

A STUDY  
OF  
GÖLLNITZ - GORDON IDENTITIES

*Thesis submitted in partial fulfillment of the requirement for  
The award of the degree of*

Master of Science  
in  
Mathematics and Computing

*Submitted by*

Richa Mittal  
301003020

UNDER  
THE GUIDANCE OF

Dr. Meenakshi Rana



School of Mathematics and Computer Applications  
Thapar University, Patiala.  
July, 2012

## CERTIFICATE

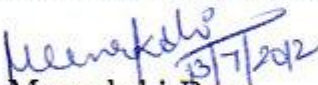
I hereby certify that the work which is being presented in the thesis entitled "A study of Göllnitz-Gordon identities" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Meenakshi Rana.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

  
Richa Mittal

Reg. No. 301003020

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

  
Dr. Meenakshi Rana

Supervisor

Assistant Professor, SMCA

Thapar University, Patiala

Countersigned by:

  
Dr. S.S. Bhatia

Professor and Head

SMCA

Thapar University, Patiala

  
Dr. S.K. Mohapatra

Dean of Academic Affairs

Thapar University

Patiala

## ACKNOWLEDGEMENT

It gives me immense pleasure to express my deep and sincere gratitude to my academic supervisor, Dr. Meenakshi Rana, Assistant Professor, School of Mathematics and Computer Applications, Thapar University, Patiala, for their patient guidance and support throughout this work. Her wide knowledge and her logical way of thinking have been great value for me. Without their encouragement and understanding it would have been impossible for me to finish this work. I was truly very fortunate to have the opportunity to work under as a student.

I wish to express my warm and sincere thanks to Dr. S.S Bhatia, Head of Department, School of Mathematics and Computer Applications, Thapar University, Patiala, for providing help and necessary facilities in the department and directly or indirectly encouraged me to work harder during the whole work.

Lastly, and more importantly, My special gratitude is due to my parents, my brother, my sister and their families for their lovely support. They have always wanted the best for me and I admire my parent's determination and sacrifice for my studies.

July 13, 2012

  
Richa Mittal

## ABSTRACT

In this thesis, we studied Combinatorial interpretation of generalized  $q$  - series given by Agarwal in 1986 (A.K. Agarwal, On a Generalized Partition Theorem, J. Indian Math. Soc. Vol. 50 (1986), pp. 185-190) using ordinary partitions. Recently Agarwal and Rana in 2009 (A.K. Agarwal and M. Rana, New Combinatorial Version of Göllnitz - Gordon identities, Utilitas Mathematica., Vol. 79(2009), pp. 145-156 ) extended the interpretation given by agarwal in 1986 using  $n$  - colour partitions which leads to an infinite family of 2 - way combinatorial identities. In some particular cases these 2 - way combinatorial identities are extended to a 3 - way combinatorial identities which gives combinatorial interpretations of Göllnitz-Gordon identities using ordinary partition discussed in Chapter 2 and using  $n$  - colour partition discussed in Chapter 3. Chapter 1 is devoted to elementary study of partition Theory.

# CONTENTS

Certificate	i
Acknowledgement	ii
Abstract	iii
I Introduction	
1.1 Partition of numbers	1
1.2 Graphical representation of partition	3
1.3 Generating function	4
1.4 Some basic theorems on partition	6
1.5 Proof of theorems for Section 1.4	7
II Combinatorial interpretation of Gollnitz - Gordon identities using ordinary partition	
2.1 Gollnitz - Gordon identities	12
2.2 Main Theorem	14
2.3 Proof of Main Theorem given in Section 2.2	14
2.4 Particular cases	18
III Combinatorial interpretation of Gollnitz - Gordon identities using coloured partition	
3.1 n - Colour Partition	21
3.2 Main Theorem	23
3.3 Proof of Main Theorem given in Section 3.2	24
3.4 Particular cases and new combinatorial identities	27
3.5 Conclusion	32
Bibliography	

# CHAPTER 1

## Introduction

Partition Theory is a rich subject, with many classical and important result which influence the development of Enumerative Combinatorics in twentieth century. It is also a collection of various terminology, notations and techniques, with a number of results rediscovered on many occasions and some fundamental bijection remain in obscurity. We find in literature, most of partition identity were first proved analytically and only much later combinatorially.

The theory of partition as a subject started with Euler's celebrated treatise[6], where Chapter 16 introduced integer partitions. Back in 1748, Euler proved a variety of partition identities, such as, "Partition of  $n$  into odd parts vs Partition of  $n$  into distinct parts". Then many mathematician proved many partition identities, included those bearing the names of Guass, Cauchy, Jacobi, Weierstrass, Sylvester, Heine, Lebesgue, MacMahon, Ramanujan. Here in this thesis we study combinatorial interpretation of famous Göllnitz Gordon identities using two different combinatorial object, that is, ordinary partition and colour partition.

### 1.1 Partition of numbers

**Definition 1.1.1** [2] A partition of a positive integer  $n$  is a finite non-increasing sequence of positive integers  $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n$  such that there sum is  $n$ . The  $a_i$  are called the parts of the partition. We denote by  $p(n)$  the numbers of partitions of  $n$ .

**Example 1.** The partitions of 7 are

$$\begin{aligned} &7, \\ &6 + 1, \\ &5 + 2, 5 + 1 + 1, \\ &4 + 3, 4 + 2 + 1, 4 + 1 + 1 + 1, \\ &3 + 3 + 1, 3 + 2 + 2, 3 + 2 + 1 + 1, 3 + 1 + 1 + 1 + 1, \\ &2 + 2 + 2 + 1, 2 + 2 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1 + 1, \\ &1 + 1 + 1 + 1 + 1 + 1 + 1 \end{aligned}$$

so,  $p(7) = 15$ .

