8.56}$$

Now, (8.48) along with (8.54) and (8.55), and with $\xi > 0$, gives

$$\sum_{i=1}^{l} \bar{\lambda}_i \Big\{\big[\big(f^i_x + \eta_i\big) - \bar{p}_i\big(g^i_x - \theta_i\big)\big] - D\big(f^i_{\dot{x}} - \bar{p}_i g^i_{\dot{x}}\big)\Big\}\Big(x(t) - \bar{x}(t)\Big) \geqq 0,\ t \in I. \tag{8.57}$$

Let $x(t) \in C_1$. Then $x(t) + \bar{x}(t) \in C_1, \quad t \in I$ and so (8.57) show that for every $x(t) \in C_1$,
$\sum_{i=1}^{l} \bar{\lambda}_i \Big\{ \big[\big(f_x^i + \eta_i\big) - \bar{p}_i\big(g_x^i - \theta_i\big)\big] - D\big(f_{\dot{x}}^i - \bar{p}_i g_{\dot{x}}^i\big) \Big\} x(t) \geqq 0, \ t \in I.$
Therefore,

$$\sum_{i=1}^{l} \bar{\lambda}_i \Big\{ \big[\big(f_x^i + \eta_i\big) - \bar{p}_i\big(g_x^i - \theta_i\big)\big] - D\big(f_{\dot{x}}^i - \bar{p}_i g_{\dot{x}}^i\big) \Big\} \in C_1^*, \quad t \in I. \tag{8.58}$$

Also, by letting $x(t) = 0$ and $x(t) = 2\bar{x}(t)$ simultaneously in (8.57), we obtain

$$\bar{x}(t)^T \sum_{i=1}^{l} \bar{\lambda}_i \Big\{ \big[\big(f_x^i + \eta_i\big) - \bar{p}_i\big(g_x^i - \theta_i\big)\big] - D\big(f_{\dot{x}}^i - \bar{p}_i g_{\dot{x}}^i\big) \Big\} = 0, \quad t \in I. \tag{8.59}$$

Thus, from (8.56), (8.58) and (8.59), it follows that $(\bar{x}, \bar{y}, \bar{p}, \bar{w} = \eta, \bar{s} = \theta)$ is a feasible solution for the dual problem $(FVD)_{\bar{\lambda}}$.
Further, from (8.45), (8.54), (8.55) and $\xi > 0, \quad t \in I$, we have for $i = 1, 2, \ldots, l$,
$\bar{\lambda}_i \bar{y} \in N_{D_i}(\bar{z}^i)$
or $\bar{y} \in N_{D_i}(\bar{z}^i)$, using $\ \bar{\lambda}_i > 0, \quad i = 1, 2, \ldots, l$.
Since D_i is a compact convex set in R^m, $\bar{y}^T \bar{z}^i = s(\bar{y} \mid D_i), \quad i = 1, 2, \ldots, l$.
Also, from (8.46), (8.54), (8.55) and $\xi > 0, \quad t \in I$, we have for $i = 1, 2, \ldots, l$,
$\bar{p}_i \bar{\lambda}_i \bar{y} \in N_{H_i}(\bar{r}^i)$
or $\bar{y} \in N_{H_i}(\bar{r}^i)$, using $\ \bar{\lambda}_i > 0, \quad i = 1, 2, \ldots, l$.
Since H_i is a compact convex set in R^m, $\bar{y}^T \bar{r}^i = s(\bar{y} \mid H_i), \quad i = 1, 2, \ldots, l$.
Thus (FVP) and $(FVD)_{\bar{\lambda}}$ have equal objective function values (i.e., $\bar{p} = \bar{q}$).
If $(\bar{x}, \bar{y}, \bar{p}, \bar{w}, \bar{s})$ is not an efficient solution of $(FVD)_{\bar{\lambda}}$, then there exists $(\bar{u}, \bar{v}, \bar{q}, \bar{w}, \bar{s})$ feasible for $(FVD)_{\bar{\lambda}}$ such that
$\bar{p} \leq \bar{q}$,
which contradicts weak duality theorem (Theorem 8.1 or 8.2). Thus $(\bar{x}, \bar{y}, \bar{p}, \bar{w}, \bar{s})$ is an efficient solution of $(FVD)_{\bar{\lambda}}$. Hence the result.

