Some Geometric Aspects on Submanifolds Theory

A Thesis

Submitted in fulfillment of the requirement

for the award of the degree of

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

by

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May, 2013

Dedicated to My Brother
Late Pawandeep Singh Chahal

CERTIFICATE

Certified that the thesis *Some Geometric Aspects on Submanifolds Theory* which is submitted by Mr. Khushwant Singh, in fulfillment of the requirement for the award of the degree of *Doctor of Philosophy* in the School of Mathematics and Computer Applications, Thapar University, Patiala is a record of the candidates own independent and original research work carried out by him under our supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any other University or Institute for the award of any degree.

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DECLARATION

I hereby declare that the work which is being presented in this thesis *Some Geometric Aspects on Submanifolds Theory* submitted by me for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. M. A. Khan, Assistant Professor, Department of Mathematics, University of Tabuk, Tabuk, Kingdom of Saudi Arabia and Dr. S. S. Bhatia, Professor, School of Mathematics and Computer Applications, Thapar University, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or Abroad and that the ideas and references cited herein have been duly acknowledged.

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4. **Khushwant Singh**, Siraj Uddin, Cenap Ozel and M. A. Khan, *A class of totally umbilical slant submanifolds of Lorentzian para-Sasakian manifolds,* International Journal of Physical Sciences, 7(10) (2012), 1526-1529.

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7. Siraj Uddin, M. A. Khan, S. H. Kon and **Khushwant Singh**, *Warped product semi-invariant submanifolds of nearly Cosymplectic manifolds,* Mathematical Problems in Engineering vol. 2011, Article ID 230374, 12 pages doi:10.1155/2011/230374.

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9. M. A. Khan, **Khushwant Singh** and V. A. Khan, *Slant Submanifolds in LP-Contact Manifolds,* Differential Geometry Dynamical Systems, 12 (2010), 102-108.

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11. M. A. Khan, **Khushwant Singh**, *Warped product Pseudo-slant submanifold of trans-Sasakian manifolds*, Thai J. Math 8(2) (2010), 263-273.

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PREFACE

To study the geometric aspects of a manifold, it is lot more convenient to first embed it into a known manifold and then study the geometry viz-a-viz that of the ambient manifold. This approach gave impetus to the study of submanifolds which later developed into a fascinating topic of study of the theory of submanifolds. A submanifold of an almost Hermitian manifold presents an interesting geometric study as the action of almost complex structure J transforms a vector to a vector perpendicular to it. The natural out come of which are three typical classes of submanifolds, namely holomorphic or invariant submanifolds (also known as almost complex submanifolds), totally real or anti-invariant submanifolds and slant submanifolds (cf. [5], [20], [21], [25], [26], [76] etc.).

The holomorphic submanifolds M are characterized by the condition $JT_xM \subseteq T_xM$ and the totally real submanifolds are characterized by $JT_xM \subseteq T_x^{\perp}M$ for all $x \in M$. That is, for a holomorphic submanifold, the action of J on a vector in the tangent space T_xM gives a vector of the same space, whereas in case of a totally real submanifold the almost complex structure J carries each tangent vector of M into the corresponding normal space of M.

The study of totally real and holomorphic submanifolds was initiated in early as 1970 (cf. [20], [21]). Since then many differential geometers have contributed interesting results (cf. [52], [76]). In 1978 A. Bejancu (cf. [3], [4]) introduced the notion of CR-submanifolds and generalized the above two classes of submanifolds. A real submanifold M of an almost Hermitian manifold is called a CR-submanifold if there exists a differentiable distribution D on M satisfying $JD_x = D_x$ and $JD_x^{\perp} \subseteq T_x^{\perp}M$ for each $x \in M$, where $D^{\perp}$ is the complementary orthogonal distribution to D. Integrability of the distributions on a CR-submanifold gives rise to CR-product submanifolds, which are CR-submanifolds that are locally Riemannian product of leaves of D and $D^{\perp}$. A lot of research has been done on CR-product submanifolds (cf. [22], [23], [30], [31], [78]). Moreover, it is proved that there do not exist non-trivial CR-product submanifolds in complex hyperbolic spaces. It was also found that S^6 does not admit a non-trivial CR-product submanifold [68]. CR-products however are obtained in complex projective spaces naturally through Segre embedding theorem. A significant contribution in the study of CR-submanifolds has been made by B.Y. Chen (cf. [22], [23]), A. Bejancu ([3], [4], [5]), K. Sekigawa [68] and A. Gray [36].

One can easily realize that a submanifold M of an almost Hermitian manifold is a holomorphic or a totally real submanifold if and only if for every non zero vector U tangent to M at any point $x \in M$, the angle between JU and T_xM is 0 or $\pi/2$ respectively. This point of view paved way for a generalization of CR-submanifold by B.Y. Chen in 1990. The new class of submanifold is known as slant submanifold [25]. They are defined as the submanifolds for which the angle between JU and the tangent space T_xM is constant. According to him, an immersed submanifold M of an almost Hermitian manifold $(\bar{M}, J, g)$ is slant if and only if the Wirtinger angle $\theta(X) \in [0, \pi/2]$ between JX and T_xM has the same value θ for any $x \in M$ and $X \in T_xM$, $X \neq 0$. The notion of slant submanifold in the sense of Chen is an evident generalization of holomorphic and totally real submanifolds.

As far as contact geometry is concerned, several results can be found in literature on the so called invariant and anti-invariant and on their generalization CR-submanifolds of almost contact manifolds (cf. [48], [63], [70], [71], [78]). A. Lotta [50], further extended this study to slant submanifolds of almost contact geometry. J. L. Cabrerizo et al. ([16], [17]) worked out a characterization for the existence of a slant submanifold in an almost contact metric manifold. Moreover, in [64] N. Papaghiuc has introduced a class of submanifolds in an almost Hermitian manifold, called the *semi-slant* submanifolds, such that the class of proper CR-submanifolds and the class of slant submanifolds appear as particular cases in the class of semi-slant submanifolds. Recently, Carriazo [18] defined and studied bi-slant immersion in almost Hermitian manifold and termed it as an anti-slant submanifold. However, the term anti-slant may suggest that the submanifold has no slant part. Hence, Sahin [67] studied and define the notion of hemi-slant submanifolds. However, in case of Lorentzian paracontact geometry [56] is concerned there is need to do a lot more for exploring the new geometric aspects of the class.

In 1969 Bishop and O'Neill [8] introduced warped product manifolds as generalization to Riemannian product manifolds. The easiest example of warped product manifolds is surface of revolution. Formally, a warped product manifold $B \times {}_fF$ of two Riemannian manifolds (B, g_B) and (F, g_F) where g_B and g_F are Riemannian metrics on B and F respectively is the product manifold $B \times F$ equipped with the Riemannian metric $g = g_B + f^2 g_F$, where f is a positive differentiable function on B. Bishop and O'Neill studied these manifolds to study and work out examples of manifolds of negative curvature. They also applied them to space time scheme. From geometric point of view the study got momentum and thrust only when B. Y. Chen initiated the study of warped product CR-submanifolds of a Kaehler manifold.

The aim of the present thesis is to extend the study of different classes of slant submanifold of almost Hermitian, almost contact and Lorentzian almost paracontact metric manifolds, in particular nearly Kaehler, nearly trans-Sasakian and LP-Sasakian manifolds respectively. We have also made an effort to study different warped product submanifolds when ambient manifold is almost contact and Lorentzian contact, in particular when it is trans-Sasakian, nearly Cosymplectic or globally framed f-manifolds with lorentz metric.

The thesis comprises of seven chapters and each chapter is divided into various sections. The mathematical relations obtained in the text have been labeled with double decimal numbering. The first figure denotes the chapter number, second represents the section and the third points out the number of definition, equation, proposition, lemma or the theorem as the case may be, e.g. Theorem 4.3.2 refers to the second theorem of section three in chapter four.

The **first chapter** is introductory and contains basic definitions and some known results, which will be used repeatedly in the subsequent chapters. However some of them might be repeated in chapters.

Slant submanifolds of a Kaehler manifold have been studied extensively in ([25], [26], [27], [28], [53], [66] etc.), whereas slant submanifolds of a nearly Kaehler manifold are yet to be explored to that extent. In **chapter II**, we have studied slant and pseudo-slant submanifold of nearly Kaehler manifolds. Slant submanifold of almost Hermitian manifolds were initiated by N. Papaghiuc. In Section 2.2, we prove that a totally umbilical proper slant submanifold of a nearly Kaehler manifold is totally geodesic if the endomorphism n is parallel. Whereas in section 2.3, we have obtained some geometrically interesting results for the totally umbilical pseudo-slant submanifolds of a nearly Kaehler manifold. The content of this chapter are published in **Kuwait Journal of Science and Engineering (accepted for publication)** and **Acta Universitatis Apulensis,** 29 (2012), 279-285.

In **chapter III**, we have introduced the study of slant submanifolds in the setting of Lorentzian Paracontact Manifolds. The study of Lorentzian almost paracontact manifold was initiated by K. Matsumoto [56]. Later on several authors studied Lorentzian almost paracontact manifolds and their different classes. We have obtained a characteristic equation of slant submanifold in the setting of LP-Contact manifold. We have also obtained a characterization for slant submanifold via some curvature tensor. Furthermore, if we use Lorentzian metric in almost contact manifolds, then the structure becomes more interesting. In the light of this, we have obtained some important results for slant submanifolds in Lorentzian almost contact manifolds. Some contents of this chapter are published in **Differential Geometry Dynamical Systems,** 12 (2010), 102-108 and some contents are communicated in **Annals of University of Craiova-Mathematics and Computer Science Series**.

In **chapter IV**, we have proved that a every totally umbilical proper slant submanifold of a Lorentzian paracontact manifold is either minimal or if it is not minimal then we derive a formula for its slant angle. The contents of chapter are published in **Annals of the University of Craiova- Mathematics and Computer Science Series**, 38, 1(2011), 49-53, **Mathematical Problems in Engineering,** vol. 2011, Article ID 516238, 9 pages, 2011. doi:10.1155/2011/516238 and **International Journal of Physical Sciences,** 7(10) (2012), 1526-1529.

Warped product manifolds are known to have applications in physics. For instance, they provide an excellent setting to model space-time near a black hole or a massive star. Recently, Chen [30] (see also [31]) studied warped product CR-submanifolds and showed that there exist no warped product CR-submanifolds of the form $M = N^{\perp} \times {}_f N^T$ such that $N^{\perp}$ is a totally real submanifold and N^T is a holomorphic submanifold of a Kaehler manifold $\tilde{M}$. Therefore he considered warped product CR-submanifold in the form $M = N^T \times {}_f N^{\perp}$ which is called CR-warped product, where N^T and $N^{\perp}$ are holomorphic and totally real submanifolds of a Kaehler manifold $\tilde{M}$.

B.Y.Chen worked out a condition forcing a CR-submanifold to be CR-product. He showed that a CR-submanifold is a CR-product if and only if $A_{JZ}X = 0$ for all $X \in D$ and $Z \in D^{\perp}$. He further proved that $\nabla P = 0$ is necessary and sufficient for a CR-submanifold to become a CR-product submanifold of a Kaehler manifold, where PU and FU denoting the tangential and normal components of JU respectively. In [23] Chen again worked out a characterization for a CR-submanifold to reduce to CR-product. To this end what remained to be explored is the characterization in terms of canonical structure T and N. In **chapter V**, we studied this problem and succeed to extend B.Y. Chen's results to the setting of nearly Cosymplectic manifold. The contents of chapter are published in **Mathematical Problems in Engineering,** vol. 2011, Article ID 230374, 12 pages doi:10.1155/2011/230374.

Chapter VI deals with the study of warped product contact CR-submanifolds of globally framed f-manifolds with Lorentz metric [15] in the particular setting of indefinite S-manifolds for both spacelike and timelike cases. We prove that if $M = N^{\perp} \times_f N^T$ is a warped CR-submanifold such that $N^{\perp}$ is ϕ-anti-invariant and N^T is ϕ-invariant, then M is a CR-product. We show that the second fundamental form of a contact CR warped product of a indefinite S-space form satisfies a geometric inequality, $\|h\|^2 \geq p\{3\|\nabla lnf\|^2 - \triangle lnf + (c+2)k + 1\}$. The contents of this chapter are communicated in **Filomat**.

Doubly warped product manifolds were introduced as generalization of warped product manifold [73].

In the Last **Chapter**, we have proved the Integrability condition of pseudo slant submanifold in trans-Sasakian manifold. Also, we obtained the warped and doubly warped product pseudo-slant submanifold of trans-Sasakian manifold. We have provide an example of pseudo-slant warped product submanifold of Kenmotsu manifold. The contents of this chapter are published in **Thai Journal of Mathematics,** 8(2) (2010), 263-273.

Towards the end, future work have been proposed. Finally at the end of the thesis, references have been reported.

Chapter I
INTRODUCTION

INTRODUCTION

The purpose of this chapter is to introduce basic concepts, preliminary notions and some fundamental results which we require for the development of the subject in the present thesis. This chapter contains brief resume of some of the known results in geometry of almost Hermitian, almost contact metric manifolds, Lorentzian almost paracontact manifolds, their allied structures and the geometry of submanifolds of these manifolds. These results are readily available in research papers and review articles (cf. [3], [22], [25], [50],[55],[56], [80],[83]) and in standard books e.g., Nomizu and Kobayashi [49], B.Y. Chen [19], D.E. Blair [13], K. Yano [77] and B. O'Neill [61] etc., nevertheless, we have collected them here to fix up our terminology and for ready references.

1.1. Structures on C^∞-Manifolds

To study the geometry of a differentiable manifold, we require a Riemannian metric on differentiable manifold i.e., a positive definite inner product on the tangent bundle of the manifold. By a *Riemannian metric* g on a manifold M, we mean a map : $p \to g_p$ where g_p is a positive definite inner product on T_pM. We require this map to be smooth (differentiable) in the sense that the function

$$p \to g_{ij}(p) = g_p\big(\frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p\big),$$

is smooth for all i, j for any chart (X, x). This smoothness condition is same as that for the map

$$p \to g_p(X_p, Y_p),$$

for all vector fields X, Y on M. On a paracompact manifold, there exist a smooth Riemannian metric. A differentiable manifold with a Riemannian metric is said to be a Riemannian manifold. Further refined information can be had by knowing additional structures on the manifold, for example almost complex, almost Kaehler, nearly Kaehler, almost contact metric, Sasakian, Kenmotsu and trans-Sasakian structures etc., (cf. [13], [42], [49] [62], [77]). In this section we briefly discuss some of these structures.

In what follows, we shall take a differentiable manifold which is connected and paracompact, so that it can always be endowed with a Riemannian metric g

and a Riemannian connection ∇.

An *almost complex structure* on a real differentiable manifold $\bar{M}$ is a tensor field J, an endomorphism of the tangent space $T_x\bar{M}$ at every point $p \in \bar{M}$, such that $J^2 = -I$ where I denotes the identity transformation on $T_x\bar{M}$. A manifold with a fixed almost complex structure is called an *almost complex manifold.* Let $\bar{M}$ be a Riemannian manifold with almost complex structure J and Hermitian metric g satisfying

$$(a) \quad J^2 = -I, \quad (b) \quad g(JX, JY) = g(X, Y) \tag{1.1.1}$$

for all $X, Y \in T\bar{M}$, by virtue of which g is called a *Hermitian metric.* An almost complex manifold (respectively, a complex manifold) with a Hermitian metric is called an *almost Hermitian manifold* (respectively, a *Hermitian manifold*).

The *fundamental 2-form* Ω of an almost Hermitian manifold $\bar{M}$ with an almost complex structure J and metric g is defined by

$$\Omega(X, Y) = g(X, JY).$$

Since g is invariant by J, so is Ω i.e.,

$$\Omega(JX, JY) = \Omega(X, Y), \tag{1.1.2}$$

for all vector fields X and Y on $\bar{M}$.

The almost complex structure J is not, in general, parallel with respect to the Riemannian connection defined by the Hermitian metric g. The following is a useful relation (cf. [35]).

$$2g((\bar{\nabla}_X J)Y, W) = d\Omega(X, Y, W) - d\Omega(X, JY, JW) - g(X, S(Y, JW)) \tag{1.1.3}$$

for any vector fields X, Y, W on $\bar{M}$, where S denotes the Nijenhuis tensor (or the torsion tensor) of J and is defined by

$$S(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y], \tag{1.1.4}$$

for all vector fields X, Y on $\bar{M}$.

It is easy to verify that the Nijenhuis tensor of J satisfies

$$S(JX, Y) = S(X, JY) = -JS(X, Y). \tag{1.1.5}$$

It is well known that vanishing of the tensor $S(X,Y)$ is the necessary and sufficient condition for an almost complex manifold to be a complex manifold [49].

If we extend the Riemannian connection $\bar{\nabla}$ to be a derivative on the tensor algebra of $\bar{M}$, then we have the following formulae.

$$(\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y, \tag{1.1.6}$$

$$(\bar{\nabla}_X \Omega)(Y,W) = g((\bar{\nabla}_X J)Y, W). \tag{1.1.7}$$

Now, we define a Kaehler structure on a Hermitian manifold as:

DEFINITION 1.1.1. A Hermitian metric on an almost complex manifold is called a *Kaehler metric* if the fundamental 2-form Ω is closed. A complex manifold equipped with a Kaehler metric is said to be a *Kaehler manifold*. Thus by formula (1.1.3), an almost complex manifold $\bar{M}$ is Kaehler if and only if

$$(\bar{\nabla}_X J)Y = 0 \tag{1.1.8}$$

or equivalently, $$\bar{\nabla}_X JY = J\bar{\nabla}_X Y$$

for all X, Y in $T\bar{M}$. In this case the connection $\bar{\nabla}$ on $\bar{M}$ is said to be the *Kaehlerian connection*.

DEFINITION 1.1.2. A Kaehler manifold $\bar{M}$ is called a *complex space form* if it is of constant holomorphic sectional curvature. We denote by $\bar{M}^m(c)$ (or simply $\bar{M}(c)$) an m-dimensional complex space form of constant holomorphic sectional curvature c. Then the curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\begin{aligned}\bar{R}(X,Y)W = \frac{c}{4}\{&g(Y,W)X - g(X,W)Y + g(JY,W)JX \\ &- g(JX,W)JY + 2g(X,JY)JW\}\end{aligned} \tag{1.1.9}$$

for any vector fields X, Y, W on $\bar{M}$.

There is another geometrically interesting and a more general class of almost Hermitian manifolds known as nearly Kaehler manifold which are defined as

DEFINITION 1.1.3. A Hermitian manifold $\bar{M}$ is said to be a *nearly Kaehler* manifold if

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0, \tag{1.1.10}$$

for all X, Y in $T\bar{M}$. Obviously (1.1.10) is equivalent to $(\bar{\nabla}_X J)X = 0$, for all $X \in T\bar{M}$.

A six dimensional unit sphere S^6 has an almost complex structure J defined by the vector cross product in the space of purely imaginary Cayley numbers. This almost complex structure is not integrable and satisfies $(\bar{\nabla}_X J)X = 0$, for any vector field X on S^6, where $\bar{\nabla}$ is the Riemannian connection on S^6. Hence S^6 is a nearly Kaehler manifold which is not Kaehler [37].

A structure on an n-dimensional manifold $\bar{M}$ given by a non null tensor field f satisfying

$$f^3 + f = 0,$$

is called an *f-structure*. Then the rank of f is a constant, say r. If $r = n$, then the f-structure gives an almost complex structure on the manifold $\bar{M}$. In this case n is even. If $\bar{M}$ is orientable and $r = n - 1$, then the f-structure on $\bar{M}$ is called an *almost contact structure*. In this case n is necessarily odd. Throughout, we denote by ϕ, the f-structure on an almost contact manifold. In the following we shall give details of an almost contact structure.

Let $\bar{M}$ be a manifold of dimension $(2m + 1)$. An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ consists of a tensor field ϕ of type $(1, 1)$, a vector field ξ (known as structure vector field), a 1-form η and a metric tensor field g on $\bar{M}$, such that

$$\left.\begin{aligned} &\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi(\xi) = 0,\ \eta \circ \phi = 0, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi) \end{aligned}\right\} \tag{1.1.11}$$

for any vector fields X, Y on $\bar{M}$, where I denotes the identity tensor and $\bar{\nabla}$ denotes the Levi-Civita connection with respect to the metric g on $\bar{M}$.

An almost contact metric structure is called a *contact metric structure* if

$$d\eta = \Phi,$$

where Φ is the fundamental 2-form defined by

$$\Phi(X, Y) = g(X, \phi Y).$$

In this case for any vector field X on $\bar{M}$, we have

$$\bar{\nabla}_X \xi = -\phi X - \phi h X, \tag{1.1.12}$$

where $h = \frac{1}{2} L_\xi \phi$, $L_\xi \phi$ being the Lie-derivative of ϕ with respect to ξ and $\bar{\nabla}$ denotes the Levi-Civita connection of $\bar{M}$. The operator h satisfies

$$g(hX,\ Y) = g(X,\ hY), \quad \phi \circ h = -h \circ \phi. \tag{1.1.13}$$

If ξ is a killing vector field with respect to g, then the contact metric structure is called a *K-contact structure*. It is known that a contact metric manifold is *K-contact* if and only if

$$\bar{\nabla}_X \xi = -\phi X, \tag{1.1.14}$$

for any $X \in T\bar{M}$.

The almost contact structure of $\bar{M}$ is said to be *normal* if,

$$S_\phi + 2d\eta \otimes \xi = 0,$$

where S_ϕ is the Nijenhuis tensor of the tensor field ϕ.

DEFINITION 1.1.4. A normal contact metric manifold is called a *Sasakian manifold.*

It is easy to show that an almost contact metric manifold is a Sasakian manifold if and only if

$$\left.\begin{aligned} (\bar{\nabla}_X \phi)Y &= g(X, Y)\xi - \eta(Y)X, \\ \bar{\nabla}_X \xi &= -\phi X \end{aligned}\right\} \tag{1.1.15}$$

for any $X, Y \in T\bar{M}$.

Following example shows the Sasakian structure:

EXAMPLE 1.1.1. We will denote by $(\mathbb{R}^{2n+1}, \phi, \xi, \eta, g)$, the manifold $\mathbb{R}^{2n+1}$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \quad \xi = 2\frac{\partial}{\partial z},$$

$$g = \eta \otimes \eta + \frac{1}{4}\{\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i)\},$$

$$\phi(\sum_{i=1}^{n}(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{n}(Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i}) + \sum_{i=1}^{n} Y_i y^i \frac{\partial}{\partial z},$$

where (x^i, y^i, z), $i = 1, \cdots, n$ are the Cartesian coordinates.

DEFINITION 1.1.5 [12]. A *Cosymplectic structure* on an almost contact manifold is a normal almost contact metric structure (ϕ, ξ, η, g) with Φ and η closed. Cosymplectic structure is characterized by the condition

$$(\bar{\nabla}_X \phi)Y = 0, \tag{1.1.16}$$

for all X, Y in $T\bar{M}$.

In [72] S. Tanno classified connected almost contact metric manifolds whose automorphism groups posses the maximum dimension. For such a manifold, the sectional curvature of a plane sections containing ξ is a constant say c. He showed that they can be divided into three classes (i) Homogenous normal contact Riemannian manifolds with $c > 0$. (ii) Global Riemannian product of a line (or a circle) with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (iii) A warped product space $\mathbb{R} \times_f \mathbb{C}^n$ if $c < 0$. It is known that manifolds of class (i) are characterized by certain tensorial equations. It admits a Sasakian structure. The manifolds of class (ii) are characterized by a tensorial relation admitting a Cosymplectic structure. Kenmotsu [42] characterized the differential geometric properties of manifolds of class (iii). The structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [42]. A manifold with a Kenmotsu structure is known as *Kenmotsu manifold*.

Kenmotsu manifolds are characterized by the following tensorial equation [42].

$$\left.\begin{aligned} &(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \\ &\bar{\nabla}_X \xi = X - \eta(X)\xi, \end{aligned}\right\} \tag{1.1.17}$$

for all $X,\ Y \in T\bar{M}$.

One of the typical examples of Kenmotsu manifold is the hyperbolic space $\bar{M}(-1)$.

Let an n-dimensional smooth connected paracompact Hausdorff manifold $\bar{M}$ be equipped with a Lorentzian metric g, i.e., g is a smooth symmetric tensor field of type $(0, 2)$ such that at every point $p \in \bar{M}$, the tensor field $g_p : T_p\bar{M} \times T_p\bar{M} \longrightarrow R$ is a non-degenerate inner-product of signature, $(-, +, \cdots, +)$, where $T_p\bar{M}$ is the tangent space of $\bar{M}$ at p and R is the real line. In other words, a matrix representation of g_p has one eigenvalue negative and all other eigenvalues positive, then $\bar{M}$ is Lorentzian manifold. A non-zero vector $X_p \in T_p\bar{M}$ is known to be spacelike, null, non-spacelike or timelike respectively according as

$$g_p(X_p, X_p) > 0, = 0, \leq 0 \ or \ < 0.$$

Let $\bar{M}$ be a n dimensional differentiable manifold. An LP-Contact structure (ϕ, ξ, η, g) on $\bar{M}$ consists of a tensor field ϕ of type$(1,1)$, a vector field ξ, a 1-form η and a metric tensor field g on $\bar{M}$ such that

$$\left.\begin{aligned} &\phi^2 = I + \eta \otimes \xi, \ \eta(\xi) = -1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0, \\ &g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \ \eta(X) = g(X, \xi) \end{aligned}\right\} \tag{1.1.18}$$

for any vector fields X, Y on $\bar{M}$, where I denotes the identity tensor and $\bar{\nabla}$ denotes the Levi-Civita connection with respect to the metric g on $\bar{M}$, we know

$$g(\phi X, Y) = g(X, \phi Y) \tag{1.1.19}$$

Moreover, if on $\bar{M}$ the following additional conditions hold:

$$\left.\begin{aligned} &(\bar{\nabla}_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \\ &\bar{\nabla}_X \xi = \phi X, \end{aligned}\right\} \tag{1.1.20}$$

for any vector fields $X, Y \in T\bar{M}$, then $\bar{M}$ is said to be an *LP-Sasakian manifold* ([56]).

Also, if on $\bar{M}$ the following additional conditions hold:

$$\left.\begin{aligned} &(\bar{\nabla}_X \phi)Y = \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \\ &\bar{\nabla}_X \xi = \beta\{X - \eta(X)\xi\} \end{aligned}\right\} \tag{1.1.21}$$

for all $X, Y \in T\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection with respect to the Lorentzian metric g, then $\bar{M}$ is said to be a *Lorentzian $\beta-$Kenmotsu.*

A Lorentzian paracontact metric structure on $\bar{M}$ is called a *Lorentzian para-Cosymplectic structure* if $\bar{\nabla}\phi = 0$, where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to g. The manifold $\bar{M}$ in this case is called a *Lorentzian para-Cosymplectic* (in brief, an *LP-Cosymplectic*) manifold. From formula $\bar{\nabla}\phi = 0$, it follows that $\bar{\nabla}_X \xi = 0$.

An almost contact manifold with Lorentzian metric is called a *Lorentzian almost contact manifold.*

Let $\bar{M}$ be a $(2n+1)-$dimensional manifold with an almost contact structure and compatible Lorentzian metric [14], $(\bar{M}, \phi, \xi, \eta, g)$ that is, ϕ is $(1,1)$ tensor field , ξ is a structure vector field, η is 1-form and g is Lorentzian metric on $\bar{M}$ satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0 \tag{1.1.22}$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = -g(X, \xi) \tag{1.1.23}$$

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$. An almost contact manifold with Lorentzian metric g is called a *Lorentzian almost contact manifold*. From (1.1.23), it follows that

$$g(\phi X, Y) = -g(X, \phi Y). \tag{1.1.24}$$

As the warped product manifolds form the core part of our study in the present thesis, now we recall some useful definitions and basic results about these product manifolds:

If M_1 and M_2 are Riemannian manifolds with Riemannian metric g_1 and g_2 respectively then the *product manifold* $(M_1 \times M_2,\ g)$ is a Riemannian manifold with the Riemannian metric g defined as

$$g(X, Y) = g_1(d\pi_1 X, d\pi_1 Y) + g_2(d\pi_2 X, d\pi_2 Y)$$

where π_i $(i = 1, 2)$ are the projection maps of M onto M_1 and M_2 respectively and $d\pi_i$ $(i = 1, 2)$ are their differentials.