**Remark 1.**  $p(n) = 0$  for  $n < 0$  because we cannot write a negative number into sum of positive integers.

Also  $p(0) = 1$ .

**Remark 2.** It is conventional to abbreviate repeated parts by the use of exponents.

**Example 2.**

$$\begin{aligned} p(1) = 1: & \quad 1 = (1); \\ p(2) = 2: & \quad 2 = (2), 1 + 1 = (1^2); \\ p(3) = 3: & \quad 3 = (3), 2 + 1 = (12), 1 + 1 + 1 = (1^3); \\ p(4) = 5: & \quad 4 = (4), 3 + 1 = (13), 2 + 2 = (2^2), \\ & \quad 2 + 1 + 1 = (1^22), 1 + 1 + 1 + 1 = (1^4). \end{aligned}$$

**Remark 3.** In the definition of partition the order does not matter.  $4 + 3$  and  $3 + 4$  are the same partition of 7. Thus a partition is an unordered collection of parts. An ordered collection is called a Composition. Thus  $4 + 3$  and  $3 + 4$  are two different compositions of 7.

## 1.2 Graphical representation of partition

**Definition 1.2.1** [2] The Ferrer's graph of a partition  $t_1, t_2, \dots, t_i$  of  $n$  is a set of rows of equi-spaced dots aligned on the left where the  $j$ th row has  $t_j$  dots.

**Example 3.** The Ferrer's graph of the partition  $5+3+2+1$  of 11 as following,

```
•   •   •   •   •
•   •   •
•   •
•
```

read this graph horizontally, we see that first row has 5 dots, second row has 3, third row has 2, fourth row has 1.

If the graph above is read vertically by columns then this represents the partition

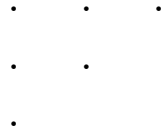
```
•   •   •   •   •
•   •   •
•   •
•
```

$4+3+2+1+1$  of 11. This new partition is called the *conjugate* of given partition.

We denote the conjugate partition of partition  $\Pi$  by  $\Pi^c$ .

**Definition 1.2.2** A partition is said to be self conjugate if it is identical with its conjugate.

**Example 4.** Consider the partition  $\Pi = 3 + 2 + 1$  of 6 then the Ferrer's graph is



now if we read this graph vertically we get the same partition  $\Pi^c = 3 + 2 + 1$  so this partition is self conjugate partition. In others words here  $\Pi = \Pi^c$ . So it is self conjugate partition.

### 1.3 Generating function

**Definition 1.3.1** The generating function  $f(q)$  for the sequences  $a_0, a_1, a_2, a_3, \dots$  is

$$f(q) = \sum_{n=0}^{\infty} a_n q^n$$

**Definition 1.3.2** The generating function for  $p(n)$  is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}$$

where  $|q| < 1$  and  $(q; q)_{\infty}$  is  $q$  - rising factorial defined by

$$(a; q)_n = \prod_{i=0}^{n-1} \frac{(1 - aq^i)}{(1 - aq^{i+n})} = \prod_{i=0}^{n-1} (1 - aq^i)$$

if  $n$  is a positive integer, then

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \dots (1 - aq^{n-1})$$

$$(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \dots$$

and when  $a = q$

$$(q; q)_n = \prod_{i=1}^n (1 - q^i).$$

**Definition 1.3.3** Let  $p$  denote the set of all partition.

**Definition 1.3.4** Let  $p(S, n)$  denote the number of partition of  $n$  that belong to a subset  $S$  of the set  $p$  of all partition.

**Remark 4.** Let  $D(n)$  denote the number of partitions of  $n$  into distinct parts, then generating function of this is

$$\sum_{n=0}^{\infty} D(n)q^n = \prod_{n=1}^{\infty} (1 + q^n) = (-q; q)_\infty.$$

**Example 5.** We might consider  $D$  the set of all partitions with distinct parts

$$p(D, 7) = 5.$$

The relevant partitions are 7, 6+1, 5+2, 4+3, 4+2+1.

**Remark 5.** Let  $O(n)$  denote the number of partitions of  $n$  into odd parts, then generating function of this is

$$\sum_{n=0}^{\infty} O(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})} = \frac{1}{(q; q^2)_{\infty}}.$$

**Example 6.** We might consider  $O$  the set of all partitions with odd parts

$$p(O, 7) = 5.$$

The relevant partitions are 7, 5+1+1, 3+3+1, 3+1+1+1, 1+1+1+1+1+1+1.

**Remark 6.** Let  $E(n)$  denote the number of partitions of  $n$  into even parts, then generating function of this is

$$\sum_{n=0}^{\infty} E(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n})} = \frac{1}{(q^2; q^2)_{\infty}}.$$

**Example 7.** We might consider  $E$  the set of all partitions with even parts

$$p(E, 7) = 0.$$

## 1.4 Some basic theorems on partition

**Theorem 1.4.1** The number of partition of  $n$  into distinct odd parts is equal to the number of self conjugate partitions of  $n$ .

**Theorem 1.4.2** The number of partition of  $n$  into distinct parts is equal to the number of partitions of  $n$  into odd parts.

**Theorem 1.4.3** The number of partitions of  $n$  into  $r$  parts is equal to the number of partitions of  $n$  in which largest part is  $r$ .

**Theorem 1.4.4** The number of partitions of  $n$  in which each part appears two, three or five times equals the number of partitions of  $n$  into parts congruent to 2, 3, 4, 6, 9 or 10 modulo 12.

## 1.5 Proof of theorems for Section 1.4

### Proof of Theorem 1.4.1

Consider a self conjugate partition of  $n$  then make the Ferrer's graph corresponding to this partition and straight each bent along the south east direction and count the nodes in each row we will get the partition of  $n$  into distinct odd parts.

Conversly, consider a partition of  $n$  into distinct odd parts then each part can be written in the form of bent that is  $2a + 1$  (at right angle) place them one after another and we see that one durfee square and 2 equal tails are obtained.

