ON A STUDY OF SOME ANALOGUES OF MINIMAL EXCLUDANT

A thesis

submitted in fulfillment of the requirements
for the award of the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

Prabh Simrat Kaur
(Regd. No. 901611008)

under the supervision of

Dr. Meenakshi Rana
(Professor)

to the

Department of Mathematics
Thapar Institute of Engineering and Technology
Patiala–147 004, Punjab, India

January, 2024

Declaration

I, Prabh Simrat Kaur, certify that the thesis titled *"On a study of some analogues of minimal excludant"* submitted for the award of the degree of Doctor of Philosophy in the Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Meenakshi Rana, Professor at Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or abroad. I further declare that I have faithfully acknowledged, given credit to and referred to the research workers wherever their works have been cited in the text and the body of the thesis.

Ms. Prabh Simrat Kaur
Regd. No. 901611008

Certificate

This is to certify that the thesis titled *"On a study of some analogues of minimal excludant"* which is being submitted by Ms. Prabh Simrat Kaur, in fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is a record of the candidate's own independent and original research work carried out under my supervision. The matter embodied in this thesis has not been submitted in part or full to any university or institute for the award of a degree.

Attestation by Thesis Supervisor:

Dr. Meenakshi Rana
Professor
Department of Mathematics
Thapar Institute of Engineering and Technology
Patiala–147 004, Punjab, India

Dated: 12 January, 2024

Acknowledgement

At this moment of accomplishment, in the first place, I would like to express my deep sense of gratitude to thesis advisor **Dr. Meenakshi Rana**. This was her expert guidance and ability to see the research problem in a unique way that helped me visualize the problem and plan the path ahead to solve it. Her invaluable suggestions and constructive criticism instilled me with morale needed to complete this task.

I extend my thanks to **Dr. Pramod Eyyunni**, **Dr. Bibekananda Maji** and **S. C. Bhoria** for their valuable ideas and words of encouragement. I also take this opportunity to heartily acknowledge the assistance of fellow researchers of the Department of Mathematics.

I am greatly indebted to my family for their unconditional support, both financially and emotionally throughout my research work. I have no words to express my gratitude to my family for their limitless sacrifices to enrich my future and lifting me uphill this phase of life.

Above all, I am thankful to Almighty for giving me a beautiful chance in my life to pursue this research work. I bow down to the will and blessings of the god in whose faith, I kept myself motivated during tough times.

......to universe

List of Publications

1. P. S. Kaur, S. C. Bhoria, P. Eyyunni and B. Maji, Minimal Excludant over Partitions into Distinct Parts. *International Journal of Number Theory*, 18(9) (2022), 2015-2028.

2. P. S. Kaur, M. Rana and P. Eyyunni, The Second Minimal Excludant and Mex Sequences . *Rocky Mountain Journal of Mathematics.* (Accepted)

Abstract

This thesis primarily studies an interesting partition statistic named as 'minimal excludant' or "mex" function. The minimal excludant, $\text{mex}(\pi)$ of an integer partition π, is the smallest positive integer that is not a part of π. The average size of this smallest gap of a partition was studied by Grabner and Knopfmacher [34]. In 2019, Andrews and Newman [8] explored the idea of minimal excludant where they define

$$\sigma\text{mex}(n) = \sum_{\pi \in \mathscr{P}(n)} \text{mex}(\pi),$$

where $\mathscr{P}(n)$ denote the collection of all integer partitions of n. We study the questions raised by Andrews and Newman for the function $\sigma\text{mex}(n)$, but restricted to partitions into distinct parts. We define the function $\sigma_d\text{mex}(n)$ by

$$\sigma_d\text{mex}(n) = \sum_{\pi \in \mathscr{D}(n)} \text{mex}(\pi),$$

where $\mathscr{D}(n)$ denote the collection of partitions of n into distinct parts and interestingly we observe that it has a nice connection with Ramanujan's function $\sigma(q)$. We also define

$$a_d(n) = \sum_{\substack{\pi \in \mathscr{D}(n) \\ \text{mex}(\pi)\ \text{odd}}} 1.$$

In fact, the generating function for $a_d(n)$ was considered by Uncu [53] in a different combinatorial context. Hence, as an application, we derive a stronger version of a result of Uncu.

In addition to the above mentioned work, a natural continuation to the study of minimal excludants is the study of second minimal excludant which we define as the second smallest integer missing from an integer partition π and it is denoted by $\text{mex}_2(\pi)$. We derive the generating function for $\sigma_2\text{mex}(n)$, where $\sigma_2\text{mex}(n) = \sum_{\pi \in \mathscr{P}(n)} \text{mex}_2(\pi)$ and along with it we also study partitions with a fixed difference between the minimal excludant and the second minimal excludant. For this, we define $\Delta_t(n)$, the number of partitions π of n with $\text{mex}_2(\pi) - \text{mex}(\pi) = t$. We derive its generating function and as special cases, we obtain interesting identities connecting $\Delta_t(n)$ to $\sigma\text{mex}(n)$ and certain restricted partition functions. This also leads us to the notion of a mex sequence and further we derive two neat identities involving the number of partitions whose mex sequence has length at least r.

Lastly, we study some more results on mex function which includes the study of generating function for $\sigma\text{meex}(n)$, defined as the sum of the smallest even integers that are missing in all the partitions of n. We also study the sum of minimal even excludant in distinct part partitions and thus found a combinatorial identity. We further study the sum of squares of minimal excludant, denoted by $\sigma\text{mex}^2(n)$ and established its relation with mex functions.

Contents

Chapter 1

Introduction

1.1 Partitions and Generating functions

Partition theory is a branch that originated from number theory, but now falls in the most youngest and active areas of discrete mathematics and counting related problems. Partitions not only have the mathematical significance, but also has been of great use in statistical mechanics [14], and describing the statistical properties of the thermodynamic equilibrium system. Partitions also appear naturally in various branches including the study of symmetric polynomials, the symmetric group and also in the representation of group theory.

In this chapter, we recall some basic definitions, notations and some results that are obligatory for the formation of the content in the current thesis.

Let us first discuss about q-series. A q-series is the one which involves summands of the form $(c;q)_n$ where

$$(c;q)_n = (1-c)(1-cq)\cdots(1-cq^{n-1}) \quad \text{for } n \geq 1, \quad |q| < 1, \tag{1.1}$$

$$(c;q)_\infty = \lim_{n\to\infty}(c;q)_n \quad \text{and} \quad (c;q)_0 = 1. \tag{1.2}$$

Ramanujan in his famous notebook [47], developed theta function in his own notion, which is denoted by $f(x,y)$ and is defined as follows:

$$f(x,y) = \sum_{n=-\infty}^{\infty} x^{\frac{n(n+1)}{2}} y^{\frac{n(n-1)}{2}}, \quad |xy| < 1. \tag{1.3}$$

The most appealing theorem in the speculation of theta function is the Jacobi triple product identity which was proved by C. G. J. Jacobi in his famous paper, *fundamenta Nova*, where the theory of elliptic functions was constructed. However, it was first proved by C. F. Gauss. The Jacobi triple product identity in Ramanujan's notion is as follows:

$$f(x;y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty, \quad |xy| < 1. \tag{1.4}$$

For $x = -q$ and $y = -q^2$, (1.3) reduces to

$$f(-q;-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

The identity obtained from (1.3) and (1.4), for $x = -q$ and $y = -q^2$ gives

$$f(-q;-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty, \quad |q| < 1.$$

is known as Euler's Pentagonal Number theorem. There are three particular cases of $f(x,y)$, in Ramanujan's notation, which are:
Ramanujan's φ-function, $\varphi(q)$, is defined by

$$\begin{aligned}\varphi(q) &= f(q;q)\\ &= \sum_{n=-\infty}^{\infty} q^{n^2}\\ &= \frac{(-q;-q)_\infty}{(q;-q)_\infty},\end{aligned} \tag{1.5}$$

Ramanujan's ψ-function, $\psi(q)$, is defined by

$$\psi(q) = f(q;q^3)$$

$$= \sum_{n=0}^{\infty} q^{n(n+1)/2} \tag{1.6}$$

$$= \frac{(q^2, q^2)_\infty}{(q; q^2)_\infty}, \tag{1.7}$$

Ramanujan's χ-function, $\chi(q)$, is defined by

$$\chi(q) = (-q; q^2)_\infty. \tag{1.8}$$

For further study on theta functions, readers can refer [50, 51]. Ramanujan [47], on page 338, has recorded eighteen values of product of theta functions. Naika et al. [44] established general theorems to evaluate explicitly the product of Ramanujan's theta functions. The authors of [43] have also found interesting arithmetic properties of Ramanujan's theta functions.

S. Ramanujan and G. H. Hardy in 1918, made the first progress in finding an asymptotic expression for $p(n)$, where $p(n)$ is the number of partitions of n, a *partition*, denoted by π, of a positive integer n is a non-increasing sequence of positive integers whose sum is n. The authors in their paper [35] developed different methods to solve this problem. The first method was elementary, the other one included Tauberian theorems, but the most efficient method was, the *circle method*, which was truly based on the representation of the coefficients of power series by mean of the Cauchy's Integral formula. They were successful in finding the exact asymptotic formula for $p(n)$ which in particular gives,

$$p(n) \approx \frac{1}{4n\sqrt{3}} exp\Big\{\pi\sqrt{\frac{2n}{3}}\Big\}. \tag{1.9}$$

Leonard Euler in 1674, gave the generating function for $p(n)$ as:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \frac{1}{(q;q)_\infty}. \tag{1.10}$$

Expanding the right side of the above expression, we have

$$\begin{aligned}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)} = \prod_{n=1}^{\infty}(1+q^n+q^{2n}+\cdots)\\
&= (1+q^{1.1}+q^{2.1}+q^{3.1}\cdots)(1+q^{1.2}+q^{2.2}+q^{3.2}\cdots)\cdots\\
&= (1+q^{1.1})+(q^{1.2}+q^{2.1})+(q^{1.3}+q^{1.2+1.1}+q^{3.1})\\
&\quad +(q^{1.4}+q^{1.3+1.1}+q^{2.2}+q^{1.2+1.2}+q^{4.1})+\cdots\\
&= p(0)+p(1)q+p(2)q^2+p(3)q^3+p(4)q^4+p(5)q^5+p(6)q^6+\cdots\\
&= 1+q+2q^2+3q^3+5q^4+7q^5+11q^6+\cdots\end{aligned}$$

Here, the number of times q^n appear is the number of partitions of n.

Example 1.1.1. *The coefficient of* q^6 *is* 11*, hence the relevant partitions of* 6 *are:* $6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1.$

1.2 Partition Statistics

Euler discovered an interesting identity which is known by his name as "Euler's Partition Identity", by placing certain restrictions on appearance of parts in a partition. He found that the number of partitions into distinct parts is equal to the number of partitions into odd parts. The following are the generating functions for distinct partitions, number of partitions into odd parts and similar such parity restrictions.

Number of partitions	Generating function
$p(n)$: ordinary partitions	$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$
$o(n)$: into odd parts	$\sum_{n=0}^{\infty} o(n)q^n = \frac{1}{(q;q^2)}_\infty$
$e(n)$: into even parts	$\sum_{n=0}^{\infty} e(n)q^n = \frac{1}{(q^2;q^2)}_\infty$
$D(n)$: into distinct parts	$\sum_{n=0}^{\infty} D(n)q^n = (-q;q)_\infty$
$Do(n)$: into distinct odd parts	$\sum_{n=0}^{\infty} Do(n)q^n = (-q;q^2)_\infty$

Identity 1 [Euler's Identity]

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})}. \tag{1.11}$$

The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

Identity 2 [Sylvester's Identity]

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty}(1+q^{2n-1}). \tag{1.12}$$

The number of partitions of n into distinct odd parts is equal to the number of self-conjugate partitions of n. A self-conjugate partition is a partition(π) which is identical with its conjugate. For example: $\pi = 3+2+1$ is a self conjugate partition of 6.

Ramanujan in 1913 found the following identities that were also discovered by Rogers in 1894 and hence known as Rogers-Ramanujan Identities (RRI's) [1, 46, 49].

Identity 3 [RRI 1]

$$\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty}(1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1}. \tag{1.13}$$

MacMahon [42] provided the partition-theoretic interpretations of the Rogers-Ramanujan identities. The above identity is interpreted as: partitions of n into summands that differ from each other by at least 2 are equinumerous with the partitions into parts of the forms $5m+1$ and $5m+4$ or we can say that the number of partitions of n with minimum difference ≥ 2 is equal to the number of partitions of n into parts which are $\equiv 1$ or $4 \pmod 5$.

Identity 4 [RRI 2]

$$\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty}(1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}. \tag{1.14}$$

Similarly, the above identity can be interpreted as: partitions of n into summands each larger than 1 which differ from each other by at least 2 are equinumerous with the partitions into parts of the forms $5m+2$ and $5m+3$ or we can say that the number of partitions of n into parts ≥ 2 and with minimum difference ≥ 2 is equal to the number of partitions of n into parts which are $\equiv 2$ or $3 \pmod 5$.

In 1961, Gordon [32] extended the combinatorial version of the Rogers-Ramanujan identities. Andrews [2] found the analytic version of the Gordon's theorem or the generalized version of Rogers-Ramanujan identities. Garsia and Milne [29] constructed a bijection between the relevant classes of partitions, with the help of Rogers-Ramanujan identities.

The above work motivated the study of restricted, unrestricted and multiplicities of a part in the partitions. The restricted partition function study has been of great interest to many mathematicians. Erdös and Lehner [24] in 1941 apparently studied the random integer partitions by a probabilistic approach. They were concerned about studying other quantities of a partition such as the average value over all partitions of n, the largest part size across all the partitions of n, the number of different part sizes etc. They further introduced the concept of uniform probability measure P on the set of all partitions of n, which stated that every sample partition is assigned the same probability, that is $\frac{1}{p(n)}$.

