RELIABILITY ANALYSIS OF AN INDUSTRIAL SYSTEM USING AN IMPROVED ARITHMETIC OPERATIONS

A Thesis
Submitted in partial fulfillment of the
requirement for the award of the degree
of
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in
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by
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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Reliability Analysis of an Industrial Systems using an Improved Arithmetic Operations**" in partial fulfillment of the requirement for the award of degree of Master of Science, School of Mathematics (SOM), Thapar University, Patiala is an authentic record of my own carried out under the supervision of Dr. Harish Garg, Assistant Professor, SoM, Thapar University Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

Today's with the growing complexities of the system, it is difficult for the system analyst to maintain the performance of the system for a longer period of time in order to increase the sustainability of the system. This is mainly due to the failure phenomenon which are occurring during the analysis as the complete information of the system is not always available. To handle this, the problems need to be set up with the approximately available data. However, fuzzy set theory is one of the successful theory to deal with such types of data. In recent times, the use of fuzzy sets has been gaining popularity and is playing an important role in the areas of engineering and management disciplines. As compared to other research domains, the fuzzy arithmetic gained great interest in scientific areas such as decision problems, reliability analysis, optimization etc. In order to perform operations on fuzzy observations, fuzzy numbers came into existence. The difference between the arithmetic operations on generalized fuzzy numbers and the traditional fuzzy numbers is that the former can deal with both non-normalized and normalized fuzzy numbers, but the later with normalized fuzzy numbers.

The research work presented in this dissertation is devoted to present an improved arithmetic operations under the fuzzy environment. As the existing work on the arithmetic operations consider the same degree of confidence level for different fuzzy numbers and hence it observed that it loss the information which cause the inexact results. So in order to avoid this and to preserve the flatness of the fuzzy numbers we derived an improved arithmetic operators such as addition, scalar multiplication, subtraction, multiplication for generalized trapezoidal (triangular) fuzzy numbers. The advantage of the proposed operations is that it preserves the flatness of the data and hence give its significance.

The present thesis is organized into four chapters including the present one that contains mainly the literature review. The rest of chapters are described below:

In **Chapter 2**, the basic and preliminaries related to the reliability theory and fuzzy set theory to be used in the subsequent chapters are given.

In **Chapter 3**, an improved arithmetic operations on generalized fuzzy numbers have been presented by overcoming the some shortcoming of the existing operations. As it has been observed that arithmetic operations of generalized trapezoidal (triangular) fuzzy numbers with function principle cause the loss of information and do not give exact results. This motivates us to correct the results of arithmetic operations of generalized trapezoidal (triangular) fuzzy numbers. Various arithmetic operations, such as addition, subtraction, multiplication, division etc are studied and illustrated with a numerical examples.

In **Chapter 4**, reliability analysis of repairable industrial systems has been analyzed by using the improved arithmetic operations for a generalized fuzzy numbers as described in Chapter 3 by considering the different degree of confidence levels. A case study from the washing unit of a paper mill, a complex repairable industrial system, has been taken for demonstrating the approach. Finally, the computed results are compared with the existing methodologies results. The presented technique utilizes uncertain data of the system and analyze its behavior with reduced level of uncertainty which makes the decision more realistic and generic for further application.

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First of all, I would like to thank the almighty for granting perseverance. I would first like to express my sincere gratitude to my honorable supervisor Dr. Harish Garg, Assistant Professor, School of Mathematics (SOM), Thapar University, Patiala, for their patient guidance and support throughout this work. I was truly very fortunate to have the opportunity to work under him as a student. It was both an honor and privilege to work with him. He also provides help in technical writing and presentation style and I found this guidance to be extremely valuable.I could not have imagined having a better advisor and mentor for my M.Sc. dissertation.

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I would like to thank my beloved parents for their years of unyielding love and encouragement. They have always wanted the best for me and I admire my parent's determination and sacrifice to put me through college.

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Chapter 1

Introduction

The objective of this work is to compute the reliability of an industrial systems more closely by utilizing uncertain, vague and imprecise data. For this some new improved arithmetic operations have been proposed under fuzzy environment. A brief literature review regarding reliability/availability evaluation under the fuzzy environment along with their arithmetic operations are given hereafter.

