SYMMETRY REDUCTION METHOD FOR EXACT SOLUTIONS OF SOME NONLINEAR SYSTEMS

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CERTIFICATE

*I hereby certify that the work presented in the thesis entitled "**Symmetry Reduction Method for Exact Solutions of Some Nonlinear Systems**" which is being submitted for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Rajesh Kumar Gupta. The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.*

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

The Thesis entitled "SYMMETRY REDUCTION METHOD FOR EXACT SOLUTIONS OF SOME NONLINEAR SYSTEMS" is attempted to find some exact solutions of nonlinear systems of partial differential equations (PDEs) governing some important physical phenomenons by using Symmetry reduction method which is based on Fréchet derivatives of the differential operators and we drive Lie algebra which then helps us to obtain the optimal system of generators. Then, we find reduced ordinary differential equations (ODEs) from given nonlinear PDEs equations and some exact solutions of them.

The Thesis has been divided into six chapters. The brief outlines of the research work presented chapter wise in the thesis is as follows:

In the first chapter, we have described the introduction of non linear partial differential equations and exact solutions. In this chapter, we discussed Lie group of transformations in preliminary material and relevant literature.

In the second chapter, we have described the detailed study of Symmetry reduction method based on Fréchet derivatives of differential operators.

In the third chapter, we apply the Symmetry reduction method based on Fréchet derivatives, for each infinitesimal generator in the optimal system of sub algebras of Drinfeld Sokolov Wilson equations, we study the reductions and some exact solutions.

In forth chapter, by using Symmetry method, we deduce the Lie symmetries and find reduced ODEs and the exact solutions of Gardner equation.

In fifth chapter, we obtain reduced ordinary equations and solutions of Fisher equation with Symmetry method of Fréchet derivatives technique.

In sixth chapter also, we apply Symmetry reduction method in order to obtain symmetries of Rayleigh equation.

It is worth to mention that all the solutions reported in the thesis are checked by Maple software.

CHAPTER 1

INTRODUCTION

Most of the problems posed by nature and which are of interest to physicists and mathematicians, are inherently nonlinear and are usually governed by a single or a system of differential equations. Physical examples of linear systems are relatively rare. Nonlinear equations are difficult to solve and the linear approximations used to describe them are often a tacit admission that the underlying equations can not be solved. In fact, the study of nonlinear systems of differential equations is regarded as a difficult and confusing endeavor. When compared with the variety of techniques available in linear system theory, the tools for analysis and design of nonlinear systems are limited to some very special categories. In a sense, nonlinear systems are in their full complexity, and so it is not surprising that there exists no general method for solving them. This is the reason when confronted with a nonlinear differential equation, the first approach usually is to linearize it; in other words, to try to avoid the nonlinear aspects of the problem completely.

Some examples of nonlinear equations are as follows:

- General relativity
- The Navier-Stokes equations of fluid dynamics
- Systems with solitons as solutions
- Nonlinear optics
- Korteweg-de Vries equation
- Nonlinear Schrödinger equation

The study of nonlinear differential equations has not only provided information about the phenomenon but has, in fact, helped in making more precise some of the concepts and theories developed in the last century mathematics. The standard strategies adopted to get the solutions of nonlinear partial differential equations (PDEs) to date are following:

i) Linearize the given set of nonlinear equations by invoking certain physical assumptions.

ii) Numerical integration of the equations under appropriate boundary conditions.

iii) To derive exact solutions of nonlinear equations.

In fact, as the applications of the first two approaches are concerned, a great deal has been contributed. The third approach is being avoided due to cumbersome and complicated calculations. But the strong desire of exact and more general solutions to nonlinear PDEs governing nonlinear phenomenon in technological enhancement and for research purpose made tremendous growth in research interest of last approach. There is much current interest in obtaining exact solutions of nonlinear PDEs; these solutions provide information about nonlinear phenomena and various aspects of the physical phenomena. These solutions, often with several important physical parameters, prove useful to discuss and examine the sensitivity of physical phenomena they describe. The exact solutions are also helpful in designing and testing of numerical algorithms.

Exact solutions for nonlinear equations are rare, and the methods, which can generate families of them, are not only increasingly popular, but increasingly sought. So far, a number of methods have been proposed to construct the exact solutions; the group theoretic methods like Lie's classical method [1, 5], nonclassical method [2, 3], Steinberg's symmetry reduction method [7]; direct method [10], modified direct method [23]; truncated Painleve approach [8]; transformation methods [20]; ansätze-based methods [9]; hyperbolic functions expansion methods [14]; elliptic functions expansion methods [13] and sine-cosine method [35] etc.

The work carried out in this thesis is devoted to the applications of some group theoretic techniques. The prime objective and motivation in carrying out the proposed study is to demonstrate the importance and efficacy of group theoretic methods over various other methods available in the literature. In the thesis, five nonlinear differential equations considered for exact solutions are: Gardner equation, Fisher equation, Drinfeld equations, Rayleigh equation.

We have used Software like Maple during the research to drive and to test the authenticity of solutions.

1.1. Literature Review

The mathematical techniques which generate a wide range of solutions and applicable to all type of nonlinear differential equations are few. The group theoretic techniques based on Lie group theory can be categorized in this class and usually these techniques produce a variety of exact solutions. The analysis and classiffication of differential equation using group theory goes back to the Norwegian mathematician Sophus Lie (1880). He developed the theory of "finite and continuous groups". Lie devoted his mathematical career to the development and application of his monumental theory of continuous groups. These groups, now universally known as Lie groups, have a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering and other mathematically based sciences.

Nevertheless, anyone who is already familiar with one of these modern manifestations of Lie group theory is perhaps surprised to learn that its original inspirational source was the field of differential equations. The entire subject lay dormant for nearly half a century until G. Birkhoff (1950) [6] called attention to the unexploited application of the lie group to differential equations. Ovsiannikov [10] and his coworkers began a systematic program of successfully applying these methods to wide range of physically important problems.

This was followed by the work of Bluman and Cole [3, 4]. Since then, the theory has witnessed a veritable explosion of research both in the application to physical systems and its development (Olver [26, 27]).

Lie’s continuous group theoretic ideas have been classified as direct methods and group theoretic methods. The direct method consists of separation of variables devised by Kline [21] and Miller [24], and dimensional analysis due to Sedov [30]. Group theoretic methods are divided into two categories namely inspectional methods and deductive methods. Inspectional methods are two fold in the sense that the first one is due to Birkhoff [6], and the other is due to Hellums and Churchill [16]. In the class of deductive procedures, there are the following techniques proposed by different authors: Nonclassical method (Bluman and Cole [3]), Classical Lie method (Olver [26]), Symmetry reduction method (Steinberg [33]) etc.

1.2. Preliminary Material

In this section, we are presenting some of basic definitions essential for the techniques to derive exact solutions of the nonlinear systems.

(i) Lie Group of Transformations

Definition 1.2.1. A *group* G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

i) Closure property. For any elements a and b of G, $\phi(a,b)$ is an element of G.

ii) Associative property. For any elements a,b,c of G:

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c).$$

iii) Identity element. There exists a unique identity element e of G such that for any element a of G:

$$\phi(a,e) = \phi(e,a) = a.$$

iv) Inverse element. For any element a of G, there exists a unique inverse element a^{-1} in G such that

$$\phi(a,a^{-1}) = \phi(a^{-1},a) = e.$$

Definition 1.2.2. Let $x = (x_1, x_2, x_3, \ldots\ldots, x_n)$ lie in a region $D \subset R^n$ the set of transformation $x^* = X(x;\varepsilon)$ defined for each x in D and parameter ε in set $S \subset R$,with $\phi(\varepsilon,\delta)$ defining a law of composition of parameter ε and δ in S, forms a *One-parameter group of transformations* on D if the following hold:

(i) For each ε in S the transformations are one to one onto D.

(ii) S with the law of composition ϕ forms a group G.

(iii) For each x in D, $x^* = x$ when $\varepsilon = \varepsilon_0$ corresponds to identity e, that is,

$$X(x;\varepsilon_0) = x.$$

(iv) If $x^* = X(x;\varepsilon)$, $x^{**} = X(x^*;\delta)$ then $x^{**} = X(x;\phi(\varepsilon,\delta))$.

.

Definition 1.2.3. A one-parameter group of transformations define a *one-parameter Lie group of transformations* if, in addition to satisfying the property (i) to (iv) of definition 1.2.2., the following hold:

(v) ε is a continuous parameter.

(vi) X is infinitely differentiable with respect to x in D and an analytic function of ε in S .

(vii) $\phi(\varepsilon,\delta)$ is an analytic function of ε and δ , $\varepsilon,\delta \in S$.

Example:-

Consider
$$x^* = x + \varepsilon$$
$$y^* = y \quad , \quad \varepsilon \in R .$$

and $\phi(\varepsilon,\delta) = \varepsilon + \delta$, this forms a one-parameter lie group of transformations.

Definition 1.2.4. Consider a one-parameter Lie group of transformations

$$x^* = X(x;\varepsilon) \qquad (1)$$

with the identity $\varepsilon = 0$ and law of composition ϕ .Expanding (1) about $\varepsilon = 0$.

Then we get

$$x^* = x + \varepsilon\left(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}\right) + \frac{1}{2}\varepsilon^2\left(\frac{\partial^2 X(x;\varepsilon)}{\partial \varepsilon^2}\Big|_{\varepsilon=0}\right) + \ldots$$
$$= x + \varepsilon\left(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}\right) + O(\varepsilon^2).$$

Let $\xi(x) = \left(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}\right)$. Then the transformation $x + \varepsilon\xi(x)$ is called the *infinitesimal transformation* of lie group of transformation (1).

The components of $\xi(x)$ is called infinitesimals of (1).

Definition 1.2.5. The *infinitesimal generator* of the one- parameter lie group of transformations (1) is the operator $X = X(x) = \xi(x).\nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}$,

where ∇ is the gradient operator $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\ldots, \frac{\partial}{\partial x_n}\right)$.

For any differentiable function $F(x) = F(x_1, x_2, \ldots\ldots, x_n)$, one has

$$XF(x) = \xi(x)\nabla F(x) = \sum_{i=1}^{n} \xi_i(x)\frac{\partial F(x)}{\partial x_i}.$$

Definition 1.2.6. An infinitely differentiable function $F(x)$ is an invariant function of lie group of transformation (1) if and only if , for any group of transformation (1) ,

$$F(x^*) \equiv F(x).$$

If $F(x)$ is invariant function of (1) , then $F(x)$ is called an invariant of (1) and $F(x)$ is said to be invariant under (1).

Definition 1.2.7. $F(x)$ is invariant under a lie group of transformations (1) if and only if

$$XF(x) \equiv 0.$$

Example:-

Let a family of circles $\omega : x^2 + y^2$ = constant.

Let the group of rotations $x_1 = x\cos\varepsilon - y\sin\varepsilon$

$$y_1 = x\sin\varepsilon + y\cos\varepsilon$$

For group of rotations infinitesimals are $(\xi_{1,}\xi_2) = (-y, x)$.

