

**SOME LINEAR PROGRAMMING TECHNIQUES
IN
FUZZY ENVIRONMENT**

A THESIS SUBMITTED TO
THAPAR UNIVERSITY
PATIALA (PUNJAB), INDIA
FOR THE DEGREE OF

**DOCTOR OF PHILOSOPHY
IN
MATHEMATICS**

SUBMITTED BY

NEHA BHATIA

(Registration no.: 900911015)



**SCHOOL OF MATHEMATICS & COMPUTER APPLICATIONS
THAPAR UNIVERSITY PATIALA-147004 (PUNJAB) INDIA**

December-2012

CERTIFICATE

This is to certify that the thesis entitled, “**Some Linear Programming Techniques in Fuzzy Environment**”, submitted by Ms. Neha Bhatia in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidate’s own work carried out by her under my supervision and guidance.

The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

Attestation by supervisor



Dr. Amit Kumar

Assistant Professor

School of Mathematics and Computer Applications

Thapar University

Patiala-147004

INDIA

DECLARATION

It is certified that the thesis is entirely my own and the ideas and references cited herein have been duly acknowledged.

Neha Bhatia

Neha Bhatia

Attestation by supervisor



Dr. Amit Kumar

Assistant Professor

School of Mathematics and Computer Applications

Thapar University

Patiala-147004

INDIA

DEDICATED

TO

THE ALMIGHTY

&

MY PARENTS

Acknowledgements

I thank the Almighty God, who gave me the opportunity and strength to carry out this work.

I feel privileged to express my sincere regards and gratitude to my supervisor Dr. Amit Kumar, Assistant Professor, School of Mathematics and Computer Applications, Thapar University, Patiala for his expert guidance, cool and benign temperament, valuable suggestions, support, advice and continuous encouragement throughout the course of my research work. The critical comments, rendered by him during the discussions are deeply appreciated.

I want to acknowledge the innocent blessings of Mehar. My supervisor, Dr. Amit Kumar, believes that Mehar is an angel for him and without Mehar blessings it would not have been possible to attain and fulfill the ideas presented in this thesis. Mehar is a lovely daughter of Parmpreet Kaur (Research Scholar under the supervision of Dr. Amit Kumar).

I am highly obliged to Dr. S. S. Bhatia, Head, School of Mathematics and Computer Applications, all the members of Doctoral Committee, faculty members and non-teaching staff for providing me encouragement and necessary facilities for carrying out my research work.

I am also grateful to Dr. P. K. Bajpai, Dean of Research and Sponsored Project, for his constant encouragement that was of great importance in the completion of the thesis.

I extend my thanks to Dr. Abhijit Mukherjee, Director, for his valuable support that made me consistent performer.

I am also grateful to my friends Priya Shahi, Manjit Verma, Puspinder Singh, Rashmi Sachdeva, Neetu and all other Research Scholars of School of Mathematics and Computer Applications for their timely help and the moral support they provided me during my research work.

I feel a deep sense of gratitude for my parents who formed part of my initial vision and inculcated in me the real value of education which really matters in life. I would like to give my special thanks to my brother and my fiancé for their constant encouragement that was of great importance in the completion of the thesis.

Patiala

September 04, 2012

Neha Bhatia

Abstract

In this thesis, shortcomings and limitations of the existing methods for solving as well as the existing methods to deal with the sensitivity analysis of such linear programming problems in which parameters or decision variables or both are represented by fuzzy numbers are pointed out. Also, some new methods are proposed to resolve these shortcomings as well as to overcome these limitations.

The chapter wise summary of the thesis is as follows:

Chapter 1

Chapter 1 is introductory in nature. In this chapter, a brief review of the work done in the literature for solving fuzzy variable linear programming problems, fuzzy number linear programming problems, fully fuzzy linear programming problems and interval-valued fuzzy number linear programming problems is presented. Also, a brief review of the work done in the literature to deal with the sensitivity analysis of fuzzy variable linear programming problems and fuzzy number linear programming problems is presented.

Chapter 2

Gani et al. [47] pointed out that although the methods, based on simplex tables, for solving fuzzy number linear programming problems and fuzzy variable linear programming problems are proposed in literature. However, no method, based on simplex tables, is proposed for solving fully fuzzy linear programming problems. To overcome this limitation, Gani et al. [47] proposed a method, based on simplex tables, for solving fully fuzzy linear programming problems. In this chapter, the shortcomings of this existing method are pointed out. Also, it is pointed out that the shortcomings in the existing method [47] are occurring due

to the use of existing multiplication of fuzzy numbers. To resolve the shortcomings of existing method [47], a new multiplication, named as Mehar multiplication, for *LR* flat fuzzy numbers is defined and it is shown that if in the existing method [47] proposed Mehar multiplication is used instead of existing multiplication then all the shortcomings of the existing method [47] are resolved.

Chapter 3

In this chapter, the limitations of the existing method [47] and the method, proposed in previous chapter, are pointed out and to overcome these limitations, a new method is proposed for solving fully fuzzy linear programming problems. To show the advantages of proposed method over existing method [47] and the method, proposed in previous chapter, a fully fuzzy linear programming which can neither be solved by the existing method [47] nor by the method proposed in previous chapter is solved by the proposed method. Also, the fully fuzzy linear programming problem, chosen in previous chapter, is solved by the proposed method.

Chapter 4

In this chapter, an alternative method is proposed for solving fully fuzzy linear programming problems and it is shown that if a fully fuzzy linear programming problem is either solved by the method proposed in previous chapter or by the method proposed in this chapter, the fuzzy optimal solution and the fuzzy optimal value will be same. However, it is much easy to apply the method proposed in this chapter, as compared to the method proposed in previous chapter for solving the fully fuzzy linear programming problems.

Chapter 5

Su [115] pointed out that the values of the membership of an element in a fuzzy set

are represented by a real number in $[0, 1]$ but a specialist is always uncertain about the values of the membership. Hence, it is better to use interval-valued fuzzy set in which the values of the membership of an element in a set is represented by the interval of possible real numbers instead of real numbers and proposed a method for solving such linear programming problems in which all the parameters except the decision variables are represented by interval-valued fuzzy numbers. In this chapter, the limitations of this method are pointed out. Also, it is pointed out that to overcome the limitations of the existing method [115] there is need to define such multiplication of two interval-valued LR flat fuzzy numbers \tilde{A} and \tilde{B} for which the property $\mathfrak{R}(\tilde{A} \otimes \tilde{B}) = \mathfrak{R}(\tilde{A})\mathfrak{R}(\tilde{B})$ is satisfied. Due to the same reason, in this chapter, a new multiplication, named as Mehar multiplication, of interval-valued LR flat fuzzy numbers is defined and using it a new method is proposed for solving the interval-valued fully fuzzy linear programming problems.

Chapter 6

To the best of our knowledge, till now neither there is any method in the literature to deal with the sensitivity analysis of fully fuzzy linear programming problems nor any method to deal with the sensitivity analysis of fully fuzzy transportation problems. In this chapter, two new methods, named as Mehar methods, are proposed for the same. To show the advantages of proposed methods over existing methods [35, 65, 95], some fuzzy sensitivity analysis problems which can not be solved by the existing methods [35, 65, 95] are solved by using the proposed method.

Chapter 7

In Chapter 5, to overcome the limitations of the existing method [115] a new method, based on proposed Mehar multiplication, is proposed for solving interval-valued fully fuzzy linear programming problems. In this chapter, a new method, based on existing

multiplication, is proposed for solving interval-valued fully fuzzy linear programming problems. Also, on the basis of proposed method, a new method is proposed to deal with the sensitivity analysis of interval-valued fully fuzzy linear programming problems.

Chapter 8

Any modification in fully fuzzy linear programming problems may or may not affect its fuzzy optimal solution. However, neither in the existing methods [35, 65, 95] nor in the methods proposed in Chapter 6 and Chapter 7, it is checked that the modifications in the fuzzy linear programming problem will affect the fuzzy optimal solution of the original fuzzy linear programming problem or not. Also, neither the existing methods [35, 65, 95] nor the methods, proposed in Chapter 6 and Chapter 7, are applicable to find the possible modifications in the fully fuzzy linear programming problem without affecting its fuzzy optimal solution. In this chapter, a new method is proposed for the same. Also, proposed method is illustrated with the help of numerical example.

Chapter 9

Finally, in this Chapter, based on the presented study future work has been suggested.

List of research papers

- 1) **Neha Bhatia**, Amit Kumar, Mehar's method for solving fuzzy sensitivity analysis problems with *LR* flat fuzzy numbers, Applied Mathematical Modelling, 36 (2012), 4087-4095. **(Impact factor: 1.579)**
- 2) **Neha Bhatia**, Amit Kumar, Sensitivity analysis for interval-valued fully fuzzy linear programming problems, Journal of Applied Research and Technology. **(Accepted)**
(Impact factor: 0.145)
- 3) **Neha Bhatia**, Amit Kumar, A new method for sensitivity analysis of fuzzy transportation problems, Journal of Intelligent and Fuzzy Systems. **(Accepted)** **(Impact factor: 0.556)**
- 4) Amit Kumar, **Neha Bhatia**, A new method for solving fuzzy sensitivity analysis problems, International Journal of Applied Science and Engineering, 9 (2011) 49-64.
- 5) Amit Kumar, **Neha Bhatia**, A new method for solving sensitivity analysis for fuzzy linear programming problems, International Journal of Applied Science and Engineering, 9 (2011) 169-176.
- 6) Amit Kumar, **Neha Bhatia**, A new method for sensitivity analysis of fuzzy linear programming problems, ASOR Bulletin, 31 (2011) 2-12.
- 7) Amit Kumar, **Neha Bhatia**, Sensitivity analysis for fuzzy linear programming problems based on interval valued fuzzy numbers, International Journal of Mathematics in Operational Research, 4 (2012) 531-547.
- 8) Amit Kumar, **Neha Bhatia**, Strict sensitivity analysis for fuzzy linear programming problem, Journal of Fuzzy Set Valued Analysis, 2012 (2012) 1-19.
- 9) Amit Kumar, **Neha Bhatia**, Sensitivity analysis for fuzzy linear programming problems, Rough Sets, Fuzzy Sets, Data Mining and Granular Computing (RSFDGrC) Lecture Notes in Computer Science, 6743 (2011) 103-110.

- 10) Amit Kumar, Neha Bhatia, Fuzzy optimal solution of fully fuzzy linear programming problems on the basis of Mehar multiplication, Journal of Optimization Theory and Applications. **(Communicated)**
- 11) Amit Kumar, Neha Bhatia, A new method for solving fully fuzzy linear programming problems, Applied Mathematical Modelling. **(Communicated)**
- 12) Amit Kumar, Neha Bhatia, An efficient method for solving fully fuzzy linear programming problems, Journal of Intelligent and Fuzzy Systems. **(Communicated)**
- 13) Amit Kumar, Neha Bhatia, Novel method for solving interval-valued fully fuzzy linear programming problems, Journal of Applied Research and Technology. **(Communicated)**
- 14) Amit Kumar, Neha Bhatia, Possible modifications in the fully fuzzy linear programming problems without affecting the fuzzy optimal solution, ASOR Bulletin. **(Communicated)**

Table of Contents

Table of contents	ix
1 Introduction	1
1.1 Literature review.....	4
1.2 Organization of the thesis.....	6
2 Fuzzy optimal solution of fully fuzzy linear programming problems on the basis of proposed Mehar multiplication	11
2.1 Preliminaries.....	11
2.1.1 Yager’s ranking approach.....	12
2.1.2 Arithmetic operations.....	14
2.2 Shortcomings of the existing method for solving fully fuzzy linear programming problems.....	14
2.3 Origin of shortcomings.....	20
2.4 Proposed Mehar multiplication and division.....	22
2.5 Proposed modifications.....	23
2.5.1 Solution of existing fully fuzzy linear programming problem.....	23
2.5.2 Solution of chosen fully fuzzy linear programming problem.....	34
2.6 Comparative study.....	41
2.7 Conclusions.....	43
3 A new method for solving fully fuzzy linear programming problems	45
3.1 Limitations of the existing method and the method porposed in previous chapter.....	45
3.2 Proposed method.....	46
3.3 Advantages of the proposed method.....	49
3.3.1 Fuzzy optimal solution of the fully fuzzy linear programming problem	

chosen in Example 2.3.....	50
3.3.2 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 3.1.....	52
3.4 Comparative study.....	55
3.5 Conclusions.....	56
4 An efficient method for solving fully fuzzy linear programming problems	57
4.1 Proposed alternative method.....	57
4.2 Advantages of the proposed alternative method.....	59
4.2.1 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 2.3.....	59
4.2.2 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 3.1.....	61
4.3 Comparative study.....	63
4.4 Conclusions.....	64
5 Novel method for solving interval-valued fully fuzzy linear programming problems	65
5.1 Preliminaries.....	65
5.1.1 Basic definition.....	65
5.1.2 Arithmetic operations.....	67
5.1.3 Comparison of interval-valued <i>LR</i> flat fuzzy numbers.....	68
5.2 Limitation of the existing method.....	69
5.3 Proposed Mehar multiplication.....	70
5.4 Proposed method.....	71
5.5 Advantages of the proposed method.....	74
5.5.1 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 5.2.....	74

5.6	Comparative study.....	77
5.7	Conclusions.....	78
6	Sensitivity analysis of fully fuzzy linear programming problems and fully fuzzy transportation problems	81
6.1	Preliminaries.....	81
6.2	Limitations of the existing methods.....	82
6.3	Proposed Mehar method for the sensitivity analysis of fully fuzzy linear programming problems.....	83
6.4	Advantages of proposed Mehar method over existing methods.....	87
6.5	Comparative study.....	94
6.6	Proposed Mehar method for the sensitivity analysis of fully fuzzy transportation problems.....	95
6.7	Illustrative example.....	101
6.8	Conclusions.....	108
7	Sensitivity analysis of interval-valued fully fuzzy linear programming problems	109
7.1	Proposed method.....	109
7.1.1	Proposed method for finding the fuzzy optimal solution of interval-valued fully fuzzy linear programming problems.....	110
7.1.2	Proposed method to deal with the sensitivity analysis of interval-valued fully fuzzy linear programming problems.....	111
7.2	Illustrative example.....	114
7.3	Conclusions.....	122
8	Possible modifications in the fully fuzzy linear programming problems without affecting the fuzzy optimal solution	123
8.1	Proposed method.....	123

8.2 Illustrative example.....	125
8.3 Conclusions.....	129
9 Future scope	131
Bibliography	133

Chapter 1

Introduction

In real life problems, the aim of each and every person is to maximize or minimize a specific objective or some specific objectives. However, to achieve the objectives of most of the real life problems there is need to fulfill some restrictions. Any real life problem whose aim is to maximize or minimize a single objective with m restrictions and n variables can be formulated into mathematical programming problem ($P_{1.1}$):

$$\begin{aligned} & \text{Maximize (or Minimize)} && f(X) \\ & \text{Subject to} && (P_{1.1}) \\ & && g_j(X) \leq (\text{or } = \text{ or } \geq) 0; j = 1, 2, \dots, m \\ & && X = [x_j]_{n \times 1} \end{aligned}$$

If all the functions $f(X)$ and $g_j(X); j = 1, 2, \dots, m$ are linear then the mathematical programming problem ($P_{1.1}$) is known as linear programming problem and can be written as:

$$\begin{aligned} & \text{Maximize (or Minimize)} && C^T X \\ & \text{Subject to} && (P_{1.2}) \\ & && AX \leq (\text{or } = \text{ or } \geq) b \\ & && X \geq 0 \end{aligned}$$

where, $C^T = [c_j]_{1 \times n}$, $X = [x_j]_{n \times 1}$, $A = [a_{ij}]_{m \times n}$, $b = [b_i]_{m \times 1}$.

In the linear programming problem ($P_{1.2}$), the values of the parameters c_j, x_j, a_{ij} and b_i are assigned by the experts. However, neither the experts nor the decision makers precisely know the values of these parameters. Due to the same reason, in the literature fuzzy numbers [134] are used to represent these parameters.

The linear programming problems in which parameters are represented by fuzzy numbers can be categorized as follows:

- 1) The linear programming problems in which the parameter c_j or the parameters a_{ij} and b_i or all the parameters c_j , a_{ij} and b_i are represented by fuzzy numbers but the decision variables x_j are represented by real numbers are known as fuzzy number linear programming problems and can be written as [83, 85, 92]:

$$(i) \text{ Maximize (or Minimize)} \quad \tilde{C}^T X$$

$$\text{Subject to} \quad (P_{1.3})$$

$$AX \leq (\text{or } = \text{ or } \geq) b$$

$$X \geq 0$$

where, $\tilde{C}^T = [\tilde{c}_j]_{1 \times n}$, $X = [x_j]_{n \times 1}$, $A = [a_{ij}]_{m \times n}$, $b = [b_i]_{m \times 1}$.

$$(ii) \text{ Maximize (or Minimize)} \quad C^T X$$

$$\text{Subject to} \quad (P_{1.4})$$

$$\tilde{A}X \leq_{\mathfrak{R}} (\text{or } =_{\mathfrak{R}} \text{ or } \geq_{\mathfrak{R}}) \tilde{b}$$

$$X \geq 0$$

where, $C^T = [c_j]_{1 \times n}$, $X = [x_j]_{n \times 1}$, $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, \mathfrak{R} is a linear ranking function.

$$(iii) \text{ Maximize (or Minimize)} \quad \tilde{C}^T X$$

$$\text{Subject to} \quad (P_{1.5})$$

$$\tilde{A}X \leq_{\mathfrak{R}} (\text{or } =_{\mathfrak{R}} \text{ or } \geq_{\mathfrak{R}}) \tilde{b}$$

$$X \geq 0$$

where, $\tilde{C}^T = [\tilde{c}_j]_{1 \times n}$, $X = [x_j]_{n \times 1}$, $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, \mathfrak{R} is a linear ranking function.

- 2) The linear programming problems in which all the decision variables x_j are represented

by fuzzy numbers are known as fuzzy variable linear programming problems and can be written as [83, 86, 91]:

$$(i) \text{ Maximize (or Minimize)} \quad \tilde{C}^T \otimes \tilde{X} \quad (P_{1.6})$$

Subject to

$$A\tilde{X} \leq_{\mathfrak{R}} \text{ (or } =_{\mathfrak{R}} \text{ or } \geq_{\mathfrak{R}}) \tilde{b}$$

where, $\tilde{C}^T = [\tilde{c}_j]_{1 \times n}$, $\tilde{X} = [\tilde{x}_j]_{n \times 1}$, $A = [a_{ij}]_{m \times n}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, $\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}$, \mathfrak{R} is a linear ranking function.

$$(ii) \text{ Maximize (or Minimize)} \quad C^T \tilde{X} \quad (P_{1.7})$$

Subject to

$$A\tilde{X} \leq_{\mathfrak{R}} \text{ (or } =_{\mathfrak{R}} \text{ or } \geq_{\mathfrak{R}}) \tilde{b}$$

where, $C^T = [c_j]_{1 \times n}$, $\tilde{X} = [\tilde{x}_j]_{n \times 1}$, $A = [a_{ij}]_{m \times n}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, $\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}$, \mathfrak{R} is a linear ranking function.

3) The linear programming problems in which all the parameters c_j , a_{ij} and b_i as well as all the decision variables x_j are represented by fuzzy numbers are known as fully fuzzy linear programming problems and can be written as [4] :

$$\text{Maximize (or Minimize)} \quad \tilde{C}^T \otimes \tilde{X} \quad (P_{1.8})$$

Subject to

$$\tilde{A} \otimes \tilde{X} \leq_{\mathfrak{R}} \text{ (or } =_{\mathfrak{R}} \text{ or } \geq_{\mathfrak{R}}) \tilde{b}$$

where, $\tilde{C}^T = [\tilde{c}_j]_{1 \times n}$, $\tilde{X} = [\tilde{x}_j]_{n \times 1}$, $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, $\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}$, \mathfrak{R} is a linear ranking function.

The membership of an element in a fuzzy set is represented by a real number in [0,1] but a specialist is always uncertain about the values of the membership. Due to the same

reason, in the literature [115] the concept of fuzzy number linear programming problems is extended by the interval-valued fuzzy number linear programming problems (linear programming problems in which the parameter c_j or the parameters a_{ij} and b_i or all the parameters c_j , a_{ij} and b_i are represented by interval-valued fuzzy numbers but the decision variables x_j are represented by real numbers).

1.1 Literature review

In this section, a brief review of the work done in the literature for solving fuzzy variable linear programming problems, fuzzy number linear programming problems, fully fuzzy linear programming problems and interval-valued fuzzy number linear programming problems is presented. Also, a brief review of the work done in the literature to deal with the sensitivity analysis of fuzzy variable linear programming problems and fuzzy number linear programming problems is presented.

Maleki et al. [86] pointed out the shortcomings of the possibilistic programming or multiobjective programming methods for solving fuzzy variable linear programming problems and introduced a new method, based on the concept of comparison of fuzzy numbers, for the same. Naseri and Ardil [91] proposed a method, based on simplex tables, for solving fuzzy variable linear programming problems. Naseri et al. [92] proposed a method, based on simplex tables, for solving fuzzy number linear programming problems.

Ganesan and Veeramani [46] proposed a new type of multiplication for symmetric trapezoidal fuzzy numbers and proposed a method, based on simplex tables, for solving fuzzy variable linear programming problems. Mahdavi-Amiri and Naseri [81] proposed some duality properties for fuzzy number linear programming problems as well as a method to find the dual of a fuzzy number linear programming problem.

Su [115] extended the concept of fuzzy number linear programming problems by the interval-valued fuzzy number linear programming problems and proposed a method for finding its optimal solution.

Allaviranloo et al. [4] proposed a method to find the fuzzy optimal solution of fully fuzzy linear programming problems by converting it into crisp linear programming problems. Gani et al. [47] pointed out that there is no method in the literature, based on simplex tables, for solving fully fuzzy linear programming problems and proposed a method for the same.

Lotfi et al. [75] proposed a method to find the fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints. Ebrahimnezad et al. [36] pointed out the limitations of the existing method [46] and proposed a new method to overcome these limitations.

Kumar et al. [69] pointed out the shortcomings of the existing method [75] and proposed a new method to overcome these shortcomings. Naseri and Ebrahimnezad [95] proposed a method, based on simplex tables, to deal with the sensitivity analysis of fuzzy variable linear programming problems. Ebrahimnezad [35] proposed a method, based on simplex tables, to deal with the sensitivity analysis of fuzzy number linear programming problems.

After reviewing the literature, it can be concluded that there are very few methods for solving fully fuzzy linear programming problems and neither there is any method to solve interval-valued fully fuzzy linear programming problems (linear programming problems in which all the parameters as well as decision variables are represented by interval-valued fuzzy numbers) nor to deal with the sensitivity analysis of fully fuzzy linear programming problems and interval-valued fully fuzzy linear programming problems.

In this thesis, new methods are proposed for solving fully fuzzy linear programming problems and interval-valued fully fuzzy linear programming problems. Also, new methods

are proposed to deal with the sensitivity analysis of fully fuzzy linear programming problems and interval-valued fully fuzzy linear programming problems.

1.2 Organization of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

Gani et al. [47] pointed out that although the methods, based on simplex tables, for solving fuzzy number linear programming problems and fuzzy variable linear programming problems are proposed in literature. However, no method, based on simplex tables, is proposed for solving fully fuzzy linear programming problems. To overcome this limitation, Gani et al. [47] proposed a method, based on simplex tables, for solving fully fuzzy linear programming problems. In this chapter, the shortcomings of this existing method are pointed out. Also, it is pointed out that the shortcomings in the existing method [47] are occurring due to the use of existing multiplication of fuzzy numbers. To resolve the shortcomings of the existing method [47], a new multiplication, named as Mehar multiplication, for *LR* flat fuzzy numbers is defined and it is shown that if in the existing method [47] proposed Mehar multiplication is used instead of existing multiplication then all the shortcomings of the existing method [47] are resolved.

Chapter 3

In this chapter, the limitations of the existing method [47] and the method, proposed in previous chapter, are pointed out and to overcome these limitations, a new method is proposed for solving fully fuzzy linear programming problems. To show the advantages of proposed method over existing method [47] and the method, proposed in previous chapter, a

fully fuzzy linear programming problem which can neither be solved by the existing method [47] nor by the method proposed in previous chapter, is solved by the proposed method. Also, the fully fuzzy linear programming problem, chosen in previous chapter, is solved by the proposed method.

Chapter 4

In this chapter, an alternative method is proposed for solving fully fuzzy linear programming problems and it is shown that if a fully fuzzy linear programming problem is either solved by the method proposed in previous chapter or by the method proposed in this chapter, the fuzzy optimal solution and the fuzzy optimal value will be same. However, it is much easy to apply the method proposed in this chapter, as compared to the method proposed in previous chapter for solving the fully fuzzy linear programming problems.

Chapter 5

Su [115] pointed out that the values of the membership of an element in a fuzzy set are represented by a real number in $[0, 1]$ but a specialist is always uncertain about the values of the membership. Hence, it is better to use interval-valued fuzzy set in which the values of the membership of an element in a set is represented by the interval of possible real numbers instead of real numbers and proposed a method for solving such linear programming problems in which all the parameters except the decision variables are represented by interval-valued fuzzy numbers. In this chapter, the limitations of this method are pointed out. Also, it is pointed out that to overcome the limitations of the existing method [115] there is need to define such multiplication of two interval-valued *LR* flat fuzzy numbers \tilde{A} and \tilde{B} for which the property $\Re(\tilde{A} \otimes \tilde{B}) = \Re(\tilde{A})\Re(\tilde{B})$ is satisfied. Due to the same reason, in this chapter, a new multiplication, named as Mehar multiplication, of interval-valued *LR* flat fuzzy numbers is defined and using it a new method is proposed for solving the interval-

valued fully fuzzy linear programming problems.

Chapter 6

To the best of our knowledge, till now neither there is any method in the literature to deal with the sensitivity analysis of fully fuzzy linear programming problems nor any method to deal with the sensitivity analysis of fully fuzzy transportation problems. In this chapter, two new methods, named as Mehar methods, are proposed for the same. To show the advantages of proposed methods over existing methods [35, 65, 95], some fuzzy sensitivity analysis problems which can not be solved by the existing methods [35, 65, 95] are solved by using the proposed method.

Chapter 7

In Chapter 5, to overcome the limitations of the existing method [115] a new method, based on proposed Mehar multiplication, is proposed for solving interval-valued fully fuzzy linear programming problems. In this chapter, a new method, based on existing multiplication, is proposed for solving interval-valued fully fuzzy linear programming problems. Also, on the basis of proposed method, a new method is proposed to deal with the sensitivity analysis of interval-valued fully fuzzy linear programming problems.

Chapter 8

Any modification in fully fuzzy linear programming problems may or may not affect its fuzzy optimal solution. However, neither in the existing methods [35, 65, 95] nor in the methods proposed in Chapter 6 and Chapter 7, it is checked that the modifications in the fuzzy linear programming problem will affect the fuzzy optimal solution of the original fuzzy linear programming problem or not. Also, neither the existing methods [35, 65, 95] nor the methods, proposed in Chapter 6 and Chapter 7, are applicable to find the possible

modifications in the fully fuzzy linear programming problem without affecting its fuzzy optimal solution. In this chapter, a new method is proposed for the same. Also, proposed method is illustrated with the help of numerical example.

Chapter 9

Finally, in this Chapter, based on the presented study future work has been suggested.

Chapter 2

Fuzzy optimal solution of fully fuzzy linear programming problems on the basis of proposed Mehar multiplication*

Gani et al. [47] pointed out that although the methods, based on simplex tables, for solving fuzzy number linear programming problems and fuzzy variable linear programming problems are proposed in literature. However, no method, based on simplex tables, is proposed for solving fully fuzzy linear programming problems. To overcome this limitation, Gani et al. [47] proposed a method, based on simplex tables, for solving fully fuzzy linear programming problems. In this chapter, the shortcomings of this existing method are pointed out. Also, it is pointed out that the shortcomings in the existing method [47] are occurring due to the use of existing multiplication of fuzzy numbers. To resolve the shortcomings of the existing method [47], a new multiplication, named as Mehar multiplication, for *LR* flat fuzzy numbers is defined and it is shown that if in the existing method [47] proposed Mehar multiplication is used instead of existing multiplication then all the shortcomings of the existing method [47] are resolved.

2.1 Preliminaries

In this section, some basic concepts which are needed in rest of the chapters are presented [33].

Fuzzy set is defined as a subset \tilde{A} of a universal set $X \subseteq \mathbb{R}$ (set of real numbers) by its membership function $\mu_{\tilde{A}}(\cdot)$, which assigns to each element $x \in \mathbb{R}$, a real number $\mu_{\tilde{A}}(x)$, in the interval $[0,1]$. The λ -cut or λ -level set of \tilde{A} , which plays an essential role in fuzzy

* The contents of this chapter are communicated in *Journal of Optimization Theory and Applications*.

optimization, is defined as an ordinary set A_λ for which the degree of its membership function exceeds the level λ . The fuzzy number is a convex normalized fuzzy set of the real line \mathbb{R} whose membership function is piecewise continuous. The set of all fuzzy numbers on \mathbb{R} is denoted by $F(\mathbb{R})$.

Definition 2.1 A function $L: [0, \infty) \rightarrow [0, 1]$ (or $R: [0, \infty) \rightarrow [0, 1]$) is said to be reference function of fuzzy number if and only if

- i) $L(0)=1$ (or $R(0)=1$)
- ii) L (or R) is non-increasing on $[0, \infty)$.

Definition 2.2 A fuzzy number \tilde{A} defined on the universal set of real numbers \mathbb{R} , denoted as $\tilde{A}=(m, n, \alpha, \beta)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\tilde{A}}(x)$ is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \leq m, \alpha > 0 \\ R\left(\frac{x-n}{\beta}\right), & x \geq n, \beta > 0 \\ 1, & m \leq x \leq n \end{cases}$$

Definition 2.3 Let $\tilde{A}=(m, n, \alpha, \beta)_{LR}$ be an LR flat fuzzy number and λ be a real number in the interval $[0, 1]$. Then, the crisp set $A_\lambda = \{x \in X : \mu_{\tilde{A}}(x) \geq \lambda\} = [m - \alpha L^{-1}(\lambda), n + \beta R^{-1}(\lambda)]$ is said to be λ -cut of \tilde{A} .

2.1.1 Yager's ranking approach

A number of ranking approaches have been proposed in the literature for comparing fuzzy numbers. However, in the whole thesis, a relatively simple computational and easily understandable existing ranking approach [132] is used for comparing fuzzy numbers.

Let $\tilde{A}=(m_1, n_1, \alpha_1, \beta_1)_{LR}$, and $\tilde{B}=(m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two *LR* flat fuzzy numbers.

Then,

- (i) $\tilde{A} \geq_{\mathfrak{R}} \tilde{B}$ if $\mathfrak{R}(\tilde{A}) \geq \mathfrak{R}(\tilde{B})$
- (ii) $\tilde{A} >_{\mathfrak{R}} \tilde{B}$ if $\mathfrak{R}(\tilde{A}) > \mathfrak{R}(\tilde{B})$
- (iii) $\tilde{A} =_{\mathfrak{R}} \tilde{B}$ if $\mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$

where, $\mathfrak{R}(\tilde{A}) = \frac{1}{2} \int_0^1 (m_1 - \alpha_1 L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_1 + \beta_1 R^{-1}(\lambda)) d\lambda$

and $\mathfrak{R}(\tilde{B}) = \frac{1}{2} \int_0^1 (m_2 - \alpha_2 L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_2 + \beta_2 R^{-1}(\lambda)) d\lambda$

Lemma 2.1 If $L(x) = R(x) = \text{maximum}\{0, 1-x^2\}$ then, $\mathfrak{R}(\tilde{A}) = \frac{1}{2} \left(m_1 + n_1 - \frac{2\alpha_1}{3} + \frac{2\beta_1}{3} \right)$ and

$\mathfrak{R}(\tilde{B}) = \frac{1}{2} \left(m_2 + n_2 - \frac{2\alpha_2}{3} + \frac{2\beta_2}{3} \right)$.

Proof: Since, $L(x) = R(x) = \text{maximum}\{0, 1-x^2\}$ so $L^{-1}(\lambda) = \sqrt{1-\lambda}$ and $R^{-1}(\lambda) = \sqrt{1-\lambda}$ where, $\lambda \in [0, 1]$.

Substituting the values of $L^{-1}(\lambda)$ and $R^{-1}(\lambda)$ in Yager's ranking formula,

$\mathfrak{R}(\tilde{A}) = \frac{1}{2} \int_0^1 (m - \alpha \sqrt{1-\lambda}) d\lambda + \frac{1}{2} \int_0^1 (n + \beta \sqrt{1-\lambda}) d\lambda$, the value of $\mathfrak{R}(\tilde{A})$ and $\mathfrak{R}(\tilde{B})$

are $\frac{1}{2} \left(m_1 + n_1 - \frac{2\alpha_1}{3} + \frac{2\beta_1}{3} \right)$ and $\frac{1}{2} \left(m_2 + n_2 - \frac{2\alpha_2}{3} + \frac{2\beta_2}{3} \right)$ respectively.

Remark 2.1 An *LR* flat fuzzy number $\tilde{A}=(m, n, \alpha, \beta)_{LR}$ is said to be zero fuzzy number if, $\mathfrak{R}(\tilde{A}) = 0$ and is denoted as $\tilde{0}$.

Remark 2.2 If $m=n$ then an *LR* flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be an *LR* fuzzy number and is denoted as $(m, m, \alpha, \beta)_{LR}$ or $(n, n, \alpha, \beta)_{LR}$ or $(m, \alpha, \beta)_{LR}$ or $(n, \alpha, \beta)_{LR}$.

Remark 2.3 If $m=n$ and $L(x)=R(x)=\text{maximum}\{0, 1-x\}$ then an *LR* flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be a triangular fuzzy number.

Remark 2.4 If $m \neq n$ and $L(x) = R(x) = \text{maximum } \{0, 1 - x\}$ then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be a trapezoidal fuzzy number.

2.1.2 Arithmetic operations

In this section, arithmetic operations between two LR flat fuzzy numbers are presented [33].

Let $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$, $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers and $\tilde{A}_3 = (m_3, n_3, \alpha_3, \beta_3)_{RL}$ be a RL flat fuzzy number. Then,

$$(i) \quad \tilde{A}_1 \oplus \tilde{A}_2 = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$$

$$(ii) \quad \tilde{A}_1 \ominus \tilde{A}_3 = (m_1 - n_3, n_1 - m_3, \alpha_1 + \beta_3, \beta_1 + \alpha_3)_{LR}$$

$$(iii) \quad \lambda \tilde{A}_1 = \begin{cases} (\lambda m_1, \lambda n_1, \lambda \alpha_1, \lambda \beta_1)_{LR}, & \lambda \geq 0 \\ (\lambda n_1, \lambda m_1, -\lambda \beta_1, -\lambda \alpha_1)_{RL}, & \lambda \leq 0 \end{cases}$$

(iv) If $m_1 - \alpha_1 \geq 0$ and $m_2 - \alpha_2 \geq 0$. Then,

$$\tilde{A}_1 \otimes \tilde{A}_2 = (m_1 m_2, n_1 n_2, m_1 \alpha_2 + \alpha_1 m_2, n_1 \beta_2 + \beta_1 n_2)_{LR}$$

(v) If $n_1 + \beta_1 \leq 0$ and $m_2 - \alpha_2 \geq 0$. Then,

$$\tilde{A}_1 \otimes \tilde{A}_2 = (-m_1 n_2, -n_1 m_2, \beta_1 m_2 - n_1 \alpha_2, \alpha_1 n_2 - m_1 \beta_2)_{LR}$$

(vi) If $m_1 - \alpha_1 \geq 0$ and $n_2 + \beta_2 \leq 0$. Then,

$$\tilde{A}_1 \otimes \tilde{A}_2 = (-n_1 m_2, -m_1 n_2, m_1 \beta_2 - \alpha_1 n_2, n_1 \alpha_2 - \beta_1 m_2)_{LR}$$

(vii) If $n_1 + \beta_1 \leq 0$ and $n_2 + \beta_2 \leq 0$. Then,

$$\tilde{A}_1 \otimes \tilde{A}_2 = (n_1 n_2, m_1 m_2, -n_1 \beta_2 - \beta_1 n_2, -m_1 \alpha_2 - \alpha_1 m_2)_{LR}$$

2.2 Shortcomings of the existing method for solving fully fuzzy linear programming problems

Gani et al. [47] pointed out that there is no method in the literature, based on simplex tables, for solving fully fuzzy linear programming problems ($P_{2.1}$) and proposed a method by replacing the crisp arithmetic operations of the existing method [61] with existing fuzzy

arithmetic operations [33] as well as by using the method, presented in Section 2.1.1, for comparing fuzzy numbers:

$$\begin{aligned} & \text{Maximize} && \sum_{j=1}^n \tilde{c}_j \otimes \tilde{x}_j \\ & \text{Subject to} && \end{aligned} \tag{P_{2.1}}$$

$$\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \leq_{\mathfrak{R}} \text{ or } =_{\mathfrak{R}} \text{ or } \geq_{\mathfrak{R}} \tilde{b}_i; i = 1, 2, \dots, m$$

$$\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}; j = 1, 2, \dots, n$$

where, \tilde{c}_j , \tilde{a}_{ij} and $\tilde{b}_i \in F(\mathbb{R})$; $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ and $F(\mathbb{R})$ is the set of all *LR* fuzzy numbers.

However, it is not genuine to use the existing method [47] due to the following reason:

In each simplex table of a crisp linear programming problem the i^{th} element of the column corresponding to the i^{th} basic variable should be one and remaining elements should be zero. In the same direction, in each simplex table of a fully fuzzy linear programming problem the rank of the i^{th} element of the column corresponding to the i^{th} fuzzy basic variable should be one and rank of remaining elements should be zero.

In this section, to show the shortcomings of the existing method [47], the existing problem [47], presented in Example 2.1, is solved by the existing method [47] and it is shown that for the elements of the simplex table, obtained by using the existing method [47], this property is not satisfying.

Example 2.1 [47, pp. 103] Consider the fully fuzzy linear programming problem:

$$\begin{aligned} & \text{Maximize} && [(5,1,2)_{LR} \otimes \tilde{x}_1 \oplus (4,1,1)_{LR} \otimes \tilde{x}_2] \\ & \text{Subject to} && \end{aligned} \tag{E_{2.1}}$$

$$(4,2,1)_{LR} \otimes \tilde{x}_1 \oplus (5,3,1)_{LR} \otimes \tilde{x}_2 \leq_{\mathfrak{R}} (24,5,8)_{LR}$$

$$(4,1,2)_{LR} \otimes \tilde{x}_1 \oplus (1,.5,1)_{LR} \otimes \tilde{x}_2 \leq_{\mathfrak{R}} (12,6,3)_{LR}$$

$$\tilde{x}_1 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_2 \geq_{\mathfrak{R}} \tilde{0}$$

where, $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$.

Solution

Using the existing method [47], the fuzzy optimal solution of fully fuzzy linear programming problem, chosen in Example 2.1, can be obtained as follows:

Step 1

Adding fuzzy slack variables \tilde{x}_3 and \tilde{x}_4 in first and second constraint respectively, the chosen fully fuzzy linear programming problem ($E_{2.1}$) is converted into fully fuzzy linear programming problem ($E_{2.2}$).

$$\text{Maximize} \quad [(5,1,2)_{LR} \otimes \tilde{x}_1 \oplus (4,1,1)_{LR} \otimes \tilde{x}_2 \oplus \tilde{0} \otimes \tilde{x}_3 \oplus \tilde{0} \otimes \tilde{x}_4]$$

$$\text{Subject to} \quad (E_{2.2})$$

$$(4,2,1)_{LR} \otimes \tilde{x}_1 \oplus (5,3,1)_{LR} \otimes \tilde{x}_2 \oplus \tilde{x}_3 =_{\mathfrak{R}} (24,5,8)_{LR}$$

$$(4,1,2)_{LR} \otimes \tilde{x}_1 \oplus (1,.5,1)_{LR} \otimes \tilde{x}_2 \oplus \tilde{x}_4 =_{\mathfrak{R}} (12,6,3)_{LR}$$

$$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \geq_{\mathfrak{R}} \tilde{0}$$

The initial fuzzy simplex table for the fully fuzzy linear programming problem ($E_{2.2}$) is:

Table 2.1 The initial fuzzy simplex table

Basic variables	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	Solution	Minimum ratio	\mathfrak{R} (Minimum ratio)
$\mathfrak{R}(\tilde{z})$	-5.25	-4	0	0	-		
\tilde{z}	$(-5, 2, 1)_{LR}$	$(-4, 1, 1)_{LR}$	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$		
\tilde{x}_3	$(4, 2, 1)_{LR}$	$(5, 3, 1)_{LR}$	$(1, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$	$(24, 5, 8)_{LR}$	$\left(6, \frac{53}{4}, 26\right)_{LR}$	9.18
\tilde{x}_4	$(4, 1, 2)_{LR}$	$(1, 5, 1)_{LR}$	$(0, 0, 0)_{LR}$	$(1, 0, 0)_{LR}$	$(12, 6, 3)_{LR}$	$\left(3, \frac{27}{2}, \frac{27}{4}\right)_{LR}$	2.43

Step 2

Since, minimum $\{\mathfrak{R}(-5, 2, 1)_{LR}, \mathfrak{R}(-4, 1, 1)_{LR}\} = \text{minimum}\{-5.25, -4\} = -5.25$, which is corresponding to the fuzzy decision variable \tilde{x}_1 . So, \tilde{x}_1 will enter into the basis. Also,

$$\text{minimum} \left\{ \mathfrak{R} \left(\frac{(24, 5, 8)_{LR}}{(4, 2, 1)_{LR}} \right), \mathfrak{R} \left(\frac{(12, 6, 3)_{LR}}{(4, 1, 2)_{LR}} \right) \right\} = \text{minimum} \left\{ \mathfrak{R} \left(6, \frac{53}{4}, 26 \right)_{LR}, \mathfrak{R} \left(3, \frac{27}{2}, \frac{27}{4} \right)_{LR} \right\}$$

$$= \text{minimum}\{9.18, 2.43\} = 2.43, \text{ which is corresponding to the fuzzy basic variable } \tilde{x}_4. \text{ So, } \tilde{x}_4$$

will leave the basis.