1.1 Review of Literature

Reliability is a popular concept that has been celebrated for years as an admirable characteristic of a person. Today reliability has grown into a universal attribute with qualitative and quantitative connotations that pervades every aspect of our present day technologically intensive world. Today with growing complexity of the repairable industrial systems along with advances in technology the job of the system analysts becomes more tedious and challenging as they have to study, characterize, measure and analyze the behavior of the systems in both a qualitative and a quantitative manner using various traditional and non-traditional techniques. But unfortunately, failure is an unavoidable phenomenon associated with technological advancement of the equipments used in these industries. These failures may be the result of human error, poor maintenance or inadequate testing/inspection. The traditional analytical techniques (mathematical and statistical models) need large amounts of data, which are difficult to obtain because of constraints (i.e. rare events of components, human errors and economic considerations) for estimation of the failure/repair characteristics of the system. If data are collected or available from resources and are used as such for analysis then they have a large amount of uncertainty because historical records can only represent the past behavior but may be unable to predict the future behavior of the equipment. However, the authors [4–7, 9–19, 21, 23, 24] have analyzed the behavior of the industrial systems using traditional arithmetic operations. Based on it, they have analyzed the behavior of the system in terms of several reliability parameters which depicts the performance and behavior of the system. But all the above existing studies do not consider the level of significance during an analysis and hence they do not conserve the flatness of the data which results the lose of significance. Therefore, there is a need for developing such type of arithmetic operations which consider the effect of different levels of significance during the analysis and will reduce the uncertainties, for each reliability index, up to a desired degree of accuracy.

1.2 Objectives of the Thesis

The main objective of this thesis is to present some new improved arithmetic operations under fuzzy environment. As the existing work on the arithmetic operations consider the same degree of confidence level for different fuzzy numbers and hence it observed that it loss the information which cause the inexact results. So in order to avoid this and to preserve the flatness of the fuzzy numbers we derived an improved arithmetic operators such as addition, scalar multiplication, subtraction, multiplication for generalized trapezoidal (triangular) fuzzy numbers. The advantage of the proposed operations is that it preserves the flatness of the data and hence give its significance.

Apart from their computation of the membership functions, reliability analysis of an industrial system by using these operations have also been studied. In it, data related to system components are extracted from the various resources and fuzzified it into triangular fuzzy numbers based on the system analyst/plant personnel suggestion. Then based on it, various reliability parameters are computed in terms of their membership functions and compare their results with the existing approaches. Also sensitivity analysis on system mean time between failures have been judged for finding the most critical combination of the parameter so as to increase the performance of the system.

1.3 Structure of the Thesis

The present thesis is organized into four chapters including the present one that contains mainly the literature review. The rest of chapters are described below:

In **Chapter 2**, the basic and preliminaries related to the reliability theory and fuzzy set theory to be used in the subsequent chapters are given.

In **Chapter 3**, an improved arithmetic operations on generalized fuzzy numbers have been presented by overcoming the some shortcoming of the existing operations. As it has been observed that arithmetic operations of generalized trapezoidal (triangular) fuzzy numbers with function principle cause the loss of information and do not give exact results. This motivates us to correct the results of arithmetic operations of generalized trapezoidal (triangular) fuzzy numbers. Various arithmetic operations, such as addition, subtraction, multiplication, division etc are studied and illustrated with a numerical examples.

In **Chapter 4**, reliability analysis of repairable industrial systems has been analyzed by using the improved arithmetic operations for a generalized fuzzy numbers as described in Chapter 3 by considering the different degree of confidence levels. A case study from the washing unit of a paper mill, a complex repairable industrial system, has been taken for demonstrating the approach. Finally, the computed results are compared with the existing methodologies results. The presented technique utilizes uncertain data of the system and analyze its behavior with reduced level of uncertainty which makes the decision more realistic and generic for further application.

Chapter 2

Preliminaries

This chapter presents some of the fundamental definitions and mathematical theory for fuzzy set theory. The focus is on defining the fuzzy set, $\alpha-$ cuts, convex and normal fuzzy set, fuzzy numbers and generalized fuzzy numbers.

