SOME STRATEGICAL METHODS FOR SOLVING DECISION-MAKING PROBLEMS UNDER FUZZY/INTUITIONISTIC FUZZY SET ENVIRONMENT

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Some Strategical Methods For Solving Decision-Making Problems Under Fuzzy/Intuitionistic Fuzzy Set Environment**" in partial fulfillment of the requirement for the award of degree of Degree of Philosophy and submitted in the School of Mathematics (SoM), Thapar Institute of Engineering & Technology, Patiala is an authentic record of my own carried out during a period from January, 2016 to April, 2019 under the supervision of Dr. Harish Garg, Assistant Professor, SoM, Thapar Institute of Engineering & Technology Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

Multiattribute group decision-making (MAGDM) problems are the imperative part of modern decision theory where a set of alternatives has to be assessed against the multiple influential attributes before the best alternative is selected. Indeed in ordinary life, "to do or not to do" is one of the foremost riddles that a person faces before jumping to action. The whole decision making (DM) process is subordinate upon the proper data being accessible to the proper individuals at the correct times. In general, in order to evaluate the given objects, a decision-maker may set some characteristic or criteria which need to be fulfilled/satisfied to select the best one(s) during solving the problems. Based on the criteria, DM problems are classified into two types, (1) decision based on the single criteria; (2) decision based two or more attributes known as multi-attribute decision-making (MADM). Due to the increasing complexity of the socioeconomic environment and the lack of knowledge, it is difficult for the decision maker to give the exact decision as there is always an imprecise, vague or uncertain information. To deal with this, the theory of the fuzzy sets (FSs) or its extensions such as intuitionistic FSs (IFSs), interval-valued IFSs (IVIFSs), Soft sets, etc., are widely used by the researchers so as to minimize the uncertainty level. In the last few decades, several types of research paid more attention to MADM or MAGDM problems in various fields. However, one of the most important factors to access the best one(s) is the considered environment, under which the decision-maker(s) have to evaluate the given alternatives. The environment considered during the DM problems may be quantitative and qualitative according to the situation of real-life problems. To address it, a concept of a linguistic variable (LV) and hence their corresponding approaches are developed by the researchers to analyze the information by using various information measures and the aggregation operators (AOs).

After this pioneering work, researchers have been engaged in extensions and applications to different disciplines. However, the most important task for the decision-maker is to rank the objects so as to obtain the desired object(s). An altar to these theories, in 1989, an uncertainty analysis theory, by combining dialectical thinking and mathematical tools, was developed by Zhao known as "set pair analysis" (SPA) and is differ from the traditional probability and fuzzy set theory in terms of considering both certainty and uncertainty as one consolidate certain-uncertain system. SPA theory studies the internal relationship between the parts of a system. The core idea of SPA is to define a set pair for two related sets and characterized them in terms of constructing the connection number (CN), which consists of "identity", "discrepancy", and"contrary" degrees. For any two dependent sets A and B of a given problem W, a set pair between them is denoted by $H(A, B)$ and having N characteristics. The CN corresponding to set pair H denoted by $\mu(H, W)$ is defined as

$$\begin{aligned} \mu(H, W) &= (S/N) + (F/N)i + (P/N)j \\ &= a + bi + cj \end{aligned}$$

where, $a(= S/N), b(= F/N)$, and $c(= P/N)$ represent the degrees of "identity", "discrepancy", and"contrary" respectively, $i \in [-1, 1]$ is the coefficient of "discrepancy degree", and $j = -1$ is the coefficient of "contrary degree" and $j = -1$. Here S, F, P denotes the identity, discrepancy, contrary characteristics respectively. It is clearly seen that $0 \leq a, b, c \leq 1$ and $a + b + c = 1$. As the complexities of the system increase day-by-day, so there is a need to plan/adopt suitable methodologies to solve the DM problems which can handle the information in a more accurate and certain manner.

The objective of this research work is to develop some new methodologies under the IFS or IVIFSs by utilizing the feature of the CNs of the SPA theory. To do it, we define the various measures and the AOs for solving the MADM and MAGDM problem where the information related to each alternative is expressed in terms of CNs from IFSs or IVIFSs. The various form of the connection number sets is defined under the different to handle the uncertain and imprecise information. The various desirable relations between the proposed measures and operators are studied. Later, based on the proposed measures, an efficient method is developed to solve the DM problems in which information related to each alternative is assessed under the consideration of the group of experts. Several real-life practical examples are taken to demonstrate the approach and compared their performance with some of the existing studies.

The present thesis is organized into nine chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of DM approaches by using several approaches is presented in the first chapter.

In **Chapter 2**, the basics and preliminaries related to the intuitionistic fuzzy sets, information measures, AOs, etc., are given.

Chapter 3 presents the novel MADM method under IVIFSs environment by integrating a TOPSIS method. For it, firstly we construct the CN of the SPA theory for each IVIFS and then based on its, we define some exponential distance measures to compute the degree of discrimination between the two IVIFSs. The supremacy of the proposed measure is also discussed. Afterward, to solve the DM problems, a novel TOPSIS method based on the proposed distance measures is developed, illustrated with a numerical example and compared their results with the several existing studies.

In **Chapter 4**, we present some axioms of the distance measures based on Hamming, Euclidean, and Hausdorff metrics whose preferences related to the attributes are made in the form of CN. Several desirable relations between the proposed measures are investigated. Later, we develop a MADM approach based on the proposed distance measures to investigate the DM problem. The effectiveness of the approach has been demonstrated through a case study and compared their studies with several existing measures.

Chapter 5 presents some novel similarity measures to measure the relative strength of the different IFSs after pointing out the weakness of the existing measures. For it, a CN set (CNS) is formulated and hence based on it, some new similarity measures between them are defined. A comparative analysis of the proposed and existing measures are formulated in terms of the counter-intuitive cases for showing the validity of it. Finally, an illustrative example is provided to demonstrate it.

Chapter 6 presents a novel correlation coefficient and weighted correlation coefficients formulation to measure the relative strength of the different IFSs in the form of the CNs.

The limitations of certain existing measures are highlighted and overcome by the proposed measure. Afterward, a DM approach is presented based on the developed measures. Two illustrative examples related to pattern recognition and medical diagnosis are taken to validate the effectiveness and applicability of the proposed method.

In **Chapter 7**, we constructed the CNs for the IFSs as well as IVIFSs and hence based on it, we constructed the TOPSIS method for solving the DM problems. The basic feature of the TOPSIS method is chosen the alternative which has the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. Afterward, based on the proposed CNs, we define two algorithms to solve the DM problems where the features are extracted either in IFNs or in IVIFNs. The validity of these proposed algorithms is tested with an example and compared it with several existing results.

Chapter 8 enhanced the LIFS with the SPA theory and hence defined the linguistic connection number (LCN) and studied their various operational laws. Based on it, we have developed various AOs namely, LCN weighted geometric (LCNWG), LCN ordered weighted geometric (LCNOWG), and LCN hybrid geometric (LCNHG) operators with LIFS environment. Also, the shortcoming of the existing operators under LIFS environment have been highlighted and overcomes by the proposed operators. Few properties of these operators have been also investigated. Further, a group DM approach has been presented, based on these operators, which has been illustrated by a numerical example to show the effectiveness and validity of the proposed approach.

Chapter 9 presents the group DM approach under the linguistic intuitionistic fuzzy (LIF) set environment. For it, we first propose a new ranking method named as possibility degree measures to compare the different LIF numbers. Further, in order to aggregate the different LIF numbers, some weighted and ordered weighted averaging AOs are proposed by using Einstein t-norm operations. The prominent characteristics of these operators are also investigated. Afterward, a MAGDM approach, based on proposed operators and the possibility degree measure, is developed under the LIFSs environment. A numerical case is taken to manifest the practicability and feasibility of the proposed group DM method.

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List of Publications

Refereed Journals

(J1) Harish Garg, Kamal Kumar, A novel exponential distance and its based TOPSIS method for interval-valued intuitionistic fuzzy sets using connection number of SPA theory, *Artificial Intelligence Review*, 53(1), pp. 595-624, 2020, doi: 10.1007/s10462-018-9668-5 (**Impact Factor: 5.095**).

(J2) Harish Garg, Kamal Kumar. Distance measures for connection number sets based on set pair analysis and its applications to decision-making process, *Applied Intelligence*, 48(10), 3346 - 3359, 2018, doi: 10.1007/s10489-018-1152-z (**Impact Factor: 2.882**)

(J3) Harish Garg, Kamal Kumar, A novel correlation coefficient of intuitionistic fuzzy sets based on the connection number of set pair analysis and its application, *Scientia Iranica*, 25(4), 2373 - 2388, 2018, doi: 10.24200/SCI.2017.4454 (**Impact Factor: 0.718**).

(J4) Harish Garg, Kamal Kumar, Group decision making approach based on possibility degree measures and the linguistic intuitionistic fuzzy aggregation operators using Einstein norm operations, *Journal of Multiple-Valued Logic and Soft Computing*, 31(1-2), 175-209, 2018, doi: 10.3934/jimo.2018162 (**Impact Factor: 0.613**)

(J5) Kamal Kumar, Harish Garg, Connection number of set pair analysis based TOPSIS method on intuitionistic fuzzy sets and their application to decision making, *Applied Intelligence*, 48(8), 2112 - 2119, 2018, doi: 10.1007/s10489-017-1067-0 (**Impact Factor: 2.882**)

(J6) Harish Garg, Kamal Kumar, An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making, *Soft Computing*, 22(15), 4959 - 4970, 2018, doi: 10.1007/s00500-018-3202-1 (**Impact Factor: 2.784**).

(J7) Kamal Kumar, Harish Garg, TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment, *Computational & Applied Mathematics*, 37(2), 1319 - 1329, 2018, doi: 10.1007/s40314-016-0402-0 (**Impact Factor: 1.260**).

(J8) Harish Garg, Kamal Kumar, Some aggregation operators for linguistic intuitionistic fuzzy set and its application to group decision-making process using the set pair analysis, *Arabian Journal for Science and Engineering*, 43 (6), 3213 - 3227, 2018, doi: 10.1007/s13369-017-2986-0 (**Impact Factor: 1.518**)

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Chapter 1

Introduction

In the real–world, every day, persons such as engineers, doctors, lawyers, scientists or HR managers face the various types of problems to complete their task successfully. Among them, so as to reach the optimal points with the desired goal, one of the important problems is to take the appropriate decision to select the best one(s), which is an integral part of daily life. To achieve it, a theory of decision–making (DM) plays an integral role in the field of the DM process. Indeed in ordinary life, "to do or not to do" is one of the foremost riddles that a person faces before jumping to action. The whole DM process is subordinate upon the proper data being accessible to the proper individuals at the correct times. Baker et al. [9] define DM process as "decision–making should start with the identification of the decision maker(s) and stakeholder(s) in the decision, reducing the possible disagreement about problem definition, requirements, goals, and criteria". Campling [13] defines "The process of DM involves making a choice among different courses of actions and entails a cycle of activities and events that begins with the identification of a problem and ends with the evaluation of implemented solutions". In short, DM is the experimental process of selecting an optimal choice(s) from multiple alternatives. The experimental process is the mental process of knowing, counting perspectives such as awareness, perception, reasoning, and judgment.

In general, in order to evaluate the given objects, a decision-maker may set some characteristic or criteria which need to be fulfilled/satisfied to select the best one(s) during solving the problems. Based on the criteria, DM problems are classified into two types, (1) decision based on the single criteria; (2) decision based two or more attribute known as multi-attribute decision-making (MADM). For example, for appointing an Assistant professor in a certain university, there is always a certain criterion such as academic record, communication skill, experience, research ability, set by the selection committee to find the best candidate for the post. A typical MADM problem is made up of taking three fundamentals ingredients that are given as :

1) **Alternatives :** It plays a key role in a MADM problem. These are the choice of action for the decision-maker. Generally, the set of alternatives is expected to be finite.

2) **Attribute :** It is a tool built for assessment and examination of alternatives during the process. Attribute offer quantifiable and integral representations of the decision maker's assessment.

3) **Decision matrix :** It is the framework whose elements represent the alternative's performances under the evaluated criteria set by the decision maker(s).

By taking the above three fundamental ingredients, decision-maker form a MADM problem which consists a set of 'm' alternatives $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$, '$n$' different attribute $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$ and form a decision matrix which is summarized as

$$R = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \ldots & \mathcal{G}_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \end{array}$$

where, a_{ij} rating value of the expert/decision-maker towards the performance of the alternative $\mathcal{A}_i (i = 1, 2, \ldots, m)$ under the criterion $\mathcal{G}_j (j = 1, 2, \ldots, n)$.

Generally, we can divide the DM problems into two parts, one is multi-objective DM (MODM) and other is multiple attribute DM (MADM) problems. In MODM problem, decision-maker(s) design the mathematical model of the given set of conflicting objects and choose the result on the basis of decision variables. On the other hand in MADM problem, decision-maker(s) choose the optimal alternative(s) or ranking the alternatives from the given set of alternatives on the basis of different attributes. MADM utilized in various fields of the real-world such as contractor selection for any task, medical diagnosis problem, pattern, etc. MADM problems may also be divided into two types, (1) when only single decision-maker involved in the DM process, and (2) when two or more decision-makers are involved in the DM process which is known as multiple attribute group DM (MAGDM) problem. In MAGDM problems, we chose the most preferred alternative(s) from the set of alternatives given by a group of decision-makers simultaneously. Group DM is normally known as aggregating distinctive individual preferences of the decision makers on the alternatives into single collective preferences. In group DM, it is assumed that each decision maker considers the same set of alternatives as well as criteria and face same standard problems during compiling the problems to select the best optimal solution(s).

In the last few decades, several types of research paid more attention to MADM or MAGDM problems in various fields. However, one of the most important factors to access the best one(s) is the considered environment, under which the decision-maker(s) have to evaluate the given alternatives. Environmental complexity is a factor that influences cognitive function. A complex environment is an environment with a large number of different possible states which come and go over time. The environment considered during the DM problems may be quantitative and qualitative according to the situation of real-life problems. An extensive literature related to various approaches in evaluation of MADM or MAGDM problems by using conventional, qualitative, quantitative and optimization techniques have been reviewed and are given section-wise hereafter.

1.1 Literature review

This section outlines the endeavor of researchers in the field of MADM or MAGDM in the hesitation and uncertainty environment. To handle the hesitation and uncertainty of the complex quantitative environment, intuitionistic fuzzy set (IFS) theory, and interval-valued IFS(IVIFS) theory are the most powerful tools introduced by the Atanassov [6] and Atanassov and Gargov [5] respectively which are the equipollent extension of the novel uncertainty theory known as fuzzy set theory [231]. On the other hand for the complex qualitative environment, Chen et al. [27] defined the linguistic IFS (LIFS) theory. Therefore, in this section, a brief literature review regarding various DM techniques under the IFS, IVIFS, and LIFS is given.

1.1.1 Review of Distance/similarity measures

Distance measurement between two sets indicates the degree of difference between the sets. The contrast of this, similarity measurement between two sets indicates the degree of similarity between the sets. A measure of distance or similarity defines the closeness of two objects or a sample. In case of a distance measure, closeness is more important when its value is reduced. However, closeness is more important for a measure of similarity when its value increases. Distance and similarity measurements are therefore dual concepts if they are normalized then: distance = 1-similarity and vice versa. For solving real-life DM problems, and hence gained much attention by researchers in recent years. For example, Szmidt and Kacprzyk [146] defined four types of distance measures, namely Hamming, Euclidean, normalized Hamming, and normalized Euclidean distance, between two IFSs. Wang and Xin [172] given some counterexamples of existing distance measures defined in [146] and hence presented axiom definition for distance measure of IFSs to solve the pattern recognition problems. Grzegorzewski [61] defined some new distance measures for IFSs and IVIFSs based on the Hausdorff metric. Xu [205] presented the distance and similarity measures for IVIFSs. Further, Park et al. [125] extended the distance measures for IVIFSs by adding the amplitude margin. Chen [24] presented some counterexamples for showing the errors in the existing distance measures of Grzegorzewski [61] and found that the inequalities of Euclidean or normalized Euclidean distance measures are not valid. Szmidt and Kacprzyk [149] address some difficulties in Hamming distance based on Hausdorff metric. Zhang and Yu [238] given some new distance measures for IFSs and interval-valued fuzzy sets that overcome the drawbacks of existing distance measures of Szmidt and Kacprzyk [146]. A three-dimensional Hausdorff distance proposed by Yang and Chiclana [219], and compare its consistency with its two-dimensional counterpart. Vasanti and Viswanadham [156] utilized the normalized Euclidean distance measure to determine student performance. Ejegwa and Modom [34] presented a new distance measure and its application to medical diagnosis problem. Gupta and Mohanty [65] derived the level of compensation numerically in MCDM problems under fuzzy environment.

Chen et al. [23] have been defined as the concept of similarity degree between two fuzzy sets. However, in 2002, Dengfeng and Chuntian [32] extended this concept for IFSs and utilized it to solve the pattern recognition problems. Later on, Mitchell [116] pointed out some errors in axiom properties of existing similarity measure given by Dengfeng and Chuntian [32], and modified it and construct an effective similarity measure. Liang and Shi [98] presented a similarity measure and a comparison analysis of it with some other existing similarity measures. Hung and Yang [80] construct a similarity measure based on Hausdorff distance, and compared with other existing similarity measures to show the effectiveness of it by taking some numerical examples. Liu [102] defined a similarity measure of IFSs that overcomes the drawbacks of the similarity measures proposed by Dengfeng and Chuntian [32]. Xu [205] defined three types of similarity measures for IFSs based on geometric distance, set-theoretic and matching function, and applied to these measures for solving MADM problems. Li et al. [97] presented a comparative study for existing similarity measure of IFSs by taking some counter-cases in pattern recognition, and this study also focused on weakness and conditions of these similarity measures where they may not work. Xia and Xu [188] proposed some similarity measures for IFSs and applied it to group DM. Ye [224] found some counterexamples of existing similarity introduced by Li et al. [97] and constructed a cosine similarity measures for IFSs with application in pattern recognition and medical diagnosis problem. Besides that, Wei, Wang and Zhang [179] defined some similarity measures for IVIFSs based on the entropy measures by using the transformation method. Singh [137] defined a cosine similarity measure for IVIFS that overcomes the drawbacks of the existing similarity measures introduced by Li et al. [97] and utilized it into pattern recognition problems. Ye [225] extended cosine similarity measure by adding hesitance degree to it and applied it for solving MADM issues. Wu et al. [187] analyzed that similarity measures, defined by Wei, Wang and Zhang [179], do not consider the hesitancy degree of IVIFSs and have some errors in the pattern recognition, therefore, a similarity measure has been constructed that overcomes them errors.

Boran and Akay [10] defined a parametric distance and similarity measures for IFSs and also presented a comparative study with the existing similarity measures [32, 62, 63, 80, 98, 116, 224] by showing some numerical counterexamples. Dugenci [33] given a novel distance measure between two IVIFSs by generalizing the distance measure defined by Boran and Akay [10] and presented few counterexamples of a well known existing measure defined by Xu [197]. Chen and Chang [18] indicates that existing similarity measures [224, 238] have the drawbacks of "the division by zero". Nguyen [120] proposed a knowledge measure of IFSs, and based on this constructed a similarity/dissimilarity measure that overcomes the drawbacks of similarity measure given by Chen and Chang [18]. Chen, Cheng and Lan [19] point out, by using some counterexamples, that the well known existing similarity measures defined by Chen and Chang [18] does not satisfy the triangular property, and presented a similarity measure based on the centroid points of transform fuzzy numbers and its application in pattern recognition problem. Besides that, Chen, Cheng and Lan [19] presented a detail comparative study of proposed measure with the existing measures [10, 18, 32, 64, 80, 98, 102, 116, 224, 238] to show the validity of proposed similarity measure. Song et al. [141] proposed a similarity measure that is directly based on the three components of IFS, and overcome the error of other existing measures. Garg [46] defined similarity and distance measures for intuitionistic multiplicative with the application into DM problems. Shen et al. [136] defined a distance measure for IFSs that overcome the drawbacks of the existing distance measure proposed by Chen, Cheng and Lan [19] and utilized it to credit risk evaluation. Recently, Luo and Zhao [112] analyzed that the existing distance measures [61, 136, 141, 146, 172, 219] have some counter-intuitive cases, and proposed a distance measures based on the binary function (strictly increasing or decreasing) and matrix norm with application in medical diagnosis problem. Ke et al. [91] given an effective distance measure, based on interval values, for IFSs by transforming the IFSs into interval-valued fuzzy sets, and a comparative study based on the numerical examples have also presented to show the validity of proposed distance measure. Rashid et al. [131] constructed a distance measure between IVIFSs for calculating the entropy measure and applied it to solve MADM problems. Jiang et al. [84] constructed a novel similarity/distance measure by intersection of transform IFSs into isosceles triangle and it overcome the drawbacks of existing measures[10, 32, 80, 98, 116, 224]. Hence, based on the above analysis, we can say that distance and similarity measures under the IFSs/IVIFSs environment are used widely for solving MADM problems, and mostly for solving pattern recognition and medical diagnosis problems. In most cases, the similarity measure is constructed from the normalized distance measure.

1.1.2 Review of TOPSIS approaches

In last two decades, various classical techniques such as AHP ("Analytical Hierarchy Process") [155], VIKOR ("VlseKriterijumska Optimizacija I Kompromisno Resenje")[122], COPRAS("Complex proportional assessment")[233], ELECTRE ("ELimination Et Choix Traduisant la REalite") [69] and TOPSIS ("Technique for Order Preference by Similarity to Ideal Solution")[82] approach are used to solve MADM/MAGDM problems. Among such techniques, TOPSIS is the most popular and dominant method whose concept is to choose the best alternative(s) which have the farthest distance from the negative ideal solution. The prominent characteristics of TOPSIS method are that they not only take into account distance/similarity between sets (as discussed in Section 1.1.1) but also examine similar or dissimilar between the sets so as to avoid to draw the conclusion based on the small distance or large similarity. By taking these advantages, several researchers have addressed the problems under the IFS/IVIFS environment. For example, Ye [221] presented the TOPSIS approach under IVIFSs environment for solving the MAGDM problems. Li [94] constructed two non-linear programming models based on TOPSIS method to solve the MADM problems. Park et al. [126] established an optimization model based on TOPSIS approach to obtain attributes weights during solving the MAGDM problems under IVIFSs environment. Tan [151] developed a MAGDM approach by combining the TOPSIS technique and Hamming distance based on Choquet integral. Zhang and Yu [237] presented a mathematical programming model based on the cross-entropy and TOPSIS method to solve MADM problems.

Bai [8] defined a TOPSIS method based on an improved score function under IVIFSs environment for tackling MADM issues. Joshi and Kumar [89] developed a TOPSIS approach for solving MADM problems by using distance measures [61] with portfolios analysis. Yue [230] and Zhao [247] defined an approach to extract the attribute weights while solving a MAGDM problem using a TOPSIS method under the IFSs and IVIFSs environment respectively. Chen [26] constructed two optimization models to obtain attribute's weights and defined a TOPSIS method by involving inclusion-comparison possibilities for solving MAGDM problems under IVIFSs environment. Zhang et al. [241] developed a technique for solving MAGDM issues under IVIFSs data by combining maximizing consensus and TOPSIS method. Onat et al. [121] defined the application of the MADM method and TOPSIS technique in the field of alternative vehicle technologies for ranking of the life-sustainability of vehicles. Chen, Cheng and Lan [19] presented a similarity measure based TOPSIS approach under IFSs environment to overcome the drawbacks of some existing methods [89].

The above-presented approaches on TOPSIS methods are based on the distance measures which do not take into account the subjective information of the attributes as well as decision maker's attitude. Therefore, to addressed these issues into the TOPSIS process, Zeng and Xiao [234] presented a hybrid TOPSIS method embedding an ordered weighted averaging (OWA) distance operator into the process under the IFSs environment. Wang, Liu, Li and Niu [168] developed a hybrid TOPSIS approach by including different types of integrating OWA aggregation to handle MAGDM problems under the IFSs environment. Wang and Chen [163] presented a MADM method, in which weights of attributes are partially known, by combining the linear programming methodology and TOPSIS techniques under the IVIFSs data. Zhang [239] presented a similarity measure [10] based TOPSIS method for solving MAGDM problems under the IVIFSs environment. Wang and Chen [162] presented an extended TOPSIS approach for IVIFSs, where the information about the alternatives values and weights of attributes are represented by IVIFNs. Further, their presented method has successfully overcome the certain drawbacks of the existing MADM methods [21, 94]. After that, Gupta et al. [68] extended the concept of Wang and Chen [162] by adding hesitancy degree into analysis for solving the MAGDM problems.

Under the LIFS environment, Ou et al. [123] presented a TOPSIS method. Kumar and Biswas [93] extended the exiting work of Li [94] by considering the third component of

IFSs into weighted distance measures and mathematical programming model. Shen et al. [136] presented a new distance measure and hence its based a TOPSIS approach for IFSs information. Recently, Memari et al. [114] used the TOPSIS approach for sustainable supplier selection under the IFSs environment. Sachdeva and Kapur [134] proposed a TOPSIS approach with entropy weight and hybrid IFSs. However, apart from them, some different kinds of MADM/MAGDM approach based on TOPSIS approach have been presented by the researchers in the literature [11, 37, 66, 67, 88, 144].

1.1.3 Review of Aggregation operators

The aggregation operators (AOs) is one of the most collective phase and process during ordering the alternatives. The basic principle of AOs is to aggregate the various values into the collective once. Since in MADM process, these always occur more than one attribute values towards a single alternative and hence the process of AOs play a significant role between them. Often, aggregation operators are based on the arithmetic, geometric, and integrals, etc. In the literature, the well-known aggregating techniques are on the basis of the weighted averaging (WA) and ordered weighted averaging (OWA) process [115, 210]. In 1995, Grabisch [60] firstly defined the concept of the fuzzy integral in the DM process. Later on, Wang and Fan [174], Xu [194] extended the OWA operator to the fuzzy environment. Also, Yager [212] generalized the concept of OWA in terms of defining the generalized OWA AOs. However, in terms of IFS, Atanassov [7], De et al. [31] presented the basic operational laws such as addition, scalar multiplication, power for IFSs. Based on these operational laws, Xu [203], Xu and Yager [207] presented the weighted, ordered weighted, and the hybrid weighted averaging and geometric AOs for different pairs of intuitionistic fuzzy numbers (IFNs). Later on, Zhao et al. [245] extended these AOs into its generalized form. However, apart from them, some other AOs by using some other families of t-norms such as Einstein, Hamacher, etc., are proposed by the researchers for a different IFNs [52, 74, 78, 99, 100, 159–161, 169, 173, 181, 184, 248]. Besides that, Wei [180] defined some induced AOs for aggregating the IFNs. Garg [40], Xia et al. [189] defined the weighted averaging and geometric AOs for the intuitionistic multiplicative environment instead of additive environment.

In the above-stated literature, it is found that their approaches are so useful during solving the MADM problems, but simultaneously their approaches are also limited to a certain extent, for example, these above-mentioned theories had developed under the assumptions that all the considered attributes are independent to each other and are uncorrelated to each other during accessing the alternatives. To address into DM problems, some various kinds of AOs are developed by the researchers such as Bonferroni mean (BM), Maclaurin symmetric mean (MSM), Choquet integral (CI), Heronian mean (HM), etc. For example, Xu et al. [193] utilized the Einstein operations and CI to developed the AOs for IFNs during solving MADM problems. Xu and Yager [208] developed BM operator by considering the interrelationship between two input argument at the same time. Qin and Liu [128] extended the BM operator characteristic, to consider two arguments, to the multi-input arguments by using the concepts of MSM and hence developed an AO based on MSM for IFNs. He, Chen, Zhau, Liu and Tao [70], He, Chen, Zhou, Han, Zhao and Liu [71] improved some existing operational laws [203, 207] by means of probability sum and hence defined some geometric interactive operational laws for IFNs. After that, Garg [45] defined some series of intuitionistic fuzzy interactive averaging (IFIA) AOs by adding the hesitance degree of each element into the analysis. Garg [39] also presented some generalized IFIA operators based on t-conorm and t-norm operations. Further, Garg [48] overcomes some shortcomings of the operators [169, 203, 248] by defining an improved Einstein operational laws and hence it based AOs. Liu and Chen [104] developed the HM operator for IFNs to solve the group DM problems.

Besides the AOs for IFNs, researchers have well focused on the AOs for IVIFNs also. For instance, Xu and Chen [200], Xu [204] presented some weighted averaging and geometric AOs for IVIFNs respectively. Also, Wei and Wang [183] presented some weighted, ordered weighted and hybrid geometric AOs for different IVIFNs information. Wang and Liu [170] presented such AOs by using Einstein t-norm operations for IVIFNs to solve MADM problems. Liu [103] aggregated the IVIFN information by using Hamacher t-norm operations. Zhou et al. [251] combined the importance of WA and the continuous OWA operators and defined Continuous-IVIFOWA operator for IVIFNs. Chen, Cheng and Tsai

[20] presented transformation techniques of IVIFNs to right-angle triangular fuzzy numbers and hence defined an AO to remove some drawbacks of the existing AOs [103, 183]. Further, he applied the presented AOs to solve MADM problems. Later on, Garg et al. [54] defined some improved interactive AOs by using Hamacher t-norm operations to resolve some deficiencies in the existing AOs [103] and hence developed an algorithm based on them to solve MADM problems. Chen and Tsai [22] defined some improved multiplication operations for IVIFNs and its based AOs to resolve the drawbacks of some existing AOs [170, 200, 204] under some certain cases.

Furthermore, in the literature, some other kinds of investigation related to AOs are found which consider the interdependency between the argument values during the DM process. In that directly, Yager [211] firstly throw a light on it by developing a power average (PA) and power OWA operators to give a more flexible opinion to the decision-makers during the process. Later on, Xu and Yager [209] extended the Yager [211] operators to the geometric ones, Under the IFS and IVIFS information, Xu [198] presented an approach to solving the MADM problems using the power AOs. Some generalized power AOs for the different IFNS are presented by Zhou et al. [250] and Zhang [244]. Liu and Li [105] utilized BM and power AOs to define some hybridized power AOs for IVIFNs to solve DM problems. To deal with a more fair decision during the process, Yager [213] defined the concepts of prioritized AOs by including the features of the prioritization between the different attributes. Based on its importance, Yu [227] presented the prioritized aggregating operators for aggregating the IFS information. Yu [226, 228] defined some generalized prioritized geometric and averaging AOs for solving group DM problems under the IFNs environment. Later on, these prioritized AOs are extended to the IVIFSs by the authors in Yu et al. [229]. Li et al. [96] modified some improved version of the AOs [229] under the IVIFS environment. Garg and Arora [58] presented generalized power AOs for the intuitionistic fuzzy soft set information. Rani and Garg [130] presented the power AOs for the complex IFSs information.

The above well-suited work is applied to solve the DM problems by collecting preference under the quantitative features only. However, to address the qualitative information into the DM process, by taking concepts of linguistic variables (LVs)Zadeh [232] has been utilized. Under this, Herrera and Herrera Viedma [72] defined some weighted AOs by taking linguistic information. Xu [201] defined some linguistic weighted geometric AO. Later on, Xu et al. [192] proposed linguistic power AOs with application in MADM issues. However, in order to address the problem of LVs under the IFS environment, Zhang [236] defined some geometric AOs for the linguistic IFNs (LIFNs). Also, Chen et al. [27] presented some weighted averaging AOs for LIFNs. Peng et al. [127] presented an AO by utilizing Frank t-norm operations and HM operators under the LIFS environment. Liu and Wang [110] presented some improved version of AOs by adding the degree of hesitance into analysis to solve the group DM problems under the LIFS environment. However, several other works have been investigated on AOs under the LIFS environment by using different operators [15, 28, 29, 35, 87, 107–109, 243]. Recently, Arora and Garg [4] developed the prioritized AOs to solve the group DM problems under the LIFS environment and hence investigated their several fundamental properties.

From the above extensive literature, we observed that both the qualitative and quantitative information is the most powerful tool to handle uncertain and fuzzy information using intuitionistic fuzzy information. In recent years, under this environment, researchers also have paid more attention to the development of AOs to solve DM problems. For example, Luo et al. [113] defined the exponential AOs based on t-conorm and t-norm operations for IFNs. Jiang et al. [85] defined PA operator based on entropy for aggregating IF information. Tao et al. [154] defined some other types of operations for IFSs based on Archimedean copulas and co-copulas. Garg [50] defined some improved operational laws for IVIFNs and hence defined some geometric AOs to modified the operators defined in the literature [103]. Liu et al. [106] defined the concept of partitioned HM AOs for LIFSs environment by considering the interactions between membership and non-membership grades. Recently, Garg [51] developed some new Aos based on the Hamacher t-norm operating with entropy weights to overcome the shortcomings of some existing AOs [78]. Tang and Meng [152] defined the operational laws for LIFNs based on the Hamacher norm and hence defined an AO for them. Wang and Mendel [171] defined some new prioritized AOs based on Lukasiewicz triangular norm to improve satisfy the monotonicity under the

IVIFS environment to solve the DM problems. Arora and Garg [4] developed the prioritized AOs for LIFNs to solve the group DM problems and hence investigated their several fundamental properties.

1.1.4 Review of Set pair analysis

In 1989, an uncertainty analysis theory, by combining dialectical thinking and mathematical tools, was developed by Zhao [246] known as set pair analysis (SPA) and is different from the traditional probability and fuzzy set theory in terms of considering both certainty and uncertainty as one consolidate certain-uncertain system. SPA theory studies the internal relationship between the parts of a system. The core idea of SPA is to define a set pair for two related sets and characterized them in terms of constructing the connection number (CN), which consists of "identity", "discrepancy", and"contrary" degrees. For any two dependent sets A and B of a given problem W, a set pair between them is denoted by $H(A, B)$ and having N characteristics. The CN corresponding to set pair H denoted by $\mu(H, W)$ is defined as

$$\begin{aligned} \mu(H,W) &= (S/N) + (F/N)i + (P/N)j \\ &= a + bi + cj \end{aligned} \tag{1.1}$$

where, $a(= S/N), b(= F/N)$, and $c(= P/N)$ represent the degrees of "identity", "discrepancy", and"contrary" respectively, $i \in [-1, 1]$ is the coefficient of "discrepancy degree", and $j = -1$ is the coefficient of "contrary degree". Here S, F, P denotes the identity, discrepancy, contrary characteristics respectively. It is clearly seen that $0 \leq a, b, c \leq 1$ and $a + b + c = 1$. Jiang et al. [86] discussed the basic concept and system information related to SPA theory, and its applications in the various fields. Yang et al. [218] defined the similarity and distance measures between the two CNs. Liu et al. [101] and Yang, Yang, Zhang and Jiao [220] defined basic operation laws, such as addition, subtraction, multiplication, division, and composition, for CNs and their properties. SPA theory has been widely applied into various fields like agriculture, military affairs, neural network forecastle, water quality assessment, and DM problems, etc. Pan et al. [124] presented

SPA assessment model based on the α-cut of triangular fuzzy number for flood lose assessment. Su et al. [143] applied the SPA theory to evaluate the urban ecosystem health levels. Wang et al. [165] applied the five-element CNs for a comprehensive evaluation of water-saving renovation in the irrigation area. Li et al. [95] presented the water quality assessment model based on the SPA theory. Wensheng and Yueqing [186] developed SPA method to assess the landslide hazard degree. Zou et al. [252] combine the AHP and SPA method with fuzzy set theory to assess the flood risk. To evaluate the crop production system, a multi-functional indicator system based on the SPA had presented by Tao et al. [153]. Yang, Liang, Singh, Wang, Zhou, Liu, Cao, Huang and Wu [216] defined the SPA model for assessing debris flow hazard.

Instead of the application of SPA theory to solve other consent, this theory is also widely applicable to solve the DM problems. Hu and Yang [77] combined the prospect theory with SPA to solve the dynamic stochastic MCDM problems under the crisp environment. Rong et al. [132] defined an entropy based on the SPA theory to determine geological disaster-risked. Rui et al. [133] proposed MAGDM method with unknown weight based on SPA under the crisp environment. Lü and Zhang [111] developed the MADM approach based on the SPA theory. Xie et al. [191] applied the SPA technique for solving the MADM problems under the interval-valued fuzzy number environment. After that, Sun et al. [145] defined SPA based lattice order DM model under crisp data. Chaokai and Meng [16] presented a novel aggregation scheme based on SPA theory. Fu and Zhou [36] presented the DM method based on SPA theory for the triangular fuzzy number environment. As the SPA theory has become increasingly habitual to the researchers in recent years, it has also been applied in other research fields such as credibility analysis of power cloud terminal[249], social-ecological system [177], and water inrush risk assessment [176].

Apart from these studies, some mathematical algorithm is also formulated by the researchers by using the SPA theory to solve different complex problems. The latest developments include Yan and Xu [214] combined the cloud model with SPA theory for hazard evaluation of biomass gasification stations. Yan and Xu [215] developed a risk evaluation model by combing the layer of protection analysis with the SPA technique. Cao et al. [14] defined a stochastic MCDM method under the IVIF environment based on

SPA theory. Gao [38] constructed a mathematical model based on the SPA to evaluate the information risk during a given task. Cui et al. [30] developed a quality evaluation model by using SPA theory to evaluate the carrying capacity and quantitative diagnosis of regional water resources. Xie and Guo [190] addressed a method to evaluate and manage the human risk factor based on SPA theory. Recently, Su et al. [142] developed a model for the evaluation and prediction of groundwater quality based on SPA and Markov chain theory, in which the SPA has been used to measure the quality of groundwater.

1.2 Gaps found in the work and motivation towards the SPA and IFS/IVIFS

In the above Section 1.1, various MADM or MAGDM techniques reviewed under the IFS or IVIFS or LIFS environment in the different fields. For solving the DM problems, the main important task for the decision-maker(s) is to choose the appropriate DM method according to the nature of the problem. Therefore, in the case of group DM, an ambiguity may occur between the decision-makers to choose the appropriate method and which can lead to extra expenditure of time, resources, and money. But the main important thing of a good DM procedure is to optimize the time and money. That's why there is a need to research the appropriate technique to handle such problems for obtaining a better result. Other than this, during the literature survey of the distance/similarity and TOPSIS approaches in Section 1.1.1 and Section 1.1.2, we found they have some gaps in DM procedure given as:

(i) In the literature, various distance/similarity measures are the exit to calculate the degree of the distinction of different IFSs/IVIFSs. In which most measures have a "blind spot" because, in certain specific situations, they may not be able to classify the results when dealing with DM problems.

(ii) TOPSIS approaches basically depend on the score functions, and various types of distance/similarity measures to find the "positive ideal solution" (PIS) and "negative ideal solution" (NIS). But in literature, existing score functions, and distance/similarity measures have deficiencies in some particular situation, therefore, they can put the adverse effect in DM procedure and can give the irrelevant result about problems. For example, existing TOPSIS methods defined in [8, 89, 126] are unable to rank the alternatives that have reported in this study.

The contrast of this, SPA is the most powerful uncertainty theory that overcomes the limitations of the traditional probability and fuzzy set theory. It also avoids the shortcomings that in the past have characterized research into the problem of uncertainty from the point of view of certainty. SPA provides the quantitative descriptions of mathematical expressions for quantitative and qualitative conversion processes. Recently, SPA theory has been widely applied in the hydrology, information management, artificial intelligence, social and economic as well as resources and environment that have been reviewed in the literature. But, there is the only a limited application of the SPA theory in DM under the IFSs/IVIFSs. There is a need to enhance the SPA theory for solving the DM problems under IFS/IVIFS/LIFS environment. Some essential tools to apply the SPA theory in DM under these environments and lack of research are given as:

(i) For solving the DM problems under the IFS/IVIFS/LIFS environment, we need a method to convert these environments into CNs environment which is the first step to operate the SPA theory under these environments.

(ii) TOPSIS method is an easier and powerful DM method for solving the MADM/MAGDM problems, but till now there is no research on TOPSIS approach based on the SPA theory and all existing TOPSIS approaches are based on the score functions and distance/similarity measures under the IFS/IVIFS environment. That's why there is a need to enhance the SPA theory in TOPSIS approach.

(iii) Measures are important tools in MADM problems to fuse the information of multiple-attribute into a collective one. However, we found some limitations in existing measures discussed above, therefore, to handle and overcome these shortcomings there is a need to extend the idea of distance/similarity measures under the SPA theory which is not reported in the literature.

1.3 Objective of the thesis

The overall aim of this research is to develop the various MADM or MAGDM methods under the different and uncertain fuzzy environment under the IFS or IVIFS environment. For it, the CNs of the SPA theory has been developed and then based on them, we develop several approaches under the different environment to solve the DM problems. By motivating from the above literature and gaps, the main objectives of the work are summarized as:

(O1) To study the various approaches, tools and technologies available for decision making problems and literature reviews for set pair analysis theory.

(O2) To develop the connection numbers based on SPA theory under the fuzzy/intuitionistic fuzzy environment.

(O3) To develop some new information measures based on connection number under the fuzzy/intuitionistic fuzzy environment and explore their application in some decision-making process.

(O4) To develop some different aggregation operators based on connection number for handling the different preferences.

1.4 Structure of the thesis

The entire thesis has been organized into nine chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of DM approaches by using several approaches is presented in the first chapter.

In **Chapter 2**, the basics and preliminaries related to the intuitionistic fuzzy sets, information measures, aggregation operators, etc., are given.

Chapter 3 presents the novel MADM method under IVIFSs environment by integrating a TOPSIS method. For it, firstly we construct the CN of the SPA theory for each IVIFS and then based on its, we define some exponential distance measures to compute the degree of discrimination between the two IVIFSs. The supremacy of the proposed measure is also discussed. Afterward, to solve the DM problems, a novel TOPSIS method based on the proposed distance measures is developed, illustrated with a numerical example and compared their results with the several existing studies.

In **Chapter 4**, we present some axioms of the distance measures based on Hamming, Euclidean, and Hausdorff metrics whose preferences related to the attributes are made in the form of CN. Several desirable relations between the proposed measures are investigated. Later, we develop a MADM approach based on the proposed distance measures to investigate the DM problem. The effectiveness of the approach has been demonstrated through a case study and compared their studies with several existing measures.

Chapter 5 presents some novel similarity measures to measure the relative strength of the different IFSs after pointing out the weakness of the existing measures. For it, a CN set (CNS) is formulated and based on it some new similarity measures are defined. A comparative analysis of the proposed and existing measures are formulated in terms of the counter-intuitive cases for showing the validity of it. Finally, an illustrative example is provided to demonstrate it.

Chapter 6 presents a novel correlation coefficient and weighted correlation coefficients formulation to measure the relative strength of the different IFSs in the form of the CNs. The limitations of certain existing measures are highlighted and overcome by the proposed measure. Afterward, a DM approach is presented based on the developed measures to solve the problems. Two illustrative examples related to pattern recognition and medical diagnosis are taken to validate the effectiveness and applicability of the proposed decision method.

In **Chapter 7**, we constructed the CNs for the IFSs as well as IVIFSs and hence based on it, we constructed the TOPSIS method for solving the DM problems. The basic feature of the TOPSIS method is chosen the alternative which has the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. Afterward, based on the proposed CNs, we define two algorithms to solve the DM problems where the features are extracted either in IFNs or in IVIFNs. The validity of these proposed algorithms is tested with an example and compared it with several existing results.

Chapter 8 enhanced the LIFS with the SPA theory and hence defined the linguistic connection number (LCN) and studied their various operational laws. Based on it, we have developed various aggregation operators namely, LCN weighted geometric (LCNWG), LCN ordered weighted geometric (LCNOWG), and LCN hybrid geometric (LCNHG) operators with LIFS environment. Also, the shortcoming of the existing operators under LIFS environment have been highlighted and overcomes by the proposed operators. Few properties of these operators have been also investigated. Further, a group DM approach has been presented, based on these operators, which has been illustrated by a numerical example to show the effectiveness and validity of the proposed approach.

Chapter 9 presents the group DM approach under the linguistic intuitionistic fuzzy (LIF) set environment. For it, we first propose a new ranking method named as possibility degree measures to compare the different LIF numbers. Further, in order to aggregate the different LIF numbers, some weighted and ordered weighted averaging aggregation operators are proposed by using Einstein t-norm operations. The prominent characteristics of these operators are also investigated. Afterward, a MAGDM approach, based on proposed operators and the possibility degree measure, is developed under the LIFSs environment. A numerical case is taken to manifest the practicability and feasibility of the proposed group DM method.

Chapter 2

Preliminaries

In this chapter, we present the basic concepts and the mathematical structure related to IFSs, IVIFSs, LIFS, aggregation operators etc., over the universal set $\mathcal{X}$.

2.1 Intuitionistic fuzzy set and interval-valued intuitionistic fuzzy set

IFSs [6] and IVIFSs [5] are the most permissible generalization of fuzzy set (FS)[231] to describe the uncertainty in the data by considering the non-membership degrees (NMDs) along with the membership degrees (MDs) such that their sum should not be greater than one. Mathematically, both these sets and their basic operations are defined as below:

Definition 2.1.1. [6] An IFS $\mathcal{A}$ in $\mathcal{X}$ is defined as an ordered pair of MD and NMD given as

$$\mathcal{A} = \{(x, u_{\mathcal{A}}(x), v_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\}, \tag{2.1}$$

where $u_{\mathcal{A}}, v_{\mathcal{A}} : \mathcal{X} \to [0,1]$ represents MD and NMD of x to $\mathcal{A}$ respectively such that for any $x \in \mathcal{X}$, $u_{\mathcal{A}}(x) + v_{\mathcal{A}}(x) \leq 1$. The pair $(u_{\mathcal{A}}(x), v_{\mathcal{A}}(x))$ of IFS is oftenly represented as (u, v) where $u \in [0,1], v \in [0,1], u + v \leq 1$ and called as IFN [197].

Definition 2.1.2. [6] For any two IFNs $\mathcal{A} = (u_{\mathcal{A}}, v_{\mathcal{A}})$ and $\mathcal{B} = (u_{\mathcal{B}}, v_{\mathcal{B}})$, we have

(i) $\mathcal{A} \subseteq \mathcal{B}$ if $u_{\mathcal{A}} \leq u_{\mathcal{B}}$ and $v_{\mathcal{A}} \geq v_{\mathcal{B}}$.

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \supseteq \mathcal{B}$.

(iii) $\mathcal{A}^c = (v_{\mathcal{A}}, u_{\mathcal{A}})$.

(iv) $\mathcal{A} \cup \mathcal{B} = (\max\{u_{\mathcal{A}}, u_{\mathcal{B}}\}, \min\{v_{\mathcal{A}}, v_{\mathcal{B}}\})$.

(v) $\mathcal{A} \cap \mathcal{B} = (\min\{u_{\mathcal{A}}, u_{\mathcal{B}}\}, \max\{v_{\mathcal{A}}, v_{\mathcal{B}}\})$.

Definition 2.1.3. [207] The score function for an IFN $\mathcal{A} = (u_{\mathcal{A}}, v_{\mathcal{A}})$ is defined as :

$$\mathcal{S}(\mathcal{A}) = u_{\mathcal{A}} - v_{\mathcal{A}}, \tag{2.2}$$

and accuracy function is defined as:

$$\mathcal{H}(\mathcal{A}) = u_{\mathcal{A}} + v_{\mathcal{A}} \tag{2.3}$$

Definition 2.1.4. [5] An IVIFS $\mathcal{A}$ in $\mathcal{X}$ is defined as

$$\mathcal{A} = \left\{(x, \widetilde{u}_{\mathcal{A}}(x), \widetilde{v}_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\right\}, \tag{2.4}$$

where $\widetilde{u}_{\mathcal{A}}(x) = [u^L_{\mathcal{A}}(x), u^U_{\mathcal{A}}(x)]$ and $\widetilde{v}_{\mathcal{A}}(x) = [v^L_{\mathcal{A}}(x), v^U_{\mathcal{A}}(x)]$ are all subsets of $[0,1]$, and represents the MDs and NMDs of x to $\mathcal{A}$ such that for any $x \in \mathcal{X}, 0 \leq u^U_{\mathcal{A}}(x) + v^U_{\mathcal{A}}(x) \leq 1$. For convenience, we denote the pair as $\mathcal{A} = ([u^L, u^U], [v^L, v^U])$ and called as IVIFN.

Definition 2.1.5. [5] Let $\mathcal{A} = ([u^L_{\mathcal{A}}, u^U_{\mathcal{A}}], [v^L_{\mathcal{A}}, v^U_{\mathcal{A}}])$ and $\mathcal{B} = ([u^L_{\mathcal{B}}, u^U_{\mathcal{B}}], [v^L_{\mathcal{B}}, v^U_{\mathcal{B}}])$ be any two IVIFNs, then

(i) $\mathcal{A} \subseteq \mathcal{B}$ if $u^L_{\mathcal{A}} \leq u^L_{\mathcal{B}}, u^U_{\mathcal{A}} \leq u^U_{\mathcal{B}}, v^L_{\mathcal{A}} \geq v^L_{\mathcal{B}}$ and $v^U_{\mathcal{A}} \geq v^U_{\mathcal{B}}$.

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \supseteq \mathcal{B}$.

(iii) $\mathcal{A}^c = ([v^L_{\mathcal{A}}, v^U_{\mathcal{A}}], [u^L_{\mathcal{A}}, u^U_{\mathcal{A}}])$.

(iv) $\mathcal{A} \cup \mathcal{B} = ([\max\{u^L_{\mathcal{A}}, u^L_{\mathcal{B}}\}, \max\{u^U_{\mathcal{A}}, u^U_{\mathcal{B}}\}], [\min\{v^L_{\mathcal{A}}, v^L_{\mathcal{B}}\}, \min\{v^U_{\mathcal{A}}, v^U_{\mathcal{B}}\}])$;

(v) $\mathcal{A} \cap \mathcal{B} = ([\min\{u^L_{\mathcal{A}}, u^L_{\mathcal{B}}\}, \min\{u^U_{\mathcal{A}}, u^U_{\mathcal{B}}\}], [\max\{v^L_{\mathcal{A}}, v^L_{\mathcal{B}}\}, \max\{v^U_{\mathcal{A}}, v^U_{\mathcal{B}}\}])$.

Definition 2.1.6. [204] The score function for an IVIFN $\mathcal{A} = ([u^L_{\mathcal{A}}, u^U_{\mathcal{A}}], [v^L_{\mathcal{A}}, v^U_{\mathcal{A}}])$ is defined as:

$$\mathcal{S}(\mathcal{A}) = \frac{u^L_{\mathcal{A}} + u^U_{\mathcal{A}} - v^L_{\mathcal{A}} - v^U_{\mathcal{A}}}{2}, \tag{2.5}$$

and accuracy function is

$$\mathcal{H}(\mathcal{A}) = \frac{u^L_{\mathcal{A}} + u^U_{\mathcal{A}} + v^L_{\mathcal{A}} + v^U_{\mathcal{A}}}{2}. \tag{2.6}$$

To order the different IFNs and/or IVIFNs, a comparison law between them is defined as follow.

Definition 2.1.7. [203] Let $\mathcal{A}$ and $\mathcal{B}$ be either two IFNs and/or IVIFNs then based on the score and accuracy functions as defined in Definitions 2.1.3 and 2.1.6, an order relation $A \succeq B$, where "$\succeq$" refer "preferred to", occurs if anyone of the following condition met.

(i) If $\mathcal{S}(\mathcal{A}) \geq \mathcal{S}(\mathcal{B})$.

(ii) $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{B})$ and $\mathcal{H}(\mathcal{A}) \geq \mathcal{H}(\mathcal{B})$.

2.2 Linguistic intuitionistic fuzzy set

In the DM process, sometimes, we are unable to rate the objects in terms of quantitative information but usually to rate them in terms of qualitative attributes. For example, to describe the intelligence of student, we generally used the word "good", "very good", "average" etc. To handle and describe such information into process, a concept of linguistic variable (LV)[232] is used, and hence their corresponding set called linguistic term set (LTS).

Definition 2.2.1. [73] Let $S = \{s_t \mid t = 0, 1, 2, \ldots, h\}$ be LTS, where s_t represents a desirable value for a LV and h is even number. Further, a LV s_t has the following properties:

(i) $s_k \leq s_t \Leftrightarrow k \leq t$.

(ii) $\text{Neg}(s_k) = s_{h-k}$.

Later on, Xu [201] extended the LTS S to continuous LTS $S_{[0,h]} = \{s_z \mid s_0 \leq s_z \leq s_h\}$, where, if $s_z \in S$, then s_z is called the original LTS, otherwise, s_z is called the virtual term.

Definition 2.2.2. [236] A LIFS '$\mathcal{A}$' in finite universe set $\mathcal{X}$ is defined as

$$\mathcal{A} = \Big\{(x, s_{\tau(x)}, s_{\theta(x)}) \mid x \in \mathcal{X}\Big\}, \tag{2.7}$$

where, $\forall\ x,\ s_\tau, s_\theta \in S_{[0,h]}$ represent linguistic MD and NMD of x to $\mathcal{A}$ respectively, such that $0 \leq \tau + \theta \leq h$. The pair (s_τ, s_θ) is called a LIFN.

Definition 2.2.3. [236] For LIFN $\mathcal{A} = (s_\tau, s_\theta)$, the score function is defined as

$$\mathcal{S}(\mathcal{A}) = s_{(h+\tau-\theta)/2} \tag{2.8}$$

and accuracy function is

$$\mathcal{H}(\mathcal{A}) = s_{\tau+\theta} \tag{2.9}$$

Definition 2.2.4. [27] For two LIFNs $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$ and $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$, we have

(i) $\mathcal{A}_1 \subseteq \mathcal{A}_2$ if $\tau_1 \leq \tau_2$ and $\theta_1 \geq \theta_2$.

(ii) $\mathcal{A}_1 = \mathcal{A}_2$ iff $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_1 \supseteq \mathcal{A}_2$.

(iii) $\mathcal{A}_1^c = (s_{\theta_1}, s_{\tau_1})$.

(iv) $\mathcal{A}_1 \cup \mathcal{A}_2 = (\max(s_{\tau_1}, s_{\tau_2}), \min(s_{\theta_1}, s_{\theta_2}))$.

(v) $\mathcal{A}_1 \cap \mathcal{A}_2 = (\min(s_{\tau_1}, s_{\tau_2}), \max(s_{\theta_1}, s_{\theta_2}))$.

Definition 2.2.5. [196] Let $\gamma_1 = [s_{\tau_1}, s_{\theta_1}]$ and $\gamma_2 = [s_{\tau_2}, s_{\theta_2}]$ two uncertain linguistic variables then possibility degree $p'(\gamma_1 \succeq \gamma_2)$ of $\gamma_1 \succeq \gamma_2$ is defined as

$$p'(\gamma_1 \succeq \gamma_2) = \min\left(\max\left(\frac{\theta_1 - \tau_2}{\theta_1 - \tau_1 + \theta_2 - \tau_2}, 0\right), 1\right)$$

2.3 Information Measures

In the literature, information measures such as distance, similarity, correlation, entropy etc., play a dominant role during the decision making process. Similarity measure and distance measure are functions that give the degree of similarity and discrimination respectively among the two objects. Entropy measure quantifies the degree of fuzziness or uncertain information and the inclusion measure between two sets gives the extent to which a set is contained in another set. All these measures of information have been extensively explored by many researchers and scholars as vital topics. Here, we have presented some overview of these measures between the different IFSs and IVIFSs. For it, let $\Phi(\mathcal{X})$ be either collection of IFSs/IVIFSs over the universal set $\mathcal{X}$.

2.3.1 Distance measures

Definition 2.3.1. [172, 197] A real-valued function $\mathcal{D}' : \Phi(X) \times \Phi(X) \to [0,1]$ is called as distance measure, if $\mathcal{D}'$ satisfies the following properties for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Phi(\mathcal{X})$:

(P1) $0 \leq \mathcal{D}'(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{D}'(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}'(\mathcal{A}, \mathcal{B}) = \mathcal{D}'(\mathcal{B}, \mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}'(\mathcal{A}, \mathcal{B}) \leq \mathcal{D}'(\mathcal{A}, \mathcal{C})$ and $\mathcal{D}'(\mathcal{B}, \mathcal{C}) \leq \mathcal{D}'(\mathcal{A}, \mathcal{C})$.

For two IFSs $\mathcal{A} = \{(x_t, u_{\mathcal{A}}(x_t), v_{\mathcal{A}}(x_t)) \mid x_t \in \mathcal{X}\}$ and $\mathcal{B} = \{(x_t, u_{\mathcal{B}}(x_t), v_{\mathcal{B}}(x_t)) \mid x_t \in \mathcal{X}\}$, some existing distance measures between them are defined as:

(i) Normalized Hamming distance [172]:

$$\mathcal{D}'_1(\mathcal{A}, \mathcal{B}) = \frac{1}{2n} \sum_{t=1}^{n} \left\{ \begin{array}{c} \mid u_{\mathcal{A}}(x_t) - u_{\mathcal{B}}(x_t) \mid + \mid v_{\mathcal{A}}(x_t) - v_{\mathcal{B}}(x_t) \mid \\ + \mid \pi_{\mathcal{A}}(x_t) - \pi_{\mathcal{B}}(x_t) \mid \end{array} \right\} \tag{2.10}$$

(ii) Normalized Euclidean distance [172]:

$$\mathcal{D}'_2(\mathcal{A}, \mathcal{B}) = \left(\frac{1}{2n} \sum_{t=1}^{n} \left\{ \begin{array}{c} \mid u_{\mathcal{A}}(x_t) - u_{\mathcal{B}}(x_t) \mid^2 + \mid v_{\mathcal{A}}(x_t) - v_{\mathcal{B}}(x_t) \mid^2 \\ + \mid \pi_{\mathcal{A}}(x_t) - \pi_{\mathcal{B}}(x_t) \mid^2 \end{array} \right\} \right)^{\frac{1}{2}} \tag{2.11}$$

(iii) Distance measure [34]:

$$\mathcal{D}'_3(\mathcal{A}, \mathcal{B}) = \tfrac{1}{2n} \sum_{t=1}^{n} \left[\begin{array}{c} |u_{\mathcal{A}}(x_t) - u_{\mathcal{B}}(x_t)| + \Big| |u_{\mathcal{A}}(x_t) - v_{\mathcal{A}}(x_t)| - |u_{\mathcal{B}}(x_t) - v_{\mathcal{B}}(x_t)| \Big| \\ + \Big| |u_{\mathcal{A}}(x_t) - \pi_{\mathcal{A}}(x_t)| - |u_{\mathcal{B}}(x_t) - \pi_{\mathcal{B}}(x_t)| \Big| \end{array} \right] \tag{2.12}$$

(iv) Hausdorff distance [149]:

$$\mathcal{D}'_4(\mathcal{A}, \mathcal{B}) = \frac{1}{n} \sum_{t=1}^{n} \max \left\{ \begin{array}{c} |u_{\mathcal{A}}(x_t) - u_{\mathcal{B}}(x_t)|, |v_{\mathcal{A}}(x_t) - v_{\mathcal{B}}(x_t)|, \\ |\pi_{\mathcal{A}}(x_t) - \pi_{\mathcal{B}}(x_t)| \end{array} \right\} \tag{2.13}$$

On the other hand, for any two IVIFSs $\mathcal{A} = \big\{(x_t, [u^L_{\mathcal{A}}(x_t), u^U_{\mathcal{A}}(x_t)], [v^L_{\mathcal{A}}(x_t), v^U_{\mathcal{A}}(x_t)]) \mid x_t \in \mathcal{X}\big\}$ and $\mathcal{B} = \big\{(x_t, [u^L_{\mathcal{B}}(x_t), u^U_{\mathcal{B}}(x_t)], [v^L_{\mathcal{B}}(x_t), v^U_{\mathcal{B}}(x_t)]) \mid x_t \in \mathcal{X}\big\}$, the existing distance measures are summarized as below.

(i) Xu [197] defined the following normalized distance measures.

(a) Hamming distance:

$$\mathcal{D}_5'(\mathcal{A},\mathcal{B}) = \frac{1}{4n}\sum_{t=1}^{n}\begin{bmatrix} \mid u_{\mathcal{A}}^{L}(x_t) - u_{\mathcal{B}}^{L}(x_t) \mid + \mid u_{\mathcal{A}}^{U}(x_t) - u_{\mathcal{B}}^{U}(x_t) \mid \\ + \mid v_{\mathcal{A}}^{L}(x_t) - v_{\mathcal{B}}^{L}(x_t) \mid + \mid v_{\mathcal{A}}^{U}(x_t) - v_{\mathcal{B}}^{U}(x_t) \mid \end{bmatrix} \tag{2.14}$$

(b) Euclidean distance:

$$\mathcal{D}_6'(\mathcal{A},\mathcal{B}) = \left(\frac{1}{4n}\sum_{t=1}^{n}\begin{bmatrix} (u_{\mathcal{A}}^{L}(x_t) - u_{\mathcal{B}}^{L}(x_t))^2 + (u_{\mathcal{A}}^{U}(x_t) - u_{\mathcal{B}}^{U}(x_t))^2 \\ +(v_{\mathcal{A}}^{L}(x_t) - v_{\mathcal{B}}^{L}(x_t))^2 + (v_{\mathcal{A}}^{U}(x_t) - v_{\mathcal{B}}^{U}(x_t))^2 \end{bmatrix}\right)^{\frac{1}{2}} \tag{2.15}$$

(c) Hausdorff Hamming distance:

$$\mathcal{D}_7'(\mathcal{A},\mathcal{B}) = \frac{1}{2n}\sum_{t=1}^{n}\max\left\{\begin{matrix} \mid u_{\mathcal{A}}^{L}(x_t) - u_{\mathcal{B}}^{L}(x_t) \mid, \mid u_{\mathcal{A}}^{U}(x_t) - u_{\mathcal{B}}^{U}(x_t) \mid, \\ \mid v_{\mathcal{A}}^{L}(x_t) - v_{\mathcal{B}}^{L}(x_t) \mid, \mid v_{\mathcal{A}}^{U}(x_t) - v_{\mathcal{B}}^{U}(x_t) \mid \end{matrix}\right\} \tag{2.16}$$

(d) Hausdorff Euclidean distance:

$$\mathcal{D}_8'(\mathcal{A},\mathcal{B}) = \left(\frac{1}{2n}\sum_{t=1}^{n}\max\left\{\begin{matrix} \left(u_{\mathcal{A}}^{L}(x_t) - u_{\mathcal{B}}^{L}(x_t)\right)^2, \left(u_{\mathcal{A}}^{U}(x_t) - u_{\mathcal{B}}^{U}(x_t)\right)^2, \\ \left(v_{\mathcal{A}}^{L}(x_t) - v_{\mathcal{B}}^{L}(x_t)\right)^2, \left(v_{\mathcal{A}}^{U}(x_t) - v_{\mathcal{B}}^{U}(x_t)\right)^2 \end{matrix}\right\}\right)^{\frac{1}{2}} \tag{2.17}$$

(ii) Park et al. [125] modified the above defined distance measures by considering the amplitude margin $w_{u_{\mathcal{A}}}(x_t) = u_{\mathcal{A}}^{U}(x_t) - u_{\mathcal{A}}^{L}(x_t)$, $w_{u_{\mathcal{B}}}(x_t) = u_{\mathcal{B}}^{U}(x_t) - u_{\mathcal{B}}^{L}(x_t)$, $w_{v_{\mathcal{A}}}(x_t) = v_{\mathcal{A}}^{U}(x_t) - v_{\mathcal{A}}^{L}(x_t)$ and $w_{v_{\mathcal{B}}}(x_t) = v_{\mathcal{B}}^{U}(x_t) - v_{\mathcal{B}}^{L}(x_t)$ as follows:

(a) Hamming distance:

$$\mathcal{D}_9'(\mathcal{A},\mathcal{B}) = \frac{1}{4n}\sum_{t=1}^{n}\begin{bmatrix} \mid u_{\mathcal{A}}^{L}(x_t) - u_{\mathcal{B}}^{L}(x_t) \mid + \mid u_{\mathcal{A}}^{U}(x_t) - u_{\mathcal{B}}^{U}(x_t) \mid \\ + \mid v_{\mathcal{A}}^{L}(x_t) - v_{\mathcal{B}}^{L}(x_t) \mid + \mid v_{\mathcal{A}}^{U}(x_t) - v_{\mathcal{B}}^{U}(x_t) \mid \\ + \mid w_{u_{\mathcal{A}}}(x_t) - w_{u_{\mathcal{B}}}(x_t) \mid + \mid w_{v_{\mathcal{A}}}(x_t) - w_{v_{\mathcal{B}}}(x_t) \mid \end{bmatrix} \tag{2.18}$$

(b) Euclidean distance:

$$\mathcal{D}_{10}'(\mathcal{A},\mathcal{B}) = \left(\tfrac{1}{4n}\sum_{t=1}^{n}\begin{bmatrix} (u_{\mathcal{A}}^{L}(x_t) - u_{\mathcal{B}}^{L}(x_t))^2 + (u_{\mathcal{A}}^{U}(x_t) - u_{\mathcal{B}}^{U}(x_t))^2 \\ + (v_{\mathcal{A}}^{L}(x_t) - v_{\mathcal{B}}^{L}(x_t))^2 + (v_{\mathcal{A}}^{U}(x_t) - v_{\mathcal{B}}^{U}(x_t))^2 \\ + (w_{u_{\mathcal{A}}}(x_t) - w_{u_{\mathcal{B}}}(x_t))^2 + (w_{v_{\mathcal{A}}}(x_t) - w_{v_{\mathcal{B}}}(x_t))^2 \end{bmatrix}\right)^{\frac{1}{2}} \tag{2.19}$$

(c) Hausdorff Hamming distance:

$$\mathcal{D}'_{11}(\mathcal{A},\mathcal{B}) = \frac{1}{2n}\sum_{t=1}^{n}\left[\begin{array}{c}\max\Big\{\mid u^{L}_{\mathcal{A}}(x_t) - u^{L}_{\mathcal{B}}(x_t)\mid, \mid u^{U}_{\mathcal{A}}(x_t) - u^{U}_{\mathcal{B}}(x_t)\mid\Big\}\\ +\max\Big\{\mid v^{L}_{\mathcal{A}}(x_t) - v^{L}_{\mathcal{B}}(x_t)\mid, \mid v^{U}_{\mathcal{A}}(x_t) - v^{U}_{\mathcal{B}}(x_t)\mid\Big\}\end{array}\right] \tag{2.20}$$

(d) Hausdorff Euclidean distance:

$$\mathcal{D}'_{12}(\mathcal{A},\mathcal{B}) = \left(\frac{1}{2n}\sum_{t=1}^{n}\left[\begin{array}{c}\Big(\max\Big\{\mid u^{L}_{\mathcal{A}}(x_t) - u^{L}_{\mathcal{B}}(x_t)\mid, \mid u^{U}_{\mathcal{A}}(x_t) - u^{U}_{\mathcal{B}}(x_t)\mid\Big\}\Big)^{2}\\ +\Big(\max\Big\{\mid v^{L}_{\mathcal{A}}(x_t) - v^{L}_{\mathcal{B}}(x_t)\mid, \mid v^{U}_{\mathcal{A}}(x_t) - v^{U}_{\mathcal{B}}(x_t)\mid\Big\}\Big)^{2}\end{array}\right]\right)^{\frac{1}{2}} \tag{2.21}$$

2.3.2 Similarity measures

Definition 2.3.2. [32] A real valued function $\mathcal{S}' : \Phi(\mathcal{X}) \times \Phi(\mathcal{X}) \to [0,1]$ is called the similarity measure, if $\mathcal{S}'$ for $\mathcal{A},\mathcal{B},\mathcal{C} \in \Phi(\mathcal{X})$ satisfies the following properties :

(P1) $0 \leq \mathcal{S}'(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{S}'(\mathcal{A},\mathcal{B}) = 1 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{S}'(\mathcal{A},\mathcal{B}) = \mathcal{S}'(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{S}'(\mathcal{A},\mathcal{B}) \geq \mathcal{S}'(\mathcal{A},\mathcal{C})$ and $\mathcal{S}'(\mathcal{B},\mathcal{C}) \geq \mathcal{S}'(\mathcal{A},\mathcal{C})$.

For any two IFSs $A = \{(x_t, u_{\mathcal{A}}(x_t), v_{\mathcal{A}}(x_t))|x_t \in \mathcal{X}\}$ and $B = \{(x_t, u_{\mathcal{B}}(x_t), v_{\mathcal{B}}(x_t))|x_t \in \mathcal{X}\}$, some existing similarity measures are given as:

(i) Based on the geometric distance model [205]:

$$\mathcal{S}'_1(\mathcal{A},\mathcal{B}) = 1 - \frac{1}{2n}\sum_{t=1}^{n}\left\{\begin{array}{c}|u_{\mathcal{A}}(x_t) - u_{\mathcal{B}}(x_t)| + |v_{\mathcal{A}}(x_t) - v_{\mathcal{B}}(x_t)|\\ +|\pi_{\mathcal{A}}(x_t) - \pi_{\mathcal{B}}(x_t)|\end{array}\right\} \tag{2.22}$$

(ii) Based on set theoretic approach [205]:

$$\mathcal{S}'_2(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{t=1}^{n}\big(\min(u_{\mathcal{A}}(x_t), u_{\mathcal{B}}(x_t)) + \min(v_{\mathcal{A}}(x_t), v_{\mathcal{B}}(x_t)) + \min(\pi_{\mathcal{A}}(x_t), \pi_{\mathcal{B}}(x_t))\big)}{\sum\limits_{t=1}^{n}\big(\max(u_{\mathcal{A}}(x_t), u_{\mathcal{B}}(x_t)) + \max(v_{\mathcal{A}}(x_t), v_{\mathcal{B}}(x_t)) + \max(\pi_{\mathcal{A}}(x_t), \pi_{\mathcal{B}}(x_t))\big)} \tag{2.23}$$

(iii) Based on the matching function [205]:

$$\mathcal{S}'_3(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{t=1}^{n}\Big(u_{\mathcal{A}}(x_t)u_{\mathcal{B}}(x_t) + v_{\mathcal{A}}(x_t)v_{\mathcal{B}}(x_t) + \pi_{\mathcal{A}}(x_t)\pi_{\mathcal{B}}(x_t)\Big)}{\max\Big\{\sum\limits_{t=1}^{n}\Big(u_{\mathcal{A}}(x_t)^2 + v_{\mathcal{A}}(x_t)^2 + \pi_{\mathcal{A}}(x_t)^2\Big), \sum\limits_{t=1}^{n}\Big(u_{\mathcal{B}}(x_t)^2 + v_{\mathcal{B}}(x_t)^2 + \pi_{\mathcal{B}}(x_t)^2\Big)\Big\}} \tag{2.24}$$

(iv) Similarity measure [81]:

$$\mathcal{S}'_4(\mathcal{A},\mathcal{B}) = \frac{1}{n}\sum_{t=1}^{n}\left(1 - \frac{1}{2}\left\{|u_{\mathcal{A}}(x_t) - u_{\mathcal{B}}(x_t)| + |v_{\mathcal{A}}(x_t) - v_{\mathcal{B}}(x_t)|\right\}\right) \tag{2.25}$$

2.3.3 Correlation coefficient

In statistical analysis, the correlation coefficients plays a vital role to measure the linear relationship between the two variables, whereas in fuzzy set theory the correlation measure determines the degree of dependency between two fuzzy sets. In this section, we briefly introduce some basic concepts about the correlation coefficient between the pairs of the IFSs.