Step 3

Since, for the next simplex table (Table 2.2) the first and second fuzzy basic variables are \tilde{x}_3 and \tilde{x}_1 respectively. So, the entries of the next fuzzy simplex table (Table 2.2) can be obtained as follows:

$$\text{Fuzzy basis matrix } \tilde{B} = \begin{bmatrix} (1, 0, 0)_{LR} & (4, 2, 1)_{LR} \\ (0, 0, 0)_{LR} & (4, 1, 2)_{LR} \end{bmatrix}, \quad |\tilde{B}| = \begin{vmatrix} (1, 0, 0)_{LR} & (4, 2, 1)_{LR} \\ (0, 0, 0)_{LR} & (4, 1, 2)_{LR} \end{vmatrix} = (4, 1, 2)_{LR}$$

$$\text{Cofactor matrix of fuzzy basis matrix } \tilde{B} = \begin{bmatrix} (4, 1, 2)_{LR} & (0, 0, 0)_{LR} \\ (-4, 1, 2)_{LR} & (1, 0, 0)_{LR} \end{bmatrix},$$

$$\text{Adj}(\tilde{B}) = \begin{bmatrix} (4, 1, 2)_{LR} & (0, 0, 0)_{LR} \\ (-4, 1, 2)_{LR} & (1, 0, 0)_{LR} \end{bmatrix}^T = \begin{bmatrix} (4, 1, 2)_{LR} & (-4, 1, 2)_{LR} \\ (0, 0, 0)_{LR} & (1, 0, 0)_{LR} \end{bmatrix}$$

$$\text{So, } \tilde{B}^{-1} = \frac{\text{Adj}(\tilde{B})}{|\tilde{B}|} = \frac{1}{(4,1,2)_{LR}} \begin{bmatrix} (4,1,2)_{LR} & (-4,1,2)_{LR} \\ (0,0,0)_{LR} & (1,0,0)_{LR} \end{bmatrix} = \begin{bmatrix} (1,0,0)_{LR} & \left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_1 :

$$\tilde{B}^{-1} \otimes \tilde{A}_1 = \begin{bmatrix} (1,0,0)_{LR} & \left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR} \end{bmatrix} \otimes \begin{bmatrix} (4,2,1)_{LR} \\ (4,1,2)_{LR} \end{bmatrix} = \begin{bmatrix} (0,13,20)_{LR} \\ \left(1, \frac{17}{4}, \frac{5}{2}\right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_2 :

$$\tilde{B}^{-1} \otimes \tilde{A}_2 = \begin{bmatrix} (1,0,0)_{LR} & \left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR} \end{bmatrix} \otimes \begin{bmatrix} (5,3,1)_{LR} \\ (1,5,1)_{LR} \end{bmatrix} = \begin{bmatrix} \left(4, \frac{25}{4}, 6\right)_{LR} \\ \left(\frac{1}{4}, \frac{9}{8}, \frac{3}{4}\right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_3 :

$$\tilde{B}^{-1} \otimes \tilde{A}_3 = \begin{bmatrix} (1,0,0)_{LR} & \left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR} \end{bmatrix} \otimes \begin{bmatrix} (1,0,0)_{LR} \\ (0,0,0)_{LR} \end{bmatrix} = \begin{bmatrix} (1,0,0)_{LR} \\ (0,0,0)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_4 :

$$\tilde{B}^{-1} \otimes \tilde{A}_4 = \begin{bmatrix} (1,0,0)_{LR} & \left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR} \end{bmatrix} \otimes \begin{bmatrix} (0,0,0)_{LR} \\ (1,0,0)_{LR} \end{bmatrix} = \begin{bmatrix} \left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR} \\ \left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR} \end{bmatrix}$$

Entries of the elements corresponding to 3rd row i.e., corresponding to \tilde{z} :

$$\tilde{c}_B \otimes \tilde{B}^{-1} \otimes \tilde{A}_1 \Theta \tilde{c}_1 = \begin{bmatrix} (0,0,0)_{LR} & (5,1,2)_{LR} \end{bmatrix} \otimes \begin{bmatrix} (0,13,20)_{LR} \\ \left(1, \frac{17}{4}, \frac{5}{2}\right)_{LR} \end{bmatrix} \Theta (5,1,2)_{LR} = \left(0, \frac{97}{4}, \frac{31}{2}\right)_{LR}$$

$$\tilde{c}_B \otimes \tilde{B}^{-1} \otimes \tilde{A}_2 \Theta \tilde{c}_2 = \begin{bmatrix} (0,0,0)_{LR} & (5,1,2)_{LR} \end{bmatrix} \otimes \begin{bmatrix} \left(4, \frac{25}{4}, 6\right)_{LR} \\ \left(\frac{1}{4}, \frac{9}{8}, \frac{3}{4}\right)_{LR} \end{bmatrix} \Theta (4,1,1)_{LR} = \left(-\frac{11}{4}, \frac{55}{8}, \frac{21}{4}\right)_{LR}$$

$$\tilde{c}_B \otimes \tilde{B}^{-1} \otimes \tilde{A}_3 \Theta \tilde{c}_3 = \left[(0,0,0)_{LR} \quad (5,1,2)_{LR} \right] \otimes \begin{bmatrix} (1,0,0)_{LR} \\ (0,0,0)_{LR} \end{bmatrix} \Theta (0,0,0)_{LR} = (0,0,0)_{LR}$$

$$\tilde{c}_B \otimes \tilde{B}^{-1} \otimes \tilde{A}_4 \Theta \tilde{c}_4 = \left[(0,0,0)_{LR} \quad (5,1,2)_{LR} \right] \otimes \begin{bmatrix} \left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR} \\ \left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR} \end{bmatrix} \Theta (0,0,0)_{LR} = \left(\frac{5}{4}, \frac{21}{4}, 3\right)_{LR}$$

$$\tilde{c}_B \otimes \tilde{B}^{-1} \otimes \tilde{b} = \left[(0,0,0)_{LR} \quad (5,1,2)_{LR} \right] \otimes \begin{bmatrix} \left(3, \frac{27}{2}, \frac{27}{4}\right)_{LR} \\ (12, 35, 68)_{LR} \end{bmatrix} = \left(15, \frac{141}{2}, \frac{159}{4}\right)_{LR}$$

Entries of the elements corresponding to 6th column i.e., corresponding to solution column:

$$\tilde{B}^{-1} \otimes \tilde{b} = \begin{bmatrix} (1,0,0)_{LR} & \left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR} \end{bmatrix} \otimes \begin{bmatrix} (24, 5, 8)_{LR} \\ (12, 6, 3)_{LR} \end{bmatrix} = \begin{bmatrix} \left(3, \frac{27}{2}, \frac{27}{4}\right)_{LR} \\ (12, 35, 68)_{LR} \end{bmatrix}$$

Table 2.2 The second fuzzy simplex table

Basic variables	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	Solution
$\mathfrak{R}(\tilde{z})$	-2.18	-3.15	0	2.75	-
\tilde{z}	$\left(0, \frac{97}{4}, \frac{31}{2}\right)_{LR}$	$\left(-\frac{11}{4}, \frac{55}{8}, \frac{21}{4}\right)_{LR}$	$(0,0,0)_{LR}$	$\left(\frac{5}{4}, \frac{21}{4}, 3\right)_{LR}$	$\left(15, \frac{141}{2}, \frac{159}{4}\right)_{LR}$
\tilde{x}_3	$(0,13,20)_{LR}$	$\left(4, \frac{25}{4}, 6\right)_{LR}$	$(1,0,0)_{LR}$	$\left(-1, \frac{9}{4}, \frac{9}{2}\right)_{LR}$	$\left(3, \frac{27}{2}, \frac{27}{4}\right)_{LR}$
\tilde{x}_1	$\left(1, \frac{17}{4}, \frac{5}{2}\right)_{LR}$	$\left(\frac{1}{4}, \frac{9}{8}, \frac{3}{4}\right)_{LR}$	$(0,0,0)_{LR}$	$\left(\frac{1}{4}, 1, \frac{1}{2}\right)_{LR}$	$(12, 35, 68)_{LR}$

Since, \tilde{x}_1 is the second fuzzy basic variable so $\mathfrak{R}\left(1, \frac{17}{4}, \frac{5}{2}\right)_{LR}$ should be 1 and $\mathfrak{R}(0,13,20)_{LR}$

as well as $\mathfrak{R}\left(0, \frac{97}{4}, \frac{31}{2}\right)_{LR}$ should be 0. However, $\mathfrak{R}\left(1, \frac{17}{4}, \frac{5}{2}\right)_{LR} = .56 \neq 1$ and

$\mathfrak{R}(0,13,20)_{LR} = 1.75 \neq 0$, $\mathfrak{R}\left(0, \frac{97}{4}, \frac{31}{2}\right)_{LR} = -2.18 \neq 0$. So, it is not genuine to use the

existing method [47] for solving fully fuzzy linear programming problems.

Remark 2.5 In the existing method [47] the following method is used for dividing two fuzzy numbers:

Let $\tilde{A}_1 = (m_1, \alpha_1, \beta_1)_{LR}$ be an *LR* fuzzy number and $\tilde{A}_2 = (m_2, \alpha_2, \beta_2)_{RL}$ be a *RL* fuzzy number. Then, $\frac{\tilde{A}_1}{\tilde{A}_2}$ can be calculated as follows:

(i) If $m_1 - \alpha_1 > 0$ and $m_2 - \alpha_2 > 0$. Then,

$$\frac{\tilde{A}_1}{\tilde{A}_2} = \left(\frac{m_1}{m_2}, \frac{m_1 + \alpha_1}{\alpha_2 + m_2}, \frac{m_1 + \beta_1}{\beta_2 + m_2} \right)_{LR}.$$

(ii) If $n_1 + \beta_1 < 0$ and $m_2 - \alpha_2 > 0$. Then,

$$\frac{\tilde{A}_1}{\tilde{A}_2} = \left(\frac{-m_1}{m_2}, \frac{\beta_1 - m_1}{\alpha_2 + m_2}, \frac{\alpha_1 - m_1}{\beta_2 + m_2} \right)_{LR}.$$

(iii) If $n_1 + \beta_1 < 0$ and $n_2 + \beta_2 < 0$. Then,

$$\frac{\tilde{A}_1}{\tilde{A}_2} = \left(\frac{m_1}{m_2}, \frac{-m_1 - \beta_1}{\beta_2 + m_2}, \frac{-m_1 - \alpha_1}{\alpha_2 + m_2} \right)_{LR}.$$

2.3 Origin of shortcomings

Let x_B and B be the basic feasible solution and basis matrix for the crisp linear programming problem:

$$\text{Maximize (or Minimize)} \quad C^T X$$

Subject to

$$AX \leq (\text{or } = \text{ or } \geq) b$$

$$X \geq 0$$

where, $C^T = [c_j]_{1 \times n}$, $X = [x_j]_{n \times 1}$, $A = [a_{ij}]_{m \times n}$, $b = [b_i]_{m \times 1}$.

Then, $x_B = B^{-1}b \Rightarrow Bx_B = b$.

$$\text{e.g., Let } x_B = \begin{bmatrix} 66 \\ 13 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -4 & 6 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 20 \\ 11 \\ 23 \end{bmatrix}.$$

$$\text{Then, } Bx_B = \begin{bmatrix} 1 & -4 & 6 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 66 \\ 13 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 11 \\ 23 \end{bmatrix} \text{ and } B^{-1}b = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 11 \\ 23 \end{bmatrix} = \begin{bmatrix} 66 \\ 13 \\ 1 \end{bmatrix} \text{ i.e., } Bx_B = b \text{ and}$$

$$B^{-1}b = x_B.$$

In the same direction, in the existing method [47] it is assumed that if \tilde{x}_B and \tilde{B} be the fuzzy basic feasible solution and the fuzzy basis matrix for the fully fuzzy linear programming problem.

$$\text{Maximize (or Minimize)} \quad \tilde{C}^T \otimes \tilde{X}$$

Subject to

$$\tilde{A} \otimes \tilde{X} \leq_{\mathfrak{R}}, =_{\mathfrak{R}}, \geq_{\mathfrak{R}} \tilde{b}$$

where, $\tilde{C}^T = [\tilde{c}_j]_{1 \times n}$, $\tilde{X} = [\tilde{x}_j]_{n \times 1}$, $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, $\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}$.

$$\text{Then, } \tilde{x}_B =_{\mathfrak{R}} \tilde{B}^{-1} \otimes \tilde{b} \Rightarrow \tilde{B} \otimes \tilde{x}_B =_{\mathfrak{R}} \tilde{b}.$$

However, $\tilde{x}_B =_{\mathfrak{R}} \tilde{B}^{-1} \otimes \tilde{b}$ does not implies $\tilde{B} \otimes \tilde{x}_B =_{\mathfrak{R}} \tilde{b}$.

e.g., Since, for the fuzzy simplex table (Table 2.2) of fully fuzzy linear programming

problem, chosen in Example 2.1, $\tilde{x}_B = \left[\left(3, \frac{27}{2}, \frac{27}{4} \right)_{LR}, (12, 35, 68)_{LR} \right]$, $\tilde{B} = \begin{bmatrix} (1, 0, 0)_{LR} & (4, 2, 1)_{LR} \\ (0, 0, 0)_{LR} & (4, 1, 2)_{LR} \end{bmatrix}$ and

$$\tilde{b} = \begin{bmatrix} (24, 5, 8)_{LR} \\ (12, 6, 3)_{LR} \end{bmatrix}, \tilde{B}^{-1} = \begin{bmatrix} (0, 0, 0)_{LR} & \left(\frac{1}{4}, 1, \frac{1}{2} \right)_{LR} \\ (1, 0, 0)_{LR} & \left(-1, \frac{9}{4}, \frac{9}{2} \right)_{LR} \end{bmatrix}, \mathfrak{R}(\tilde{x}_B) = \begin{bmatrix} \mathfrak{R} \left(3, \frac{27}{2}, \frac{27}{4} \right)_{LR} \\ \mathfrak{R}(12, 35, 68)_{LR} \end{bmatrix} = \begin{bmatrix} 1.31 \\ 20.25 \end{bmatrix} \text{ and}$$

$$\mathfrak{R}(\tilde{b}) = \begin{bmatrix} \mathfrak{R}(24, 5, 8)_{LR} \\ \mathfrak{R}(12, 6, 3)_{LR} \end{bmatrix} = \begin{bmatrix} 24.75 \\ 11.25 \end{bmatrix}$$

$$\text{So, } \tilde{B} \otimes \tilde{x}_B = \begin{bmatrix} (1, 0, 0)_{LR} & (4, 2, 1)_{LR} \\ (0, 0, 0)_{LR} & (4, 1, 2)_{LR} \end{bmatrix} \otimes \begin{bmatrix} \left(3, \frac{27}{2}, \frac{27}{4} \right)_{LR} \\ (12, 35, 68)_{LR} \end{bmatrix} = \begin{bmatrix} \left(51, \frac{355}{2}, \frac{1163}{4} \right)_{LR} \\ (48, 152, 320)_{LR} \end{bmatrix}$$

$$\text{and } \tilde{B}^{-1} \otimes \tilde{b} = \begin{bmatrix} (0, 0, 0)_{LR} & \left(\frac{1}{4}, 1, \frac{1}{2} \right)_{LR} \\ (1, 0, 0)_{LR} & \left(-1, \frac{9}{4}, \frac{9}{2} \right)_{LR} \end{bmatrix} \otimes \begin{bmatrix} (24, 5, 8)_{LR} \\ (12, 6, 3)_{LR} \end{bmatrix} = \begin{bmatrix} \left(3, \frac{27}{2}, \frac{27}{4} \right)_{LR} \\ (12, 26, 59)_{LR} \end{bmatrix}$$

$$\text{Now, } \mathfrak{R}(\tilde{B}^{-1} \otimes \tilde{b}) = \begin{bmatrix} \mathfrak{R}\left(3, \frac{27}{2}, \frac{27}{4}\right)_{LR} \\ \mathfrak{R}(12, 26, 59)_{LR} \end{bmatrix} = \begin{bmatrix} 1.31 \\ 20.25 \end{bmatrix} = \mathfrak{R}(\tilde{x}_B)$$

$$\text{But, } \mathfrak{R}(\tilde{B} \otimes \tilde{x}_B) = \begin{bmatrix} \mathfrak{R}\left(51, \frac{355}{2}, \frac{1163}{4}\right)_{LR} \\ \mathfrak{R}(48, 152, 320)_{LR} \end{bmatrix} = \begin{bmatrix} 79.68 \\ 90 \end{bmatrix} \neq \mathfrak{R}(\tilde{b})$$

i.e., $\tilde{B}^{-1} \otimes \tilde{b} =_{\mathfrak{R}} \tilde{x}_B$ does not implies $\tilde{B} \otimes \tilde{x}_B =_{\mathfrak{R}} \tilde{b}$.

2.4 Proposed Mehar multiplication and division

All the shortcomings in the existing method [47], discussed in Section 2.3, are occurring due to the assumption that $\mathfrak{R}(\tilde{A} \otimes \tilde{B}) = \mathfrak{R}(\tilde{A})\mathfrak{R}(\tilde{B})$ which is not valid in general.

In this section, such a multiplication and division for *LR* flat fuzzy numbers, named as Mehar multiplication and Mehar division, by modifying the existing multiplication and division [31] is defined so that the property $\mathfrak{R}(\tilde{A} \otimes \tilde{B}) = \mathfrak{R}(\tilde{A})\mathfrak{R}(\tilde{B})$ will always be satisfied.

If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two *LR* flat fuzzy numbers then,

$$\tilde{A}_1 \otimes_M \tilde{A}_2 = \begin{cases} \left(m_1 \mathfrak{R}(\tilde{A}_2), n_1 \mathfrak{R}(\tilde{A}_2), \alpha_1 \mathfrak{R}(\tilde{A}_2), \beta_1 \mathfrak{R}(\tilde{A}_2) \right)_{LR}, & \mathfrak{R}(\tilde{A}_2) \geq 0 \\ \left(n_1 \mathfrak{R}(\tilde{A}_2), m_1 \mathfrak{R}(\tilde{A}_2), -\beta_1 \mathfrak{R}(\tilde{A}_2), -\alpha_1 \mathfrak{R}(\tilde{A}_2) \right)_{RL}, & \mathfrak{R}(\tilde{A}_2) \leq 0 \end{cases}$$

$$\tilde{A}_1 /_M \tilde{A}_2 = \begin{cases} \left(\frac{m_1}{\mathfrak{R}(\tilde{A}_2)}, \frac{n_1}{\mathfrak{R}(\tilde{A}_2)}, \frac{\alpha_1}{\mathfrak{R}(\tilde{A}_2)}, \frac{\beta_1}{\mathfrak{R}(\tilde{A}_2)} \right)_{LR}, & \mathfrak{R}(\tilde{A}_2) > 0 \\ \left(\frac{n_1}{\mathfrak{R}(\tilde{A}_2)}, \frac{m_1}{\mathfrak{R}(\tilde{A}_2)}, \frac{-\beta_1}{\mathfrak{R}(\tilde{A}_2)}, \frac{-\alpha_1}{\mathfrak{R}(\tilde{A}_2)} \right)_{RL}, & \mathfrak{R}(\tilde{A}_2) < 0 \end{cases}$$

where, $\mathfrak{R}(\tilde{A}_2) = \frac{1}{2} \int_0^1 (m_2 - \alpha_2 L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_2 + \beta_2 R^{-1}(\lambda)) d\lambda$.

Remark 2.6 Proposed Mehar multiplication and division is basically the product and division of a fuzzy number with the Yager's rank of other. (The details about Mehar are mentioned in the second paragraph of acknowledgement).

Remark 2.7 Let the i^{th} fuzzy constraint of a fully fuzzy linear programming problem be

$\sum_{j=1}^n \tilde{a}_{ij} \otimes_M \tilde{x}_j \leq_{\mathfrak{R}} \tilde{b}_i$ where $\tilde{b}_i \geq_{\mathfrak{R}} \tilde{0}$. Then, a fuzzy number \tilde{s}_i , $\tilde{s}_i \geq_{\mathfrak{R}} \tilde{0}$ such that

$\sum_{j=1}^n \tilde{a}_{ij} \otimes_M \tilde{x}_j \oplus \tilde{s}_i =_{\mathfrak{R}} \tilde{b}_i$ is called a fuzzy slack variable.

Remark 2.8 Let the i^{th} fuzzy constraint of a fully fuzzy linear programming problem be

$\sum_{j=1}^n \tilde{a}_{ij} \otimes_M \tilde{x}_j \geq_{\mathfrak{R}} \tilde{b}_i$ where $\tilde{b}_i \geq_{\mathfrak{R}} \tilde{0}$. Then, a fuzzy number \tilde{s}_i , $\tilde{s}_i \geq_{\mathfrak{R}} \tilde{0}$ such that

$\sum_{j=1}^n \tilde{a}_{ij} \otimes_M \tilde{x}_j \ominus \tilde{s}_i =_{\mathfrak{R}} \tilde{b}_i$ is called a fuzzy surplus variable.

2.5 Proposed modifications

If in the existing method [47], presented in Section 2.2, Mehar multiplication and division, defined in Section 2.4, is used instead of existing multiplication and division [33] then all the shortcomings, pointed out in Section 2.2 and Section 2.3, will be resolved.

To illustrate the existing method [47] with proposed modifications, the existing fully fuzzy linear programming problem, presented in Example 2.1, is solved. Also, a fully fuzzy linear programming problem in which all the parameters and decision variables are represented by *LR* flat fuzzy numbers is solved by using the existing method [47] with proposed modifications.

2.5.1 Solution of existing fully fuzzy linear programming problem

On using the proposed Mehar multiplication instead of existing multiplication the fully fuzzy linear programming problem, chosen in Example 2.1, can be written as $(E_{2.3})$ and can be solved as follows:

$$\text{Maximize} \quad [(5,1,2)_{LR} \otimes_M \tilde{x}_1 \oplus (4,1,1)_{LR} \otimes_M \tilde{x}_2]$$

$$\text{Subject to} \quad (E_{2.3})$$

$$(4,2,1)_{LR} \otimes_M \tilde{x}_1 \oplus (5,3,1)_{LR} \otimes_M \tilde{x}_2 \leq_{\mathfrak{R}} (24,5,8)_{LR}$$

$$(4,1,2)_{LR} \otimes_M \tilde{x}_1 \oplus (1,.5,1)_{LR} \otimes_M \tilde{x}_2 \leq_{\mathfrak{R}} (12,6,3)_{LR}$$

$$\tilde{x}_1 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_2 \geq_{\mathfrak{R}} \tilde{0}$$

where, $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$.

Step 1

Adding fuzzy slack variables \tilde{x}_3 and \tilde{x}_4 in first and second constraint respectively, the fully fuzzy linear programming problem ($E_{2.3}$) can be written as:

$$\text{Maximize} \quad [(5,1,2)_{LR} \otimes_M \tilde{x}_1 \oplus (4,1,1)_{LR} \otimes_M \tilde{x}_2 \oplus \tilde{0} \otimes_M \tilde{x}_3 \oplus \tilde{0} \otimes_M \tilde{x}_4]$$

$$\text{Subject to} \quad (E_{2.4})$$

$$(4,2,1)_{LR} \otimes_M \tilde{x}_1 \oplus (5,3,1)_{LR} \otimes_M \tilde{x}_2 \oplus \tilde{x}_3 =_{\mathfrak{R}} (24,5,8)_{LR}$$

$$(4,1,2)_{LR} \otimes_M \tilde{x}_1 \oplus (1,.5,1)_{LR} \otimes_M \tilde{x}_2 \oplus \tilde{x}_4 =_{\mathfrak{R}} (12,6,3)_{LR}$$

$$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \geq_{\mathfrak{R}} \tilde{0}$$

where, \tilde{x}_3, \tilde{x}_4 are fuzzy slack variables.

The initial fuzzy simplex table for the fully fuzzy linear programming problem ($E_{2.4}$) is:

Table 2.3 The initial fuzzy simplex table

Basic variables	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	Solution	Minimum ratio	\Re (Minimum ratio)
$\Re(\tilde{z})$	-5.25	-4	0	0	-		
\tilde{z}	$(-5, 2, 1)_{LR}$	$(-4, 1, 1)_{LR}$	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$		
\tilde{x}_3	$(4, 2, 1)_{LR}$	$(5, 3, 1)_{LR}$	$(1, 0, 0)_{LR}$	$(0, 0, 0)_{LR}$	$(24, 5, 8)_{LR}$	$\left(\frac{96}{15}, \frac{20}{15}, \frac{32}{15}\right)_{LR}$	26.4
\tilde{x}_4	$(4, 1, 2)_{LR}$	$(1, .5, 1)_{LR}$	$(0, 0, 0)_{LR}$	$(1, 0, 0)_{LR}$	$(12, 6, 3)_{LR}$	$\left(\frac{48}{17}, \frac{24}{17}, \frac{12}{17}\right)_{LR}$	10.58

Step 2

Since, $\text{minimum} \{ \mathfrak{R}(-5, 2, 1)_{LR}, \mathfrak{R}(-4, 1, 1)_{LR} \} = \text{minimum} \{-5.25, -4\} = -5.25$, which is corresponding to the fuzzy decision variable \tilde{x}_1 . So, \tilde{x}_1 will enter into the basis. Also,

$$\text{minimum} \left\{ \mathfrak{R} \left(\frac{(24, 5, 8)_{LR}}{(4, 2, 1)_{LR}} \right), \mathfrak{R} \left(\frac{(12, 6, 3)_{LR}}{(4, 1, 2)_{LR}} \right) \right\} = \text{minimum} \left\{ \mathfrak{R} \left(\frac{96}{15}, \frac{20}{15}, \frac{32}{15} \right)_{LR}, \mathfrak{R} \left(\frac{48}{17}, \frac{24}{17}, \frac{12}{17} \right)_{LR} \right\}$$

$\text{minimum} \{26.4, 10.54\} = 10.54$, which is corresponding to the fuzzy basic variable \tilde{x}_4 . So,

\tilde{x}_4 will leave the basis.

Step 3

Since, for the next simplex table (Table 2.4) the first and second fuzzy basic variables are \tilde{x}_3 and \tilde{x}_1 respectively. So, the entries of the next fuzzy simplex table (Table 2.4) can be obtained as follows:

$$\text{Fuzzy basis matrix } \tilde{B} = \begin{bmatrix} (1, 0, 0)_{LR} & (4, 2, 1)_{LR} \\ (0, 0, 0)_{LR} & (4, 1, 2)_{LR} \end{bmatrix}, \quad |\tilde{B}| = \begin{vmatrix} (1, 0, 0)_{LR} & (4, 2, 1)_{LR} \\ (0, 0, 0)_{LR} & (4, 1, 2)_{LR} \end{vmatrix} = (4, 1, 2)_{LR}$$

$$\text{Cofactor matrix of fuzzy basis matrix } \tilde{B} = \begin{bmatrix} (4, 1, 2)_{LR} & (0, 0, 0)_{LR} \\ (-4, 1, 2)_{LR} & (1, 0, 0)_{LR} \end{bmatrix},$$

$$\text{Adj}(\tilde{B}) = \begin{bmatrix} (4, 1, 2)_{LR} & (0, 0, 0)_{LR} \\ (-4, 1, 2)_{LR} & (1, 0, 0)_{LR} \end{bmatrix}^T = \begin{bmatrix} (4, 1, 2)_{LR} & (-4, 1, 2)_{LR} \\ (0, 0, 0)_{LR} & (1, 0, 0)_{LR} \end{bmatrix}$$

$$\text{So, } \tilde{B}^{-1} = \frac{\text{Adj}(\tilde{B})}{|\tilde{B}|} = \frac{1}{(4, 1, 2)_{LR}} \begin{bmatrix} (4, 1, 2)_{LR} & (-4, 1, 2)_{LR} \\ (0, 0, 0)_{LR} & (1, 0, 0)_{LR} \end{bmatrix} = \begin{bmatrix} (1, 0, 0)_{LR} & \left(-\frac{15}{17}, 0, 0 \right)_{LR} \\ (0, 0, 0)_{LR} & \left(\frac{4}{17}, 0, 0 \right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_1 :

$$\tilde{B}^{-1} \otimes \tilde{A}_1 = \begin{bmatrix} (1, 0, 0)_{LR} & \left(-\frac{15}{17}, 0, 0 \right)_{LR} \\ (0, 0, 0)_{LR} & \left(\frac{4}{17}, 0, 0 \right)_{LR} \end{bmatrix} \otimes \begin{bmatrix} (4, 2, 1)_{LR} \\ (4, 1, 2)_{LR} \end{bmatrix} = \begin{bmatrix} \left(\frac{8}{17}, \frac{64}{17}, \frac{32}{17} \right)_{LR} \\ \left(\frac{16}{17}, \frac{4}{17}, \frac{8}{17} \right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_2 :

$$\tilde{B}^{-1} \otimes_M \tilde{A}_2 = \begin{bmatrix} (1,0,0)_{LR} & \left(-\frac{15}{17}, 0, 0\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{4}{17}, 0, 0\right)_{LR} \end{bmatrix} \otimes_M \begin{bmatrix} (5,3,1)_{LR} \\ (1,5,1)_{LR} \end{bmatrix} = \begin{bmatrix} \left(\frac{70}{17}, \frac{66}{17}, \frac{49}{34}\right)_{LR} \\ \left(\frac{4}{17}, \frac{2}{17}, \frac{4}{17}\right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_3 :

$$\tilde{B}^{-1} \otimes_M \tilde{A}_3 = \begin{bmatrix} (1,0,0)_{LR} & \left(-\frac{15}{17}, 0, 0\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{4}{17}, 0, 0\right)_{LR} \end{bmatrix} \otimes_M \begin{bmatrix} (1,0,0)_{LR} \\ (0,0,0)_{LR} \end{bmatrix} = \begin{bmatrix} (1,0,0)_{LR} \\ (0,0,0)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_4 :

$$\tilde{B}^{-1} \otimes_M \tilde{A}_4 = \begin{bmatrix} (1,0,0)_{LR} & \left(-\frac{15}{17}, 0, 0\right)_{LR} \\ (0,0,0)_{LR} & \left(\frac{4}{17}, 0, 0\right)_{LR} \end{bmatrix} \otimes_M \begin{bmatrix} (0,0,0)_{LR} \\ (1,0,0)_{LR} \end{bmatrix} = \begin{bmatrix} \left(-\frac{15}{17}, 0, 0\right)_{LR} \\ \left(\frac{4}{17}, 0, 0\right)_{LR} \end{bmatrix}$$

Entries of the elements corresponding to 3rd row i.e., corresponding to \tilde{z} :

$$\tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{A}_1 \Theta \tilde{c}_1 = \left[(0,0,0)_{LR} \ (5,1,2)_{LR} \right] \otimes_M \begin{bmatrix} \left(\frac{8}{17}, \frac{64}{17}, \frac{32}{17}\right)_{LR} \\ \left(\frac{16}{17}, \frac{4}{17}, \frac{8}{17}\right)_{LR} \end{bmatrix} \Theta (5,1,2)_{LR} =$$

$$\left(-\frac{1}{17}, \frac{55}{17}, \frac{59}{17} \right)_{LR}$$

$$\tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{A}_2 \Theta \tilde{c}_2 = \left[(0,0,0)_{LR} \ (5,1,2)_{LR} \right] \otimes_M \begin{bmatrix} \left(\frac{70}{17}, \frac{66}{17}, \frac{49}{34}\right)_{LR} \\ \left(\frac{4}{17}, \frac{2}{17}, \frac{4}{17}\right)_{LR} \end{bmatrix} \Theta (4,1,1)_{LR} =$$

$$\left(-\frac{47}{17}, \frac{55}{34}, \frac{38}{17} \right)_{LR}$$

$$\tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{A}_3 \Theta \tilde{c}_3 = \left[(0,0,0)_{LR} \ (5,1,2)_{LR} \right] \otimes_M \begin{bmatrix} (1,0,0)_{LR} \\ (0,0,0)_{LR} \end{bmatrix} \Theta (0,0,0)_{LR} = (0,0,0)_{LR}$$

$$\tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{A}_4 \Theta \tilde{c}_4 = \left[(0,0,0)_{LR} \quad (5,1,2)_{LR} \right] \otimes_M \begin{bmatrix} \left(-\frac{15}{17}, 0, 0 \right)_{LR} \\ \left(\frac{4}{17}, 0, 0 \right)_{LR} \end{bmatrix} \Theta (0,0,0)_{LR} = \left(\frac{21}{17}, 0, 0 \right)_{LR}$$

$$\tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{b} = \left[(0,0,0)_{LR} \quad (5,1,2)_{LR} \right] \otimes_M \begin{bmatrix} \left(\frac{228}{17}, \frac{130}{17}, \frac{226}{17} \right)_{LR} \\ \left(\frac{48}{17}, \frac{24}{17}, \frac{12}{17} \right)_{LR} \end{bmatrix} = \left(\frac{252}{17}, \frac{126}{17}, \frac{63}{17} \right)_{LR}$$

Entries of the elements corresponding to 6th column i.e., corresponding to solution column:

$$\tilde{B}^{-1} \otimes_M \tilde{b} = \begin{bmatrix} (1,0,0)_{LR} & \left(-\frac{15}{17}, 0, 0 \right)_{LR} \\ 0,0,0)_{LR} & \left(\frac{4}{17}, 0, 0 \right)_{LR} \end{bmatrix} \otimes_M \begin{bmatrix} (24,5,8)_{LR} \\ (12,6,3)_{LR} \end{bmatrix} = \begin{bmatrix} \left(\frac{228}{17}, \frac{130}{17}, \frac{226}{17} \right)_{LR} \\ \left(\frac{48}{17}, \frac{24}{17}, \frac{12}{17} \right)_{LR} \end{bmatrix}$$

Table 2.4 The second fuzzy simplex table

Basic variables	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	Solution	Minimum ratio	\mathfrak{R} (Minimum ratio)
$\mathfrak{R}(\tilde{z})$	0	-10.44	0	1.23	-		
\tilde{z}	$\left(-\frac{1}{17}, \frac{55}{17}, \frac{59}{17}\right)_{LR}$	$\left(-\frac{47}{17}, \frac{55}{34}, \frac{38}{17}\right)_{LR}$	$(0, 0, 0)_{LR}$	$\left(\frac{21}{17}, 0, 0\right)_{LR}$	$\left(\frac{252}{17}, \frac{126}{17}, \frac{63}{17}\right)_{LR}$		
\tilde{x}_3	$\left(\frac{8}{17}, \frac{64}{17}, \frac{32}{17}\right)_{LR}$	$\left(\frac{70}{17}, \frac{66}{17}, \frac{49}{17}\right)_{LR}$	$(1, 0, 0)_{LR}$	$\left(-\frac{15}{17}, 0, 0\right)_{LR}$	$\left(\frac{228}{17}, \frac{136}{17}, \frac{226}{17}\right)_{LR}$	$\left(\frac{1824}{477}, \frac{1040}{477}, \frac{1808}{477}\right)_{LR}$	4.22
\tilde{x}_1	$\left(\frac{16}{17}, \frac{4}{17}, \frac{8}{17}\right)_{LR}$	$\left(\frac{4}{17}, \frac{2}{17}, \frac{4}{17}\right)_{LR}$	$(0, 0, 0)_{LR}$	$\left(\frac{4}{17}, 0, 0\right)_{LR}$	$\left(\frac{48}{17}, \frac{24}{17}, \frac{12}{17}\right)_{LR}$	$\left(\frac{96}{17}, \frac{48}{17}, \frac{24}{17}\right)_{LR}$	5.29

Step 4

Since $\Re\left(-\frac{47}{17}, \frac{55}{34}, \frac{38}{17}\right)_{LR} = -10.44$, which is corresponding to the fuzzy decision variable

\tilde{x}_2 . So, \tilde{x}_2 will enter into the basis. Also,

$$\text{minimum} \left\{ \Re \left(\frac{\left(\frac{228}{17}, \frac{136}{17}, \frac{226}{17} \right)_{LR}}{\left(\frac{70}{17}, \frac{66}{17}, \frac{49}{17} \right)_{LR}} \right), \Re \left(\frac{\left(\frac{48}{17}, \frac{24}{17}, \frac{12}{17} \right)_{LR}}{\left(\frac{4}{17}, \frac{2}{17}, \frac{4}{17} \right)_{LR}} \right) \right\} = \text{minimum} \left\{ \Re \left(\frac{1824}{477}, \frac{1040}{477}, \frac{1808}{477} \right)_{LR}, \Re \left(\frac{96}{17}, \frac{48}{17}, \frac{24}{17} \right)_{LR} \right\} = \text{minimum} \{4.22, 5.29\} = 4.22, \text{ which is corresponding to the fuzzy basic}$$

variable \tilde{x}_3 . So, \tilde{x}_3 will leave the basis.