Erdös and Szalay [25], studied repetitions or multiplicities of different parts for the first time in a random partition. Szekeres [52] modified the result of Erdös and Lehner and further studied the joint distribution of length and also explored other analogous problem for partitions into unequal parts. Further, Wilf [55] found a derivation on the expected number of different part sizes. Goh and Schmutz [31], extended the work of Wilf and hence they were succeeded to find the expected number of distinct part sizes in a particular partition. This concept was further taken into consideration by Fristedt [28], who considered other parameters of a partition.

Corteel et al. [20], showed that a randomly selected part of a random partition has multiplicity d with probability tending to $\frac{1}{d(d+1)}$. As an important step of the result, they provided an asymptotic formula for the average number of parts of multiplicity d. Corteel and Hitchzenko [19] have discussed the multiplicities in overpartitions with the help of asymptotic and probabilistic arguments. In [54], the authors considered the distribution of the longest run (i.e. the largest occurring multiplicity) of a random partition. They showed that the multiplicities in a partition correspond to gaps in the conjugate, thus it is equivalent to study the largest gap (or ascent) in a partition. Hirschhorn [36, 37] has studied mean and variance of the parts in partitions. Extending the work of Hirschhorn, Goyal and Rana [33] have studied mean and variance of a particular part across all the ordinary partitions and n-color partitions.

Knopfmacher and Munagi [41] studied p-successions in integer partitions, where for fixed integer p, a p-succession is defined as a pair of adjacent parts in a partition such that $b_{i+1} - b_i = p$. The authors of [22] studied a bijective map in which multiplicities of the parts from integer partitions are mapped to the multiplicities of prime factors of natural numbers. Finally, another paper by Grabner and Knopfmacher [34] that discusses various partition statistics based on gaps deserves to be mentioned. Thus, instead of only focusing at parts in partition, they started looking at gaps, that is, the part sizes which do not appear in the partition. This lead to the study of smallest gap known as minimal excludant.

1.3 Minimal and Maximal Excludant

Grabner and Knopfmacher [34] studied an interesting partition statistic under the name 'smallest gap'. They defined the smallest gap of an integer partition as the least integer missing from the partition. Fraenkel and Peled [27] introduced the concept of a *minimal excludant* on a set S of positive integers, namely, the least positive integer missing from the set, denoted by "mex(S)". Recently, in 2019, Andrews and Newman [8] explored the idea of minimal excludant or "mex" function for an integer partition π of a positive integer n, mex(π), as the smallest positive integer that is not a part of π. They naturally generalized this concept to other arithmetic progressions in their papers [7, 8]. They define

$$\sigma\text{mex}(n) = \sum_{\pi \in \mathscr{P}(n)} \text{mex}(\pi), \tag{1.15}$$

where $\mathscr{P}(n)$ denotes the collection of all integer partitions of n. They also studied another arithmetic function, namely,

$$a(n) = \sum_{\substack{\pi \in \mathscr{P}(n) \\ \text{mex}(\pi)\ \text{odd}}} 1.$$

Andrews and Newman proved the following theorem in [8].

Theorem 1.3.1. *[8] We have*

$$\sigma\text{mex}(n) = D_2(n), \tag{1.16}$$

where $D_2(n)$ is the number of partitions of n into distinct parts using two colors, where we refer colors as 2-copies of relevant part.

Example 1.3.1. *Consider the following table for* $n = 5$, $D_2(5) = 14$, *with relevant partitions as:* 5_1, 5_2, $4_1 1_1$, $4_2 1_1$, $4_1 1_2$, $4_2 1_2$, $3_1 2_1$, $3_2 2_1$, $3_1 2_2$, $3_2 2_2$, $3_1 1_1 1_2$, $3_2 1_1 1_2$, $2_1 2_2 1_1$ *and* $2_1 2_2 1_2$, *and* $\sigma\text{mex}(5) = 14$ *where*

Relevant partitions for $n = 5$	*mex*(π)
5	1
$4 + 1$	2
$3 + 2$	1
$3 + 1 + 1$	2
$2 + 2 + 1$	3
$2 + 1 + 1 + 1$	3
$1 + 1 + 1 + 1 + 1$	2
	$\sigma\text{mex}(5) = 14$

Proof. Define $M(z, q)$ to be the double series such that the coefficient of $z^m q^n$ depicts the number of partitions π of n with $mex(\pi)$=m.

Clearly,

$$\begin{aligned}
M(z,q) &= \sum_{m=1}^{\infty} \frac{z^m q^{1+2+\cdots+(m-1)}}{\prod\limits_{\substack{n=1\\ n\neq m}}^{\infty}(1-q^n)}\\
&= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty} z^m q^{\binom{m}{2}}(1-q^m)
\end{aligned}$$

Differentiating both sides with respect to z and putting $z=1$, we get

$$\begin{aligned}
\sum_{n\geq 0}\sigma\text{mex}(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty} mq^{\binom{m}{2}}(1-q^m)\\
&= \frac{1}{(q;q)_\infty}\left(\sum_{m=1}^{\infty} mq^{\binom{m}{2}} - \sum_{m=1}^{\infty}(m-1)q^{\binom{m}{2}}\right)\\
&= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty} q^{\binom{m+1}{2}}\\
&= \frac{(q^2;q^2)_\infty}{(q;q)_\infty(q;q^2)_\infty} \qquad \text{(by [5, p. 23, eq. (2.2.13)])}\\
&= (-q;q)_\infty^2 \qquad \text{(by [5, p. 5, eq. (1.2.5)])}\\
&= \sum_{n\geq 0} D_2(n)q^n.
\end{aligned}$$

□

Andrews and Newman, in same paper [8], provided an alternate proof of above theorem 1.3.1 given below:

Proof. Consider $M_i(n)$ to be the number of partitions of n such that $\text{mex}(\pi) > i$. Then we claim that

$$M_i(n) = p\left(n - \frac{i(i+1)}{2}\right), \tag{1.17}$$

because each partition π counted by $M_i(n)$ must consist summands $1,2,3,\ldots,i$ (and the sum of these parts is $i(i+1)/2$) with remaining parts an arbitrary partition of $n - i(i+1)/2$.

Furthermore,

$$\sigma\text{mex}(n) = \sum_{i\geq 0} M_i(n),$$

because each partition with $mex(\pi)= m$ gets counted once by each $M_0(n)$, $M_1(n)$, $\cdots$, $M_{m-1}(n)$ and hence this sum is actually the sum of all the mex values for the partitions of n. We are ready to obtain the generating function of $\sigma\text{mex}(n)$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \sigma\text{mex}(n)q^n &= \sum_{n=0}^{\infty} q^n \sum_{i=0}^{\infty} M_i(n)q^n \\ &= \sum_{n=0}^{\infty} q^n \sum_{i\geq 0} p\left(n - \frac{i(i+1)}{2}\right) \\ &= \sum_{n=0}^{\infty} p(n)q^n \sum_{i\geq 0} q^{i(i+1)/2} \\ &= \frac{1}{(q;q)_\infty} \sum_{m=0}^{\infty} q^{m(m+1)/2} \end{aligned} \tag{1.18}$$

As we can see in the first proof, this expression equals $(-q;q)_\infty^2$. □

Interestingly, the generating function on the smallest gap, in fact, was first obtained by Grabner and Knopfmacher in their work [34, Theorem 3]. They were also able to obtain the following Hardy-Ramanujan-Rademacher type exact formula for $\sigma\text{mex}(n)$:

$$\sigma\text{mex}(n) = \frac{\pi}{2\sqrt{6\left(n+\frac{1}{12}\right)}} \sum_{k=1}^{\infty} \frac{A_{2k-1}(n)}{2k-1} I_1\left(\pi\frac{\sqrt{2\left(n+\frac{1}{12}\right)}}{\sqrt{3}(2k-1)}\right), \tag{1.19}$$

where

$$A_k(n) = \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} \exp\left(2\pi i\left(s(h,k) - s(2h,k) - \frac{hn}{k}\right)\right),$$

and $s(h,k)$ denotes the Dedekind sum, and I_1 denotes the modified Bessel function of the first kind with index 1. Moreover, $\sigma\text{mex}(n)$ satisfies the following asymptotic formula:

$$\sigma\mathrm{mex}(n) \sim \frac{1}{4\sqrt[4]{6n^3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty. \tag{1.20}$$

Moreover, a combinatorial proof of (1.16) has been obtained by Ballantine and Merca [10]. Ancillary to it, they were able to prove several results on the truncated series formulas which involved the function $\sigma\mathrm{mex}(n)$. In the proof, the authors made use of the fact that

$$\sigma\mathrm{mex}(n) = \sum_{k\geq 0} p\left(n - k(k+1)/2\right), \tag{1.21}$$

where $p(n)$ usually denotes the number of partitions of n.

In [9], the authors put a restriction on mex-function and defined $M_k(n)$ as the number of partitions of n such that k is the least positive integer which is not a part and there exists more parts $> k$ than there are parts $< k$.
For $k = 1$, $M_1(0) = 0$ and, if $n > 0$, $M_1(n)$ is the number of partitions of n, such that it does not contain 1 as a part. Hence, for $n > 0$, $M_1(n) = p(n) - p(n-1)$.
Then, from (1.21), we obtain

$$\sigma\mathrm{mex}(n) - \sigma\mathrm{mex}(n-1) - \delta(n) = \sum_{j\geq 0} M_1\left(n - j(j+1)/2\right),$$

where δ is the characteristic function of the set of triangular numbers,

$$\delta(n) = \begin{cases} 1, & \textit{if } n{=}m(m{+}1)/2, \quad m \in \mathbb{Z}, \\ 0, & \textit{otherwise.} \end{cases}$$

Furthermore, the idea of minimal excludant was studied more intensively by putting restriction on the parts of the partitions in a specific arithmetic progression. As an example, the authors of [8] defined $\mathrm{moex}(\pi)$ to be the smallest odd integer that is not a part of π and thus proved the following theorem.

Theorem 1.3.2. *[8] We have*

$$\sum_{n\geq 0} \sigma\text{moex}(n) q^n = (-q;q)_\infty (-q;q^2)^2{}_\infty, \tag{1.22}$$

where

$$\sigma\text{moex}(n) = \sum_{\pi \in \mathscr{P}(n)} \text{moex}(\pi)$$

Analogues of the minimal excludant have begun to appear in the literature, Andrews and Newman in their subsequent paper [7], defined $\text{mex}_{A,a}(\pi)$ to be the smallest positive integer congruent to a modulo A that is not a part of π.
Let $p_{A,a}(n)$ denote the number of partitions π of n, such that

$$mex_{A,a}(\pi) \equiv A \pmod{2A}, \tag{1.23}$$

and $\overline{p}_{A,a}(n)$ denote the number of partitions π of n, such that

$$mex_{A,a}(\pi) \equiv A + a \pmod{2A}, \tag{1.24}$$

If $p(n)$ denotes the number of partitions of n, then

$$p(n) = p_{A,a}(n) + \overline{p}_{A,a}(n) \tag{1.25}$$

In addition to it, authors of [7] proved certain theorems and found their connections with the rank and the crank of a partition, where the *rank* of a partition is the difference between the largest part and the number of parts of the partition. Likewise, the *crank* of a partition is the largest part if there are no ones as parts, and otherwise it is the number of parts larger than the number of ones minus the number of ones.

The authors of [7], defined $p_{1,1}(n)$ to be the number of partitions of n with non-negative crank and $p_{3,3}(n)$ to be the number of partitions of n with rank ≥ -1. Sellers and da Silva [21] in their paper, presented parity considerations for the functions, namely, $p_{1,1}(n)$ and $p_{3,3}(n)$ corresponding to modulo 2. With the help of Euler's

Pentagonal Number theorem, the authors of [21] proved the following theorem:

Theorem 1.3.3. *[21] For all* $n \geq 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} p_{1,1}(n) \equiv (q^2;q^2)_\infty \equiv \sum_{k=-\infty}^{\infty} q^{k(3k-1)} \pmod 2 \\ \equiv \sum_{k=1}^{\infty} q^{k(3k+1)} + \sum_{k=0}^{\infty} q^{k(3k-1)} \pmod 2, \end{aligned} \tag{1.26}$$

Interestingly, the authors of [21] also found that $p_{3,3}(n) \equiv a_3(n) \pmod 2$ for all $n \geq 0$ where $a_3(n)$ is the generating function for the number of 3-core partitions, which is given as:

$$\sum_{n=0}^{\infty} a_3(n) q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \tag{1.27}$$

Hirschhorn and Sellers [38] worked on closed form for $a_3(n)$. For t-core partitions, readers can refer [26, 45]. By using the theory of modular forms, Barman and Singh [11, 12], and Chakraborty and Ray [17] obtained appealing congruence properties and density results for mex-related partition functions. Hopkins et al. [39] extended the relation between the crank and mex function and further defined the mex function $mex_j(\pi)$ to be the least integer greater than j that is not a part of π, where j is a positive integer.

Bhoria et al. [16] gave an analogue of Franklin's identity and thus discovered a generalization of the minimal excludant, defined by the r-chain mex, where the r-chain mex of a partition is the least positive integer k such that the integers k, $k+1$, …, $k+r+1$ do not occur as parts in the partition. Also Dhar, Mukhopadhyay and Sarma [23] studied different identities of $p_{A,a}(n)$ functions and further found bounds on the statistics rank and crank of partition functions. Using the theory of Hecke eigenforms, Ray [48] deduced multiplicative formulas and infinite families of congruences for $\sigma\text{mex}(n)$.

Chern in his recent paper [18], defined *maximal excludant* "maex(π)" as the largest non-negative integer smaller than the largest part of π that is not a part of π.

Chern defined

$$\sigma\mathrm{maex}(n) = \sum_{\pi \in \mathscr{P}(n)} \mathrm{maex}(\pi).$$

Example 1.3.2. *Consider the following table for $n = 5$,*

Relevant partitions for $n = 5$	*maex*(π)
5	4
$4+1$	3
$3+2$	1
$3+1+1$	2
$2+2+1$	0
$2+1+1+1$	0
$1+1+1+1+1$	0

He successfully obtained the following generating function identity for $\sigma\mathrm{maex}(n)$ in [18].