In [8] R.L. Bishop and B. O'Neill introduced the notion of warped product manifolds by homothetically warping the product metric of a product manifold $B \times F$ on the fibers $p \times F$ for each $p \in B$. This generalized product metric appears in differential geometric studies in a natural way. For example, a surface of revolution is a warped product manifold. They defined these product manifolds as follows:

DEFINITION 1.1.6 [8]. Let $(B,\ g_B)$ and $(F,\ g_F)$ be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and f, a positive differentiable function on B. The warped product of B and F is the Riemannian manifold

$$B \times {}_f F = (B \times F,\ g),$$

where

$$g = g_B + f^2 g_F.$$

More explicitly if X is tangent to $M = B \times {}_f F$ at (p, q) then

$$\|X\|^2 = \|d\pi_1 X\|^2 + f^2(p)\|d\pi_2 X\|^2,$$

where the function f is known as the warping function.

Doubly warped product manifolds were introduced as a generalization to warped product manifolds by B. Unal [73]. Doubly warped product manifolds are defined as:

DEFINITION 1.1.7 [73]. A *doubly warped product manifold* of B and F, denoted as ${}_f B \times_b F$ is endowed with a metric g defined as

$$g = f^2 g_B + b^2 g_F,$$

where $b : B \to \mathbb{R}^+$ and $f : F \to \mathbb{R}^+$ are smooth maps and $g_B,\ g_F$ are the metrics on the Riemannian manifolds B and F respectively. If either $b = 1$ or $f = 1$, but not both, then we obtain a *(single)* warped product. If both $b = 1$ and $f = 1$, then we have a *product manifold.* If neither b nor f is constant, then we call it *non trivial doubly warped product manifold.*

DEFINITION 1.1.8. A warped product manifold is said to be trivial if its warping function f is constant. In this case, the warped product manifold $B \times {}_f F$ is locally a Riemannian product $B \times F^f$, where F^f is the manifold with Riemannian metric $f^2 g_F$ which is homothetic to the original metric g_F on F.

THEOREM 1.1.1 [40]. *If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non trivial vector subbundles such that E_1 is spherical and its orthogonal complement E_0 is autoparallel, then the manifold M is locally isometric to a warped product $N_0 \times {}_f N_1$.*

EXAMPLE 1.1.2. A surface of revolution is a warped product with leaves as the different positions of the rotated curve and fibers the circles of revolution. Explicitly, if M is obtained by revolving a plane curve C about an axis in $\mathbb{R}^3$ and $f : C \to \mathbb{R}^+$, gives distance to the axis, then the surface $M = C \times {}_f S^1(1)$ is a warped product manifold. Here $S^1(1)$ denotes the unit circle.

EXAMPLE 1.1.3. In spherical coordinates the line element of $\mathbb{R}^3 - \{0\}$ is

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Setting $r = 1$ gives the line element of the unit sphere S^2. Evidently $\mathbb{R}^3 - \{0\}$ is diffeomorphic to $\mathbb{R}^3 - \{0\}$ can be identified with the warped product $\mathbb{R}^+ \times {}_r S^2$. In $\mathbb{R}^3 - \{0\}$ the leaves are the rays from the origin and the fibers are the spheres $S^2(r)$, $r > 0$.

In general, $\mathbb{R}^n - \{0\}$ is naturally isometric to $\mathbb{R}^+ \times_r S^{n-1}$.

EXAMPLE 1.1.4. The standard spacetime models of the universe are warped products, as are the simplest models of neighborhoods of stars and black holes.

1.2. Submanifolds Theory

If an n-dimensional differentiable manifold M admits an immersion

$$f : M \to \bar{M}$$

into an m-dimensional differentiable manifold $\bar{M}$, then M is said to be a *submanifold* of $\bar{M}$. Clearly $n \leq m$. If $\bar{M}$ is a Riemannian manifold with a Riemannian metric g, then M also admits a Riemannian metric induced from $\bar{M}$ which is denoted by the same symbol g. The immersion f is said to be an *isometric immersion* if the differentiable map $f_* : TM \to T\bar{M}$ preserves the Riemannian metric, that is for any $X, Y \in TM$,

$$g(f_*X, f_*Y) = g(X, Y). \tag{1.2.1}$$

We shall identify TM with f_*TM through the isomorphism f_*. Hence a tangent vector in $T\bar{M}$ tangent to M, shall mean tangent vector which is the image of an element in TM under f_*. More generally, a C^∞-cross section of the restriction of $T\bar{M}$ on M shall be called a vector field of $\bar{M}$ on M. Those tangent vectors of $T\bar{M}$ which are normal to TM form the normal bundle $T^\perp M$ of M. Hence for every point $x \in M$, the tangent space $T_{f(x)}\bar{M}$ of $\bar{M}$ admits the following decomposition

$$T_{f(x)}\bar{M} = T_xM \oplus T_x^\perp M.$$

The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connection ∇ and $\nabla^\perp$ on TM and $T^\perp M$ respectively governed by the Gauss and Wiengarten formulae Y.i.z.

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.2.2}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{1.2.3}$$

where $X,\ Y$ are tangent vector fields on M and $N \in T^{\perp}M$. A_N and h are second fundamental forms and are related by

$$g(h(X,Y),N) = g(A_N X, Y). \tag{1.2.4}$$

Looking into the Gauss formula, we observe that one can classify the submanifolds, by putting conditions on h as follows:

DEFINITION 1.2.1. A submanifold for which the second fundamental form h is identically zero is called a *totally geodesic submanifold*.

DEFINITION 1.2.2. A submanifold is called *totally umbilical* if its second fundamental form h satisfies

$$h(X,Y) = g(X,Y)H, \tag{1.2.5}$$

for any $X,\ Y \in TM$ and $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$ is the *mean curvature vector*, where n is the dimension of M and $\{e_i\}$ is a local orthonormal frame of vector fields on M.

DEFINITION 1.2.3. A submanifold is called a *minimal submanifold* if the mean curvature vector vanishes identically i.e., $H = 0$.

For the second fundamental form h, we define the covariant differentiation $\bar{\nabla}$ with respect to the connection in $T\bar{M}$ by

$$(\bar{\nabla}_X h)(Y,W) = \nabla_X^{\perp}(h(Y,W)) - h(\nabla_X Y, W) - h(Y, \nabla_X W), \tag{1.2.6}$$

for any vector field X, Y and W tangent to M.

Let $\bar{R}$ and R denote the curvature tensor of the connection on $\bar{M}$ and M respectively, then the equation of Gauss, Coddazi and Ricci are given by

$$R(X,Y,W,Z) = \bar{R}(X,Y,W,Z) + g(h(X,Z),h(Y,W)) \\ -g(h(X,W),h(Y,Z)), \tag{1.2.7}$$

$$(\bar{R}(X,Y)W)^{\perp} = (\bar{\nabla}_X h)(Y,W) - (\bar{\nabla}_Y h)(X,W), \tag{1.2.8}$$

$$\bar{R}(X,Y,N_1,N_2) = R^{\perp}(X,Y,N_1,N_2) - g([A_{N_1}, A_{N_2}]X, Y), \tag{1.2.9}$$

where $R(X,Y,W,Z) = g(R(X,Y)W,Z)$. In (1.2.8), $(\bar{R}(X,Y)W)^{\perp}$ denotes the normal component of $\bar{R}$. X, Y, W and Z are vector fields tangent to M and N_1, N_2 are vector fields normal to M.

DEFINITION 1.2.4. A vector sub-bundle ν of the normal bundle $T^{\perp}M$ is said to be *parallel* (in the normal bundle) if

$$\nabla^{\perp}_{X} N \in \nu$$

for any $X \in TM$ and any local cross section N in ν.

A submanifold M of $\bar{M}$ is called *auto-parallel* if for each vector $X \in T_xM$ and for each curve γ in M starting from x, the parallel displacement of X along γ (with respect to the affine connection $\bar{\nabla}$ of $\bar{M}$) yields a vector field tangent to M. Thus, a distribution D on a manifold $\bar{M}$ is *auto-parallel* if $\nabla_X Y \in D$ for each X, Y in D. In general, if D and $\bar{D}$ are two distributions defined on a manifold $\bar{M}$, we say that D is *$\bar{D}$-parallel* if for all $X \in \bar{D}$ and $Y \in D$ we have $\nabla_X Y \in D$. If D is D parallel then it is auto parallel. D is called *X-parallel* for some $X \in TM$ and for all $Y \in D$ we have $\nabla_X Y \in D$. D is said to be *parallel* if for all $X \in TM$ and $Y \in D$, $\nabla_X Y \in D$.

If a distribution D on M is auto parallel, then it is clearly integrable and by Gauss formula D is totally geodesic in M. If D is parallel, then the orthogonal complementary distribution D' is also parallel, which implies that D is parallel if and only if D' is parallel. In this case M is locally the product of the leaves of D and D'.

REMARK 1.2.1. In view of the above observation, throughout the auto parallelism of a distribution on a manifold $\bar{M}$ and totally geodesicness of its leaves in M are treated equivalent and the two terms are used interchangeably.

On a submanifold M of an almost Hermitian manifold $\bar{M}$, for any vector field X tangent to M, we put

$$JX = TX + NX, \tag{1.2.10}$$

where TX and NX are the tangential and normal components of JX respectively. Then T is an endomorphism of the tangent bundle TM and N is normal valued one form on TM.

Similarly for any vector field N normal to M, if we put

$$JN = tN + nN, \tag{1.2.11}$$

where tN and nN are tangential and normal components of JN respectively, then n can be treated as an endomorphism of the normal bundle $T^{\perp}M$ and t, a tangent bundle valued 1-form on $T^{\perp}M$.

The covariant differentiation of the operator T, N, t and n are defined respectively as

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{1.2.12}$$

$$(\bar{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{1.2.13}$$

$$(\bar{\nabla}_X t)N = \nabla_X tN - t\nabla^{\perp}_X N, \tag{1.2.14}$$

$$(\bar{\nabla}_X n)N = \nabla^{\perp}_X nN - n\nabla^{\perp}_X N, \tag{1.2.15}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$.

Furthermore, for any $X, Y \in TM$, let us decompose $(\bar{\nabla}_X J)Y$ into tangential and normal parts as

$$(\bar{\nabla}_X J)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y. \tag{1.2.16}$$

By making use of equations (1.2.2), (1.2.3), (1.2.4), (1.2.10) and (1.2.11), we may obtain that

$$\mathcal{P}_X Y = (\bar{\nabla}_X T)Y - A_{NV}X - th(X, Y), \tag{1.2.17}$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X N)Y + h(X, TV) - nh(X, Y). \tag{1.2.18}$$

Similarly for $N \in T^{\perp}M$, denoting by $\mathcal{P}_X N$ and $\mathcal{Q}_X N$ respectively the tangential and normal parts of $(\bar{\nabla}_X J)N$, we find that

$$\mathcal{P}_X N = (\bar{\nabla}_X t)N + TA_N X - A_{nN}X, \tag{1.2.19}$$

$$\mathcal{Q}_X N = (\bar{\nabla}_X n)N + h(tN, X) + NA_N X. \tag{1.2.20}$$

The following properties of $\mathcal{P}$ and $\mathcal{Q}$ are used in our subsequent discussion and can be verified through straightforward computations:

$(\mathbf{p_1})$ (i) $\quad \mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W,$

(ii) $\quad \mathcal{Q}_{X+Y}W = \mathcal{Q}_X W + \mathcal{Q}_Y W.$

$$(\mathbf{p_2})\quad \text{(i)}\qquad \mathcal{P}_X(Y+W) = \mathcal{P}_XY + \mathcal{P}_XW,$$

$$\text{(ii)}\qquad \mathcal{Q}_X(Y+W) = \mathcal{Q}_XY + \mathcal{Q}_XW.$$

$$(\mathbf{p_3})\quad \text{(i)}\qquad g(\mathcal{P}_XY,\ W) = -g(Y, \mathcal{P}_XW),$$

$$\text{(ii)}\qquad g(\mathcal{Q}_XY,\ N) = -g(Y, \mathcal{P}_XN).$$

$$(\mathbf{p_4})\qquad \mathcal{P}_XJY + \mathcal{Q}_XJY = -J(\mathcal{P}_XY + \mathcal{Q}_XY).$$

However, on a submanifold M of an almost contact manifold $(\bar{M}, \phi, \xi, \eta)$, for any $X \in TM$, we also denote the tangential and normal components of ϕX by TX and NX respectively. Similarly the tangential and normal components of ϕN, for $N \in T^{\perp}M$ are denoted by tN and nN respectively i.e., we write

$$\phi X = TX + NX \tag{1.2.21}$$

and

$$\phi N = tN + nN. \tag{1.2.22}$$

The covariant differentiation of the operators T, N, t and n are defined in the same manner as in equations (1.2.12) to (1.2.15).

In view of equation (1.2.12), the Nijenhuis tensor S_T of T is also expressed as

$$S_T(X,Y) = (\bar{\nabla}_{TX}T)Y - (\bar{\nabla}_{TV}T)X + T(\bar{\nabla}_YT)X - T(\bar{\nabla}_XT)Y, \tag{1.2.23}$$

for all $X, Y \in TM$.

1.3. Submanifolds of Almost Hermitian, Almost Contact Metric Manifolds

Since the present thesis deals with the submanifold theory, we recall, in this section, some of the basic notions and results about submanifolds of almost Hermitian, almost contact metric manifolds, which are relevant for the subsequent chapters.

On an almost Hermitian manifold $\bar{M}$,

$$g(JX, JY) = g(X, Y),$$

for all vector fields X, Y on $\bar{M}$. In other words,

$$g(JX, X) = 0,$$

i.e., $JX \perp X$ for each vector field X on $\bar{M}$. Hence for a submanifold M of $\bar{M}$ if $X \in T_xM$, JX may or may not belong to T_xM. Thus the action of the almost complex structure J on the tangent vectors of the submanifold of the almost Hermitian manifold gives rise to its classification into invariant and anti-invariant submanifolds. These submanifolds are defined as follows:

DEFINITION 1.3.1. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be *invariant* (or *holomorphic*) submanifold if

$$J(T_xM) = T_xM, \text{ for all } x \in M.$$

DEFINITION 1.3.2. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a *totally real* (or *anti-holomorphic*) if

$$J(T_xM) \subseteq T_x^{\perp}M, \text{ for all } x \in M.$$

In 1978, A. Bejancu (cf. [3], [4]) considered a new class of submanifolds of an almost Hermitian manifold of which the above classes namely invariant and totally real submanifolds are particular cases and named these submanifolds as CR-submanifolds. That is, a CR-submanifold provides a single setting to study the invariant and anti-invariant submanifolds of an almost Hermitian manifold. Formally a CR-submanifold of an almost Hermitian manifold is defined as follows:

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g and M, a Riemannian submanifold immersed in $\bar{M}$. At each point $x \in M$, let D_x be the maximal holomorphic subspace of the tangent space T_xM i.e.,

$$D_x = T_xM \cap JT_xM.$$

If the dimension of D_x is same for all $x \in M$, we get a holomorphic distribution D on M.

DEFINITION 1.3.3 [3]. A submanifold is said to be a *CR-submanifold* of an almost Hermitian manifold $\bar{M}$ if there exists on M a C^{∞}-holomorphic distribution D such that its orthogonal complementary distribution $D^{\perp}$ is totally real i.e., $JD_x^{\perp} \subseteq T_x^{\perp}M$, for all $x \in M$.

For a CR-submanifold M of an almost Hermitian manifold $\bar{M}$, the tangent bundle TM and normal bundle $T^{\perp}M$ are decomposed as

$$TM = D \oplus D^{\perp}$$

and

$$T^{\perp}M = JD^{\perp} \oplus \nu,$$

where ν is the orthogonal complementary distribution to $JD^{\perp}$ and is invariant under J.

It is easy to observe that $TX \in D$ and $NX \in JD^{\perp}$, $\forall\ X \in TM$. Similarly $tN \in D^{\perp}$ and $nN \in \nu$, for all $N \in T^{\perp}M$.

DEFINITION 1.3.4. A CR-submanifold M is said to be *proper* if neither D nor $D^{\perp} = 0$. Obviously if $D = 0$, then M is *totally real submanifold* and if $D^{\perp} = 0$, then M is a *holomorphic submanifold*.

DEFINITION 1.3.5. A CR-submanifold is called an *anti-holomorphic* submanifold if $JD_p^{\perp} = T_p^{\perp}M$, for all $p \in M$.

DEFINITION 1.3.6. A CR-submanifold M is called a *CR-product* if it is locally a Riemannian product of a holomorphic submanifold N_T and a totally real submanifold $N_{\perp}$.

From the above definition it follows that on a CR-product submanifold, the leaves of D and $D^{\perp}$ are totally geodesic in M and vice-versa.

We know that the leaves of a distribution D on a manifold M are totally geodesic in M if and only if $\nabla_X Y \in D$ for all $X, Y \in D$. Thus in the setting of CR-submanifold of an almost Hermitian manifold, the leaves of D are totally geodesic in M if and only if

$$\nabla_X Y \in D \tag{1.3.1}$$

for all $X, Y \in D$, which is equivalent to the condition

$$\nabla_X W \in D^{\perp} \tag{1.3.2}$$

for $X \in D$ and $Z, W \in D^{\perp}$. Similarly, for the totally geodesicness of the leaves of $D^{\perp}$, the conditions

$$\nabla_Z W \in D^{\perp} \tag{1.3.3}$$

and

$$\nabla_Z X \in D \tag{1.3.4}$$

for X in D and Z, W in $D^{\perp}$, are equivalent.

From the definition (1.3.6), a CR-submanifold is a CR-product if and only if the leaves of D and $D^{\perp}$ are totally geodesic in M. Hence combining (1.3.1) and

(1.3.4), we conclude that a CR-submanifold of an almost Hermitian manifold is a CR-product if and only if

$$\nabla_U X \in D \tag{1.3.5}$$

for all $X \in TM$. Similarly, combining (1.3.2) and (1.3.3), it is concluded that a CR-submanifold is a CR-product if and only if

$$\nabla_U Z \in D^\perp. \tag{1.3.6}$$

Since,

$$g(\nabla_U X, Z) = 0 \iff g(X, \nabla_U Z) = 0.$$

Therefore the conditions (1.3.5) and (1.3.6) are equivalent.

DEFINITION 1.3.7 [22]. A CR-submanifold M in a Kaehler manifold is said to be *mixed foliate* if

(i) D is integrable and

(ii) $h(D, D^\perp) = 0$.

If the ambient space $\bar{M}$ of a CR-submanifold M is a Kaehler manifold, then the integrability conditions for the distributions D and $D^\perp$ obtained by A. Bejancu and B.Y. Chen (cf. [3], [4], [22], [23]) are:

THEOREM 1.3.1 [22]. *The holomorphic distribution D on a CR-submanifold of a Kaehler manifold $\bar{M}$ is integrable if and only if*

$$g(h(X, JY), JZ) = g(h(JX, Y), JZ),$$

for any vector fields $X, Y \in D$ and $Z \in D^\perp$.

THEOREM 1.3.2 [22]. *The totally real distribution $D^\perp$ on a CR-submanifold in a Kaehler manifold is integrable.*

With regard to the geometry of D and those of $D^\perp$, we have

THEOREM 1.3.3 [22]. *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then*

(i) *The leaves of D are totally geodesic in M if and only if*

$$g(h(D, D), JD^\perp) = 0,$$

(ii) *The leaves of* $D^\perp$ *are totally geodesic in* M *if and only if*

$$g(h(D, D^\perp), JD^\perp) = 0.$$

If a CR-submanifold M is locally a Riemannian product of the leaves of D and $D^\perp$, then M is said to be a *CR-product* submanifold. B.Y. Chen obtained the following characterization for M to be a CR-product in a Kaehler manifold:

THEOREM 1.3.4 [22]. *A CR-submanifold* M *in a Kaehler manifold* $\bar{M}$ *is a CR-product if and only if*

$$A_{JD^\perp} D = 0.$$

Another equivalent condition for CR-product is obtained in the form of the following theorem:

THEOREM 1.3.5 [22]. *A CR-submanifold of a Kaehler manifold* $\bar{M}$ *is a CR-product if and only if*

$$\bar{\nabla} T = 0.$$

The condition $\bar{\nabla} T = 0$ was further extended by Chen [23] for general Riemannian submanifolds of almost Hermitian manifolds and is given as:

THEOREM 1.3.6 [25]. *Let* M *be a submanifold of an almost Hermitian manifold* $\bar{M}$. *Then* $\bar{\nabla} T = 0$ *if and only if* M *is locally the Riemannian product* $M_1 \times M_2 \times \cdots \times M_k$. *Where each* M_i *is either a Kaehler submanifold, a totally real submanifold or a Kaehlerian slant submanifold.*

We now define slant submanifolds of almost Hermitian manifolds.

Let M be an n-dimensional submanifold of an almost Hermitian manifold $\bar{M}$ with almost complex structure J and Hermitian metric g. To extend the notion of CR-submanifold to slant submanifold it is imperative to relook at the notions of holomorphic and totally real submanifolds. To this end, we can say that a submanifold of an almost Hermitian manifold is a holomorphic submanifold if and only if every nonzero vector X tangent to M at $x \in M$, the angle between JX and T_xM is equal to zero. Similarly M is totally real if and only if for every non zero vector $X \in T_xM$ the angle between JX and T_xM is $\pi/2$. As before, for any vector field X tangent to M, we put

$$JX = TX + NX,$$

where TX and NX denote respectively the tangential and normal components of JX.

For each non zero vector X tangent to M at x, the angle $\theta(X)$ between JX and T_xM is called the *Wirtinger angle* of X. It is easy to observe that Wirtinger angle $\theta(X)$ of X is infact the angle between JX and TX. An immersion $f : M \to \bar{M}$ is called a *general slant immersion* if the angle $\theta(X)$ is constant (i.e., independent of the choice of $x \in M$ and $X \in T_xM$). Holomorphic and totally real immersions are slant immersions with Wirtinger angle θ equal to 0 and $\pi/2$ respectively.

A general slant immersion which is not holomorphic is called a *slant immersion*. The Wirtinger angle θ is called the *slant angle* of the slant immersion. If M is a slant submanifold of an almost Hermitian manifold $\bar{M}$, then we have (cf. [26])

$$T^2 = -(\cos^2\theta)I, \tag{1.3.7}$$

where θ is the Wirtinger angle of M in $\bar{M}$. From (1.3.7) it can be easily seen that

$$g(TX, TY) = \cos^2\theta g(X, Y), \tag{1.3.8}$$

$$g(NX, NY) = \sin^2\theta g(X, Y), \tag{1.3.9}$$

for $X,\ Y$ tangent to M.

Various aspects of slant submanifolds of a Kaehler manifold have been investigated by B. Y. Chen [25], Chen and Tazawa (c.f., [27],[28]), Maeda and ohnita [53]. Before introducing slant submanifolds of contact manifolds, we first recall the notions of invariant and anti-invariant submanifolds of contact manifolds.

A submanifold M of an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ is said to be *invariant submanifold* if $\phi T_xM \subseteq T_xM,\ \forall\ x \in M$. And a submanifold M of an almost contact metric manifold is said to be *anti-invariant submanifold* if $\phi T_xM \subseteq T_x^{\perp}M,\ \forall\ x \in M$.

The study of the differential geometry of semi-invariant or contact CR-submanifolds, as a generalization of invariant and anti-invariant submanifolds of an almost contact metric manifold was initiated by A. Bejancu and N. Papaghiuc [7] and was followed by several geometers (cf. [48], [57], [78] etc.).

Throughout, for a submanifold M of an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$, we assume that the structure vector field ξ is tangential to the submanifold M and therefore the tangent bundle TM is decomposed as

$$TM = \mathcal{D} \oplus \langle \xi \rangle,$$

where $\langle \xi \rangle$ is the one dimensional distribution on M spanned by structure vector field ξ.

DEFINITION 1.3.8. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a *contact CR-submanifold* (or *semi-invariant submanifold*) if there exists a pair of orthogonal distributions $(D,\ D^{\perp})$ satisfying the conditions

(i) $TM = D \oplus D^{\perp} \oplus \langle \xi \rangle$,

(ii) The distribution D is invariant by ϕ i.e., $\phi D_x = D_x,\ \ x \in M$,

(iii) The distribution $D^{\perp}$ is anti-invariant i.e., $\phi D_x^{\perp} \subset T_x^{\perp} M,\ \ x \in M$,

where TM and $T^{\perp}M$ denote the tangent and normal bundle to M respectively. It follows that the normal bundle splits as

$$T^{\perp}M = \phi D^{\perp} \oplus \nu, \tag{1.3.10}$$

where ν is the invariant sub-bundle of $T^{\perp}M$ by ϕ. If $D = \{0\}$ (resp. $D^{\perp} = \{0\}$), then M is said to be an *anti-invariant* (resp. *invariant*) submanifold. We say that M is a proper contact CR-submanifold (or semi-invariant submanifold) if it is neither invariant nor anti-invariant.

REMARK 1.3.1. Let M be a semi-invariant submanifold of an almost contact metric manifold $\bar{M}$ then, we have

(i) For any $X \in TM$, $TX \in D$ and $NX \in \phi D^{\perp}$,

(ii) For any $N \in T^{\perp}M$, $tN \in D^{\perp}$ and $nN \in \nu$.

The notion of slant submanifolds is extended to the setting of almost contact metric manifolds by A. Lotta [50]. Recently, J.L. Cabrerizo et al. [17] and A. Carriazo et al. [18] studied these submanifolds in a more specialized setting of Sasakian and K-contact manifolds.

Let $\bar{M}$ be an almost contact metric manifold with structure (ϕ, ξ, η, g) and M an immersed submanifold of $\bar{M}$. For any $x \in M$ and $X \in T_xM$, if the vectors X and ξ are linearly independent, the angle $\theta(X) \in [0, \pi/2]$, between ϕX and T_xM is well defined. If $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_xM$, then M is said to be *slant* in $\bar{M}$. The constant angle $\theta(X)$ is then called the *slant* *angle* of M in $\bar{M}$, which in short, we denote by sla(M).

In particular, anti-invariant submanifolds of a Sasakian manifold are slant submanifolds with slant angle equal to $\pi/2$. A. Lotta [50] proved that slant submanifolds of a Sasakian manifold with slant angle equal to zero are invariant submanifolds. This fact is not trivial, since by definition an invariant submanifold must have odd dimension and the characteristic vector field ξ of the ambient manifold is required to be tangent to the submanifold. More generally, Lotta showed that these properties are always satisfied by any non anti-invariant slant submanifold of a contact metric manifold.

It can be seen with the help of (1.1.11) and (1.2.21) that

$$g(\phi X, Y) = -g(X, \phi Y) \tag{1.3.11}$$

i.e.,

$$g(TX, Y) = -g(X, TY), \tag{1.3.12}$$

which implies that

$$g(T^2X, Y) = g(X, T^2Y), \tag{1.3.13}$$

for all $X, Y \in TM$ that is T^2 (which will be denote by Q) is a self adjoint endomorphism on TM. it is also easy to observe that, the eigen value of Q belong to $[-1, 0]$ and that each non-vanishing eigen value of Q has even multiplicity.

The covariant derivative of endomorphism Q is defined as:

$$(\bar{\nabla}_X Q)Y = \nabla_X QY - Q\nabla_X Y, \tag{1.3.14}$$

for all $X, Y \in TM$.

The following are some important results of A. Lotta [50], which will be used in the subsequent chapters.

THEOREM 1.3.7 [50]. *Let $x \in M$ and $X \in T_xM$ be an eigen vector of Q with eigen value $\lambda(X)$. Suppose that X is linearly independent from ξ_x. Then*

$$\cos\theta(X) = \sqrt{-\lambda(X)}\frac{|X|}{|\phi X|}. \tag{1.3.15}$$

THEOREM 1.3.8 [50]. *Suppose that M is slant in $\bar{M}$ and $\theta = Sla(M) \neq \pi/2$. Then, for any $x \in M$, Q admits the real number $-\cos^2\theta$ as the only non vanishing eigen value. Moreover the related eigen space H satisfies $H \subset D$ where*

$D = span(\xi_x)^{\perp} \subset T_xM$.