Infinitesimal transformation is $x_1 = x - y\varepsilon$

$$y_1 = y + x\varepsilon \cdot$$

The infinitesimal generator is given by $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

Invariant condition $X\omega = 0$.

Hence, equation of family of circles is invariant under given transformations.

Definition 1.2.8. Point Transformations and Prolongations

In later chapters, we will be concerned with the determination of one-parameter Lie groups of transformations admitted by a given system S of differential equations. A one-parameter (ε) Lie group of point transformations is a group of transformations of the form

$$x^* = X(x,u;\varepsilon) \qquad (1.2.8a)$$

$$u^* = U(x,u;\varepsilon), \qquad (1.2.8b)$$

acting on the space of $n+m$ variables

$$x = (x_1, x_2, \ldots, x_n)$$

$$u = (u^1, u^2, \ldots, u^m),$$

where x represents n independent variables and u denotes m dependent variables.

A Lie group of point transformations (1.2.8) admitted by S maps any solution $u=\Theta(x)$ of S into a one-parameter family of solutions $u=\phi(x;\varepsilon)$ of S. Equivalently, a Lie group of point transformations (1.2.8) leaves S invariant in the sense that the form of S is unchanged in terms of the transformed variables (1.2.8) for any solution $u=\Theta(x)$ of S.

Let ∂u denotes the set of nm coordinates corresponding to all first order partial derivatives of u with respect to x:

$$\partial u=\left(\frac{\partial u^1}{\partial x_1},\frac{\partial u^1}{\partial x_2},\cdots,\frac{\partial u^1}{\partial x_n},\frac{\partial u^2}{\partial x_1},\frac{\partial u^2}{\partial x_2},\cdots\frac{\partial u^2}{\partial x_n},\cdots\frac{\partial u^m}{\partial x_1},\frac{\partial u^m}{\partial x_2},\cdots\frac{\partial u^m}{\partial x_n}\right). \qquad (1.2.9)$$

In general, for $k\geq 1$, let $\partial^k u$ denote the set of coordinates

$$u^{\mu}_{i_1 i_2\cdots i_k}=\frac{\partial^k u^{\mu}}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}} ,$$

with $\mu=1,2,\ldots m$ and $i_j=1,2,\ldots,n,\ \ j=1,2,\ldots,k$ corresponding to all kth-order partial derivatives of u with respect to x.

It turns out that the natural transformation of partial derivatives of the dependent variables leads successively to extensions (prolongations) of a one-parameter Lie group of transformations (1.2.8) acting on (x,u)-space to one-parameter Lie groups of transformations acting on $(x,u,\partial u)$-space, $(x,u,\partial u,\partial^2 u)$-space, **...** $(x,u,\partial u,\partial^2 u,\ldots\partial^k u)$-space for any $k>2$. [For a given system S of differential equations, k would be the order of the highest order derivative appearing in S]. Then the infinitesimal transformations of (1.2.8) is naturally extended successively to infinitesimal transformations acting on $(x,u,\partial u,\partial^2 u,\ldots,\partial^l u)$-space, $l=1,2,\ldots,k.$

Definition 1.2.9. Multiparameter Lie Groups of Transformations

Consider an *r-parameter Lie group of point transformations*

$$\overline{x}^*=\overline{X}(\overline{x};\overline{\varepsilon}), \qquad (1.2.10)$$

with $\overline{x}=(x_1,x_2,\ldots,x_n)$ and parameters $\overline{\varepsilon}=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r)$. Let the law of composition of parameters be denoted by

$$\varphi(\overline{\varepsilon},\overline{\delta})=(\phi_1(\overline{\varepsilon},\overline{\delta}),\phi_2(\overline{\varepsilon},\overline{\delta}),\ldots,\phi_r(\overline{\varepsilon},\overline{\delta})),$$

with $\bar{\delta} = (\delta_1, \delta_2, \ldots, \delta_r)$ where $\varphi(\bar{\varepsilon}, \bar{\delta})$ satisfies the group axioms with $\bar{\varepsilon} = \bar{0}$ corresponding to the identity $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_r = 0$, and $\varphi(\bar{\varepsilon}, \bar{\delta})$ is assumed to be analytic in its domain of definition.

Let the infinitesimal matrix $\Xi(\bar{x})$ be the $r \times n$ matrix with entries

$$\xi_{\alpha j}(\bar{x}) = \left.\frac{\partial \bar{x}^*{}_j}{\partial \varepsilon_\alpha}\right|_{\varepsilon=0} = \left.\frac{\partial X_j(\bar{x}; \bar{\varepsilon})}{\partial \varepsilon_\alpha}\right|_{\varepsilon=0}, \quad \alpha = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, n. \qquad (1.2.11)$$

Definition 1.2.10. The *infinitesimal generator* X_α, corresponding to the parameter ε_α of the r-parameter Lie group of transformations (1.2.10), is given by

$$\mathrm{X}_\alpha = \sum_{j=1}^{n} \xi_{\alpha j}(\bar{x}) \frac{\partial}{\partial x_j}, \quad \alpha = 1, 2, \ldots, r. \qquad (1.2.12)$$

Definition 1.2.11. For an r-parameter Lie group of transformations (1.2.10) with infinitesimal generators $\mathrm{X}_\alpha, \alpha = 1, 2, \ldots, r$ defined by (1.2.11) and (1.2.12), the *commutator* (Lie Bracket) of X_α and X_β is a first order operator defined by

$$[\mathrm{X}_\alpha, \mathrm{X}_\beta] = \mathrm{X}_\alpha \mathrm{X}_\beta - \mathrm{X}_\beta \mathrm{X}_\alpha = \sum_{i,j=1}^{n} \left[\left(\xi_{\alpha i}(\bar{x}) \frac{\partial}{\partial x_i} \right) \left(\xi_{\beta j}(\bar{x}) \frac{\partial}{\partial x_j} \right) - \left(\xi_{\beta i}(\bar{x}) \frac{\partial}{\partial x_i} \right) \left(\xi_{\alpha j}(\bar{x}) \frac{\partial}{\partial x_j} \right) \right]$$

$$= \sum_{j=1}^{n} \eta_j(\bar{x}) \frac{\partial}{\partial x_j}, \qquad (1.2.13a)$$

where

$$\eta_j(\bar{x}) = \sum_{i=1}^{n} \left(\xi_{\alpha i}(\bar{x}) \frac{\partial \xi_{\beta j}(\bar{x})}{\partial x_i} - \xi_{\beta i}(\bar{x}) \frac{\partial \xi_{\alpha j}(\bar{x})}{\partial x_i} \right). \qquad (1.2.13b)$$

It immediately follows that

$$[\mathrm{X}_\alpha, \mathrm{X}_\beta] = -[\mathrm{X}_\beta, \mathrm{X}_\alpha]. \qquad (1.2.14)$$

Theorem 1.2.2. *The commutator of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator. In particular,*

$$[\mathrm{X}_\alpha, \mathrm{X}_\beta] = \sum_{\gamma=1}^{r} C_{\alpha\beta}^{\gamma} \mathrm{X}_\gamma, \qquad (1.2.15)$$

where the coefficients $C_{\alpha\beta}^{\gamma}$ are constants called structure constants, $\alpha, \beta, \gamma = 1, 2, \ldots, r$.

Definition 1.2.12. Equations (1.2.15) are called the *commutation relations* of the r-parameter Lie group of transformations (1.2.10) with the infinitesimal generators (1.2.12).

For any three infinitesimal generators X_α, X_β and X_γ, by direct computation one can show that Jacobi's identity holds:

$$[\mathrm{X}_\alpha,[\mathrm{X}_\beta,\mathrm{X}_\gamma]]+[\mathrm{X}_\beta,[\mathrm{X}_\gamma,\mathrm{X}_\alpha]]+[\mathrm{X}_\gamma,[\mathrm{X}_\alpha,\mathrm{X}_\beta]]=0. \qquad (1.2.16)$$

Definition 1.2.13. A *Lie algebra* **L** is a vector space over $\Re$ or C with a bilinear bracket operation (the commutator) satisfying the properties (1.2.14), (1.2.16) and, most important, (1.2.15). In particular, the set of infinitesimal generators $\{\mathrm{X}_\alpha\}, \alpha=1,2,\ldots,r,$ of an r-parameter Lie group of transformations (1.2.10) forms an r-dimensional Lie algebra over $\Re$.

Proposition. *Let G be a Lie group with Lie algebra **L**. For each vector $v\in \boldsymbol{L}$, the adjoint vector ad v at $w\in \boldsymbol{L}$ is*

$$ad\ v\big|_w=[w,v]=-[v,w].$$

The adjoint representation $\mathrm{Ad}\,G$ of the underlying Lie group can be reconstructed either by integrating the system of linear ordinary differential equations

$$\frac{dw}{d\varepsilon}=\ \mathrm{ad}\ v\big|_w,\quad w(0)=w_0,$$

with solution

$$w(\varepsilon)=Ad(\exp(\varepsilon v))w_0,$$

$$Ad(\exp(\varepsilon v))w_0=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\ (\mathrm{ad}\ v)^n(w_0)$$

$$=w_0-\varepsilon[v,w_0]+\frac{\varepsilon^2}{2}[v,[v,w_0]]-\cdots. \qquad (1.2.17)$$

Definition 1.2.14. Optimal system

An Optimal system of one-parameter subgroups is a list of conjugacy inequivalent one-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of sub algebras. For one dimensional sub algebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation.

CHAPTER 2

METHODOLOGY

In this thesis, we deal with the methods of group invariant solutions, based on the theory of continuous group of transformations, better known as 'Lie groups', acting on the space of independent and dependent variables of the system. The method is due originally to Sophus Lie [22], who unified and extended the bewildering special methods of integration of differential equations. Through the constructive procedures Lie established that, in the case of ordinary differential equations (ODEs), invariance under one-parameter symmetry group implies that the order of the equation can be reduced by one.

Lie's work for ordinary differential equations examines in a systematic and comprehensive way a wide spectrum of topics such as integrating factors, separable and homogeneous equations, reduction of order, methods of undetermined coefficients and variation of parameters, Euler equation and homogeneous equations with constant coefficients. Further for linear partial differential equations, Lie has established that the invariance under continuous group of transformations leads directly to superposition of solutions in terms of transformations.

The work put up in this thesis has primarily been based on certain concepts of group symmetry through the applications of Symmetry Reduction method.

By symmetry group of a single or a system of partial differential equations, we mean a continuous group of transformations acting on the space of independent and dependent variables which leaves the equation(s) invariant. The solutions of partial differential equation(s) are all found by solving a reduced system of differential equations involving fewer independent variables. Thus, in particular, the solutions to a partial differential equation in two independent variables which is invariant under one parameter symmetry group can be found by solving a 'reduced' ordinary differential equation.

As mentioned earlier the work comprising this thesis is based primarily on the applications of symmetry reduction method. The problems are dealt-with in two phasesin the first, the symmetries of the system under investigation are derived using Symmetry reduction method and then in the second phase, after successful deduction of the reduced systems of ordinary differential equations, find some exact solutions.