Theorem 1.3.4. *[18] We have*

$$\begin{aligned}\sum_{n=1}^{\infty} \sigma\mathrm{maex}(n) q^n &= \sum_{n=1}^{\infty} \frac{k}{(q;q)_{k-1}} \sum_{m=1}^{\infty} q^{m(k+1)}(-q;q)_{m-1} \\ &= \frac{1}{(q;q)_\infty}\left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} q^n (q^2;q^2)_{n-1}\right) \\ &= \frac{1}{(q;q)_\infty}\left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n}\right).\end{aligned} \tag{1.28}$$

Furthermore, the author of [18] proved that

$$\sum_{n=1}^{\infty} \sigma L(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n},$$

where $\sigma L(n)$ is the sum of the largest parts of all partitions of n, and

$$\sum_{n=1}^{\infty}(\sigma L(n) - \sigma maex(n))q^n = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty} q^n(q^2;q^2)_{n-1}.$$

Also, Chern [18, Theorem 1.3] established the following asymptotic formula for $\sigma\mathrm{maex}(n)$. More precisely, he showed that

$$\sigma L(n) - \sigma\mathrm{maex}(n) \sim \frac{1}{4\sqrt{3n}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

and

$$\sigma\mathrm{maex}(n) \sim \sigma L(n), \quad \text{as } n \to \infty.$$

In this thesis, our focus is on the study of some analogues of minimal excludant. Chapter 2 is devoted to the study of the sum of the minimal excludant for restricted partitions. Chapter 3 includes the study of second minimal excludant and analyze its relationship with minimal excludant and related mex sequences. Chapter 4 deals with the study of sum of squares of minimal excludant and sum of minimal even excludant in distinct part partitions and thus exploring a combinatorial identity.

Chapter 2

Minimal Excludant over partitions into distinct parts

2.1 Introduction

In this chapter, we study the function $\sigma_d\mathrm{mex}(n)$, defined by

$$\sigma_d\mathrm{mex}(n) = \sum_{\pi \in \mathscr{D}(n)} \mathrm{mex}(\pi), \tag{2.1}$$

where $\mathscr{D}(n)$ denotes the collection of partitions of n into distinct parts. We also study the partition statistic $a_d(n)$ such that

$$a_d(n) = \sum_{\substack{\pi \in \mathscr{D}(n) \\ \mathrm{mex}(\pi)\ \mathrm{odd}}} 1.$$

We further study the function $\sigma_d\mathrm{moex}(n)$, given by

$$\sigma_d\mathrm{moex}(n) = \sum_{\pi \in \mathscr{D}(n)} \mathrm{moex}(\pi).$$

In fact, the generating function for $a_d(n)$ was considered by Uncu [53] in a different combinatorial context. In the next section, we state the main results.

The contents of this chapter have been published in *International Journal of Number Theory*, 18(9), 2015-2028, 2022.

2.2 Main Results

For our main results, we use Ramanujan's q-series, $\sigma(q)$, given by

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n}. \tag{2.2}$$

For more details related to $\sigma(q)$, refer [3, 4, 6]. Interestingly, the generating function for $\sigma_d\mathrm{mex}(n)$ is directly connected to $\sigma(q)$ as stated below:

Theorem 2.2.1. *We have*

$$\sum_{n=0}^{\infty} \sigma_d\mathrm{mex}(n) q^n = (-q;q)_\infty \sigma(q). \tag{2.3}$$

The next result gives us an asymptotic formula for $\sigma_d\mathrm{mex}(n)$.

Theorem 2.2.2. *We have*

$$\sigma_d\mathrm{mex}(n) \sim \frac{\exp\left(\pi\sqrt{\frac{n}{3}}\right)}{2(3n^3)^{1/4}}, \quad \text{as } n \to \infty. \tag{2.4}$$

Now we will state a result due to Uncu [53, Theorem 3.2].

Theorem 2.2.3. *Let $U(n)$ be the sequence of numbers defined by*

$$\sum_{n=0}^{\infty} U(n) q^n = (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(-q;q)_n}.$$

Then $U(n) \geq 0$ for all $n \geq 0$.

Interestingly, we observe that the generating function for $U(n)$ and $a_d(n)$ are indeed the same.

Theorem 2.2.4. *We have*

$$\sum_{n=0}^{\infty} U(n) q^n = \sum_{n=0}^{\infty} a_d(n) q^n = (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(-q;q)_n}. \tag{2.5}$$

An immediate consequence of this result is:

Corollary 2.2.5. *For any $n \geq 0$, we have*

$$U(n) > 0 \quad \textit{except for } n = 1.$$

As introduced in chapter 1, Andrews and Newman defined $\operatorname{moex}(\pi)$ to be the smallest odd integer missing from π. This naturally led them to define the function $\sigma\operatorname{moex}(n)$, given by

$$\sigma\operatorname{moex}(n) = \sum_{\pi \in \mathscr{P}(n)} \operatorname{moex}(\pi).$$

In this chapter, we analogously define a quantity for partitions into distinct parts and study its generating function. Define

$$\sigma_d\operatorname{moex}(n) = \sum_{\pi \in \mathscr{D}(n)} \operatorname{moex}(\pi).$$

Theorem 2.2.6. *The generating function for $\sigma_d\operatorname{moex}(n)$ is*

$$\begin{aligned} \sum_{n=0}^{\infty} \sigma_d\operatorname{moex}(n) q^n &= (-q;q)_\infty \left(1 + 2\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n}\right) \\ &= (-q;q)_\infty \left(1 + 2\sum_{n=1}^{\infty} (-1)^{n-1} q^n (q^2;q^2)_{n-1}\right). \end{aligned}$$

In other words,

$$\sum_{n=0}^{\infty} \sigma_d\operatorname{moex}(n) q^n = (-q;q)_\infty \left(1 + \sigma^*(-q)\right), \tag{2.6}$$

where

$$\sigma^*(q) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n}.$$

In literature, the q-series $\sigma(q)$ and $\sigma^*(q)$ are found to appear simultaneously at many places. Andrews, Dyson and Hickerson [6] proved that the coefficients of these two q-series are very small and related to the arithmetic of the quadratic real field $\mathbb{Q}(\sqrt{6})$.

Next, similar to the definition (2.1), we define $\sigma_d\text{maex}(n)$ as

$$\sigma_d\text{maex}(n) = \sum_{\pi\in\mathscr{D}(n)} \text{maex}(n).$$

The next result provides us a generating function identity for $\sigma_d\text{maex}(n)$.

Theorem 2.2.7. *We have*

$$\sum_{n=0}^{\infty} \sigma_d\text{maex}(n)q^n = \sum_{k=1}^{\infty} k(-q;q)_{k-1} \sum_{m=1}^{\infty} q^{\frac{m(m+1)}{2}+km}. \tag{2.7}$$

One can easily observe that Theorem 2.2.7 is an analogue of Chern's identity [18, (2)]. Before going on to the proofs of our main results, we establish some preliminaries in the next section.

2.3 Preliminaries

For the proof of the asymptotic formula for $\sigma_d\text{mex}(n)$, namely, Theorem 2.2.2, we derive a monotonic property of $\sigma_d\text{mex}(n)$, which is itself of independent interest.

Proposition 2.3.1. *For $n \geq 7$, we have that*

$$\sigma_d\text{mex}(n+1) > \sigma_d\text{mex}(n).$$

Proof. We consider two cases depending on whether n is a triangular number or not.
Case 1: Suppose $n \geq 7$ is not a triangular number, that is, $n \neq 1+2+\cdots+k$ for any positive integer k. We define a map f from $\mathscr{D}(n)$ to $\mathscr{D}(n+1)$ as follows: Consider a distinct parts partition π of n. It will be of the form

$$\pi = (a_1, a_2, \ldots, a_k) \quad \text{with } a_1 > a_2 > \cdots > a_k \geq 1,\ a_1 + a_2 + \cdots + a_k = n. \tag{2.8}$$

Then let $f(\pi) := \pi' = (a_1+1, a_2, \ldots, a_k)$. Clearly, this is a distinct parts partition of $n+1$. Also see that $(a_1+1) - a_2 \geq 2$, i.e., a partition of the form $f(\pi)$ has a gap of at least *two* between its largest part and the next largest part. Moreover, f preserves minimal excludants, that is $\text{mex}(\pi) = \text{mex}(\pi^{'})$. This is because as n is not a triangular number, $\pi = (a_1, \ldots, a_k)$ cannot be the partition $(k, k-1, \ldots, 1)$ and consequently, there exists an i, $1 \leq i \leq k$ such that $a_i - a_{i+1} > 1$ with the interpretation that $a_{k+1} = 0$. Choosing i_0 to be the largest integer i for which $a_i - a_{i+1} > 1$ implies that $\text{mex}(\pi) = \text{mex}(\pi^{'}) = a_{i_0+1} + 1$.

For example, if $n = 9$, then the partition $\pi_1 = (4, 3, 2)$ satisfies $k = i_0 = 3$ with $\text{mex}(\pi_1) = \text{mex}(f(\pi_1)) = 1$ and the partition $\pi_2 = (8, 1)$ satisfies $k = 2, i_0 = 1$ with $\text{mex}(\pi_2) = \text{mex}(f(\pi_2)) = 2$. Thus, we have matched up the minimal excludants of partitions in $\mathscr{D}(n)$ with those of a certain collection of partitions in $\mathscr{D}(n+1)$ and hence $\sigma_d\text{mex}(n) \leq \sigma_d\text{mex}(n+1)$. We next exhibit a partition in $\mathscr{D}(n+1)$ that does not lie in the image of f, thus proving that the inequality $\sigma_d\text{mex}(n+1) > \sigma_d\text{mex}(n)$ holds. The idea is to get hold of a partition π in which both $\ell(\pi)$ and $\ell(\pi) - 1$ occur as parts so that it cannot be in the image of f.

If n is odd: Consider the partition $\lambda_1 = \left(\frac{n+1}{2}, \frac{n-1}{2}, 1\right)$ of $n+1$. Note that $\frac{n+1}{2} - \frac{n-1}{2} = 1$ and that λ_1 has distinct parts as $(n-1)/2 > 1$.

If n is even: Look at the partition $\lambda_2 = \left(\frac{n}{2} + 1, \frac{n}{2}\right)$ of $n+1$, which is again in $\mathscr{D}(n+1) \setminus f(\mathscr{D}(n))$.

Case 2: Assume that $n \geq 7$ is a triangular number, so that $n = k + (k-1) + \cdots + 1$ for a unique positive integer k. As in Case 1, for any π in $\mathscr{D}(n)$ other than the partition $\mu = (k, k-1, \ldots, 1)$, the partition $f(\pi)$ lying in $\mathscr{D}(n+1)$ will have the same minimal excludant as π. But for μ, whose minimal excludant is $k+1$, we see that $\text{mex}(f(\mu)) = k$. This means that $\sum_{\pi \in \mathscr{D}(n)} \text{mex}(\pi) = 1 + \sum_{\pi \in f(\mathscr{D}(n))} \text{mex}(\pi)$. In other words, to prove the required inequality we have to show that the contribution of the partitions from $\mathscr{D}(n+1) \setminus f(\mathscr{D}(n))$ to the sum of minimal excludants is at least 2. We once again proceed according to the parity of n.

If n is odd: Consider the partition $\nu_1 = \left(\frac{n+1}{2}, \frac{n-1}{2}, 1\right)$ of $n+1$. Observe that $(n-1)/2 > 2$ and so $\nu_1 \in \mathscr{D}(n+1) \setminus f(\mathscr{D}(n))$ with its minimal excludant being 2.

If n is even: In this case, we look at two partitions given by $\nu_2 = (\frac{n}{2} + 1, \frac{n}{2})$ and $\nu_3 = (\frac{n}{2}, \frac{n}{2} - 1, 2)$. Clearly, $\nu_2 \in \mathscr{D}(n+1) \setminus f(\mathscr{D}(n))$ with minimal excludant 1. Again, note that ν_3 has distinct parts since $\frac{n}{2} - 1 > 2$ and hence $\text{mex}(\nu_3) = 1$.

With this, we complete the proof of the proposition. ☐

In the upcoming section, we provide the proofs of our results.