THEOREM 1.3.9 [50]. *Let M be a slant submanifold of dimension m of an almost contact metric manifold $\bar{M}$ and suppose $sla(M) \neq \pi/2$. Then we have*

$$m \text{ is even} \Longleftrightarrow \xi \text{is orthogonal to } M,$$

$$m \text{ is odd} \Longleftrightarrow \xi \text{is tangent to } M.$$

THEOREM 1.3.10 [50]. *Let M be an immersed submanifold of a contact metric manifold $\bar{M}$. If ξ is orthogonal to M, then M is anti-invariant.*

The following theorem provides a useful characterization for the existence of a slant distribution on a contact metric manifold.

THEOREM 1.3.11 [17]. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{1.3.16}$$

Furthermore, in such case, if θ is slant angle, then $\lambda = \cos^2\theta$.

Following relations are straight forward consequence of equation (1.3.16)

$$\left.\begin{aligned} g(TU, TV) &= \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)] \\ g(NU, NV) &= \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)] \end{aligned}\right\} \tag{1.3.17}$$

for any X, Y tangent to M.

Recently, N. Papaghiuc [64] introduced the notion of semi-slant submanifolds of an almost Hermitian manifold as a generalized version of CR-submanifolds. J.L. Cabrerizo et al. [16] studied this class of submanifolds in an almost contact metric manifold. Therefore, semi-invariant and contact slant submanifolds, become particular cases of these submanifolds.

DEFINITION 1.3.9. Given a submanifold M, isometrically immersed in almost Hermitian manifold $(\bar{M}, J, g)$, a differentiable distribution D on M is said to be a *slant distribution* if for any non zero vector $X \in D_x$, $x \in M$, the angle between JX and the vector space D_x is constant i.e., it is independent of the choice of $x \in M$ and $X \in D_x$. This constant angle is called the slant angle of the slant distribution D. M is said to be a *Semi-slant submanifold* if there exist on M two differentiable distributions D and D_θ such that

$$TM = D \oplus D_\theta,$$

where D is a holomorphic distribution i.e., $JD = D$ and D_θ is a slant distribution with the angle $\theta \neq 0$.

Now, if $\bar{M}$ is an almost contact metric manifold with contact metric structure (ϕ, ξ, η, g), then a submanifold M of $\bar{M}$ is said to be *semi-slant submanifold* if there exist two orthogonal distributions D and D_θ on M such that

(i) TM admits the orthogonal direct decomposition $TM = D \oplus D_\theta \oplus \langle\xi\rangle$,

(ii) The distribution D is invariant under ϕ i.e., $\phi D \subseteq D$,

(iii) The distribution D_θ is slant with slant angle $\theta \neq 0$.

If we denote the dimension of D by d_1 and D_θ by d_2 then we find the following cases

(a) If $d_2 = 0$, then M is an invariant submanifold.

(b) If $d_1 = 0$ and $\theta = \pi/2$, then M is an anti-invariant submanifold.

(c) If $d_1 = 0$ and $\theta \neq \pi/2$, then M is a proper slant submanifold with slant angle θ.

Similarly, we define anti-slant submanifold as follows:

DEFINITION 1.3.10. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be *anti-slant submanifold* if there exist two orthogonal distributions D_1 and D_2 such that

(i) $TM = D_1 \oplus D_2 \oplus \langle\xi\rangle$,

(ii) D_1 is anti-invariant i.e., $\phi D_1 \subseteq T^\perp M$,

(iii) D_2 is slant with slant angle $\theta \neq \pi/2$.

Chapter II

SOME SLANT AND BI-SLANT SUBMANIFOLDS OF ALMOST HERMITIAN MANIFOLDS

SOME SLANT AND BI-SLANT SUBMANIFOLDS OF ALMOST HERMITIAN MANIFOLDS

2.1. Introduction

The notion of slant submanifolds of an almost Hermitian manifold was introduced by B.Y. Chen (cf. [25,26]). These submanifolds are the generalization of both holomorphic and totally real submanifolds of an almost Hermitian manifold with an almost complex structure J. A nearly Kaehler structure on a manifold provides an interesting study with differential geometric point of view (cf. [36], [37]). Consequently, the study of submanifolds of a nearly Kaehler manifold vis-a-vis that of a Kaehler manifold assumes significance in general, then the study of totally umbilical CR-submanifolds of a nearly Kaehler manifold has been studied in [43]. Later on, V.A. Khan and M.A. Khan extended this study to the semi-slant submanifold of a nearly Kaehler manifold [45]. Recently, B. Sahin proved every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic. In this chapter we have extended the results of B. Sahin [66] in this more general setting of nearly Kaehler manifolds.

Bi-slant submanifolds of an almost Hermitian manifold were introduced as a natural generalization of semi-slant submanifolds by A. Carriazo [18]. The class of bi-slant submanifolds includes complex, totally real and CR-submanifolds. One of the most important bi-slant submanifolds of an almost Hermitian manifold is anti-slant submanifold which is studied by A. Carriazo [18]. Later, he called these submanifolds as pseudo slant submanifolds (*also known as Hemi-slant submanifolds* [67]). We have extended the study of these submanifolds to the setting of nearly Kaehler manifolds and have obtained some basic and interesting results, some of which are incorporated in Section 2.3 and a classification theorem is proved for these submanifolds.

2.2. Totally umbilical propar Slant submanifolds of Nearly Kaehler Manifolds

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This section deals with totally umbilical proper slant submanifolds of nearly Kaehler manifolds. Throughout the section, we assume $\bar{M}$ to be a nearly Kaehler manifold and M a slant submanifold of $\bar{M}$. Let $\bar{M}$ be a Riemannian manifold with almost complex structure J and Hermitian metric g satisfying

$$(a) \quad J^2 = -I, \quad (b) \quad g(JX, JY) = g(X, Y) \tag{2.2.1}$$

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ is the tangent bundle of $\bar{M}$. If the almost complex structure J satisfies

$$(\bar{\nabla}_X J)X = 0 \tag{2.2.2}$$

for any $X \in T\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection on $T\bar{M}$ then $\bar{M}$ is said to have a *nearly Kaehler structure*. In this case $\bar{M}$ is a *nearly Kaehler* manifold. Equation (2.2.2) is equivalent to $(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0$. Obviously, every Kaehler manifold is a nearly Kaehler. The geometric meaning of the nearly Kaehler condition is that geodesic are holomorphically planer curves. So far as non Kaehler nearly Kaehler manifolds are concerned, one of the most prominent example is that of S^6 on a 6-dimensional unit sphere S^6 [36].

For each non-zero vector X tangent to M at x, the angle $\theta(X)$ between JX and T_xM is called *Wirtinger angle* of X. It is easy to observe that Wirtinger angle $\theta(X)$ of X is in fact, the angle between JX and TX. An immersion $f : M \longrightarrow \bar{M}$ is called a *general slant immersion* if the angle $\theta(X)$ is constant (i.e., independent of the choice of $x \in M$ and $X \in T_xM$). Holomorphic and totally real immersions with Wirtinger angle $\theta = 0$ and $\pi/2$. A general slant immersion which is not holomorphic and totally real is called proper slant immersion.

A submanifold M of almost Hermitian manifold $\bar{M}$ is slant submanifold of $\bar{M}$ if and only if [25]

$$T^2 = \lambda I \tag{2.2.3}$$

for some real number $\lambda \in [-1, 0]$, where I is the Identity transformation of the tangent bundle TM of the submanifold M. Moreover, if M is a slant submanifold and θ is the slant angle of M, then $\lambda = -\cos^2\theta$. Hence, for a slant submanifold we have

$$g(TX, TX) = \cos^2\theta g(X, Y) \tag{2.2.4}$$

$$g(NX, NY) = \sin^2\theta g(X, Y) \tag{2.2.5}$$

for any $X, Y \in TM$.

Let M be a proper slant submanifold of an almost Hermitian manifold $\bar{M}$, then NT_xM is a subspace of $T_x^{\perp}M$. Thus for any $x \in M$, we decompose the normal space as

$$T^{\perp}M = NTM \oplus \mu$$

where μ is an invariant normal subbundle orthogonal to NTM.

On a submanifold M of a nearly Kaehler manifold $\bar{M}$, it follows from (2.2.2) that

$$(a)\ \ \mathcal{P}_X Y + \mathcal{P}_Y X = 0, \qquad (b)\ \ \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0 \tag{2.2.6}$$

for any $X, Y \in TM$.

THEOREM 2.2.1. *Let M be a totally umbilical proper slant submanifold of a nearly Kaehler manifold $\bar{M}$. Then the following conditions are equivalent*

(i) The submanifold M has a nearly Kaehler structure (T, g)

(ii) $H \in \mu$

where H is the mean curvature vector of M.

Proof: As M is a totally umbilical proper slant submanifold then for any $X, Y \in TM$, we have

$$h(TX, TY) = g(TX, TY)H.$$

Using (1.2.2) and (2.2.4), we obtain

$$\bar{\nabla}_{TX}TY - \nabla_{TX}TY = \cos^2\theta g(X, Y)H.$$

Then from equation (1.2.10), we get

$$\bar{\nabla}_{TX}JY - \bar{\nabla}_{TX}NY - \nabla_{TX}TY = \cos^2\theta g(X, Y)H.$$

Thus on using covariant derivative of J, we arrive at

$$(\bar{\nabla}_{TX}J)Y + J\bar{\nabla}_{TX}Y - \bar{\nabla}_{TX}NY - \nabla_{TX}TY = \cos^2\theta g(X, Y)H.$$

Using (1.2.2) and (1.2.3), we get

$$\begin{aligned} \cos^2\theta g(X, Y)H = {} & (\bar{\nabla}_{TX}J)Y + J(\nabla_{TX}Y + h(TX, Y)) \\ & + A_{NY}TX - \nabla^{\perp}_{TX}NY - \nabla_{TX}TY. \end{aligned} \tag{2.2.7}$$

Interchanging Y with TY in (2.2.7), we obtain

$$\cos^2\theta g(X, TY)H = (\bar{\nabla}_{TX}J)TY + J(\nabla_{TX}TY + h(TX, TY))$$

$$+ A_{NTY}TX - \nabla^{\perp}_{TX}NTY - \nabla_{TX}T^2Y.$$

Then taking account of equations (1.2.5), (2.2.3) and (2.2.4) we deduce that

$$\cos^2\theta g(X,TY)H = (\bar{\nabla}_{TX}J)TY + J\nabla_{TX}TY + \cos^2\theta g(X,Y)JH$$
$$+ A_{NTY}TX - \nabla^{\perp}_{TX}NTY - \cos^2\theta\nabla_{TX}Y. \quad (2.2.8)$$

Similarly, we can obtain

$$\cos^2\theta g(Y,TX)H = (\bar{\nabla}_{TY}J)TX + J\nabla_{TY}TX + \cos^2\theta g(X,Y)JH$$
$$+ A_{NTX}TY - \nabla^{\perp}_{TY}NTX - \cos^2\theta\nabla_{TY}X. \quad (2.2.9)$$

Adding the equations (2.2.8) and (2.2.9), we get

$$\cos^2\theta\{g(Y,TX)+g(X,TY)\}H = (\bar{\nabla}_{TY}J)TX+(\bar{\nabla}_{TX}J)TY$$
$$+ J(\nabla_{TY}TX + \nabla_{TX}TY) + 2\cos^2\theta g(X,Y)JH$$
$$+ A_{NTX}TY + A_{NTY}TX - \nabla^{\perp}_{TY}NTX - \nabla^{\perp}_{TX}NTY$$
$$- \cos^2\theta(\nabla_{TY}X + \nabla_{TX}Y). \quad (2.2.10)$$

Then from (2.2.1) and (1.1.8), we obtain

$$0 = J(\nabla_{TX}TY + \nabla_{TY}TX) + A_{NTX}TY + A_{NTY}TX - \nabla^{\perp}_{TY}NTX$$
$$- \nabla^{\perp}_{TX}NTY + 2\cos^2\theta g(X,Y)JH - \cos^2\theta(\nabla_{TX}Y + \nabla_{TY}X).$$

Using (1.2.10) and then equating the tangential components, we get

$$0 = T(\nabla_{TX}TY + \nabla_{TY}TX) + 2\cos^2\theta g(X,Y)tH$$
$$+ A_{NTY}TX + A_{NTX}TY - \cos^2\theta(\nabla_{TX}Y + \nabla_{TY}X). \quad (2.2.11)$$

As $\theta \neq \pi/2$, interchanging X by TX and Y by TY in (2.2.11) we obtain that

$$0 = \cos^4\theta\{T(\nabla_XY + \nabla_YX) + 2g(X,Y)tH + A_{NTY}TX$$
$$+A_{NTX}TY - \nabla_XTY - \nabla_YTX\}. \quad (2.2.12)$$

On the other hand, for any $U \in TM$, we have

$$g(A_{NTY}TX, U) = g(h(TX,U), NTY).$$

Using (2.2.1), we get

$$g(A_{NTY}TX, U) = -g(Jh(TX,U), TY).$$

Then from equations (1.2.5) and (1.2.10), we obtain that

$$g(A_{NTY}TX, U) = -g(TX, U)g(tH, TY).$$

That is

$$A_{NTY}TX = -TXg(tH, TY). \tag{2.2.13}$$

Using this fact the equation (2.2.12) takes the form

$$0 = \cos^4\theta\{T(\nabla_X Y + \nabla_Y X) + 2g(X, Y)tH - TXg(tH, TY)$$
$$-TYg(tH, TX) - \nabla_X TY - \nabla_Y TX\}.$$

In particular, we obtain from the above equation that

$$0 = 2\cos^4\theta\{T\nabla_X X + g(X, X)tH - TXg(tH, TX) - \nabla_X TX\}. \tag{2.2.14}$$

Since $\theta \neq \pi/2$, then from (2.2.14), it follows that

$$\nabla_X TX - T\nabla_X X = \|X\|^2 tH - TXg(tH, TX). \tag{2.2.15}$$

From the above equation (2.2.15) it can be seen as if $H \in \mu$ and then $(\bar{\nabla}_X T)X = 0$ and vice-versa. This completes the proof of the theorem.

THEOREM 2.2.2. *A totally umbilical proper slant submanifold M of a nearly Kaehler manifold $\bar{M}$ is totally geodesic if n is parallel and $H \in \mu$.*

Proof: For any $X, Y \in TM$, we have

$$(\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y.$$

Using (1.2.2) and (1.2.10) we get

$$\mathcal{P}_X Y + \mathcal{Q}_X Y = \bar{\nabla}_X TY + \bar{\nabla}_X NY - T\nabla_X Y - N\nabla_X Y - Jh(X, Y).$$

Again from (1.2.2) and (1.2.3), we obtain

$$\mathcal{P}_X Y + \mathcal{Q}_X Y = \nabla_X TY + h(X, TY) - A_{NY}X$$
$$+ \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y - Jh(X, Y).$$

Then from (1.2.5), we derive

$$\mathcal{P}_X Y + \mathcal{Q}_X Y = \nabla_X TY + g(X, TY)H - A_{NY}X$$
$$+ \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y - g(X, Y)JH. \tag{2.2.16}$$

Taking the product in (2.2.16) with JH and using Theorem 2.2.1 (equivalent conditions), we deduce that

$$g(\mathcal{Q}_X Y, JH) = g(X, TY)g(H, JH) + g(\nabla^{\perp}_X NY, JH) \\ - g(N\nabla_X Y, JH) - g(X, Y)g(JH, JH).$$

Using (2.2.1) and again Theorem 2.2.1 (equivalent conditions), we obtain

$$g(\mathcal{Q}_X Y, JH) = g(\nabla^{\perp}_X NY, JH) - g(X, Y)\|H\|^2.$$

Thus by (1.2.3), we get

$$g(\mathcal{Q}_X Y, JH) = g(\bar{\nabla}_X NY, JH) - g(X, Y)\|H\|^2. \tag{2.2.17}$$

Similarly, we can obtain

$$g(\mathcal{Q}_Y X, JH) = g(\bar{\nabla}_Y NX, JH) - g(X, Y)\|H\|^2. \tag{2.2.18}$$

Adding the equations (2.2.17) and (2.2.18) and on applying (2.2.6)(b), we arrive at

$$g(\bar{\nabla}_X NY + \bar{\nabla}_Y NX, JH) = 2g(X, Y)\|H\|^2. \tag{2.2.19}$$

Now for any $X \in TM$, we have

$$\bar{\nabla}_X JH = (\bar{\nabla}_X J)H + J\bar{\nabla}_X H.$$

Using (1.2.3), (1.2.10) and (1.2.11) we get

$$-A_{JH}X + \nabla^{\perp}_X JH = \mathcal{P}_X H + \mathcal{Q}_X H - TA_H X - NA_H X + t\nabla^{\perp}_X H + n\nabla^{\perp}_X H. \tag{2.2.20}$$

Taking the product in (2.2.20) with NY for any $Y \in TM$ and using the fact that $n\nabla^{\perp}_X H \in \mu$ the above equation gives

$$g(\nabla^{\perp}_X JH, NY) = -g(NA_H X, NY) + g(\mathcal{Q}_X H, NY). \tag{2.2.21}$$

Then from (2.2.5), we get

$$g(\nabla^{\perp}_X JH, NY) = -\sin^2\theta g(A_H X, Y) + g(\mathcal{Q}_X H, NY).$$

Since $H \in \mu$ (by Theorem 2.2.1) then using (1.2.3) and (1.2.4), we obtain

$$g(\bar{\nabla}_X NY, JH) = \sin^2\theta g(h(X, Y), H) - g(\mathcal{Q}_X H, NY).$$

Thus from (1.2.5), we derive

$$g(\bar{\nabla}_X NY, JH) = \sin^2\theta g(X, Y)\|H\|^2 - g(\mathcal{Q}_X H, NY). \tag{2.2.22}$$

Similarly, we obtain

$$g(\bar{\nabla}_Y NX, JH) = \sin^2\theta g(X,Y)\|H\|^2 - g(\mathcal{Q}_Y H, NX). \tag{2.2.23}$$

On addition of the equations (2.2.22) and (2.2.23), we get

$$\begin{aligned} g(\bar{\nabla}_X NY + \bar{\nabla}_Y NX, JH) = {} & 2\sin^2\theta g(X,Y)\|H\|^2 \\ & - g(\mathcal{Q}_X H, NY) - g(\mathcal{Q}_Y H, NX). \end{aligned} \tag{2.2.24}$$

From equation (2.2.19) and (2.2.24) we derive

$$2g(X,Y)\|H\|^2 = 2\sin^2\theta g(X,Y)\|H\|^2 - g(\mathcal{Q}_X H, NY) - g(\mathcal{Q}_Y H, NX).$$

That is,

$$2\cos^2\theta g(X,Y)\|H\|^2 + g(\mathcal{Q}_X H, NY) + g(\mathcal{Q}_Y H, NX) = 0. \tag{2.2.25}$$

Now, from equation (1.2.21) we have

$$g(\mathcal{Q}_X H, NY) = g((\bar{\nabla}_X n)H, NY) + g(h(tH, X), NY) + g(NA_H X, NY),$$

for any $X, Y \in TM$. Using the assumption that n is parallel and Theorem 2.2.1 (equivalent conditions), then in view of the equation (2.2.5), we obtain that

$$g(\mathcal{Q}_X H, NY) = \sin^2\theta g(A_H X, Y). \tag{2.2.26}$$

Then from (1.2.4) and (1.2.5), we get

$$g(\mathcal{Q}_X H, NY) = \sin^2\theta g(X,Y)\|H\|^2. \tag{2.2.27}$$

Using this fact in equation (2.2.25), we obtain

$$g(X,Y)\|H\|^2 = 0. \tag{2.2.28}$$

It follows from (2.2.28) that $H = 0$, i.e., M is totally geodesic. This proves the theorem completely.

2.3. Pseudo-slant submanifolds of Nearly Kaehler Manifolds

Recently, V.A. Khan and M.A. Khan [46] studied pseudo-slant submanifolds of a Sasakian manifold. The purpose of the present section is to study totally umbilical pseudo-slant submanifolds of nearly Kaehler manifolds.

Before dealing with the pseudo-slant submanifolds of $\bar{M}$, we first recall the definition.

A distribution D on a submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be *slant distribution* if for each $U \in D_x$. The angle θ between JU and D_x is constant i.e., independent of $x \in M$ and $U \in D_x$. A submanifold M of $\bar{M}$ is said to be *slant submanifold* if the tangent bundle TM of M is slant.

A *pseudo-slant submanifold* M of $\bar{M}$ is a submanifold which admits two orthogonal complementary distributions D_θ and $D^\perp$ such that D_θ is slant and $D^\perp$ is totally real i.e., $JD^\perp \subseteq T^\perp M$ and D_θ is slant with slant angle $\theta \neq \pi/2$.

Let TM denote the tangent bundle on M. If μ is the invariant subspace of the normal bundle $T^\perp M$. Then, in the case of pseudo-slant submanifold the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = \mu \oplus ND_\theta \oplus ND^\perp.$$

First we shall discuss the integrability of involved distributions.

PROPOSITION 2.3.1. *Let M be a pseudo-slant submanifold of a nearly Kaehler manifold then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$A_{JY}X - A_{JX}Y + 2\mathcal{P}_X Y \in D^\perp$$

for all $X, Y \in D^\perp$.

Proof. For any $X, Y \in D^\perp$ and $Z \in D_\theta$, then by (1.2.10), we have

$$g([X,Y],TZ) = g(\bar{\nabla}_Y JX - (\bar{\nabla}_Y J)X - \bar{\nabla}_X JY + (\bar{\nabla}_X J)Y, Z)$$

Using (1.2.2) and (1.2.3) the above equation reduced to

$$g([X,Y],TZ) = g(A_{JY}X - A_{JX}Y + 2\mathcal{P}_X Y, Z). \tag{2.3.1}$$

Thus the assertion follows from (2.3.1).

PROPOSITION 2.3.2. *Let M be a pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$, then the distribution D_θ is integrable if and only if*

$$h(Z,TW) - h(W,TZ) + \nabla^\perp_Z NW - \nabla^\perp_W NZ - 2\mathcal{Q}_Z W \in ND_\theta.$$

Proof. For any $X \in D^\perp$ and $Z, W \in D_\theta$, then from equations (1.2.10), (1.2.2), (1.2.3) and the normal components of $(\bar{\nabla}_Z J)W$ we have

$$g(N[Z,W], NX) = g(h(Z,TW) - h(W,TZ) \\ + \nabla^\perp_Z NW - \nabla^\perp_W NZ - 2\mathcal{Q}_Z W, NX). \tag{2.3.2}$$

The result follows from equation (2.3.2) and the fact that ND_θ and $ND^\perp$ are orthogonal.

As M is a totally umbilical pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$. We consider M as a pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$ and D_θ and $D^\perp$ are integrable distributions corresponding to the slant and totally real submanifolds of $\bar{M}$, respectively. Now, for any $U \in TM$, we have $(\bar{\nabla}_U J)U = 0$, using this fact, we get

$$(\bar{\nabla}_Z J)Z = 0 \tag{2.3.3}$$

for any $Z \in D^\perp$. Therefore the tangential and normal parts of the above equation are $\mathcal{P}_Z Z = 0$ and $\mathcal{Q}_Z Z = 0$, respectively. From (1.2.17) and the tangential component of (2.3.3), we obtain

$$\mathcal{P}_Z Z = 0 = (\bar{\nabla}_Z T)Z - A_{NZ}Z - th(Z,Z).$$

That is,

$$(\bar{\nabla}_Z T)Z = A_{NZ}Z + th(Z,Z). \tag{2.3.4}$$

As M is totally umbilical and $Z \in D^\perp$, $TZ = 0$, using this fact and equation (1.2.12), the above equation takes the form

$$T\nabla_Z Z = -g(H, NZ)Z - |Z|^2 tH. \tag{2.3.5}$$

Taking the product in equation (2.3.5) with $W \in D^\perp$, we obtain

$$g(H, NZ)g(Z,W) + |Z|^2 g(tH, W) = 0. \tag{2.3.6}$$

Thus the equation (2.3.6) has a solution if one of the following holds:

(a) $dim D^\perp = 1$

(b) $H \in \mu$.

Now, we are in the position to prove our main result.

THEOREM 2.3.1. *Let M be a totally umbilical pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$. Then atleast one of the following statements is true*

(i) M is totally real submanifold,

(ii) The slant distribution D_θ has a nearly Kaehler structure (T, g),

(iii) M is totally geodesic submanifold when n is parallel,

(iv) $dim D^\perp = 1$.

Proof. If $D_\theta = \{0\}$, then by definition M is totally real which is case (i). If $D_\theta \neq \{0\}$ and $H \in \mu$, then by Theorem 2.2.1, there is a nearly Kaehler structure (T, g) on D_θ, this is case (ii). Moreover, if $H \in \mu$ and n is parallel then by Theorem (2.2.2) M is totally geodesic, which proves case (iii). Finally, if $H \notin \mu$, then equation (2.3.6) has a solution if $dim D^\perp = 1$ which is case (iv). Thus the theorem is proved completely.

Chapter III

SLANT SUBMANIFOLDS IN LORENTZIAN PARACONTACT GEOMETRY

SLANT SUBMANIFOLDS IN LORENTZIAN PARACONTACT GEOMETRY

3.1. Introduction

The study of Lorentzian almost paracontact manifold was initiated by K. Matsumoto (cf. [55],[56]). Later on several authors studied Lorentzian almost paracontact manifolds and their different classes including those of [33].

The term *"Slant submanifolds"* have been introduced by B. Y. Chen in the paper *"Slant Immersions"* ([25]), in which slant submanifold of almost Hermitian manifolds and Kaehler manifolds are studied. According to Chen an immersed submanifold N of an almost Hermitian manifold (M, J, g) is slant if the Wirtinger angle $J(X) \in [0, \frac{\pi}{2}]$ between JX and T_xN has the same value θ for any $x \in N$ and $X \in T_xN$, $X \neq 0$.

The notion of slant submanifold in the sense of Chen is an evident generalization of the important notion of *holomorphic* and *totally real* submanifolds. The idea was followed up by many geometers (cf., [17],[18],[28],[53] etc.). A. Lotta [50] extended the notion of slant immersion to the setting of almost contact metric manifolds and obtained results of fundamental importance. J.L. Cabrerizo et.al. [17] studied the geometry of slant submanifolds in more specialized settings of K-contact and Sasakian manifolds.

As far as LP-Contact geometry is concerned several results can be found in literature on the so called *anti-invariant* and *invariant* submanifolds of various types of LP-Contact manifolds. Our aim in the present chapter is to extend the study of slant submanifold to the setting of LP-Contact manifolds. Furthermore, we have extended these results in the setting of Lorentzian almost contact manifolds.

3.2. Slant immersions in LP-contact Manifolds

[1]Results of this chapter are published in Differential Geometry Dynamical Systems and some results are under consideration in Annals of University of Craiova-Mathematics and Computer Science Series.

Throughout, for a submanifold M of a LP-contact manifold (see equation 1.1.18), we assume that the structure vector field ξ is tangential to the submanifold M and therefore the tangent bundle TM is decomposed as

$$TM = D \oplus < \xi >$$

where $< \xi >$ is the one dimensional distribution on M spanned by the structure vector field ξ, also we consider that $g(X, X) \geq 0 \ \forall \ X \in TM \backslash \langle \xi \rangle$.

Let M be an immersed submanifold of $\bar{M}$. For any $x \in M$ and $X \in T_x M$, if the vector X and ξ are linearly independent, the angle $\theta(X) \in [0, \pi/2]$ between ϕX and $T_x M$ is well defined, if $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_x M$, then M is *slant* in $\bar{M}$. The constant angle $\theta(X)$ is then called the slant angle of M in $\bar{M}$ and which in short we denote by $Sla(M)$.

For any $x \in M$ taking $X \in T_x M$ we put $\phi X = TX + NX$, $TX \in T_x M$ and $NX \in {T_x}^{\perp} M$ thus defining endomorphism $T : T_x M \to T_x M$ whose square T^2 will be denoted by Q. Then tensor field on M of type $(1, 1)$ determined by these endomorphisms will be denoted by the same letters T and Q.