2.1 Reliability Aspects

This chapter introduces basic concepts related to reliability engineering and soft-computing knowledge of each of which is essential for the research in this area. Basically the purpose of reliability engineering is to develop methods and tools to evaluate and demonstrate reliability, maintainability, availability, and safety of systems and their components/equipments, as well as to support design/production engineers in building in these characteristics. These main aspects of reliability engineering of a system i.e. reliability, maintainability and availability are described briefly in the subsequent subsections.

2.1.1 Reliability

Reliability is a characteristic of an item(component or system), expressed by the probability that the item (component/system) will perform its required function under given conditions for a stated time interval [3]. From a qualitative point of view, reliability can be defined as the ability of the item to remain functional. Quantitatively, reliability specifies the probability that no operational interruptions will occur during a stated time interval. Mathematically,if we define continuous random variable T to be the time to failure of the component/system; $T \geq 0$, then the basic reliability function $R(t)$, is defined for time to failure of the system (or subsystem) as

$$R(t) = Pr(T \geq t) = 1 - \int_0^t f(u)\ du \tag{2.1.1}$$

where $R(t) \geq 0$, $R(0) = 1$, and $\lim_{t \to \infty} R(t) = 0$ and $f(t)$ failure probability density function.

In addition to the probability function, there is another function, called the failure rate or hazard rate function, is often used in reliability. It provides an instantaneous (at time t) rate of failure. The conditional probability of a failure in the time interval from t to $t + \delta t$ given that the system has survived to time t is

$$Pr\{t \leq T \leq t + \delta t \mid T \geq t\} = \frac{R(t) - R(t + \delta t)}{R(t)} \tag{2.1.2}$$

then $\dfrac{R(t) - R(t + \delta t)}{R(t)\delta t}$ is the conditional probability of failure per unit of time (failure rate). The rule of conditional probability therefore dictates that:

$$\lambda(t) = \frac{-dR(t)}{dt} \cdot \frac{1}{R(t)} = \frac{f(t)}{R(t)} \tag{2.1.3}$$

then $\lambda(t)$ is known as the instantaneous hazard rate or failure rate function. Based on these hazard rate function, the reliability function can be derived as

$$R(t) = \exp\left[- \int_0^t \lambda(u)\ du \right] \tag{2.1.4}$$

The mean time to failure(MTTF) of the system is defined as

$$MTTF = \int_0^\infty R(t)dt \tag{2.1.5}$$

2.1.2 Availability

Availability is defined as the probability of a product or system working satisfactorily at any given point of time when used under the given conditions of use [8]. Thus availability signifies the probability that the system is available and is working satisfactorily at a given point of time. Availability is a more meaningful parameter of performance of a maintained system than reliability. However, reliability and maintainability are related to availability and are two important design parameters in establishing the availability of equipment. Similar to the reliability function, it also gives a probability that a system will be available to function at the given time t. At any given time t, the system will be operational if the following conditions are met:

$$A_s(t) = \frac{\mu_s}{\lambda_s + \mu_s} + \frac{\lambda_s}{\lambda_s + \mu_s} e^{-(\lambda_s + \mu_s)t} \tag{2.1.6}$$

where, λ_s and μ_s are respectively the failure and repair rates of the system.

The steady state availability, A, of a system is the limit of the instantaneous availability function as time approaches to infinity (∞) and is represented by

$$A = \lim_{t \to \infty} A_s(t) = \frac{\text{MTBF}}{\text{MTBF} + \text{MTTR}} \tag{2.1.7}$$

where, MTTR and MTBF are the mean time to repair and the mean time between failures of the system/component respectively.

2.2 Basic concepts on fuzzy set theory

2.2.1 Classical Sets (Crisp Sets)

Basically, a set is defined as a collection of objects, which share certain characteristics. A classical set is a collection of distinct objects. The classical set is defined in such a way that the universe of discourse is split into two groups: members and nonmembers. Consider an object x in a crisp set A. This object x is either a member or a nonmember of the given set A. In case of crisp sets, no partial membership exists. This binary issue of membership can be represented mathematically by the indicator function,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{2.2.1}$$

where χ_A is the membership in set A for element x in the universe. The membership concept represents mapping from an element x in universe X to one of the two elements in universe Y (either to element 0 or 1). There exist a function-theoretic set called value set for any set A defined on universe X, based on the mapping of characteristic function, The whole set is assigned a membership value 1, and the null set is assigned a membership value 0.