Theorem 8.4 (Converse duality). Let $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{w}, \bar{s})$ be a weakly efficient solution for (FVD). Assume that

(i) $\Big[\psi(t)^T \Big\{ \sum_{i=1}^{l} \bar{\lambda}_i \big[\big(f_{xx}^i - \bar{q}_i g_{xx}^i\big) - D\big(f_{\dot{x}x}^i - \bar{q}_i g_{\dot{x}x}^i\big)\big] \Big\} + D\Big[\psi(t)^T \sum_{i=1}^{l} \bar{\lambda}_i \big\{ D(f_{\dot{x}\dot{x}}^i - \bar{q}_i g_{\dot{x}\dot{x}}^i) \big\} \Big] +$

$$D^2\Big\{-\psi(t)^T\sum_{i=1}^{l}\bar{\lambda}_i\big(f^i_{\dot{x}\dot{x}}-\bar{q}_i g^i_{\dot{x}\dot{x}}\big)\Big\}\Big]\psi(t)dt=0$$

implies $\psi(t)=0,\ \forall\ t\in I$, and

(ii) the set of vectors $\left\{\big(f^i_x+\bar{w}^i\big)-\bar{q}_i\big(g^i_x-\bar{s}^i\big)-D\big(f^i_{\dot{x}}-\bar{q}_i g^i_{\dot{x}}\big)\right\}_{i=1}^{l}$ is linearly independent.

Then there exist $\bar{z}^i\in R^m,\ \bar{r}^i\in R^m,\ i=1,2,\ldots,l$, such that $(\bar{u},\bar{v},\bar{q},\bar{z},\bar{r})$ is feasible for $(FVP)_{\bar{\lambda}}$, and the objective values of $(FVP)_{\bar{\lambda}}$ and (FVD) are equal. Furthermore, if the hypotheses of a weak duality theorem are satisfied for all feasible solutions of $(FVP)_{\bar{\lambda}}$ and $(FVD)_{\bar{\lambda}}$, then $(\bar{u},\bar{v},\bar{q},\bar{z},\bar{r})$ is an efficient solution of $(FVP)_{\bar{\lambda}}$.

Proof. Follows on the lines of Theorem 8.3.

8.4 Self duality

A mathematical programming problem is said to be self dual, if it is formally identical with its dual that is, the dual can be recast in the form of the primal. In general, the problem (NKFVP) is not self dual. In order to establish the self duality some conditions are required. Assume that x and y have the same dimension, $B_i=D_i,\ E_i=H_i$, $C_1=C_2=C,\ C_1^*=C_2^*=C^*,\ f^i$ and $g^i,\ i\in L$, are skew symmetric and symmetric respectively. Further assume that

$g(t,u,\dot{u},v,\dot{v})+s(v\mid H)-u^Ts\ \ =g(t,v,\dot{v},u,\dot{u})-s(v\mid H)+u^Ts.$

The function $f^i(t,u,\dot{u},v,\dot{v}),\ i\in L$, is said to be skew symmetric, if

$f^i(t,u,\dot{u},v,\dot{v})=-f^i(t,v,\dot{v},u,\dot{u}),\ i\in L,\ t\in I,$

for all u and v in the domain of f^i,

and the function $g^i(t,u,\dot{u},v,\dot{v}),\ i\in L$, is said to be symmetric, if

$g^i(t,u,\dot{u},v,\dot{v})=g^i(t,v,\dot{v},u,\dot{u}),\ i\in L,\ t\in I,$

for all u and v in the domain of g^i.

By recasting the dual problem (VD) as minimization problem, we have minimize

$$\left[\frac{\displaystyle\int_a^b \left\{ -f(t,u,\dot{u},v,\dot{v}) + s(v \mid D) - u^T w \right\} dt}{\displaystyle\int_a^b \left\{ g(t,u,\dot{u},v,\dot{v}) + s(v \mid H) - u^T s \right\} dt}\right]$$

$$= \left[\frac{\displaystyle\int_a^b \left\{ -f^1(t,u,\dot{u},v,\dot{v}) + s(v \mid D_1) - u^T w^1 \right\} dt}{\displaystyle\int_a^b \left\{ g^1(t,u,\dot{u},v,\dot{v}) + s(v \mid H_1) - u^T s^1 \right\} dt}\right.$$