It is easy to prove that for every $x \in M$, $g(TX, Y) = g(X, TY)$, which implies that Q is symmetric. Moreover, in the following steps we can prove that the eigenvalue of Q always belong to $[0, 1]$, for any vector $X \in T_x M - < \xi >$

$$g(QX, X) = \|TX\|^2$$

but,

$$\|TX\| \leq \|\phi X\| = \|X\|$$

therefore,

$$g(QX, X) = \lambda(X) \|X\|^2$$

where $0 \leq \lambda(X) \leq 1$ and λ depends on X. In other words, each eigen value of Q lies in $[0, 1]$.

THEOREM 3.2.1. *Let $x \in M$ and $X \in T_x M$ be an eigenvector of Q with eigenvalue $\lambda(X)$. Suppose that X is linearly independent from ξ_x, then,*

$$\cos \theta(X) = \sqrt{\lambda(X)} \frac{|X|}{|\phi X|} \tag{3.2.1}$$

Proof. We have $|TX|^2 = g(TX, TX) = \lambda(X) |X|^2$, by definition of $\theta(X)$ we get

$$\cos\theta(X) = \frac{g(\phi X, TX)}{|TX||\phi X|} = \sqrt{\lambda(X)}\frac{|X|}{|\phi X|}$$

which proves the theorem.

LEMMA 3.2.1. *Let M be a slant submanifold of $LP-$contact manifold $\bar{M}$ and $\theta = Sla(M) \neq \pi/2$, then Q admits the real number $\cos^2\theta$ as the only non-vanishing eigen value, for any $x \in M$. Moreover, the related eigen space H satisfies $H \subset D$, where $D = Span(\xi_x)^{\perp} \subset T_x\bar{M}$.*

Proof. Let $x \in M$, from equation (3.2.1) $Ker(Q) \neq T_xM$, otherwise $Sla(M) = \pi/2$ which contradict the assumption. So let λ be an arbitrary non-vanishing eigen value of Q and let H be the corresponding eigen space, Now we have $dim(D) = n-1$ and $dim(H)$ is even, If n is even or odd in both cases $dim(H \cap D) \geq 1$. Let $X \in H \cap D$ is a unit vector, then ϕX is also unit vector then from equation (3.2.1)

$$\cos\theta = \sqrt{\lambda(X)},$$

which proves the first assertion. Moreover, for any $X \in H$, formula (3.2.1) yields $|\phi X| = |X|$ which allows us to conclude that $g(X, \xi) = 0$, hence $H \subset D$.

We have noted that, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which neither invariant nor anti-invariant is called a proper slant immersion. In case of invariant submanifold $T = \phi$ and so

$$T^2 = \phi^2 = I + \eta \otimes \xi.$$

While in case of anti-invariant submanifold, $T^2 = 0$. Infact, we have the following general result which characterize slant immersions.

THEOREM 3.2.2. *Let M be a submanifold of an LP-Contact manifold $\bar{M}$ such that $\xi \in TM$. Then, M is slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(I + \eta \otimes \xi) \tag{3.2.2}$$

Furthermore, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Proof. The necessary condition is obvious, we have to prove the sufficient condition, suppose that there exist a constant λ such that $T^2 = \lambda(I + \eta \otimes \xi)$, then for any $X \in TM - <\xi>$, we have

$$\cos\theta(X) = \frac{g(\phi X, TX)}{|TX||\phi X|} = \frac{g(X, T^2X)}{|TX||\phi X|}$$
$$= \lambda\frac{g(X, \phi^2 X)}{|TX||\phi X|} = \lambda\frac{|\phi X|^2}{|TX||\phi X|}$$
$$\cos\theta(X) = \lambda\frac{|\phi X|}{|TX|}$$

also $\cos\theta(X) = \frac{|TX|}{|\phi X|}$ therefore $\lambda = \cos^2\theta$. Hence, $\theta(X)$ is constant so M is slant.

Now, we have the following corollary, which can be easily verified.

COROLLARY 3.2.1. *Let M be a slant submanifold of a LP-contact manifold $\bar{M}$ with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(TX, TY) = \cos^2\theta(g(X, Y) + \eta(X)\eta(Y)) \tag{3.2.3}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) + \eta(X)\eta(Y)). \tag{3.2.4}$$

3.3. Slant immersions in LP-Sasakian Manifolds

In this section, we will study slant submanifolds of LP-Sasakian manifolds. We recall some important formulae first.

Let $\bar{M}$ be an n- dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g) (see equation 1.1.20), then we have (cf. [34], [55])

$$\left.\begin{array}{l} \bar{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \\ \bar{R}(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \\ \bar{R}(\xi, X)\xi = X + \eta(X)\xi, \end{array}\right\} \tag{3.3.1}$$

where $\bar{R}(X, Y)Z$ is the Riemannian curvature tensor.

THEOREM 3.3.1. *Let M be a slant submanifold of a LP-Sasakian manifold $\bar{M}$. Then Q is parallel if and only if M is anti-invariant.*

Proof. Let θ be slant angle of M in $\bar{M}$, then for any X, Y in TM by equation (3.2.2)

$$T^2Y = QY = \cos^2\theta(Y + \eta(Y)\xi) \tag{3.3.2}$$

$$Q\nabla_X Y = \cos^2\theta(\nabla_X Y + \eta(\nabla_X Y)\xi) \tag{3.3.3}$$

By taking covariant derivative of (3.3.2) with respect to $X \in TM$, we get

$$\nabla_X QY = \cos^2\theta(\nabla_X Y + \eta(\nabla_X Y)\xi + g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi) \tag{3.3.4}$$

from (3.3.3) and (3.3.4)

$$(\bar{\nabla}_X Q)Y = \cos^2\theta(g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi)$$

on using $\nabla_X \xi = TX$

$$(\bar{\nabla}_X Q)Y = \cos^2\theta(g(Y, TX)\xi + \eta(Y)TX) \tag{3.3.5}$$

assertion follows on using equation (3.3.5).

Now, we investigate the existence of a slant submanifold via some curvature measures of the submanifold. To this end, first some formulae for the curvature tensor are obtained in the next lemma.

LEMMA 3.3.1. *Let M be an immersed submanifold of LP-Sasakian manifold $\bar{M}$ such that ξ is tangent to M, then for any $X, Y \in TM$.*

$$R(X, Y)\xi = (\nabla_X T)Y - (\nabla_Y T)X \tag{3.3.6}$$

where ∇, R are the Levi-Civita connection and the curvature tensor field associated to the metric induced by $\bar{M}$ on M. Moreover,

$$R(\xi, X)\xi = QX + (\nabla_\xi T)X \tag{3.3.7}$$

$$R(X, \xi, X, \xi) = g(QX, X). \tag{3.3.8}$$

Proof. We have $\bar{\nabla}_X \xi = \phi X$, using equation (1.1.2)

$$\phi X = \nabla_X \xi + h(X, \xi). \tag{3.3.9}$$

Now, by definition of T, we get

$$TX = \nabla_X \xi$$

hence

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi.$$

Similarly,

$$(\nabla_Y T)X = \nabla_Y TX - T\nabla_Y X = \nabla_Y \nabla_X \xi - \nabla_{\nabla_Y X}\xi$$

,

substituting these equations in equation (3.3.1) it is easy to get(3.3.6). Rewriting (3.3.6) for $X = \xi$ and $Y = X$ and using (3.3.9), we obtain

$$R(\xi, X)\xi = (\nabla_\xi T)X - (\nabla_X T)\xi = (\nabla_\xi T)X + QX.$$

Now taking scaler product of above equation by X, since

$$g((\nabla_\xi T)X, X) = g(\nabla_\xi X, TX) - g(TX, \nabla_\xi X) = 0$$

which proves (3.3.8).

THEOREM 3.3.2. *Let M be immersed submanifold of a LP-Sasakian manifold $\bar{M}$ such that the characteristic vector field ξ of $\bar{M}$ is tangent to M. If $\theta \in (0, \frac{\pi}{2})$, then following statements are equivalent.*

(a) M is slant with slant angle θ

(b) For any $x \in M$ the sectional curvature of any 2-plane of T_xM containing ξ_x equals $\cos^2\theta$.

Proof. Let statement (a) be true, then for any $X \perp \xi$ by equation (3.2.2)

$$QX = \cos^2\theta X$$

which by virtue of (3.3.8) yields

$$R(X, \xi, X, \xi) = \cos^2\theta \tag{3.3.10}$$

and (b) is proved

Conversely, assuming (b) is true, then for any $X \in TM$, we may write

$$X = X_\xi + X_\xi{}^\perp \tag{3.3.11}$$

where $X_\xi = \eta(X)\xi$ and $X_\xi{}^\perp$ is the component of X perpendicular to the ξ, using (3.3.10) and (3.3.11)

$$\frac{R(X_\xi{}^\perp, \xi, X_\xi{}^\perp, \xi)}{|X_\xi|^2} = \cos^2\theta$$

$$R(X_\xi{}^\perp, \xi, X_\xi{}^\perp, \xi) = \cos^2\theta |X_\xi|^2. \tag{3.3.12}$$

Let X be a unit vector such that $QX = 0$. Then from (3.3.6) and (3.3.12)

$$\cos^2\theta |X_\xi{}^\perp|^2 = 0, \tag{3.3.13}$$

if $\cos\theta \neq 0$, then from the above equation $X = X_\xi$. This proves that at each point $x \in M$,

$$Ker(Q) = \langle \xi_x \rangle. \tag{3.3.14}$$

More generally, Let A be the matrix of the endomorphism Q at $x \in M$, then for a unit vector field X on M, $QX = AX$, and as $Q(X_\xi) = 0, X = X_\xi$. Then by (3.3.8) and (3.3.12)

$$A = \cos^2\theta I \tag{3.3.15}$$

Choosing $\lambda = \cos^2\theta$, we conclude that for any $x \in M$, This fact together with (3.3.14) and theorem (3.2.2) verifies that M is slant in $\bar{M}$ with slant angle θ.

Finally, suppose $\cos\theta = 0$ and X is an arbitrary unit vector field such that $QX = \lambda X$ where $\lambda \in C^\infty(M)$. Then, from (3.3.8) and (3.3.12) $g(QX, X) = 0$ that is $\lambda = 0$ and therefore $Q = 0$ which means M is anti-invariant.

3.4. Slant immersions in Lorentzian Almost Contact Manifolds

In this section we study slant submanifolds of Lorentzian almost contact manifolds. We have taken the submanifold as a space like and then defined the slant angle on a submanifold and thus we extended the results of section (3.2) in this new setting.

An almost contact manifold with Lorentzian metric is called a *Lorentzian almost contact manifold* (see equation (1.1.22) and equation (1.1.23)). Let M be a submanifold of a Lorentzian manifold $\bar{M}$ such that for all $X \in TM$, $g(X, X) > 0$ or $g(X, X) = 0$ i.e., all the tangent vectors on M are Space like or null like, we shall call these type of submanifolds as *space like* and also we assume that the structure vector field ξ is tangent to the submanifold M.

DEFINITION 3.4.1. *Let M be an immersed submanifold of $\bar{M}$ and for any $x \in M$ and $X \in T_xM$, if the vector field X and ξ are linearly independent then the angle $\theta(X) \in [0, \pi/2]$ between ϕX and T_xM is well defined, if $\theta(X)$ does not depend upon the choice of $x \in M$ and $X \in T_xM$, then M is slant in $\bar{M}$. The constant angle $\theta(X)$ is then called the slant angle of M in $\bar{M}$ and which in short we denote by Sla(M). The tangent bundle TM at every point $x \in M$ is decomposed as*

$$TM = D \oplus \langle \xi \rangle$$

where $\langle \xi \rangle$ is the one dimensional distribution orthogonal to the slant distribution D on M and spanned by the structure vector field ξ.

Now, for any $x \in M$ taking $X \in T_X M$ we put $\phi X = TX + NX$ where $TX \in T_X M$ and $NX \in T_x^{\perp} M$. Thus defining an endomorphism $T : T_x M \longrightarrow T_x M$, whose square T^2 will be denoted by Q. Then tensor fields on M of the type $(1,1)$ determined by their endomorphisms shall be denoted by same letters T and Q. It is easy to show that for every $x \in M$, $g(TX, Y) = -g(X, TY)$, which implies that Q is anti-symmetric. Moreover, in the following steps we can prove that the eigen value of Q always belong to $[-1, 0]$. For any $X \in T_x M - < \xi >$, we get

$$g(QX, X) = -\|TX\|^2$$

but,

$$\|TX\| \leq \|QX\|$$

$$\|TX\| \leq \mu \|X\|, \quad \text{and} \quad \mu \in [0, 1],$$

thus we obtain

$$g(QX, X) = -\mu(X)\|X\|^2,$$

that is,

$$g(QX, X) = \lambda(X)\|X\|^2,$$

where $-1 \leq \lambda(X) \leq 0$ and λ depends on X. In other words, each eigen value of Q lies in $[-1, 0]$ and each eigen value has even multiplicity.

Now, we have the following theorem.

THEOREM 3.4.1. *Let $x \in M$ and $X \in T_x M$ be an eigenvector of Q with eigenvalue $\lambda(X)$. Suppose X is linearly independent from ξ_x, then,*

$$\cos\theta(X) = \sqrt{-\lambda(X)} \frac{\|X\|}{\|\phi X\|}. \tag{3.4.1}$$

Proof. For any $X \in TM$ we have

$$\|TX\|^2 = g(TX, TX) = -\lambda(X)\|X\|^2. \tag{3.4.2}$$

On the other hand by definition of $\theta(X)$ we have

$$\begin{aligned}\cos\theta(X) &= \frac{g(\phi X, TX)}{\|TX\|\|\phi X\|} \\ &= \frac{g(TX, TX)}{\|TX\|\|\phi X\|} = -\lambda(X)\frac{\|X\|^2}{\|TX\|\|\phi X\|}.\end{aligned}$$

Then from (3.4.2), we obtain that

$$\cos(\theta)(X) = \sqrt{-\lambda(X)}\frac{\|X\|}{\|\phi X\|}.$$

This completes the proof.

The following characterization theorem gives the existence of eigen values of the endomorphism Q.

THEOREM 3.4.2. *Let M be a space like slant submanifold of a Lorentzian almost contact manifold $\bar{M}$ and $\theta = Sla(M) \neq \pi/2$, then Q admits the real number $-\cos^2\theta$ as the only non-vanishing eigen value, for any $x \in M$. Moreover the related eigen space H satisfies $H \subset D$, where $D = Span(\xi_x)^{\perp} \subset T_xM$.*

Proof. Let $x \in M$, from equation (3.4.1) $Ker(Q) \neq T_xM$, otherwise $Sla(M) = \pi/2$ which contradict the assumption. So let λ be an arbitrary non-vanishing eigen value of Q and let H be the corresponding eigen space. Now, we have $dim(D) = 2n$ and $dim(H)$ is even, which shows that $dim(H \cap D) \geq 1$. Let $X \in H \cap D$ is a unit vector, then ϕX is also unit vector then from equation (3.1) we obtain

$$\cos\theta = \sqrt{-\lambda(X)},$$

which proves the first part. Moreover, for any $X \in H$, formula (3.1) yields $\|\phi X\| = \|X\|$ which imply that $g(X, \xi) = 0$, hence $H \subset D$.

We have noted that, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion. In case of invariant submanifold $T = \phi$ and so

$$T^2 = \phi^2 = -I + \eta \otimes \xi.$$

While in case of anti-invariant submanifold, $T^2 = 0$. In fact, we have the following general result which characterize slant immersion.

THEOREM 3.4.3. *Let M be a space like submanifold of a Lorentzian manifold $\bar{M}$ such that $\xi \in TM$. Then, M is slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{3.4.3}$$

Furthermore, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Proof. Necessary condition is obvious, we have to prove the sufficient condition, suppose that there exist a constant λ such that $T^2 = \lambda(-I + \eta \otimes \xi)$, then for any $X \in TM - \langle \xi \rangle$, we have

$$\cos\theta(X) = \frac{g(\phi X, TX)}{\|TX\|\|\phi X\|}$$

$$= -\frac{g(X, T^2X)}{\|TX\|\|\phi X\|} = \lambda\frac{\|\phi X\|}{\|TX\|}. \tag{3.4.4}$$

On the other hand $\cos\theta(X) = \frac{\|TX\|}{\|\phi X\|}$ and so by using (3.4.4) we obtain that $\lambda = \cos^2\theta$. Hence, $\theta(X)$ is a constant angle of M i.e, M is slant.

Now, we have the following corollary, which can be easily verified.

COROLLARY 3.4.1. *Let M be a space like slant submanifold of a Lorentzian manifold $\bar{M}$ with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{3.4.5}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{3.4.6}$$

Proof. For any $X, Y \in TM$, then by using (1.1.24), we have

$$g(X, TY) = -g(TX, Y).$$

Substituting Y by TY in the above equation we get

$$g(TX, TY) = -g(X, T^2Y).$$

Then by virtue of (3.4.3), we obtain (3.4.5). The proof of (3.4.6) follows from (1.1.23).

Chapter IV

SOME SLANT AND BI-SLANT SUBMANIFOLDS OF LORENTZIAN PARACONTACT MANIFOLDS

SOME SLANT AND BI-SLANT SUBMANIFOLDS OF LORENTZIAN PARACONTACT MANIFOLDS

4.1. Introduction

Totally umbilical CR-submanifolds of a Kaehler manifold have been considered by Bejancu [5]. Roughly speaking, a submanifold of a Riemannian manifold is totally umbilical, or simply umbilical, if it is equally curved in all tangent directions. More precisely, an isometric immersion $f : M^m \longrightarrow M^n$ between Riemannian manifolds is umbilical if there exists a normal vector field ζ along f such that its second fundamental form $\alpha_f : TM \times TM \longrightarrow N^f M$ with values in the normal bundle satisfies $\alpha_f(X, Y) =< X, Y > \zeta$ for all $X, Y \in TM$.

Umbilical submanifolds are the simplest submanifolds after the totally geodesic ones (for which the second fundamental form vanishes identically), and their knowledge sheds light on the geometry of the ambient space. Apart from space forms, however, there are few Riemannian manifolds for which umbilical submanifolds are classified. The class of totally umbilical submanifolds includes the class of all totally geodesic submanifolds and extrinsic spheres. The class of totally umbilical submanifolds with constant mean curvature lies between these two classes.

In this chapter, we give some new results by taking a special kind of slant submanifold which is totally umbilical and obtain a necessary and sufficient conditions on a totally umbilical slant submanifold M of an LP-contact manifold $\bar{M}$. Finally, we prove that a totally umbilical proper slant submanifold M of $\bar{M}$ either minimal (in case of totally umbilical submanifolds minimality implies totally geodesicness) or if it is not minimal (totally geodesic) then we have derived a formula for its slant angle.

A manifold $\bar{M}$ with Lorentzian paracontact metric structure (ϕ, ξ, η, g) satisfying $(\bar{\nabla}_X \phi)Y = 0$ is called an LP-Cosymplectic manifold, where $\bar{\nabla}$ is the Levi-Civita connection corresponding to the Lorentzian metric g on $\bar{M}$.

[1]Results of this chapter are published in *Annals of the University of Craiova- Mathematics and Computer Science Series, Mathematical Problems in Engineering and International Journal of Physical Sciences.*

Let M be a submanifold of a Lorentzian almost paracontact manifold $\bar{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g). Let the induced metric on M also be denoted by g. Then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4.1.1}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{4.1.2}$$

for any X, Y in TM and N in $T^{\perp}M$, where TM is the Lie algebra of vector field in M and $T^{\perp}M$ is the set of all vector fields normal to M. $\nabla^{\perp}$ is the connection in the normal bundle, h the second fundamental form and A_N is the Weingarten endomorphism associated with N. It is easy to see that

$$g(A_N X, Y) = g(h(X, Y), N). \tag{4.1.3}$$

For any $X \in TM$, we write

$$\phi X = TX + NX, \tag{4.1.4}$$

where TX is the tangential component and NX is the normal component of ϕX.

Similarly for $N \in T^{\perp}M$, we write

$$\phi N = tN + nN, \tag{4.1.5}$$

where tN is the tangential component and nN is the normal component of ϕN.

The covariant derivatives of the tensor fields ϕ, T and N are defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \qquad \forall\ X, Y \in T\bar{M} \tag{4.1.6}$$

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \qquad \forall\ X, Y \in TM \tag{4.1.7}$$

$$(\bar{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \qquad \forall\ X, Y \in TM. \tag{4.1.8}$$

Moreover, for an LP-Cosymplectic manifold we have

$$(\bar{\nabla}_X T)Y = A_{NY}X + th(X, Y), \tag{4.1.9}$$

$$(\bar{\nabla}_X N)Y = nh(X, Y) - h(X, TY). \tag{4.1.10}$$

A submanifold M is said to be *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \tag{4.1.11}$$

where H is the mean curvature vector. Furthermore, if $h(X,Y) = 0$ for all $X, Y \in TM$, then M is said to be *totally geodesic* and if $H = 0$ then M is *minimal* in $\bar{M}$.

A submanifold M of a Lorentzian Paracontact manifold $\bar{M}$ is said to be a *slant submanifold* if for any $x \in M$ and $X \in T_xM - \langle\xi\rangle$, the angle between ϕX and T_xM is constant. The constant angle $\theta \in [0, \pi/2]$ is then called *slant angle* of M. The tangent bundle TM of M is decomposed as

$$TM = D \oplus \langle\xi\rangle \tag{4.1.12}$$

where the orthogonal complementary distribution D of $\langle\xi\rangle$ is known as the *slant distribution* on M. If μ is an invariant subspace of the normal bundle $T^{\perp}M$, then

$$T^{\perp}M = NTM \oplus \mu. \tag{4.1.13}$$

In the previous chapter, we have proved the following theorem for a slant submanifold M of a Lorentzian Paracontact manifold $\bar{M}$ with slant angle θ.

THEOREM 4.1.1. *(theorem 3.2.2. of previous chapter) Let M be a submanifold of an LP-contact manifold $\bar{M}$ such that $\xi \in TM$. Then, M is slant submanifold if and only if there exist a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{4.1.14}$$

Furthermore, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Thus, we have the following consequences of formula (4.1.14),

$$g(TX, TX) = \cos^2\theta[g(X,Y) + \eta(X)\eta(Y)] \tag{4.1.15}$$

$$g(NX, NY) = \sin^2\theta[g(X,Y) + \eta(X)\eta(Y)] \tag{4.1.16}$$

for any $X, Y \in TM$.

4.2. Totally umbilical propar Slant submanifolds of LP-Cosymplectic Manifolds

In this section, we consider M as a totally umbilical proper slant submanifold of an LP-Cosymplectic manifold $\bar{M}$. Such submanifolds, we always consider tangent to the structure vector field ξ. We study proper slant submanifolds of an LP-Cosymplectic manifold.

THEOREM 4.2.1. *A non trivial totally umbilical proper slant submanifold M of an LP-Cosymplectic manifold $\bar{M}$ is either totally geodesic or if it is not totally geodesic then the slant angle $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$, for any $X, Y \in TM$.*

Proof. For any $X, Y \in TM$, the equation (4.1.9) gives

$$(\bar{\nabla}_X T)Y = A_{NY}X + th(X,Y).$$

Taking the product with ξ and using (4.1.7), we obtain

$$g(\nabla_X TY, \xi) = g(A_{NY}X, \xi) + g(th(X,Y), \xi).$$

Using (4.1.3) and the fact that M is totally umbilical, the above equation takes the form

$$-g(TY, \nabla_X \xi) = g(H, NY)\eta(X) + g(X,Y)g(tH, \xi).$$

Then from the characteristic equation of LP-Cosymplectic manifold, we obtain

$$0 = g(H, NY)\eta(X). \tag{4.2.1}$$

Thus from (4.2.1), it follows that either $H \in \mu$ or M is trivial.

Now, for an LP-Cosymplectic manifold, we have from (4.1.6),

$$\bar{\nabla}_X \phi Y = \phi \bar{\nabla}_X Y$$

for any $X, Y \in TM$. From (4.1.1) and (4.1.4), we obtain

$$\bar{\nabla}_X TY + \bar{\nabla}_X NY = \phi(\nabla_X Y + h(X,Y)).$$

Again using (4.1.1), (4.1.2) and (4.1.4), we get

$$\nabla_X TY + h(X, TY) - A_{NY}X + \nabla^{\perp}_X NY = T\nabla_X Y + N\nabla_X Y + \phi h(X,Y).$$

As M is totally umbilical, then

$$\nabla_X TY + h(X, TY) - A_{NY}X + \nabla^{\perp}_X NY = T\nabla_X Y + N\nabla_X Y + g(X,Y)\phi H.$$

Taking product with ϕH and using the fact that $H \in \mu$, we obtain

$$g(h(X,TY), \phi H) + g(\nabla^{\perp}_X NY, \phi H) = g(N\nabla_X Y, \phi H) + g(X,Y)g(\phi H, \phi H).$$

Then from (1.1.18) and (4.1.11), we get

$$g(X, TY)g(H, \phi H) + g(\nabla^{\perp}_X NY, \phi H) = g(N\nabla_X Y, \phi H) + g(X,Y)\|H\|^2.$$

Again, using (1.1.18) and the fact that $H \in \mu$, then ϕH is also lies in μ, thus we obtain

$$g(\nabla^{\perp}_X NY, \phi H) = g(X, Y)\|H\|^2.$$

Then from (4.1.2), we derive

$$g(\bar{\nabla}_X NY, \phi H) = g(X, Y)\|H\|^2. \tag{4.2.2}$$

Now for any $X \in TM$, we have

$$(\bar{\nabla}_X \phi)H = \bar{\nabla}_X \phi H - \phi \bar{\nabla}_X H.$$

Using the fact that as $\bar{M}$ is an LP-Cosymplectic manifold, we obtain

$$\bar{\nabla}_X \phi H = \phi \bar{\nabla}_X H.$$

Using (4.1.2), (4.1.4),and (4.1.5), we obtain

$$-A_{\phi H}X + \nabla^{\perp}_X \phi H = -TA_H X - NA_H X + t\nabla^{\perp}_X H + n\nabla^{\perp}_X H. \tag{4.2.3}$$

Taking the product in (4.2.3) with NY for any $Y \in TM$ and using the fact $n\nabla^{\perp}_X H \in \mu$, the above equation gives

$$g(\nabla^{\perp}_X \phi H, NY) = -g(NA_H X, NY).$$

Using (4.1.16), we obtain

$$g(\bar{\nabla}_X NY, \phi H) = \sin^2\theta[g(A_H X, Y) + \eta(A_H X)\eta(Y)],$$

Then from (4.1.3) and (4.1.11), we get

$$g(\bar{\nabla}_X NY, \phi H) = \sin^2\theta[g(X, Y) + \eta(X)\eta(Y)]\|H\|^2. \tag{4.2.4}$$

Thus, from (4.2.2) and (4.2.4), we derive

$$[\cos^2\theta g(X, Y) - sin^2\theta\eta(X)\eta(Y)]\|H\|^2 = 0. \tag{4.2.5}$$

Hence, equation (4.2.5) gives either $H = 0$, or if $H \neq 0$, then the slant angle of M is $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$. This proves the theorem completely.

4.3. Hemi Slant submanifolds of LP-Cosymplectic Manifolds

In the following section, we assume M a hemi-slant submanifold of an LP-Cosymplectic manifold $\bar{M}$ such that the structure vector field ξ tangent to M. First, we define a hemi-slant submanifold [67] and then we obtain the integrability conditions of the involved distributions D_θ and $D^\perp$ in the definition of a hemi-slant submanifold M of an LP-Cosymplectic manifold $\bar{M}$.

DEFINITION 4.3.1. *A submanifold M of an LP-contact manifold $\bar{M}$ is said to be a hemi-slant submanifold if there exist two orthogonal complementary distributions D_θ and $D^\perp$ satisfying:*

(i) $TM = D_\theta \oplus D^\perp \oplus \langle \xi \rangle$

(ii) *D_θ is a slant distribution with slant angle $\theta \neq \pi/2$*

(iii) *$D^\perp$ is totally real i.e., $\phi D^\perp \subseteq T^\perp M$.*

If μ is an invariant subspace of the normal bundle $T^\perp M$, then in case of hemi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = ND_\theta \oplus ND^\perp \oplus \mu. \tag{4.3.1}$$

In the following we obtain the integrability conditions of involved distributions in the definition of hemi-slant submanifold.