2.2.2 Fuzzy Sets

Fuzzy sets may be viewed as an extension and generalization of the basic concepts of crisp sets [25, 27]. An important property of fuzzy set is that it allows partial membership. A fuzzy set is a set having degree of membership between 1 and 0. The membership in a fuzzy set need not be complete, i.e., member of one fuzzy set can also be a member of other fuzzy set in the same universe. Zadeh [25] extended the notion of valuation set $\{1, 0\}$ (definitely in / definitely out) to the interval of real values (degree of membership) between 1 and 0, denoted as [0, 1], where 0.0 represents the absolutely false and 1.0 represents absolutely truth. The fuzzy set A in the universe of discourse U is defined as a set of ordered pairs $(x, \mu_A(x))$, i.e.

$$A = \{(x, \mu_A(x) \mid x \in U\} \tag{2.2.2}$$

where $\mu_A(x)$ is the degree of membership of x in fuzzy set A and it indicates the degree that x belongs to A. Clearly $\mu_A(x) \in [0, 1]$.

2.2.3 Membership functions

Membership function defines the fuzziness in a fuzzy set irrespective of the elements in the set, which are discrete or continuous. For a fuzzy set A a membership function, denoted by $\mu_A(\cdot)$ maps U to the membership space M, i.e. $\mu_A : U \to M$. The membership value ranges in the interval [0, 1] i.e. the range of the membership function is a subset of the non-negative real numbers whose supremum is finite.

The three main basic features involved in characterizing membership function are the following[25].

(i) **Core:** The core of a membership function for some fuzzy set A is defined as that region of universe that is characterized by complete membership in the set A. The core has elements x of the universe such that

$$\mu_A(x) = 1 \tag{2.2.3}$$

The core of a fuzzy set may be an empty set.

(ii) **Support:** The support of a membership function for a fuzzy set A is defined as that region of universe that is characterized by a nonzero membership in the set A. The support comprises elements x of the universe such that

$$\mu_A(x) > 0 \tag{2.2.4}$$

(iii) **Boundary:** The boundary of a membership function for a fuzzy set A is defined as the region of universe that contains a nonzero but not a complete membership. In other words the boundary comprises those elements x of the universe such that

$$0 < \mu_A(x) < 1 \tag{2.2.5}$$

The boundary elements are those which possess partial membership in the fuzzy set A.

2.2.4 $\alpha-$ cuts

α-cut is one of the most significant and extensively used concept in fuzzy set theory which was introduced by Zadeh [26]. When we want to exhibit an element $x \in U$ that typically belongs to a fuzzy set A, we may demand that its membership value be greater than some threshold $\alpha \in [0, 1]$.

For a fuzzy set A,

$$\begin{aligned} A_\alpha &= \{x \mid \mu_A(x) > \alpha\}; \qquad \alpha \in [0,1) \\ A_\alpha &= \{x \mid \mu_A(x) \geq \alpha\}; \qquad \alpha \in [0,1) \end{aligned}$$

are called strong $\alpha-$ cut and weak $\alpha-$ cut respectively.

2.2.5 Convex fuzzy set

A fuzzy set $A = \langle (x, \mu_A(x)) \mid x \in U \rangle$ is said to be convex fuzzy set [27] if the following inequality has been hold for $x_1, x_2 \in U$

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geqslant \min[\mu_A(x_1), \mu_A(x_2)]$$

If above inequality does not hold then that it is said to be non-convex fuzzy set.

2.2.6 Normal fuzzy set

A fuzzy set is said to be normal fuzzy set [27] if there exist at least one element in the universal set U such that their corresponding membership function is unity.