$$\left., \ldots, \frac{\displaystyle\int_a^b \left\{ -f^l(t,u,\dot{u},v,\dot{v}) + s(v \mid D_l) - u^T w^l \right\} dt}{\displaystyle\int_a^b \left\{ g^l(t,u,\dot{u},v,\dot{v}) + s(v \mid H_l) - u^T s^l \right\} dt}\right]$$

subject to

$$u(a) = 0 = u(b), \quad v(a) = 0 = v(b),$$

$$\dot{u}(a) = 0 = \dot{u}(b), \quad \dot{v}(a) = 0 = \dot{v}(b),$$

$$\sum_{i=1}^{l} \lambda_i \left\{ \left[f_x^i - D f_{\dot{x}}^i + w^i \right] - \left[g_x^i - D g_{\dot{x}}^i - s^i \right] \frac{M^i(u,v)}{L^i(u,v)} \right\} \in C^*, \quad t \in I,$$

$$u^T \sum_{i=1}^{l} \lambda_i \left\{ \left[f_x^i - D f_{\dot{x}}^i + w^i \right] - \left[g_x^i - D g_{\dot{x}}^i - s^i \right] \frac{M^i(u,v)}{L^i(u,v)} \right\} \leqq 0, \quad t \in I,$$

$$\lambda > 0, \quad v(t) \in C, \quad t \in I,$$

$$w^i \in B_i, \quad s^i \in E_i, \quad i = 1, 2, \ldots, l.$$

Since f^i and g^i, $i \in L$ are skew symmetric and symmetric, respectively. Consequently, it follows that

$f_x^i(t,u,\dot{u},v,\dot{v}) = -f_y^i(t,v,\dot{v},u,\dot{u}),\ f_{\dot{x}}^i(t,u,\dot{u},v,\dot{v}) = -f_{\dot{y}}^i(t,v,\dot{v},u,\dot{u}),\ i \in L,\ t \in I,$

and $g_x^i(t,u,\dot{u},v,\dot{v}) = g_y^i(t,v,\dot{v},u,\dot{u}),\ g_{\dot{x}}^i(t,u,\dot{u},v,\dot{v}) = g_{\dot{y}}^i(t,v,\dot{v},u,\dot{u}),\ i \in L,\ t \in I.$

On using the skew symmetry and symmetry of f^i and g^i, $i \in L$, respectively, and the aforementioned assumptions, the above problem becomes

$(VD)^*$

minimize

$$\left[\frac{\displaystyle\int_a^b \left\{ f(t,v,\dot{v},u,\dot{u}) + s(v \mid D) - u^T w \right\} dt}{\displaystyle\int_a^b \left\{ g(t,v,\dot{v},u,\dot{u}) - s(v \mid H) + u^T s \right\} dt}\right]$$

$$= \Big[\frac{\displaystyle\int_a^b \Big\{f^1(t,v,\dot v,u,\dot u)+s(v\mid D_1)-u^Tw^1\Big\}dt}{\displaystyle\int_a^b\Big\{g^1(t,v,\dot v,u,\dot u)-s(v\mid H_1)+u^Ts^1\Big\}dt},\dots,\frac{\displaystyle\int_a^b \Big\{f^l(t,v,\dot v,u,\dot u)+s(v\mid D_l)-u^Tw^l\Big\}dt}{\displaystyle\int_a^b\Big\{g^l(t,v,\dot v,u,\dot u)-s(v\mid H_l)+u^Ts^l\Big\}dt}\Big]$$

subject to $u(a)=0=u(b),\quad v(a)=0=v(b),$

$$\dot u(a)=0=\dot u(b),\quad \dot v(a)=0=\dot v(b),$$

$$-\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot y}^i-w^i\big]-\big[g_y^i-Dg_{\dot y}^i+s^i\big]\frac{M^i(v,u)}{L^i(v,u)}\Big\}\in C^*,\quad t\in I,$$

$$u^T\sum_{i=1}^{l}\lambda_i\Big\{\big[f_y^i-Df_{\dot y}^i-w^i\big]-\big[g_y^i-Dg_{\dot y}^i+s^i\big]\frac{M^i(v,u)}{L^i(v,u)}\Big\}\geqq 0,\quad t\in I,$$

$$\lambda>0,\quad v(t)\in C,\quad t\in I,$$

$$w^i\in B_i,\quad s^i\in E_i,\quad i=1,2,\dots,l,$$

which is formally identical to (VP), i.e., the objectives, the constraint functions and the initial conditions of (VP) and $(VD)^*$ are identical. Therefore (VP) is self dual. Thus, if $(\bar x,\bar y,\bar p,\bar\lambda,\bar z,\bar r)$ is feasible for (VP), then $(\bar y,\bar x,\bar p,\bar\lambda,\bar z,\bar r)$ is feasible for (VD) and conversely.