PROPOSITION 4.3.1. *Let M be a hemi-slant submanifold of an LP-Cosymplectic manifold $\bar{M}$, then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$A_{NZ}W = A_{NW}Z \tag{4.3.2}$$

for any $Z, W \in D^\perp$.

Proof. For any $Z, W \in D^\perp$, we have

$$\phi[Z, W] = \phi\bar{\nabla}_Z W - \phi\bar{\nabla}_W Z.$$

Using (4.1.6), we obtain

$$\phi[Z, W] = \bar{\nabla}_Z \phi W - \bar{\nabla}_W \phi Z.$$

Then from (4.1.2), we derive

$$\phi[Z, W] = -A_{NW}Z + \nabla^\perp_Z NW + A_{NZ}W - \nabla^\perp_W NZ. \tag{4.3.3}$$

As $D^\perp$ is an anti-invariant distribution, then the tangential part of (4.3.3) should be identically zero, hence we obtain the required result.

PROPOSITION 4.3.2. *Let M be a hemi-slant submanifold of an LP-Cosymplectic manifold $\bar{M}$, then the invariant distribution $D_\theta \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(h(X,TY) - h(Y,TX) + \nabla^{\perp}_{X}NY - \nabla^{\perp}_{Y}NX, NZ) = 0,$$

for any $X,\ Y \in D_\theta \oplus \langle\xi\rangle$ and $Z \in D^{\perp}$.

Proof. For any $X,\ Y \in D_\theta \oplus \langle\xi\rangle$, we have

$$\phi[X,Y] = \phi\bar{\nabla}_X Y - \phi\bar{\nabla}_Y X.$$

The from (4.1.6) and the fact that $\bar{M}$ is LP-Cosymplectic, we obtain

$$\phi[X,Y] = \bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X.$$

Using (4.1.4), we get

$$\phi[X,Y] = \bar{\nabla}_X TY + \bar{\nabla}_X NY - \bar{\nabla}_Y TX - \bar{\nabla}_Y NX.$$

Thus from (4.1.1) and (4.1.2), we derive

$$\begin{aligned}\phi[X,Y] = \nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_{X}NY \\ -\nabla_Y TX - h(Y,TX) + A_{NX}Y - \nabla^{\perp}_{Y}NX. \end{aligned} \tag{4.3.4}$$

Taking the product in (4.3.4) with NZ, for any $Z \in D^{\perp}$, we obtain

$$g(\phi[X,Y], NZ) = g(h(X,TY) + \nabla^{\perp}_{X}NY - h(Y,TX) - \nabla^{\perp}_{Y}NX, NZ). \tag{4.3.5}$$

Thus the assertion follows from (4.3.5) after using (1.1.18) and the fact that ξ is tangential to D_θ.

Now, we consider M as a totally umbilical hemi-slant submanifold of an LP-Cosymplectic manifold $\bar{M}$. For any $X, Y \in TM$, we have

$$\bar{\nabla}_X \phi Y = \phi\bar{\nabla}_X Y.$$

Using this fact, if we take for any $Z, W \in D^{\perp}$, then from (4.1.1) and (4.1.2), the above equation takes the form

$$-A_{NW}Z + \nabla^{\perp}_{Z}NW = \phi(\nabla_Z W + h(Z,W)).$$

Thus on using (4.1.4) and (4.1.5), we obtain

$$-A_{NW}Z + \nabla^{\perp}_{Z}NW = T\nabla_Z W + N\nabla_Z W + th(Z,W) + nh(Z,W). \tag{4.3.6}$$

equating the tangential components, we get

$$T\nabla_Z W = -A_{NW}Z - th(Z, W).$$

Taking the product with $V \in D^{\perp}$, we obtain

$$g(T\nabla_Z W, V) = -g(A_{NW}Z, V) - g(th(Z, W), V).$$

Using (1.1.18), (4.1.3) and the fact that $TW = 0$, for any $W \in D^{\perp}$. Thus, the above equation takes the form

$$0 = g(h(Z, V), NW) + g(th(Z, W), V).$$

As M is totally umbilical, hence we derive

$$0 = g(Z, V)g(H, NW) + g(Z, W)g(tH, V). \tag{4.3.7}$$

Thus, the equation (4.3.7) has a solution if either $Z = W = V = \xi$, i.e. $dimD^{\perp} = 1$ or $H \in \mu$ or $D^{\perp} = \{0\}$. Hence, we state the following theorem:

THEOREM 4.3.1. *Let M be a totally umbilical hemi-slant submanifold of an LP-Cosymplectic manifold $\bar{M}$. Then atleast one of the following statement is true:*

(i) *The dimension of anti-invariant distribution is one, i.e., $dimD^{\perp} = 1$.*

(ii) *The mean curvature vector $H \in \mu$.*

(iii) *M is proper slant submanifold of $\bar{M}$.*

4.4. Totally umbilical propar Slant submanifolds of LP-Sasakian Manifolds

Now in this section we shall discuss slant submanifolds of LP-Sasakian manifolds. We consider M as a proper slant submanifold of a Lorentzian para-Sasakian manifold $\bar{M}$. Throughout, the structure vector field ξ is assumed to be tangential to M, otherwise M is simply anti-invariant.

A manifold $\bar{M}$ with Lorentzian paracontact metric structure (ϕ, ξ, η, g) satisfies the following additional conditions

$$\left.\begin{array}{l}(\bar{\nabla}_X\phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \\ \bar{\nabla}_X\xi = \phi X,\end{array}\right\} \tag{4.4.1}$$

for any vector fields $X, Y \in T\bar{M}$, then $\bar{M}$ is said to be an *LP-Sasakian manifold* ([55]).

Now, for any $X, Y \in TM$ on using (4.4.1) and (4.1.1) we may obtain

$$(a)\ \ \nabla_X \xi = TX, \quad (b)\ \ h(X, \xi) = NX. \tag{4.4.2}$$

On using (4.4.1), (4.1.1), (4.1.2), (4.1.3), (4.1.5), (4.1.7) and (4.1.8), we obtain

$$(\bar{\nabla}_X T)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi + A_{NY}X + th(X, Y) \tag{4.4.3}$$

$$(\bar{\nabla}_X N)Y = -h(X, TY) + nh(X, Y). \tag{4.4.4}$$

In, the following theorems we consider M as a totally umbilical proper slant submanifold of an LP-Sasakian manifold $\bar{M}$.

THEOREM 4.4.1 *Let M be a totally umbilical slant submanifold of an LP-Sasakian manifold $\bar{M}$, then the following statements are equivalent*

(i) $H \in \mu$

(ii) *M is an invariant submanifold of $\bar{M}$ or M is trivial.*

Proof. For any $X, Y \in TM$ then from (4.4.3), we have

$$(\bar{\nabla}_X T)Y = A_{NY}X + th(X, Y) + g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{4.4.5}$$

Taking the product in (4.4.5) with ξ, we obtain

$$g(\nabla_X TY, \xi) = g(h(X, \xi), NY) + g(th(X, Y), \xi) - g(X, Y)$$
$$+\eta(X)\eta(Y) - 2\eta(X)\eta(Y).$$

As M is a totally umbilical slant submanifold of $\bar{M}$, then from (4.1.11), the above equation takes the form

$$-g(TY, \nabla_X \xi) = g(H, NY)\eta(X) + g(X, Y)g(tH, \xi) - g(X, Y) - \eta(X)\eta(Y).$$

Using (4.1.15), we get

$$\cos^2\theta[g(X, Y) + \eta(X)\eta(Y)] = -g(H, NY)\eta(X) + g(X, Y) + \eta(X)\eta(Y).$$

The above equation can be written as

$$\sin^2\theta[g(X, Y) + \eta(X)\eta(Y)] = g(H, NY)\eta(X). \tag{4.4.6}$$

If $H \in \mu$, then from (4.1.13), the right hand side of (4.4.6) is identically zero hence (ii) holds. Conversely, if (ii) holds then from (4.4.6), we get $H \in \mu$. This completes the proof of the theorem.

THEOREM 4.4.2 *Let M be a totally umbilical proper slant submanifold of an LP-Sasakian manifold $\bar{M}$ such that $H \in \mu$. Then at least one of the following statements is true*

(i) *M is totally geodesic*

(ii) *If M is proper slant, then the slant angle $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$*

for any $X, Y \in TM$.

Proof. For any $X, Y \in TM$, we have

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

From (4.1.1) and (4.1.4), we obtain

$$\bar{\nabla}_X TY + \bar{\nabla}_X NY - \phi(\nabla_X Y + h(X,Y)) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

Again using (4.1.1), (4.1.2) and (4.1.4), we get

$$\begin{aligned} g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi &= \nabla_X TY + h(X,TY) \\ &\quad - A_{NY}X + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y - \phi h(X,Y). \end{aligned}$$

As M is totally umbilical proper slant, then

$$\begin{aligned} g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi &= \nabla_X TY + g(X,TY)H - A_{NY}X \\ &\quad + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y - g(X,Y)\phi H. \quad (4.4.7) \end{aligned}$$

Taking the product in (4.4.7) with ϕH, we obtain

$$g(X,TY)g(H,\phi H) + g(\nabla^{\perp}_X NY, \phi H) = g(N\nabla_X Y, \phi H) + g(X,Y)g(\phi H, \phi H).$$

Using (1.1.18) and the fact that $H \in \mu$, we get

$$g(\nabla^{\perp}_X NY, \phi H) = g(X,Y)\|H\|^2.$$

Then from (4.1.2), we derive

$$g(\bar{\nabla}_X NY, \phi H) = g(X,Y)\|H\|^2. \quad (4.4.8)$$

Now, for any $X \in TM$, we have

$$(\bar{\nabla}_X \phi)H = \bar{\nabla}_X \phi H - \phi \bar{\nabla}_X H.$$

Using (4.4.1) and the fact that $H \in \mu$, we obtain

$$0 = \bar{\nabla}_X \phi H - \phi \bar{\nabla}_X H.$$

Using (4.1.1), (4.1.2), (4.1.4) and (4.1.5), we obtain

$$-A_{\phi H}X + \nabla^{\perp}_X \phi H = -TA_H X - NA_H X + t\nabla^{\perp}_X H + n\nabla^{\perp}_X H. \tag{4.4.9}$$

Taking the product in (4.4.9) with NY for any $Y \in TM$ and using the fact $n\nabla^{\perp}_X H \in \mu$, the above equation gives

$$g(\nabla^{\perp}_X \phi H, NY) = -g(NA_H X, NY).$$

Using (4.1.16), we obtain that

$$g(\bar{\nabla}_X NY, \phi H) = \sin^2\theta[g(A_H X, Y) + \eta(A_H X)\eta(Y)],$$

that is,

$$g(\bar{\nabla}_X NY, \phi H) = \sin^2\theta[g(X, Y) + \eta(X)\eta(Y)]\|H\|^2. \tag{4.4.10}$$

Thus, from (4.4.8) and (4.4.10), we derive

$$[\cos^2\theta g(X, Y) - \sin^2\theta\eta(X)\eta(Y)]\|H\|^2 = 0. \tag{4.4.11}$$

Since M is proper slant submanifold then it follows from (4.4.11) that either $H = 0$, i.e., M is minimal or $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$. This proves the theorem completely.

Similar calculations leads us to the following important result for totally umbilical proper slant submanifolds of a Lorentzian β-Kenmotsu Manifold (see equation 1.1.21):

THEOREM 4.4.3. *Let M be a totally umbilical submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$, then atleast one of the following statements is true*

(i) $H \in \mu$

(ii) *M is trivial.*

THEOREM 4.4.4. *Let M be a non trivial totally umbilical proper slant submanifold of a Lorentzian $\beta-$Kenmotsu manifold $\bar{M}$. Then at least one of the following statements is true*

(i) *M is totally geodesic*

(ii) *If M is not minimal, then the slant angle of M is $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$*

for any $X, Y \in TM$.

Chapter V

WARPED PRODUCT SEMI-INVARIANT SUBMANIFOLDS OF NEARLY COSYMPLECTIC MANIFOLDS

WARPED PRODUCT SEMI-INVARIANT SUBMANIFOLDS OF NEARLY COSYMPLECTIC MANIFOLDS

5.1. Introduction

R.L. Bishop and B. O'Neill [8] introduced warped product metric on a product manifold $B \times F$ by homothetically warping the product metric on the fibers $p \times F$ for each $p \in B$. Such metrics are not only seen in differential geometric studies but are also used to model space time near black holes or massive stars. For instance, the best relativistic model of the Schwarzschild space time describing the neighborhood of bodies with large gravitational fields is a warped product manifold. For these applications the study of warped product manifolds assumed significance. Recently, the study of differential geometry of warped product manifolds got impetus with B.Y. Chen's work on warped product CR-submanifolds of a Kaehler manifold (cf. [30], [31], [32] etc.). So far as the existence of these warped products are concerned, he proved that there does not exist a proper warped product CR-submanifold $N^{\perp} \times {}_f N^T$ of a Kaehler manifold, where $N^{\perp}$ and N^T are totally real and holomorphic submanifolds respectively of the Kaehler manifold. Recently, K. A. Khan and others studied the warped product submanifolds of Cosymplectic manifolds. They obtained some interesting properties of these warped product submanifolds with differential geometric point of view and provided example of these warped product submanifolds (cf. [44]). We have extended this study in nearly Cosymplectic manifolds.

In this chapter, we have generalized the results of Chen's (cf. [30],[31]) in this more general setting of nearly Cosymplectic manifolds and have shown that the warped product in the form $M = N^{\perp} \times {}_f N^T$ is simply Riemannian product of $N^{\perp}$ and N^T where $N^{\perp}$ is an anti-invariant submanifold and N^T is an invariant submanifold of a nearly Cosymplectic manifold $\bar{M}$. Thus we consider the warped product submanifold of the type $M = N^T \times {}_f N^{\perp}$ by reversing the two factor $N^{\perp}$ and N^T, and simply will be called *warped product semi-invariant submanifold*. Thus, we derive the integrability of the involved distributions in the warped product and obtain a characterization result.

[1]Results of this chapter are published in Mathematical Problems in Engineering.

5.2. Some Basic Results

Almost contact manifolds with Killing structure tensors were defined in [10] as nearly Cosymplectic manifolds, and it was shown normal nearly Cosymplectic manifolds are Cosymplectic (see also [11]). Later on, Blair [12] studied nearly Cosymplectic structure (ϕ, ξ, η, g) on a manifold $\bar{M}$ with η closed from the topological viewpoint.

An almost contact structure (ϕ, ξ, η) is said to be *normal* if the almost complex structure J on the product manifold $\bar{M} \times \mathbb{R}$ given by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi,\ \eta(X)\frac{d}{dt}),$$

where f is a $C^{\infty}-$function on $\bar{M} \times \mathbb{R}$ has no torsion i.e., J is integrable, the condition for normality in terms of $\phi,\ \xi$ and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally the *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be *Cosymplectic*, if it is normal and both Φ and η are closed [12]. The structure is said to be *nearly Cosymplectic* if ϕ is Killing, i.e., if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 0, \tag{5.2.1}$$

for any $X,\ Y \in T\bar{M}$, where $T\bar{M}$ is the tangent bundle of $\bar{M}$ and $\bar{\nabla}$ denotes the Riemannian connection of the metric g. Equation (5.2.1) is equivalent to $(\bar{\nabla}_X \phi)X = 0$, for each $X \in T\bar{M}$. The structure is said to be *closely Cosymplectic* if ϕ is Killing and η is closed. It is well known that an almost contact metric manifold is *Cosymplectic* if and only if $\bar{\nabla}\phi$ vanishes identically, i.e., $(\bar{\nabla}_X \phi)Y = 0$ and $\bar{\nabla}_X \xi = 0$.

PROPOSITION 5.2.1 [12]. *On a nearly Cosymplectic manifold the vector field ξ is Killing.*

From the above proposition we have $\bar{\nabla}_X \xi = 0$, for any vector field X tangent to $\bar{M}$, where $\bar{M}$ is a nearly Cosymplectic manifold.

Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\bar{M}$, then for every $x \in M$ there exists a maximal invariant subspace denoted by $\mathcal{D}_x$ of the tangent space T_xM of M. If the dimension of $\mathcal{D}_x$ is the same for all value of $x \in M$, then $\mathcal{D}_x$ gives an invariant distribution $\mathcal{D}$ on M.

A submanifold M of an almost contact metric manifold $\bar{M}$ is called *semi-invariant* submanifold if there exists on M a differentiable invariant distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant, i.e.,

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$,

(ii) $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$,

(iii) $\phi(\mathcal{D}_x^{\perp}) \subset T_x^{\perp}M$

for any $x \in M$, where $T_x^{\perp}M$ denotes the orthogonal space of T_xM in $T_x\bar{M}$. A semi-invariant submanifold is called *anti-invariant* if $\mathcal{D}_x = \{0\}$ and *invariant* if $\mathcal{D}_x^{\perp} = \{0\}$, respectively for any $x \in M$. It is called *proper semi-invariant* submanifold if neither $\mathcal{D}_x = \{0\}$ nor $\mathcal{D}_x^{\perp} = \{0\}$, for every $x \in M$.

Let M be a semi-invariant submanifold of an almost contact metric manifold $\bar{M}$. Then, $N(T_xM)$ is a subspace of $T_x^{\perp}M$. Then for every $x \in M$, there exists an invariant subspace ν_x of $T_x\bar{M}$ such that

$$T_x^{\perp}M = N(T_xM) \oplus \nu_x. \tag{5.2.2}$$

A semi-invariant submanifold M of an almost contact metric manifold $\bar{M}$ is called *Riemannian product* if the invariant distribution $\mathcal{D}$ and anti-invariant distribution $\mathcal{D}^{\perp}$ are totally geodesic distributions in M.

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. The *warped product* of N_1 and N_2 is the product manifold $N_1 \times {}_fN_2 = (N_1 \times N_2,\ g)$, where

$$g = g_1 + f^2 g_2.$$

where f is called the *warping function* of the warped product. The warped product $N_1 \times {}_fN_2$ is said to be *trivial* or simply Riemannian product if the warping function f is constant. This means that the Riemannian product is a special case of warped product.

We recall the following general results obtained by Bishop and O'Neill [8] for warped product manifolds.

LEMMA 5.2.1. *Let $M = N_1 \times {}_fN_2$ be a warped product manifold with the warping function f, then*

(i) $\nabla_X Y \in TN_1$ *for each* $X, Y \in TN_1$,

(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$, *for each* $X \in TN_1$ *and* $Z \in TN_2$,

(iii) $\nabla_Z W = \nabla_Z^{N_2} W - \frac{g(Z,W)}{f} gradf$

where ∇ *and* ∇^{N_2} *denote the Levi-Civita connections on* M *and* N_2*, respectively.*

From (ii) of above lemma we can see that

$$\nabla_X Y = \nabla_Y X = (X \ln f)Y \tag{5.2.3}$$

for any vector fields X tangent to B and Y tangent to F.

In the above Lemma *gradf* is the gradient of the function f defined by $g(gradf, U) = Uf$, for each $U \in TM$. From the Lemma 5.2.1, we have on a warped product manifold $M = N_1 \times {}_f N_2$

(i) N_1 is totally geodesic in M.

(ii) N_2 is totally umbilical in M.

Now, we denote by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\bar{\nabla}_X \phi)Y$, i.e.,

$$(\bar{\nabla}_X \phi)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y \tag{5.2.4}$$

for all $X, Y \in TM$. Making use of equations (1.2.2), (1.2.3),(1.2.12),(1.2.13) and (1.2.21),(1.2.22), the following relations may easily be obtained,

$$\mathcal{P}_X Y = (\bar{\nabla}_X T)Y - A_{NY} X - th(X, Y) \tag{5.2.5}$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X F)Y + h(X, TY) - nh(X, Y). \tag{5.2.6}$$

It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$, which we enlist here for later use

(p_1) (i) $\mathcal{P}_{X+Y} W = \mathcal{P}_X W + \mathcal{P}_Y W$, (ii) $\mathcal{Q}_{X+Y} W = \mathcal{Q}_X W + \mathcal{Q}_Y W$,

(p_2) (i) $\mathcal{P}_X (Y + W) = \mathcal{P}_X Y + \mathcal{P}_X W$, (ii) $\mathcal{Q}_X (Y + W) = \mathcal{Q}_X Y + \mathcal{Q}_X W$,

(p_3) $g(\mathcal{P}_X Y,\ W) = -g(Y, \mathcal{P}_X W)$

for all $X, Y, W \in TM$.

On a submanifold M of a nearly Cosymplectic manifold $\bar{M}$, we obtain from equations (5.2.1) and (5.2.4) that

$$(i)\ \mathcal{P}_X Y + \mathcal{P}_Y X = 0, \quad (ii)\ \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0 \tag{5.2.7}$$

for any $X,\ Y \in TM$.

5.3. Warped Product Semi-invariant submanifolds of Nearly Cosymplectic Manifolds

Through out the section we consider the submanifold M of a nearly Cosymplectic manifold $\bar{M}$ such that the structure vector field ξ is tangent to M. First, we prove that the warped product $M = N_1 \times {}_f N_2$ is trivial when ξ is tangent to N_2, where N_1 and N_2 are Riemannian submanifolds of a nearly Cosymplectic manifold $\bar{M}$. Thus, we consider the warped product $M = N_1 \times {}_f N_2$, when ξ is tangent to the submanifold N_1. We have the following non-existence theorem.

THEOREM 5.3.1. *A warped product submanifold $M = N_1 \times {}_f N_2$ of a nearly Cosymplectic manifold $\bar{M}$ is usual Riemannian product if the structure vector field ξ is tangent to N_2, where N_1 and N_2 are the Riemannian submanifolds of $\bar{M}$.*

Proof. For any $X \in TN_1$ and ξ is tangent to N_2, we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

Using the fact that ξ is killing on a nearly Cosymplectic manifold (see Proposition 5.2.1) and Lemma 5.2.1 (ii), we get

$$0 = (X \ln f)\xi + h(X, \xi). \tag{5.3.1}$$

Equating the tangential component of (5.3.1), we obtain $X \ln f = 0$, for all $X \in TN_1$, i.e., f is constant function on N_1. Thus, M is Riemannian product. This proves the theorem.

Now, the other case of warped product $M = N_1 \times {}_f N_2$ when $\xi \in TN_1$, where N_1 and N_2 are Riemannian submanifolds of $\bar{M}$. For any $X \in TN_2$, we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

By Proposition 5.2.1, and Lemma 5.2.1 (ii), we obtain

$$(i)\ \ \xi \ln f = 0, \qquad (ii)\ \ h(X, \xi) = 0. \tag{5.3.2}$$

Thus, we consider the warped product semi-invariant submanifolds of a nearly Cosymplectic manifold $\bar{M}$ of the types:

(i) $M = N^{\perp} \times {}_f N^T$,

(ii) $M = N^T \times {}_f N^\perp$,

where N^T and $N^\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. In the following theorem we prove that the warped product semi-invariant submanifold of the type (i) is CR-product.

THEOREM 5.3.2. *The warped product semi-invariant submanifold $M = N^\perp \times {}_f N^T$ of a nearly Cosymplectic manifold $\bar{M}$ is a usual Riemannian product of $N^\perp$ and N^T, where $N^\perp$ and N^T are anti-invariant and invariant submanifolds of $\bar{M}$, respectively.*

Proof. When $\xi \in TN^T$, then by Theorem 5.3.1, M is a Riemannian product. Thus, we consider $\xi \in TN^\perp$. For any $X \in TN^T$ and $Z \in TN^\perp$, we have

$$g(h(X, \phi X), NZ) = g(h(X, \phi X), \phi Z) = g(\bar{\nabla}_X \phi X, \phi Z)$$
$$= g(\phi \bar{\nabla}_X X, \phi Z) + g((\bar{\nabla}_X \phi)X, \phi Z).$$

From the structure equation of nearly Cosymplectic, the second term of right hand side vanishes identically. Thus from (1.1.11), we derive

$$g(h(X, \phi X), NZ) = g(\bar{\nabla}_X X, Z) - \eta(Z) g(\bar{\nabla}_X X, \xi)$$
$$= -g(X, \bar{\nabla}_X Z) + \eta(Z) g(X, \bar{\nabla}_X \xi).$$

Then from (1.2.2), Lemma 5.2.1 (ii) and Proposition 5.2.1, we obtain

$$g(h(X, \phi X), NZ) = -(Z \ln f)\|X\|^2. \tag{5.3.3}$$

Interchanging X by ϕX in (5.3.3) and using the fact that $\xi \in TN^\perp$, we obtain

$$g(h(X, \phi X), NZ) = (Z \ln f)\|X\|^2. \tag{5.3.4}$$

It follows from (5.3.3) and (5.3.4) that $Z \ln f = 0$, for all $Z \in TN^\perp$. Also, from (5.3.2) we have $\xi \ln f = 0$. Thus, the warping function f is constant. This completes the proof of the theorem.

From the above theorem we have seen that the warped product of the type $M = N^\perp \times {}_f N^T$ is a usual Riemannian product of an anti-invariant submanifold $N^\perp$ and an invariant submanifold N^T of a nearly Cosymplectic manifold $\bar{M}$. Since both $N^\perp$ and N^T are totally geodesic in M, then M is CR-product. Now, we study the warped product semi-invariant submanifold $M = N^T \times {}_f N^\perp$ of a nearly Cosymplectic manifold $\bar{M}$.

THEOREM 5.3.3. *Let $M = N^T \times {}_f N^\perp$ be a warped product semi-invariant submanifold of a nearly Cosymplectic manifold $\bar{M}$. Then the invariant distribution $\mathcal{D}$ and the anti-invariant distribution $\mathcal{D}^\perp$ are always integrable.*

Proof. For any $X, Y \in \mathcal{D}$, we have

$$N[X,Y] = N\nabla_X Y - N\nabla_Y X.$$

Using (1.2.13), we obtain

$$N[X,Y] = (\bar{\nabla}_X N)Y - (\bar{\nabla}_Y N)X.$$

Then by (5.2.6), we derive

$$N[X,Y] = \mathcal{Q}_X Y - h(X,TY) + nh(X,Y) - \mathcal{Q}_Y X + h(Y,TX) - nh(X,Y).$$

Thus from (5.2.7)(ii), we get

$$N[X,Y] = 2\mathcal{Q}_X Y + h(Y,TX) - h(X,TY). \tag{5.3.5}$$

Now, for any $X, Y \in \mathcal{D}$, we have

$$h(X,TY) + \nabla_X TY = \bar{\nabla}_X TY = \bar{\nabla}_X \phi Y.$$

Using the covariant derivative property of $\bar{\nabla}\phi$, we obtain

$$h(X,TY) + \nabla_X TY = (\bar{\nabla}_X \phi)Y + \phi\bar{\nabla}_X Y.$$

Then by equations (1.2.2) and (5.2.4), we get

$$h(X,TY) + \nabla_X TY = \mathcal{P}_X Y + \mathcal{Q}_X Y + \phi(\nabla_X Y + h(X,Y)).$$

Since N^T is totally geodesic in M (see Lemma 5.2.1(i)), then using (1.2.2) and (1.2.22), we obtain

$$h(X,TY) + \nabla_X TY = \mathcal{P}_X Y + \mathcal{Q}_X Y + T\nabla_X Y + th(X,Y) + nh(X,Y). \tag{5.3.6}$$

Equating the normal components of (5.3.6), we get

$$h(X,TY) = \mathcal{Q}_X Y + nh(X,Y). \tag{5.3.7}$$

Similarly, we obtain

$$h(Y,TX) = \mathcal{Q}_Y X + nh(X,Y). \tag{5.3.8}$$

Then from (5.3.7) and (5.3.8), we arrive at

$$h(Y,TX) - h(X,TY) = \mathcal{Q}_Y X - \mathcal{Q}_X Y.$$

Hence, using (5.2.7)(ii), we get

$$h(Y, TX) - h(X, TY) = -2\mathcal{Q}_X Y. \tag{5.3.9}$$

Thus, it follows from (5.3.5) and (5.3.9) that $N[X, Y] = 0$, for all $X, Y \in \mathcal{D}$. This proves the integrability of $\mathcal{D}$. Now, for the integrability of $\mathcal{D}^\perp$, we consider any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$, we have

$$g([Z, W], X) = g(\bar{\nabla}_Z W - \bar{\nabla}_W Z, X).$$

$$= -g(\nabla_Z X, W) + g(\nabla_W X, Z).$$

Using Lemma 5.2.1 (ii), we obtain

$$g([Z, W], X) = -(X \ln f)g(Z, W) + (X \ln f)g(Z, W) = 0. \tag{5.3.10}$$

Thus from (5.3.10), we conclude that $[Z, W] \in \mathcal{D}^\perp$, for each $Z, W \in \mathcal{D}^\perp$. Hence, the theorem is proved completely.