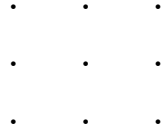
Thus, by reading partition from this Ferrer's graph we get a self conjugate partition of  $n$ .

So, we get one-one correspondence between number of self conjugate partition of  $n$  into distinct odd parts.

Hence, we get the number of partitions of  $n$  into distinct odd parts is equal to the number of self conjugates partitions of  $n$ .



**Example 8.** Consider the partition of 9 into distinct and odd parts  $5+3+1$ . We draw its Ferrer's graph and bent along the south east direction



we see that one Durfee square and reading this Ferrer's graph we get a self conjugate partition of 9.

Hence, the number of partition of  $n$  into distinct odd parts is equal to the number of self conjugates partition of  $n$ .

**Proof of Theorem 1.4.2**

Given a partition of  $n$  into distinct parts replace each even parts by its two halves and repeat this process till even parts are left. Finally arrange the parts in non-increasing order. This will be a partition of  $n$  into odd parts.

Conversely, for a partition into odd parts go on adding two equal parts until there are no repetitions. Finally arrange the parts in non-increasing order. This gives rise to a partition into distinct parts.

**Example 9.** Consider the partition of 26 into distinct parts

$$\begin{aligned}
 26 &= 10 + 8 + 5 + 3 \\
 &\rightarrow 5 + 5 + 4 + 4 + 5 + 3 \\
 &\rightarrow 5 + 5 + 2 + 2 + 2 + 2 + 5 + 3 \\
 &\rightarrow 5 + 5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 5 + 3 \\
 &\rightarrow 5 + 5 + 5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
 \end{aligned}$$

Conversely

$$26 = 5 + 5 + 5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

$$\rightarrow 10 + 5 + 3 + 2 + 2 + 2 + 2$$

$$\rightarrow 10 + 5 + 3 + 4 + 4$$

$$\rightarrow 10 + 5 + 3 + 8$$

$$\rightarrow 10 + 8 + 5 + 3$$

### Proof of Theorem 1.4.3

Consider a partition  $\Pi$  of  $n$  into  $r$  parts and draw its Ferrer's graph. Now read this Ferrer's graph vertically we get conjugate partition of  $\Pi$  which is a partition of  $n$  in which largest part is  $r$ .

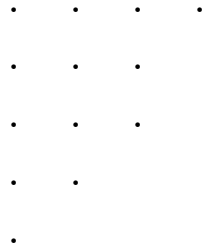
• • • • • • • •  
• • • • • •  
• • • • •

and so on

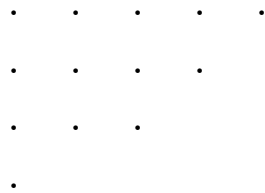
• • • •  
• • •

Consider a partition  $\Pi$  in which largest part is  $r$ . Draw its Ferrer's graph now read the graph vertically we get conjugate partition of  $\Pi$  which is a partition of  $n$  into  $r$  parts. So, we get one-one correspondence between the partitions of  $n$  into  $r$  parts and the partition of  $n$  in which largest part is  $r$ . Hence, we get the number of partitions of  $n$  into  $r$  parts is equal to the number of partitions of  $n$  in which largest part is  $r$ .

**Example 10.** Consider the partition  $\Pi = 4+3+3+2+1$  of 13 then the Ferrer's graph is



read vertically, we get conjugate partition of  $\Pi$



and the new partition becomes  $5+4+3+1$  in this partition we see the number of parts is equal to the largest part in the original partition.

#### Proof of Theorem 1.4.4

The Generating function of number of partitions  $n$  which each part appears two, three or five times is

$$\begin{aligned}
 \prod_{n=0}^{\infty} (1 + q^{2n} + q^{3n} + q^{5n}) &= \prod_{n=0}^{\infty} (1 + q^{2n} + q^{3n}(1 + q^{2n})) \\
 &= \prod_{n=0}^{\infty} (1 + q^{2n})(1 + q^{3n}) \\
 &= \prod_{n=0}^{\infty} \frac{(1 - q^{4n})(1 - q^{6n})}{(1 - q^{2n})(1 - q^{3n})} \\
 &= \prod_{n=0}^{\infty} \frac{(1 - q^{4n})(1 - q^{6n})}{(1 - q^{4n})(1 - q^{4n+2})(1 - q^{6n})(1 - q^{6n+3})}
 \end{aligned}$$

$$\begin{aligned}
&= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{4n+2})(1 - q^{6n+3})} \\
&= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{12n+2})(1 - q^{12n+3})(1 - q^{12n+6})(1 - q^{12n+9})(1 - q^{12n+10})}
\end{aligned}$$

is the Generating function of number of partitions of  $n$  into parts congruent to 2, 3, 6, 9 or 10 modulo 12.

## CHAPTER 2

### Combinatorial interpretation of Göllnitz - Gordon identities using ordinary partition.

#### 2.1 Göllnitz - Gordon identities

The following two Theorems,

**Theorem 2.1.1** The number of partitions of  $n$  into parts differing by at least 2 among which no two consecutive even numbers appear is equal to the number of partitions of  $n$  into parts which are congruent to 1, 4 or 7 (mod 8).