2.4 Proofs of the main results

2.4.1 Proof of Theorem 2.2.1

First proof of Theorem 2.2.1

Let $p_d^{mex}(m,n)$ be the number of partitions of n into distinct parts whose minimal excludant is m. Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p_d^{mex}(m,n)z^m q^n &= \sum_{m=1}^{\infty} z^m q^1\cdot q^2\cdots q^{m-1}\prod_{k=m+1}^{\infty}(1+q^k)\\ &= \sum_{m=1}^{\infty} z^m q^{\binom{m}{2}}\prod_{k=m+1}^{\infty}(1+q^k)\\ &= (-q;q)_\infty\sum_{m=1}^{\infty}\frac{z^m q^{\binom{m}{2}}}{(-q;q)_m}. \qquad (2.9)\end{aligned}$$

Differentiating both sides of (2.9) with respect to z and putting $z=1$, we get

$$\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} m p_d^{mex}(m,n)\right)q^n = (-q;q)_\infty\sum_{m=1}^{\infty}\frac{m q^{\binom{m}{2}}}{(-q;q)_m}.$$

But $\sum_{m=1}^{\infty} m p_d^{mex}(m,n) = \sigma_d\text{mex}(n)$, the sum of minimal excludants in all the partitions of n into distinct parts. Thus,

$$\sum_{n=0}^{\infty}\sigma_d\text{mex}(n)q^n = (-q;q)_\infty\sum_{m=1}^{\infty}\frac{m q^{\binom{m}{2}}}{(-q;q)_m}. \qquad (2.10)$$

We now show that the sum on the right hand side of (2.10) is nothing but Ramanujan's series $\sigma(q)$. Start with

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=0}^{\infty}\frac{(n+1)-n}{(-q;q)_n}q^{n(n+1)/2}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(n+1)q^{n(n+1)/2}}{(-q;q)_n} - \sum_{n=0}^{\infty} \frac{nq^{n(n+1)/2}}{(-q;q)_n} \\
&= \sum_{n=1}^{\infty} \frac{nq^{n(n-1)/2}}{(-q;q)_{n-1}} - \sum_{n=1}^{\infty} \frac{nq^{n(n+1)/2}}{(-q;q)_n} \\
&= \sum_{n=1}^{\infty} \frac{nq^{n(n-1)/2}}{(-q;q)_{n-1}} \left(1 - \frac{q^n}{1+q^n}\right) \\
&= \sum_{n=1}^{\infty} \frac{nq^{n(n-1)/2}}{(-q;q)_n}.
\end{aligned}
$$

Therefore, from (2.10), we deduce that

$$\sum_{n=0}^{\infty} \sigma_d\mathrm{mex}(n)q^n = (-q;q)_\infty \sigma(q). \tag{2.11}$$

□

We now give an alternate proof of (2.11), on the lines of Andrews and Newman's second proof of the generating function for $\sigma\mathrm{mex}(n)$. (see [8, p. 251])

Second proof of Theorem 2.2.1

Let $\mathscr{D}_i(n)$ denote the collection of partitions of n into distinct parts for which $\mathrm{mex}(\pi) > i$ and $|\mathscr{D}_i(n)|$ is the number of partitions in $\mathscr{D}_i(n)$. Then we claim that

$$|\mathscr{D}_i(n)| = p_d\left(n - \frac{i(i+1)}{2},\ i\right), \tag{2.12}$$

where $p_d(m,\ i)$ denotes the number of partitions of m into distinct parts with smallest part greater than i. To see this, start with a distinct parts partition π of n with $\mathrm{mex}(\pi) > i$. By the definition of minimal excludant, the integers 1 through i must all occur as parts in π. Moreover, since π is a distinct parts partition, each of the numbers 1 to i appears exactly once in π. Subtract the quantity $1 + 2 + \cdots + i$ from π. This gives a distinct parts partition π' of $n - (1 + 2 + \cdots + i)$ (since we began with a distinct parts partition π, removing some parts from it doesn't affect its distinct nature). Now, since π has only one copy of each of 1 to i, π' will not have any parts less than or equal to i. Therefore π' is a distinct parts partition of $n-\frac{i(i+1)}{2}$ with smallest part greater than i.

Conversely, starting with a distinct parts partition λ of $n-\frac{i(i+1)}{2}$ with $s(\pi)>i$, we add the quantity $1+2+\cdots+i$ to λ to get a distinct parts partition λ' (since λ had no parts less than or equal to i) with the integers 1 to i all occurring as parts. This means that $\text{mex}(\lambda')>i$. Hence, this bijection proves the claim in (2.12).

From the definition of $|\mathscr{D}_i(n)|$, $\sigma_d\text{mex}(n)$ can be expressed as

$$\sigma_d\text{mex}(n)=\sum_{i=0}^{\infty}|\mathscr{D}_i(n)|, \tag{2.13}$$

since each distinct parts partition π with $\text{mex}(\pi)=i$ is counted i times on the right hand side of (2.13), once in each of $|\mathscr{D}_0(n)|,|\mathscr{D}_1(n)|,\ldots,|\mathscr{D}_{i-1}(n)|$. On the left hand side of (2.13), we add together the minimal excludants over all the distinct parts partitions, thus each distinct parts partition π contributes a weight $\text{mex}(\pi)$ to it. Thus, on both sides of equation (2.13), each distinct parts partition contributes the same number and hence the identity holds.

Now, the generating function of distinct parts partitions with $s(\pi)>i$ is simply

$$\sum_{n=0}^{\infty}p_d(n,\ i)q^n=(-q^{i+1};q)_\infty.$$

Therefore, $|\mathscr{D}_i(n)|$, which is the number of distinct parts partitions of $n-\frac{i(i+1)}{2}$ with smallest part greater than i, will be the coefficient of $q^{n-\frac{i(i+1)}{2}}$ in $(-q^{i+1};q)_\infty$. Equivalently, this is the coefficient of q^n in $q^{\frac{i(i+1)}{2}}(-q^{i+1};q)_\infty$.

Thus,

$$\sum_{n=0}^{\infty}|\mathscr{D}_i(n)|q^n=q^{\frac{i(i+1)}{2}}(-q^{i+1};q)_\infty=(-q;q)_\infty\frac{q^{\frac{i(i+1)}{2}}}{(-q;q)_i}.$$

We are ready to obtain the generating function of $\sigma_d\text{mex}(n)$. Starting with (2.13), we get

$$\sum_{n=0}^{\infty}\sigma_d\text{mex}(n)q^n=\sum_{n=0}^{\infty}q^n\sum_{i=0}^{\infty}|\mathscr{D}_i(n)|=\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}|\mathscr{D}_i(n)|q^n$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} (-q;q)_\infty \frac{q^{\frac{i(i+1)}{2}}}{(-q;q)_i} \\
&= (-q;q)_\infty \sum_{i=0}^{\infty} \frac{q^{\frac{i(i+1)}{2}}}{(-q;q)_i} = (-q;q)_\infty \sigma(q).
\end{aligned}
$$

□

2.4.2 Proof of Theorem 2.2.2

We first state an important asymptotic result for coefficients of a power series, due to Ingham [40].

Proposition 2.4.1. *Let* $A(q) = \sum_{n=0}^{\infty} a(n) q^n$ *be a power series with radius of convergence* 1. *Assume that* $\{a(n)\}$ *is a weakly increasing sequence of non-negative real numbers. If there are constants* $\alpha, \beta \in \mathbb{R}$, *and* $C > 0$ *such that*

$$A(e^{-t}) \sim \alpha \, t^\beta \exp\left(\frac{C}{t}\right), \quad \text{as } t \to 0^+,$$

then we have

$$a(n) \sim \frac{\alpha}{2\sqrt{\pi}} \frac{C^{\frac{2\beta+1}{4}}}{n^{\frac{2\beta+3}{4}}} \exp\left(2\sqrt{Cn}\right), \quad \text{as } n \to \infty. \tag{2.14}$$

Proof of Theorem 2.2.2. Recall that the generating function for $\sigma_d\mathrm{mex}(n)$ is given by Theorem 2.2.1, namely,

$$\sum_{n=0}^{\infty} \sigma_d\mathrm{mex}(n) q^n = (-q;q)_\infty \sigma(q) := B(q).$$

In a famous work on quantum modular forms, Zagier [56, p. 7] pointed out that, for $t \to 0^+$,

$$\sigma\left(e^{-t}\right) = 2 - 2t + 5t^2 - \frac{55}{3}t^3 + \frac{1073}{12}t^4 - \frac{32671}{60}t^5 + \cdots . \tag{2.15}$$

Now, using the transformation formula for the Dedekind eta function, one can show that, for $t \to 0^+$,

$$\frac{1}{\left(e^{-t}; e^{-t}\right)_\infty} \sim \sqrt{\frac{t}{2\pi}} \exp\left(\frac{\pi^2}{6t}\right). \tag{2.16}$$

Euler's identity suggests that

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}. \tag{2.17}$$

Therefore, in view of (2.16) and (2.17), we can derive, for $t \to 0^+$,

$$\left(-e^{-t}; e^{-t}\right)_\infty \sim \frac{1}{\sqrt{2}} \exp\left(\frac{\pi^2}{12t}\right). \tag{2.18}$$

Finally, combining (2.15) and (2.18), we arrive at

$$B\left(e^{-t}\right) \sim \sqrt{2} \exp\left(\frac{\pi^2}{12t}\right), \quad \text{as } t \to 0^+.$$

Again, by Proposition 2.3.1 we know that the sequence $\{\sigma_d\text{mex}(n)\}$, for $n \geq 7$, is an increasing sequence of positive integers. Now we are ready to invoke Proposition 2.4.1, with $\alpha = \sqrt{2}, \beta = 0$ and $C = \dfrac{\pi^2}{12}$. Substituting these constants in (2.14), we obtain

$$\sigma_d\text{mex}(n) \sim \frac{1}{2(3n^3)^{1/4}} \exp\left(\pi\sqrt{\frac{n}{3}}\right), \quad \text{as } n \to \infty.$$

This finishes the proof. □

In the next subsection, we provide proofs of other results.

2.4.3 Proofs of other results

Proof of Theorem 2.2.4. Recall that $a_d(n)$ counts the number of distinct parts partitions of n with an odd minimal excludant. So the least integer missing from such a partition can only be of the form $2n+1$ for some $n \geq 0$. And all the integers from 1 through $2n$ should occur exactly once and for the integers greater than $2n+1$, they may occur at most once. Putting this together, we may write

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p_d^{omex}(m,n)z^m q^n = \sum_{k=0}^{\infty} z^{2k+1}q^1\cdot q^2\cdots q^{2k}\prod_{\ell=2k+2}^{\infty}(1+q^{\ell}), \tag{2.19}$$

where $p_d^{omex}(m,n)$ denotes the number of distinct parts partitions of n with an odd minimal excludant m. Putting $z=1$ in (2.19), we get

$$\sum_{n=0}^{\infty} a_d(n)q^n = \sum_{k=0}^{\infty} q^{\binom{2k+1}{2}}\prod_{\ell=2k+2}^{\infty}(1+q^{\ell}) = (-q;q)_{\infty}\sum_{n=0}^{\infty}\frac{q^{\binom{2n+1}{2}}}{(-q;q)_{2n+1}}. \tag{2.20}$$

Consider the rightmost sum in (2.20). Rewriting it, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{\binom{2n+1}{2}}}{(-q;q)_{2n+1}} &= \sum_{n=0}^{\infty}\frac{q^{1+\cdots+2n}}{(1+q)\cdots(1+q^{2n+1})}\\ &= \sum_{n=0}^{\infty}\frac{q^{1+\cdots+2n}}{(1+q)\cdots(1+q^{2n})}\left\{1-\frac{q^{2n+1}}{1+q^{2n+1}}\right\}\\ &= \sum_{n=0}^{\infty}\frac{q^{1+\cdots+2n}}{(1+q)\cdots(1+q^{2n})} - \sum_{n=0}^{\infty}\frac{q^{1+\cdots+(2n+1)}}{(1+q)\cdots(1+q^{2n+1})}\\ &= \sum_{n=0}^{\infty}\frac{q^{\binom{2n+1}{2}}}{(-q;q)_{2n}} - \sum_{n=0}^{\infty}\frac{q^{\binom{2n+2}{2}}}{(-q;q)_{2n+1}}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(-q;q)_n}.\end{aligned}$$

Putting this in (2.20), we see that

$$\sum_{n=0}^{\infty} a_d(n)q^n = (-q;q)_{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(-q;q)_n}.$$

This completes the proof. □

Proof of Corollary 2.2.5. We note that $a_d(n)$ is non-negative for all $n \geq 0$, since it counts certain kind of partitions. Moreover, for $n > 1$, we always have a partition of n into distinct parts with an odd minimal excludant, namely, the partition n, where the minimal excludant is 1. So $a_d(n) > 0$ for all $n > 1$, and hence by (2.5), we conclude that $U(n) > 0$ for all $n > 1$. □

Uncu [53, Theorem 3.2], remarks that the infinite series in (2.5) is a false theta function studied by Rogers. Further, in the same paper, he gives a combinatorial explanation of the fact that the coefficients on the right hand side of (2.5) are non-negative. But, by Corollary 2.2.5, via our interpretation in terms of minimal excludant, we have shown that all but one of the coefficients, namely $a_d(1)$, are infact positive.

Proof of Theorem 2.2.6. Let $p_d^{moex}(m,n)$ denote the number of distinct parts partitions π of n with $\operatorname{moex}(\pi) = m$. Consider the following double sum

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p_d^{moex}(m,n) z^m q^n &= (-q^2;q^2)_\infty \sum_{k=0}^{\infty} z^{2k+1} q^{1+3+\cdots+(2k-1)}(-q^{2k+3};q^2)_\infty \\ &= (-q^2;q^2)_\infty \sum_{k=0}^{\infty} z^{2k+1} q^{k^2}(-q^{2k+3};q^2)_\infty \\ &= (-q;q^2)_\infty(-q^2;q^2)_\infty \sum_{k=0}^{\infty} \frac{z^{2k+1}q^{k^2}(-q^{2k+3};q^2)_\infty}{(-q;q^2)_\infty} \\ &= (-q;q)_\infty \sum_{k=0}^{\infty} \frac{z^{2k+1}q^{k^2}}{(-q;q^2)_{k+1}}.\end{aligned}$$

Differentiate with respect to z and then put $z = 1$ to get

$$\begin{aligned}\sum_{n=0}^{\infty} \sigma_d \mathrm{moex}(n) q^n &= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(2n+1)q^{n^2}}{(-q;q^2)_{n+1}} \\ &= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(2n+1)q^{n^2}}{(-q;q^2)_{n}} \left\{1 - \frac{q^{2n+1}}{1+q^{2n+1}}\right\} \\ &= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(2n+1)q^{n^2}}{(-q;q^2)_{n}} - (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(2n+1)q^{(n+1)^2}}{(-q;q^2)_{n+1}} \\ &= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(2n+1)q^{n^2}}{(-q;q^2)_{n}} - (-q;q)_\infty \sum_{n=1}^{\infty} \frac{(2n-1)q^{n^2}}{(-q;q^2)_{n}} \\ &= (-q;q)_\infty + 2(-q;q)_\infty \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n}.\end{aligned}$$

Alternatively, using Chern's result [18, Proposition 2.1] with $x = 1, y = -1$ gives us

$$\sum_{n=0}^{\infty} \sigma_d \mathrm{moex}(n) q^n = (-q;q)_\infty + 2(-q;q)_\infty \sum_{n=1}^{\infty} (-1)^{n-1} q^n (q^2;q^2)_{n-1}.$$

□

Proof of Theorem 2.2.7. Suppose π_d is a distinct parts partition of n with maximal excludant k. Note that $k \geq 1$, since partitions with maximal excludant 0 do not contribute to the sum $\sum_{\pi \in \mathscr{D}(n)} \text{maex}(\pi) = \sigma_d\text{maex}(n)$. We can divide π_d into two components, $\pi_d{}'$ and $\pi_d{}''$:
The first component $\pi_d{}'$ is a distinct parts partition with parts $\leq k-1$; and the second one $\pi_d{}''$ is a gapfree distinct parts partition with $s(\pi) = k+1$, i.e., each integer between $s(\pi)$ and $\ell(\pi)$ also occurs as a part. Observe that the second component $\pi_d{}''$ upon conjugation gives a gapfree partition in which the smallest part $s(\pi) = 1$ and the largest part $\ell(\pi)$ appears exactly $k+1$ times and all other parts appear exactly once. We consider a two variable generating function $D(z,q)$ for $p_{d,k}(n)$, the number of distinct parts partitions of n with maximal excludant k. In $D(z,q)$, the exponent of z indicates the maximal excludant of a partition π_d into distinct parts, and the exponent of q, as always, keeps track of the number being partitioned by π_d.

$$D(z,q) := \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} p_{d,k}(n) z^k q^n = \sum_{k=1}^{\infty}(-q;q)_{k-1} z^k \sum_{m=1}^{\infty} q^{1+2+\ldots+(m-1)+(k+1)m}.$$

Now differentiating $D(z,q)$ with respect to z and substituting $z=1$, we get the generating function for $\sigma_d\text{maex}(n)$,

$$\sum_{n=0}^{\infty} \sigma_d\text{maex}(n) q^n = \sum_{k=1}^{\infty} k(-q;q)_{k-1} \sum_{m=1}^{\infty} q^{\frac{m(m+1)}{2}+km}.$$

□

2.5 Concluding Remarks

Inspired by the work of Andrews and Newman, in this chapter, we studied the minimal excludant over partitions into distinct parts. We have proved that the generating function for $\sigma_d\text{mex}(n)$ is the product of the generating function for distinct parts partition function and Ramanujan's well-known q-series $\sigma(q)$. We also established a

Hardy-Ramanujan type asymptotic formula for $\sigma_d\mathrm{mex}(n)$. It would be interesting to find a Hardy-Ramanujan-Rademacher type exact formula for $\sigma_d\mathrm{mex}(n)$, analogous to the result (1.19) of Grabner and Knopfmacher for $\sigma\mathrm{mex}(n)$.