Step 5

Since, for the next simplex table (Table 2.5) the first and second fuzzy basic variables \tilde{x}_2 and \tilde{x}_1 respectively. So, the entries of the next fuzzy simplex table (Table 2.5) can be obtained as

follows:

$$\text{Fuzzy basic matrix } \tilde{B} = \begin{bmatrix} (5,3,1)_{LR} & (4,2,1)_{LR} \\ (1,.5,1)_{LR} & (4,1,2)_{LR} \end{bmatrix}, |\tilde{B}| = \begin{vmatrix} (5,3,1)_{LR} & (4,2,1)_{LR} \\ (1,.5,1)_{LR} & (4,1,2)_{LR} \end{vmatrix} = \left(\frac{670}{40}, \frac{555}{40}, \frac{179}{40} \right)_{LR}$$

$$\text{Cofactor matrix of fuzzy basic matrix } \tilde{B} = \begin{bmatrix} (4,1,2)_{LR} & (-1,1,.5)_{LR} \\ (-4,1,2)_{LR} & (5,3,1)_{LR} \end{bmatrix},$$

$$\text{Adj}(\tilde{B}) = \begin{bmatrix} (4,1,2)_{LR} & (-1,1,.5)_{LR} \\ (-4,1,2)_{LR} & (5,3,1)_{LR} \end{bmatrix}^T = \begin{bmatrix} (4,1,2)_{LR} & (-4,1,2)_{LR} \\ (-1,1,.5)_{LR} & (5,3,1)_{LR} \end{bmatrix}$$

So,

$$\tilde{B}^{-1} = \frac{\text{Adj}(\tilde{B})}{|\tilde{B}|} = \frac{1}{\left(\frac{670}{40}, \frac{555}{40}, \frac{179}{40} \right)_{LR}} \begin{bmatrix} (4,1,2)_{LR} & (-4,1,2)_{LR} \\ (-1,1,.5)_{LR} & (5,3,1)_{LR} \end{bmatrix} = \begin{bmatrix} \left(\frac{136}{477}, 0, 0 \right)_{LR} & \left(-\frac{120}{477}, 0, 0 \right)_{LR} \\ \left(-\frac{1224}{16218}, 0, 0 \right)_{LR} & \left(\frac{1080}{16218}, 0, 0 \right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_1 :

$$\tilde{B}^{-1} \otimes_M \tilde{A}_1 = \begin{bmatrix} \left(\frac{136}{477}, 0, 0\right)_{LR} & \left(-\frac{120}{477}, 0, 0\right)_{LR} \\ \left(-\frac{1224}{16218}, 0, 0\right)_{LR} & \left(\frac{1080}{16218}, 0, 0\right)_{LR} \end{bmatrix} \otimes_M \begin{bmatrix} (4, 2, 1)_{LR} \\ (4, 1, 2)_{LR} \end{bmatrix} = \begin{bmatrix} \left(\frac{64}{477}, \frac{512}{477}, \frac{256}{477}\right)_{LR} \\ \left(\frac{14688}{16218}, \frac{15660}{16218}, \frac{12240}{16218}\right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_2 :

$$\tilde{B}^{-1} \otimes_M \tilde{A}_2 = \begin{bmatrix} \left(\frac{136}{477}, 0, 0\right)_{LR} & \left(-\frac{120}{477}, 0, 0\right)_{LR} \\ \left(-\frac{1224}{16218}, 0, 0\right)_{LR} & \left(\frac{1080}{16218}, 0, 0\right)_{LR} \end{bmatrix} \otimes_M \begin{bmatrix} (5, 3, 1)_{LR} \\ (1, .5, 1)_{LR} \end{bmatrix} = \begin{bmatrix} \left(\frac{560}{477}, \frac{528}{477}, \frac{196}{477}\right)_{LR} \\ \left(-\frac{1224}{16218}, \frac{3672}{16218}, \frac{8568}{16218}\right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_3 :

$$\tilde{B}^{-1} \otimes_M \tilde{A}_3 = \begin{bmatrix} \left(\frac{136}{477}, 0, 0\right)_{LR} & \left(-\frac{120}{477}, 0, 0\right)_{LR} \\ \left(-\frac{1224}{16218}, 0, 0\right)_{LR} & \left(\frac{1080}{16218}, 0, 0\right)_{LR} \end{bmatrix} \otimes_M \begin{bmatrix} (1, 0, 0)_{LR} \\ (0, 0, 0)_{LR} \end{bmatrix} = \begin{bmatrix} \left(\frac{136}{477}, 0, 0\right)_{LR} \\ \left(-\frac{1224}{16218}, 0, 0\right)_{LR} \end{bmatrix}$$

Entries of the column corresponding to fuzzy decision variable \tilde{x}_4 :

$$\tilde{B}^{-1} \otimes_M \tilde{A}_4 = \begin{bmatrix} \left(\frac{136}{477}, 0, 0\right)_{LR} & \left(-\frac{120}{477}, 0, 0\right)_{LR} \\ \left(-\frac{1224}{16218}, 0, 0\right)_{LR} & \left(\frac{1080}{16218}, 0, 0\right)_{LR} \end{bmatrix} \otimes_M \begin{bmatrix} (0, 0, 0)_{LR} \\ (1, 0, 0)_{LR} \end{bmatrix} = \begin{bmatrix} \left(-\frac{120}{477}, 0, 0\right)_{LR} \\ \left(\frac{1080}{16218}, 0, 0\right)_{LR} \end{bmatrix}$$

Entries of the elements corresponding to 3rd row i.e., corresponding to \tilde{z} :

$$\begin{aligned} \tilde{c}_B \otimes \tilde{B}^{-1} \otimes \tilde{A}_1 \Theta \tilde{c}_1 &= \left[(4, 1, 1)_{LR} \quad (5, 1, 2)_{LR} \right] \otimes \begin{bmatrix} \left(\frac{64}{477}, \frac{512}{477}, \frac{256}{477}\right)_{LR} \\ \left(\frac{14688}{16218}, \frac{15660}{16218}, \frac{12240}{16218}\right)_{LR} \end{bmatrix} \Theta (5, 1, 2)_{LR} \\ &= \left(-\frac{4726}{16218}, \frac{184283}{16218}, \frac{115294}{16218} \right)_{LR} \end{aligned}$$

$$\begin{aligned} \tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{A}_2 \Theta \tilde{c}_2 &= \left[(4,1,1)_{LR} \ (5,1,2)_{LR} \right] \otimes_M \left[\begin{array}{c} \left(\frac{560}{477}, \frac{528}{477}, \frac{196}{477} \right)_{LR} \\ \left(-\frac{1224}{16218}, \frac{3672}{16218}, \frac{8568}{16218} \right)_{LR} \end{array} \right] \Theta (4,1,1)_{LR} \\ &= \left(\frac{4862}{16218}, \frac{37602}{16218}, \frac{87856}{16218} \right)_{LR} \end{aligned}$$

$$\begin{aligned} \tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{A}_3 \Theta \tilde{c}_3 &= \left[(4,1,1)_{LR} \ (5,1,2)_{LR} \right] \otimes_M \left[\begin{array}{c} \left(\frac{136}{477}, 0, 0 \right)_{LR} \\ \left(-\frac{1224}{16218}, 0, 0 \right)_{LR} \end{array} \right] \Theta (0,0,0)_{LR} = \\ &\left(\frac{12070}{16218}, 0, 0 \right)_{LR} \end{aligned}$$

$$\begin{aligned} \tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{A}_4 \Theta \tilde{c}_4 &= \left[(4,1,1)_{LR} \ (5,1,2)_{LR} \right] \otimes_M \left[\begin{array}{c} \left(-\frac{120}{477}, 0, 0 \right)_{LR} \\ \left(\frac{1080}{16218}, 0, 0 \right)_{LR} \end{array} \right] \Theta (0,0,0)_{LR} = \\ &\left(\frac{10650}{16218}, 0, 0 \right)_{LR} \end{aligned}$$

$$\begin{aligned} \tilde{c}_B \otimes_M \tilde{B}^{-1} \otimes_M \tilde{b} &= \left[(0,0,0)_{LR} \ (5,1,2)_{LR} \right] \otimes_M \left[\begin{array}{c} \left(\frac{1824}{477}, \frac{1040}{477}, \frac{1808}{477} \right)_{LR} \\ \left(\frac{29376}{16218}, \frac{39168}{16218}, \frac{20808}{16218} \right)_{LR} \end{array} \right] = \\ &\left(\frac{277440}{16218}, \frac{180608}{16218}, \frac{266696}{16218} \right)_{LR} \end{aligned}$$

Entries of the elements corresponding to 6th column i.e., corresponding to solution column:

$$\tilde{B}^{-1} \otimes_M \tilde{b} = \left[\begin{array}{cc} \left(\frac{136}{477}, 0, 0 \right)_{LR} & \left(-\frac{120}{477}, 0, 0 \right)_{LR} \\ \left(-\frac{1224}{16218}, 0, 0 \right)_{LR} & \left(\frac{1080}{16218}, 0, 0 \right)_{LR} \end{array} \right] \otimes_M \left[\begin{array}{c} (24, 5, 8)_{LR} \\ (12, 6, 3)_{LR} \end{array} \right] = \left[\begin{array}{c} \left(\frac{1824}{477}, \frac{1040}{477}, \frac{1808}{477} \right)_{LR} \\ \left(\frac{29376}{16218}, \frac{39168}{16218}, \frac{20808}{16218} \right)_{LR} \end{array} \right]$$

Table 2.5 The fuzzy optimal table

Basic variables	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	Solution
$\mathfrak{R}(\tilde{z})$	0	10.44	0	1.23	–
\tilde{z}	$\left(-\frac{4726}{16218}, \frac{184283}{16218}, \frac{115294}{16218}\right)_{LR}$	$\left(\frac{4862}{16218}, \frac{37602}{16218}, \frac{87856}{16218}\right)_{LR}$	$\left(\frac{12070}{16218}, 0, 0\right)_{LR}$	$\left(\frac{10650}{16218}, 0, 0\right)_{LR}$	$\left(\frac{277440}{16218}, \frac{180608}{16218}, \frac{266696}{16218}\right)_{LR}$
\tilde{x}_2	$\left(\frac{64}{477}, \frac{512}{477}, \frac{256}{477}\right)_{LR}$	$\left(\frac{560}{477}, \frac{528}{477}, \frac{196}{477}\right)_{LR}$	$\left(\frac{136}{477}, 0, 0\right)_{LR}$	$\left(-\frac{120}{477}, 0, 0\right)_{LR}$	$\left(\frac{1824}{477}, \frac{1040}{477}, \frac{1808}{477}\right)_{LR}$
\tilde{x}_1	$\left(\frac{14688}{16218}, \frac{15660}{16218}, \frac{12240}{16218}\right)_{LR}$	$\left(-\frac{1224}{16218}, \frac{3672}{16218}, \frac{8568}{16218}\right)_{LR}$	$\left(-\frac{1224}{16218}, 0, 0\right)_{LR}$	$\left(\frac{1080}{16218}, 0, 0\right)_{LR}$	$\left(\frac{29376}{16218}, \frac{39168}{16218}, \frac{20808}{16218}\right)_{LR}$

Since, there does not exist any \tilde{z} such that $\mathfrak{R}(\tilde{z}) < 0$ so the fuzzy simplex table (Table 2.5) is the fuzzy optimal table. Also, the fuzzy optimal solution of the chosen fully fuzzy linear programming problem is:

$$\tilde{x}_1 =_{\mathfrak{R}} \left(\frac{29376}{16218}, \frac{39168}{16218}, \frac{20808}{16218} \right)_{LR}, \tilde{x}_2 =_{\mathfrak{R}} \left(\frac{1824}{477}, \frac{1040}{477}, \frac{1808}{477} \right)_{LR}$$

and the fuzzy optimal value is:

$$\tilde{z} =_{\mathfrak{R}} \left(\frac{277440}{16218}, \frac{180608}{16218}, \frac{266696}{16218} \right)_{LR}.$$

2.5.2 Solution of chosen fully fuzzy linear programming problem

Using the existing method [47] with proposed modification, the fuzzy optimal solution of the chosen fully fuzzy linear programming problem, chosen in Example 2.3, in which all the parameters as well as decision variables are represented by *LR* flat fuzzy numbers can be obtained as follows:

Example 2.3 Consider the following fully fuzzy linear programming problem:

$$\text{Maximize} \quad [(2,3,2,5)_{LR} \otimes_M \tilde{x}_1 \oplus (1,2,1,3)_{LR} \otimes_M \tilde{x}_2 \oplus (3,4,3,4)_{LR} \otimes_M \tilde{x}_3]$$

$$\text{Subject to} \quad (E_{2.5})$$

$$\left(2, 3, 1, \frac{1}{2} \right)_{LR} \otimes_M \tilde{x}_1 \oplus \left(1, 3, \frac{1}{2}, \frac{3}{2} \right)_{LR} \otimes_M \tilde{x}_2 \oplus (3,4,1,1)_{LR} \otimes_M \tilde{x}_3 \leq_{\mathfrak{R}} (5,7,2,2)_{LR}$$

$$(2,3,2,5)_{LR} \otimes_M \tilde{x}_1 \oplus \left(1, 3, 1, \frac{1}{2} \right)_{LR} \otimes_M \tilde{x}_2 \leq_{\mathfrak{R}} (3,5,1,1)_{LR}$$

$$\tilde{x}_1 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_2 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_3 \geq_{\mathfrak{R}} \tilde{0}$$

where, $L(x) = \text{maximum}\{0, 1 - x^2\}$, $R(x) = \text{maximum}\{0, 1 - x^2\}$.

Step 1

Adding fuzzy slack variables \tilde{x}_4 and \tilde{x}_5 in first and second constraint respectively, the chosen fully fuzzy linear programming problem ($E_{2.5}$) can be written as:

$$\text{Maximize} \quad [(2,3,2,5)_{LR} \otimes_M \tilde{x}_1 \oplus (1,2,1,3)_{LR} \otimes_M \tilde{x}_2 \oplus (3,4,3,4)_{LR} \otimes_M \tilde{x}_3 \oplus \tilde{0} \otimes \tilde{x}_4 \oplus \tilde{0} \otimes \tilde{x}_5]$$

$$\text{Subject to} \quad (E_{2.6})$$

$$\left(2,3,1,\frac{1}{2}\right)_{LR} \otimes_M \tilde{x}_1 \oplus \left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \otimes_M \tilde{x}_2 \oplus (3,4,1,1)_{LR} \otimes_M \tilde{x}_3 \oplus \tilde{x}_4 =_{\mathfrak{R}} (5,7,2,2)_{LR}$$

$$(2,3,2,5)_{LR} \otimes_M \tilde{x}_1 \oplus \left(1,3,1,\frac{1}{2}\right)_{LR} \otimes_M \tilde{x}_2 \oplus \tilde{x}_5 =_{\mathfrak{R}} (3,5,1,1)_{LR}$$

$$\tilde{x}_1 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_2 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_3 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_4 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_5 \geq_{\mathfrak{R}} \tilde{0}$$

where, \tilde{x}_4, \tilde{x}_5 are fuzzy slack variables, $L(x)=R(x)=\text{maximum}\{0,1-x^2\}$.

The initial fuzzy simplex table for the fully fuzzy linear programming problem $(E_{2.6})$ is:

Table 2.6 The initial fuzzy simplex table

Basis	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	\tilde{x}_5	Solution	Minimum ratio	\Re (Minimum ratio)
$\Re(\tilde{z})$	$-\frac{7}{2}$	$-\frac{13}{6}$	$-\frac{23}{6}$	0	0	– $(0, 0, 0, 0)_{LR}$		
\tilde{z}	$(-3, -2, 5, 2)_{LR}$	$(-2, -1, 3, 1)_{LR}$	$(-4, -3, 4, 3)_{LR}$	$\tilde{0}$	$\tilde{0}$			
\tilde{x}_4	$\left(2, 3, 1, \frac{1}{2}\right)_{LR}$	$\left(1, 3, \frac{1}{2}, \frac{3}{2}\right)_{LR}$	$(3, 4, 1, 1)_{LR}$	$(1, 1, 1, 1)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(5, 7, 2, 2)_{LR}$	$\left(\frac{10}{7}, 2, \frac{4}{7}, \frac{4}{7}\right)_{LR}$	$\frac{12}{7}$
\tilde{x}_5	$(2, 3, 2, 5)_{LR}$	$\left(1, 3, 1, \frac{1}{2}\right)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(1, 1, 1, 1)_{LR}$	$(3, 5, 1, 1)_{LR}$	–	–

Since, $\text{minimum}\{\mathfrak{R}(-3, -2, 5, 2)_{LR}, \mathfrak{R}(-2, -1, 3, 1)_{LR}, \mathfrak{R}(-4, -3, 4, 3)_{LR}\} = \text{minimum}\{-3.5, -2.1, -3.8\} = -3.8$, which is corresponding to the fuzzy decision variable \tilde{x}_3 . So, \tilde{x}_3 will enter into

the basis. Also, $\text{minimum}\left\{\mathfrak{R}\left(\frac{(5, 7, 2, 2)_{LR}}{(3, 7, 1, 1)_{LR}}\right)\right\} = \mathfrak{R}\left(\frac{10}{7}, 2, \frac{4}{7}, \frac{4}{7}\right)_{LR} = \frac{12}{7}$, which is

corresponding to the fuzzy basic variable \tilde{x}_4 . So, \tilde{x}_4 will leave the basis.

Step 3

Since, for the next simplex table (Table 2.7) the first and second fuzzy basic variables are

\tilde{x}_3, \tilde{x}_5 . So, next table will be:

Table 2.7 The second fuzzy simplex table

Basic variables	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	\tilde{x}_5	Solution	Minimum ratio	\mathfrak{R} (Minimum ratio)
$\mathfrak{R}(\tilde{z})$	$-\frac{17}{18}$	$\frac{7}{18}$	0	0	0	—		
\tilde{z}	$\left(-\frac{17}{21}, -\frac{9}{7}, \frac{128}{21}, \frac{107}{42}\right)_{LR} \left(-\frac{19}{21}, -\frac{16}{7}, \frac{149}{42}, \frac{37}{14}\right)_{LR} \left(-\frac{5}{7}, \frac{29}{21}, \frac{107}{21}, \frac{86}{21}\right)_{LR} \left(\frac{23}{21}, \frac{23}{21}, \frac{23}{21}, \frac{23}{21}\right)_{LR} \tilde{0}$					$(0,0,0,0)_{LR}$		
\tilde{x}_3	$\left(\frac{4}{7}, \frac{6}{7}, \frac{2}{7}, \frac{1}{7}\right)_{LR}$	$\left(\frac{2}{7}, \frac{6}{7}, \frac{1}{7}, \frac{3}{7}\right)_{LR}$	$\left(\frac{6}{7}, \frac{8}{7}, \frac{2}{7}, \frac{2}{7}\right)_{LR}$	$\left(\frac{2}{7}, \frac{2}{7}, \frac{2}{7}, \frac{2}{7}\right)_{LR}$	$(0,0,0,0)_{LR}$	$\left(\frac{10}{7}, 2, \frac{4}{7}, \frac{4}{7}\right)_{LR}$	$\left(\frac{15}{7}, 3, \frac{6}{7}, \frac{6}{7}\right)_{LR}$	$\frac{18}{7}$
\tilde{x}_5	$(2,3,2,5)_{LR}$	$\left(1,3,1, \frac{1}{2}\right)_{LR}$	$(0,0,0,0)_{LR}$	$(0,0,0,0)_{LR}$	$(1,1,1,1)_{LR}$	$(3,5,1,1)_{LR}$	$\left(\frac{6}{7}, \frac{10}{7}, \frac{2}{7}, \frac{2}{7}\right)_{LR}$	$\frac{8}{7}$

Step 4

Since $\Re\left(-\frac{17}{21}, -\frac{9}{7}, \frac{128}{21}, \frac{107}{42}\right)_{LR} = -0.94$, which is corresponding to the fuzzy decision

variable \tilde{x}_1 . So, \tilde{x}_1 will enter into the basis. Also,

$$\text{minimum} \left\{ \Re \left(\frac{\left(\frac{10}{7}, 4, \frac{4}{7}, \frac{4}{7} \right)_{LR}}{\left(\frac{4}{7}, \frac{6}{7}, \frac{2}{7}, \frac{1}{7} \right)_{LR}}, \Re \left(\frac{(3, 5, 1, 1)_{LR}}{(2, 3, 2, 5)_{LR}} \right) \right\} = \text{minimum} \left\{ \Re \left(\frac{15}{7}, 3, \frac{6}{7}, \frac{6}{7} \right)_{LR}, \Re \left(\frac{6}{7}, \frac{10}{7}, \frac{2}{7}, \frac{2}{7} \right)_{LR} \right\}$$

$\text{minimum}\{2.5, 1.1\} = 1.1$, which is corresponding to the fuzzy basic variable \tilde{x}_5 . So, \tilde{x}_5 will

leave the basis.

Step 5

Since, for the next simplex table (Table 2.8) the first and second fuzzy basic variables are

\tilde{x}_3, \tilde{x}_1 so next table will be:

Table 2.8 The fuzzy optimal table

Basic variables	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	\tilde{x}_5	Solution
$\mathfrak{R}(\tilde{z})$	0	$\frac{7}{18}$	0	$\frac{23}{21}$	$\frac{17}{63}$	-
\tilde{z}	$\left(-1, -\frac{356}{126}, \frac{1480}{126}, \frac{1135}{126}\right)_{LR}$	$\left(-\frac{44}{21}, -\frac{47}{9}, \frac{619}{126}, \frac{488}{126}\right)_{LR}$	$\left(-\frac{5}{7}, \frac{29}{21}, \frac{107}{21}, \frac{86}{21}\right)_{LR}$	$\left(\frac{23}{21}, \frac{23}{21}, \frac{23}{21}, \frac{23}{21}\right)_{LR}$	$\left(\frac{17}{63}, \frac{17}{63}, \frac{109}{63}, \frac{109}{63}\right)_{LR}$	$\left(\frac{608}{126}, \frac{860}{126}, \frac{494}{126}, \frac{494}{126}\right)_{LR}$
\tilde{x}_3	$\left(0, \frac{10}{21}, \frac{26}{21}, \frac{11}{21}\right)_{LR}$	$\left(-\frac{6}{21}, \frac{14}{21}, \frac{5}{21}, \frac{13}{21}\right)_{LR}$	$\left(\frac{6}{7}, \frac{8}{7}, \frac{2}{7}, \frac{2}{7}\right)_{LR}$	$\left(\frac{2}{7}, \frac{2}{7}, \frac{2}{7}, \frac{2}{7}\right)_{LR}$	$\left(-\frac{4}{21}, \frac{4}{21}, \frac{4}{21}, \frac{4}{21}\right)_{LR}$	$\left(\frac{10}{21}, \frac{30}{21}, \frac{16}{21}, \frac{16}{21}\right)_{LR}$
\tilde{x}_1	$\left(\frac{12}{21}, \frac{18}{21}, \frac{12}{21}, \frac{30}{21}\right)_{LR}$	$\left(\frac{6}{21}, \frac{18}{21}, \frac{6}{21}, \frac{3}{21}\right)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(0, 0, 0, 0)_{LR}$	$\left(\frac{6}{21}, \frac{6}{21}, \frac{6}{21}, \frac{6}{21}\right)_{LR}$	$\left(\frac{18}{21}, \frac{30}{21}, \frac{6}{21}, \frac{6}{21}\right)_{LR}$

Since, there does not exist any \tilde{z} such that $\mathfrak{R}(\tilde{z}) < 0$ so the fuzzy simplex table (Table 2.8) will be the fuzzy optimal table. Also, the fuzzy optimal solution of the chosen fully fuzzy linear programming problem is:

$$\tilde{x}_1 =_{\mathfrak{R}} \left(\frac{18}{21}, \frac{30}{21}, \frac{6}{21}, \frac{6}{21} \right)_{LR}, \quad \tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR}, \quad \tilde{x}_3 =_{\mathfrak{R}} \left(\frac{10}{21}, \frac{30}{21}, \frac{16}{21}, \frac{16}{21} \right)_{LR}$$

and the fuzzy optimal value is $\tilde{z} =_{\mathfrak{R}} \left(\frac{608}{126}, \frac{860}{126}, \frac{494}{126}, \frac{494}{126} \right)_{LR}$.

2.6 Comparative study

The results of an existing fuzzy number linear programming problem, two existing fuzzy variable linear programming problems, an existing fully fuzzy linear programming problem as well as a fully fuzzy linear programming problem, chosen in Example 2.3, obtained by using the existing methods [46, 91, 92] and the existing method [47] with proposed modifications are shown in Table 2.9.

Table 2.9 Fuzzy optimal value

Example	Existing method [92]	Existing method [91]	Existing method [46]	Existing method [47] with proposed modification
8.1 [92, pp. 287]	$\left(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7} \right)_{LR}$	Not applicable	Not applicable	$\left(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7} \right)_{LR}$
8.1 [91, pp. 202]	Not applicable	$(8, 16, 4, 12)_{LR}$	Not applicable	$(8, 16, 4, 12)_{LR}$
2.4 [46, pp. 313]	Not applicable	Not applicable	$\left(\frac{94235}{169}, \frac{120265}{169}, \frac{19819}{169}, \frac{19819}{169} \right)_{LR}$	$\left(\frac{94235}{169}, \frac{120265}{169}, \frac{19819}{169}, \frac{19819}{169} \right)_{LR}$
Section 6 [47, pp. 103]	Not applicable	Not applicable	Not applicable	$\left(\frac{15043}{100}, \frac{93}{4}, \frac{11797}{80} \right)_{LR}$
2.1 [47, pp. 103]	Not applicable	Not applicable	Not applicable	$\left(\frac{277440}{16218}, \frac{180608}{16218}, \frac{266696}{16218} \right)_{LR}$
2.3	Not applicable	Not applicable	Not applicable	$\left(\frac{608}{126}, \frac{860}{126}, \frac{494}{126}, \frac{494}{126} \right)_{LR}$

The results, presented in Table 2.9, can be explained as follows:

- (i) The existing method [92] can be used only to find the optimal solution of fuzzy number linear programming problem $(P_{1.3})$. Since, the existing problem [92, Example 8.1 pp. 287] is a fuzzy number linear programming problem of type $(P_{1.3})$ so it can be solved by using the existing method [92, Example 8.1 pp. 287]. However, the existing problem [91, Example 8.1 pp. 202], the existing problem [46, Example 2.4 pp. 313], the existing problem [47, Section 6 pp. 103] and the problem chosen in Example 2.3, are not fuzzy number linear programming problem of type $(P_{1.3})$. So, none of these problems can be solved by the existing method [92].
- (ii) The existing method [91] can be used only to find the fuzzy optimal solution of fuzzy variable linear programming problem $(P_{1.7})$. Since, the existing problem [91, Example 8.1 pp. 202] is a fuzzy variable linear programming problem of type $(P_{1.7})$ so it can be solved by using the existing method [91]. However, the existing problem [92, Example 8.1 pp. 287], the existing problem [46, Example 2.4 pp. 313], the existing problem [47, Section 6 pp. 103] and the problem chosen in Example 2.3, are not fuzzy variable linear programming problem of type $(P_{1.7})$. So, none of these problems can be solved by the existing method [91].
- (iii) The existing method [46] can be used only to find the fuzzy optimal solution of such fuzzy variable linear programming problem in which all the parameters as well as the decision variables except a_{ij} are represented by symmetric trapezoidal fuzzy numbers. Since, in the existing problem [46, Example 2.4 pp. 313] all the parameters as well as the decision variables except a_{ij} are represented by symmetric trapezoidal fuzzy numbers so it can be solved by using the existing method [46]. However, the existing problem [92, Example 8.1 pp. 287], the existing problem [91, Example 8.1 pp. 202], the existing problem [47, Section 6 pp. 103] and the problem chosen in Example 2.3, are neither the

parameters nor the decision variables are represented by symmetric trapezoidal fuzzy numbers. So, none of these problems can be solved the existing method [46].

- (iv) Since, the existing method [47] with proposed modifications can be used to find the fuzzy optimal solution of fully fuzzy linear programming problems and fuzzy number linear programming problems as well as fuzzy variable linear programming problems are special cases of fully fuzzy linear programming problems. So, all the existing as well as chosen problems can be solved by the existing method [47] with proposed modifications.

2.7 Conclusions

On the basis of presented study, it can be concluded that the existing method [47] should not be used for solving fully fuzzy linear programming problems. However, the existing method [47] with proposed modifications can be used for the same.

Chapter 3

A new method for solving fully fuzzy linear programming problems[†]

In this chapter, the limitations of the existing method [47] and the method, proposed in previous chapter, are pointed out and to overcome these limitations, a new method is proposed for solving fully fuzzy linear programming problems. To show the advantages of proposed method over existing method [47] and the method, proposed in previous chapter, a fully fuzzy linear programming which can neither be solved by the existing method [47] nor by the method proposed in previous chapter, is solved by the proposed method. Also, the fully fuzzy linear programming problem, chosen in previous chapter, is solved by the proposed method.

3.1 Limitations of the existing method and the method proposed in previous chapter

For solving the fully fuzzy linear programming problems by using the existing method [47] as well as by the method proposed in previous chapter there is need to subtract as well as to divide LR flat fuzzy numbers. Since, in the literature [33] it is pointed out that only RL flat fuzzy numbers can be subtracted as well as divided from LR flat fuzzy numbers. So, all such fully fuzzy linear programming problems, in which the parameters are represented by such LR flat fuzzy numbers for which $L(x) \neq R(x)$ can neither be solved by the existing method [47] nor by the method proposed in previous chapter e.g., the fully fuzzy linear programming problem, chosen in Example 3.1, in which all the parameters are represented by such LR flat fuzzy numbers for which $L(x) \neq R(x)$ can neither be solved by the existing method [47] nor by the method proposed in previous chapter.

[†] The contents of this chapter are communicated in *Applied Mathematical Modelling*.

Example 3.1 Consider the following fully fuzzy linear programming problem

$$\text{Maximize} \quad [(2,3,2,5)_{LR} \otimes_M \tilde{x}_1 \oplus (1,2,1,3)_{LR} \otimes_M \tilde{x}_2 \oplus (3,4,3,4)_{LR} \otimes_M \tilde{x}_3]$$

$$\text{Subject to} \quad (E_{3.1})$$

$$\left(2,3,1,\frac{1}{2}\right)_{LR} \otimes_M \tilde{x}_1 \oplus \left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \otimes_M \tilde{x}_2 \oplus (3,4,1,1)_{LR} \otimes_M \tilde{x}_3 \leq_{\mathfrak{R}} (5,7,2,2)_{LR},$$

$$(2,3,2,5)_{LR} \otimes_M \tilde{x}_1 \oplus \left(1,3,1,\frac{1}{2}\right)_{LR} \otimes_M \tilde{x}_2 \leq_{\mathfrak{R}} (3,5,1,1)_{LR},$$

$$\tilde{x}_1 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_2 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_3 \geq_{\mathfrak{R}} \tilde{0}.$$

where, $L(x) = \text{maximum}\{0, 1 - x^2\}$, $R(x) = \text{maximum}\{0, 1 - x\}$.

3.2 Proposed method

In this section, to overcome the limitations of the existing method [47] and the method proposed in previous chapter, discussed in Section 3.1, a new method is proposed for solving fully fuzzy linear programming problems ($P_{3.1}$).

$$\text{Maximize (or Minimize)} \quad \sum_{j=1}^n \tilde{c}_j \otimes_M \tilde{x}_j,$$

$$\text{Subject to} \quad (P_{3.1})$$

$$\sum_{j=1}^n \tilde{a}_{ij} \otimes_M \tilde{x}_j \leq_{\mathfrak{R}} \text{ or } = \text{ or } \geq_{\mathfrak{R}} \tilde{b}_i; i = 1, 2, \dots, m$$

$$\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}, j = 1, 2, \dots, n.$$

where, \tilde{c}_j , \tilde{b}_i and \tilde{a}_{ij} are LR flat fuzzy numbers, \mathfrak{R} is a linear ranking function.

The steps of the proposed method are as follows:

Step 1

Since, for the ranking function \mathfrak{R} the property $\mathfrak{R}\left(\sum_{j=1}^n \tilde{c}_j\right) = \sum_{j=1}^n \mathfrak{R}(\tilde{c}_j)$ is satisfied so the fully

fuzzy linear programming problem ($P_{3.1}$) can be written as:

$$\text{Maximize (or Minimize)} \quad \sum_{j=1}^n \mathfrak{R}(\tilde{c}_j \otimes_M \tilde{x}_j),$$

$$\text{Subject to} \quad (P_{3.2})$$

$$\sum_{j=1}^n \mathfrak{R}(\tilde{a}_{ij} \otimes_M \tilde{x}_j) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b}_i); i = 1, 2, \dots, m$$

$$\mathfrak{R}(\tilde{x}_j) \geq 0, j = 1, 2, \dots, n.$$

Step 2

Since, for the Mehar multiplication, proposed in previous chapter, the property $\mathfrak{R}(\tilde{A} \otimes_M \tilde{B}) = \mathfrak{R}(\tilde{A})\mathfrak{R}(\tilde{B})$ is satisfied. So, the crisp linear programming problem $(P_{3.2})$ can be written as:

$$\text{Maximize (or Minimize)} \quad \sum_{j=1}^n \mathfrak{R}(\tilde{c}_j)\mathfrak{R}(\tilde{x}_j),$$

$$\text{Subject to} \quad (P_{3.3})$$

$$\sum_{j=1}^n \mathfrak{R}(\tilde{a}_{ij})\mathfrak{R}(\tilde{x}_j) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b}_i); i = 1, 2, \dots, m$$

$$\mathfrak{R}(\tilde{x}_j) \geq 0, j = 1, 2, \dots, n.$$

Step 3

Assuming $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$, $\tilde{c}_j = (m_j', n_j', \alpha_j', \beta_j')_{LR}$, $\tilde{b}_i = (m_i'', n_i'', \alpha_i'', \beta_i'')_{LR}$ and

$\tilde{a}_{ij} = (m_{ij}''', n_{ij}''', \alpha_{ij}''', \beta_{ij}''')_{LR}$ the problem $(P_{3.3})$ can be written as:

Maximize (or Minimize)

$$\sum_{j=1}^n \mathfrak{R}(m_j', n_j', \alpha_j', \beta_j')_{LR} \mathfrak{R}(m_j, n_j, \alpha_j, \beta_j)_{LR},$$

$$\text{Subject to} \quad (P_{3.4})$$

$$\sum_{j=1}^n \mathfrak{R}(m_{ij}''', n_{ij}''', \alpha_{ij}''', \beta_{ij}''')_{LR} \mathfrak{R}(m_j, n_j, \alpha_j, \beta_j)_{LR} \leq \text{or} = \text{or} \geq \mathfrak{R}(m_i'', n_i'', \alpha_i'', \beta_i'')_{LR}; i = 1, 2, \dots, m$$

$$\mathfrak{R}(m_j, n_j, \alpha_j, \beta_j)_{LR} \geq 0, j = 1, 2, \dots, n.$$

Step 4

Using the existing ranking formula [132], $\mathfrak{R}(\tilde{A}) = \mathfrak{R}(m, n, \alpha, \beta) = \frac{1}{2} \int_0^1 (m - \alpha L^{-1}(\lambda)) d\lambda$

$+ \frac{1}{2} \int_0^1 (n + \beta R^{-1}(\lambda)) d\lambda$, the problem $(P_{3.4})$ can be written as:

Maximize (or Minimize)

$$\sum_{j=1}^n \left(\frac{1}{2} \int_0^1 (m_j - \alpha_j L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_j + \beta_j R^{-1}(\lambda)) d\lambda \right) \left(\frac{1}{2} \int_0^1 (m_j - \alpha_j L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_j + \beta_j R^{-1}(\lambda)) d\lambda \right)$$

Subject to (P_{3.5})

$$\sum_{j=1}^n \left(\frac{1}{2} \int_0^1 (m_j - \alpha_j L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_j + \beta_j R^{-1}(\lambda)) d\lambda \right) \left(\frac{1}{2} \int_0^1 (m_j - \alpha_j L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_j + \beta_j R^{-1}(\lambda)) d\lambda \right) \leq \text{or } = \text{or } \geq \frac{1}{2} \int_0^1 (m_i - \alpha_i L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_i + \beta_i R^{-1}(\lambda)) d\lambda; i = 1, 2, \dots, m$$

$$\frac{1}{2} \int_0^1 (m_j - \alpha_j L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_j + \beta_j R^{-1}(\lambda)) d\lambda \geq 0.$$

Step 5

Solve the crisp linear programming problem $(P_{3.5})$ to find its optimal solution

$$\{m_j, n_j, \alpha_j, \beta_j\}.$$

Step 6

Put the values of $m_j, n_j, \alpha_j, \beta_j$, obtained from Step 5, in $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$ to find the

fuzzy optimal solution $\{\tilde{x}_j, j = 1, 2, \dots, n\}$ of fully fuzzy linear programming problem $(P_{3.1})$.

Step 7

Substitute the values of $\{\tilde{x}_j; j = 1, 2, \dots, n\}$, obtained from Step 6, in $\sum_{j=1}^n \tilde{c}_j \otimes_M \tilde{x}_j$, to find the

fuzzy optimal value of fully fuzzy linear programming problem $(P_{3.1})$.

Lemma 3.1: If for any LR flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$, $L(x) = \text{maximum}\{0, 1 - x^2\}$

and $R(x) = \text{maximum}\{0, 1-x\}$ then, $\mathfrak{R}(\tilde{A}) = \frac{1}{2} \left(m+n - \frac{2\alpha}{3} + \frac{\beta}{2} \right)$.

Proof: Since, $L(x) = \text{maximum}\{0, 1-x^2\}$ and $R(x) = \text{maximum}\{0, 1-x\}$ so $L^{-1}(\lambda) = \sqrt{1-\lambda}$ and $R^{-1}(\lambda) = 1-\lambda$ where, $\lambda \in [0,1]$.

Substituting the values of $L^{-1}(\lambda)$ and $R^{-1}(\lambda)$ in Yager's ranking formula,

$$\mathfrak{R}(\tilde{A}) = \frac{1}{2} \int_0^1 (m - \alpha\sqrt{1-\lambda}) d\lambda + \frac{1}{2} \int_0^1 (n + \beta(1-\lambda)) d\lambda, \quad \text{the value of } \mathfrak{R}(\tilde{A}) \text{ is}$$

$$\frac{1}{2} \left(m+n - \frac{2\alpha}{3} + \frac{\beta}{2} \right).$$

3.3 Advantages of the proposed method

The advantages of the method, proposed in this chapter, over the existing method [47] and the method, proposed in previous chapter, are as follows:

- (i) As discussed in Section 3.1, the existing method [47] and the method, proposed in previous chapter, can not be used for solving such fully fuzzy linear programming problems in which parameters are represented by such LR flat fuzzy numbers for which $L(x) \neq R(x)$. Since, in the proposed method neither there is need to subtract nor to divide two LR flat fuzzy numbers. So, the method, proposed in this chapter, can be used for solving these fully fuzzy linear programming problems.
- (ii) For solving the fully fuzzy linear programming problems by using the existing method [47] and the method proposed in previous chapter there is need to apply the fuzzy arithmetic operations for finding the elements of each simplex table. While, for solving the same problem by using the proposed method there is need to apply the crisp arithmetic operations for finding the elements of each simplex table.

Since, the number of simplex tables in both the methods are same. While, it is much easy to apply the crisp arithmetic operations as compared to the fuzzy arithmetic operations. So, it is much easy to apply the method proposed in this chapter as compared to the existing

method [47] and method proposed in previous chapter.

To show the advantage of the method, proposed in Section 3.2, over the existing method [47] and the method, proposed in previous chapter, the fully fuzzy linear programming problem, chosen in Example 2.3, which can be solved by the method proposed in previous chapter as well as the fully fuzzy linear programming problem, chosen in Example 3.1, which can neither be solved by the existing method [47] nor by the method, proposed in previous chapter, are solved by using the proposed method.

3.3.1 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 2.3

Using the method, proposed in Section 3.2, the fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 2.3, can be obtained as follows:

Step 1

Using Step 1 of the method, proposed in Section 3.2, the fully fuzzy linear programming problem ($E_{2.5}$) can be written as:

Maximize

$$\mathfrak{R}((2, 3, 2, 5)_{LR} \otimes_M \tilde{x}_1) + \mathfrak{R}((1, 2, 1, 3)_{LR} \otimes_M \tilde{x}_2) + \mathfrak{R}((3, 4, 3, 4)_{LR} \otimes_M \tilde{x}_3)$$

Subject to

($E_{3.2}$)

$$\mathfrak{R}\left(\left(2, 3, 1, \frac{1}{2}\right)_{LR} \otimes_M \tilde{x}_1\right) + \mathfrak{R}\left(\left(1, 3, \frac{1}{2}, \frac{3}{2}\right)_{LR} \otimes_M \tilde{x}_2\right) + \mathfrak{R}((3, 4, 1, 1)_{LR} \otimes_M \tilde{x}_3) \leq \mathfrak{R}(5, 7, 2, 2)_{LR}$$

$$\mathfrak{R}((2, 3, 2, 5)_{LR} \otimes_M \tilde{x}_1) + \mathfrak{R}\left(\left(1, 3, 1, \frac{1}{2}\right)_{LR} \otimes_M \tilde{x}_2\right) \leq \mathfrak{R}(3, 5, 1, 1)_{LR}$$

$$\tilde{x}_1 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_2 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_3 \geq_{\mathfrak{R}} \tilde{0}.$$

where, $L(x) = \text{maximum}\{0, 1 - x^2\}$, $R(x) = \text{maximum}\{0, 1 - x^2\}$.

Step 2

Using Step 2 of the method, proposed in Section 3.2, the crisp linear programming problem

$(E_{3.2})$ can be written as:

Maximize

$$\mathfrak{R}(2, 3, 2, 5)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}(1, 2, 1, 3)_{LR} \mathfrak{R}(\tilde{x}_2) + \mathfrak{R}(3, 4, 3, 4)_{LR} \mathfrak{R}(\tilde{x}_3)$$

Subject to

$(E_{3.3})$

$$\mathfrak{R}\left(2, 3, 1, \frac{1}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}\left(1, 3, \frac{1}{2}, \frac{3}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_2) + \mathfrak{R}(3, 4, 1, 1)_{LR} \mathfrak{R}(\tilde{x}_3) \leq \mathfrak{R}(5, 7, 2, 2)_{LR}$$

$$\mathfrak{R}(2, 3, 2, 5)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}\left(1, 3, 1, \frac{1}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_2) \leq \mathfrak{R}(3, 5, 1, 1)_{LR}$$

$$\mathfrak{R}(\tilde{x}_1) \geq 0, \mathfrak{R}(\tilde{x}_2) \geq 0, \mathfrak{R}(\tilde{x}_3) \geq 0.$$

Step 3

Assuming $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$, the problem $(E_{3.3})$ can be written as:

Maximize

$$\mathfrak{R}(2, 3, 2, 5)_{LR} \mathfrak{R}(m_1, n_1, \alpha_1, \beta_1)_{LR} + \mathfrak{R}(1, 2, 1, 3)_{LR} \mathfrak{R}(m_2, n_2, \alpha_2, \beta_2)_{LR} + \mathfrak{R}(3, 4, 3, 4)_{LR} \mathfrak{R}(m_3, n_3, \alpha_3, \beta_3)_{LR}$$

Subject to

$(E_{3.4})$

$$\mathfrak{R}\left(2, 3, 1, \frac{1}{2}\right)_{LR} \mathfrak{R}(m_1, n_1, \alpha_1, \beta_1)_{LR} + \mathfrak{R}\left(1, 3, \frac{1}{2}, \frac{3}{2}\right)_{LR} \mathfrak{R}(m_2, n_2, \alpha_2, \beta_2)_{LR} + \mathfrak{R}(3, 4, 1, 1)_{LR} \mathfrak{R}(m_3, n_3, \alpha_3, \beta_3)_{LR} \leq \mathfrak{R}(5, 7, 2, 2)_{LR}$$

$$\mathfrak{R}(m_3, n_3, \alpha_3, \beta_3)_{LR} \leq \mathfrak{R}(5, 7, 2, 2)_{LR}$$

$$\mathfrak{R}(2, 3, 2, 5)_{LR} \mathfrak{R}(m_1, n_1, \alpha_1, \beta_1)_{LR} + \mathfrak{R}\left(1, 3, 1, \frac{1}{2}\right)_{LR} \mathfrak{R}(m_2, n_2, \alpha_2, \beta_2)_{LR} \leq \mathfrak{R}(3, 5, 1, 1)_{LR}$$

$$\mathfrak{R}(m_1, n_1, \alpha_1, \beta_1)_{LR} \geq 0, \mathfrak{R}(m_2, n_2, \alpha_2, \beta_2)_{LR} \geq 0, \mathfrak{R}(m_3, n_3, \alpha_3, \beta_3)_{LR} \geq 0.$$

Step 4

Using Lemma 2.1, the problem $(E_{3.4})$ can be written as:

Maximize

$$\left(\frac{7}{2} m_1 + \frac{7}{2} n_1 + \frac{7}{3} \beta_1 - \frac{7}{3} \alpha_1 + \frac{13}{6} m_2 + \frac{13}{6} n_2 + \frac{13}{9} \beta_2 - \frac{13}{9} \alpha_2 + \frac{23}{6} m_3 + \frac{23}{6} n_3 + \frac{23}{9} \beta_3 - \frac{23}{9} \alpha_3 \right)$$

Subject to

$(E_{3.5})$

$$\frac{7}{3}m_1 + \frac{7}{3}n_1 + \frac{14}{9}\beta_1 - \frac{14}{9}\alpha_1 + \frac{7}{3}m_2 + \frac{7}{3}n_2 + \frac{14}{9}\beta_2 - \frac{14}{9}\alpha_2 + \frac{7}{2}m_3 + \frac{7}{2}n_3 + \frac{7}{3}\beta_3 - \frac{7}{3}\alpha_3 \leq 12$$

$$\frac{7}{2}m_1 + \frac{7}{2}n_1 + \frac{7}{3}\beta_1 - \frac{7}{3}\alpha_1 + \frac{11}{6}m_2 + \frac{11}{6}n_2 + \frac{11}{9}\beta_2 - \frac{11}{9}\alpha_2 \leq 8$$

$$m_1 + n_1 - \frac{2\alpha_1}{3} + \frac{2\beta_1}{3} \geq 0$$

$$m_2 + n_2 - \frac{2\alpha_2}{3} + \frac{2\beta_2}{3} \geq 0$$

$$m_3 + n_3 - \frac{2\alpha_3}{3} + \frac{2\beta_3}{3} \geq 0$$

Step 5

The optimal solution of the crisp linear programming problem ($E_{3.5}$) is:

$$m_1 = 0.85, m_2 = 0, m_3 = 0.47, n_1 = 1.42, n_2 = 0, n_3 = 1.42, \alpha_1 = 0.28, \alpha_2 = 0, \alpha_3 = 0.28, \beta_1 = 0.28, \beta_2 = 0, \beta_3 = 0.28.$$

Step 6

Putting the values of $m_j, n_j, \alpha_j, \beta_j; j=1,2,3$ in $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$ the fuzzy optimal solution of fully fuzzy linear programming problem ($E_{2.5}$) is:

$$\tilde{x}_1 =_{\mathfrak{R}} (0.85, 1.42, 0.28, 0.28)_{LR}, \tilde{x}_2 =_{\mathfrak{R}} \tilde{0}, \tilde{x}_3 =_{\mathfrak{R}} (0.47, 1.42, 0.76, 0.76)_{LR}.$$

Step 7

Substituting the values of \tilde{x}_1, \tilde{x}_2 and \tilde{x}_3 , obtained from Step 6, in $[(2,3,2,5)_{LR} \otimes_M \tilde{x}_1 \oplus (1,2,1,3)_{LR} \otimes_M \tilde{x}_2 \oplus (3,4,3,4)_{LR} \otimes_M \tilde{x}_3]$ the fuzzy optimal value of fully fuzzy linear programming problem ($E_{2.5}$) is $(4.82, 6.82, 3.92, 3.92)_{LR}$.