**Theorem 2.1.2** The number of partitions of  $n$  into parts differing by at least 2 among which no two consecutive even numbers appear and with each part being at least equal to 3 is equal to the number of partitions of  $n$  into parts which are congruent to 3, 4 or 5(mod 8).

give combinatorial interpretations of Göllnitz - Gordon identities which are due to Göllnitz and were included in his (1960) unpublished honors baccalaureate thesis. However, essentially no one knew about the result until Gordon [9] (1965) independently rediscovered them and are commonly referred as Göllnitz - Gordon identities. Another Theorem,

**Theorem 2.1.3** The number of partitions of  $n$  into parts differing by at least 2 among which no two consecutive odd numbers appear and with each part being at least equal to 2 is equal to the number of partitions of  $n$  into parts which are congruent to 2, 3 or 7(mod 8).

is due to Göllnitz [8]. The analytic counterparts of the Göllnitz - Gordon partition identities and the Göllnitz identity are the  $q$  - series identities,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_n} &= \frac{1}{(q; q^8)_{\infty} (q^4; q^8)_{\infty} (q^7; q^8)_{\infty}} & (2.1.1) \\ &= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 2q^6 + 3q^7 + 4q^8 + 5q^9 + 5q^{10} + \dots \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_n} &= \frac{1}{(q^3; q^8)_{\infty} (q^4; q^8)_{\infty} (q^5; q^8)_{\infty}} & (2.1.2) \\ &= 1 + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + \dots \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+1)}}{(q^2; q^2)_n} &= \frac{1}{(q; q^8)_{\infty} (q^5; q^8)_{\infty} (q^6; q^8)_{\infty}} & (2.1.3) \\ &= 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 2q^8 + 3q^9 + 4q^{10} + \dots \end{aligned}$$

These analytic identities were published by Slater in 1952, Equation (2.1.1) is number 36 and Equation (2.1.2) is number 34 in Slater paper [13]. The Equation (2.1.3) is Corollary 2.7., page 21, in [4]. However, it has recently been discovered by Sills [11] that two analytic identities equivalent to the analytic Göllnitz - Gordon

identities were recorded by Ramanujan in his lost notebook, and thus that Ramanujan knew these identities more than 30 years before Slater rediscovered them (Andrews and Berndt (2008) page 37 [5]).

## 2.2 Main Theorem

In [1] Agarwal proved the following theorem:

**Theorem 2.2.1** Given positive integer  $k$ , let  $A_k(n)$  denote the number of partitions of  $n$  in which each part  $\geq k$ , minimal difference  $\geq 2$  between the parts, consecutive odd integers are not allowed if  $k$  is even and consecutive even integers are not allowed if  $k$  is odd. Then

$$\sum_{n=0}^{\infty} A_k(n)q^n = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n(n+k-1)}. \quad (2.2.1)$$

## 2.3 Proof of Main Theorem given in Section 2.2

Let  $A_k(m, n)$  denote the number of the partitions of the type enumerated by  $A_k(n)$  with added restriction that there be exactly  $m$  parts. We shall first prove the identity

$$A_k(m, n) = A_k(m-1, n-k-2m+2) + A_k(m-1, n-k-4m+3) + A_k(m, n-2m). \quad (2.3.1)$$

We split the partitions enumerated by  $A_k(m, n)$  into three classes:

- (i) those that have least part equal to  $k$
- (ii) those that have least part equal to  $k+1$
- (iii) those that have least part greater than or equal to  $k+2$

we now transform the partitions in class (i) by deleting the least part  $k$  and then subtracting 2 from all the remaining parts. This produces a partition of  $n-k-$

$2(m - 1)$  into exactly  $m - 1$  parts, each of which  $\geq k$ ; furthermore, since this transformation does not disturb the inequalities between the parts, we see that the transformed partition is of the type enumerated by

$$A_k(m - 1, n - k - 2m + 2)$$

next we transform the partitions in class (ii) by deleting the least part  $k + 1$  and then subtracting 4 from all the remaining parts. This produces a partition of

$$n - (k + 1) - 4(m - 1) = n - k - 4m + 3$$

into  $m - 1$  parts, each of which  $\geq k$  (since originally the second smallest part was  $\geq k + 4$ . Note that originally  $k + 2$  could not be the second smallest part because of the minimal difference between the parts  $\geq 2$ . Also  $k + 3$  could not be the second smallest part as  $k + 1$  and  $k + 3$  are consecutive odd if  $k$  is even and consecutive even if  $k$  is odd.) furthermore, since the inequalities between the parts are not disturbed, we see that the transformed partition is of the type enumerated by

$$A_k(m - 1, n - k - 4m + 3)$$

finally, we transform the partitions in class (iii) by subtracting 2 from each part. This produces a partition of  $n - 2m$  into  $m$  parts, each of which  $\geq k$ , as in the first two cases, here too, the inequalities between the parts are not disturbed, we see that the transformed partition is of the type enumerated by  $A_k(m, n - 2m)$ . The above transformation establish a bijection between the partitions enumerated by  $A_k(m, n)$  and those enumerated by

$$A_k(m, n) = A_k(m - 1, n - k - 2m + 2) + A_k(m - 1, n - k - 4m + 3) + A_k(m, n - 2m)$$

let

$$f_k(z, q) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_k(m, n) z^m q^n$$

then identity (2.3.1) implies that

$$\begin{aligned} f_k(z, q) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (A_k(m-1, n-k-2m+2) + A_k(m-1, n-k-4m+3) + A_k(m, n-2m)) z^m q^n \\ &= zq^k \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_k(m-1, n-k-2m+2) (zq^2)^{m-1} q^{n-k-2m+2} \\ &\quad + zq^{k+1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_k(m-1, n-k-4m+3) (zq^4)^{m-1} q^{n-k-4m+3} \\ &\quad + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_k(m, n-2m) (zq^2)^m q^{n-2m} \\ &= z_k q^k f_k(zq^2; q) + zq^{k+1} f_k(zq^4; q) + f_k(zq^2; q) \end{aligned} \tag{2.3.2}$$

setting

$$f_k(z, q) = \sum_{n=0}^{\infty} \alpha(n, k; q) z^n$$

and then comparing the coefficients of  $z^n$  on each side of (2.3.2), we see that

$$\alpha(n, k; q) = q^{2n-2+k} \alpha(n-1, k; q) + q^{4n-3+k} \alpha(n-1, k; q) + q^{2n} \alpha(n, k; q)$$

therefore

$$\alpha(n, k; q) = \frac{(1 + q^{2n-1})q^{2n-2+k}}{1 - q^{2n}} \alpha(n-1, k; q) \tag{2.3.3}$$

iterating (2.3.3)  $n$  – times and observing that

$$\alpha(0, k; q) = 1$$

we get that

$$\alpha(n, k; q) = \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n(n+k-1)}$$

therefore

$$f_k(z, q) = \sum_{n=0}^{\infty} \alpha(n, k; q) z^n = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n(n+k-1)} z^n$$

now

$$\sum_{n=0}^{\infty} A_k(n) q^n = \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} A_k(m, n) \right] q^n$$

$$= f_k(1, q)$$

$$= \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+k-1)}}{(q^2; q^2)_n}$$