We also examined $a_d(n)$, which counts the number of distinct parts partitions with an odd minimal excludant. Quite surprisingly, we have observed that the generating function for $a_d(n)$ has been studied by Uncu in a different context, which immediately improved Uncu's result [53, Theorem 3.2]. Subsequently, we studied $\sigma_d\mathrm{moex}(n)$ and its generating function has been expressed as the product of the generating function for the distinct parts partition function and $1 + \sigma^*(-q)$. It is interesting that the function $\sigma^*(q)$ has mostly been seen to appear in the vicinity of $\sigma(q)$.

Chapter 3

The second minimal excludant and mex sequences

3.1 Introduction

Motivated by the work done in chapter 2, where we looked at restricted versions of $\sigma\text{mex}(n)$ type functions, in the present chapter, we study a natural continuation to the study of minimal excludants. In particular, we study second minimal excludant, denoted by $\text{mex}_2(\pi)$, and its generating function along with the mex sequences. We further study partitions with a fixed difference between the minimal excludant and the second minimal excludant. For this, we define $\Delta_t(n)$, the number of partitions π of n with $\text{mex}_2(\pi) - \text{mex}(\pi) = t$. We derive its generating function and as special cases, obtain interesting identities connecting $\Delta_t(n)$ to $\sigma\text{mex}(n)$ and certain restricted partition functions. Before going ahead, let us discuss preliminaries and some basic definitions required for our main results.

3.2 Preliminaries and Basic Definitions

We recall the q-binomial theorem which is given by [5, Equation (2.2.1), p. 17]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \quad (|z| < 1) \tag{3.1}$$

The contents of this chapter have been accepted for publication in *Rocky Mountain Journal of Mathematics.*

and a version of Heine's transformation for q-hypergeometric series, given below [30, Equation (III.1), p. 241]:

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right] = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} {}_2\phi_1\left[\begin{matrix} \frac{c}{b}, & z \\ & az \end{matrix}; q, b\right], \tag{3.2}
$$

where the q-hypergeometric series ${}_{r+1}\phi_r$ is defined to be

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_{r+1};q)_n}{(q;q)_n (b_1;q)_n \cdots (b_r;q)_n} z^n. \tag{3.3}
$$

We also require the following result which is due to Andrews and Newman [8, Theorem 1.1]. We have the ensuing identity for the generating function of $\sigma\text{mex}(n)$:

$$
\sum_{n=0}^{\infty} \sigma\text{mex}(n) q^n = \frac{\psi(q)}{(q;q)_\infty}, \tag{3.4}
$$

where

$$
\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}. \tag{3.5}
$$

A natural continuation to the study of minimal exludants is the second minimal exludant which we define as follows:

Definition 3.2.1 (Second minimal excludant)**.** *The second smallest integer missing from an integer partition π is known as the second minimal excludant, denoted by* $\text{mex}_2(\pi)$*.*

We analogously define the $\sigma_2\text{mex}(n)$ function as

$$
\sigma_2\text{mex}(n) = \sum_{\pi \in \mathscr{P}(n)} \text{mex}_2(\pi). \tag{3.6}
$$

Example 3.2.1. *Consider the following table for $n = 5$, where*

Relevant partitions for $n = 5$	$mex_2(\pi)$
5	2
$4+1$	3
$3+2$	4
$3+1+1$	4
$2+2+1$	4
$2+1+1+1$	4
$1+1+1+1+1+1$	3
	$\sigma\text{mex}_2(5) = 24$

We derive the generating function for $\sigma_2\text{mex}(n)$ and study partitions with a fixed difference between the minimal excludant and the second minimal excludant. For this, we define $\Delta_t(n)$, the number of partitions π of n with $\text{mex}_2(\pi) - \text{mex}(\pi) = t$.

Starting with the observation that $\Delta_1(n)$ enumerates the partitions of n in which the minimal excludant and the second minimal excludant are consecutive integers, we are naturally led to examine the longest sequence of missing integers in a partition, starting from the minimal excludant.

Definition 3.2.2 (Mex sequence)**.** *The mex sequence of a partition is the longest sequence of consecutive missing integers in the partition, starting from its minimal excludant.*

Example 3.2.2. *The mex sequences of some partitions of* 6 *are tabulated below:*

Partition	*Minimal excludant*	*Mex sequence*	*Length of mex sequence*
6	1	$(1,2,3,4,5)$	5
$5+1$	2	$(2,3,4)$	3
$4+2$	1	(1)	1
$4+1+1$	2	$(2,3)$	2
$3+3$	1	$(1,2)$	2
$3+1+1+1$	2	(2)	1
$2+2+2$	1	(1)	1

We may note that, the mex sequence of a partition can be *infinitely long*. For instance, the partitions of 6 not alluded to in the table above, namely, $3+2+1,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1$ all have such mex sequences.

3.3 Main Results

Following is our first main result:

Theorem 3.3.1. *The generating function of* $\sigma_2\text{mex}(n)$ *is given by*

$$\sum_{n=0}^{\infty} \sigma_2\text{mex}(n) q^n = \frac{1}{(q;q)_\infty} \left\{ \frac{1}{(1-q)} - \sum_{s=0}^{\infty} (s-1) q^{\binom{s+1}{2}} \right\}. \tag{3.7}$$

As outlined in Section 3.1, for a positive integer t, we define:

$$\Delta_t(n) = \text{number of partitions } \pi \text{ of } n \text{ satisfying } \text{mex}_2(\pi) - \text{mex}(\pi) = t. \tag{3.8}$$

Example 3.3.1. *See that* $\Delta_1(5) = 5$*, as the relevant partitions of* 5 *are* $5,\ 4+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$*, whereas* $\Delta_2(5) = 1$*, by taking into consideration the partition* $3+1+1$*.*

The generating function for $\Delta_t(n)$ has a nice representation in terms of a "tail" of $\psi(q)$.

Theorem 3.3.2. *Let* t *be a positive integer, then*

$$\sum_{n=0}^{\infty} \Delta_t(n) q^n = \frac{q^{t-1}}{(q^2;q)_\infty} \sum_{r=0}^{\infty} q^{\binom{r+t}{2}}. \tag{3.9}$$

Corollary 3.3.3. *For* $t = 1, 2$*, the above gives us the partition identities listed below.*

$$\Delta_1(n) = \sigma\text{mex}(n) - \sigma\text{mex}(n-1), \tag{3.10}$$

$$\Delta_2(n) = \sigma\text{mex}(n-1) - \sigma\text{mex}(n-2) - p(n|\textit{exactly one } 1). \tag{3.11}$$

Here, $p(n|\text{condition})$ means the number of partitions of n satisfying the condition appearing after the | symbol. For instance, $p(n|\textit{exactly one } r)$ counts the number of partitions of n in which the integer r appears exactly once.

Now we discuss the results pertaining to mex sequences. Let $\mathscr{M}_r(n)$ denote the set of partitions of n whose mex sequence has length *at least* r. We also put

$$|\mathscr{M}_r(n)| = p_r^{\text{mex}}(n),$$

where $p_r^{\text{mex}}(n)$ enumerates the number of partitions of n whose corresponding mex sequences have length at least r. The generating function for $p_r^{\text{mex}}(n)$ and its consequences form an important component of the present work. Interestingly, depending on the parity of r, this gives us two elegant partition identities for $p_r^{\text{mex}}(n)$.

Firstly, observe that $p_1^{\text{mex}}(n)$ is simply the partition function $p(n)$, since the mex sequence of every partition has length at least 1. From the definition of $\Delta_1(n)$ in (3.8), one can also deduce that $p_2^{\text{mex}}(n) = \Delta_1(n)$.

Example 3.3.2. *Consider the set* $\mathscr{M}_3(6)$*, which consists of the partitions* $6,\ 5+1,\ 3+2+1,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1$*, and so* $p_3^{\text{mex}}(6) = 6$*. Similarly, the partitions belonging to the set* $\mathscr{M}_4(6)$ *are* $6,\ 3+2+1,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1$*, which gives us* $p_4^{\text{mex}}(6) = 5$*. Note that* $\mathscr{M}_4(6) \subset \mathscr{M}_3(6)$ *and hence,* $p_4^{\text{mex}}(6) \leq p_3^{\text{mex}}(6)$*.*

In fact, from the definition of the sets $\mathscr{M}_r(n)$, it follows that $\mathscr{M}_{r+1}(n) \subset \mathscr{M}_r(n)$, and consequently

$$p_{r+1}^{\text{mex}}(n) \leq p_r^{\text{mex}}(n)\ \forall\ r \geq 1. \tag{3.12}$$

At the other end of the spectrum, it is interesting to see what happens for "large" values of r. The next result speaks to this:

Proposition 3.3.4. *Let r, n be positive integers and suppose $D(n)$ denotes the number of distinct parts partitions of n. Then*

$$p_r^{\text{mex}}(n) = D(n) \quad \Longleftrightarrow \quad r \geq n.$$

We now turn to the generating function for $p_r^{\mathrm{mex}}(n)$, and see that it has a succinct representation in terms of q-products.

Theorem 3.3.5. *If r is a positive integer, then*

$$\sum_{n=0}^{\infty} p_r^{\mathrm{mex}}(n) q^n = \frac{1}{(q;q^2)_\infty (q^{r+1};q^2)_\infty}. \tag{3.13}$$

Indeed, with $r = 1$ this gives the generating function for $p_1^{\mathrm{mex}}(n)$ to be $1/(q;q)_\infty$, the generating function for $p(n)$, as it should be. Also, as $r \to \infty$ the right hand side of (3.13) tends to $1/(q;q^2)_\infty = (-q;q)_\infty$, the generating function of distinct parts partitions, which is in agreement with Proposition 3.3.4. Interpreting Theorem 3.3.5 combinatorially leads to two identities, based on the parity of r.

Corollary 3.3.6. *Define $p_e^{>r}(n)$ to be the number of partitions of n in which no even integer less than r is allowed to be a part and $p_{o,2}^{>r}(n)$ is the number of partitions of n into odd parts where parts greater than r come in two colors. Then,*

- *For odd integers r, we have $p_r^{\mathrm{mex}}(n) = p_e^{>r}(n)$ and*
- *For even integers r, we have $p_r^{\mathrm{mex}}(n) = p_{o,2}^{>r}(n)$.*

Example 3.3.3. *We have $p_2^{\mathrm{mex}}(5) = 5$, because the relevant partitions are 5, $4+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$. The partitions of 5 into odd parts where parts greater than 2 come in two colors: 5_1, 5_2, 3_1+1+1, 3_2+1+1, $1+1+1+1+1$, are also five in number. Next, 5, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$ are the four partitions of 5 with mex sequence of length at least 3. We also see that there are four partitions of 5 with no '2's, namely, 5, $4+1$, $3+1+1$, $1+1+1+1+1$.*

Remark 3.3.1. *Observe that we can also calculate the number of partitions of n whose mex sequence has length r. Such partitions are precisely the members of the set $\mathscr{M}_r(n) \setminus \mathscr{M}_{r+1}(n)$ and thus total $p_r^{\mathrm{mex}}(n) - p_{r+1}^{\mathrm{mex}}(n)$ in number. From the example above, we see that there is $p_2^{\mathrm{mex}}(5) - p_3^{\mathrm{mex}}(5) = 1$ partition of 5 with mex sequence of length two, namely, $4+1$.*

Finally, as a consequence of Theorem 3.3.5 for $r = 2$, we obtain a q-product representation for $\psi(q)$, which is usually derived using Jacobi's triple product identity [15, Equation (1.3.14), p. 11].