Definition 2.3.3. [59] For any two IFSs $\mathcal{A} = \{(x_t, u_{\mathcal{A}}(x_t), v_{\mathcal{A}}(x_t)) \mid x_t \in \mathcal{X}\}$ and $\mathcal{B} = \{(x_t, u_{\mathcal{B}}(x_t),\, v_{\mathcal{B}}(x_t)) \mid x_t \in \mathcal{X}\}$ in universe of discourse $\mathcal{X}$, the informational intuitionistic energies of two IFSs $\mathcal{A}$ and $\mathcal{B}$ are defined as:

$$\begin{aligned} E_{IFS}(\mathcal{A}) &= \sum_{t=1}^{n}\left(u^2_{\mathcal{A}}(x_t) + v^2_{\mathcal{A}}(x_t)\right) \\ E_{IFS}(\mathcal{B}) &= \sum_{t=1}^{n}\left(u^2_{\mathcal{B}}(x_t) + v^2_{\mathcal{B}}(x_t)\right) \end{aligned}$$

The correlation of IFSs $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$C_{IFS}(\mathcal{A},\mathcal{B}) = \sum_{t=1}^{n}\left(u_{\mathcal{A}}(x_t)u_{\mathcal{B}}(x_t) + v_{\mathcal{A}}(x_t)v_{\mathcal{B}}(x_t)\right) \tag{2.26}$$

Clearly, it is seen that the function $C_{IFS}(\mathcal{A},\mathcal{B})$ over two IFSs $\mathcal{A}$ and $\mathcal{B}$ satisfies the following properties:

(i) $C_{IFS}(\mathcal{A},\mathcal{A}) = E_{IFS}(\mathcal{A})$.

(ii) $C_{IFS}(\mathcal{A},\mathcal{B}) = C_{IFS}(\mathcal{B},\mathcal{A})$.

Thus, based on the functions $C_{IFS}(\mathcal{A},\mathcal{B})$ and informational energies $E_{IFS}(\mathcal{A})$, $E_{IFS}(\mathcal{B})$ between two IFSs $\mathcal{A}$ and $\mathcal{B}$, the general definition of correlation coefficient between them is defined as

$$K_{IFS}(\mathcal{A},\mathcal{B}) = \frac{C_{IFS}(\mathcal{A},\mathcal{B})}{\sqrt{E_{IFS}(\mathcal{A})\cdot E_{IFS}(\mathcal{B})}} \tag{2.27}$$

Definition 2.3.4. For any two IFSs $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{X}$, the correlation coefficient K_{IFS} satisfies the following properties:

(P1) $0 \leq K_{IFS}(\mathcal{A},\mathcal{B}) \leq 1$;

(P2) $K_{IFS}(\mathcal{A},\mathcal{B}) = K_{IFS}(\mathcal{B},\mathcal{A})$;

(P3) $K_{IFS}(\mathcal{A},\mathcal{B}) = 1$ if $\mathcal{A} = \mathcal{B}$.

Under the IFSs environment, some of the existing correlation coefficient between two IFSs $\mathcal{A}$ and $\mathcal{B}$ are defined as:

(i) Correlation coefficient of Gerstenkorn and Manko [59]:

$$K_{IFS_1}(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{t=1}^{n} \left(u_{\mathcal{A}}(x_t)u_{\mathcal{B}}(x_t) + v_{\mathcal{A}}(x_t)v_{\mathcal{B}}(x_t)\right)}{\sqrt{\sum\limits_{t=1}^{n} \left(u_{\mathcal{A}}^2(x_t) + v_{\mathcal{A}}^2(x_t)\right) \cdot \sum\limits_{t=1}^{n} \left(u_{\mathcal{B}}^2(x_t) + v_{\mathcal{B}}^2(x_t)\right)}} \tag{2.28}$$

(ii) Correlation coefficient of Zeng and Li [235] :

$$K_{IFS_2}(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{t=1}^{n} \left(u_{\mathcal{A}}(x_t)u_{\mathcal{B}}(x_t) + v_{\mathcal{A}}(x_t)v_{\mathcal{B}}(x_t) + \pi_{\mathcal{A}}(x_t)\pi_{\mathcal{B}}(x_t)\right)}{\sqrt{\sum\limits_{t=1}^{n} \left(u_{\mathcal{A}}^2(x_t) + v_{\mathcal{A}}^2(x_t) + \pi_{\mathcal{A}}^2(x_t)\right) \cdot \sum\limits_{t=1}^{n} \left(u_{\mathcal{B}}^2(x_t) + v_{\mathcal{B}}^2(x_t) + \pi_{\mathcal{B}}^2(x_t)\right)}} \tag{2.29}$$

(iii) Correlation coefficient defined by Xu et al. [206] :

$$K_{IFS_3}(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{t=1}^{n} \left(u_{\mathcal{A}}(x_t)u_{\mathcal{B}}(x_t)+v_{\mathcal{A}}(x_t)v_{\mathcal{B}}(x_t)+\pi_{\mathcal{A}}(x_t)\pi_{\mathcal{B}}(x_t)\right)}{\max\left\{\sum\limits_{t=1}^{n} \left(u_{\mathcal{A}}^2(x_t)+v_{\mathcal{A}}^2(x_t)+\pi_{\mathcal{A}}^2(x_t)\right), \sum\limits_{t=1}^{n} \left(u_{\mathcal{B}}^2(x_t)+v_{\mathcal{B}}^2(x_t)+\pi_{\mathcal{B}}^2(x_t)\right)\right\}} \tag{2.30}$$

(iv) Correlation coefficient of Szmidt and Kacprzyk [150]:

$$K_{IFS_4}(\mathcal{A},\mathcal{B}) = \frac{1}{3}[r_1(\mathcal{A},\mathcal{B}) + r_2(\mathcal{A},\mathcal{B}) + r_3(\mathcal{A},\mathcal{B})], \tag{2.31}$$

where:

$$\begin{aligned}
r_1(\mathcal{A},\mathcal{B}) &= \frac{\sum\limits_{t=1}^{n}(u_{\mathcal{A}}(x_t)-\overline{u_{\mathcal{A}}})(u_{\mathcal{B}}(x_t)-\overline{u_{\mathcal{B}}})}{\sqrt{\sum\limits_{t=1}^{n}(u_{\mathcal{A}}(x_t)-\overline{u_{\mathcal{A}}})^2\cdot\sum\limits_{t=1}^{n}(u_{\mathcal{B}}(x_t)-\overline{u_{\mathcal{B}}})^2}},\\
r_2(\mathcal{A},\mathcal{B}) &= \frac{\sum\limits_{t=1}^{n}(v_{\mathcal{A}}(x_t)-\overline{v_{\mathcal{A}}})(v_{\mathcal{B}}(x_t)-\overline{v_{\mathcal{B}}})}{\sqrt{\sum\limits_{t=1}^{n}(v_{\mathcal{A}}(x_t)-\overline{v_{\mathcal{A}}})^2\cdot\sum\limits_{t=1}^{n}(v_{\mathcal{B}}(x_t)-\overline{v_{\mathcal{B}}})^2}},\\
r_3(\mathcal{A},\mathcal{B}) &= \frac{\sum\limits_{t=1}^{n}(\pi_{\mathcal{A}}(x_t)-\overline{\pi_{\mathcal{A}}})(\pi_{\mathcal{B}}(x_t)-\overline{\pi_{\mathcal{B}}})}{\sqrt{\sum\limits_{t=1}^{n}(\pi_{\mathcal{A}}(x_t)-\overline{\pi_{\mathcal{A}}})^2\cdot\sum\limits_{t=1}^{n}(\pi_{\mathcal{B}}(x_t)-\overline{\pi_{\mathcal{B}}})^2}},
\end{aligned}$$

and

$$\overline{u_{\mathcal{A}}}=\frac{1}{n}\sum_{t=1}^{n}u_{\mathcal{A}}(x_t),\qquad \overline{u_{\mathcal{B}}}=\frac{1}{n}\sum_{t=1}^{n}u_{\mathcal{B}}(x_t),\qquad \overline{v_{\mathcal{A}}}=\frac{1}{n}\sum_{t=1}^{n}v_{\mathcal{A}}(x_t),$$
$$\overline{v_{\mathcal{B}}}=\frac{1}{n}\sum_{t=1}^{n}v_{\mathcal{B}}(x_t),\qquad \overline{\pi_{\mathcal{A}}}=\frac{1}{n}\sum_{t=1}^{n}\pi_{\mathcal{A}}(x_t),\qquad \overline{\pi_{\mathcal{B}}}=\frac{1}{n}\sum_{t=1}^{n}\pi_{\mathcal{B}}(x_t).$$

2.4 Archimedean t-norms and Archimedean t-conorms

A t-norm (fuzzy intersection)[92, 117] ‘$\mathcal{T}$’ is a binary operation on $[0,1]$ i.e.

$$\mathcal{T}:[0,1]\times[0,1]\to[0,1] \tag{2.32}$$

defined by

$$(\mathcal{B}_1\cap\mathcal{B}_2)(x)=\mathcal{T}\left(\mathcal{B}_1(x),\mathcal{B}_2(x)\right)\qquad \forall x\in[0,1] \tag{2.33}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are arbitrary fuzzy sets. Further, the mapping $\mathcal{T}$ satisfies the following axioms for all $a,b,d\in[0,1]$

Axiom 1: $\mathcal{T}(a,1)=a$ (Boundary Condition)

Axiom 2: If $b\leq d$ then $\mathcal{T}(a,b)\leq\mathcal{T}(a,d)$ (Monotonicity)

Axiom 3: $\mathcal{T}(a,b)=\mathcal{T}(b,a)$ (Commutativity)

Axiom 4: $\mathcal{T}(a, \mathcal{T}(b, d)) = \mathcal{T}(\mathcal{T}(a, b), d)$ (Associativity)

Alternatively, a T-conorm (fuzzy union)[92, 117] 'S' is also a binary operation on $[0, 1]$ given by

$$S : [0, 1] \times [0, 1] \to [0, 1] \tag{2.34}$$

defined by

$$(\mathcal{B}_1 \cup \mathcal{B}_2)(x) = S\left(\mathcal{B}_1(x), \mathcal{B}_2(x)\right) \qquad \forall x \in [0, 1] \tag{2.35}$$

Also, the mapping 'S' further satisfies the boundary, monotonicity, commutativity and associativity conditions.

The relation between 'S' and '$\mathcal{T}$' norms is given as

$$S(a, b) = 1 - \mathcal{T}(1 - a, 1 - b) \quad \forall a, b \in [0, 1] \tag{2.36}$$

A class of fuzzy intersection (t-norm) is obtained if t-norm also satisfies the additional axioms [92, 117], i.e.,

Axiom 5: $\mathcal{T}$ is continuous function (Continuity)

Axiom 6: $\mathcal{T}(a, a) < a$ (Subidempotency)

Axiom 7: If $a_1 < a_2$ and $b_1 < b_2$ implies $\mathcal{T}(a_1, b_1) < \mathcal{T}(a_2, b_2)$ (Strict monotonicity)

Similarly, for t-conorm, Axiom 6 is replaced by $S(a, a) > a$ and is called superidempotency.

A continuous t-norm that satisfy the subidempotency i.e., $\mathcal{T}(a, a) < a$ is called an Archimedean t-norm(AT)[117]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-norm. On the other hand, a continuous t-conorm that satisfy the superidempotency i.e. $S(a, a) > a$ is called an Archimedean t-conorm(AC)[92, 117]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-conorm.

Furthermore, strict AT and AC can be expressed in the form of continuous function $y : [0, 1) \to [0, 1]$ and $z : (0, 1] \to [0, 1]$ respectively for $a, b \in [0, 1]$ as

$$T(a, b) = z^{-1}(z(a) + z(b)) \quad \text{and} \quad S(a, b) = y^{-1}(y(a) + y(b))$$

where z(or y) is a decreasing(or increasing) function with $z(1) = 0$, $y(0) = 0$ and $z(a) = y(1 - a)$. However, some standard union and intersection form for $a, b \in [0, 1]$ are defined as [92, 117]:

(i) Standard intersection and union

$$\mathcal{T}(a,b) = \min(a,b) \qquad ; \qquad S(a,b) = \max(a,b)$$

(ii) Algebraic product and algebraic sum

$$\mathcal{T}(a,b) = ab \qquad ; \qquad S(a,b) = a + b - ab$$

(iii) Bounded Difference and Sum

$$\mathcal{T}(a,b) = \max(0, a + b - 1) \qquad ; \qquad S(a,b) = \min(1, a + b)$$

(iv) Drastic intersection and union

$$\mathcal{T}(a,b) = \begin{cases} a & ;\text{when } b = 1 \\ b & ;\text{when } a = 1 \\ 0 & ;\text{otherwise} \end{cases} \quad ; \quad S(a,b) = \begin{cases} a & ;\text{when } b = 0 \\ b & ;\text{when } a = 0 \\ 1 & ;\text{otherwise} \end{cases}$$

(v) Yagar class of t-norm and t-conorm

$$\begin{aligned} &\mathcal{T}(a,b) = 1 - \min\left(1, [(1-a)^p + (1-b)^p]^{1/p}\right); \\ &S(a,b) = \min\left[1, (a^p + b^p)^{1/p}\right] \end{aligned}$$

where $p > 0$.

Apart from them, some other well known ATs and ACs with their generator function are summarized in Table 2.1 [92, 117].

Table 2.1: Some AT and AC with their relative additive generators

Name	T-norm	Additive generator	S-norm	Additive generator
	$\mathcal{T}(a,b)$	$z(t)$	$S(a,b)$	$y(t)$
Algebraic	ab	$-\log(t)$	$a+b-ab$	$-\log(1-t)$
Einstein	$\frac{ab}{1+(1-a)(1-b)}$	$\log\left(\frac{2-t}{t}\right)$	$\frac{a+b}{1+ab}$	$\log\left(\frac{1+t}{1-t}\right)$
Hamacher ($\gamma > 0$)	$\frac{ab}{\gamma+(1-\gamma)(a+b-ab)}$	$\log\left(\frac{\gamma+(1-\gamma)t}{t}\right)$	$\frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab}$	$\log\left(\frac{\gamma+(1-\gamma)(1-t)}{1-t}\right)$
Frank($\eta > 1$)	$\log_\eta\left(1+\frac{(\eta^a-1)(\eta^b-1)}{\eta-1}\right)$	$-\log\left(\frac{\eta-1}{\eta^t-1}\right)$	$\log_\eta\left(1+\frac{(\eta^{1-a}-1)(\eta^{1-b}-1)}{\eta-1}\right)$	$-\log\left(\frac{\eta-1}{\eta^{1-t}-1}\right)$

2.5 Aggregation operator

The operations AT and AC are commonly used to combine several fuzzy sets into a single set by using desirable pattern is called as aggregation operators (AOs).

Definition 2.5.1. An aggregation operator on 'n' fuzzy sets ($n \geq 2$) is defined as

$$\mathcal{F} : [0,1]^n \to [0,1] \tag{2.37}$$

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_n$ be first 'n' fuzzy sets defined on $\mathcal{X}$, then function '$\mathcal{F}$' produce an aggregate fuzzy set $\mathcal{B}$ by operating the membership degrees of these sets for each $x \in \mathcal{X}$, i.e.,

$$\mathcal{B}(x) = \mathcal{F}\left(\mathcal{B}_1(x), \mathcal{B}_2(x), \ldots, \mathcal{B}_n(x)\right) \tag{2.38}$$

Further, an aggregation operator $\mathcal{F}$ must satisfies the three axiomatic conditions.

Axiom 1: $\mathcal{F}(0,0,\ldots,0) = 0$ and $\mathcal{F}(1,1,\ldots,1) = 1$ (Boundary condition)

Axiom 2: If $a_i \leq b_i$ for $i = 1, 2, \ldots n$, then $\mathcal{F}(a_1, a_2, \ldots, a_n) \leq \mathcal{F}(b_1, b_2, \ldots, b_n)$ (Monotonicity)

Axiom 3: $\mathcal{F}$ is continuous. (Continuity)

Based on the AT and AC operation as defined in Table 2.1, researchers have presented several kinds of AOs under the fuzzy sets and its extensions, which are defined as follows.

Definition 2.5.2. [207] For any two IFNs $\mathcal{A} = (u_{\mathcal{A}}, v_{\mathcal{A}})$ and $\mathcal{B} = (u_{\mathcal{B}}, v_{\mathcal{B}})$, some basic operational laws by using Algebraic AT and AC operations are defined for a real number $\lambda > 0$ as :

(i) $\mathcal{A} \oplus \mathcal{B} = (u_{\mathcal{A}} + u_{\mathcal{B}} - u_{\mathcal{A}} u_{\mathcal{B}}, v_{\mathcal{A}} v_{\mathcal{B}})$.

(ii) $\mathcal{A} \otimes \mathcal{B} = (u_{\mathcal{A}} u_{\mathcal{B}}, v_{\mathcal{A}} + v_{\mathcal{B}} - v_{\mathcal{A}} v_{\mathcal{B}})$.

(iii) $\lambda \mathcal{A} = (1 - (1 - u_{\mathcal{A}})^{\lambda}, v_{\mathcal{A}}^{\lambda})$.

(iv) $A^{\lambda} = (u_{\mathcal{A}}^{\lambda}, 1 - (1 - v_{\mathcal{A}})^{\lambda})$.

Based on these operations, Xu [203], Xu and Yager [207] defined the certain geometric and averaging AOs for the collection of IFNs $\mathcal{A}_t = (u_t, v_t); t = 1, 2, \ldots, n,$ which are defined as follows:

$$\text{IFWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\prod_{t=1}^{n} u_t^{\omega_t}, 1 - \prod_{t=1}^{n}(1 - v_t)^{\omega_t}\right) \tag{2.39}$$

$$\text{IFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(1 - \prod_{t=1}^{n}(1 - u_t)^{\omega_t}, \prod_{t=1}^{n} v_t^{\omega_t},\right) \tag{2.40}$$

where $\omega_t > 0$ be the weight vectors of $\mathcal{A}_t$ with $\sum\limits_{t=1}^{n} \omega_t = 1$. Then, IFWG and IFWA represents the intuitionistic fuzzy weighted geometric and averaging operator respectively.

Later on, Wang and Liu [169, 173] utilize Einstein AT and AC operations to define some basic operational laws for IFNs, which are defined as follows.

Definition 2.5.3. [169, 173] For any two IFNs $\mathcal{A} = (u_{\mathcal{A}}, v_{\mathcal{A}})$ and $\mathcal{B} = (u_{\mathcal{B}}, v_{\mathcal{B}})$ and a real number $\lambda > 0$, the operational laws based on Einstein AT and AC operations are defined as

(i) $\mathcal{A} \oplus \mathcal{B} = \left(\dfrac{u_{\mathcal{A}} + u_{\mathcal{B}}}{1 + u_{\mathcal{A}} u_{\mathcal{B}}}, \dfrac{v_{\mathcal{A}} v_{\mathcal{B}}}{1 + (1 - v_{\mathcal{A}})(1 - v_{\mathcal{B}})}\right)$.

(ii) $\mathcal{A} \otimes \mathcal{B} = \left(\dfrac{u_{\mathcal{A}} u_{\mathcal{B}}}{1 + (1 - u_{\mathcal{A}})(1 - u_{\mathcal{B}})}, \dfrac{v_{\mathcal{A}} + v_{\mathcal{B}}}{1 + v_{\mathcal{A}} v_{\mathcal{B}}}\right)$.

(iii) $\lambda \mathcal{A} = \left(\dfrac{(1 + u_{\mathcal{A}})^{\lambda} - (1 - u_{\mathcal{A}})^{\lambda}}{(1 + u_{\mathcal{A}})^{\lambda} + (1 - u_{\mathcal{A}})^{\lambda}}, \dfrac{2 v_{\mathcal{A}}^{\lambda}}{v_{\mathcal{A}}^{\lambda} + (2 - v_{\mathcal{A}})^{\lambda}}\right)$.

(iv) $\mathcal{A}^{\lambda} = \left(\dfrac{2 u_{\mathcal{A}}^{\lambda}}{u_{\mathcal{A}}^{\lambda} + (2 - u_{\mathcal{A}})^{\lambda}}, \dfrac{(1 + v_{\mathcal{A}})^{\lambda} - (1 - v_{\mathcal{A}})^{\lambda}}{(1 + v_{\mathcal{A}})^{\lambda} + (1 - v_{\mathcal{A}})^{\lambda}}\right)$.

Based on these operations, some weighted averaging and geometric AOs, represented by IFWEA and IFWEG respectively, are defined as [169, 173]:

$$\text{IFWEA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\frac{\prod\limits_{t=1}^{n}(1 + u_t)^{\omega_t} - \prod\limits_{t=1}^{n}(1 - u_t)^{\omega_t}}{\prod\limits_{t=1}^{n}(1 + u_t)^{\omega_t} + \prod\limits_{t=1}^{n}(1 - u_t)^{\omega_t}}, \frac{2\prod\limits_{t=1}^{n} v_t^{\omega_t}}{\prod\limits_{t=1}^{n} v_t^{\omega_t} + \prod\limits_{t=1}^{n}(2 - v_t)^{\omega_t}}\right) \tag{2.41}$$

$$\text{IFWEG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\frac{2\prod\limits_{t=1}^{n} u_t^{\omega_t}}{\prod\limits_{t=1}^{n} u_t^{\omega_t} + \prod\limits_{t=1}^{n}(2 - u_t)^{\omega_t}}, \frac{\prod\limits_{t=1}^{n}(1 + v_t)^{\omega_t} - \prod\limits_{t=1}^{n}(1 - v_t)^{\omega_t}}{\prod\limits_{t=1}^{n}(1 + v_t)^{\omega_t} + \prod\limits_{t=1}^{n}(1 - v_t)^{\omega_t}}\right) \tag{2.42}$$

Later on, Garg [48] presented an improved version of these AOs by adding hesitation degree into the analysis. Their proposed AO denoted by IFEIWA is defined as

$$\begin{aligned}&\text{IFEIWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\ =\;&\left(\frac{\prod\limits_{t=1}^{n}(1+u_t)^{\omega_t}-\prod\limits_{t=1}^{n}(1-u_t)^{\omega_t}}{\prod\limits_{t=1}^{n}(1+u_t)^{\omega_t}+\prod\limits_{t=1}^{2}(1-u_t)^{\omega_t}},\frac{2\left(\prod\limits_{t=1}^{n}(1-u_t)^{\omega_t}-\prod\limits_{t=1}^{n}(1-u_t-v_t)^{\omega_t}\right)}{\prod\limits_{t=1}^{n}(1+u_t)^{\omega_t}+\prod\limits_{t=1}^{n}(1-u_t)^{\omega_t}}\right)\end{aligned} \tag{2.43}$$

Also, Garg [39] presented the generalized intuitionistic fuzzy interactive geometric AOs denoted by GIFEWGIA and defined as

$$\text{GIFEWGIA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)=\left(\frac{E^{1/\lambda}-F^{1/\lambda}}{E^{1/\lambda}+F^{1/\lambda}},\frac{2[E^{1/\lambda}-G^{1/\lambda}]}{E^{1/\lambda}+F^{1/\lambda}}\right), \tag{2.44}$$

where,

$$\begin{aligned}E&=\prod_{t=1}^{n}\left\{\begin{array}{c}(1+u_t)^{\lambda}+3(1-u_t)^{\lambda}\\ -2(1-u_t-v_t)^{\lambda}\end{array}\right\}^{\omega_t}+3\prod_{t=1}^{n}\left\{\begin{array}{c}(1+u_t)^{\lambda}-(1-u_t)^{\lambda}\\ +2(1-u_t-v_t)^{\lambda}\end{array}\right\}^{\omega_t}-2\prod_{t=1}^{n}\{2(1-u_t-v_t)^{\lambda}\}^{\omega_t},\\ F&=\prod_{t=1}^{n}\left\{\begin{array}{c}(1+u_t)^{\lambda}+3(1-u_t)^{\lambda}\\ -2(1-u_t-v_t)^{\lambda}\end{array}\right\}^{\omega_t}-\prod_{t=1}^{n}\left\{\begin{array}{c}(1+u_t)^{\lambda}-(1-u_t)^{\lambda}\\ +2(1-u_t-v_t)^{\lambda}\end{array}\right\}^{\omega_t}+2\prod_{t=1}^{n}\{2(1-u_t-v_t)^{\lambda}\}^{\omega_t},\\ G&=2\prod_{t=1}^{n}\left\{2(1-u_t-v_t)^{\lambda}\right\}^{\omega_t}.\end{aligned}$$

However, on the different collections of the IVIFNs $\mathcal{A}_t=([u_t^L,u_t^U],[v_t^L,v_t^U])$, $(t=1,2,\ldots,n)$ and by using Algebraic AT and AC operations, [200, 204] defined some weighted average and geometric AOs which are represented as

$$\begin{aligned}&\text{IVIFWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\ =\;&\left(\left[1-\prod_{t=1}^{n}(1-u_t^L)^{\omega_t},1-\prod_{t=1}^{n}(1-u_t^U)^{\omega_t}\right],\left[\prod_{t=1}^{n}(v_t^L)^{\omega_t},\prod_{t=1}^{n}(v_t^U)^{\omega_t}\right]\right)\end{aligned} \tag{2.45}$$

and

$$\begin{aligned}&\text{IVIFWG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n)\\ =\;&\left(\left[\prod_{t=1}^{n}(u_t^L)^{\omega_t},\prod_{t=1}^{n}(u_t^U)^{\omega_t}\right],\left[1-\prod_{t=1}^{n}(1-v_t^L)^{\omega_t},1-\prod_{t=1}^{n}(1-v_t^U)^{\omega_t}\right]\right)\end{aligned} \tag{2.46}$$

On the other hand, Liu [103] utilized the Hamacher AT and AC operations to define the geometric AOs for a collection of IVIFNs as

$$\begin{aligned}&\text{IVIFHWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\\ &= \left(\left[\begin{array}{c} \dfrac{\gamma \prod\limits_{t=1}^{n} (u_t^L)^{\omega_t}}{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)u_t^L)^{\omega_t} + (\gamma - 1) \prod\limits_{t=1}^{n} (u_t^L)^{\omega_t}}, \\ \dfrac{\gamma \prod\limits_{t=1}^{n} (u_t^U)^{\omega_t}}{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)u_t^U)^{\omega_t} + (\gamma - 1) \prod\limits_{t=1}^{n} (u_t^U)^{\omega_t}} \end{array} \right], \left[\begin{array}{c} \dfrac{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^L)^{\omega_t} - \prod\limits_{t=1}^{n} (1 - v_t^L)^{\omega_t}}{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^L)^{\omega_t} + (\gamma - 1) \prod\limits_{t=1}^{n} (1 - v_t^L)^{\omega_t}}, \\ \dfrac{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^U)^{\omega_t} - \prod\limits_{t=1}^{n} (1 - v_t^U)^{\omega_t}}{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^U)^{\omega_t} + (\gamma - 1) \prod\limits_{t=1}^{n} (1 - v_t^U)^{\omega_t}} \end{array} \right] \right) \end{aligned} \tag{2.47}$$

Further, in order to improve the performance of these AOs, Garg [50] defined an improved geometric AOs by adding the degree of hesitancy between the pairs of the MDs and NMDs as follows

$$\begin{aligned}&\text{IIFHIWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)\\ &= \left(\left[\begin{array}{c} \dfrac{\gamma \left\{ \left(\prod\limits_{t=1}^{n} (1 - v_t^L)^{\omega_t} - \prod\limits_{t=1}^{n} (1 - u_t^L - v_t^L)^{\omega_t} \right) \right\}}{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^L)^{\omega_t} + (\gamma - 1) \prod\limits_{t=1}^{n} (1 - v_t^L)^{\omega_t}}, \\ \dfrac{\gamma \left\{ \left(\prod\limits_{t=1}^{n} (1 - v_t^U)^{\omega_t} - \prod\limits_{t=1}^{n} (1 - u_t^U - v_t^U)^{\omega_t} \right) \right\}}{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^U)^{\omega_t} + (\gamma - 1) \prod\limits_{t=1}^{n} (1 - v_t^U)^{\omega_t}} \end{array} \right], \left[\begin{array}{c} \dfrac{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^L)^{\omega_t} - \prod\limits_{t=1}^{n} (1 - v_t^L)^{\omega_t}}{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^L)^{\omega_t} + (\gamma - 1) \prod\limits_{t=1}^{2} (1 - v_t^L)^{\omega_t}}, \\ \dfrac{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^U)^{\omega_t} - \prod\limits_{t=1}^{n} (1 - v_t^U)^{\omega_t}}{\prod\limits_{t=1}^{n} (1 + (\gamma - 1)v_t^U)^{\omega_t} + (\gamma - 1) \prod\limits_{t=1}^{2} (1 - v_t^U)^{\omega_t}} \end{array} \right] \right) \end{aligned} \tag{2.48}$$

As similar to IFSs/IVIFSs and the features of AT and AC operations as defined in Table 2.1, many scholars have presented the operational laws for the LIFSs also.

Definition 2.5.4. [236] For any three LIFNs $\mathcal{A} = (s_\tau, s_\theta)$, $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$, $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$ and real number $\lambda > 0$, the basic operational laws based on Algebraic AT and AC operations are defined as

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(s_{\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{h}}, s_{\frac{\theta_1 \theta_2}{h}} \right)$.

(ii) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \left(s_{\frac{\tau_1 \tau_2}{h}}, s_{\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{h}} \right)$.

(iii) $\lambda \mathcal{A} = \left(s_{h - h(1 - \tau/h)^\lambda}, s_{h(\theta/h)^\lambda} \right)$.

(iv) $\mathcal{A}^\lambda = \left(s_{h(\theta/h)^\lambda}, s_{h - h(1 - \tau/h)^\lambda} \right)$

Based on these operational laws, Chen et al. [27], Zhang [236] discussed the weighted averaging and geometric AOs for a collection of LIFNs $\mathcal{A}_t = (s_{\theta_t}, s_{\tau_t}), (t = 1, 2, \ldots, n)$ as follows.

$$\text{LIFWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(s_{h\left(1-\prod\limits_{t=1}^{n}\left(1-\frac{\tau_t}{h}\right)^{\omega_t}\right)}, s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{\theta_t}{h}\right)^{\omega_t}\right)} \right) \tag{2.49}$$

and

$$\text{LIFWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{\tau_t}{h}\right)^{\omega_t}\right)}, s_{h\left(1-\prod\limits_{t=1}^{n}\left(1-\frac{\theta_t}{h}\right)^{\omega_t}\right)} \right) \tag{2.50}$$

where $\omega_t > 0$ be the weight vectors of LIFNs $\mathcal{A}_t$ with $\sum\limits_{t=1}^{n} \omega_t = 1$, LIFWA and LIFWG denotes the linguistic intuitionistic fuzzy weighted averaging and geometric AOs respectively.

Furthermore, the ordered weighted averaging and geometric AOs under LIFS environment are defined as

$$\text{LIFOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(s_{h\left(1-\prod\limits_{t=1}^{n}\left(1-\frac{\tau_{\sigma(t)}}{h}\right)^{\omega_t}\right)}, s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{\theta_{\sigma(t)}}{h}\right)^{\omega_t}\right)} \right) \tag{2.51}$$

and

$$\text{LIFOWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{\tau_{\sigma(t)}}{h}\right)^{\omega_t}\right)}, s_{h\left(1-\prod\limits_{t=1}^{n}\left(1-\frac{\theta_{\sigma(t)}}{h}\right)^{\omega_t}\right)} \right) \tag{2.52}$$

Recently, Arora and Garg [4] presented some prioritized weighted averaging and geometric AOs, denoted by LIFPWA and LIFPWG respectively, for different LIFNs by considering the prioritization factor between the pairs of the LIFNs, and are defined as

$$\text{LIFPWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(s_{h\left(1-\prod\limits_{t=1}^{n}\left(1-\frac{\tau_t}{h}\right)^{\frac{T_t}{\sum_{t=1}^{n} T_t}}\right)}, s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{\theta_t}{h}\right)^{\frac{T_t}{\sum_{t=1}^{n} T_t}}\right)} \right) \tag{2.53}$$

and

$$\text{LIFPWG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{\tau_t}{h}\right)^{\frac{T_t}{\sum_{t=1}^{n} T_t}}\right)}, s_{h\left(1-\prod\limits_{t=1}^{n}\left(1-\frac{\theta_t}{h}\right)^{\frac{T_t}{\sum_{t=1}^{n} T_t}}\right)} \right) \tag{2.54}$$

2.6 SPA theory

Zhao [246] presented a theory, known as SPA, for handling the uncertainty by defining a connection degree between the features of the set pair. SPA is a updated uncertainty theory to study both certainty and uncertainty as one consolidate certain-uncertain system. The significance and applications of SPA theory are reviewed in Section 1.1.4 of Chapter 1 and here some basic definitions related to the CNs are addressed.

Definition 2.6.1. For two interrelated sets $\mathcal{P}$ and $\mathcal{Q}$, the set pair between them is denoted by $\mathcal{H}(\mathcal{P}, \mathcal{Q})$ for a given problem $\mathcal{W}$. A connection number (CN), represented by $\mu(\mathcal{H}, \mathcal{W})$, is defined as

$$\mu(\mathcal{H}, \mathcal{W}) = (S/N) + (F/N)i + (P/N)j \tag{2.55}$$

where N is the total number of features, in which S represents the "identity" features, P be the "contrary" features and $F = N - S - P$ be the "discrepancy" features of sets $\mathcal{P}$ and $\mathcal{Q}$. If we take $S/N = a$ (identity degree), $F/N = b$ (discrepancy degree) and $P/N = c$ (contrary degree) then Eq. (2.55) becomes

$$\mu(\mathcal{H}, \mathcal{W}) = a + bi + cj \tag{2.56}$$

Here $i \in [-1, 1]$ is coefficient of "discrepancy degree" $j = -1$ is the coefficient of "contrary degree". It is clearly seen that $0 \leq a, b, c \leq 1$ and $a + b + c = 1$.

Definition 2.6.2. [16] For any two CNs $\mu_1 = a_1 + b_1 i + c_1 j$ and $\mu_2 = a_2 + b_2 i + c_2 j$, some basic relations between them are given as

(i) $\mu_1 = \mu_2 \Leftrightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2$.

(ii) $\mu_1 \preceq \mu_2$ if $a_1 \leq a_2, c_1 \geq c_2$.

(iii) $\mu_1 \oplus \mu_2 = \frac{(a_1+a_2)}{2} + \frac{(b_1+b_2)}{2} i + \frac{(c_1+c_2)}{2} j$.

(iv) $\mu_1 \otimes \mu_2 = (a_1 a_2) + (a_1 b_2 + b_1 a_2 + b_1 b_2 + b_1 c_2 + c_1 b_2)i + (a_1 c_2 + c_1 a_2 + c_1 c_2)j$.

(v) $\mu_1^\lambda = a_1^\lambda + (1 - (1 - b_1)^\lambda)i + ((a_1 + c_1)^\lambda - a_1^\lambda)j, \qquad \lambda > 0$.

and are also CNs.

Definition 2.6.3. [217] Let $\mu = a + bi + cj$ be a CN, then relatively certainty probability power $P(\mu)$ of the CN μ is defined as

$$P(\mu) = \frac{2a}{b+c} - \frac{c}{a+b}. \tag{2.57}$$

The larger the value of $P(\mu)$, the greater is the CN μ.

2.7 Description of the multiattribute decision making

In this thesis, we have presented various algorithms to solve the MADM problem by utilizing the input features of the IFS or IVIFSs. For it, the brief description of the MADM process is given as below.

Assume that there are m alternatives, denoted by $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$, which are evaluated under the set of n attributes, denoted by $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$, and give their preferences either in the form of the IFNs $\widetilde{\alpha}_{kt} = (u_{kt}, v_{kt})$ or in IVIFSs $\widetilde{\alpha}_{kt} = ([u^L_{kt}, u^U_{kt}], [v^L_{kt}, v^U_{kt}]))$; $k = 1, 2, \ldots, m; t = 1, 2, \ldots, n$ where u_{kt} (or $([u^L_{kt}, u^U_{kt}])$) be the degree that the alternative $\mathcal{A}_k$ satisfies the attribute $\mathcal{G}_t$ and v_{kt} (or $([v^L_{kt}, v^U_{kt}])$) be the dissatisfaction degree of the alternative $\mathcal{A}_k$ under the attribute $\mathcal{G}_t$ such that $u_{kt}, v_{kt} \in [0,1]$, $u_{kt} + v_{kt} \leq 1$ or $u^L_{kt}, v^L_{kt}, u^U_{kt}, v^U_{kt} \in [0,1]$, $u^U_{kt} + v^U_{kt} \leq 1$. Thus, the rating values corresponding to each alternative are represented in the form of IFNs over the universal set $\mathcal{X}$ as follows:

$$\mathcal{A}_k = \{(\mathcal{G}_t, u_{kt}(\mathcal{G}_t), v_{kt}(\mathcal{G}_t)) \mid \mathcal{G}_t \in \mathcal{G}\},$$

while under IVIFNs, these rating values are represented as

$$\mathcal{A}_k = \{(\mathcal{G}_t, [u^L_{kt}(\mathcal{G}_t), u^U_{kt}(\mathcal{G}_t)], [v^L_{kt}(\mathcal{G}_t), v^U_{kt}(\mathcal{G}_t)]) \mid \mathcal{G}_t \in \mathcal{G}\}$$

where $k = 1, 2, \ldots, m; t = 1, 2, \ldots, n$. Thus, the complete decision matrix of the given alternative is summarized as $\mathcal{M} = (\widetilde{\alpha}_{kt})_{m \times n}$. Let $\omega_t (t = 1, 2, \ldots, n)$ be the weight of the criteria $\mathcal{G}_t$ such that $\omega_t > 0$ and $\sum\limits_{t=1}^{n} \omega_t = 1$. In order to access these given alternatives, a set of 'l' decision makers $\mathcal{DM} = \{\mathcal{DM}^{(1)}, \mathcal{DM}^{(2)}, \ldots, \mathcal{DM}^{(l)}\}$ are invited to evaluate them and find the finest alternative(s). The general procedure to access the best alternative(s) are summarized in the following steps:

Step 1: Discretion the attribute set $\mathcal{G}$ into two disjoint sets namely the benefit $(\mathcal{F}_1)$ and the cost $(\mathcal{F}_2)$ attributes, if required, such that $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{G}$, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ and $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{G}$.

Step 2: Normalize the rating values of the cost type attributes into the benefit type by using the following normalization equation

$$\alpha_{kt} = \begin{cases} \widetilde{\alpha}_{kt} & ; \text{ if } t \in \mathcal{F}_1 \\ \widetilde{\alpha}^c_{kt} & ; \text{ if } t \in \mathcal{F}_2 \end{cases} \tag{2.58}$$

where $\widetilde{\alpha}^c_{kt}$ represent the complement of the number $\widetilde{\alpha}_{kt}$ under the considered environment.

Step 3: Aggregate the preference values of the all the alternatives into the overall collective values by using appropriate method/technique.

Step 4: Utilize the appropriate defuzzification method to compute the crisp value of the alternatives.

Step 5: Rank the alternatives based on the defuzzified values and select the best one(s) according to them.

Chapter 3

Novel Exponential distances and its based TOPSIS method for interval-valued intuitionistic fuzzy sets[1]

In this chapter, we present a novel multi-attribute decision making (MADM) method under IVIFSs environment by integrating a Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method. For it, firstly we construct the CN of the SPA theory for each IVIFS and then based on its, we define some exponential distance measures to compute the degree of discrimination between the two IVIFSs. The supremacy of the proposed measure is also discussed. Afterward, to solve the decision-making problems, a novel TOPSIS method based on the proposed distance measures is developed, illustrated with a numerical example and compared their results with the several existing studies.

3.1 Introduction

In a practice, due to the increasing complexity of the environment in the decision-making process and the problem itself, decision-makers (DMs) no longer are satisfied with numerical values to express decision making information. Therefore, researchers have tried to express this fuzziness in human cognition using fuzzy sets (FSs) [231], intuitionistic

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fuzzy sets (IFSs) [6] and interval-valued IFSs (IVIFSs) [5] which can more easily describe the fuzzy information by using membership degree (MD) and non-membership degree (NMD). In the literature, numerable attempts have been made by different researchers in processing the information values using aggregation operators [2, 48, 90, 207], distance measures[25, 126], possibility measures [157, 242], score and accuracy functions [8, 41, 135, 139], TOPSIS method [47, 136] under IFS, IVIFS environments. Of all these concepts, one of the significant ways to solve such type of problems is using the concept of distance measure.

However, apart from that, TOPSIS [82] is a well-known MADM method whose aim is to choose the best alternative whose distance from its best solution is the shortest. After their existence, numerous attempts are made by the researchers to apply it under the IFS and IVIFS environment. For instance, Szmidt and Kacprzyk [146] defined the concept of a distance measure between the IFSs. Park et al. [126] presented the concept of the TOPSIS to develop a method for solving MADM problems under IVIFS environment. Singh and Garg [138] presented some distance measures for type-2 IFSs. Dugenci [33] presented a distance measure for IVIFS and their applications to MADM with incomplete weight information. Ye [221] developed a group MADM method with interval-valued intuitionistic fuzzy numbers to solve the partner selection problem of a virtual enterprise under incomplete information environment. Wang and Gong [166] presented a method to solve the multiattribute decision-making (MADM) problems by using SPA theory. Cao et al. [14] presented the stochastic decision-making using the SPA theory under the IVIFS environment.

Presently, distance measure and its based approach are one of the most important measures which will help not only in comparing one data entity with others but also show the extent of association between them. Furthermore, to provide more liberty to the decision-makers, it is prudent to ask the experts to describe their decisions by means of intervals. For it, IVIFSs has important ability to model the imprecise and ambiguous information in real-world applications than the existing theories such as FSs, IFSs. Therefore, keeping the advantages of these sets, it is necessary to extend the distance measures to process the IVIFNs by using CNs of the SPA theory and hence to develop TOPSIS method for solving some MADM methods. These considerations have led us to consider the main objectives for this chapter: (1) to define some new distance measures named as exponential distance measures for the different CNs under IVIFS environment, (2) to show the advantages of the proposed measures with respect to the existing measures, (3) to develop an algorithm to solve the decision-making problems based on proposed measures; (4) to present some example in which relevance of the preferences in IF decision problems is made explicit.

3.2 Connection number sets for IVIFSs

In this section, we formulate the two distinctive CNs corresponding to IVIFNs by combining the concept of score and accuracy functions. The major advantage of this formulated CNs is to consider the degree of certainty and instability into one framework. For it, firstly we assume that $\mathcal{A}$ and $\mathcal{B}$ are any two IVIFNs which satisfy the conditions that $\mathcal{S}(\mathcal{A}) \neq \mathcal{S}(\mathcal{B})$ and $\mathcal{H}(\mathcal{A}) \neq \mathcal{H}(\mathcal{B})$, where $\mathcal{S}(\cdot)$ and $\mathcal{H}(\cdot)$ represents the score and accuracy functions given in Eq. (2.5) and Eq. (2.6) respectively of Chapter 2 for an IVIFN. Then, the CNs corresponding to set $\mathcal{A}$ and $\mathcal{B}$ are defined as below.

Definition 3.2.1. If $\mathcal{A} = \left\{(x_t, [u^L_{\mathcal{A}}(x_t), u^U_{\mathcal{A}}(x_t)], [v^L_{\mathcal{A}}(x_t), v^U_{\mathcal{A}}(x_t)]) \mid x_t \in \mathcal{X}\right\}$ be IVIFS defined over $\mathcal{X}$, then we define the CN set (CNS) corresponding to $\mathcal{A}$ as follows

$$\mathcal{A} = \left\{\big(x_t, a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_{\mathcal{A}}(x_t)j\big) \mid x_t \in \mathcal{X}\right\}, \tag{3.1}$$

where "identity", "discrepancy" and "contrary" degrees of the set are respectively defined as

$$\begin{aligned}
a_{\mathcal{A}}(x_t) &= \frac{\big(u^L_{\mathcal{A}}(x_t) + u^U_{\mathcal{A}}(x_t)\big)\big(2 - v^L_{\mathcal{A}}(x_t) - v^U_{\mathcal{A}}(x_t)\big)}{4};\\
b_{\mathcal{A}}(x_t) &= \frac{1 + \big(1 - u^L_{\mathcal{A}}(x_t) - u^U_{\mathcal{A}}(x_t)\big)\big(1 - v^L_{\mathcal{A}}(x_t) - v^U_{\mathcal{A}}(x_t)\big)}{2};\\
c_{\mathcal{A}}(x_t) &= \frac{\big(v^L_{\mathcal{A}}(x_t) + v^U_{\mathcal{A}}(x_t)\big)\big(2 - u^L_{\mathcal{A}}(x_t) - u^U_{\mathcal{A}}(x_t)\big)}{4}.
\end{aligned}$$

On the other hand, if in some situations the score values of any two IVIFNs say, $\mathcal{A}$ and $\mathcal{B}$ are equal i.e., $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{B})$ provided either $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{B})$ or $\mathcal{H}(\mathcal{A}) \neq \mathcal{H}(\mathcal{B})$ satisfied, then we compute the CNs corresponding to the IVIFNs by incorporating the degree of hesitance into analysis. For it, we define CNS for IVIFS according to Definition 3.2.2 which is stated as follows.

Definition 3.2.2. If $\mathcal{A} = \left\{(x_t, [u^L_{\mathcal{A}}(x_t), u^U_{\mathcal{A}}(x_t)], [v^L_{\mathcal{A}}(x_t), v^U_{\mathcal{A}}(x_t)]) \mid x_t \in \mathcal{X}\right\}$ be IVIFS defined over $\mathcal{X}$, then we define the CN set (CNS) corresponding to $\mathcal{A}$ as follows

$$\mathcal{A} = \{(x_t, a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_{\mathcal{A}}(x_t)j) \mid x_t \in \mathcal{X}\}, \tag{3.2}$$

where "identity", "discrepancy" and "contrary" degrees of the set are respectively defined as

$$\begin{aligned}
a_{\mathcal{A}}(x_t) &= \frac{\left(u^L_{\mathcal{A}}(x_t)\left(1 - u^U_{\mathcal{A}}(x_t) - v^U_{\mathcal{A}}(x_t)\right) + u^U_{\mathcal{A}}(x_t)\left(1 - u^L_{\mathcal{A}}(x_t) - v^L_{\mathcal{A}}(x_t)\right)\right)\left(2 - v^L_{\mathcal{A}}(x_t) - v^U_{\mathcal{A}}(x_t)\right)}{4}; \\
c_{\mathcal{A}}(x_t) &= \frac{\left(v^L_{\mathcal{A}}(x_t)\left(1 - u^U_{\mathcal{A}}(x_t) - v^U_{\mathcal{A}}(x_t)\right) + v^U_{\mathcal{A}}(x_t)\left(1 - u^L_{\mathcal{A}}(x_t) - v^L_{\mathcal{A}}(x_t)\right)\right)\left(2 - u^L_{\mathcal{A}}(x_t) - u^U_{\mathcal{A}}(x_t)\right)}{4}; \\
b_{\mathcal{A}}(x_t) &= 1 - a_{\mathcal{A}}(x_t) - c_{\mathcal{A}}(x_t).
\end{aligned}$$

In order to justify, whether the CNSs defined in Definition 3.2.1 and 3.2.2 are valid CNSs. For it, we prove the results in the following theorems.

Theorem 3.2.1. For an IVIFS, the set defined in Definition 3.2.1 is a valid CNS over the universal set $\mathcal{X}$.

Proof. To prove that set defined in Eq. (3.1) is a valid CNS for an IVIFS $\mathcal{A} = \{(x_t,$ $[u^L_{\mathcal{A}}(x_t)\ , u^U_{\mathcal{A}}(x_t)]$, $[v^L_{\mathcal{A}}(x_t)\ , v^U_{\mathcal{A}}(x_t)]) \mid x_t \in \mathcal{X}\}$, we need to show that it satisfy the following properties:

(P1) $0 \leq a_{\mathcal{A}}(x_t), b_{\mathcal{A}}(x_t), c_{\mathcal{A}}(x_t) \leq 1$;

(P2) $a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t) + c_{\mathcal{A}}(x_t) = 1$

Since $\mathcal{A}$ is an IVIFS, so we have

(P1) For any $x_t \in \mathcal{X}$, we get $0 \leq u^L_{\mathcal{A}}(x_t), u^U_{\mathcal{A}}(x_t), v^L_{\mathcal{A}}(x_t), v^U_{\mathcal{A}}(x_t) \leq 1$ and $0 \leq u^U_{\mathcal{A}}(x_t) + v^U_{\mathcal{A}}(x_t) \leq 1$. Therefore, $0 \leq u^L_{\mathcal{A}}(x_t) + u^U_{\mathcal{A}}(x_t) \leq 2$ and $0 \leq v^L_{\mathcal{A}}(x_t) + v^U_{\mathcal{A}}(x_t) \leq 2$ which implies that $0 \leq (u^L_{\mathcal{A}}(x_t) + u^U_{\mathcal{A}}(x_t))(2 - v^L_{\mathcal{A}}(x_t) - v^U_{\mathcal{A}}(x_t)) \leq 4$. Thus,

$$0 \leq \frac{(u^L_{\mathcal{A}}(x_t) + u^U_{\mathcal{A}}(x_t))(2 - v^L_{\mathcal{A}}(x_t) - v^U_{\mathcal{A}}(x_t))}{4} \leq 1$$

and hence $0 \leq a_{\mathcal{A}}(x_t) \leq 1$. Similarly, we can obtain $0 \leq c_{\mathcal{A}}(x_t) \leq 1$.

On the other hand, we have

$$-1 \leq 1 - u_{\mathcal{A}}^L(x_t) - u_{\mathcal{A}}^U(x_t) \leq 1 \text{ and } -1 \leq 1 - v_{\mathcal{A}}^L(x_t) - v_{\mathcal{A}}^U(x_t) \leq 1$$

which implies that

$$\begin{aligned}
& -1 \leq (1 - u_{\mathcal{A}}^L(x_t) - u_{\mathcal{A}}^U(x_t))(1 - v_{\mathcal{A}}^L(x_t) - v_{\mathcal{A}}^U(x_t)) \leq 1 \\
\Rightarrow \quad & 0 \leq 1 + (1 - u_{\mathcal{A}}^L(x_t) - u_{\mathcal{A}}^U(x_t))(1 - v_{\mathcal{A}}^L(x_t) - v_{\mathcal{A}}^U(x_t)) \leq 2 \\
\Rightarrow \quad & 0 \leq \frac{1 + (1 - u_{\mathcal{A}}^L(x_t) - u_{\mathcal{A}}^U(x_t))(1 - v_{\mathcal{A}}^L(x_t) - v_{\mathcal{A}}^U(x_t))}{2} \leq 1 \\
\Rightarrow \quad & 0 \leq b_{\mathcal{A}}(x_t) \leq 1.
\end{aligned}$$

Hence (P1) holds.

(P2) By using the Definition 3.2.1, we have,

$$\begin{aligned}
& a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t) + c_{\mathcal{A}}(x_t) \\
= \; & \frac{\left\{\begin{array}{l} (u_{\mathcal{A}}^L(x_t) + u_{\mathcal{A}}^U(x_t))(2 - v_{\mathcal{A}}^L(x_t) - v_{\mathcal{A}}^U(x_t)) + 2 + 2(1 - u_{\mathcal{A}}^L(x_t) - u_{\mathcal{A}}^U(x_t)) \\ (1 - v_{\mathcal{A}}^L(x_t) - v_{\mathcal{A}}^U(x_t)) + (v_{\mathcal{A}}^L(x_t) + v_{\mathcal{A}}^U(x_t))(2 - u_{\mathcal{A}}^L(x_t) - u_{\mathcal{A}}^U(x_t)) \end{array}\right\}}{4} \\
= \; & \frac{\left\{\begin{array}{l} 2(u_{\mathcal{A}}^L(x_t) + u_{\mathcal{A}}^U(x_t)) - (u_{\mathcal{A}}^L(x_t) + u_{\mathcal{A}}^U(x_t))(v_{\mathcal{A}}^L(x_t) + v_{\mathcal{A}}^U(x_t)) + 2 \\ + 2\left(1 - (u_{\mathcal{A}}^L(x_t) + u_{\mathcal{A}}^U(x_t)) - (v_{\mathcal{A}}^L(x_t) + v_{\mathcal{A}}^U(x_t)) + (v_{\mathcal{A}}^L(x_t) + v_{\mathcal{A}}^U(x_t))\right) \\ + 2(v_{\mathcal{A}}^L(x_t) + v_{\mathcal{A}}^U(x_t)) - (v_{\mathcal{A}}^L(x_t) + v_{\mathcal{A}}^U(x_t))(u_{\mathcal{A}}^L(x_t) + u_{\mathcal{A}}^U(x_t)) \end{array}\right\}}{4} \\
= \; & 1
\end{aligned}$$

$\therefore$ (P2) also holds.

Hence, the set defined in Definition 3.2.1 is a valid CNS. ☐

Theorem 3.2.2. For an IVIFS, the set defined in Definition 3.2.2 is a valid CNS over the universal set $\mathcal{X}$.

Proof. As similar to Theorem 3.2.1, so we omit here. ☐

The illustrations of above defined CNs are given as below:

Example 3.2.1. Let $\mathcal{A} = ([0.7, 0.8], [0.1, 0.2])$ and $\mathcal{B} = ([0.4, 0.5], [0.3, 0.4])$ be two IVIFNs. By definition of score and accuracy function of IVIFNs, we have $\mathcal{S}(\mathcal{A}) = 0.6$ and $\mathcal{S}(\mathcal{B}) = 0.1$. So by ranking order, we have $\mathcal{A} \succ \mathcal{B}$. In order to justify the proposed CNs is able to rank the given numbers, we first compute the CNs corresponding to $\mathcal{A}$ and $\mathcal{B}$ by using Definition 3.2.1 and hence get

$$\begin{aligned}
a_{\mathcal{A}} &= \frac{(u^L_{\mathcal{A}} + u^U_{\mathcal{A}})(2 - v^L_{\mathcal{A}} - v^U_{\mathcal{A}})}{4} = \frac{(0.7 + 0.8)(2 - 0.1 - 0.2)}{4} = 0.6375 \\
b_{\mathcal{A}} &= \frac{1 + (1 - u^L_{\mathcal{A}} - u^U_{\mathcal{A}})(1 - v^L_{\mathcal{A}} - v^U_{\mathcal{A}})}{2} = \frac{1 + (1 - 0.7 - 0.8)(1 - 0.1 - 0.2)}{2} = 0.3250 \\
c_{\mathcal{A}} &= \frac{(v^L_{\mathcal{A}} + v^U_{\mathcal{A}})(2 - u^L_{\mathcal{A}} - u^U_{\mathcal{A}})}{4} \frac{(0.1 + 0.2)(2 - 0.7 - 0.8)}{4} = 0.0375
\end{aligned}$$

Thus, CN of the set $\mathcal{A}$ is $\mu_{\mathcal{A}} = 0.6375 + 0.3250i + 0.0375j$. Similarly, we can get CN of the set $\mathcal{B}$ is $\mu_{\mathcal{B}} = 0.2925 + 0.5150i + 0.1925j$. Since $a_{\mathcal{A}} > a_{\mathcal{B}}$ and $c_{\mathcal{A}} < c_{\mathcal{B}}$ so we get $\mathcal{A} \succ \mathcal{B}$ and it coincides with the ranking order of IVIFNs.

Example 3.2.2. Let $\mathcal{A} = ([0.2, 0.3], [0.4, 0.5])$ and $\mathcal{B} = ([0.1, 0.4], [0.3, 0.6])$ be two IVIFNs. By using Eq. (2.5) we get $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{B}) = -0.2000$ and thus, to construct the CN corresponding to them, we utilize the Eq. (3.2) and get

$$\begin{aligned}
a_{\mathcal{A}} &= \frac{\big(u^L_{\mathcal{A}}(1 - u^U_{\mathcal{A}} - v^U_{\mathcal{A}}) + u^U_{\mathcal{A}}(1 - u^L_{\mathcal{A}} - v^L_{\mathcal{A}})\big)(2 - v^L_{\mathcal{A}} - v^U_{\mathcal{A}})}{4} \\
&= \frac{\big(0.2(1 - 0.3 - 0.5) + 0.3(1 - 0.2 - 0.4)\big)(2 - 0.4 - 0.5)}{4} \\
&= 0.0440 \\
c_{\mathcal{A}} &= \frac{\big(v^L_{\mathcal{A}}(1 - u^U_{\mathcal{A}} - v^U_{\mathcal{A}}) + v^U_{\mathcal{A}}(1 - u^L_{\mathcal{A}} - v^L_{\mathcal{A}})\big)(2 - u^L_{\mathcal{A}} - u^U_{\mathcal{A}})}{4} \\
&= \frac{\big(0.4(1 - 0.3 - 0.5) + 0.5(1 - 0.2 - 0.4)\big)(2 - 0.2 - 0.3)}{4} \\
&= 0.1050 \\
b_{\mathcal{A}} &= 1 - a_{\mathcal{A}} - c_{\mathcal{A}} \\
&= 0.8510
\end{aligned}$$

Hence, CN corresponding to $\mathcal{A}$ is given as $\mu_{\mathcal{A}} = 0.0440 + 0.8510i + 0.1050j$. Similarly, we can obtain CN corresponding to $\mathcal{B}$ is $\mu_{\mathcal{B}} = 0.0660 + 0.7990i + 0.1350j$.

3.3 Proposed exponential distance measures

In this section, based on the proposed CNSs, we define some new exponential distance measures for IVIFSs. For it, we consider $F(\mathcal{X})$ be the family of CNSs over the universal set $\mathcal{X}$.

Definition 3.3.1. Let $\mathcal{A} = \big\{(x_t, a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_{\mathcal{A}}(x_t)j) \mid x_t \in \mathcal{X}\big\}$ and $\mathcal{B} = \big\{(x_t, a_{\mathcal{B}}(x_t) + b_{\mathcal{B}}(x_t)i + c_{\mathcal{B}}(x_t)j) \mid x_t \in \mathcal{X}\big\}$ be two CNSs corresponding to the two different IVIFSs, then we define the following distance measures:

(i) Normalized exponential Hamming distance :

$$\mathcal{D}_1(\mathcal{A},\mathcal{B}) = \frac{1-\exp\left\{-\frac{1}{3}\sum\limits_{t=1}^{n}\left[\begin{array}{c}\left|\sqrt{a_{\mathcal{A}}(x_t)}-\sqrt{a_{\mathcal{B}}(x_t)}\right|+\left|\sqrt{b_{\mathcal{A}}(x_t)}-\sqrt{b_{\mathcal{B}}(x_t)}\right|\\ +\left|\sqrt{c_{\mathcal{A}}(x_t)}-\sqrt{c_{\mathcal{B}}(x_t)}\right|\end{array}\right]\right\}}{1-\exp\left(-n\right)} \tag{3.3}$$

(ii) Normalized exponential Euclidean distance:

$$\mathcal{D}_2(\mathcal{A},\mathcal{B}) = \frac{1-\exp\left\{-\left(\frac{1}{3}\sum\limits_{t=1}^{n}\left[\begin{array}{c}\left|\sqrt{a_{\mathcal{A}}(x_t)}-\sqrt{a_{\mathcal{B}}(x_t)}\right|^2+\left|\sqrt{b_{\mathcal{A}}(x_t)}-\sqrt{b_{\mathcal{B}}(x_t)}\right|^2\\ +\left|\sqrt{c_{\mathcal{A}}(x_t)}-\sqrt{c_{\mathcal{B}}(x_t)}\right|^2\end{array}\right]\right)^{\frac{1}{2}}\right\}}{1-\exp\left(-\sqrt{n}\right)} \tag{3.4}$$

Theorem 3.3.1. Let $\mathcal{A},\mathcal{B},\mathcal{C} \in F(\mathcal{X})$ be three CNSs defined over the universal set $\mathcal{X}$ then the distance measures $\mathcal{D}_q$ $(q = 1, 2)$ satisfy the following properties:

(P1) $0 \leq \mathcal{D}_q(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{D}_q(\mathcal{A},\mathcal{B}) = 0 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}_q(\mathcal{A},\mathcal{B}) = \mathcal{D}_q(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}_q(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_q(\mathcal{A},\mathcal{C})$ and $\mathcal{D}_q(\mathcal{B},\mathcal{C}) \leq \mathcal{D}_q(\mathcal{A},\mathcal{C})$.

Proof. Let $\mathcal{A} = \big\{(x_t, a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_A(x_t)j) \mid x_t \in \mathcal{X}\big\}$, $\mathcal{B} = \big\{(x_t, a_{\mathcal{B}}(x_t) + b_{\mathcal{B}}(x_t)i + c_B(x_t)j) \mid x_t \in \mathcal{X}\big\}$ and $\mathcal{C} = \big\{(x_t, a_{\mathcal{C}}(x_t) + b_{\mathcal{C}}(x_t)i + c_C(x_t)j) \mid x_t \in \mathcal{X}\big\}$ are three CNSs over the universal set $\mathcal{X}$, then for $q = 1, 2$, we have

(P1) It is clear that $\mathcal{D}_q \geq 0$, so we need prove only $\mathcal{D}_q \leq 1$. Since $\mathcal{A}$ and $\mathcal{B}$ are CNSs, so by Theorem 3.2.1, we get

$$0 \leq \mid \sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)} \mid^q \leq 1;$$

$$0 \leq \mid \sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)} \mid^q \leq 1;$$

$$\text{and} \qquad 0 \leq \mid \sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)} \mid^q \leq 1.$$

Thus, we have

$$\left(\frac{1}{3}\sum_{t=1}^{n}\left[\begin{array}{c}\left|\sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)}\right|^q + \left|\sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)}\right|^q \\ + \left|\sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)}\right|^q\end{array}\right]\right)^{\frac{1}{q}} \leq (n)^{\frac{1}{q}}$$

$$\Rightarrow \quad \exp\left\{-\left(\frac{1}{3}\sum_{t=1}^{n}\left[\begin{array}{c}\left|\sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)}\right|^q + \left|\sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)}\right|^q \\ + \left|\sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)}\right|^q\end{array}\right]\right)^{\frac{1}{q}}\right\} \geq \exp\left(-(n)^{\frac{1}{q}}\right)$$

$$\Rightarrow \quad 1 - \exp\left\{-\left(\frac{1}{3}\sum_{t=1}^{n}\left[\begin{array}{c}\left|\sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)}\right|^q + \left|\sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)}\right|^q \\ + \left|\sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)}\right|^q\end{array}\right]\right)^{\frac{1}{q}}\right\} \leq 1 - \exp\left(-(n)^{\frac{1}{q}}\right)$$

$$\Rightarrow \quad \frac{1 - \exp\left\{-\left(\frac{1}{3}\sum_{t=1}^{n}\left[\begin{array}{c}\left|\sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)}\right|^q + \left|\sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)}\right|^q \\ + \left|\sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)}\right|^q\end{array}\right]\right)^{\frac{1}{q}}\right\}}{1 - \exp\left(-(n)^{\frac{1}{q}}\right)} \leq 1$$

$$\Rightarrow \quad \mathcal{D}_q(\mathcal{A}, \mathcal{B}) \leq 1$$

Hence $0 \leq \mathcal{D}_q \leq 1$.

(P2) For two CNSs $\mathcal{A}$ and $\mathcal{B}$. Assume that $\mathcal{D}_q(\mathcal{A}, \mathcal{B}) = 0$ then

$$\Leftrightarrow \quad \exp\left\{-\left(\frac{1}{3}\sum_{t=1}^{n}\left[\begin{array}{c}\mid \sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)} \mid^q + \mid \sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)} \mid^q \\ + \mid \sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)} \mid^q\end{array}\right]\right)^{\frac{1}{q}}\right\} = 1$$

$$\Leftrightarrow \quad \frac{1}{3}\sum_{t=1}^{n}\left(\begin{array}{c}\mid \sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)} \mid^q + \mid \sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)} \mid^q \\ + \mid \sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)} \mid^q\end{array}\right) = 0$$

$$\Leftrightarrow \quad a_{\mathcal{A}}(x_t) = a_{\mathcal{B}}(x_t), b_{\mathcal{A}}(x_t) = b_{\mathcal{B}}(x_t), c_{\mathcal{A}}(x_t) = c_{\mathcal{B}}(x_t)$$

$$\Leftrightarrow \quad A = B$$

(P3) Since for any two real numbers a and b, we have $\mid a-b \mid = \mid b-a \mid$. Thus, (P3) holds.

(P4) Since $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, therefore $a_{\mathcal{A}}(x_t) \leq a_{\mathcal{B}}(x_t) \leq a_{\mathcal{C}}(x,t), b_{\mathcal{A}}(x_t) \geq b_{\mathcal{B}}(x_t) \geq b_{\mathcal{C}}(x_t)$ and $c_{\mathcal{A}}(x_t) \geq c_{\mathcal{B}}(x_t) \geq c_{\mathcal{C}}(x_t)$ which implies that

$$\begin{aligned}
& \mid \sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)} \mid^q \leq \mid \sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{C}}(x_t)} \mid^q; \\
& \mid \sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)} \mid^q \leq \mid \sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{C}}(x_t)} \mid^q; \\
\text{and} \quad & \mid \sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)} \mid^q \leq \mid \sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{C}}(x_t)} \mid^q .
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \mathcal{D}_q(\mathcal{A},\mathcal{B}) \\
= & \frac{1-\exp\left\{-\left(\frac{1}{3}\sum_{t=1}^{n}\left[\begin{array}{c} \mid \sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{B}}(x_t)} \mid^q + \mid \sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{B}}(x_t)} \mid^q \\ + \mid \sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{B}}(x_t)} \mid^q \end{array}\right]\right)^{\frac{1}{q}}\right\}}{1-\exp(-n^{1/q})} \\
\leq & \frac{1-\exp\left\{-\left(\frac{1}{3}\sum_{t=1}^{n}\left[\begin{array}{c} \mid \sqrt{a_{\mathcal{A}}(x_t)} - \sqrt{a_{\mathcal{C}}(x_t)} \mid^q + \mid \sqrt{b_{\mathcal{A}}(x_t)} - \sqrt{b_{\mathcal{C}}(x_t)} \mid^q \\ + \mid \sqrt{c_{\mathcal{A}}(x_t)} - \sqrt{c_{\mathcal{C}}(x_t)} \mid^q \end{array}\right]\right)^{\frac{1}{q}}\right\}}{1-\exp(-n^{1/q})} \\
= & \mathcal{D}_q(\mathcal{A},\mathcal{C})
\end{aligned}$$

Similarly, we obtain $\mathcal{D}_q(\mathcal{B},\mathcal{C}) \leq \mathcal{D}_q(\mathcal{A},\mathcal{C})$.

Hence, $\mathcal{D}_q$ $(q=1,2)$ are the valid distance measures. □

In order to illustrate the working of these measures, we explain them with the help of numerical example as follows.

Example 3.3.1. Let $\mathcal{A} = \{(x_1, 0.2+0.3i+0.5j), (x_2, 0.4+0.3i+0.3j), (x_3, 0.6+0.1i+0.3j)\}$ and $\mathcal{B} = \{(x_1, 0.4+0.1i+0.5j), (x_2, 0.3+0.5i+0.2j), (x_3, 0.7+0.2i+0.1j)\}$ be two CNSs.