This completes the proof of the Theorem 2.2.1.

## 2.4 Particular cases

If  $k = 1$ , then Theorem 2.2.1 in view of Identity 2.1.1 becomes

**Corollary 1.**

$$A_1(v) = D_1(v) \tag{2.4.1}$$

where  $A_1(v)$  is the number of partitions of  $n$  into parts differing by at least 2 among which no two consecutive even numbers appear.

where  $D_1(v)$  is the number of partitions of  $n$  into parts which are congruent to 1, 4 or 7(mod 8).

**Example 11.** The following table gives the relevant partition of  $A_1(v)$  and  $D_1(v)$  when  $v = 1$  to 10.

$v$	$A_1(v)$	Relevant partition	$D_1(v)$	Relevant partition
1	1	1	1	1
2	1	2	1	$1^2$
3	1	3	1	$1^3$
4	2	4, 3+1	2	4, $1^4$
5	2	5, 4+1	2	$14^1$ , $1^5$
6	2	6, 5+1	2	$1^24$ , $1^6$
7	3	7, 6+1, 5+2	3	7, $1^34$ , $1^7$
8	4	8, 7+1, 6+2, 5+3	4	$17^1$ , $4^2$ , $1^44$ , $1^8$
9	5	9, 8+1, 7+2, 6+3, 5+3+1	5	9, $1^27$ , $14^2$ , $1^54$ , $1^9$
10	5	10, 9+1, 8+2, 7+3, 6+3+1	5	$1^37$ , $1^24^2$ , $1^64$ , $1^46$ , $1^{10}$

If  $k = 2$ , then Theorem 2.2.1 in view of Identity 2.1.3 becomes

**Corollary 2.**

$$A_2(v) = D_2(v) \tag{2.4.2}$$

where  $A_2(v)$  is the number of partitions of  $n$  into parts differing by at least 2 among which no two consecutive odd numbers appear and with each part being at least equal to 2.

where  $D_2(v)$  is the number of partitions of  $n$  into parts which are congruent to 2, 3 or  $7 \pmod{8}$ .

**Example 12.** The following table gives the relevant partition of  $A_2(v)$  and  $D_2(v)$  when  $v = 1$  to 10.

$v$	$A_2(v)$	Relevant partition	$D_2(v)$	Relevant partition
1	0	-	0	-
2	1	2	1	2
3	1	3	1	3
4	1	4	1	$2^2$
5	1	5	1	23
6	2	6, 4+2	2	$3^2, 2^3$
7	2	7, 5+2	2	7, $2^23$
8	2	8, 6+2	2	$23^2, 2^4$
9	3	9, 7+2, 6+3	3	$27^1, 3^3, 2^33$
10	4	10, 8+2, 7+3, 6+4	4	10, $37^1, 2^5, 2^23^2$

If  $k = 3$ , then Theorem 2.2.1 in view of Identity 2.1.2 becomes

**Corollary 3.**

$$A_3(v) = D_3(v) \tag{2.4.3}$$

where  $A_3(v)$  is the number of partitions of  $n$  into parts differing by at least 2 among which no two consecutive even numbers appear and with each part being at least equal to 3.

where  $D_3(v)$  is the number of partitions of  $n$  into parts which are congruent to 3, 4 or 5(mod 8).

**Example 13.** The following table gives the relevant partition of  $A_3(v)$  and  $D_3(v)$  when  $v = 1$  to 10.

$v$	$A_3(v)$	Relevant partition	$D_3(v)$	Relevant partition
1	0	-	0	-
2	0	-	0	-
3	1	3	1	3
4	1	4	1	4
5	1	5	1	5
6	1	6	1	$3^2$
7	1	7	1	$34^1$
8	2	8, 5+3	2	$35^1, 4^2$
9	2	9, 6+3	2	$45^1, 3^3$
10	2	10, 7+3	2	$5^2, 3^24$

## CHAPTER 3

### Combinatorial interpretation of Göllnitz - Gordon identities using coloured partition.

#### 3.1 $n$ - Colour partition

**Definition 3.1.1** [2] An  $n$ -colour partition of a positive integer is a partition in which a part of size  $n$ , can come in  $n$  different colors denoted by subscripts  $n_1, n_2, n_3 \dots n_n$  and parts satisfy the order  $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 \dots$ .

**Example 14.** The  $n$ -colour partition of 3 are

$$3_1, 3_2, 3_3$$

$$2_1 + 1_1, 2_2 + 1_1$$

$$1_1 + 1_1 + 1_1$$

if  $P(v)$  denotes the number of  $n$ -colour partitions of  $v$ , then by using the standard techniques of partition theory it can be shown that

$$1 + \sum_{v=1}^{\infty} P(v)q^v = \prod_{n=1}^{\infty} (1 - q^n)^{-n}.$$

**Definition 3.1.2** [2] Let  $\Pi = (a_1)_{b_1} + (a_2)_{b_2} + \dots + (a_r)_{b_r}$  be an  $n$ -color partition of  $v$ . We call  $(a_i)_{a_i - b_i + 1}$  the conjugate of  $(a_i)_{b_i}$ . An  $n$ -color partition  $v$  obtained from  $\Pi$  by replacing each of its parts by its conjugate will be called the conjugate of  $\Pi$  and will be denoted by  $\Pi^c$ .