2.1. Symmetry Reduction Method

A technique that has found an important place in literature on group theoretic methods for the determination of the solutions of a single or system of nonlinear partial differential equations is due to Steinberg and is termed as Symmetry Reduction Method. Though the technique relies heavily on the theory of sophisticated use of nonlinear operators yet it has been cast in a form that it is easy to utilize by specialist and non-specialist alike. The algorithmic representation of the method makes the concepts clear and straight forward. Further, it bears a close relationship to the method of separation of variables in the case of linear equations.

This method is utilized in chapters 3-6 to investigate symmetries and reductions of Drinfeld equations, Gardner equation, Fisher equation, and Rayleigh equation. The advantage in this approach is that it not only furnishes the group infinitesimals comparatively easily but, also often renders symmetries more generalized than the classical Lie method.

The analytical execution of the technique can be thought of as following of the three steps:

a) Find the symmetries of the differential equations

b) Determine the canonical coordinates for symmetry or assume a separable form for the differential equation

c) Find the reduced problem in terms of the canonical coordinates

For determining the symmetry operator of a system of differential equations, we need to proceed as follows:

Let us consider a system of k nonlinear partial differential equations in k dependent variables $\bar{u} = (u_1, u_2, \ldots, u_k)$ and $n+1$ independent variables $(t, \bar{x}) = (t, x_1, x_2, \ldots, x_n)$. Let us assume that our system can be written in terms of *nonlinear differential operator* $\overline{N} = (N_1, N_2, \ldots, N_k)$ as follows:

$$\overline{N}(\overline{u}) \equiv \frac{\partial^p \overline{u}}{\partial t^p} - \overline{H}(\overline{u}) = \overline{0}\,, \tag{2.1.1}$$

where $\overline{u} = \overline{u}(t, \overline{x})$. $\overline{H}$ may be defined on the space of t, $\overline{x}$, $\overline{u}$ and any derivative of $\overline{u}$ as long as the derivatives of $\overline{u}$ don't contain more than $p-1$ derivatives of t. $\overline{H}$ can be nonlinear.

Next, we define *symmetry operator* $\overline{S} = (S_1, S_2, \ldots, S_k)$ for the system (2.1.1) called infinitesimal symmetries. These symmetries are quasi-linear partial differential operators of first order and consequently must have the form

$$\overline{S} \equiv A(t, \overline{x}, \overline{u}) \frac{\partial \overline{u}}{\partial t} + \sum_{i=1}^{n} B_i(t, \overline{x}, \overline{u}) \frac{\partial \overline{u}}{\partial x_i} + \overline{C}(t, \overline{x}, \overline{u}), \tag{2.1.2}$$

where $\overline{C} = (C_1, C_2, \ldots, C_k)$.

Fréchet derivative $\overline{F} = (F_1, F_2, \ldots, F_k)$ of $\overline{N}$ at $\overline{u} = (u_1, u_2, \ldots, u_k)$ in the direction of $\overline{v} = (v_1, v_2, \ldots, v_k)$ is given by

$$\overline{F}(\overline{N}, \overline{u}, \overline{v}) = \frac{d}{d\varepsilon}[\overline{N}(\overline{u} + \varepsilon \overline{v})]\Big|_{\varepsilon=0}. \tag{2.1.3}$$

a) The method mainly consists of determining the coefficients A, B_i, $i = 1, 2, \ldots, n$ and C_j, $j = 1, 2, \ldots, k$ in the symmetry operator $\overline{S}$, we need to proceed as follows:

We first find Fréchet derivative $\overline{F} = (F_1, F_2, \ldots, F_k)$ of $\overline{N}(\overline{u}) = (N_1, N_2, \ldots, N_k)$ by the equations (2.1.3), then $\overline{v} = (v_1, v_2, \ldots, v_k)$ is substituted by $\overline{S} = (S_1, S_2, \ldots, S_k)$ in order to evaluate them in the direction of the symmetry operator.

$$\overline{F}(\overline{N}, \overline{u}, \overline{S}) = \frac{d}{d\varepsilon}[\overline{N}(\overline{u} + \varepsilon \overline{S})]\Big|_{\varepsilon=0}. \tag{2.1.4}$$

For invariance of the system (2.1.1), we require that the Fréchet derivative (2.1.4) must vanish on the solution set of (2.1.1) in the direction of the symmetry operator $\overline{S}$. That is, we must have

$$\overline{F}(\overline{N}, \overline{\eta}, \overline{S})\Big|_{\overline{N}=\overline{0}} = \overline{O}\,. \tag{2.1.5}$$

For this we substitute $\overline{H}(\overline{u})$ for $\dfrac{\partial^p \overline{u}}{\partial t^p}$ in (2.1.5). The equations (2.1.5) when expanded, result in to polynomial expressions in various partial derivatives of $\overline{u}$. Equating the various coefficients of these derivative terms, we will get a set of linear partial differential equations called "determining equations" for the group infinitesimals $A, B_i, i = 1, 2, \ldots, n$ and $C_j, j = 1, 2, \ldots, k$. Solve the resulting "determining equations" for symmetries of the system (2.1.1).

b) Once this resulting set of partial differential equations is solved for coefficients of $\overline{S}$. The associated Lie algebra of infinitesimal symmetries of (2.1.1) is then the set of vector fields of the form

$$V \equiv A(t, \overline{x}, \overline{u}) \frac{\partial}{\partial t} + \sum_{i=1}^{n} B_i(t, \overline{x}, \overline{u}) \frac{\partial}{\partial x_i} - \sum_{j=1}^{k} C_j(t, \overline{x}, \overline{u}) \frac{\partial}{\partial u_j}. \qquad (2.1.6)$$

Or, equivalently the one-parameter group of point transformations of (2.1.1) is as follows:

$$t^* = t + \varepsilon A(t, \overline{x}, \overline{u}) + O(\varepsilon^2)$$

$$\overline{x}^* = \overline{x} + \varepsilon \overline{B}(t, \overline{x}, \overline{u}) + O(\varepsilon^2)$$

$$\overline{u}^* = \overline{u} - \varepsilon \overline{C}(t, \overline{x}, \overline{u}) + O(\varepsilon^2),$$

where $\overline{u}^* = (u_1^*, u_2^*, \ldots, u_k^*)$. Using the infinitesimal generators (2.1.6), one can obtain a reduction of system (2.1.1) to a system with number of independent variables one less than the original one. For this, first we solve the "characteristic equations"

$$\frac{dt}{A} = \frac{dx_1}{B_1} = \frac{dx_2}{B_2} = \cdots = \frac{dx_n}{B_n} = \frac{du_1}{-C_1} = \frac{du_2}{-C_2} = \cdots = \frac{du_k}{-C_k}.$$

From these equations, we obtain the canonical coordinates (similarity form of the solution in terms of new independent variables).

c) Change the system (2.1.1) in these new coordinates to get the reduced form of the problem.

CHAPTER 3

DRINFELD SOKOLOV WILSON SYSTEM

Drinfeld Sokolov Wilson System:

$$u_t = 3ww_x$$
$$w_t = 2w_{xxx} + 2uw_x + u_x w.$$

Zhao Xue-Qin and Zhi Hong-Yan [37] solved the Drinfeld sokolov Wilson equation by F-expansion method. It is also solved for Darboux Transformation and Explicit Solutions by Gen Xian-Guo and Wu Li-hua [15]. Analytical solutions of Drinfeld equation also found by Erik Sweet and Robert A.van Gorder [12].

3.1. Lie symmetries

Let us consider this equation

$$u_t = 3ww_x$$
$$w_t = 2w_{xxx} + 2uw_x + u_x w. \qquad (3.1.1)$$

Let the system (3.1.1) is defined in terms of non linear operators N_1 and N_2 as follows:

$$N_1 \equiv u_t - 3ww_x = 0$$
$$N_2 \equiv w_t - 2w_{xxx} - 2uw_x - u_x w = 0. \qquad (3.1.2)$$

The symmetry operator $\overline{S} = (S_1, S_2)$ for the system (3.1.1) is:

$$S_1(u) \equiv A(\overline{X},\overline{\eta})\frac{\partial u}{\partial t} + B(\overline{X},\overline{\eta})\frac{\partial u}{\partial x} + C(\overline{X},\overline{\eta})$$
$$S_2(w) \equiv A(\overline{X},\overline{\eta})\frac{\partial w}{\partial t} + B(\overline{X},\overline{\eta})\frac{\partial w}{\partial x} + D(\overline{X},\overline{\eta}), \qquad (3.1.3)$$

with $\overline{X} = (t, x)$, $\overline{\eta} = (u, w)$.

The Fréchet derivative of nonlinear operators N_1 and N_2 obtained in the direction of the symmetry operator $\overline{S} = (S_1, S_2)$ are given, respectively, by the following

$$F_1(N_1,\overline{\eta},\overline{S}) = \frac{d}{d\varepsilon}[N_1(\overline{\eta}+\varepsilon\overline{S})]\Big|_{\varepsilon=0} = [S_1]_t - 3\left(w[S_2]_x + [S_2]w_x\right) = 0 \qquad (3.1.4)$$
$$F_2(N_2,\overline{\eta},\overline{S}) = \frac{d}{d\varepsilon}[N_2(\overline{\eta}+\varepsilon\overline{S})]\Big|_{\varepsilon=0} = [S_2]_t - 2[S_2]_{xxx} - 2\left(u[S_2]_x + [S_1]w_x\right)$$
$$-\left(u_x[S_2] + [S_1]_x w\right) = 0. \qquad (3.1.5)$$

In equation (3.1.4), on replacing S_1 and S_2 with the help of equations (3.1.3):

$$(Au_t + Bu_x + C)_t - 3w(Aw_t + Bw_x + D)_t - 3w_x(Aw_t + Bw_x + D) = 0. \qquad (3A.1.4)$$

$$[A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t$$
$$-3w\left([A]_x w_t + Aw_{tx} + [B]_x w_x + Bw_{xx} + [D]_x\right) - 3w_x w_t A$$
$$-3Bw_x^2 - 3Dw_x = 0. \qquad (3B.1.4)$$