Then by using normalized exponential hamming distance defined in Eq. (3.3), we get

$$\begin{aligned}
&\mathcal{D}_1(\mathcal{A},\mathcal{B})\\
=\ & \frac{1-\exp\left\{-\frac{1}{3}\begin{pmatrix} \mid\sqrt{0.2}-\sqrt{0.4}\mid + \mid\sqrt{0.3}-\sqrt{0.1}\mid + \mid\sqrt{0.5}-\sqrt{0.5}\mid \\ + \mid\sqrt{0.4}-\sqrt{0.3}\mid + \mid\sqrt{0.3}-\sqrt{0.5}\mid + \mid\sqrt{0.3}-\sqrt{0.2}\mid \\ + \mid\sqrt{0.6}-\sqrt{0.7}\mid + \mid\sqrt{0.1}-\sqrt{0.2}\mid + \mid\sqrt{0.3}-\sqrt{0.1}\mid \end{pmatrix}\right\}}{1-\exp(-3)}\\
=\ & \frac{1-\exp\left\{-\frac{1}{3}\begin{pmatrix} 0.1857+0.2315+0.0000+0.0847+0.1594 \\ +0.1005+0.0621+0.1310+0.2315 \end{pmatrix}\right\}}{0.9502}\\
=\ & 0.3436
\end{aligned}$$

On the other hand, by using normalized exponential Euclidean distance defined in Eq. (3.4), we get

$$\begin{aligned}
&\mathcal{D}_2(\mathcal{A},\mathcal{B})\\
=\ & \frac{1-\exp\left\{-\left(\frac{1}{3}\begin{bmatrix} \mid\sqrt{0.2}-\sqrt{0.4}\mid^2 + \mid\sqrt{0.3}-\sqrt{0.1}\mid^2 + \mid\sqrt{0.5}-\sqrt{0.5}\mid^2 \\ + \mid\sqrt{0.4}-\sqrt{0.3}\mid^2 + \mid\sqrt{0.3}-\sqrt{0.5}\mid^2 + \mid\sqrt{0.3}-\sqrt{0.2}\mid^2 \\ + \mid\sqrt{0.6}-\sqrt{0.7}\mid^2 + \mid\sqrt{0.1}-\sqrt{0.2}\mid^2 + \mid\sqrt{0.3}-\sqrt{0.1}\mid^2 \end{bmatrix}\right)^{1/2}\right\}}{1-\exp(-\sqrt{3})}\\
=\ & \frac{1-\exp\left\{-\frac{1}{3}\begin{pmatrix} 0.0343+0.0536+0.0000+0.0072+0.0254 \\ +0.0101+0.0039+0.0172+0.0536 \end{pmatrix}\right\}}{0.8231}\\
=\ & 0.2796
\end{aligned}$$

However, there are various practical problems in which the weights of the parameter plays a dominant role during the decision-making process. In order to handle it, weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_t)^T$ of the parameters are taken in to the account such that $\omega_t > 0, \sum\limits_{t=1}^{n} \omega_t = 1$. Then, we define the weighted exponential distance measures for measuring the different CNSs as follows :

(i) Weighted normalized exponential Hamming distance :

$$\mathcal{D}_3(\mathcal{A},\mathcal{B}) = \frac{1-\exp\left\{-\frac{1}{3}\sum\limits_{t=1}^{n}\omega_t\left[\begin{array}{c}\left|\sqrt{a_{\mathcal{A}}(x_t)}-\sqrt{a_{\mathcal{B}}(x_t)}\right|+\left|\sqrt{b_{\mathcal{A}}(x_t)}-\sqrt{b_{\mathcal{B}}(x_t)}\right|\\ +\left|\sqrt{c_{\mathcal{A}}(x_t)}-\sqrt{c_{\mathcal{B}}(x_t)}\right|\end{array}\right]\right\}}{1-\exp\left(-n\right)} \tag{3.5}$$

(ii) Weighted normalized exponential Euclidean distance:

$$\mathcal{D}_4(\mathcal{A},\mathcal{B}) = \frac{1-\exp\left\{-\left(\frac{1}{3}\sum\limits_{t=1}^{n}\omega_t\left[\begin{array}{c}\left|\sqrt{a_{\mathcal{A}}(x_t)}-\sqrt{a_{\mathcal{B}}(x_t)}\right|^2+\left|\sqrt{b_{\mathcal{A}}(x_t)}-\sqrt{b_{\mathcal{B}}(x_t)}\right|^2\\ +\left|\sqrt{c_{\mathcal{A}}(x_t)}-\sqrt{c_{\mathcal{B}}(x_t)}\right|^2\end{array}\right]\right)^{\frac{1}{2}}\right\}}{1-\exp\left(-\sqrt{n}\right)} \tag{3.6}$$

Theorem 3.3.2. Let $\mathcal{A},\mathcal{B},\mathcal{C} \in F(\mathcal{X})$ be the three CNSs defined over the universal set $\mathcal{X}$, then the above defined distance measures $\mathcal{D}_q, (q = 3, 4)$ satisfies the following properties:

(P1) $0 \leq \mathcal{D}_q(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{D}_q(\mathcal{A},\mathcal{B}) = 0 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}_q(\mathcal{A},\mathcal{B}) = \mathcal{D}_q(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}_q(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_q(\mathcal{A},\mathcal{C})$ and $\mathcal{D}_q(\mathcal{B},\mathcal{C}) \leq \mathcal{D}_q(\mathcal{A},\mathcal{C})$.

Proof. For CNSs $\mathcal{A}$, $\mathcal{B}$ and ω_t is the weight vector of the x_t s.t. $\omega_t > 0$ and $\sum\limits_{t=1}^{n}\omega_t = 1$, we have

$$\begin{aligned}
&\frac{1}{3}\sum_{t=1}^{n}\omega_t\left(\begin{array}{c}\mid\sqrt{a_{\mathcal{A}}(x_t)}-\sqrt{a_{\mathcal{B}}(x_t)}\mid+\mid\sqrt{b_{\mathcal{A}}(x_t)}-\sqrt{b_{\mathcal{B}}(x_t)}\mid\\ +\mid\sqrt{c_{\mathcal{A}}(x_t)}-\sqrt{c_{\mathcal{B}}(x_t)}\mid\end{array}\right)\\
&\qquad\leq\frac{1}{3}\sum_{t=1}^{n}\left(\begin{array}{c}\mid\sqrt{a_{\mathcal{A}}(x_t)}-\sqrt{a_{\mathcal{B}}(x_t)}\mid+\mid\sqrt{b_{\mathcal{A}}(x_t)}-\sqrt{b_{\mathcal{B}}(x_t)}\mid\\ +\mid\sqrt{c_{\mathcal{A}}(x_t)}-\sqrt{c_{\mathcal{B}}(x_t)}\mid\end{array}\right)\\
\Rightarrow\quad&1-\exp\left\{-\frac{1}{3}\sum_{t=1}^{n}\omega_t\left(\begin{array}{c}\mid\sqrt{a_{\mathcal{A}}(x_t)}-\sqrt{a_{\mathcal{B}}(x_t)}\mid+\mid\sqrt{b_{\mathcal{A}}(x_t)}-\sqrt{b_{\mathcal{B}}(x_t)}\mid\\ +\mid\sqrt{c_{\mathcal{A}}(x_t)}-\sqrt{c_{\mathcal{B}}(x_t)}\mid\end{array}\right)\right\}\\
&\qquad\leq 1-\exp\left\{-\frac{1}{3}\sum_{t=1}^{n}\omega_t\left(\begin{array}{c}\mid\sqrt{a_{\mathcal{A}}(x_t)}-\sqrt{a_{\mathcal{B}}(x_t)}\mid+\mid\sqrt{b_{\mathcal{A}}(x_t)}-\sqrt{b_{\mathcal{B}}(x_t)}\mid\\ +\mid\sqrt{c_{\mathcal{A}}(x_t)}-\sqrt{c_{\mathcal{B}}(x_t)}\mid\end{array}\right)\right\}\\
\Rightarrow\quad&\mathcal{D}_3(\mathcal{A},\mathcal{B})\leq\mathcal{D}_1(\mathcal{A},\mathcal{B})
\end{aligned}$$

As $0 \leq \mathcal{D}_1(\mathcal{A},\mathcal{B}) \leq 1$ which implies that $\mathcal{D}_3(\mathcal{A},\mathcal{B})$ satisfies the (P1). The proof of (P2)-(P4) is similar to the Theorem 3.3.1. Therefore, the weighted distance $\mathcal{D}_3$ is a valid distance measures. Similarly, we can prove that $\mathcal{D}_4$ is also valid distance measure. □

3.3.1 Advantages of the proposed distance measures

Here, we present some counterexamples to demonstrate the advantages of the proposed distance measures with regard to the existing measures [125, 197], and found that the proposed distance measures work well where these existing measures are unable to classify the objects.

Example 3.3.2. Consider a decision matrix $\mathcal{M}$ consists of two unknown patterns $\mathcal{A}$ and $\mathcal{B}$ which are evaluated under three different attributes. The rating values of these patterns are classified in terms of IVIFNs and are summarized as

$$M = \begin{array}{c} \\ \mathcal{A} \\ \mathcal{B} \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} ([0.7,0.8],[0.1,0.2]) & ([0.5,0.6],[0.3,0.4]) & ([0.4,0.5],[0.3,0.4]) \\ ([0.6,0.7],[0.0,0.1]) & ([0.4,0.5],[0.2,0.3]) & ([0.5,0.6],[0.3,0.4]) \end{pmatrix} \end{array} \tag{3.7}$$

Consider a known pattern $\mathcal{P}$ whose representation is given under IVIFS and is as follows:

$$P = \big\{(x_1,[0.4,0.5],[0.3,0.4]),(x_2,[0.5,0.6],[0.2,0.3]),(x_3,[0.3,0.4],[0.4,0.5])\big\}$$

The target of this considered example is to choose the best unknown pattern which classifies with the sample $\mathcal{P}$. To analyze it, if we apply existing distance measures as defined in Eqs. (2.14)-(2.17) of Chapter 2, then the measurement values corresponding to them are summarized as below.

Pair	$\mathcal{D}'_5$	$\mathcal{D}'_6$	$\mathcal{D}'_7$	$\mathcal{D}'_8$
$(\mathcal{P},\mathcal{A})$	0.1500	0.1780	0.2000	0.2160
$(\mathcal{P},\mathcal{B})$	0.1500	0.1780	0.2000	0.2160

It is clearly seen from these results that these existing measures are unable to classify the samples because all distance measures have produced equivalent results. Also, it is clearly seen that $\mathcal{A}$ and $\mathcal{B}$ are not same i.e., $\mathcal{A} \neq \mathcal{B}$.

However, if we compute CN decision matrix corresponding to Eq. (3.7), then we get

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A} \\ \mathcal{B} \\ \\ \mathcal{A} \\ \mathcal{B} \end{array} \begin{array}{c} \\ \left(\begin{array}{cc} \begin{array}{c} x_1 \end{array} & \begin{array}{c} x_2 \end{array} \\ 0.6375 + 0.3250i + 0.0375j & 0.3575 + 0.4850i + 0.1575j \\ 0.6175 + 0.3650i + 0.0175j & 0.3375 + 0.5250i + 0.1375j \\ & x_3 \\ & 0.3375 + 0.5250i + 0.1375j \\ & 0.3575 + 0.4850i + 0.1575j \end{array}\right) \end{array} \tag{3.8}$$

while CNS corresponding to known pattern $\mathcal{P}$ is

$$\mathcal{P} = \left\{ \begin{array}{c} (x_1, 0.2925 + 0.5150i + 0.1925j), (x_2, 0.4125 + 0.4750i + 0.1125j), \\ (x_3, 0.1925 + 0.5150i + 0.2925j) \end{array} \right\}$$

Based on these information and by utilizing the proposed distance measure $\mathcal{D}_1$, we get $\mathcal{D}_1(\mathcal{P}, \mathcal{A}) = 0.3188$ and $\mathcal{D}_1(\mathcal{P}, \mathcal{B}) = 0.3282$. Since $\mathcal{D}_1(\mathcal{P}, \mathcal{A}) < \mathcal{D}_1(\mathcal{P}, \mathcal{B})$, therefore sample $\mathcal{P}$ classify with the pattern $\mathcal{A}$. On the other hand, if we apply exponential Euclidean distance measures as defined in the Eq. (3.4) on the data set then corresponding measurement values are $\mathcal{D}_2(\mathcal{P}, \mathcal{A}) = 0.2784, \mathcal{D}_2(\mathcal{P}, \mathcal{B}) = 0.2880$. Thus, sample $\mathcal{P}$ classify with $\mathcal{A}$. Therefore, it successfully classifying the pattern and hence overcome the drawback of the existing distance measures.

Example 3.3.3. Consider an MADM problem consists of two alternatives $\mathcal{A}_1$ and $\mathcal{A}_2$ which are evaluated under the five different attributes denoted by x_1, x_2, x_3, x_4 and x_5 by an expert under IVIFS environment. An expert gives their preferences which are summarized as below

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \left(\begin{array}{ccc} x_1 & x_2 & x_3 \\ ([0.4, 0.5], [0.2, 0.3]) & ([0.3, 0.4], [0.0, 0.1]) & ([0.9, 0.9], [0.0, 0.0]) \\ ([0.6, 0.7], [0.1, 0.2]) & ([0.3, 0.4], [0.0, 0.2]) & ([0.8, 0.9], [0.0, 0.1]) \\ & x_4 & x_5 \\ & ([0.4, 0.5], [0.3, 0.4]) & ([0.6, 0.7], [0.0, 0.1]) \\ & ([0.5, 0.6], [0.2, 0.3]) & ([0.8, 0.8], [0.0, 0.0]) \end{array}\right)$$

The target of this problem is to find an alternative which has a close resemblance with the desired alternative $\mathcal{P}$ whose preference value is summarized by $\mathcal{P} = \{(x_1, [0.6, 0.8], [0.0, 0.2]), (x_2, [0.5, 0.6], [0.2, 0.3]), (x_3, [0.6, 0.7], [0.0, 0.1]), (x_4, [0.4, 0.5], [0.3, 0.4]), (x_5, [0.3, 0.4], [0.4, 0.5])\}$. For it, we utilize the distance measures $\mathcal{D}'_5$ and $\mathcal{D}'_7$ as defined in Eqs. (2.18) and (2.20) respectively [125] of Chapter 2, then we get

$$\mathcal{D}'_5(\mathcal{P}, \mathcal{A}_1) = \mathcal{D}'_5(\mathcal{P}, \mathcal{A}_2) = 0.2$$

and

$$\mathcal{D}'_7(\mathcal{P}, \mathcal{A}_1) = \mathcal{D}'_7(\mathcal{P}, \mathcal{A}_2) = 0.2.$$

From these measurement values, we conclude that measures defined by Park et al. [125] are unable to find out the best one among them.

On the other hand, if construct the CNSs from the given rating values by using the proposed Definition (3.2.1) and (3.2.2) then we get

$$\begin{aligned}
\mathcal{A}_1 &= \big\{(x_1, 0.3375 + 0.5250i + 0.1375j), (x_2, 0.20425 + 0.7730i + 0.0325j), \\
&\quad (x_3, 0.9000 + 0.1000i + 0.0000j), (x_4, 0.06175 + 0.8970i + 0.1925j), \\
&\quad (x_5, 0.6175 + 0.3650i + 0.0175j)\big\} \\
\mathcal{P} &= \big\{(x_1, 0.6300 + 0.3400i + 0.0300j), (x_2, 0.08625 + 0.8890i + 0.02475j), \\
&\quad (x_3, 0.6175 + 0.3650i + 0.0175j), (x_4, 0.06175 + 0.8970i + 0.04125j), \\
&\quad (x_5, 0.1925 + 0.5150i + 0.2925j)\big\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_2 &= \big\{(x_1, 0.5525 + 0.3950i + 0.0525j), (x_2, 0.3150 + 0.6200i + 0.0650j), \\
&\quad (x_3, 0.8075 + 0.1850i + 0.0075j), (x_4, 0.4125 + 0.4750i + 0.1125j), \\
&\quad (x_5, 0.8000 + 0.2000i + 0.0000j)\big\} \\
\mathcal{P} &= \big\{(x_1, 0.6300 + 0.3400i + 0.0300j), (x_2, 0.4125 + 0.4750i + 0.1125j), \\
&\quad (x_3, 0.6175 + 0.3650i + 0.0175j), (x_4, 0.2925 + 0.5150i + 0.1925j), \\
&\quad (x_5, 0.1925 + 0.5150i + 0.2925j)\big\}
\end{aligned}$$

Therefore, by applying the proposed distance measure $\mathcal{D}_1$ as given in Eq. (3.3) on it, we get $\mathcal{D}_1(\mathcal{P}, \mathcal{A}_1) = 0.5284$ and $\mathcal{D}_1(\mathcal{P}, \mathcal{A}_2) = 0.5303$. Since $\mathcal{D}_1(\mathcal{P}, \mathcal{A}_1) > \mathcal{D}_1(\mathcal{P}, \mathcal{A}_2)$ and concluded that the alternative $\mathcal{A}_2$ has met with with alternative $\mathcal{P}$.

Example 3.3.4. Consider a decision-making problem which is the essence of providing sufficient information of students for proper career choice cannot be overemphasized. This is dominant because the several problems in the lack of proper career guidance faced by students are of great significance in their career choice and efficiency. To face these problems, a secondary student takes the help of the student counseling center which provides the sufficient information to a student on career determination and helps them to choose a suitable career path in which student can highest grow. Consider that a counseling center taken the five different subjects namely English (x_1), Mathematics (x_2), Physics (x_3), Biology (x_4) and Chemistry (x_5) related to the careers determination in the field of Medical ($\mathcal{A}$) and Engineering ($\mathcal{B}$). An expert gives their preferences to each subject related to career goals under the IVIFNs and their corresponding decision matrix is summarized as follow.

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A} \\ \mathcal{B} \\ \\ \mathcal{A} \\ \mathcal{B} \end{array} \left(\begin{array}{ccc} x_1 & x_2 & x_3 \\ ([0.8,0.9],[0.0,0.1]) & ([0.7,0.8],[0.0,0.1]) & ([0.7,0.8],[0.0,0.1]) \\ ([0.6,0.7],[0.1,0.2]) & ([0.4,0.5],[0.3,0.4]) & ([0.8,0.9],[0.0,0.1]) \\ & x_4 & x_5 \\ & ([0.4,0.5],[0.2,0.3]) & ([0.7,0.7],[0.2,0.2]) \\ & ([0.5,0.6],[0.2,0.3]) & ([0.8,0.8],[0.0,0.0]) \end{array} \right) \tag{3.9}$$

Assume that the unknown student $\mathcal{P}$ goes to the counseling center for taking the help to choose his suitable career and give their rating values towards the interest of the subject $x_t (t = 1, 2, \ldots, 5)$ in terms of IVIFNs which are summarized as below

$$\mathcal{P} = \left\{ \begin{array}{l} (x_1, [0.4,0.5],[0.2,0.3]), (x_2, [0.5,0.6],[0.1,0.2]), (x_3, [0.4,0.5],[0.2,0.3]), \\ (x_4, [0.3,0.4],[0.4,0.5]), (x_5, [0.7,0.7],[0.2,0.2])\} \end{array} \right\}$$

The aim of the problem is to determine the most expected career of the student in the fields $\mathcal{A}$ or $\mathcal{B}$. To achieve it, if we apply existing distance measures $\mathcal{D}'_6$ and $\mathcal{D}'_8$ as defined in

Eqs. (2.15) and (2.17) respectively of Chapter 2, then we get their measurement values are $\mathcal{D}'_6(\mathcal{P},\mathcal{A}) = \mathcal{D}'_6(\mathcal{P},\mathcal{B}) = 0.2704$ and $\mathcal{D}'_8(\mathcal{P},\mathcal{A}) = \mathcal{D}'_8(\mathcal{P},\mathcal{B}) = 0.2704$. From it, we conclude that both the measures give equal values and hence distance measures $\mathcal{D}'_6$ and $\mathcal{D}'_8$ defined by the Park et al. [125] are unable to classify the student to get the most suitable career for the student $\mathcal{P}$.

On the other hand, we use the proposed distance measure to choose the most suitable career field for the student $\mathcal{P}$. For it, we first construct the CN decision matrix of the considered data given the Eq. (3.9) and are summarized as below

$$\mathcal{M} = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ \mathcal{A} & 0.8075 + 0.1850i + 0.0075j & 0.7125 + 0.2750i + 0.0125j) & 0.7125 + 0.2750i + 0.0125j \\ \mathcal{B} & 0.5525 + 0.3950i + 0.0525j & 0.2925 + 0.5150i + 0.1925j & 0.8075 + 0.1850i + 0.0075j \\ & & x_3 & x_5 \\ \mathcal{A} & & 0.3375 + 0.5250i + 0.1375j & 0.0560 + 0.9380i + 0.0600j \\ \mathcal{B} & & 0.4125 + 0.4750i + 0.1125j & 0.8000 + 0.2000i + 0.0000j \end{array}$$

and

$$P = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ \mathcal{A} & 0.3375 + 0.5250i + 0.1375j & 0.4675 + 0.4650i + 0.0675j & 0.3375 + 0.5250i + 0.1375j \\ \mathcal{B} & 0.3375 + 0.5250i + 0.1375j & 0.4675 + 0.4650i + 0.0675j & 0.3375 + 0.5250i + 0.1375j \\ & & x_4 & x_5 \\ \mathcal{A} & & 0.1925 + 0.5150i + 0.2925j & 0.0560 + 0.9380i + 0.0600j \\ \mathcal{B} & & 0.1925 + 0.5150i + 0.2925j & 0.5600 + 0.3800i + 0.0600j \end{array}$$

Then, we have applied the proposed distance measure $\mathcal{D}_1$ on it and get $\mathcal{D}_1(\mathcal{P},\mathcal{A}) = 0.5550$ and $\mathcal{D}_1(\mathcal{P},\mathcal{B}) = 0.5908$. Since $\mathcal{D}_1(\mathcal{P},\mathcal{A}) < \mathcal{D}_1(\mathcal{P},\mathcal{B})$ and hence, the most suitable career for the student $\mathcal{P}$ is Medical i.e., $\mathcal{A}$. However, if we apply distance measures $\mathcal{D}_2$, defined in the Eq. (3.4), on given data, then we get measurement values are $\mathcal{D}_2(\mathcal{P},\mathcal{A}) = 0.3946$ and $\mathcal{D}_2(\mathcal{P},\mathcal{B}) = 0.3962$. Again, from these values, we compute that student $\mathcal{P}$ classify with the Medical field again.

Hence, from the above examples, it has been understood that the proposed distance measures are suitably working in those cases where the existing distance measures fail to classify the objects. In the next, by taking the support of the proposed measures, we will introduce a TOPSIS approach to solve the MADM problems under IVIFSs environment.

3.4 TOPSIS based distance approach for solving the MADM problem under IVIFS environment

The general description of the MADM problem is given in the Section 2.7 of Chapter 2. Under the IVIFS environment, the interval-valued intuitionistic fuzzy decision matrix $\mathcal{M} = (\widetilde{\alpha}_{kt})_{m\times n}$, where $\widetilde{\alpha}_{kt} = ([\widetilde{u}^L_{kt}, \widetilde{u}^U_{kt}], [\widetilde{v}^L_{kt}, \widetilde{v}^U_{kt}])$ and hence the following steps have been summarized for describing the TOPSIS approach based on the proposed distance measures as:

Step 1: In order to balance the physical dimensions of these rating values, matrix $\mathcal{M}$ is converted into its normalized matrix $\mathcal{R} = (\alpha_{kt})_{m\times n}$ where $\alpha_{kt} = ([u^L_{kt}, u^U_{kt}], [v^L_{kt}, v^U_{kt}])$ $(k = 1, 2, \ldots, m; t = 1, 2, \ldots, n)$ is given as :

$$\alpha_{kt} = \begin{cases} ([\widetilde{u}^L_{kt}, \widetilde{u}^U_{kt}],\ [\widetilde{v}^L_{kt}, \widetilde{v}^U_{kt}]) & ;\ \text{for } \mathcal{F}_1 \text{ attribute} \\ ([\widetilde{v}^L_{kt}, \widetilde{v}^U_{kt}],\ [\widetilde{u}^L_{kt}, \widetilde{u}^U_{kt}]) & ;\ \text{for } \mathcal{F}_2 \text{ attribute} \end{cases} \tag{3.10}$$

Step 2: Compute the positive ideal alternative (PIA) and negative ideal alternative (NIA) denoted by $\mathcal{R}^+ = (\alpha^+_t)_{1\times n}$ and $\mathcal{R}^- = (\alpha^-_t)_{1\times n}$ respectively from the decision matrix $\mathcal{R}$ as follows.

$$\alpha^+_t = ([(u^L_t)^+, (u^U_t)^+], [(v^L_t)^+, (v^U_t)^+]) \tag{3.11}$$

where $(u^L_t)^+ = \max\limits_k\{u^L_{kt}\}$, $(u^U_t)^+ = \max\limits_k\{u^U_{kt}\}$, $(v^L_t)^+ = \min\limits_k\{v^L_{kt}\}$, $(v^U_t)^+ = \min\limits_k\{v^U_{kt}\}$. and

$$\alpha^-_t = ([(u^L_t)^-, (u^U_t)^-], [(v^L_t)^-, (v^U_t)^-]) \tag{3.12}$$

where $(u^L_t)^- = \min\limits_k\{u^L_{kt}\}$, $(u^U_t)^- = \min\limits_k\{u^U_{kt}\}$, $(v^L_t)^- = \max\limits_k\{v^L_{kt}\}$ and $(v^U_t)^- = \max\limits_k\{v^U_{kt}\}$.

Step 3: Check, for all k, t, whether $\mathcal{S}(\alpha_{kt}) \neq \mathcal{S}(\alpha^+_t); \mathcal{H}(\alpha_{kt}) \neq \mathcal{H}(\alpha^+_t)$ holds or not. If it holds, then compute the CN decision matrices $\mathcal{R}' = (a_{kt} + b_{kt}i + c_{kt}j)_{m\times n}$ and $\mathcal{A}^+ = (a^+_{kt} + b^+_{kt}i + c^+_{kt}j)_{m\times n}$ corresponding to alternatives $\mathcal{A}_k$ and $\mathcal{A}^+$ respectively, by using Definition 3.2.1 otherwise compute these matrices by using Definition

3.2.2. Similarly, compute the CN decision matrices $\mathcal{R}' = (a_{kt} + b_{kt}i + c_{kt}j)_{m\times n}$ and $\mathcal{A}^- = (a^-_{kt} + b^-_{kt}i + c^-_{kt}j)_{m\times n}$ corresponding to alternatives $\mathcal{A}_k$ and $\mathcal{A}^-$ respectively by using Definition 3.2.1 or Definition 3.2.2.

Step 4: Compute the distance measures, $\mathfrak{D}^+(\mathcal{A}_k, \mathcal{A}^+)$ and $\mathfrak{D}^-(\mathcal{A}_k, \mathcal{A}^-)$, of each alternative $\mathcal{A}_k$ $(k = 1, 2, \ldots, m)$ from its PIA and NIA respectively by using exponential distance measures under CNs. For instance, if we utilize weighted exponential Hamming distance then we get

$$\mathfrak{D}^+(\mathcal{A}_k, \mathcal{A}^+) = \frac{1 - \exp\left\{-\frac{1}{3}\sum\limits_{t=1}^{n} \omega_t \left[\left|\sqrt{a_{kt}} - \sqrt{a^+_{kt}}\right| + \left|\sqrt{b_{kt}} - \sqrt{b^+_{kt}}\right| + \left|\sqrt{c_{kt}} - \sqrt{c^+_{kt}}\right|\right]\right\}}{1 - \exp(-n)} \tag{3.13}$$

and

$$\mathfrak{D}^-(\mathcal{A}_k, \mathcal{A}^-) = \frac{1 - \exp\left\{-\frac{1}{3}\sum\limits_{t=1}^{n} \omega_t \left[\left|\sqrt{a_{kt}} - \sqrt{a^-_{kt}}\right| + \left|\sqrt{b_{kt}} - \sqrt{b^-_{kt}}\right| + \left|\sqrt{c_{kt}} - \sqrt{c^-_{kt}}\right|\right]\right\}}{1 - \exp(-n)} \tag{3.14}$$

On the other hand, if we utilize the weighted exponential Euclidean distance then we get

$$\mathfrak{D}^+(\mathcal{A}_k, \mathcal{A}^+) = \frac{1 - \exp\left\{-\left(\frac{1}{3}\sum\limits_{t=1}^{n} \omega_t \left[\left|\sqrt{a_{kt}} - \sqrt{a^+_{kt}}\right|^2 + \left|\sqrt{b_{kt}} - \sqrt{b^+_{kt}}\right|^2 + \left|\sqrt{c_{kt}} - \sqrt{c^+_{kt}}\right|^2\right]\right)^{\frac{1}{2}}\right\}}{1 - \exp(-\sqrt{n})} \tag{3.15}$$

and

$$\mathfrak{D}^-(\mathcal{A}_k, \mathcal{A}^-) = \frac{1 - \exp\left\{-\left(\frac{1}{3}\sum\limits_{t=1}^{n} \omega_t \left[\left|\sqrt{a_{kt}} - \sqrt{a^-_{kt}}\right|^2 + \left|\sqrt{b_{kt}} - \sqrt{b^-_{kt}}\right|^2 + \left|\sqrt{c_{kt}} - \sqrt{c^-_{kt}}\right|^2\right]\right)^{\frac{1}{2}}\right\}}{1 - \exp(-\sqrt{n})} \tag{3.16}$$

Step 5: The optimal degree of an alternative $\mathcal{A}_k$ $(k = 1, 2, \ldots, m)$, which is closest with respect to the PIA $\mathcal{A}^+$ is defined as follows:

$$\mathfrak{R}(\mathcal{A}_k) = \frac{\mathfrak{D}^-(\mathcal{A}_k, \mathcal{A}^-)}{\mathfrak{D}^-(\mathcal{A}_k, \mathcal{A}^-) + \mathfrak{D}^+(\mathcal{A}_k, \mathcal{A}^+)} \tag{3.17}$$

where $\mathfrak{D}^+(\mathcal{A}_k, \mathcal{A}^+) \neq 0$, $0 \leq \mathfrak{R}(\mathcal{A}_k) \leq 1$ and hence, choose the best alternative(s) according to the decreasing order of the value of $\mathfrak{R}(\mathcal{A}_k)$.

The complete flow chart of the proposed approach is given in Fig. 3.1.

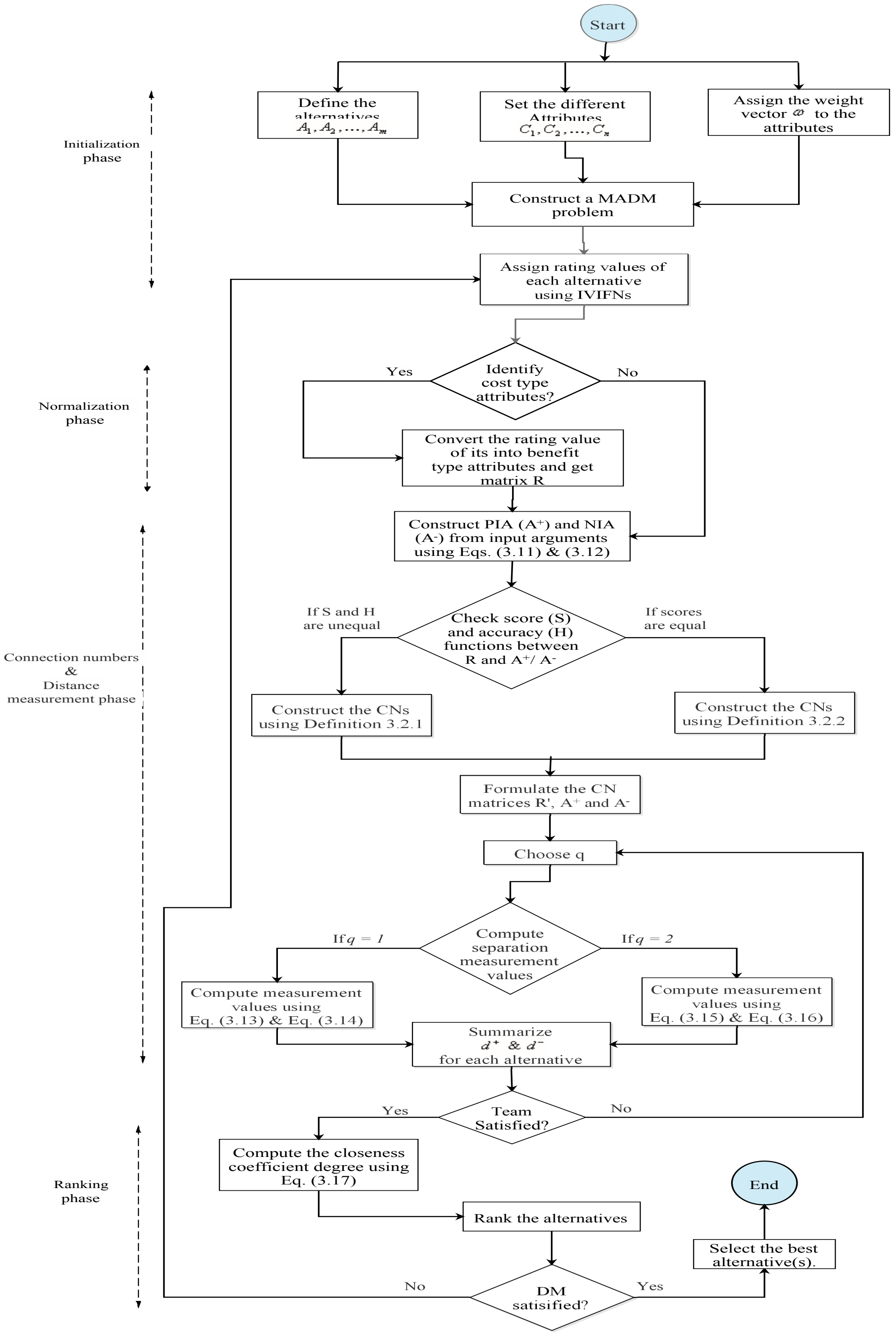

Figure 3.1: Flow chart of the proposed approach

3.5 Application of the proposed TOPSIS approach

In this section, the above-mentioned approach has been illustrated with a real-life decision-making problem under the IVIFS environment and obtained results have been compared with the other existing results.

3.5.1 Illustrative example

Jharkhand is the eastern state of the India, which has the 40 percent mineral resources of the country and second leading state of the mineral wealth after Chhatisgarh state. It is also known for its vast forest resources. Jamshedpur, Bokaro and Dhanbad cities of the Jharkhand are famous for industries in all over the world. After that, it is the widespread poverty state of the India because it is the primarily a rural state as 76 percent of the population live in the villages which depend on the agriculture and wages. Only 30 percent villages are connected by roads while only 55 percent villages have access to electricity and other facilities. But in the today's life, every one is changing fast to himself for a better life, therefore, every one moves to the urban cities for a better job. To stop this emigration, Jharkhand government wants to setup the industries based on the agriculture in the rural areas. For this, government have been organized MOMENTUM JHARKHAND global investor submit 2017 in Ranchi to invite the companies for investment in the rural areas. The government announced the various facilities for setup the five food processing parks in the rural areas and consider the six attributes required for company selection to setup them, namely, project cost ($\mathcal{G}_1$), completion time ($\mathcal{G}_2$), technical capability ($\mathcal{G}_3$), financial status ($\mathcal{G}_4$), company background ($\mathcal{G}_5$), reference from previous project ($\mathcal{G}_6$) and assign the weights of relative importance of each attributes. The weight vector corresponding to it has been assigned as $\omega = (0.3, 0.2, 0.15, 0.1, 0.15, 0.1)^T$ on the basis of decision maker's preferences. The four companies taken as in the form of the alternatives, namely, Surya Food and Agro Pvt. Ltd. ($\mathcal{A}_1$), Mother Dairy Fruit and Vegetable Pvt. Ltd. ($\mathcal{A}_2$), Parle Products Ltd. ($\mathcal{A}_3$), Heritage Food Ltd. ($\mathcal{A}_4$) interested for these projects. Then the main object of the government is to choose the best company among them for the task. In order to fulfill it, they have evaluated these and gave their preferences in the term of

IVIFNs which are summarized in Table 3.1. Then, the following steps of the proposed approach are executed to find the best alternative(s).

Table 3.1: Rating values for each alternative using IVIFNs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$([0.7,0.8],[0.0,0.1])$	$([0.5,0.6],[0.2,0.4])$	$([0.7,0.8],[0.1,0.2])$	$([0.2,0.5],[0.1,0.2])$	$([0.2,0.3],[0.6,0.7])$	$([0.4,0.5],[0.3,0.4])$
$\mathcal{A}_2$	$([0.2,0.6],[0.2,0.3])$	$([0.5,0.7],[0.1,0.2])$	$([0.4,0.5],[0.2,0.4])$	$([0.8,0.9],[0.1,0.1])$	$([0.4,0.5],[0.2,0.3])$	$([0.6,0.8],[0.1,0.2])$
$\mathcal{A}_3$	$([0.7,0.8],[0.1,0.2])$	$([0.3,0.6],[0.1,0.2])$	$([0.5,0.8],[0.1,0.2])$	$([0.8,0.9],[0.0,0.1])$	$([0.4,0.5],[0.3,0.4])$	$([0.8,0.9],[0.0,0.1])$
$\mathcal{A}_4$	$([0.3,0.4],[0.3,0.4])$	$([0.1,0.2],[0.7,0.8])$	$([0.5,0.8],[0.1,0.2])$	$([0.6,0.7],[0.1,0.2])$	$([0.6,0.6],[0.1,0.2])$	$([0.3,0.4],[0.2,0.3])$

Step 1: Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are only the attributes of the cost types, so by using Eq. (3.10), the normalized decision-matrix is summarized in Table 3.2.

Table 3.2: Normalized rating values of IVIFNs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$([0.0,0.1],[0.7,0.8])$	$([0.2,0.4],[0.5,0.6])$	$([0.7,0.8],[0.1,0.2])$	$([0.2,0.5],[0.1,0.2])$	$([0.2,0.3],[0.6,0.7])$	$([0.4,0.5],[0.3,0.4])$
$\mathcal{A}_2$	$([0.2,0.3],[0.2,0.6])$	$([0.1,0.2],[0.5,0.7])$	$([0.4,0.5],[0.2,0.4])$	$([0.8,0.9],[0.1,0.1])$	$([0.4,0.5],[0.2,0.3])$	$([0.6,0.8],[0.1,0.2])$
$\mathcal{A}_3$	$([0.1,0.2],[0.7,0.8])$	$([0.1,0.2],[0.3,0.6])$	$([0.5,0.8],[0.1,0.2])$	$([0.8,0.9],[0.0,0.1])$	$([0.4,0.5],[0.3,0.4])$	$([0.8,0.9],[0.0,0.1])$
$\mathcal{A}_4$	$([0.3,0.4],[0.3,0.4])$	$([0.7,0.8],[0.1,0.2])$	$([0.5,0.8],[0.1,0.2])$	$([0.6,0.7],[0.1,0.2])$	$([0.6,0.6],[0.1,0.2])$	$([0.3,0.4],[0.2,0.3])$

Step 2: By using Eqs. (3.11) and (3.12), the PIA and NIA are computed as

$$\mathcal{R}^+ = \left\{ \begin{array}{c} (\mathcal{G}_1,([0.3,0.4],[0.2,0.4])),(\mathcal{G}_2,([0.7,0.8],[0.1,0.2])),(\mathcal{G}_3,([0.7,0.8],[0.1,0.2])), \\ (\mathcal{G}_4,([0.8,0.9],[0.0,0.1])),(\mathcal{G}_5,([0.6,0.6],[0.1,0.2])),(\mathcal{G}_6,([0.8,0.9],[0.0,0.1])) \end{array} \right\}$$

and

$$\mathcal{R}^- = \left\{ \begin{array}{c} (\mathcal{G}_1,([0.0,0.1],[0.7,0.8])),(\mathcal{G}_2,([0.1,0.2],[0.5,0.7])),(\mathcal{G}_3,([0.4,0.5],[0.2,0.4])) \\ (\mathcal{G}_4,([0.2,0.5],[0.1,0.2])),(\mathcal{G}_5,([0.2,0.3],[0.6,0.7])),(\mathcal{G}_6,([0.3,0.4],[0.3,0.4])) \end{array} \right\}$$

Step 3: Check the score and accuracy values between the given arguments and $\mathcal{R}^+$. Based on its, compute the CNs by using Definition 3.2.1 and Definition 3.2.2 and their results are summarized in Table 3.3. Similarly, we compute the CN decision matrix for a given input and $\mathcal{R}^-$ by using Definition 3.2.1 and Definition 3.2.2 and their corresponding results are summarized in Table 3.4.

Table 3.3: Connection number decision-matrix $\mathcal{R}'$ and $\mathcal{A}^+$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
$\mathcal{A}_1$	$0.0125 + 0.2750i + 0.7125j$	$0.1350 + 0.4800i + 0.3850j$	$0.0680 + 0.9270i + 0.0050j$
$\mathcal{A}_2$	$0.1500 + 0.5500i + 0.3000j$	$0.0600 + 0.4300i + 0.5100j$	$0.3150 + 0.5200i + 0.1650j$
$\mathcal{A}_3$	$0.0375 + 0.3250i + 0.6375j$	$0.0825 + 0.5350i + 0.3825j$	$0.5525 + 0.3950i + 0.0525j$
$\mathcal{A}_4$	$0.2275 + 0.5450i + 0.2275j$	$0.0680 + 0.9270i + 0.0050j$	$0.5525 + 0.3950i + 0.0525j$
	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$0.2975 + 0.6050i + 0.0975j$	$0.0875 + 0.4250i + 0.4875j$	$0.2925 + 0.5150i + 0.1925j$
$\mathcal{A}_2$	$0.7650 + 0.2200i + 0.0150j$	$0.3375 + 0.5250i + 0.1375j$	$0.5950 + 0.3600i + 0.0450j$
$\mathcal{A}_3$	$0.0855 + 0.9130i + 0.0015j$	$0.2925 + 0.5150i + 0.1925j$	$0.0855 + 0.9130i + 0.0015j$
$\mathcal{A}_4$	$0.5525 + 0.3950i + 0.0525j$	$0.1275 + 0.8565i + 0.0160j$	$0.2625 + 0.5750i + 0.1625j$
		CNs corresponding to PIA	
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
$\mathcal{A}_1$	$0.2450 + 0.5600i + 0.1950j$	$0.6375 + 0.3250i + 0.0375j$	$0.0680 + 0.9270i + 0.0050j$
$\mathcal{A}_2$	$0.2450 + 0.5600i + 0.1950j$	$0.6375 + 0.3250i + 0.0375j$	$0.6375 + 0.3250i + 0.0375j$
$\mathcal{A}_3$	$0.2450 + 0.5600i + 0.1950j$	$0.6375 + 0.3250i + 0.0375j$	$0.6375 + 0.3250i + 0.0375j$
$\mathcal{A}_4$	$0.2450 + 0.5600i + 0.1950j$	$0.0680 + 0.9270i + 0.0050j$	$0.6375 + 0.3250i + 0.0375j$
	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$0.8075 + 0.1850i + 0.0075j$	$0.5100 + 0.4300i + 0.0600j$	$0.8075 + 0.1850i + 0.0075j$
$\mathcal{A}_2$	$0.8075 + 0.1850i + 0.0075j$	$0.5100 + 0.4300i + 0.0600j$	$0.8075 + 0.1850i + 0.0075j$
$\mathcal{A}_3$	$0.0855 + 0.9130i + 0.0015j$	$0.5100 + 0.4300i + 0.0600j$	$0.0855 + 0.9130i + 0.0015j$
$\mathcal{A}_4$	$0.8075 + 0.1850i + 0.0075j$	$0.1275 + 0.8565i + 0.0160j$	$0.8075 + 0.1850i + 0.0075j$

Table 3.4: Connection number decision-matrix $\mathcal{R}'$ and $\mathcal{A}^-$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
$\mathcal{A}_1$	$0.0038 + 0.8490i + 0.1473j$	$0.1350 + 0.4800i + 0.3850j$	$0.6375 + 0.3250i + 0.0375j$
$\mathcal{A}_2$	$0.1500 + 0.5500i + 0.3000j$	$0.0180 + 0.8418i + 0.1403j$	$0.0840 + 0.8665i + 0.0495j$
$\mathcal{A}_3$	$0.0375 + 0.3250i + 0.6375j$	$0.0825 + 0.5350i + 0.3825j$	$0.5525 + 0.3950i + 0.0525j$
$\mathcal{A}_4$	$0.2275 + 0.5450i + 0.2275j$	$0.6375 + 0.3250i + 0.0375j$	$0.5525 + 0.3950i + 0.0525j$
	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$0.1743 + 0.7705i + 0.0553j$	$0.0105 + 0.9370i + 0.0525j$	$0.2925 + 0.5150i + 0.1925j$
$\mathcal{A}_2$	$0.7650 + 0.2200i + 0.0150j$	$0.3375 + 0.5250i + 0.1375j$	$0.5950 + 0.3600i + 0.0450j$
$\mathcal{A}_3$	$0.8075 + 0.1850i + 0.0075j$	$0.2925 + 0.5150i + 0.1925j$	$0.8075 + 0.1850i + 0.0075j$
$\mathcal{A}_4$	$0.5525 + 0.3950i + 0.0525j$	$0.5100 + 0.4300i + 0.0600j$	$0.2625 + 0.5750i + 0.1625j$
		CNs corresponding to NIA	
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
$\mathcal{A}_1$	$0.0038 + 0.8490i + 0.1473j$	$0.0600 + 0.4300i + 0.5100j$	$0.3150 + 0.5200i + 0.1650j$
$\mathcal{A}_2$	$0.0125 + 0.2750i + 0.7125j$	$0.0180 + 0.8418i + 0.1403j$	$0.0840 + 0.8665i + 0.0495j$
$\mathcal{A}_3$	$0.0125 + 0.2750i + 0.7125j$	$0.0600 + 0.4300i + 0.5100j$	$0.3150 + 0.5200i + 0.1650j$
$\mathcal{A}_4$	$0.0125 + 0.2750i + 0.7125j$	$0.0600 + 0.4300i + 0.5100j$	$0.3150 + 0.5200i + 0.1650j$
	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$0.1743 + 0.7705i + 0.0553j$	$0.0105 + 0.9370i + 0.0525j$	$0.2275 + 0.5450i + 0.2275j$
$\mathcal{A}_2$	$0.2975 + 0.6050i + 0.0975j$	$0.0875 + 0.4250i + 0.4875j$	$0.2275 + 0.5450i + 0.2275j$
$\mathcal{A}_3$	$0.2975 + 0.6050i + 0.0975j$	$0.0875 + 0.4250i + 0.4875j$	$0.2275 + 0.5450i + 0.2275j$
$\mathcal{A}_4$	$0.2975 + 0.6050i + 0.0975j$	$0.0875 + 0.4250i + 0.4875j$	$0.2275 + 0.5450i + 0.2275j$

Step 4: Without loss of generality, here we utilized the exponential Euclidean distances as described in Eq. (3.15) and Eq. (3.16) to compute the measurement values of each alternative $\mathcal{A}_k$ $(k = 1, 2, 3, 4)$ from $\mathcal{A}^+$ and $\mathcal{A}^-$. The computed results are

$$\mathfrak{D}^+(\mathcal{A}_1, \mathcal{A}^+) = 0.2986, \mathfrak{D}^+(\mathcal{A}_2, \mathcal{A}^+) = 0.2223, \mathfrak{D}^+(\mathcal{A}_3, \mathcal{A}^+) = 0.2380, \mathfrak{D}^+(\mathcal{A}_4, \mathcal{A}^+) = 0.1271,$$

$$\mathfrak{D}^-(\mathcal{A}_1, \mathcal{A}^-) = 0.0941, \mathfrak{D}^-(\mathcal{A}_2, \mathcal{A}^-) = 0.2084, \mathfrak{D}^-(\mathcal{A}_3, \mathcal{A}^-) = 0.1903, \mathfrak{D}^-(\mathcal{A}_4, \mathcal{A}^-) = 0.2910.$$

Step 5: The optimal degrees for the alternatives are obtained by using Eq. (3.17) as

$$\mathfrak{R}(\mathcal{A}_1) = 0.2396, \quad \mathfrak{R}(\mathcal{A}_2) = 0.4838, \quad \mathfrak{R}(\mathcal{A}_3) = 0.4443, \quad \mathfrak{R}(\mathcal{A}_4) = 0.6960,$$

and hence the ranking order of these alternatives is $\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$. Therefore, $\mathcal{A}_4$ is the best alternative among the others.

3.5.2 Comparative study

To show the efficiency and validity of the developed method with respect to the existing methods [8, 20, 25, 41, 135, 136, 139, 158, 167, 242] under the IVIFSs environment, we perform an analysis over the considered data by these existing methods. The results corresponding to these existing methods are summarized in Table 3.5. From this analysis, it is noted that the final ranking order of the alternatives is similar to that of the proposed approach. But simultaneously it is also noted that the computational procedure of the proposed approach is completely different from these existing approaches. For instance, in [41, 135, 139] approaches, researchers applied the AOs to aggregate the information of the alternatives into collective one and then by using the score and accuracy function respectively of IVIFN to rank the alternatives. On the other hand, in [158, 242] approaches, the authors used the possibility degree method to rank the alternatives. However, in [25] approach, the authors have used the likelihood-based comparison approach for solving the MADM approaches. In [8, 20, 136, 167] applied the TOPSIS approaches for ranking of alternatives under the IVIFNs environment.

Apart from its, the computational procedure to compute the PIA and NIA in the existing and proposed approach is entirely different. For instance, in the existing TOPSIS

Table 3.5: Comparative study with some existing approaches

Existing methods	Overall value of the alternative				Ranking order
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	
Garg [41]	0.1466	0.4653	0.3970	0.6828	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Sahin [135]	0.2195	0.4445	0.3880	0.6547	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Sivaraman et al. [139]	0.1016	0.2695	0.2337	0.4551	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Bai [8]	0.3245	0.4174	0.3938	0.6018	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Zhang et al. [242]	0.1394	0.2653	0.2229	0.3724	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Wan and Li [158]	0.2499	0.2392	0.2583	0.2526	$\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2$
Chen [25]	0.1760	0.2558	0.2290	0.3392	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Shen et al. [136]	-0.2865	-0.0206	-0.0563	0.3116	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Wang et al. [167]	0.4035	0.5008	0.4805	0.5870	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Chen, Cheng and Tsai [20]	0.4212	0.4935	0.4862	0.5795	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$

method, the PIA and NIA for a collection of IVIFNs $A = ([u^L_{\mathcal{A}}, u^U_{\mathcal{A}}], [v^L_{\mathcal{A}}, v^U_{\mathcal{A}}])$ are computed by using the following equation

$$\mathcal{A}^+ = ([u^-_{\mathcal{A}}, u^+_{\mathcal{A}}], [v^-_{\mathcal{A}}, v^+_{\mathcal{A}}]) \text{ and } \mathcal{A}^- = ([U^-_{\mathcal{A}}, U^+_{\mathcal{A}}], [V^-_{\mathcal{A}}, V^+_{\mathcal{A}}])$$

where $u^- = \max\{u^L_{\mathcal{A}}\}$, $u^+ = \max\{u^U_{\mathcal{A}}\}$, $v^- = \min\{v^L_{\mathcal{A}}\}$, $v^+ = \min\{v^U_{\mathcal{A}}\}$ and $U^- = \min\{u^L_{\mathcal{A}}\}$, $U^+ = \min\{u^U_{\mathcal{A}}\}$, $V^- = \max\{v^L_{\mathcal{A}}\}$, $V^+ = \max\{v^U_{\mathcal{A}}\}$. Thus, it is clearly seen that $u^-_{\mathcal{A}}$ is computed only based on $u^L_{\mathcal{A}}$ and are independent of other degrees. Similarly for others. Therefore there is an information loss during the computational process. On the other hand, the major advantages of the proposed approach are to consider the much more information more precisely and objectively related to the object to reduce the information loss. Also, the presented study considers the CNs which integrate the degrees of uncertainty and certainty as one consolidated system rather than the individual as in IVIFS theory. Further, the presented study considers all the degrees of the IF information during the analysis, while the above existing approaches consider mostly only the membership and non-membership degrees during the analysis. Lastly, it has been seen from the Examples 3.3.2-3.3.4 that the existing distance measures are unable to classify the alternatives under some certain cases. Hence, the proposed technique can suitably utilize to solve the decision-making problem than the other existing measures.

3.5.3 Superiority of the proposed TOPSIS method

In this section, we have presented some counter-examples which show that the existing TOPSIS approaches [8, 126] under the IVIFS environment fails to rank the alternatives while the proposed TOPSIS approach can overcome their drawbacks.

Example 3.5.1. Consider a MADM problem in which there are two alternatives $\mathcal{A}_1$ and $\mathcal{A}_2$ which are evaluated by an expert under the set of three attributes $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ whose weight vectors are $\omega = (0.3, 0.4, 0.3)^T$. Then the objective of the problem is to find the best alternative under the given set. In order to do so, an expert evaluated these alternatives and gives their preferences in the terms of IVIFNs, which are summarized as follows:

$$\mathcal{V} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1 & \qquad\qquad\qquad\mathcal{G}_2 & \qquad\qquad\qquad\mathcal{G}_3 \end{array} \\ \begin{pmatrix} ([0.3,0.6],[0.3,0.4]) & ([0.2,0.3],[0.4,0.5]) & ([0.2,0.4],[0.5,0.6]) \\ ([0.4,0.5],[0.3,0.5]) & ([0.2,0.3],[0.1,0.7]) & ([0.3,0.3],[0.5,0.7]) \end{pmatrix} \end{array} \tag{3.18}$$

Based on this decision-matrix, if we utilized the existing TOPSIS approach [8] to find the best alternative, then the following steps of their approach are executed to find the best alternative(s).

Step 1: The information related to the alternatives are represented in Eq. (3.18).

Step 2: Construct the score matrix $\mathcal{M}$ based on the score function $I(\alpha) = \frac{a+b+a(1-a-c)+b(1-b-d)}{2}$ corresponding to the IVIFN $\alpha = ([a,b],[c,d])$ (for more detail, we refer to Bai [8]) and hence we get

$$\mathcal{M} = \begin{pmatrix} 0.5100 & 0.3200 & 0.3300 \\ 0.5100 & 0.3200 & 0.3300 \end{pmatrix}$$

Step 3: Based on this matrix, the separation measurement values of the alternatives $\mathcal{A}_k (k = 1, 2)$ from its ideal values are computed as $d_1^+(\mathcal{A}_1, \mathcal{A}^+) = 0.3688$, $d_2^+(\mathcal{A}_2, \mathcal{A}^+) = 0.3688$, $d_1^-(\mathcal{A}_1, \mathcal{A}^-) = 0.2227$, and $d_2^+(\mathcal{A}_2, \mathcal{A}^-) = 0.2227$.

Step 4: The relative closeness of each alternative $\mathcal{A}_k (k = 1, 2)$ is computed as

$$\mathfrak{C}(\mathcal{A}_1) = \frac{d_1^-(\mathcal{A}_1, \mathcal{A}^-)}{d_1^-(\mathcal{A}_1, \mathcal{A}^-) + d_1^+(\mathcal{A}_1, \mathcal{A}^+)} = 0.3765$$

and

$$\mathfrak{C}(\mathcal{A}_2) = \frac{d_2^-(\mathcal{A}_2, \mathcal{A}^-)}{d_2^-(\mathcal{A}_2, \mathcal{A}^-) + d_2^+(\mathcal{A}_2, \mathcal{A}^+)} = 0.3765.$$

Since $\mathfrak{C}(\mathcal{A}_1) = \mathfrak{C}(\mathcal{A}_2)$ so we conclude that the existing TOPSIS approach is unable to rank the given alternatives.

On the other hand, if we apply the proposed TOPSIS approach to the above consider data, then the following steps are followed:

Step 1: As all attributes are of the same type, then there is no need of the normalization process.

Step 2: The PIA and NIA are computed from the given arguments by using Eqs. (3.11) and (3.12) as $\mathcal{R}^+ = \{(\mathcal{G}_1, ([0.4, 0.6], [0.3, 0.4])), (\mathcal{G}_2, ([0.2, 0.3], [0.1, 0.5])), (\mathcal{G}_3, ([0.3, 0.4], [0.5, 0.6]))\}$ and $\mathcal{R}^- = \{(\mathcal{G}_1, ([0.3, 0.5], [0.3, 0.5])), (\mathcal{G}_2, ([0.2, 0.3], [0.4, 0.7])), (\mathcal{G}_3, ([0.2, 0.3], [0.5, 0.7]))\}$.

Step 3: Based on the score and accuracy values of given arguments and $\mathcal{R}^+$ as well as $\mathcal{R}^-$, we construct the CNs of each alternative by using Definition 3.2.1 or Definition 3.2.2, as:

$$\mathcal{R}' = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \end{array} \\ \begin{pmatrix} 0.2925 + 0.5150\imath + 0.1925\jmath & 0.1375 + 0.5250\imath + 0.3375\jmath & 0.1350 + 0.4800\imath + 0.3850 \\ 0.2700 + 0.5100\imath + 0.2200\jmath & 0.1500 + 0.5500\imath + 0.3000\jmath & 0.1200 + 0.4600\imath + 0.4200 \end{pmatrix}; \end{array}$$

$$\mathcal{A}^+ = \left\{ \begin{array}{c} (\mathcal{G}_1, 0.3250 + 0.5000\imath + 0.1750\jmath), (\mathcal{G}_2, 0.1750 + 0.6000\imath + 0.2250\jmath) \\ (\mathcal{G}_3, 0.1575 + 0.4850\imath + 0.3575\jmath) \end{array} \right\}$$

and

$$\mathcal{A}^- = \left\{ \begin{array}{c} (\mathcal{G}_1, 0.2400 + 0.5200\imath + 0.2400\jmath), (\mathcal{G}_2, 0.1125 + 0.4750\imath + 0.4125\jmath) \\ (\mathcal{G}_3, 0.1000 + 0.4500\imath + 0.4500\jmath) \end{array} \right\}$$

Step 4: Based on these ideal values, the Euclidean distance measures by using the Eqs. (3.15) and (3.16) of each alternative are calculated as $\mathfrak{D}^+(\mathcal{A}_1, \mathcal{A}^+) = 0.0584$, $\mathfrak{D}^+(\mathcal{A}_2, \mathcal{A}^+) = 0.0537$, $\mathfrak{D}^-(\mathcal{A}_1, \mathcal{A}^-) = 0.0521$, $\mathfrak{D}^-(\mathcal{A}_2, \mathcal{A}^-) = 0.0557$.

Step 5: The optimal degree for each alternative is calculated by using Eq. (3.17) as, $\mathfrak{R}(\mathcal{A}_1) = 0.4715$ and $\mathfrak{R}(\mathcal{A}_2) = 0.5091$. Since $\mathfrak{R}(\mathcal{A}_1) < \mathfrak{R}(\mathcal{A}_2)$ and hence, conclude that $\mathcal{A}_2$ is the best alternative.

Therefore, the proposed TOPSIS approach is suitably working in those situations where the existing TOPSIS approach fails.

Example 3.5.2. Consider an another MADM problem with two alternatives $\mathcal{A}_1$ and $\mathcal{A}_2$ which are evaluated under the set of different attributes $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ whose weight vector is $\omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$. An expert evaluated these alternatives and give their preferences in the term of IVIFNs, are represented in the form of decision-matrix, denoted by $\mathcal{M}$, which are summarized as follows:

$$\mathcal{D} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \left(\begin{array}{cc} \mathcal{G}_1 & \mathcal{G}_2 \\ ([0.8750, 0.9360], [0.0080, 0.0270]) & ([0.7840, 0.9360], [0.0270, 0.0640]) \\ ([0.7840, 0.8750], [0.0270, 0.0640]) & ([0.7840, 0.9360], [0.0080, 0.0270]) \\ & \mathcal{G}_3 \\ & ([0.6570, 0.7840], [0.0080, 0.0270]) \\ & ([0.7840, 0.8750], [0.0080, 0.0270]) \end{array}\right) \tag{3.19}$$

Now, we want to find the best alternative based on these decision-matrix. For this, if we utilized the existing TOPSIS approach as presented by Park et al. [126], then following steps have been executed as follows:

Step 1: Based on the decision-matrix $\mathcal{M}$, we compute $\mathcal{R} = \omega \otimes \mathcal{M}$, the weighted collective decision-matrix as

$$\mathcal{R} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1\quad & \quad\mathcal{G}_2\quad & \quad\mathcal{G}_3\quad \end{array} \\ \begin{bmatrix} ([0.5, 0.6], [0.2, 0.3]) & ([0.4, 0.6], [0.3, 0.4]) & ([0.3, 0.4], [0.2, 0.3]) \\ ([0.4, 0.5], [0.3, 0.4]) & ([0.4, 0.6], [0.2, 0.3]) & ([0.4, 0.5], [0.2, 0.3]) \end{bmatrix} \end{array}$$

Step 2: The positive and negative ideals of these two alternatives are computed as follows:

$$\mathcal{A}^+ = \{([0.5, 0.6], [0.2, 0.3]), ([0.4, 0.6], [0.2, 0.3]), ([0.4, 0.5], [0.2, 0.3])\}$$
$$\text{and} \quad \mathcal{A}^- = \{([0.4, 0.5], [0.3, 0.4]), ([0.4, 0.6], [0.3, 0.4]), ([0.3, 0.4], [0.2, 0.3])\}$$

Step 3: Separation measure of the alternatives from its ideal alternatives are computed as $d(\mathcal{A}_1, \mathcal{A}^-) = d(\mathcal{A}_2, \mathcal{A}^-) = 0.0333$ and $d(\mathcal{A}_1, \mathcal{A}^+) = d(\mathcal{A}_2, \mathcal{A}^+) = 0.0333$.

Step 4: The relative closeness $\mathfrak{C}(.)$ of each alternative is $\mathfrak{C}(\mathcal{A}_1) = 0.50$ and $\mathfrak{C}(\mathcal{A}_2) = 0.50$. Since $\mathfrak{C}(\mathcal{A}_1) = \mathfrak{C}(\mathcal{A}_2)$ and hence, we conclude that the existing TOPSIS approach [126] can not rank the given alternatives.

On the other hand, if we utilized the proposed TOPSIS approach to the above consider data for ranking the alternatives, then we have executed the following steps as follows:

Step 1: Since all the attributes of the same type, therefore no need of the normalization process.

Step 2: The PIA and NIA are computed from the given arguments by using Eqs. (3.11) and (3.12) as $\mathcal{R}^+ = \{(\mathcal{G}_1, ([0.8750, 0.9360], [0.00800.0270])), (\mathcal{G}_2, ([0.7840, 0.9360], [0.0080, 0.0270])), (\mathcal{G}_3, ([0.7840, 0.8750], [0.0080, 0.0270]))\}$ and $\mathcal{R}^- = \{(\mathcal{G}_1, ([0.7840, 0.8750], [0.0270, 0.0640])), (\mathcal{G}_2, ([0.7840, 0.9360], [0.0270, 0.0640])), (\mathcal{G}_3, ([0.6570, 0.7840], [0.0080, 0.0270]))\}$.

Step 3: CNs of each alternative, $\mathcal{R}^+$ and $\mathcal{R}^-$ are computed based on the Definition 3.2.1 or Definition 3.2.2 and summarized in the form of decision matrix $\mathcal{R}'$, $\mathcal{A}^+$ and $\mathcal{A}^-$ respectively as follows:

$$\mathcal{R}' = \begin{array}{cccc} & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \\ \mathcal{A}_1 & 0.0697 + 0.9301i + 0.0002j & 0.8209 + 0.1728i + 0.0064j & 0.7079 + 0.2872i + 0.0049j \\ \mathcal{A}_2 & 0.7918 + 0.2005i + 0.0078j & 0.1099 + 0.8897i + 0.0004j & 0.1272 + 0.8723i + 0.0005j \end{array}$$

$$\mathcal{A}^+ = \begin{array}{cccc} & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \\ \mathcal{A}_1 & 0.0697 + 0.9301i + 0.0002j & 0.8450 + 0.1526i + 0.0024j & 0.8150 + 0.1820i + 0.0030j \\ \mathcal{A}_2 & 0.8897 + 0.1087i + 0.0017j & 0.1099 + 0.8897i + 0.0004j & 0.1272 + 0.8723i + 0.0005j \end{array}$$

and

$$\mathcal{R}' = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \end{array} \\ \begin{pmatrix} 0.8897 + 0.1087i + 0.0017j & 0.0844 + 0.9147i + 0.0008j & 0.1900 + 0.8085i + 0.0015j \\ 0.1017 + 0.8971i + 0.0012j & 0.8450 + 0.1526i + 0.0024j & 0.8150 + 0.1820i + 0.0030j \end{pmatrix} \end{array}$$

$$\mathcal{A}^- = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \end{array} \\ \begin{pmatrix} 0.7918 + 0.2005i + 0.0078j & 0.0844 + 0.9147i + 0.0008j & 0.1900 + 0.8085i + 0.0015j \\ 0.1017 + 0.8971i + 0.0012j & 0.8209 + 0.1728i + 0.0064j & 0.7079 + 0.2872i + 0.0049j \end{pmatrix} \end{array}$$

Step 4: Without loss of generality, we compute the Euclidean distance measures by using Eqs. Eqs. (3.15) and (3.16) of each alternative from its ideal alternatives. The computed results from them are summarized as

$$\mathfrak{D}^+(\mathcal{A}_1, \mathcal{A}^+) = 0.0526, \qquad \mathfrak{D}^+(\mathcal{A}_2, \mathcal{A}^+) = 0.0546,$$
$$\mathfrak{D}^-(\mathcal{A}_1, \mathcal{A}^-) = 0.0546, \qquad \mathfrak{D}^-(\mathcal{A}_2, \mathcal{A}^-) = 0.0526$$

Step 5: The optimal degree of each alternative is computed by using Eq. (3.17) as $\mathfrak{R}(\mathcal{A}_1) = 0.5093$ and $\mathfrak{R}(\mathcal{A}_2) = 0.4906$. Since $\mathfrak{R}(\mathcal{A}_1) > \mathfrak{R}(\mathcal{A}_2)$ and hence, we conclude that the alternative $\mathcal{A}_1$ is the best alternative.

Therefore, from the results of two above examples, it is clear that the proposed TOPSIS approach is more suitable and effective comparing with the other existing TOPSIS approaches [8, 126].

3.6 Conclusion

In this chapter, some new exponential distance measures by using the CNs for IVIFSs have been proposed with some desirable properties. Some counterexamples are presented of existing distance measures defined in [125, 197] and also applied proposed distance measures on them. From results, it has been concluded that proposed distance measures classify the different IVIFNs where the existing measures are unable. Further, based on the proposed measures, a TOPSIS approach has been presented for solving the MADM problems by using the CNs of corresponding IVIFNs. The applicability of the proposed

TOPSIS approach has been illustrated through a real-life numerical example and results are compared with the results of the several other existing methods. Also, the superiority of the proposed approach has been validated by giving some examples to show that the proposed approach is suitable working in those cases when the existing TOPSIS approaches [8, 126] under IVIFS environment fail. From the experimental and comparative study, it has been concluded that the proposed approach provides an effective way of solving the MADM problems under the IVIFNs environment where the existing methods fail to rank the alternatives. Other than this, proposed is different from the existing approaches and based on CNs which integrate the degrees of uncertainty and certainty as one consolidated system rather than the individual as in IVIFS theory.

Chapter 4

Connection numbers based distance measures for IFSs and its applications to decision-making process[1]

In this chapter, we present some axioms of the distance measures based on Hamming, Euclidean, and Hausdorff metrics whose preferences related to the attributes are made in the form of CN. Several desirable relations between the proposed measures are investigated. Later, we develop a MADM approach based on the proposed distance measures to investigate the decision-making problem. The effectiveness of the approach has been demonstrated through a case study and compared their studies with several existing measures.

4.1 Introduction

As decision-making environments and problems have become increasingly complex and uncertain, MADM problems with intuitionistic fuzzy information have gradually received more attention. But almost all of the methods for dealing with such problems are based on prospect theory. In prospect hypothesis, it is troublesome for the decision makers to give an exact reference point in the real decision-making process because of the huge many-sided

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quality of the framework step by step. Besides, several parameters need to be adjusted during the process which makes it difficult as it depends on the psychological behavior of the decision-makers. In contrast to this, set pair analysis (SPA) [246] is one of the most precise ways to deal with the uncertainties in the information, as it has no prerequisite to decide a reference point or a parameter. The fundamental rule of the SPA is to construct a function called as connection number (CN), which describe the certainty and uncertainty as one consolidated system considering three aspects of identity, discrepancy, and contrary degrees. Zhang [240] presented a transforming technique to convert IFNs into CNs and built up a decision-making approach in light of it. Jiang et al. [86] discuss the basic concept of the SPA theory. Hu and Yang [77] proposed a dynamic stochastic MADM in view of cumulative prospect theory and SPA. Xie et al. [191] presented a CN under interval-valued fuzzy set by taking the positive and negative ideal scheme. The complete literature related to the MADM problem under the different environment is summarized in Chapter 1.

In light of the above perceptions, it has been concluded that the SPA stands above amongst the existing techniques to deal with the uncertainties in the framework. However, little research is done under this environment. The gap in the research propels us to apply SPA theory under the IFS environment for solving decision-making problems. Propelled by this, the present investigation builds up a theory of information measure by proposing some families of the distance measures in view of the CN of the SPA to solve the intuitionistic fuzzy decision-making problem. The main contributions of this work are outlined underneath:

(i) To feature the deficiencies of the different existing distance measures under the intuitionistic fuzzy environment through illustrative cases.

(ii) To conquer the weaknesses of the current measures, this chapter characterizes some new series of distance measures in view of the CN of the SPA under the intuitionistic fuzzy environment. The different desirable relations between them have been developed.

(iii) The proposed technique acquaints another representation which expresses the intuitionistic fuzzy data by utilizing CNs. In the strategy, CNs are obtained specifically from the IFNs with no loss of fuzzy information. Besides, in light of it, a MADM approach has been introduced for tackling the decision-making issues.

(iv) Rather than the current intuitionistic fuzzy MADM approach, the proposed approach based on the SPA does not consider the arrangement of a reference point or parameter esteems. Further, a comparative study with existing distance measures of IFSs has been set up to demonstrate the benefits of the proposed distance measures.

4.2 Proposed distance measure based on CNs

In this section, we have presented a family of the distance measures based on the Hamming, Euclidean, and Hausdorff metrics, to find the best alternative under the SPA theory.

Before defining it, we firstly construct the CN for the IFS $\mathcal{A} = (u, v)$ which is characterized by the three components, namely "identity", "discrepancy" and "contrary" and are defined as follows.

Let $A = \{(x_t, u_{\mathcal{A}}(x_t), v_{\mathcal{A}}(x_t)) \mid x_t \in \mathcal{X}\}$ be the IFSs where $u_{\mathcal{A}}(x_t)$ and $v_{\mathcal{A}}(x_t)$ represents the degree of membership and nonmembership value respectively of x to $\mathcal{A}$ such that $u_{\mathcal{A}}, v_{\mathcal{A}} \in [0, 1]$ and $u_{\mathcal{A}} + v_{\mathcal{A}} \leq 1$ for all $x_t \in \mathcal{X}$, then CNS corresponding to the set $\mathcal{A}$ is defined as

$$\mathcal{A} = \left\{\left(x_t, a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_{\mathcal{A}}(x_t)j\right) \mid x_t \in \mathcal{X}\right\}, \tag{4.1}$$

where $a_{\mathcal{A}}(x_t) = u_{\mathcal{A}}(x_t)(1 - v_{\mathcal{A}}(x_t))$ is the "identity", $b_{\mathcal{A}}(x_t) = 1 - u_{\mathcal{A}}(x_t)(1 - v_{\mathcal{A}}(x_t)) - v_{\mathcal{A}}(x_t)(1 - u_{\mathcal{A}}(x_t))$ is the "discrepancy" and $c_{\mathcal{A}}(x_t) = v_{\mathcal{A}}(x_t)(1 - u_{\mathcal{A}}(x_t))$ is the "contrary" degrees.

In order to justify, whether the CNSs defined in Eq. (4.1) is valid CNSs. For it, we prove the results in the following theorems.

Theorem 4.2.1. For an IFS, the set defined in Eq. (4.1) is a valid CNS over the universal set $\mathcal{X}$.