**Example 15.** If we consider  $\Pi = 5_2 + 3_1$ , an  $n$ -colour partition of 8, then  $\Pi^c = 5_{5-2+1} + 3_{3-1+1} = 5_4 + 3_3$ .

**Definition 3.1.3** An  $n$ -colour partition  $\Pi$  to be called self conjugate if it is identical with its conjugate  $\Pi^c$ .

**Example 16.** Consider  $\Pi = 5_3 + 3_2 + 1_1$  be an  $n$ -colour partition of 9.

$$\begin{aligned}\Pi^c &= 5_{5-3+1} + 3_{3-2+1} + 1_{1-1+1} \\ &= 5_3 + 3_2 + 1_1\end{aligned}$$

so  $\Pi^c = \Pi$

thus  $5_3 + 3_2 + 1_1$  is a self conjugate  $n$ -colour partition of 9.

**Definition 3.1.4** [2] The weighted difference of any pair of parts  $m_i, n_j$  is defined by  $m - n - i - j$  and is denoted by  $((m_i - n_j))$ .

**Theorem 3.1.5** Let  $A(v)$  denote the number of  $n$ -colour self-conjugate partitions of  $v$  such that each part is self conjugate. Let  $B(v)$  denote the number of ordinary partitions of  $v$  into odd parts then  $A(v) = B(v)$  for all  $v \geq 0$ .

**Proof.** Let  $\Pi$  be an  $n$ -colour partition enumerated by  $A(v)$ . Then in each part  $m_i$  of it,  $m$  must be odd, because  $m_i = m_{m-i+1} \implies m = 2i - 1$ . Thus if we ignore the subscripts of all parts in  $\Pi$ , we get a unique ordinary partition of  $v$  into odd parts.

Conversely, if we consider an ordinary partition of  $v$  into odd parts and replace each part  $2a - 1$  by  $(2a - 1)_a$ . We get a unique partition enumerated by  $A(v)$

hence

$$1 + \sum_{v=1}^{\infty} A(v)q^v = \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}}.$$

**Example 17.** The relevant example is  $A(3) = 2$ , since the relevant partitions are  $3_2$ , and  $1_1 1_1 1_1$ .

Also,  $B(3) = 2$ , since in this case the relevant partitions are  $3$ ,  $1^3$ .

### 3.2 Main Theorem

In an attempt to unify the Göllnitz - Gordon partition functions appearing in Theorem 2.1.1, 2.1.2 and the Göllnitz partition function of Theorem 2.1.3, Agarwal in [1] defined a more generalized partition function  $A_k(v)$  given in Theorem 2.2.1 in Chapter 2. Obviously,  $A_1(v)$  and  $A_3(v)$  are Göllnitz - Gordon functions of Theorem 2.1.1, 2.1.2 and  $A_2(v)$  is Göllnitz function appearing in Theorem 2.1.3.

In [3] Agarwal and Rana interpret the right hand side of (2.2.1) as a generating function of a  $n$ -colour partition function. This results in an infinite family of 2 - way combinatorial identities. In some particular cases we get even 3 - way combinatorial identities. We discuss three such cases and obtain new combinatorial versions of each of the Theorem 2.1.1, 2.1.2 and 2.1.3.

**Theorem 3.2.1** Given a positive integer  $k$ , let  $B_k(v)$  denote the number of  $n$ -colour partitions of  $v$  such that parts are greater than or equal to  $k$ , parts used are of the type  $(2l - 1)_1$  and  $(2l)_2$  if  $k$  is odd,  $(2l - 1)_2$  and  $(2l)_1$  if  $k$  is even. The weighted difference between any two parts is non-negative and even. Then

$$A_k(v) = B_k(v) \tag{3.2.1}$$

for all  $v$ , where  $A_k(v)$  is in last Theorem 2.2.1

### 3.3 Proof of Main Theorem given in Section 3.2

We shall prove that

$$\sum_{v=0}^{\infty} B_k(v)q^v = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n(n+k-1)} \quad (3.3.1)$$

Let  $B_k(m, v)$  denote the number of the partitions of the type enumerated by  $B_k(v)$  into exactly  $m$  parts. We shall first prove the identity,

$$B_k(m, v) = B_k(m-1, v-k-2m+2) + B_k(m-1, v-k-4m+3) + B_k(m, v-2m). \quad (3.3.2)$$

We give the proof of (3.3.1) for odd  $k$  as the proof for even  $k$  is similar.

We split the partitions enumerated by  $B_k(m, v)$  into three classes:

- (i) those that have least part equal to  $k_1$
- (ii) those that have least part equal to  $(k+1)_2$
- (iii) those that have least part greater than or equal to  $(k+2)_1$

We now transform the partitions in class (i) by deleting the least part  $k_1$  and then subtracting 2 from all the remaining parts ignoring the subscripts. This produces a partition of  $v-k-2(m-1)$  into exactly  $m-1$  parts, each of which  $\geq k_1$  (since originally the second smallest part was  $\geq (k+2)_1$ ).