3.3.2 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in

Example 3.1

Using the method, proposed in Section 3.2, the fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 3.1, can be obtained as follows:

Step 1

Using Step 1 of the method, proposed in Section 3.2, the fully fuzzy linear programming problem ($E_{3,1}$) can be written as:

Maximize

$$\mathfrak{R}((2,3,2,5)_{LR} \otimes_M \tilde{x}_1) + \mathfrak{R}((1,2,1,3)_{LR} \otimes_M \tilde{x}_2) + \mathfrak{R}((3,4,3,4)_{LR} \otimes_M \tilde{x}_3)$$

Subject to ($E_{3,6}$)

$$\mathfrak{R}\left(\left(2,3,1,\frac{1}{2}\right)_{LR} \otimes_M \tilde{x}_1\right) + \mathfrak{R}\left(\left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \otimes_M \tilde{x}_2\right) + \mathfrak{R}((3,4,1,1)_{LR} \otimes_M \tilde{x}_3) \leq \mathfrak{R}(5,7,2,2)_{LR}$$

$$\mathfrak{R}((2,3,2,5)_{LR} \otimes_M \tilde{x}_1) + \mathfrak{R}\left(\left(1,3,1,\frac{1}{2}\right)_{LR} \otimes_M \tilde{x}_2\right) \leq \mathfrak{R}(3,5,1,1)_{LR}$$

$$\mathfrak{R}(\tilde{x}_1) \geq 0, \mathfrak{R}(\tilde{x}_2) \geq 0, \mathfrak{R}(\tilde{x}_3) \geq 0$$

where, $L(x) = \text{maximum}\{0, 1 - x^2\}$, $R(x) = \text{maximum}\{0, 1 - x\}$.

Step 2

Using Step 2 of the method, proposed in Section 3.2, the crisp linear programming problem ($E_{3,6}$) can be written as:

Maximize

$$\mathfrak{R}(2,3,2,5)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}(1,2,1,3)_{LR} \mathfrak{R}(\tilde{x}_2) + \mathfrak{R}(3,4,3,4)_{LR} \mathfrak{R}(\tilde{x}_3)$$

Subject to ($E_{3,7}$)

$$\mathfrak{R}\left(2,3,1,\frac{1}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}\left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_2) + \mathfrak{R}(3,4,1,1)_{LR} \mathfrak{R}(\tilde{x}_3) \leq \mathfrak{R}(5,7,2,2)_{LR}$$

$$\mathfrak{R}(2,3,2,5)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}\left(1,3,1,\frac{1}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_2) \leq \mathfrak{R}(3,5,1,1)_{LR}$$

$$\mathfrak{R}(\tilde{x}_1) \geq 0, \mathfrak{R}(\tilde{x}_2) \geq 0, \mathfrak{R}(\tilde{x}_3) \geq 0.$$

Step 3

Assuming $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$, the problem ($E_{3,7}$) can be written as:

Maximize

$$\mathfrak{R}(2,3,2,5)_{LR} \mathfrak{R}(m_1, n_1, \alpha_1, \beta_1)_{LR} + \mathfrak{R}(1,2,1,3)_{LR} \mathfrak{R}(m_2, n_2, \alpha_2, \beta_2)_{LR} + \mathfrak{R}(3,4,3,4)_{LR} \mathfrak{R}(m_3, n_3, \alpha_3, \beta_3)_{LR}$$

Subject to

(E_{3,8})

$$\mathfrak{R}\left(2,3,1,\frac{1}{2}\right)_{LR} \mathfrak{R}(m_1, n_1, \alpha_1, \beta_1)_{LR} + \mathfrak{R}\left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \mathfrak{R}(m_2, n_2, \alpha_2, \beta_2)_{LR} + \mathfrak{R}(3,4,1,1)_{LR}$$

$$\mathfrak{R}(m_3, n_3, \alpha_3, \beta_3)_{LR} \leq \mathfrak{R}(5,7,2,2)_{LR}$$

$$\mathfrak{R}(2,3,2,5)_{LR} \mathfrak{R}(m_1, n_1, \alpha_1, \beta_1)_{LR} + \mathfrak{R}\left(1,3,1,\frac{1}{2}\right)_{LR} \mathfrak{R}(m_2, n_2, \alpha_2, \beta_2)_{LR} \leq \mathfrak{R}(3,5,1,1)_{LR}$$

$$\mathfrak{R}(m_1, n_1, \alpha_1, \beta_1)_{LR} \geq 0, \mathfrak{R}(m_2, n_2, \alpha_2, \beta_2)_{LR} \geq 0, \mathfrak{R}(m_3, n_3, \alpha_3, \beta_3)_{LR} \geq 0.$$

Step 4

Using Lemma 3.1, the problem (E_{3,8}) can be written as:

Maximize

$$\frac{1}{2} \left(\frac{37}{6} m_1 + \frac{37}{6} n_1 + \frac{37}{12} \beta_1 - \frac{37}{9} \alpha_1 + \frac{23}{6} m_2 + \frac{23}{6} n_2 + \frac{23}{12} \beta_2 - \frac{23}{9} \alpha_2 + 7m_3 + 7n_3 + \frac{7}{2} \beta_3 - \frac{14}{3} \alpha_3 \right)$$

Subject to

(E_{3,9})

$$\frac{55}{12} m_1 + \frac{55}{12} n_1 + \frac{55}{24} \beta_1 - \frac{55}{18} \alpha_1 + \frac{53}{12} m_2 + \frac{53}{12} n_2 + \frac{53}{24} \beta_2 - \frac{53}{18} \alpha_2 + \frac{41}{6} m_3 + \frac{41}{6} n_3 + \frac{41}{12} \beta_3 - \frac{41}{9} \alpha_3 \leq \frac{35}{3}$$

$$\frac{37}{6} m_1 + \frac{37}{6} n_1 + \frac{37}{12} \beta_1 - \frac{37}{9} \alpha_1 + \frac{43}{12} m_2 + \frac{43}{12} n_2 + \frac{43}{24} \beta_2 - \frac{43}{18} \alpha_2 \leq \frac{47}{6}$$

$$m_1 + n_1 - \frac{2\alpha_1}{3} + \frac{\beta_1}{2} \geq 0$$

$$m_2 + n_2 - \frac{2\alpha_2}{3} + \frac{\beta_2}{2} \geq 0$$

$$m_3 + n_3 - \frac{2\alpha_3}{3} + \frac{\beta_3}{2} \geq 0.$$

Step 5

The optimal solution of the crisp linear programming problem ($E_{3,9}$) is:

$$m_1 = 0.95, m_2 = 0, m_3 = 0, n_1 = 0.95, n_2 = 0, n_3 = 0, \alpha_1 = 0.95, \alpha_2 = 0, \alpha_3 = 0, \beta_1 = 0, \beta_2 = 0, \beta_3 = 1.71.$$

Step 6

Putting the values of $m_j, n_j, \alpha_j, \beta_j; j = 1, 2, 3$ in $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$ the fuzzy optimal solution of fully fuzzy linear programming problem ($E_{3,1}$) is:

$$\tilde{x}_1 =_{\mathfrak{R}} (0.95, 0.95, 0.95, 0)_{LR}, \tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR}, \tilde{x}_3 =_{\mathfrak{R}} (0, 0, 0, 1.71)_{LR}.$$

Step 7

Substituting the values of \tilde{x}_1, \tilde{x}_2 and \tilde{x}_3 , obtained from Step 6, in $[(2, 3, 2, 5)_{LR} \otimes_M \tilde{x}_1 \oplus (1, 2, 1, 3)_{LR} \otimes_M \tilde{x}_2 \oplus (3, 4, 3, 4)_{LR} \otimes_M \tilde{x}_3]$ the fuzzy optimal value of fully fuzzy linear programming problem ($E_{3,1}$) is $(2.52, 3.57, 2.52, 4.83)_{LR}$.

3.4 Comparative study

The results of the fully fuzzy linear programming problem, chosen in Example 2.3 and Example 3.1, obtained by using the method proposed in previous chapter and the method proposed in this chapter are shown in Table 3.1.

Table 3.1 Fuzzy optimal solution

Example	Method proposed in previous chapter	Method proposed in this chapter
2.3	$\tilde{x}_1 =_{\mathfrak{R}} (0.85, 1.42, 0.28, 0.28)_{LR},$ $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR},$ $\tilde{x}_3 =_{\mathfrak{R}} (0.47, 1.42, 0.76, 0.76)_{LR}$	$\tilde{x}_1 =_{\mathfrak{R}} (0.85, 1.42, 0.28, 0.28)_{LR},$ $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR},$ $\tilde{x}_3 =_{\mathfrak{R}} (0.47, 1.42, 0.76, 0.76)_{LR}$
3.1	Not applicable	$\tilde{x}_1 =_{\mathfrak{R}} (0.95, 0.95, 0.95, 0)_{LR},$ $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR},$ $\tilde{x}_3 =_{\mathfrak{R}} (0, 0, 0, 1.71)_{LR}$

The results, presented in Table 3.1, can be explained as follows:

- (i) In the fully fuzzy linear programming problem, chosen in Example 2.3, all the parameters as well as decision variables are represented by such LR flat fuzzy numbers for which $L(x) = R(x)$. While, in the fully fuzzy linear programming problem, chosen in Example 3.1, all the parameters as well as decision variables are represented by such LR flat fuzzy numbers for which $L(x) \neq R(x)$. So, due to the limitations of the method proposed in previous chapter, discussed in Section 3.1, the fully fuzzy linear programming problem, chosen in Example 2.3, can be solved by the method proposed in previous chapter but the fully fuzzy linear programming problem, chosen in Example 3.1, can not be solved by the method proposed in previous chapter.
- (ii) As discussed in Section 3.3, that the method proposed in this chapter can be used for solving fully fuzzy linear programming problems in which all the parameters as well as decision variables are represented by such LR flat fuzzy numbers for which $L(x) = R(x)$ and $L(x) \neq R(x)$. So both the fully fuzzy linear programming problems, chosen in Example 2.3 and Example 3.1, can be solved by the method proposed in this chapter.

3.5 Conclusions

On the basis of presented study, it can be concluded that all the fully fuzzy linear programming problems which can be solved by using the existing method [47] and the method proposed in the previous chapter can also be solved by using the method proposed in this chapter. However, there exist several fully fuzzy linear programming problems which can be solved by the method proposed in this chapter but can neither be solved by using the existing method [47] nor by the method proposed in previous chapter. Hence, it is better to use the method, proposed in this chapter, as compared to the existing method [47] and the method proposed in the previous chapter for solving fully fuzzy linear programming problems.

Chapter 4

An efficient method for solving fully fuzzy linear programming problems[‡]

In this chapter, an alternative method is proposed for solving fully fuzzy linear programming problems and it is shown that if a fully fuzzy linear programming problem is either solved by the method proposed in previous chapter or by the method proposed in this chapter, the fuzzy optimal solution and the fuzzy optimal value will be same. However, it is much easy to apply the method, proposed in this chapter, as compared to the method proposed in previous chapter for solving the fully fuzzy linear programming problems.

4.1 Proposed alternative method

In this section, an alternative method is proposed for solving fully fuzzy linear programming problem ($P_{3.1}$).

The steps of the proposed method are as follows:

Step 1

Use Step 1 to Step 3 of the method proposed in previous chapter to convert the fully fuzzy linear programming problem ($P_{3.1}$) into the crisp linear programming problem ($P_{4.1}$).

Maximize (or Minimize)

$$\sum_{j=1}^n \mathfrak{R}(m_j', n_j', \alpha_j', \beta_j')_{LR} \mathfrak{R}(m_j, n_j, \alpha_j, \beta_j)_{LR},$$

Subject to ($P_{4.1}$)

$$\sum_{j=1}^n \mathfrak{R}(m_{ij}''', n_{ij}''', \alpha_{ij}''', \beta_{ij}''')_{LR} \mathfrak{R}(m_j, n_j, \alpha_j, \beta_j)_{LR} \leq \text{or} = \text{or} \geq \mathfrak{R}(m_i'', n_i'', \alpha_i'', \beta_i'')_{LR}; i = 1, 2, \dots, m$$

$$(m_j, n_j, \alpha_j, \beta_j)_{LR} \geq_{\mathfrak{R}} \tilde{0}, j = 1, 2, \dots, n.$$

[‡] The contents of this chapter are communicated in *Journal of Intelligent and Fuzzy Systems*.

Step 2

Assuming $\Re(\tilde{x}_j) = \Re(m_j, n_j, \alpha_j, \beta_j)_{LR} = x_j, j = 1, 2, \dots, n$ the problem $(P_{4.1})$ can be written as:

Maximize (or Minimize)

$$\sum_{j=1}^n \Re(m_j, n_j, \alpha_j, \beta_j)_{LR} x_j$$

Subject to (P_{4.2})

$$\sum_{j=1}^n \Re(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR} x_j \leq \text{or} = \text{or} \geq \Re(m_i, n_i, \alpha_i, \beta_i)_{LR}; i = 1, 2, \dots, m$$

$$x_j \geq 0; j = 1, 2, \dots, n.$$

Step 3

Using the existing ranking formula [132], $\Re(\tilde{A}) = \Re(m, n, \alpha, \beta) = \frac{1}{2} \int_0^1 (m - \alpha L^{-1}(\lambda)) d\lambda +$

$\frac{1}{2} \int_0^1 (n + \beta R^{-1}(\lambda)) d\lambda$, the problem $(P_{4.2})$ can be written as:

Maximize (or Minimize)

$$\sum_{j=1}^n \left(\frac{1}{2} \int_0^1 (m_j - \alpha_j L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_j + \beta_j R^{-1}(\lambda)) d\lambda \right) x_j$$

Subject to (P_{4.3})

$$\sum_{j=1}^n \left(\frac{1}{2} \int_0^1 (m_{ij} - \alpha_{ij} L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_{ij} + \beta_{ij} R^{-1}(\lambda)) d\lambda \right) x_j \leq \text{or} = \text{or} \geq \frac{1}{2} \int_0^1 (m_i - \alpha_i L^{-1}(\lambda)) d\lambda + \frac{1}{2} \int_0^1 (n_i + \beta_i R^{-1}(\lambda)) d\lambda; i = 1, 2, \dots, m$$

$$x_j \geq 0; j = 1, 2, \dots, n.$$

Step 4

Solve the crisp linear programming problem $(P_{4.3})$ to find its optimal solution $\{x_j\}$.

Step 5

Since, every set of *LR* flat fuzzy numbers $\{\tilde{x}_j; j = 1, 2, \dots, n : \Re(\tilde{x}_j) = x_j\}$ will be the fuzzy

optimal solution of the fully fuzzy linear programming problem $(P_{3.1})$. So, find a set of *LR*

flat fuzzy numbers $\{\tilde{x}_j ; j = 1, 2, \dots, n : \mathfrak{R}(\tilde{x}_j) = x_j\}$.

Step 6

Put $\{\tilde{x}_j ; j = 1, 2, \dots, n\}$, obtained from Step 5, in $\sum_{j=1}^n \tilde{c}_j \otimes_M \tilde{x}_j$, to find the fuzzy optimal value of fully fuzzy linear programming problem $(P_{3.1})$.

4.2 Advantages of the proposed alternative method

To find the fuzzy optimal solution of fully fuzzy linear programming problems by the method proposed in previous chapter as well as the method proposed in this chapter, there is need to convert the fully fuzzy linear programming problems into the crisp linear programming problems. Since, the number of variables in the crisp linear programming problem obtained by the method proposed in previous chapter is four times the number of variables in crisp linear programming problem obtained by the method proposed in this chapter. So, it is much easy to apply the method proposed in this chapter as compared to the method proposed in previous chapter for solving the fully fuzzy linear programming problems.

To illustrate the proposed method, both the fully fuzzy linear programming problems chosen in Example 2.3 and Example 3.1, are solved by the method proposed in this chapter.

4.2.1 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in

Example 2.3

Using the method, proposed in Section 4.1, the fuzzy optimal solution of fully fuzzy linear programming problem, chosen in Example 2.3, can be obtained as follows:

Step 1

Using Step 1 of the method, proposed in Section 4.1, the fully fuzzy linear programming problem $(E_{2.5})$ can be written as:

Maximize

$$\mathfrak{R}(2,3,2,5)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}(1,2,1,3)_{LR} \mathfrak{R}(\tilde{x}_2) + \mathfrak{R}(3,4,3,4)_{LR} \mathfrak{R}(\tilde{x}_3)$$

Subject to (E_{4.1})

$$\mathfrak{R}\left(2,3,1,\frac{1}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}\left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_2) + \mathfrak{R}(3,4,1,1)_{LR} \mathfrak{R}(\tilde{x}_3) \leq \mathfrak{R}(5,7,2,2)_{LR}$$

$$\mathfrak{R}(2,3,2,5)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}\left(1,3,1,\frac{1}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_2) \leq \mathfrak{R}(3,5,1,1)_{LR}$$

$$\mathfrak{R}(\tilde{x}_1) \geq 0, \mathfrak{R}(\tilde{x}_2) \geq 0, \mathfrak{R}(\tilde{x}_3) \geq 0.$$

Step 2

Assuming $\mathfrak{R}(\tilde{x}_j) = \mathfrak{R}(m_j, n_j, \alpha_j, \beta_j)_{LR} = x_j$, the problem (E_{4.1}) can be written as:

Maximize

$$\mathfrak{R}(2,3,2,5)_{LR} x_1 + \mathfrak{R}(1,2,1,3)_{LR} x_2 + \mathfrak{R}(3,4,3,4)_{LR} x_3$$

Subject to (E_{4.2})

$$\mathfrak{R}\left(2,3,1,\frac{1}{2}\right)_{LR} x_1 + \mathfrak{R}\left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} x_2 + \mathfrak{R}(3,4,1,1)_{LR} x_3 \leq \mathfrak{R}(5,7,2,2)_{LR}$$

$$\mathfrak{R}(2,3,2,5)_{LR} x_1 + \mathfrak{R}\left(1,3,1,\frac{1}{2}\right)_{LR} x_2 \leq \mathfrak{R}(3,5,1,1)_{LR}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Step 3

Using Lemma 2.1, the problem (E_{4.2}) can be written as:

Maximize

$$\left(\frac{21}{6} x_1 + \frac{13}{6} x_2 + \frac{23}{6} x_3 \right)$$

Subject to (E_{4.3})

$$\frac{7}{3} x_1 + \frac{7}{3} x_2 + \frac{7}{2} x_3 \leq 12$$

$$\frac{7}{2} x_1 + \frac{11}{6} x_2 \leq 8$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Step 4

The optimal solution of the crisp linear programming problem ($E_{4.3}$) is:

$$x_1 = 1.13, x_2 = 0, x_3 = 0.94.$$

Step 5

Since, $x_1 = 1.13, x_2 = 0$ and $x_3 = 0.94$. So, every set of *LR* flat fuzzy numbers $\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 : \mathfrak{R}(\tilde{x}_1) = 1.13, \mathfrak{R}(\tilde{x}_2) = 0, \mathfrak{R}(\tilde{x}_3) = 0.94\}$ will be the fuzzy optimal solution of the chosen problem. e.g., $\{\tilde{x}_1 = (0.85, 1.42, 0.28, 0.28)_{LR}, \tilde{x}_2 = \tilde{0}, \tilde{x}_3 = (0.47, 1.42, 0.76, 0.76)_{LR}\}$ is one of the fuzzy optimal solutions of the chosen problem.

Step 6

Substituting the values of \tilde{x}_1, \tilde{x}_2 and \tilde{x}_3 , obtained from Step 5, in $[(2, 3, 2, 5)_{LR} \otimes_M \tilde{x}_1 \oplus (1, 2, 1, 3)_{LR} \otimes_M \tilde{x}_2 \oplus (3, 4, 3, 4)_{LR} \otimes_M \tilde{x}_3]$ the fuzzy optimal value of fully fuzzy linear programming problem ($E_{2.5}$) is $(4.82, 6.82, 3.92, 3.92)_{LR}$.

4.2.2 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in

Example 3.1

Using the method, proposed in Section 4.1, the fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 3.1, can be obtained as follows:

Step 1

Using Step 1 of the method, proposed in Section 4.1, the fully fuzzy linear programming problem ($E_{3.1}$) can be written as:

Maximize

$$\mathfrak{R}(2, 3, 2, 5)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}(1, 2, 1, 3)_{LR} \mathfrak{R}(\tilde{x}_2) + \mathfrak{R}(3, 4, 3, 4)_{LR} \mathfrak{R}(\tilde{x}_3)$$

Subject to

($E_{4.4}$)

$$\mathfrak{R}\left(2,3,1,\frac{1}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}\left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_2) + \mathfrak{R}(3,4,1,1)_{LR} \mathfrak{R}(\tilde{x}_3) \leq \mathfrak{R}(5,7,2,2)_{LR}$$

$$\mathfrak{R}(2,3,2,5)_{LR} \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}\left(1,3,1,\frac{1}{2}\right)_{LR} \mathfrak{R}(\tilde{x}_2) \leq \mathfrak{R}(3,5,1,1)_{LR}$$

$$\mathfrak{R}(\tilde{x}_1) \geq 0, \mathfrak{R}(\tilde{x}_2) \geq 0, \mathfrak{R}(\tilde{x}_3) \geq 0.$$

Step 2

Assuming $\mathfrak{R}(\tilde{x}_j) = \mathfrak{R}(m_j, n_j, \alpha_j, \beta_j)_{LR} = x_j$, the problem ($E_{4.4}$) can be written as:

Maximize

$$\mathfrak{R}(2,3,2,5)_{LR} x_1 + \mathfrak{R}(1,2,1,3)_{LR} x_2 + \mathfrak{R}(3,4,3,4)_{LR} x_3$$

Subject to

($E_{4.5}$)

$$\mathfrak{R}\left(2,3,1,\frac{1}{2}\right)_{LR} x_1 + \mathfrak{R}\left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} x_2 + \mathfrak{R}(3,4,1,1)_{LR} x_3 \leq \mathfrak{R}(5,7,2,2)_{LR}$$

$$\mathfrak{R}(2,3,2,5)_{LR} x_1 + \mathfrak{R}\left(1,3,1,\frac{1}{2}\right)_{LR} x_2 \leq \mathfrak{R}(3,5,1,1)_{LR}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Step 3

Using Lemma 3.1, the problem ($E_{4.5}$) can be written as:

Maximize

$$\left(\frac{37}{12} x_1 + \frac{23}{12} x_2 + \frac{7}{2} x_3 \right)$$

Subject to

($E_{4.6}$)

$$\frac{55}{12} x_1 + \frac{53}{12} x_2 + \frac{41}{6} x_3 \leq \frac{35}{3}$$

$$\frac{37}{6} x_1 + \frac{43}{12} x_2 \leq \frac{47}{6}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Step 4

The optimal solution of the crisp linear programming problem ($E_{4.6}$) is:

$$x_1 = 0.63, x_2 = 0, x_3 = 0.42.$$

Step 5

Since, $x_1 = 0.63, x_2 = 0$ and $x_3 = 0.42$. So, every set of LR flat fuzzy numbers $\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 : \mathfrak{R}(\tilde{x}_1) = 0.63, \mathfrak{R}(\tilde{x}_2) = 0, \mathfrak{R}(\tilde{x}_3) = 0.85\}$ will be the fuzzy optimal solution of the chosen problem. e.g., $\{\tilde{x}_1 = (0.95, 0.95, 0.95, 0)_{LR}, \tilde{x}_2 = (0, 0, 0, 0)_{LR}, \tilde{x}_3 = (0, 0, 0, 1.71)_{LR}\}$ is one of the fuzzy optimal solutions of the chosen problem.

Step 6

Substituting the values of \tilde{x}_1, \tilde{x}_2 and \tilde{x}_3 , obtained from Step 5, in $[(2, 3, 2, 5)_{LR} \otimes_M \tilde{x}_1 \oplus (1, 2, 1, 3)_{LR} \otimes_M \tilde{x}_2 \oplus (3, 4, 3, 4)_{LR} \otimes_M \tilde{x}_3]$ the fuzzy optimal value of fully fuzzy linear programming problem ($E_{3.1}$) is $(2.52, 3.57, 2.52, 4.83)_{LR}$.

4.3 Comparative study

The results of the fully fuzzy linear programming problems chosen in Example 2.3 and Example 3.1, obtained by using the method proposed in previous chapter and the method proposed in this chapter, are shown in Table 4.1. Also, the number of variables in the crisp linear programming problem corresponding to the fully fuzzy linear programming problem obtained with the help of method, proposed in previous chapter, and the method, proposed in this chapter, are shown in Table 4.2.

Table 4.1 Fuzzy optimal solution

Example	Method proposed in previous chapter	Method proposed in this chapter
2.3	$\tilde{x}_1 =_{\mathfrak{R}} (0.85, 1.42, 0.28, 0.28)_{LR}$, $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0)_{LR}$, $\tilde{x}_3 =_{\mathfrak{R}} (0.47, 1.42, 0.76, 0.76)_{LR}$	$\tilde{x}_1 =_{\mathfrak{R}} (0.85, 1.42, 0.28, 0.28)_{LR}$, $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0)_{LR}$, $\tilde{x}_3 =_{\mathfrak{R}} (0.47, 1.42, 0.76, 0.76)_{LR}$
3.1	$\tilde{x}_1 =_{\mathfrak{R}} (0.95, 0.95, 0.95, 0)_{LR}$, $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR}$, $\tilde{x}_3 =_{\mathfrak{R}} (0, 0, 0, 1.71)_{LR}$	$\tilde{x}_1 =_{\mathfrak{R}} (0.95, 0.95, 0.95, 0)_{LR}$, $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR}$, $\tilde{x}_3 =_{\mathfrak{R}} (0, 0, 0, 1.71)_{LR}$

Table 4.2 Number of variables

Example	Number of variables in the crisp linear programming problem obtained by the method proposed in previous chapter	Number of variables in the crisp linear programming problem obtained by the method proposed in this chapter
2.3	12	3
3.1	12	3

It is obvious from Table 4.1 and Table 4.2 that on solving the fully fuzzy linear programming problems by using the method, proposed in previous chapter, and the method, proposed in this chapter, same fuzzy optimal solution is obtained. However, the number of crisp variables in the crisp linear programming problem corresponding to fully fuzzy linear programming problem, obtained by using the method, proposed in previous chapter, are more than the number of variables in the crisp linear programming problem corresponding to fully fuzzy linear programming problem obtained by using the method proposed in this chapter.

4.4 Conclusions

On the basis of presented study, it can be concluded that it is better to use the method proposed in this chapter as compared to the method proposed in previous chapter for solving fully fuzzy linear programming problems.

Chapter 5

Novel method for solving interval-valued fully fuzzy linear programming problems[§]

Su [115] pointed out that the values of the membership of an element in a fuzzy set are represented by a real number in $[0, 1]$ but a specialist is always uncertain about the values of the membership. Hence, it is better to use interval-valued fuzzy set in which the values of the membership of an element in a set is represented by the interval of possible real numbers instead of real numbers and proposed a method for solving such linear programming problems in which all the parameters except the decision variables are represented by interval-valued fuzzy numbers. In this chapter, the limitations of this method are pointed out. Also, it is pointed out that to overcome the limitations of the existing method [115] there is need to define such multiplication of two interval-valued LR flat fuzzy numbers \tilde{A} and \tilde{B} for which the property $\Re(\tilde{A} \otimes \tilde{B}) = \Re(\tilde{A})\Re(\tilde{B})$ is satisfied. Due to the same reason, in this chapter, a new multiplication, named as Mehar multiplication, of interval-valued LR flat fuzzy numbers is defined and using it a new method is proposed for solving the interval-valued fully fuzzy linear programming problems.

5.1 Preliminaries

In this section, some basic definitions, arithmetic operations and an existing method for comparing interval-valued LR flat fuzzy numbers are presented.

5.1.1 Basic definitions

In this section, some basic definitions are presented [115].

[§] The contents of this chapter are communicated in *Journal of Applied Research and Technology*.

Definition 5.1 If the membership function of the fuzzy set $\tilde{A}=(m, n, \alpha, \beta)_{LR}$ on \mathbb{R} is

$$\mu_{\tilde{A}}(x)=\begin{cases} \lambda L\left(\frac{m-x}{\alpha}\right), & x \leq m, \alpha > 0 \\ \lambda R\left(\frac{x-n}{\beta}\right), & x \geq n, \beta > 0 \\ \lambda, & m \leq x \leq n \end{cases}$$

$m < n$ then \tilde{A} is called λ LR flat fuzzy number, $0 < \lambda \leq 1$ and is denoted as

$$\tilde{A}=(m, n, \alpha, \beta; \lambda)_{LR}.$$

Let $F_N(\lambda)$ be the family of all λ LR flat fuzzy numbers, i.e.

$$F_N(\lambda)=\{(m, n, \alpha, \beta; \lambda)_{LR} : \forall m < n, \beta \geq 0, \alpha \geq 0, m, n, \alpha, \beta \in \mathbb{R}\}, \quad 0 < \lambda \leq 1.$$

Definition 5.2 An interval-valued fuzzy set \tilde{A} on \mathbb{R} is given by $\tilde{A}=\{(x, [\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x)])$

$: x \in \mathbb{R}\}$, where $\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x) \in [0, 1]$ and $\mu_{\tilde{A}^L}(x) \leq \mu_{\tilde{A}^U}(x) \forall x \in \mathbb{R}$ and is denoted as

$$\tilde{A}=[\tilde{A}^L, \tilde{A}^U].$$

This means that the grade of membership of x belongs to the interval $[\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x)]$, the

least grade of membership at x is $\mu_{\tilde{A}^L}(x)$ and greatest grade of membership at x is $\mu_{\tilde{A}^U}(x)$.

Let

$$\mu_{\tilde{A}^L}(x)=\begin{cases} \lambda L\left(\frac{m-x}{\alpha'}\right), & x \leq m, \alpha' > 0 \\ \lambda R\left(\frac{x-n}{\beta'}\right), & x \geq n, \beta' > 0 \\ \lambda, & m \leq x \leq n \\ 0, & \text{otherwise} \end{cases}$$

Then, $\tilde{A}^L=(m, n, \alpha', \beta'; \lambda)_{LR}$.

Let

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \rho L\left(\frac{m-x}{\alpha}\right), & x \leq m, \alpha > 0 \\ \rho R\left(\frac{x-n}{\beta}\right), & x \geq n, \beta > 0 \\ \rho & , \quad m \leq x \leq n \\ 0 & , \quad otherwise \end{cases}$$

Then, $\tilde{A}^U = (m, n, \alpha, \beta; \rho)_{LR}$.

Here $0 < \lambda \leq \rho \leq 1$ and $m - \alpha < m - \alpha' < m < n < n + \beta' < n + \beta$. The interval-valued *LR* flat fuzzy number $\tilde{A} = [(m, n, \alpha', \beta'; \lambda)_{LR}, (m, n, \alpha, \beta; \rho)_{LR}]$ is called (λ, ρ) interval-valued *LR* flat fuzzy number.

Remark 5.1 If $\lambda = \rho$, $\alpha = \alpha'$ and $\beta = \beta'$ then $\tilde{A} = [\tilde{A}^L, \tilde{A}^U]$ is reduced to λ *LR* flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta; \lambda)_{LR}$.

5.1.2 Arithmetic operations

In this section, arithmetic operations between (λ, ρ) interval-valued *LR* flat fuzzy numbers are presented [115].

Let $\tilde{A} = [(m_1, n_1, \alpha_1', \beta_1'; \lambda)_{LR}, (m_1, n_1, \alpha_1, \beta_1; \rho)_{LR}]$ and $\tilde{B} = [(m_2, n_2, \alpha_2', \beta_2'; \lambda)_{LR}, (m_2, n_2, \alpha_2, \beta_2; \rho)_{LR}]$ be two interval-valued *LR* flat fuzzy numbers and $\tilde{C} = [(m_3, n_3, \alpha_3', \beta_3'; \lambda)_{RL}, (m_3, n_3, \alpha_3, \beta_3; \rho)_{RL}]$ be an interval-valued *RL* flat fuzzy number. Then,

- (i) $\tilde{A} \oplus \tilde{B} = [(m_1 + m_2, n_1 + n_2, \alpha_1' + \alpha_2', \beta_1' + \beta_2'; \lambda)_{LR}, (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2; \rho)_{LR}]$
- (ii) $\tilde{A} \ominus \tilde{C} = [(m_1 - n_3, n_1 - m_3, \alpha_1' + \beta_3', \beta_1' + \alpha_3'; \lambda)_{LR}, (m_1 - n_3, n_1 - m_3, \alpha_1 + \beta_3, \beta_1 + \alpha_3; \rho)_{LR}]$
- (iii) $\tilde{A} \otimes \tilde{B} = [(m_1 m_2, n_1 n_2, m_2 \alpha_1' + m_1 \alpha_2' - \alpha_1' \alpha_2', n_2 \beta_1' + n_1 \beta_2' + \beta_2' \beta_1'; \lambda)_{LR}, (m_1 m_2, n_1 n_2, m_2 \alpha_1 + m_1 \alpha_2 - \alpha_1 \alpha_2, n_2 \beta_1 + n_1 \beta_2 + \beta_2 \beta_1; \rho)_{LR}]$
- (iv) If $k \geq 0$ then $k\tilde{A} = [(km_1, kn_1, k\alpha_1', k\beta_1'; \lambda)_{LR}, (km_1, kn_1, k\alpha_1, k\beta_1; \rho)_{LR}]$
- (v) If $k \leq 0$ then $k\tilde{A} = [(kn_1, km_1, -k\beta_1', -k\alpha_1'; \lambda)_{RL}, (kn_1, km_1, -k\beta_1, -k\alpha_1; \rho)_{RL}]$

5.1.3 Comparison of interval-valued LR flat fuzzy numbers

In this section, the existing method [115] for comparing interval-valued fuzzy numbers is presented.

Let $\tilde{A}=[(m_1, n_1, \alpha_1', \beta_1'; \lambda)_{LR}, (m_1, n_1, \alpha_1, \beta_1; \rho)_{LR}]$ and $\tilde{B}=[(m_2, n_2, \alpha_2', \beta_2'; \lambda)_{LR}, (m_2, n_2, \alpha_2, \beta_2; \rho)_{LR}]$ are two (λ, ρ) interval-valued LR flat fuzzy numbers. Then,

- (i) $\tilde{A} >_{\mathfrak{R}} \tilde{B}$ if $\mathfrak{R}(\tilde{A}) > \mathfrak{R}(\tilde{B})$
- (ii) $\tilde{A} =_{\mathfrak{R}} \tilde{B}$ if $\mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$
- (iii) $\tilde{A} <_{\mathfrak{R}} \tilde{B}$ if $\mathfrak{R}(\tilde{A}) < \mathfrak{R}(\tilde{B})$

where,

$$\begin{aligned} \mathfrak{R}(\tilde{A}) = & \frac{1}{\lambda} \int_0^{\lambda} \frac{1}{4} \left[2m_1 + 2n_1 + \beta_1' R^{-1}\left(\frac{\alpha}{\lambda}\right) + \beta_1 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_1' L^{-1}\left(\frac{\alpha}{\lambda}\right) - \alpha_1 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha + \\ & \frac{1}{\rho - \lambda} \int_{\lambda}^{\rho} \frac{1}{2} \left[m_1 + n_1 + \beta_1 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_1 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha \end{aligned} \quad (5.1)$$

$$\begin{aligned} \mathfrak{R}(\tilde{B}) = & \frac{1}{\lambda} \int_0^{\lambda} \frac{1}{4} \left[2m_2 + 2n_2 + \beta_2' R^{-1}\left(\frac{\alpha}{\lambda}\right) + \beta_2 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_2' L^{-1}\left(\frac{\alpha}{\lambda}\right) - \alpha_2 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha + \\ & \frac{1}{\rho - \lambda} \int_{\lambda}^{\rho} \frac{1}{2} \left[m_2 + n_2 + \beta_2 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_2 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha \end{aligned} \quad (5.2)$$

Lemma 5.1 If $L(x) = R(x) = \text{Maximum}\{0, 1-x\}$ then,

$$\mathfrak{R}(\tilde{A}) = \frac{1}{8} \left(8m_1 + 8n_1 + \beta_1' - \alpha_1' + \left(4 - \frac{3\lambda}{\rho}\right)(\beta_1 - \alpha_1) \right).$$

Proof: Since, $L(x) = R(x) = \text{Maximum}\{0, 1-x\}$ so $L^{-1}(\lambda) = R^{-1}(\lambda) = 1 - \lambda$ where, $\lambda \in [0, 1]$.

Substituting the values of $L^{-1}(\lambda)$ and $R^{-1}(\lambda)$ in ranking formula,

$$\begin{aligned} \mathfrak{R}(\tilde{A}) = & \frac{1}{\lambda} \int_0^{\lambda} \frac{1}{4} \left[2m_1 + 2n_1 + \beta_1' R^{-1}\left(\frac{\alpha}{\lambda}\right) + \beta_1 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_1' L^{-1}\left(\frac{\alpha}{\lambda}\right) - \alpha_1 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha + \frac{1}{\rho - \lambda} \\ & \int_{\lambda}^{\rho} \frac{1}{2} \left[m_1 + n_1 + \beta_1 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_1 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha = \frac{1}{8} \left(8m_1 + 8n_1 + \beta_1' - \alpha_1' + \left(4 - \frac{3\lambda}{\rho}\right)(\beta_1 - \alpha_1) \right). \end{aligned}$$

5.2 Limitation of the existing method

The existing method [115] can be used only for solving fuzzy linear programming problems ($P_{5.1}$) in which all the parameters except the decision variables are represented by interval-valued fuzzy numbers. However, the same method can not be used for solving interval-valued fully fuzzy linear programming problems ($P_{5.2}$).

$$(i) \text{ Maximize (or Minimize) } \sum_{j=1}^n \tilde{c}_j x_j$$

Subject to ($P_{5.1}$)

$$\sum_{j=1}^n \tilde{a}_{ij} x_j \leq_{\mathfrak{R}} \text{ or } = \text{ or } \geq_{\mathfrak{R}} \tilde{b}_i; i = 1, 2, \dots, m$$

$$\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}, j = 1, 2, \dots, n$$

where, \tilde{c}_j , \tilde{b}_i and \tilde{a}_{ij} are interval-valued fuzzy numbers, \mathfrak{R} is a linear ranking function.

Example 5.1 [115 pp. 103] Consider the interval-valued fuzzy linear programming problem:

Maximize

$$[(19, 25, 33; .9), (18, 25, 34; 1)]x_1 \oplus [(44, 48, 54; .9), (43, 48, 56; 1)]x_2$$

Subject to ($E_{5.1}$)

$$[(14, 15, 17; .9), (10, 15, 18; 1)]x_1 \oplus [(25, 30, 34; .9), (23, 30, 38; 1)]x_2 \leq_{\mathfrak{R}} [(430, 450, 480; .9), (420, 450, 520; 1)]$$

$$[(21, 24, 26; .9), (20, 24, 33; 1)]x_1 \oplus [(4, 6, 8; .9), (2, 6, 11; 1)]x_2 \leq_{\mathfrak{R}} [(220, 240, 290; .9), (180, 240, 300; 1)]$$

$$[(17, 21, 22; .9), (16, 21, 26; 1)]x_1 \oplus [(12, 14, 19; .9), (8, 14, 22; 1)]x_2 \leq_{\mathfrak{R}} [(270, 280, 310; .9), (230, 280, 320; 1)]$$

$$x_j \geq 0; j = 1, 2$$

$$(ii) \text{ Maximize (or Minimize) } \sum_{j=1}^n \tilde{c}_j \otimes_M \tilde{x}_j$$

Subject to ($P_{5.2}$)

$$\sum_{j=1}^n \tilde{a}_{ij} \otimes_M \tilde{x}_j \leq_{\mathfrak{R}} \text{or} = \text{or} \geq_{\mathfrak{R}} \tilde{b}_i; i = 1, 2, \dots, m$$

$$\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}, j = 1, 2, \dots, n$$

where, $\tilde{c}_j = [(c_j, c_j', \delta_{j2}, \delta_{j3}; \lambda)_{LR}, (c_j, c_j', \delta_{j1}, \delta_{j4}; \rho)_{LR}]$, $\tilde{a}_{ij} = [(a_{ij}, a_{ij}', \delta_{ij2}, \delta_{ij3}; \lambda)_{LR}, (a_{ij}, a_{ij}', \delta_{ij1}, \delta_{ij4}; \rho)_{LR}]$, $\tilde{b}_i = [(b_i, b_i', \alpha_{i2}, \alpha_{i3}; \lambda)_{LR}, (b_i, b_i', \alpha_{i1}, \alpha_{i4}; \rho)_{LR}]$, $\tilde{x}_j = [(m_j, n_j, \alpha_j', \beta_j'; \lambda)_{LR}, (m_j, n_j, \alpha_j, \beta_j; \rho)_{LR}]$, $0 < \delta_{j2} < \delta_{j1} < c_j < c_j'$, $0 < \delta_{j3} < \delta_{j4}$, $0 < \delta_{ij2} < \delta_{ij1} < a_{ij} < a_{ij}'$, $0 < \delta_{ij3} < \delta_{ij4}$, $0 < \alpha_{i2} < \alpha_{i1} < b_i < b_i'$, $0 < \alpha_{i3} < \alpha_{i4}$, $0 < \alpha_j' < \alpha_j < m_j < n_j$, $0 < \beta_j' < \beta_j$; $j = 1, 2, \dots, n$; $0 < \lambda < 1$; $i = 1, 2, \dots, m$.