Proof. To prove that set defined in Eq. (4.1) is a valid CNS for an IFS $\mathcal{A} = \{(x_t, u_{\mathcal{A}}(x_t), v_{\mathcal{A}}(x_t)) \mid x_t \in \mathcal{X}\}$, we need to show that it satisfy the following properties:

(P1) $0 \leq a_{\mathcal{A}}(x_t), b_{\mathcal{A}}(x_t), c_{\mathcal{A}}(x_t) \leq 1$;

(P2) $a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t) + c_{\mathcal{A}}(x_t) = 1$

Since $\mathcal{A}$ is an IFS, so we have

(P1) For any $x_t \in \mathcal{X}$, we get $0 \leq u_{\mathcal{A}}(x_t), v_{\mathcal{A}}(x_t) \leq 1$. Therefore, $0 \leq u_{\mathcal{A}}(x_t)(1-v_{\mathcal{A}}(x_t) \leq 1$, $0 \leq v_{\mathcal{A}}(x_t)(1-u_{\mathcal{A}}(x_t)) \leq 1$ which implies that $0 \leq a_{\mathcal{A}}(x_t), c_{\mathcal{A}}(x_t) \leq 1$. Similarly, we can obtain $0 \leq b_{\mathcal{A}}(x_t) \leq 1$. Hence, (P1) holds.

(P2) By definition of $a_{\mathcal{A}}$, $b_{\mathcal{A}}$ and $c_{\mathcal{A}}$, we can easily compute that $a_{\mathcal{A}} + b_{\mathcal{A}} + c_{\mathcal{A}} = 1$. Thus, (P2) also holds.

Hence, the set defined in Eq. (4.1) is a valid CNS. □

In the following, we have presented the distance measures between the non-zero CNs for an IFS. Let $F(\mathcal{X})$ be the family of CNSs over the universe of discourse $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, then the distance measure on $F(\mathcal{X})$ is defined as

Definition 4.2.1. A real valued function $\mathcal{D} : F(\mathcal{X}) \times F(\mathcal{X}) \to [0, 1]$ is called the distance measure if $\mathcal{D}$ satisfies the following properties for $\mathcal{A}, \mathcal{B} \in F(\mathcal{X})$ as:

(P1) $0 \leq \mathcal{D}(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{D}(\mathcal{A}, \mathcal{B}) = 0 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}(\mathcal{A}, \mathcal{B}) = \mathcal{D}(\mathcal{B}, \mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}(\mathcal{A}, \mathcal{B}) \leq \mathcal{D}(\mathcal{A}, \mathcal{C})$ and $\mathcal{D}(\mathcal{B}, \mathcal{C}) \leq \mathcal{D}(\mathcal{A}, \mathcal{C})$, for $\mathcal{C} \in F(\mathcal{X})$.

Let $\mathcal{A} = \{(x_t, a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_{\mathcal{A}}(x_t)j) \mid x_t \in \mathcal{X}\}$ and $\mathcal{B} = \{(x_t, a_{\mathcal{B}}(x_t) + b_{\mathcal{B}}(x_t)i + c_B(x_t)j) \mid x_t \in \mathcal{X}\}$ be two CNSs of the family $F(\mathcal{X})$, then we define the following distance measures.

(i) The Hamming distance

$$\mathcal{D}_1(\mathcal{A}, \mathcal{B}) = \frac{1}{3} \sum_{t=1}^{n} \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array} \right\} \tag{4.2}$$

(ii) The normalized Hamming distance :

$$\mathcal{D}_2(\mathcal{A},\mathcal{B}) = \frac{1}{3n}\sum_{t=1}^{n}\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array}\right\} \tag{4.3}$$

(iii) The Euclidean distance :

$$\mathcal{D}_3(\mathcal{A},\mathcal{B}) = \left[\frac{1}{3}\sum_{t=1}^{n}\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2 + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2 \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \end{array}\right\}\right]^{\frac{1}{2}} \tag{4.4}$$

(iv) The normalized Euclidean distance :

$$\mathcal{D}_4(\mathcal{A},\mathcal{B}) = \left[\frac{1}{3n}\sum_{t=1}^{n}\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2 + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2 \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \end{array}\right\}\right]^{\frac{1}{2}} \tag{4.5}$$

Proposition 4.2.1. The above defined distance measures $\mathcal{D}_k(\mathcal{A},\mathcal{B}), (k = 2, 4)$ between two CNSs $\mathcal{A}$ and $\mathcal{B}$ satisfies the following properties:

(P1) $0 \leq \mathcal{D}_k(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{D}_k(\mathcal{A},\mathcal{B}) = 0 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}_k(\mathcal{A},\mathcal{B}) = \mathcal{D}_k(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}_k(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_k(\mathcal{A},\mathcal{C})$ and $\mathcal{D}_k(\mathcal{B},\mathcal{C}) \leq \mathcal{D}_k(\mathcal{A},\mathcal{C})$ for $\mathcal{C} \in F(\mathcal{X})$.

Proof. Let $F(\mathcal{X})$ be the family of CNSs over $\mathcal{X}$ and $\mathcal{A},\mathcal{B},\mathcal{C} \in F(\mathcal{X})$. Then for $q = 1, 2$, we have

(P1) Since $\mathcal{A}$ and $\mathcal{B}$ are CNSs, we have $0 \leq a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t) \leq 1, 0 \leq b_{\mathcal{B}}(x_t), b_{\mathcal{B}}(x_t) \leq 1$ and $0 \leq c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t) \leq 1$ which implies that

$$\begin{aligned} & 0 \leq |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q \leq 1, \\ & 0 \leq |b_{\mathcal{B}}(x_t) - b_{\mathcal{B}}(x_t)|^q \leq 1, \\ \text{and} \qquad & 0 \leq |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \leq 1 \end{aligned}$$

and thus, $0 \leq |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \leq 3$, which implies that $0 \leq \mathcal{D}_k(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) For two CNSs $\mathcal{A}$ and $\mathcal{B}$, we have

$$\begin{aligned}
& \mathcal{D}_k(\mathcal{A},\mathcal{B}) = 0 \\
\Leftrightarrow\ & \sum_{t=1}^{n} \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \end{array} \right\} = 0 \\
\Leftrightarrow\ & a_{\mathcal{A}}(x_t) = a_{\mathcal{B}}(x_t), b_{\mathcal{A}}(x_t) = b_{\mathcal{B}}(x_t), c_{\mathcal{A}}(x_t) = c_{\mathcal{B}}(x_t) \\
\Leftrightarrow\ & \mathcal{A} = \mathcal{B}
\end{aligned}$$

(P3)

$$\begin{aligned}
\mathcal{D}_k(\mathcal{A},\mathcal{B}) &= \left[\frac{1}{3n} \sum_{t=1}^{n} \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \end{array} \right\} \right]^{\frac{1}{q}} \\
&= \left[\frac{1}{3n} \sum_{t=1}^{n} \left\{ \begin{array}{c} |a_{\mathcal{B}}(x_t) - a_{\mathcal{A}}(x_t)|^q + |b_{\mathcal{B}}(x_t) - b_{\mathcal{A}}(x_t)|^q \\ + |c_{\mathcal{B}}(x_t) - c_{\mathcal{A}}(x_t)|^q \end{array} \right\} \right]^{\frac{1}{q}} \\
&= \mathcal{D}_k(\mathcal{B},\mathcal{A})
\end{aligned}$$

(P4) $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, therefore $a_{\mathcal{A}}(x_t) \leq a_{\mathcal{B}}(x_t) \leq a_{\mathcal{C}}(x_t), b_{\mathcal{A}}(x_t) \geq b_{\mathcal{B}}(x_t) \geq b_{\mathcal{C}}(x,t)$ and $c_{\mathcal{A}}(x_t) \geq c_{\mathcal{B}}(x_t) \geq c_{\mathcal{C}}(x_t)$ which implies that

$$\begin{aligned}
& |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q \leq |a_{\mathcal{A}}(x_t) - a_{\mathcal{C}}(x_t)|^q \quad ; \\
& |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q \leq |b_{\mathcal{A}}(x_t) - b_{\mathcal{C}}(x_t)|^q \quad ; \\
\text{and} \qquad & |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \leq |c_{\mathcal{A}}(x_t) - c_{\mathcal{C}}(x_t)|^q
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left(\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \end{array} \right) \\
& \qquad \leq \left(\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{C}}(x_t)|^q + |b_{\mathcal{A}}(x_t) - b_{\mathcal{C}}(x_t)|^q \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{C}}(x_t)|^q \end{array} \right) \\
\Rightarrow \quad & \mathcal{D}_k(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_k(\mathcal{A},\mathcal{C})
\end{aligned}$$

Similarly, we obtain $\mathcal{D}_k(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_k(\mathcal{A},\mathcal{C})$.

Thus, $\mathcal{D}_k(\mathcal{A},\mathcal{B})$, $(k=2,4)$ are valid distance measures. $\square$

Furthermore, in order to stress the importance of the proposed distance measures with respect to the existing ones, we illustrate it with some numerical examples as below.

Example 4.2.1. Let $\mathcal{X}=\{x_1,x_2,\ldots,x_5\}$ be the universe of discourse and consider two known patterns $\mathcal{A}=\{(x_1,0.5,0.3),\ (x_2,0.5,\ 0.2),\ (x_3,0.9,\ 0.0),\ (x_4,0.5,0.4),\ (x_5,0.7,0.1)\}$ and $\mathcal{B}=\{(x_1,0.7,0.2),\ (x_2,0.5,\ 0.4),\ (x_3,0.9,\ 0.1),\ (x_4,0.6,0.3),\ (x_5,0.8,0.0)\}$ defined over $\mathcal{X}$ as IFSs. Consider an unknown pattern $\mathcal{P}\in F(\mathcal{X})$ which will be recognized, where $P=\{(x_1,0.7,0.1),\ (x_2,0.6,0.3),\ (x_3,0.7,0.1),\ (x_4,0.5,\ 0.4),\ (x_5,0.4,0.5)\}$, then the target of this problem is to classify the pattern $\mathcal{P}$ in one of the classes $\mathcal{A}$ and $\mathcal{B}$. For this, if we apply the measures [81, 149, 172] given in Eqs. (2.10), (2.13) and (2.25) then we get

$$\mathcal{D}_1'(\mathcal{P},\mathcal{A})=\mathcal{D}_1'(\mathcal{P},\mathcal{B})=0.2000;\quad \mathcal{D}_4^H(\mathcal{P},\mathcal{A})=\mathcal{D}_4^H(\mathcal{P},\mathcal{B})=0.2000;$$
$$\text{and } \mathcal{S}_4'(\mathcal{P},\mathcal{A})=S(\mathcal{P},\mathcal{B})=0.1600.$$

Hence, we conclude that these existing measures are unable to classify the pattern $\mathcal{P}$ with $\mathcal{A}$ and $\mathcal{B}$.

On the other hand, if we applied the proposed distance measures $\mathcal{D}_2$ and $\mathcal{D}_4$ on the data set. To do so, we first convert the given IFSs into its equivalent CNs by using Eq. (4.1) and hence get $\mathcal{A}=\{(x_1,0.35+0.50i+0.15j),\ (x_2,0.40+0.50i+0.10j),\ (x_3,0.90+0.10i+0.00j),\ (x_4,0.42+0.46i+0.12j),\ (x_5,0.80+0.20i+0.00j)\}$, $\mathcal{B}=\{(x_1,0.56+0.38i+0.06j),\ (x_2,0.30+0.50i+0.20j),\ (x_3,0.81+0.18i+0.01j),\ (x_4,0.42+0.46i+0.12j),\ (x_5,0.80+0.20i+0.00j)\}$, and $P=\{(x_1,0.63+0.34i+0.03j),\ (x_2,0.42+0.46i+0.12j),\ (x_3,0.63+0.34i+0.03j),\ (x_4,0.30+0.50i+0.20j),\ (x_5,0.20+0.50i+0.30j)\}$. Then by using Eq. (4.3) and Eq. (4.5), we get

$$\mathcal{D}_2(\mathcal{P},\mathcal{A})=0.1360, \mathcal{D}_2(\mathcal{P},\mathcal{B})=0.1453,$$
$$\text{and}\qquad \mathcal{D}_4(\mathcal{P},\mathcal{A})=0.1890, \mathcal{D}_4(\mathcal{P},\mathcal{B})=0.2083.$$

Therefore, the proposed distance measures classify the unknown pattern $\mathcal{P}$ to the pattern $\mathcal{A}$.

Example 4.2.2. Consider an IFS defined on the universal set $\mathcal{X} = \{x_1, x_2, \ldots, x_5\}$ given by $\mathcal{P} = \{(x_1,\ 0.5,\ 0.3),\ (x_2,\ 0.6,\ 0.2),\ (x_3,\ 0.5,\ 0.3),\ (x_4, 0.4, 0.5),\ (x_5, 0.7, 0.2)\}$. Assume that there are two classes $\mathcal{A}$ and $\mathcal{B}$ of IFSs given by $\mathcal{A} = \{(x_1, 0.9, 0.1),\ (x_2, 0.8, 0.1),\ (x_3, 0.8, 0.1),\ (x_4, 0.5, 0.3),\ (x_5, 0.7, 0.2)\}$ and $\mathcal{B} = \{(x_1, 0.7, 0.2),\ (x_2, 0.5,\ 0.4),\ (x_3, 0.9,\ 0.1),\ (x_4, 0.6, 0.3),\ (x_5, 0.8, 0.0)\}$. Now, if we utilize the distance measures as proposed by Ejegwa and Modom [34], Wang and Xin [172] defined in Eqs. (2.11) and (2.12) of Chapter 2 for classifying the set $\mathcal{P}$ with $\mathcal{A}$ and $\mathcal{B}$, then we get their corresponding measurement values as $\mathcal{D}'_2(\mathcal{P}, \mathcal{A}) = \mathcal{D}'_2(\mathcal{P}, \mathcal{B}) = 0.2236$ and $\mathcal{D}'_3(\mathcal{P}, \mathcal{A}) = \mathcal{D}'_3(\mathcal{P}, \mathcal{B}) = 0.3800$. Thus, these measures are again unable to classify $\mathcal{P}$ with either $\mathcal{A}$ or $\mathcal{B}$.

On the other hand, the CNs corresponding to the IFSs $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{P}$ are computed by Eq. (4.1) and we get $\mathcal{P} = \{(x_1, 0.35 + 0.50i + 0.15j),\ (x_2, 0.48 + 0.44i + 0.08j),\ (x_3, 0.35 + 0.50i + 0.15j),\ (x_4, 0.20 + 0.50i + 0.30j),\ (x_5, 0.56 + 0.38i + 0.06j)\}$, $\mathcal{A} = \{(x_1, 0.81 + 0.18i + 0.01j),\ (x_2, 0.72 + 0.26i + 0.02j),\ (x_3, 0.72 + 0.26i + 0.02j),\ (x_4, 0.35 + 0.50i + 0.15j),\ (x_5, 0.56 + 0.38i + 0.06j)\}$ and $\mathcal{B} = \{(x_1, 0.56 + 0.38i + 0.06j),\ (x_2, 0.30 + 0.50i + 0.20j),\ (x_3, 0.81 + 0.18i + 0.01j),\ (x_4, 0.42 + 0.46i + 0.12j),\ (x_5, 0.80 + 0.20i + 0.00j)\}$. Thus, by using Eq. (4.5) we get $\mathcal{D}_4(\mathcal{P}, \mathcal{A}) = 0.2135$, $\mathcal{D}_4(\mathcal{P}, \mathcal{B}) = 0.2044$ and hence a set $\mathcal{P}$ is classified with the class $\mathcal{B}$.

Thus, it follows that the proposed distance measures works well over the certain existing measures to solve the problems.

Next, we show that the above defined measure $\mathcal{D}_k(\mathcal{A}_1, \mathcal{A}_2)$ for $k = 1, 2, 3, 4$ satisfies the certain properties which are stated as below.

Proposition 4.2.2. For two CNSs $\mathcal{A}$ and $\mathcal{B}$, measures $\mathcal{D}_1$ and $\mathcal{D}_3$ satisfy the following inequalities

(i) $\mathcal{D}_1 \leq n$

(ii) $\mathcal{D}_3 \leq \sqrt{n}$

Proof. It can be easily obtain that $\mathcal{D}_1(\mathcal{A}, \mathcal{B}) = n\mathcal{D}_2(\mathcal{A}, \mathcal{B})$ and thus by Proposition 4.2.1, we get $0 \leq \mathcal{D}_1(\mathcal{A}, \mathcal{B}) \leq n$. Similarly we can obtain $0 \leq \mathcal{D}_3(\mathcal{A}, \mathcal{B}) \leq n^{1/2}$. □

Proposition 4.2.3. For two CNSs $\mathcal{A}$ and $\mathcal{B}$, the measures $\mathcal{D}_1, \mathcal{D}_3$ and $\mathcal{D}_2, \mathcal{D}_4$ satisfies the following inequalities:

(i) $\mathcal{D}_3 \leq \sqrt{\mathcal{D}_1}$

(ii) $\mathcal{D}_4 \leq \sqrt{\mathcal{D}_2}$

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two CNSs. Since $0 \leq a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t) \leq 1, 0 \leq b_{\mathcal{B}}(x_t), b_{\mathcal{B}}(x_t) \leq 1$ and $0 \leq c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t) \leq 1$. Then by using the property that $\xi^2 \leq \xi$ if $\xi \in [0,1]$ we have,

$$\begin{aligned} & \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid^2 \leq \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid, \\ & \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid^2 \leq \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ \text{and} \quad & \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid^2 \leq \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{aligned}$$

which implies that

$$\begin{aligned} & \left(\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid^2 + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid^2 \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid^2 \end{array} \right) \\ & \qquad \leq \left(\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array} \right) \\ \Rightarrow \quad & \left[\frac{1}{3} \sum_{t=1}^{n} \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2 + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2 \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \end{array} \right\} \right]^{\frac{1}{2}} \\ & \qquad \leq \left[\frac{1}{3} \sum_{t=1}^{n} \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array} \right\} \right]^{\frac{1}{2}} \\ \Rightarrow \quad & \mathcal{D}_3(\mathcal{A}, \mathcal{B}) \leq \sqrt{\mathcal{D}_1(\mathcal{A}, \mathcal{B})} \end{aligned}$$

Since, $\mathcal{A}$ and $\mathcal{B}$ are arbitrary CNSs and thus $\mathcal{D}_3 \leq \sqrt{\mathcal{D}_1}$ is true for all CNSs. Similarly, we can prove the other part. □

Next, we define a weighted distance and the normalized weighted distance between the two CNSs $\mathcal{A}$ and $\mathcal{B}$ by taking the weights $\omega_t > 0$ of the elements $x_t \in \mathcal{X}$ with $\sum\limits_{t=1}^{n} \omega_t = 1$.

(i) The weighted Hamming distance :

$$\mathcal{D}_5(\mathcal{A},\mathcal{B}) = \frac{1}{3}\sum_{t=1}^{n}\omega_t\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|+|b_{\mathcal{A}}(x_t)-b_{\mathcal{B}}(x_t)| \\ +|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)| \end{array}\right\} \tag{4.6}$$

(ii) The weighted Euclidean distance :

$$\mathcal{D}_6(\mathcal{A},\mathcal{B}) = \left[\frac{1}{3}\sum_{t=1}^{n}\omega_t\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|^2+|b_{\mathcal{A}}(x_t)-b_{\mathcal{B}}(x_t)|^2 \\ +|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)|^2 \end{array}\right\}\right]^{\frac{1}{2}} \tag{4.7}$$

(iii) The normalized weighted Hamming distance :

$$\mathcal{D}_7(\mathcal{A},\mathcal{B}) = \frac{1}{3n}\sum_{t=1}^{n}\omega_t\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|+|b_{\mathcal{A}}(x_t)-b_{\mathcal{B}}(x_t)| \\ +|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)| \end{array}\right\} \tag{4.8}$$

(iv) The normalized weighted Euclidean distance :

$$\mathcal{D}_8(\mathcal{A},\mathcal{B}) = \left[\frac{1}{3n}\sum_{t=1}^{n}\omega_t\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|^2+|b_{\mathcal{A}}(x_t)-b_{\mathcal{B}}(x_t)|^2 \\ +|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)|^2 \end{array}\right\}\right]^{\frac{1}{2}} \tag{4.9}$$

It is easy to check that the weighted distance $\mathcal{D}_k(\mathcal{A},\mathcal{B})(k=7,8)$ between CNSs $\mathcal{A}$ and $\mathcal{B}$ also satisfy the above properties (P1)-(P4) as given in Definition 4.2.1. Especially, when $\omega_t = 1/n$, for $t = 1,2,\ldots,n$, then Eqs. (4.6) and (4.7) reduce to Eqs. (4.3) and (4.5) respectively.

Proposition 4.2.4. The measures $\mathcal{D}_7$ and $\mathcal{D}_2$ satisfy the inequality $\mathcal{D}_7 \leq \mathcal{D}_2$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two CNSs then by definition of $\mathcal{D}_7$ and by using $\omega_t \leq 1$ for all t, we have

$$\begin{aligned}
\mathcal{D}_7(\mathcal{A},\mathcal{B}) &= \frac{1}{3n}\sum_{t=1}^{n}\omega_t\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|+|b_{\mathcal{A}}(x_t)-b_{\mathcal{B}}(x_t)| \\ +|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)| \end{array}\right\} \\
&\leq \frac{1}{3n}\sum_{t=1}^{n}\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|+|b_{\mathcal{A}}(x_t)-b_{\mathcal{B}}(x_t)| \\ +|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)| \end{array}\right\} \\
&= \mathcal{D}_2(\mathcal{A},\mathcal{B})
\end{aligned}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are arbitrary and hence $\mathcal{D}_7 \leq \mathcal{D}_2$ is true for all CNSs. □

Proposition 4.2.5. The weighted measures $\mathcal{D}_k(\mathcal{A},\mathcal{B}),(k=7,8)$ also satisfy the following properties for $\mathcal{A},\mathcal{B}\in F(\mathcal{X})$:

(P1) $0\leq \mathcal{D}_k(\mathcal{A},\mathcal{B})\leq 1$.

(P2) $\mathcal{D}_k(\mathcal{A},\mathcal{B})=0\Leftrightarrow \mathcal{A}=\mathcal{B}$.

(P3) $\mathcal{D}_k(\mathcal{A},\mathcal{B})=\mathcal{D}_k(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A}\subseteq\mathcal{B}\subseteq\mathcal{C}$ then $\mathcal{D}_k(\mathcal{A},\mathcal{B})\leq \mathcal{D}_k(\mathcal{A},\mathcal{C})$ and $\mathcal{D}_k(\mathcal{B},\mathcal{C})\leq \mathcal{D}_k(\mathcal{A},\mathcal{C})$, for $\mathcal{C}\in F(\mathcal{X})$.

Proof. Since $\omega_t>0$ and $\sum\limits_{t=1}^{n}\omega_t=1$, then we can easily get $0\leq \mathcal{D}_7(\mathcal{A},\mathcal{B})\leq \mathcal{D}_2(\mathcal{A},\mathcal{B})\leq 1$. Thus, $\mathcal{D}_7(\mathcal{A},\mathcal{B})$ satisfies (P1). The proofs of (P2)-(P4) are similar to the Proposition 4.2.1. Similarly we can have it for $\mathcal{D}_8$. ☐

Proposition 4.2.6. For two CNSs $\mathcal{A}$ and $\mathcal{B}$, the measures $\mathcal{D}_1,\mathcal{D}_5$ and $\mathcal{D}_6$ satisfy the following inequalities:

(i) $\mathcal{D}_5\leq\mathcal{D}_1$

(ii) $\mathcal{D}_6\leq\sqrt{\mathcal{D}_5}$

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ be two CNSs and $\omega_t>0$ such that $\sum\limits_{t=1}^{n}\omega_t=1$, then we have

(i) By definition of $\mathcal{D}_5$, we get

$$\begin{aligned}
\mathcal{D}_5(\mathcal{A},\mathcal{B}) &= \frac{1}{3}\sum_{t=1}^{n}\omega_t\big\{|a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|+|b_{\mathcal{A}}(x_t)-b_{\mathcal{B}}(x_t)|+|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)|\big\}\\
&= \frac{1}{3}\Bigg(\omega_1\Bigg\{|a_{\mathcal{A}}(x_1)-a_{\mathcal{B}}(x_1)|+|b_{\mathcal{A}}(x_1)-b_{\mathcal{B}}(x_1)|+|c_{\mathcal{A}}(x_1)-c_{\mathcal{B}}(x_1)|\Bigg\}\\
&\quad+\omega_2\Bigg\{|a_{\mathcal{A}}(x_2)-a_{\mathcal{B}}(x_2)|+|b_{\mathcal{A}}(x_2)-b_{\mathcal{B}}(x_2)|+|c_{\mathcal{A}}(x_2)-c_{\mathcal{B}}(x_2)|\Bigg\}\\
&\quad+\ldots+\\
&\quad+\omega_n\Bigg\{|a_{\mathcal{A}}(x_n)-a_{\mathcal{B}}(x_n)|+|b_{\mathcal{A}}(x_n)-b_{\mathcal{B}}(x_n)|+|c_{\mathcal{A}}(x_n)-c_{\mathcal{B}}(x_n)|\Bigg\}\Bigg)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{3}\Bigg(\Big\{|a_{\mathcal{A}}(x_1)-a_{\mathcal{B}}(x_1)|+|b_{\mathcal{A}}(x_1)-b_{\mathcal{B}}(x_1)|+|c_{\mathcal{A}}(x_1)-c_{\mathcal{B}}(x_1)|\Big\}\\
&\quad+\Big\{|a_{\mathcal{A}}(x_2)-a_{\mathcal{B}}(x_2)|+|b_{\mathcal{A}}(x_2)-b_{\mathcal{B}}(x_2)|+|c_{\mathcal{A}}(x_2)-c_{\mathcal{B}}(x_2)|\Big\}\\
&\quad+\ldots+\\
&\quad+\Big\{|a_{\mathcal{A}}(x_n)-a_{\mathcal{B}}(x_n)|+|b_{\mathcal{A}}(x_n)-b_{\mathcal{B}}(x_n)|+|c_{\mathcal{A}}(x_n)-c_{\mathcal{B}}(x_n)|\Big\}\Bigg)\\
&= \mathcal{D}_1(\mathcal{A},\mathcal{B})
\end{aligned}$$

i.e., $\mathcal{D}_5(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_1(\mathcal{A},\mathcal{B})$. As $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, so we get $\mathcal{D}_5 \leq \mathcal{D}_1$.

(ii) As $0 \leq a_{\mathcal{A}}(x_t) \leq 1$ and $0 \leq a_{\mathcal{B}}(x_t) \leq 1$ and thus $\mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid^2 \leq \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid$ and therefore,

$$\begin{aligned}
\mathcal{D}_6(\mathcal{A},\mathcal{B}) &= \left[\frac{1}{3}\sum_{t=1}^{n}\omega_t\left\{\begin{matrix}|a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|^2+|b_{\mathcal{B}}(x_t)-b_{\mathcal{B}}(x_t)|^2\\ +\,|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)|^2\end{matrix}\right\}\right]^{\frac{1}{2}}\\
&\leq \left[\frac{1}{3}\sum_{t=1}^{n}\omega_t\left\{\begin{matrix}|a_{\mathcal{A}}(x_t)-a_{\mathcal{B}}(x_t)|+|b_{\mathcal{B}}(x_t)-b_{\mathcal{B}}(x_t)|\\ +\,|c_{\mathcal{A}}(x_t)-c_{\mathcal{B}}(x_t)|\end{matrix}\right\}\right]^{\frac{1}{2}}\\
&= \sqrt{\mathcal{D}_5(\mathcal{A},\mathcal{B})}
\end{aligned}$$

As $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, so we get $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_5}$.

□

Proposition 4.2.7. For two CNSs $\mathcal{A}$ and $\mathcal{B}$, the measures $\mathcal{D}_1, \mathcal{D}_3$ and $\mathcal{D}_6$ satisfy the following inequalities:

(i) $\mathcal{D}_6 \leq \mathcal{D}_3$

(ii) $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_1}$

Proof. The proof follows from the above proposition, so we omit it here. □

Hausdorff distance between two non-empty closed and bounded sets is a measure of resemblance between them. For example, consider $\mathcal{A} = [x_1, x_2]$ and $\mathcal{B} = [y_1, y_2]$ in the Euclidean domain R, the Hausdroff distance in additive set environment is given by

$$\mathcal{D}^H(\mathcal{A},\mathcal{B}) = \max\Big\{ \mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid \Big\}$$

Now, for two CNSs $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, we propose the following Hausdorff distance measures:

(i) Hausdorff normalized Hamming distance :

$$\mathcal{D}_1^H(\mathcal{A},\mathcal{B}) = \frac{1}{3n}\sum_{t=1}^{n} \max\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array}\right\} \tag{4.10}$$

(ii) Hausdorff weighted Hamming distance :

$$\mathcal{D}_2^H(\mathcal{A},\mathcal{B}) = \frac{1}{3}\sum_{t=1}^{n} \omega_t \max\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array}\right\} \tag{4.11}$$

(iii) Hausdorff normalized Euclidean distance:

$$\mathcal{D}_3^H(\mathcal{A},\mathcal{B}) = \left[\frac{1}{3n}\sum_{t=1}^{n} \max\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \end{array}\right\}\right]^{\frac{1}{2}} \tag{4.12}$$

(iv) Hausdorff weighted Euclidean distance :

$$\mathcal{D}_4^H(\mathcal{A},\mathcal{B}) = \left[\frac{1}{3}\sum_{t=1}^{n} \omega_t \max\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \end{array}\right\}\right]^{\frac{1}{2}} \tag{4.13}$$

Especially, when $\omega_t = 1/n, \forall t = 1, 2, \ldots, n$, then Eqs. (4.11) and (4.13) reduce to Eqs. (4.10) and (4.12) respectively.

Proposition 4.2.8. For $\mathcal{A}, \mathcal{B} \in F(\mathcal{X})$, the distance measures $\mathcal{D}_k^H(\mathcal{A},\mathcal{B}), (k = 1, 3)$ defined in Eqs. (4.10) and (4.12) also satisfy the following properties.

(P1) $0 \le \mathcal{D}_k^H(\mathcal{A},\mathcal{B}) \le 1$.

(P2) $\mathcal{D}_k^H(\mathcal{A},\mathcal{B}) = 0 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}_k^H(\mathcal{A},\mathcal{B}) = \mathcal{D}_k^H(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}_k^H(\mathcal{A},\mathcal{B}) \le \mathcal{D}_k^H(\mathcal{A},\mathcal{C})$ and $\mathcal{D}_k^H(\mathcal{B},\mathcal{C}) \le \mathcal{D}_k^H(\mathcal{A},\mathcal{C})$, for $\mathcal{C} \in F(\mathcal{X})$.

Proof. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in F(\mathcal{X})$ be three CNSs. Then for $q = 1, 2$, we have

(P1) Since $\mathcal{A}$ and $\mathcal{B}$ be two CNSs, so using definition of CNs, we have

$$\begin{aligned} & 0 \leq \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid^q \leq 1; \\ & 0 \leq \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid^q \leq 1; \\ \text{and} \qquad & 0 \leq \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid^q \leq 1 \end{aligned}$$

and thus,

$$0 \leq \max \left\{ \begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid^q, \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid^q, \\ \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid^q \end{array} \right\} \leq 1.$$

Therefore, $0 \leq \mathcal{D}_k^H(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) For two CNSs $\mathcal{A}$ and $\mathcal{B}$,

$$\begin{aligned} & \mathcal{D}_k^H(\mathcal{A}, \mathcal{B}) = 0 \\ \Leftrightarrow \quad & \max \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \end{array} \right\} = 0 \\ \Leftrightarrow \quad & a_{\mathcal{A}}(x_t) = a_{\mathcal{B}}(x_t), b_{\mathcal{A}}(x_t) = b_{\mathcal{B}}(x_t), c_{\mathcal{A}}(x_t) = c_{\mathcal{B}}(x_t). \end{aligned}$$

Therefore $\mathcal{A} = \mathcal{B}$.

(P3) For two CNSs $\mathcal{A}$ and $\mathcal{B}$,

$$\begin{aligned} \mathcal{D}_k^H(\mathcal{A}, \mathcal{B}) &= \left[\frac{1}{3n} \sum_{t=1}^{n} \max \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \end{array} \right\} \right]^{\frac{1}{q}} \\ &= \left[\frac{1}{3n} \sum_{t=1}^{n} \max \left\{ \begin{array}{c} |a_{\mathcal{B}}(x_t) - a_{\mathcal{A}}(x_t)|^q, |b_{\mathcal{B}}(x_t) - b_{\mathcal{A}}(x_t)|^q, \\ |c_{\mathcal{B}}(x_t) - c_{\mathcal{A}}(x_t)|^q \end{array} \right\} \right]^{\frac{1}{q}} \\ &= \mathcal{D}_k^H(\mathcal{B}, \mathcal{A}) \end{aligned}$$

(P4) Since $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ which implies that

$$\begin{aligned} & a_{\mathcal{A}}(x_t) \leq a_{\mathcal{B}}(x_t) \leq a_{\mathcal{C}}(x, t); \\ & b_{\mathcal{A}}(x_t) \geq b_{\mathcal{B}}(x_t) \geq b_{\mathcal{C}}(x, t); \\ \text{and} \qquad & c_{\mathcal{A}}(x_t) \geq c_{\mathcal{B}}(x_t) \geq c_{\mathcal{C}}(x_t) \end{aligned}$$

Therefore $|a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q \leq |a_{\mathcal{A}}(x_t) - a_{\mathcal{C}}(x_t)|^q, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q \leq |b_{\mathcal{A}}(x_t) - b_{\mathcal{C}}(x_t)|^q, |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \leq |c_{\mathcal{A}}(x_t) - c_{\mathcal{C}}(x_t)|^q$ and hence

$$\begin{aligned} & \max\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^q, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^q, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^q \end{array}\right\} \\ & \qquad \leq \max\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{C}}(x_t)|^q, |b_{\mathcal{A}}(x_t) - b_{\mathcal{C}}(x_t)|^q, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{C}}(x_t)|^q \end{array}\right\} \\ \Rightarrow \quad & \mathcal{D}_k^H(\mathcal{A},\mathcal{B}) \leq \mathcal{D}_k^H(\mathcal{A},\mathcal{C}) \end{aligned}$$

Similarly, we can get $\mathcal{D}_k^H(\mathcal{B},\mathcal{C}) \leq \mathcal{D}_k^H(\mathcal{A},\mathcal{C})$.

Hence, $\mathcal{D}_k^H(\mathcal{A},\mathcal{B})(k = 1, 3)$ are the valid distance measures. □

Proposition 4.2.9. For $\mathcal{A},\mathcal{B} \in F(\mathcal{X})$, the distance measures $\mathcal{D}_k^H(\mathcal{A},\mathcal{B}), (k = 2, 4)$ defined in Eqs. (4.11) and (4.13) also satisfy the properties (P1)-(P4).

Proposition 4.2.10. For two CNSs $\mathcal{A}$ and $\mathcal{B}$, the measures $\mathcal{D}_1^H$ and $\mathcal{D}_2$ satisfy the inequality $\mathcal{D}_1^H \leq \mathcal{D}_2$.

Proof. Since for any positive numbers $h_t, t = 1, 2 \ldots n, \sum_t h_t \geq \max_t\{h_t\}$. Thus, for any two CNSs $\mathcal{A}$ and $\mathcal{B}$, we have

$$\begin{aligned} \mathcal{D}_2(\mathcal{A},\mathcal{B}) &= \frac{1}{3n}\sum_{t=1}^{n}\left\{\begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| \\ + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array}\right\} \\ &\geq \frac{1}{3n}\sum_{t=1}^{n}\max\left\{\begin{array}{l} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array}\right\} \\ &= \mathcal{D}_1^H(\mathcal{A},\mathcal{B}) \end{aligned}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are arbitrary CNSs, so we have $\mathcal{D}_1^H \leq \mathcal{D}_2$. □

Proposition 4.2.11. For two CNSs $\mathcal{A}$ and $\mathcal{B}$, the measures $\mathcal{D}_2^H$ and $\mathcal{D}_5$, $\mathcal{D}_3^H$ and $\mathcal{D}_4$, $\mathcal{D}_4^H$ and $\mathcal{D}_6$ satisfy the inequalities $\mathcal{D}_2^H \leq \mathcal{D}_5$, $\mathcal{D}_3^H \leq \mathcal{D}_4$ and $\mathcal{D}_4^H \leq \mathcal{D}_6$ respectively.

Proof. As similar to above proposition, and hence we omit here. □

Proposition 4.2.12. Distance measures $\mathcal{D}_1^H$ and $\mathcal{D}_3^H$ satisfy the inequality $\mathcal{D}_3^H \leq \sqrt{\mathcal{D}_1^H}$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two CNSs, we have $0 \leq a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t) \leq 1, 0 \leq b_{\mathcal{B}}(x_t), b_{\mathcal{B}}(x_t) \leq 1$ and $0 \leq c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t) \leq 1$. Then we have,

$$\begin{aligned} & |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2 \leq |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|; \\ & |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2 \leq |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|; \\ \text{and} \quad & |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \leq |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{aligned}$$

Therefore,

$$\begin{aligned} & \max \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \end{array} \right\} \\ & \qquad \leq \max \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array} \right\} \\ \Rightarrow \quad & \frac{1}{3n} \sum_{t=1}^{n} \max \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \end{array} \right\} \\ & \qquad \leq \frac{1}{3n} \sum_{t=1}^{n} \max \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array} \right\} \\ \Rightarrow \quad & \left[\frac{1}{3n} \sum_{t=1}^{n} \max \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|^2, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|^2, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|^2 \end{array} \right\} \right]^{1/2} \\ & \qquad \leq \left[\frac{1}{3n} \sum_{t=1}^{n} \max \left\{ \begin{array}{c} |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)|, |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)|, \\ |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \end{array} \right\} \right]^{1/2} \\ \Rightarrow \quad & \mathcal{D}_3^H(\mathcal{A}, \mathcal{B}) \leq \sqrt{\mathcal{D}_1^H(\mathcal{A}, \mathcal{B})} \end{aligned}$$

Since, $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, so we have $\mathcal{D}_3^H \leq \sqrt{\mathcal{D}_1^H}$. □

Proposition 4.2.13. Distance measures $\mathcal{D}_2^H$ and $\mathcal{D}_4^H$ satisfy the inequality $\mathcal{D}_4^H \leq \sqrt{\mathcal{D}_2^H}$.

Proof. Follows from proposition 4.2.12. □

Proposition 4.2.14. The distance measures $\mathcal{D}_1, \mathcal{D}_2^H$ and $\mathcal{D}_5$ satisfy the following inequalities :

(i) $\mathcal{D}_1 \geq \frac{\mathcal{D}_5+\mathcal{D}_2^H}{2}$ and $\mathcal{D}_1 \geq \sqrt{\mathcal{D}_5 \cdot \mathcal{D}_2^H}$

(ii) $\mathcal{D}_2 \geq \frac{\mathcal{D}_7+\mathcal{D}_1^H}{2}$ and $\mathcal{D}_2 \geq \sqrt{\mathcal{D}_7 \cdot \mathcal{D}_1^H}$

Proof. Since $\mathcal{D}_2^H \leq \mathcal{D}_1$ and $\mathcal{D}_5 \leq \mathcal{D}_1$. So by adding these inequalities, we get $\frac{\mathcal{D}_5+\mathcal{D}_2^H}{2} \leq \mathcal{D}_1$. On the other hand, by multiplying these, we get $\sqrt{\mathcal{D}_5 \cdot \mathcal{D}_2^H} \leq \mathcal{D}_1$.

Also, $\mathcal{D}_1^H \leq \mathcal{D}_2$ and $\mathcal{D}_7 \leq \mathcal{D}_2$. So by adding these inequalities, we get $\frac{\mathcal{D}_7+\mathcal{D}_1^H}{2} \leq \mathcal{D}_2$. On the other hand, by multiplying these, we get $\sqrt{\mathcal{D}_7 \cdot \mathcal{D}_1^H} \leq \mathcal{D}_2$. □

Proposition 4.2.15. The distance measures $\mathcal{D}_3, \mathcal{D}_4^H$ and $\mathcal{D}_6$ satisfy the following inequalities :

(i) $\frac{\mathcal{D}_6+\mathcal{D}_4^H}{2} \leq \mathcal{D}_3$

(ii) $\sqrt{\mathcal{D}_6 \cdot \mathcal{D}_4^H} \leq \mathcal{D}_3$

Proof. Follows from the above Proposition 4.2.14. □

4.3 Decision making approach based on the proposed distances

The general description of the MADM problem is given in the Section 2.7 of Chapter 2 under the IFS environment. In the following, we develop an approach based on the proposed measures with intuitionistic fuzzy information, which involve the following steps:

Step 1: Arrange the rating values of the alternatives in terms of IFNs and represent them as a intuitionistic fuzzy decision matrix $\mathcal{M}$.

Step 2: Normalize the decision matrix $\mathcal{M}$, if required, by using Eq. (2.58) of Chapter 2 and hence obtained the normalized decision matrix $\mathcal{R}$.

Step 3: Convert the matrix $\mathcal{R}$ into CN matrix $\mathcal{H}$ as

$$\mathcal{H} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \cdots & \mathcal{G}_n \end{array} \\ \begin{pmatrix} a_{11}+b_{11}i+c_{11}j & a_{12}+b_{12}i+c_{12}j & \cdots & a_{1n}+b_{1n}i+c_{1n}j \\ a_{21}+b_{21}i+c_{21}j & a_{22}+b_{22}i+c_{22}j & \cdots & a_{2n}+b_{2n}i+c_{2n}j \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1}i+c_{m1}j & a_{m2}+b_{m2}i+c_{m2}j & \cdots & a_{mn}+b_{mn}i+c_{mn}j \end{pmatrix} \end{array}$$

where a_{kt}, b_{kt} and c_{kt} are computed by using Eq. (4.1) as $a_{kt} = u_{kt}(1 - v_{kt})$, $b_{kt} = 1 - u_{kt}(1 - v_{kt}) - v_{kt}(1 - u_{kt})$ and $c_{kt} = v_{kt}(1 - u_{kt})$ for $k = 1, 2, \ldots, m; t = 1, 2, \ldots, n$.

Step 4: Utilize the proposed distance measures '$\mathcal{D}$' to compute the overall value of each alternative.

Step 5: Rank the alternatives and select the best one(s) according to the minimum value of $\arg\min\{d\}$.

4.4 Illustrative example

In this section, the above-mentioned approach is demonstrated with a numerical example and compared with several existing approaches.

4.4.1 A case study

Indian rural society is changing and transforming in many aspects, including jobs, business structures, transportation facilities, and communication system. Therefore, the people of the rural area migrate to the big cities to find more opportunities. To stop this immigration, the Indian government wants to provide all the facilities in these rural areas. In this context, the Indian government has taken a considerable number of road building projects either to preserve the roads which are already built or to undertake the new roads. For this, the Indian government had been issued the global tender to select the contractor for these projects in the newspaper and considered the four attributes required for its namely, contractor background experience ($\mathcal{G}_1$), technical capability ($\mathcal{G}_2$), tender price ($\mathcal{G}_3$), and completion time ($\mathcal{G}_4$), and assign the weights of a relative importance of each attribute as $\omega = (0.25, 0.25, 0.40, 0.10)^T$ on the basis of decision maker's preferences. The six contractors (i.e. alternatives) namely, Jaihind Road Builders private (Pvt.) limited (Ltd.) ($\mathcal{A}_1$), J.K. Construction ($\mathcal{A}_2$), Build quick Infrastructure Pvt. Ltd. ($\mathcal{A}_3$), Relcon Infra projects Ltd. ($\mathcal{A}_4$), Tata Infrastructure Ltd. ($\mathcal{A}_5$) and Birla Pvt. Ltd. ($\mathcal{A}_6$) bid for these projects. Then, the aim of the government is to recognize the best contractor among $\mathcal{A}_k, k = 1, 2, \ldots, 6$ which classify with the known desired goals of the contractor, denoted by $\mathcal{B}$, where

$$B = \{(\mathcal{G}_1, 0.7, 0.2), (\mathcal{G}_2, 0.6, 0.4), (\mathcal{G}_3, 0.5, 0.5), (\mathcal{G}_4, 0.7, 0.1)\}$$

Then, the following steps have been executed based on the proposed approach to find the best contractor for the required project.

Step 1: The evaluation values given by an expert, assigned by the Government, to evaluate these given six alternatives $\mathcal{A}_k, (k = 1, 2, \ldots, 6)$ are summarized in Table 4.1 in terms of IFNs.

Table 4.1: Input data of the alternatives in terms of IFNs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	(0.6, 0.3)	(0.7, 0.2)	(0.2, 0.8)	(0.1, 0.9)
$\mathcal{A}_2$	(0.6, 0.2)	(0.7, 0.2)	(0.1, 0.8)	(0.1, 0.8)
$\mathcal{A}_3$	(0.9, 0.1)	(0.8, 0.1)	(0.2, 0.7)	(0.2, 0.8)
$\mathcal{A}_4$	(0.7, 0.3)	(0.7, 0.2)	(0.1, 0.8)	(0.1, 0.8)
$\mathcal{A}_5$	(0.7, 0.2)	(0.7, 0.3)	(0.2, 0.8)	(0.2, 0.7)
$\mathcal{A}_6$	(0.7, 0.3)	(0.8, 0.1)	(0.0, 0.8)	(0.2, 0.8)

Step 2: Since $\mathcal{G}_3$ and $\mathcal{G}_4$, are the cost types attributes, so normalized decision matrix R is obtained by Eq. (2.58) and resultant matrix is given in Table 4.2.

Step 3: Construct the CN to each normalized IFNs by using Eq. (4.1) and the corresponding decision matrix is given in Table 4.3, while the CNS to the known contractor

Table 4.2: Normalzied matrix of the alternatives in terms of IFNs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	(0.6, 0.3)	(0.7, 0.2)	(0.8, 0.2)	(0.9, 0.1)
$\mathcal{A}_2$	(0.6, 0.2)	(0.7, 0.2)	(0.8, 0.1)	(0.8, 0.1)
$\mathcal{A}_3$	(0.9, 0.1)	(0.8, 0.1)	(0.7, 0.2)	(0.8, 0.2)
$\mathcal{A}_4$	(0.7, 0.3)	(0.7, 0.2)	(0.8, 0.1)	(0.8, 0.1)
$\mathcal{A}_5$	(0.7, 0.2)	(0.7, 0.3)	(0.8, 0.2)	(0.7, 0.2)
$\mathcal{A}_6$	(0.7, 0.3)	(0.8, 0.1)	(0.8, 0.0)	(0.8, 0.2)

$\mathcal{B}$ is given as

$$\mathcal{B} = \left\{ \begin{array}{l} (\mathcal{G}_1, 0.56 + 0.38i + 0.06j), (\mathcal{G}_2, 0.36 + 0.48i + 0.16j), \\ (\mathcal{G}_3, 0.25 + 0.50i + 0.25j), (\mathcal{G}_4, 0.49 + 0.42i + 0.09) \end{array} \right\}$$

Table 4.3: CN matrix of the alternatives

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	0.42 + 0.46i+ 0.12j	0.56+ 0.38i+ 0.06j	0.64+ 0.32i+ 0.04j	0.81+ 0.18i+0.01j
$\mathcal{A}_2$	0.48 + 0.44i+ 0.08j	0.56+ 0.38i+ 0.06j	0.72+ 0.26i+ 0.02j	0.72+ 0.26i+0.02j
$\mathcal{A}_3$	0.81 + 0.18i+ 0.01j	0.72+ 0.26i+ 0.02j	0.63+ 0.34i+ 0.03j	0.64+ 0.32i+0.04j
$\mathcal{A}_4$	0.49 + 0.42i+ 0.09j	0.56+ 0.38i+ 0.06j	0.72+ 0.26i+ 0.02j	0.72+ 0.26i+0.02j
$\mathcal{A}_5$	0.56 + 0.38i+ 0.06j	0.49+ 0.42i+ 0.09j	0.64+ 0.32i+ 0.04j	0.56+ 0.38i+0.06j
$\mathcal{A}_6$	0.49 + 0.42i+ 0.09j	0.72+ 0.26i+ 0.02j	0.80+ 0.20i+ 0.00j	0.64+ 0.32i+0.04j

Step 4: Utilize the proposed distance measures to compute the measurement values of $\mathcal{A}_k (k = 1, 2, \ldots, 6)$ from $\mathcal{B}$. The computed values are summarized in Table 4.4.

Step 5: From Table 4.4, it is found that $\mathcal{A}_5$, i.e., Tata infrastructure Ltd. contractor is the best one which recognize with the $\mathcal{B}$.

4.4.2 Comparative study with the existing measures

In order to validate the feasibility of the proposed distance measures, a comparative analysis has been conducted by using existing methods of the IFS proposed by the authors [34, 81, 172, 235] to the considered data. The results corresponding to each measure are computed and presented in Table 4.5. From this table, it is observed that the best alternative remains same i.e., $\mathcal{A}_5$ with the proposed one which validates the proposed approach, but the proposed one have several advantages over the existing ones. Also, the existing measures are unable to do the proper ranking for some alternatives. For example, in Wang and Xin [172], their approach is unable to distinguish between the contractors $\mathcal{A}_2, \mathcal{A}_4$ and $\mathcal{A}_3, \mathcal{A}_6$. Similarly, in Hung and Yang [81] approach, their measure is unable to rank the patterns $\mathcal{A}_3$ and $\mathcal{A}_6$. Hence, the existing measures are unable to justify their approach to solve these type of problems more accurately while the proposed measures have successfully overcome its drawbacks.

4.5 Conclusion

In this chapter, we present some series of distance measures based on the Hamming, Euclidean, and Hausdorff metric by using the CNSs of IFSs. To do so, we first define the CNS corresponding to IFSs and study their properties. Further, based on this CNS, we define various measures between the pairs of the IFSs and investigate their several correlations to shows its stability. The advantages of the proposed distance measures are explained with the help of some counter-intuitive examples. Later, we developed a novel MADM method based on the proposed measures to rank the alternatives and illustrate with a numerical example to demonstrates that the current measures under intuitionistic fuzzy information fail to deal with the circumstances while the proposed measure can work beautifully on such situations. From the study, it has been inferred that proposed work gives another simple approach to deal with the uncertainties and vagueness in the data and subsequently gives an alternative method to solve the decision-making problems.

Table 4.4: Computed results by the proposed distance measures

Measures	Measurement values of $\mathcal{B}$ from						Ranking order
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	
$\mathcal{D}_1$	0.6067	0.5600	0.6733	0.5533	**0.3933**	0.6667	$\mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_6 \succ \mathcal{A}_3$
$\mathcal{D}_2$	0.1517	0.1400	0.1683	0.1383	**0.0983**	0.1667	$\mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_6 \succ \mathcal{A}_3$
$\mathcal{D}_3$	0.3543	0.3726	0.4170	0.3712	**0.2952**	0.4693	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
$\mathcal{D}_4$	0.1772	0.1863	0.2085	0.1856	**0.1476**	0.2346	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
$\mathcal{D}_1^H$	0.0758	0.0700	0.0842	0.0692	**0.0492**	0.0833	$\mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_6 \succ \mathcal{A}_3$
$\mathcal{D}_3^H$	0.1426	0.1515	0.1676	0.1511	**0.1204**	0.1909	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
$\mathcal{D}_5$	0.1727	0.1780	0.2043	0.1763	**0.1303**	0.2197	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
$\mathcal{D}_6$	0.1997	0.2248	0.2331	0.2243	**0.1812**	0.2789	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
$\mathcal{D}_7$	0.0432	0.0445	0.0511	0.0441	**0.0326**	0.0549	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
$\mathcal{D}_8$	0.0999	0.1124	0.1166	0.1121	**0.0906**	0.1394	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
$\mathcal{D}_2^H$	0.0216	0.0223	0.0255	0.0220	**0.0163**	0.0275	$\mathcal{A}_5 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_6 \succ \mathcal{A}_3$
$\mathcal{D}_4^H$	0.0811	0.0916	0.0939	0.0915	**0.0739**	0.1135	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$

Table 4.5: Comparative analysis results with the existing measures

	Measures	Measurement values of $\mathcal{B}$ from						Ranking order
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	
Wang and Xin [172]	Hamming distance	0.200	0.200	0.2750	0.200	0.1250	0.2750	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_2 = \mathcal{A}_4 \succ \mathcal{A}_3 = \mathcal{A}_6$
Wang and Xin [172]	Euclidean distance	0.2062	0.2121	0.2500	0.2121	0.1658	0.2739	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 = \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
Zeng and Li [235]	Correlation coefficient	0.9296	0.9191	0.9029	0.9233	0.9518	0.8773	$\mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_6$
Hung and Yang [81]	Similarity measure	0.8375	0.8500	0.800	0.8500	0.8875	0.8000	$\mathcal{A}_5 \succ \mathcal{A}_2 = \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3 = \mathcal{A}_6$

Chapter 5

Connection numbers based similarity measures for intuitionistic fuzzy sets and their application[1]

In this chapter, we develop some novel similarity measures to measure the relative strength of the different IFSs after pointing out the weakness of the existing measures. For it, a connection number, the main component of SPA theory is formulated in the form of the degrees of identity, discrepancy, and contrary. Then, based on it some new similarity and weighted similarity measures between the connection number sets (CNSs) are defined. A comparative analysis of the proposed and existing measures are formulated in terms of the counter-intuitive cases for showing the validity of it. Finally, an illustrative example is provided to demonstrate it.

5.1 Introduction

Decision making (DM) is the process to choose or to select the best alternative based on their attributes. Many DM processes are based on a single attribute but some DM processes are based on more than one attribute known as MADM). Traditionally, DM information had been assumed to be determinable and clear; however, these properties

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[1] The content of this chapter is published as "An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making", *Soft Computing*, 22(15), 4959 - 4970, 2018 (**Impact Factor: 2.784**).

have not been observed. To address it completely, FSs and its extensions such as intuitionistic fuzzy sets (IFSs) [6], interval-valued intuitionistic fuzzy sets (IVIFSs) [5] are proposed by the scholars to handle the uncertain information. Considering the margin of error in constructing membership in FSs, IFSs add two other functions: a non-membership function and a hesitancy function.IVIFSs extend IFSs to an interval form. Over the last decade, numerable attempts have been made by different researchers in processing the information values using different aggregation operators under IFS and IVIFS environments [1, 39, 41, 48, 56, 57, 83, 90, 118, 207].

Apart from the aggregation process, the information measures such as entropy, similarity, distance or correlation measures also a key attraction of the many researchers from a different point of view. In that direction, Szmidt and Kacprzyk [146] defined the concept of a distance measure between the IFSs. Hung and Yang [80] presented the similarity measures between the two different IFSs based on Hausdorff distance. Szmidt and Kacprzyk [147] presented the distance and similarity measures between the IFSs. Grzegorzewski [61] presented the distance measures between the IFSs and IVIFSs based on the Hausdorff metric. Hung and Yang [81] presented the similarity measures between the IFSs. Xu et al. [206] presented the clustering algorithm based on the correlation coefficient for different IFSs. Ejegwa and Modom [34] presented some new type of distance measures for different IFSs and applied them to solve the medical diagnosis problems. Garg et al. [53] presented a generalized entropy measure of order α and degree β under the IFS environment and applied to solve the DM problems. Singh and Garg [138] developed the distance measures between the type-2 intuitionistic fuzzy sets. Garg [46] presented the distance and similarity measures for the intuitionistic multiplicative preference relation and apply it to solve the DM problems from the field of pattern recognition and medical diagnosis. Arora and Garg [3] presented some correlation coefficients under the intuitionistic fuzzy soft set environment. Apart from these in the past few decades, the DM problems have gained great attention from the researchers [32, 55, 129, 148, 172, 205] and different algorithms have been proposed.

To enrich the theories of the similarity measures under the intuitionistic fuzzy sets and by keeping the advantages of the CNs of the SPA theory, this chapter develops the families of the similarity measures based on the connection number of the SPA to solve the DM problems in which preferences related to different alternatives are taken in the form of the IFNs. In order to achieve it, firstly the shortcoming of the existing similarities measures under the IFSs environment has been highlighted and then connection numbers corresponding to it have been proposed for overcoming these shortcomings. The various desirable properties of the proposed measures are investigated in details. Finally, the proposed measures are applied to the DM process and the advantages of the proposed measures over the existing measure under IFS environment have been discussed.

5.2 Similarity measures between the CNSs

In this section, we define the similarity measures for the CNSs based on the distance model and set-theoretic approach. For it, let $F(\mathcal{X})$ be the collections of the CNS over the universal set $\mathcal{X}$.

5.2.1 Based on the distance model

Let $\mathcal{A} = \{(x_t; a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_{\mathcal{A}}(x_t)j) \mid x_t \in \mathcal{X}\}$ and $\mathcal{B} = \{(x_t; a_{\mathcal{B}}(x_t) + b_{\mathcal{B}}(x_t)i + c_{\mathcal{B}}(x_t)j) \mid x_t \in \mathcal{X}\}$ be two CNSs. Then, the similarity measure, denoted by $\mathcal{S}_1$, between them is defined as follows :

$$\mathcal{S}_1(\mathcal{A},\mathcal{B}) = \frac{1}{n}\sum_{t=1}^{n}\left(1 - \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\}\right) \tag{5.1}$$

Proposition 5.2.1. Let $\mathcal{A},\mathcal{B},\mathcal{C} \in F(\mathcal{X})$ be three CNSs defined on $\mathcal{X}$, then the function $\mathcal{S}_1$ defined in Eq. (5.1) satisfy the following properties:

(P1) $0 \le \mathcal{S}_1(\mathcal{A},\mathcal{B}) \le 1$.

(P2) $\mathcal{S}_1(\mathcal{A},\mathcal{B}) = 1 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{S}_1(\mathcal{A},\mathcal{B}) = \mathcal{S}_1(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{S}_1(\mathcal{A},\mathcal{B}) \ge \mathcal{S}_1(\mathcal{A},\mathcal{C})$ and $\mathcal{S}_1(\mathcal{B},\mathcal{C}) \ge \mathcal{S}_1(\mathcal{A},\mathcal{C})$.

Proof. Let $\mathcal{A} = \{(x_t; a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_{\mathcal{A}}(x_t)j) \mid x_t \in \mathcal{X}\}$, $\mathcal{B} = \{(x_t; a_{\mathcal{B}}(x_t) + b_{\mathcal{B}}(x_t)i + c_{\mathcal{B}}(x_t)j) \mid x_t \in \mathcal{X}\}$ and $\mathcal{C} = \{(x_t; a_{\mathcal{C}}(x_t) + b_{\mathcal{C}}(x_t)i + c_{\mathcal{C}}(x_t)j) \mid x_t \in \mathcal{X}\}$ be three CNSs defined on $\mathcal{X}$.

(P1) As $\mathcal{A}$ and $\mathcal{B}$ are CNSs, so we have $0 \leq |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| \leq 1$, $0 \leq |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| \leq 1$ and $0 \leq |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \leq 1$ which implies that

$$\begin{aligned} & 0 \leq \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\} \leq 1 \\ \Leftrightarrow \quad & 0 \leq \frac{1}{n}\sum_{t=1}^{n}\left(1 - \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\}\right) \leq 1 \end{aligned}$$

Therefore, $0 \leq \mathcal{S}_1(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) For two CNSs $\mathcal{A}$ and $\mathcal{B}$. Assume that $\mathcal{S}_1(\mathcal{A}, \mathcal{B}) = 1$, then

$$\begin{aligned} \Leftrightarrow \quad & \sum_{t=1}^{n}\left(1 - \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\}\right) = n \\ \Leftrightarrow \quad & \sum_{t=1}^{n}\frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\} = 0 \\ \Leftrightarrow \quad & a_{\mathcal{A}}(x_t) = a_{\mathcal{B}}(x_t), b_{\mathcal{A}}(x_t) = b_{\mathcal{B}}(x_t), c_{\mathcal{A}}(x_t) = c_{\mathcal{B}}(x_t) \quad \text{for all } t \\ \Leftrightarrow \quad & \mathcal{A} = \mathcal{B}. \end{aligned}$$

(P3) Since for any two numbers a and b, we have $|a - b| = |b - a|$. Thus, we set

$$\begin{aligned} \mathcal{S}_1(\mathcal{A}, \mathcal{B}) &= \frac{1}{n}\sum_{t=1}^{n}\left(1 - \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\}\right) \\ &= \frac{1}{n}\sum_{t=1}^{n}\left(1 - \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{B}}(x_t) - a_{\mathcal{A}}(x_t) \mid + \mid b_{\mathcal{B}}(x_t) - b_{\mathcal{A}}(x_t) \mid \\ + \mid c_{\mathcal{B}}(x_t) - c_{\mathcal{A}}(x_t) \mid \end{array}\right\}\right) \\ &= \mathcal{S}_1(\mathcal{B}, \mathcal{A}) \end{aligned}$$

(P4) Since $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, therefore $a_{\mathcal{A}}(x_t) \leq a_{\mathcal{B}}(x_t) \leq a_{\mathcal{C}}(x_t), b_{\mathcal{A}}(x_t) \geq b_{\mathcal{B}}(x_t) \geq b_{\mathcal{C}}(x_t)$ and $c_{\mathcal{A}}(x_t) \geq c_{\mathcal{B}}(x_t) \geq c_{\mathcal{C}}(x_t)$ for all $x_t \in \mathcal{X}$ which implies that $\mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid \leq \mid$

$a_{\mathcal{A}}(x_t) - a_{\mathcal{C}}(x_t) \mid$, $\mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \leq \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{C}}(x_t) \mid$ and $\mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \leq \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{C}}(x_t) \mid$. Therefore, we have

$$\begin{aligned}
& \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\} \\
& \qquad \leq \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{C}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{C}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{C}}(x_t) \mid \end{array}\right\} \\
\Rightarrow \quad & 1 - \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\} \\
& \qquad \geq 1 - \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{C}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{C}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{C}}(x_t) \mid \end{array}\right\} \\
\Rightarrow \quad & \mathcal{S}_1(\mathcal{A}, \mathcal{B}) \geq \mathcal{S}_1(\mathcal{A}, \mathcal{C}).
\end{aligned}$$

Similarly $\mathcal{S}_1(\mathcal{B}, \mathcal{C}) \geq \mathcal{S}_1(\mathcal{A}, \mathcal{C})$. Hence $\mathcal{S}_1$ is the valid similarity measure. □

Example 5.2.1. Let $\mathcal{A} = \{(x_1; 0.2+0.3i+0.5j), (x_2; 0.4+0.3i+0.3j), (x_3; 0.6+0.1i+0.3j)\}$ and $\mathcal{B} = \{(x_1; 0.4+0.1i+0.5j), (x_2; 0.3+0.5i+0.2j), (x_3; 0.7+0.2i+0.1j)\}$ be two CNSs. Then we have

$$\begin{aligned}
\mathcal{S}_1(\mathcal{A}, \mathcal{B}) &= \frac{1}{3}\left(\begin{array}{c} 1 - \frac{1}{3}\Big\{|0.2 - 0.4| + |0.3 - 0.1| + |0.5 - 0.5|\Big\} \\ +1 - \frac{1}{3}\Big\{|0.4 - 0.3| + |0.3 - 0.5| + |0.3 - 0.2|\Big\} \\ +1 - \frac{1}{3}\Big\{|0.6 - 0.7| + |0.1 - 0.2| + |0.3 - 0.1|\Big\} \end{array}\right) \\
&= 0.8666
\end{aligned}$$

5.2.2 Based on the set-theoretic approach

Now, here we defined the another similarity measure $\mathcal{S}_2$ between the CNSs based on the set theoretic approach for a collection of the two CNSs $\mathcal{A}$ and $\mathcal{B}$ as follows

$$\mathcal{S}_2(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{t=1}^{n}\left(\begin{array}{c}\min\left(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)\right) + \min\left(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)\right)\\ + \min\left(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t)\right)\end{array}\right)}{\sum\limits_{t=1}^{n}\left(\begin{array}{c}\max(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \max(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t))\\ + \max(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t))\end{array}\right)} \tag{5.2}$$

Proposition 5.2.2. The similarity measure $\mathcal{S}_2$ between the CNSs $\mathcal{A}$ and $\mathcal{B}$ satisfies the following properties for $\mathcal{A},\mathcal{B},\mathcal{C} \in F(\mathcal{X})$:

(P1) $0 \leq \mathcal{S}_2(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{S}_2(\mathcal{A},\mathcal{B}) = 1 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{S}_2(\mathcal{A},\mathcal{B}) = \mathcal{S}_1(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{S}_2(\mathcal{A},\mathcal{B}) \geq \mathcal{S}_2(\mathcal{A},\mathcal{C})$ and $\mathcal{S}_2(\mathcal{B},\mathcal{C}) \geq \mathcal{S}_2(\mathcal{A},\mathcal{C})$.

Proof. Let $\mathcal{A},\mathcal{B},\mathcal{C} \in F(\mathcal{X})$ be three CNSs.

(P1) By definition of CNSs, we have $0 \leq a_{\mathcal{A}}(x_t)$, $a_{\mathcal{B}}(x_t)$, $a_{\mathcal{C}}(x_t)$, $b_{\mathcal{A}}(x_t)$, $b_{\mathcal{B}}(x_t)$, $b_{\mathcal{C}}(x_t)$, $c_{\mathcal{A}}(x_t)$, $c_{\mathcal{B}}(x_t)$, $c_{\mathcal{C}}(x_t) \leq 1$. Therefore, $\min(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) \leq \max(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t))$, $\min(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) \leq \max(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t))$ and $\min(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t)) \leq \max(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t))$. Therefore, we have

$$0 \leq \frac{\min(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \min(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) + \min(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t))}{\max(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \max(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) + \max(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t))} \leq 1$$

and hence $0 \leq \mathcal{S}_2(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) Let

$$\begin{aligned} & \mathcal{S}_2(\mathcal{A},\mathcal{B}) = 1 \\ \Leftrightarrow\ & \frac{\sum\limits_{t=1}^{n}\Big(\min(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \min(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) + \min(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t))\Big)}{\sum\limits_{t=1}^{n}\Big(\max(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \max(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) + \max(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t))\Big)} = 1 \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow\ \min(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \min(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) + \min(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t)) \\
&\qquad\qquad = \max(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \max(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) + \max(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t)) \\
&\Leftrightarrow\ a_{\mathcal{A}}(x_t) = a_{\mathcal{B}}(x_t), b_{\mathcal{A}}(x_t) = b_{\mathcal{B}}(x_t), c_{\mathcal{A}}(x_t) = c_{\mathcal{B}}(x_t) \\
&\Leftrightarrow\ \mathcal{A} = \mathcal{B}.
\end{aligned}$$

(P3) Since, we have

$$\begin{aligned}
\mathcal{S}_2(\mathcal{A},\mathcal{B}) &= \frac{\sum_{t=1}^{n}\begin{pmatrix} \min(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \min(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) \\ + \min(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t)) \end{pmatrix}}{\sum_{t=1}^{n}\begin{pmatrix} \max(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \max(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) \\ + \max(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t)) \end{pmatrix}} \\
&= \frac{\sum_{t=1}^{n}\begin{pmatrix} \min(a_{\mathcal{B}}(x_t), a_{\mathcal{A}}(x_t)) + \min(b_{\mathcal{B}}(x_t), b_{\mathcal{A}}(x_t)) \\ + \min(c_{\mathcal{B}}(x_t), c_{\mathcal{A}}(x_t)) \end{pmatrix}}{\sum_{t=1}^{n}\begin{pmatrix} \max(a_{\mathcal{B}}(x_t), a_{\mathcal{A}}(x_t)) + \max(b_{\mathcal{B}}(x_t), b_{\mathcal{A}}(x_t)) \\ + \max(c_{\mathcal{B}}(x_t), c_{\mathcal{A}}(x_t)) \end{pmatrix}} \\
&= \mathcal{S}_2(\mathcal{B},\mathcal{A})
\end{aligned}$$

(P4) Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $a_{\mathcal{A}}(x_t) \leq a_{\mathcal{B}}(x_t) \leq a_{\mathcal{C}}(x_t), b_{\mathcal{A}}(x_t) \geq b_{\mathcal{B}}(x_t) \geq b_{\mathcal{C}}(x_t)$ and $c_{\mathcal{A}}(x_t) \geq c_{\mathcal{B}}(x_t) \geq c_{\mathcal{C}}(x_t), \forall x_t \in \mathcal{X}$. It follows that

$$\begin{aligned}
&\frac{\min(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \min(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) + \min(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t))}{\max(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \max(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) + \max(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t))} \\
&\qquad = \frac{a_{\mathcal{A}}(x_t) + b_{\mathcal{B}}(x_t) + c_{\mathcal{B}}(x_t)}{a_{\mathcal{B}}(x_t) + b_{\mathcal{A}}(x_t) + c_{\mathcal{A}}(x_t)} \\
\text{and}\ &\frac{\min(a_{\mathcal{A}}(x_t), a_{\mathcal{C}}(x_t)) + \min(b_{\mathcal{A}}(x_t), b_{\mathcal{C}}(x_t)) + \min(c_{\mathcal{A}}(x_t), c_{\mathcal{C}}(x_t))}{\max(a_{\mathcal{A}}(x_t), a_{\mathcal{C}}(x_t)) + \max(b_{\mathcal{A}}(x_t), b_{\mathcal{C}}(x_t)) + \max(c_{\mathcal{A}}(x_t), c_{\mathcal{C}}(x_t))} \\
&\qquad = \frac{a_{\mathcal{A}}(x_t) + b_{\mathcal{C}}(x_t) + c_{\mathcal{C}}(x_t)}{a_{\mathcal{C}}(x_t) + b_{\mathcal{A}}(x_t) + c_{\mathcal{A}}(x_t)} \\
\Rightarrow\ &\frac{a_{\mathcal{A}}(x_t) + b_{\mathcal{B}}(x_t) + c_{\mathcal{B}}(x_t)}{a_{\mathcal{B}}(x_t) + b_{\mathcal{A}}(x_t) + c_{\mathcal{A}}(x_t)} \geq \frac{a_{\mathcal{A}}(x_t) + b_{\mathcal{C}}(x_t) + c_{\mathcal{C}}(x_t)}{a_{\mathcal{C}}(x_t) + b_{\mathcal{A}}(x_t) + c_{\mathcal{A}}(x_t)} \\
\Rightarrow\ &\mathcal{S}_2(\mathcal{A},\mathcal{B}) \geq \mathcal{S}_2(\mathcal{A},\mathcal{C})
\end{aligned}$$

Similarly, we can obtain $\mathcal{S}_2(\mathcal{B},\mathcal{C}) \geq \mathcal{S}_2(\mathcal{A},\mathcal{C})$.