Obviously this transformation does not disturb the weighted difference condition between the parts and so the transformed partition is of the type enumerated by

$$B_k(m-1, v-k-2m+2)$$

next, we transform the partitions in class (ii) by deleting the least part  $(k+1)_2$  and then subtracting 4 from all the remaining parts ignoring the subscripts. This produces a partition of

$$v - (k+1) - 4(m-1) = v - k - 4m + 3$$

into  $m - 1$  parts, each of which  $\geq k_1$  (since originally the second smallest part was  $\geq (k + 4)_1$ ). Note that originally  $(k + 2)_1$  and  $(k + 3)_2$  could not be the smallest part because of the weighted difference condition. Furthermore, since the weighted difference condition between the parts are not disturbed, we see that the transformed partition is of the type enumerated by

$$B_k(m - 1, v - k - 4m + 3)$$

finally, we transform the partitions in class (iii) by subtracting 2 from each part ignoring the subscripts. This produces a partition of  $v - 2m$  into  $m$  parts, each of which  $\geq k_1$ , since the weighted difference condition between the parts are not disturbed. We see that the transformed partition is of the type enumerated by  $B_k(m, v - 2m)$ . The above transformation establish a bijection between the partitions enumerated by  $B_k(m, v)$  and those enumerated by

$$B_k(m, v) = B_k(m - 1, v - k - 2m + 2) + B_k(m - 1, v - k - 4m + 3) + B_k(m, v - 2m)$$

This proves the identity (3.3.2) for odd  $k$ .

let

$$g_k(z, q) = \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} B_k(m, v) z^m q^v \tag{3.3.3}$$

using (3.3.2) in (3.3.3), we get the following q-functional equation

$$g_k(z, q) = z_k q^k g_k(zq^2; q) + zq^{k+1} g_k(zq^4; q) + g_k(zq^2; q). \tag{3.3.4}$$

setting

$$g_k(z, q) = \sum_{n=0}^{\infty} \beta_k(n; q) z^n$$

and then comparing the coefficients of  $z^n$  on each side of (3.3.4), we get

$$\beta_k(n; q) = q^{2n-2+k} \beta_k(n-1; q) + q^{4n-3+k} \beta_k(n-1; q) + q^{2n} \beta_k(n; q)$$

therefore

$$\beta_k(n; q) = \frac{(1 + q^{2n-1})q^{2n-2+k}}{1 - q^{2n}} \beta_k(n-1; q)$$

(3.3.5)

iterating (3.3.5)  $n$  - times and observing that

$$\beta_k(0; q) = 1$$

we see that

$$\beta(n, k; q) = \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n(n+k-1)}$$

therefore

$$g_k(z, q) = \sum_{n=0}^{\infty} \beta_k(n; q) z^n = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n(n+k-1)} z^n.$$

now

$$\begin{aligned}
\sum_{n=0}^{\infty} B_k(n)q^n &= \sum_{v=0}^{\infty} \left[ \sum_{m=0}^{\infty} B_k(m, v) \right] q^v \\
&= g_k(1, q) \\
&= \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+k-1)}}{(q^2; q^2)_n}
\end{aligned}$$

This completes the proof of the theorem (3.2.1).

### 3.4 Particular cases and new combinatorial identities

For  $k = 1$ , in view of (2.1.1) our Theorem 3.2.1 reduces to:

**Corollary 1.**

$$A_1(v) = B_1(v) = D_1(v) \tag{3.4.1}$$

where  $A_1(v)$  is as given in Theorem 2.2.1.

where  $D_1(v)$  is the number of partitions of  $n$  into parts which are congruent to 1, 4, 7(mod 8).

(3.4.1) gives us following three identities in the usual sense,

$$A_1(v) = B_1(v) \tag{3.4.1}_a$$

$$A_1(v) = D_1(v) \tag{3.4.1}_b$$

$$B_1(v) = D_1(v) \tag{3.4.1}_c$$

The case  $(3.4.1)_b$  of (3.4.1) is the first Göllnitz -Gordon identity, that is, Theorem 2.1.1. The other two identities induced by (3.4.1) are new combinatorial versions of it.

**Example 18.** The following table gives the relevant partition of  $A_1(v)$ ,  $B_1(v)$  and  $D_1(v)$  when  $v = 1$  to 10.

$v$	$A_1(v)$	Relevant partitions	$B_1(v)$	Relevant partitions	$D_1(v)$	Relevant partitions
1	1	1	1	$1_1$	1	1
2	1	2	1	$2_2$	1	$1^2$
3	1	3	1	$3_1$	1	$1^3$
4	2	4, 13	2	$4_2, 3_1 1_1$	2	4, $1^4$
5	2	5, 14	2	$5_1, 4_2 1_1$	2	$14^1, 1^5$
6	2	6, 15	2	$6_2, 5_1 1_1$	2	$41^2, 1^6$
7	3	7, 16, 25	3	$7_1, 6_2 1_1, 5_1 2_2$	3	7, $41^3, 1^7$
8	4	8, 17, 26, 35	4	$8_2, 7_1 1_1, 6_2 2_2,$ $5_1 3_1$	4	$17^1, 4^2, 41^4,$ $1^8$
9	5	9, 18, 27, 36, 135	5	$9_1, 8_2 1_1, 7_1 2_2,$ $6_2 3_1, 5_1 3_1 1_1$	5	9, $71^2, 14^2,$ $41^5, 1^9$
10	5	10, 19, 28, 37, 136	5	$10_2, 9_1 1_1, 8_2 2_2,$ $7_1 3_1, 6_2 3_1 1_1$	5	$71^3, 61^4, 4^2 1^2,$ $41^6, 1^{10}$

For  $k = 2$ , in view of (2.1.3) we get the following three way identity:

**Corollary 2.**

$$A_2(v) = B_2(v) = D_2(v) \tag{3.4.2}$$

where  $A_2(v)$  is as given in Theorem 2.2.1.

where  $D_2(v)$  is the number of partitions of  $n$  into parts which are congruent to 2, 3, 7(mod 8).

(3.4.2) gives us following three identities in the usual sense,

$$A_2(v) = B_2(v) \tag{3.4.2}_a$$

$$A_2(v) = D_2(v) \tag{3.4.2}_b$$

$$B_2(v) = D_2(v) \tag{3.4.2}_c$$

The case (3.4.2)<sub>b</sub> of (3.4.2) is the Göllnitz -Gordon identity, that is, Theorem 2.1.3.