Example 5.2 Consider the interval-valued fully fuzzy linear programming problem:

Maximize

$$([(25, 27, 6, 8; 9)_{LR}, (25, 27, 7, 9; 1)_{LR}] \otimes_M \tilde{x}_1 \oplus [(48, 50, 4, 6; 9)_{LR}, (48, 50, 5, 8; 1)_{LR}] \otimes_M \tilde{x}_2)$$

Subject to

(E_{5.2})

$$[(15, 17, 1, 2; 9)_{LR}, (15, 17, 5, 3; 1)_{LR}] \otimes_M \tilde{x}_1 \oplus [(30, 32, 5, 4; 9)_{LR}, (30, 32, 7, 8; 1)_{LR}] \otimes_M \tilde{x}_2 \leq_{\mathfrak{R}} [(450, 470, 20, 30; 9)_{LR}, (450, 470, 30, 70; 1)_{LR}]$$

$$[(24, 26, 3, 2; 9)_{LR}, (24, 26, 4, 9; 1)_{LR}] \otimes_M \tilde{x}_1 \oplus [(6, 8, 2, 2; 9)_{LR}, (6, 8, 4, 5; 1)_{LR}] \otimes_M \tilde{x}_2 \leq_{\mathfrak{R}} [(240, 260, 20, 50; 9)_{LR}, (240, 260, 60, 60; 1)_{LR}]$$

$$[(21, 23, 4, 1; 9)_{LR}, (21, 23, 5, 5; 1)_{LR}] \otimes_M \tilde{x}_1 \oplus [(14, 16, 2, 5; 9)_{LR}, (14, 16, 6, 8; 1)_{LR}] \otimes_M \tilde{x}_2 \leq_{\mathfrak{R}} [(280, 300, 10, 30; 9)_{LR}, (280, 300, 50, 40; 1)_{LR}]$$

$$\tilde{x}_1 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_2 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_3 \geq_{\mathfrak{R}} \tilde{0}$$

where, $L(x) = R(x) = \text{Maximum}\{0, 1 - x\}$.

5.3 Proposed Mehar multiplication

In this section, a new multiplication, named as Mehar multiplication, by modifying the multiplication of *LR* flat fuzzy numbers defined in Chapter 2, is defined for interval-valued *LR* flat fuzzy numbers.

If $\tilde{A} = [(m_1, n_1, \alpha_1', \beta_1'; \lambda)_{LR}, (m_1, n_1, \alpha_1, \beta_1; \rho)_{LR}]$ and $\tilde{B} = [(m_2, n_2, \alpha_2', \beta_2'; \lambda)_{LR}, (m_2, n_2, \alpha_2, \beta_2; \rho)_{LR}]$ are two interval-valued *LR* flat fuzzy numbers. Then,

$$\tilde{A} \otimes_M \tilde{B} = \begin{cases} [(m_1 \mathfrak{R}(\tilde{B}), n_1 \mathfrak{R}(\tilde{B}), \alpha_1 \mathfrak{R}(\tilde{B}), \beta_1 \mathfrak{R}(\tilde{B}); \lambda)_{LR}, (m_1 \mathfrak{R}(\tilde{B}), n_1 \mathfrak{R}(\tilde{B}), \alpha_1 \mathfrak{R}(\tilde{B}), \beta_1 \mathfrak{R}(\tilde{B}); \rho)_{LR}], & \mathfrak{R}(\tilde{B}) \geq 0 \\ [(n_1 \mathfrak{R}(\tilde{B}), m_1 \mathfrak{R}(\tilde{B}), -\beta_1 \mathfrak{R}(\tilde{B}), -\alpha_1 \mathfrak{R}(\tilde{B}); \lambda)_{RL}, (n_1 \mathfrak{R}(\tilde{B}), m_1 \mathfrak{R}(\tilde{B}), -\beta_1 \mathfrak{R}(\tilde{B}), -\alpha_1 \mathfrak{R}(\tilde{B}); \rho)_{RL}], & \mathfrak{R}(\tilde{B}) \leq 0 \end{cases}$$

$$\text{where, } \mathfrak{R}(\tilde{B}) = \frac{1}{\lambda} \int_0^\lambda \frac{1}{4} \left[2m_2 + 2n_2 + \beta_2 R^{-1}\left(\frac{\alpha}{\lambda}\right) + \beta_2 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_2 L^{-1}\left(\frac{\alpha}{\lambda}\right) - \alpha_2 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha + \frac{1}{\rho - \lambda}$$

$$\int_\lambda^\rho \frac{1}{2} \left[m_2 + n_2 + \beta_2 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_2 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha.$$

5.4 Proposed method

In this section, a new method, by modifying the method proposed in previous chapter, is proposed for solving interval-valued fully fuzzy linear programming problem ($P_{5.2}$).

The steps of the proposed method are as follows:

Step 1

Since, for the ranking function \mathfrak{R} the property $\mathfrak{R}\left(\sum_{j=1}^n \tilde{c}_j\right) = \sum_{j=1}^n \mathfrak{R}(\tilde{c}_j)$ is satisfied so the fully

fuzzy linear programming problem ($P_{5.2}$) can be written as:

$$\text{Maximize (or Minimize)} \quad \sum_{j=1}^n \mathfrak{R}(\tilde{c}_j \otimes_M \tilde{x}_j),$$

$$\text{Subject to} \quad (P_{5.3})$$

$$\sum_{j=1}^n \mathfrak{R}(\tilde{a}_{ij} \otimes_M \tilde{x}_j) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b}_i); i = 1, 2, \dots, m$$

$$\tilde{x}_j \geq_{\mathfrak{R}} \tilde{0}, j = 1, 2, \dots, n.$$

Step 2

Since, for the Mehar multiplication, proposed in Section 5.3, the property

$\mathfrak{R}(\tilde{A} \otimes_M \tilde{B}) = \mathfrak{R}(\tilde{A})\mathfrak{R}(\tilde{B})$ is satisfied. So, the crisp linear programming problem ($P_{5.3}$) can be

written as:

$$\text{Maximize (or Minimize)} \quad \sum_{j=1}^n \mathfrak{R}(\tilde{c}_j)\mathfrak{R}(\tilde{x}_j),$$

Subject to (P_{5.4})

$$\sum_{j=1}^n \mathfrak{R}(\tilde{a}_{ij})\mathfrak{R}(\tilde{x}_j) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b}_i); i = 1, 2, \dots, m$$

$$\mathfrak{R}(\tilde{x}_j) \geq \tilde{0}, j = 1, 2, \dots, n.$$

Step 3

Assuming $\tilde{c}_j = [(c_j, c_j', \delta_{j2}, \delta_{j3}; \lambda)_{LR}, (c_j, c_j', \delta_{j1}, \delta_{j4}; \rho)_{LR}]$, $\tilde{a}_{ij} = [(a_{ij}, a_{ij}', \delta_{ij2}, \delta_{ij3}; \lambda)_{LR}, (a_{ij}, a_{ij}', \delta_{ij1}, \delta_{ij4}; \rho)_{LR}]$, $\tilde{b}_i = [(b_i, b_i', \alpha_{i2}, \alpha_{i3}; \lambda)_{LR}, (b_i, b_i', \alpha_{i1}, \alpha_{i4}; \rho)_{LR}]$, the problem (P_{5.4}) can be written as:

$$\text{Maximize (or Minimize)} \quad \sum_{j=1}^n \mathfrak{R}[(c_j, c_j', \delta_{j2}, \delta_{j3}; \lambda)_{LR}, (c_j, c_j', \delta_{j1}, \delta_{j4}; \rho)_{LR}] \mathfrak{R}(\tilde{x}_j)$$

Subject to (P_{5.5})

$$\sum_{j=1}^n \mathfrak{R}[(a_{ij}, a_{ij}', \delta_{ij2}, \delta_{ij3}; \lambda)_{LR}, (a_{ij}, a_{ij}', \delta_{ij1}, \delta_{ij4}; \rho)_{LR}] \mathfrak{R}(\tilde{x}_j) \leq \text{or} = \text{or} \geq \mathfrak{R}[(b_i, b_i', \alpha_{i2}, \alpha_{i3}; \lambda)_{LR},$$

$$(b_i, b_i', \alpha_{i1}, \alpha_{i4}; \rho)_{LR}]; i = 1, 2, \dots, m$$

$$\mathfrak{R}(\tilde{x}_j) \geq \tilde{0}, j = 1, 2, \dots, n.$$

Step 4

Assuming $\mathfrak{R}(\tilde{x}_j) = x_j$ the problem (P_{5.5}) can be written as:

$$\text{Maximize (or Minimize)} \quad \sum_{j=1}^n \mathfrak{R}[(c_j, c_j', \delta_{j2}, \delta_{j3}; \lambda)_{LR}, (c_j, c_j', \delta_{j1}, \delta_{j4}; \rho)_{LR}] x_j$$

Subject to (P_{5.6})

$$\sum_{j=1}^n \mathfrak{R}[(a_{ij}, a_{ij}', \delta_{ij2}, \delta_{ij3}; \lambda)_{LR}, (a_{ij}, a_{ij}', \delta_{ij1}, \delta_{ij4}; \rho)_{LR}] x_j \leq \text{or} = \text{or} \geq \mathfrak{R}[(b_i, b_i', \alpha_{i2}, \alpha_{i3}; \lambda)_{LR},$$

$$(b_i, b_i', \alpha_{i1}, \alpha_{i4}; \rho)_{LR}]; i = 1, 2, \dots, m$$

$$x_j \geq 0, j = 1, 2, \dots, n.$$

Step 5

Using the ranking formula $\Re(\tilde{A}) = \Re[(m_1, n_1, \alpha_1', \beta_1'; \lambda)_{LR}, (m_1, n_1, \alpha_1, \beta_1; \rho)_{LR}] = \frac{1}{\lambda}$

$$\int_0^\lambda \frac{1}{4} \left[2m_1 + 2n_1 + \beta_1' R^{-1}\left(\frac{\alpha}{\lambda}\right) + \beta_1 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_1' L^{-1}\left(\frac{\alpha}{\lambda}\right) - \alpha_1 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha + \frac{1}{\rho - \lambda} \int_\lambda^\rho \frac{1}{2} \left[m_1 + n_1 + \beta_1 R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_1 L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha$$

the problem $(P_{5.6})$ can be written as:

Maximize (or Minimize)

$$\frac{1}{2} \sum_{j=1}^n \left(\frac{1}{\lambda} \int_0^\lambda \frac{1}{4} \left[2c_j + 2c_j' + \delta_{j3} R^{-1}\left(\frac{\alpha}{\lambda}\right) + \delta_{j4} R^{-1}\left(\frac{\alpha}{\rho}\right) - \delta_{j2} L^{-1}\left(\frac{\alpha}{\lambda}\right) - \delta_{j1} L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha + \frac{1}{\rho - \lambda} \int_\lambda^\rho \frac{1}{2} \left[c_j + c_j' + \delta_{j4} R^{-1}\left(\frac{\alpha}{\rho}\right) - \delta_{j1} L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha \right) x_j$$

Subject to (P_{5.7})

$$\sum_{j=1}^n \left(\frac{1}{\lambda} \int_0^\lambda \frac{1}{4} \left[2a_{ij} + 2a_{ij}' + \delta_{ij3} R^{-1}\left(\frac{\alpha}{\lambda}\right) + \delta_{ij4} R^{-1}\left(\frac{\alpha}{\rho}\right) - \delta_{ij2} L^{-1}\left(\frac{\alpha}{\lambda}\right) - \delta_{ij1} L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha + \frac{1}{1 - \lambda} \int_\lambda^1 \frac{1}{2} \left[m_1 + n_1 + \delta_{ij4} R^{-1}\left(\frac{\alpha}{\rho}\right) - \delta_{ij1} L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha \right) x_j \leq \text{or } = \text{or } \geq \frac{1}{\lambda} \int_0^\lambda \frac{1}{4} \left[2b_i + 2b_i' + \alpha_{i3} R^{-1}\left(\frac{\alpha}{\lambda}\right) + \alpha_{i4} R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_{i2} L^{-1}\left(\frac{\alpha}{\lambda}\right) - \alpha_{i1} L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha + \frac{1}{1 - \lambda} \int_\lambda^1 \frac{1}{2} \left[b_i + b_i' + \alpha_{i4} R^{-1}\left(\frac{\alpha}{\rho}\right) - \alpha_{i1} L^{-1}\left(\frac{\alpha}{\rho}\right) \right] d\alpha ; i = 1, 2, \dots, m$$

$$x_j \geq 0.$$

Step 6

Solve the crisp linear programming problem $(P_{5.7})$ to find its optimal solution $\{x_j\}$.

Step 7

Since, every set of interval-valued LR flat fuzzy numbers $\{\tilde{x}_j; j = 1, 2, \dots, n : \Re(\tilde{x}_j) = x_j\}$ will be the fuzzy optimal solution of the interval-valued fully fuzzy linear programming problem $(P_{5.2})$. So, find a set of interval-valued LR flat fuzzy numbers $\{\tilde{x}_j; j = 1, 2, \dots, n : \Re(\tilde{x}_j) = x_j\}$.

Step 8

Put $\{\tilde{x}_j; j = 1, 2, \dots, n\}$, obtained from Step 7, in $\sum_{j=1}^n \tilde{c}_j \otimes_M \tilde{x}_j$, to find the fuzzy optimal value

of the interval-valued fully fuzzy linear programming problem $(P_{5.2})$.

5.5 Advantages of the proposed method

The advantages of the method, proposed in this chapter over the existing method [115] and the methods proposed in the previous chapters, is that all the problems which can be solved by the existing method [115] and the method proposed in previous chapters can also be solved by the method proposed in this chapter. However, there exist several problems which can be solved by the method proposed in this chapter but can neither be solved by the existing method [115] nor by the method proposed in previous chapter.

To show the advantages of the proposed method, the interval-valued fully fuzzy linear programming problem, chosen in Example 5.2, which can neither be solved by the existing method [115] nor by the method, proposed in previous chapter, is solved by the proposed method.

5.5.1 Fuzzy optimal solution of the interval-valued fully fuzzy linear programming problem chosen in Example 5.2

The fuzzy optimal solution of the interval-valued fully fuzzy linear programming problem, chosen in Example 5.2, can be obtained as follows:

Step 1

Using the Step 1 of the method, proposed in Section 5.4, the interval-valued fully fuzzy linear programming problem $(E_{5.2})$ can be written as:

Maximize

$$\Re([(25, 27, 6, 8; .9)_{LR}, (25, 27, 7, 9; 1)_{LR}] \otimes_M \tilde{x}_1) + \Re([(48, 50, 4, 6; .9)_{LR}, (48, 50, 5, 8; 1)_{LR}] \otimes_M \tilde{x}_2)$$

Subject to

$(E_{5.3})$

$$\begin{aligned} &\Re([(15, 17, 1, 2; .9)_{LR}, (15, 17, 5, 3; 1)_{LR}] \otimes_M \tilde{x}_1) + \Re([(30, 32, 5, 4; .9)_{LR}, (30, 32, 7, 8; 1)_{LR}] \otimes_M \tilde{x}_2) \\ &\leq \Re([(450, 470, 20, 30; .9)_{LR}, (450, 470, 30, 70; 1)_{LR}]) \end{aligned}$$

$$\begin{aligned} & \mathfrak{R}([(24, 26, 3, 2; .9)_{LR}, (24, 26, 4, 9; 1)_{LR}] \otimes_M \tilde{x}_1) + \mathfrak{R}([(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}] \otimes_M \tilde{x}_2) \\ & \leq \mathfrak{R}[(240, 260, 20, 50; .9)_{LR}, (240, 260, 60, 60; 1)_{LR}] \end{aligned}$$

$$\begin{aligned} & \mathfrak{R}([(21, 23, 4, 1; .9)_{LR}, (21, 23, 5, 5; 1)_{LR}] \otimes_M \tilde{x}_1) + \mathfrak{R}([(14, 16, 2, 5; .9)_{LR}, (14, 16, 6, 8; 1)_{LR}] \otimes_M \tilde{x}_2) \\ & \leq \mathfrak{R}[(280, 300, 10, 30; .9)_{LR}, (280, 300, 50, 40; 1)_{LR}] \end{aligned}$$

$$\tilde{x}_1 \geq_{\mathfrak{R}} \tilde{0}, \tilde{x}_2 \geq_{\mathfrak{R}} \tilde{0}.$$

Step 2

Using Step 2 of the method, proposed in Section 5.4, the crisp linear programming problem

($E_{5.3}$) can be written as:

Maximize

$$\mathfrak{R}[(25, 27, 6, 8; .9)_{LR}, (25, 27, 7, 9; 1)_{LR}] \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}[(48, 50, 4, 6; .9)_{LR}, (48, 50, 5, 8; 1)_{LR}] \mathfrak{R}(\tilde{x}_2)$$

Subject to

($E_{5.4}$)

$$\begin{aligned} & \mathfrak{R}[(15, 17, 1, 2; .9)_{LR}, (15, 17, 5, 3; 1)_{LR}] \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}[(30, 32, 5, 4; .9)_{LR}, (30, 32, 7, 8; 1)_{LR}] \mathfrak{R}(\tilde{x}_2) \\ & \leq \mathfrak{R}[(450, 470, 20, 30; .9)_{LR}, (450, 470, 30, 70; 1)_{LR}] \end{aligned}$$

$$\begin{aligned} & \mathfrak{R}[(24, 26, 3, 2; .9)_{LR}, (24, 26, 4, 9; 1)_{LR}] \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}[(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}] \mathfrak{R}(\tilde{x}_2) \\ & \leq \mathfrak{R}[(240, 260, 20, 50; .9)_{LR}, (240, 260, 60, 60; 1)_{LR}] \end{aligned}$$

$$\begin{aligned} & \mathfrak{R}[(21, 23, 4, 1; .9)_{LR}, (21, 23, 5, 5; 1)_{LR}] \mathfrak{R}(\tilde{x}_1) + \mathfrak{R}[(14, 16, 2, 5; .9)_{LR}, (14, 16, 6, 8; 1)_{LR}] \mathfrak{R}(\tilde{x}_2) \\ & \leq \mathfrak{R}[(280, 300, 10, 30; .9)_{LR}, (280, 300, 50, 40; 1)_{LR}] \end{aligned}$$

$$\mathfrak{R}(\tilde{x}_1) \geq \tilde{0}, \mathfrak{R}(\tilde{x}_2) \geq \tilde{0}.$$

Step 3

Assuming $\mathfrak{R}(\tilde{x}_j) = \mathfrak{R}(m_j, n_j, \alpha_j, \beta_j)_{LR} = x_j$, the problem ($E_{5.4}$) can be written as:

Maximize

$$\mathfrak{R}[(25, 27, 6, 8; .9)_{LR}, (25, 27, 7, 9; 1)_{LR}] x_1 \oplus \mathfrak{R}[(48, 50, 4, 6; .9)_{LR}, (48, 50, 5, 8; 1)_{LR}] x_2$$

Subject to

($E_{5.5}$)

$$\begin{aligned} & \mathfrak{R}[(15, 17, 1, 2; .9)_{LR}, (15, 17, 5, 3; 1)_{LR}] x_1 \oplus \mathfrak{R}[(30, 32, 5, 4; .9)_{LR}, (30, 32, 7, 8; 1)_{LR}] x_2 \\ & \leq \mathfrak{R}[(450, 470, 20, 30; .9)_{LR}, (450, 470, 30, 70; 1)_{LR}] \end{aligned}$$

$$\mathfrak{R}[(24, 26, 3, 2; .9)_{LR}, (24, 26, 4, 9; 1)_{LR}]x_1 \oplus \mathfrak{R}[(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}]x_2 \\ \leq \mathfrak{R}[(240, 260, 20, 50; .9)_{LR}, (240, 260, 60, 60; 1)_{LR}]$$

$$\mathfrak{R}[(21, 23, 4, 1; .9)_{LR}, (21, 23, 5, 5; 1)_{LR}]x_1 \oplus \mathfrak{R}[(14, 16, 2, 5; .9)_{LR}, (14, 16, 6, 8; 1)_{LR}]x_2 \\ \leq \mathfrak{R}[(280, 300, 10, 30; .9)_{LR}, (280, 300, 50, 40; 1)_{LR}]$$

$$x_1 \geq 0, x_2 \geq 0.$$

Step 4

Using Lemma 5.1 the problem ($E_{5,5}$) can be written as:

Maximize

$$\frac{1}{16}(420.6x_1 + 789.9x_2)$$

Subject to

($E_{5,6}$)

$$254.4x_1 + 496.3x_2 \leq 7422$$

$$405.5x_1 + 113.3x_2 \leq 4030$$

$$349x_1 + 245.6x_2 \leq 4647$$

$$x_1 \geq 0, x_2 \geq 0.$$

Step 5

The optimal solution of the crisp linear programming problem ($E_{5,6}$) is: $x_1 = 12.72$, $x_2 = 4.37$.

Step 6

Since, $x_1 = 12.72$ and $x_2 = 4.37$ So, every set of interval-valued LR flat fuzzy numbers

$\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 : \mathfrak{R}(\tilde{x}_1) = 1.13, \mathfrak{R}(\tilde{x}_2) = 0, \mathfrak{R}(\tilde{x}_3) = 0.94\}$ will be the fuzzy optimal solution of the

chosen problem. e.g., $\{\tilde{x}_1 = [(5, 7.72, 3, 3; .9)_{LR}, (5, 7.72, 4, 4; 1)_{LR}], \tilde{x}_2 = [(2, 2.37, 2, 2; .9)_{LR}, (2,$

$2.37, 4, 4; 1)_{LR}]\}$ is one of the fuzzy optimal solutions of the chosen problem.

Step 7

Substituting the values of \tilde{x}_1 and \tilde{x}_2 , obtained from Step 6, in

$([(25, 27, 6, 8; .9)_{LR}, (25, 27, 7, 9; 1)_{LR}] \otimes_M \tilde{x}_1 \oplus [(48, 50, 4, 6; .9)_{LR}, (48, 50, 5, 8; 1)_{LR}] \otimes_M \tilde{x}_2)$ the

fuzzy optimal value of the interval-valued fully fuzzy linear programming problem ($E_{5.2}$) is $[(527.76, 561.94, 93.8, 127.98; .9)_{LR}, (527.76, 561.94, 110.89, 149.44; 1)_{LR}]$.

5.6 Comparative study

The results of the fully fuzzy linear programming problems, chosen in Example 2.3 and Example 3.1, as well as the results of existing interval-valued fuzzy number linear programming problem, presented in Example 5.1 and the interval-valued fully fuzzy linear programming problem chosen in Example 5.2, obtained by using the method proposed in previous chapter and the method, proposed in Section 5.4, are shown in Table 5.1.

Table 5.1 Fuzzy optimal solution

Example	Method proposed in previous chapter	Existing method [115]	Method proposed in this chapter
2.3	$\tilde{x}_1 =_{\mathfrak{R}} (0.85, 1.42, 0.28, 0.28)_{LR}$, $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0)_{LR}$, $\tilde{x}_3 =_{\mathfrak{R}} (0.47, 1.42, 0.76, 0.76)_{LR}$	Not applicable	$\tilde{x}_1 =_{\mathfrak{R}} (0.85, 1.42, 0.28, 0.28)_{LR}$, $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0)_{LR}$, $\tilde{x}_3 =_{\mathfrak{R}} (0.47, 1.42, 0.76, 0.76)_{LR}$
3.1	$\tilde{x}_1 =_{\mathfrak{R}} (0.95, 0.95, 0.95, 0)_{LR}$, $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR}$, $\tilde{x}_3 =_{\mathfrak{R}} (0, 0, 0, 1.71)_{LR}$	Not applicable	$\tilde{x}_1 =_{\mathfrak{R}} (0.95, 0.95, 0.95, 0)_{LR}$, $\tilde{x}_2 =_{\mathfrak{R}} (0, 0, 0, 0)_{LR}$, $\tilde{x}_3 =_{\mathfrak{R}} (0, 0, 0, 1.71)_{LR}$
5.1	Not applicable	$x_1 = 475, x_2 = 1261$	$x_1 = 475, x_2 = 1261$
5.2	Not applicable	Not applicable	$\tilde{x}_1 =_{\mathfrak{R}} [(5, 7.72, 3, 3; .9)_{LR}, (5, 7.72, 4, 4; 1)_{LR}]$, $\tilde{x}_2 =_{\mathfrak{R}} [(2, 2.37, 2, 2; .9)_{LR}, (2, 2.37, 4, 4; 1)_{LR}]$,

The results, presented in Table 5.1, can be explained as follows:

- (i) The method, proposed in Chapter 4, can be used for solving fully fuzzy linear programming problems but can neither be used for solving interval-valued fuzzy linear programming problems nor for solving interval-valued fully fuzzy linear programming problems. Since, the problem, chosen in Example 2.3 and the problem, chosen in Example 3.1, are fully fuzzy linear programming problems so these problems can be solved by the method proposed in previous chapter. However, the existing problem,

presented in Example 5.1, and the problem, chosen in Example 5.2, are interval-valued fuzzy number linear programming problem and interval-valued fully fuzzy linear programming problem respectively. So, none of these problems can be solved by the method proposed in Chapter 4.

- (ii) The existing method [115] can be used for solving such interval-valued fuzzy number linear programming problems in which all the parameters except decision variables are represented by interval-valued fuzzy numbers but can neither be used for solving fully fuzzy linear programming problems nor for solving interval-valued fully fuzzy linear programming problems. Since, the existing problem, presented in Example 5.1, is an interval-valued fuzzy number linear programming problem so it can be solved by the existing method. However, the existing problem, chosen in Example 2.3, the problem, chosen in Example 3.1 and the problem, chosen in Example 5.2, are fully fuzzy linear programming problems and interval-valued fully fuzzy linear programming problems respectively. So, none of these problems can be solved by the existing method [115].
- (iii) The method, proposed in this chapter, can be used for solving fully fuzzy linear programming problems. Since, fully fuzzy linear programming problems and interval-valued fuzzy number linear programming problems are specific type of interval-valued fully fuzzy linear programming problems. So, all the existing as well as chosen problems can be solved by the method proposed in this chapter.

5.7 Conclusions

On the basis of presented study, it can be concluded that all the interval-valued fuzzy linear programming problems which can be solved by using the existing method [115] can also be solved by using the method proposed in this chapter. However, there exists several interval-valued fully fuzzy linear programming problems which can be solved by the method proposed in this chapter but can neither be solved by the using existing method [115] nor by the method proposed in previous chapters. Hence, it is better to use the method proposed in

this chapter as compared to the existing method [115] and the method proposed in the previous chapters for solving interval-valued fully fuzzy linear programming problems.

Chapter 6

Sensitivity analysis of fully fuzzy linear programming problems and fully fuzzy transportation problems**

To the best of our knowledge, till now neither there is any method in the literature to deal with the sensitivity analysis of fully fuzzy linear programming problems nor any method to deal with the sensitivity analysis of fully fuzzy transportation problems. In this chapter, two new methods, named as Mehar methods, are proposed for the same. To show the advantages of proposed methods over existing methods, some fuzzy sensitivity analysis problems which can not be solved by the existing methods [35, 65, 95] are solved by using the proposed method.

6.1 Preliminaries

In this section, some basic concepts which are needed in rest of the chapters are presented [33].

Definition 6.1 An *LR* flat fuzzy number $\tilde{A}=(m,n,\alpha,\beta)_{LR}$ is said to be zero *LR* flat fuzzy number if and only if $m=0, n=0, \alpha=0$ and $\beta=0$.

Definition 6.2 Two *LR* flat fuzzy numbers $\tilde{A}_1=(m_1,n_1,\alpha_1,\beta_1)_{LR}$, $\tilde{A}_2=(m_2,n_2,\alpha_2,\beta_2)_{LR}$ are said to be equal i.e., $\tilde{A}_1=\tilde{A}_2$ if and only if $m_1=m_2, n_1=n_2, \alpha_1=\alpha_2$ and $\beta_1=\beta_2$.

Definition 6.3 An *LR* flat fuzzy number $\tilde{A}=(m,n,\alpha,\beta)_{LR}$ is said to be non-negative *LR* flat fuzzy number if and only if $m-\alpha \geq 0$.

** Some contents of this chapter are published in *Applied Mathematical Modelling* 36 (2012) 4087-4095 and remaining are accepted for publication in *Journal of Intelligent and Fuzzy Systems*.

6.2 Limitations of the existing methods

In this section, the limitations of the existing methods [35, 65, 95] are pointed out.

The existing methods [65, 95] can be used only to deal with the sensitivity analysis of such fuzzy variable linear programming problems ($P_{6.1}$) in which the decision vector (X) and requirement vector (b) are represented by fuzzy numbers and rest of the parameters are represented by real numbers.

$$\text{Maximize (or Minimize)} \quad (C\tilde{X})$$

Subject to

$$A\tilde{X} =_{\mathfrak{R}} \tilde{b} \quad (P_{6.1})$$

$$\tilde{X} \geq_{\mathfrak{R}} \tilde{0}.$$

where, $C = [c_j]_{1 \times n}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, $\tilde{X} = [\tilde{x}_j]_{n \times 1}$, $A = [a_{ij}]_{m \times n}$, \mathfrak{R} is a linear ranking function.

Similarly, the existing method [35] can be used to deal with the sensitivity analysis of such fuzzy number linear programming problems ($P_{6.2}$) in which only the decision variables (X) are represented by real numbers and rest of the parameters are represented by fuzzy numbers.

$$\text{Maximize (or Minimize)} \quad (\tilde{C}X)$$

Subject to

$$\tilde{A}X =_{\mathfrak{R}} \tilde{b} \quad (P_{6.2})$$

$$X \geq 0.$$

where, $\tilde{C} = [\tilde{c}_j]_{1 \times n}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, $X = [x_j]_{n \times 1}$, $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$, \mathfrak{R} is a linear ranking function.

However, none of the existing methods [35, 65, 95] can be used to deal with the sensitivity analysis of fully fuzzy linear programming problem ($P_{6.3}$).

$$\text{Maximize (or Minimize)} \quad (\tilde{C}^T \otimes \tilde{X})$$

Subject to

$$\tilde{A} \otimes \tilde{X} \leq_{\mathfrak{R}} \text{ or } =_{\mathfrak{R}} \text{ or } \geq_{\mathfrak{R}} \tilde{b} \quad (P_{6.3})$$

where, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$, $\tilde{X} = [\tilde{x}_j]_{n \times 1}$, $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$, $\tilde{C}^T = [\tilde{c}_j]_{1 \times n}$, \mathfrak{R} is a linear ranking function, $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$ are non-negative LR flat fuzzy decision variables, $\tilde{c}_j = (m''_j, n''_j, \alpha''_j, \beta''_j)_{LR}$, $\tilde{b}_i = (m'_i, n'_i, \alpha'_i, \beta'_i)_{LR}$ and $\tilde{a}_{ij} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ are either non-negative or non-positive LR flat fuzzy numbers.

6.3 Proposed Mehar method for the sensitivity analysis of fully fuzzy linear programming problems

In this section, to overcome the limitations of the existing methods [35, 65, 95], discussed in Section 6.1, a new method, named as Mehar method, is proposed to deal with the sensitivity analysis of fully fuzzy linear programming problems ($P_{6.3}$). The same method can be used to deal with the sensitivity analysis of fuzzy linear programming problems ($P_{6.1}$) and ($P_{6.2}$).

The steps of proposed method are as follows:

Step 1

Use the following steps of existing method [4] to find the fuzzy optimal solution $\{\tilde{x}_j\}$ and fuzzy optimal value ($\tilde{C}^T \otimes \tilde{X}$) of the fully fuzzy linear programming problem ($P_{6.3}$).

Step 1(a)

Convert the fully fuzzy linear programming problem ($P_{6.3}$) into the crisp linear programming problem ($P_{6.4}$).

$$\text{Maximize (or Minimize)} \quad \mathfrak{R}(\tilde{C}^T \otimes \tilde{X})$$

Subject to

$$\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq \text{or } = \text{or } \geq \mathfrak{R}(\tilde{b}) \quad (P_{6.4})$$

$$m_j - \alpha_j \geq 0, m_j \leq n_j, \alpha_j \geq 0, \beta_j \geq 0 \quad j = 1, 2, 3, \dots, n.$$

Step 1(b)

Solve the crisp linear programming problem $(P_{6.4})$, obtained in Step 1, to find the optimal solution $\{m_j, n_j, \alpha_j, \beta_j\}$.

Step 1(c)

Find the fuzzy optimal solution $\tilde{X} = [\tilde{x}_j]_{n \times 1}$, by putting the values of m_j, n_j, α_j and β_j obtained from Step 1(b), in $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$, and the fuzzy optimal value by putting the values of \tilde{X} in $\tilde{C}^T \otimes \tilde{X}$.

Step 2

If any change is made in the chosen fully fuzzy linear programming problem $(P_{6.3})$ then the optimal solution of the resulting fully fuzzy linear programming problem can be obtained as follows:

Check that which of the following case is to be considered:

1. Change in fuzzy cost vector,
2. Change in fuzzy requirement vector,
3. Addition of a new fuzzy variable,
4. Addition of new fuzzy constraint,
5. Change in fuzzy coefficient matrix of the constraints.

Case 1: Change in fuzzy cost vector

If the cost vector \tilde{C}^T changes to \tilde{C}'^T in the given fully fuzzy linear programming problem $(P_{6.3})$ then replace $\Re(\tilde{C}^T \otimes \tilde{X})$ by $\Re(\tilde{C}'^T \otimes \tilde{X})$ in crisp linear programming $(P_{6.4})$ to obtain $(P_{6.5})$ and Go to Step 3.

Maximize (or Minimize) $\Re(\tilde{C}'^T \otimes \tilde{X})$

Subject to (P_{6.5})

$$\Re(\tilde{A} \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \Re(\tilde{b})$$

$$m_j - \alpha_j \geq 0, m_j \leq n_j, \alpha_j \geq 0, \beta_j \geq 0 \quad j=1,2,3,\dots,n.$$

Case 2: Change in fuzzy requirement vector

If the change in fuzzy requirement vector is made i.e., \tilde{b} is changed to \tilde{b}' in $(P_{6.3})$ then replace $\mathfrak{R}(\tilde{b})$ by $\mathfrak{R}(\tilde{b}')$ in crisp linear programming problem $(P_{6.4})$ to obtain $(P_{6.6})$ and Go to Step 3.

$$\begin{aligned} \text{Maximize (or Minimize)} \quad & \mathfrak{R}(\tilde{C}^T \otimes \tilde{X}) \\ \text{Subject to} \quad & \end{aligned} \tag{P_{6.6}}$$

$$\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b}')$$

$$m_j - \alpha_j \geq 0, m_j \leq n_j, \alpha_j \geq 0, \beta_j \geq 0 \quad j=1,2,3,\dots,n.$$

Case 3: Addition of a new fuzzy variable

Suppose a new non-negative fuzzy variable, say \tilde{x}_{n+1} , is added in $(P_{6.3})$. Assume that if \tilde{c}_{n+1} is cost and \tilde{A}_{n+1} is the column associated with \tilde{x}_{n+1} , then replace $\mathfrak{R}(\tilde{A} \otimes \tilde{X})$ by $\mathfrak{R}(\tilde{A} \otimes \tilde{X} \oplus \tilde{A}_{n+1} \otimes \tilde{x}_{n+1})$ and $\mathfrak{R}(\tilde{C}^T \otimes \tilde{X})$ by $\mathfrak{R}(\tilde{C}^T \otimes \tilde{X} \oplus \tilde{c}_{n+1} \otimes \tilde{x}_{n+1})$ in $(P_{6.4})$ to obtain the crisp linear programming problem $(P_{6.7})$ and Go to Step 3.

$$\begin{aligned} \text{Maximize (or Minimize)} \quad & \mathfrak{R}(\tilde{C}^T \otimes \tilde{X} \oplus \tilde{c}_{n+1} \otimes \tilde{x}_{n+1}) \\ \text{Subject to} \quad & \end{aligned} \tag{P_{6.7}}$$

$$\mathfrak{R}(\tilde{A} \otimes \tilde{X} \oplus \tilde{A}_{n+1} \otimes \tilde{x}_{n+1}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b})$$

$$m_j - \alpha_j \geq 0, m_j \leq n_j, \alpha_j \geq 0, \beta_j \geq 0 \quad j=1,2,3,\dots,n+1.$$

Case 4: Addition of a new fuzzy constraint

Suppose a new fuzzy constraint is added in the original fully fuzzy linear programming problem $(P_{6.3})$ then replace $\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b})$ by $\mathfrak{R}(\tilde{A}' \otimes \tilde{X}) \leq \text{or} = \text{or}$

$\geq \mathfrak{R}(\tilde{b}')$ in $(P_{6.4})$ to obtain crisp linear programming problem $(P_{6.8})$ and Go to Step 3.

$$\text{Maximize (or Minimize)} \quad \mathfrak{R}(\tilde{C}^T \otimes \tilde{X})$$

$$\text{Subject to} \quad (P_{6.8})$$

$$\mathfrak{R}(\tilde{A}' \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b}')$$

$$m_j - \alpha_j \geq 0, m_j \leq n_j, \alpha_j \geq 0, \beta_j \geq 0 \quad j = 1, 2, 3, \dots, n.$$

Case 5: Change in fuzzy coefficient matrix of the constraints

Suppose if the column of the coefficient matrix of the constraints, corresponding to the fuzzy variable \tilde{x}_j , is changed from \tilde{A}_j to \tilde{A}_j' in the original fuzzy linear programming problem $(P_{6.3})$ then replace $\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b})$ by $\mathfrak{R}(\tilde{A}' \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b})$ in $(P_{6.4})$ to obtain new crisp linear programming problem $(P_{6.9})$ and Go to Step 3.

$$\text{Maximize (or Minimize)} \quad \mathfrak{R}(\tilde{C}^T \otimes \tilde{X})$$

$$\text{Subject to} \quad (P_{6.9})$$

$$\mathfrak{R}(\tilde{A}' \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b})$$

$$m_j - \alpha_j \geq 0, m_j \leq n_j, \alpha_j \geq 0, \beta_j \geq 0 \quad j = 1, 2, 3, \dots, n.$$

Step 3

Apply the existing sensitivity analysis technique [116] to find the optimal solution of $(P_{6.5})$, $(P_{6.6})$, $(P_{6.7})$, $(P_{6.8})$ and $(P_{6.9})$ with the help of optimal solution of $(P_{6.4})$ and use Step 1(c) of the proposed method to find the fuzzy optimal solution of the resulting fully fuzzy linear programming problem.

Remark 6.1 The other cases i.e., deletion of fuzzy decision variables, deletion of fuzzy constraints, simultaneous change in coefficients of the fuzzy decision variables in objective function and fuzzy requirement vectors etc. can also be solved by using the proposed Mehar method.

6.4 Advantages of proposed Mehar method over existing methods

The main advantage of proposed Mehar method over existing methods [35, 65, 95] is that there may exist several fuzzy sensitivity analysis problems, as discussed in Section 6.1, which may or may not be solved by the existing methods but can be solved by using the proposed Mehar method.

To show the advantages of the proposed Mehar method over existing methods [35, 65, 95], the fuzzy sensitivity analysis problem, chosen in Example 6.1, which can't be solved by using any of the existing methods [35, 65, 95] is solved by using the proposed Mehar method.

Example 6.1 Consider the following fully fuzzy linear programming problem,

$$\begin{aligned} \text{Maximize} \quad & [(2,3,2,5)_{LR} \otimes \tilde{x}_1 \oplus (1,2,1,3)_{LR} \otimes \tilde{x}_2 \oplus (3,4,3,4)_{LR} \otimes \tilde{x}_3] \\ \text{Subject to} \quad & \hspace{20em} (E_{6.1}) \end{aligned}$$

$$\begin{aligned} & \left(2,3,1,\frac{1}{2}\right)_{LR} \otimes \tilde{x}_1 \oplus \left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \otimes \tilde{x}_2 \oplus (3,4,1,1)_{LR} \otimes \tilde{x}_3 \leq_{\mathfrak{R}} (5,7,2,2)_{LR} \\ & (2,3,2,5)_{LR} \otimes \tilde{x}_1 \oplus \left(1,3,1,\frac{1}{2}\right)_{LR} \otimes \tilde{x}_2 \leq_{\mathfrak{R}} (3,5,1,1)_{LR} \end{aligned}$$

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are non-negative *LR* flat fuzzy decision variables

where, $L(x) = \text{maximum}\{0, 1 - x^2\}$, $R(x) = \text{maximum}\{0, 1 - x\}$.

- (i) Discuss the effect of changing the cost coefficients $(2,3,2,5)_{LR}$, $(1,2,1,3)_{LR}$ and $(3,4,3,4)_{LR}$ of the fuzzy decision variables \tilde{x}_1, \tilde{x}_2 and \tilde{x}_3 to $(2,3,1,4)_{LR}$, $(2,3,1,4)_{LR}$ and $(3,4,1,5)_{LR}$ respectively on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.
- (ii) Discuss the effect of changing the fuzzy requirement vector from $(5,7,2,2)_{LR}$, $(3,5,1,1)_{LR}$ to $(5,8,2,2)_{LR}$, $(3,6,2,2)_{LR}$ on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.