In equation (3.1.5), on replacing S_1 and S_2 with the help of equations (3.1.3)

$$(Aw_t + Bw_x + D)_t - 2(Aw_t + Bw_x + D)_{xxx} - 2u(Aw_t + Bw_x + D)_x$$
$$- 2(Au_t + Bu_x + C)w_x - u_x(Aw_t + Bw_x + D) - w(Au_t + Bu_x + C)_x = 0.$$
$$(3A.1.5)$$

$$[A]_t w_t + Aw_{tt} + [B]_t w_x + Bw_{xt} + [D]_t - 2\left(3[A]_{xx} w_{tx} + 3[A]_x w_{txx} + Aw_{txxx} + [A]_{xxx} w_t\right)$$
$$- 2\left(3[B]_{xx} w_{xx} + 3[B]_x w_{xxx} + Bw_{xxxx} + [B]_{xxx} w_x\right)$$
$$- 2[D]_{xxx} - 2u\left([A]_x w_t + Aw_{tx} + [B]_x w_x + Bw_{xx} + [D]_x\right)$$
$$- 2Aw_x u_t - 2Bu_x w_x - 2Cw_x - Au_x w_t - Bu_x w_x - Du_x$$
$$- w\left([A]_x u_t + Au_{tx} + [B]_x u_x + Bu_{xx} + [C]_x\right) = 0, \qquad (3B.1.5)$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc. in equation (3B.1.4) and (3*B*.1.5) are represent the total differentiation with respect to suffices and are given by the following expressions:

$$[A]_t = A_t + A_u u_t + A_w w_t$$
$$[A]_x = A_x + A_u u_x + A_w w_x$$
$$[A]_{xx} = A_{xx} + 2A_{xu}u_x + 2A_{xw}w_x + A_{uu}u_x^2 + A_{ww}w_x^2 + 2A_{uw}u_x w_x + A_u u_{xx} + A_w w_{xx}$$
$$[A]_{xxx} = A_{xxx} + 3A_{xxu}u_x + 3A_{xxv}v_x + 3A_{xu}u_{xx} + 33A_{xv}v_{xx} + 3A_{uux}u_x^2 + 3A_{vvx}v_x^2$$
$$+ 3A_{uu}u_x u_{xx} + 3A_{vv}v_x v_{xx} + 6A_{uvx}u_x v_x + 3A_{uv}u_{xx}v_x + 3A_{uv}u_x v_{xx}$$
$$+ A_u u_{xxx} + A_v v_{xxx} + A_{uuu}u_x^3 + A_{vvv}v_x^3 + 3A_{uvv}u_x v_x^2 + 3A_{uuv}u_x^2 v_x.$$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{xxx}, [C]_t, [C]_x, [C]_{xx}, [C]_{xxx}$ and for $[D]_t, [D]_x, [D]_{xx}, [D]_{xxx}$.

The invariance equation applied on equation (3.1.1a) leads to the following simplified set of determining equations for the group infinitesimals *A*, *B*, *C* and *D* which are obtained after equating the coefficients of various derivative terms to zero:

$$B_w - 3wA_u = 0$$
$$C_w - 3wA_x = 0$$

$$3wA_t + 3wC_u + 2uC_w - 6uwA_x - 3wB_x - 3wD_w - 3D = 0$$
$$B_t + wC_w - 3w^2A_x - 3wD_u = 0$$
$$C_t - 3wD_x = 0. \qquad (3.1.6)$$

Similarly, the equation (3.1.1b), under the condition of invariance, brings-in the following additional equations:

$$A_x = A_u = A_w = 0$$
$$B_u = B_w = 0$$
$$D_u = D_{ww} = 0$$
$$2A_t - 6B_x = 0$$
$$B_{xx} - D_{xw} = 0$$
$$2uA_t + B_t - 2B_{xxx} - 6D_{xxw} - 2uB_x - 2C - wC_w = 0$$
$$wA_t + wD_w - D - wB_x - wC_u = 0$$
$$D_t - 2D_{xxx} - 2uD_x - wC_x = 0. \qquad (3.1.7)$$

Combine the system (3.1.6) and (3.1.7) then we have:

$$A_x = A_u = A_w = 0 \Rightarrow A = A(t)$$
$$B_u = B_w = B_t = 0 \Rightarrow B = B(x)$$
$$C_w = 0 \Rightarrow C = C(u,x,t)$$
$$D_{ww} = 0 \Rightarrow D = D_1(x,t)w + D_2(x,t)$$
$$2A_t - 6B_x = 0 \qquad (a.1)$$
$$B_{xx} - D_{xw} = 0 \qquad (a.2)$$
$$2uA_t + B_t - 2B_{xxx} - 6D_{xxw} - 2uB_x - 2C - wC_w = 0 \qquad (a.3)$$
$$D_t - 2D_{xxx} - 2uD_x - wC_x = 0 \qquad (a.4)$$
$$wA_t + wD_w - D - wB_x - wC_u = 0 \qquad (a.5)$$
$$3wA_t + 3wC_u - 3wB_x - 3wD_w - 3D = 0 \qquad (a.6)$$
$$C_t - 3wD_x = 0. \qquad (a.7) \qquad (3.1.8)$$

From $(a.1)$, Differentiate w.r.to. t:- $A_{tt} = 0$

So, $A = k_1t + k_2$.

Differentiate w.r.to. x :- $B_{xx} = 0$

So, $B = k_3x + k_4$.

From $(a.2)$, $D_{xw} = 0$ $\quad (\because\ B_{xx} = 0)$

So, $D = D_1(t)w + D_2(x,t)$.

Our system (3.1.8) becomes:-

$$A = k_1 t + k_2$$
$$B = k_3 x + k_4$$
$$C = C(u,x,t), \quad C_w = 0$$
$$D = D_1(t)w + D_2(x,t)$$
$$k_1 - 3k_3 = 0 \qquad (b.1)$$
$$2uk_1 - 2uk_3 - 2C = 0 \qquad (b.2)$$
$$D_{1t}w + D_{2t} - 2D_{2xxx} - 2uD_{2x} - wC_x = 0 \qquad (b.3)$$
$$k_1 w + wD_1 - (D_1 w + D_2) - wk_3 - wC_u = 0 \qquad (b.4)$$
$$3k_1 w + 3wC_u - 3wk_3 - 3wD_1 - 3(D_1 w + D_2) = 0 \qquad (b.5)$$
$$C_t - 3wD_{2x} = 0. \qquad (b.6) \quad (3.1.9)$$

From $(b.3)$, coeff. of w :- $D_{1t} - C_x = 0$.

coeff. of u :- $D_{2x} = 0$.

constant :- $D_{2t} = 0$.

From $(b.4)$, coeff. Of w :- $k_1 - k_3 - C_u = 0$.

constant :- $D_2 = 0$.

From $(b.5)$, coeff. of w :- $k_1 - k_3 + C_u - 2D_1 = 0$.

From $(b.6)$, constant :- $C_t = 0$.

Our system (3.1.9) becomes:-

$$A = k_1 t + k_2$$
$$B = k_3 x + k_4$$
$$C = C(u,x), \quad C_w = C_t = 0$$
$$D = D_1(t)w$$
$$k_1 - 3k_3 = 0 \qquad (c.1)$$
$$2uk_1 - 2uk_3 - 2C = 0 \qquad (c.2)$$
$$D_{1t} - C_x = 0 \qquad (c.3)$$
$$k_1 - k_3 - C_u = 0 \qquad (c.4)$$
$$k_1 - k_3 + C_u - 2D_1 = 0. \qquad (c.5) \quad (3.1.10)$$

From $(c.4)$, Differentiate w.r.to. u :- $C_{uu} = 0$

So, $C = C_1(x)u + C_2(x)$.

Substitute the value of C in above system (3.1.10):-

$$k_1 - 3k_3 = 0 \qquad (d.1)$$
$$2uk_1 - 2uk_3 - 2(C_1u + C_2) = 0 \qquad (d.2)$$
$$D_{1t} - (C_{1x}u + C_{2x}) = 0 \qquad (d.3)$$
$$k_1 - k_3 - C_1 = 0 \qquad (d.4)$$
$$k_1 - k_3 + C_1 - 2D_1 = 0. \qquad (d.5) \quad (3.1.11)$$

From $(d.2)$, coeff. of u:- $k_1 - k_3 - C_1 = 0 \Rightarrow C_1 = k_1 - k_3$.

constant :- $C_2 = 0$.

From $(d.3)$, coeff. of u :- $C_{1x} = 0$.

constant:- $D_{1t} = 0$.

From $(d.5)$, substitute the value of C_1 then $D_1 = k_1 - k_3$.

System (3.1.11) becomes:-

$$k_1 - 3k_3 = 0$$
$$k_1 - k_3 = 0.$$

By solving these, $k_1 = 0$ and $k_3 = 0$.

So, $C_1 = 0$ and $D_1 = 0$.

So infinitesimals *A*, *B*, *C* and *D* are:

$$A = k_2$$
$$B = k_4$$
$$C = 0$$
$$D = 0. \qquad (3.1.12)$$

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C} = \frac{dw}{-D}. \qquad (3.1.13)$$

Thus, it is easily seen that the application of symmetry method to equations (3.1.1) leads to a two-parameter Lie group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$V_1 = \frac{\partial}{\partial t}$$
$$V_2 = \frac{\partial}{\partial x}. \qquad (3.1.14)$$

3.2. Optimal system

The commutator table-3.2.1 and adjoint table-3.2.2 for Lie algebra (3.1.14) can easily constructed as follows:

The commutator (lie bracket) of X_α and X_β is a first order operator defined by

$\left[X_\alpha, X_\beta\right] = X_\alpha X_\beta - X_\beta X_\alpha$.

Commutator table-3.2.1

comm	V_1	V_2
V_1	0	0
V_2	0	0

Adjoint table using:-

$$Ad(\exp \varepsilon V_i)V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i,[V_i,V_j]] + ...$$

Adjoint table-3.2.2

Ad	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

We deduce an optimal system of sub algebra with their corresponding generators as follows:

(i) $V_1 + mV_2$

(ii) V_2,

where m is arbitrary constant.

3.3. Reductions and Exact solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of PDEs (3.1.1) into ODEs in terms of similarity variable ξ and the new dependent variables- F and G, are obtained using the auxiliary equations (3.1.13). Some exact solutions of each reduced system are then attempted.

Generator (i)

The generator (i) in the optimal system defines the similarity variable and similarity solution as follows:

$V_1 + mV_2$.