Hence $\mathcal{S}_2$ is a valid measure. □

However, there are various practical problems in which the weight of the parameter plays a dominant role during the DM process. In order to handle it, weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_t)$ of the parameters are taken in to the account such that $\omega_t > 0$, $\sum\limits_{t=1}^{n} \omega_t = 1$. Then, we define the weighted similarity measures for measuring the different CNSs as follows :

(i) Weighted similarity measure based on the distance model :

$$\mathcal{S}_1^{\omega}(\mathcal{A},\mathcal{B}) = \sum_{t=1}^{n} \omega_t \left(1 - \frac{1}{3}\left\{\begin{array}{c} \mid a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t) \mid + \mid b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t) \mid \\ + \mid c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t) \mid \end{array}\right\}\right) \tag{5.3}$$

(ii) Weighted similarity measure based on the set theoretic approach :

$$\mathcal{S}_2^{\omega}(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{t=1}^{n} \omega_t \left(\begin{array}{c} \min(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \min(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) \\ + \min(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t)) \end{array}\right)}{\sum\limits_{t=1}^{n} \omega_t \left(\begin{array}{c} \max(a_{\mathcal{A}}(x_t), a_{\mathcal{B}}(x_t)) + \max(b_{\mathcal{A}}(x_t), b_{\mathcal{B}}(x_t)) \\ + \max(c_{\mathcal{A}}(x_t), c_{\mathcal{B}}(x_t)) \end{array}\right)} \tag{5.4}$$

Especially, when $\omega_t = \frac{1}{n}, \forall t = 1, 2, \ldots, n$, then Eqs. (5.3) and (5.4) reduce to Eqs. (5.1) and (5.2) respectively.

Proposition 5.2.3. The similarity measures $\mathcal{S}_1^{\omega}$ defined in the Eq. (5.3) also satisfies the following properties for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in F(\mathcal{X})$:

(P1) $0 \leq \mathcal{S}_1^{\omega}(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{S}_1^{\omega}(\mathcal{A},\mathcal{B}) = 1 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{S}_1^{\omega}(\mathcal{A},\mathcal{B}) = \mathcal{S}_1^{\omega}(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{S}_1^{\omega}(\mathcal{A},\mathcal{B}) \geq \mathcal{S}_1^{\omega}(\mathcal{A},\mathcal{C})$ and $\mathcal{S}_1^{\omega}(\mathcal{B},\mathcal{C}) \geq \mathcal{S}_1^{\omega}(\mathcal{A},\mathcal{C})$.

Proof. Since ω_t is the weight vector of the x_t such that $\omega_t \in (0,1]$ and $0 \leq |a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| \leq 1, 0 \leq |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| \leq 1$, and $0 \leq |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)| \leq 1$. Thus, we have $0 \leq \frac{1}{3}\Big\{|a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|\Big\} \leq 1 \Rightarrow 0 \leq \Big(1 - \frac{1}{3}\Big\{|a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| + |b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|\Big\}\Big) \leq 1 \Rightarrow 0 \leq \sum\limits_{t=1}^{n} \omega_t \Big(1 - \frac{1}{3}\Big\{|a_{\mathcal{A}}(x_t) - a_{\mathcal{B}}(x_t)| +$

$|b_{\mathcal{A}}(x_t) - b_{\mathcal{B}}(x_t)| + |c_{\mathcal{A}}(x_t) - c_{\mathcal{B}}(x_t)|\Big\}\Big) \leq \sum\limits_{t=1}^{n} \omega_t = 1 \Rightarrow 0 \leq \mathcal{S}_1^{\omega}(\mathcal{A}, \mathcal{B}) \leq 1$. Hence (P1) satisfies. The other properties can be easily proved as the Proposition 5.2.1. ☐

Proposition 5.2.4. The weighted similarity measures $\mathcal{S}_2^{\omega}$ satisfies the following properties.

(P1) $0 \leq \mathcal{S}_2^{\omega}(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{S}_2^{\omega}(\mathcal{A}, \mathcal{B}) = 1 \Leftrightarrow \mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{S}_2^{\omega}(\mathcal{A}, \mathcal{B}) = \mathcal{S}_2^{\omega}(\mathcal{B}, \mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{S}_2^{\omega}(\mathcal{A}, \mathcal{B}) \geq \mathcal{S}_2^{\omega}(\mathcal{A}, \mathcal{C})$ and $\mathcal{S}_2^{\omega}(\mathcal{B}, \mathcal{C}) \geq \mathcal{S}_2^{\omega}(\mathcal{A}, \mathcal{C})$.

Proof. Omitted. ☐

5.3 Drawbacks of the existing measures and superiority of the proposed measures

In this section, we have presented some counterexamples in order to show that existing similarity measures under IFS environment, defined in Section 2.3.2 of Chapter 2, are unable to rank the different alternatives. Further, we have validated it by the proposed similarity measures that it overcomes the flaws of the existing methods and able to the rank the alternatives.

Example 5.3.1. Consider two unknown patterns $\mathcal{A}$ and $\mathcal{B}$ defined over the $\mathcal{X} = \{x_1, x_2, \ldots, x_5\}$ as

$$\begin{aligned}
\mathcal{A} &= \{(x_1; 0.5, 0.3), (x_2; 0.5, 0.2), (x_3; 0.9, 0.0), (x_4; 0.5, 0.4), (x_5; 0.7, 0.1)\} \\
\mathcal{B} &= \{(x_1; 0.7, 0.2), (x_2; 0.5, 0.4), (x_3; 0.9, 0.1), (x_4; 0.6, 0.3), (x_5; 0.8, 0.0)\}
\end{aligned}$$

Suppose known pattern $\mathcal{P}$ is represented by $\mathcal{P} = \{(x_1; 0.7,\ 0.1), (x_2; 0.6,\ 0.3), (x_3; 0.7,\ 0.1), (x_4; 0.5,\ 0.4), (x_5; 0.4,\ 0.5)\}$, then the target of this problem is to classify the known pattern $\mathcal{P}$ with the unknown patterns.

For it, if we apply similarity measure $\mathcal{S}_1'$ as defined in Eq. (2.22) of Chapter 2, then we get $\mathcal{S}_1'(\mathcal{P},\mathcal{A}) = \mathcal{S}_1'(\mathcal{P},\mathcal{B}) = 0.80 \Rightarrow \mathcal{A} = B$. But $\mathcal{A} \neq \mathcal{B}$, which is illogical. On the other hand, if apply the proposed similarity measures $\mathcal{S}_1$ and $\mathcal{S}_2$ to it then, we are able to get the proper ordering between $\mathcal{A}$ and $\mathcal{B}$. This has been explained as below.

For it, firstly, CNSs corresponding to the IFSs $\mathcal{A}, \mathcal{B}$ and $\mathcal{P}$ are formulated by using the Eq. (4.1) of Chapter 4 and thus we get

$$\begin{aligned}
\mathcal{A} &= \left\{ \begin{array}{l} (x_1; 0.35 + 0.50i + 0.15j), (x_2; 0.40 + 0.50i + 0.10j), (x_3; 0.90 + 0.10i + 0.00j), \\ (x_4; 0.30 + 0.50i + 0.20j), (x_5; 0.63 + 0.34i + 0.03j) \end{array} \right\} \\
\mathcal{B} &= \left\{ \begin{array}{l} (x_1; 0.56 + 0.38i + 0.06j), (x_2; 0.30 + 0.50i + 0.20j), (x_3; 0.81 + 0.18i + 0.01j), \\ (x_4; 0.42 + 0.46i + 0.12j), (x_5; 0.80 + 0.20i + 0.00j) \end{array} \right\} \\
\mathcal{P} &= \left\{ \begin{array}{l} (x_1; 0.63 + 0.34i + 0.03j), (x_2; 0.42 + 0.46i + 0.12j), (x_3; 0.63 + 0.34i + 0.03j), \\ (x_4; 0.30 + 0.50i + 0.20j), (x_5; 0.20 + 0.50i + 0.30j) \end{array} \right\}
\end{aligned}$$

Therefore, by using Eq. (5.1), we have $\mathcal{S}_1(\mathcal{P},\mathcal{A}) = 0.8640$, $\mathcal{S}_1(\mathcal{P},\mathcal{B}) = 0.8547$. Hence, $\mathcal{A} \succ \mathcal{B}$ where "$\succ$" refers "preferred to", and concluded that the known pattern $\mathcal{P}$ classifies with $\mathcal{A}$. Furthermore, if we apply similarity measure $\mathcal{S}_2$ on it, as defined in Eq. (5.2), then we get $\mathcal{S}_2(\mathcal{P},\mathcal{A}) = 0.6611$, $\mathcal{S}_2(\mathcal{P},\mathcal{B}) = 0.6420$ and hence known pattern $\mathcal{P}$ classifies with $\mathcal{A}$.

Example 5.3.2. Consider three unknown pattern $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ defined over the $X = \{x_1, x_2, x_3\}$ as

$$\begin{aligned}
\mathcal{A} &= \{(x_1; 0.4, 0.5), (x_2; 0.7, 0.1), (x_3; 0.3, 0.3)\} \\
\mathcal{B} &= \{(x_1; 0.5, 0.4), (x_2; 0.7, 0.2), (x_3; 0.4, 0.3)\} \\
\text{and} \quad \mathcal{C} &= \{(x_1; 0.4, 0.5), (x_2; 0.7, 0.1), (x_3; 0.4, 0.3)\}
\end{aligned}$$

Suppose known pattern $\mathcal{P}$ is represented by $\mathcal{P} = \{(x_1; 0.1, 0.1), (x_2; 1.0, 0.0), (x_3; 0.0, 1.0)\}$, then target is to classifies the unknown patterns $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ with the known pattern $\mathcal{P}$. For it, if we apply similarity measures $\mathcal{S}_2'$ and $\mathcal{S}_3'$ as defined in Eqs. (2.23) and (2.24) of Chapter 2 respectively, then we get $\mathcal{S}_2'(\mathcal{P},\mathcal{A}) = \mathcal{S}_2'(\mathcal{P},\mathcal{B}) = \mathcal{S}_2'(\mathcal{P},\mathcal{C}) = 0.2766$ and $\mathcal{S}_3'(\mathcal{P},\mathcal{A}) = \mathcal{S}_3'(\mathcal{P},\mathcal{B}) = \mathcal{S}_3'(\mathcal{P},\mathcal{C}) = 0.4398$ which implies that $\mathcal{A} = \mathcal{B} = \mathcal{C}$. But clearly seen from the data sets that $\mathcal{A} \neq \mathcal{B} \neq \mathcal{C}$, and hence the existing measures are unable to choose the best one(s).

On the other hand, if apply the proposed similarity measures $\mathcal{S}_1$ and $\mathcal{S}_2$ to compute the classifying pattern with $\mathcal{P}$ then, for it, firstly we convert these IFSs into CNSs by using the Eq. (4.1) of Chapter 4 and get

$$\begin{aligned}
\mathcal{A} &= \{(x_1; 0.20 + 0.50i + 0.30j), (x_2; 0.63 + 0.34i + 0.03j), (x_3; 0.21 + 0.58i + 0.21j)\} \\
\mathcal{B} &= \{(x_1; 0.30 + 0.50i + 0.20j), (x_2; 0.56 + 0.38i + 0.06j), (x_3; 0.28 + 0.54i + 0.18j)\} \\
\mathcal{C} &= \{(x_1; 0.20 + 0.50i + 0.30j), (x_2; 0.63 + 0.34i + 0.03j), (x_3; 0.28 + 0.54i + 0.18j)\} \\
\mathcal{P} &= \{(x_1; 0.09 + 0.82i + 0.09j), (x_2; 1.00 + 0.00i + 0.00j), (x_3; 0.00 + 0.00i + 1.00j)\}.
\end{aligned}$$

Thus by using Eq. (5.1), we get $\mathcal{S}_1(\mathcal{P}, \mathcal{A}) = 0.6711$, $\mathcal{S}_1(\mathcal{P}, \mathcal{B}) = 0.6489$, $\mathcal{S}_1(\mathcal{P}, \mathcal{C}) = 0.6644$ and hence $\mathcal{A} \succ \mathcal{C} \succ \mathcal{B}$ where "$\succ$" refers "preferred to". Therefore, the known pattern $\mathcal{P}$ classifies with $\mathcal{A}$. However, if we apply Eq. (5.2) on it then their corresponding measures values are $\mathcal{S}_2(\mathcal{P}, \mathcal{A}) = 0.3393$, $\mathcal{S}_2(\mathcal{P}, \mathcal{B}) = 0.3100$, $\mathcal{S}_2(\mathcal{P}, \mathcal{C}) = 0.3304$ and hence known pattern $\mathcal{P}$ classifies with $\mathcal{A}$.

Example 5.3.3. Consider two unknown patterns $\mathcal{A}$ and $\mathcal{B}$ defined over the $\mathcal{X} = \{x_1, x_2, \ldots, x_5\}$ as

$$\mathcal{A} = \{(x_1; 0.4, 0.5), (x_2; 0.4, 0.4), (x_3; 0.7, 0.0), (x_4; 0.4, 0.6), (x_5; 0.6, 0.3)\}$$

and

$$\mathcal{B} = \{(x_1; 0.6, 0.4), (x_2; 0.4, 0.6), (x_3; 0.9, 0.1), (x_4; 0.5, 0.5), (x_5; 0.7, 0.2)\}$$

whose target is to choose the best one among them which classify with the known pattern $\mathcal{P}$ whose representation is given by $\mathcal{P} = \{(x_1; 0.7, 0.1), (x_2; 0.6, 0.3), (x_3; 0.7, 0.1), (x_4; 0.5, 0.4), (x_5; 0.4, 0.5)\}$. Now, if we apply existing similarity measure $\mathcal{S}'_4$ defined in Eq. (2.25) of Chapter 2, then we get $\mathcal{S}'_4(\mathcal{P}, \mathcal{A}) = \mathcal{S}'_4(\mathcal{P}, \mathcal{B}) = 0.8200$. As both measurement values are equal and hence $\mathcal{A} = \mathcal{B}$. But $\mathcal{A} \neq \mathcal{B}$, which is illogical.

On the other hand, if we obtain the CNSs corresponding to these patterns $\mathcal{A}, \mathcal{B}$ and

$\mathcal{P}$ then we get

$$\mathcal{A} = \left\{ \begin{array}{l} (x_1; 0.20 + 0.50i + 0.30j), (x_2; 0.24 + 0.52i + 0.24j), (x_3; 0.70 + 0.30i + 0.00j), \\ (x_4; 0.16 + 0.48i + 0.36j), (x_5; 0.42 + 0.46i + 0.12j) \end{array} \right\}$$

$$\mathcal{B} = \left\{ \begin{array}{l} (x_1; 0.36 + 0.48i + 0.16j), (x_2; 0.16 + 0.48i + 0.36j), (x_3; 0.81 + 0.18i + 0.01j), \\ (x_4; 0.25 + 0.50i + 0.25j), (x_5; 0.56 + 0.38i + 0.06j) \end{array} \right\}$$

and

$$\mathcal{P} = \left\{ \begin{array}{l} (x_1; 0.35 + 0.50i + 0.15j), (x_2; 0.48 + 0.44i + 0.08j), (x_3; 0.35 + 0.50i + 0.15j), \\ (x_4; 0.20 + 0.50i + 0.30j), (x_5; 0.56 + 0.38i + 0.06j) \end{array} \right\}$$

Thus by using Eq. (5.1), we get $\mathcal{S}_1(\mathcal{P}, \mathcal{A}) = 0.8587$, $\mathcal{S}_1(\mathcal{P}, \mathcal{B}) = 0.8507$ and hence $\mathcal{A} \succ \mathcal{B}$, which implies that the known pattern $\mathcal{P}$ classifies with $\mathcal{A}$. Also, proposed similarity measure $\mathcal{S}_2$ on this data given in Eq. (5.2), we get $\mathcal{S}_2(\mathcal{P}, \mathcal{A}) = 0.6502$, $\mathcal{S}_2(\mathcal{P}, \mathcal{B}) = 0.6340$ and hence known pattern $\mathcal{P}$ classifies with $\mathcal{A}$.

5.4 Proposed MADM method based on similarity measures

The general description of the MADM problem is given in the Section 2.7 of Chapter 2 under the IFS environment. In the following, we develop an approach based on the proposed measures with intuitionistic fuzzy information, which involve the following steps:

Step 1: Arrange the information about the alternatives in a decision matrix $\mathcal{M}$ as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \cdots & \mathcal{G}_n \end{array} \\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \end{array} \tag{5.5}$$

Step 2: Construct its equivalent CN matrix for decision matrix $\mathcal{M}$ by using Eq. (4.1) of Chapter 4.

Step 3: Utilize the proposed similarity measures '$\mathcal{S}$' to compute the overall value of each alternative.

Step 4: Rank the alternatives by finding the maximum value of $r_k = \arg \max\limits_{1 \le k \le m} \{\mathcal{S}\}$ and select the best one(s).

5.5 Illustrative examples

In this section, the above-defined similarity measures based method has been applied to solve a real-life problem under the IFS environment and obtained results have been compared with the other existing results. A case study from the field of career determination of the student has been taken for an illustrative purpose which can be read as:

5.5.1 A Case study related to the career determination of a student

The essence of providing sufficient information of students for proper career choice cannot be overemphasized. This is dominant because the several problems in the lack of proper career guidance faced by students are of great significance in their career choice and efficiency. To face these problems, a secondary student takes the help of the student counseling center which provides the sufficient information to a student on career determination and helps them to choose a suitable career path in which student can highest grow. Consider that a counseling center taken the five different subjects namely English$(\mathcal{Q}_1)$, Mathematics$(\mathcal{Q}_2)$, Biology$(\mathcal{Q}_3)$, Physics$(\mathcal{Q}_4)$, and Chemistry$(\mathcal{Q}_5)$ related to the careers determination in the field of Medicine$(\mathcal{J}_1)$, Pharmacy$(\mathcal{J}_2)$, Surgery$(\mathcal{J}_3)$, and Anatomy$(\mathcal{J}_4)$. Assume that an unknown student $\mathcal{Y}$ goes to the counseling center for taking the help to choose his suitable career. The aim of the problem is to determine the most expected career of the student $\mathcal{Y}$ in the $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ and $\mathcal{J}_4$. In order to achieve it, the following steps of the proposed approach are utilized which are summarized as follows.

Step 1: An expert gives their preference to each subject related to the career goals under the IFN and their corresponding intuitionistic fuzzy decision matrix has been summarized in Eq. (5.6) as follows.

$$\mathcal{D} = \begin{array}{c} \\ \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \mathcal{J}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 & \mathcal{Q}_4 & \mathcal{Q}_5 \end{array} \\ \begin{pmatrix} (0.8,0.1) & (0.7,0.2) & (0.9,0.0) & (0.6,0.3) & (0.8,0.1) \\ (0.9,0.1) & (0.8,0.0) & (0.8,0.1) & (0.5,0.3) & (0.7,0.2) \\ (0.5,0.3) & (0.5,0.2) & (0.9,0.0) & (0.5,0.4) & (0.7,0.1) \\ (0.7,0.2) & (0.5,0.4) & (0.9,0.1) & (0.6,0.3) & (0.8,0.0) \end{pmatrix} \end{array} \tag{5.6}$$

Also, an unknown student $\mathcal{Y}$ give their rating values towards the interest of the subject $\mathcal{Q}_t (t = 1, 2, \ldots, 5)$ in the form of IFN which are summarized as below

$$\mathcal{Y} = \{(\mathcal{Q}_1; 0.6, 0.3), (\mathcal{Q}_2; 0.5, 0.4), (\mathcal{Q}_3; 0.6, 0.2), (\mathcal{Q}_4; 0.5, 0.3), (\mathcal{Q}_5; 0.5, 0.5)\}$$

Step 2: CNSs corresponding to the preferences of the decision matrix $\mathcal{D}$ and student $\mathcal{Y}$ are formulated by Eq. (4.1) of Chapter 4 and their results are summarized as follows:

$$\mathcal{M} = \left(\begin{array}{cccc} & \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 \\ \mathcal{J}_1 & 0.72 + 0.26i + 0.02j & 0.56 + 0.38i + 0.06j & 0.90 + 0.10i + 0.00j \\ \mathcal{J}_2 & 0.81 + 0.18i + 0.01j & 0.72 + 0.26i + 0.02j & 0.72 + 0.26i + 0.02j \\ \mathcal{J}_3 & 0.35 + 0.50i + 0.15j & 0.40 + 0.50i + 0.10j & 0.90 + 0.10i + 0.00j \\ \mathcal{J}_4 & 0.56 + 0.38i + 0.06j & 0.30 + 0.50i + 0.20j & 0.81 + 0.18i + 0.01j \\ & \mathcal{Q}_4 & \mathcal{Q}_5 & \\ \mathcal{J}_1 & 0.42 + 0.46i + 0.12j & 0.72 + 0.26i + 0.02j & \\ \mathcal{J}_2 & 0.35 + 0.50i + 0.15j & 0.56 + 0.38i + 0.06j & \\ \mathcal{J}_3 & 0.30 + 0.50i + 0.20j & 0.63 + 0.34i + 0.03j & \\ \mathcal{J}_4 & 0.42 + 0.46i + 0.12j & 0.80 + 0.20i + 0.00j & \end{array}\right)$$

and

$$\mathcal{Y} = \left\{ \begin{array}{c} (\mathcal{Q}_1; 0.42 + 0.46i + 0.12j), (\mathcal{Q}_2; 0.30 + 0.50i + 0.20j), (\mathcal{Q}_3; 0.48 + 0.44i + 0.08j), \\ (\mathcal{Q}_4; 0.35 + 0.50i + 0.15j), (\mathcal{Q}_5; 0.25 + 0.50i + 0.25j) \end{array} \right\}$$

Step 3a: By using the proposed similarity measures $\mathcal{S}_1$ between $\mathcal{Y}$ and $\mathcal{J}_h (h = 1, 2, 3, 4)$ given in Eq. (5.1), the measurement values are obtained as $\mathcal{S}_1(\mathcal{J}_1, \mathcal{Y}) = 0.7973$, $\mathcal{S}_1(\mathcal{J}_2, \mathcal{Y}) = 0.8187$, $\mathcal{S}_1(\mathcal{J}_3, \mathcal{Y}) = 0.8640$ and $\mathcal{S}_1(\mathcal{J}_4, \mathcal{Y}) = 0.8547$. On the other hand, if we use similarity measure $\mathcal{S}_2$ given in Eq. (5.2), then we get their measurement values are $\mathcal{S}_2(\mathcal{J}_1, \mathcal{Y}) = 0.5337$, $\mathcal{S}_2(\mathcal{J}_2, \mathcal{Y}) = 0.5723$, $\mathcal{S}_2(\mathcal{J}_3, \mathcal{Y}) = 0.6611$ and $\mathcal{S}_2(\mathcal{J}_4, \mathcal{Y}) = 0.6420$.

Step 3b: On the other hand, if we assign the weight vector $\omega = (0.15, 0.10, 0.30, 0.20, 0.25)^T$ corresponding to $\mathcal{Q}_t(t = 1, 2, 3, 4, 5)$, then by utilizing the proposed weighted similarity measures $\mathcal{S}_1^{\omega}$ and $\mathcal{S}_2^{\omega}$ defined in the Eqs. (5.3) and (5.4), respectively, their corresponding measurement values are obtained as $\mathcal{S}_1^{\omega}(\mathcal{J}_1, \mathcal{Y}) = 0.7810$, $\mathcal{S}_1^{\omega}(\mathcal{J}_2, \mathcal{Y}) = 0.8333$, $\mathcal{S}_1^{\omega}(\mathcal{J}_3, \mathcal{Y}) = 0.8323$, $\mathcal{S}_1^{\omega}(\mathcal{J}_4, \mathcal{Y}) = 0.3190$ and $\mathcal{S}_2^{\omega}(\mathcal{J}_1, \mathcal{Y}) = 0.5055$, $\mathcal{S}_2^{\omega}(\mathcal{J}_2, \mathcal{Y}) = 0.6000$, $\mathcal{S}_2^{\omega}(\mathcal{J}_3, \mathcal{Y}) = 0.5981$, $\mathcal{S}_2^{\omega}(\mathcal{J}_4, \mathcal{Y}) = 0.5729$.

Step 4: From these computed results, we compute the ranking order of the alternatives is $\mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_1$, where "$\succ$" refers to "preferred to" when $\mathcal{S}_1$ or $\mathcal{S}_2$ measure is utilized. On the other hand, corresponding to assigned weight vector, the ranking order of the alternatives is $\mathcal{J}_2 \succ \mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_1$. Hence, we conclude the most suitable career for the student $\mathcal{Y}$ is Surgery ($\mathcal{J}_3$) or Pharmacy ($\mathcal{J}_2$) according to the use of the proposed measure.

5.5.2 Comparative studies

In order to validate the feasibility of the proposed similarity measures, a comparative analysis has been conducted by using various existing approaches [34, 81, 172, 206] under the IFS environment to the considered data. The results corresponding to these approaches are summarized below.

(i) If we apply the Hamming distance $d_1(.)$, as proposed by Wang and Xin [172], to the considered problem for finding the most suitable career of the student $\mathcal{Y}$, then their corresponding measurement values with respect to the career $\mathcal{J}_h(h = 1, 2, 3, 4)$ are $d_1(\mathcal{J}_1, \mathcal{Y}) = 0.2400$, $d_1(\mathcal{J}_2, \mathcal{Y}) = 0.2200$, $d_1(\mathcal{J}_3, \mathcal{Y}) = 0.2200$, $d_1(\mathcal{J}_4, \mathcal{Y}) = 0.2000$. Thus, the ranking corresponding to it is $\mathcal{J}_4 \succ \mathcal{J}_2 = \mathcal{J}_3 \succ \mathcal{J}_1$. From it, we conclude that $\mathcal{J}_4$ is the best one. But their approach is the limited as it can not be distinguish between the $\mathcal{J}_2$ and $\mathcal{J}_3$.

(ii) If we apply the similarity measure $\mathcal{S}'_4$ defined in the Eq. (2.25) of Chapter 2, then their measurement values are $\mathcal{S}'_4(\mathcal{J}_1, \mathcal{Y}) = 0.7900$, $\mathcal{S}'_4(\mathcal{J}_2, \mathcal{Y}) = 0.8100$, $\mathcal{S}'_4(\mathcal{J}_3, \mathcal{Y}) = 0.8500$, $\mathcal{S}'_4(\mathcal{J}_4, \mathcal{Y}) = 0.8500$. Therefore, the ranking order corresponding to it is $\mathcal{J}_3 = \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_1$. From it, we conclude that $\mathcal{J}_3$ is the best one. But their approach is also limited as it can not be distinguish between the $\mathcal{J}_3$ and $\mathcal{J}_4$.

(iii) If we apply the Euclidean distance $d_2(.)$, as proposed by Wang and Xin [172], then the measurement values are $d_2(\mathcal{J}_1, \mathcal{Y}) = 0.2408$, $d_2(\mathcal{J}_2, \mathcal{Y}) = 0.2280$, $d_2(\mathcal{J}_3, \mathcal{Y}) = 0.2236$, $d_2(\mathcal{J}_4, \mathcal{Y}) = 0.2366$. Thus, the ranking order of the alternatives is $\mathcal{J}_3 \succ \mathcal{J}_2 \succ \mathcal{J}_4 \succ \mathcal{J}_1$. From it, we conclude that $\mathcal{J}_3$ is the best one and it concide with the our proposed results.

(iv) If we apply the distance $d_3(.)$, as proposed by Ejegwa and Modom [34] to the considered data, then their measurement values of the alternatives are $d_3(\mathcal{J}_1, \mathcal{Y}) = 0.4400$, $d_3(\mathcal{J}_2, \mathcal{Y}) = 0.4000$, $d_3(\mathcal{J}_3, \mathcal{Y}) = 0.3000$, $d_3(\mathcal{J}_4, \mathcal{Y}) = 0.3200$. Thus, the ranking order corresponding to it is $\mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_1$ and hence conclude that $\mathcal{J}_3$ is the best one and it coincide with the our proposed results.

(v) If we apply the correlation coefficient $\mathcal{K}(.)$, as proposed by Xu et al. [206], then their measurement values are $\mathcal{K}(\mathcal{J}_1, \mathcal{Y}) = 0.7580$, $\mathcal{K}(\mathcal{J}_2, \mathcal{Y}) = 0.7745$, $\mathcal{K}(\mathcal{J}_3, \mathcal{Y}) = 0.8346$, $\mathcal{K}(\mathcal{J}_4, \mathcal{Y}) = 0.7808$. Thus, the ranking order corresponding to it is $\mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_1$ and conclude that $\mathcal{J}_3$ is the best one and it again coincide with the our proposed results.

Furthermore, if we assign the weight vector $\omega = (0.15, 0.10, 0.30, 0.20, 0.25)^T$ corresponding to $\mathcal{Q}_t (t = 1, 2, 3, 4, 5)$, then the comparison analysis with the existing approaches [34, 81, 172, 206] under the IFS environment is conducted to the considered data. The results corresponding to these approaches are summarized in Table 5.1. From it, we conclude that the best alternative coincides with the proposed approach results. Hence, we can say that the proposed similarity measures are able to solve the DM problem in a better way under IFS environment where the existing studies fail to rank the alternatives.

Table 5.1: Comparative study with existing approaches

	Measure values of Y with careers				
Methods	J_1	J_2	J_3	J_4	Ranking
Hamming distance [172]	0.0520	0.0420	0.0490	0.0500	$J_2 \succ J_3 \succ J_4 \succ J_1$
Euclidean distance [172]	0.1145	0.0959	0.1082	0.1200	$J_2 \succ J_3 \succ J_1 \succ J_4$
New distance [34]	0.0940	0.0740	0.0700	0.0800	$J_3 \succ J_2 \succ J_4 \succ J_1$
Similarity measure [81]	0.1555	0.1650	0.1645	0.1630	$J_2 \succ J_3 \succ J_4 \succ J_1$
Correlation coefficient [206]	0.7386	0.7943	0.7920	0.7377	$J_2 \succ J_3 \succ J_1 \succ J_4$
Proposed similarity $\mathcal{S}_1^\omega$	0.7810	0.8333	0.8323	0.8190	$J_2 \succ J_3 \succ J_4 \succ J_1$
Proposed similarity $\mathcal{S}_2^\omega$	0.5055	0.6000	0.5981	0.5729	$J_2 \succ J_3 \succ J_4 \succ J_1$

5.6 Conclusion

Measures play an important role in the DM process to compare the objects under any theory. SPA is the updated uncertainty theory, which deals with the uncertainty based on the three components, namely “identity”, “discrepancy” and “contrary” degrees of the CN. Under this environment, in this chapter, some new similarity and weighted similarity measures are proposed under the intuitionistic fuzzy set environment by using the connection number of the SPA theory. It is observed from the present study that under some certain cases, the existing measures given by the authors [34, 81, 172, 205, 206] under IFS environment are unable to classify the alternatives, but the proposed measures are well suited for them. Then, based on the proposed measures, a DM approach is presented to solve the multi-attribute DM problems. A case study from the carrier counseling is taken to illustrate the approach and compare their results with some of the existing approaches. From the studies, it is concluded that proposed work provides a new and easy way to handle the uncertainty and vagueness in the data and hence provides an alternative way to solve the DM problem under the IFS environment.

Chapter 6

Novel correlation coefficients of intuitionistic fuzzy sets based on the CNs and its application[1]

In this chapter, we present a novel correlation coefficient and weighted correlation coefficients formulation to measure the relative strength of the different IFSs in the form of the CNs. The limitations of certain existing measures are highlighted and overcome by the proposed measure. Afterward, a DM approach is presented based on the developed measures to solve the problems. Two illustrative examples related to pattern recognition and medical diagnosis are taken to validate the effectiveness and applicability of the proposed decision method.

6.1 Introduction

In statistical analysis, the correlation coefficients play a vital role to measure the linear relationship between the two variables, whereas in fuzzy set theory the correlation measure determines the degree of dependency between two fuzzy sets. In that direction, Hung and Wu [79] firstly defined the correlation coefficient for fuzzy numbers. Gerstenkorn and Manko [59] introduced the correlation coefficients of IFSs which was later analyzed by Szmidt and Kacprzyk [150]. Hong [76] studied the correlation coefficient of fuzzy numbers

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by applying T_w (the weakest t-norm)- based algebraic operations. Wang and Li [164] introduced the correlation of interval-valued fuzzy numbers. Furthermore, Gerstenkorn and Manko [59] introduced the correlation coefficient of IFSs, whereas Bustince and Burillo [12] discussed the concepts of correlation and correlation coefficient of IVIFSs. Hong [75] generalizes the concepts of correlation and correlation coefficient of IVIFSs to a general probability space and extends the results of Bustince and Burillo [12]. Xu et al. [206] define a correlation coefficient of IFSs from the set-theoretic viewpoint. Garg [44] defined the concepts of correlation and correlation coefficients of Pythagorean fuzzy sets. Garg [49] presented the correlation coefficients for the intuitionistic multiplicative sets and applied them to solve the problems of pattern recognition and decision making.

From these studies, it has been observed that their proposed correlation coefficients have several drawbacks. For instance, the existing correlation coefficients measures are independent of the degree of the membership or non-membership when one of the degrees of membership and other non-membership is zero in IFSs. Also, the effect of the change of the degree of the membership function remains unaffected on their index values. Therefore, it gives an inconsistent and unable to rank the different IFSs on the respective scales. Although, the above studies have widely used by the researchers' credibility is not guaranteed. In order to handle the uncertainties in a more precise way, Zhao [246] introduced the set pair analysis (SPA) for handling the uncertainties in the data in which certainty and uncertainty are studied as one system.

To overcome the certain drawbacks of the existing studies under the correlation coefficient measure, in this chapter we developed the correlation coefficient measures based on the connection number of the SPA to solve the multi-criteria group DM problems in which preferences related to different alternatives are taken in the form of the IFNs. In order to achieve it, firstly the shortcoming of the existing correlation coefficient measures under the IFSs environment has been highlighted and then connection numbers corresponding to IFNs have been proposed for overcoming this shortcoming. Based on these connection numbers, , as well as the covariance between the two CNs, have been defined and hence correlation coefficient and a weighted correlation coefficient have been proposed. Then, we utilize it to solve the problem of MCDM from the fields of pattern recognition and medical diagnosis to validate the effectiveness and applicability of the proposed decision method.

6.2 Correlation coefficients between the CNSs

In this section, we define some new correlation coefficients between the pairs of CNSs defined over the universal set $\mathcal{X}$.

Let $\mathcal{A} = \{(x_t, u_{\mathcal{A}}(x_t), v_{\mathcal{A}}(x_t)) \mid x_t \in \mathcal{X}\}$ be an IFS in the universe of discourse $\mathcal{X}$ and its corresponding connection number is defined in Eq. (4.1) of Chapter 4. Then, we define

$$\mathcal{E}(\mathcal{A}) = \sum_{t=1}^{n} \left(a^2(x_t) + b^2(x_t) + c^2(x_t)\right) \tag{6.1}$$

which is called the informational energy of the set $\mathcal{A}$.

Assume that two CNSs $\mathcal{A} = \{(x_t, a_{\mathcal{A}}(x_t) + b_{\mathcal{A}}(x_t)i + c_{\mathcal{A}}(x_t)j) \mid x_t \in \mathcal{X}\}$ and $\mathcal{B} = \{(x_t, a_{\mathcal{B}}(x_t) + b_{\mathcal{B}}(x_t)i + c_{\mathcal{B}}(x_t)j) \mid x_t \in \mathcal{X}\}$ defined over $\mathcal{X}$ where $a_{\mathcal{A}}(x_t)$, $a_{\mathcal{B}}(x_t)$, $b_{\mathcal{A}}(x_t)$, $b_{\mathcal{B}}(x_t)$, $c_{\mathcal{A}}(x_t)$, $c_{\mathcal{B}}(x_t) \in [0, 1]$ for every $x_t \in \mathcal{X}$. Then, we define the following so-called correlation of the CNs $\mathcal{A}$ and $\mathcal{B}$:

$$\mathfrak{C}(\mathcal{A}, \mathcal{B}) = \sum_{t=1}^{n} \left[a_{\mathcal{A}}(x_t) \cdot a_{\mathcal{B}}(x_t) + b_{\mathcal{A}}(x_t) \cdot b_{\mathcal{B}}(x_t) + c_{\mathcal{A}}(x_t) \cdot c_{\mathcal{B}}(x_t)\right] \tag{6.2}$$

From the above, it is obvious that the correlation of CNs satisfies the following properties:

(P1) $\mathfrak{C}(\mathcal{A}, \mathcal{A}) = \mathcal{E}(\mathcal{A})$

(P2) $\mathfrak{C}(\mathcal{A}, \mathcal{B}) = \mathfrak{C}(\mathcal{B}, \mathcal{A})$

Definition 6.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be the sets of CNs corresponding to the IFSs on a universe of discourse $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ denoted by $\mathcal{A} = \{(x_t, a_{\mathcal{A}}(x_t), b_{\mathcal{A}}(x_t), c_{\mathcal{A}}(x_t)) \mid x_t \in \mathcal{X}\}$ and $\mathcal{B} = \{(x_t, a_{\mathcal{B}}(x_t), b_{\mathcal{B}}(x_t), c_{\mathcal{B}}(x_t)) \mid x_t \in \mathcal{X}\}$, respectively. Then, the correlation coefficient between $\mathcal{A}$ and $\mathcal{B}$ is given by

$$\begin{aligned} \mathcal{K}_1(\mathcal{A}, \mathcal{B}) &= \frac{\mathfrak{C}(\mathcal{A}, \mathcal{B})}{\sqrt{\mathcal{E}(\mathcal{A}) \cdot \mathcal{E}(\mathcal{B})}} \\ &= \frac{\sum\limits_{t=1}^{n} \left[a_{\mathcal{A}}(x_t) \cdot a_{\mathcal{B}}(x_t) + b_{\mathcal{A}}(x_t) \cdot b_{\mathcal{B}}(x_t) + c_{\mathcal{A}}(x_t) \cdot c_{\mathcal{B}}(x_t)\right]}{\sqrt{\sum\limits_{t=1}^{n} \left(a_{\mathcal{A}}^2(x_t) + b_{\mathcal{A}}^2(x_t) + c_{\mathcal{A}}^2(x_t)\right)} \sqrt{\sum\limits_{t=1}^{n} \left(a_{\mathcal{B}}^2(x_t) + b_{\mathcal{B}}^2(x_t) + c_{\mathcal{B}}^2(x_t)\right)}} \end{aligned} \tag{6.3}$$

Theorem 6.2.1. For any two sets of CN $\mathcal{A}$ and $\mathcal{B}$ defined over the universe of discourse $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$, the correlation coefficient $\mathcal{K}_1$ satisfies the following properties.

(P1) $0 \leq \mathcal{K}_1(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) = \mathcal{K}_1(\mathcal{B}, \mathcal{A})$.

(P3) $\mathcal{A} = B \Leftrightarrow \mathcal{K}_1(\mathcal{A}, \mathcal{B}) = 1$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ for a connection number $\mathcal{C}$ then $\mathcal{K}_1(\mathcal{A}, \mathcal{C}) \leq \mathcal{K}_1(\mathcal{A}, \mathcal{B})$ and $\mathcal{K}_1(\mathcal{A}, \mathcal{C}) \leq \mathcal{K}_1(\mathcal{B}, \mathcal{C})$.

Proof. Consider two IFSs and their corresponding CNs $\mathcal{A}$ and $\mathcal{B}$ defined according to Eq. (4.1) of Chapter 4, we have

(P1) The inequality $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) \geq 0$ is evident, so we will prove $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) \leq 1$.

$$\mathcal{K}_1(\mathcal{A}, \mathcal{B}) \quad = \quad \frac{\sum\limits_{t=1}^{n} \big(a_{\mathcal{A}}(x_t) \cdot a_{\mathcal{B}}(x_t) + b_{\mathcal{A}}(x_t) \cdot b_{\mathcal{B}}(x_t) + c_{\mathcal{A}}(x_t) \cdot c_{\mathcal{B}}(x_t)\big)}{\sqrt{\sum\limits_{t=1}^{n} \big(a^2_{\mathcal{A}}(x_t) + b^2_{\mathcal{A}}(x_t) + c^2_{\mathcal{A}}(x_t)\big) \cdot \sum\limits_{t=1}^{n} \big(a^2_{\mathcal{B}}(x_t) + b^2_{\mathcal{B}}(x_t) + c^2_{\mathcal{B}}(x_t)\big)}}$$

By Cauchy-Schwarz inequality, we have $(x_1y_1 + x_2y_2 + \ldots + x_ny_n)^2 \leq (x_1^2 + y_1^2 + \ldots + x_n^2) \cdot (y_{\mathcal{A}}^2 + y_2^2 + \ldots + y_n^2)$ where $(x_1 + x_2 + \ldots + x_n) \in \mathbb{R}^n$ and $(y_1 + y_2 + \ldots + y_n) \in \mathbb{R}^n$, we get

$$\mathcal{K}_1(\mathcal{A}, \mathcal{B}) \leq \frac{\left(\sum\limits_{t=1}^{n} a^2_{\mathcal{A}}(x_t) \sum\limits_{t=1}^{n} a^2_{\mathcal{B}}(x_t)\right)^{\frac{1}{2}} + \left(\sum\limits_{t=1}^{n} b^2_{\mathcal{A}}(x_t) \sum\limits_{t=1}^{n} b^2_{\mathcal{B}}(x_t)\right)^{\frac{1}{2}} + \left(\sum\limits_{t=1}^{n} c^2_{\mathcal{A}}(x_t) \sum\limits_{t=1}^{n} c^2_{\mathcal{B}}(x_t)\right)^{\frac{1}{2}}}{\sqrt{\left(\sum\limits_{t=1}^{n} a^2_{\mathcal{A}}(x_t) + \sum\limits_{t=1}^{n} b^2_{\mathcal{A}}(x_t) + \sum\limits_{t=1}^{n} c^2_{\mathcal{A}}(x_t)\right)} \cdot \sqrt{\left(\sum\limits_{t=1}^{n} a^2_{\mathcal{B}}(x_t) + \sum\limits_{t=1}^{n} b^2_{\mathcal{B}}(x_t) + \sum\limits_{t=1}^{n} c^2_{\mathcal{B}}(x_t)\right)}}$$

Let the following notations

$$\sum_{t=1}^{n} a^2_{\mathcal{A}}(x_t) = \xi_1, \qquad \sum_{t=1}^{n} b^2_{\mathcal{A}}(x_t) = \rho_1, \qquad \sum_{t=1}^{n} c^2_{\mathcal{A}}(x_t) = \eta_1$$

$$\sum_{t=1}^{n} a^2_{\mathcal{B}}(x_t) = \xi_2, \qquad \sum_{t=1}^{n} b^2_{\mathcal{B}}(x_t) = \rho_2, \qquad \sum_{t=1}^{n} c^2_{\mathcal{B}}(x_t) = \eta_2$$

So the above inequality becomes:

$$\mathcal{K}_1(\mathcal{A}, \mathcal{B}) \quad \leq \quad \frac{\sqrt{\xi_1\xi_2} + \sqrt{\rho_1\rho_2} + \sqrt{\eta_1\eta_2}}{\sqrt{\xi_1 + \rho_1 + \eta_1} \cdot \sqrt{\xi_2 + \rho_2 + \eta_2}}.$$

Squaring on both sides, we get:

$$\mathcal{K}_1^2(\mathcal{A},\mathcal{B}) \;\leq\; \frac{\left(\sqrt{\xi_1\xi_2}+\sqrt{\rho_1\rho_2}+\sqrt{\eta_1\eta_2}\right)^2}{(\xi_1+\rho_1+\eta_1)\cdot(\xi_2+\rho_2+\eta_2)}$$

Subtracting 1 on both sides, and we get

$$\begin{aligned} & \mathcal{K}_1^2(\mathcal{A},\mathcal{B})-1 \\ \leq\; & \frac{-\left(\left(\sqrt{\xi_1\rho_2}-\sqrt{\rho_1\xi_2}\right)^2+\left(\sqrt{\rho_1\eta_2}-\sqrt{\eta_1\rho_2}\right)^2+\left(\sqrt{\xi_1\eta_2}-\sqrt{\eta_1\xi_2}\right)^2\right)}{(\xi_1+\rho_1+\eta_1)\cdot(\xi_2+\rho_2+\eta_2)} \\ \leq\; & 0 \end{aligned}$$

Thus, we have $0 \leq \mathcal{K}_1(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) It is straightforward.

(P3) Sufficient condition is obviously true. Now, necessary condition is proven.

So, let equality sign holds in the proof of (P1) if we have $a_{\mathcal{A}}(x_t) = ha_{\mathcal{B}}(x_t)$, $b_{\mathcal{A}}(x_t) = hb_{\mathcal{B}}(x_t)$ and $c_{\mathcal{A}}(x_t) = hc_{\mathcal{B}}(x_t)$, for some positive real number h. However, according to the condition of CN, we have $a_{\mathcal{A}}(x_t)+b_{\mathcal{A}}(x_t)+c_{\mathcal{A}}(x_t) = 1 = a_{\mathcal{B}}(x_t)+b_{\mathcal{B}}(x_t)+c_{\mathcal{B}}(x_t)$ which implies that $h = 1$. Hence, $\mathcal{A} = \mathcal{B}$.

(P4) $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ which implies that $a_{\mathcal{A}}(x_t) \leq a_{\mathcal{B}}(x_t) \leq a_3(x_t)$, $b_{\mathcal{A}}(x_t) \geq b_{\mathcal{B}}(x_t) \geq b_3(x_t)$ and $c_{\mathcal{A}}(x_t) \geq c_{\mathcal{B}}(x_t) \geq c_3(x_t)$. Geometrically, if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then the angle between $\mathcal{A}$ and $\mathcal{C}$ should be larger than the angle between $\mathcal{A}$ and $\mathcal{B}$ and the angle between $\mathcal{B}$ and $\mathcal{C}$ for any element x_t and is decreasing function within the interval $[0, \pi/2]$. Thus, the relation $\mathcal{K}_1(\mathcal{A},\mathcal{C}) \leq \mathcal{K}_1(\mathcal{A},\mathcal{B})$ and $\mathcal{K}_1(\mathcal{A},\mathcal{C}) \leq \mathcal{K}_1(\mathcal{B},\mathcal{C})$ can be obtained from Eq. (6.3).

$\square$

Example 6.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be any two sets of CNs defined in $\mathcal{X} = \{x_1, x_2, x_3\}$ such that $\mathcal{A} = \{(x_1, 0.25+0.35i+0.40j), (x_2, 0.30+0.45i+ 0.25j), (x_3, 0.60+0.20i + 0.20j)\}$, $B = \{(x_1, 0.75+ 0.15i+0.10j), (x_2,\ 0.55+ 0.35i+0.10j), (x_3, 0.40+0.20i+0.40j)\}$. Then, using Eq. (6.1), the informational energy of $\mathcal{A}$ is written as follows:

$$\begin{aligned}
\mathcal{E}(\mathcal{A}) &= \sum_{t=1}^{n}\left(a^2_{\mathcal{A}}(x_t)+b^2_{\mathcal{A}}(x_t)+c^2_{\mathcal{A}}(x_t)\right)\\
&= (0.25^2+0.35^2+0.40^2)+(0.30^2+0.45^2+0.25^2)+(0.60^2+0.20^2+0.20^2)\\
&= 1.1400
\end{aligned}$$

and the informational energy of $\mathcal{B}$ is given as $\mathcal{E}(\mathcal{B}) = 1.3900$. Now, by using Eq. (6.2), the correlation between the sets $\mathcal{A}$ and $\mathcal{B}$ is written as

$$\begin{aligned}
\mathfrak{C}(\mathcal{A},\mathcal{B}) &= \sum_{t=1}^{n}\left[a_{\mathcal{A}}(x_t)\cdot a_{\mathcal{B}}(x_t)+b_{\mathcal{A}}(x_t)\cdot b_{\mathcal{B}}(x_t)+c_{\mathcal{A}}(x_t)\cdot c_{\mathcal{B}}(x_t)\right]\\
&= (0.25\times 0.75+0.35\times 0.15+0.40\times 0.10)+(0.30\times 0.55+0.45\times 0.35\\
&\quad +0.25\times 0.10)+(0.60\times 0.40+0.20\times 0.20+0.20\times 0.40)\\
&= 0.9875
\end{aligned}$$

Hence, the correlation coefficient between the sets of CNs $\mathcal{A}$ and $\mathcal{B}$ is given by

$$\mathcal{K}_1(\mathcal{A},\mathcal{B}) = \frac{\mathfrak{C}(\mathcal{A},\mathcal{B})}{\sqrt{\mathcal{E}(\mathcal{A})\cdot\mathcal{E}(\mathcal{B})}} = \frac{0.9875}{1.1400\times 1.3900} = 0.7845$$

Furthermore, in order to stress the importance of the proposed correlation coefficient with respect to the existing ones, the correlation measure has been computed for the certain examples, where the existing measures have fails to rank the alternatives, as below.

Example 6.2.2. Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be three IFSs defined in $\mathcal{X} = \{x_1, x_2\}$ by: $\mathcal{A}_1 = \{(x_1, 0.10, 0.15),\ (x_2, 0.10, 0.15)\}$, $\mathcal{A}_2 = \{(x_1, 0.20, 0.30),\ (x_2, 0.20, 0.30)\}$ and $\mathcal{A}_3 = \{(x_1$,0.30, 0.45), (x_2, 0.30, 0.45)}. Then we can check easily by using existing correlation coefficient given in Eq. (2.28) of Chapter 2 that $K_{IFS_1}(\mathcal{A}_1,\mathcal{A}_2) = K_{IFS_1}(\mathcal{A}_2,\mathcal{A}_3) = K_{IFS_1}(\mathcal{A}_3,\mathcal{A}_1)$. On the other hand, if we apply the proposed $\mathcal{K}_1$ correlation coefficient to the above data set then we get $\mathcal{K}_1(\mathcal{A}_1,\mathcal{A}_2) = 0.9759$, $\mathcal{K}_1(\mathcal{A}_2,\mathcal{A}_3) = 0.9839$ and $\mathcal{K}_1(\mathcal{A}_3,\mathcal{A}_1) = 0.9215$. Thus, the proposed measure classifies the IFSs and hence able to identify the best one.

Example 6.2.3. Consider two IFNs $\mathcal{A} = (0.4, 0)$ and $\mathcal{B} = (0, 0.5)$ then the correlation coefficient between them, by using Eq. (2.28) of Chapter 2, is $K_{IFS_1}(\mathcal{A},\mathcal{B}) =$

$\frac{(0.4)(0)+(0.5)(0)}{\sqrt{0.4^2+0^2}\sqrt{0.5^2+0^2}} = 0$. On the other hand, if we replace the IFN $\mathcal{A}$ with $\mathcal{A}_1 = (0.6, 0)$ then the the correlation coefficient between $\mathcal{A}_1$ and $\mathcal{B}$ is $K_{IFS_1}(\mathcal{A}_1, \mathcal{B}) = \frac{(0.6)(0)+(0)(0.5)}{\sqrt{0.6^2+0^2}\sqrt{0^2+0.5^2}} = 0$. Hence, the existing index is independent of the change of the membership degree in IFS.

However, if we apply the proposed correlation coefficient measure on it i.e., by taking two IFNs $\mathcal{A} = (0.4, 0)$ and $\mathcal{B} = (0, 0.5)$ then $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) = 0.5883$, while for sets $\mathcal{A}_1$ and $\mathcal{B}$, we have $\mathcal{K}_1(A_{\mathcal{A}}, \mathcal{B}) = 0.3922$. Hence, the proposed measure simultaneously consider the effect of the change of the membership degree in IFS during the analysis.

Example 6.2.4. Let two IFNs $\mathcal{A} = \langle 0.4, 0.3\rangle$ and $\mathcal{B} = \langle 0, 0.5\rangle$ then the existing correlation coefficient becomes

$$K_{IFS_1}(\mathcal{A}, \mathcal{B}) = \frac{(0.4)(0) + (0.3)(0.5)}{\sqrt{0.4^2 + 0.3^2}\sqrt{0^2 + 0.5^2}} = 0.6$$

On the other hand, if we replace IFNs $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A}_1 = \langle 0.1, 0.4898\rangle$ and $\mathcal{B}_1 = \langle 0.45189, 0.21398\rangle$ respectively, then the correlation coefficient becomes

$$K_{IFS_1}(\mathcal{A}_1, \mathcal{B}_1) = \frac{(0.1)(0.45189) + (0.4898)(0.21398)}{\sqrt{0.1^2 + 0.4898^2}\sqrt{0.45189^2 + 0.21398^2}} = 0.6$$

Hence, it has been concluded that by changing the degree of membership and non-membership degree of IFSs, the correlation coefficient remain same, therefore it is inconsistent and hence unable to rank the alternative.

However, if we apply the proposed measures on these data sets then we get $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) = 0.8026$ and $\mathcal{K}_1(A_1, B_1) = 0.7865$. Therefore, the proposed measure have sufficiently considered the impact of the change of the degree of membership and non-membership on to the correlation coefficient measure and hence overcome the shortcoming of the existing measures.

Definition 6.2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be the set of two CNs then the correlation coefficient is defined as

$$\begin{aligned}\mathcal{K}_2(\mathcal{A}, \mathcal{B}) &= \frac{\mathfrak{C}(\mathcal{A}, \mathcal{B})}{\max\{\mathcal{E}(\mathcal{A}), \mathcal{E}(\mathcal{B})\}} \\ &= \frac{\sum\limits_{t=1}^{n}\left[a_{\mathcal{A}}(x_t)\cdot a_{\mathcal{B}}(x_t) + b_{\mathcal{A}}(x_t)\cdot b_{\mathcal{B}}(x_t) + c_{\mathcal{A}}(x_t)\cdot c_{\mathcal{B}}(x_t)\right]}{\max\left\{\sum\limits_{t=1}^{n}\left(a^2_{\mathcal{A}}(x_t) + b^2_{\mathcal{A}}(x_t) + c^2_{\mathcal{A}}(x_t)\right), \sum\limits_{t=1}^{n}\left(a^2_{\mathcal{B}}(x_t) + b^2_{\mathcal{B}}(x_t) + c^2_{\mathcal{B}}(x_t)\right)\right\}} \qquad (6.4)\end{aligned}$$

Theorem 6.2.2. The correlation coefficient, $\mathcal{K}_2(\mathcal{A},\mathcal{B})$, between the two sets of CNs $\mathcal{A}$ and $\mathcal{B}$ satisfies the following properties:

(P1) $0 \leq \mathcal{K}_2(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\mathcal{K}_2(\mathcal{A},\mathcal{B}) = \mathcal{K}_2(\mathcal{B},\mathcal{A})$.

(P3) $\mathcal{A} = \mathcal{B} \Leftrightarrow \mathcal{K}_2(\mathcal{A},\mathcal{B}) = 1$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ for a connection number $\mathcal{C}$ then $\mathcal{K}_2(\mathcal{A},\mathcal{C}) \leq \mathcal{K}_2(\mathcal{A},\mathcal{B})$ and $\mathcal{K}_2(\mathcal{A},\mathcal{C}) \leq \mathcal{K}_2(\mathcal{B},\mathcal{C})$.

Proof. Properties (P2), (P3) and (P4) are straight forward; thus we omit them here. Also $\mathcal{K}_2(\mathcal{A},\mathcal{B}) \geq 0$ is evident. Now, from Theorem 6.2.1, we have $(\mathfrak{C}(\mathcal{A},\mathcal{B}))^2 \leq \mathcal{E}(\mathcal{A}) \cdot \mathcal{E}(\mathcal{B})$. Therefore, $\mathfrak{C}(\mathcal{A},\mathcal{B}) \leq \max\{\mathcal{E}(\mathcal{A}), \mathcal{E}(\mathcal{B})\}$ and thus $\mathcal{K}_2(\mathcal{A},\mathcal{B}) \leq 1$. ☐

In the following, a weighted correlation coefficient is developed between the CNs by considering the weight ω_t of the element $x_t \in \mathcal{X}(t = 1, 2, \ldots, n)$ such that $\sum\limits_{t=1}^{n} \omega_t = 1$.

$$\begin{aligned}
\mathcal{K}_3(\mathcal{A},\mathcal{B}) &= \frac{\mathcal{C}_\omega(\mathcal{A},\mathcal{B})}{\sqrt{\mathcal{E}_\omega(A) \cdot \mathcal{E}_\omega(B)}} \\
&= \frac{\sum\limits_{t=1}^{n} \omega_t \left(a_{\mathcal{A}}(x_t) \cdot a_{\mathcal{B}}(x_t) + b_{\mathcal{A}}(x_t) \cdot b_{\mathcal{B}}(x_t) + c_{\mathcal{A}}(x_t) \cdot c_{\mathcal{B}}(x_t)\right)}{\sqrt{\sum\limits_{t=1}^{n} \omega_t \left(a_{\mathcal{A}}^2(x_t) + b_{\mathcal{A}}^2(x_t) + c_{\mathcal{A}}^2(x_t)\right) \cdot \sum\limits_{t=1}^{n} \omega_t \left(a_{\mathcal{B}}^2(x_t) + b_{\mathcal{B}}^2(x_t) + c_{\mathcal{B}}^2(x_t)\right)}} \qquad (6.5)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_4(\mathcal{A},\mathcal{B}) &= \frac{\mathcal{C}_\omega(\mathcal{A},\mathcal{B})}{\max\{\mathcal{E}_\omega(A), \mathcal{E}_\omega(B)\}} \\
&= \frac{\sum\limits_{t=1}^{n} \omega_t (a_{\mathcal{A}}(x_t) \cdot a_{\mathcal{B}}(x_t) + b_{\mathcal{A}}(x_t) \cdot b_{\mathcal{B}}(x_t) + c_{\mathcal{A}}(x_t) \cdot c_{\mathcal{B}}(x_t))}{\max\left\{\sum\limits_{t=1}^{n} \omega_t \left(a_{\mathcal{A}}^2(x_t) + b_{\mathcal{A}}^2(x_t) + c_{\mathcal{A}}^2(x_t)\right), \sum\limits_{t=1}^{n} \omega_t \left(a_{\mathcal{B}}^2(x_t) + b_{\mathcal{B}}^2(x_t) + c_{\mathcal{B}}^2(x_t)\right)\right\}} \qquad (6.6)
\end{aligned}$$

It can be easily deduced that if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then Eqs. (6.5) and (6.6) reduce to the correlation coefficient defined in Eqs. (6.3) and (6.4) respectively. It is easy to see that the weighted correlation coefficients $\mathcal{K}_3(\mathcal{A},\mathcal{B})$ and $\mathcal{K}_4(\mathcal{A},\mathcal{B})$ between the sets of CNs satisfies the property of $0 \leq \mathcal{K}_3(\mathcal{A},\mathcal{B}) \leq 1$ and $0 \leq \mathcal{K}_4(\mathcal{A},\mathcal{B}) \leq 1$.

Theorem 6.2.3. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $x_t, (t = 1, 2, \ldots, n)$ such that $\omega_t > 0$ and $\sum\limits_{t=1}^{n} \omega_t = 1$. Then, the weighted correlation coefficient, $\mathcal{K}_k(\mathcal{A}, \mathcal{B}), (k = 3, 4)$, between the set of CNs $\mathcal{A}$ and $\mathcal{B}$ defined by Eqs. (6.5) and (6.6) satisfies the following properties:

(P1) $0 \leq \mathcal{K}_k(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{K}_k(\mathcal{A}, \mathcal{B}) = \mathcal{K}_k(\mathcal{B}, \mathcal{A})$.

(P3) $\mathcal{A} = B \Leftrightarrow \mathcal{K}_k(\mathcal{A}, \mathcal{B}) = 1$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ for a connection number $\mathcal{C}$ then $\mathcal{K}_k(\mathcal{A}, \mathcal{C}) \leq \mathcal{K}_k(\mathcal{A}, \mathcal{B})$ and $\mathcal{K}_k(\mathcal{A}, \mathcal{C}) \leq \mathcal{K}_k(\mathcal{B}, \mathcal{C})$.

Proof. Similarly follows from the above theorem; hence, it is omitted here. □

6.3 Drawbacks of the existing correlation coefficients

In this section, we show some limitations in existing methods of correlation coefficients for Atanassov IFSs. Advantages of proposed correlation coefficients have been discussed. To show the limitations of existing correlation coefficients, a few examples are presented in this section. For convenience, assume that the weights of elements $x_t \in \mathcal{X}$ are all equal.

6.3.1 Zeng and Li [235] correlation coefficient

Assume that there are three patterns defined in the form of IFS with the degree of membership and non-membership $(u(x_t), v(x_t))$ in $\mathcal{X} = \{x_1, x_2, x_3\}$ which are defined as

$$\mathcal{A}_1 = \{(x_1, 0.4, 0.5), (x_2, 0.7, 0.1), (x_3, 0.3, 0.3)\}$$
$$\mathcal{A}_2 = \{(x_1, 0.5, 0.4), (x_2, 0.7, 0.2), (x_3, 0.4, 0.3)\}$$
$$\mathcal{A}_3 = \{(x_1, 0.4, 0.5), (x_2, 0.7, 0.1), (x_3, 0.4, 0.3)\}$$

Assume that a sample of $\mathcal{B} = \{(x_1, 0.1, 0.1), (x_2, 1.0, 0.0), (x_3, 0.0, 1.0)\}$ is given.

Zeng and Li [235] defined the correlation measure between the IFSs $\mathcal{A}$ and $\mathcal{B}$ which is defined in Eq. (2.29) of Chapter 2. Hence, accordingly, the suitable classifier for sample $\mathcal{B}$

from $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ patterns can be identified using Eq. (2.29). The results corresponding to each pattern are obtained as

$$K_{IFS_2}(A_{\mathcal{A}}, B) = K_{IFS_2}(A_{\mathcal{B}}, B) = K_{IFS_2}(A_3, B) = 0.6262$$

On the other hand, to identify which pattern belongs to sample $\mathcal{B}$ by using Eq. (6.3), we convert IFNs of the given patterns into the CNs according to Eq. (4.1) of Chapter 4 as follows:

$$\begin{aligned}
\mathcal{A}_1 &= \{(x_1, 0.20 + 0.50i + 0.30j), (x_2, 0.63 + 0.34i + 0.03j), (x_3, 0.21 + 0.58i + 0.21j)\} \\
\mathcal{A}_2 &= \{(x_1, 0.30 + 0.50i + 0.20j), (x_2, 0.56 + 0.38i + 0.06j), (x_3, 0.28 + 0.54i + 0.18j)\} \\
\mathcal{A}_3 &= \{(x_1, 0.20 + 0.50i + 0.30j), (x_2, 0.63 + 0.34i + 0.03j), (x_3, 0.28 + 0.54i + 0.18j)\} \\
\text{and} \quad \mathcal{B} &= \{(x_1, 0.09 + 0.82i + 0.09j), (x_2, 1.00 + 0.00i + 0.00j), (x_3, 0.00 + 0.00i + 1.00j)\}
\end{aligned}$$

Then based on these CNs, Eq. (6.3) is utilized, and the measurement values are obtained as follows:

$$\mathcal{K}_1(\mathcal{A}_1, \mathcal{B}) = 0.6879, \qquad \mathcal{K}_1(\mathcal{A}_2, \mathcal{B}) = 0.6534, \qquad \mathcal{K}_1(\mathcal{A}_3, \mathcal{B}) = 0.6777$$

Thus, based on these results, Zeng and Li [235] correlation coefficient can not classify to this sample. On the other hand, the CN theory are able to accommodate the required information about the sample and hence find that the sample $\mathcal{B}$ belongs to the pattern $A_{\mathcal{A}}$ according to the recognition principle.

6.3.2 Szmidt and Kacprzyk [150] correlation coefficient

Szmidt and Kacprzyk [150] propose a correlation coefficient for two Atanassov's IFSs, $\mathcal{A}$ and $\mathcal{B}$, so that we could express not only a relative strength but also a positive or negative relationship between $\mathcal{A}$ and $\mathcal{B}$. Suppose that we have a random sample $x_1, x_2, \ldots, x_n \in \mathcal{X}$, then the correlation coefficient $K_{IFS_4}(\mathcal{A}, \mathcal{B})$ is defined in Eq. (2.31) of Chapter 2. Based on its formulation, it has been shown that it is not able to find the good correlation between the IFSs. For instance, consider $\mathcal{A}$ and $\mathcal{B}$ as two Atanassov's IFSs in $\mathcal{X} = \{x_1, x_2, x_3\}$ defined as

$$\begin{aligned}
\mathcal{A} &= \{(x_1, 0.1, 0.2), (x_2, 0.2, 0.1), (x_3, 0.29, 0.0)\}, \\
\mathcal{B} &= \{(x_1, 0.1, 0.3), (x_2, 0.2, 0.2), (x_3, 0.29, 0.1)\}
\end{aligned}$$

Then by apply Eq. (2.31) of Chapter 2, we get $K_{IFS_4}(\mathcal{A}, \mathcal{B}) = 1$ which means that these two IFSs are perfectly correlated, which is not true as seen from the structure of sets $\mathcal{A}$ and $\mathcal{B}$. On the other hand, if we apply our proposed method to find the correlation coefficient between $\mathcal{A}$ and $\mathcal{B}$ by using Eq. (6.3) and Eq. (6.4), then we get $\mathcal{K}_1(\mathcal{A}, \mathcal{B}) = 0.9922$ and $\mathcal{K}_2(\mathcal{A}, \mathcal{B}) = 0.9298$ respectively. These results shows that $\mathcal{A}$ and $\mathcal{B}$ are not perfectly correlated.

6.3.3 Xu et al. [206] correlation coefficient

Xu et al. [206] defined the correlation coefficient in intuitionistic fuzzy sets environment by considering the degree of membership, non-membership and the degree of hesitation between the two intuitionistic sets $\mathcal{A}$ and $\mathcal{B}$. The expression of their proposed correlation coefficient is defined in Eq. (2.30) of Chapter 2.

Consider the following three patterns $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ which are represented in the form of IFSs:

$$\begin{aligned} \mathcal{A}_1 &= \{(x_1, 0.4, 0.5), (x_2, 0.7, 0.1), (x_3, 0.3, 0.3)\} \\ \mathcal{A}_2 &= \{(x_1, 0.5, 0.4), (x_2, 0.7, 0.2), (x_3, 0.4, 0.3)\} \\ \text{and} \quad \mathcal{A}_3 &= \{(x_1, 0.4, 0.5), (x_2, 0.7, 0.1), (x_3, 0.4, 0.3)\} \end{aligned}$$

Assume that a sample $\mathcal{B} = \{(x_1, 0.1, 0.1),\ (x_2, 1, 0),\ (x_3, 0, 1)\}$ is given. Then, by using correlation coefficient of Xu et al. [206] defined in Eq. (2.30) of Chapter 2, the following results are obtained:

$$K_{IFS_3}(\mathcal{A}_1, \mathcal{B}) = K_{IFS_3}(\mathcal{A}_1, \mathcal{B}) = K_{IFS_3}(\mathcal{A}_3, \mathcal{B}) = 0.4398$$

Thus, based on their results, it has been concluded that their correlation coefficient can not be used to classify the sample $\mathcal{B}$ with three patterns $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$.

To remove this problem, the intuitionistic fuzzy values of the given patterns and sample are converted into the CNs according to the Eq. (4.1) of Chapter 4 as follows:

$$\begin{aligned} \mathcal{A}_1 &= \{(x_1, 0.20 + 0.50i + 0.30j), (x_2, 0.63 + 0.34i + 0.03j), (x_3, 0.21 + 0.58i + 0.21j)\} \\ \mathcal{A}_2 &= \{(x_1, 0.30 + 0.50i + 0.20j), (x_2, 0.56 + 0.38i + 0.06j), (x_3, 0.28 + 0.54i + 0.18j)\} \end{aligned}$$

$$\begin{aligned}\mathcal{A}_3 &= \{(x_1, 0.20 + 0.50i + 0.30j), (x_2, 0.63 + 0.34i + 0.03j), (x_3, 0.28 + 0.54i + 0.18j)\}\\ \text{and} \quad \mathcal{B} &= \{(x_1, 0.09 + 0.82i + 0.09j), (x_3, 1.00 + 0.00i + 0.00j), (x_3, 0.00 + 0.00i + 1.00j)\}\end{aligned}$$

Thus, by using Eq. (6.4), we get

$$\mathcal{K}_2(\mathcal{A}_1, \mathcal{B}) = 0.4817, \qquad \mathcal{K}_2(\mathcal{A}_2, \mathcal{B}) = 0.4445, \qquad \mathcal{K}_2(\mathcal{A}_3, \mathcal{B}) = 0.4705$$

Thus, it has been concluded that the sample $\mathcal{B}$ is classify with the pattern $\mathcal{A}_1$.

6.4 Decision-making approach based on the proposed correlation coefficients

The general description of the MADM problem is given in the Section 2.7 of Chapter 2 under the IFS environment. Then, the following steps have been proposed, based on the proposed correlation coefficients, to find the best alternative(s).

Step 1: Obtain the CNs $\mu_{kt} = a_{kt} + b_{kt}i + c_{kt}j$ corresponding to each rating information of IFNs $\alpha_{kt} = (u_{kt}, v_{kt})$ where $a_{kt} = u_{kt}(1 - v_{kt})$, $b_{kt} = 1 - u_{kt}(1 - v_{kt}) - v_{kt}(1 - u_{kt})$ and $c_{kt} = v_{kt}(1 - u_{kt})$ are obtained accordingly to Eq. (4.1) of Chapter 4.

Step 2: Compute the correlation coefficient measures either by $\mathcal{K}_1(\mathcal{A}_k, \mathcal{A}^*)$ or $\mathcal{K}_2(\mathcal{A}_k, \mathcal{A}^*)$ or $\mathcal{K}_3(\mathcal{A}_k, \mathcal{A}^*)$ or $\mathcal{K}_4(\mathcal{A}_k, \mathcal{A}^*)$ between the alternatives $\mathcal{A}_k (k = 1, 2, \ldots, m)$ and the ideal alternative $\mathcal{A}^*$ by using Eqs. (6.3)–(6.6) respectively. Here, $\mathcal{A}^*$ is the ideal alternative whose connection number is taken as $\mu_{\mathcal{A}^*} = 1 + 0i + 0j$.

Step 3: Choose the best alternative(s) according to the maximum value of $\mathcal{K}(\mathcal{A}_k, \mathcal{A}^*)$.

6.5 Illustrative Example

In this section, an example related to the DM problem, from the field of pattern recognition and medical diagnosis, are taken to demonstrate as well as the effectiveness of the proposed method.