Theorem 8.5 (Self duality). If $f(t,x,\dot x,y,\dot y)$ is skew-symmetric and $g(t,x,\dot x,y,\dot y)$ is symmetric along with the assumptions $m=n,\ B=D,\ E=H,\ C_1=C_2=C,\ C_1^*=C_2^*=C^*$ and $g(t,u,\dot u,v,\dot v)+s(v\mid H)-u^Ts=g(t,v,\dot v,u,\dot u)-s(v\mid H)+u^Ts$, then (VP) is self dual. Also, if (VP) and (VD) are dual variational problems, and $(x^*,y^*,p^*,\lambda^*,w^*,s^*)$ is a joint weakly efficient solution, then so is $(y^*,x^*,p^*,\lambda^*,w^*,s^*)$ and

$$\frac{\displaystyle\int_a^b\Big\{f(t,x^*,\dot x^*,y^*,\dot y^*)-s(y^*\mid B)+x^{*T}w^*\Big\}dt}{\displaystyle\int_a^b\Big\{g(t,x^*,\dot x^*,y^*,\dot y^*)+s(y^*\mid E)-x^{*T}s^*\Big\}dt}=0.$$

Proof. Since $(x^*,y^*,p^*,\lambda^*,w^*,s^*)$ is a joint weakly efficient solution of (VP) and (VD), the objective functional values are equal to

$$\frac{\displaystyle\int_a^b\Big\{f(t,x^*,\dot x^*,y^*,\dot y^*)-s(y^*\mid B)+x^{*T}w^*\Big\}dt}{\displaystyle\int_a^b\Big\{g(t,x^*,\dot x^*,y^*,\dot y^*)+s(y^*\mid E)-x^{*T}s^*\Big\}dt}.$$

As (VP) is a self dual, it follows that $(x^*, y^*, p^*, \lambda^*, w^*, s^*)$ is feasible for (VP) iff $(y^*, x^*, p^*, \lambda^*, w^*, s^*)$ is feasible for (VD). Thus, weak efficiency of $(x^*, y^*, p^*, \lambda^*, w^*, s^*)$ for (VP) implies the weak efficiency of $(y^*, x^*, p^*, \lambda^*, w^*, s^*)$ for (VD) and conversely. Also the two objective values are equal to

$$\frac{\displaystyle\int_a^b \left\{ f(t, y^*, \dot{y}^*, x^*, \dot{x}^*) + s(y^* \mid B) - x^{*T} w^* \right\} dt}{\displaystyle\int_a^b \left\{ g(t, y^*, \dot{y}^*, x^*, \dot{x}^*) - s(y^* \mid E) + x^{*T} s^* \right\} dt}.$$

Thus, we have

$$\begin{aligned}
&\frac{\displaystyle\int_a^b \left\{ f(t, x^*, \dot{x}^*, y^*, \dot{y}^*) - s(y^* \mid B) + x^{*T} w^* \right\} dt}{\displaystyle\int_a^b \left\{ g(t, x^*, \dot{x}^*, y^*, \dot{y}^*) + s(y^* \mid E) - x^{*T} s^* \right\} dt} \\
&\qquad = \frac{\displaystyle\int_a^b \left\{ f(t, y^*, \dot{y}^*, x^*, \dot{x}^*) + s(y^* \mid B) - x^{*T} w^* \right\} dt}{\displaystyle\int_a^b \left\{ g(t, y^*, \dot{y}^*, x^*, \dot{x}^*) - s(y^* \mid E) + x^{*T} s^* \right\} dt} \\
&\qquad = -\frac{\displaystyle\int_a^b \left\{ f(t, x^*, \dot{x}^*, y^*, \dot{y}^*) - s(y^* \mid B) + x^{*T} w^* \right\} dt}{\displaystyle\int_a^b \left\{ g(t, x^*, \dot{x}^*, y^*, \dot{y}^*) + s(y^* \mid E) - x^{*T} s^* \right\} dt},
\end{aligned}$$