Corollary 3.3.7. *Let $\psi(q)$ be the theta function of Ramanujan defined in (3.5). Then,*

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

3.4 Proofs of the main results

Proof of Theorem 3.3.1. If $A_{r,s}(n)$ denotes the number of partitions of n with minimal excludant r and second minimal excludant s, we can write

$$\begin{aligned}\sum_{n=0}^{\infty} A_{r,s}(n)q^n &= \frac{q^1}{1-q^1}\cdots\frac{q^{r-1}}{1-q^{r-1}}\cdot\frac{q^{r+1}}{1-q^{r+1}}\cdots\frac{q^{s-1}}{1-q^{s-1}}\cdot\frac{1}{1-q^{s+1}}\cdot\frac{1}{1-q^{s+2}}\cdots\text{to }\infty\\ &= \frac{q^{\binom{s}{2}-r}(1-q^r)(1-q^s)}{(q;q)_\infty}. \qquad (3.14)\end{aligned}$$

We introduce two parameters z and w and let the exponents of z and w keep track of the minimal excludant and the second minimal excludant of a partition respectively. Note that the second minimal excludant s of a partition is *at least two* and the minimal excludant then ranges between 1 and $s-1$. We thus have the three parameter generating function for the number of partitons with a specified minimal excludant and second minimal excludant as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{s=2}^{\infty}\sum_{r=1}^{s-1} A_{r,s}(n)z^r w^s q^n &= \sum_{s=2}^{\infty}\sum_{r=1}^{s-1} z^r w^s \sum_{n=0}^{\infty} A_{r,s}(n)q^n \qquad (3.15)\\ &= \frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty}\sum_{r=1}^{s-1} z^r w^s q^{\binom{s}{2}-r}(1-q^r)(1-q^s), \qquad (3.16)\end{aligned}$$

by (3.14).

If we put $z = 1$ in the left side of (3.15), we get

$$\sum_{n=0}^{\infty}\sum_{s=2}^{\infty}\left(\sum_{r=1}^{s-1} A_{r,s}(n)\right) w^s q^n = \sum_{n=0}^{\infty}\sum_{s=2}^{\infty} p^{\mathrm{mex}_2}(s,n)w^s q^n, \qquad (3.17)$$

where $p^{\mathrm{mex}_2}(s,n)$ is the number of partitions of n with second minimal excludant s. This is because in $\sum_{r=1}^{s-1} A_{r,s}(n)$, we are summing over all possible values of r for a given s. Now, put $z=1$ in the right hand side of (3.16) obtaining

$$\begin{aligned}
&\frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty}\sum_{r=1}^{s-1} q^{\binom{s}{2}-r}(1-q^r)(1-q^s)w^s \\
&= \frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s}{2}}\sum_{r=1}^{s-1}\frac{1-q^r}{q^r} \\
&= \frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s}{2}}\{q^{-1}+q^{-2}+\cdots+q^{-(s-1)}-(s-1)\} \\
&= \frac{1}{(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s}{2}}\left\{\frac{1-q^{s-1}-q^{s-1}(1-q)(s-1)}{q^{s-1}(1-q)}\right\} \\
&= \frac{1}{(1-q)(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s-1}{2}}\{1+(s-1)q^s-sq^{s-1}\}.
\end{aligned}$$

But by (3.17), this gives us

$$\sum_{n=0}^{\infty}\sum_{s=2}^{\infty} p^{\mathrm{mex}_2}(s,n)w^s q^n = \frac{1}{(1-q)(q;q)_\infty}\sum_{s=2}^{\infty} w^s(1-q^s)q^{\binom{s-1}{2}}\{1+(s-1)q^s-sq^{s-1}\}. \tag{3.18}$$

Note that differentiating the left hand side of (3.18) with respect to w and putting $w=1$, we arrive at

$$\sum_{n=0}^{\infty}\left(\sum_{s=2}^{\infty} s p^{\mathrm{mex}_2}(s,n)\right)q^n = \sum_{n=0}^{\infty}\sigma\mathrm{mex}_2(n)q^n, \tag{3.19}$$

since each partition of n contributes s, its second minimal excludant, to the sum $\sum_{s=2}^{\infty} s p^{\mathrm{mex}_2}(s,n)$. Now differentiate the right hand side of (3.18) with respect to w and put $w=1$ to get:

$$\frac{1}{(1-q)(q;q)_\infty}\sum_{s=2}^{\infty} s(1-q^s)q^{\binom{s-1}{2}}\{1+(s-1)q^s-sq^{s-1}\}. \tag{3.20}$$

From (3.18), (3.19) and (3.20), we obtain

$$\sum_{n=0}^{\infty}\sigma\mathrm{mex}_2(n)q^n = \frac{1}{(1-q)(q;q)_\infty}\sum_{s=2}^{\infty} s(1-q^s)q^{\binom{s-1}{2}}\{1+(s-1)q^s-sq^{s-1}\}. \tag{3.21}$$

We start by re-indexing the sum on the right hand side of (3.21) as shown:

$$\begin{aligned}
&\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}}\{1+sq^{s+1}-(s+1)q^{s}\}\\
&=\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}}\{1+sq^{s}(q-1)-q^{s}\}\\
&=\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}}-(1-q)\sum_{s=1}^{\infty}s(s+1)(1-q^{s+1})q^{\binom{s+1}{2}}-\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s+1}{2}}.
\end{aligned} \tag{3.22}$$

We shall consider the three terms in the right hand side above one by one. First, start with

$$\begin{aligned}
\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}}&=\sum_{s=1}^{\infty}(s+1)q^{\binom{s}{2}}-q\sum_{s=1}^{\infty}(s+1)q^{\binom{s+1}{2}}\\
&=2+\sum_{s=2}^{\infty}(s+1)q^{\binom{s}{2}}-q\sum_{s=2}^{\infty}sq^{\binom{s}{2}}\\
&=2+\sum_{s=2}^{\infty}q^{\binom{s}{2}}+(1-q)\sum_{s=2}^{\infty}sq^{\binom{s}{2}}=\sum_{s=0}^{\infty}q^{\binom{s}{2}}+(1-q)\sum_{s=2}^{\infty}sq^{\binom{s}{2}}.
\end{aligned} \tag{3.23}$$

Next, look at the second term in (3.22):

$$\begin{aligned}
(1-q)\sum_{s=1}^{\infty}s(s+1)(1-q^{s+1})q^{\binom{s+1}{2}}&=(1-q)\sum_{s=1}^{\infty}s(s+1)\left(q^{\binom{s+1}{2}}-q^{\binom{s+2}{2}}\right)\\
&=(1-q)\sum_{s=1}^{\infty}s(s+1)q^{\binom{s+1}{2}}-(1-q)\sum_{s=2}^{\infty}s(s-1)q^{\binom{s+1}{2}}\\
&=2q(1-q)+2(1-q)\sum_{s=2}^{\infty}sq^{\binom{s+1}{2}}\\
&=2(1-q)\sum_{s=1}^{\infty}sq^{\binom{s+1}{2}}=2(1-q)\sum_{s=2}^{\infty}(s-1)q^{\binom{s}{2}}.
\end{aligned} \tag{3.24}$$

And the last term in (3.22) is

$$\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s+1}{2}}=\sum_{s=2}^{\infty}s(1-q^{s})q^{\binom{s}{2}}$$

$$\begin{aligned}&= \sum_{s=2}^{\infty} s\left(q^{\binom{s}{2}} - q^{\binom{s+1}{2}}\right) = \sum_{s=2}^{\infty} sq^{\binom{s}{2}} - \sum_{s=3}^{\infty}(s-1)q^{\binom{s}{2}}\\ &= 2q + \sum_{s=3}^{\infty} q^{\binom{s}{2}} = q + \sum_{s=2}^{\infty} q^{\binom{s}{2}}. \qquad (3.25)\end{aligned}$$

Putting (3.23), (3.24), and (3.25) into (3.22), we obtain

$$\begin{aligned}&\sum_{s=1}^{\infty}(s+1)(1-q^{s+1})q^{\binom{s}{2}}\{1+sq^{s+1}-(s+1)q^{s}\}\\ &= \sum_{s=0}^{\infty} q^{\binom{s}{2}} + (1-q)\sum_{s=2}^{\infty} sq^{\binom{s}{2}} - 2(1-q)\sum_{s=2}^{\infty}(s-1)q^{\binom{s}{2}} - q - \sum_{s=2}^{\infty} q^{\binom{s}{2}}\\ &= 2-q-(1-q)\sum_{s=2}^{\infty}(s-2)q^{\binom{s}{2}} = 2-q-(1-q)\sum_{s=0}^{\infty} sq^{\binom{s+2}{2}}. \qquad (3.26)\end{aligned}$$

Substituting (3.26) back into (3.21), we finally arrive at

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma\mathrm{mex}_2(n)q^n &= \frac{1}{(1-q)(q;q)_\infty}\left\{2-q-(1-q)\sum_{s=0}^{\infty} sq^{\binom{s+2}{2}}\right\}\\ &= \frac{1}{(1-q)(q;q)_\infty}\left\{1-(1-q)\sum_{s=0}^{\infty}(s-1)q^{\binom{s+1}{2}}\right\}\\ &= \frac{1}{(q;q)_\infty}\left\{\frac{1}{(1-q)} - \sum_{s=0}^{\infty}(s-1)q^{\binom{s+1}{2}}\right\}, \qquad (3.27)\end{aligned}$$

which is precisely (3.7). ☐

Proof of Theorem 3.3.2. As we have seen in the proof of Theorem 3.3.1, the generating function for the number of partitions with minimal excludant r and second minimal excludant s is

$$\frac{q^{\binom{s}{2}-r}(1-q^r)(1-q^s)}{(q;q)_\infty}. \quad (\text{for } 1 \le r \le s-1)$$

We are interested in the generating function of $\Delta_t(n)$, the number of partitions π of n with $\mathrm{mex}_2(\pi) - \mathrm{mex}(\pi) = t$. Suppose that the minimal excludant equals r for some positive integer r. Then the generating function for partitions with minimal excludant r and second minimal excludant $r+t$ is given by

$$\frac{q^{\binom{r+t}{2}-r}(1-q^r)(1-q^{r+t})}{(q;q)_\infty}. \tag{3.28}$$

For keeping track of all partitions with $\mathrm{mex}_2(\pi) - \mathrm{mex}(\pi) = t$, we need to sum expressions of the form in (3.28) as r runs over the positive integers. Therefore,

$$\sum_{n=0}^{\infty} \Delta_t(n) q^n = \frac{1}{(q;q)_\infty} \sum_{r=1}^{\infty} q^{\binom{r+t}{2}-r}(1-q^r)(1-q^{r+t}). \tag{3.29}$$

Starting with the right side of (3.29), we have

$$\begin{aligned}
\sum_{r=1}^{\infty} q^{\binom{r+t}{2}-r}(1-q^r)(1-q^{r+t}) &= \sum_{r=0}^{\infty} q^{\binom{r+1+t}{2}-r-1}(1-q^{r+1})(1-q^{r+1+t}) \\
&= \sum_{r=0}^{\infty} q^{\binom{r+1+t}{2}-r-1} - q^t \sum_{r=0}^{\infty} q^{\binom{r+1+t}{2}} - \sum_{r=0}^{\infty} q^{\binom{r+1+t}{2}} \\
&+ \sum_{r=0}^{\infty} q^{\binom{r+1+t}{2}+r+t+1} \\
&= q^{t-1} \sum_{r=0}^{\infty} q^{\binom{r+1+t}{2}-(r+t)} - q^t \left(-q^{\binom{t}{2}} + \sum_{r=0}^{\infty} q^{\binom{r+t}{2}} \right) \\
&- \sum_{r=0}^{\infty} q^{\binom{r+1+t}{2}} + \sum_{r=0}^{\infty} q^{\binom{r+2+t}{2}} \\
&= q^{t-1} \sum_{r=0}^{\infty} q^{\binom{r+t}{2}} + q^{\binom{t+1}{2}} - q^t \sum_{r=0}^{\infty} q^{\binom{r+t}{2}} - q^{\binom{t+1}{2}} \\
&= (q^{t-1} - q^t) \sum_{r=0}^{\infty} q^{\binom{r+t}{2}}.
\end{aligned} \tag{3.30}$$

Hence, putting the information from (3.30) in (3.29), we finally get

$$\sum_{n=0}^{\infty} \Delta_t(n) q^n = \frac{q^{t-1}}{(q^2;q)_\infty} \sum_{r=0}^{\infty} q^{\binom{r+t}{2}}.$$

□

Proof of Corollary 3.3.3. Put $t=1$ in Theorem 3.3.2 to see that

$$\sum_{n=0}^{\infty} \Delta_1(n) q^n = \frac{1}{(q^2;q)_\infty} \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{\psi(q)}{(q^2;q)_\infty}, \tag{3.31}$$

by (3.5). Using (3.4), we can write the rightmost expression in (3.31) in terms of the generating function of $\sigma\mathrm{mex}(n)$, which gives us

$$\sum_{n=0}^{\infty} \Delta_1(n)q^n = (1-q)\sum_{n=0}^{\infty} \sigma\mathrm{mex}(n)q^n = \sum_{n=0}^{\infty} \left(\sigma\mathrm{mex}(n) - \sigma\mathrm{mex}(n-1)\right) q^n.$$

From this, we readily derive (3.10).