6.5.1 Example 1: Pattern Recognition

Consider a three known pattern $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ represented in the form of IFSs on $\mathcal{X} = \{x_1, x_2, x_3\}$ as

$$\begin{aligned} \mathcal{A}_1 &= \{(x_1, 1.0, 0.0), (x_2, 0.8, 0.0), (x_3, 0.7, 0.1)\} \\ \mathcal{A}_2 &= \{(x_1, 0.8, 0.1), (x_2, 1.0, 0.0), (x_3, 0.9, 0.0)\} \\ \text{and} \quad \mathcal{A}_3 &= \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.0), (x_3, 1.0, 0.0)\} \end{aligned}$$

Consider an unknown pattern $\mathcal{B}$ which will be recognized, where

$$B = \{(x_1, 0.5, 0.3), (x_2, 0.6, 0.2), (x_3, 0.8, 0.1)\}$$

The target of this problem is to classify the pattern $\mathcal{B}$ in one of the classes $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$. To do so, firstly, the IFSs are converted into CNs of given patterns, then sample and their corresponding numbers are summarized as below.

$$\begin{aligned} \mathcal{A}_1 &= \{(x_1, 1.00 + 0.00i + 0.00j), (x_2, 0.80 + 0.20i + 0.00j), (x_3, 0.63 + 0.34i + 0.03j)\} \\ \mathcal{A}_2 &= \{(x_1, 0.72 + 0.26i + 0.02j), (x_2, 1.00 + 0.00i + 0.00j), (x_3, 0.90 + 0.10i + 0.00j)\} \\ \mathcal{A}_3 &= \{(x_1, 0.48 + 0.44i + 0.08j), (x_2, 0.80 + 0.20i + 0.00j), (x_3, 1.00 + 0.00i + 0.00j)\} \\ \text{and} \ \mathcal{B} &= \{(x_1, 0.35 + 0.50i + 0.15j), (x_2, 0.48 + 0.44i + 0.08j), (x_3, 0.72, 0.26i + 0.02j)\} \end{aligned}$$

Now, the proposed correlation coefficient indexes, $\mathcal{K}_1$ and $\mathcal{K}_2$ have been computed from $\mathcal{B}$ to $\mathcal{A}_k (k = 1, 2, 3)$ and their results are given as follows.

$$\begin{aligned} &\mathcal{K}_1(\mathcal{A}_1, \mathcal{B}) = 0.7755; \quad \mathcal{K}_1(\mathcal{A}_2, \mathcal{B}) = 0.8350, \quad \mathcal{K}_1(\mathcal{A}_3, \mathcal{B}) = 0.9223 \\ &\mathcal{K}_2(\mathcal{A}_1, \mathcal{B}) = 0.6221; \quad \mathcal{K}_2(\mathcal{A}_2, \mathcal{B}) = 0.6395; \quad \mathcal{K}_2(\mathcal{A}_3, \mathcal{B}) = 0.7544 \end{aligned}$$

Thus, from these two proposed correlation coefficient indexes, it can be conclude that the pattern $\mathcal{B}$ belongs to the pattern $\mathcal{A}_3$.

On the other hand, if it is assumed that weight of x_1, x_2 and x_3 are 0.5, 0.3 and 0.2, respectively, then correlation coefficient $\mathcal{K}_3$ and $\mathcal{K}_4$ can be utilized for obtaining the most suitable pattern as

$$\begin{aligned} &\mathcal{K}_3(\mathcal{A}_1, \mathcal{B}) = 0.7104; \quad \mathcal{K}_3(\mathcal{A}_2, \mathcal{B}) = 0.8129; \quad \mathcal{K}_3(\mathcal{A}_3, \mathcal{B}) = 0.9262 \\ &\mathcal{K}_4(\mathcal{A}_1, \mathcal{B}) = 0.5270; \quad \mathcal{K}_4(\mathcal{A}_2, \mathcal{B}) = 0.6224; \quad \mathcal{K}_4(\mathcal{A}_3, \mathcal{B}) = 0.7842 \end{aligned}$$

Thus, ranking order of the three patterns is $\mathcal{A}_3, \mathcal{A}_2$ and $\mathcal{A}_1$ and hence $\mathcal{A}_3$ is the most desirable pattern to be classified with $\mathcal{B}$.

In order to compare the performance of the proposed approach with some existing approaches under the IFS environment, we conducted a comparative analysis based on the different approaches as given by the authors in [32, 34, 81, 102, 206, 223, 235]. The results corresponding to these are summarized in Table 6.1. From this table, it is clearly seen that the pattern $\mathcal{B}$ belongs to $\mathcal{A}_3$ and coincides with the proposed measure results.

Table 6.1: Comparative analysis of Pattern recognition example

	Methods	Measurement values of $\mathcal{B}$ from			Ranking order
		$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	
Zeng and Li [235]	Correlation coefficient	0.8882	0.9291	0.9710	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Ye [223]	Cosine similarity measure	0.9353	0.9519	0.9724	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Dengfeng and Chuntian [32]	Similarity measure	0.74	0.78	0.84	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Liu [102]	Similarity measure	0.72	0.74	0.84	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1$
Xu et al. [206]	Correlation coefficient	0.7252	0.7177	0.8113	$\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2$
Ejegwa and Modom [34]	Distance measure	0.3200	0.3200	0.2000	$\mathcal{A}_3 \succ \mathcal{A}_2 = \mathcal{A}_1$
Hung and Yang [81]	Similarity measure	0.4700	0.4700	0.5100	$\mathcal{A}_3 \succ \mathcal{A}_2 = \mathcal{A}_1$

6.5.2 Example 2: Medical diagnosis

Consider the dataset taken from Son and Thong [140] with four patients namely $\mathcal{P} = \{\text{Al}, \text{Bob}, \text{Joe}, \text{Ted}\}$, five symptoms $\mathcal{S} = \{s_1$ (Temperature), s_2 (HeadAche), s_3 (Stomach Pain), s_4 (Cough), s_5 (Chest pain)$\}$ and five diseases $Q = \{\mathcal{Q}_1$(Viral fever), $\mathcal{Q}_2$(Malaria), $\mathcal{Q}_3$(Typhoid), $\mathcal{Q}_4$(Stomach Problem), $\mathcal{Q}_5$ (Chest problem)$\}$. The values of it are represented in the form of the IFNs and are given in Table 6.2.

Then, the target of this problem is to classify the patients $\mathcal{P} = \{\text{Al}, \text{Bob}, \text{Joe}, \text{Ted}\}$ with one of the diagnoses $\mathcal{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5\}$ respectively. To do so, we construct CNs of each IFN by using Eq. (4.1) of Chapter 4 and then, the obtained values are summarized in Table 6.3.

Now, by taking these information and the developed correlation coefficient $\mathcal{K}_1$ as given

Table 6.2: Rating values of the patients with symptoms and the symptoms with diseases

Patients ↓	s_1	s_2	s_3	s_4	s_5
Al	(0.8, 0.1)	(0.6, 0.1)	(0.2, 0.8)	(0.6, 0.1)	(0.1, 0.6)
Bob	(0.0, 0.8)	(0.4, 0.4)	(0.6, 0.1)	(0.1, 0.7)	(0.1, 0.8)
Joe	(0.8, 0.1)	(0.8, 0.1)	(0.0, 0.6)	(0.2, 0.7)	(0.0, 0.5)
Ted	(0.6, 0.1)	(0.5, 0.4)	(0.3, 0.4)	(0.7, 0.2)	(0.3, 0.4)
Symptoms ↓	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$
s_1	(0.4, 0.0)	(0.7, 0.0)	(0.3, 0.3)	(0.1, 0.7)	(0.1, 0.8)
s_2	(0.3, 0.5)	(0.2, 0.6)	(0.6, 0.1)	(0.2, 0.4)	(0.0, 0.8)
s_3	(0.1, 0.7)	(0.0, 0.9)	(0.2, 0.7)	(0.8, 0.0)	(0.2, 0.8)
s_4	(0.4, 0.3)	(0.7, 0.0)	(0.2, 0.6)	(0.2, 0.7)	(0.2, 0.8)
s_5	(0.1, 0.7)	(0.1, 0.8)	(0.1, 0.9)	(0.2, 0.7)	(0.8, 0.1)

Table 6.3: CNs for patients with symptoms and the symptoms with diseases

Patients ↓	s_1	s_2	s_3	s_4	s_5
Al	0.72+0.26i+0.02j	0.54+0.42i+0.04j	0.04+0.32i+0.64j	0.54+0.42i+0.04j	0.04+0.42i+0.54j
Bob	0.00+0.20i+0.80j	0.24+0.52i+0.24j	0.54+0.42i+0.04j	0.03+0.34i+0.63j	0.02+0.26i+0.72j
Joe	0.72+0.26i+0.02j	0.72+0.26i+0.02j	0.00+0.40i+0.60j	0.06+0.38i+0.56j	0.00+0.50i+0.50j
Ted	0.54+0.42i+0.04j	0.30+0.50i+0.20j	0.18+0.54i+0.28j	0.56+0.38i+0.06j	0.18+0.54i+0.28j
Symptoms ↓	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$
s_1	0.40+0.60i+0.00j	0.70+0.30i+0.00j	0.21+0.58i+0.21j	0.03+0.34i+0.63j	0.02+0.26i+0.72j
s_2	0.15+0.50i+0.35j	0.08+0.44i+0.48j	0.54+0.42i+0.04j	0.12+0.56i+0.32j	0.00+0.20i+0.80j
s_3	0.03+0.34i+0.63j	0.00+0.10i+0.90j	0.06+0.38i+0.56j	0.80+0.20i+0.00j	0.04+0.32i+0.64j
s_4	0.28+0.54i+0.18j	0.70+0.30i+0.00j	0.08+0.44i+0.48j	0.06+0.38i+0.56j	0.04+0.32i+0.64j
s_5	0.03+0.34i+0.63j	0.02+0.26i+0.72j	0.01+0.18i+0.81j	0.06+0.38i+0.56j	0.72+0.26i+0.02j

in Eq. (6.3), the following indices have been computed corresponding to it:

$$
\begin{array}{c|ccccc|}
 & \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 & \mathcal{Q}_4 & \mathcal{Q}_5 \\
\text{Al} & 0.8790 & \mathbf{0.8899} & 0.8102 & 0.4788 & 0.3930 \\
\text{Bob} & 0.6165 & 0.4308 & 0.7599 & \mathbf{0.9541} & 0.6226 \\
\text{Joe} & 0.8061 & 0.7054 & \mathbf{0.8696} & 0.5279 & 0.4697 \\
\text{Ted} & \mathbf{0.8695} & 0.7949 & 0.7362 & 0.6033 & 0.4966
\end{array}
$$

On the other hand, if weights 0.15, 0.25, 0.20, 0.15 and 0.25 are assigned to $\mathcal{Q}_i(i =$

$1, 2, \ldots, 5$) respectively, then by applying Eq. (6.5), the following values are obtained:

$$\begin{array}{c|ccccc|} & \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 & \mathcal{Q}_4 & \mathcal{Q}_5 \\ \text{Al} & 0.8806 & \mathbf{0.8691} & 0.8455 & 0.5275 & 0.3913 \\ \text{Bob} & 0.6741 & 0.5074 & 0.7809 & \mathbf{0.9530} & 0.5566 \\ \text{Joe} & 0.8016 & 0.7025 & \mathbf{0.8787} & 0.5411 & 0.4323 \\ \text{Ted} & \mathbf{0.8654} & 0.7741 & 0.7425 & 0.6501 & 0.5182 \end{array}$$

Based on this analysis, it has been concluded that **Al** suffers from “Malaria”, **Bob** from “Stomach problems”, **Joe** from “Typhoid” and **Ted** from “Viral Fever”.

To check the performance of the developed method with some existing methods [34, 81, 206, 235], we applied them on to considered data and their obtained results are summarized in Table 6.4. From this table, it has been concluded by all these existing methods, the patient **Al** suffers from “Malaria”, **Bob** from “Stomach problems”, **Joe** from “Typhoid” and **Ted** from “Viral Fever” and it coincides with the proposed measures result.

Table 6.4: Measurement values of each patient with respect to the diseases of the existing methods

	By Zeng and Li [235] approach					By Xu et al. [206] approach				
	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$
Al	0.8360	**0.8914**	0.8122	0.4833	0.4499	0.7721	**0.8377**	0.7941	0.4779	0.4048
Bob	0.3474	0.2892	0.5525	**0.7846**	0.5449	0.2857	0.2741	0.4810	**0.6910**	0.5394
Joe	0.7583	0.6816	**0.8113**	0.5167	0.4661	0.6806	0.6591	**0.7708**	0.4965	0.4315
Ted	**0.8608**	0.8493	0.7447	0.6135	0.5647	**0.8190**	0.7013	0.6692	0.5451	0.4464
	By Ejegwa and Modom [34] approach					By Hung and Yang [81] approach				
	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$
Al	0.3200	**0.2600**	0.3200	0.4200	0.4400	0.8100	**0.8200**	0.8000	0.5400	0.5000
Bob	0.3700	0.3700	0.3700	**0.3300**	0.4500	0.6700	0.5400	0.7400	**0.9000**	0.6400
Joe	0.4400	0.4400	**0.4200**	0.4800	0.4600	0.7500	0.6800	**0.8200**	0.6000	0.5400
Ted	**0.3800**	0.3900	0.4400	0.4800	0.6000	**0.8000**	0.7700	0.7100	0.6300	0.5500

6.5.3 Advantages of proposed method

According to the above comparison analysis, the proposed method for addressing the DM problems has the following advantages.

(i) The existing studies based on the correlation measures [150, 206, 235] and their corresponding MADM approach fails to classify the objects under some certain cases, as demonstrated in Examples 6.2.2 - 6.2.4. However, the proposed measures have successfully overcome their shortcoming and hence the proposed methods based on the correlation coefficients give sufficient results to the decision makers to select the finest one(s).

(ii) The presented approach utilized the CNs to describe the information by considering the uncertain and uncertain information into a single platform, which results avoids the loss of information. Thus, the proposed measures based on CNs are so beneficial to handle the DM problems.

(iii) The results obtained by the proposed methods might be more accurate for it takes the hesitation degree into account. The proposed correlation coefficient is more generalized and suitable to solve the real-life problem more accurately than the existing ones.

6.6 Conclusion

The present chapter deals with the correlation coefficients by utilizing the concept of the CNs of the SPA theory. To do so, we firstly define the CN for an IFN and hence based on it, we construct the correlation coefficient between the pairs of the IFNs. The several shortcomings of the existing measures are highlighted and overcome by the proposed measures. The proposed correlation coefficients have suitably utilized the degree of certainty and uncertainties by the “identity”, “discrepancy” and “contrary” degrees in terms of the connection number. Later, we stated a weighted correlation coefficient measures to deal with the situations where the elements are correlated to each other. Finally, two illustrative examples from the field of pattern recognition and medical diagnosis are taken to demonstrate the approach. From the results, it has been concluded that the proposed measures are so beneficial while solving the DM problems as compared to some existing measures based approaches.

Chapter 7

Connection numbers based TOPSIS method under intuitionistic and interval-valued intuitionistic fuzzy sets[1]

In this chapter, we constructed the CNs for the IFSs as well as IVIFSs and hence based on it, we constructed the TOPSIS method for solving the DM problems. The basic feature of the TOPSIS method is chosen the alternative which has the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. Afterward, based on the proposed CNs, we define two algorithms to solve the DM problems where the features are extracted either in IFNs or in IVIFNs. The validity of these proposed algorithms is tested with an example and compared it with several existing results.

7.1 Introduction

The literature on the TOPSIS approaches are summarized in Section 1.1.2 of Chapter 1. TOPSIS is a well-known DM method. The basic features of this method are chosen the alternative in such a way that they have the shortest distance from the positive ideal solution

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[1] The content of this chapter is published as "Connection number of set pair analysis based TOPSIS method on intuitionistic fuzzy sets and their application to decision making", *Applied Intelligence*, 48(8), 2112 - 2119, 2018 (**Impact Factor: 2.882**) and "TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment", *Computational & Applied Mathematics*, 37(2), 1319 - 1329, 2018 (**Impact Factor: 1.260**)

and the farthest distance from the negative ideal solution. After their existence, many researchers are paying more attention to it and have widely applied in fuzzy multi-attribute DM problems. Chen [17] extended the TOPSIS approach to fuzzy DM by defining the crisp Euclidean distance between the two fuzzy numbers. In the context of IVIFS, Park et al. [126] extended the TOPSIS method from the intuitionistic fuzzy set to the IVIFSs in which all the preference information provided by the decision-makers are in the form of interval-valued intuitionistic fuzzy numbers (IVIFNs). Li [94] developed a TOPSIS-based nonlinear programming method for MADM problems. Ye [222] presented a novel accuracy method for solving the multi-criteria fuzzy DM problems under the IVIFSs environment. Bai [8] presented an interval-valued intuitionistic fuzzy TOPSIS approach based on an improved score function. Garg [41] presented the generalized improved score function for ranking the IVIFSs and showed that score function as proposed by [8] is a special case of it.

In light of the features of TOPSIS and CNs of the SPA theory, the objective of this chapter is to extend the existing theories based on CNs. For it, we firstly define the CNs for IFSs as well as IVIFSs and hence define a TOPSIS method under both the environment. The features of the proposed CNs are also discussed. Later, based on the construction of the CNs, we define two different approaches by utilizing the features of IFNs and IVIFNs respectively. The prominent characteristics of the methods are that they not only take into account the distance between the sets but also examine similar or dissimilar between them so as to avoid to draw the conclusion based on the small distance or large similarity. The presented approaches are illustrated with numerical examples to validate them. Further, the performance of the algorithms is compared with the several existing studies to support the work.

7.2 Connection number based TOPSIS method

The general description of the DM problem is given in the Section 2.7 of Chapter 2. The rating values corresponding to each alternative are represented in the decision matrix $\mathcal{M} = (\widetilde{\alpha}_{kt})_{m\times n}$. Then we have presented two different approaches for solving the MADM problem based on the TOPSIS method which is described as follows:

7.2.1 Proposed approach for IFSs

Hwang and Yoon [82] introduced the TOPSIS method to find out the best alternative based on the shortest distance from an ideal solution. In this section, TOPSIS method has been presented under IFS environment by using SPA and its connection number. For it, if an expert evaluate the given alternatives under the IFSs environment, then the decision matrix $\mathcal{M}$ is called as intuitionistic fuzzy decision matrix where $\widetilde{\alpha}_{kt} = (\widetilde{u}_{kt}, \widetilde{v}_{kt})$ for $k = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, n$. In order to balance the physical dimension, normalization process has been carried out by using Eq. (2.58) of Chapter 2 and hence obtain the decision matrix $\mathcal{R} = (r_{kt}) = (u_{kt}, v_{kt})$.

Based on the matrix $\mathcal{R}$, the positive ideal scheme (PIS) and negative ideal scheme (NIS) of the alternatives are computed and are denoted by $\mathcal{A}^+ = (u_t^+, v_t^+) = (\max\limits_k u_{kt}, \min\limits_k v_{kt})$ and $\mathcal{A}^- = (u_t^-, v_t^-) = (\min\limits_k u_{kt}, \max\limits_k v_{kt})$ respectively, for all $t = 1, 2, \ldots, n$. Thus, CN between the pairs of $(\mathcal{R}, \mathcal{A}^+)$ and $(\mathcal{R}, \mathcal{A}^-)$ are denoted by μ_{kt}^+ and μ_{kt}^- respectively, which are defined as

$$\mu_{kt}^+ = a_{kt}^+ + c_{kt}^+ j \tag{7.1}$$

$$\text{and} \quad \mu_{kt}^- = a_{kt}^- + c_{kt}^- j \tag{7.2}$$

where $a_{kt}^+ = \left(\frac{u_{kt}}{u_t^+} \times \frac{v_t^+}{v_{kt}}\right)$ and $a_{kt}^- = \left(\frac{u_t^-}{u_{kt}} \times \frac{v_{kt}}{v_t^-}\right)$ are identity degrees with proximity to PIS and NIS respectively, while $c_{kt}^+ = \left(\frac{u_t^+ - u_{kt}}{u_t^+} \times \frac{v_{kt} - v_t^+}{v_{kt}}\right)$ and $c_{kt}^- = \left(\frac{u_{kt} - u_t^-}{u_{kt}} \times \frac{v_t^- - v_{kt}}{v_t^-}\right)$ are contrary degrees which is remote from PIS and NIS respectively.

Now, based on these ideal schemes, the CN of the alternatives proximity to PIS and remote from NIS is defined as

$$\mu_{kt} = a_{kt} + c_{kt} j \tag{7.3}$$

where $a_{kt} = a_{kt}^+ \times c_{kt}^-$ represents the "identity degree" proximity to PIS or remote from NIS while $c_{kt} = c_{kt}^+ \times a_{kt}^-$ represents the "contrary degree" which is remote from PIS and proximity to NIS. Thus, the CNs for each alternative $\mathcal{A}_k (k = 1, 2, \ldots, m)$ are obtained under the set of attribute weights $\omega_t, t = 1, 2, \ldots, n$ as

$$\mu_{\mathcal{A}_k} = a_k + c_k j \tag{7.4}$$

where $a_k = \sum\limits_{t=1}^{n} \omega_t a_{kt}$ represents the overall "identity degree" between the alternative $\mathcal{A}_k$ and PIS, while $c_k = \sum\limits_{t=1}^{n} \omega_t c_{kt}$ represents the overall "contrary degree" between the alternative $\mathcal{A}_k$ and NIS. Hence, the relative closeness degree of an alternative $\mathcal{A}_k$ is defined as:

$$\Re(\mu_{\mathcal{A}_k}) = \frac{a_k}{a_k + c_k}; \qquad \text{provided} \;\; c_k \neq 0 \tag{7.5}$$

In a nutshell, the following steps are summarized based on the above theory to solve the MADM problems by using proposed TOPSIS approach.

Step 1: Normalize the decision matrix, if needed, by using Eq. (2.58) of Chapter 2 for each alternative.

Step 2: Determine the PIS and NIS of the alternatives.

Step 3: Utilize Eq. (7.3) to determine the CN of each alternative proximity to PIS and remote from NIS.

Step 4: Determine the relative weighted CN by using Eq. (7.4).

Step 5: Compute the closeness degrees $\Re(\mu_{\mathcal{A}_k})$ by Eq. (7.5) and hence choose the best alternative(s) accordingly.

7.2.2 Proposed approach for IVIFSs

In this section, TOPSIS method has been presented under IVIFS environment. To do so, assume that rating values $\widetilde{\alpha}_{kt} = ([\widetilde{u}_{kt}^L, \widetilde{u}_{kt}^U], [\widetilde{v}_{kt}^L, \widetilde{v}_{kt}^U])$ is represented in the IVIFNs. Then, these values are normalized into decision matrix $\mathcal{R} = (r_{ij}) = ([u_{kt}^L, u_{kt}^U], [v_{kt}^L, v_{kt}^U])$ where r_{ij} are calculated by following rules with the conditions that $u_{kt}^L \neq 0$.

For benefit type attributes

$$\left.\begin{array}{ll} u_{kt}^{L}=\dfrac{\widetilde{u}_{kt}^{L}}{\sqrt{\sum\limits_{k=1}^{m}(2-\widetilde{v}_{kt}^{L}-\widetilde{v}_{kt}^{U})^{2}}}, & u_{kt}^{U}=\dfrac{\widetilde{u}_{kt}^{U}}{\sqrt{\sum\limits_{k=1}^{m}(2-\widetilde{v}_{kt}^{L}-\widetilde{v}_{kt}^{U})^{2}}} \\ v_{kt}^{L}=1-\dfrac{(1-\widetilde{v}_{kt}^{L})}{\sqrt{\sum\limits_{k=1}^{m}(\widetilde{u}_{kt}^{L}+\widetilde{u}_{kt}^{U})^{2}}}, & v_{kt}^{U}=1-\dfrac{(1-\widetilde{v}_{kt}^{U})}{\sqrt{\sum\limits_{k=1}^{m}(\widetilde{u}_{kt}^{L}+\widetilde{u}_{kt}^{U})^{2}}} \end{array}\right\} \tag{7.6}$$

For cost type attributes

$$\left.\begin{array}{ll} u_{kt}^{L}=\dfrac{(1-\widetilde{v}_{kt}^{L})^{-1}}{\sqrt{\sum\limits_{k=1}^{m}((\widetilde{u}_{kt}^{L})^{-1}+(\widetilde{u}_{kt}^{U})^{-1})^{2}}}, & v_{kt}^{L}=1-\dfrac{(\widetilde{u}_{kt}^{L})^{-1}}{\sqrt{\sum\limits_{k=1}^{m}((1-\widetilde{v}_{kt}^{L})^{-1}+(1-\widetilde{v}_{kt}^{U})^{-1})^{2}}} \\ u_{kt}^{U}=\dfrac{(1-\widetilde{v}_{kt}^{U})^{-1}}{\sqrt{\sum\limits_{k=1}^{m}((\widetilde{u}_{kt}^{L})^{-1}+(\widetilde{u}_{kt}^{U})^{-1})^{2}}}, & v_{kt}^{U}=1-\dfrac{(\widetilde{u}_{kt}^{U})^{-1}}{\sqrt{\sum\limits_{k=1}^{m}((1-\widetilde{v}_{kt}^{L})^{-1}+(1-\widetilde{v}_{kt}^{U})^{-1})^{2}}} \end{array}\right\} \tag{7.7}$$

Based on the matrix $\mathcal{R}$, PIS and NIS of the alternative $\mathcal{A}_k$ are determined as $\mathcal{A}^+ = ([u_t^{L+}, u_t^{U+}], [v_t^{L+}, v_t^{U+}]) = ([\max\limits_k u_{kt}^L, \max\limits_k u_{kt}^U],\ [\min\limits_k v_{kt}^L, \min\limits_k v_{kt}^U])$ and $\mathcal{A}^- = ([u_t^{L-}, u_t^{U-}], [v_t^{L-}, v_t^{U-}]) = ([\min\limits_k u_{kt}^L, \min\limits_k u_{kt}^U], [\max\limits_k v_{t+}^U, \max\limits_k v_{kt}^U])$ respectively for all $t = 1, 2, \ldots, n$. Thus, the CN between the matrix $\mathcal{R}$ and ideal schemes, denoted by μ_{kt}^+ and μ_{kt}^- respectively, are computed as

$$\mu_{kt}^{+} = a_{kt}^{+} + c_{kt}^{+} j \tag{7.8}$$

$$\text{and} \qquad \mu_{kt}^{-} = a_{kt}^{-} + c_{kt}^{-} j \tag{7.9}$$

where the identity degrees which proximity to PIS and NIS respectively, are defined as

$$a_{kt}^{+} = \left(\frac{u_{kt}^{L}+u_{kt}^{U}}{u_{t}^{L+}+u_{t}^{U+}} \times \frac{v_{t}^{L+}+v_{t}^{U+}}{v_{kt}^{L}+v_{kt}^{U}}\right) \tag{7.10}$$

$$\text{and} \quad a_{kt}^{-} = \left(\frac{u_{t}^{L-}+u_{t}^{U-}}{u_{kt}^{L}+u_{kt}^{U}} \times \frac{v_{kt}^{L}+v_{kt}^{U}}{v_{t}^{L-}+v_{t}^{U-}}\right) \tag{7.11}$$

while contrary degrees which remote to PIS and NIS are defined as

$$c_{kt}^{+} = \left(\frac{(u_{t}^{L+}+u_{t}^{U+})-(u_{kt}^{L}+u_{kt}^{U})}{(u_{t}^{L+}+u_{t}^{U+})} \times \frac{(v_{kt}^{L}+v_{kt}^{U})-(v_{t}^{L+}+v_{t}^{U+})}{(v_{kt}^{L}+v_{kt}^{U})}\right) \tag{7.12}$$

$$\text{and} \quad c_{kt}^{-} = \left(\frac{(u_{kt}^{L}+u_{kt}^{U})-(u_{t}^{L-}+u_{t}^{U-})}{(u_{kt}^{L}+u_{kt}^{U})} \times \frac{(v_{t}^{L-}+v_{t}^{U-})-(v_{kt}^{L}+v_{kt}^{U})}{(v_{t}^{L-}+v_{t}^{U-})}\right) \tag{7.13}$$

respectively.

Now, the CN of the alternative proximity to PIS and remote from NIS are computed by Eq. (7.3) and the relative weighted CN are defined by Eq. (7.4). Hence, finally rank the alternatives based on the minimum value of the closeness degree as defined in Eq. (7.5).

Therefore, in a nutshell, the following steps are summarized based on the above theory to solve the MADM problems by using proposed TOPSIS approach under IVIFSs environment.

Step 1: Computed the normalized decision matrix by using Eqs. (7.6) and (7.7) for each alternative.

Step 2: Determine the PIS and NIS of the alternatives by using Eqs. (7.10) - (7.13).

Step 3: Utilize Eq. (7.3) to determine the CN of each alternative proximity to PIS and remote from NIS.

Step 4: Determine the relative weighted CN by using Eq. (7.4).

Step 5: Compute the closeness degrees $\Re(\mu_{\mathcal{A}_k})$ by Eq. (7.5) and hence choose the best alternative(s) accordingly.

7.3 Illustrative example for IFSs

The approaches described in Section 7.2.1 have been illustrated with a practical example of the DM which can be read as:

7.3.1 A case study

The Kedarnath valley, along with other parts of the state of Uttarakhand in northern India, was hit with an unprecedented flash floods in 2013. Large number of roads, which connect the Kedarnath valley to the other parts of Uttarakhand, had been destroyed in this flood. In this context, Uttrakhand government had to take a considerable number of road building projects either to maintain the roads already built or to undertake new roads.

These projects were carried out by a limited number of well-established contractors, and the selection process was on the basis of bid price alone. In recent years, the use of MADM methods have been demanded for increased project complexity, technical capability, higher performance, safety and financial requirements. For this, Uttarakhand government issued the notice in the newspapers, and considered the six attribute required for contractor selection, namely, tender price ($\mathcal{G}_1$), completion time ($\mathcal{G}_2$), technical capability ($\mathcal{G}_3$), financial status ($\mathcal{G}_4$), contractor background ($\mathcal{G}_5$), reference from previous project ($\mathcal{G}_6$) and assigned the weights of relative importance of each attributes as $\omega = (0.3, 0.2, 0.15, 0.1, 0.15, 0.1)^T$ on the basis of decision maker's preferences. The four contractors taken as in the form of the alternatives, namely, Jaihind Road Builders Pvt. Ltd. ($\mathcal{A}_1$), J.K. Construction ($\mathcal{A}_2$), Buildquick Infrastructure Pvt. Ltd. ($\mathcal{A}_3$), Relcon Intraprojects Ltd. ($\mathcal{A}_4$) bid for these projects. Then, the objective of the Government is to choose the best contractor among them for the task. In order to fulfill it, they evaluated these and gave their preferences in term of intuitionistic fuzzy numbers which are summarized in Table 7.1.

Table 7.1: Decision matrix of the alternatives in the form of IFNs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$(0.3, 0.7)$	$(0.5, 0.4)$	$(0.8, 0.2)$	$(0.5, 0.2)$	$(0.8, 0.1)$	$(0.6, 0.4)$
$\mathcal{A}_2$	$(0.5, 0.3)$	$(0.8, 0.2)$	$(0.5, 0.4)$	$(0.9, 0.1)$	$(0.6, 0.3)$	$(0.7, 0.2)$
$\mathcal{A}_3$	$(0.5, 0.4)$	$(0.9, 0.1)$	$(0.8, 0.2)$	$(0.9, 0.1)$	$(0.8, 0.2)$	$(0.6, 0.2)$
$\mathcal{A}_4$	$(0.6, 0.2)$	$(0.4, 0.3)$	$(0.8, 0.2)$	$(0.7, 0.2)$	$(0.4, 0.4)$	$(0.2, 0.8)$

The steps of the approach described in Section 7.2.1 are executed as follows:

Step 1: Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are the cost types attributes, so by using Eq. (2.58) of Chapter 2 we get the normalized decision-matrix which is summarized in Table 7.2.

Step 2: Based on PIS and NIS, the CNs of each alternative are computed by Eq. (7.3) and the obtained results are summarized in Table 7.3.

Step 3: The PIS and NIS of the alternative are evaluated and are given as

$$\begin{aligned} \mathcal{A}^+ &= \{(0.7, 0.3), (0.4, 0.4), (0.8, 0.2), (0.9, 0.1), (0.8, 0.1)(0.7, 0.2)\} \\ \mathcal{A}^- &= \{(0.2, 0.6), (0.1, 0.9), (0.5, 0.4), (0.5, 0.2), (0.4, 0.4), (0.2, 0.8)\} \end{aligned}$$

Table 7.2: Normalized decision matrix of IFSs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$(0.7, 0.3)$	$(0.4, 0.5)$	$(0.8, 0.2)$	$(0.5, 0.2)$	$(0.8, 0.1)$	$(0.6, 0.4)$
$\mathcal{A}_2$	$(0.3, 0.5)$	$(0.2, 0.8)$	$(0.5, 0.4)$	$(0.9, 0.1)$	$(0.6, 0.3)$	$(0.7, 0.2)$
$\mathcal{A}_3$	$(0.4, 0.5)$	$(0.1, 0.9)$	$(0.8, 0.2)$	$(0.9, 0.1)$	$(0.8, 0.2)$	$(0.6, 0.2)$
$\mathcal{A}_4$	$(0.2, 0.6)$	$(0.3, 0.4)$	$(0.8, 0.2)$	$(0.7, 0.2)$	$(0.4, 0.4)$	$(0.2, 0.8)$

Step 4: The weighted CN by taking weight vector ω are computed by Eq. (7.4) and get $\mu_{\mathcal{A}_1} = 0.2591 + 0.0234j$; $\mu_{\mathcal{A}_2} = 0.0860 + 0.1009j$, $\mu_{\mathcal{A}_3} = 0.1205 + 0.1048j$ and $\mu_{\mathcal{A}_4} = 0.0837 + 0.2249j$.

Step 5: The closeness degree of the alternatives are computed by Eq. (7.5) and get $\mathfrak{R}(\mu_{\mathcal{A}_1}) = 0.9171$, $\mathfrak{R}(\mu_{\mathcal{A}_2}) = 0.4600$, $\mathfrak{R}(\mu_{\mathcal{A}_3}) = 0.5350$ and $\mathfrak{R}(\mu_{\mathcal{A}_4}) = 0.2712$ and hence conclude that the best alternative is $\mathcal{A}_1$.

7.3.2 Comparative study

In order to compare the performance of the proposed approach with some existing approaches under the IFS environment, we conducted a comparative analysis based on different approaches as given by the authors in [8, 41–43, 89, 135, 139, 203, 207]. The results corresponding to these are summarized in Table 7.4. From these comparative studies, it has been concluded that the results, computed by the existing approaches, coincide with the proposed one which validates the proposed approach. Therefore, the proposed technique can be suitably utilized to solve the DM problem better than the other existing measures.

Table 7.3: Connection number of each alternative

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$0.3571+0.0000j$	$0.2667+0.0000j$	$0.1875+0.0000j$	$0.0000+0.2222j$	$0.3750+0.0000j$	$0.1429+0.0119j$
$\mathcal{A}_2$	$0.0143+0.1270j$	$0.0139+0.1111j$	$0.0000+0.1875j$	$0.2222+0.0000j$	$0.0208+0.0833j$	$0.5357+0.0000j$
$\mathcal{A}_3$	$0.0286+0.0714j$	$0.0000+0.4167j$	$0.1875+0.0000j$	$0.2222+0.0000j$	$0.1250+0.0000j$	$0.4286+0.0000j$
$\mathcal{A}_4$	$0.0000+0.3571j$	$0.2778+0.0000j$	$0.1875+0.0000j$	$0.0000+0.0794j$	$0.0000+0.3750j$	$0.0000+0.5357j$

Table 7.4: Comparative analysis

	Overall values of the alternatives				Ranking order
Xu [203]	0.4074	0.1425	0.2925	0.0435	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
Xu and Yager [207]	0.3122	-0.0973	-0.1147	-0.1527	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$
Joshi and Kumar [89]	0.7886	0.4423	0.5637	0.3793	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
Garg [41]	0.6647	0.4419	0.4393	0.3948	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Sahin [135]	0.6647	0.4419	0.4393	0.3948	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Bai [8]	0.6561	0.4415	0.4931	0.4042	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
Sivaraman et al. [139]	0.5247	0.3020	0.3085	0.2538	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
Garg [43]	-0.2260	-0.4172	-0.2389	-0.5215	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
Garg [42]	0.2907	-0.0928	-0.1105	-0.1309	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4$

7.3.3 Superiority of the proposed approach

In this section, we present some counterexamples which show that the existing TOPSIS methods under the IFS environment fail to rank the given alternatives while the proposed approach can overcome their shortcoming.

Example 7.3.1. Consider a DM problem in which there are two alternatives denoted by $\mathcal{A}_1$ and $\mathcal{A}_2$ which are evaluated by an expert under the set of three different attributes denoted by $\mathcal{G}_1$,$\mathcal{G}_2$ and $\mathcal{G}_3$. The objective of the problem is to find out the best alternative under the given set. In order to do so, an expert evaluated these alternatives and gave their preferences in terms of intuitionistic fuzzy numbers which are summarized as follows:

$$D = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1 \quad & \quad\mathcal{G}_2\quad & \quad\mathcal{G}_3\quad \end{array} \\ \begin{bmatrix} (0.5, 0.2) & (0.6, 0.3) & (0.3, 0.2) \\ (0.4, 0.3) & (0.6, 0.2) & (0.4, 0.3) \end{bmatrix} \end{array} \tag{7.14}$$

Based on this decision-matrix, by utilizing the existing TOPSIS approach [11] to find out the best alternative, the following steps are to be executed as:

Step 1: The information related to the alternatives is represented in the form of the decision matrix D as given in Eq. (7.14).

Step 2: The positive and negative ideal solutions of these two alternatives are found as $\mathcal{A}^+ = \{(0.5, 0.2), (0.6,\ 0.2),\ (0.4, 0.2)\}$ and $\mathcal{A}^- = \{(0.4, 0.3),\ (0.6, 0.3),\ (0.3, 0.3)\}$ respectively.

Step 3: Based on these values, the distance measure values between the alternatives $\mathcal{A}_i (i = 1, 2)$ from its ideals values are computed as $d(\mathcal{A}_1, \mathcal{A}^+) = 0.0667$, $d(\mathcal{A}_2, \mathcal{A}^+)= 0.0667$, $d(\mathcal{A}_1, \mathcal{A}^-) = 0.0667$ and $d(\mathcal{A}_2, \mathcal{A}^-) = 0.0667$.

Step 4: The relative closeness coefficient of each alternative is $C(\mathcal{A}_1) = \frac{d(\mathcal{A}_1,\mathcal{A}^-)}{d(\mathcal{A}_1,\mathcal{A}^-)+d(\mathcal{A}_1,\mathcal{A}^+)}$ $= 0.5$ and $C(\mathcal{A}_2) = \frac{d(\mathcal{A}_2,\mathcal{A}^-)}{d(\mathcal{A}_2,\mathcal{A}^-)+d(\mathcal{A}_2,\mathcal{A}^+)} = 0.5$. Since, $C(\mathcal{A}_1) = C(\mathcal{A}_2)$ and hence we conclude that the existing TOPSIS approach is unable to rank the given alternatives.

On the other hand, if we utilize the proposed approach for above considered data, then we get the relative closeness degrees of each alternative i.e., $\Re(\mu_{\mathcal{A}_1}) = 1$ and $\Re(\mu_{\mathcal{A}_2}) = 0$. Since $\Re(\mu_{\mathcal{A}_1}) > \Re(\mu_{\mathcal{A}_2})$ and hence conclude that the alternative $\mathcal{A}_1$ is better than $\mathcal{A}_2$. Therefore, the proposed approach is suitably working in those cases where the existing TOPSIS method fails.

Example 7.3.2. Consider another DM problem with two alternatives $\mathcal{A}_1$ and $\mathcal{A}_2$ which are evaluated under the set of the different attributes $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ whose weight vector is $\omega = (0.24, 0.40, 0.36)^T$. An expert evaluated these alternatives and gave their preferences in terms of intuitionistic fuzzy numbers, which are represented in the form of decision-matrix $\mathcal{M} = (\alpha_{kt})_{2\times 3}$, as follows:

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1 \quad\quad & \quad\quad \mathcal{G}_2 \quad\quad & \quad\quad \mathcal{G}_3 \quad \end{array} \\ \begin{bmatrix} (0.510, 0.361) & (0.660, 0.250) & (0.530, 0.290) \\ (0.532, 0.400) & (0.360, 0.520) & (0.760, 0.120) \end{bmatrix} \end{array}$$

If we utilize the intuitionistic fuzzy weighted averaging (IFWA) operators [203] to aggregate these alternatives, then the aggregating values of IFNs obtained are $(0.5829, 0.2880)$ and $(0.5829, 0.2880)$ respectively, of the alternatives $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus, we get the same values for both the alternatives and hence decision-makers will be unable to choose the best one for their decision.

On the other hand, if we apply the proposed DM method on this data, then the relative closeness degrees corresponding to alternative $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\Re(\mu_{\mathcal{A}_1}) = 0.5965$ and $\Re(\mu_{\mathcal{A}_2}) = 0.4035$ respectively. Hence, it is concluded that the alternative $\mathcal{A}_1$ is better than that of alternative $\mathcal{A}_2$.

7.4 Illustrative example for IVIFSs

The approaches described in Section 7.2.2 have been illustrated with a practical example of the DM which can be read as:

7.4.1 A case study

In order to demonstrate the approach, an illustrative example has been taken from the field of the investment where an investor wants to invest his/her money in a certain company. After careful analysis of the market, he/she considers the four possible alternatives, denoted by $\mathcal{A}_i (i = 1, 2, \ldots, 4)$; (i) $\mathcal{A}_1$ is a computer company, (ii) $\mathcal{A}_2$ is a furniture company, (iii) $\mathcal{A}_3$ is a car company, (iv) $\mathcal{A}_4$ is a chemical company. In evaluating these alternatives, the investor has summarized the ability of these companies with six attributes denoted by $\mathcal{G}_j (j = 1, 2, \ldots, 6)$ where $\mathcal{G}_1$: ability to bear risk, $\mathcal{G}_2$: expected benefit, $\mathcal{G}_3$: technical ability, $\mathcal{G}_4$: management capability, $\mathcal{G}_5$: competitive power on the market, $\mathcal{G}_6$: organizational culture. Let $\omega = (0.22, 0.07, 0.14, 0.02, 0.18, 0.37)^T$ be their corresponding weight vector and assume that expert offers their own opinion in the form of the IVIFNs which are summarized in Table 7.5.

Then, the steps of the approach described in Section 7.2.2 are executed as below:

Step 1: Normalized the data of Table 7.5 by Eqs. (7.6) and (7.7) and represented their results in Table 7.6.

Step 2: The PIS and NIS of the alternative are computed by Eqs. (7.10)-(7.13) and get

$$\begin{aligned}
\mathcal{A}^+ &= \big\{([0.2218, 0.2218], [0.5053, 0.5603]), ([0.2476, 0.2785], [0.5968, 0.6371]), \\
&\quad ([0.2147, 0.2454], [0.6454, 0.6848]), ([0.2251, 0.2532], [0.6456, 0.6810]), \\
&\quad ([0.2166, 0.2476], [0.5786, 0.6207]), ([0.1769, 0.2476], [0.5223, 0.5754])\big\} \\
\mathcal{A}^- &= \big\{([0.0739, 0.1109], [0.7801, 0.8351]), ([0.0928, 0.1238], [0.7177, 0.7581]), \\
&\quad ([0.1227, 0.1534], [0.6848, 0.7636]), ([0.0563, 0.1407], [0.6810, 0.7164]), \\
&\quad ([0.0619, 0.1238], [0.7050, 0.7471]), ([0.0354, 0.0708], [0.8408, 0.8939])\big\}
\end{aligned}$$

Step 3: The CNs of each alternative is constructed by using Eq. (7.3) and their corresponding values are summarized in Table 7.7.

Step 4: The weighted CNs of each alternative are computed by using Eq. (7.4) and get

$$\begin{aligned}
\mu_{\mathcal{A}_1} &= 0.0872 + 0.0840j, \qquad \mu_{\mathcal{A}_2} = 0.1245 + 0.0235j, \\
\mu_{\mathcal{A}_3} &= 0.1024 + 0.0071j, \qquad \mu_{\mathcal{A}_4} = 0.0467 + 0.2149j
\end{aligned}$$

Step 5: The overall performance value of each alternative $\mathcal{A}_k (k = 1, 2, 3, 4)$ is computed by using Eq. (7.5) as

$$\Re(\mu_{\mathcal{A}_1}) = 0.4962, \quad \Re(\mu_{\mathcal{A}_2}) = 0.8410, \quad \Re(\mu_{\mathcal{A}_3}) = 0.9352, \quad \Re(\mu_{\mathcal{A}_4}) = 0.1786$$

and hence the best alternative is $\mathcal{A}_3$.

7.4.2 Comparative study

In order to compare the performance of the proposed approach with the existing approaches as proposed by the authors [8, 119, 139, 182, 199, 200, 222] on IVIFSs, the analysis has been conducted and their overall performance values corresponding to each alternative is summarized in Table 7.8. Thus, from the comparative studies, it has been concluded that the results computed by the existing approaches coincide with the proposed one which validates the proposed approach.

7.5 Conclusion

The objective of this chapter is to present a CN of SPA based on TOPSIS methods for solving the DM problem under the IFSs and IVIFSs environment. In the evaluation process, based on the set pairs of the collective information data and positive, negative ideal schemes, connection number of each alternative with respect to attribute is constructed which is proximate to PIS and remote from NIS. Based on these CNs, a weighted connection number of each alternative scheme is determined to rank the alternatives based on closeness degree. The methods have been validated through a case study and comparison with the other existing techniques. From the results and corresponding comparative studied, it has been observed that ranking is identical and proposed DM method based on SPA can equivalently solve the DM problem efficiently and more suitable to manipulate the real world problem based on IVIFS. The superiority of the proposed methods over the several existing methods has also been demonstrated with some numerical examples.

Table 7.5: Rating values of the alternatives in terms of IVIFNs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$([0.2, 0.3], [0.6, 0.7])$	$([0.4, 0.5], [0.3, 0.4])$	$([0.7, 0.8], [0.1, 0.2])$	$([0.2, 0.5], [0.1, 0.2])$	$([0.7, 0.8], [0.0, 0.1])$	$([0.5, 0.6], [0.2, 0.4])$
$\mathcal{A}_2$	$([0.4, 0.5], [0.2, 0.3])$	$([0.6, 0.8], [0.1, 0.2])$	$([0.4, 0.5], [0.2, 0.4])$	$([0.8, 0.9], [0.1, 0.1])$	$([0.2, 0.6], [0.2, 0.3])$	$([0.5, 0.7], [0.1, 0.2])$
$\mathcal{A}_3$	$([0.4, 0.5], [0.3, 0.4])$	$([0.8, 0.9], [0.0, 0.1])$	$([0.5, 0.8], [0.1, 0.2])$	$([0.8, 0.9], [0.0, 0.1])$	$([0.7, 0.8], [0.1, 0.2])$	$([0.3, 0.6], [0.1, 0.2])$
$\mathcal{A}_4$	$([0.6, 0.6], [0.1, 0.2])$	$([0.3, 0.4], [0.2, 0.3])$	$([0.5, 0.8], [0.1, 0.2])$	$([0.6, 0.7], [0.1, 0.2])$	$([0.3, 0.4], [0.3, 0.4])$	$([0.1, 0.2], [0.7, 0.8])$

Table 7.6: Normalized decision matrix in terms of IVIFNs

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
$\mathcal{A}_1$	$([0.0739, 0.1109], [0.7801, 0.8351])$	$([0.1238, 0.1547], [0.7177, 0.7581])$	$([0.2147, 0.2454], [0.6454, 0.6848])$
$\mathcal{A}_2$	$([0.1478, 0.1848], [0.5603, 0.6152])$	$([0.1857, 0.2476], [0.6371, 0.6774])$	$([0.1227, 0.1534], [0.6848, 0.7636])$
$\mathcal{A}_3$	$([0.1478, 0.1848], [0.6152, 0.6702])$	$([0.2476, 0.2785], [0.5968, 0.6371])$	$([0.1534, 0.2454], [0.6454, 0.6848])$
$\mathcal{A}_4$	$([0.2218, 0.2218], [0.5053, 0.5603])$	$([0.0928, 0.1238], [0.6774, 0.7177])$	$([0.1534, 0.2454], [0.6454, 0.6848])$
	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$([0.0563, 0.1407], [0.6810, 0.7164])$	$([0.2166, 0.2476], [0.5786, 0.6207])$	$([0.1769, 0.2123], [0.5754, 0.6816])$
$\mathcal{A}_2$	$([0.2251, 0.2532], [0.6810, 0.6810])$	$([0.0619, 0.1857], [0.6628, 0.7050])$	$([0.1769, 0.2123], [0.5754, 0.6816])$
$\mathcal{A}_3$	$([0.2251, 0.2532], [0.6456, 0.6810])$	$([0.2166, 0.2476], [0.6207, 0.6628])$	$([0.1061, 0.2123], [0.5223, 0.5754])$
$\mathcal{A}_4$	$([0.1688, 0.1970], [0.6810, 0.7164])$	$([0.0928, 0.1238], [0.7050, 0.7471])$	$([0.0354, 0.0708], [0.8408, 0.8939])$

Table 7.7: Connection number of each alternative

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$
$\mathcal{A}_1$	$0.000 + 0.3389j$	$0.0000 + 0.1004j$	$0.0326 + 0.0000j$	$0.0000 + 0.0486j$	$0.1045 + 0.0000j$	$0.1603 + 0.0038j$
$\mathcal{A}_2$	$0.0823 + 0.0161j$	$0.0422 + 0.0081j$	$0.0000 + 0.0516j$	$0.0145 + 0.0000j$	$0.0068 + 0.0677j$	$0.2754 + 0.0000j$
$\mathcal{A}_3$	$0.0564 + 0.0323j$	$0.0964 + 0.0000j$	$0.0218 + 0.0000j$	$0.0298 + 0.0000j$	$0.0651 + 0.0000j$	$0.1836 + 0.0000j$
$\mathcal{A}_4$	$0.1985 + 0.0000j$	$0.0000 + 0.1076j$	$0.0218 + 0.0000j$	$0.0000 + 0.0105j$	$0.0000 + 0.1327j$	$0.0000 + 0.4952j$

Table 7.8: Comparative study

	Method	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	Ranking
Wei, Wang and Lin [182]	Correlation	0.2549	0.3422	0.1734	0.6035	$\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
Ye [222]	Novel accuracy	0.3066	0.2778	0.3172	0.1691	$\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
Bai [8]	Improved score function	0.5577	0.5707	0.5909	0.4798	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
Nayagam et al. [119]	New novel accuracy function	0.3266	0.4070	0.4651	-0.0171	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$
Sivaraman et al. [139]	New score function	0.3711	0.3583	0.3693	0.3597	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2$
Xu and Chen [199]	Averaging operator	0.4167	0.3381	0.4945	0.0955	$\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4$
Xu and Chen [200]	Geometric operator	0.1488	0.2933	0.3554	-0.1754	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$

Chapter 8

Linguistic connection number based aggregation operators for linguistic intuitionistic fuzzy set and its application[1]

In this chapter, we have enhanced the LIFS with the SPA theory and hence defined the linguistic connection number (LCN) and studied their various operational laws. Based on it, we have developed various aggregation operators namely, LCN weighted geometric (LCNWG), LCN ordered weighted geometric (LCNOWG), and LCN hybrid geometric (LCNHG) operators with LIFS environment. Also, the shortcoming of the existing operators under LIFS environment have been highlighted and overcomes by the proposed operators. Few properties of these operators have been also investigated. Further, a group DM approach has been presented, based on these operators, which has been illustrated by a numerical example to show the effectiveness and validity of the proposed approach.

8.1 Introduction

The overview of the aggregation operators to solve the DM problems is summarized in Section 1.1.3 of Chapter 1. As the study presented in the previous chapters are based on the analysis by utilizing the quantitative aspects only. But, in the real world, the

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decision maker may think to use of linguistic variables Zadeh [232] to represent their rating values towards the object. For instance, in order to measure the honesty of a person, he may use the some of the linguistic terms such as “extremely honest”, “honest”, “honest at a time”, “dishonest”, etc. Under such cases, linguistic information [73] provides us a more degree of freedom to analyze the imprecise and vague information. In the field of the aggregation process for the linguistic information, the linguistic weighted and ordered weighted averaging and geometric aggregation operators have been proposed by Xu [195, 196]. After their work, Chen et al. [27] extended these to linguistic intuitionistic fuzzy sets (LIFSs) by taking linguistic degrees of membership and non-membership and hence proposed their corresponding aggregation operators. Also, Zhang [236] defined some series of aggregation operators for LIFSs information. Peng et al. [127] defined Frank operations and Frank Heronian mean operator with application to evaluating coal mine safety under LIFSs environment. The above-mentioned work and their theories have been widely used by the researchers, but credibility is not guaranteed.

In order to address the problems and to handle the information in a more precise manner, in this chapter, we have extended the theory of the linguistic intuitionistic fuzzy set under the SPA theory. In order to achieve it, firstly linguistic connection number (LCN) has been proposed under the IFSs environment and hence discuss some of its operational laws. Further, based on their operational laws, some geometric aggregation operators under the linguistic intuitionistic fuzzy set environment namely, linguistic connection number weighted geometric (LCNWG), linguistic connection number ordered weighted geometric (LCNOWG), and linguistic connection number hybrid geometric (LCNHG) operators are proposed. Some of the desirable properties of these operators have been investigated. As far as we know, under the SPA, aggregation operators based on the CN and linguistic information have not been reported in the existing literature. Therefore, there is a growing interest in the direction of the aggregation operators under this linguistic fuzzy environment. Thus, under the linguistic intuitionistic fuzzy environment, the objective of this chapter is divided into four parts:

i) to propose LCN by combining the features of LIFSs and CN of SPA theories,

ii) to introduce some basic operational laws of LCN,

iii) to develop the various geometric aggregation operators under LIFSs environment and their properties, and

iv) to establish a MAGDM method based on these proposed operators.

8.2 Shortcoming of the existing operators

From the existing LIFWG operator under the LIFS environment, it has been observed that they have some sort of deficiencies, which has been explained with some counterexamples as follows:

Example 8.2.1. Let $S_{[0,8]} = \{s_z \mid s_0 \leq s_z \leq s_8\}$ be a continuous linguistic term set and $A = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ be a collection of LIFNs, where $\mathcal{A}_1 = (s_4, s_4), \mathcal{A}_2 = (s_2, s_5)$, and $\mathcal{A}_3 = (s_0, s_8)$ be three LIFNs and $\omega = (0.3, 0.5, 0.2)^T$ is the weight vector of these numbers. By applying the LIFWG operator as defined in Eq. (2.50) of Chapter 2 to aggregate these LIFNs, we get LIFWG$(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = (s_0, s_8)$. On other hand, if we take another collections of LIFNs $B = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, where $\mathcal{B}_1 = (s_3, s_4), \mathcal{B}_2 = (s_0, s_8)$, and $\mathcal{B}_3 = (s_2, s_2)$ corresponding to the same weight set, then we get LIFWG$(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) = (s_0, s_8)$. Thus, the score and accuracy functions, for A and B are same and cannot rank the alternatives. Therefore, it is difficult to choose the best alternatives among the existing ones by using the LIFWG operator.

Example 8.2.2. Let $S_{[0,8]}$ be a continuous linguistic term set and let $A = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$ be a collection of LIFNs, where $\mathcal{A}_1 = (s_4, s_3)$, $\mathcal{A}_2 = (s_2, s_4), \mathcal{A}_3 = (s_5, s_1)$ and $\mathcal{A}_4 = (s_4, s_2)$ be four LIFNs and $\omega = (0.3, 0.2, 0.3, 0.2)^T$ be associate weight vector of these four LIFNs. If we aggregate these LIFNs by using LIFWG operator, then we get LIFWG$(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) = (s_{3.7233}, s_{2.5139})$. On other hand, if we change the non-membership degree of each LIFN and get a another collection $B = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$, where $\mathcal{B}_1 = (s_4, s_1)$, $\mathcal{B}_2 = (s_2, s_2)$, $\mathcal{B}_3 = (s_5, s_3)$ and $\mathcal{B}_4 = (s_4, s_4)$, then LIFWG$(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4) = (s_{3.7233}, s_{2.5139})$. Thus, we conclude that by changing the degree of non-membership values of the LIFNs, the corresponding effect on the aggregated non-membership degree is independent. Hence score and accuracy functions, for A and B are same and cannot rank the alternatives. Therefore, it is difficult to choose the best alternative among the existing ones by using the LIFWG operator.

Therefore, it has been analyzed that the existing LIFWG operator is invalid to rank the alternatives and hence there is a necessity to pay more attention to this issue and to need other measuring functions. For this, the new aggregation operator has been introduced by using the Set pair analysis (SPA) theory.

8.3 Linguistic connection number

In this section, we have presented the concept of the linguistic connection number (LCN) for the LIFNs using the SPA theory.

Definition 8.3.1. Let $S_{[0,h]}$ be a continuous linguistic term set. A linguistic connection number is defined as

$$\mu = s_a + s_b i + s_c j \tag{8.1}$$

where $s_a, s_b, s_c \in S_{[0,h]}$ represents the "identity", "discrepancy" and "contrary" degrees respectively such that $0 \leq a, b, c \leq h$ and $a + b + c = h$; $i \in [-1, 1]$ and $j = -1$ are the coefficients of "discrepancy and contrary" degrees respectively.

Definition 8.3.2. Let $\mu = s_a + s_b i + s_c j$ be a LCN, then relatively certainty probability power $P(\mu)$ of the LCN μ is defined as

$$P(\mu) = s_{\left(h + \frac{2a}{b+c} - \frac{c}{a+b}\right)/2}. \tag{8.2}$$

The larger the value of $P(\mu)$, the greater the LCN μ.

Definition 8.3.3. Let $X = \{x_1, x_2, \ldots, x_n\}$ be finite universal set and $\mathcal{A}_t = (s_{\tau_t}, s_{\theta_t})$ $t = 1, 2, \ldots, n$ be any collection of LIFNs. Then, their corresponding LCN μ_t defined as

$$\mu_t = s_{a_t} + s_{b_t} i + s_{c_t} j \tag{8.3}$$

where $a_t = \frac{\tau_t(h-\theta_t)}{h}$ "identity", $b_t = h - \frac{\tau_t(h-\theta_t)}{h} - \frac{\theta_t(h-\tau_t)}{h}$ "discrepancy" and $c_t = \frac{\theta_t(h-\tau_t)}{h}$ "contrary" degrees.

Definition 8.3.4. Let $\mu_1 = s_{a_1} + s_{b_1}i + s_{c_1}j$ and $\mu_2 = s_{a_2} + s_{b_2}i + s_{c_2}j$ be two LCNs, then

(i) $\mu_1 = \mu_2 \Leftrightarrow s_{a_1} = s_{a_2}, s_{b_1} = s_{b_2}, s_{c_1} = s_{c_2} \Leftrightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2$;

(ii) $\mu_1 \leq \mu_2$ if $s_{a_1} \leq s_{a_2}, s_{c_1} \geq s_{c_2} \Leftrightarrow a_1 \leq a_2, c_1 \geq c_2$;

(iii) $\mu_1^c = s_{c_1} + s_{b_1}i + s_{a_1}j$;

(iv) $\mu_1 \cup \mu_2 = \max\{s_{a_1}, s_{a_2}\} + \big(s_h - \max\{s_{a_1}, s_{a_2}\} - \min\{s_{c_1}, s_{c_2}\}\big)i + \big(\min\{s_{c_1}, s_{c_2}\}\big)j$;

(v) $\mu_1 \cap \mu_2 = \min\{s_{a_1}, s_{a_2}\} + (s_h - \min\{s_{a_1}, s_{a_2}\} - \max\{s_{c_1}, s_{c_2}\})i + (\max\{s_{c_1}, s_{c_2}\})j$.

Definition 8.3.5. Let $\mu_1 = s_{a_1} + s_{b_1}i + s_{c_1}j$ and $\mu_2 = s_{a_2} + s_{b_2}i + s_{c_2}j$ be two LCNs, and $\lambda > 0$, then some operations between μ_1 and μ_2 defined as follows:

(i) $\mu_1 \otimes \mu_2 = s_{h\left(\frac{a_1 a_2}{h^2}\right)} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)\left(1-\frac{b_2}{h}\right)\right)}i + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)\left(\frac{a_2}{h}+\frac{c_2}{h}\right)-\frac{a_1 a_2}{h^2}\right)}j$

(ii) $\mu_1^\lambda = s_{h\left(\frac{a_1}{h}\right)^\lambda} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^\lambda\right)}i + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^\lambda-\left(\frac{a_1}{h}\right)^\lambda\right)}j$

Theorem 8.3.1. If μ_1 and μ_2 be two LCNs, then operations defined in Definition 8.3.5 are also LCNs.

Proof. Since $\mu_1 = s_{a_1} + s_{b_1}i + s_{c_1}j$ and $\mu_2 = s_{a_2} + s_{b_2}i + s_{c_2}j$ be two LCNs, therefore $0 \leq a_1, b_1, c_1, a_2, b_2, c_2 \leq h$ and $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = h$. Then we have

(i) Since $0 \leq a_1, a_2 \leq h$ which implies that $0 \leq h\left(\frac{a_1 a_2}{h^2}\right) \leq h$. Similarly, for $0 \leq b_1, b_2 \leq h$ and $0 \leq c_1, c_2 \leq h$, we have

$$\begin{aligned} & 0 \leq \left(1 - \frac{b_1}{h}\right), \left(1 - \frac{b_2}{h}\right) \leq 1 \\ \Rightarrow \quad & 0 \leq \left(1 - \frac{b_1}{h}\right)\left(1 - \frac{b_2}{h}\right) \leq 1 \\ \Rightarrow \quad & 0 \leq h\left(1 - \left(1 - \frac{b_1}{h}\right)\left(1 - \frac{b_2}{h}\right)\right) \leq h \end{aligned}$$

and

$$0 \leq h\left(\left(\frac{a_1}{h} + \frac{c_1}{h}\right)\left(\frac{a_2}{h} + \frac{c_2}{h}\right) - \frac{a_1 a_2}{h^2}\right) \leq h.$$

Now,

$$\begin{aligned} & h\left(\frac{a_1a_2}{h^2}\right)+h\left(1-\left(1-\frac{b_1}{h}\right)\left(1-\frac{b_2}{h}\right)\right)+h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)\left(\frac{a_2}{h}+\frac{c_2}{h}\right)-\frac{a_1a_2}{h^2}\right) \\ = \; & h\left(\frac{a_1a_2}{h^2}+1-\left(\frac{a_1}{h}+\frac{c_1}{h}\right)\left(\frac{a_2}{h}+\frac{c_2}{h}\right)+\left(\frac{a_1}{h}+\frac{c_1}{h}\right)\left(\frac{a_2}{h}+\frac{c_2}{h}\right)+\frac{a_1a_2}{h^2}\right) \\ = \; & h \end{aligned}$$

Hence $\mu_1 \otimes \mu_2$ is also a LCN.

(ii) For $\lambda > 0$, we have $0 \le h\left(\frac{a_1}{h}\right)^\lambda \le h, 0 \le h\left(1-\left(1-\frac{b_1}{h}\right)^\lambda\right) = h\left(1-\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^\lambda\right) \le h, 0 \le h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^\lambda - \left(\frac{a_1}{h}\right)^\lambda\right) \le h$ and

$$\begin{aligned} & h\left(\frac{a_1}{h}\right)^\lambda + h\left(1-\left(1-\frac{b_1}{h}\right)^\lambda\right)+h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^\lambda-\left(\frac{a_1}{h}\right)^\lambda\right) \\ = \; & h\left(\left(\frac{a_1}{h}\right)^\lambda+1-\left(1-\frac{b_1}{h}\right)^\lambda+\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^\lambda-\left(\frac{a_1}{h}\right)^\lambda\right) \\ = \; & h\left(\left(\frac{a_1}{h}\right)^\lambda+1-\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^\lambda+\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^\lambda-\left(\frac{a_1}{h}\right)^\lambda\right) \\ = \; & h \end{aligned}$$

Hence, μ_1^λ is also a LCN.

□

Theorem 8.3.2. If μ_1 and μ_2 be two LCNs with $\lambda, \lambda_1, \lambda_2 > 0$ be three real numbers. Then, the following properties hold:

(i) $\mu_1 \otimes \mu_2 = \mu_2 \otimes \mu_1$.

(ii) $\mu_1^{\lambda_1} \otimes \mu_1^{\lambda_2} = \mu_1^{\lambda_1+\lambda_2}$.

(iii) $\mu_1^\lambda \otimes \mu_2^\lambda = (\mu_1 \otimes \mu_2)^\lambda$.

(iv) $\mu_1^c \cup \mu_2^c = (\mu_1 \cap \mu_2)^c$.

(v) $\mu_1^c \cap \mu_2^c = (\mu_1 \cup \mu_2)^c$.

(vi) $(\mu_1 \cup \mu_2) \cap \mu_2 = \mu_2$.

(vii) $(\mu_1 \cap \mu_2) \cup \mu_2 = \mu_2$.

(viii) $\mu_1 \cup \mu_2 = \mu_2 \cup \mu_1$.

(ix) $\mu_1 \cap \mu_2 = \mu_2 \cap \mu_1$.

Proof. Here, we shall prove only the parts (i)-(iii), while rest can be proven similarly.

(i) Let $\mu_1 = s_{a_1} + s_{b_1} i + s_{c_1} j$ and $\mu_2 = s_{a_2} + s_{b_2} i + s_{c_2} j$ be two LCNs, then we have

$$\begin{aligned}
\mu_1 \otimes \mu_2 &= s_{\left(\frac{a_1 a_2}{h}\right)} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)\left(1-\frac{b_2}{h}\right)\right)} i + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)\left(\frac{a_2}{h}+\frac{c_2}{h}\right)-\frac{a_1 a_2}{h^2}\right)} j \\
&= s_{\left(\frac{a_2 a_1}{h}\right)} + s_{h\left(1-\left(1-\frac{b_2}{h}\right)\left(1-\frac{b_1}{h}\right)\right)} i + s_{h\left(\left(\frac{a_2}{h}+\frac{c_2}{h}\right)\left(\frac{a_1}{h}+\frac{c_1}{h}\right)-\frac{a_2 a_1}{h^2}\right)} j \\
&= \mu_2 \otimes \mu_1
\end{aligned}$$

(ii) For $\lambda_1, \lambda_2 > 0$, we have

$$\mu_1^{\lambda_1} = s_{h\left(\frac{a_1}{h}\right)^{\lambda_1}} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^{\lambda_1}\right)} i + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^{\lambda_1}-\left(\frac{a_1}{h}\right)^{\lambda_1}\right)} j$$

and

$$\mu_1^{\lambda_2} = s_{h\left(\frac{a_1}{h}\right)^{\lambda_2}} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^{\lambda_2}\right)} i + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^{\lambda_2}-\left(\frac{a_1}{h}\right)^{\lambda_2}\right)} j$$

then,

$$\begin{aligned}
\mu_1^{\lambda_1} \otimes \mu_1^{\lambda_2} &= s_{h\left(\frac{a_1}{h}\right)^{\lambda_1}\left(\frac{a_1}{h}\right)^{\lambda_2}} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^{\lambda_1}\left(1-\frac{b_1}{h}\right)^{\lambda_2}\right)} i \\
&\qquad + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^{\lambda_1}\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^{\lambda_2}-\left(\frac{a_1}{h}\right)^{\lambda_1}\left(\frac{a_1}{h}\right)^{\lambda_2}\right)} j \\
&= s_{h\left(\frac{a_1}{h}\right)^{\lambda_1+\lambda_2}} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^{\lambda_1+\lambda_2}\right)} i \\
&\qquad + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^{\lambda_1+\lambda_2}-\left(\frac{a_1}{h}\right)^{\lambda_1+\lambda_2}\right)} j \\
&= \mu_1^{\lambda_1+\lambda_2}
\end{aligned}$$

(iii) For $\lambda > 0$, we have

$$\mu_1^{\lambda} = s_{h\left(\frac{a_1}{h}\right)^{\lambda}} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^{\lambda}\right)} i + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^{\lambda}-\left(\frac{a_1}{h}\right)^{\lambda}\right)} j$$

and

$$\mu_2^{\lambda} = s_{h\left(\frac{a_2}{h}\right)^{\lambda}} + s_{h\left(1-\left(1-\frac{b_2}{h}\right)^{\lambda}\right)} i + s_{h\left(\left(\frac{a_2}{h}+\frac{c_2}{h}\right)^{\lambda}-\left(\frac{a_2}{h}\right)^{\lambda}\right)} j.$$

Therefore,

$$\begin{aligned}
\mu_1^\lambda \otimes \mu_2^\lambda &= s_{h\left(\frac{a_1}{h}\right)^\lambda\left(\frac{a_2}{h}\right)^\lambda} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^\lambda\left(1-\frac{b_2}{h}\right)^\lambda\right)} i + \\
&\qquad s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^\lambda\left(\frac{a_2}{h}+\frac{c_2}{h}\right)^\lambda-\left(\frac{a_1}{h}\right)^\lambda\left(\frac{a_2}{h}\right)^\lambda\right)} j \\
&= s_{h\left(\frac{a_1a_2}{h^2}\right)^\lambda} + s_{h\left(1-\left(\left(1-\frac{b_1}{h}\right)\left(1-\frac{b_2}{h}\right)\right)^\lambda\right)} i + \\
&\qquad s_{h\left(\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)\left(\frac{a_2}{h}+\frac{c_2}{h}\right)\right)^\lambda-\left(\frac{a_1a_2}{h^2}\right)^\lambda\right)} j \\
&= (\mu_1 \otimes \mu_2)^\lambda.
\end{aligned}$$

□

8.4 Aggregation operators for linguistic connection numbers

In this section, we develop some new aggregation operators namely, LCN weighted geometric (LCNWG), LCN ordered weighted geometric (LCNOWG) and LCN hybrid geometric (LCNHG) operators, and some desirable properties based on these. For convenience, let $X = \{x_1, x_2, \dots x_n\}$ be finite universal sets and Ω be the set of all LCNs $\mu = s_a + s_b i + s_c j$.

8.4.1 LCN weighted geometric operator

Here, we shall propose weighted geometric operator to aggregate the LCNs.