LEMMA 5.3.1 *Let $M = N^T \times {}_f N^\perp$ be a warped product submanifold of a nearly Cosymplectic manifold $\bar{M}$. If $X, Y \in TN^T$ and $Z, W \in TN^\perp$, then*

(i) $g(\mathcal{P}_X Y, Z) = g(h(X, Y), NZ) = 0$,

(ii) $g(\mathcal{P}_X Z, W) = g(h(X, Z), NW) - g(h(X, W), NZ) = -(\phi X \ln f)g(Z, W) - g(h(X, Z), NW)$,

(iii) $g(h(\phi X, Z), NZ) = (X \ln f)\|Z\|^2$.

Proof. For a warped product manifold $M = N^T \times {}_f N^\perp$, we have N^T is totally geodesic in M, then by (1.2.12) $(\bar{\nabla}_X T)Y \in TN^T$, for any $X, Y \in TN^T$ and therefore from equation (5.2.5), we get

$$g(\mathcal{P}_X Y, Z) = -g(th(X, Y), Z) = g(h(X, Y), NZ). \tag{5.3.11}$$

The left hand side of (5.3.11) is skew symmetric in X and Y whereas the right hand side is symmetric in X and Y, which proves (i). Now, from equations (1.2.12) and (5.2.5), we have

$$\mathcal{P}_X Z = -T\nabla_X Z - A_{NZ} X - th(X, Z). \tag{5.3.12}$$

for any $X \in TN^T$ and $Z \in TN^\perp$. Using Lemma 5.2.1 (ii), the first term of right hand side is zero. Thus, taking the product with $W \in TN^\perp$, we obtain

$$g(\mathcal{P}_X Z, W) = -g(A_{NZ} X, W) - g(th(X, Z), W).$$

Then by (1.1.11) and (1.2.5), we get

$$g(\mathcal{P}_X Z, W) = -g(h(X, W), NZ) + g(h(X, Z), NW).$$

Which proves the first equality of (ii). Again, from equations (1.2.12) and (5.2.5), we have

$$\mathcal{P}_Z X = \nabla_Z TX - T\nabla_Z X - Bh(X, Z).$$

Thus using Lemma 5.2.1(ii), we derive

$$\mathcal{P}_Z X = (TX \ln f)Z - th(X, Z). \tag{5.3.13}$$

Taking inner product with $W \in TN^\perp$ and using (1.1.11), we obtain

$$g(\mathcal{P}_Z X, W) = (\phi X \ln f)g(Z, W) + g(h(X, Z), NW).$$

Then from equation (5.2.7)(i), we get

$$g(\mathcal{P}_X Z, W) = -(\phi X \ln f)g(Z, W) - g(h(X, Z), NW).$$

This is the second equality of (ii). Now, from equations (5.3.12) and (5.3.13), we have

$$\mathcal{P}_X Z + \mathcal{P}_Z X = -T\nabla_X Z - A_{NZ}X + (TX \ln f)Z - 2th(X, Z).$$

Left hand side and the first term of right hand side are zero on using (5.2.7)(i) and Lemma 5.2.1(i), respectively. Thus the above equation takes the form

$$(TX \ln f)Z = A_{NZ}X + 2th(X, Z).$$

Taking the product with Z and on using (1.1.11) and (1.2.5), we get

$$(\phi X \ln f)\|Z\|^2 = g(h(X, Z), NZ) - 2g(h(X, Z), NZ) = -g(h(X, Z), NZ).$$

Interchanging X by ϕX and using (1.1.11), we obtain

$$\{-X + \eta(X)\xi\} \ln f\|Z\|^2 = -g(h(\phi X, Z), NZ).$$

Thus by (5.3.2)(i), the above equation reduces to

$$(X \ln f)\|Z\|^2 = g(h(\phi X, Z), NZ).$$

This proves the lemma completely.

THEOREM 5.3.4 *A proper semi-invariant submanifold M of a nearly Cosymplectic manifold $\bar{M}$ is locally a semi-invariant warped product if and only if the shape operator of M satisfies*

$$A_{\phi Z}X = -(\phi X\mu)Z, \quad X \in \mathcal{D} \oplus \langle\xi\rangle, \quad Z \in \mathcal{D}^\perp \tag{5.3.14}$$

for some function μ *on* M *satisfying* $V(\mu) = 0$ *for each* $V \in \mathcal{D}^{\perp}$.

Proof If $M = N^T \times {}_f N^{\perp}$ is a warped product semi-invariant submanifold, then by Lemma 5.3.1(*iii*), we obtain (5.3.14). In this case $\mu = \ln f$.

Conversely, suppose M is a semi-invariant submanifold of a nearly Cosymplectic manifold $\bar{M}$ satisfying the equation (5.3.14), then

$$g(h(X,Y), \phi Z) = g(A_{\phi Z}X, Y) = -(\phi X\mu)g(Y,Z) = 0. \tag{5.3.15}$$

Now, from (1.2.2) and the property of covariant derivative of $\bar{\nabla}$, we have

$$\begin{aligned} g(h(X,Y), \phi Z) &= g(\bar{\nabla}_X Y, \phi Z) = -g(\phi\bar{\nabla}_X Y, Z) \\ &= -g(\bar{\nabla}_X \phi Y, Z) + g((\bar{\nabla}_X \phi)Y, Z). \end{aligned}$$

Then from (1.2.2), (5.2.4) and (5.3.15), the above equation takes the form

$$g(\nabla_X TY, Z) = g(\mathcal{P}_X Y, Z).$$

Using (1.2.12) and (5.2.5), we obtain

$$g(\nabla_X TY, Z) = g(\nabla_X TY, Z) - g(T\nabla_X Y, Z) - g(th(X,Y), Z).$$

Thus by (1.1.11), the above equation reduces to

$$g(T\nabla_X Y, Z) = g(h(X,Y), \phi Z).$$

Hence using (1.2.5) and (5.3.14), we get

$$g(T\nabla_X Y, Z) = g(A_{\phi Z}X, Y) = 0.$$

Which implies $\nabla_X Y \in \mathcal{D} \oplus \langle\xi\rangle$, i.e., $\mathcal{D} \oplus \langle\xi\rangle$ is integrable and its leaves are totally geodesic in M. Now, for any $Z, W \in \mathcal{D}^{\perp}$ and $X \in \mathcal{D} \oplus \langle\xi\rangle$, we have

$$\begin{aligned} g(\nabla_Z W, \phi X) &= g(\bar{\nabla}_Z W, \phi X) = -g(\phi\bar{\nabla}_Z W, X) \\ &= g((\bar{\nabla}_Z \phi)W, X) - g(\bar{\nabla}_Z \phi W, X). \end{aligned}$$

Then, using (1.2.3) and (5.2.4), we obtain

$$g(\nabla_Z W, \phi X) = g(\mathcal{P}_Z W, X) + g(A_{\phi W}Z, X).$$

Thus from (1.2.5) and the property (p_3), we arrive at

$$g(\nabla_Z W, \phi X) = -g(W, \mathcal{P}_Z X) + g(h(Z,X), \phi W).$$

Again using (1.2.5) and (5.2.7)(i), we get

$$g(\nabla_Z W, \phi X) = g(\mathcal{P}_X Z, W) + g(A_{\phi W} X, Z). \tag{5.3.16}$$

On the other hand, from (1.2.12) and (5.2.5), we have

$$\mathcal{P}_X Z = T\nabla_X Z - A_{NZ} X - th(X, Z).$$

Taking the product with $W \in \mathcal{D}^\perp$ and using (5.3.14), we obtain

$$g(\mathcal{P}_X Z, W) = g(T\nabla_X Z, W) + (\phi X\mu)g(Z, W) + g(h(X, Z), NW).$$

The first term of right hand side of above equation is zero using the fact that $TW = 0$, for any $W \in \mathcal{D}^\perp$. Again using (1.2.5), we get

$$g(\mathcal{P}_X Z, W) = (\phi X\mu)g(Z, W) + g(A_{\phi W} X, Z).$$

Thus from (5.3.14), we derive

$$g(\mathcal{P}_X Z, W) = (\phi X\mu)g(Z, W) - (\phi X\mu)g(Z, W) = 0. \tag{5.3.17}$$

Then from (5.3.14), (5.3.16) and (5.3.17), we obtain

$$g(\nabla_Z W, \phi X) = -(\phi X\mu)g(Z, W). \tag{5.3.18}$$

Let $N^\perp$ be a leaf of $\mathcal{D}^\perp$ and $h^\perp$ be the second fundamental form of the immersion of $N^\perp$ into M. Then for any $Z,\ W \in \mathcal{D}^\perp$, we have

$$g(h^\perp(Z, W), \phi X) = g(\nabla_Z W, \phi X). \tag{5.3.19}$$

Hence, from (5.3.18) and (5.3.19), we conclude that

$$g(h^\perp(Z, W), \phi X) = -(\phi X\mu)g(Z, W). \tag{5.3.20}$$

This means that integral manifold $N^\perp$ of $\mathcal{D}^\perp$ is totally umbilical in M. Since the anti-invariant distribution $\mathcal{D}^\perp$ of a semi-invariant submanifold M is always integrable (Theorem 5.3.3) and $V(\mu) = 0$ for each $V \in \mathcal{D}^\perp$, which implies that the integral manifold of $\mathcal{D}^\perp$ is an extrinsic sphere in M, that is it is totally umbilical and its mean curvature vector field is non-zero and parallel along $N^\perp$. Hence by virtue of result [10], M is locally a warped product $N^T \times_f N^\perp$, where N^T and $N^\perp$ denote the integral manifolds of the distributions $\mathcal{D} \oplus \langle\xi\rangle$ and $\mathcal{D}^\perp$, respectively and f is the warping function. Thus the theorem is proved.

Chapter VI

WARPED PRODUCT CONTACT CR-SUBMANIFOLDS OF GLOBALLY FRAMED f-MANIFOLDS WITH LORENTZ METRIC

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6.1. Introduction

Recently, Chen [30] (see also [31]) studied warped product CR-submanifolds and showed that there exist no warped product CR-submanifolds of the form $M = N^{\perp} \times {}_f N^T$ such that $N^{\perp}$ is a totally real submanifold and N^T is a holomorphic submanifold of a Kaehler manifold $\tilde{M}$. Therefore he considered warped product CR-submanifold in the form $M = N^T \times {}_f N^{\perp}$ which is called CR-warped product, where N^T and $N^{\perp}$ are holomorphic and totally real submanifolds of a Kaehler manifold $\tilde{M}$.

In this chapter, we study warped product CR-submanifolds of globally framed f-manifolds in the particular setting of indefinite S-manifolds for both spacelike and timelike cases. We prove that if $M = N^{\perp} \times_f N^T$ is a warped CR-submanifold such that $N^{\perp}$ is ϕ-anti-invariant and N^T is ϕ-invariant, then M is a CR-product. We show that the second fundamental form of a contact CR warped product of a indefinite S space form satisfies a geometric inequality, $\|h\|^2 \geq p\{3\|\nabla lnf\|^2 - \triangle lnf + (c+2)k + 1\}$.

6.2. CR-submanifolds of indefinite S-manifolds

A manifold $\tilde{M}$ is called a globally framed f-manifold (briefly $g.f.f$-manifold) if it is endowed with a non null $(1,1)-$tensor field ϕ of constant rank, such that $ker\ \phi$ is parallelizable i.e. there exist global vector field ξ_α, such that $\alpha = \{1, ..., s\}$, and 1-form η^α, satisfying [34]

$$\phi^2 = -I + \eta^\alpha \otimes \xi_\alpha \quad and \quad \eta^\alpha(\xi_\beta) = \delta^\alpha{}_\beta. \tag{6.2.1}$$

A $g.f.f$-manifold $(\tilde{M}^{2n+s}, \phi, \eta^\alpha, \xi_\alpha)$, such that $\alpha \in \{1, ..., s\}$, is said to be an indefinite $g.f.f$-manifold if g is a semi-Riemannian metric satisfying the following compatibility condition

[1] Results of this chapter are communicated in Filomat.

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{6.2.2}$$

for any vector fields X, Y, where $\epsilon_\alpha = \pm 1$ according to whether ξ_α is spacelike or timelike. Then for any $\alpha \in \{1, ..., s\}$,

$$\eta^\alpha(X) = \epsilon_\alpha g(X, \xi_\alpha).$$

Note: We will consider $\alpha \in \{1, ..., s\}$ throughout the chapter.

An indefinite $g.f.f$-manifold is an indefinite S-manifold if it is normal and $d\eta^\alpha = \Phi$, for any $\alpha \in \{1, ..., s\}$, where $\Phi(X, Y) = g(X, \phi Y)$ for any $X, Y \in \chi(M^{2n+s})$. The Levi-Civita connection of an indefinite S-manifold satisfies

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\bar{\xi}_\alpha + \bar{\eta}^\alpha(Y)\phi^2(X), \tag{6.2.3}$$

where $\tilde{\xi} = \sum\limits_{\alpha=1}^{s} \xi_\alpha$ and $\tilde{\eta} = \epsilon_\alpha \eta^\alpha$. Note that for $s = 1$, indefinite S-manifold becomes indefinite Sasaki manifold.

From (6.2.3), it follows that $\nabla_X \xi_\alpha = -\epsilon_\alpha \phi X$ and $ker\phi$ is an integrable flat distribution since $\nabla_{\xi_\alpha} \xi_\beta = 0$, for any $\alpha, \beta \in \{1, ..., s\}$. A $g.f.f$-manifold is subject to the topological condition: It has to be either non compact or compact with vanishing Euler characteristic, since it admits never vanishing vector fields. This implies that such a $g.f.f$-manifold always admit Lorentz metrics.

An indefinite S-manifold $(\tilde{M}, \phi, \xi_\alpha, \eta^\alpha, g)$ is said to be an indefinite $S-$space if the ϕ-sectional curvature $H_p(X)$ is constant, for any point and any ϕ-plane. In particular, in [15] it is proved that an indefinite S-manifold $(M, \phi, \xi_\alpha, \eta^\alpha, g)$ is an indefinite $S-$space form with $H_p(X) = c$ if and only if the Riemannian $(0, 4)-$type curvature tensor field R is given by

$$\begin{aligned}
R(X, Y, Z, W) = &-\frac{c + 3\epsilon}{4}\{g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\} \\
&- \frac{c - \epsilon}{4}\{\Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) \\
&+ 2\Phi(X, Y)\Phi(W, Z)\} - \{\tilde{\eta}(W)\tilde{\eta}(X)g(\phi Z, \phi Y) \\
&- \tilde{\eta}(W)\tilde{\eta}(Y)g(\phi Z, \phi X) + \tilde{\eta}(Y)\tilde{\eta}(Z)g(\phi W, \phi X) \\
&- \tilde{\eta}(Z)\tilde{\eta}(X)g(\phi W, \phi Y)\},
\end{aligned} \tag{6.2.4}$$

for any vector fields X,Y,Z and W on M, where $\epsilon = \sum\limits_{\alpha=1}^{s} \epsilon_\alpha$.

Let M be a real $m-$dimensional submanifold of $\tilde{M}^{2n+s}$, tangent to the global vector field ξ_α. We shall need the Gauss and Weigarten formulae

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{6.2.5}$$

for any $X, Y \in \chi(M)$ and $N \in \Gamma^\infty(T(M)^\perp)$, where $\nabla^\perp$ is the connection on the normal bundle $T(M)^\perp$, h is the second fundamental form and A_N is the Weingarten map associated with the vector field $N \in T(M)^\perp$ as

$$g(A_N X, Y) = \tilde{g}(h(X,Y), N).$$

For any $X \in \chi(M)$ we set $TX = tan(\phi X)$ and $NX = nor(\phi X)$, where tan_x and nor_x are the natural projections associated to the direct sum decomposition

$$T_x(\tilde{M}) = T_x(M) \oplus T(M)_x{}^\perp, \quad x \in M.$$

Then T is an endomorphism of the tangent bundle of $T(M)$ and N is a normal bundle valued $1-$form on M. Since ξ_α is tangent to M, we get

$$T\xi_\alpha = 0, \quad N\xi_\alpha = 0, \quad \nabla_X \xi_\alpha = TX, \quad h(X, \xi_\alpha) = NX.$$

Similarly, for a normal vector field N, we put $tN = tan(\phi N)$ and $nN = nor(\phi N)$ for the tangential and normal part of ϕN, respectively.

The covariant derivative of the morphisms T and N are defined respectively as

$$(\nabla_U T)V = \nabla_U TV - T\nabla_U V, \quad (\nabla_U N)V = \nabla^\perp_U NV - N\nabla_U V,$$

for $U, V \in \chi(M)$. On using equation (6.2.3) and (6.2.5) we get

$$(\nabla_U T)V = g(TU,TV)\xi_\alpha + \eta^\alpha(U)\eta^\alpha(V)\xi_\alpha - \eta^\alpha(V)U + th(U,V) + A_{NV}U \tag{6.2.6}$$

and

$$(\nabla_U N)V = g(NU,NV)\xi_\alpha + nh(U,V) + h(U,TV). \tag{6.2.7}$$

The Riemannian curvature tensor R of M is given by

$$R_{XY}Z = \frac{3\epsilon^2 + c\epsilon - 4}{4}\{g(Y,Z)\eta^\alpha(X)\xi_\alpha - g(X,Z)\eta^\alpha(Y)\xi_\alpha \tag{6.2.8}$$

$$+\eta^\alpha(Y)\eta^\alpha(Z)X - \eta^\alpha(X)\eta^\alpha(Z)Y + \frac{c+3\epsilon}{4}\{g(X,Z)Y - g(Y,Z)X\}$$

$$- \frac{c-\epsilon}{4}\{g(Z,TY)TX - g(Z,TX)TY - 2g(X,TY)TZ\},$$

for all X,Y,Z vector fields on M, we recall the equation of Gauss and Codazzi, respectively

$$\tilde{g}(\tilde{R}_{XY}Z,W) = g(R_{XY}Z,W) - g(h(X,W),h(Y,Z)) + g(h(Y,W),h(X,Z)) \tag{6.2.9}$$

$$(\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z) = (\tilde{R}_{XY}Z)^{\perp}, \tag{6.2.10}$$

where $(\nabla)h$ the covariant derivative of the second fundamental form is given by

$$((\tilde{\nabla}_X h)(Y,Z) = \bar{\nabla}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z), \tag{6.2.11}$$

for all $X,Y,Z \in TM$. Codazzi equation becomes

$$(\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z) = \frac{c-\epsilon}{4}\{g(TX,Z)NY - g(TY,Z)NX + 2g(X,TY)NZ\}. \tag{6.2.12}$$

LEMMA 6.2.1 *Let M^m be a submanifold of an indefinite S-space form $\tilde{M}^{2m+s}(c)$ tangent to the global vector field ξ_α with $c \neq 1,-1$ according to whether ξ_α is spacelike or timelike. If the second fundamental form h of M^m satisfies the classical Codazzi equation then M^m is $\phi-$invariant or $\phi-$anti-invariant.*

Proof. By using (6.2.12) and (6.2.13), we get

$$\{g(TX,Z)NY - g(TY,Z)NX + 2g(X,TY)NZ\} = 0, \tag{6.2.14}$$

for all $X,Y,Z \in T(M)$. By contradiction, let there exist $U_x \in T_x(M)$ such that $TU_x \neq 0$ and $NU_x \neq 0$. From (6.2.14), we deduce $2g(U_x,TU_x)NU_x = 0$, which is false. Therefore, for $U_x \in T_x(M)$, we have either $TU_x = 0$ or $NU_x = 0$. It can be also proved that we can not have $U_x, V_x \in T_x(M)$ such that $TU_x \neq 0$, $NU_x = 0$, $TV_x = 0$ and $NV_x \neq 0$. Therefore either $T = 0$ or $N = 0$ which completes the statement.

LEMMA 6.2.2 *Let M^m be a contact CR-submanifold of an indefinite S-manifold $\tilde{M}^{2n+s}$. Then for any $Z,W \in D^{\perp}$, we have*

$$A_{NZ}W + A_{NW}Z = \eta^\alpha(Z)W + \eta^\alpha(W)Z - 2\eta^\alpha(W)\eta^\alpha(Z)\xi_\alpha. \tag{6.2.15}$$

Proof. Proof is straightforward and can be obtained by using equation (6.2.3) and (6.2.5).

Clearly, for $\xi_\alpha \in D$ we also have

$$A_{NZ}W + A_{NW}Z = 0, \tag{6.2.16}$$

where $W, Z \in D^\perp$.

By easy calculations we can get the following lemma:

LEMMA 6.2.3 *Let M^m be a contact CR-submanifold of an indefinite S-manifold $\tilde{M}^{2n+s}$ with $\xi_\alpha \in D$. Then the following are equivalent*

(i) $h(X, TY) = h(TX, Y) \quad \forall\, X, Y \in D$,

(ii) $\tilde{g}(h(X, TY), \phi Z) = \tilde{g}(h(TX, Y), \phi Z) \quad \forall\, X, Y \in D, \quad \forall\, Z \in D^\perp$,

(iii) D is completely Integrable.

For a leaf of anti-invariant distribution $D^\perp$. We prove the following

PROPOSITION 6.2.1 *Let M^m be a contact CR-submanifold of an indefinite S-manifold $\tilde{M}^{2n+s}$. Then any leaf of $D^\perp$ is totally geodesic in M^m if and only if*

$$g(h(D, D^\perp), \phi D^\perp) = 0. \tag{6.2.17}$$

Proof. By hypothesis

$$g(T\nabla_W Z, Y) = -g(A_{NZ}W, Y) - g(th(Z, W), Y) = -g(h(Y, W), NZ), \tag{6.2.18}$$

for any $Y \in D$, $Z, W \in D^\perp$. Then

$$g(\nabla_W Z, TY) = -g(h(Y, W), NZ). \tag{6.2.19}$$

Consequently, $\nabla_W Z \in D^\perp$ if and only if (6.2.17) holds.

REMARK 6.2.1 *Let ν be the complementary orthogonal subbundle of $\phi D^\perp$ in the normal bundle $T(M)^\perp$. Thus we have the following direct sum decomposition*

$$T(M)^\perp = \phi D^\perp \oplus \nu. \tag{6.2.20}$$

Similarly, we can also prove that, $h(X, Y) \in \nu$ and $\phi h(X, Y) = h(X, TY)$ for all X, Y tangent to N^T. On N^T we have an induced indefinite S-structure.

LEMMA 6.2.4 *Let M^m be a contact CR-submanifold of an indefinite S-manifold $\tilde{M}^{2n+s}$ with $\xi_\alpha \in D$. Then for all $X, Y \in D$, we have $\phi h(X, Y) \in \phi D^\perp \oplus \nu$.*

Proof. From above remark 6.2.1 it follows that $\phi\nu = \nu$. Since h is normal to M^m and $\eta^\alpha(D^\perp) = 0$. We easily get the result.

A contact CR-submanifold M^m of an indefinite S-manifold is called contact CR-product if it is locally a Riemannian product of a $\phi-$invariant submanifold N^T tangent to ξ_α and a totally real submanifold $N^\perp$ of $\tilde{M}^{2n+s}$.

THEOREM 6.2.1 *Let M^m be a contact CR-submanifold of an indefinite S-manifold $\tilde{M}^{2n+s}$ and set $\xi_\alpha \in D$. Then M^m is contact CR-product if and only if T satisfies*

$$(\nabla_U T)V = g(TU_D, TV)\xi_\alpha - \eta^\alpha(V)U_D + \eta^\alpha(U_D)\eta^\alpha(V)\xi_\alpha, \tag{6.2.21}$$

where U, V tangent to M^m and we are taking U_D as the component of D.

Proof. Since $\phi \equiv T$ on N^T, due to indefinite S-structure of $\tilde{M}^{2n+s}$ using the Gauss formula we get

$$(\nabla_X T)Y = g(TX, TY)\xi_\alpha - \eta^\alpha(Y)X + \eta^\alpha(X)\eta^\alpha(Y)\xi_\alpha - h(X, TY) + \phi h(X, Y),$$

for any $X, Y \in N^T$. Taking the components in D one gets

$$(\nabla_X T)Y = g(TX, TY)\xi_\alpha - \eta^\alpha(Y)X + \eta^\alpha(X)\eta^\alpha(Y)\xi_\alpha. \tag{6.2.22}$$

Consider now $Z \in N^\perp$ and $Y \in N^T$. Similarly, we can prove

$$(\nabla_Z T)Y = -\eta^\alpha(Y)Z, \tag{6.2.23}$$

as consequence

$$h(Z, TY) = \phi h(Z, Y) + \eta(Y)Z, \quad \forall\, Y \in N^T, \;\; Z \in N^\perp.$$

Now it is easy to show that $(\nabla_U T)Z = 0$ for all $U \in \chi(M)$, $Z \in D^\perp$ and hence the conclusion.

Conversely, consider (6.2.21) exists. Let $U = X$, $V = Z$ with $X \in D$ and $Z \in D^\perp$. The relation (6.2.21) becomes $(\nabla_X T)Z = 0$ and by using (6.2.6) we obtain $th(X, Z) = -A_{NZ}X$. Considering $U = Z, V = X$ (with X, Z as above) we obtain $(\nabla_Z T)X = -\eta^\alpha(X)Z$. Thus one gets

$$A_{NZ}X = \eta^\alpha(X)Z, \tag{6.2.24}$$

for all $X \in D$ and $Z \in D^\perp$. After the computations we obtain $\tilde{g}(h(X, TY) - h(TX, Y)) = 0$. Let $X \in H(M)$, $Z, W \in D^\perp$. Due to (6.2.24) we have

$$\tilde{g}(h(X, Z), \phi W) = \tilde{g}(A_{NW}X, Z) = g(\eta^\alpha(X)W, Z) = \eta^\alpha(X)g(W, Z) = 0.$$

Thus by virtue of proposition (6.2.1), $N^{\perp}$ is totally geodesic in M^m. Let now $X, Y \in D$, from (6.2.21) and (6.2.6) we obtain $th(X,Y) = 0$. If $Z \in D^{\perp}$ we have

$$0 = \tilde{g}(th(X,Y), Z) = \tilde{g}(\tilde{\nabla}_X Y, \phi Z) = -\tilde{g}(Y, (\tilde{\nabla}_X \phi)Z) - \tilde{g}(Y, \phi\tilde{\nabla}_X Z) = -\tilde{g}(\phi Y, \nabla_X Z),$$

replacing Y by ϕY one obtains $\tilde{g}(Y, \nabla_X Z) = 0$ for all $X, Y \in D$ and $Z \in D^{\perp}$. It follows that $g(\nabla_X Y, Z) = 0$ which means that N^T is also totally geodesic in M^m. We may conclude that M^m is a contact CR-product in $\tilde{M}^{2n+s}$.

PROPOSITION 6.2.2 *Let M^m be a contact CR-submanifold of an indefinite S-manifold $\tilde{M}^{2n+s}$ with $\xi_\alpha \in D$. Then M^m is a contact CR-product if and only if*

$$A_{\phi Z} X = \eta^\alpha(X) Z, \tag{6.2.25}$$

for all $X \in D$ and $Z \in D^{\perp}$.