Now to prove (3.11), start by setting $t=2$ in Theorem 3.3.2 to get

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_2(n)q^n &= \frac{q}{(q^2;q)_\infty}\sum_{r=0}^{\infty} q^{\binom{r+2}{2}} \\ &= \frac{q}{(q^2;q)_\infty}(\psi(q)-1) = (q-q^2)\frac{\psi(q)}{(q;q)_\infty} - \frac{q}{(q^2;q)_\infty}, \qquad (3.32)\end{aligned}$$

where we again invoked (3.5) between the expressions in the first and second lines above. Now, another application of (3.4) gives us

$$(q-q^2)\frac{\psi(q)}{(q;q)_\infty} = (q-q^2)\sum_{n=0}^{\infty} \sigma\mathrm{mex}(n)q^n = \sum_{n=0}^{\infty}\{\sigma\mathrm{mex}(n-1) - \sigma\mathrm{mex}(n-2)\}q^n. \quad (3.33)$$

Also, observe that $\frac{q}{(q^2;q)_\infty}$ is the generating function for partitions with exactly one 1. Combining this knowledge along with (3.33), then substituting in (3.32) and comparing the coefficients of q^n at the two extremes furnishes us the required identity:

$$\Delta_2(n) + p(n|\mathit{exactly\ one}\ 1) = \sigma\mathrm{mex}(n-1) - \sigma\mathrm{mex}(n-2).$$

□

Proof of Proposition 3.3.4. We begin by taking a note of the structure of partitions π of n with infinitely long mex sequences. This happens precisely when no integer greater than the minimal excludant can occur as a part in π. Hence these partitions must be 'gap-free' with smallest part 1, i.e., every part between 1 and the largest part must also occur as parts. Denoting the set of such partitions of n by $\mathscr{P}^*(n)$, we hence see that $\mathscr{P}^*(n) \subset \mathscr{M}_r(n)$ for all positive integers r. Thus,

$$p_r^{\mathrm{mex}}(n) \geq |\mathscr{P}^*(n)|, \quad \forall\, r \geq 1. \tag{3.34}$$

Next, we claim that $p_n^{\mathrm{mex}}(n) = |\mathscr{P}^*(n)|$. Suppose that $\mu \in \mathscr{M}_n(n)$ and $\mathrm{mex}(\mu) = r(\geq 1)$. Then, $r, r+1, \ldots, r+n-1$ do not occur in μ. But an integer $m \geq r+n$ also cannot occur in μ, a partition of n, because $r+n \geq n+1$. Thus the parts in μ, possibly with repetitions, are $1, 2, \ldots, r-1$. (each of them occurs at least once since $\mathrm{mex}(\mu) = r$) This means that $\mu \in \mathscr{P}^*(n)$ and we conclude $\mathscr{M}_n(n) \subset \mathscr{P}^*(n)$, which gives us $p_n^{\mathrm{mex}}(n) = |\mathscr{P}^*(n)|$. As already observed in (3.12) in Section 3.3, we know that for fixed n, $p_r^{\mathrm{mex}}(n)$ is a non-increasing function of r. Therefore, using this along with (3.34) gives us

$$|\mathscr{P}^*(n)| \leq p_r^{\mathrm{mex}}(n) \leq p_n^{\mathrm{mex}}(n) = |\mathscr{P}^*(n)|, \quad \forall\, r \geq n.$$

So, for all $r \geq n$, we have showed that $p_r^{\mathrm{mex}}(n) = |\mathscr{P}^*(n)|$. We next show that if $r < n$, then $p_r^{\mathrm{mex}}(n) > |\mathscr{P}^*(n)|$. Assume that $n > 1$ as the proposition is readily seen to hold for $n = 1$. Consider the partition $\mu_0 = n$ of n, which has the mex sequence $(1, 2, \ldots, n-1)$ of length $n-1$. Since $r \leq n-1$, we deduce that $\mu_0 \in \mathscr{M}_r(n)$. But note that as $n > 1$ we have that $\mu_0 \notin \mathscr{P}^*(n)$, and consequently $p_r^{\mathrm{mex}}(n) > |\mathscr{P}^*(n)|$. Thus, we have established that $p_r^{\mathrm{mex}}(n) = |\mathscr{P}^*(n)| \iff r \geq n$. The proof of the proposition follows because $\mathscr{P}^*(n)$ is equinumerous with the set of distinct parts partitions of n, as can be seen by the bijection of conjugation between the two sets. ☐

Proof of Theorem 3.3.5. Suppose the minimal excludant in a partition is $k+1$ with the integers $k+2, \ldots, k+r$ also not occurring as parts. The integers $k+r+1$ and upwards may or may not occur as parts. Also, note that k is a non-negative integer (as the minimal excludant can be 1). Then we can begin to write the generating function for $p_r^{\mathrm{mex}}(n)$ in the following manner:

$$\sum_{n=0}^{\infty} p_r^{\mathrm{mex}}(n) q^n = \sum_{k=0}^{\infty} \frac{q^1}{1-q^1} \times \cdots \times \frac{q^k}{1-q^k} \times \frac{1}{1-q^{k+r+1}} \times \ldots \quad \text{to } \infty$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(q;q)_k} \cdot \frac{1}{(q^{k+r+1};q)_\infty} \\
&= \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}(q^{k+1};q)_r}{(q;q)_\infty} \\
&= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(q;q)_{k+r}}{(q;q)_k} q^{k(k+1)/2} \\
&= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(q;q)_r (q^{r+1};q)_k}{(q;q)_k} q^{k(k+1)/2} \\
&= \frac{1}{(q^{r+1};q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{r+1};q)_k}{(q;q)_k} q^{k(k-1)/2} \cdot q^k \\
&= \frac{1}{(q^{r+1};q)_\infty} \lim_{A\to 0} \sum_{k=0}^{\infty} \frac{(-1/A;q)_k (q^{r+1};q)_k}{(q;q)_k (0;q)_k} (Aq)^k. \qquad (3.35)
\end{aligned}$$

Note that the sum in (3.35) can be written as ${}_2\phi_1 \left[\begin{matrix} -1/A, & q^{r+1} \\ & 0 \end{matrix} ; q, Aq \right]$, using the notation in (3.3). It then changes as follows, by setting $a = -1/A,\ b = q^{r+1}, c = 0$ and $z = Aq$ in Heine's transformation (3.2),

$$\begin{aligned}
\frac{1}{(q^{r+1};q)_\infty} \lim_{A\to 0} \sum_{k=0}^{\infty} \frac{(-1/A;q)_k (q^{r+1};q)_k}{(q;q)_k (0;q)_k} (Aq)^k &= \frac{1}{(q^{r+1};q)_\infty} \lim_{A\to 0} \frac{(q^{r+1};q)_\infty (-q;q)_\infty}{(Aq;q)_\infty} \\
&\quad \sum_{n=0}^{\infty} \frac{(Aq;q)_n}{(q;q)_n (-q;q)_n} q^{(r+1)n} \\
&= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{(r+1)n}}{(q;q)_n (-q;q)_n} \\
&= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{1}{(q^2;q^2)_n} q^{(r+1)n}. \qquad (3.36)
\end{aligned}$$

The sum in (3.36) can be written as a q-product by first replacing q by q^2 in q-binomial theorem (3.1), and then setting $a = 0$ and $z = q^{r+1}$ in it. This gives us

$$\sum_{n=0}^{\infty} \frac{1}{(q^2;q^2)_n} q^{(r+1)n} = \frac{1}{(q^{r+1};q^2)_\infty}.$$

Putting this in (3.36), we finally obtain the following from (3.35):

$$\sum_{n=0}^{\infty} p_r^{\text{mex}}(n) q^n = (-q;q)_\infty \frac{1}{(q^{r+1};q^2)_\infty} = \frac{1}{(q;q^2)_\infty} \cdot \frac{1}{(q^{r+1};q^2)_\infty},$$

where the rightmost equality follows by Euler's partition theorem $(-q;q)_\infty = 1/(q;q^2)_\infty$.

☐

Proof of Corollary 3.3.6. We proceed with the proof in two directions depending on the parity of r. Recall from Theorem 3.3.5 that the generating function for $p_r^{\mathrm{mex}}(n)$ is $\dfrac{1}{(q;q^2)_\infty (q^{r+1};q^2)_\infty}$.

If r **is odd:** In this case, the numbers $r+1,\ r+3,\dots$ are all even and hence $\dfrac{1}{(q;q^2)_\infty} \times \dfrac{1}{(q^{r+1};q^2)_\infty}$ is the generating function for partitions where no even part less than r is allowed. Hence, $p_r^{\mathrm{mex}}(n) = p_e^{>r}(n)$.

If r **is even:** This time around, the integers $r+1,\ r+3,\dots$ are all odd and therefore, $\dfrac{1}{(q;q^2)_\infty} \times \dfrac{1}{(q^{r+1};q^2)_\infty}$ represents partitions into odd parts where parts greater than r come in two colors. Thus, $p_r^{\mathrm{mex}}(n) = p_{o,2}^{>r}(n)$. ☐

Proof of Corollary 3.3.7. By Theorem 3.3.2, we know that the generating function for $\Delta_1(n)$ is $\dfrac{\psi(q)}{(q^2;q)_\infty}$. On the other hand, from Theorem 3.3.5, we have seen that $\dfrac{1}{(q;q^2)_\infty (q^3;q^2)_\infty}$ generates the numbers $p_2^{\mathrm{mex}}(n)$. Since $p_2^{\mathrm{mex}}(n) = \Delta_1(n)$, we get

$$
\begin{aligned}
\frac{\psi(q)}{(q^2;q)_\infty} &= \frac{1}{(q;q^2)_\infty (q^3;q^2)_\infty} \\
\implies \frac{\psi(q)}{(q;q)_\infty} &= \frac{1}{(q;q^2)_\infty^2},
\end{aligned}
$$

where we divided throughout by $1-q$ in the last step. This finally gives

$$
\psi(q) = \frac{(q;q)_\infty}{(q;q^2)_\infty^2} = \frac{(q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty^2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.
$$

☐

3.5 Concluding Remarks

In this chapter, we introduced the concept of second minimal excludant in an integer partition. We found its generating function and linked it to minimal excludants via the function $\Delta_t(n)$. The generating function for $\Delta_t(n)$, when interpreted combinatorially, gave rise to nice identities connecting it to $\sigma\mathrm{mex}(n)$ and certain restricted partition functions. We also defined the mex sequence of a partition and discovered an elegant q-product expression for the generating function of a related function, namely, $p_r^{\mathrm{mex}}(n)$. And this gives rise to the following natural question:

Question 1. *It would be highly desirable to get a bijective proof of the identities for* $p_r^{\mathrm{mex}}(n)$ *in Corollary 3.3.6.*

Chapter 4

Some more results on Mex function

4.1 Introduction

The primary goal of this chapter is to study some more results on minimal excludant of a partition. In chapter 1, we studied $\operatorname{moex}(\pi)$, to be the smallest odd integer that is not a part of π, and

$$\sigma\operatorname{moex}(n) = \sum_{\pi\in\mathscr{P}(n)} \operatorname{moex}(\pi)$$

We also noted that,

$$\sum_{n\geq 0} \sigma\operatorname{moex}(n)q^n = (-q;q)_\infty(-q;q^2)_\infty^2. \tag{4.1}$$

Here, in this chapter we analogously define minimal even excludant denoted by $\operatorname{meex}(\pi)$, to be the smallest even integer that is not a part of π. Thus,

$$\sigma\operatorname{meex}(n) = \sum_{\pi\in\mathscr{P}(n)} \operatorname{meex}(\pi)$$

We explore the relation between $\sigma\operatorname{moex}(n)$, studied in [8] and $\sigma\operatorname{meex}(n)$ as defined above. We also study the sum of minimal even excludant in partitions into distinct parts and thus constructed a combinatorial identity. We further study the sum of squares of minimal excludant, denoted by $\sigma\operatorname{mex}^2(n)$ and established its connection with the mex functions.

4.2 Minimal Even Excludant in Ordinary Partitions

In the following theorem, we present the generating function for $\sigma\text{meex}(n)$, the sum of the smallest even integers that are missing in all the partitions of n.

Theorem 4.2.1. *We have*

$$\sum_{n\geq 0} \sigma\text{meex}(n)q^n = 2(-q;q)_\infty(-q^2;q^2)_\infty^2. \tag{4.2}$$

Proof.

$$\begin{aligned}\sum_{n\geq 0}\sum_{m\geq 2} p^{meex}(m,n)z^m q^n &= \frac{1}{(q;q^2)_\infty}\sum_{m\geq 0} z^{2m+2}\frac{q^2}{1-q^2}\times\frac{q^4}{1-q^4}\times\cdots\times\frac{q^{2m}}{1-q^{2m}}\\ &\quad\times\frac{1}{1-q^{2m+4}}\times\frac{1}{1-q^{2m+6}}\times\cdots \text{ and so on}\\ &= \frac{1}{(q;q)_\infty}\sum_{m\geq 0} z^{2m+2}q^{m(m+1)}(1-q^{2m+2}). \end{aligned}\tag{4.3}$$

As before, we differentiate with respect to z and put $z=1$ to get

$$\begin{aligned}\sum_{n\geq 0}\sigma\text{meex}(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{m\geq 0}(2m+2)q^{m(m+1)}(1-q^{2m+2})\\ &= \frac{1}{(q;q)_\infty}\sum_{m\geq 0}(2m+2)\left\{q^{m(m+1)}-q^{(m+1)(m+2)}\right\}\\ &= \frac{1}{(q;q)_\infty}\left\{\sum_{m\geq 0}(2m+2)q^{m(m+1)}-\sum_{m\geq 1}2mq^{m(m+1)}\right\}\\ &= \frac{1}{(q;q)_\infty}\left\{2+2\sum_{m\geq 1}q^{m(m+1)}\right\}\\ &= \frac{2}{(q;q)_\infty}\sum_{m\geq 0}q^{m(m+1)}\\ &= \frac{2\psi(q^2)}{(q;q)_\infty}, \end{aligned}\tag{4.4}$$

where $\psi(q)=\sum_{n\geq 0} q^{n(n+1)/2}$ is one of Ramanujan's theta functions. We know from the

Jacobi triple product identity that

$$\psi(q) = (-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty. \tag{4.5}$$

Putting this in (4.4), we see that

$$\begin{aligned}\sum_{n\geq 0}\sigma\text{meex}(n)q^n = \frac{2\psi(q^2)}{(q;q)_\infty} &= \frac{2}{(q;q)_\infty}(-q^2;q^8)_\infty(-q^6;q^8)_\infty(q^8;q^8)_\infty\\ &= \frac{2}{(q;q)_\infty}(-q^2;q^4)_\infty(q^8;q^8)_\infty\\ &= 2\frac{(q^4;q^8)_\infty(q^8;q^8)_\infty}{(q;q)_\infty(q^2;q^4)_\infty}\\ &= 2\frac{(q^4;q^4)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty}\\ &= 2\frac{(q^2;q^2)_\infty(-q^2;q^2)^2_\infty}{(q;q)_\infty}\\ &= 2(-q;q)_\infty(-q^2;q^2)^2_\infty. \end{aligned}\tag{4.6}$$

Therefore, from (4.1) and (4.6), we have

$$\begin{aligned}\left(\sum_{n\geq 0}\sigma\text{meex}(n)q^n\right)\left(\sum_{m\geq 0}\sigma\text{moex}(m)q^m\right) &= \left(2(-q;q)_\infty(-q^2;q^2)^2_\infty\right)\left((-q;q)_\infty(-q;q^2)^2_\infty\right)\\ &= 2(-q;q)^4_\infty. \end{aligned}\tag{4.7}$$

This relation between generating functions can be expressed as the following identity (by comparing coefficients on both sides):

$$\sum_{j=0}^{n}\sigma\text{meex}(j)\sigma\text{moex}(n-j) = 2D_4(n), \quad n\geq 0 \tag{4.8}$$

where $D_4(n)$ is the number of partitions of n into distinct parts using four colors. □

Note: The above expression (4.8) can be expressed in the form of a combinatorial identity, such that,

$$A_1(n) = 2D_4(n), \quad n\geq 0,$$

where $A_1(n)=\sum_{j=0}^{n}\sigma\text{meex}(j)\sigma\text{moex}(n-j)$.