Definition 8.4.1. Let $\mu_t = s_{a_t} + s_{b_t} i + s_{c_t} j$, $(t = 1, 2, \dots, n)$ be a collection of LCNs. The linguistic connection number weighted geometric (LCNWG) operator is a mapping $\mathrm{LCNWG} : \Omega^n \to \Omega$, such that

$$\mathrm{LCNWG}_\omega(\mu_1, \mu_2, \dots \mu_n) = \mu_1^{\omega_1} \otimes \mu_2^{\omega_2} \otimes \dots \otimes \mu_n^{\omega_n} \tag{8.4}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of μ_t $(t = 1, 2, \dots, n)$, and $\omega_t > 0$, $\sum\limits_{t=1}^{n} \omega_t = 1$. Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the LCNWG operator is reduced to linguistic connection number geometric (LCNG) operator, which is defined as

$$\mathrm{LCNG}_\omega(\mu_1, \mu_2, \dots \mu_n) = (\mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_n)^{\frac{1}{n}}$$

Theorem 8.4.1. For a collection of "n" LCNs $\mu_t = s_{a_t} + s_{b_t} i + s_{c_t} j$ and $\omega_t > 0$ be the weight vector of μ_t with $\sum\limits_{t=1}^{n} \omega_t = 1$, then their aggregated value by using Definition 8.4.1 is also a LCN, and is given as

$$\begin{aligned} \mathrm{LCNWG}_{\omega}(\mu_1, \mu_2, \ldots \mu_n) &= s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t}\right)} + s_{h\left(1-\prod\limits_{t=1}^{n}\left(1-\frac{b_t}{h}\right)^{\omega_t}\right)} i \\ &\quad + s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{a_t}{h}+\frac{c_t}{h}\right)^{\omega_t} - \prod\limits_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t}\right)} j \end{aligned} \tag{8.5}$$

Proof. The first result holds immediately from the Definition 8.3.5. Now by using the principal of mathematical induction, we prove the Eq. (8.5).

Firstly, for $n = 2$, by Definition 8.3.5, we have

$$\begin{aligned} \mu_1^{\omega_1} &= s_{h\left(\frac{a_1}{h}\right)^{\omega_1}} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^{\omega_1}\right)} i + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^{\omega_1}-\left(\frac{a_1}{h}\right)^{\omega_1}\right)} j \\ \text{and } \mu_2^{\omega_2} &= s_{h\left(\frac{a_2}{h}\right)^{\omega_2}} + s_{h\left(1-\left(1-\frac{b_2}{h}\right)^{\omega_2}\right)} i + s_{h\left(\left(\frac{a_2}{h}+\frac{c_2}{h}\right)^{\omega_2}-\left(\frac{a_2}{h}\right)^{\omega_2}\right)} j \end{aligned}$$

then,

$$\begin{aligned} \mathrm{LCNWG}_{\omega}(\mu_1, \mu_2) &= \mu_1^{\omega_1} \otimes \mu_2^{\omega_2} \\ &= s_{h\left(\frac{a_1}{h}\right)^{\omega_1}\left(\frac{a_2}{h}\right)^{\omega_2}} + s_{h\left(1-\left(1-\frac{b_1}{h}\right)^{\omega_1}\left(1-\frac{b_2}{h}\right)^{\omega_2}\right)} i \\ &\quad + s_{h\left(\left(\frac{a_1}{h}+\frac{c_1}{h}\right)^{\omega_1}\left(\frac{a_2}{h}+\frac{c_2}{h}\right)^{\omega_2}-\left(\frac{a_1}{h}\right)^{\omega_1}\left(\frac{a_2}{h}\right)^{\omega_2}\right)} j \end{aligned}$$

Hence, result is true for $n = 2$.

Assume that Eq. (8.5) holds for $n = k$, that is,

$$\begin{aligned} & \mathrm{LCNWG}_{\omega}(\mu_1, \mu_2, \ldots \mu_k) \\ = \; & \mu_1^{\omega_1} \otimes \mu_2^{\omega_2} \otimes \ldots \otimes \mu_k^{\omega_k} \\ = \; & s_{h\left(\prod\limits_{t=1}^{k}\left(\frac{a_t}{h}\right)^{\omega_t}\right)} + s_{h\left(1-\prod\limits_{t=1}^{k}\left(1-\frac{b_t}{h}\right)^{\omega_t}\right)} i + s_{h\left(\prod\limits_{t=1}^{k}\left(\frac{a_t}{h}+\frac{c_t}{h}\right)^{\omega_t}-\prod\limits_{t=1}^{k}\left(\frac{a_t}{h}\right)^{\omega_t}\right)} j \end{aligned}$$

then, for $n = k + 1$, by Definition 8.3.5, we have

$$\begin{aligned} & \mathrm{LCNWG}_{\omega}(\mu_1, \mu_2, \ldots \mu_{k+1}) \\ = \; & \mathrm{LCNWG}_{\omega}(\mu_1, \mu_2, \ldots \mu_k) \otimes \mu_{k+1}^{\omega_{k+1}} \end{aligned}$$

$$\begin{aligned} &= s_{h\left(\prod_{t=1}^{k}\left(\frac{a_t}{h}\right)^{\omega_t}\left(\frac{a_{k+1}}{h}\right)^{\omega_{k+1}}\right)} + s_{h\left(1-\prod_{t=1}^{k}\left(1-\frac{b_t}{h}\right)^{\omega_t}\left(1-\frac{b_{k+1}}{h}\right)^{\omega_{k+1}}\right)} i \\ &\quad + s_{h\left(\prod_{t=1}^{k}\left(\frac{a_t}{h}+\frac{c_t}{h}\right)^{\omega_t}\left(\frac{a_{k+1}}{h}+\frac{c_{k+1}}{h}\right)^{\omega_{k+1}}-\prod_{t=1}^{k}\left(\frac{a_t}{h}\right)^{\omega_t}\left(\frac{a_{k+1}}{h}\right)^{\omega_{k+1}}\right)} j \\ &= s_{h\left(\prod_{t=1}^{k+1}\left(\frac{a_t}{h}\right)^{\omega_t}\right)} + s_{h\left(1-\prod_{t=1}^{k+1}\left(1-\frac{b_t}{h}\right)^{\omega_t}\right)} i + s_{h\left(\prod_{t=1}^{k+1}\left(\frac{a_t}{h}+\frac{c_t}{h}\right)^{\omega_t}-\prod_{t=1}^{k+1}\left(\frac{a_t}{h}\right)^{\omega_t}\right)} j \end{aligned}$$

i.e., Eq. (8.5) holds for $n = k + 1$. Hence the theorem. □

Example 8.4.1. Let $S_{[0,8]} = \left\{ s_z \mid s_0 \leq s_z \leq s_8 \right\}$ be a continuous LTS and let $\mu_1 = s_1 + s_3 i + s_4 j$, $\mu_2 = s_3 + s_2 i + s_3 j$ and $\mu_3 = s_4 + s_3 i + s_1 j$ be three LCNs and $\omega = (0.3, 0.4, 0.3)^T$ be the weight vector of $\mu_t (t = 1, 2, 3)$, then

$$\begin{aligned} &\mathrm{LCNWG}_{\omega}(\mu_1, \mu_2, \mu_3) \\ &= s_{8\left(\prod_{t=1}^{3}\left(\frac{a_t}{8}\right)^{\omega_t}\right)} + s_{8\left(1-\prod_{t=1}^{3}\left(1-\frac{b_t}{8}\right)^{\omega_t}\right)} i + s_{8\left(\prod_{t=1}^{3}\left(\frac{a_t}{8}+\frac{c_t}{8}\right)^{\omega_t}-\prod_{t=1}^{3}\left(\frac{a_t}{8}\right)^{\omega_t}\right)} j \\ &= s_{8\left(\left(\frac{1}{8}\right)^{0.3}\times\left(\frac{3}{8}\right)^{0.4}\times\left(\frac{4}{8}\right)^{0.3}\right)} + s_{8\left(1-\left(1-\frac{3}{8}\right)^{0.3}\times\left(1-\frac{2}{8}\right)^{0.4}\times\left(1-\frac{3}{8}\right)^{0.3}\right)} i \\ &\qquad + s_{8\left(\left(\frac{1}{8}+\frac{4}{8}\right)^{0.3}\times\left(\frac{3}{8}+\frac{3}{8}\right)^{0.4}\times\left(\frac{4}{8}+\frac{1}{8}\right)^{0.3}-\left(\frac{1}{8}\right)^{0.3}\times\left(\frac{3}{8}\right)^{0.4}\times\left(\frac{4}{8}\right)^{0.3}\right)} j \\ &= s_{2.3522} + s_{2.6217} i + s_{3.0261} j \end{aligned}$$

Now based on the proposed operator, it has been analyzed that it successfully overcome the drawbacks of the existing operators of the LIFS theory as described in Examples 8.2.1 and 8.2.2.

Example 8.4.2. If LCNWG_{ω} operator utilized to aggregate the LIFNs as given in the Example 8.2.1. For this, firstly we convert the LIFNs to its LCNs by using the Definition 8.3.3 and their values summarized as $\mu_{\mathcal{A}_1} = s_2 + s_4 i + s_2 j$, $\mu_{\mathcal{A}_2} = s_{0.75} + s_{3.5} i + s_{3.75} j$, $\mu_{\mathcal{A}_3} = s_0 + s_0 i + s_8 j$ and $\mu_{\mathcal{B}_1} = s_{1.5} + s_4 i + s_{2.5} j$, $\mu_{\mathcal{B}_2} = s_0 + s_0 i + s_8 j$, $\mu_{\mathcal{B}_3} = s_{1.5} + s_5 i + s_{1.5} j$. Then $\mathrm{LCNWG}_{\omega}(\mu_{\mathcal{A}_1}, \mu_{\mathcal{A}_2}, \mu_{\mathcal{A}_3}) = s_0 + s_{3.1265} i + s_{4.8735} j$ and $\mathrm{LCNWG}_{\omega}(\mu_{\mathcal{B}_1}, \mu_{\mathcal{B}_2}, \mu_{\mathcal{B}_3}) = s_0 + s_{2.6594} i + s_{5.3406} j$. Thus, by using the Eq. (8.2), we get $P(A) = s_{3.2206}$ and $P(B) = s_{2.9959}$. Since $P(A) > P(B)$, therefore $A \succ B$.

Example 8.4.3. If we considered the data set as given in the Example 8.2.2 to find out the best one among them, then the proposed operator LCNWG_{ω} have been used to it. In order to achieve it, the LCNs corresponding to LIFNs are computed, by using the Definition

8.3.3 as $\mu_{\mathcal{A}_1} = s_{2.5} + s_4 i + s_{1.5} j$, $\mu_{\mathcal{A}_2} = s_1 + s_4 i + s_3 j$, $\mu_{\mathcal{A}_3} = s_{4.375} + s_{3.25} i + s_{0.3750} j$, $\mu_{\mathcal{A}_4} = s_3 + s_4 i + s_1 j$ and $\mu_{\mathcal{B}_1} = s_{3.5} + s_4 i + s_{0.5} j$, $\mu_{\mathcal{B}_2} = s_{1.5} + s_5 i + s_{1.5} j$, $\mu_{\mathcal{B}_3} = s_{3.125} + s_{3.75} i + s_{1.125} j$, $\mu_{\mathcal{B}_4} = s_2 + s_4 i + s_2 j$. Then $\text{LCNWG}_\omega(\mu_{\mathcal{A}_1}, \mu_{\mathcal{A}_2}, \mu_{\mathcal{A}_3}, \mu_{\mathcal{A}_4}) = s_{2.5533} + s_{3.7884} i + s_{1.6583} j$ and $\text{LCNWG}_\omega(\mu_{\mathcal{B}_1}, \mu_{\mathcal{B}_2}, \mu_{\mathcal{B}_3}, \mu_{\mathcal{B}_4}) = s_{2.5533} + s_{4.1543} + s_{1.2924} j$. Hence, by using the Eq. (8.2), we get $P(A) = s_{4.3380}$ and $P(B) = s_{4.3724}$. Since $P(A) < P(B)$, therefore $A \prec B$.

Hence, from the above study, it is deduced that the proposed geometric aggregation operator is suitably working in those cases where the existing aggregation operators of the LIFS theory fail to classify the objects.

Theorem 8.4.2. Let $\mu_t = s_{a_t} + s_{b_t} i + s_{c_t} j$, $(t = 1, 2, \ldots, n)$ be a collection of LCNs, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of μ_t $(t = 1, 2, \ldots, n)$, with $\omega_t > 0$, $\sum\limits_{t=1}^{n} \omega_t = 1$, then LCNWG_ω operator satisfies the following properties:

(P1) **(Idempotency)** If all μ_t $(t = 1, 2, \ldots, n)$ are equal, i.e. $\mu_t = \mu$, $\forall\ t$, then

$$\text{LCNWG}_\omega(\mu_1, \mu_2, \ldots \mu_n) = \mu$$

(P2) **(Boundedness)** Let $\mu^- = s_{a^-} + (s_{b^-}) i + (s_{c^-}) j$ and $\mu^+ = s_{a^+} + (s_{b^+}) i + (s_{c^+}) j$, where $a^- = \min\limits_t \{a_t\}$, $c^- = \max\limits_t \{a_t + c_t\} - \min\limits_t \{a_t\}$, $b^- = h - a^- - c^-$, $a^+ = \max\limits_t \{a_t\}$, $c^+ = \min\limits_t \{a_t + c_t\} - \max\limits_t \{a_t\}$, and $b^+ = h - a^+ - c^+$, then

$$\mu^- \leq \text{LCNWG}_\omega(\mu_1, \mu_2, \ldots \mu_n) \leq \mu^+$$

(P3) **(Monotonicity)** If $\mu_t \leq \mu_t^*$, $(t = 1, 2, \ldots, n)$, then

$$\text{LCNWG}_\omega(\mu_1, \mu_2, \ldots \mu_n) \leq \text{LCNWG}_\omega(\mu_1^*, \mu_2^*, \ldots \mu_n^*)$$

Proof. For LCNs, we have

(P1) Let $\mu_t = s_{a_t} + s_{b_t} i + s_{c_t} j$ for $t = 1, 2, \ldots, n$ and $\mu = s_a + s_b i + s_c j$ be LCNs. Assume that $\mu_t = \mu$, $\forall t$ which implies that $a_t = a$, $b_t = b$ and $c_t = t$ for all t, then

$$\begin{aligned} & \text{LCNWG}_\omega(\mu_1, \mu_2, \ldots \mu_n) \\ = \ & s_{h\left(\prod\limits_{t=1}^{n} \left(\frac{a_t}{h}\right)^{\omega_t}\right)} + s_{h\left(1 - \prod\limits_{t=1}^{n} \left(1 - \frac{b_t}{h}\right)^{\omega_t}\right)} i + s_{h\left(\prod\limits_{t=1}^{n} \left(\frac{a_t}{h} + \frac{c_t}{h}\right)^{\omega_t} - \prod\limits_{t=1}^{n} \left(\frac{a_t}{h}\right)^{\omega_t}\right)} j \end{aligned}$$

$$
\begin{aligned}
&= s_{h\left(\prod_{t=1}^{n}\left(\frac{a}{h}\right)^{\omega_t}\right)} + s_{h\left(1-\prod_{t=1}^{n}\left(1-\frac{b}{h}\right)^{\omega_t}\right)} i + s_{h\left(\prod_{t=1}^{n}\left(\frac{a}{h}+\frac{c}{h}\right)^{\omega_t}-\prod_{t=1}^{n}\left(\frac{a}{h}\right)^{\omega_t}\right)} j \\
&= s_{h\left(\frac{a}{h}\right)^{\sum_{t=1}^{n}\omega_t}} + s_{h\left(1-\left(1-\frac{b}{h}\right)^{\sum_{t=1}^{n}\omega_t}\right)} i + s_{h\left(\left(\frac{a}{h}+\frac{c}{h}\right)^{\sum_{t=1}^{n}\omega_t}-\left(\frac{a}{h}\right)^{\sum_{t=1}^{n}\omega_t}\right)} j \\
&= s_{h\left(\frac{a}{h}\right)} + s_{h\left(1-\left(1-\frac{b}{h}\right)\right)} i + s_{h\left(\frac{a}{h}+\frac{c}{h}-\frac{a}{h}\right)} j \\
&= s_a + s_b i + s_c j \\
&= \mu
\end{aligned}
$$

(P2) For LCNs $\mu_t = s_{a_t} + s_{b_t} i + s_{c_t} j$, and their corresponding weight vector be $\omega_t > 0$ with $\sum_{t=1}^{n} \omega_t = 1$. Assume that $a^- = \min_t\{a_t\}$, $c^- = \max_t\{a_t + c_t\} - \min_t\{a_t\}$, $a^+ = \max_t\{a_t\}$, and $c^+ = \min_t\{a_t + c_t\} - \max_t\{a_t\}$, we have

$$
\begin{aligned}
& \min_t\{a_t\} \le a_t \le \max_t\{a_t\} \\
\Rightarrow \quad & \left(\frac{\min_t\{a_t\}}{h}\right)^{\omega_t} \le \left(\frac{a_t}{h}\right)^{\omega_t} \le \left(\frac{\max_t\{a_t\}}{h}\right)^{\omega_t} \\
\Rightarrow \quad & \prod_{t=1}^{n}\left(\frac{\min_t\{a_t\}}{h}\right)^{\omega_t} \le \prod_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t} \le \prod_{t=1}^{n}\left(\frac{\max_t\{a_t\}}{h}\right)^{\omega_t} \\
\Rightarrow \quad & \frac{\min_t\{a_t\}}{h} \le \prod_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t} \le \frac{\max_t\{a_t\}}{h} \\
\Rightarrow \quad & \min_t\{a_t\} \le h\prod_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t} \le \max_t\{a_t\} \\
i.e., \quad & a^- \le h\prod_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t} \le a^+
\end{aligned}
\tag{8.6}
$$

On the other hand, we have

$$
\begin{aligned}
& \min_t\{a_t + c_t\} \le a_t + c_t \le \max_t\{a_t + c_t\} \\
\Rightarrow \quad & \frac{\min_t\{a_t + c_t\}}{h} \le \frac{a_t + c_t}{h} \le \frac{\max_t\{a_t + c_t\}}{h} \\
\Rightarrow \quad & \frac{\min_t\{a_t + c_t\}}{h} \le \prod_{t=1}^{n}\left(\frac{a_t + c_t}{h}\right)^{\omega_t} \le \frac{\max_t\{a_t + c_t\}}{h}
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow \quad & \frac{\min_t\{a_t+c_t\}}{h} - \frac{\max_t\{a_t\}}{h} \leq \prod_{t=1}^{n}\left(\frac{a_t+c_t}{h}\right)^{\omega_t} - \prod_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t} \\
& \leq \frac{\max_t\{a_t+c_t\}}{h} - \frac{\min_t\{a_t\}}{h} \\
\Rightarrow \quad & \min_t\{a_t+c_t\} - \max_t\{a_t\} \leq h\prod_{t=1}^{n}\left(\frac{a_t+c_t}{h}\right)^{\omega_t} - h\prod_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t} \\
& \leq \max_t\{a_t+c_t\} - \min_t\{a_t\} \\
\textit{i.e.,} \quad & c^+ \leq h\prod_{t=1}^{n}\left(\frac{a_t+c_t}{h}\right)^{\omega_t} - h\prod_{t=1}^{n}\left(\frac{a_t}{h}\right)^{\omega_t} \leq c^- \qquad (8.7)
\end{aligned}$$

Therefore, by using Eq. (8.6) and Eq. (8.7) and by the Definition 8.3.4, we have

$$\mu^- \leq \mathrm{LCNWG}_\omega(\mu_1, \mu_2, \ldots \mu_n) \leq \mu^+$$

(P3) Proof follows as similar as that of (P2) and so we omit here.

□

8.4.2 LCN ordered weighted geometric operator

In this section, we extend the idea of LCNWG$_\omega$ aggregation into the LCNOWG$_w$ operator.

Definition 8.4.2. Let $\mu_t = s_{a_t} + s_{b_t}i + s_{c_t}j$, $(t = 1, 2, \ldots, n)$ be a collection of LCNs, then LCNOWG$_w$ operator of dimension n is a mapping LCNOWG$_w : \Omega^n \to \Omega$, which is define as follows:

$$\mathrm{LCNOWG}_w(\mu_1, \mu_2, \ldots \mu_n) = \mu_{\sigma(1)}^{w_1} \otimes \mu_{\sigma(2)}^{w_2} \otimes \ldots \otimes \mu_{\sigma(n)}^{w_n} \qquad (8.8)$$

where $\sigma : (1, 2, \ldots, n) \to 1, 2, \ldots, n)$ is the permutation map, $\mu_{\sigma(t)} = s_{a_{\sigma(t)}} + s_{b_{\sigma(t)}}i + s_{c_{\sigma(t)}}j$ is the t^{th} largest value of the μ_t $(t = 1, 2, \ldots, n)$ and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\mu_{\sigma(t)}$ $(t = 1, 2, \ldots, n)$ with $w_t > 0$, $\sum\limits_{t=1}^{n} w_t = 1$. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the LCNOWG$_w$ operator is reduced to linguistic connection number geometric (LCNG) operator.

Theorem 8.4.3. Let $\mu_t = s_{a_t} + s_{b_t} i + s_{c_t} j$, $(t = 1, 2, \ldots, n)$ be a collection of LCNs, then their aggregated value by using the LCNOWG$_w$ operator is also a LCN, and

$$\begin{aligned} & \mathrm{LCNOWG}_w(\mu_1, \mu_2, \ldots \mu_n) \\ = \quad & s_{h\left(\prod\limits_{t=1}^{n} \left(\frac{a_{\sigma(t)}}{h}\right)^{w_t}\right)} + s_{h\left(1-\prod\limits_{t=1}^{n} \left(1-\frac{b_{\sigma(t)}}{h}\right)^{w_t}\right)} i + s_{h\left(\prod\limits_{t=1}^{n} \left(\frac{a_{\sigma(t)}}{h}+\frac{c_{\sigma(t)}}{h}\right)^{w_t} - \prod\limits_{t=1}^{n} \left(\frac{a_{\sigma(t)}}{h}\right)^{w_t}\right)} j \end{aligned} \tag{8.9}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of LCNOWG$_w$ operator, and $w_t > 0$, $\sum\limits_{t=1}^{n} w_t = 1$.

Proof. The proof of this theorem is similar to the Theorem 8.4.1. □

Similar to Theorem 8.4.2, LCNOWG$_w$ operator is also satisfies the same properties, which are defined as follows:

Theorem 8.4.4. Let $\mu_t = s_{a_t} + s_{b_t} i + s_{c_t} j$, $(t = 1, 2, \ldots, n)$ be a collection of LCNs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the LCNOWG$_w$ operator, with $w_t > 0$, $\sum\limits_{t=1}^{n} w_t = 1$, then LCNOWG$_w$ operator satisfies the following properties:

(P1) **(Idempotency)** If all μ_t $(t = 1, 2, \ldots, n)$ are equal, i.e. $\mu_t = \mu$, for all t, then

$$\mathrm{LCNOWG}_w(\mu_1, \mu_2, \ldots \mu_n) = \mu$$

(P2) **(Boundedness)** Let $\mu^- = s_{a^-} + (s_{b^-})i + (s_{c^-})j$ and $\mu^+ = s_{a^+} + (s_{b^+})i + (s_{c^+})j$, where $a^- = \min\limits_t\{a_t\}$, $c^- = \max\limits_t\{a_t + c_t\} - \min\limits_t\{a_t\}$, $b^- = h - a^- - c^-$, $a^+ = \max\limits_t\{a_t\}$, $c^+ = \min\limits_t\{a_t + c_t\} - \max\limits_t\{a_t\}$, and $b^+ = h - a^+ - c^+$, then

$$\mu^- \leq \mathrm{LCNOWG}_w(\mu_1, \mu_2, \ldots \mu_n) \leq \mu^+$$

(P3) **(Monotonicity)** If $\mu_t \leq \mu_t^*$, $\forall t$ then

$$\mathrm{LCNOWG}_w(\mu_1, \mu_2, \ldots \mu_n) \leq \mathrm{LCNOWG}_w(\mu_1^*, \mu_2^*, \ldots \mu_n^*)$$

Proof. As similar to Theorem 8.4.2, so it is omit here. □

8.4.3 LCN hybrid geometric operator

In this section, we extend the idea of LCNWG$_\omega$ and LCNOWG$_w$ aggregation into the LCNHG$_{w,\omega}$ operator, which weights both the LCN and its ordered position.

Definition 8.4.3. Let $\mu_t = s_{a_t} + s_{b_t}i + s_{c_t}j$, $(t = 1, 2, \ldots, n)$ be a collection of LCNs, then LCNHG$_{w,\omega}$ operator of dimension n is a mapping LCNHG$_{w,\omega} : \Omega^n \rightarrow \Omega$, which is define as follows:

$$\text{LCNHG}_{w,\omega}(\mu_1, \mu_2, \ldots \mu_n) = \dot{\mu}_{\sigma(1)}^{w_1} \otimes \dot{\mu}_{\sigma(2)}^{w_2} \otimes \ldots \otimes \dot{\mu}_{\sigma(n)}^{w_n} \tag{8.10}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weight vector of LCNHG operator with $w_t > 0$ and $\sum\limits_{t=1}^{n} w_t = 1$, and $\dot{\mu}_{\sigma(t)} = s_{\dot{a}_{\sigma(t)}} + s_{\dot{b}_{\sigma(t)}}i + s_{\dot{c}_{\sigma(t)}}j$ is the t^{th} largest of the weighted LCNs $\dot{\mu}_t = \mu_t^{n\omega_t}$ $(1, 2, \ldots, n)$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of μ_t $(1, 2, \ldots, n)$.

Theorem 8.4.5. Let $\mu_t = s_{a_t} + s_{b_t}i + s_{c_t}j$, $(t = 1, 2, \ldots, n)$ be a collection of LCNs, then their aggregated value by using the LCNHG$_{w,\omega}$ operator is also a LCN, and

$$\begin{aligned} &\text{LCNHG}_{w,\omega}(\mu_1, \mu_2, \ldots \mu_n) \qquad (8.11) \\ = \;& s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{\dot{a}_{\sigma(t)}}{h}\right)^{w_t}\right)} + s_{h\left(1-\prod\limits_{t=1}^{n}\left(1-\frac{\dot{b}_{\sigma(t)}}{h}\right)^{w_t}\right)} i + s_{h\left(\prod\limits_{t=1}^{n}\left(\frac{\dot{a}_{\sigma(t)}}{h}+\frac{\dot{c}_{\sigma(t)}}{h}\right)^{w_t}-\prod\limits_{t=1}^{n}\left(\frac{\dot{a}_{\sigma(t)}}{h}\right)^{w_t}\right)} j \end{aligned}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weight vector of LCNHG operator satisfying $w_t > 0$ and $\sum\limits_{t=1}^{n} w_t = 1$, and $\dot{\mu}_{\sigma(t)}$ is the t^{th} largest of the weighted LCNs $\dot{\mu}_t = \mu_t^{n\omega_t}$ $(1, 2, \ldots, n)$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of μ_t $(1, 2, \ldots, n)$.

Proof. The proof of this theorem is similar to the Theorem 8.4.1. ☐

Theorem 8.4.6. The LCNWG$_\omega$ and LCNOWG$_w$ operators are the special cases of the LCNHG$_{w,\omega}$ operator.

Proof. Let $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\begin{aligned} \text{LCNHG}_{w,\omega}(\mu_1, \mu_2, \ldots \mu_n) &= \dot{\mu}_{\sigma(1)}^{w_1} \otimes \dot{\mu}_{\sigma(2)}^{w_2} \otimes \ldots \otimes \dot{\mu}_{\sigma(n)}^{w_n} \\ &= \left(\dot{\mu}_{\sigma(1)} \otimes \dot{\mu}_{\sigma(2)} \otimes \ldots \otimes \dot{\mu}_{\sigma(n)}\right)^{\frac{1}{n}} \\ &= \mu_1^{\omega_1} \otimes \mu_2^{\omega_2} \otimes \ldots \otimes \mu_n^{\omega_n} \\ &= \text{LCNWG}_\omega(\mu_1, \mu_2, \ldots \mu_n) \end{aligned}$$

if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\mu}_t = \mu_t \ (1, 2, \ldots, n)$, thus

$$\begin{aligned}
\text{LCNHG}_{w,\omega}(\mu_1, \mu_2, \ldots \mu_n) &= \dot{\mu}_{\sigma(1)}^{w_1} \otimes \dot{\mu}_{\sigma(2)}^{w_2} \otimes \ldots \otimes \dot{\mu}_{\sigma(n)}^{w_n} \\
&= \mu_{\sigma(1)}^{w_1} \otimes \mu_{\sigma(2)}^{w_2} \otimes \ldots \otimes \mu_{\sigma(n)}^{w_n} \\
&= \text{LCNOWG}_w(\mu_1, \mu_2, \ldots \mu_n)
\end{aligned}$$

□

8.5 Proposed MADM method under linguistic intuitionistic fuzzy set environment

In this section, we develop a novel method to solve the MADM problem under the LIFS environment. The general description of the MADM process is given in Section 2.7 of Chapter 2, where information about the alternatives are assessed by the l decision makers, denoted by $\mathcal{DM}^{(1)}, \mathcal{DM}^{(2)}, \ldots, \mathcal{DM}^{(l)}$ and represented their values in the form of decision matrix $\mathcal{M}^{(p)} = (\widetilde{\alpha}_{kt}^{(p)})_{m \times n}$ for $k = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ and $p = 1, 2, \ldots, l$. Then, based on the description and the proposed AOs, we develop an approach to find the finest alternative(s) whose steps are summarized as follows:

Step 1: Normalize the matrix $\mathcal{M}^{(p)}$ into decision matrix $\mathcal{R}^{(p)} = (\alpha_{kt}^{(p)})_{m \times n}$ by using Eq. (2.58) of Chapter 2.

Step 2: Construct LCN decision matrix $\mathcal{L}^{(p)} = (\mu_{kt}^{(p)})_{m \times n}$ corresponding to each decision maker $\mathcal{DM}^{(p)}$, where $\mu_{kt}^{(p)} = s_{a_{kt}}^{(p)} + s_{b_{kt}}^{(p)} i + s_{c_{kt}}^{(p)} j$ and $a_{kt}^{(p)}, b_{kt}^{(p)}$ and $c_{kt}^{(p)}$ are computed by using Definition 8.3.3 as $a_{kt}^{(p)} = \frac{u_{kt}^{(p)}(h - v_{kt}^{(p)})}{h}$, $b_{kt}^{(p)} = h - \frac{u_{kt}^{(p)}(h - v_{kt}^{(p)})}{h} - \frac{v_{kt}^{(p)}(h - u_{kt}^{(p)})}{h}$, and $c_{kt}^{(p)} = \frac{v_{kt}^{(p)}(h - u_{kt}^{(p)})}{h}$.

Step 3: Aggregate the different decision maker preferences matrix $\mathcal{L}^{(p)}$ into the matrix $\mathcal{L} = (\mu_{kt})_{m \times n}$ by using LCNOWG_w operator, where μ_{kt} are computed as

$$\begin{aligned}
\mu_{kt} &= s_{a_{kt}} + s_{b_{kt}} i + s_{c_{kt}} j \\
&= \text{LCNOWG}_w\left(\mu_{kt}^{\sigma(1)}, \mu_{kt}^{\sigma(2)}, \ldots, \mu_{kt}^{\sigma(l)}\right) \\
&= s_{h\left(\prod\limits_{p=1}^{l}\left(\frac{a_{kt}^{\sigma(p)}}{h}\right)^{w_p}\right)} + s_{h\left(1 - \prod\limits_{p=1}^{l}\left(1 - \frac{b_{kt}^{\sigma(p)}}{h}\right)^{w_p}\right)} i + s_{h\left(\prod\limits_{p=1}^{l}\left(\frac{a_{kt}^{\sigma(p)}}{h} + \frac{c_{kt}^{\sigma(p)}}{h}\right)^{w_p} - \prod\limits_{p=1}^{l}\left(\frac{a_{kt}^{\sigma(p)}}{h}\right)^{w_p}\right)} j
\end{aligned}$$

where $w^p > 0$ is the weight vector of decision maker $\mathcal{DM}^{(p)}$ with $\sum\limits_{p=1}^{l} w_p = 1$ and $\mu_{kt}^{\sigma(p)} = s_{a_{kt}^{\sigma(p)}} + s_{b_{kt}^{\sigma(p)}} i + s_{c_{kt}^{\sigma(p)}} j$ is the p^{th} largest value of the $\mu_{kt}^{(p)}$ $(p = 1, 2, \ldots, l)$.

Step 4: Utilize LCNWG$_\omega$ operator with weights $\omega_t > 0$ to aggregate the preferences of the matrix $\mathcal{L}$ into the collective LCN μ_k of each alternative $\mathcal{A}_k$, $(k = 1, 2, \ldots, m)$, where

$$\begin{aligned} \mu_k &= s_{a_k} + s_{b_k} i + s_{c_k} j \\ &= \text{LCNWG}_\omega(\mu_{k1}, \mu_{k2}, \ldots, \mu_{kn}) \\ &= s_{h\left(\prod\limits_{t=1}^{n} \left(\frac{a_{kt}}{h}\right)^{\omega_t}\right)} + s_{h\left(1 - \prod\limits_{t=1}^{n} \left(1 - \frac{b_{kt}}{h}\right)^{\omega_t}\right)} i + s_{h\left(\prod\limits_{t=1}^{n} \left(\frac{a_{kt}}{h} + \frac{c_{kt}}{h}\right)^{\omega_t} - \prod\limits_{t=1}^{n} \left(\frac{a_{kt}}{h}\right)^{\omega_t}\right)} j \end{aligned}$$

Step 5: Compute the relatively certainty probability power $P(\mu_k)$ of each obtained LCN μ_k as follows:

$$P(\mu_k) = s_{\left(h + \frac{2a_k}{b_k + c_k} - \frac{c_k}{a_k + b_k}\right)/2} \tag{8.12}$$

Step 6: According to the values of $P(\mu_k)$, rank the given alternatives $\mathcal{A}_k, k = 1, 2, \ldots, m$ and select the best one(s).

8.6 Illustrative example

The above-mentioned approach is demonstrated with a numerical example under the LIFS environment and obtained results have been compared with the other existing results.

8.6.1 A case study

Indian rural society is changing and transforming in many aspects, including jobs, business structures, transportation facilities, and communication system. Therefore, the people of the rural area migrate to the big cities to find the more opportunities. To stop this immigration, Indian government wants to provide all the facilities in these rural areas. In this context, the Indian government has taken a considerable number of road building projects either to preserve the roads which are already built or to undertake the new roads. For this, the Indian government had been issued the global tender to select the contractor for these projects in the newspaper and considered the four attributes required for its namely, contractor background experience ($\mathcal{G}_1$), technical capability ($\mathcal{G}_2$), financial status ($\mathcal{G}_3$), and number of projects handled ($\mathcal{G}_4$), and assign the weights of a relative importance of each attribute as $\omega = (0.4, 0.25, 0.2, 0.15)^T$ on the basis of decision maker's preferences. The four contractors (i.e. alternatives) namely, Jaihind Road Builders private (Pvt.) limited (Ltd.) ($\mathcal{A}_1$), J.K. Construction ($\mathcal{A}_2$), Build quick Infrastructure Pvt. Ltd. ($\mathcal{A}_3$), and Relcon Infra projects Ltd. ($\mathcal{A}_4$) bid for these projects. Then, the aim of the government is to recognize the best contractor among $\mathcal{A}_k, k = 1, 2, 3, 4$ which classify with the known desired goals of the contractor. In order to fulfill it, three experts d_1, d_2 and d_3 are invited to given their preferences on each attribute in the term of LIFNs according to linguistic term set $S = \big\{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slightly poor}, s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extremely good}\big\}$ and their preferences values are summarized in the form of decision-matrix $\mathcal{M}^{(p)}$ $(p = 1, 2, 3)$, represented in Table 8.1.

Table 8.1: Decision matrices $\mathcal{M}$ corresponding to each decision maker in terms of LIFNs

$\mathcal{DM}$		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{DM}^{(1)}$	$\mathcal{A}_1$	(s_7, s_1)	(s_6, s_2)	(s_4, s_3)	(s_7, s_1)
	$\mathcal{A}_2$	(s_5, s_3)	(s_5, s_2)	(s_4, s_3)	(s_3, s_1)
	$\mathcal{A}_3$	(s_7, s_1)	(s_5, s_1)	(s_5, s_2)	(s_5, s_3)
	$\mathcal{A}_4$	(s_6, s_2)	(s_4, s_3)	(s_5, s_1)	(s_7, s_1)
$\mathcal{DM}^{(2)}$	$\mathcal{A}_1$	(s_5, s_2)	(s_2, s_1)	(s_3, s_4)	(s_2, s_5)
	$\mathcal{A}_2$	(s_2, s_3)	(s_3, s_3)	(s_1, s_2)	(s_3, s_3)
	$\mathcal{A}_3$	(s_3, s_3)	(s_3, s_5)	(s_6, s_1)	(s_2, s_6)
	$\mathcal{A}_4$	(s_3, s_2)	(s_2, s_4)	(s_2, s_1)	(s_3, s_4)
$\mathcal{DM}^{(3)}$	$\mathcal{A}_1$	(s_5, s_3)	(s_4, s_4)	(s_7, s_1)	(s_5, s_1)
	$\mathcal{A}_2$	(s_4, s_2)	(s_4, s_2)	(s_6, s_2)	(s_5, s_2)
	$\mathcal{A}_3$	(s_6, s_1)	(s_7, s_1)	(s_6, s_1)	(s_5, s_2)
	$\mathcal{A}_4$	(s_6, s_1)	(s_3, s_4)	(s_5, s_2)	(s_4, s_3)

Then, the steps of the proposed approach are executed as follows to find the best alternative(s).

Step 1: Since all attributes are of the same types, so there is no need of the normalization process.

Step 2: By using Definition 8.3.3, the LCNs towards the rating values of each decision maker are computed and summarized in Tables 8.2.

Step 3: By taking the weight vector $w = (0.243, 0.514, 0.243)^T$ for decision-makers computed by using normal distribution method [202], the aggregated LCNs decision-matrix $\mathcal{L} = (\mu_{kt})_{4\times 4}$ are obtained by LCNOWG$_w$ operator. The results of them are summarized in Table 8.3.

Step 4: By utilizing LCNWG$_\omega$ operator to aggregate the collective values of each alternative and the results of them are given as

$$\mu_1 = s_{3.1125} + s_{3.6575}i + s_{1.2300}j, \qquad \mu_2 = s_{2.5012} + s_{4.1179}i + s_{1.3809}j$$

$$\mu_3 = s_{3.7105} + s_{2.9832}i + s_{1.3063}j, \qquad \mu_4 = s_{2.8386} + s_{3.6766}i + s_{1.4848}j$$

Step 5: By Eq. (8.12), we get $P(\mu_1) = s_{4.5460}$, $P(\mu_2) = s_{4.3506}$, and $P(\mu_3) = s_{4.7674}$, $P(\mu_4) = s_{4.4360}$.

Step 6: Since $P(\mu_3) > P(\mu_1) > P(\mu_4) > P(\mu_2)$, therefore $\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2$. Hence $\mathcal{A}_3$ is the best alternative.

8.6.2 Comparative study

Zhang [236] proposed an approach for solving the DM problems based on the LIFWG and LIFOWG operators under the linguistic intuitionistic fuzzy environment. For more details, see Eqs. (2.49) - (2.52) of Chapter 2. Now, if we utilize these operators to aggregate and rank the given alternatives, then the executions are summarized in the following steps.

Step 1: Applying the LIFOWG operator (for more details, we refer to [236]) to aggregate the decision maker preferences into the collective one, then the resultant matrix is summarized in Table 8.4.

Step 2: Aggregate the information of Table 8.4 by using LIFWG operator (expression is given in Eq. (2.50) of Chapter 2) and the computed results are given as $\alpha_1 = (s_{4.3465}, s_{2.2714})$, $\alpha_2 = (s_{3.5413}, s_{2.3495})$, $\alpha_3 = (s_{5.0208}, s_{2.0878})$, $\alpha_4 = (s_{4.1043}, s_{2.4672})$.

Step 3: Utilize Eq. (2.8) of Chapter 2 to compute the score values of each number and hence we get $\mathcal{S}(\alpha_1) = s_{5.0376}$, $\mathcal{S}(\alpha_2) = s_{4.5959}$, $\mathcal{S}(\alpha_3) = s_{5.4665}$, $\mathcal{S}(\alpha_4) = s_{4.8186}$. Since $\mathcal{S}(\alpha_3) > \mathcal{S}(\alpha_1) > \mathcal{S}(\alpha_4) > \mathcal{S}(\alpha_2)$, therefore $\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2$. Hence, $\mathcal{A}_3$ is the best alternative.

Thus, from this study, it has been concluded that the results computed by the existing approach coincides with the proposed one which validates the proposed approach.

8.6.3 Validity test

In order to demonstrate the feasibility of the proposed approach, the approach has been tested on certain test criteria which has corroborated by Wang and Triantaphyllou [175] as follows:

Test criterion 1: "An effective MAGDM method does not change the index of the best alternative by replacing a non optimal alternative with a worse alternative, provided the relative weighted criteria remains unchanged."

Test criterion 2: "Method should possess transitive nature."

Test criterion 3: "If we decomposed a given MAGDM problem into the sub DM problems and the same MAGDM method is utilized on sub problems to rank the alternatives, then the combined ranking of the alternatives should be identical to the ranking of un-decomposed DM problem".

In the below, we have validated these test criteria on our proposed MCDM method.

Test by applying criterion 1

Under this test, we replace the membership degrees of non-optimal alternative $\mathcal{A}_2$ with the worse alternative $\mathcal{A}'_2$ in the original original decision matrix for each decision maker $\mathcal{DM}^{(p)}(p = 1, 2, 3)$. The rating values of $\mathcal{A}'_2$ is chosen as an arbitrary and their values are summarized in Table 8.5.

Then, by applying the proposed MAGDM approach to transform data and hence obtain the relatively certainty probability power for each candidate $\mathcal{A}_k$ $(k = 1, 2, 3, 4)$ as $P(\mathcal{A}_1) = s_{4.5460}$, $P(\mathcal{A}'_2) = s_{3.9370}$, $P(\mathcal{A}_3) = s_{4.7674}$ and $P(\mathcal{A}_4) = s_{4.4360}$. Since $P(\mathcal{A}_3) > P(\mathcal{A}_1) > P(\mathcal{A}_4) > P(\mathcal{A}'_2)$ and hence the ranking order of the alternatives is $\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}'_2$ which is similar to the ranking order of original problem. Therefore, the best candidate remains same i.e., $\mathcal{A}_3$ and it validate the test criterion 1.

Test by criteria 2 and 3

Under this test, if we decomposed the given problem into the sub-problems $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4\}$, $\{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_4\}$ and $\{\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$ and the procedure steps of the approach has been applied, then we get the ranking order of these smaller problems is $\mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2$, $\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_4$ and $\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2$ respectively. Therefore, by combining these, we get the overall ranking order of the alternatives is $\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2$ which is same as that of the original ranking order and show transitive property. Thus, the proposed MAGDM approach is valid under the test criteria 2 and 3.

8.7 Conclusion

Set pair analysis theory is more effective uncertainty theory compare to other uncertainty theory as vague, fuzzy and intuitionistic fuzzy theory. So, in this chapter, we have been presented the concept of LCN by integrating the connection number and linguistic approach with some operational laws and their properties based on these. Furthermore, we have defined some geometric aggregating operators for aggregating LCNs information such as LCNWG, LCNOWG, and LCNHG operators. Also, some desirable properties of these operators such as idempotency, monotonicity, boundedness are studied. Later, we developed an efficient algorithm to solve the MAGDM problems under the LIFS environment and illustrate with a numerical example. From the computed results and their corresponding comparative study, it has been observed that the developed approach is more stable for solving the problems.

Table 8.2: LCN decision matrices $\mathcal{L}$ corresponding to each decision maker

		$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{DM}^{(1)}$	$\mathcal{A}_1$	$s_{6.125} + s_{1.750}i + s_{0.125}j$	$s_{4.500} + s_{3.000}i + s_{0.500}j$	$s_{2.500} + s_{4.000}i + s_{1.500}j$	$s_{6.125} + s_{1.750}i + s_{0.125}j$
	$\mathcal{A}_2$	$s_{3.125} + s_{3.750}i + s_{1.125}j$	$s_{3.750} + s_{3.500}i + s_{0.750}j$	$s_{2.500} + s_{4.000}i + s_{1.500}j$	$s_{2.625} + s_{4.750}i + s_{0.625}j$
	$\mathcal{A}_3$	$s_{6.125} + s_{1.750}i + s_{0.125}j$	$s_{4.375} + s_{3.250}i + s_{0.375}j$	$s_{3.750} + s_{3.500}i + s_{0.750}j$	$s_{3.125} + s_{3.750}i + s_{1.125}j$
	$\mathcal{A}_4$	$s_{4.500} + s_{3.000}i + s_{0.500}j$	$s_{2.500} + s_{4.000}i + s_{1.500}j$	$s_{4.375} + s_{3.250}i + s_{0.375}j$	$s_{6.125} + s_{1.750}i + s_{0.125}j$
$\mathcal{DM}^{(2)}$	$\mathcal{A}_1$	$s_{3.750} + s_{3.500}i + s_{0.750}j$	$s_{1.750} + s_{5.500}i + s_{0.750}j$	$s_{1.500} + s_{4.000}i + s_{2.500}j$	$s_{0.750} + s_{3.500}i + s_{3.750}j$
	$\mathcal{A}_2$	$s_{1.250} + s_{4.500}i + s_{2.250}j$	$s_{1.875} + s_{4.250}i + s_{1.875}j$	$s_{0.750} + s_{5.500}i + s_{1.750}j$	$s_{1.875} + s_{4.250}i + s_{1.875}j$
	$\mathcal{A}_3$	$s_{1.875} + s_{4.250}i + s_{1.875}j$	$s_{1.125} + s_{3.750}i + s_{3.125}j$	$s_{5.250} + s_{2.500}i + s_{0.250}j$	$s_{0.500} + s_{3.000}i + s_{4.500}j$
	$\mathcal{A}_4$	$s_{2.250} + s_{4.500}i + s_{1.250}j$	$s_{1.000} + s_{4.000}i + s_{3.000}j$	$s_{1.750} + s_{5.500}i + s_{0.750}j$	$s_{1.500} + s_{4.000}i + s_{2.500}j$
$\mathcal{DM}^{(3)}$	$\mathcal{A}_1$	$s_{3.125} + s_{3.750}i + s_{1.125}j$	$s_{2.000} + s_{4.000}i + s_{2.000}j$	$s_{6.125} + s_{1.750}i + s_{0.125}j$	$s_{4.375} + s_{3.250}i + s_{0.375}j$
	$\mathcal{A}_2$	$s_{3.000} + s_{4.000}i + s_{1.000}j$	$s_{3.000} + s_{4.000}i + s_{1.000}j$	$s_{4.500} + s_{3.000}i + s_{0.500}j$	$s_{3.750} + s_{3.500}i + s_{0.750}j$
	$\mathcal{A}_3$	$s_{5.250} + s_{2.500}i + s_{0.250}j$	$s_{6.125} + s_{1.750}i + s_{0.125}j$	$s_{5.250} + s_{2.500}i + s_{0.250}j$	$s_{3.750} + s_{3.500}i + s_{0.750}j$
	$\mathcal{A}_4$	$s_{5.250} + s_{2.500}i + s_{0.250}j$	$s_{1.500} + s_{4.000}i + s_{2.500}j$	$s_{3.750} + s_{3.500}i + s_{0.750}j$	$s_{2.500} + s_{4.000}i + s_{1.500}j$

Table 8.3: Aggregated LCNs of the decision makers by LCNOWA operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$s_{4.0417} + s_{3.1933}i + s_{0.7650}j$	$s_{2.2741} + s_{4.6834}i + s_{1.0425}j$	$s_{2.7454} + s_{3.5418}i + s_{1.7128}j$	$s_{3.0929} + s_{2.9887}i + s_{1.9184}j$
$\mathcal{A}_2$	$s_{2.4493} + s_{4.0702}i + s_{1.4805}j$	$s_{2.8253} + s_{3.9479}i + s_{1.2268}j$	$s_{2.1523} + s_{4.2329}i + s_{1.6148}j$	$s_{2.6379} + s_{4.3581}i + s_{1.0040}j$
$\mathcal{A}_3$	$s_{4.2439} + s_{2.8307}i + s_{0.9254}j$	$s_{3.4132} + s_{3.0578}i + s_{1.5290}j$	$s_{4.8378} + s_{2.7618}i + s_{0.4004}j$	$s_{2.0926} + s_{3.5170}i + s_{2.3904}j$
$\mathcal{A}_4$	$s_{3.9476} + s_{3.3077}i + s_{0.7447}j$	$s_{1.5389} + s_{4.0000}i + s_{2.4611}j$	$s_{3.2350} + s_{4.0474}i + s_{0.7177}j$	$s_{2.7454} + s_{3.5418}i + s_{1.7128}j$

Table 8.4: Aggregated values by existing LIFOWG operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$(s_{5.4260}, s_{2.0409})$	$(s_{3.0912}, s_{2.1147})$	$(s_{4.2732}, s_{2.8604})$	$(s_{4.3429}, s_{2.3026})$
$\mathcal{A}_2$	$(s_{3.5683}, s_{2.5088})$	$(s_{3.9378}, s_{2.2600})$	$(s_{3.1517}, s_{2.5367})$	$(s_{3.3965}, s_{1.7867})$
$\mathcal{A}_3$	$(s_{5.2634}, s_{1.5496})$	$(s_{4.7926}, s_{2.3026})$	$(s_{5.7400}, s_{1.2574})$	$(s_{4.0019}, s_{3.8168})$
$\mathcal{A}_4$	$(s_{5.0699}, s_{1.7710})$	$(s_{2.9154}, s_{3.7771})$	$(s_{4.0019}, s_{1.5332})$	$(s_{4.2732}, s_{2.8604})$

Table 8.5: Rating values of the worse alternative $\mathcal{A}'_2$ for each decision maker

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{DM}^{(1)}$	(s_3, s_3)	(s_3, s_3)	(s_2, s_4)	(s_2, s_2)
$\mathcal{DM}^{(2)}$	(s_1, s_4)	(s_2, s_5)	(s_1, s_4)	(s_2, s_4)
$\mathcal{DM}^{(3)}$	(s_3, s_3)	(s_2, s_4)	(s_4, s_3)	(s_3, s_4)

Chapter 9

Group decision making approach based on possibility degree measures and the linguistic aggregation operators using Einstein norm operations[1]

In this chapter, we present a group DM approach under the LIFS environment. For it, we first propose a new ranking method named as possibility degree measures to compare the different linguistic intuitionistic fuzzy (LIF) numbers. Further, to aggregate the different LIF numbers, some weighted and ordered weighted averaging AOs are proposed by using Einstein t-norm operations. The prominent characteristics of these operators are also investigated. Afterward, a MAGDM approach, based on proposed operators and the possibility degree measure, is developed under the LIFSs environment. A numerical case is taken to manifest the practicability and feasibility of the proposed group DM method.

9.1 Introduction

MAGDM is one of the hot topics in the DM field to choose the best alternative to the set of feasible one. In the DM process, we usually face much uncertain, incomplete and even

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unknown information. In order to handle the uncertainties of the data, it is quite popular that the assessments are presented by a fuzzy set or extensions of the fuzzy set which can be classified into two types. One is based on the quantitative fuzzy sets [6, 231] and the other is the qualitative fuzzy sets that are usually denoted by the linguistic variables [232].

It is evident from the existing theories that they have widely used by the researchers, but in the process of the ranking the different IFSs, several researchers were paying more attention on to the score and possibility measures. Under this environment, Wei and Tang [178] presented the possibility degree measure for IFNs. Wan and Dong [157] presented a possibility degree based method to solve the MADM problem. Garg [41] presented a generalized improved score function of ranking the interval-valued IFSs. Out of that, possibility measures is one of the most successful ways to compare the two different objects. Possibility degree between the two objects reflects the probability of one object to another object and hence can be used to compare the objects.

All the above existing theories deal the uncertain information only by quantitative aspects. But in real-life problems, there are many attribute values which are qualitative in nature and cannot be expressed by a numeric value. Thus, LV provides us more degree of freedom to analyzed the imprecise and vague information. In the aspects of the ranking order, Xu [196] presented a DM approach based on aggregating operators and possibility degree method under the uncertain linguistic information. Wan and Dong [157] proposes a new ranking method of interval-valued IFNs based on the possibility degree measures from the probability viewpoint and their corresponding DM method. However, in the aspects of the aggregation process, Zhang [236] defined the concept of the linguistic intuitionistic fuzzy set (LIFS) by considering the linguistic degrees of the membership and non-membership during the analysis. Chen et al. [27] presented a group DM approach based on the averaging and geometric operators for the LIFS.

Since the most important aspects of the DM process are to aggregate the information by suitable AOs and a ranking method to rank them. So motivated by this, in the present chapter, we enhance the study under the LIFS environment by proposing a new ranking method called as possibility degree measure to rank the different LIFNs which overcomes the drawbacks of some existing ranking methods. Few properties of the measures are also investigated. Furthermore, by using the Einstein t-norm operations, two new LIF AOs named as LIF Einstein weighted averaging and LIF Einstein ordered weighted averaging operators are proposed to aggregate the different preferences of the decision makers. Some of the desirable properties are proved. In the nutshell, under the LIFS environment, the objective of this chapter is divided into four parts:

i) to introduce the new ranking method named as possibility degree measure to compare the different LIFNs,

ii) to develop some new LIF weighted averaging AOs by using Einstein t-norm operations,

iii) to establish a group DM approach based on the proposed operators and ranking method, and

iv) to illustrate the approach with a numerical example and validate it with some validity and comparative studies.

9.2 Proposed new possibility degree measure

In this section, we present a new possibility degree measure to compare the two different LIFNs $\mathcal{A}_t (t = 1, 2)$ defined over the universal set $\mathcal{X}$.

9.2.1 Possibility degree measure

Definition 9.2.1. Let $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$ and $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$ be two LIFNs, then the possibility degree measure, $p(\mathcal{A}_1 \succeq \mathcal{A}_2)$, of $\mathcal{A}_1 \succeq \mathcal{A}_2$ is proposed as

$$p(\mathcal{A}_1 \succeq \mathcal{A}_2) = \min\left(\max\left(\frac{h + \tau_1 - 2\tau_2 - \theta_2}{\pi_1 + \pi_2}, 0\right), 1\right), \tag{9.1}$$

where either $\pi_1 \neq 0$ or $\pi_2 \neq 0$.

On the other hand, if $\pi_1 = \pi_2 = 0$, then we define

$$p(\mathcal{A}_1 \succeq \mathcal{A}_2) = \begin{cases} 1 & ; \tau_1 > \tau_2 \\ 0 & ; \tau_1 < \tau_2 \\ 0.5 & ; \tau_1 = \tau_2 \end{cases} \tag{9.2}$$

Theorem 9.2.1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two LIFNs, then

(i) $0 \leq p(\mathcal{A}_1 \succeq \mathcal{A}_2) \leq 1$.

(ii) $p(\mathcal{A}_1 \succeq \mathcal{A}_2) = 0.5$ if $\mathcal{A}_1 = \mathcal{A}_2$.

(iii) $p(\mathcal{A}_1 \succeq \mathcal{A}_2) + p(\mathcal{A}_2 \succeq \mathcal{A}_1) = 1$.

Proof. We will prove conclusions (i), (ii) and (iii), respectively.

(i) $p(\mathcal{A}_1 \succeq \mathcal{A}_2) \geq 0$ is obvious, so we need to prove only $p(\mathcal{A}_1 \succeq \mathcal{A}_2) \leq 1$. For it, we take

$$x = \frac{h + \tau_1 - 2\tau_2 - \theta_2}{\pi_1 + \pi_2}$$

Then the following cases are arising:

(a) If $x \geq 1$, then

$$\begin{aligned} p(\mathcal{A}_1 \succeq \mathcal{A}_2) &= \min(\max(x, 0), 1) \\ &= \min(x, 1) = 1 \end{aligned}$$

(b) If $0 < x < 1$, then

$$\begin{aligned} p(\mathcal{A}_1 \succeq \mathcal{A}_2) &= \min(\max(x, 0), 1) \\ &= \min(x, 1) = x \end{aligned}$$

(c) If $x \leq 0$, then

$$\begin{aligned} p(\mathcal{A}_1 \succeq \mathcal{A}_2) &= \min(\max(x, 0), 1) \\ &= \min(0, 1) = 0 \end{aligned}$$

Hence, in all cases, we have $0 \leq p(\mathcal{A}_1 \succeq \mathcal{A}_2) \leq 1$.

(ii) Let $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$, $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$ be two LIFNs. If $\mathcal{A}_1 = \mathcal{A}_2$ which implies that

$\tau_1 = \tau_2$ and $\theta_1 = \theta_2$, then, Eq. (9.1) becomes

$$\begin{aligned} p(\mathcal{A}_1 \succeq \mathcal{A}_2) &= \min\left(\max\left(\frac{h+\tau_1-2\tau_2-\theta_2}{\pi_1+\pi_2}, 0\right), 1\right) \\ &= \min\left(\max\left(\frac{h+\tau_1-2\tau_1-\theta_1}{\pi_1+\pi_1}, 0\right), 1\right) \\ &= \min\left(\max\left(\frac{\pi_1}{2\pi_1}, 0\right), 1\right) \\ &= \min(\max(0.5, 0), 1) \\ &= 0.5 \end{aligned}$$

(iii) For two LIFNs $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$ and $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$, we take

$$x = \frac{h+\tau_1-2\tau_2-\theta_2}{\pi_1+\pi_2},$$

and

$$y = \frac{h+\tau_2-2\tau_1-\theta_1}{\pi_1+\pi_2}$$

such that

$$\begin{aligned} x + y &= \frac{h+\tau_1-2\tau_2-\theta_2}{\pi_1+\pi_2} + \frac{h+\tau_2-2\tau_1-\theta_1}{\pi_1+\pi_2} \\ &= \frac{h+\tau_1-2\tau_2-\theta_2+h+\tau_2-2\tau_1-\theta_1}{\pi_1+\pi_2} \\ &= \frac{\pi_1+\pi_2}{\pi_1+\pi_2} = 1 \end{aligned}$$

Then, the following three cases are arising:

(a) If $x \leq 0$ and $y \geq 1$ then

$$\begin{aligned} & p(\mathcal{A}_1 \succeq \mathcal{A}_2) + p(\mathcal{A}_2 \succeq \mathcal{A}_1) \\ = \ & \min\left(\max\left(x, 0\right), 1\right) + \min\left(\max\left(y, 0\right), 1\right) \\ = \ & \min(0, 1) + \min(y, 1) \\ = \ & 0 + 1 = 1 \end{aligned}$$

(b) If $0 < x, y < 1$ then

$$\begin{aligned} & p(\mathcal{A}_1 \succeq \mathcal{A}_2) + p(\mathcal{A}_2 \succeq \mathcal{A}_1) \\ = \; & \min\left(\max\left(x, 0\right), 1\right) + \min\left(\max\left(y, 0\right), 1\right) \\ = \; & \min(x, 1) + \min(y, 1) \\ = \; & x + y = 1 \end{aligned}$$

(c) If $x \geq 1$ and $y \leq 0$ then

$$\begin{aligned} & p(\mathcal{A}_1 \succeq \mathcal{A}_2) + p(\mathcal{A}_2 \succeq \mathcal{A}_1) \\ = \; & \min\left(\max\left(x, 0\right), 1\right) + \min\left(\max\left(y, 0\right), 1\right) \\ = \; & \min(x, 1) + \min(0, 1) \\ = \; & 1 + 0 = 1 \end{aligned}$$

Hence, in all cases, we have $p(\mathcal{A}_1 \succeq \mathcal{A}_2) + p(\mathcal{A}_2 \succeq \mathcal{A}_1) = 1$. □

Theorem 9.2.2. For two LIFNs $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$ and $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$, the proposed possibility degree measure $p(\mathcal{A}_1 \succeq \mathcal{A}_2)$ satisfies the following characteristics:

(i) $p(\mathcal{A}_1 \succeq \mathcal{A}_2) = 1$ if $\tau_1 - \pi_1 \geq \tau_2$;

(ii) $p(\mathcal{A}_1 \succeq \mathcal{A}_2) = 0$ if $\tau_2 - \pi_2 \geq \tau_1$.

Proof. For two LIFNs $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$ and $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$, we have

(i) Let $\tau_1 - \pi_1 \geq \tau_2$ then we have

$$\begin{aligned} \frac{h + \tau_1 - 2\tau_2 - \theta_2}{\pi_1 + \pi_2} &= \frac{\tau_1 - \tau_2 + 1 - \tau_2 - \theta_2}{\pi_1 + \pi_2} \\ &= \frac{\tau_1 - \tau_2 + \pi_2}{\pi_1 + \pi_2} \\ &\geq \frac{\tau_2 + \pi_1 - \tau_2 + \pi_2}{\pi_1 + \pi_2} \\ &= 1 \end{aligned}$$

Therefore, $\min\left(\max\left(\frac{h+\tau_1-2\tau_2-\theta_2}{\pi_1+\pi_2}, 0\right), 1\right) = 1$. Hence $p(\mathcal{A}_1 \succeq \mathcal{A}_2) = 1$.

(ii) Let $\tau_2 - \pi_2 \geq \tau_1$ then we have

$$\begin{aligned}
\frac{h+\tau_1-2\tau_2-\theta_2}{\pi_1+\pi_2} &= \frac{\tau_1-\tau_2+1-\tau_2-\theta_2}{\pi_1+\pi_2} \\
&= \frac{\tau_1-\tau_2+\pi_2}{\pi_1+\pi_2} \\
&\leq \frac{\tau_1-\tau_1}{\pi_1+\pi_2} \\
&= 0
\end{aligned}$$

Therefore, $\min\left(\max\left(\frac{h+\tau_1-2\tau_2-\theta_2}{\pi_1+\pi_2}, 0\right), 1\right) = 0$. Hence $p(\mathcal{A}_1 \succeq \mathcal{A}_2) = 0$.

□

Example 9.2.1. Let $\mathcal{A}_1 = (s_4, s_2)$ and $\mathcal{A}_2 = (s_3, s_2)$ be two LIFNs and $S_{[0,8]}$ be continuous LTS, then

$$\begin{aligned}
p(\mathcal{A}_1 \succeq \mathcal{A}_2) &= \min\left(\max\left(\frac{8+4-2\times 3-2}{2+3}, 0\right), 1\right) \\
&= \min\left(\max\left(0.8, 0\right), 1\right) \\
&= 0.8
\end{aligned}$$

Further, to rank the different LIFNs, the inclusion-comparison probability of LIFNs $\mathcal{A}_t \succeq \mathcal{A}_j$; $t, j \in \{1, 2, \ldots, n\}$ is denoted by $p(\mathcal{A}_t \succeq \mathcal{A}_j)$ and their corresponding likelihood possibility degree matrix is denoted by $P = (p_{tj})_{n\times n}$ where $p_{tj} = p(\mathcal{A}_t \succeq \mathcal{A}_j); (t, j = 1, 2, \ldots, n)$ given by

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

Now, the ranking value which represents the optimal degrees of the membership for the numbers $\mathcal{A}_t (t = 1, 2, \ldots, n)$ is given as follows:

$$r_t = \frac{1}{n(n-1)}\left(\sum_{j=1}^{n} p_{tj} + \frac{n}{2} - 1\right) \tag{9.3}$$

Then, the ranking order of all alternatives $\mathcal{A}_t, (t = 1, 2, \ldots, n)$ is found according to decreasing order of the values of r_t's and hence choose the best alternative.

Example 9.2.2. Let $S_{[0,8]}$ be LTS and $\mathcal{A}_1 = (s_4, s_3)$, $\mathcal{A}_2 = (s_5, s_2)$, $\mathcal{A}_3 = (s_3, s_2)$ and $\mathcal{A}_4 = (s_2, s_4)$ four LIFNs and, then for ranking to these LIFNs, we construct the possibility degree matrix as

$$P = \begin{pmatrix} 0.5000 & 0.0000 & 1.0000 & 1.0000 \\ 1.0000 & 0.5000 & 1.0000 & 1.0000 \\ 0.0000 & 0.0000 & 0.5000 & 0.6000 \\ 0.0000 & 0.0000 & 0.4000 & 0.5000 \end{pmatrix}$$

Based on this matrix, the optimal degrees of the membership for $\mathcal{A}_t (t = 1, 2, 3, 4)$ are computed by using Eq. (9.3) as $r_1 = 0.2917$, $r_2 = 0.3750$, $r_3 = 0.1750$ and $r_4 = 0.1583$. Since $r_2 > r_1 > r_3 > r_4$ and thus, the ranking order of these numbers is $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_4$.

9.2.2 Advantages of the proposed possibility degree

Also, it is observed from this proposed measure that it is suitable working under those cases also, where the existing measures for ranking the LIFNs such as score function as proposed by Chen et al. [27] or possibility measure, as proposed by Xu [196] fails to classify the numbers. The advantages of the proposed measures are demonstrated through some illustrative examples as follows.

Example 9.2.3. Consider two LIFNs $\mathcal{A}_1 = (s_4, s_2)$ and $\mathcal{A}_2 = (s_3, s_1)$ and the target of the problem is to identify the best one among them. For it, if we apply the score function as proposed by Chen et al.[27] given in Eq.(2.8) of Chapter 2, then we get $\mathcal{S}(\mathcal{A}_1) = \mathcal{S}(\mathcal{A}_2) = 2$. Hence, we are unable to classify the LIFNs $\mathcal{A}_1$ and $\mathcal{A}_2$. On the other hand, if we apply the proposed possibility degree measure on these numbers to rank it, then we get the possibility degree matrix as

$$P = \begin{pmatrix} 0.5000 & 0.8333 \\ 0.1667 & 0.5000 \end{pmatrix}$$

The optimal value of the membership for the given LIFNs is obtained by using Eq. (9.3) and get $r_1 = 0.6667$ and $r_2 = 0.3333$. Since $r_1 > r_2$, thus we can conclude that LIFN $\mathcal{A}_1$ is superior to the LIFN $\mathcal{A}_2$ i.e., $\mathcal{A}_1 \succ \mathcal{A}_2$ where "$\succ$" refers "preferred to".

Example 9.2.4. Let $S_{[0,8]}$ be LTS and $\mathcal{A}_1 = (s_5, s_2)$ and $\mathcal{A}_2 = (s_4, s_1)$ be two LIFNs defined over it. The uncertain linguistic numbers corresponding to these LIFNs are obtained as $\mathcal{A}_1 = [s_5, s_6]$ and $\mathcal{A}_2 = [s_4, s_7]$. Thus, corresponding to these uncertain linguistic numbers, we compute the possibility degree measure as defined in Definition 2.2.5 of Chapter 2 and we get the measurement values as $p'(\mathcal{A}_1 \succeq \mathcal{A}_2) = p'(\mathcal{A}_2 \succeq \mathcal{A}_1) = 0.5$. Therefore, the possibility degree matrix is obtained as

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

Now, the optimal value of the membership is obtained by using Eq. (9.3) and get $r_1 = r_2 = 0.5000$. Since $r_1 = r_2$ but it is clearly seen that $\mathcal{A}_1 \neq \mathcal{A}_2$. Thus, we conclude the existing measure is unable to classify the numbers.

On the other hand, if we utilize the proposed possibility degree measure on these numbers, then we get the possibility degree matrix as

$$P = \begin{pmatrix} 0.5 & 1.0 \\ 0.0 & 0.5 \end{pmatrix}$$

Thus, from it we conclude that $\mathcal{A}_1 \succ \mathcal{A}_2$.

In the nutshells, we can conclude that for any two different LIFNs with equal score values then the possibility degree measure between them is always 0.5 and hence we cannot rank such numbers by using existing measures. On the other hand, by using the proposed possibility degree measure, we conclude it reasonably classifies the different numbers.

9.3 LIF Aggregation operators using Einstein norm operations

In this section, we define some new LIF weighted averaging AOs for LIFNs by using Einstein norm operations.

9.3.1 Einstein operational laws of LIFNs

In it, we introduce the Einstein operations on LIFNs and analyze some desirable properties of these operations.

Definition 9.3.1. Let $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$ and $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$ be two LIFNs defined over $S_{[0,h]}$ and $\lambda > 0$ be a real number. The operational laws between the pairs of LIFNs by using Einstein t-norm operations (as defined in Table 2.1 of Chapter 2) are defined as

(i) $\mathcal{A}_1 \vee \mathcal{A}_2 = (\max\{s_{\tau_1}, s_{\tau_2}\}, \min\{s_{\theta_1}, s_{\theta_2}\})$.

(ii) $\mathcal{A}_1 \wedge \mathcal{A}_2 = (\min\{s_{\tau_1}, s_{\tau_2}\}, \max\{s_{\theta_1}, s_{\theta_2}\})$.

(iii) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \left(s_{h\left(\frac{\tau_1/h+\tau_2/h}{1+(\tau_1/h)\cdot(\tau_2/h)}\right)}, s_{h\left(\frac{(\theta_1/h)\cdot(\theta_2/h)}{1+(1-\theta_1/h)\cdot(1-\theta_2/h)}\right)}\right)$.

(iv) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \left(s_{h\left(\frac{(\tau_1/h)\cdot(\tau_2/h)}{1+(1-\tau_1/h)\cdot(1-\tau_2/h)}\right)}, s_{h\left(\frac{(\theta_1/h)+(\theta_2/h)}{1+(\theta_1/h)\cdot(\theta_2/h)}\right)}\right)$.

(v) $\lambda \mathcal{A}_1 = \left(s_{h\left(\frac{(1+\tau_1/h)^\lambda-(1-\tau_1/h)^\lambda}{(1+\tau_1/h)^\lambda+(1-\tau_1/h)^\lambda}\right)}, s_{h\left(\frac{2(\theta_1/h)^\lambda}{(2-\theta_1/h)^\lambda+(\theta_1/h)^\lambda}\right)}\right)$.

(vi) $\mathcal{A}_1^\lambda = \left(s_{h\left(\frac{2(\tau_1/h)^\lambda}{(2-\tau_1/h)^\lambda+(\tau_1/h)^\lambda}\right)}, s_{h\left(\frac{(1+\theta_1/h)^\lambda-(1-\theta_1/h)^\lambda}{(1+\theta_1/h)^\lambda+(1-\theta_1/h)^\lambda}\right)}\right)$.