We use values of system (3.1.14), and use characteristic equation we have:

$$\frac{dt}{1} = \frac{dx}{m} = \frac{du}{0} = \frac{dw}{0}.$$

Then by solving this characteristic equation:

$$\xi = mt - x$$
$$u(x,t) = F(\xi)$$
$$w(x,t) = G(\xi). \qquad (3.3.1)$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (3.1.1) reduces to following system of ODEs:

$$3GG' + mF' = 0$$
$$2G''' + 2FG' + F'G + mG' = 0. \qquad (3.3.2)$$

By using Maple software, we get some exact solutions of (3.1.16):

$$F(\xi) = c_3, G(\xi) = c_4$$

$$F(\xi) = -c_2{}^2 - \frac{1}{2}m + 3c_2{}^2 \sec h(c_1 + c_2\xi)^2,\ G(\xi) = \sqrt{-2m}\, c_2 \sec h(c_1 + c_2\xi)$$

$$F(\xi) = -c_2{}^2 - \frac{1}{2}m + 3c_2{}^2 \sec h(c_1 + c_2\xi)^2,\ G(\xi) = -\sqrt{-2m}\, c_2 \sec h(c_1 + c_2\xi)$$

$$F(\xi) = 2c_2{}^2 - \frac{1}{2}m - 3c_2{}^2 \tan h(c_1 + c_2\xi)^2,\ G(\xi) = \sqrt{2m}\, c_2 \tan h(c_1 + c_2\xi)$$

$$F(\xi) = 2c_2{}^2 - \frac{1}{2}m - 3c_2{}^2 \tan h(c_1 + c_2\xi)^2,\ G(\xi) = -\sqrt{2m}\, c_2 \tan h(c_1 + c_2\xi).$$

Thus, on using $F(\xi)$ and $G(\xi)$ in (3.3.1) then the following solution of the system (3.1.1) is obtained

$$u(x,t) = c_3,\ w(x,t) = c_4;$$

$$u(x,t) = -c_2{}^2 - \frac{1}{2}m + 3c_2{}^2 \sec h(c_1 + c_2(mt - x))^2,$$

$$w(x,t) = \sqrt{-2m}\, c_2 \sec h(c_1 + c_2(mt - x));$$

$$u(x,t) = -c_2{}^2 - \frac{1}{2}m + 3c_2{}^2 \sec h(c_1 + c_2(mt - x))^2,$$

$$w(x,t) = -\sqrt{-2m}\, c_2 \sec h(c_1 + c_2(mt - x));$$

$$u(x,t) = 2c_2^{\,2} - \frac{1}{2}m - 3c_2^{\,2} \tan h(c_1 + c_2(mt - x))^2,$$
$$w(x,t) = \sqrt{2m}\, c_2 \tan h(c_1 + c_2(mt - x)); \qquad (3.3.3)$$
$$u(x,t) = 2c_2^{\,2} - \frac{1}{2}m - 3c_2^{\,2} \tan h(c_1 + c_2(mt - x))^2,$$
$$w(x,t) = -\sqrt{2m}\, c_2 \tan h(c_1 + c_2(mt - x)), \qquad (3.3.4)$$

where c_1, c_2 and m are constants.

Generator (ii)

The generator (ii) in the optimal system defines the similarity variable and similarity solution by similar process used in generator (i) as follows:

$$\xi = t$$
$$u(x,t) = F(\xi)$$
$$w(x,t) = G(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (3.1.1) reduces to following system of ODEs:

$$F' = 0$$
$$G' = 0.$$

So, we have $F(\xi) = a$

$G(\xi) = b.$

Then $u(x,t) = a$

$w(x,t) = b,$

where a and b are constants.

CHAPTER 4

GARDNER EQUATION

The mathematical theory of nonlinear evolution equations, starting from the Korteweg-de-Vries (KDV) equation and modified Korteweg-de-Vries (mKDV) equation, is an area of research for the past few decades [11, 31, 32, 36]. In particular,the Gardner equation [34] that is also known as the mixed KDV-mKDV equation.

This Gardner equation shows up, particularly, in the context of internal gravity waves in a density-stratified ocean. This is commonly described by the KDV equations and its versions with small nonlinrearity. This lead to study of Gardner equation.

$$u_t = 6(u+\varepsilon^2 u^2)u_x + u_{xxx}.$$

4.1. Vector fields and Optimal system

Consider Gardner equation

$$u_t = 6(u+\varepsilon^2 u^2)u_x + u_{xxx}. \qquad (4.1.1)$$

Let the system (4.1.1) is defined in terms of non linear operator N_1 as follows:

$$N_1 \equiv u_t - 6(u+\varepsilon^2 u^2)u_x - u_{xxx} = 0. \qquad (4.1.2)$$

The symmetry operator $\overline{S} = (S_1)$ for the equation (4.1.1) is:

$$S_1(u) \equiv A(\overline{X},\overline{\eta})\frac{\partial u}{\partial t} + B(\overline{X},\overline{\eta})\frac{\partial u}{\partial x} + C(\overline{X},\overline{\eta}), \qquad (4.1.3)$$

with $\overline{X} = (t,x)$, $\overline{\eta} = (u)$.

The Fréchet derivative of nonlinear operator N_1 obtained in the direction of the symmetry operator $\overline{S} = (S_1)$ are given, respectively, by the following:

$$F_1(N_1,\overline{\eta},\overline{S}) = \frac{d}{d\varepsilon}[N_1(\overline{\eta}+\varepsilon\overline{S})]\Big|_{\varepsilon=0} = [S_1]_t - 6\left(u[S_1]_x + [S_1]u_x\right)$$
$$-6\varepsilon^2\left(u^2[S_1]_x + 2u[S_1]u_x\right) - [S_1]_{xxx} = 0. \qquad (4.1.4)$$

In equation (4.1.4), on replacing S_1 with the help of equation (4.1.3):

$$(Au_t + Bu_x + C)_t - 6\left(u(Au_t + Bu_x + C)_x + (Au_t + Bu_x + C)u_x\right)$$
$$-6\varepsilon^2\left(u^2(Au_t + Bu_x + C)_x + 2uu_x(Au_t + Bu_x + C)\right) - (Au_t + Bu_x + C)_{xxx} = 0.$$

$$(4A.1.4)$$

$$[A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t$$
$$-6\left(u\left([A]_x u_t + Au_{tx} + [B]_x u_x + Bu_{xx} + [C]_x\right) + u_x(Au_t + Bu_x + C)\right)$$
$$-6\varepsilon^2\left(u^2\left([A]_x u_t + Au_{tx} + [B]_x u_x + Bu_{xx} + [C]_x\right) + 2uu_x(Au_t + Bu_x + C)\right)$$
$$-\begin{pmatrix} 3[A]_{xx} u_{tx} + 3[A]_x u_{txx} + Au_{txxx} + [A]_{xxx} u_t + 3[B]_{xx} u_{xx} + 3[B]_x u_{xxx} \\ + Bu_{xxxx} + [B]_{xxx} u_x + [C]_{xxx} \end{pmatrix} = 0, \quad (4B.1.4)$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc. in equation (4*B*.1.4) are represent the total differentiation with respect to suffices and are given by the following expressions:

$$[A]_t = A_t + A_u u_t$$
$$[A]_x = A_x + A_u u_x$$
$$[A]_{xx} = A_{xx} + 2A_{ux} u_x + A_u u_{xx} + A_{uu} u_x^2$$
$$[A]_{xxx} = A_{xxx} + 3A_{uxx} u_x + 3A_{ux} u_{xx} + A_u u_{xxx} + 3A_{uux} u_x^2 + 3A_{uu} u_{xx} u_x + A_{uuu} u_x^3.$$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{xxx}$ and for $[C]_t, [C]_x, [C]_{xx}, [C]_{xxx}$.

The invariance equation applied on equation (4.1.1) leads to the following simplified set of determining equations for the group infinitesimals *A*, *B* and *C* which are obtained after equating the coefficients of various derivative terms to zero:

$$A_x = A_u = 0$$
$$B_u = 0$$
$$C_{uu} = 0$$
$$A_t + 2C_u + 3B_x = 0 \quad (a.1)$$
$$C_{ux} + B_{xx} = 0 \quad (a.2)$$
$$6uA_t + 6\varepsilon^2 u^2 A_t + B_t - 6uB_x - 6C - 6\varepsilon^2 u^2 B_x - 12\varepsilon^2 uC + B_{xxx} - 3C_{uxx} = 0 \quad (a.3)$$
$$C_t - 6uC_x - 6\varepsilon^2 u^2 C_x + C_{xxx} = 0. \quad (a.4) \quad (4.1.5)$$

From $(a.1)$, Differentiate w.r.to. x :- $2C_{ux} - 3B_{xx} = 0.$ $(a.5)$

Solve$(a.2)$ and $(a.5)$ then we have $B_{xx} = 0,\ C_{ux} = 0.$

If $B_{xx} = 0$ then $B = B_1(t)x + B_2(t).$

System (4.1.5) becomes:

$$A_x = A_u = 0 \Rightarrow A = A(t)$$
$$B_u = 0, B_{xx} = 0 \Rightarrow B = B_1(t)x + B_2(t)$$
$$C_{uu} = 0, C_{ux} = 0 \Rightarrow C = C_1(t)u + C_2(t)$$
$$A_t + 2C_u + 3B_x = 0 \quad (b.1)$$
$$6uA_t + 6\varepsilon^2 u^2 A_t + B_t - 6uB_x - 6C - 6\varepsilon^2 u^2 B_x - 12\varepsilon^2 uC = 0 \quad (b.2)$$
$$C_t - 6uC_x - 6\varepsilon^2 u^2 C_x + C_{xxx} = 0. \quad (b.3) \quad (4.1.6)$$

Put values of B and C then system (4.1.6) becomes:

$$A_t + 2C_1 + 3B_1 = 0 \qquad (c.1)$$

$$6uA_t + 6\varepsilon^2u^2A_t + B_{1t}x + B_{2t} - 6uB_1 - 6(C_1u + C_2) - 6\varepsilon^2u^2B_1$$
$$-12\varepsilon^2u(C_1u + C_2) = 0 \qquad (c.2)$$

$$C_{1t}u + C_{2t} = 0. \qquad (c.3) \quad (4.1.7)$$

From $(c.3)$, coeff. of u :- $C_{1t} = 0$.

constant :- $C_{2t} = 0$.

Then $C_1 = k_1$ and $C_2 = k_2$.

So, $C = k_1u + k_2$.

Put value of C in $(c.1)$ and $(c.2)$, then we have:

$$A_t + 2k_1 + 3B_1 = 0 \qquad (d.1)$$

$$6uA_t + 6\varepsilon^2u^2A_t + B_{1t}x + B_{2t} - 6uB_1 - 6(k_1u + k_2) - 6\varepsilon^2u^2B_1$$
$$-12\varepsilon^2u(k_1u + k_2) = 0. \qquad (d.2)$$

From (d.2), coeff of u^2 :- $A_t - 2k_1 - B_1 = 0$.

coeff of u :- $A_t - k_1 - B_1 - 2\varepsilon^2k_2 = 0$.

constant :- $B_{2t} - 6k_2 = 0$.

So, $B_2 = 6k_2t + k_3$.

coeff of x :- $B_{1t} = 0$.

So, $B_1 = k_4$.

From $(d.1)$, Differentiate w.r.to. t :- $A_{tt} = 0$.

So, $A = k_4t + k_5$,

where k_1, k_2, k_3, k_4 and k_5 are constants.

System (4.1.7) becomes:

$$A = k_4t + k_5$$
$$B = k_4 + 6k_2t + k_3$$
$$C = k_1u + k_2$$
$$k_1 = k_4$$
$$k_1 = 0$$
$$k_2 = 0.$$

From these values, we get infinitesimals A, B and C:

$$A = k_5$$
$$B = k_3$$
$$C = 0. \qquad (4.1.8)$$

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A}=\frac{dx}{B}=\frac{du}{-C}. \qquad (4.1.9)$$

Thus, it is easily seen that the application of symmetry method to equations (4.1.1) leads to a two-parameter Lie group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$V_1=\frac{\partial}{\partial x}$$

$$V_2=\frac{\partial}{\partial t}. \qquad (4.1.10)$$

Now we calculate Optimal system:

The commutator table-4.1 and adjoint table-4.2 for Lie algebra (4.1.10) can easily constructed as follows:

The commutator (lie bracket) of X_α and X_β is a first order operator defined by $\left[X_\alpha,X_\beta\right]=X_\alpha X_\beta-X_\beta X_\alpha$.