2.2.7 Fuzzy number

A convex, normal membership function on the real line $\mathbb{R}$ is called a fuzzy number [27] i.e., if its membership function is piecewise continuous and there exist at least one $x_0 \in U$ such that $\mu_A(x_0) = 1$. The corresponding membership function defined on $[a, b] \neq \emptyset$ is given as

$$\mu_A(x) = \begin{cases} f(x) & ; \quad x \in (-\infty, a) \\ 1 & ; \quad x = [a, b] \\ g(x) & ; \quad x \in (b, \infty) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

where f and g are monotonic, continuous from the right and left, nondecreasing and nonincreasing functions such that $f(x) = 0$ for $x \in (-\infty, \omega_1)$ and $g(x) = 0$ for $x \in (\omega_2, \infty)$.

2.2.8 Triangular fuzzy number

A fuzzy number $A = \langle a, b, c\rangle$ is called a triangular fuzzy number [27] if its membership function μ_A is given by

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & ; \quad a \leq x < b \\ 1 & ; \quad x = b \\ \frac{c-x}{c-b} & ; \quad b \leq x < c \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The α-cut of fuzzy number (a, b, c) is defined below and shown graphically in Fig. 2.1 whose interval of confidence are defined are

$$A_\alpha = [a^{(\alpha)}, c^{(\alpha)}] = [(b-a)\alpha + a, -(c-b)\alpha + c] \tag{2.2.6}$$

The basic arithmetic operations, i.e. addition, subtraction, multiplication and division on two TFNs $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$ where $a_i \geqslant 0$, $i = 1, 2$ are defined as

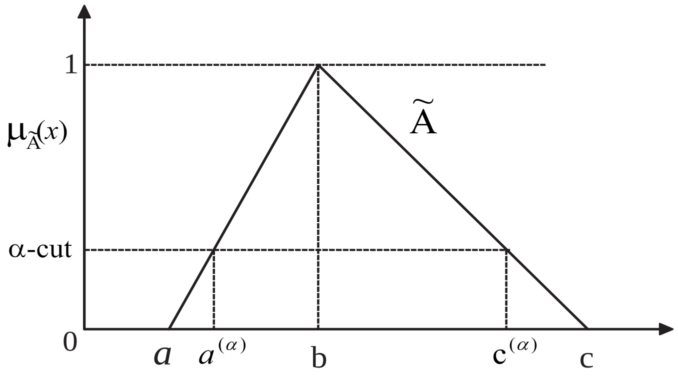


Figure 2.1: A fuzzy set A

(i) Addition : $A + B = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

(ii) Subtraction: $A - B = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$

(iii) Multiplication: $A \cdot B = (a_1 a_2, b_1 b_2, c_1 c_2)$

(iv) Division: $A \div B = (a_1/c_2, b_1/b_2, c_1/a_2)$ if $a_2 > 0$

It is clear that the multiplication and division of two TFNs is not again a TFN with linear sides but it is a new fuzzy number with parabolic sides.

2.2.9 Trapezoidal fuzzy number

A fuzzy number $A = \langle a, b, c, d \rangle$ is called a trapezoidal fuzzy number if its membership function μ_A is given by

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & ; \quad a \leq x < b \\ 1 & ; \quad b \leq x < c \\ \frac{d-x}{d-c} & ; \quad c \leq x < d \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The $\alpha-$ cut of the number $A = \langle a, b, c, d \rangle$ is the closed interval

$$A_\alpha = [A^L_\alpha, A^R_\alpha] = [a + \alpha(b-a), d - \alpha(d-c)] \quad , \quad \alpha \in (0, 1]$$

2.3 Generalized fuzzy number

A fuzzy number $A = \langle (a_1, a_2, a_3; \omega) \mid a_i \in \mathbb{R} \rangle$, is said to be generalized fuzzy number if there corresponding membership function $\mu_A(x) : \mathbb{R} \to [0, 1]$ satisfies the following properties.

(i) It is continuous.

(ii) It is zero for all $x \in (-\infty, a_1] \cup [a_3, \infty)$.

(iii) It is strictly increasing on $[a_1, a_2]$ and strictly decreasing on $[a_2, a_3]$.

(iv) $\mu_A(x) = \omega$ for all $x = a_2$ where $0 < \omega \leqslant 1$.