Theorem 9.3.1. If $\mathcal{A}_1$ and $\mathcal{A}_2$ be two LIFNs, then operations defined in Definition 9.3.1 are also LIFNs.

Proof. Let $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$ and $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$ be two LIFNs which gives that $0 \leq \tau_1, \tau_2 \leq h$, $0 \leq \theta_1, \theta_2 \leq h$, $0 \leq \tau_1 + \theta_1 \leq h$ and $0 \leq \tau_2 + \theta_2 \leq h$. Therefore, we conclude that $\frac{(\tau_1/h)+(\tau_2/h)}{1+(\tau_1/h)\cdot(\tau_2/h)} \in [0,1]$ and $\frac{(\theta_1/h)\cdot(\theta_2/h)}{1+(1-\theta_1/h)\cdot(1-\theta_2/h)} \in [0,1]$. Furthermore,

$$\begin{aligned}
& h\left(\frac{(\tau_1/h)+(\tau_2/h)}{1+(\tau_1/h)\cdot(\tau_2/h)}\right) + h\left(\frac{(\theta_1/h)\cdot(\theta_2/h)}{1+(1-\theta_1/h)\cdot(1-\theta_2/h)}\right) \\
\leq \;& h\left(\frac{(\tau_1/h)+(\tau_2/h)}{1+(\tau_1/h)\cdot(\tau_2/h)}\right) + h\left(\frac{(1-\tau_1/h)\cdot(1-\tau_2/h)}{1+(\tau_1/h)\cdot(\tau_2/h)}\right) \\
= \;& h\left(\frac{(\tau_1/h)+(\tau_2/h)+(1-\tau_1/h)\cdot(1-\tau_2/h)}{1+(\tau_1/h)\cdot(\tau_2/h)}\right) \\
= \;& h.
\end{aligned}$$

Hence, $\mathcal{A}_1 \oplus \mathcal{A}_2$ is LIFN. Similarly, we can proven for the remaining parts. □

Theorem 9.3.2. Let $\mathcal{A} = (s_\tau, s_\theta)$, $\mathcal{A}_1 = (s_{\tau_1}, s_{\theta_1})$, $\mathcal{A}_2 = (s_{\tau_2}, s_{\theta_2})$ be three LIFNs, and $\lambda, \lambda_1, \lambda_2 > 0$ be three real numbers, then

(i) $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$.

(ii) $\lambda(\mathcal{A}_1 \oplus \mathcal{A}_2) = \lambda\mathcal{A}_1 \oplus \lambda\mathcal{A}_2$.

(iii) $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = (\lambda_1 + \lambda_2)\mathcal{A}$.

(iv) $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{A}_2 \otimes \mathcal{A}_1$.

(v) $\mathcal{A}^{\lambda_1} \otimes \mathcal{A}^{\lambda_2} = \mathcal{A}^{\lambda_1+\lambda_2}$.

(vi) $\mathcal{A}_1^\lambda \otimes \mathcal{A}_2^\lambda = (\mathcal{A}_1 \otimes \mathcal{A}_2)^\lambda$.

Proof. Here, we proof the parts (i)-(iii), while others can be proven similarly.

(i) For two LIFNs $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

$$\begin{aligned}
\mathcal{A}_1 \oplus \mathcal{A}_2 &= \left(s_{h\left(\frac{\tau_1/h+\tau_2/h}{1+(\tau_1/h)\cdot(\tau_2/h)}\right)}, s_{h\left(\frac{(\theta_1/h)\cdot(\theta_2/h)}{1+(1-\theta_1/h)\cdot(1-\theta_2/h)}\right)}\right) \\
&= \left(s_{h\left(\frac{\tau_2/h+\tau_1/h}{1+(\tau_2/h)\cdot(\tau_1/h)}\right)}, s_{h\left(\frac{(\theta_2/h)\cdot(\theta_1/h)}{1+(1-\theta_2/h)\cdot(1-\theta_1/h)}\right)}\right) \\
&= \mathcal{A}_2 \oplus \mathcal{A}_1
\end{aligned}$$

(ii) For a real number $\lambda > 0$ and two LIFNs $\mathcal{A}_1$, $\mathcal{A}_2$, we have

$$\begin{aligned}
\lambda\mathcal{A}_1 &= \left(s_{h\left(\frac{(1+\tau_1/h)^\lambda-(1-\tau_1/h)^\lambda}{(1+\tau_1/h)^\lambda+(1-\tau_1/h)^\lambda}\right)}, s_{h\left(\frac{2(\theta_1/h)^\lambda}{(2-\theta_1/h)^\lambda+(\theta_1/h)^\lambda}\right)}\right) \\
&= \left(s_{h\left(\frac{a_1-b_1}{a_1+b_1}\right)}, s_{h\left(\frac{2c_1}{d_1+c_1}\right)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\lambda\mathcal{A}_2 &= \left(s_{h\left(\frac{(1+\tau_2/h)^\lambda-(1-\tau_2/h)^\lambda}{(1+\tau_2/h)^\lambda+(1-\tau_2/h)^\lambda}\right)}, s_{h\left(\frac{2(\theta_2/h)^\lambda}{(2-\theta_2/h)^\lambda+(\theta_2/h)^\lambda}\right)}\right) \\
&= \left(s_{h\left(\frac{a_2-b_2}{a_2+b_2}\right)}, s_{h\left(\frac{2c_2}{d_2+c_2}\right)}\right)
\end{aligned}$$

where $a_1 = (1 + \tau_1/h)^\lambda$, $b_1 = (1 - \tau_1/h)^\lambda$, $c_1 = (\theta_1/h)^\lambda$, $d_1 = (2 - \theta_1/h)^\lambda$, $a_2 = (1 + \tau_2/h)^\lambda$, $b_2 = (1 - \tau_2/h)^\lambda$, $c_2 = (\theta_2/h)^\lambda$ and $d_2 = (2 - \theta_2/h)^\lambda$. So, by using additional operational laws, we have

$$\begin{aligned}
\lambda\mathcal{A}_1 \oplus \lambda\mathcal{A}_2 &= \left(s_{h\left(\frac{a_1-b_1}{a_1+b_1}\right)}, s_{h\left(\frac{2c_1}{d_1+c_1}\right)}\right) \oplus \left(s_{h\left(\frac{a_2-b_2}{a_2+b_2}\right)}, s_{h\left(\frac{2c_2}{d_2+c_2}\right)}\right) \\
&= \left(s_{h\left(\frac{\frac{a_1-b_1}{a_1+b_1}+\frac{a_2-b_2}{a_2+b_2}}{1+\frac{a_1-b_1}{a_1+b_1}\cdot\frac{a_2-b_2}{a_2+b_2}}\right)}, s_{h\left(\frac{\cdot\frac{2c_1}{d_1+c_1}\cdot\frac{2c_2}{d_2+c_2}}{1+\left(1-\frac{2c_1}{d_1+c_1}\right)\cdot\left(1-\frac{2c_2}{d_2+c_2}\right)}\right)}\right)
\end{aligned}$$

$$\begin{aligned}
&= \left(s_{h\left(\frac{a_1a_2-b_1b_2}{a_1a_2+b_1b_2}\right)}, s_{h\left(\frac{2c_1c_2}{d_1d_2+c_1c_2}\right)}\right) \\
&= \left(\begin{array}{l} s_{h\left(\frac{(1+\tau_1/h)^\lambda(1+\tau_2/h)^\lambda-(1-\tau_1/h)^\lambda(1-\tau_2/h)^\lambda}{(1+\tau_1/h)^\lambda(1+\tau_2/h)^\lambda+(1-\tau_1/h)^\lambda(1-\tau_2/h)^\lambda}\right)}, \\ \\ s_{h\left(\frac{2(\theta_1/h)^\lambda(\theta_2/h)^\lambda}{(2-\theta_1/h)^\lambda(2-\theta_2/h)^\lambda+(\theta_1/h)^\lambda(\theta_2/h)^\lambda}\right)} \end{array}\right)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{A}_1 \oplus \mathcal{A}_2 &= \left(s_{h\left(\frac{\tau_1/h+\tau_2/h}{1+(\tau_1/h)\cdot(\tau_2/h)}\right)}, s_{h\left(\frac{(\theta_1/h)\cdot(\theta_2/h)}{1+(1-\theta_1/h)\cdot(1-\theta_2/h)}\right)}\right) \\
&= \left(s_{h\left(\frac{a-b}{a+b}\right)}, s_{h\left(\frac{2c}{d+c}\right)}\right)
\end{aligned}$$

where $a = (1+\tau_1/h)\cdot(1+\tau_2/h)$, $b = (1-\tau_1/h)\cdot(1-\tau_2/h)$, $c = (\theta_1/h)\cdot(\theta_2/h)$ and $d = (2-\theta_1/h)\cdot(2-\theta_2/h)$. Then, by using Einstein operational laws, we have

$$\begin{aligned}
\lambda(\mathcal{A}_1 \oplus \mathcal{A}_2) &= \lambda\left(s_{h\left(\frac{a-b}{a+b}\right)}, s_{h\left(\frac{2c}{d+c}\right)}\right) \\
&= \left(s_{h\left(\frac{\left(1+\frac{a-b}{a+b}\right)^\lambda-\left(1-\frac{a-b}{a+b}\right)^\lambda}{\left(1+\frac{a-b}{a+b}\right)^\lambda+\left(1-\frac{a-b}{a+b}\right)^\lambda}\right)}, s_{h\left(\frac{2\cdot\left(\frac{2c}{d+c}\right)^\lambda}{\left(2-\frac{2c}{d+c}\right)^\lambda+\left(\frac{2c}{d+c}\right)^\lambda}\right)}\right) \\
&= \left(s_{h\left(\frac{a^\lambda-b^\lambda}{a^\lambda+b^\lambda}\right)}, s_{h\left(\frac{2\cdot c^\lambda}{d^\lambda+c^\lambda}\right)}\right) \\
&= \left(\begin{array}{l} s_{h\left(\frac{(1+\tau_1/h)^\lambda(1+\tau_2/h)^\lambda-(1-\tau_1/h)^\lambda(1-\tau_2/h)^\lambda}{(1+\tau_1/h)^\lambda(1+\tau_2/h)^\lambda+(1-\tau_1/h)^\lambda(1-\tau_2/h)^\lambda}\right)}, \\ \\ s_{h\left(\frac{2(\theta_1/h)^\lambda(\theta_2/h)^\lambda}{(2-\theta_1/h)^\lambda(2-\theta_2/h)^\lambda+(\theta_1/h)^\lambda(\theta_2/h)^\lambda}\right)} \end{array}\right)
\end{aligned}$$

Hence, $\lambda(\mathcal{A}_1 \oplus \mathcal{A}_2) = \lambda\mathcal{A}_1 \oplus \lambda\mathcal{A}_2$.

(iii) For real numbers $\lambda_1 > 0, \lambda_2 > 0$,

$$\begin{aligned}
\lambda_1\mathcal{A} &= \left(s_{h\left(\frac{a_1-b_1}{a_1+b_1}\right)}, s_{h\left(\frac{2c_1}{d_1+c_1}\right)}\right) \\
\text{and} \quad \lambda_2\mathcal{A} &= \left(s_{h\left(\frac{a_2-b_2}{a_2+b_2}\right)}, s_{h\left(\frac{2c_2}{d_2+c_2}\right)}\right)
\end{aligned}$$

where $a_i = (1+\tau/h)^{\lambda_i}, b_i = (1-\tau/h)^{\lambda_i}, c_i = (\theta/h)^{\lambda_i}, d_i = (2-\theta/h)^{\lambda_i}$ for $i = 1, 2$.

Therefore,

$$\begin{aligned}
\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} &= \left(s_{h\left(\frac{a_1-b_1}{a_1+b_1}\right)}, s_{h\left(\frac{2c_1}{d_1+c_1}\right)}\right) \oplus \left(s_{h\left(\frac{a_2-b_2}{a_2+b_2}\right)}, s_{h\left(\frac{2c_2}{d_2+c_2}\right)}\right) \\
&= \left(s_{h\left(\frac{\frac{a_1-b_1}{a_1+b_1}+\frac{a_2-b_2}{a_2+b_2}}{1+\frac{a_1-b_1}{a_1+b_1}\cdot\frac{a_2-b_2}{a_2+b_2}}\right)}, s_{h\left(\frac{4\frac{c_1c_2}{(d_1+c_1)\cdot(d_2+c_2)}}{1+(1-\frac{2c_1}{d_1+c_1})\cdot(1-\frac{2c_2}{d_2+c_2})}\right)}\right) \\
&= \left(s_{h\left(\frac{a_1\cdot a_2-b_1\cdot b_2}{a_1\cdot a_2+b_1\cdot b_2}\right)}, s_{h\left(\frac{2(c_1\cdot c_2)}{d_1\cdot d_2+c_1\cdot c_2}\right)}\right) \\
&= \left(s_{h\left(\frac{(1+\tau/h)^{\lambda_1+\lambda_2}-(1-\tau/h)^{\lambda_1+\lambda_2}}{(1+\tau/h)^{\lambda_1+\lambda_2}+(1-\tau/h)^{\lambda_1+\lambda_2}}\right)}, s_{h\left(\frac{2(\theta/h)^{\lambda_1+\lambda_2}}{(2-\theta/h)^{\lambda_1+\lambda_2}+(\theta/h)^{\lambda_1+\lambda_2}}\right)}\right) \\
&= (\lambda_1+\lambda_2)\mathcal{A}
\end{aligned}$$

Hence, $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = (\lambda_1+\lambda_2)\mathcal{A}$.

$\square$

Theorem 9.3.3. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two LIFNs then

(i) $\mathcal{A}_1^c \wedge \mathcal{A}_2^c = (\mathcal{A}_1 \vee \mathcal{A}_2)^c$.

(ii) $\mathcal{A}_1^c \vee \mathcal{A}_2^c = (\mathcal{A}_1 \wedge \mathcal{A}_2)^c$.

(iii) $\mathcal{A}_1^c \oplus \mathcal{A}_2^c = (\mathcal{A}_1 \otimes \mathcal{A}_2)^c$.

(iv) $\mathcal{A}_1^c \otimes \mathcal{A}_2^c = (\mathcal{A}_1 \oplus \mathcal{A}_2)^c$.

(v) $(\mathcal{A}_1 \vee \mathcal{A}_2) \oplus (\mathcal{A}_1 \wedge \mathcal{A}_2) = \mathcal{A}_1 \oplus \mathcal{A}_2$.

(vi) $(\mathcal{A}_1 \vee \mathcal{A}_2) \otimes (\mathcal{A}_1 \wedge \mathcal{A}_2) = \mathcal{A}_1 \otimes \mathcal{A}_2$.

Proof. The proof is trivial, so it is omitted here. $\square$

Theorem 9.3.4. Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be three LIFNs then

(i) $(\mathcal{A}_1 \vee \mathcal{A}_2) \wedge \mathcal{A}_3 = (\mathcal{A}_1 \wedge \mathcal{A}_3) \vee (\mathcal{A}_2 \wedge \mathcal{A}_3)$.

(ii) $(\mathcal{A}_1 \wedge \mathcal{A}_2) \vee \mathcal{A}_3 = (\mathcal{A}_1 \vee \mathcal{A}_3) \wedge (\mathcal{A}_2 \vee \mathcal{A}_3)$.

(iii) $(\mathcal{A}_1 \vee \mathcal{A}_2) \oplus \mathcal{A}_3 = (\mathcal{A}_1 \oplus \mathcal{A}_3) \vee (\mathcal{A}_2 \oplus \mathcal{A}_3)$.

(iv) $(\mathcal{A}_1 \wedge \mathcal{A}_2) \oplus \mathcal{A}_3 = (\mathcal{A}_1 \oplus \mathcal{A}_3) \wedge (\mathcal{A}_2 \oplus \mathcal{A}_3)$.

(v) $(\mathcal{A}_1 \vee \mathcal{A}_2) \otimes \mathcal{A}_3 = (\mathcal{A}_1 \otimes \mathcal{A}_3) \vee (\mathcal{A}_2 \otimes \mathcal{A}_3)$.

(vi) $(\mathcal{A}_1 \wedge \mathcal{A}_2) \otimes \mathcal{A}_3 = (\mathcal{A}_1 \otimes \mathcal{A}_3) \wedge (\mathcal{A}_2 \otimes \mathcal{A}_3)$.

Proof. The proof is trivial, so it is omitted here. □

9.3.2 LIF Einstein arithmetic aggregation operators

In this section, we investigate averaging AOs with the help of Einstein operations to aggregate the collections of LIFNs.

Definition 9.3.2. Let Ω be the set of LIFNs and LIF Einstein weighted averaging (LIFEWA) operator is a mapping given by LIFEWA: $\Omega^n \longrightarrow \Omega$ and defined as

$$\text{LIFEWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2 \oplus \ldots \oplus \omega_n\mathcal{A}_n, \tag{9.4}$$

where ω_t is be the weight of $\mathcal{A}_t$ such that $\omega_t > 0$ and $\sum\limits_{t=1}^{n} \omega_t = 1$.

Theorem 9.3.5. For LIFNs $\mathcal{A}_t = (s_{\tau_t}, s_{\theta_t})(t = 1, 2, \ldots, n)$, the aggregated value by LIFEWA operator remains LIFN and is given by

$$\text{LIFEWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(s_{h\left(\frac{\prod_{t=1}^{n}(1+\tau_t/h)^{\omega_t} - \prod_{t=1}^{n}(1-\tau_t/h)^{\omega_t}}{\prod_{t=1}^{n}(1+\tau_t/h)^{\omega_t} + \prod_{t=1}^{n}(1-\tau_t/h)^{\omega_t}}\right)}, s_{h\left(\frac{2\prod_{t=1}^{n}(\theta_t/h)^{\omega_t}}{\prod_{t=1}^{n}(2-\theta_t/h)^{\omega_t} + \prod_{t=1}^{n}(\theta_t/h)^{\omega_t}}\right)} \right) \tag{9.5}$$

Proof. The first result holds immediately from the Theorem 9.3.1. To prove Eq. (9.5), we apply the following steps of the mathematical induction on n.

Step 1: Let $n = 2$, we have

$$\begin{aligned} \omega_1\mathcal{A}_1 &= \left(s_{h\left(\frac{(1+\tau_1/h)^{\omega_1} - (1-\tau_1/h)^{\omega_1}}{(1+\tau_1/h)^{\omega_1} + (1-\tau_1/h)^{\omega_1}}\right)}, s_{h\left(\frac{2(\theta_1/h)^{\omega_1}}{(2-\theta_1/h)^{\omega_1} + (\theta_1/h)^{\omega_1}}\right)} \right) \\ \text{and} \qquad \omega_2\mathcal{A}_2 &= \left(s_{h\left(\frac{(1+\tau_2/h)^{\omega_2} - (1-\tau_2/h)^{\omega_2}}{(1+\tau_2/h)^{\omega_2} + (1-\tau_2/h)^{\omega_2}}\right)}, s_{h\left(\frac{2(\theta_2/h)^{\omega_2}}{(2-\theta_2/h)^{\omega_2} + (\theta_2/h)^{\omega_2}}\right)} \right) \end{aligned}$$

which implies that

$$\begin{aligned}
\mathrm{LIFEWA}(\mathcal{A}_1,\mathcal{A}_2) &= \omega_1\mathcal{A}_1 \oplus \omega_2\mathcal{A}_2 \\
&= \left(\begin{array}{l}
s_{h\left(\frac{\frac{(1+\tau_1/h)^{\omega_1}-(1-\tau_1/h)^{\omega_1}}{(1+\tau_1/h)^{\omega_1}+(1-\tau_1/h)^{\omega_1}}+\frac{(1+\tau_2/h)^{\omega_2}-(1-\tau_2/h)^{\omega_2}}{(1+\tau_2/h)^{\omega_2}+(1-\tau_2/h)^{\omega_2}}}{1+\frac{(1+\tau_1/h)^{\omega_1}-(1-\tau_1/h)^{\omega_1}}{(1+\tau_1/h)^{\omega_1}+(1-\tau_1/h)^{\omega_1}}\cdot\frac{(1+\tau_2/h)^{\omega_2}-(1-\tau_2/h)^{\omega_2}}{(1+\tau_2/h)^{\omega_2}+(1-\tau_2/h)^{\omega_2}}}\right)}, \\
s_{h\left(\frac{\frac{2(\theta_1/h)^{\omega_1}}{(2-\theta_1/h)^{\omega_1}+(\theta_1/h)^{\omega_1}}\cdot\frac{2(\theta_2/h)^{\omega_2}}{(2-\theta_2/h)^{\omega_2}+(\theta_2/h)^{\omega_2}}}{1+\left(1-\frac{2(\theta_1/h)^{\omega_1}}{(2-\theta_1/h)^{\omega_1}+(\theta_1/h)^{\omega_1}}\right)\left(1-\frac{2(\theta_2/h)^{\omega_2}}{(2-\theta_2/h)^{\omega_2}+(\theta_2/h)^{\omega_2}}\right)}\right)}
\end{array}\right) \\
&= \left(s_{h\left(\frac{\prod_{t=1}^{2}(1+\tau_t/h)^{\omega_t}-\prod_{t=1}^{2}(1-\tau_t/h)^{\omega_t}}{\prod_{t=1}^{2}(1+\tau_t/h)^{\omega_t}+\prod_{t=1}^{2}(1-\tau_t/h)^{\omega_t}}\right)}, s_{h\left(\frac{2\prod_{t=1}^{2}(\theta_t/h)^{\omega_t}}{\prod_{t=1}^{2}(2-\theta_t/h)^{\omega_t}+\prod_{t=1}^{2}(\theta_t/h)^{\omega_t}}\right)} \right)
\end{aligned}$$

Hence, Eq. (9.5) holds for $n = 2$.

Step 2: Let Eq. (9.5) holds for $n = k$, that is

$$\mathrm{LIFEWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k) = \left(s_{h\left(\frac{\prod_{t=1}^{k}(1+\tau_t/h)^{\omega_t}-\prod_{t=1}^{k}(1-\tau_t/h)^{\omega_t}}{\prod_{t=1}^{k}(1+\tau_t/h)^{\omega_t}+\prod_{t=1}^{k}(1-\tau_t/h)^{\omega_t}}\right)}, s_{h\left(\frac{2\prod_{t=1}^{k}(\theta_t/h)^{\omega_t}}{\prod_{t=1}^{k}(2-\theta_t/h)^{\omega_t}+\prod_{t=1}^{k}(\theta_t/h)^{\omega_t}}\right)} \right)$$

Step 3: Now, for $n = k+1$, by Definition 9.3.1, we have

$$\begin{aligned}
&\mathrm{LIFEWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots\mathcal{A}_{k+1}) = \mathrm{LIFEWA}(\mathcal{A}_1,\mathcal{A}_2,\ldots\mathcal{A}_k) \oplus \omega_{k+1}\mathcal{A}_{k+1} \\
&= \left(\begin{array}{l}
s_{h\left(\frac{\prod_{t=1}^{k}(1+\tau_t/h)^{\omega_t}-\prod_{t=1}^{k}(1-\tau_t/h)^{\omega_t}}{\prod_{t=1}^{k}(1+\tau_t/h)^{\omega_t}+\prod_{t=1}^{k}(1-\tau_t/h)^{\omega_t}}\right)}, \\
s_{h\left(\frac{2\prod_{t=1}^{k}(\theta_t/h)^{\omega_t}}{\prod_{t=1}^{k}(2-\theta_t/h)^{\omega_t}+\prod_{t=1}^{k}(\theta_t/h)^{\omega_t}}\right)}
\end{array}\right) \oplus \left(\begin{array}{l}
s_{h\left(\frac{(1+\tau_{k+1}/h)^{\omega_{k+1}}-(1-\tau_{k+1}/h)^{\omega_{k+1}}}{(1+\tau_{k+1}/h)^{\omega_{k+1}}+(1-\tau_{k+1}/h)^{\omega_{k+1}}}\right)}, \\
s_{h\left(\frac{2(\theta_{k+1}/h)^{\omega_{k+1}}}{(2-\theta_{k+1}/h)^{\omega_{k+1}}+(\theta_{k+1}/h)^{\omega_{k+1}}}\right)}
\end{array}\right) \\
&= \left(\begin{array}{l}
s_{h\left(\frac{\frac{\prod_{t=1}^{k}(1+\tau_t/h)^{\omega_t}-\prod_{t=1}^{k}(1-\tau_t/h)^{\omega_t}}{\prod_{t=1}^{k}(1+\tau_t/h)^{\omega_t}+\prod_{t=1}^{k}(1-\tau_t/h)^{\omega_t}}+\frac{(1+\tau_{k+1}/h)^{\omega_{k+1}}-(1-\tau_{k+1}/h)^{\omega_{k+1}}}{(1+\tau_{k+1}/h)^{\omega_{k+1}}+(1-\tau_{k+1}/h)^{\omega_{k+1}}}}{1+\frac{\prod_{t=1}^{k}(1+\tau_t/h)^{\omega_t}-\prod_{t=1}^{k}(1-\tau_t/h)^{\omega_t}}{\prod_{t=1}^{k}(1+\tau_t/h)^{\omega_t}+\prod_{t=1}^{k}(1-\tau_t/h)^{\omega_t}}\cdot\frac{(1+\tau_{k+1}/h)^{\omega_{k+1}}-(1-\tau_{k+1}/h)^{\omega_{k+1}}}{(1+\tau_{k+1}/h)^{\omega_{k+1}}+(1-\tau_{k+1}/h)^{\omega_{k+1}}}}\right)}, \\
s_{h\left(\frac{\frac{2\prod_{t=1}^{k}(\theta_t/h)^{\omega_t}}{\prod_{t=1}^{k}(2-\theta_t/h)^{\omega_t}+\prod_{t=1}^{k}(\theta_t/h)^{\omega_t}}\cdot\frac{2(\theta_{k+1}/h)^{\omega_{k+1}}}{(2-\theta_{k+1}/h)^{\omega_{k+1}}+(\theta_{k+1}/h)^{\omega_{k+1}}}}{1+\left(1-\frac{2\prod_{t=1}^{k}(\theta_t/h)^{\omega_t}}{\prod_{t=1}^{k}(2-\theta_t/h)^{\omega_t}+\prod_{t=1}^{k}(\theta_t/h)^{\omega_t}}\right)\left(1-\frac{2(\theta_{k+1}/h)^{\omega_{k+1}}}{(2-\theta_{k+1}/h)^{\omega_{k+1}}+(\theta_{k+1}/h)^{\omega_{k+1}}}\right)}\right)}
\end{array}\right)
\end{aligned}$$

$$= \left(s_{h\left(\frac{\prod_{t=1}^{k+1}(1+\tau_t/h)^{\omega_t}-\prod_{t=1}^{k+1}(1-\tau_t/h)^{\omega_t}}{\prod_{t=1}^{k+1}(1+\tau_t/h)^{\omega_t}+\prod_{t=1}^{k+1}(1-\tau_t/h)^{\omega_t}}\right)}, s_{h\left(\frac{2\prod_{t=1}^{k+1}(\theta_t/h)^{\omega_t}}{\prod_{t=1}^{k+1}(2-\theta_t/h)^{\omega_t}+\prod_{t=1}^{k+1}(\theta_t/h)^{\omega_t}}\right)} \right)$$

Thus, results holds for $n = k + 1$ also. Hence, by the principle of mathematical induction, result is true for all positive integer n. □

The above proposed operator is illustrated with a numerical example as follows.

Example 9.3.1. Let $S_{[0,8]}$ be a continuous LTS and $\mathcal{A}_1 = (s_4, s_2)$, $\mathcal{A}_2 = (s_3, s_1)$, $\mathcal{A}_3 = (s_1, s_4)$ be three LIFNs with weight vector $\omega = (0.35, 0.40, 0.25)^T$. Based on this information, we get

$$\prod_{t=1}^{3}(1+\tau_t/8)^{\omega_t} = (1+4/8)^{0.35}\cdot(1+3/8)^{0.4}\cdot(1+1/8)^{0.25} = 1.3482;$$
$$\prod_{t=1}^{3}(1-\tau_t/8)^{\omega_t} = (1-4/8)^{0.35}\cdot(1-3/8)^{0.4}\cdot(1-1/8)^{0.25} = 0.6288;$$
$$\prod_{t=1}^{3}(\theta_t/8)^{\omega_t} = (2/8)^{0.35}\cdot(1/8)^{0.4}\cdot(4/8)^{0.25} = 0.2253;$$
$$\prod_{t=1}^{3}(2-\theta_t/8)^{\omega_t} = (2-2/8)^{0.35}\cdot(2-1/8)^{0.4}\cdot(2-4/8)^{0.25} = 1.7310.$$

Therefore,

$$\begin{aligned} &\text{LIFEWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \\ = &\left(s_{8\left(\frac{\prod_{t=1}^{3}(1+\tau_t/8)^{\omega_t}-\prod_{t=1}^{3}(1-\tau_t/8)^{\omega_t}}{\prod_{t=1}^{3}(1+\tau_t/8)^{\omega_t}+\prod_{t=1}^{3}(1-\tau_t/8)^{\omega_t}}\right)}, s_{8\left(\frac{2\prod_{t=1}^{3}(\theta_t/8)^{\omega_t}}{\prod_{t=1}^{3}(2-\theta_t/8)^{\omega_t}+\prod_{t=1}^{3}(\theta_t/8)^{\omega_t}}\right)} \right) \\ = &\left(s_{8\left(\frac{1.3482-0.6288}{1.3482+0.6288}\right)}, s_{8\left(\frac{2\times 0.2253}{1.7310+0.2253}\right)} \right) \\ = &\left(s_{2.9111}, s_{1.8428} \right) \end{aligned}$$

Further, it is observed that the proposed LIFEWA operator satisfies the following properties for a collection of LIFNs $\mathcal{A}_t (t = 1, 2, \ldots, n)$.

Theorem 9.3.6. If $\mathcal{A}_t = (s_\tau, s_\theta)$ for all t then

$$\text{LIFEWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) = (s_\tau, s_\theta).$$

This property is called as Idempotency.

Proof. For a collection of LIFNs $\mathcal{A}_t = (s_{\tau_t}, s_{\theta_t})(t = 1, 2, \ldots, n)$. If we assume $\mathcal{A}_t = (s_\tau, s_\theta)$ for all t and ω_t is the normalized weight of them. Then, from Eq. (9.5) we have

$$\begin{aligned}
& \mathrm{LIFEWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \\
= & \left(s_{h\left(\frac{\prod_{t=1}^{n}(1+\tau/h)^{\omega_t} - \prod_{t=1}^{n}(1-\tau/h)^{\omega_t}}{\prod_{t=1}^{n}(1+\tau/h)^{\omega_t} + \prod_{t=1}^{n}(1-\tau/h)^{\omega_t}}\right)}, s_{h\left(\frac{2\prod_{t=1}^{n}(\theta/h)^{\omega_t}}{\prod_{t=1}^{n}(2-\theta/h)^{\omega_t} + \prod_{t=1}^{n}(\theta/h)^{\omega_t}}\right)} \right) \\
= & \left(s_{h\left(\frac{(1+\tau/h)^{\sum_{t=1}^{n}\omega_t} - (1-\tau/h)^{\sum_{t=1}^{n}\omega_t}}{(1+\tau/h)^{\sum_{t=1}^{n}\omega_t} + (1-\tau/h)^{\sum_{t=1}^{n}\omega_t}}\right)}, s_{h\left(\frac{2(\theta/h)^{\sum_{t=1}^{n}\omega_t}}{(2-\theta/h)^{\sum_{t=1}^{n}\omega_t} + (\theta/h)^{\sum_{t=1}^{n}\omega_t}}\right)} \right) \\
= & \left(s_{h\left(\frac{(1+\tau/h)-(1-\tau/h)}{(1+\tau/h)+(1-\tau/h)}\right)}, s_{h\left(\frac{2(\theta/h)}{(2-\theta/h)+(\theta/h)}\right)} \right) \\
= & (s_\tau, s_\theta)
\end{aligned}$$

□

Theorem 9.3.7. Let $\gamma^- = (s_{\tau^-}, s_{\theta^-})$ and $\gamma^+ = (s_{\tau^+}, s_{\theta^+})$, where $\tau^- = \min_t\{\tau_t\}$, $\theta^- = \max_t\{\theta_t\}$, $\tau^+ = \max_t\{\tau_t\}$, and $\theta^+ = \min_t\{\theta_t\}$, then we have

$$\gamma^- \leq \mathrm{LIFEWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) \leq \gamma^+.$$

This property is called as Boundedness.

Proof. Let $g(x) = \frac{1-x}{1+x}$, $x \in [0, 1]$, then $g'(x) = \frac{-2}{(1+x)^2} < 0$, which implies $g(x)$ is decreasing function on $[0, 1]$. Since $\tau^- \leq \tau_t \leq \tau^+$, $\forall t$, then $g(\tau^+/h) \leq g(\tau_t/h) \leq g(\tau^-/h)$. Therefore, we have

$$\begin{aligned}
& \left(\frac{1-\tau^+/h}{1+\tau^+/h}\right) \leq \left(\frac{1-\tau_t/h}{1+\tau_t/h}\right) \leq \left(\frac{1-\tau^-/h}{1+\tau^-/h}\right) \\
\Leftrightarrow\ & \prod_{t=1}^{n}\left(\frac{1-\tau^+/h}{1+\tau^+/h}\right)^{\omega_t} \leq \prod_{t=1}^{n}\left(\frac{1-\tau_t/h}{1+\tau_t/h}\right)^{\omega_t} \leq \prod_{t=1}^{n}\left(\frac{1-\tau^-/h}{1+\tau^-/h}\right)^{\omega_t} \\
\Leftrightarrow\ & 1+\left(\frac{1-\tau^+/h}{1+\tau^+/h}\right) \leq 1+\prod_{t=1}^{n}\left(\frac{1-\tau_t/h}{1+\tau_t/h}\right)^{\omega_t} \leq 1+\left(\frac{1-\tau^-/h}{1+\tau^-/h}\right) \\
\Leftrightarrow\ & \left(\frac{2}{1+\tau^+/h}\right) \leq 1+\prod_{t=1}^{n}\left(\frac{1-\tau_t/h}{1+\tau_t/h}\right)^{\omega_t} \leq \left(\frac{2}{1+\tau^-/h}\right) \\
\Leftrightarrow\ & \left(\frac{1+\tau^-/h}{2}\right) \leq \frac{1}{1+\prod_{t=1}^{n}\left(\frac{1-\tau_t/h}{1+\tau_t/h}\right)^{\omega_t}} \leq \left(\frac{2}{1+\tau^+/h}\right)
\end{aligned}$$

$$\Leftrightarrow \quad \frac{\tau^-}{h} \leq \frac{2}{1+\prod\limits_{t=1}^{n}\left(\frac{1-\tau_t/h}{1+\tau_t/h}\right)^{\omega_t}} - 1 \leq \frac{\tau^+}{h}$$

$$\Leftrightarrow \quad \tau^- \leq h\left(\frac{\prod\limits_{t=1}^{n}(1+\tau_t/h)^{\omega_t} - \prod\limits_{t=1}^{n}(1-\tau_t/h)^{\omega_t}}{\prod\limits_{t=1}^{n}(1+\tau_t/h)^{\omega_t} + \prod\limits_{t=1}^{n}(1-\tau_t/h)^{\omega_t}}\right) \leq \tau^+$$

$$\Leftrightarrow \quad s_{\tau^-} \leq s_{h\left(\frac{\prod_{t=1}^{n}(1+\tau_t/h)^{\omega_t} - \prod_{t=1}^{n}(1-\tau_t/h)^{\omega_t}}{\prod_{t=1}^{n}(1+\tau_t/h)^{\omega_t} + \prod_{t=1}^{n}(1-\tau_t/h)^{\omega_t}}\right)} \leq s_{\tau^+}$$

On the other hand, if we consider a function $f(y) = \frac{2-y}{y}$, $y \in (0,1]$, then $f'(x) = \frac{-2}{y^2} < 0$, which implies $f(y)$ is decreasing function on $(0,1]$. Since $\theta^+ \leq \theta_t \leq \theta^-$, $\forall t$, then $g(\theta^-/h) \leq g(\theta_t/h) \leq g(\theta^+/h)$. Therefore, we have

$$\left(\frac{2-\theta^-/h}{\theta^-/h}\right) \leq \left(\frac{2-\theta_t/h}{\theta_t/h}\right) \leq \left(\frac{2-\theta^+/h}{\theta^+/h}\right)$$

$$\Leftrightarrow \quad \prod_{t=1}^{n}\left(\frac{2-\theta^-/h}{\theta^-/h}\right)^{\omega_t} \leq \prod_{t=1}^{n}\left(\frac{2-\theta_t/h}{\theta_t/h}\right)^{\omega_t} \leq \prod_{t=1}^{n}\left(\frac{2-\theta^+/h}{\theta^+/h}\right)^{\omega_t}$$

$$\Leftrightarrow \quad \left(\frac{2-\theta^-/h}{\theta^-/h}\right)^{\sum\limits_{t=1}^{n}\omega_t} \leq \prod_{t=1}^{n}\left(\frac{2-\theta_t/h}{\theta_t/h}\right)^{\omega_t} \leq \left(\frac{2-\theta^+/h}{\theta^+/h}\right)^{\sum\limits_{t=1}^{n}\omega_t}$$

$$\Leftrightarrow \quad \left(\frac{2-\theta^-/h}{\theta^-/h}\right) \leq \prod_{t=1}^{n}\left(\frac{2-\theta_t/h}{\theta_t/h}\right)^{\omega_t} \leq \left(\frac{2-\theta^+/h}{\theta^+/h}\right)$$

$$\Leftrightarrow \quad \left(\frac{2}{\theta^-/h}\right) \leq \prod_{t=1}^{n}\left(\frac{2-\theta_t/h}{\theta_t/h}\right)^{\omega_t} + 1 \leq \left(\frac{2}{\theta^+/h}\right)$$

$$\Leftrightarrow \quad \frac{\theta^+}{h} \leq \frac{2}{\prod\limits_{t=1}^{n}\left(\frac{2-\theta_t/h}{\theta_t/h}\right)^{\omega_t} + 1} \leq \frac{\theta^-}{h}$$

$$\Leftrightarrow \quad \theta^+ \leq h\left(\frac{2\prod\limits_{t=1}^{n}(\theta_t/h)^{\omega_t}}{\prod\limits_{t=1}^{n}(2-\theta_t/h)^{\omega_t} + \prod\limits_{t=1}^{n}(\theta_t/h)^{\omega_t}}\right) \leq \theta^-$$

$$\Leftrightarrow \quad s_{\theta^+} \leq s_{h\left(\frac{2\prod_{t=1}^{n}(\theta_t/h)^{\omega_t}}{\prod_{t=1}^{n}(2-\theta_t/h)^{\omega_t} + \prod_{t=1}^{n}(\theta_t/h)^{\omega_t}}\right)} \leq s_{\theta^-}$$

Hence, according to Definition 2.2.4 of Chapter 2, we obtain

$$\gamma^- \leq \text{LIFEWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) \leq \gamma^+.$$

$\square$

Theorem 9.3.8. Let $\mathcal{A}_t = (s_{\tau_t}, s_{\theta_t})$ and $\mathcal{A}'_t = (s_{\tau'_t}, s_{\theta'_t})$ for $t = 1, 2, \ldots, n$ be two LIFNs such that $s_{\tau_t} \leq s_{\tau'_t}$ and $s_{\theta_t} \geq s_{\theta'_t}$ for all t, then we have

$$\text{LIFEWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) \leq \text{LIFEWA}(\mathcal{A}'_1, \mathcal{A}'_2, \ldots \mathcal{A}'_n).$$

Proof. Since $s_{\tau_t} \leq s_{\tau'_t}$ and $s_{\theta_t} \geq s_{\theta'_t}$ for all t, so by using boundedness property, we can obtain the desired result. □

The LIFEWA operator does not take into account the idea of OWA in the information fusion process. In the following, we first introduce the new concept of Linguistic intuitionistic fuzzy Einstein ordered weighted averaging (LIFEOWA) operator and then illustrate it with a numerical example.

Definition 9.3.3. Let $\mathcal{A}_t = (s_{\tau_t}, s_{\theta_t})$ for $t = 1, 2, \ldots, n$ be a family of LIFNs and $w = (w_1, w_2 \ldots, w_n)^T$ is the associated weight vector such that $w_t > 0$ and $\sum_{t=1}^{n} w_t = 1$; then, a LIFEOWA operator of dimension n is a mapping LIFEOWA : $\Omega^n \longrightarrow \Omega$, and

$$\text{LIFEOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) = w_1\mathcal{A}_{\sigma(1)} \oplus w_2\mathcal{A}_{\sigma(2)} \oplus \ldots \oplus w_n\mathcal{A}_{\sigma(n)}, \tag{9.6}$$

where σ is the permutation of $(1, 2, \ldots, n)$ such that $\mathcal{A}_{\sigma(t-1)} \geq \mathcal{A}_{\sigma(t)}$ for $t = 2, 3, \ldots, n$.

Based on the Einstein operational rules of the LIFNs, the LIFEOWA operator can be transformed into the following form.

Theorem 9.3.9. For a collection of LIFNs $\mathcal{A}_t = (s_{\tau_t}, s_{\theta_t})(t = 1, 2, \ldots, n)$, the aggregated value by using LIFEOWA operator remains LIFN and is given by

$$\begin{aligned} &\text{LIFEOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) \\ = &\left(s_{h\left(\frac{\prod_{t=1}^{n}\left(1+\tau_{\sigma(t)}/h\right)^{w_t} - \prod_{t=1}^{n}\left(1-\tau_{\sigma(t)}/h\right)^{w_t}}{\prod_{t=1}^{n}\left(1+\tau_{\sigma(t)}/h\right)^{w_t} + \prod_{t=1}^{n}\left(1-\tau_{\sigma(t)}/h\right)^{w_t}}\right)}, s_{h\left(\frac{2\prod_{t=1}^{n}\left(\theta_{\sigma(t)}/h\right)^{w_t}}{\prod_{t=1}^{n}\left(2-\theta_{\sigma(t)}/h\right)^{w_t} + \prod_{t=1}^{n}\left(\theta_{\sigma(t)}/h\right)^{w_t}}\right)} \right) \end{aligned} \tag{9.7}$$

Proof. The proof of this theorem is similar to that of Theorem 9.3.5 and hence it is omitted here. □

Example 9.3.2. Let $S_{[0,8]}$ be a continuous LTS and $\mathcal{A}_1 = (s_3, s_4)$, $\mathcal{A}_2 = (s_4, s_2)$, $\mathcal{A}_3 = (s_2, s_3)$ be three LIFNs with their associated weight vector $w = (0.3, 0.4, 0.3)^T$. To aggregate these numbers by LIFEOWA operator, we firstly apply the proposed possibility degree measure to find the t^{th} largest LIFN. For it, we construct possibility degree matrix as

$$p = \begin{pmatrix} 0.5000 & 0.3333 & 1.0000 \\ 0.6667 & 0.5000 & 1.0000 \\ 0.0000 & 0.0000 & 0.5000 \end{pmatrix}.$$

The optimal degrees of the membership for these $\mathcal{A}_t (t = 1, 2, 3)$ are obtained by using Eq. (9.3) as $r_1 = 0.3889$, $r_2 = 0.4444$, and $r_3 = 0.1667$. Since $r_2 > r_1 > r_3$ and thus, the ranking order of these numbers is $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$ which give us that $\mathcal{A}_{\sigma(1)} = \mathcal{A}_2, \mathcal{A}_{\sigma(2)} = \mathcal{A}_1$ and $\mathcal{A}_{\sigma_{(3)}} = \mathcal{A}_3$. Based on these permuted numbers, we get

$$\begin{aligned}
&\prod_{t=1}^{3} \left(1 + \tau_{\sigma(t)}/8\right)^{w_t} = (1 + 4/8)^{0.3} \cdot (1 + 3/8)^{0.4} \cdot (1 + 2/8)^{0.3} = 1.3482; \\
&\prod_{t=1}^{3} \left(1 - \tau_{\sigma(t)}/8\right)^{w_t} = (1 - 4/8)^{0.3} \cdot (1 - 3/8)^{0.4} \cdot (1 - 2/8)^{0.3} = 0.6288; \\
&\prod_{t=1}^{3} \left(\theta_{\sigma(t)}/8\right)^{w_t} = (2/8)^{0.3} \cdot (4/8)^{0.4} \cdot (3/8)^{0.25} = 0.2253; \\
&\prod_{t=1}^{3} \left(2 - \theta_{\sigma(t)}/8\right)^{w_t} = (2 - 2/8)^{0.3} \cdot (2 - 4/8)^{0.4} \cdot (2 - 3/8)^{0.3} = 1.7310.
\end{aligned}$$

Therefore, by Eq. (9.7) we get

$$\begin{aligned}
& \text{LIFEOWA}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \\
= & \left(s_{8\left(\frac{\prod_{t=1}^{3}\left(1+\tau_{\sigma(t)}/8\right)^{w_t} - \prod_{t=1}^{3}\left(1-\tau_{\sigma(t)}/8\right)^{w_t}}{\prod_{t=1}^{n}\left(1+\tau_{\sigma(t)}/8\right)^{w_t} + \prod_{t=1}^{3}\left(1-\tau_{\sigma(t)}/8\right)^{w_t}}\right)}, s_{8\left(\frac{2\prod_{t=1}^{3}\left(\theta_{\sigma(t)}/8\right)^{w_t}}{\prod_{t=1}^{n}\left(2-\theta_{\sigma(t)}/8\right)^{w_t} + \prod_{t=1}^{3}\left(\theta_{\sigma(t)}/8\right)^{w_t}}\right)} \right) \\
= & \left(s_{8\left(\frac{1.3716-0.6174}{1.3716+0.6174}\right)}, s_{8\left(\frac{2\times 0.3725}{1.6092+0.3725}\right)} \right) \\
= & \left(s_{3.0335}, s_{3.0079} \right)
\end{aligned}$$

As similar to those of the LIFEWA operator, the LIEOWA operator also satisfies the same properties, which are stated without proof, as follows.

Theorem 9.3.10. For a collection of LIFNs $\mathcal{A}_t (t = 1, 2, \ldots, n)$ and $w_t > 0$ be their associated weight vector such that $\sum\limits_{t=1}^{n} w_t = 1$, LIFEOWA have the following properties:

(P1) (Idempotency) If $\mathcal{A}_t = (s_\tau, s_\theta)$ for all t, then

$$\text{LIFEOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) = (s_\tau, s_\theta)$$

(P2) (Boundedness) Let $\gamma^- = (s_{\tau^-}, s_{\theta^-})$ and $\gamma^+ = (s_{\tau^+}, s_{\theta^+})$, where $\tau^- = \min\limits_t\{\tau_t\}$, $\theta^- = \max\limits_t\{\theta_t\}$, $\tau^+ = \max\limits_t\{\tau_t\}$, and $\theta^+ = \min\limits_t\{\theta_t\}$, then we have

$$\gamma^- \leq \text{LIFEOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) \leq \gamma^+.$$

(P3) (Monotonicity) If $\mathcal{A}_t = (s_{\tau_t}, s_{\theta_t})$ and $\mathcal{A}'_t = (s_{\tau'_t}, s_{\theta'_t})$ for $t = 1, 2, \ldots, n$ be two LIFNs such that $s_{\tau_t} \leq s_{\tau'_t}$ and $s_{\theta_t} \geq s_{\theta'_t}$ for all t, then we have

$$\text{LIFEOWA}(\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n) \leq \text{LIFEOWA}(\mathcal{A}'_1, \mathcal{A}'_2, \ldots \mathcal{A}'_n).$$

9.4 Proposed MAGDM approach under the LIFS environment

In this section, we present a group DM approach based on proposed operators and possibility degree method measure to rank the alternatives under the LIFSs environment.

The description of the MAGDM problem is given in Section 2.7 of Chapter 2 with l decision makers. Assume that $S_{[0,h]}$ be the LTS of the LVs and each decision maker evaluates the given alternatives in terms of LIFNs and their corresponding decision matrices are represented as $\mathcal{M}^{(q)} = \left(s_{\widetilde{\tau}^{(q)}_{kt}}, s_{\widetilde{\theta}^{(q)}_{kt}}\right)$ where $\widetilde{\tau}^{(q)}_{kt}, \widetilde{\theta}^{(q)}_{kt} \in [0, h]$ and $\widetilde{\tau}^{(q)}_{kt} + \widetilde{\theta}^{(q)}_{kt} \leq h$ where $k = 1, 2, \ldots, m; t = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, l$. Then, the proposed method is summarized in the following steps to solve the problem.

Step 1: Normalize the information, if required, by using Eq. (2.58) of Chapter 2 and get the normalized matrix $\mathcal{R}^{(q)}$.

Step 2: Utilize the matrices $\mathcal{R}^{(q)}(q = 1, 2, \ldots, l)$ and LIFEOWA operator to aggregate all the preference values of the 'l' decision makers into the collective decision matrix

$\mathcal{R} = (\mathcal{A}_{kt})$, where $A_{kt} = (s_{\tau_{kt}}, s_{\theta_{kt}})$ and are computed as

$$\begin{aligned}\mathcal{A}_{kt} &= \text{LIFEOWA}(\mathcal{A}_{kt}^{(1)}, \mathcal{A}_{kt}^{(2)}, \ldots, \mathcal{A}_{kt}^{(l)}) \\ &= \left(s_{h\left(\frac{\prod_{q=1}^{n} \left(1+\tau_{kt}^{\sigma(q)}/h\right)^{w^{(q)}} - \prod_{q=1}^{n} \left(1-\tau_{kt}^{\sigma(q)}/h\right)^{w^{(q)}}}{\prod_{q=1}^{n} \left(1+\tau_{kt}^{\sigma(q)}/h\right)^{w^{(q)}} + \prod_{q=1}^{n} \left(1-\tau_{kt}^{\sigma(q)}/h\right)^{w^{(q)}}} \right)}, s_{h\left(\frac{2\prod_{q=1}^{n} \left(\theta_{kt}^{\sigma(q)}/h\right)^{w^{(q)}}}{\prod_{q=1}^{n} \left(2-\theta_{kt}^{\sigma(q)}/h\right)^{w^{(q)}} + \prod_{q=1}^{n} \left(\theta_{kt}^{\sigma(q)}/h\right)^{w^{(q)}}} \right)} \right)\end{aligned} \tag{9.8}$$

where σ is the permutation of $(1, 2, \ldots, l)$ and $w^{(q)}$ is the weight vector of decision maker $\mathcal{DN}^{(q)}$.

Step 3: Aggregate the values of $\mathcal{A}_{kt}(t = 1, 2, \ldots, n)$ into the collective LIFN $\mathcal{A}_k(k = 1, 2, \ldots, m)$ by using LIFEWA operator where $\mathcal{A}_k = (s_{\tau_k}, s_{\theta_k})$ and are given as

$$\begin{aligned}\mathcal{A}_{k} &= \text{LIFEWA}(\mathcal{A}_{k1}, \mathcal{A}_{k2}, \ldots, \mathcal{A}_{kn}) \\ &= \left(s_{h\left(\frac{\prod_{t=1}^{n} (1+\tau_{kt}/h)^{\omega_t} - \prod_{t=1}^{n} (1-\tau_{kt}/h)^{\omega_t}}{\prod_{t=1}^{n} (1+\tau_{kt}/h)^{\omega_t} + \prod_{t=1}^{n} (1-\tau_{kt}/h)^{\omega_t}} \right)}, s_{h\left(\frac{2\prod_{t=1}^{n} (\theta_{kt}/h)^{\omega_t}}{\prod_{t=1}^{n} (2-\theta_{kt}/h)^{\omega_t} + \prod_{t=1}^{n} (\theta_{kt}/h)^{\omega_t}} \right)} \right)\end{aligned} \tag{9.9}$$

Step 4: Compute the possibility degree matrix $P = (p_{kj})_{m\times m}$ as

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix}$$

where $p_{kj} = p(\mathcal{A}_k \succeq \mathcal{A}_j)$ as defined as

(i) If either $\pi_k \neq 0$ or $\pi_j \neq 0$, then

$$p(\mathcal{A}_k \succeq \mathcal{A}_j) = \min\left(\max\left(\frac{h + \tau_k - 2\tau_j - \theta_j}{\pi_k + \pi_j}, 0 \right), 1 \right), \tag{9.10}$$

(ii) If $\pi_k = \pi_j = 0$, then

$$p(\mathcal{A}_k \succeq \mathcal{A}_j) = \begin{cases} 1 & ; \tau_k > \tau_j \\ 0 & ; \tau_k < \tau_j \\ 0.5 & ; \tau_k = \tau_j \end{cases} \tag{9.11}$$

Step 5: The optimal value of the membership for alternative $\mathcal{A}_k (k = 1, 2, \ldots, m)$ is computed by using

$$r_k = \frac{1}{m(m-1)} \left(\sum_{j=1}^{m} p_{kj} + \frac{m}{2} - 1 \right) \tag{9.12}$$

and hence rank the alternatives based on the descending values of it.

9.5 Illustrative example

In this section, the above mentioned MAGDM approach is illustrated by a problem of the global supplier(GS) adapted from Chen et al. [27] under the LIFSs environment.

9.5.1 Case study

A manufacturing company wants to find the best GS for buying its vital products which are utilized in assembling process. For it, they have considered the four different suppliers taken as an alternative denoted by $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_4$ and are evaluated under the five attributes namely $\mathcal{G}_1$ ("cost of product"), $\mathcal{G}_2$ ("quality of product"), $\mathcal{G}_3$ ("risk factor"), $\mathcal{G}_4$("profile of supplier") and $\mathcal{G}_5$ ("performance of supplier") with weight vector $\omega = (0.25, 0.2, 0.15, 0.18, 0.22)^T$. To select the best GS, company invite the four experts $\mathcal{DM}^{(1)}, \mathcal{DM}^{(2)}$, $\mathcal{DM}^{(3)}$ and $\mathcal{DM}^{(4)}$ to given their opinion on alternatives under each attribute in the term of LIFNs according to linguistic term set $S = \{s_0 =$ "extremely poor", $s_1 =$ "very poor", $s_2 =$ "poor", $s_3 =$ "slightly poor", $s_4 =$ "fair", $s_5 =$ "slightly good", $s_6 =$ "good", $s_7 =$ "very good", $s_8 =$ "extremely good"$\}$. Then, the following steps of the proposed approach are executed as.

Step 1: The rating values for the alternatives given by each decision maker under LIFSs environment are summarized in Table 9.1.

Step 2: Since $\mathcal{G}_1$ and $\mathcal{G}_3$ are the cost type attributes, the normalized decision matrices are summarized in Table 9.2.

Table 9.1: Rating information of each alternative by the decision makers under LIFS environment

	$\mathcal{DM}^{(1)}$					$\mathcal{DM}^{(2)}$				
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{A}_1$	(s_1, s_7)	(s_6, s_2)	(s_3, s_4)	(s_7, s_1)	(s_5, s_2)	(s_1, s_7)	(s_4, s_4)	(s_2, s_6)	(s_5, s_2)	(s_3, s_5)
$\mathcal{A}_2$	(s_2, s_6)	(s_5, s_2)	(s_1, s_6)	(s_6, s_2)	(s_7, s_1)	(s_1, s_7)	(s_5, s_1)	(s_1, s_6)	(s_5, s_2)	(s_4, s_3)
$\mathcal{A}_3$	(s_1, s_6)	(s_5, s_3)	(s_1, s_7)	(s_5, s_1)	(s_3, s_4)	(s_2, s_5)	(s_6, s_1)	(s_1, s_7)	(s_5, s_3)	(s_4, s_4)
$\mathcal{A}_4$	(s_2, s_5)	(s_7, s_1)	(s_3, s_4)	(s_6, s_1)	(s_4, s_4)	(s_2, s_6)	(s_4, s_3)	(s_2, s_5)	(s_7, s_1)	(s_5, s_3)
	$\mathcal{DM}^{(3)}$					$\mathcal{DM}^{(4)}$				
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{A}_1$	(s_1, s_6)	(s_5, s_2)	(s_4, s_3)	(s_7, s_1)	(s_5, s_2)	(s_3, s_5)	(s_4, s_4)	(s_1, s_7)	(s_5, s_1)	(s_4, s_2)
$\mathcal{A}_2$	(s_1, s_7)	(s_6, s_2)	(s_1, s_7)	(s_6, s_2)	(s_5, s_1)	(s_1, s_6)	(s_7, s_1)	(s_1, s_6)	(s_5, s_2)	(s_6, s_1)
$\mathcal{A}_3$	(s_3, s_5)	(s_5, s_2)	(s_1, s_6)	(s_4, s_3)	(s_3, s_1)	(s_2, s_5)	(s_3, s_4)	(s_2, s_6)	(s_3, s_3)	(s_5, s_2)
$\mathcal{A}_4$	(s_2, s_6)	(s_7, s_1)	(s_1, s_5)	(s_5, s_2)	(s_4, s_4)	(s_3, s_4)	(s_5, s_1)	(s_2, s_4)	(s_6, s_2)	(s_5, s_2)

Table 9.2: Normalized rating information of each alternative under LIFS environment

	$\mathcal{DM}^{(1)}$					$\mathcal{DM}^{(2)}$				
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{A}_1$	(s_7, s_1)	(s_6, s_2)	(s_4, s_3)	(s_7, s_1)	(s_5, s_2)	(s_7, s_1)	(s_4, s_4)	(s_6, s_2)	(s_5, s_2)	(s_3, s_5)
$\mathcal{A}_2$	(s_6, s_2)	(s_5, s_2)	(s_6, s_1)	(s_6, s_2)	(s_7, s_1)	(s_7, s_1)	(s_5, s_1)	(s_6, s_1)	(s_5, s_2)	(s_4, s_3)
$\mathcal{A}_3$	(s_6, s_1)	(s_5, s_3)	(s_7, s_1)	(s_5, s_1)	(s_3, s_4)	(s_5, s_2)	(s_6, s_1)	(s_7, s_1)	(s_5, s_3)	(s_4, s_4)
$\mathcal{A}_4$	(s_5, s_2)	(s_7, s_1)	(s_4, s_3)	(s_6, s_1)	(s_4, s_4)	(s_6, s_2)	(s_4, s_3)	(s_5, s_2)	(s_7, s_1)	(s_5, s_3)
	$\mathcal{DM}^{(3)}$					$\mathcal{DM}^{(4)}$				
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{A}_1$	(s_6, s_1)	(s_5, s_2)	(s_3, s_4)	(s_7, s_1)	(s_5, s_2)	(s_5, s_3)	(s_4, s_4)	(s_7, s_1)	(s_5, s_1)	(s_4, s_2)
$\mathcal{A}_2$	(s_7, s_1)	(s_6, s_2)	(s_7, s_1)	(s_6, s_2)	(s_5, s_1)	(s_6, s_1)	(s_7, s_1)	(s_6, s_1)	(s_5, s_2)	(s_6, s_1)
$\mathcal{A}_3$	(s_5, s_3)	(s_5, s_2)	(s_6, s_1)	(s_4, s_3)	(s_3, s_1)	(s_5, s_2)	(s_3, s_4)	(s_6, s_2)	(s_3, s_3)	(s_5, s_2)
$\mathcal{A}_4$	(s_6, s_2)	(s_7, s_1)	(s_5, s_1)	(s_5, s_2)	(s_4, s_4)	(s_4, s_3)	(s_5, s_1)	(s_4, s_2)	(s_6, s_2)	(s_5, s_2)

Step 3: By taking weight vector $w = (0.25, 0.3, 0.2, 0.25)^T$ of the decision makers and the LIFEOWA operator as given in Eq. (9.8), the aggregated values of the decision makers are summarized in Table 9.3.

Step 4: By attribute weights $\omega = (0.25, 0.2, 0.15, 0.18, 0.22)^T$ and LIFEWA operator given in Eq. (9.9), we aggregate all the values of Table 9.3 corresponding to each alternative. The resultant values are computed as $\mathcal{A}_1 = (s_{5.6443}, s_{1.8676})$, $\mathcal{A}_2 = (s_{6.1088}, s_{1.3452})$, $\mathcal{A}_3 = (s_{5.0478}, s_{2.0008})$, $\mathcal{A}_4 = (s_{5.4552}, s_{1.9451})$.

Step 5: Utilize Eq. (9.1) to construct the possibility degree matrix $P = (p_{kj})_{4\times4}$ as:

$$P = \begin{pmatrix} 0.5000 & 0.0788 & 1.0000 & 0.7252 \\ 0.9212 & 0.5000 & 1.0000 & 1.0000 \\ 0.0000 & 0.0000 & 0.5000 & 0.1240 \\ 0.2748 & 0.0000 & 0.8760 & 0.5000 \end{pmatrix}$$

Step 6: By Eq. (9.12), we obtain the optimal values of the alternatives $\mathcal{A}_k (k = 1, 2, 3, 4)$ as $r_1 = 0.2753$, $r_2 = 0.3684$, $r_3 = 0.1353$ and $r_4 = 0.2209$. Since $r_2 > r_1 > r_4 > r_3$, thus, the ranking order of the given alternatives is $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$. Hence, $\mathcal{A}_2$ is the best alternative for the desired project.

Table 9.3: Aggregate values of decision makers by LIFEOWA operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{A}_1$	$(s_{6.4678}, s_{1.3337})$	$(s_{4.8873}, s_{2.7677})$	$(s_{5.4814}, s_{2.2143})$	$(s_{6.3296}, s_{1.1528})$	$(s_{4.4058}, s_{2.4415})$
$\mathcal{A}_2$	$(s_{6.6260}, s_{1.1528})$	$(s_{5.9554}, s_{1.4226})$	$(s_{6.3103}, s_{1.0000})$	$(s_{5.5908}, s_{2.0000})$	$(s_{5.7857}, s_{1.3337})$
$\mathcal{A}_3$	$(s_{5.2814}, s_{1.9198})$	$(s_{4.8773}, s_{2.3074})$	$(s_{6.6260}, s_{1.1528})$	$(s_{4.3624}, s_{2.1950})$	$(s_{3.8554}, s_{2.4448})$
$\mathcal{A}_4$	$(s_{5.3966}, s_{2.2194})$	$(s_{6.1870}, s_{1.3337})$	$(s_{4.5757}, s_{1.7790})$	$(s_{6.1204}, s_{1.4728})$	$(s_{4.5757}, s_{3.0530})$

9.5.2 Validity test

Some DM methods deal with irregularities like they may give different ranking results with exactly same numerical data in DM problem or ranking results may change by changing some of the uncertain data in a decision problem. So, DM method does not give reliable results all the time, therefore, there is need to explore the validity and reliability of the

DM methods. For this, Wang and Triantaphyllou [175] proposed the following testing criteria:

Test criterion 1: "An effective MAGDM method does not change the index of the best alternative by replacing a non-optimal alternative with a worse alternative without shifting the corresponding importance of every decision attribute".

Test criterion 2: "To an effective MAGDM method must be satisfied transitive property".

Test criterion 3: "If we decomposed a MAGDM problem into the sub DM problems and same MAGDM method is utilized on subproblems to rank alternatives, the collective ranking of alternatives must be identical to the ranking of un-decomposed DM problem".

The applicability of test criteria to proposed DM method is discussed below:

Validity test by criterion 1

Under this test, if we replace the rating values of any non-optimal alternative with its worse rating values, then the ranking order of the best alternative remains unchanged. In order to validate it, we arbitrary choose the non-optimal alternative $\mathcal{A}_3$ and its rating values corresponding to each decision maker is replacing with its any worse rating values which are denoted by $\mathcal{A}_3'$. The rating value of $\mathcal{A}_3'$ is summarized in Table 9.4.

Table 9.4: Rating value of worse alternative $\mathcal{A}_3'$ for each decision maker

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{DM}^{(1)}$	(s_5, s_2)	(s_3, s_4)	(s_5, s_2)	(s_3, s_3)	(s_2, s_5)
$\mathcal{DM}^{(2)}$	(s_3, s_4)	(s_5, s_2)	(s_5, s_2)	(s_3, s_4)	(s_2, s_5)
$\mathcal{DM}^{(3)}$	(s_3, s_4)	(s_4, s_3)	(s_4, s_3)	(s_2, s_5)	(s_1, s_3)
$\mathcal{DM}^{(4)}$	(s_3, s_4)	(s_2, s_5)	(s_4, s_4)	(s_2, s_4)	(s_3, s_4)

Now, by applying the steps of the proposed approach to this transferred problem and by using Eq. (9.12) we can obtain the optimal values of the membership for the alternatives are $r_1 = 0.2753$, $r_2 = 0.3684$, $r_3 = 0.1250$ and $r_4 = 0.2312$ respectively. Therefore, the ranking order of the given alternative is $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3'$. Hence, the best alternative $\mathcal{A}_2$ is similar as obtained with the original DM problem and it validate the *test 1.*

Validity test by criteria 2 and 3

Under this test, we decomposed the original DM problem into three sub DM problems which contain the alternatives $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4\}$, $\{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_4\}$ and $\{\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$. Now, by applying the proposed DM approach to each subproblem individually then we get the ranking order of the membership corresponding to each problem as $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4$, $\mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$ and $\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$ respectively. Therefore, after combining together, the ranking order of the alternatives of these smaller problems is obtained as $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$ which is similar to the ranking order of original problem. Thus, we conclude that the proposed approach show the transitive property and hence proposed MAGDM approach validate the *test criteria 2 and 3*.

9.5.3 Comparative study

In order to manifest the efficiency of the proposed MAGDM approach, we compare our proposed approach result with the existing DM approach based on the LIFOWA operator as proposed by Zhang [236] on to the considered data under the LIFS environment. Under it, the following steps of their approach are executed, on the normalized data set represent in Step 2 of the proposed approach, as follows:

Step 1: Aggregate the different rating values of each decision maker corresponding to weight vector $w = (0.25, 0.3, 0.2, 0.25)^T$ by using LIFOWA operator as mentioned in Eq. (2.51) of Chapter 2. The corresponding aggregated values are summarized in Table 9.5.

Table 9.5: Aggregate decision maker preferences matrix by LIFOWA operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
$\mathcal{A}_1$	$(s_{6.4882}, s_{1.3161})$	$(s_{4.9145}, s_{2.7321})$	$(s_{5.5708}, s_{2.1689})$	$(s_{6.3605}, s_{1.1892})$	$(s_{4.3895}, s_{2.5149})$
$\mathcal{A}_2$	$(s_{6.6340}, s_{1.1892})$	$(s_{5.9816}, s_{1.4641})$	$(s_{6.3182}, s_{1.0000})$	$(s_{5.5997}, s_{2.0000})$	$(s_{5.8311}, s_{1.3161})$
$\mathcal{A}_3$	$(s_{5.2892}, s_{1.8612})$	$(s_{4.9199}, s_{2.1689})$	$(s_{6.6340}, s_{1.1892})$	$(s_{4.3895}, s_{2.2795})$	$(s_{3.7915}, s_{2.2191})$
$\mathcal{A}_4$	$(s_{5.4207}, s_{2.2134})$	$(s_{6.2383}, s_{1.3161})$	$(s_{4.5854}, s_{1.8612})$	$(s_{6.1388}, s_{1.3660})$	$(s_{4.7763}, s_{2.9130})$

Step 2: Utilize the LIFWA operator (defined in Eq. (2.50) of Chapter 2) to get the collective values of each alternative $\mathcal{A}_k$ $(k = 1, 2, 3, 4)$. The obtained values of each

alternative are $\mathcal{A}_1 = (s_{5.6993}, s_{1.8587})$, $\mathcal{A}_2 = (s_{6.1329}, s_{1.3563})$, $\mathcal{A}_3 = (s_{5.0894}, s_{1.9345})$ and $\mathcal{A}_4 = (s_{5.5313}, s_{1.8929})$.

Step 3: By using Eq. 2.8 mentioned in Chapter 2 to compute the score values of the obtained $\mathcal{A}_k$ and we get $I(\mathcal{S}(\mathcal{A}_1)) = 5.9203$, $I(\mathcal{S}(\mathcal{A}_2)) = 6.3883$, $I(\mathcal{S}(\mathcal{A}_3)) = 5.5775$ and $I(\mathcal{S}(\mathcal{A}_4)) = 5.8192$, where $I(\mathcal{S}(\cdot))$ denotes the subscript values of $\mathcal{S}(\cdot)$. Since $I(\mathcal{S}(\mathcal{A}_2)) > I(\mathcal{S}(\mathcal{A}_1)) > I(\mathcal{S}(\mathcal{A}_4)) > I(\mathcal{S}(\mathcal{A}_3))$ therefore we can obtain that $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$. Hence, $\mathcal{A}_2$ is the best alternative and it coincides with the proposed approach result.

Apart from this approach, under the LIFSs environment if we utilize some others existing work of the researchers as given in [27, 107–110, 185] to the considered problem, then the overall rating values of the membership for the given alternative are summarized in Table 9.6. From this table, we conclude from the ranking order that the best alternative for the required task remains same i.e., $\mathcal{A}_2$ and hence validates the efficiency of the proposed approach.

Table 9.6: Comparative study with some existing approaches

	Overall value of alternative				Ranking order
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	
Wei et al. [185]	-7.0150	-6.7997	-7.0895	-7.0394	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Liu et al. [108]	-6.6800	-6.3730	-6.6898	-6.5573	$\mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
Chen et al. [27]	3.6892	4.7491	3.0774	3.4175	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Liu and Qin [109]	-7.0266	-6.8203	-7.0861	-7.0429	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Liu and Wang [110]	3.2376	4.1290	2.2313	2.8980	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Liu and Liu [107]	4.0703	5.1682	3.6197	4.0493	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$

9.6 Conclusion

LIFS is the generalization of IFS in which membership and nonmembership degree represented by the linguistic terms for better dealing with fuzzy information under the qualitative aspect. Under these aspects, in this chapter, we enrich the theory by defining the

AOs to aggregate the LIFNs by using Einstein AT and AC operations. Based on it, we develop some new LIF Einstein weighted averaging and ordered weighted averaging AOs to aggregate the different preference values of the LIFNs. The various properties such as idempotency, boundedness, monotonicity are discussed in details. Further, to rank the different LIFNs, a new possibility degree measure is presented which overcome the drawbacks of the certain existing ranking methods. Later, we utilize the proposed AOs and the possibility degree measures to define a novel MAGDM approach to solve the group DM problems. The developed method is explained with the help of numerical examples and illustrate the feasibility of the proposed approach. A comparative study with some existing operators shows that the proposed operators and their corresponding techniques provide a more stable and practical nature to the decision-maker during the aggregation process.

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