Commutator table-4.1

comm	V_1	V_2
V_1	0	0
V_2	0	0

Adjoint table using:-

$$Ad(\exp\varepsilon V_i)V_j=V_j-\varepsilon[V_i,V_j]+\frac{\varepsilon^2}{2}[V_i,[V_i,V_j]]+\dots$$

Adjoint table-4.2

Ad	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

We deduce an optimal system of sub algebra with their corresponding generators as follows:

$(i)\ V_1 + mV_2$

$(ii)\ V_2$,

where m is arbitrary constant.

4.2. Reduced ODEs and Exact solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of PDEs (4.1.1) into ODEs in terms of similarity variable ξ and the new dependent variables- F is obtained using the auxiliary equations (4.1.9). Some exact solutions of each reduced system are then attempted.

Vector field (i)

The vector field (i) in the optimal system defines the similarity variable and similarity solution as follows:

$V_1 + mV_2.$

We use values of system (4.1.10), and use characteristic equation we have:

$$\frac{dx}{1} = \frac{dt}{m} = \frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$$\xi = mx - t$$

$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (4.1.1) reduces to following system of ODEs:

$$6mFF' + 6\varepsilon^2 mF^2 F' + m^3 F''' + F' = 0. \qquad (4.2.1)$$

By using Maple software, we get some exact solutions of (4.2.1):

$$F(\xi) = c_3$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2 + 3m)}\sec h\left(c_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2 + 3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\operatorname{sec}h\left(c_1+\frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\operatorname{sec}h\left(c_1+\frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1-\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1-\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1-\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1-\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1+\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1+\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right).$$

The solution of the system (4.1.1) is given by

$$u(x,t)=c_3$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\operatorname{sec}h\left(c_1-\frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right) \tag{4.2.2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\operatorname{sec}h\left(c_1-\frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right) \tag{4.2.3}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\operatorname{sec}h\left(c_1+\frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

(4.2.4)

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\operatorname{sec}h\left(c_1+\frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

(4.2.5)

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1-\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1-\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1-\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1-\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1+\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}\tanh\left(c_1+\frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right),$$

where ε, c_1 and m are constants.

Vector field (ii)

For this vector field the associated the similarity variable and similarity solution as follows: V_2.

We use values of system (4.1.10), and use characteristic equation we have:

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$$\xi = x$$

$$u(x,t)=F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (4.1.1) reduces to following system of ODEs:

$$6FF' + 6\varepsilon^2 F^2 F' + F''' = 0. \tag{4.2.6}$$

By using Maple software, we get some exact solutions of (4.2.6):

$$F(\xi) = c_3,\; F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{6}\,\mathrm{sec}\,h\left(-c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{6}\,\mathrm{sec}\,h\left(-c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{6}\,\mathrm{sec}\,h\left(c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{6}\,\mathrm{sec}\,h\left(c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{3}\tanh\left(-c_1 + \frac{1}{2}\frac{\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{3}\tanh\left(-c_1 + \frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{3}\tanh\left(c_1 + \frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{3}\tanh\left(c_1 + \frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}.$$

Thus, the following solution of the system (4.1.1) is obtained

$$u(x,t)=c_3,\ u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{6}\operatorname{sec}h\left(-c_1+\frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{6}\operatorname{sec}h\left(-c_1+\frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{6}\operatorname{sec}h\left(c_1+\frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{6}\operatorname{sec}h\left(c_1+\frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{3}\tanh\left(-c_1+\frac{1}{2}\frac{\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{3}\tanh\left(-c_1+\frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{3}\tanh\left(c_1+\frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{3}\tanh\left(c_1+\frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2},$$

where ε and c_1 are constants.

CHAPTER 5

FISHER EQUATION

In mathematics, Fisher equation, also known as the Fisher-Kolmogorov equation, named after R. A. Fisher and A. N. Kolmogorov is the partial differential equation.

$u_t = u(1-u) + u_{xx}$.

Fisher equation describes the evolution of a density function $u(x,t)$. This is commonly used in mathematical biology of population dynamics models and also has applications in other fields. Fisher equation is studied by Hayes [17] for traveling wave solutions in 1991. Kaliappan P. [19] examine the Fisher equation for an exact analytical solutions for traveling wave. In this the reduced ordinary differential equation of the equation of study is investigated for the Painlevé property.

5.1 Essential fields and Optimal system

Consider Fisher Equation

$$u_t = u(1-u) + u_{xx}. \tag{5.1.1}$$

Let the system (5.1.1) is defined in terms of non linear operator N_1 as follows:

$$N_1 \equiv u_t - u(1-u) - u_{xx} = 0. \tag{5.1.2}$$

The symmetry operator $\overline{S} = (S_1)$ for the equation (5.1.1) is:

$$S_1(u) \equiv A(\overline{X},\overline{\eta})\frac{\partial u}{\partial t} + B(\overline{X},\overline{\eta})\frac{\partial u}{\partial x} + C(\overline{X},\overline{\eta}), \tag{5.1.3}$$

with $\overline{X} = (t,x)$, $\overline{\eta} = (u)$.

The Fréchet derivative of nonlinear operator N_1 obtained in the direction of the symmetry operator $\overline{S} = (S_1)$ are given, respectively, by the following:

$$F_1(N_1,\overline{\eta},\overline{S}) = \frac{d}{d\varepsilon}[N_1(\overline{\eta} + \varepsilon\overline{S})]\Big|_{\varepsilon=0} = [S_1]_t - [S_1] + 2u[S_1] - [S_1]_{xx} = 0. \tag{5.1.4}$$

In equation (5.1.4), on replacing S_1 with the help of equation (5.1.3):

$$(Au_t + Bu_x + C)_t - (Au_t + Bu_x + C) + 2u(Au_t + Bu_x + C) - (Au_t + Bu_x + C)_{xx} = 0. \tag{5A.1.4}$$

$$[A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t - Au_t - Bu_x - C + 2uAu_t + 2uBu_x + 2uC$$
$$- [A]_{xx} u_t - 2[A]_x u_{tx} - Au_{txx} - [B]_{xx} u_x - 2[B]_x u_{xx} - Bu_{xxx} - [C]_{xx} = 0, \quad (5B.1.4)$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc. in equation (5*B*.1.4) are represent the total differentiation with respect to suffices and are given by the following expressions:

$$[A]_t = A_t + A_u u_t$$
$$[A]_x = A_x + A_u u_x$$
$$[A]_{xx} = A_{xx} + 2A_{ux} u_x + A_u u_{xx} + A_{uu} u_x^2 .$$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}$ and for $[C]_t, [C]_x, [C]_{xx}$.

The invariance equation applied on equation (5.1.1) leads to the following simplified set of determining equations for the group infinitesimals *A*, *B* and *C* which are obtained after equating the coefficients of various derivative terms to zero:

$$A_x = A_u = 0$$
$$B_u = 0$$
$$C_{uu} = 0$$
$$A_t - 2B_x = 0 \quad (a.1)$$
$$B_t - 2C_{ux} - B_{xx} = 0 \quad (a.2)$$
$$(u - u^2)A_t + C_t + C_u(u - u^2) - C + 2uC - C_{xx} = 0 = 0. \quad (a.3) \quad (5.1.5)$$

From $(a.1)$ Differentiate w.r.to. x :- $B_{xx} = 0$ $(a.5)$

So, $B = B_1(t)x + B_2(t)$.

Above system (5.1.5) of equations becomes:

$$A_x = A_u = 0 \Rightarrow A = A(t)$$
$$B_u = 0, B_{xx} = 0 \Rightarrow B = B_1(t)x + B_2(t)$$
$$C_{uu} = 0 \Rightarrow C = C_1(x,t)u + C_2(x,t)$$
$$A_t - B_x = 0 \quad (b.1)$$
$$B_t - 2C_{ux} = 0 \quad (b.2)$$
$$(u - u^2)A_t + C_t + C_u(u - u^2) - C + 2uC - C_{xx} = 0. \quad (b.3) \quad (5.1.6)$$

Put values of *B* and *C* then system (5.1.6) becomes:

$$A_t - B_1 = 0 \quad (c.1)$$
$$B_{1t}x + B_{2t} - 2C_{1x} = 0 \quad (c.2)$$
$$(u - u^2)A_t + C_{1t}u + C_{2t} + C_1(u - u^2) - (C_1 u + C_2) + 2u(C_1 u + C_2)$$
$$- C_{1xx}u - C_{2xx} = 0. \quad (c.3) \quad (5.1.7)$$

From $(c.3)$, coeff. of u :- $A_t + C_{1t} + 2C_2 - C_{1xx} = 0.$ $(c.4)$

coeff. of u^2 :- $C_1 - A_t = 0.$ $(c.5)$

constant :- $C_{2t} - C_2 - C_{2xx} = 0.$ $(c.6)$

From $(c.5)$, Differential w.r.to. x :- $C_{1x} = 0.$

Then from $(c.4)$, $A_t + C_{1t} + 2C_2 = 0.$ $(c.7)$

From $(c.2)$, coeff. of x :- $B_{1t} = 0$

So, $B_1 = k_1.$

constant:- $B_{2t} = 0$

So, $B_2 = k_2.$

Value of B is: $B = k_1 x + k_2.$

From $(c.1)$, Differentiate w.r.to. t :- $A_{tt} = 0$

So, $A = k_3 t + k_4.$

From $(c.4)$, Differential w.r.to. x :- $C_{1t} = 0.$

From $(c.5)$, Differentiate w.r.to. t :- $C_{2x} = 0$

So, $C_1 = k_5.$ $(\because C_{1x} = C_{1t} = 0)$

System (5.1.7) becomes:-

$$A = k_3 t + k_4$$
$$B = k_1 x + k_2$$
$$C = k_5 u + C_2(t)$$
$$A_t - B_1 = 0 \qquad (d.1)$$
$$A_t + 2C_2 = 0 \qquad (d.2)$$
$$C_1 - A_t = 0 \qquad (d.3)$$
$$C_{2t} - C_2 = 0. \qquad (d.4) \quad (5.1.8)$$

From $(d.2)$, Differentiate w.r.to. t :- $C_{2t} = 0$

So, $C_2 = k_6.$ $(\because C_{2x} = C_{2t} = 0)$

and $C = k_5 u + k_6,$

where k_1, k_2, k_3, k_4, k_5 and k_6 are constants.

Put values of A, B and C in $(d.1)$, $(d.2)$, $(d.3)$ and $(d.4)$, then we have:

$$k_3 - k_1 = 0$$
$$k_3 + 2k_6 = 0$$
$$k_5 - k_3 = 0$$
$$k_6 = 0.$$

Solve these equations, we have:

$k_1 = k_3 = k_5 = k_6 = 0$.