$\Bigg($ using the skew symmetry of f and the symmetry of g along with the assumptions of $B = D,\ E = H$, and

$$g(t, u, \dot{u}, v, \dot{v}) + s(v \mid H) - u^T s = g(t, v, \dot{v}, u, \dot{u}) - s(v \mid H) + u^T s, \Bigg).$$

Therefore,

$$\frac{\displaystyle\int_a^b \left\{ f(t, x^*, \dot{x}^*, y^*, \dot{y}^*) - s(y^* \mid B) + x^{*T} w^* \right\} dt}{\displaystyle\int_a^b \left\{ g(t, x^*, \dot{x}^*, y^*, \dot{y}^*) + s(y^* \mid E) - x^{*T} s^* \right\} dt} = 0.$$

Hence the result.

Appendix A

A Note On Second-Order Symmetric Duality[1]

A.1 Introduction

Yang et al. [140] studied duality relations for the following pair of multiobjective second-order symmetric dual programs:

Primal problem

$$\text{(MP)} \quad \min_{x,y,\lambda,p} \quad F_P(x,y,\lambda,p) = f(x,y) - y^T\nabla_y(\lambda^T f)(x,y)e_k - y^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k - \tfrac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k$$

$$\text{subject to} \quad \nabla_y(\lambda^T f)(x,y) + \nabla_{yy}(\lambda^T f)(x,y)p \leqq 0,$$

$$\lambda > 0, \qquad \lambda^T e_k = 1,$$

Dual problem

$$\text{(MD)} \quad \max_{u,v,\lambda,r} \quad F_D(u,v,\lambda,r) = f(u,v) - u^T\nabla_u(\lambda^T f)(u,v)e_k - u^T(\nabla_{uu}(\lambda^T f)(u,v)r)e_k - \tfrac{1}{2}r^T(\nabla_{uu}(\lambda^T f)(u,v)r)e_k$$

$$\text{subject to} \quad \nabla_u(\lambda^T f)(u,v) + \nabla_{uu}(\lambda^T f)(u,v)r \geqq 0,$$

$$\lambda > 0, \qquad \lambda^T e_k = 1,$$

where $f : R^n \times R^m \longrightarrow R^k, p \in R^m, r \in R^n, \lambda \in R^k$ and $e_k = (1,1,\ldots,1)^T \in R^k$. For other notations and definitions, we refer to Yang et al. [140]. After establishing a weak duality theorem under second-order F-convexity assumptions, they proved the

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[1] A part of this chapter has appeared in *European Journal of Operational Research* 201 (2010) 649-651.

following strong duality theorem:

Theorem A.1. Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ be a weak efficient solution of (MP). Assume that
(i) the matrix $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is nonsingular,
(ii) the vectors $\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})$ are linearly independent,
(iii) the vector $\bar{p}^T \nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p}) = 0$ implies that $\bar{p} = 0$, and
(iv) $\bar{p} \neq 0$ implies $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} \neq 0$.
Assume further that the assumptions of Theorem 1 in [140] are satisfied. Then the objective values of (MP) and (MD) are equal, and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r} = 0)$ is an efficient solution of (MD).

In the proof of strong duality theorem, the authors claim $\bar{p} = 0$. To prove this, they have taken two cases. However, in the second case, if $\bar{\alpha} \neq (\bar{\alpha}^T e_k)\bar{\lambda}$, then using the hypotheses (ii) and (iv) in

$$\nabla_y(\nabla_{yy}(\lambda^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} = \frac{-2}{(\alpha^T e_k)} \nabla_y f(\bar{x}, \bar{y})(\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}) \tag{A.1}$$

does not imply $\bar{p} = 0$. Since if we suppose that $\bar{p} \neq 0$, then using the above assumptions we should obtain a contradiction. However, from assumption (iv) and (A.1), we obtain
$\nabla_y f(\bar{x}, \bar{y})(\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}) \neq 0$,
which along with the given assumptions does not prove the contradiction. Hence, the proof of the Theorem 2 in [140] seems to be erroneous.

In the present note, we have given an appropriate modification for this deficiency contained in Theorem A.1.

A.2 Strong duality

In this section, we present a strong duality theorem which is a correction of Theorem A.1. The assumption (iii) in Theorem A.1 has been replaced by assumption (iii) below in Theorem A.2. On the same lines, the converse duality theorem (Theorem 3 in [140]) can be rectified.