- (iii) Discuss the effect of addition of a new non-negative LR flat fuzzy decision variable \tilde{x}_4 with cost $(2,3,2,5)_{LR}$ and column vectors $[(2,4,1,3)_{LR}, (1,3,1,1)_{LR}]^T$ on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.
- (iv) Discuss the effect of changing column of coefficient matrix of the constraints corresponding to an LR flat fuzzy variable \tilde{x}_2 by $[(2,4,1,3)_{LR}, (3,4,1,5)_{LR}]^T$ on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.
- (v) Discuss the effect of addition of a new fuzzy constraint $(2,3,1,4)_{LR} \otimes \tilde{x}_1 \oplus (1,2,1,3)_{LR} \otimes \tilde{x}_2 \leq_{\mathfrak{R}} (2,4,1,1)_{LR}$ on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.

Solution: The solution of fully fuzzy sensitivity analysis problem, chosen in Example 6.1, by using the proposed Mehar method can be obtained as follows:

Step 1

Using Step 1 of the proposed Mehar method the fuzzy optimal solution and fuzzy optimal value of the fully fuzzy linear programming problem, chosen in Example 6.1, can be obtained as follows:

Step 1(a)

Assuming $\tilde{x}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$, $\tilde{x}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ and $\tilde{x}_3 = (m_3, n_3, \alpha_3, \beta_3)_{LR}$ and using

Step 1(a) of the proposed method the fully fuzzy linear programming problem ($E_{6.1}$) is converted into the crisp linear programming problem ($E_{6.2}$).

$$\text{Maximize} \quad \left(\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{7n_2}{2} + \frac{5\beta_2}{2} + m_3 + 6n_3 + 4\beta_3 \right)$$

$$\text{Subject to} \quad (E_{6.2})$$

$$\frac{4m_1}{3} + \frac{13n_1}{4} + \frac{7\beta_1}{4} - \frac{2\alpha_1}{3} + \frac{2m_2}{3} + \frac{15n_2}{4} + \frac{9\beta_2}{4} - \frac{\alpha_2}{3} + \frac{7m_3}{3} + \frac{9n_3}{2} + \frac{5\beta_3}{2} - \frac{4\alpha_3}{3} \leq \frac{35}{3}$$

$$\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{13n_2}{4} + \frac{7\beta_2}{4} \leq \frac{47}{6}$$

$$m_1 - \alpha_1 \geq 0, m_2 - \alpha_2 \geq 0, m_3 - \alpha_3 \geq 0$$

$$m_1 \leq n_1, m_2 \leq n_2, m_3 \leq n_3$$

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0$$

$$\beta_1 \geq 0, \beta_2 \geq 0, \beta_3 \geq 0.$$

Step 1(b)

The optimal solution of the crisp linear programming problem ($E_{6.2}$) is:

$$m_1 = 0, m_2 = 0, m_3 = 0, n_1 = 0, n_2 = 0, n_3 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \beta_1 = 1.957, \beta_2 = 0, \beta_3 = 3.293$$

and the optimal value is 21.005.

Step 1(c)

Using Step 1(c) of the proposed method the fuzzy optimal solution of the fully fuzzy linear programming problem ($E_{6.1}$) is:

$$\tilde{x}_1 = (0, 0, 0, 1.957)_{LR}, \tilde{x}_2 = (0, 0, 0, 0)_{LR}, \tilde{x}_3 = (0, 0, 0, 3.293)_{LR} \text{ and the fuzzy optimal value is: } (0, 0, 0, 42)_{LR}.$$

Step 2

Using Step 2 of the proposed method the fuzzy optimal solution and fuzzy optimal value of the resulting fully fuzzy linear programming problem can be obtained as follows:

- (i) Since, the cost coefficients corresponding to the fuzzy decision variables \tilde{x}_1, \tilde{x}_2 and \tilde{x}_3 changes to $(2, 3, 1, 4)_{LR}, (2, 3, 1, 4)_{LR}$ and $(3, 4, 1, 5)_{LR}$ respectively in the original fully fuzzy linear programming problem so using Case 1 of the proposed method replace the crisp linear programming problem ($E_{6.2}$) by ($E_{6.3}$).

Maximize

$$\left(\frac{4m_1}{3} + 5n_1 - \frac{2\alpha_1}{3} + \frac{7\beta_1}{2} + \frac{4m_2}{3} + 5n_2 - \frac{2\alpha_2}{3} + \frac{7\beta_2}{2} + \frac{7m_3}{3} + \frac{13n_3}{2} + \frac{9\beta_3}{2} - \frac{4\alpha_3}{3} \right)$$

Subject to (E_{6.3})

$$\frac{4m_1}{3} + \frac{13n_1}{4} + \frac{7\beta_1}{4} - \frac{2\alpha_1}{3} + \frac{2m_2}{3} + \frac{15n_2}{4} + \frac{9\beta_2}{4} - \frac{\alpha_2}{3} + \frac{7m_3}{3} + \frac{9n_3}{2} + \frac{5\beta_3}{2} - \frac{4\alpha_3}{3} \leq \frac{35}{3}$$

$$\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{13n_2}{4} + \frac{7\beta_2}{4} \leq \frac{47}{6}$$

$$m_1 - \alpha_1 \geq 0, m_2 - \alpha_2 \geq 0, m_3 - \alpha_3 \geq 0$$

$$m_1 \leq n_1, m_2 \leq n_2, m_3 \leq n_3$$

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0$$

$$\beta_1 \geq 0, \beta_2 \geq 0, \beta_3 \geq 0.$$

Applying the existing sensitivity analysis technique [116] the optimal solution of crisp linear programming problem (E_{6.3}) is:

$$m_1 = 0, m_2 = 0, m_3 = 0, n_1 = 0, n_2 = 0, n_3 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \beta_1 = 1.957, \beta_2 = 0, \beta_3 = 3.293$$

and the optimal value is 21.673.

Using Step 1(c) of the proposed method the fuzzy optimal solution of the resulting fully fuzzy linear programming problem is:

$$\tilde{x}_1 = (0, 0, 0, 1.957)_{LR}, \tilde{x}_2 = (0, 0, 0, 0)_{LR}, \tilde{x}_3 = (0, 0, 0, 3.293)_{LR} \text{ and the fuzzy optimal value is } (0, 0, 0, 43.336)_{LR}.$$

- (ii) Since, the requirement vector is changed from $(5, 7, 2, 2)_{LR}$, $(3, 5, 1, 1)_{LR}$ to $(5, 8, 2, 2)_{LR}$, $(3, 6, 2, 2)_{LR}$ in the original fully fuzzy linear programming problem (E_{6.1}) so using Case 2 of the proposed method replace $\mathfrak{R}(5, 7, 2, 2)_{LR}$, $\mathfrak{R}(3, 5, 1, 1)_{LR}$ by $\mathfrak{R}(5, 8, 2, 2)$, $\mathfrak{R}(3, 6, 2, 2)$ i.e., $\frac{35}{3}$, $\frac{47}{6}$ by $\frac{38}{3}$, $\frac{26}{3}$ respectively in (E_{6.2}).

$$\text{Maximize} \quad \left(\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{7n_2}{2} + \frac{5\beta_2}{2} + m_3 + 8n_3 + 8\beta_3 \right)$$

Subject to (E_{6.4})

$$\frac{4m_1}{3} + \frac{13n_1}{4} + \frac{7\beta_1}{4} - \frac{2\alpha_1}{3} + \frac{2m_2}{3} + \frac{15n_2}{4} + \frac{9\beta_2}{4} - \frac{\alpha_2}{3} + \frac{7m_3}{3} + \frac{9n_3}{2} + \frac{5\beta_3}{2} - \frac{4\alpha_3}{3} \leq \frac{38}{3}$$

$$\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{13n_2}{4} + \frac{7\beta_2}{4} \leq \frac{26}{3}$$

$$m_1 - \alpha_1 \geq 0, m_2 - \alpha_2 \geq 0, m_3 - \alpha_3 \geq 0$$

$$m_1 \leq n_1, m_2 \leq n_2, m_3 \leq n_3$$

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0$$

$$\beta_1 \geq 0, \beta_2 \geq 0, \beta_3 \geq 0.$$

Applying the existing sensitivity analysis technique [116] the optimal solution of crisp linear programming problem ($E_{6.4}$) is:

$$m_1 = 0, m_2 = 0, m_3 = 0, n_1 = 0, n_2 = 0, n_3 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \beta_1 = 2.165, \beta_2 = 0, \beta_3 = 3.548$$

and the optimal value is 22.854.

Using Step 1(c) of the proposed method the fuzzy optimal solution of the resulting fully fuzzy linear programming problems is:

$$\tilde{x}_1 = (0, 0, 0, 2.165)_{LR}, \tilde{x}_2 = (0, 0, 0, 0)_{LR}, \tilde{x}_3 = (0, 0, 0, 3.548)_{LR} \text{ and the fuzzy optimal value is } (0, 0, 0, 45.705)_{LR}.$$

- (iii) Since, a new non-negative *LR* flat fuzzy decision variable \tilde{x}_4 with cost $(2, 3, 2, 5)_{LR}$ and column vectors $[(2, 4, 1, 3)_{LR}, (1, 3, 1, 1)_{LR}]^T$ is added in the original fully fuzzy linear programming problem ($E_{6.1}$) so using Case 3 of the proposed method replace the crisp linear programming problem ($E_{6.2}$) by ($E_{6.5}$).

$$\text{Maximize} \left(\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{7n_2}{2} + \frac{5\beta_2}{2} + m_3 + 6n_3 + 4\beta_3 + \frac{2m_4}{3} + \frac{11n_4}{2} + 4\beta_4 \right)$$

Subject to ($E_{6.5}$)

$$\frac{4m_1}{3} + \frac{13n_1}{4} + \frac{7\beta_1}{4} - \frac{2\alpha_1}{3} + \frac{2m_2}{3} + \frac{15n_2}{4} + \frac{9\beta_2}{4} - \frac{\alpha_2}{3} + \frac{7m_3}{3} + \frac{9n_3}{2} + \frac{5\beta_3}{2} - \frac{4\alpha_3}{3} + \frac{4m_4}{3} + \frac{15n_4}{2}$$

$$+\frac{7\beta_4}{2}-\frac{2\alpha_4}{3}\leq\frac{35}{3}$$

$$\frac{2m_1}{3}+\frac{11n_1}{2}+4\beta_1+\frac{m_2}{3}+\frac{13n_2}{4}+\frac{7}{4}\beta_2+\frac{m_4}{3}+\frac{7n_4}{2}+2\beta_4\leq\frac{47}{6}$$

$$m_1-\alpha_1\geq 0, m_2-\alpha_2\geq 0, m_3-\alpha_3\geq 0, m_4-\alpha_4\geq 0$$

$$m_1\leq n_1, m_2\leq n_2, m_3\leq n_3, m_4\leq n_4$$

$$\alpha_1\geq 0, \alpha_2\geq 0, \alpha_3\geq 0, \alpha_4\geq 0$$

$$\beta_1\geq 0, \beta_2\geq 0, \beta_3\geq 0, \beta_4\geq 0$$

Applying the existing sensitivity analysis technique [116] the optimal solution of crisp linear programming problem ($E_{6.5}$) is:

$$m_1=0, m_2=0, m_3=0, m_4=0, n_1=0, n_2=0, n_3=0, n_4=0, \alpha_1=0, \alpha_2=0, \alpha_3=0, \alpha_4=0, \beta_1=1.957, \beta_2=0, \beta_3=3.293, \beta_4=0 \text{ and the optimal value is } 21.005.$$

Using Step 1(c) of the proposed method the fuzzy optimal solution of the resulting fully fuzzy linear programming problems is:

$$\tilde{x}_1=(0,0,0,1.957)_{LR}, \tilde{x}_2=(0,0,0,0)_{LR}, \tilde{x}_3=(0,0,0,3.293)_{LR}, \tilde{x}_4=(0,0,0,0)_{LR} \text{ and the fuzzy optimal value is } (0,0,0,42)_{LR}.$$

- (iv) Since, the column of the constraint matrix corresponding to an LR flat fuzzy decision variable \tilde{x}_2 is changed to $[(2,4,1,3)_{LR}, (3,4,1,5)_{LR}]^T$ in the original fuzzy linear programming problem ($E_{6.1}$) so using Case 4 of the proposed method replace the crisp linear programming problem ($E_{6.2}$) by ($E_{6.6}$).

$$\text{Maximize} \quad \left(\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{7n_2}{2} + \frac{5\beta_2}{2} + m_3 + 6n_3 + 4\beta_3 \right)$$

Subject to ($E_{6.6}$)

$$\frac{4m_1}{3} + \frac{13n_1}{4} + \frac{7\beta_1}{4} - \frac{2\alpha_1}{3} + \frac{4m_2}{3} + \frac{15n_2}{2} + \frac{7\beta_2}{2} - \frac{2\alpha_2}{3} + \frac{7m_3}{3} + \frac{9n_3}{2} + \frac{5\beta_3}{2} - \frac{4\alpha_3}{3} \leq \frac{35}{3}$$

$$\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{7m_2}{3} + \frac{13n_2}{2} + \frac{9\beta_2}{2} - \frac{4\alpha_2}{3} \leq \frac{47}{6}$$

$$m_1 - \alpha_1 \geq 0, m_2 - \alpha_2 \geq 0, m_3 - \alpha_3 \geq 0$$

$$m_1 \leq n_1, m_2 \leq n_2, m_3 \leq n_3$$

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0$$

$$\beta_1 \geq 0, \beta_2 \geq 0, \beta_3 \geq 0.$$

Applying the existing sensitivity analysis technique [116] the optimal solution of crisp linear programming problem ($E_{6.6}$) is:

$$m_1 = 0, m_2 = 0, m_3 = 0, n_1 = 0, n_2 = 0, n_3 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \beta_1 = 1.957, \beta_2 = 0, \beta_3 = 3.293$$

and the optimal value is 21.005 .

Using Step 1(c) of the proposed method the fuzzy optimal solution of the resulting fully fuzzy linear programming problems is:

$$\tilde{x}_1 = (0, 0, 0, 1.957)_{LR}, \tilde{x}_2 = (0, 0, 0, 0)_{LR}, \tilde{x}_3 = (0, 0, 0, 3.293)_{LR} \text{ and the fuzzy optimal value is } (0, 0, 0, 42)_{LR}.$$

(v) Since, a new fuzzy constraint $(2, 3, 1, 4)_{LR} \otimes \tilde{x}_1 \oplus (1, 2, 1, 3)_{LR} \otimes \tilde{x}_2 \leq_{\mathfrak{R}} (2, 4, 1, 1)_{LR}$ is added to the original fuzzy linear programming problem ($E_{6.1}$) so using Case 5 of the proposed

method add the constraint $\frac{4m_1}{3} + \frac{13n_1}{4} + \frac{7\beta_1}{4} - \frac{2\alpha_1}{3} + \frac{m_2}{3} + \frac{7n_2}{2} + \frac{5\beta_2}{4} \leq \frac{35}{6}$ to ($E_{6.7}$).

$$\text{Maximize} \quad \left(\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{7n_2}{2} + \frac{5\beta_2}{2} + m_3 + 6n_3 + 4\beta_3 \right)$$

Subject to ($E_{6.9}$)

$$\frac{4m_1}{3} + \frac{13n_1}{4} + \frac{7}{4}\beta_1 - \frac{2\alpha_1}{3} + \frac{2m_2}{3} + \frac{15n_2}{4} + \frac{9\beta_2}{4} - \frac{1\alpha_2}{3} + \frac{7m_3}{3} + \frac{9n_3}{2} + \frac{5\beta_3}{2} - \frac{4\alpha_3}{3} \leq \frac{35}{3}$$

$$\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{13n_2}{4} + \frac{7\beta_2}{4} \leq \frac{47}{6}$$

$$\frac{4m_1}{3} + 5n_1 + \frac{7\beta_1}{2} - \frac{2\alpha_1}{3} + \frac{m_2}{3} + \frac{7n_2}{2} + \frac{5\beta_2}{4} \leq \frac{35}{6}$$

$$m_1 - \alpha_1 \geq 0, m_2 - \alpha_2 \geq 0, m_3 - \alpha_3 \geq 0$$

$$m_1 \leq n_1, m_2 \leq n_2, m_3 \leq n_3$$

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0$$

$$\beta_1 \geq 0, \beta_2 \geq 0, \beta_3 \geq 0.$$

Applying the existing sensitivity analysis technique [116] the optimal solution of the crisp linear programming problem ($E_{6,9}$) is:

$$m_1 = 0, m_2 = 0, m_3 = 0, n_1 = 0, n_2 = 0, n_3 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \beta_1 = 1.665, \beta_2 = 0, \beta_3 = 3.498$$

and the optimal value is 20.654 .

Using Step 1(c) of the proposed method the fuzzy optimal solution of the resulting fully fuzzy linear programming problems is:

$$\tilde{x}_1 = (0, 0, 0, 1.665)_{LR}, \tilde{x}_2 = (0, 0, 0, 0)_{LR}, \tilde{x}_3 = (0, 0, 0, 3.498)_{LR} \text{ and the fuzzy optimal value is } (0, 0, 0, 41.304)_{LR}.$$

6.5 Comparative study

To compare the proposed method with the existing methods [35, 65, 95] the results of fuzzy sensitivity analysis problems, obtained by using the existing methods [35, 65, 95] as well as the proposed Mehar method, are shown in Table 6.1

Table 6.1 Comparison of proposed Mehar method with the existing methods

Example	Existing method [35]	Existing methods [65, 95]	Proposed Mehar method
3.1 [35, pp.1882]	Applicable	Not Applicable	Applicable
1 [65, pp. 263]	Not applicable	Applicable	Applicable
6.1	Not applicable	Not applicable	Applicable

The results, presented in Table 6.1, can be explained as follows:

- (a) Since, in the existing fuzzy sensitivity analysis problem [35, Example 3.1, pp. 1882] only the decision variables are represented by real numbers and rest of the parameters are represented by triangular or trapezoidal fuzzy numbers. So, as discussed in Section 6.2 and Section 6.3, this problem can be solved by using the proposed Mehar method but can't be solved by using the existing methods [65, 95].
- (b) Since, in the existing fuzzy sensitivity analysis problem [65, Example 1, pp. 263] the decision variables and right-hand side vectors are represented by fuzzy numbers and rest of the parameters are represented by real numbers. So, as discussed in Section 6.2 and Section 6.3, this problem can be solved by using the proposed Mehar method but can't be solved by using the existing method [35].
- (c) Since, in the fuzzy sensitivity analysis problem, chosen in Example 6.1, all the parameters as well as decision variables are represented by *LR* flat fuzzy numbers. So, as discussed in Section 6.2 and Section 6.3, this problem can only be solved by using the proposed Mehar method but not by using any of the existing methods [35, 65, 95].

6.6 Proposed Mehar method for the sensitivity analysis of fully fuzzy transportation problems

To the best of our knowledge there is no method in the literature to deal with the sensitivity analysis of fully fuzzy transportation problems. In this section, a new method, named as Mehar method, is proposed to deal with the sensitivity analysis of fully fuzzy transportation problems.

The steps of the proposed method are as follows:

Step 1

Use the following steps of the existing method [68] to find the fuzzy optimal solution and

fuzzy optimal value of the balanced fully fuzzy transportation problem represented by Table 6.2.

Table 6.2 Tabular representation of balanced fully fuzzy transportation problem

Destinations→	D_1	D_2	...	D_j	...	D_n	Fuzzy availability
Sources ↓							
S_1	\tilde{c}_{11}	\tilde{c}_{12}	...	\tilde{c}_{1j}	...	\tilde{c}_{1n}	\tilde{a}_1
⋮	⋮	⋮	...	⋮	...	⋮	⋮
S_i	\tilde{c}_{i1}	\tilde{c}_{i2}	...	\tilde{c}_{ij}	...	\tilde{c}_{in}	\tilde{a}_i
⋮	⋮	⋮	...	⋮	...	⋮	⋮
S_m	\tilde{c}_{m1}	\tilde{c}_{m2}	...	\tilde{c}_{mj}	...	\tilde{c}_{mn}	\tilde{a}_m
Fuzzy demand	\tilde{b}_1	\tilde{b}_2	...	\tilde{b}_j	...	\tilde{b}_n	$\sum_{i=1}^m \tilde{a}_i = \sum_{j=1}^n \tilde{b}_j$

where, m : total number of sources;

n : total number of destinations;

$\tilde{a}_i = (m_i, n_i, \alpha_i, \beta_i)_{LR}$: the fuzzy availability of the product at i^{th} source;

$\tilde{b}_j = (m'_j, n'_j, \alpha'_j, \beta'_j)_{LR}$: the fuzzy demand of the product at j^{th} destination;

$\tilde{c}_{ij} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$: the fuzzy cost for transporting one unit quantity of the product from i^{th} source to j^{th} destination;

$\sum_{i=1}^m \tilde{a}_i$: total fuzzy availability of the product; $\sum_{j=1}^n \tilde{b}_j$: total fuzzy demand of the

product.

Step 1(a)

Split Table 6.2 into four balanced crisp transportation tables, Table 6.3, Table 6.4, Table 6.5 and Table 6.6 respectively.

Table 6.3 Tabular representation of first balanced crisp transportation problem

Destinations→ Sources ↓	D_1	D_2	...	D_j	...	D_n	Availability
S_1	η_{11}	η_{12}	...	η_{1j}	...	η_{1n}	$m_1 - \alpha_1$
\vdots	\vdots	\vdots	...	\vdots	...	\vdots	\vdots
S_i	η_{i1}	η_{i2}	...	η_{ij}	...	η_{in}	$m_i - \alpha_i$
\vdots	\vdots	\vdots	...	\vdots	...	\vdots	\vdots
S_m	η_{m1}	η_{m2}	...	η_{mj}	...	η_{mn}	$m_m - \alpha_m$
Demand	$m'_1 - \alpha'_1$	$m'_2 - \alpha'_2$...	$m'_j - \alpha'_j$...	$m'_n - \alpha'_n$	$\sum_{i=1}^m (m_i - \alpha_i) = \sum_{j=1}^n (m'_j - \alpha'_j)$

where, $\eta_{ij} = \frac{1}{2}((m_{ij} + n_{ij}) - \alpha_{ij} \int_0^1 L^{-1}(\lambda) d\lambda + \beta_{ij} \int_0^1 R^{-1}(\lambda) d\lambda)$, $i=1,2,\dots,m$ and $j=1,2,\dots,n$

Table 6.4 Tabular representation of second balanced crisp transportation problem

Destinations→ Sources ↓	D_1	D_2	...	D_j	...	D_n	Availability
S_1	ρ_{11}	ρ_{12}	...	ρ_{1j}	...	ρ_{1n}	α_1
\vdots	\vdots	\vdots	...	\vdots	...	\vdots	\vdots
S_i	ρ_{i1}	ρ_{i2}	...	ρ_{ij}	...	ρ_{in}	α_i
\vdots	\vdots	\vdots	...	\vdots	...	\vdots	\vdots
S_m	ρ_{m1}	ρ_{m2}	...	ρ_{mj}	...	ρ_{mn}	α_m
Demand	α'_1	α'_2	...	α'_j	...	α'_n	$\sum_{i=1}^m \alpha_i = \sum_{j=1}^n \alpha'_j$

where, $\rho_{ij} = \frac{1}{2}((m_{ij} + n_{ij}) - m_{ij} \int_0^1 L^{-1}(\lambda) d\lambda + \beta_{ij} \int_0^1 R^{-1}(\lambda) d\lambda)$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$

Table 6.5 Tabular representation of third balanced crisp transportation problem

Destinations → Sources ↓	D_1	D_2	...	D_j	...	D_n	Availability
S_1	δ_{11}	δ_{12}	...	δ_{1j}	...	δ_{1n}	$n_1 - m_1$
⋮	⋮	⋮	...	⋮	...	⋮	⋮
S_i	δ_{i1}	δ_{i2}	...	δ_{ij}	...	δ_{in}	$n_i - m_i$
⋮	⋮	⋮	...	⋮	...	⋮	⋮
S_m	δ_{m1}	δ_{m2}	...	δ_{mj}	...	δ_{mn}	$n_m - m_m$
Demand	$n'_1 - m'_1$	$n'_2 - m'_2$...	$n'_j - m'_j$...	$n'_n - m'_n$	$\sum_{i=1}^m (n_i - m_i) = \sum_{j=1}^n (n'_j - m'_j)$

where, $\delta_{ij} = \frac{1}{2}(n_{ij} + \beta_{ij} \int_0^1 R^{-1}(\lambda) d\lambda)$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$

Table 6.6 Tabular representation of fourth balanced crisp transportation problem

Destinations → Sources ↓	D_1	D_2	...	D_j	...	D_n	Availability
S_1	ξ_{11}	ξ_{12}	...	ξ_{1j}	...	ξ_{1n}	β_1
⋮	⋮	⋮	...	⋮	...	⋮	⋮
S_i	ξ_{i1}	ξ_{i2}	...	ξ_{ij}	...	ξ_{in}	β_i
⋮	⋮	⋮	...	⋮	...	⋮	⋮
S_m	ξ_{m1}	ξ_{m2}	...	ξ_{mj}	...	ξ_{mn}	β_m
Demand	β'_1	β'_2	...	β'_j	...	β'_n	$\sum_{i=1}^m \beta_i = \sum_{j=1}^n \beta'_j$

where, $\xi_{ij} = \frac{1}{2}((n_{ij} + \beta_{ij}) \int_0^1 R^{-1}(\lambda) d\lambda)$, $i=1,2,\dots,m$ and $j=1,2,\dots,n$.

Step 1(b)

Find the optimal solution $m''_{ij}, \alpha''_{ij}, n''_{ij} - m''_{ij}$ and β''_{ij} by solving the balanced crisp transportation problems, represented by Table 6.3, Table 6.4, Table 6.5 and Table 6.6, respectively.

Step 1(c)

Find the values of $m''_{ij}, n''_{ij}, \alpha''_{ij}$ and β''_{ij} by solving the equations, obtained in Step 1(b).

Step 1(d)

Find the fuzzy optimal solution by putting the values of $m''_{ij}, n''_{ij}, \alpha''_{ij}$ and β''_{ij} in $\tilde{x}_{ij} = (m''_{ij}, n''_{ij}, \alpha''_{ij}, \beta''_{ij})_{LR}$.

Step 2

If any change is made in the chosen fully fuzzy transportation problem then the optimal solution of the resulting fully fuzzy transportation problem can be obtained as follows:

Check that which of the following case is to be considered:

1. Change in the fuzzy transportation cost.
2. Increase (or Decrease) in the fuzzy availability and fuzzy demand.

Case 1: Change in the fuzzy transportation cost

If in the original fully fuzzy transportation problem the fuzzy cost for transporting one unit quantity of the product from i^{th} source to j^{th} destination $\tilde{c}_{ij} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ is changed to $\tilde{c}'_{ij} = (m'_{ij}, n'_{ij}, \alpha'_{ij}, \beta'_{ij})_{LR}$ then replace the crisp transportation costs $\eta_{ij}, \rho_{ij}, \delta_{ij}$ and ξ_{ij} of Table 6.3, Table 6.4, Table 6.5 and Table 6.6, by $\eta'_{ij}, \rho'_{ij}, \delta'_{ij}$ and ξ'_{ij} respectively.

where, $\eta'_{ij} = \frac{1}{2}((m'_{ij} + n'_{ij}) - \alpha'_{ij} \int_0^1 L^{-1}(\lambda) d\lambda + \beta'_{ij} \int_0^1 R^{-1}(\lambda) d\lambda)$, $i=1,2,\dots,m$ and

$j=1,2,\dots,n$,

$\rho'_{ij} = \frac{1}{2}((m'_{ij} + n'_{ij}) - m'_{ij} \int_0^1 L^{-1}(\lambda) d\lambda + \beta'_{ij} \int_0^1 R^{-1}(\lambda) d\lambda)$, $i=1,2,\dots,m$ and $j=1,2,\dots,n$,

$\delta'_{ij} = \frac{1}{2}(n'_{ij} + \beta'_{ij} \int_0^1 R^{-1}(\lambda) d\lambda)$, $i=1,2,\dots,m$ and $j=1,2,\dots,n$,

$\xi'_{ij} = \frac{1}{2}((n'_{ij} + \beta'_{ij}) \int_0^1 R^{-1}(\lambda) d\lambda)$, $i=1,2,\dots,m$ and $j=1,2,\dots,n$.

Case 2: Increase (or decrease) in both fuzzy availability \tilde{a}_i and fuzzy demand \tilde{b}_j

If in the original fully fuzzy transportation problem, both fuzzy availability \tilde{a}_i and fuzzy demand \tilde{b}_j is increased (or decreased) by $(m,n,\alpha,\beta)_{LR}$ then replace the crisp availabilities $m_i - \alpha_i, \alpha_i, n_i - m_i$ and β_i of Table 6.3, Table 6.4, Table 6.5 and Table 6.6 by $m_i + m - \alpha_i - \alpha, \alpha_i + \alpha, n_i + n - m_i - m$ and $\beta_i + \beta$ respectively. Also, replace the crisp demands $m'_j - \alpha'_j, \alpha'_j, n'_j - m'_j$ and β'_j of Table 6.3, Table 6.4, Table 6.5 and Table 6.6, by $m'_j + m - \alpha'_j - \alpha, \alpha'_j + \alpha, n'_j + n - m'_j - m$ and $\beta'_j + \beta$ respectively.

Step 3

Use the existing sensitivity analysis technique [129] to find the optimal solution of modified balanced crisp transportation tables, with the help of optimal solution of balanced crisp transportation tables, Table 6.3, Table 6.4, Table 6.5 and Table 6.6, obtained in Step 1(b), of the method, proposed in Section 6.6.

Step 4

Use Step 1(d) of the method, proposed in Section 6.6, to find the fuzzy optimal solution of modified fully fuzzy transportation problem with the help of optimal solution of modified balanced crisp transportation problems.

6.7 Illustrative example

In this section, proposed method is illustrated with the help of a numerical example.

Example 6.2 [68 pp. 88]: Considered the balanced fully fuzzy transportation problem represented by Table 6.7.

Table 6.7 Tabular representation of the balanced fully fuzzy transportation problem

	D_1	D_2	D_3	D_4	\tilde{a}_i
S_1	$(20, 30, 10, 10)_{LR}$	$(60, 70, 10, 20)_{LR}$	$(90, 110, 10, 10)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(90, 90, 20, 10)_{LR}$
S_2	$(70, 80, 10, 10)_{LR}$	$(80, 100, 10, 20)_{LR}$	$(30, 50, 10, 10)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(60, 70, 20, 10)_{LR}$
S_3	$(0, 0, 0, 0)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(0, 0, 0, 0)_{LR}$	$(0, 10, 0, 40)_{LR}$
\tilde{b}_j	$(40, 50, 10, 20)_{LR}$	$(30, 40, 10, 10)_{LR}$	$(50, 50, 10, 30)_{LR}$	$(30, 30, 10, 0)_{LR}$	$\sum_{i=1}^3 \tilde{a}_i = \sum_{j=1}^4 \tilde{b}_j$

- (i) Find the fuzzy optimal solution of resulting fuzzy transportation problem obtained by changing the fuzzy cost for transporting one unit quantity of the product \tilde{c}_{11} from $(20, 30, 10, 10)_{LR}$ to $(60, 70, 10, 10)_{LR}$ in the original fuzzy transportation problem.
- (ii) Find the fuzzy optimal solution of resulting fuzzy transportation problem with the increase in both fuzzy availability \tilde{a}_2 and fuzzy demand \tilde{b}_1 by $(30, 40, 10, 10)_{LR}$.

Solution: The solution of fully fuzzy sensitivity analysis problem, chosen in Example 6.2, by using the proposed method can be obtained as follows:

Step 1

Using Step 1 of the method, proposed in Section 6.6, the fuzzy optimal solution of the fully fuzzy transportation problem, chosen in Example 6.2, can be obtained as follows:

Step 1(a)

Using Step 1(a) of the method, proposed in Section 6.6, split the Table 6.7 into four balanced

crisp transportation tables, Table 6.8, Table 6.9, Table 6.10 and Table 6.11 respectively.

Table 6.8 Tabular representation of first balanced crisp transportation problem

	D_1	D_2	D_3	D_4	$m_i - \alpha_i$
S_1	25	67.5	100	0	70
S_2	75	92.5	40	0	40
S_3	0	0	0	0	0
$m'_j - \alpha'_j$	30	20	40	20	

Table 6.9 Tabular representation of second balanced crisp transportation problem

	D_1	D_2	D_3	D_4	α_i
S_1	22.5	55	80	0	20
S_2	60	75	35	0	20
S_3	0	0	0	0	0
α'_j	10	10	10	10	

Table 6.10 Tabular representation of third balanced crisp transportation problem

	D_1	D_2	D_3	D_4	$n_j - m_j$
S_1	17.5	40	57.5	0	0
S_2	42.5	55	27.5	0	10
S_3	0	0	0	0	10
$n'_j - m'_j$	10	10	0	0	

Table 6.11. Tabular representation of fourth balanced crisp transportation problem

	D_1	D_2	D_3	D_4	β_i
S_1	10	22.5	30	0	10
S_2	22.5	30	15	0	10
S_3	0	0	0	0	40
β'_j	20	10	30	0	

Step 1(b)

The optimal solution of balanced crisp transportation problem, represented by Table 6.8, is:

$$m_{11} - \alpha_{11} = 30, m_{12} - \alpha_{12} = 20, m_{13} - \alpha_{13} = 0, m_{14} - \alpha_{14} = 20, m_{21} - \alpha_{21} = 0, m_{22} - \alpha_{22} = 0, m_{23} - \alpha_{23} = 40, m_{24} - \alpha_{24} = 0, m_{31} - \alpha_{31} = 0, m_{32} - \alpha_{32} = 0, m_{33} - \alpha_{33} = 0, m_{34} - \alpha_{34} = 0$$

The optimal solution of the balanced crisp transportation problem, represented by Table 6.9, is:

$$\alpha_{11} = 10, \alpha_{12} = 10, \alpha_{13} = 0, \alpha_{14} = 0, \alpha_{21} = 0, \alpha_{22} = 0, \alpha_{23} = 10, \alpha_{24} = 10, \alpha_{31} = 0, \alpha_{32} = 0, \alpha_{33} = 0, \alpha_{34} = 0$$

The optimal solution of the balanced crisp transportation problem, represented by Table 6.10, is:

$$n_{11} - m_{11} = 0, n_{12} - m_{12} = 0, n_{13} - m_{13} = 0, n_{14} - m_{14} = 0, n_{21} - m_{21} = 10, n_{22} - m_{22} = 0, n_{23} - m_{23} = 0, n_{24} - m_{24} = 0, n_{31} - m_{31} = 0, n_{32} - m_{32} = 10, n_{33} - m_{33} = 0, n_{34} - m_{34} = 0$$

The optimal solution of the balanced crisp transportation problem, represented by Table 6.11, is:

$$\beta_{11} = 10, \beta_{12} = 0, \beta_{13} = 0, \beta_{14} = 0, \beta_{21} = 0, \beta_{22} = 0, \beta_{23} = 10, \beta_{24} = 0, \beta_{31} = 10, \beta_{32} = 10, \beta_{33} = 20, \beta_{34} = 0$$

Step 1(c)

Using Step 1(d) of the method, proposed in Section 6.6, the fuzzy optimal solution of fully fuzzy transportation problem represented by Table 6.7 is:

$$\tilde{x}_{11} = (40, 40, 10, 10)_{LR}, \tilde{x}_{12} = (30, 30, 10, 0)_{LR}, \tilde{x}_{13} = (0, 0, 0, 0)_{LR}, \tilde{x}_{14} = (20, 20, 0, 0)_{LR},$$

$$\tilde{x}_{21} = (0, 10, 0, 0)_{LR}, \tilde{x}_{22} = (0, 0, 0, 0)_{LR}, \tilde{x}_{23} = (50, 50, 10, 10)_{LR}, \tilde{x}_{24} = (10, 10, 10, 0)_{LR},$$

$$\tilde{x}_{31} = (0, 0, 0, 10)_{LR}, \tilde{x}_{32} = (0, 10, 0, 10)_{LR}, \tilde{x}_{33} = (0, 0, 0, 20)_{LR}, \tilde{x}_{34} = (0, 0, 0, 0)_{LR}$$

Step 2

Using Step 2 of the method, proposed in Section 6.6, the fuzzy optimal solution of modified fully fuzzy transportation problem can be obtained as follows:

- (i) Since the fuzzy cost for transporting one unit quantity of the product is changed from $\tilde{c}_{11} = (20, 30, 10, 10)_{LR}$ to $\tilde{c}'_{11} = (60, 70, 10, 10)_{LR}$ in the original fuzzy transportation problem so using Case 1 of Step 1 of the proposed method Table 6.12, Table 6.13, Table 6.14 and Table 6.15 can be obtained by replacing the crisp transportation costs 25, 22.5, 17.5 and 10 of Table 6.8, Table 6.9, Table 6.10 and Table 6.11 by 65, 52.5, 37.5 and 20 respectively.

Table 6.12 Tabular representation of first balanced crisp transportation problem

	D_1	D_2	D_3	D_4	$m_i - \alpha_i$
S_1	65	67.5	100	0	70
S_2	75	92.5	40	0	40
S_3	0	0	0	0	0
$m'_j - \alpha'_j$	30	20	40	20	

Table 6.13 Tabular representation of second balanced crisp transportation problem

	D_1	D_2	D_3	D_4	α_i
S_1	52.7	55	80	0	20
S_2	60	75	35	0	20
S_3	0	0	0	0	0
α'_j	10	10	10	10	

Table 6.14 Tabular representation of third balanced crisp transportation problem

	D_1	D_2	D_3	D_4	$n_j - m_j$
S_1	37.5	40	57.5	0	0
S_2	42.5	55	27.5	0	10
S_3	0	0	0	0	10
$n'_j - m'_j$	10	10	0	0	

Table 6.15 Tabular representation of fourth balanced crisp transportation problem

	D_1	D_2	D_3	D_4	β_i
S_1	20	22.5	30	0	10
S_2	22.5	30	15	0	10
S_3	0	0	0	0	40
β'_j	20	10	30	0	

Applying the existing sensitivity analysis technique [129], the optimal solution of modified balanced crisp transportation tables with the help of optimal solution of balanced crisp transportation Table 6.8, Table 6.9, Table 6.10 and Table 6.11 are as follows:

The optimal solution of the balanced crisp transportation problem, represented by Table 6.12 is:

$$m_{11} - \alpha_{11} = 30, m_{12} - \alpha_{12} = 20, m_{13} - \alpha_{13} = 0, m_{14} - \alpha_{14} = 20, m_{21} - \alpha_{21} = 0, m_{22} - \alpha_{22} = 0, m_{23} - \alpha_{23} = 40, m_{24} - \alpha_{24} = 0, m_{31} - \alpha_{31} = 0, m_{32} - \alpha_{32} = 0, m_{33} - \alpha_{33} = 0, m_{34} - \alpha_{34} = 0$$

The optimal solution of the balanced crisp transportation problem, represented by Table 6.13 is:

$$\alpha_{11} = 0, \alpha_{12} = 10, \alpha_{13} = 0, \alpha_{14} = 10, \alpha_{21} = 10, \alpha_{22} = 0, \alpha_{23} = 10, \alpha_{24} = 0, \alpha_{31} = 0, \alpha_{32} = 0, \alpha_{33} = 0, \alpha_{34} = 0$$

The optimal solution of the balanced crisp transportation problem, represented by Table 6.14 is:

$$n_{11} - m_{11} = 0, n_{12} - m_{12} = 0, n_{13} - m_{13} = 0, n_{14} - m_{14} = 0, n_{21} - m_{21} = 10, n_{22} - m_{22} = 0, n_{23} - m_{23} = 0, n_{24} - m_{24} = 0, n_{31} - m_{31} = 0, n_{32} - m_{32} = 10, n_{33} - m_{33} = 0, n_{34} - m_{34} = 0$$

The optimal solution of the balanced crisp transportation problem, represented by Table 6.15 is:

$$\beta_{11} = 10, \beta_{12} = 0, \beta_{13} = 0, \beta_{14} = 0, \beta_{21} = 0, \beta_{22} = 0, \beta_{23} = 10, \beta_{24} = 0, \beta_{31} = 10, \beta_{32} = 10, \beta_{33} = 20, \beta_{34} = 0$$

Using Step 1(d) of the method, proposed in Section 6.6, the fuzzy optimal solution of modified fully fuzzy transportation problem is:

$$\begin{aligned} \tilde{x}_{11} &= (40, 0, 10, 10)_{LR}, \tilde{x}_{12} = (30, 30, 10, 0)_{LR}, \tilde{x}_{13} = (0, 0, 0, 0)_{LR}, \tilde{x}_{14} = (20, 0, 0, 0)_{LR}, \\ \tilde{x}_{21} &= (0, 0, 10, 0)_{LR}, \tilde{x}_{22} = (0, 0, 0, 0)_{LR}, \tilde{x}_{23} = (50, 50, 10, 10)_{LR}, \tilde{x}_{24} = (0, 0, 0, 0)_{LR}, \\ \tilde{x}_{31} &= (0, 0, 0, 10)_{LR}, \tilde{x}_{32} = (0, 10, 0, 10)_{LR}, \tilde{x}_{33} = (0, 0, 0, 20)_{LR}, \tilde{x}_{34} = (0, 0, 0, 0)_{LR} \end{aligned}$$

and the minimum total fuzzy transportation cost is $(5700, 4600, 3400, 2200)_{LR}$

- (ii) Since, in the original fuzzy transportation problem both fuzzy availability \tilde{a}_2 and fuzzy demand \tilde{b}_1 are increased by $(30, 40, 10, 10)_{LR}$ so using Case 2 of Step 1 of the proposed method Table 6.16, Table 6.17, Table 6.18 and Table 6.19 is obtained by replacing crisp availabilities 40, 20, 10 and 10 by 60, 30, 20 and 20 respectively and replacing crisp demands 30, 10, 10 and 20 by 50, 20, 20 and 30 respectively.