From these values, we get infinitesimals *A*, *B* and *C:*

$$A = k_4$$
$$B = k_2$$
$$C = 0. \qquad (5.1.9)$$

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C}. \qquad (5.1.10)$$

Thus, it is easily seen that the application of symmetry method to equations (5.1.1) leads to a two-parameter Lie group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$V_1 = \frac{\partial}{\partial x}$$
$$V_2 = \frac{\partial}{\partial t}. \qquad (5.1.11)$$

For Optimal system, the commutator table-5.1.1 and adjoint table-5.1.2 for Lie algebra (5.1.7) can easily constructed as follows:

The commutator (lie bracket) of X_α and X_β is a first order operator defined by

$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha$.

Commutator table-5.1.1

comm	V_1	V_2
V_1	0	0
V_2	0	0

Adjoint table-5.1.2

Ad	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

We deduce an optimal system of sub algebra with their corresponding generators as follows:

$(i)\ V_1 + mV_2$

$(ii)\ V_2$,

where m is arbitrary constant.

5.2. Reduced ODEs and Exact solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of PDEs (5.1.1) into ODEs in terms of similarity variable ξ and the new dependent variables- F is obtained using the auxiliary equations (5.1.10). Some exact solutions of each reduced system are then attempted.

Essential field (i)

The Essential vector field (i) in the optimal system the similarity variable and similarity solution as follows:

$V_1 + mV_2$.

We use values of system (5.1.11), and use characteristic equation we have:

$$\frac{dx}{1} = \frac{dt}{m} = \frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$\xi = mx - t$

$u(x,t) = F(\xi)$.

Using the similarity variable, the forms of similarity solutions, the system of PDEs (5.1.1) reduces to following system of ODEs:

$$F - F^2 + m^2 F'' + F' = 0. \qquad (5.2.1)$$

Solution of this ODE is $F(\xi) = 1$.

Thus, the following solution of the system (5.1.1) is obtained $u(x,t) = 1$.

Essential field (ii)

The similarity variable and similarity solution of this Essential vector field as follows:

V_2.

We use values of system (5.1.11), and use characteristic equation we have:

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$$\xi = x$$
$$u(x,t)=F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (5.1.1) reduces to following system of ODEs:

$$F-F^2+F''=0. \qquad (5.2.2)$$

By using maple software, we get some exact solutions of (5.2.2):

$$F(\xi)=1,\ F(\xi)=1-\frac{3}{2}\sec h(-c_1+\frac{1}{2}\xi)^2$$
$$F(\xi)=1-\frac{3}{2}\sec h(c_1+\frac{1}{2}\xi)^2,\ F(\xi)=\frac{3}{2}\sec h(-c_1+\frac{1}{2}i\xi)^2$$
$$F(\xi)=\frac{3}{2}\sec h(c_1+\frac{1}{2}i\xi)^2, F(\xi)=-\frac{1}{2}+\frac{3}{2}\tanh(-c_1+\frac{1}{2}\xi)^2$$
$$F(\xi)=-\frac{1}{2}+\frac{3}{2}\tanh(c_1+\frac{1}{2}\xi)^2,\ F(\xi)=\frac{3}{2}-\frac{3}{2}\tanh(-c_1+\frac{1}{2}i\xi)^2$$
$$F(\xi)=\frac{3}{2}-\frac{3}{2}\tanh(c_1+\frac{1}{2}i\xi)^2.$$

Thus, the following solution of the system (5.1.1) is obtained

$$u(x,t)=1,\ u(x,t)=1-\frac{3}{2}\sec h(-c_1+\frac{1}{2}x)^2$$
$$u(x,t)=1-\frac{3}{2}\sec h(c_1+\frac{1}{2}x)^2,\ u(x,t)=\frac{3}{2}\sec h(-c_1+\frac{1}{2}ix)^2$$
$$u(x,t)=\frac{3}{2}\sec h(c_1+\frac{1}{2}ix)^2, u(x,t)=-\frac{1}{2}+\frac{3}{2}\tanh(-c_1+\frac{1}{2}x)^2$$
$$u(x,t)=-\frac{1}{2}+\frac{3}{2}\tanh(c_1+\frac{1}{2}x)^2,\ u(x,t)=\frac{3}{2}-\frac{3}{2}\tanh(-c_1+\frac{1}{2}ix)^2$$
$$u(x,t)=\frac{3}{2}-\frac{3}{2}\tanh(c_1+\frac{1}{2}ix)^2,$$

where c_1 is constant.

CHAPTER 6

RAYLEIGH EQUATION

Rayleigh equation

$u_{tt} - u_{xx} = \varepsilon(u_t - u_t^3).$

Rayleigh surface wave equation is particular important in seismology, acoustic, geophisics and electronics applications. Resolution of Rayleigh equation [18] has been the subject of intensive studies [25, 28], which was discovered as late as the 19th century ([29], Raleigh, 1887).

6.1. Symmetries

Let us consider this equation

$$u_{tt} - u_{xx} = \varepsilon(u_t - u_t^3). \tag{6.1.1}$$

Let the system (6.1.1) is defined in terms of non linear operator N_1 as follows:

$$N_1 \equiv u_{tt} - u_{xx} - \varepsilon(u_t - u_t^3) = 0. \tag{6.1.2}$$

The symmetry operator $\overline{S} = (S_1)$ for the equation (6.1.1) is:

$$S_1(u) \equiv A(\overline{X},\overline{\eta})\frac{\partial u}{\partial t} + B(\overline{X},\overline{\eta})\frac{\partial u}{\partial x} + C(\overline{X},\overline{\eta}), \tag{6.1.3}$$

with $\overline{X} = (t,x)$, $\overline{\eta} = (u)$.

The Fréchet derivative of nonlinear operator N_1 obtained in the direction of the symmetry operator $\overline{S} = (S_1)$ are given, respectively, by the following:

$$F_1(N_1,\overline{\eta},\overline{S}) = \frac{d}{d\varepsilon}[N_1(\overline{\eta} + \varepsilon\overline{S})]\Big|_{\varepsilon=0} = [S_1]_{ttt} - [S_1]_{xx} - \varepsilon[S_1]_t + 3\varepsilon[S_1]_t u_t^2 = 0. \tag{6.1.4}$$

In equation (6.1.4), on replacing S_1 with the help of equation (6.1.3):

$$\begin{aligned}
&[A]_{tt}u_t + 2[A]_t u_{tt} + Au_{ttt} + [B]_{tt}u_x + 2[B]_t u_{xt} + Bu_{xtt} + [C]_{tt} \\
&\quad - [A]_{xx}u_t - 2[A]_x u_{tx} - Au_{txx} - [B]_{xx}u_x - 2[B]_x u_{xx} - Bu_{xxx} - [C]_{xx} \\
&\quad - \varepsilon\left([A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t\right) \\
&\quad + 3\varepsilon u_t^2\left([A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t\right) = 0,
\end{aligned} \tag{6A.1.4}$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc. in equation (6*A*.1.4) are represent the total differentiation with respect to suffices and are given by the following expressions:

$$[A]_x = A_x + A_u u_x$$
$$[A]_t = A_t + A_u u_t$$
$$[A]_{xx} = A_{xx} + 2A_{ux} u_x + A_u u_{xx} + A_{uu} u_x^2$$
$$[A]_{tt} = A_{tt} + 2A_{ut} u_t + A_u u_{tt} + A_{uu} u_t^2 .$$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{tt}$ and for $[C]_t, [C]_x, [C]_{xx}, [C]_{tt}$. The invariance equation applied on equation (5.1.1) leads to the following simplified set of determining equations for the group infinitesimals A, B and C which are obtained after equating the coefficients of various derivative terms to zero:

$$A_x = A_u = 0$$
$$B_u = B_t = 0, B_{xx} = 0$$
$$C_u = C_t = 0, C_{xx} = 0$$
$$A_t - B_x = 0$$
$$A_{tt} + \varepsilon A_t = 0. \qquad (6.1.5)$$

Then above system (6.1.5) becomes:

$$A_x = A_u = 0 \Rightarrow A = A(t)$$
$$B_u = B_t = 0, B_{xx} = 0 \Rightarrow B = k_1 x + k_2$$
$$C_u = C_t = 0, C_{xx} = 0 \Rightarrow C = k_3 x + k_4$$
$$A_t - B_x = 0 \qquad (a.1)$$
$$A_{tt} + \varepsilon A_t = 0, \qquad (a.2) \quad (6.1.6)$$

where k_1, k_2, k_3 and k_4 are constants.

Put values of *A* and *B* in $(a.1)$and $(a.2)$, then we have:

$$A_t = k_1 \Rightarrow A = k_1 t + k_5$$
$$k_1 = 0.$$

So, $A = k_5$

$$B = k_2$$
$$C = k_3 x + k_4. \qquad (6.1.7)$$

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C}. \qquad (6.1.8)$$

Thus, it is easily seen that the application of symmetry method to equations (6.1.1) leads to a four-parameter Lie group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$V_1 = \frac{\partial}{\partial t}$$
$$V_2 = \frac{\partial}{\partial x}$$
$$V_3 = -x\frac{\partial}{\partial u}$$
$$V_4 = -\frac{\partial}{\partial u}. \qquad (6.1.9)$$

6.2. Optimal system

The commutator table-6.2.1 and adjoint table-6.2.2 for Lie algebra (6.1.9) can easily constructed as follows:

The commutator (lie bracket) of X_α and X_β is a first order operator defined by $\left[X_\alpha, X_\beta\right] = X_\alpha X_\beta - X_\beta X_\alpha$.

Commutator table-6.2.1

comm	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	V_4	0
V_3	0	$-V_4$	0	0
V_4	0	0	0	0

Adjoint table-6.2.2

Ad	V_1	V_2	V_3	V_4
V_1	V_1	V_2	V_3	V_4
V_2	V_1	V_2	$V_3 - \varepsilon V_4$	V_4
V_3	V_1	$V_2 + \varepsilon V_4$	V_3	V_4
V_4	V_1	V_2	V_3	V_4

We deduce an optimal system of sub algebra with their corresponding generators as follows:

$(i)\, mV_1 + nV_2 + pV_3$
$(ii)\, \alpha V_1 + \beta V_2 + V_3$
$(iii)\, \rho V_1 + \nu V_2$
$(iv)\, V_1,$

where m, n, p, α, β and ρ are arbitrary constants.

6.3. Reductions and Exact solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of PDEs (6.1.1) into ODEs in terms of similarity variable ξ and the new dependent variables- F is obtained using the auxiliary equations (6.1.8). Some exact solutions of each reduced system are then attempted.

Generator (i)

The generator (i) in the optimal system defines the similarity variable and similarity solution as follows:

$mV_1 + nV_2 + pV_3.$

We use values of system (6.1.9), and use characteristic equation we have:

$$\frac{dx}{n} = \frac{dt}{n} = \frac{du}{-xp}.$$

Then by solving this characteristic equation, we have:

$$\xi = nt - mx$$

$$u(x,t) = F(\xi) - \frac{p}{n}\frac{x^2}{2}.$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (6.1.1) reduces to following system of ODEs:

$$(n^2 - m^2)F'' + \frac{p}{n} - n\varepsilon F' + n^3 \varepsilon F'^3 = 0. \qquad (6.3.1)$$

In this case we are able to find only reductions.