Theorem A.2. (Strong duality). Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ be a weak efficient solution of (MP). Assume that

(i) the matrix $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is nonsingular,

(ii) the vectors $\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})$ are linearly independent,

(iii) the vector $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} \notin \operatorname{span}\{\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\}\backslash\{0\}$, and

(iv) $\bar{p} \neq 0$ implies $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} \neq 0$.

Assume further that the assumptions of Theorem 1 in [140] are satisfied. Then the objective values of (MP) and (MD) are equal, and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r} = 0)$ is an efficient solution of (MD).

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is a weak efficient solution of (MP), by the Fritz John optimality conditions [46], there exist $\bar{\alpha} \in R^k$, $\bar{\beta} \in R^m$, $\bar{\mu} \in R$ and $\bar{\omega} \in R^k$ such that

$$\begin{aligned}&\nabla_x(\bar{\alpha}^T f)(\bar{x}, \bar{y}) + \nabla_{xy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}) \\ &\qquad + \nabla_x\{\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p}\}(\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p})) = 0,\end{aligned} \tag{A.2}$$

$$\begin{aligned}&\nabla_y f(\bar{x}, \bar{y})(\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}) + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}) - (\bar{\alpha}^T e_k) \\ &\qquad \times \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p} + \nabla_y\{\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p}\}(\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p})) = 0,\end{aligned} \tag{A.3}$$

$$\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})) = 0, \tag{A.4}$$

$$\begin{aligned}&\nabla_y^T f(\bar{x}, \bar{y})(\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}) - \bar{\omega} + \bar{\mu} e_k + \{(\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p}))^T \\ &\qquad\qquad + \nabla_{yy} f_1(\bar{x}, \bar{y})\bar{p}, \ldots, (\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \frac{1}{2}\bar{p}))^T \nabla_{yy} f_k(\bar{x}, \bar{y})\bar{p}\} = 0,\end{aligned} \tag{A.5}$$

$$\bar{\beta}^T\{\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p}\} = 0, \tag{A.6}$$

$$\bar{\omega}^T \bar{\lambda} = 0, \tag{A.7}$$

$$(\bar{\alpha}, \bar{\beta}, \bar{\omega}) \geqq 0, \qquad (\bar{\alpha}, \bar{\beta}, \bar{\omega}, \bar{\mu}) \neq 0. \tag{A.8}$$

Since $\bar{\lambda} > 0$ and $\bar{\omega} \geqq 0$, (A.7) yields $\bar{\omega} = 0$.

Using Hypothesis (i) in (A.4), we get

$$\bar{\beta} = (\bar{\alpha}^T e_k)(\bar{y} + \bar{p}). \tag{A.9}$$

If $\bar{\alpha} = 0$, then (A.9) yields $\bar{\beta} = 0$ and (A.5) gives $\bar{\mu} = 0$. Hence $(\bar{\alpha}, \bar{\beta}, \bar{\omega}, \bar{\mu}) = 0$, contradicting (A.8). Therefore, $\bar{\alpha} \geq 0$ or

$$\bar{\alpha}^T e_k \neq 0. \tag{A.10}$$

Now, we claim that $\bar{p} = 0$. Indeed, if $\bar{p} \neq 0$, then Hypothesis (iv) implies
$\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} \neq 0$.
Using (A.9) and (A.10) in (A.3), we obtain
$\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})\bar{p} = \dfrac{-2}{\bar{\alpha}^T e_k} \nabla_y f(\bar{x}, \bar{y})(\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda})$,
which contradicts the Hypothesis (iii). Hence, $\bar{p} = 0$.
The rest of the proof follows on the lines of Theorem 2 in Yang et al. [140].

Remark A.1. For single objective problems, the authors in [137] have considered the assumption

$$\nabla_y(\nabla_{yy} f(\bar{x}, \bar{y})\bar{p})\bar{p} = 0 \Rightarrow \bar{p} = 0.$$

It may be noted that this can also be obtained from the assumptions (iii) and (iv) in Theorem A.2. above as

$$\begin{aligned} &\nabla_y(\nabla_{yy} f(\bar{x}, \bar{y})\bar{p})\bar{p} \notin \operatorname{span}(\nabla_y f(\bar{x}, \bar{y}))/\{0\} \\ &\Rightarrow \nabla_y(\nabla_{yy} f(\bar{x}, \bar{y})\bar{p})\bar{p} = 0 \\ &\Rightarrow \bar{p} = 0. \end{aligned}$$

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