If $\omega = 1$ then it is said to be normal fuzzy number else it is generalized fuzzy number. A fuzzy number $A = (a_1, a_2, a_3; \omega)$, is called a generalized triangular fuzzy number if its membership function is defined as

$$\mu_A(x) = \begin{cases} \omega\left(\dfrac{x - a_1}{a_2 - a_1}\right) & ; \text{ if } a_1 \leqslant x < a_2 \\ \omega & ; \text{ if } x = a_2 \\ \omega\left(\dfrac{a_3 - x}{a_3 - a_2}\right) & ; \text{ if } a_2 \leqslant x < a_3 \\ 0 & ; \text{ if otherwise} \end{cases}$$

The $\alpha-$ cut of the number $A = \langle a_1, a_2, a_3; \omega \rangle$ is the closed interval

$$A_\alpha = [A^L_\alpha, A^R_\alpha] = [a_1 + \frac{\alpha}{\omega}(a_2 - a_1), a_3 - \frac{\alpha}{\omega}(a_3 - a_2)] \quad , \quad \alpha \in (0, \omega]$$

Chapter 3

Improved arithmetic operations on generalized Trapezoidal (Triangular) fuzzy numbers

In this chapter, an improved arithmetic operations on generalized fuzzy numbers have been presented by overcoming the some shortcoming of the existing operations. As it has been observed that arithmetic operations of generalized trapezoidal (triangular) fuzzy numbers with function principle cause the loss of information and do not give exact results. This motivates us to correct the results of arithmetic operations of generalized trapezoidal (triangular) fuzzy numbers.

3.1 Fuzzy arithmetic operations

Consider two generalized trapezoidal fuzzy number $A_1 = (a_1, b_1, c_1, d_1; \omega_1)$ and $A_2 = (a_2, b_2, c_2, d_2; \omega_2)$ then some basic arithmetic operation between them have been defined as [5]

- Addition of A_1 and A_2:

$$A_1 + A_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min(\omega_1, \omega_2))$$

- Subtraction of A_1 and A_2:

$$A_1 - A_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \min(\omega_1, \omega_2))$$

- Multiplication of A_1 and A_2:

$$A_1 \cdot A_2 = (a_1a_2, b_1b_2, c_1c_2, d_1d_2; \min(\omega_1, \omega_2)) \quad ; \text{if } a_1, a_2 > 0$$

- Division of A_1 and A_2:

$$A_1/A_2 = (a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2; \min(\omega_1, \omega_2))$$

As various authors [4–7, 9–11, 17–19, 21, 23, 24] have used these operations for analyzing the system behavior in different environment, but it has been observed that they have some shortcomings which have been illustrated with the help of following example:

Example 3.1.1.

Consider the generalized triangular fuzzy number $A = (0.5, 0.6, 0.7; 0.5)$ and $B = (0.6, 0.7, 0.8; 0.9)$ as shown in Fig. 3.1. After performing the above operations for addition then we get $A + B = (1.1, 1.3, 1.5; 0.5)$ and hence the resultant number becomes a generalized triangular fuzzy number. But it has been observed from the figure that if we take 0.5 (= $\min(0.5, 0.9)$) cut of B, then the fuzzy number B is transformed into a generalized trapezoidal fuzzy number. Therefore, the triangular fuzzy number convert into the trapezoidal fuzzy number and hence flatness of the numbers are not preserve in Chen [5] operations. Hence, it is necessary to preserve this flatness into the generalized fuzzy number. Thus the existing approach loses its significance.

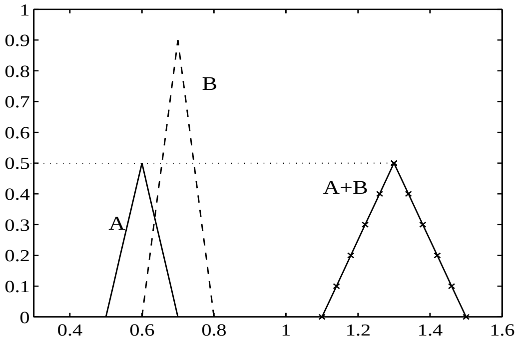


Figure 3.1: Sum of two generalized triangular fuzzy numbers

Example 3.1.2.