Generator (ii)

The generator (ii) in the optimal system defines the similarity variable and similarity solution as follows:

$\alpha V_1 + \beta V_2 + V_3$.

We use values of system (6.1.9), and use characteristic equation we have:

$$\frac{dx}{\alpha} = \frac{dt}{\beta} = \frac{du}{-x}.$$

Then by solving this characteristic equation, we have:

$$\xi = \alpha x - \beta t$$

$$u(x,t) = F(\xi) - \frac{1}{\beta}\frac{x^2}{2}.$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (6.1.1) reduces to following system of ODEs:

$$(\beta^2 - \alpha^2)F'' + \frac{1}{\beta} - \beta\varepsilon F' + \beta^3\varepsilon F'^3 = 0. \qquad (6.3.2)$$

In this case also we are able to find only reduations.

Generator (iii)

The generator (iii) in the optimal system defines the similarity variable and similarity solution as follows:

$\rho V_1 + \nu V_2$.

We use values of system (6.1.9), and use characteristic equation we have:

$$\frac{dx}{\rho} = \frac{dt}{\nu} = \frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$$\xi = \nu t - \rho x$$

$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (6.1.1) reduces to following system of ODEs:

$$(\nu^2 - \rho^2)F'' - \nu\varepsilon F' + \nu^3\varepsilon F'^3 = 0. \qquad (6.3.3)$$

Solution of this ODE (6.3.3) is $F(\xi) = \frac{1}{c_3}$.

So, $u(x,t) = \frac{1}{c_3}$,

where c_3 is constant.

Generator (iv)

The generator (iv) in the optimal system defines the similarity variable and similarity solution as follows: V_1.

We use values of system (6.1.9), and use characteristic equation we have:

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$$\xi = x$$
$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (6.1.1) reduces to following system of ODEs:

$$F'' = 0.$$

So, $F(\xi) = a\xi + b$.

Where a and b are arbitrary constants.

So, $u(x,t) = ax + b$,

where a and b are constants.

CONCLUSION

In summary, the Symmetry reduction method based on Fréchet derivatives of differential operators is utilized to investigate the symmetries and invariant solutions of Drinfeld Sokolov Wilson equations, Gardner equation, Fisher equation and Rayleigh equation. The infinitesimal generators in optimal system of sub algebras is considered for reductions and exact solution. The various exact solutions are in hyperbolic functions. The exact solutions reported in the theses have indeed been found to satisfies (3.1), (4.1), (5.1) on maple software. In Rayleigh equation, we are able to find only reductions. So we can not find solutions of this equation. All these solutions of equations which are reported in the thesis are physically important.

Some plots to have an idea about the nature of the solutions are obtained in the thesis.

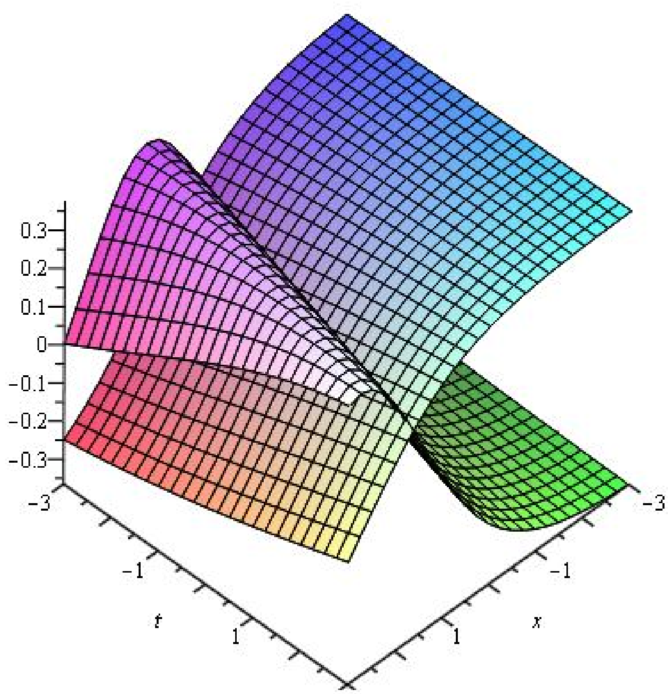

Figure (1)

In section 3.3. for solution (3.3.3), with $m=\frac{1}{4}, c_1=1$ and $c_2=\frac{1}{2}$.

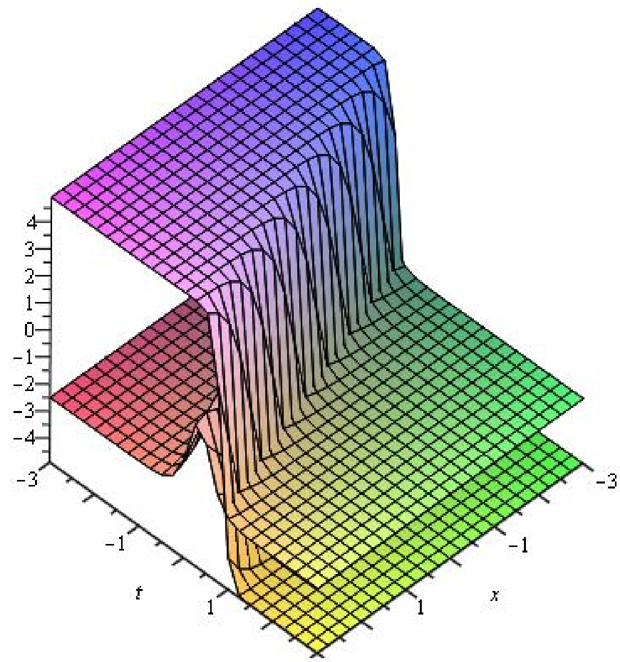

Figure (2)

In section 3.3. for solution (3.3.4), with $m = 3, c_1 = 2$ and $c_2 = 1$.

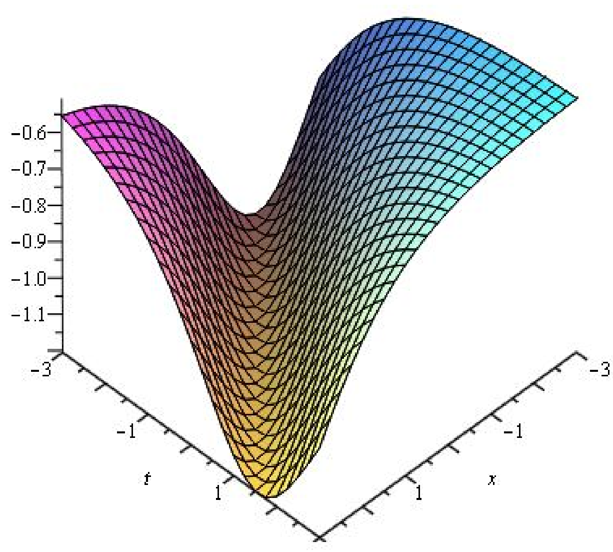

Figure (3)

In section 4.2. for solution (4.2.2), with $m = 1, \varepsilon = 1$ and $c_1 = 1$.

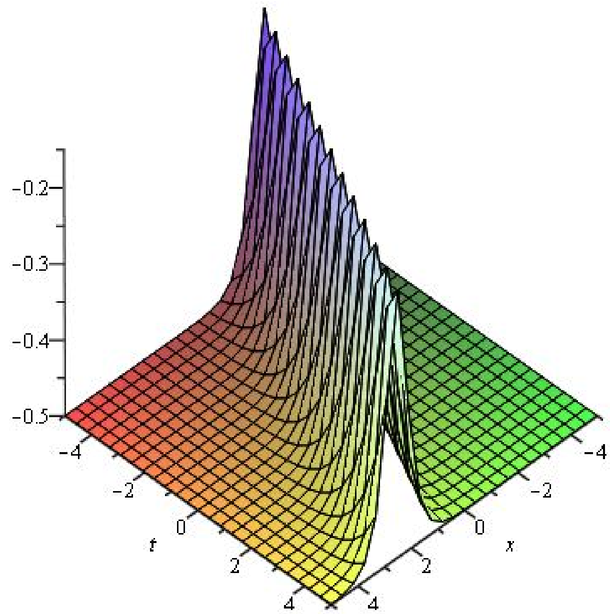

Figure (4)

In section 4.2. for solution (4.2.3), with $m = 2, \varepsilon = 1$ and $c_1 = \frac{1}{8}$.

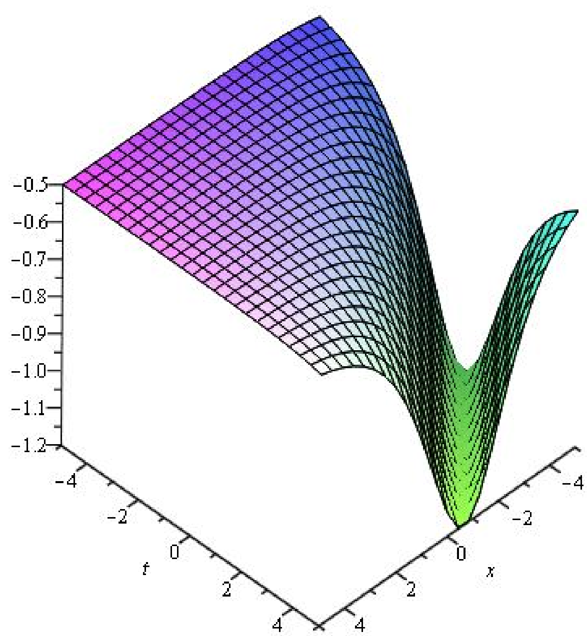

Figure (5)

In section 4.2. for solution (4.2.4), with $m = 1, \varepsilon = 1$ and $c_1 = 4$.

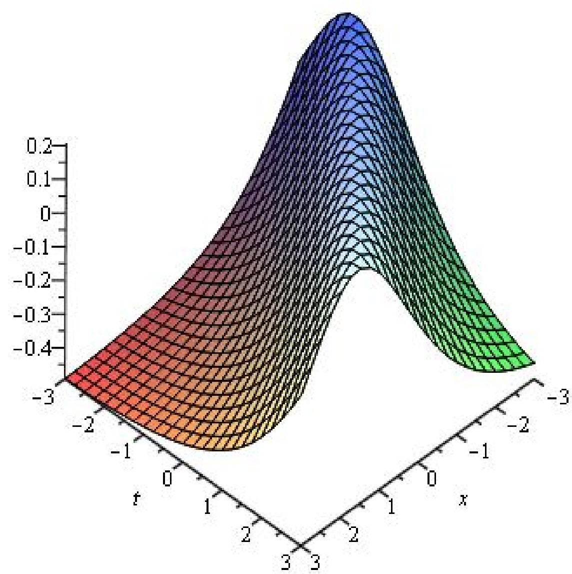

Figure (6)

In section 4.2. for solution (4.2.5), with $m=1, \varepsilon=1$ and $c_1=1$.

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