4.3 Minimal Even Excludant in Partitions into Distinct Parts

We start this section by defining bipartition. Consider the frequency notation for partitions (π_1, π_2) as

$$\Big(\pi_1, \pi_2\Big) = \Big(({a_1}^{f_1}, {a_2}^{f_2}, \ldots, {a_l}^{f_l}), ({b_1}^{g_1}, {b_2}^{g_2}, \ldots, {b_s}^{g_s})\Big)$$

where $f_i \geq 0$ and $g_j \geq 0 \ \ \forall\ i$ and j, and

$$a_1 > a_2 > \ldots > a_l, \qquad b_1 > b_2 > \ldots > b_s.$$

Let us now look at $\sigma_d\text{meex}(n)$, the sum of the smallest even integers that are missing in all the partitions into distinct parts of n. With similar reasoning as in the case of $\sigma_d\text{moex}(n)$, if $p_d^{meex}(m,n)$ denote the number of partitions into distinct parts of n with $\text{meex}(\pi) = m$, then we have

$$\begin{aligned}\sum_{n\geq 0}\sum_{m\geq 2} p_d^{meex}(m,n)z^m q^n &= (-q;q^2)_\infty \sum_{n\geq 1} z^{2n} q^{2+4+\cdots+(2n-2)}(1+q^{2n+2})(1+q^{2n+4})\cdots \\ &= (-q;q^2)_\infty \sum_{n\geq 1} z^{2n} q^{n^2-n}(-q^{2n+2};q^2)_\infty \\ &= (-q;q)_\infty \sum_{n\geq 1} \frac{z^{2n}q^{n^2-n}}{(-q^2;q^2)_n}.\end{aligned}$$

Now differentiate with respect to z and then put $z=1$ to get

$$\begin{aligned}\sum_{n\geq 0}\sigma_d\text{meex}(n)q^n &= (-q;q)_\infty \sum_{n\geq 1}\frac{2nq^{n^2-n}}{(-q^2;q^2)_n} \\ &= (-q;q)_\infty \sum_{n\geq 1}\frac{2nq^{n^2-n}}{(-q^2;q^2)_{n-1}} - (-q;q)_\infty \sum_{n\geq 1}\frac{2nq^{n^2+n}}{(-q^2;q^2)_n}\end{aligned}$$

$$= (-q;q)_\infty \sum_{n\geq 0} \frac{(2n+2)q^{n^2+n}}{(-q^2;q^2)_n} - (-q;q)_\infty \sum_{n\geq 1} \frac{2nq^{n^2+n}}{(-q^2;q^2)_n}$$

$$= 2(-q;q)_\infty \sum_{n\geq 0} \frac{q^{n^2+n}}{(-q^2;q^2)_n}. \tag{4.9}$$

and we have the following results.

Theorem 4.3.1. *For $n \geq 0$, $B_1(n)$ generates twice of the partition pairs (π_1, π_2) such that*

(1) π_1 is a partition into distinct parts.

(2) π_2 is a partition into even parts without gaps.

Each partition is counted with weight $(-1)^t$, where $t = \sum_{i\geq 1}(g_i - 1)$. Then,

$$\sum_{n=0}^{\infty} B_1(n)q^n = 2(-q;q)_\infty \sum_{n\geq 0} \frac{q^{n^2+n}}{(-q^2;q^2)_n}.$$

Theorem 4.3.2. *For $n \geq 0$,*

$$B_1(n) = C_1(n),$$

where

$$C_1(n) = \sum_{m\geq 2} p_d^{meex}(m,n)z^m$$

and $B_1(n)$ is defined in Theorem 4.3.1.

Example 4.3.1. *Consider the partitions of $n = 7$ as*

$$((7),\phi), ((6,1),\phi), ((5,2),\phi), ((4,3),\phi), ((4,2,1),\phi), ((1),(2^3)), ((1),(4,2)),$$

$$((3),(2^2)), ((3,2),(2)), ((2,1),(2^2)), ((4,1),(2)), ((5),(2)).$$

Total weight attached to the partition is 8. And $B_1(n) = 2 \times 8 = 16$. Now consider the minimal even excludant for partitions of 7 into distinct parts as:

Relevant distinct partitions for $n = 7$	*meex*(π)
7	*2*
6+1	*2*
5+2	*4*
4+3	*2*
4+2+1	*6*
	$\sigma_d\mathrm{meex}(7) = 16$

4.4 Sum of squares of minimal excludants, $\sigma\mathrm{mex}^2(n)$

Define $\sigma\mathrm{mex}^2(n) = \sum_{\pi\in\mathscr{P}(n)} \mathrm{mex}^2(\pi)$, the sum of squares of minimal excludants ranging over all partitions of n. We know that

$$\sum_{n\geq 0}\sum_{m\geq 1} p^{mex}(m,n)z^m q^n = \sum_{m\geq 1} z^m \frac{q^1}{1-q^1}\times\frac{q^2}{1-q^2}\times\cdots\times\frac{q^{m-1}}{1-q^{m-1}} \tag{4.10}$$

$$\frac{1}{1-q^{m+1}}\times\frac{1}{1-q^{m+2}}\times\cdots \quad \text{and so on}$$

$$= \frac{1}{(q;q)_\infty}\sum_{m\geq 1} z^m q^{m(m-1)/2}(1-q^m)$$

$$= \frac{1}{(q;q)_\infty}\sum_{m\geq 1} z^m \left\{q^{\binom{m}{2}} - q^{\binom{m+1}{2}}\right\}. \tag{4.11}$$

Differentiate the L.H.S. of the above equation in (4.11) with respect to z twice and put $z = 1$ to get

$$\sum_{n\geq 0}\sum_{m\geq 1} m(m-1)p^{mex}(m,n)q^n = \sum_{n\geq 0}\sum_{m\geq 1} m^2 p^{mex}(m,n)q^n - \sum_{n\geq 0}\sum_{m\geq 1} m p^{mex}(m,n)q^n$$

$$= \sum_{n\geq 0}\sigma\mathrm{mex}^2(n)q^n - \sum_{n\geq 0}\sigma\mathrm{mex}(n)q^n. \tag{4.12}$$

Now, differentiating the R.H.S. in (4.11), we obtain after simplification

$$\frac{1}{(q;q)_\infty}\sum_{m\geq 1} m^2\left\{q^{\binom{m}{2}} - q^{\binom{m+1}{2}}\right\} - \frac{1}{(q;q)_\infty}\sum_{m\geq 1} m\left\{q^{\binom{m}{2}} - q^{\binom{m+1}{2}}\right\}. \tag{4.13}$$

Noting that the rightmost sum in (4.13) is simply the generating function of $\sigma\mathrm{mex}(n)$, we deduce, after comparing (4.12) and (4.13), that

$$\sum_{n\geq 0}\sigma\mathrm{mex}^2(n)q^n = \frac{1}{(q;q)_\infty}\sum_{m\geq 1} m^2\left\{q^{\binom{m}{2}} - q^{\binom{m+1}{2}}\right\}. \tag{4.14}$$

We modify the R.H.S. above into a simpler form:

$$\begin{aligned}
&\frac{1}{(q;q)_\infty}\sum_{m\geq 1} m^2\left\{q^{\binom{m}{2}} - q^{\binom{m+1}{2}}\right\}\\
&= \frac{1}{(q;q)_\infty}\left(\sum_{m\geq 1} m^2 q^{\binom{m}{2}} - \sum_{m\geq 2}(m-1)^2 q^{\binom{m}{2}}\right)\\
&= \frac{1}{(q;q)_\infty}\left(1 + \sum_{m\geq 2}(2m-1)q^{\binom{m}{2}}\right) = \frac{1}{(q;q)_\infty}\sum_{m\geq 1}(2m-1)q^{\binom{m}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n\geq 0}\sigma\mathrm{mex}^2(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{m\geq 0}(2m+1)q^{\binom{m+1}{2}}\\
&= \frac{2}{(q;q)_\infty}\sum_{m\geq 0} m q^{\binom{m+1}{2}} + \frac{1}{(q;q)_\infty}\sum_{m\geq 0} q^{\binom{m+1}{2}}.
\end{aligned} \tag{4.15}$$

Note: The sum on the R.H.S. in the first equality above is the unsigned version of the sum appearing in the famous Jacobi's identity.

We now connect the generating functions of $\sigma\mathrm{mex}_2(n)$ and $\sigma\mathrm{mex}^2(n)$. The idea comes from observing the closely related sums in (3.27) and (4.15), namely, $\sum\limits_{m\geq 0}(m-1)q^{\binom{m+1}{2}}$ and $\sum\limits_{m\geq 0} m q^{\binom{m+1}{2}}$ respectively. From (4.15), we see that

$$\frac{2}{(q;q)_\infty}\sum_{m\geq 0} m q^{\binom{m+1}{2}} = \sum_{n\geq 0}\sigma\mathrm{mex}^2(n)q^n - \frac{1}{(q;q)_\infty}\sum_{m\geq 0} q^{\binom{m+1}{2}}.$$

But the rightmost term above is simply the generating function of $\sigma\mathrm{mex}(n)$ shown in (1.18). Therefore,

$$\frac{1}{(q;q)_\infty}\sum_{m\geq 0} m q^{\binom{m+1}{2}} = \frac{1}{2}\sum_{n\geq 0}\{\sigma\mathrm{mex}^2(n) - \sigma\mathrm{mex}(n)\}q^n. \tag{4.16}$$

Now, coming to $\frac{1}{(q;q)_\infty}\sum_{m\geq 0}(m-1)q^{\binom{m+1}{2}}$, this can be written as

$$\frac{1}{(q;q)_\infty}\sum_{m\geq 0} mq^{\binom{m+1}{2}} - \frac{1}{(q;q)_\infty}\sum_{m\geq 0} q^{\binom{m+1}{2}}.$$

Using (4.16) and the fact that the rightmost term above is the generating function of $\sigma\mathrm{mex}(n)$, we have

$$\begin{aligned}\frac{1}{(q;q)_\infty}\sum_{m\geq 0}(m-1)q^{\binom{m+1}{2}} &= \frac{1}{2}\sum_{n\geq 0}\{\sigma\mathrm{mex}^2(n)-\sigma\mathrm{mex}(n)\}q^n - \sum_{n\geq 0}\sigma\mathrm{mex}(n)q^n \\ &= \frac{1}{2}\sum_{n\geq 0}\{\sigma\mathrm{mex}^2(n)-3\sigma\mathrm{mex}(n)\}q^n. \qquad (4.17)\end{aligned}$$

We are now ready to link this to $\sigma\mathrm{mex}_2(n)$. Recall (3.27), namely,

$$\sum_{n\geq 0}\sigma\mathrm{mex}_2(n)q^n = \frac{1}{(1-q)(q;q)_\infty} - \frac{1}{(q;q)_\infty}\sum_{s\geq 0}(s-1)q^{\binom{s+1}{2}}.$$

Hence, using (4.17), we obtain

$$\sum_{n\geq 0}\sigma\mathrm{mex}_2(n)q^n = \frac{1}{(1-q)(q;q)_\infty} - \frac{1}{2}\sum_{n\geq 0}\{\sigma\mathrm{mex}^2(n)-3\sigma\mathrm{mex}(n)\}q^n.$$

We can rewrite this as

$$\sum_{n\geq 0}\{2\sigma\mathrm{mex}_2(n)+\sigma\mathrm{mex}^2(n)-3\sigma\mathrm{mex}(n)\}q^n = \frac{2}{(1-q)(q;q)_\infty}. \qquad (4.18)$$

But, note that the quantity on the R.H.S. in (4.18) is the generating function of $2p_1(n)$, where $p_1(n)$ is the number of partitions of n where the integer 1 can appear in 2 colors. But proceeding either combinatorially or referring to Hirschhorn's article (*'The number of 1's in the partitions of n'*), we have $p_1(n) = p(0)+p(1)+\cdots+p(n)$. Therefore, we get the following identity connecting the mex functions we have encountered so far:

$$2\sigma\mathrm{mex}_2(n)+\sigma\mathrm{mex}^2(n)-3\sigma\mathrm{mex}(n) = 2\{p(0)+p(1)+\cdots+p(n)\}. \qquad (4.19)$$

As an example, consider $n = 4$. It is easily seen that $\sigma\text{mex}_2(4) = 16$, $\sigma\text{mex}^2(4) = 19$, $\sigma\text{mex}(n) = 9$ and $p(0) + \cdots + p(4) = 12$ and the two sides in (4.19) match for $n = 4$.

4.5 Concluding Remarks

In this chapter, we found the generating function for $\sigma\text{meex}(n)$ and $\sigma_d\text{meex}(n)$. We further found a combinatorial identity between the sum of minimal even excludant in distinct part partitions and bipartitions. We also study the sum of squares of minimal excludant, denoted by $\sigma\text{mex}^2(n)$ and thus established its connection with mex functions.

Recently in [13], the authors studied the k^{th} moments of minimal excludant and their sum. It is worthwhile to extend relation of Section 4.4 in connection with possible k^{th} minimal excludant in a particular partition.

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