Consider the generalized triangular fuzzy numbers $A_1 = (0.2, 0.4, 0.6; 0.5)$, $A_2 = (0.5, 0.7, 0.9; 0.7)$ and $A_3 = (0.5, 0.7, 0.9; 0.9)$ as in Fig. 3.2. From it, it has been seen that $A_2 \in A_3$. Then, we $A_1 + A_2 \in A_1 + A_3$. On the other hand, if we use Chen [5] method, we have $A_1 + A_2 = (0.7, 1.1, 1.5; 0.5)$ and $A_1 + A_3 = (0.7, 1.1, 1.5; 0.5)$. Thus $A_1 + A_2 \equiv A_1 + A_3$ which violate the fact that $A_1 + A_2 \in A_1 + A_3$. Therefore, Chen [5] method cannot consistency calculate the arithmetic operations between generalized fuzzy numbers.

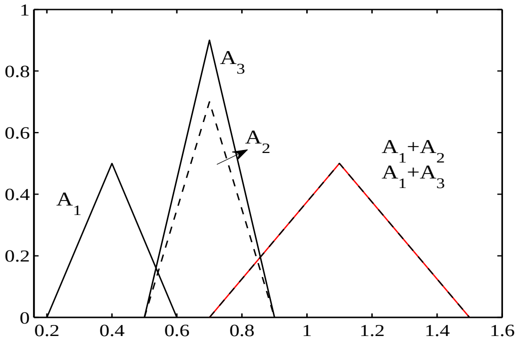


Figure 3.2: Sum of two generalized triangular fuzzy numbers

3.2 Proposed Arithmetic operations between generalized fuzzy numbers

In this section, an improved arithmetic operations have been proposed between generalized fuzzy numbers by using $\alpha-$ cuts. Let $A = (a_1, a_2, a_3, a_4; \omega_A)$ and $B = (b_1, b_2, b_3, b_4; \omega_B)$ be two generalized trapezoidal fuzzy numbers which membership functions μ_A and μ_B respectively, which can be written as

$$\mu_A(x) = \begin{cases} \omega_A\left(\dfrac{x - a_1}{a_2 - a_1}\right) & a_1 \le x \le a_2 \\ \omega_A & a_2 \le x \le a_3 \\ \omega_A\left(\dfrac{a_4 - x}{a_4 - a_3}\right) & a_3 \le x \le a_4 \end{cases} \quad ; \quad \mu_B(x) = \begin{cases} \omega_B\left(\dfrac{x - b_1}{b_2 - b_1}\right) & b_1 \le x \le b_2 \\ \omega_B & b_2 \le x \le b_3 \\ \omega_B\left(\dfrac{b_4 - x}{b_4 - b_3}\right) & b_3 \le x \le b_4 \end{cases}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are real numbers, $0 \le \omega_A, \omega_B \le 1$ such that $\omega_A \le \omega_B$. Take $\omega = \min(\omega_A, \omega_B)$, then make a ω cut of the fuzzy number B such that B will transform into a new generalized fuzzy number as $B^* = (b_1, b_2^*, b_3^*, b_4; \omega)$, where $b_2^* = b_1 + \omega\left(\dfrac{b_2 - b_1}{\omega_B}\right)$ and $b_3^* = b_4 - \omega\left(\dfrac{b_4 - b_3}{\omega_B}\right)$ for membership functions. Clearly if $\omega_A = \omega_B$ then $\omega = \omega_A = \omega_B$, $b_2^* = b_2, b_3^* = b_3$ and hence the new generalized fuzzy number B^* is same as that of generalized fuzzy number B. Now the $\alpha-$ cut of the fuzzy number A and B^* becomes

$$\begin{aligned} A_\alpha &= \left[a_1 + \alpha\left(\frac{a_2 - a_1}{\omega_A}\right), a_4 - \alpha\left(\frac{a_4 - a_3}{\omega_A}\right)\right] \quad \forall \alpha \in [0, \omega_A] \quad 0 \le \omega_A \le 1 \\ B_\alpha &= \left[b_1 + \alpha\left(\frac{b_2^* - b_1}{\omega}\right), b_4 - \alpha\left(\frac{b_4 - b_3^*}{\omega}\right)\right] \quad \forall \alpha \in [0, \omega] \quad 0 \le \omega \le 1 \end{aligned}$$

Thus, based on these $\alpha-$ cuts, we compute improved arithmetic operations namely, addition, substraction, scalar multiplication, division etc., between the two generalized fuzzy numbers.