Table 6.16 Tabular representation of first balanced crisp transportation problem

	D_1	D_2	D_3	D_4	$m_i - \alpha_i$
S_1	25	67.5	100	0	70
S_2	75	92.5	40	0	60
S_3	0	0	0	0	0
$m'_j - \alpha'_j$	50	20	40	20	

Table 6.17 Tabular representation of second balanced crisp transportation problem

	D_1	D_2	D_3	D_4	α_i
S_1	22.5	55	80	0	20
S_2	60	75	35	0	30
S_3	0	0	0	0	0
α'_j	20	10	10	10	

Table 6.18 Tabular representation of third balanced crisp transportation problem

	D_1	D_2	D_3	D_4	$n_j - m_j$
S_1	17.5	40	57.5	0	0
S_2	42.5	55	27.5	0	20
S_3	0	0	0	0	10
$n'_j - m'_j$	20	10	0	0	

Table 6.19 Tabular representation of fourth balanced crisp transportation problem

	D_1	D_2	D_3	D_4	β_i
S_1	10	22.5	30	0	10
S_2	22.5	30	15	0	20
S_3	0	0	0	0	40
β'_j	30	10	30	0	

Applying the existing sensitivity analysis technique [129], the optimal solution of modified balanced crisp transportation tables with the help of optimal solution of balanced crisp transportation Table 6.8, Table 6.9, Table 6.10 and Table 6.11 are as follows:

The optimal solution of crisp transportation Table 6.16 is:

$$m_{11} - \alpha_{11} = 50, m_{12} - \alpha_{12} = 20, m_{13} - \alpha_{13} = 0, m_{14} - \alpha_{14} = 0, m_{21} - \alpha_{21} = 0, m_{22} - \alpha_{22} = 0, m_{23} - \alpha_{23} = 40, m_{24} - \alpha_{24} = 20, m_{31} - \alpha_{31} = 0, m_{32} - \alpha_{32} = 0, m_{33} - \alpha_{33} = 0, m_{34} - \alpha_{34} = 0$$

The optimal solution of crisp transportation Table 6.17 is:

$$\alpha_{11} = 20, \alpha_{12} = 0, \alpha_{13} = 0, \alpha_{14} = 0, \alpha_{21} = 0, \alpha_{22} = 10, \alpha_{23} = 10, \alpha_{24} = 10, \alpha_{31} = 0, \alpha_{32} = 0, \alpha_{33} = 0, \alpha_{34} = 0$$

The optimal solution of crisp transportation Table 6.18 is:

$$n_{11} - m_{11} = 0, n_{12} - m_{12} = 0, n_{13} - m_{13} = 0, n_{14} - m_{14} = 0, n_{21} - m_{21} = 20, n_{22} - m_{22} = 0, n_{23} - m_{23} = 0, n_{24} - m_{24} = 0, n_{31} - m_{31} = 0, n_{32} - m_{32} = 10, n_{33} - m_{33} = 0, n_{34} - m_{34} = 0$$

The optimal solution of crisp transportation Table 6.19 is:

$$\beta_{11} = 10, \beta_{12} = 0, \beta_{13} = 0, \beta_{14} = 0, \beta_{21} = 0, \beta_{22} = 0, \beta_{23} = 20, \beta_{24} = 0, \beta_{31} = 20, \beta_{32} = 10, \beta_{33} = 10, \beta_{34} = 0.$$

Using Step 1(d) of the method, proposed in Section 6.6, the fuzzy optimal solution of modified fully fuzzy transportation problem is:

$$\begin{aligned} \tilde{x}_{11} &= (70, 70, 20, 10)_{LR}, \tilde{x}_{12} = (20, 20, 10, 0)_{LR}, \tilde{x}_{13} = (0, 0, 0, 0)_{LR}, \tilde{x}_{14} = (0, 0, 0, 0)_{LR}, \\ \tilde{x}_{21} &= (0, 0, 0, 0)_{LR}, \tilde{x}_{22} = (10, 10, 10, 0)_{LR}, \tilde{x}_{23} = (50, 50, 10, 20)_{LR}, \tilde{x}_{24} = (30, 30, 10, 0)_{LR}, \\ \tilde{x}_{31} &= (0, 0, 0, 20)_{LR}, \tilde{x}_{32} = (0, 10, 0, 10)_{LR}, \tilde{x}_{33} = (0, 0, 0, 10)_{LR}, \tilde{x}_{34} = (0, 0, 0, 0)_{LR} \end{aligned}$$

and the minimum total fuzzy transportation cost is $(4900, 7000, 3600, 4100)_{LR}$.

6.8 Conclusions

On the basis of presented study, it can be concluded that neither there is any method in the literature to deal with the sensitivity analysis of fully fuzzy linear programming problems nor to deal with the sensitivity analysis of fully fuzzy transportation problems. In this chapter, new methods are proposed for the same.

Chapter 7

Sensitivity analysis of interval-valued fully fuzzy linear programming problems^{††}

In Chapter 5, to overcome the limitations of the existing method [115] a new method, based on proposed Mehar multiplication, is proposed for solving interval-valued fully fuzzy linear programming problems. In this chapter, a new method, based on existing multiplication, is proposed for solving interval-valued fully fuzzy linear programming problems. Also, on the basis of proposed method, a new method is proposed to deal with the sensitivity analysis of interval-valued fully fuzzy linear programming problems.

7.1 Proposed methods

To the best of our knowledge neither there is any method in the literature to find the fuzzy optimal solution nor to deal with the sensitivity analysis of interval-valued fully fuzzy linear programming problem $(P_{7.1})$. So, in this section, two new methods, one for finding the fuzzy optimal solution and another for dealing with the sensitivity analysis of interval-valued fully fuzzy linear programming problem $(P_{7.1})$ are proposed.

$$\begin{aligned} & \text{Maximize (or Minimize)} && (\tilde{C}^T \otimes \tilde{X}) \\ & \text{Subject to} && (P_{7.1}) \\ & && \tilde{A} \otimes \tilde{X} \leq_{\mathfrak{R}} \text{ or } =_{\mathfrak{R}} \text{ or } \geq_{\mathfrak{R}} \tilde{b} \\ & && \tilde{X} = [\tilde{x}_j]_{n \times 1} \end{aligned}$$

where, \tilde{x}_j are non-negative interval-valued *LR* flat fuzzy decision variables.

^{††} The contents of this chapters are accepted for publication in *Journal of Applied Research and Technology*.

and $\tilde{b} = [\tilde{b}_k]_{m \times 1}$, $\tilde{A} = [\tilde{a}_{kj}]_{m \times n}$, $\tilde{C}^T = [\tilde{c}_j]_{1 \times n}$, $\tilde{c}_j = [(c_j, c_j', \delta_{j2}, \delta_{j3}; \lambda)_{LR}, (c_j, c_j', \delta_{j1}, \delta_{j4}; \rho)_{LR}]$, $\tilde{a}_{kj} = [(a_{kj}, a_{kj}', \delta_{kj2}, \delta_{kj3}; \lambda)_{LR}, (a_{kj}, a_{kj}', \delta_{kj1}, \delta_{kj4}; \rho)_{LR}]$, $\tilde{b}_k = [(b_k, b_k', \alpha_{k2}, \alpha_{k3}; \lambda)_{LR}, (b_k, b_k', \alpha_{k1}, \alpha_{k4}; \rho)_{LR}]$, $\tilde{x}_j = [(m_j, n_j, \alpha_j', \beta_j'; \lambda)_{LR}, (m_j, n_j, \alpha_j, \beta_j; \rho)_{LR}]$, $0 < \delta_{j2} < \delta_{j1} < c_j < c_j', 0 < \delta_{j3} < \delta_{j4}, 0 < \delta_{kj2} < \delta_{kj1} < a_{kj} < a_{kj}', 0 < \delta_{kj3} < \delta_{kj4}, 0 < \alpha_{k2} < \alpha_{k1} < b_k < b_k', 0 < \alpha_{k3} < \alpha_{k4}; 0 < \alpha_j' < \alpha_j < m_j < n_j, 0 < \beta_j' < \beta_j; j=1, 2, \dots, n; 0 < \lambda < 1; k=1, 2, \dots, m$.

7.1.1 Proposed method for finding the fuzzy optimal solution of interval-valued fully fuzzy linear programming problems

In this section, a new method is proposed to find the fuzzy optimal solution (\tilde{X}) and fuzzy optimal value ($\tilde{C}^T \otimes \tilde{X}$) of interval-valued fully fuzzy linear programming problem ($P_{7.1}$).

The steps of the proposed method are as follows:

Step 1

Convert the chosen interval-valued fully fuzzy linear programming problem ($P_{7.1}$) into the crisp linear programming problem ($P_{7.2}$).

$$\text{Maximize (or Minimize)} \quad \mathfrak{R}(\tilde{C}^T \otimes \tilde{X})$$

Subject to

$$\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b})$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2, \dots, n.$$

or

$$\text{Maximize (or Minimize)} \quad \left(\frac{1}{16} \sum_{j=1}^n \begin{bmatrix} 8c_j m_j + 8c_j' n_j + n_j \delta_{j3} + c_j' \beta_j' + \beta_j' \delta_{j3} - c_j \alpha_j' \\ -m_j \delta_{j2} + \alpha_j' \delta_{j2} + (4-3\lambda)(n_j \delta_{j4} + c_j' \beta_j + \delta_{j4} \beta_j) \\ -c_j \alpha_j - m_j \delta_{j1} + \alpha_j \delta_{j1} \end{bmatrix} \right)$$

Subject to ($P_{7.2}$)

$$\sum_{j=1}^n \left[\begin{array}{l} 8a_{kj} m_j + 8a_{kj}' n_j + n_j \delta_{kj3} + a_{kj}' \beta_j' + \beta_j' \delta_{kj3} - a_{kj} \alpha_j' - m_j \delta_{kj2} + \alpha_j' \delta_{kj2} \\ + (4-3\lambda)(n_j \delta_{kj4} + a_{kj}' \beta_j + \delta_{kj4} \beta_j - a_{kj} \alpha_j - m_j \delta_{kj1} + \alpha_j \delta_{kj1}) \end{array} \right] \leq \text{or} =$$

$$\text{or} \geq \sum_{k=1}^m [8b_k + 8b_k' + \alpha_{k3} - \alpha_{k2} + (4-3\lambda)(\alpha_{k4} - \alpha_{k1})]$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2, \dots, n.$$

Step 2

Solve the crisp linear programming problem $(P_{7.2})$, to find the optimal solution

$$\{n_j, m_j, \alpha_j, \alpha_j', \beta_j, \beta_j'; j = 1, 2, \dots, n\}.$$

Step 3

Find the fuzzy optimal solution $\tilde{X} = [\tilde{x}_j]_{n \times 1}$ by putting the values of $n_j, m_j, \alpha_j, \alpha_j', \beta_j$ and β_j' , obtained from Step 2, in $\tilde{x}_j = [(m_j, n_j, \alpha_j', \beta_j'; \lambda)_{LR}, (m_j, n_j, \alpha_j, \beta_j; \rho)_{LR}]$ and the fuzzy optimal value by putting the values of \tilde{X} in $\tilde{C}^T \otimes \tilde{X}$.

7.1.2 Proposed method to deal with the sensitivity analysis of interval-valued fully fuzzy linear programming problems

In this section, a new method is proposed to deal with the sensitivity analysis of interval-valued fully fuzzy linear programming problem $(P_{7.1})$.

The steps of the proposed method are as follows:

Step 1

Check that which of the following case is to be considered

1. Change in fuzzy cost vector,
2. Change in fuzzy requirement vector,
3. Addition of a new fuzzy variable,
4. Addition of new fuzzy constraint,
5. Change in fuzzy coefficient matrix of the constraints.

Case 1: Change in the fuzzy cost vector

If the cost vector \tilde{C}^T changes to \tilde{C}'^T in the original interval-valued fully fuzzy linear programming problem $(P_{7.1})$ then replace $\Re(\tilde{C}^T \otimes \tilde{X})$ by $\Re(\tilde{C}'^T \otimes \tilde{X})$ in crisp linear programming $(P_{7.2})$ to obtain $(P_{7.3})$ and Go to Step 2.

Maximize (or Minimize) $\mathfrak{R}(\tilde{C}^T \otimes \tilde{X})$

Subject to (P_{7.3})

$$\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b})$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2, \dots, n.$$

Case 2: Change in fuzzy requirement vector

If the change in fuzzy requirement vector is made i.e., \tilde{b} is changed to \tilde{b}' in original interval-valued fully fuzzy linear programming problem (P_{7.1}) then replace $\mathfrak{R}(\tilde{b})$ by $\mathfrak{R}(\tilde{b}')$ in crisp linear programming problem (P_{7.2}) to obtain (P_{7.4}) and Go to Step 2.

Maximize (or Minimize) $\mathfrak{R}(\tilde{C}^T \otimes \tilde{X})$

Subject to (P_{7.4})

$$\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b}')$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2, \dots, n.$$

Case 3: Addition of a new fuzzy decision variable

Suppose a new non-negative interval-valued LR flat fuzzy decision variable, say \tilde{x}_{n+1} , is added in original interval-valued fully fuzzy linear programming problem (P_{7.1}) Assume that if \tilde{c}_{n+1} is cost and \tilde{A}_{n+1} is the column associated with \tilde{x}_{n+1} then replace $\mathfrak{R}(\tilde{A} \otimes \tilde{X})$ by $\mathfrak{R}(\tilde{A} \otimes \tilde{X} \oplus \tilde{A}_{n+1} \otimes \tilde{x}_{n+1})$ and $\mathfrak{R}(\tilde{C}^T \otimes \tilde{X})$ by $\mathfrak{R}(\tilde{C}^T \otimes \tilde{X} \oplus \tilde{c}_{n+1} \otimes \tilde{x}_{n+1})$ in (P_{7.2}) to obtain the crisp linear programming problem (P_{7.5}) and Go to Step 2.

Maximize (or Minimize) $\mathfrak{R}(\tilde{C}^T \otimes \tilde{X} \oplus \tilde{c}_{n+1} \otimes \tilde{x}_{n+1})$

Subject to (P_{7.5})

$$\mathfrak{R}(\tilde{A} \otimes \tilde{X} \oplus \tilde{A}_{n+1} \otimes \tilde{x}_{n+1}) \leq \text{or} = \text{or} \geq \mathfrak{R}(\tilde{b})$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2, \dots, n, n+1.$$

Case 4: Addition of a new fuzzy constraint

Suppose a new fuzzy constraint is added in the original interval-valued fully fuzzy linear programming problem $(P_{7.1})$ then replace $\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq_{or} =_{or} \geq \mathfrak{R}(\tilde{b})$ by $\mathfrak{R}(\tilde{A}' \otimes \tilde{X}) \leq_{or} =_{or} \geq \mathfrak{R}(\tilde{b}')$ in $(P_{7.2})$ to obtain crisp linear programming problem $(P_{7.6})$ and Go to Step 2.

$$\begin{aligned} & \text{Maximize (or Minimize)} && \mathfrak{R}(\tilde{C}^T \otimes \tilde{X}) \\ & \text{Subject to} && \end{aligned} \tag{P_{7.6}}$$

$$\mathfrak{R}(\tilde{A}' \otimes \tilde{X}) \leq_{or} =_{or} \geq \mathfrak{R}(\tilde{b}')$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2, \dots, n.$$

Case 5: Change in fuzzy constraint matrix

Suppose the column of a constraint matrix, corresponding to the fuzzy decision variable \tilde{x}_j , is changed from \tilde{A}_j to \tilde{A}_j' in the original interval-valued fully fuzzy linear programming problem then replace $\mathfrak{R}(\tilde{A} \otimes \tilde{X}) \leq_{or} =_{or} \geq \mathfrak{R}(\tilde{b})$ by $\mathfrak{R}(\tilde{A}' \otimes \tilde{X}) \leq_{or} =_{or} \geq \mathfrak{R}(\tilde{b})$ in $(P_{7.2})$ to obtain new crisp linear programming problem $(P_{7.7})$ and Go to Step 2.

$$\begin{aligned} & \text{Maximize (or Minimize)} && \mathfrak{R}(\tilde{C}^T \otimes \tilde{X}) \\ & \text{Subject to} && \end{aligned} \tag{P_{7.7}}$$

$$\mathfrak{R}(\tilde{A}' \otimes \tilde{X}) \leq_{or} =_{or} \geq \mathfrak{R}(\tilde{b})$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2, \dots, n.$$

Step 2

Apply the existing sensitivity analysis technique [116] to find the optimal solution of $(P_{7.3})$, $(P_{7.4})$, $(P_{7.5})$, $(P_{7.6})$ and $(P_{7.7})$ with the help of optimal solution of $(P_{7.2})$ and use Step 3 of the method, proposed in Section 7.1.1, to find the fuzzy optimal solution and fuzzy optimal

value of the resulting interval-valued fully fuzzy linear programming problem.

Remark 7.1 The other cases i.e., deletion of fuzzy decision variables, deletion of fuzzy constraints, simultaneous change in coefficients of fuzzy decision variables in the objective function and fuzzy requirement vectors etc. can also be solved by using the proposed method.

7.2 Illustrative example

In this section, proposed method is illustrated with the help of a numerical example:

Example 7.1 Consider the interval-valued fully fuzzy linear programming problem:

$$\text{Maximize } \left(\begin{array}{l} [(25, 27, 6, 8; .9)_{LR}, (25, 27, 7, 9; 1)_{LR}] \otimes [(m_1, n_1, \alpha_1', \beta_1'; .9)_{LR}, (m_1, n_1, \alpha_1, \beta_1; 1)_{LR}] \oplus \\ [(48, 50, 4, 6; .9)_{LR}, (48, 50, 5, 8; 1)_{LR}] \otimes [(m_2, n_2, \alpha_2', \beta_2'; .9)_{LR}, (m_2, n_2, \alpha_2, \beta_2; 1)_{LR}] \end{array} \right)$$

Subject to (E_{7.1})

$$\begin{aligned} & [(15, 17, 1, 2; .9)_{LR}, (15, 17, 5, 3; 1)_{LR}] \otimes [(m_1, n_1, \alpha_1', \beta_1'; .9)_{LR}, (m_1, n_1, \alpha_1, \beta_1; 1)_{LR}] \oplus \\ & [(30, 32, 5, 4; .9)_{LR}, (30, 32, 7, 8; 1)_{LR}] \otimes [(m_2, n_2, \alpha_2', \beta_2'; .9)_{LR}, (m_2, n_2, \alpha_2, \beta_2; 1)_{LR}] \\ & \leq [(450, 470, 20, 30; .9)_{LR}, (450, 470, 30, 70; 1)_{LR}] \end{aligned}$$

$$\begin{aligned} & [(24, 26, 3, 2; .9)_{LR}, (24, 26, 4, 9; 1)_{LR}] \otimes [(m_1, n_1, \alpha_1', \beta_1'; .9)_{LR}, (m_1, n_1, \alpha_1, \beta_1; 1)_{LR}] \oplus \\ & [(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}] \otimes [(m_2, n_2, \alpha_2', \beta_2'; .9)_{LR}, (m_2, n_2, \alpha_2, \beta_2; 1)_{LR}] \leq \\ & [(240, 260, 20, 50; .9)_{LR}, (240, 260, 60, 60; 1)_{LR}] \end{aligned}$$

$$\begin{aligned} & [(21, 23, 4, 1; .9)_{LR}, (21, 23, 5, 5; 1)_{LR}] \otimes [(m_1, n_1, \alpha_1', \beta_1'; .9)_{LR}, (m_1, n_1, \alpha_1, \beta_1; 1)_{LR}] \oplus \\ & [(14, 16, 2, 5; .9)_{LR}, (14, 16, 6, 8; 1)_{LR}] \otimes [(m_2, n_2, \alpha_2', \beta_2'; .9)_{LR}, (m_2, n_2, \alpha_2, \beta_2; 1)_{LR}] \leq \\ & [(280, 300, 10, 30; .9)_{LR}, (280, 300, 50, 40; 1)_{LR}] \end{aligned}$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2, \dots, n$$

where, $L(x) = R(x) = \text{Maximum}\{0, 1 - x\}$.

- (i) Discuss the effect of changing the fuzzy cost coefficients corresponding to the fuzzy decision variables \tilde{x}_1 and \tilde{x}_2 to $[(30, 32, 6, 8; .9)_{LR}, (30, 32, 7, 9; 1)_{LR}]$ and $[(50, 52, 4, 6; .9)_{LR}, (50, 52, 5, 8; 1)_{LR}]$ respectively on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.

- (ii) Discuss the effect of changing the fuzzy requirement vector from $[(450, 470, 20, 30; .9)_{LR}, (450, 470, 30, 70; 1)_{LR}], [(240, 260, 20, 50; .9)_{LR}, (240, 260, 60, 60; 1)_{LR}], [(280, 300, 10, 30; .9)_{LR}, (280, 300, 50, 40; 1)_{LR}]$ to $[(500, 550, 20, 30; .9)_{LR}, (500, 550, 30, 70; 1)_{LR}], [(300, 350, 20, 50; .9)_{LR}, (300, 350, 60, 60; 1)_{LR}]$ and $[(400, 450, 10, 30; .9), (400, 450, 50, 40; 1)_{LR}]$ respectively on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.
- (iii) Discuss the effect of addition of a new non-negative interval-valued LR flat fuzzy decision variable \tilde{x}_3 with cost $[(50, 52, 4, 6; .9)_{LR}, (50, 52, 5, 8; 1)_{LR}]$ and column vectors $[[[(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}], [(14, 16, 2, 5; .9)_{LR}, (14, 16, 6, 8; 1)_{LR}], [(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}]]^T$ on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.
- (iv) Discuss the effect of changing the column of fuzzy constraint matrix corresponding to the fuzzy decision variable \tilde{x}_2 by $[[[(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}], [(14, 16, 2, 5; .9)_{LR}, (14, 16, 6, 8; 1)_{LR}], [(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}]]^T$ on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.
- (v) Discuss the effect of addition of a new fuzzy constraint $[(24, 26, 3, 2; .9)_{LR}, (24, 26, 4, 9; 1)_{LR}] \otimes \tilde{x}_1 \oplus [(14, 16, 2, 5; .9)_{LR}, (14, 16, 6, 8; 1)_{LR}] \otimes \tilde{x}_2 \leq [(500, 550, 20, 30; .9)_{LR}, (500, 550, 30, 70; 1)_{LR}]$ on the fuzzy optimal solution and fuzzy optimal value of resulting fully fuzzy linear programming problem.

Solution: Using the method, proposed in Section 7.1.1, the fuzzy optimal solution and fuzzy optimal value of the interval-valued fully fuzzy linear programming problem, chosen in Example 7.1, can be obtained as follows:

Step 1

Using Step 1 of proposed method, the fully fuzzy linear programming problem $(E_{7.1})$ can be converted into the crisp linear programming problem $(E_{7.2})$.

$$\text{Maximize} \quad \frac{1}{16} \left[\begin{array}{l} 248.9m_1 + 235.7n_1 + 35\beta_1' + 46.8\beta_1 - 19\alpha_1' - 23.4\alpha_1 + \\ 373.5m_2 + 416.4n_2 + 56\beta_2' + 75.4\beta_2 - 44\alpha_2' - 55.9\alpha_2 \end{array} \right]$$

Subject to (E_{7.2})

$$112.5m_1 + 141.9n_1 + 19\beta_1' + 26\beta_1 - 14\alpha_1' - 13\alpha_1 + 225m_2 + 268n_2 + 36\beta_2' + 50\beta_2 - 25\alpha_2' - 29.9\alpha_2 \leq 7422$$

$$183.8m_1 + 221.7n_1 + 28\beta_1' + 45.5\beta_1 - 21\alpha_1' - 26\alpha_1 + 40.8m_2 + 72.5n_2 + 10\beta_2' + 16.9\beta_2 - 4\alpha_2' - 2.6\alpha_2 \leq 4030$$

$$157.5m_1 + 191.5n_1 + 24\beta_1' + 36.4\beta_1 - 17\alpha_1' - 20.8\alpha_1 + 102m_2 + 143.7n_2 + 21\beta_2' + 31.5\beta_2 - 12\alpha_2' - 10.4\alpha_2 \leq 4647$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2.$$

Step 2

The optimal solution of the crisp linear programming problem (E_{7.2}) is:

$$m_1 = 0, n_1 = 0, \alpha_1' = 0, \beta_1' = 22.38, \alpha_1 = 0, \beta_1 = 22.38, m_2 = 12.97, n_2 = 12.97, \alpha_2' = 0, \beta_2' = 0, \alpha_2 = 0, \beta_2 = 0.$$

Step 3

Using Step 3 of the method, proposed in Section 7.1.1, the fuzzy optimal solution of chosen interval-valued fully fuzzy linear programming problem (E_{7.1}) is:

$$\tilde{x}_1 = [(0, 0, 0, 22.38; .9)_{LR}, (0, 0, 0, 22.38; 1)_{LR}], \tilde{x}_2 = [(12.97, 12.97, 0, 0; .9)_{LR}, (12.97, 12.97, 0, 0; 1)_{LR}]$$

and the fuzzy optimal value is:

$$[(622.5, 648.5, 51.82, 861.12; .9)_{LR}, (622.5, 648.5, 64.85, 909.44; 1)_{LR}].$$

Using the method, proposed in Section 7.2.2, the fuzzy optimal solution and fuzzy optimal value of the resulting fully fuzzy linear programming problem can be obtained as follows:

- (i) Since the fuzzy cost coefficients corresponding to the fuzzy decision variables \tilde{x}_1 and \tilde{x}_2 changes to $[(30, 32, 6, 8; .9)_{LR}, (30, 32, 7, 9; 1)_{LR}]$ and $[(50, 52, 4, 6; .9)_{LR}, (50, 52, 5, 8; 1)_{LR}]$

respectively in the original fuzzy linear programming problem ($E_{7.1}$) so using Case 1 of the method proposed in Section 7.2.2 replace the crisp linear programming problem ($E_{7.2}$) by ($E_{7.3}$).

$$\begin{aligned} \text{Maximize} \quad & \frac{1}{16} \left[224.9m_1 + 275.7n_1 + 35\beta_1' + 46.8\beta_1 - 19\alpha_1' - 23.4\alpha_1 + \right. \\ & \left. 389.5m_2 + 432.4n_2 + 56\beta_2' + 75.4\beta_2 - 44\alpha_2' - 55.9\alpha_2 \right] \\ \text{Subject to} \quad & \end{aligned} \tag{E_{7.3}}$$

$$112.5m_1 + 141.9n_1 + 19\beta_1' + 26\beta_1 - 14\alpha_1' - 13\alpha_1 + 225m_2 + 268n_2 + 36\beta_2' + 50\beta_2 - 25\alpha_2' - 29.9\alpha_2 \leq 7422$$

$$183.8m_1 + 221.7n_1 + 28\beta_1' + 45.5\beta_1 - 21\alpha_1' - 26\alpha_1 + 40.8m_2 + 72.5n_2 + 10\beta_2' + 16.9\beta_2 - 4\alpha_2' - 2.6\alpha_2 \leq 4030$$

$$157.5m_1 + 191.5n_1 + 24\beta_1' + 36.4\beta_1 - 17\alpha_1' - 20.8\alpha_1 + 102m_2 + 143.7n_2 + 21\beta_2' + 31.5\beta_2 - 12\alpha_2' - 10.4\alpha_2 \leq 4647$$

$$\begin{aligned} n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \\ \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2. \end{aligned}$$

Applying the existing sensitivity analysis technique [116], the optimal solution of crisp linear programming problem ($E_{7.3}$) is:

$$\begin{aligned} m_1 = 4.01, n_1 = 4.01, \alpha_1' = 4.01, \beta_1' = 4.01, \alpha_1 = 4.01, \beta_1 = 4.01, m_2 = 12.83, n_2 = 12.83, \\ \alpha_2' = 0, \beta_2' = 0, \alpha_2 = 0, \beta_2 = 0. \end{aligned}$$

Using Step 3 of the method, proposed in Section 7.1.1, the fuzzy optimal solution of resulting interval-valued fully fuzzy linear programming problem is:

$$\tilde{x}_1 = [(4.01, 4.01, 4.01, 4.01; .9)_{LR}, (4.01, 4.01, 4.01, 4.01; 1)_{LR}],$$

$$\tilde{x}_2 = [(12.83, 12.83, 0, 0; .9)_{LR}, (12.83, 12.83, 0, 0; 1)_{LR}] \text{ and the fuzzy optimal value is:}$$

$$[(761.8, 795.48, 171.62, 1860.42; .9)_{LR}, (761.8, 795.48, 184.45, 1894.1; 1)_{LR}].$$

- (ii) Since, the requirement vector is changed from $[(450, 470, 20, 30; .9)_{LR}, (450, 470, 30, 70; 1)_{LR}]$, $[(240, 260, 20, 50; .9)_{LR}, (240, 260, 60, 60; 1)_{LR}]$, $[(280, 300, 10, 30; .9)_{LR}, [(280, 300, 50, 40; 1)_{LR}]$ to $[(500, 550, 20, 30; .9)_{LR}, (500, 550, 30, 70; 1)_{LR}]$, $[(300, 350, 20, 50; .9)_{LR}, (300, 350, 60,$

$60;1)_{LR}]$ and $[(400,450,10,30;.9),(400,450, 50,40;1)_{LR}]$ respectively in the original fully fuzzy linear programming problem $(E_{7.1})$ so using Case 2 of the method proposed in Section 7.2.2 replace the crisp linear programming problem $(E_{7.2})$ by $(E_{7.4})$.

$$\begin{aligned} \text{Maximize} \quad & \frac{1}{16} \left[248.9m_1 + 235.7n_1 + 35\beta_1' + 46.8\beta_1 - 19\alpha_1' - 23.4\alpha_1 + \right. \\ & \left. 373.5m_2 + 416.4n_2 + 56\beta_2' + 75.4\beta_2 - 44\alpha_2' - 55.9\alpha_2 \right] \\ \text{Subject to} \quad & \hspace{20em} (E_{7.4}) \end{aligned}$$

$$112.5m_1 + 141.9n_1 + 19\beta_1' + 26\beta_1 - 14\alpha_1' - 13\alpha_1 + 225m_2 + 268n_2 + 36\beta_2' + 50\beta_2 - 25\alpha_2' - 29.9\alpha_2 \leq 8462$$

$$183.8m_1 + 221.7n_1 + 28\beta_1' + 45.5\beta_1 - 21\alpha_1' - 26\alpha_1 + 40.8m_2 + 72.5n_2 + 10\beta_2' + 16.9\beta_2 - 4\alpha_2' - 2.6\alpha_2 \leq 5230$$

$$157.5m_1 + 191.5n_1 + 24\beta_1' + 36.4\beta_1 - 17\alpha_1' - 20.8\alpha_1 + 102m_2 + 143.7n_2 + 21\beta_2' + 31.5\beta_2 - 12\alpha_2' - 10.4\alpha_2 \leq 7377$$

$$\begin{aligned} n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \\ \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2. \end{aligned}$$

Applying the existing sensitivity analysis technique [116] the optimal solution of crisp linear programming problem $(E_{7.4})$ is

$$m_1 = 0, n_1 = 0, \alpha_1' = 0, \beta_1' = 15.07, \alpha_1 = 0, \beta_1 = 15.07, m_2 = 12.37, n_2 = 12.37, \alpha_2' = 0, \beta_2' = 0, \alpha_2 = 0, \beta_2 = 0.$$

Using Step 3 of the method, proposed in Section 7.1.1, the fuzzy optimal solution of resulting interval-valued fully fuzzy linear programming problem is:

$$\begin{aligned} \tilde{x}_1 &= [(0, 0, 0, 15.07;.9)_{LR}, (0, 0, 0, 15.07;.9)_{LR}], \\ \tilde{x}_2 &= [(12.37, 12.37, 0, 0;.9)_{LR}, (12.37, 12.37, 0, 0;1)_{LR}] \text{ and the fuzzy optimal value is:} \end{aligned}$$

$$[(593.76, 618.5, 49.48, 74.22;.9)_{LR}, (593.76, 618.5, 61.85, 98.96;1)_{LR}].$$

(iii) Since, a new non-negative interval-valued fuzzy decision variable \tilde{x}_3 with cost

$$[(50, 52, 4, 6;.9)_{LR}, (50, 52, 5, 8;1)_{LR}] \text{ and column vectors } [[(6, 8, 2, 2;.9)_{LR},$$

$$(6, 8, 4, 5;1)_{LR}], [(14, 16, 2, 5;.9)_{LR}, (14, 16, 6, 8;1)_{LR}], [(6, 8, 2, 2;.9)_{LR}, (6, 8, 4, 5;1)_{LR}]]^T \text{ is added}$$

in the original fully fuzzy linear programming problem ($E_{7.1}$) so using Case 3 of the method proposed in Section 7.1.1, replace the crisp linear programming problem ($E_{7.2}$) by ($E_{7.5}$).

$$\begin{aligned} & \text{Maximize} \quad \frac{1}{16} \left[\begin{array}{l} 248.9m_1 + 235.7n_1 + 35\beta_1' + 46.8\beta_1 - 19\alpha_1' - 23.4\alpha_1 + \\ 373.5m_2 + 416.4n_2 + 56\beta_2' + 75.4\beta_2 - 44\alpha_2' - 55.9\alpha_2 + \\ 389.5m_3 + 432.4n_3 + 58\beta_3' + 78\beta_3 - 46\alpha_3' - 58.5\alpha_3 \end{array} \right] \\ & \text{Subject to} \end{aligned} \tag{E_{7.5}}$$

$$112.5m_1 + 141.9n_1 + 19\beta_1' + 26\beta_1 - 14\alpha_1' - 13\alpha_1 + 225m_2 + 268n_2 + 36\beta_2' + 50\beta_2 - 25\alpha_2' - 29.9\alpha_2 + 40.8m_3 + 72.5n_3 + 10\beta_3' + 16.9\beta_3 - 4\alpha_3' - 2.6\alpha_3 \leq 7422$$

$$183.8m_1 + 221.7n_1 + 28\beta_1' + 45.5\beta_1 - 21\alpha_1' - 26\alpha_1 + 40.8m_2 + 72.5n_2 + 10\beta_2' + 16.9\beta_2 - 4\alpha_2' - 2.6\alpha_2 + 102.2m_3 + 143.7n_3 + 21\beta_3' + 31.5\beta_3 - 12\alpha_3' - 10.4\alpha_3 \leq 4030$$

$$157.5m_1 + 191.5n_1 + 24\beta_1' + 36.4\beta_1 - 17\alpha_1' - 20.8\alpha_1 + 102m_2 + 143.7n_2 + 21\beta_2' + 31.5\beta_2 - 12\alpha_2' - 10.4\alpha_2 + 40.8m_3 + 72.5n_3 + 10\beta_3' + 16.9\beta_3 - 4\alpha_3' - 2.6\alpha_3 \leq 4647$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; \\ j = 1, 2, 3.$$

Applying the existing sensitivity analysis technique [116] the optimal solution of crisp linear programming problem ($E_{7.5}$) is:

$$m_1 = 0, n_1 = 0, \alpha_1' = 0, \beta_1' = 0, \alpha_1 = 0, \beta_1 = 0, m_2 = 12.67, n_2 = 12.67, \alpha_2' = 0, \beta_2' = 0, \alpha_2 = 0, \beta_2 = 0, m_3 = 10.21, n_3 = 10.21, \alpha_3' = 0, \beta_3' = 0, \alpha_3 = 0, \beta_3 = 0.$$

Using Step 3 of the method, proposed in Section 7.1.1, the fuzzy optimal solution of resulting interval-valued fully fuzzy linear programming problem is:

$$\tilde{x}_1 = [(0, 0, 0, 0; .9)_{LR}, (0, 0, 0, 0; .9)_{LR}], \tilde{x}_2 = [(12.67, 12.67, 0, 0; .9)_{LR}, (12.67, 12.67, 0, 0; 1)_{LR}],$$

$$\tilde{x}_3 = [(10.21, 10.21, 0, 0; .9)_{LR}, (10.21, 10.21, 0, 0; 1)_{LR}] \text{ and the fuzzy optimal value is:}$$

$$[(608.16, 633.5, 50.68, 76.02; .9)_{LR}, (608.16, 633.5, 63.35, 101.36; 1)_{LR}].$$

- (iv) Since, the column of the constraint matrix corresponding to the interval-valued fuzzy variable \tilde{x}_2 is changed to $[(6, 8, 2, 2; .9)_{LR}, (6, 8, 4, 5; 1)_{LR}], [(14, 16, 2, 5; .9)_{LR}, (14, 16, 6, 8;$

1)_{LR}], [(6, 8, 2, 2; 9)_{LR}, (6, 8, 4, 5; 1)_{LR}]]^T in the original interval-valued fully fuzzy linear programming problem ($E_{7.1}$) so using Case 4 of the proposed method replace the crisp linear programming problem ($E_{7.2}$) by ($E_{7.6}$).

$$\text{Maximize } \frac{1}{16} \left[\begin{array}{l} 248.9m_1 + 235.7n_1 + 35\beta_1' + 46.8\beta_1 - 19\alpha_1' - 23.4\alpha_1 + \\ 373.5m_2 + 416.4n_2 + 56\beta_2' + 75.4\beta_2 - 44\alpha_2' - 55.9\alpha_2 \end{array} \right]$$

Subject to ($E_{7.6}$)

$$112.5m_1 + 141.9n_1 + 19\beta_1' + 26\beta_1 - 14\alpha_1' - 13\alpha_1 + 40.8m_2 + 72.5n_2 + 10\beta_2' + 16.9\beta_2 - 4\alpha_2' - 2.6\alpha_2 \leq 7422$$

$$183.8m_1 + 221.7n_1 + 28\beta_1' + 45.5\beta_1 - 21\alpha_1' - 26\alpha_1 + 102m_2 + 143.7n_2 + 21\beta_2' + 31.5\beta_2 - 12\alpha_2' - 10.4\alpha_2 \leq 4030$$

$$157.5m_1 + 191.5n_1 + 24\beta_1' + 36.4\beta_1 - 17\alpha_1' - 20.8\alpha_1 + 40.8m_2 + 72.5n_2 + 10\beta_2' + 16.9\beta_2 - 4\alpha_2' - 2.6\alpha_2 \leq 4647$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2.$$

Applying the existing sensitivity analysis technique [116] the optimal solution of crisp linear programming problem ($E_{7.6}$) is:

$$m_1 = 0, n_1 = 0, \alpha_1' = 0, \beta_1' = 0, \alpha_1 = 0, \beta_1 = 0, m_2 = 15.87, n_2 = 15.87, \alpha_2' = 0, \beta_2' = 0, \alpha_2 = 0, \beta_2 = 0.$$

Using Step 3 of the method, proposed in Section 7.1.1, the fuzzy optimal solution of resulting interval-valued fully fuzzy linear programming problem is:

$$\tilde{x}_1 = [(0, 0, 0, 0; 9)_{LR}, (0, 0, 0, 0; 9)_{LR}], \tilde{x}_2 = [(15.87, 15.87, 0, 0; 9)_{LR}, (15.87, 15.87, 0, 0; 1)_{LR}]$$

and the fuzzy optimal value is:

$$[(761.76, 793.5, 63.48, 95.22; 9)_{LR}, (761.76, 793.5, 79.35, 126.96; 1)_{LR}].$$

- (v) Since, a new fuzzy constraint $[(24, 26, 3, 2; 9)_{LR}, (24, 26, 4, 9; 1)_{LR}] \otimes \tilde{x}_1 \oplus [(14, 16, 2, 5; 9)_{LR}, (14, 16, 6, 8; 1)_{LR}] \otimes \tilde{x}_2 \leq [(500, 550, 20, 30; 9)_{LR}, (500, 550, 30, 70; 1)_{LR}]$ is added to the

original fully fuzzy linear programming problem ($E_{7.1}$) so using Case 5 of the proposed method replace the crisp linear programming problem ($E_{7.2}$) by crisp linear programming problem ($E_{7.7}$).

$$\text{Maximize } \frac{1}{16} \left[\begin{array}{l} 248.9m_1 + 235.7n_1 + 35\beta_1' + 46.8\beta_1 - 19\alpha_1' - 23.4\alpha_1 + \\ 373.5m_2 + 416.4n_2 + 56\beta_2' + 75.4\beta_2 - 44\alpha_2' - 55.9\alpha_2 \end{array} \right]$$

Subject to ($E_{7.7}$)

$$112.5m_1 + 141.9n_1 + 19\beta_1' + 26\beta_1 - 14\alpha_1' - 13\alpha_1 + 225m_2 + 268n_2 + 36\beta_2' + 50\beta_2 - 25\alpha_2' - 29.9\alpha_2 \leq 7422$$

$$183.8m_1 + 221.7n_1 + 28\beta_1' + 45.5\beta_1 - 21\alpha_1' - 26\alpha_1 + 40.8m_2 + 72.5n_2 + 10\beta_2' + 16.9\beta_2 - 4\alpha_2' - 2.6\alpha_2 \leq 4030$$

$$157.5m_1 + 191.5n_1 + 24\beta_1' + 36.4\beta_1 - 17\alpha_1' - 20.8\alpha_1 + 102m_2 + 143.7n_2 + 21\beta_2' + 31.5\beta_2 - 12\alpha_2' - 10.4\alpha_2 \leq 4647$$

$$183.8m_1 + 221.7n_1 + 28\beta_1' + 45.5\beta_1 - 21\alpha_1' - 26\alpha_1 + 102m_2 + 143.7n_2 + 21\beta_2' + 31.5\beta_2 - 12\alpha_2' - 10.4\alpha_2 \leq 8462$$

$$n_j - m_j \geq 0, \alpha_j - \alpha_j' \geq 0, \beta_j - \beta_j' \geq 0, m_j - \alpha_j \geq 0, \alpha_j \geq 0, \alpha_j' \geq 0, \beta_j \geq 0, \beta_j' \geq 0; j = 1, 2.$$

Applying the existing sensitivity analysis technique [116] the optimal solution of crisp linear programming problem ($E_{7.7}$) is:

$$m_1 = 0, n_1 = 0, \alpha_1' = 0, \beta_1' = 22.36, \alpha_1 = 0, \beta_1 = 22.36, m_2 = 12.97, n_2 = 12.97, \alpha_2' = 0, \beta_2' = 0, \alpha_2 = 0, \beta_2 = 0.$$

Using Step 3 of the method, proposed in Section 7.1.1, the fuzzy optimal solution of resulting interval-valued fully fuzzy linear programming problem is:

$$\tilde{x}_1 = [(0, 0, 0, 22.36; .9)_{LR}, (0, 0, 0, 22.36; .9)_{LR}],$$

$$\tilde{x}_2 = [(12.97, 12.97, 0, 0; .9)_{LR}, (12.97, 12.97, 0, 0; 1)_{LR}] \text{ the fuzzy optimal value is:}$$

$$[(622.56, 648.5, 51.88, 860.42; .9)_{LR}, (622.56, 648.5, 64.85, 908.72; 1)_{LR}].$$

7.3 Conclusions

On the basis of the presented study, it can be concluded that neither the existing method [115] nor the methods, proposed in previous chapters, can be used for solving interval-valued fully fuzzy linear programming problems and to deal with the sensitivity analysis of interval-valued fully fuzzy linear programming problems. While, the methods proposed in this chapter can be used for the same.