The other two identities induced by (3.4.2) are new combinatorial versions of it.

**Example 18.** The following table gives the relevant partition of  $A_2(v)$ ,  $B_2(v)$  and  $D_2(v)$  when  $v = 1$  to 10.

$v$	$A_2(v)$	Relevant partitions	$B_2(v)$	Relevant partitions	$D_2(v)$	Relevant partitions
1	0	-	0	-	0	-
2	1	2	1	$2_1$	1	2
3	1	3	1	$3_2$	1	3
4	1	4	1	$4_1$	1	$2^2$
5	1	5	1	$5_2$	1	$23^1$
6	2	6, 24	2	$6_1, 4_12_1$	2	$3^2, 2^3$
7	2	7, 25	2	$7_2, 5_22_1$	2	7, $2^23$
8	2	8, 26	2	$8_1, 6_12_1$	2	$23^2, 2^4$
9	3	9, 27, 36	3	$9_2, 7_22_1, 6_13_2$	3	$27^1, 3^3, 2^33$
10	4	10, 28, 37, 46	4	$10_1, 8_12_1, 7_23_2, 6_14_1$	4	10, $37^1, 2^23^2, 2^5$

For  $k = 3$ , in view of (2.1.2) we get the following three way identity:

**Corollary 3.**

$$A_3(v) = B_3(v) = D_3(v) \tag{3.4.3}$$

where  $A_3(v)$  is as given in Theorem 2.2.1.

where  $D_3(v)$  is the number of partitions of  $n$  into parts which are congruent to 3, 4, 5(mod 8).

(3.4.3) gives us following three identities in the usual sense,

$$A_3(v) = B_3(v) \tag{3.4.3}_a$$

$$A_3(v) = D_3(v) \tag{3.4.3}_b$$

$$B_3(v) = D_3(v) \tag{3.4.3}_c$$

The case (3.4.3)<sub>b</sub> of (3.4.3) is the second Göllnitz -Gordon identity, that is, Theorem 2.1.2. The other two identities induced by (3.4.3) are new combinatorial versions of it.

**Example 18.** The following table gives the relevant partition of  $A_3(v)$ ,  $B_3(v)$  and  $D_3(v)$  when  $v = 1$  to 10.

$v$	$A_3(v)$	Relevant partitions	$B_3(v)$	Relevant partitions	$D_3(v)$	Relevant partitions
1	0	-	0	-	0	-
2	0	-	0	-	0	-
3	1	3	1	$3_1$	1	3
4	1	4	1	$4_2$	1	4
5	1	5	1	$5_1$	1	5
6	1	6	1	$6_2$	1	$3^2$
7	1	7	1	$7_1$	1	$34^1$
8	2	8, 35	2	$8_2, 5_13_1$	2	$35^1, 4^2$
9	2	9, 36	2	$9_1, 6_23_1$	2	$45^1, 3^3$
10	2	10, 37	2	$10_2, 7_13_1$	2	$5^2, 43^2$

### 3.5 Conclusion

In this thesis we studied combinatorially Göllnitz-Gordon identities using Ordinary partition and  $n$ -colour partition. These identities have also been interpreted with other combinatorially tools such as Frobenius partition and Lattice path [3], [12]. The generalization of these identities is given in [4].

## Bibliography

1. Agarwal, A.K., On a Generalized Partition Theorem, J.Indian Math. Soc., Vol. 50(1986), pp. 185-190.
2. Agarwal, A.K., Padmavathama, Subbarao M.V., Partition Theory, Atma Ram Sons, Chandigarh, India, 2005.
3. Agarwal, A.K. and Rana M., New Combinatorial Version of Göllnitz-Gordon Identities, Utilitas Mathematica, Vol. 79(2009), pp. 145-156.
4. Andrews, G.E., The Theory of Partition, Encyclopedia of Mathematics and its Application, Addison-Wesley, 1976.
5. Andrews, G.E. and Berndt, B.C., Ramanujan's Lost Notebook, Part 2, New York: Springer, 2008.
6. Euler, L., Introductio in Analysin Infinitorum Tomus Primus, Macum-Michaelem Bousquet, Lausanne, 1748.
7. Göllnitz, H., Einfache Partition (unpublished), Diplomarbeit W.S., Gottingen, 1960 pp. 65.
8. Göllnitz, H., Partition mit Differenzenbedingungen, J. Reine Angew. Math., Vol. 225(1967), pp. 154-190.
9. Gordon, B., Some Continued Fraction of the Rogers-Ramanujan Type, Duke J. Math., Vol. 32(1965), pp. 741-748.
10. Hardy, G.H. and Wright, E.M., An Introduction to the Theory of Numbers, 5th ed., Oxford University Press, London, New York., 1979.
11. McLaughlin, J., Sills, A. V. and Zimmer, P., Dynamic Survey DS15: Rogers-Ramanujan-Slater Type Identities, Electronic J. Combinatorics, Vol. DS15(2008), pp. 1-59.
12. Rana, M. and Agarwal, A.K., On an Extension of a Combinatorial Identity, Proc. Indian Acad. Sci. (Math. Sci) Vol. 119, No.1(2009), pp. 1-7.

- 13.** Slater, L.J., Further Identities of the Rogers-Ramanujan type, Proc. London Math. Soc., Vol. 2, 54(1952), pp. 147-167.
- 14.** Subbarao, M.V., Combinatorial Proof of Some Identities, Proc. Washington State Univ. Conf. Number Theory(1971), pp. 80-91.
- 15.** Subbarao, M.V., Some Rogers-Ramanujan type Partition Theorems, Pacific J. Math., Vol. 120(1985), pp. 431-435.