Proof. Suppose that (6.2.25) holds. We have

$$\tilde{g}(h(X,Z), \phi W) = \tilde{g}(A_{\phi W} X, Z) = \eta^\alpha(X) g(Z,W) = 0, \quad \forall X \in H(M),\ \forall Z, W \in D^{\perp}.$$

From proposition 6.2.1 we get that $N^{\perp}$ (the integral manifold of $D^{\perp}$) is totally geodesic in M^m. Consider now $X, Y \in D$ and $Z \in D^{\perp}$. We have

$$\tilde{g}(h(X, \phi Y), \phi Z) = \tilde{g}(A_{\phi Z} X, \phi Y) = \tilde{g}(\eta^\alpha(X) Z, \phi Y) = 0.$$

Similarly $\tilde{g}(h(Y, \phi X), \phi Z) = 0$ and by lemma 6.2.3 it follows that D is completely integrable. To prove that N^T (the integral manifold of D) is totally geodesic in M^m we will prove that $\nabla_X Y$ belongs to N^T for all X, Y tangent to N^T. We have $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z)$. On the other hand, from the hypothesis $\tilde{g}(h(X,Y), \phi Z) = 0$. Then

$$\tilde{g}(h(X,Y), \phi Z) = -\tilde{g}(Y, \tilde{\nabla}_X(\phi Z)) = -\tilde{g}(\phi Y, \tilde{\nabla}_X Z) = -\tilde{g}(\phi Y, \nabla_X Z).$$

So, we obtain $g(\phi Y, \nabla_X Z) = 0$, for all $X, Y \in D$ and for all $Z \in D^{\perp}$. But $g(\xi_\alpha, \nabla_X Z) = 0$ and hence $g(Y, \nabla_X Z) = 0$. We may conclude now that $\nabla_X Y \in N^T$ for all $X, Y \in N^T$. Therefore the two integral manifolds N^T and $N^{\perp}$ are both totally geodesic in M^m Consequently, M^m is locally a Riemannian product of N^T and $N^{\perp}$.

Conversely, from the totally geodesy of N^T and $N^{\perp}$, using the Gauss formula we get $\tilde{g}(\tilde{\nabla}_X Y, \phi Z) = \tilde{g}(h(X,Y), \phi Z)$ with $X, Y \in D$ and $Z \in D^{\perp}$. The right side is exactly $g(A_{\phi Z} X, Y)$ while the left side equals to $-\tilde{g}(\phi Y, \tilde{\nabla}_X Z) = g(\nabla_X(\phi Y), Z) = 0$.

It follows that $A_{\phi Z}X \in D^{\perp}$. Again by using the Gauss formula we obtain after the computations $\eta^{\alpha}(X)g(Z,W) = \tilde{g}(A_{\phi Z}X, W)$. Taking into account that $A_{\phi Z}X \in D^{\perp}$ it follows $A_{\phi Z}X = \eta^{\alpha}(X)Z$. This completes the proof.

Now we will prove the geometrical description of contact CR-products in indefinite S-space forms.

THEOREM 6.2.2 *Let M^m be a generic, simply connected contact CR-submanifold of an indefinite S-space form $\tilde{M}^{2n+s}(c)$, If M^m is a contact CR-product then*

(i) For spacelike global vector field ξ_{α}

$$(\frac{c+3}{4})(g(TX,TZ)NY - g(TY,TZ)NX) = 0.$$

(ii) For timelike global vector field ξ_{α}

$$(\frac{c+5}{4})(g(TX,TZ)NY - g(TY,TZ)NX) = 0.$$

Therefore if $c \neq -3, -5$ and M^m is a ϕ-anti-invariant submanifold of $\tilde{M}^{2n+s}$, then M^m is locally a Riemannian product of an integral curve of ξ_{α} and a totally real submanifold $N^{\perp}$ of $\tilde{M}^{2n+s}$ and if $c = -3, -5$, then M^m is locally a Riemannian product of N^T and $N^{\perp}$.

Proof. Since M^m is generic it follows that $\xi_{\alpha} \in D$. By remark 6.2.1 we have $h(X,Y) = 0$ for all $X, Y \in D$ and $A_{NZ}X = \eta^{\alpha}(X)Z$ for all $X \in D$ and $Z \in D^{\perp}$. Sice $T(M)^{\perp} = \phi D^{\perp}$ and $h \in T(M)^{\perp}$ by using the Weingarten formula we immediately see that $g(h(X,Z), \phi W) = g(A_{\phi W}X, Z) = \eta^{\alpha}(X)g(W,Z)$.

Consequently $h(X,Z) = \eta^{\alpha}(X)\phi Z$ for all $X \in D$ and $Z \in D^{\perp}$.

By making use of (6.2.11) we obtain for $X, U, V \in T(M)$.

$$\begin{aligned}(\nabla_X h)(U,TV) &= -h(U,(\nabla_X T)V + T\nabla_X V) \qquad (6.2.26)\\ &= g(U, g(TX,TV)\xi_{\alpha} - \eta^{\alpha}(V)X + \eta^{\alpha}(X)\eta^{\alpha}(V)\xi_{\alpha})\\ &= g(TX,TV)\eta^{\alpha}(U) - \eta^{\alpha}(V)g(U,X) + \eta^{\alpha}(X)\eta^{\alpha}(V)\eta^{\alpha}(U),\end{aligned}$$

hence we get

$$(\nabla_X h)(U,TV) = g(TX,TV)NU - \eta^{\alpha}(V)g(U,X) + \eta^{\alpha}(X)\eta^{\alpha}(V)\eta^{\alpha}(U) \quad (6.2.27)$$

Substitute in (6.2.12) Z by TZ (with $Z \in T(M)$ arbitrary) the following identity holds:

$$(\nabla_X h)(Y, TZ) - (\nabla_Y h)(X, TZ) = \frac{c-\epsilon}{4}\{g(TX, TZ)NY - g(TY, TZ)NX\},$$

where $\epsilon = \pm 1$ according to whether ξ_α is spacelike or timelike. Combining with (6.2.27) above relation yields to

$$g(TX, TZ)NY - g(TY, TZ)NX = \frac{c-\epsilon}{4}\{g(TX, TZ)NY - g(TY, TZ)NX\},$$

which is equivalent to

$$(\frac{c-\epsilon}{4} + 1)(g(TX, TZ)NY - g(TY, TZ)NX) = 0. \tag{6.2.28}$$

Now we have to discuss two situations: $\epsilon = \pm 1$. For spacelike global vector field $\epsilon = +1$, above equation becomes

$$(\frac{c+3}{4})(g(TX, TZ)NY - g(TY, TZ)NX) = 0. \tag{6.2.29}$$

For timelike global vector field $\epsilon = -1$, the above equation becomes

$$(\frac{c+5}{4})(g(TX, TZ)NY - g(TY, TZ)NX) = 0. \tag{6.2.30}$$

Now we discuss the two cases

Case I. For $c \neq -3, -5$, From the equation (6.2.29) and (6.2.30) we obtain $g(TY, TZ)NX - g(TX, TZ)NY = 0$, for all $X, Y, Z \in T(M)$: Since M^m is generic we have $N \neq 0$ and it is not difficult to prove that $T = 0$, thus M^m is ϕ-anti-invariant. Moreover, by theorem 6.2.1 we can say that M^m is a contact CR-product of an integral curve of ξ_α and a totally real submanifold $N^\perp$ of $\tilde{M}^{2n+s}$.

Case II. For $c = -3, -5$ from (6.2.29) and (6.2.30). M is a contact CR-product of the invariant submanifold N^T and the anti-invariant submanifold $N^\perp$: Since N^T is totally geodesic in M^m and $h(X, Y) = 0$ for all $X, Y \in D$ then N^T is totally geodesic in $\tilde{M}^{2n+s}$. Thus, we can use the well known result that M^m has constant $\phi-$ sectional curvature, then M^m is simply connected and hence M^m is the Riemannian product of N^T and $N^\perp$.

Let $\tilde{H}_h(U, V)$ be the ϕ-holomorphic bisectional curvature of the plane $U \wedge V$, i.e.

$$\tilde{H}_B(U, V) = \tilde{R}(\phi U, U; \phi V, V) \qquad for \;\; U, V \in T(M).$$

We prove the following important lemmas for later use.

LEMMA 6.2.5 *Let M^m be a contact CR-product of a indefinite S-manifold $\tilde{M}^{2n+s}$. Then, for any unit vector fields $X \in D$ and $Z \in D^\perp$, then:*

$$\tilde{g}(h(\nabla_{\phi X}X, \phi Z), Z) = -s, \quad \tilde{g}(h(\nabla_X \phi X, \phi Z), Z) = s, \tag{6.2.31}$$

$$\tilde{g}(h(X, \nabla_{\phi X}\phi Z), Z) = 0, \quad \tilde{g}(h(\phi X, \nabla_X \phi Z), Z) = 0. \tag{6.2.32}$$

Proof. By theorem (6.2.1)

$$\tilde{g}(h(\nabla_{\phi X}X, \phi Z), Z) = \sum_{\alpha=1}^{s} \eta^\alpha(\nabla_{\phi X}X) g(\phi^2 Z, Z) = -\sum_{\alpha=1}^{s} g(X, \nabla_{\phi X}\xi_\alpha) g(Z, Z)$$

$$= -\sum_{\alpha=1}^{s} g(\phi X, \phi X) = -s,$$

which is first part of (6.2.31). We also have

$$\tilde{g}(h(\nabla_X \phi X, \phi Z), Z) = \sum_{\alpha=1}^{s} \eta^\alpha(\nabla_X \phi X) g(Z, Z) = -\sum_{\alpha=1}^{s} g(\phi X, \nabla_X \xi_\alpha) g(Z, Z)$$

$$= \sum_{\alpha=1}^{s} g(\phi X, \phi X) = s,$$

which is other part of (6.2.31). Finally

$$\tilde{g}(h(X, \nabla_{\phi X}\phi Z), Z) = \sum_{\alpha=1}^{s} \eta^\alpha(X) g(\nabla_{\phi X}\phi Z, Z) = 0,$$

$$\tilde{g}(h(\phi X, \nabla_X \phi Z), Z) = \sum_{\alpha=1}^{s} \eta^\alpha(\phi X) g(\nabla_X \phi Z, Z) = 0,$$

which are (6.2.32).

LEMMA 6.2.6 *Let M^m be a contact CR-product of a indefinite S-manifold $\tilde{M}^{2n+s}$. Then, for any unit vector fields $X \in D$ and $Z \in D^\perp$ we have*

$$\tilde{H}_B(X, Z) = 2s - 2\|h(X, Z)\|^2 \tag{6.2.33}$$

Proof. We know

$$\tilde{R}(\phi X, X, \phi Z, Z) = \tilde{g}((\nabla^h{}_{\phi X} h)(X, \phi Z) - (\nabla^h{}_X h)(\phi X, \phi Z), Z)$$

by using (6.2.11) and above lemma, we gets

$$\begin{aligned}\tilde{R}(\phi X, X, \phi Z, Z) &= \tilde{g}(\tilde{\nabla}_{\phi X} h(X, \phi Z) - h(\nabla_{\phi X} X, \phi Z) - h(X, \nabla_{\phi X}\phi Z), Z)\\ &\quad - \tilde{g}(\tilde{\nabla}_X h(\phi X, \phi Z) - h(\nabla_X \phi X, \phi Z) - h(\phi X, \nabla_X \phi Z), Z)\\ &= 2s - \tilde{g}(h(X, \phi Z), \nabla_{\phi X} Z) - \tilde{g}(h(X, \phi Z), h(\phi X, Z))\\ &\quad + g(h(\phi X, \phi Z), \nabla_X Z) + g(h(\phi X, \phi Z), h(X, Z)).\end{aligned}$$

Now, using the fact that $\nabla_X Z$ and $\nabla_{\phi X} Z$ belong to $D^{\perp}$, therefore above equation becomes

$$\tilde{R}(\phi X, X, \phi Z, Z) = 2s - 2\|h(X, Z\|^2,$$

this ends the proof.

We come to know that $\tilde{H}_B(U, \xi_\alpha) = 0$ and $h(U, \xi_\alpha) = \phi U$, So, when we will refer to the ϕ-holomorphic bisectional curvature of the plane $U \wedge V$, we intend that this plane is orthogonal ξ_α. Thus for X in the above lemma we can suppose that it belongs to $H(M)$.

PROPOSITION 6.2.3 *Let $\tilde{M}^{2n+s}(c)$ be a indefinite S-space form and let X, Z be two unit vector fields orthogonal to global vector filed ξ_α. Then the ϕ-holomorphic bisectional curvature of the plane $X \wedge Z$ is given by*

(i) For spacelike global vector field ξ_α

$$\tilde{H}_B(X, Z) = \frac{c-1}{4} g(\phi X, Z) - \frac{c+3}{4} g(\phi X, Z)^2 + c. \tag{6.2.34}$$

(ii) For timelike global vector field ξ_α

$$\tilde{H}_B(X, Z) = \frac{c+1}{4} g(\phi X, Z) - \frac{c-3}{4} g(\phi X, Z)^2 + c. \tag{6.2.35}$$

COROLLARY 6.2.1 *Let $\tilde{M}^{2n+s}(c)$ be a indefinite S-space form and let $X \in H(M)$ and $Z \in D^{\perp}$ be unit vector fields orthogonal to global vector filed ξ_α. Then the ϕ-holomorphic bisectional curvature of the plane $X \wedge Z$ for spacelike and timelike global vector field ξ_α is given by*

$$\tilde{H}_B(X, Z) = c. \tag{6.2.36}$$

THEOREM 6.2.3 *Let $\tilde{M}^{2m+s}(c)$ be a indefinite S-space form and let $M = N^T \times N^{\perp}$ be a contact CR-product in $\tilde{M}^{2n+s}$. Then the norm of the second fundamental form of M satisfies the inequality*

$$\|h\|^2 \geq ((3c + 8s - 3\epsilon)p + 2s)q, \tag{6.2.37}$$

where $\epsilon = \pm 1$ according to whether global vector field ξ_α is spacelike or timelike. The equality sign holds if and only if both N^T and $N^\perp$ are totally geodesic.

Proof. For $X \in H(M)$ and $Z \in D^\perp$ we have $\|h(X,Z)\|^2 = \frac{1}{4}(3c + 8s - 3\epsilon)$.

Now, we choose a local field of orthonormal frames

$$\{X_1, ..., X_p, X_{p+1} = \phi X_1, ..., X_{2p} = \phi X_p, X_{2p+1} = Z_1, ..., X_n = Z_q,$$

$$X_{n+1} = \phi Z_1, ..., X_{n+q} = \phi Z_q, X_{n+q+1}, ..., X_2 m, \xi_1, ..., \xi_s\}$$

on $\tilde{M}^{2n+s}(c)$ in such a way that $\{X_1, ..., X_{2p}\}$ is a local frame field on D and $\{Z_1, ..., Z_q\}$ is a local frame field on $D^\perp$. Thus

$$\|h\|^2 = \|h(D,D)\|^2 + \|h(D^\perp, D^\perp\|^2 + 2\|h(D, D^\perp\|^2 \geq 2\|h(D, D^\perp\|^2$$

$$= 2(\sum_{i=1}^{2p}\sum_{j=1}^{q} \|h(X_i, Z_j)\|^2 + \sum_{\alpha=1}^{s}\sum_{j=1}^{q} \|h(\xi_\alpha, Z_j\|^2)$$

$$= ((3c + 8s - 3\epsilon)p + 2s)q,$$

where X_i and Z_j are orthonormal basis in $H(M)$ and $D^\perp$ respectively. The equality sign holds if and only if $h(D,D) = 0$ and $h(D^\perp, D^\perp) = 0$, which is equivalent to the totally geodesy of N^T and $N^\perp$.

6.3. CR-WARPED PRODUCT SUBMANIFOLDS OF INDEFINITE S-MANIFOLDS

The main purpose of this section is devoted to the presentation of some properties of warped product contact CR-submanifolds in indefinite S-manifolds.

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. The *warped product* of N_1 and N_2 is the product manifold $N_1 \times {}_f N_2 = (N_1 \times N_2,\ g)$, where

$$g = g_1 + f^2 g_2, \tag{6.3.1}$$

where f is called the *warping function* of the warped product. The warped product $N_1 \times {}_f N_2$ is said to be *trivial* or simply Riemannian product if the warping function f is constant. This means that the Riemannian product is a special case of warped product.

THEOREM 6.3.1 *Let $\tilde{M}^{2n+s}$ be a indefinite S-manifold and let $M = N^{\perp} \times_f N^T$ be a warped product CR-submanifold such that $N^{\perp}$ is a totally real submanifold and N^T is ϕ-holomorphic (invariant) of $\tilde{M}^{2n+s}$. Then M is a CR-product.*

Proof. Let X be tangent to N^T and let Z be a vector field tangent to $N^{\perp}$. From the lemma 5.2.1 we find that

$$\nabla_X Z = (Zlnf)X. \qquad (6.3.2))$$

Now we have two cases either ξ_α is tangent to N^T or ξ_α is tangent to $N^{\perp}$.

Case I. ξ_α is tangent to N^T. Take $X = \xi_\alpha$. Since $\nabla_Z \xi_\alpha = -TZ = 0$ and $\nabla_Z \xi_\alpha = \nabla_{\xi_\alpha} Z$ (ξ_α is tangent to N^T while Z is tangent to $N^{\perp}$) one gets $0 = Z(lnf)\xi_\alpha$ and hence $Z(lnf) = 0$ for all Z tangent to $N^{\perp}$. Consequently f is constant and thus the warped product above is nothing but a Riemannian product.

Case II. Now we will consider the other case, ξ_α is tangent to $N^{\perp}$. Similarly, take $Z = \xi_\alpha$. Since $\nabla_X \xi_\alpha = -\epsilon_\alpha TX = -\epsilon_\alpha \phi X$ it follows $-\epsilon_\alpha \phi X = (\xi_\alpha lnf)X$. But this is impossible if $dimN^T \neq 0$.

REMARK 6.3.1: *There do not exist warped product CR-submanifolds in the form $N^{\perp} \times_f N^T$ other than CR-products such that N^T is a ϕ-invariant submanifold and $N^{\perp}$ is a totally real submanifold of $\tilde{M}$.*

From now on we will consider warped product CR-submanifolds in the form $N^T \times_f N^{\perp}$.

We can say that *A contact CR-submanifold M of a indefinite S-manifold $\tilde{M}^{2n+s}$, tangent to the structure vector field ξ_α is called a contact CR-warped product if it is the warped product $N^T \times_f N^{\perp}$ of an invariant submanifold N^T, tangent to ξ_α and a totally real submanifold $N^{\perp}$ of $\tilde{M}^{2n+s}$, where f is the warping function.*

LEMMA 6.3.1 *Let M^m be a contact CR-submanifold in indefinite S-manifold $\tilde{M}^{2n+s}$, such that $\xi_\alpha \in D$. Then we have*

$$g(\nabla_U X, Z) = -\tilde{g}(\phi A_{\phi Z} U, X), \quad \forall\ X \in D,\ \ \forall\ Z \in D^{\perp},\ \ \forall\ U \in T(M), \quad (6.3.3)$$

$$A_{\phi\mu} X + A_\mu \phi X = 0 \quad \forall\ X \in D,\ \ \forall\ \mu \in \nu. \qquad (6.3.4)$$

Proof. We have $\tilde{g}(\phi A_{\phi Z} U, X) = \tilde{g}(A_{\phi Z} U, \phi X) = \tilde{g}(\nabla^{\perp}{}_U \phi Z - \tilde{\nabla}_U \phi Z, \phi X) = -\tilde{g}(\phi \tilde{\nabla}_U Z, \phi X) = -\tilde{g}(\tilde{\nabla}_U Z, X) + \eta^\alpha(\tilde{\nabla}_U Z)\eta^\alpha(X) = -g(\nabla_U Z, X) - \tilde{g}(Z, \tilde{\nabla}_U \xi)\eta^\alpha(X) =$

$-g(\nabla_U Z, X) + g(Z, \phi U)\eta^\alpha(X) = -g(\nabla_U Z, X)$. So, equation (6.3.3) follows.

For the proof of the equation (6.3.4) we have $g(A_{\phi\mu}X, U) = -g(\tilde{\nabla}_X \phi\mu, U) = g(\mu, \phi\tilde{\nabla}_X U)$ and $g(A_\mu U, \phi X) = -g(\mu, \phi\tilde{\nabla}_U X)$ with $U \in T(M)$. It follow that $A_{\phi\mu}X + A_\mu \phi X = 0$, $\forall\ X \in D$, $\forall\ \mu \in \nu$.

LEMMA 6.3.2 *If $M = N^T \times_f N^\perp$ is a contact CR-warped product in a indefinite S-manifold $\tilde{M}^{2n+s}$ the for X tangent to N^T and Z, W tangent to $N^\perp$ we have*

$$g(h(D, D), \phi D^\perp) = 0 \tag{6.3.5}$$

$$\xi_\alpha(f) = 0 \tag{6.3.6}$$

$$g(h(\phi X, Z), \phi W) = (X ln f) g(Z, W) \tag{6.3.7}$$

Proof. For any $X, Y \in D$ and $Z \in D^\perp$, we have

$$g(h(X, Y), \phi Z) = g(\tilde{\nabla}_X Y, \phi Z) = -g(\phi Y, \tilde{\nabla}_X Z) = g(\tilde{\nabla}_X \phi Y, Z) = 0.$$

We know that $\nabla_U \xi_\alpha = -\epsilon_\alpha TU$. It follows that $\nabla_Z \xi_\alpha = 0$ for all Z tangent to $N^\perp$. Using lemma 6.3.1 and theorem 6.3.1 we get equation (6.3.6).

From equation (6.3.2) it follows that

$$g(h(\phi X, Z), \phi W) = g(A_{\phi W} Z, \phi X) = -g(\nabla_Z W, X) = X(ln f) g(Z, W).$$

Hence proved.

THEOREM 6.3.2 *The necessary and sufficient condition for a strictly proper CR-submanifold M of a indefinite S-manifold $\tilde{M}^{2n+s}$, tangent to the structure vector field ξ_α to be locally a contact CR-warped product is that*

$$A_{\phi Z} X = (-(\phi X)(\mu) - \eta^\alpha(X)) Z + \eta^\alpha(X)\eta^\alpha(Z)\xi_\alpha \tag{6.3.8}$$

for some function μ on M satisfying $W\mu = 0$ for all $W \in D^\perp$.

Proof. Let $M = N^T \times_f N^\perp$ be a locally contact CR-warped product.

Consider $X, Y \in D$, $Z \in D^\perp$. We can easily get $g(A_{\phi Z} X, Y) = 0$, which shows that $A_{\phi Z} X$ belongs to $D^\perp$.

Now take any $W \in D^\perp$, we get

$$g(A_{\phi Z} X, W) = (-(\phi X)(\mu) - \eta^\alpha(X)) g(W, Z) + \eta^\alpha(X)\eta^\alpha(Z)\eta^\alpha(W)$$

hence the result where $\mu = lnf$.

Conversely, Let

$$A_{\phi Z}X = (-(\phi X)(\mu) - \eta^{\alpha}(X))Z + \eta^{\alpha}(X)\eta^{\alpha}(Z)\xi_{\alpha}.$$

We get easily that $\tilde{g}(h(\phi X, Y), \phi Z) = 0$,

Also

$$\tilde{g}(h(X, W), \phi Z) = (-(\phi X)(\mu) - \eta^{\alpha}(X))Z + \eta^{\alpha}(X)\eta^{\alpha}(Z)\xi_{\alpha},$$

where $X, Y \in D$ and $Z, W \in D^{\perp}$. In the above equation replacing X by ϕX, we obtain

$$\tilde{g}(h(\phi X, W), \phi Z) = (X(\mu) - \eta^{\alpha}(X)\xi_{\alpha}(\mu))g(Z, W).$$

So, if we take $X \in H(M)$ it becomes $\tilde{g}(h(\phi X, W), \phi Z) = (X(\mu)g(Z, W)$ and if $X = \xi_{\alpha}$ we get $\tilde{g}(h(\phi X, W), \phi Z) = (1 - \delta_{\alpha}{}^{\alpha})\xi_{\alpha}(\mu)g(Z, W)$.

Now, we will prove the main theorems of the chapter.

Let M be a (pseudo-)Riemannian k-manifold with inner product $<, >$ and $e_1, ..., e_k$ be an orthonormal frame fields on M. For differentiable function ϕ on M, the gradient $\nabla\phi$ and the Laplacian $\triangle\phi$ of ϕ are defined respectively by

$$< \nabla\phi, X >= X(\phi), \tag{6.3.9}$$

$$\triangle\phi = \sum_{j=1}^{k}\{(\nabla_{e_j}e_j)\phi - e_je_j(\phi)\} = -div\nabla\phi \tag{6.3.10}$$

for vector field X tangent to M, where ∇ is the Riemannian connection on M. As consequence, we have

$$\|\nabla\phi\|^2 = \sum_{j=1}^{k}(e_j(\phi))^2. \tag{6.3.11}$$

THEOREM 6.3.3 *Let $M = N^T \times_f N^{\perp}$ be a contact CR-warped product of a indefinite S-space form $\tilde{M}^{2n+s}(c)$. Then the second fundamental form of M satisfies the following inequality*

$$\|h\|^2 \geq p\{3\|\nabla lnf\|^2 - \triangle lnf + (c+2)k + 1\}. \tag{6.3.12}$$

Proof. We have

$$\|h(D, D^{\perp})\|^2 = \sum_{j=1}^{k}\sum_{i=1}^{p}\|h(X_j, Z_i)\|^2, \tag{6.3.13}$$

where X_j for $\{j = 1, ..., k\}$ and Z_α for $\alpha = \{1, ..., p\}$ are orthonormal frames on N^T and $N^\perp$, respectively. On N^T we will consider a ϕ-adapted orthonormal frame, namely $\{e_j, \phi e_j, \xi_\alpha\}$, where $\{j = 1, ..., k\}, \{\alpha = 1, ..., s\}$.