Chapter 8

Possible modifications in the fully fuzzy linear programming problems without affecting the fuzzy optimal solution^{‡‡}

Any modification in fully fuzzy linear programming problems may or may not affect on its fuzzy optimal solution. However, neither in the existing methods [35, 65, 95] nor in the methods proposed in Chapter 6 and Chapter 7, it is checked that the modifications in the fuzzy linear programming problem will affect on the fuzzy optimal solution of the original fuzzy linear programming problem or not. Also, neither the existing methods [35, 65, 95] nor the methods, proposed in Chapter 6 and Chapter 7, are applicable to find the possible modifications in the fully fuzzy linear programming problem without affecting its fuzzy optimal solution. In this chapter, a new method is proposed for the same, Also, proposed method is illustrated with the help of numerical example.

8.1 Proposed method

In this section, a new method is proposed to find the possible modifications in the fully fuzzy linear programming problem ($P_{6.3}$) without affecting its fuzzy optimal solution.

Step 1

Using Step 1 of the method, proposed in Chapter 6, find the fuzzy optimal solution and fuzzy optimal value of fully fuzzy linear programming problem ($P_{6.3}$).

Step 2

Check that which of the following case is to be considered:

^{‡‡} The contents of this chapter are communicated in *ASOR Bulletin*.

Case 1: Variation in the fuzzy cost

All such fuzzy cost \tilde{c}_j' which can be replaced by \tilde{c}_j corresponding to the fuzzy variable \tilde{x}_j in fuzzy linear programming problem $(P_{6.3})$, without any change in its fuzzy optimal solution can be obtained as follows:

Let us assume that the fuzzy cost \tilde{c}_j corresponding to the fuzzy decision variable $\tilde{x}_j = (m_j, n_j, \alpha_j, \beta_j)_{LR}$ is changed to \tilde{c}_j' in $(P_{6.3})$. Then, in the crisp linear programming problem $(P_{6.4})$ the cost corresponding to the variables m_j, n_j, α_j and β_j and hence the relative cost corresponding to some or all the non-basic variables of the optimal basis of $(P_{6.4})$ will be affected. If \tilde{c}_j' in $(P_{6.3})$ is chosen in such a manner that the relative costs corresponding to all the non basic variables of the optimal basis of $(P_{6.4})$ are greater than or equal to zero then the optimal solution of crisp linear programming problem $(P_{6.4})$ and hence the fuzzy optimal solution of $(P_{6.3})$ will remain unaffected.

Case 2: Variation in fuzzy requirement vector

All such fuzzy right hand side entries \tilde{b}_j of fuzzy requirement vector $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_j, \dots, \tilde{b}_m)^T$ which can be replaced by \tilde{b}_j' in fully fuzzy linear programming problem $(P_{6.3})$, without any change in its fuzzy optimal basis can be obtained as follows:

Let us assume that the right hand side entry \tilde{b}_j of the fuzzy requirement vector $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_j, \dots, \tilde{b}_m)^T$ is changed to \tilde{b}_j' in fully fuzzy linear programming problem $(P_{6.3})$. Then in the crisp linear programming problem $(P_{6.4})$ the right hand side vector corresponding to the basic variables m_j, n_j, α_j and β_j and hence the right hand side vector corresponding to some or all the basic variables of the optimal basis of $(P_{6.4})$ will be affected. If \tilde{b}_j' is chosen in such a manner that all the entries of the right hand side vector corresponding to all the

basic variables of the optimal basis of $(P_{6.4})$ are greater than or equal to zero then the optimal basis of $(P_{6.4})$ and hence the fuzzy optimal basis of $(P_{6.3})$ will remain unaffected.

Case 3: Addition of a fuzzy constraint

Suppose a new fuzzy constraint is added to the fully fuzzy linear programming problem $(P_{6.3})$, the current fuzzy optimal solution will remain same for the new problem as well if the fuzzy constraint to be added is satisfied by the current fuzzy optimal solution of $(P_{6.3})$ i.e., on adding all such fuzzy constraints that are satisfied by the current fuzzy optimal solution of $(P_{6.3})$, there will be no change in fuzzy optimal solution of fully fuzzy linear programming problem $(P_{6.3})$.

8.2 Illustrative example

In this section, to illustrate the proposed method the fuzzy sensitivity analysis problem is solved.

Example 8.1 Consider the following fully fuzzy linear programming problem

$$\text{Maximize} \quad [(2, 3, 2, 5)_{LR} \otimes \tilde{x}_1 \oplus (1, 2, 1, 3)_{LR} \otimes \tilde{x}_2 \oplus (3, 4, 3, 4)_{LR} \otimes \tilde{x}_3]$$

$$\text{Subject to} \quad (E_{8.1})$$

$$\left(2, 3, 1, \frac{1}{2}\right)_{LR} \otimes \tilde{x}_1 \oplus \left(1, 3, \frac{1}{2}, \frac{3}{2}\right)_{LR} \otimes \tilde{x}_2 \oplus (3, 4, 1, 1)_{LR} \otimes \tilde{x}_3 \leq_{\mathfrak{R}} (5, 7, 2, 2)_{LR}$$

$$(2, 3, 2, 5)_{LR} \otimes \tilde{x}_1 \oplus \left(1, 3, 1, \frac{1}{2}\right)_{LR} \otimes \tilde{x}_2 \leq_{\mathfrak{R}} (3, 5, 1, 1)_{LR}$$

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are non-negative *LR* flat fuzzy numbers

where, $L(x) = \text{maximum}\{0, 1 - x^2\}$, $R(x) = \text{maximum}\{0, 1 - x\}$.

- (i) Within what range the cost coefficient corresponding to the fuzzy decision variables \tilde{x}_2 varies so that the optimality remains unaffected.
- (ii) Within what range the first component of fuzzy requirement vector varies so that the optimality remains unaltered.

(iii) Check whether the addition of a new fuzzy constraint $(2,3,1,4)_{LR} \otimes \tilde{x}_1 \oplus (1,2,1,3)_{LR} \otimes \tilde{x}_2 \leq_{\mathfrak{R}} (2,4,1,1)_{LR}$ will effect the optimality of the given fully fuzzy linear programming problem or not.

Solution: The optimal solution of the chosen fully fuzzy linear programming problem $(E_{8.1})$ by using the proposed method can be obtained as follows:

Assuming $\tilde{x}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$, $\tilde{x}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ and $\tilde{x}_3 = (m_3, n_3, \alpha_3, \beta_3)_{LR}$, the fully fuzzy linear programming problem $(E_{8.1})$ can be written as:

$$\begin{aligned} \text{Maximize} \quad & [(2,3,2,5)_{LR} \otimes (m_1, n_1, \alpha_1, \beta_1)_{LR} \oplus (1,2,1,3)_{LR} \otimes (m_2, n_2, \alpha_2, \beta_2)_{LR} \oplus (3,4,3,4)_{LR} \\ & \otimes (m_3, n_3, \alpha_3, \beta_3)_{LR}] \end{aligned}$$

Subject to (E_{8.2})

$$\begin{aligned} & \left(2,3,1,\frac{1}{2}\right)_{LR} \otimes (m_1, n_1, \alpha_1, \beta_1)_{LR} \oplus \left(1,3,\frac{1}{2},\frac{3}{2}\right)_{LR} \otimes (m_2, n_2, \alpha_2, \beta_2)_{LR} \oplus (3,4,1,1)_{LR} \otimes (m_3, n_3, \alpha_3, \beta_3) \\ & \leq_{\mathfrak{R}} (5,7,2,2)_{LR} \end{aligned}$$

$$(2,3,2,5)_{LR} \otimes (m_1, n_1, \alpha_1, \beta_1)_{LR} \oplus \left(1,3,1,\frac{1}{2}\right)_{LR} \otimes (m_2, n_2, \alpha_2, \beta_2)_{LR} \leq_{\mathfrak{R}} (3,5,1,1)_{LR}$$

$(m_1, n_1, \alpha_1, \beta_1)_{LR}$, $(m_2, n_2, \alpha_2, \beta_2)_{LR}$ and $(m_3, n_3, \alpha_3, \beta_3)_{LR}$ are non-negative LR flat fuzzy numbers.

Step 1

Using Step 1(a) of the method, proposed in Chapter 6, the fully fuzzy linear programming problem $(E_{8.2})$ can be converted into the crisp linear programming problem $(E_{8.3})$.

$$\text{Maximize} \quad \left(\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{7n_2}{2} + \frac{5\beta_2}{2} + m_3 + 6n_3 + 4\beta_3 \right)$$

Subject to (E_{8.3})

$$\frac{4m_1}{3} + \frac{13n_1}{4} + \frac{7\beta_1}{4} - \frac{2\alpha_1}{3} + \frac{2m_2}{3} + \frac{15n_2}{4} + \frac{9\beta_2}{4} - \frac{\alpha_2}{3} + \frac{7m_3}{3} + \frac{9n_3}{2} + \frac{5\beta_3}{2} - \frac{4\alpha_3}{3} \leq \frac{35}{3}$$

$$\frac{2m_1}{3} + \frac{11n_1}{2} + 4\beta_1 + \frac{m_2}{3} + \frac{13n_2}{4} + \frac{7\beta_2}{4} \leq \frac{47}{6}$$

$$m_1 - \alpha_1 \geq 0, m_2 - \alpha_2 \geq 0, m_3 - \alpha_3 \geq 0$$

$$m_1 \leq n_1, m_2 \leq n_2, m_3 \leq n_3$$

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0$$

$$\beta_1 \geq 0, \beta_2 \geq 0, \beta_3 \geq 0.$$

The optimal solution of the crisp linear programming problem ($E_{8.3}$) is:

$$m_1 = 0, m_2 = 0, m_3 = 0, n_1 = 0, n_2 = 0, n_3 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \beta_1 = 1.957, \beta_2 = 0, \beta_3 = 3.293$$

and the optimal value is 21.005 .

Using Step 1(c) of the method, proposed in Chapter 6, the fuzzy optimal solution of the fully fuzzy linear programming problem ($E_{8.1}$) is:

$$\tilde{x}_1 = (0, 0, 0, 1.957)_{LR}, \tilde{x}_2 = (0, 0, 0, 0)_{LR}, \tilde{x}_3 = (0, 0, 0, 3.293)_{LR} \text{ and the fuzzy optimal value is: } (0, 0, 0, 42)_{LR}.$$

Case 1: Variation in the fuzzy cost

Suppose, the fuzzy cost $\tilde{c}_2 = (1, 2, 1, 3)_{LR}$ corresponding to the fuzzy decision variable \tilde{x}_2 in fully fuzzy linear programming problem ($E_{8.1}$) is replaced by a non-negative LR flat fuzzy number $\tilde{c}_2' = (m, n, \alpha, \beta)_{LR}$. Thus by using Case 1 of the proposed method, the cost corresponding to the variables m_2, n_2, α_2 and β_2 in crisp linear programming problem ($E_{8.3}$)

changes to $m - \frac{2}{3}\alpha$, $n + \frac{\beta}{2}$, $-\frac{2}{3}m + \frac{2}{3}\alpha$ and $\frac{n}{2} + \frac{\beta}{2}$ respectively and hence the relative cost

corresponding to some of the non basic variable of the optimal basis changes to

$$-m + \frac{2}{3}\alpha - n - \frac{\beta}{2} + 16.32, -\frac{m}{3} - n - \frac{\beta}{2} + 21.28, m - \frac{2}{3}\alpha + n + \frac{\beta}{2} - 16.32, \frac{m}{3} + n + \frac{\beta}{2} - 15.28, n + \beta - 16.48 \text{ and } -n - \frac{\beta}{2} + 13.92 .$$

Thus the optimal solution of ($E_{8.3}$) and hence the fuzzy optimal solution of fully fuzzy linear

programming problem $(E_{8.1})$ will remain same if the relative costs corresponding to the non basic variables of the optimal basis of $(E_{8.3})$ are greater than or equal to zero i.e., if it satisfies the following inequalities:

$$m - \frac{2}{3}\alpha + n + \frac{\beta}{2} \leq 16.32,$$

$$\frac{m}{3} + n + \frac{\beta}{2} \leq 21.28,$$

$$m - \frac{2}{3}\alpha + n + \frac{\beta}{2} \geq 16.32,$$

$$\frac{m}{3} + n + \frac{\beta}{2} \geq 15.28,$$

$$n + \beta \geq 16.48, n + \frac{\beta}{2} \leq 13.92,$$

$$m \leq n, m - \alpha \geq 0, \alpha \geq 0, \beta \geq 0.$$

e.g., Since, $\tilde{c}_2' = (m, n, \alpha, \beta)_{LR} = (2.40, 13.92, 0, 0)_{LR}$ satisfies all the inequalities so on replacing the fuzzy cost $\tilde{c}_2 = (1, 2, 1, 3)_{LR}$ by $\tilde{c}_2' = (2.40, 13.92, 0, 0)_{LR}$ the fuzzy optimal solution of fully fuzzy linear programming problem $(E_{8.1})$ will remain unaffected.

Case 2: Variation in fuzzy requirement vector \tilde{b}

Suppose the right hand side entry $\tilde{b}_1 = (5, 7, 2, 2)_{LR}$ of the fuzzy requirement vector $\tilde{b} = (\tilde{b}_1, \tilde{b}_2)^T$ is replaced by $\tilde{b}_1' = (m, n, \alpha, \beta)_{LR}$ in fully fuzzy linear programming problem $(E_{8.1})$. Thus using Case 2 of the proposed method, in the crisp linear programming $(E_{8.3})$ the right hand side vector corresponding to the basic variable β_2 changes to $.4m + .4n - \frac{.8\alpha}{3} + .2\beta - 1.40$.

Thus the optimal solution of $(E_{8.3})$ and hence the fuzzy optimal solution of fully fuzzy linear programming problem $(E_{8.1})$ will remain same if all the entries of the right hand side vector corresponding to all the basic variables of the optimal basis of $(E_{8.3})$ are greater than or equal to zero i.e., if it satisfies the following inequalities:

$$.4m + .4n - \frac{.8\alpha}{3} + .2\beta - 1.40 \geq 0,$$

$$m \leq n, \quad m - \alpha \geq 0, \quad \alpha \geq 0, \quad \beta \geq 0.$$

e.g., Since, $\tilde{b}_1' = (m, n, \alpha, \beta)_{LR} = (3, 4, 2, 2)_{LR}$ satisfies all the inequalities so on replacing the fuzzy right hand side entry $\tilde{b}_1 = (5, 7, 2, 2)_{LR}$ by $\tilde{b}_1' = (3, 4, 2, 2)_{LR}$, the fuzzy optimal basis of fully fuzzy linear programming problem $(E_{8.1})$ will remain unaffected.

Case 3: Addition of a fuzzy constraint

Suppose, a new fuzzy constraint $(2, 3, 1, 4)_{LR} \otimes \tilde{x}_1 \oplus (1, 2, 1, 3)_{LR} \otimes \tilde{x}_2 \leq_{\mathfrak{R}} (2, 4, 1, 1)_{LR}$ is added to the original fuzzy linear programming problem $(E_{8.1})$. Since this fuzzy constraint is satisfied by the current optimal solution of $(E_{8.1})$ so the fuzzy optimal solution of $(E_{8.1})$ remains unchanged.

Remark 8.1: In the same manner, the possible modification in the interval-valued fully fuzzy linear programming problem can also be easily calculated.

8.3 Conclusions

On the basis of presented study it can be concluded that neither the existing methods [35, 65, 95] nor the methods, proposed in Chapter 6 and Chapter 7, can be used to find the possible modifications in the fully fuzzy linear programming problem without affecting its fuzzy optimal solution. While, the methods, proposed in this chapter, can be used for the same.

Chapter 9

Future scope

The following may be treated as future directions:

In the literature, there was no method to deal with the sensitivity analysis of fully fuzzy linear programming problems and interval-valued fully fuzzy linear programming problems. So, in Chapter 6 and in Chapter 7, new methods are proposed for the same. However, to apply the methods, proposed in Chapter 6 and Chapter 7 there is need to convert the fully fuzzy linear programming problems into crisp linear programming problems.

In future, it may be tried to develop such new methods which can be used directly to deal with the sensitivity analysis of fully fuzzy linear programming problems and interval-valued fully fuzzy linear programming problems without converting it into crisp linear programming problems.

Bibliography

- [1] R. P. Aggarwal, I. Ahmad, A Jayswal, Higher-order symmetric duality in nondifferentiable multiobjective programming problems involving generalized cone convex functions, *Mathematical and Computer Modelling*, 52 (2010) 1644-1650.
- [2] I. Ahmad, T. R Gulati, Mixed type duality for multiobjective variational problems with generalized (F, ρ) -convexity, *Journal of Mathematical Analysis and Applications*, 306 (2005) 669-683.
- [3] T. Allahviranloo, F.H. Lotfi, L. Alizadeh, N. Kiani, Degeneracy in fuzzy linear programming problems, *The Journal of Fuzzy Mathematics*, 17 (2009) 389-402.
- [4] T. Allahviranloo, F.H. Lotfi, M.K. Kiasary, N.A. Kiani, L. Alizadeh, Solving fully fuzzy linear programming problem by the ranking function, *Applied Matematical Sciences*, 2 (2008) 19-32.
- [5] T. Allahviranloo, N. Mikaeilvand, F.H. Lotfi, M.K. Kiasari, Fully fuzzy linear programming problem with positive or negative core, *Far East Journal of Applied Mathematics*, 33 (2008) 337-350.
- [6] T. Allahviranloo, K.H. Shamsolkotabi, N.A. Kiani, L. Alizadeh, Fuzzy integer linear programming problems, *International Journal of Contemporary Mathematical Sciences*, 2 (2007) 167-181.
- [7] N.M. Amiri, S.H. Nasser, Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables, *Fuzzy Sets and Systems*, 158 (2007) 1961-1978.
- [8] A. Baykasoglu, T. Gocken, A direct solution approach to fuzzy mathematical programs with fuzzy decision variables, *Expert Systems with Applications*, 39 (2012) 1972-1978.
- [9] M.S. Bazaraa, J.J Jarvis, H.D Sherali, *Linear Programming and Network Flows*, Second Edition, Wiley, New York, 1990.

- [10] C.R. Bector, S. Chandra, On duality in linear programming under fuzzy environment, *Fuzzy Sets and Systems*, 125 (2002) 317-325.
- [11] C.R. Bector, S. Chandra, *Fuzzy Mathematical Programming and Fuzzy Matrix Games*, Springer-Verlag, 2005.
- [12] R.E. Bellman, L.A. Zadeh, Decision making in a fuzzy environment, *Management Science*, 17 (1970) 141-164.
- [13] J.J. Buckley, Possibilistic linear programming with triangular fuzzy numbers, *Fuzzy Sets and Systems*, 26 (1988) 135-138.
- [14] J.J. Buckley, Solving possibilistic linear programming problems, *Fuzzy Sets and Systems*, 31 (1989) 329-341.
- [15] J.J. Buckley, Fuzzy genetics algorithm and applications, *Fuzzy Sets and Systems*, 61 (1994) 129-136.
- [16] J.J. Buckley, Joint solution to fuzzy programming problems, *Fuzzy Sets and Systems*, 72 (1995) 215-220.
- [17] J. Buckley, T. Feuring, Evolutionary algorithm solution to fuzzy problems: Fuzzy linear programming, *Fuzzy Sets and Systems*, 109 (2000) 35-53.
- [18] J.M. Cadenas, J.L. Verdegay, Using fuzzy numbers in linear programming, *IEEE Transactions on Systems, Man and Cybernetics-Part B: Cybernetics*, 27 (1997) 1016-1022.
- [19] Q. Cai, Z. Hao, S. Pan, The solution of linear programming with LR fuzzy numbers in objective function, *Proceedings Second International Conference on Fuzzy Information and Engineering*, 40 (2007) 988-999.
- [20] L.M. Campos, A. Gonzalez, A subjective approach for ranking fuzzy numbers, *Fuzzy Sets and Systems*, 29 (1989) 145-153.
- [21] L. Campos, J.L. Verdegay, Linear programming problems and ranking of fuzzy numbers, *Fuzzy Sets and Systems*, 32 (1989) 1-11.

- [22] C. Carlsson, P. Korhonen, A parametric approach to fuzzy linear programming, *Fuzzy Sets and Systems*, 20 (1986) 17-30.
- [23] S. Chanas, The use of parametric programming in fuzzy linear programming, *Fuzzy Sets and Systems*, 11 (1983) 229-241.
- [24] S. Chanas, M. Delgado, J.L. Verdegay, M.A. Vila, Ranking fuzzy interval numbers in the setting of random sets, *Information Sciences*, 69 (1993) 201-217.
- [25] J. Chiang, Fuzzy linear programming based on statistical confidence interval and interval-valued fuzzy set, *European Journal of Operational Research*, 129 (2001) 65-86.
- [26] K. Dahiya, V. Verma, Positive sensitivity analysis in linear programming with bounded variables, *ASOR Bulletin*, 26 (2007) 2-26.
- [27] G.B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, New Jersey, 1963.
- [28] M. Dehghan, B. Hashemi, M. Ghatee, Computational methods for solving fully fuzzy linear systems, *Applied Mathematics and Computation*, 179 (2006) 328-343.
- [29] M. Dehghan, B. Hasemi, M. Ghatee, Solution of the fully fuzzy linear systems using iterative techniques, *Chaos Solitons Fractals*, 34 (2007) 316-336.
- [30] M. Delgado, J.L. Verdegay, M.A. Vila, A general model for fuzzy linear programming, *Fuzzy Sets and Systems*, 29 (1989) 21-29.
- [31] D.S. Dinagar, K. Palanivel, The transportation problem in fuzzy environment, *International Journal of Algorithms, Computing and Mathematics*, 2 (2009) 65-71.
- [32] U. Diwekar, *Introduction to Applied Optimization*, Second Edition, Springer-Verlag, Cambridge, MA, 2008.
- [33] D. Dubois, H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980.
- [34] D. Dutta, J.R. Rao, R.N. Tiwari, Sensitivity analysis in fuzzy linear fractional programming problem, *Fuzzy Sets and Systems*, 48 (1992) 211-216.

- [35] A. Ebrahimnejad, Sensitivity analysis in fuzzy number linear programming problems, *Mathematical and Computer Modelling*, 53 (2011) 1878-1888.
- [36] A. Ebrahimnejad, Some new results in linear programs with trapezoidal fuzzy numbers: Finite convergence of the Ganesan and Veeramani's method and a fuzzy revised simplex method, *Applied Mathematical Modelling*, 35 (2011) 4526-4540.
- [37] A. Ebrahimnejad, A primal-dual simplex algorithm for solving linear programming problems with symmetric trapezoidal fuzzy numbers, *Applied Mathematics*, 2 (2011) 676-684.
- [38] A. Ebrahimnejad, S.H. Nasseri, Using complementary slackness property to solve linear programming with fuzzy parameters, *Fuzzy Information and Engineering*, 1 (2009) 233-245.
- [39] A. Ebrahimnejad, S.H. Nasseri, A dual simplex method for bounded linear programmes with fuzzy numbers, *International Journal of Mathematics in Operational Research*, 2 (2010) 762-779.
- [40] A. Ebrahimnejad, S.H. Nasseri, F.H. Lotfi, Bounded linear programs with trapezoidal fuzzy numbers, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 18 (2010) 269-286.
- [41] A. Ebrahimnejad, S.H. Nasseri, F.H. Lotfi, M. Soltanifar, A primal-dual method for linear programming problems with fuzzy variables, *European Journal of Industrial Engineering*, 4 (2010) 189-209.
- [42] A. Ebrahimnejad, S.H. Nasseri, S.M. Mansourzadeh, Bounded primal simplex algorithm for bounded linear programming with fuzzy cost coefficients, *International Journal of Operations Research and Information Systems*, 2 (2011) 100-124.
- [43] S.C. Fang, C.F. Hu, H.F. Wang, S.Y. Wu, Linear programming with fuzzy coefficients in constraints, *Computers and Mathematics with Applications*, 37 (1999) 63-76.
- [44] R. Fuller, On stability in fuzzy linear programming problems, *Fuzzy Sets and Systems*, 30

(1989) 339-344.

- [45] R. Fuller, H.J. Zimmermann, Fuzzy reasoning for solving fuzzy mathematical programming problems, *Fuzzy Sets and Systems*, 60 (1993) 121-133.
- [46] K. Ganesan, P. Veeramani, Fuzzy linear programs with trapezoidal fuzzy numbers, *Annals of Operations Research*, 143 (2006) 305-315.
- [47] A.N. Gani, C. Duraisamy, C. Veeramani, A note on fuzzy linear programming problem using LR fuzzy number, *International Journal of Algorithms, Computing and Mathematics*, 2 (2009) 93-106.
- [48] T.R. Gulati, S.K. Gupta, I. Ahmad, Second-order symmetric duality with cone constraints, *Journal of Computational and Applied Mathematics*, 220 (2008) 347-354.
- [49] P. Gupta, D. Bhatia, Sensitivity analysis in fuzzy multi-objective linear fractional programming problem, *Fuzzy Sets and Systems*, 122 (2001) 229-236.
- [50] P. Gupta, M.K. Mehlawat, An algorithm for a fuzzy transportation problem to select a new type of coal for a steel manufacturing unit, *TOP*, 15 (2007) 114-137.
- [51] P. Gupta, M.K. Mehlawat, Bector-Chandra type duality in fuzzy linear programming with exponential membership functions, *Fuzzy Sets and Systems*, 160 (2009) 3290-3308.
- [52] H. Hamacher, H. Leberling, H.J. Zimmermann, Sensitivity analysis in fuzzy linear programming, *Fuzzy Sets and Systems*, 1 (1978) 269-281.
- [53] S.H. Hashemi, M. Modarres, E. Nasrabadi, M.M. Nasrabadi, Fully fuzzified linear programming, solution and duality, *Journal of Intelligent and Fuzzy Systems*, 17 (2006) 253-261.
- [54] A. Hatami-Marbini, M. Tavana, An extension of the linear programming method with fuzzy parameters, *International Journal of Mathematics in Operational Research*, 3 (2011) 44-55.
- [55] R. Jain, Decision making in the presence of fuzzy variables, *IEEE Transactions on Systems, Man and Cybernetics*, 6 (1976) 698-703.

- [56] S. Jain, A. Mangal, P.R. Parihar, Solution of fuzzy linear fractional programming problem, *OPSEARCH*, 48 (2011) 129-135.
- [57] M. Jayalakshmi, P. Pandian, A new method for finding an optimal fuzzy solution for fully fuzzy linear programming problems, *International Journal of Engineering Research and Applications*, 2 (2012) 247-254.
- [58] M. Jimenez, M. Arenas, A. Bilbao, M. V. Rodriguez, Linear programming with fuzzy parameters: An interactive method resolution, *European Journal of Operational Research*, 177 (2007) 1599-1609.
- [59] M. Jimenez, J.M. Cadenas, J.L. Verdegay, G. Sanchez, Solving fuzzy optimization problems by evolutionary algorithms, *Information Sciences*, 152 (2003) 303-311.
- [60] X. Jiuping, A kind of fuzzy linear programming problems based on interval-valued fuzzy sets, *Applied Mathematics-A Journal of Chinese Universities*, 15 (2000) 65-72.
- [61] H. S. Kasana, K. D. Kumar, *Introductory Operations Research: Theory and Applications*, Springer-Verlag, Berlin Heidelberg, New York, 2004.
- [62] A. Kaufmann, M.M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Applications*, Van Nostrand Reinhold, New York, 1985.
- [63] A. Kaufmann, M.M. Gupta, *Fuzzy Mathematical Models in Engineering and Management Science*, Amsterdam Netherland, Elsevier, 1988.
- [64] Kavita, S.P. Yadav, S. Kumar, A multi-criteria interval-valued intuitionistic fuzzy group decision making for supplier selection with TOPSIS method, *Lecture Notes in Computer Science*, 5908 (2009) 303-312.
- [65] B. Kheirfam, F. Hasani, Sensitivity analysis for fuzzy linear programming problems with fuzzy variables, *Advanced Modeling and Optimization*, 12 (2010) 257-272.
- [66] K. Kim, K.S. Park, Ranking fuzzy number with index of optimism, *Fuzzy Sets and Systems*, 35 (1990) 143-150.
- [67] G.J. Klir, B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice-Hall,

Inc., New Jersey, USA, 1996.

- [68] A Kumar, A Kaur, Application of classical transportation methods to find the fuzzy optimal solution of fuzzy transportation problems, *Fuzzy Information and Engineering*, 1 (2011) 81-99.
- [69] A. Kumar, J. Kaur, P. Singh, A new method for solving fully fuzzy linear programming problems, *Applied Mathematical Modelling*, 35 (2011) 817-823.
- [70] Y.J. Lai, C.L. Hwang, *Fuzzy Mathematical Programming: Methods and Applications*, Springer-Verlag, Heidelberg, 1992.
- [71] Y.J. Lai, C.L. Hwang, A new approach to some possibilistic linear programming problem, *Fuzzy Sets and Systems*, 49 (1992) 121-133.
- [72] F.T. Lin, M.G. Lee, Fuzzy constraints linear programming: A genetic algorithm approach, *International Journal of Innovative Computing, Information and Control*, 3 (2007) 71-83.
- [73] T.S. Liou, M.J. Wang, Ranking fuzzy numbers with integral value, *Fuzzy Sets and Systems*, 50 (1992) 247-255.
- [74] H. Liu, A. Abraham, An hybrid fuzzy variable neighborhood particle swarm optimization algorithm for solving quadratic assignment problem, *Journal of Universal Computer Science*, 13 (2007) 1309-1331.
- [75] F.H. Lotfi, T. Allahviranloo, M.A. Jondabeha, L. Alizadeh, Solving a fully fuzzy linear programming using lexicography method and fuzzy approximate solution, *Applied Mathematical Modelling*, 33 (2009) 3151-3156.
- [76] F.H. Lotfi, M.A. Jondabeh, M. Faizrahnemoon, Sensitivity analysis in fuzzy environment, *Applied Mathematical Sciences*, 4 (2010) 1635-1646.
- [77] M.K. Luhandjula, Linear programming under randomness and fuzziness, *Fuzzy Sets and Systems*, 10 (1983) 45-55.
- [78] M.K. Luhandjula, Linear programming with a possibilistic objective function, *European Journal of Operational Research*, 31 (1987) 110-117.

- [79] M. Maa, A. Kandel, M. Friedman, A new approach for defuzzification, *Fuzzy Sets and Systems*, 111 (2000) 351-356.
- [80] N. Mahdavi-Amiri, S.H. Nasser, Duality in fuzzy variable linear programming, *World Academy of Science, Engineering and Technology*, 6 (2005) 115-117.
- [81] N. Mahdavi-Amiri, S.H. Nasser, Duality in fuzzy number linear programming by the use of a certain linear ranking function, *Applied Mathematics and Computation*, 180 (2006) 206-216.
- [82] N. Mahdavi-Amiri, S.H. Nasser, Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables, *Fuzzy Sets and Systems*, 158 (2007) 1961-1978.
- [83] N. Mahdavi-Amiri, S.H. Nasser, A. Yazdani, Fuzzy primal simplex algorithms for solving fuzzy linear programming problems, *Iranian Journal of Operations Research*, 1 (2009) 68-84.
- [84] H.R. Maleki, Ranking functions and their applications to fuzzy linear programming, *Far East Journal of Mathematical Sciences*, 4 (2002) 283-301.
- [85] H.R. Maleki, M. Mashinchi, Fuzzy number linear programming: A probabilistic approach, *Journal of Applied Mathematics and Computing*, 15 (2004) 333-341.
- [86] H.R. Maleki, M. Tata, M. Mashinchi, Linear programming with fuzzy variables, *Fuzzy Sets and Systems*, 109 (2000) 21-33.
- [87] C. Mohan, S.K. Verma, Interactive algorithms using fuzzy concepts for solving mathematical models of real life optimization problems, In J.L. Verdegay (Ed.), *Fuzzy Sets Based Heuristics for Optimization*, Springer-Verlag, Berlin, Heidelberg, (2003) 123-137.
- [88] B.K. Mohanty, A procedure for measuring uncertainties due to lack of information in a fuzzy game theory problem, *International Journal of Systems Science*, 25 (1994) 2309-2317.

- [89] Y. Nakahara, M. Gen, A method for solving linear programming problems with triangular fuzzy coefficients using new ranking index, *Computers and Industrial Engineering*, 25 (1993) 1-4.
- [90] S.H. Nasser, A new method for solving fuzzy linear programming by solving linear programming, *Applied Mathematical Sciences*, 2 (2008) 2473-2480.
- [91] S.H. Nasser, E. Ardil, Simplex method for fuzzy variable linear programming problems, *World Academy of Science, Engineering and Technology*, 8 (2005) 198-202.
- [92] S.H. Nasser, E. Ardil, A. Yazdani, R. Zaefarian, Simplex method for solving linear programming problems with fuzzy numbers, *World Academy of Science, Engineering and Technology*, 10 (2005) 284-288.
- [93] S.H. Nasser, A. Ebrahimnejad, A fuzzy dual simplex method for a fuzzy number linear programming problem, *Advances in Fuzzy Sets and Systems*, 5 (2010) 81-95.
- [94] S.H. Nasser, A. Ebrahimnejad, A fuzzy primal simplex algorithm and its application for solving exible linear programming problems, *European Journal of Industrial Engineering*, 4 (2010) 372-389.
- [95] S.H. Nasser, A. Ebrahimnejad, Sensitivity analysis on linear programming problems with trapezoidal fuzzy variables, *International Journal of Operation Research and Information Systems*, 2 (2011) 22-39.
- [96] S.H. Nasser, A. Ebrahimnejad, S. Mizuno, Duality in fuzzy linear programming with symmetric trapezoidal numbers, *Applications and Applied Mathematics*, 5 (2010) 1467-1482.
- [97] S.H. Nasser, B. Khabiri, Revised fuzzy simplex algorithm for linear programming problems with fuzzy variables using linear ranking functions, *International Journal of Mathematics and Computation*, 6 (2010) 44-54.
- [98] S.H. Nasser, B. Khabiri, A revised simplex algorithm for fuzzy number linear programming problems using linear ranking functions, *International Journal of*

Mathematics and Computation, 8 (2010) 114-126.

- [99] S.H. Nasseri, N. Mahdavi-Amiri, Some duality results on linear programming problems with symmetric fuzzy numbers, *Fuzzy Information and Engineering*, 1 (2009) 61-68.
- [100] H.M. Nehi, H.R. Maleki, M. Mashinchi, Solving fuzzy number linear programming problem by lexicographic ranking function, *Italian Journal of Pure and Applied Mathematics*, 15 (2004) 9-20.
- [101] P. Pandian, M. Jayalakshmi, A new method for solving integer linear programming problems with fuzzy variables, *Applied Mathematical Sciences*, 4 (2010) 997-1004.
- [102] R.E. Precup, S. Preitl, Stability and sensitivity analysis of fuzzy control systems, *Mechatronics Applications*, 3 (2006) 61-76.
- [103] J. Ramik, Duality in fuzzy linear programming: Some new concepts and results, *Fuzzy Optimization and Decision Making*, 4 (2005) 25-39.
- [104] J. Ramik, J. Rimanek, Inequality relation between fuzzy numbers and its use in fuzzy optimization, *Fuzzy Sets and Systems*, 16 (1985) 123-138.
- [105] L.V. Reddy, R.N. Mukherjee, Composite nonsmooth multiobjective programs with (V, ρ) - invexity, *Journal of Mathematical Analysis and Applications*, 235 (1999) 567-577.
- [106] L.V. Reddy, R.N. Mukherjee, Some results on mathematical programming with generalized ratio invexity, *Journal of Mathematical Analysis and Applications*, 240 (1999) 299-310.
- [107] H. Rommelfanger, Interactive decision making in fuzzy linear optimization problems, *European Journal of Operational Research*, 41 (1989) 210-217.
- [108] H. Rommelfanger, Fuzzy linear programming and applications, *European Journal of Operational Research*, 92 (1996) 512-527.
- [109] H. Rommelfanger, R. Hanuscheck, J. Wolf, Linear programming with fuzzy objectives, *Fuzzy Sets and Systems*, 29 (1989) 31-48.

- [110] M. Sakawa, H. Yano, An interactive fuzzy satisfying method for multi-objective linear fractional programming problems, *Fuzzy Sets and Systems*, 28 (1988) 129-144.
- [111] A. Sengupta, T.K. Pal, On comparing interval numbers, *European Journal of Operational Research*, 127 (2000) 28-43.
- [112] T. Shaocheng, Interval number and fuzzy number linear programmings, *Fuzzy Sets and Systems*, 66 (1994) 301-306.
- [113] I.M. Stancu-Minasian, B. Pop, On a fuzzy set approach to solving multiple objective linear fractional programming problem, *Fuzzy Sets and Systems*, 134 (2003) 397-405.
- [114] T. Shaocheng, Interval number and fuzzy number linear programmings, *Fuzzy Sets and Systems*, 66 (1994) 301-306.
- [115] J.S. Su, Fuzzy programming based on interval-valued fuzzy numbers and ranking, *International Journal of Contemporary Mathematical Sciences*, 2 (2007) 393-410.
- [116] H.A. Taha, *Operations Research: An Introduction*, Prentice-Hall, New Jersey, 2003.
- [117] H. Tanaka, K. Asai, Fuzzy linear programming problems with fuzzy numbers, *Fuzzy Sets and Systems*, 13 (1984) 1-10.
- [118] H. Tanaka, H. Ichihashi, K. Asai, A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers, *Control and Cybernetics*, 13 (1984) 185-194.
- [119] H. Tanaka, H. Ichihashi, K. Asai, A value of information in fuzzy linear programming problems via sensitivity analysis, *Fuzzy Sets and Systems*, 18 (1986) 119-129.
- [120] H. Tanaka, T. Okuda, K. Asai, On fuzzy mathematical programming, *Journal of Cybernetics*, 3 (1973) 37-46.
- [121] S. Tong, Interval number and fuzzy number linear programming, *Fuzzy Sets and Systems*, 66 (1994) 301-306.
- [122] J.L. Verdegay, A dual approach to solve the fuzzy linear programming problems, *Fuzzy Sets and Systems*, 14 (1984) 131-141.
- [123] J.L. Verdegay, Application of fuzzy optimization in operational research, *Control and*

Cybernetics, 13 (1984) 229-239.

- [124] J.L. Verdegay, C.G. Aguado, On the sensitivity of membership function for fuzzy linear programming problem, *Fuzzy Sets and Systems*, 56 (1993) 47-49.
- [125] A.K. Verma, A. Srividya, B.C. Deka, Composite system reliability assessment using fuzzy linear programming, *Electric Power Systems Research*, 73 (2005) 143-149.
- [126] D. Wang, An inexact approach for linear programming problems with fuzzy objective and resources, *Fuzzy Sets and Systems*, 89 (1997) 61-68.
- [127] Z.X. Wang, Y.J. Liu, Z.P. Fan, B. Feng, Ranking LR fuzzy number based on deviation degree, *Information Sciences*, 179 (2009) 2070-2077.
- [128] B. Werners, An interactive fuzzy programming system, *Fuzzy Sets and Systems*, 23 (1987) 131-147.
- [129] W. L. Winston (2007). *Operations Research: Applications and Algorithms*, fourth edition, Thomson, Australia.
- [130] H.C. Wu, Optimality conditions for linear programming problems with fuzzy coefficients, *Computers and Mathematics with Applications*, 55 (2008) 2807-2822.
- [131] R.R. Yager, Mathematical programming with fuzzy constraints and preference on the objective, *Kybernetes*, 8 (1979) 285-291.
- [132] R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, *Information Sciences*, 24 (1981) 143-161.
- [133] Y. Yuan, Criteria for evaluating fuzzy ranking methods, *Fuzzy Sets and Systems*, 44 (1991) 139-157.
- [134] L.A. Zadeh, Fuzzy sets, *Information and Control*, 8 (1965) 338-353.
- [135] L.A. Zadeh, Fuzzy algorithms, *Information and Control*, 12 (1968) 94-102.
- [136] H.J. Zimmermann, Description and optimization of fuzzy systems, *International Journal of General Systems*, 2 (1976) 209-215.
- [137] H.J. Zimmermann, Fuzzy programming and linear programming with several objective

functions, *Fuzzy Sets and Systems*, 1 (1978) 45-55.

- [138] H.J. Zimmermann, Fuzzy mathematical programming, *Computers and Operations Research*, 10 (1983) 291-298.
- [139] H.J. Zimmermann, *Fuzzy Sets, Decision Making and Expert Systems*, Kluwer Academic Publishers, Boston, MA, 1987.
- [140] H.J. Zimmermann, *Fuzzy Set Theory and Its Applications*, Fourth Edition, Kluwer Academic Publisher, Dordrecht, 2001.