We have to evaluate $\|h(X, Z)\|^2$ with $X \in D$ and $Z \in D^\perp$. The second fundamental form $h(X, Z)$ is normal to M so, it splits into two orthogonal components

$$h(X, Z) = h_{\phi D^\perp}(X, Z) + h_\nu(X, Z), \tag{6.3.14}$$

where $h_{\phi D^\perp}(X, Z) \in \phi D^\perp$ and $h_\nu(X, Z) \in \nu$. So

$$\|h(X, Z)\|^2 = \|h_{\phi D^\perp}(X, Z)\|^2 + \|h_\nu(X, Z)\|^2. \tag{6.3.15}$$

If $X = \xi_\alpha$, we have $h(\xi_\alpha, Z) = -\phi Z$. Hence

$$h_{\phi D^\perp}(\xi_\alpha, Z) = -\phi Z, \quad h_\nu(\xi_\alpha, Z) = 0. \tag{6.3.16}$$

Consider now $X \in H(M)$ and let's compute the norm of the $\phi D^\perp$-component of $h(X, Z)$. We have

$$\|h_{\phi D^\perp}(X, Z)\|^2 =< h_{\phi D^\perp}(X, Z), h(X, Z) > .$$

By using relation (6.3.7), after the computations, we obtain

$$\|h_{\phi D^\perp}(X, Z)\|^2 = (\phi X(lnf))^2 \|Z\|^2.$$

So

$$\|h_{\phi D^\perp}(e_j, Z_i)\|^2 = (\phi e_j(lnf))^2, \quad \|h_{\phi D^\perp}(\phi e_j, Z_i)\|^2 = (e_j(lnf))^2. \tag{6.3.17}$$

On the other hand, from (6.3.11) we have

$$\|\nabla lnf\|^2 = \sum_{j=1}^{k} (e_j(lnf))^2 + \sum_{j=1}^{k} (\phi e_j(lnf))^2. \tag{6.3.18}$$

Since $\xi_\alpha(lnf) = 0$. Finally we can compute the norm $\|h_{\phi D^\perp}(D, D^\perp)\|^2$. Thus $\|h_{\phi D^\perp}(D, D^\perp)\|^2 = \sum_{j=1}^{k}\sum_{i=1}^{p}\{\|h_{\phi D^\perp}(e_j, Z_i)\|^2 + \|h_{\phi D^\perp}(\phi e_j, Z_i)\|^2\} + \sum_{\alpha=1}^{s}\sum_{i=1}^{p}\|h_{\phi D^\perp}(\xi_\alpha, Z_i)\|^2 = \sum_{i=1}^{p}\|\nabla lnf\|^2 + \sum_{i=1}^{p}\|\phi Z_i\|^2$. Since $\|\phi Z_i\|^2 = 1$ we can conclude that

$$\|h_{\phi D^\perp}(D, D^\perp)\|^2 = \{\sum_{i=1}^{p} \|\nabla lnf\|^2 + 1\}p. \tag{6.3.19}$$

Now we will compute the norm of the ν-component of $h(X,Z)$. We have

$$\|h_\nu(X,Z)\|^2 =< h_\nu(X,Z), h(X,Z) >=< A_{h_\nu(X,Z)}X, Z >,$$

by using lemma (6.3.1) we can write $A_{h_\nu(X,Z)}X = A_{\phi h_\nu(X,Z)}(\phi X)$ so,

$$\|h_\nu(X,Z)\|^2 =< \phi h(X,Z) - \phi h_{\phi D^\perp}(X,Z), h(\phi X, Z) > .$$

Since $\phi h_{\phi D^\perp}(X,Z)$ belongs to $D^\perp$ we obtain

$$\|h_\nu(X,Z)\|^2 = \tilde{g}(\phi h(X,Z), h(\phi X, Z)), \quad \forall\ X \in H(M),\ \ Z \in D^\perp. \qquad (6.3.20)$$

Consider the tensor field $\tilde{H}_B$. As we already have seen

$$\tilde{H}_B(X,Z) =< (\nabla_{\phi X})h(X,Z) - (\nabla_X h)(\phi X, Z), \phi Z >, \quad \forall\ X \in H(M),\ \ Z \in D^\perp.$$

Using the definition of ∇h, developing the expression above we obtain

$$\tilde{H}_B(X,Z) =< \nabla^\perp{}_{\phi X} h(X,Z), \phi Z > - < h(\nabla_{\phi X}X, Z), \phi Z > - < h(X, \nabla_{\phi X}Z), \phi Z >$$
$$- < \nabla^\perp{}_X h(\phi X, Z), \phi Z > + < h(\nabla_X \phi X, Z), \phi Z > + < h(\phi X, \nabla_X Z), \phi Z >,$$

after using lemma 6.3.2 and theorem 6.3.2, we get

$$\tilde{H}_B(X,Z) = \|Z\|^2\{(\phi X(lnf))^2 - (\phi X)(\phi X(lnf)) + (Xlnf)^2 - X(Xlnf)$$
$$+ (\phi\nabla_{\phi X}X)(lnf) - \|X\|^2 - (\phi\nabla_X \phi X)(lnf) - \|X\|^2$$
$$+ (\phi X(lnf))^2 + (X(lnf))^2\} + 2 < \phi h(X,Z), h(\phi X, Z) >,$$

which becomes

$$\tilde{H}_B(X,Z) = \|Z\|^2\{2(\phi X(lnf))^2 - (\phi X)^2(lnf) + 2(Xlnf)^2 - X^2(lnf)$$
$$+ ((\phi\nabla_{\phi X}X) - (\phi X \nabla_X \phi X))(lnf) - 2\|X\|^2\} + 2\|h_\nu(X,Z)\|^2. \quad (6.3.21)$$

We can easily prove that

$$(\phi\nabla_{\phi X}X)(lnf) = (\nabla_{\phi X}\phi X)(lnf), \qquad (\phi\nabla_X \phi X)(lnf) = -\nabla_X X(lnf). \quad (6.3.22)$$

Using equations (6.3.21) and (6.3.22), we get

$$\tilde{H}_B(X,Z) = \|Z\|^2\{2(\phi X(lnf))^2 - (\phi X)^2(lnf) + 2(Xlnf)^2 - X^2(lnf) \quad (6.3.23)$$
$$+ (\nabla_{\phi X}(\phi X) + \nabla_X X)(lnf) - 2\|X\|^2\} + 2\|h_\nu(X,Z)\|^2.$$

Using orthonormal frames, we have

$$\tilde{H}_B(e_j, Z_i) = \|Z_i\|^2\{2(\phi e_j(lnf))^2 - (\phi e_j)^2(lnf) + 2(e_j lnf)^2 - {e_j}^2(lnf) \quad (6.3.24)$$
$$+ (\nabla_{\phi e_j}(\phi e_j) + \nabla_{e_j} e_j)(lnf) - 2\|X_j\|^2\} + 2\|h_\nu(e_j, Z_i)\|^2.$$

Similarly,

$$\tilde{H}_B(\phi e_j, Z_i) = \|Z_i\|^2\{2(e_j(lnf))^2 - {e_j}^2(lnf) + 2(\phi e_j lnf)^2 - (\phi e_j)^2(lnf) \quad (6.3.25)$$
$$+ (\nabla_{e_j} e_j + \nabla_{\phi e_j}(\phi e_j))(lnf) - 2\|X_j\|^2\} + 2\|h_\nu(\phi e_j, Z_i)\|^2.$$

On the other hand we have

$$\triangle(lnf) = \sum_{j=1}^{k}\{(\nabla_{e_j} e_j)(lnf) - {e_j}^2(lnf)\} + \sum_{j=1}^{k}\{(\nabla_{\phi e_j}\phi e_j)(lnf) - \phi {e_j}^2(lnf)\}$$

by using (6.3.11), we get

$$2\|\nabla lnf\|^2 = 2\sum_{j=1}^{k}(e_j(lnf))^2 + 2\sum_{j=1}^{k}(\phi e_j(lnf))^2.$$

Since $\xi_\alpha lnf = 0$. Taking the sum of (6.3.24) and (6.3.25) we get

$$2\sum_{j=1}^{k}\sum_{i=1}^{p}\{\|h_\nu(e_j, Z_i)\|^2 + \|h_\nu(\phi e_j, Z_i)\|^2\} = \sum_{j=1}^{k}\sum_{i=1}^{p}\{\tilde{H}_B(e_j, Z_i) + \tilde{H}_B(\phi e_j, Z_i)\}$$
$$- 2p\triangle(lnf) + 4kp + 4\|\nabla lnf\|^2,$$

by using proposition 6.3.3, we have

$$\sum_{j=1}^{k}\sum_{i=1}^{p}\{\|h_\nu(e_j, Z_i)\|^2 + \|h_\nu(\phi e_j, Z_i)\|^2\} = ckp - p\triangle(lnf) + 2kp + 2p\|\nabla lnf\|^2. \quad (6.3.26)$$

Now from (6.3.15), (6.3.19) and (6.3.26) we conclude that h satisfies the inequality.

Chapter VII

WARPED AND DOUBLY WARPED PRODUCT PSEUDO-SLANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLDS

WARPED AND DOUBLY WARPED PRODUCT PSEUDO-SLANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLDS

7.1. Introduction

In chapter 6 we have studied warped product contact CR-submanifolds of globally framed f-manifolds with Lorentz metric. In this chapter, we change the ambient space to trans-Sasakian manifold and study another important class of warped product bi-slant submanifolds, namely warped product pseudo-slant submanifolds (*also known as Hemi-slant submanifolds* [67]). we have also studied doubly warped products which were introduced as generalization of warped product submanifolds by B. Unel [73]. A doubly warped product (M, g) is a product manifold of the form $M =_f B \times_b F$ with the metric $g = f^2 g_B \oplus b^2 g_F$, where $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ are smooth maps and g_B, g_F are the metric on the Riemannian manifolds B and F respectively. If either $b = 1$ or $f = 1$, but not both, then we obtain a (*single*) warped product. If both $b = 1$ and $f = 1$, then we have a product manifold. If neither b nor f is constant, then we have a non trivial doubly warped product.

If $X \in T(B)$ and $Z \in T(F)$, then the Levi-Civita connection is

$$\nabla_X Z = Z(lnf)X + X(lnb)Z. \tag{7.1.1}$$

In the present chapter, we have studied the warped and doubly warped pseudo-slant submanifolds of trans-Sasakian manifolds and obtained some interesting properties of these submanifolds with differential geometric point of view and provided an example of these warped product submanifolds.

An almost contact metric structure on a manifold $\bar{M}$ is called a trans-Sasakian structure if the product manifold $\bar{M} \times R$ belongs to class $\mathcal{W}_4$. J. C. Marrero [54] studied the tensor equations characterizing the trans-Sasakian structure. D. E. Blair [12] and J. A. Oubina [62] showed that an almost contact metric manifold $\bar{M}$ with a contact metric structure (ϕ, ξ, η, g) is trans-Sasakian if

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{7.1.2}$$

[1]Results of this chapter are Published in Thai Journal of Mathematics.

for some smooth functions α and β on $\bar{M}$ and ∇ being the Levi-Civita connection of $\bar{M}$. The trans-Sasakian structure on a manifold defined by equation (7.1.2) are termed as structure of type (α, β). From the formula (7.1.2) it follows that

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{7.1.3}$$

We note that trans-Sasakian structure of type $(0, 0)$ are Cosymplectic [12]. Trans-Sasakian structure of type $(0, \beta)$ are β-Kenmotsu [42] and trans-Sasakian structure of type $(\alpha, 0)$ are α-Sasakian [42]. Their tensorial equations are respectively given by

$$(\bar{\nabla}_X \phi)Y = 0$$

,

$$(\bar{\nabla}_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

and

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X)$$

$X, Y \in T\bar{M}$.

NOTE 7.1.1. Throughout, the structure vector field ξ is assumed to be tangential to M, for otherwise M is simply anti-invariant by virtue of theorem 1.3.10.

Now, Let M be a submanifold immersed in $\bar{M}$. For any $X \in TM$, applying (1.2.2) and (1.2.6) we have,

$$\nabla_X \xi + h(X, \xi) = -\alpha TX - \alpha NX + \beta(X - \eta(X)\xi).$$

Comparing tangential and normal parts in the above equation we find that

$$(a)\ \ \nabla_X \xi = -\alpha TX + \beta(X - \eta(X)\xi) \quad (b)\ \ h(X, \xi) = -\alpha NX. \tag{7.1.4}$$

If $(\bar{M}, \phi, \xi, \eta, g)$ is a trans-Sasakian manifold, then $(\bar{M} \times R, J, g_1)$ belongs to the class $\mathcal{W}_\triangle$ of almost Hermitian manifold. The almost complex structure J and Hermitian metric g_1 on $\bar{M} \times R$ are defined as under.

A vector field on $\bar{M} \times R$ has the form $(X, a\frac{d}{dt})$, where X is a vector field tangential to $\bar{M}$, t denotes the coordinate function on R and a is a differentiable function on $\bar{M} \times R$. The almost complex structure J on $\bar{M} \times R$ is defined as

$$J(X, a\frac{d}{dt}) = (\phi X - a\xi, \eta(X)\frac{d}{dt}). \tag{7.1.5}$$

The product metric g_1 on $\bar{M} \times R$ is given by

$$g_1((X, a\frac{d}{dt}), (Y, b\frac{d}{dt})) = g(X, Y) + ab, \tag{7.1.6}$$

which is compatible with the almost complex structure J on $\bar{M} \times R$ turning $\bar{M} \times R$, into a Hermitian Manifold.

7.2. Pseudo-slant Submanifolds of Trans-Sasakian Manifolds

Our aim in the present section is to study pseudo-slant submanifolds of a trans-Sasakian manifold and obtain integrability conditions for the distributions on these submanifolds. To begin with, we elaborate as to how, a pseudo-slant submanifold of an almost Hermitian manifold is obtained via a slant submanifold of an almost contact metric manifold.

Let $\bar{M}$ be an almost contact metric manifold with an almost structure (ϕ, ξ, η, g) and M be a submanifold isometrically immersed into $\bar{M}$ with structure vector field ξ tangent to M. D denote the orthogonal complement of $< \xi >$ i.e.,

$$TM = D \oplus < \xi > .$$

The following theorem provides a mechanism of obtaining a pseudo-slant submanifold of $\bar{M} \times R$ from a slant submanifold of $\bar{M}$.

THEOREM 7.2.1. *Let M be a non anti-invariant even dimensional slant submanifold of an almost contact metric manifold $\bar{M}$. Then $M \times R$ is a pseudo-slant submanifold of $\bar{M} \times R$.*

Proof. Set, $D^{\perp} = \{(0, \frac{d}{dt})\}$ and $D_{\theta} = \{(X, 0) | X \in D\}$.

It is easy to check that $D^{\perp}$ and D_{θ} are orthogonal with respect to the product metric g_1 on $M \times R$ (see equation (7.1.6) and,

$$T(M \times R) = D^{\perp} \oplus D_{\theta}.$$

Moreover, by equation (7.1.5),

$$J(0, \frac{d}{dt}) = (-\xi, 0).$$

$D^{\perp}$ is a totally real distribution by Theorem 1.3.9 and D_{θ} defines a slant distribution on $M \times R$ in the sense of N. Papaghiuc [64]. Therefore by definition, $M \times R$ is a pseudo-slant submanifold of $\bar{M} \times R$.

As a special case of bi-slant submanifolds of an almost contact metric manifold, we study pseudo-slant submanifold which are formally defined as follows:

DEFINITION 7.2.1. We say that M is a pseudo-slant submanifold of an almost contact metric manifold $\bar{M}$, if there exist two orthogonal distributions $D^{\perp}$ and D_θ on M such that,

(i) $TM = D^{\perp} \oplus D_\theta \oplus < \xi >$,

(ii) The distribution $D^{\perp}$ is anti-invariant *i.e.*, $\phi D^{\perp} \subseteq T^{\perp}M$,

(iii) The distribution D_θ is slant with slant angle $\theta \neq \pi/2$.

Suppose that M is a pseudo-slant submanifold of an almost contact metric manifold. Then, for any $X \in TM$, put

$$X = P_1X + P_2X + \eta(X)\xi \tag{7.2.1}$$

where $P_i = (i = 1, 2)$ are projection maps on the distributions $D^{\perp}$ and D_θ. Now operating ϕ on both sides of equation (7.2.1), and using equation (1.2.16), we find that

$$TX = TP_2X, \qquad NX = NP_1X + NP_2X, \tag{7.2.2}$$

$$\phi P_1X = NP_1X, \quad TP_1X = 0, \tag{7.2.3}$$

$$TP_2X \in D_\theta \tag{7.2.4}$$

Since D_θ is a slant distribution, by Theorem 1.3.11

$$T^2X = -\cos^2\theta X, \tag{7.2.5}$$

for any $X \in D_\theta$.

The above condition leads to the following characterization for a slant submanifold M to be a pseudo-slant submanifold in an almost contact metric manifold.

THEOREM 7.2.2. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$, such that $\xi \in TM$. Then M is a pseudo-slant submanifold if and only if there exist a constant $\lambda \in (0, 1]$ such that*

(i) $D = \{X \in \mathcal{D} | T^2X = -\lambda X\}$ is a distribution,

(ii) For any $X \in TM$, orthogonal to D, $TX = 0$.

Furthermore, in this case $\lambda = \cos^2\theta$, where θ denotes the slant angle of the distribution D.

Proof. If we put $\lambda = \cos^2\theta$, then it follows from (7.2.4) and (7.2.5) that $D = D_\theta$. Conversely, we consider the orthogonal direct decomposition $TM = D \oplus D^\perp \oplus <\xi>$. It is evident that $TD \subseteq D$. Hence, by statement (ii) it is clear that $D^\perp$ is an anti-invariant distribution. Finally, Theorem 1.3.11 and the statement (i) imply that D is slant distribution, with slant angle θ satisfying $\lambda = \cos^2\theta$.

Now, we will discuss the integrability of distribution on a pseudo-slant submanifold of a trans-Sasakian manifold.

If μ is the invariant sub bundle of the normal bundle $T^\perp M$ then, in the case of pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = \mu \oplus ND^\perp \oplus ND_\theta. \tag{7.2.6}$$

As $D^\perp$ and D_θ are orthogonal distribution on M, $g(Z, X) = 0$ for each $Z \in D^\perp$ and $X \in D_\theta$. Thus, by equations (1.2.21) and (1.3.11), we may write

$$g(NZ, NX) = g(\phi Z, \phi X) = g(Z, X) = 0.$$

That means the decomposition (7.2.6) is an orthogonal direct decomposition.

For this setting following lemmas play an important role in working out the integrability conditions of the canonical distributions on M.

LEMMA 7.2.1 *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then*

$$A_{\phi Y}X = A_{\phi X}Y \tag{7.2.7}$$

for all $X, Y \in D^\perp$.

Proof. For any X, Y in $D^\perp$ and Z in TM, using (1.2.4),(1.3.11),(7.1.3) and (1.2.2) we find that

$$\begin{aligned} g(A_{\phi Y}X, Z) &= -g(\phi\bar{\nabla}_Z X, Y) \\ &= -g(\bar{\nabla}_Z \phi X - (\bar{\nabla}_z \phi)X, Y). \end{aligned}$$

on applying equations (7.1.2) and (1.2.3) the above equation yields

$$g(A_{\phi Y}X, Z) = g(A_{\phi X}Y, Z).$$

the results follows from the above equation.

LEMMA 7.2.3. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$, $\alpha \neq 0$, then for any $X, Y \in D^{\perp} \oplus D_{\theta}$*

$$g([X, Y], \xi) = 2\alpha g(TX, Y) \tag{7.2.8}$$

The proof of equation (7.2.8) is straightforward and may be obtained by using (7.1.4)(a).

PROPOSITION 7.2.1. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then, anti-invariant distribution $D^{\perp}$ is integrable.*

Proof. For any $X, Y \in D^{\perp}$ and $Z \in D_{\theta}$, by (7.2.1)

$$g([X, Y], TP_2Z) = -g(\phi[X, Y], P_2Z)$$

which on using (7.1.2) and (1.2.3), gives

$$g([X, Y], TP_2Z) = g(A_{\phi Y}X - A_{\phi X}Y, P_2Z)$$

Now, the integrability of the distribution $D^{\perp}$ follows on using equation (7.2.7) and (7.2.8).

LEMMA 7.2.3. *Let M be a pseudo-slant submanifold of trans-Sasakian manifold $\bar{M}$, with $\alpha \neq 0$. Then, the slant distribution D_{θ} is not integrable.*

The proof of the lemma is straightforward in view of equation (7.2.8) and the definition of pseudo-slant submanifolds.

PROPOSITION 7.2.2. *Let M be a pseudo-slant submanifold of $\bar{M}$, with $\alpha \neq 0$, then the distribution $D_{\theta} \oplus < \xi >$ is integrable if and only if*

$$h(Z, TW) - h(W, TZ) + \nabla^{\perp}_{X} NW - \nabla^{\perp}_{W} NZ$$

lies in ND_{θ} for each $Z, W \in D_{\theta} \oplus < \xi >$.

Proof. Making use of equations (1.2.21),(7.1.2),(1.2.2) and (1.2.3), we obtain

$$g(N[Z, W], NX) = g(h(Z, TW) - h(W, TZ) + \nabla^{\perp}_{X} NW - \nabla^{\perp}_{W} NZ, NX)$$

for each $X \in D^{\perp}$ and $Z, W \in D_{\theta}$. The Result follows on using the fact that $ND^{\perp}$ and ND_{θ} are mutually perpendicular.

Now we have the following consequence of above result.

COROLLARY 7.2.2 *Let M be a pseudo-slant submanifold of a β-Kenmotsu manifold or Cosymplectic manifold, then the slant distribution D_θ is integrable if and only if*

$$h(Z,TW) - h(W,TZ) + \nabla_X{}^{\perp}NW - \nabla_W{}^{\perp}NZ$$

lies in ND_θ for each $Z, W \in D_\theta$.

Now, in the following sections we shall study doubly warped product and warped product submanifolds of trans-Sasakian manifolds.

7.3 Warped and DoublyWarped Product Pseudo-slant Submanifolds of Trans-Sasakian Manifolds

In this section we study warped product pseudo-slant submanifolds of trans-Sasakian manifolds. Throughout the section, $\bar{M}$ trans-Sasakian manifold and N^θ and $N^\perp$ are slant and anti-invariant submanifolds of $\bar{M}$, respectively.

Let $M = {}_{f_2}N^\perp \times {}_{f_1}N^\theta$ be doubly warped product pseudo-slant submanifolds of trans-Sasakian manifold $\bar{M}$. Such submanifolds are always tangent to the structure vector field ξ.We distinguish 2 cases

(i) ξ tangent to $N^\perp$,
(ii) ξ tangent to N^θ.

NOTE 7.3.1. By Lemma (7.2.3), in case of trans-Sasakian manifold with $\alpha \neq 0$, The slant distribution D_θ is not integrable, due to this there does not exist slant submanifold N^θ, but $D_\theta \oplus < \xi >$ is integrable, so in view of this remark we cannot take ξ tangential to $N^\perp$, hence there is only possibility of case (ii).

THEOREM 7.3.1. *Let $\bar{M}$ be trans-Sasakian manifold, with $\alpha \neq 0$, then there do not exist doubly warped product submanifolds $M = {}_{f_2}N^\perp \times {}_{f_1}N^\theta$ in $\bar{M}$, such that $N^\perp$ is anti-invariant and N^θ is proper slant submanifold of $\bar{M}$, ξ tangential to N^θ.*

Proof. Let $M = {}_{f_2}N^\perp \times {}_{f_1}N^\theta$ be doubly warped product pseudo-slant submanifold of trans-Sasakian manifold $\bar{M}$, ξ tangent to N^θ then, for any $Z \in TN^\perp$

$$\nabla_Z\xi = Z(lnf_1)\xi + \xi(lnf_2)Z \tag{7.3.1}$$

Also, by equation (7.1.3) and (1.2.2), we have

$$-\alpha\phi Z + \beta(Z - \eta(Z)\xi) = \bar{\nabla}_Z\xi = \nabla_Z\xi + h(Z,\xi) \tag{7.3.2}$$

$$-\alpha NZ + \beta Z = \nabla_Z\xi + h(Z,\xi) \tag{7.3.3}$$

This means that

$$\nabla_Z\xi = \beta Z \tag{7.3.4}$$

$$h(Z,\xi) = -\alpha NZ. \tag{7.3.5}$$

Using equation (7.3.1) and (7.3.4), we get

$$Z(lnf_1)\xi + \xi(lnf_2)Z = \beta Z \tag{7.3.6}$$

By the orthogonality of two distribution, we get

$$Z(lnf_1) = 0 \tag{7.3.7}$$

$$\xi(lnf_2) = \beta \tag{7.3.8}$$

Now (7.3.7) yields, f_1 is constant. So, there does not exist doubly warped product pseudo-slant submanifold of the form ${}_{f_2}N^{\perp} \times {}_{f_1}N^{\theta}$, with ξ tangent to N^{θ}.

In view of the physical applications of the warped product manifolds (cf. [30],[60] etc.), the question of existence or non-existence of warped product submanifolds assumes significance. We turn to study warped product pseudo-slant submanifolds in the following:
Let $M = N^{\perp} \times_f N^{\theta}$ be a warped product pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$ with $\alpha \neq 0$. Such submanifolds are always tangent to the structure vector field ξ. Again, we have two cases

(i) ξ tangent to $N^{\perp}$,
(ii) ξ tangent to N^{θ}.

In view of above Note 7.3.1 we have the following result.

THEOREM 7.3.2. *Let $\bar{M}$ be a trans-Sasakian manifold, $\alpha \neq 0$, then there do not exist warped product submanifolds $M = N^{\perp} \times_f N^{\theta}$ in $\bar{M}$ such that $N^{\perp}$ is anti-invariant submanifold and N^{θ} is a proper slant submanifold of $\bar{M}$, ξ tangent to N^{θ}.*

Proof. By equation (5.2.3),

$$\nabla_X Z = \nabla_Z X = (Zlnf)X \tag{7.3.9}$$

for any vector fields $X \in N^\theta$ and $Z \in N^\perp$.

Now, for $\xi \in N^\theta$

$$\nabla_Z \xi = (Zlnf)\xi \tag{7.3.10}$$

Also, by equation (7.1.3) and (1.2.2), we have

$$-\alpha\phi Z + \beta(Z - \eta(Z)\xi) = \bar{\nabla}_Z \xi = \nabla_Z \xi + h(Z,\xi) \tag{7.3.11}$$

$$\nabla_Z \xi + h(Z,\xi) = -\alpha NZ + \beta Z \tag{7.3.12}$$

from (7.3.12), we get

$$\nabla_Z \xi = \beta Z \tag{7.3.13}$$

$$h(Z,\xi) = -\alpha NZ \tag{7.3.14}$$

Thus (7.3.10) and (7.3.13) imply $(Zlnf) = 0$, for all $Z \in N^\perp$, it follows that f is constant. Warped product submanifolds do not exists. Hence, the proof is complete.

7.4. Pseudo-slantWarped and DoublyWarped Product Submanifolds of Trans-Sasakian Manifolds

We begin the section by proving the non-existence of pseudo-slant doubly warped product submanifolds of trans-Sasakian manifolds. As earlier in view of Note 7.3.1, there is only case in which we can take ξ tangential to N^θ.

THEOREM 7.4.1 *There is no proper pseudo-slant doubly warped product submanifolds in trans-Sasakian manifolds, with $\alpha \neq 0$.*

Proof. Let $M =_{f_1} N^\theta \times_{f_2} N^\perp$ be a pseudo-slant Doubly warped product submanifold in trans-Sasakian manifold $\bar{M}$, where N^θ and $N^\perp$ are proper slant and anti-invariant submanifolds respectively.
Due to Note 7.3.1, we can not take ξ tangential to $N^\perp$, So taking ξ tangential to N^θ, Now, for any $X \in N^\theta$ and $Z \in N^\perp$, by equation (7.1.3) and (1.2.2), we have

$$\nabla_Z \xi = \beta Z \tag{7.4.1}$$

Also, from (7.1.1), we get

$$\nabla_Z \xi = Z(lnf_1)\xi + \xi(lnf_2)Z. \tag{7.4.2}$$

It follows from (7.4.1) and (7.4.2)

$$Z(lnf_1) = 0, \tag{7.4.3}$$

$$\xi(lnf_2) = \beta \tag{7.4.4}$$

from (7.4.4),$Z(lnf_1) = 0$, shows that f is constant. Therefore there does not exist pseudo-slant doubly warped product submanifold in trans-Sasakian manifold.

Now, for pseudo-slant warped product submanifold in trans-Sasakian manifold. In view of Note 7.3.1, we cannot take ξ tangential to $N^\perp$. So, the remaining case is case ξ tangential to N^θ.

THEOREM 7.4.2. *Let $\bar{M}$ be trans-Sasakian manifold, with $\alpha \neq 0$, then there exit $M = N^\theta \times_f N^\perp$ pseudo-slant warped product submanifold, such that N^θ is a proper slant submanifold tangent to ξ and $N^\perp$ is an anti-invariant submanifold of $\bar{M}$.*

Proof. For any vector fields $X \in N^\theta$ and $Z \in N^\perp$, using equation(5.2.3), for $\xi \in N^\theta$ we have

$$\nabla_Z \xi = (\xi lnf)Z. \tag{7.4.5}$$

Now, by structure equation (7.1.3) and (1.2.2), we have

$$\nabla_Z \xi = \beta Z \tag{7.4.6}$$

$$h(Z, \xi) = -\alpha NZ \tag{7.4.7}$$

these equations imply, $\xi lnf = \beta$ for all $Z \in N^\perp$. Therefore in this case warped product do exist.

In particular, we can obtain an example of pseudo-slant submanifold in the setting of Kenmotsu manifold as follows.

EXAMPLE 7.4.1. *Consider the complex space C^4 with the Usual Kaehler Structure and real global coordinates $(x^1, y^1, x^2, y^2, x^3, y^3, x^4, y^4)$.Let $\bar{M} = R \times_f C^4$ be the warped product between the real line R and C^4, where the warping function e^z, where z being the global coordinates in R, then $\bar{M}$ is a Kenmotsu manifold [58]. Now defining orthogonal basis*

$e_1 = \partial/\partial x^1$, $e_2 = \partial/\partial y^3$, $e_3 = \cos\theta\ \partial/\partial y^4 - \sin\theta\ \partial/\partial x^4$, $e_4 = \cos\theta\ \partial/\partial x^4 + \sin\theta\ \partial/\partial y^4$ *and* $e_5 = \partial/\partial z$

Distribution $D_\theta =< e_3, e_4 >$ and $D^\perp =< e_5, e_1, e_2 >$ are integrable and denoted by N^θ and $N^\perp$, then $M = N^\perp \times_f N^\theta$ is a pseudo-slant warped product submanifold isometrically immersed in $\bar{M}$, here the warping function is $f = e^z$.

Future Work

Slant and bi-slant submanifolds of various setting of almost contact, Lorentzian Paracontact and Lorentzian almost contact manifolds are yet to explore.

Warped product manifolds are known to have applications in physics. For instance, they provide an excellent setting to model space-time near a black hole or a massive star. One can explore more general warped product submanifold than CR warped products manifold, for example Warped and Doubly warped product and their geometric inequalities in different settings namely LP-Contact and Lorentzian almost contact Manifolds. As warping function is always a solution of a second degree equation so the study may find useful applications in physics and different branches of engineering K

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