Theorem 3.2.1. *Addition of two generalized fuzzy numbers $A = (a_1, a_2, a_3, a_4; \omega_A)$ and $B = (b_1, b_2, b_3, b_4; \omega_B)$ with two different confidence levels generates a trapezoidal fuzzy numbers $C = A + B = (c_1, c_2, c_3, c_4; \omega)$ where*

$$\begin{aligned} c_1 &= a_1 + b_1 \\ c_2 &= b_1 + a_2 + \omega(b_2 - b_1)/\omega_B \\ c_3 &= b_4 + a_3 - \omega(b_4 - b_3)/\omega_B \\ c_4 &= a_4 + b_4 \end{aligned}$$

Proof. Let A and B be two generalized fuzzy numbers with different confidence levels such that $\omega_A < \omega_B$. Take $\omega = \min(\omega_A, \omega_B)$ i.e. $\omega = \omega_A$ then $\alpha-$ cut of A and B are

$$\begin{aligned} A_\alpha &= \left[a_1 + \alpha\left(\frac{a_2 - a_1}{\omega_A}\right), a_4 - \alpha\left(\frac{a_4 - a_3}{\omega_A}\right)\right] \quad \forall \alpha \in [0, \omega_A] \quad 0 \le \omega_A \le 1 \\ B_\alpha &= \left[b_1 + \alpha\left(\frac{b_2^* - b_1}{\omega}\right), b_4 - \alpha\left(\frac{b_4 - b_3^*}{\omega}\right)\right] \quad \forall \alpha \in [0, \omega] \quad 0 \le \omega \le 1 \end{aligned}$$

Let $C = A + B = \{x \mid x \in C_\alpha\}$ for all $\alpha \in [0, \omega]$. Here $C_\alpha = [C^L(\alpha), C^U(\alpha)]$ be its $\alpha-$ cuts such that $C^L(\alpha) = A^L(\alpha) + B^L(\alpha)$ and $C^U(\alpha) = A^U(\alpha) + B^U(\alpha)$ i.e.

$$\begin{aligned} C_\alpha &= [A^L(\alpha) + B^L(\alpha), A^U(\alpha) + B^U(\alpha)] \\ &= \left[a_1 + \alpha\left(\frac{a_2 - a_1}{\omega}\right) + b_1 + \alpha\left(\frac{b_2^* - b_1}{\omega}\right), a_4 - \alpha\left(\frac{a_4 - a_3}{\omega}\right) + b_4 - \alpha\left(\frac{b_4 - b_3^*}{\omega}\right)\right] \\ &= \left[a_1 + b_1 + \alpha\left(\frac{a_2 - a_1}{\omega_A} + \frac{b_2^* - b_1}{\omega}\right), a_4 + b_4 - \alpha\left(\frac{a_4 - a_3}{\omega_A} + \frac{b_4 - b_3^*}{\omega}\right)\right] \end{aligned}$$

Now,

$$\begin{aligned} a_1 + b_1 + \alpha\left(\frac{a_2 - a_1}{\omega_A} + \frac{b_2^* - b_1}{\omega}\right) - x &= 0 \qquad \text{and} \\ a_4 + b_4 - \alpha\left(\frac{a_4 - a_3}{\omega_A} + \frac{b_4 - b_3^*}{\omega}\right) - x &= 0 \end{aligned}$$

Therefore, the left and right membership function of C is

$$f_C^L(x) = \frac{x-a_1-b_1}{\frac{a_2-a_1}{\omega_A}+\frac{b_2^*-b_1}{\omega}} \quad ; \quad f_C^R(x) = \frac{a_4+b_4-x}{\frac{a_4-a_3}{\omega_A}+\frac{b_4-b_3^*}{\omega}}$$

Since $\omega = \omega_A$ and $b_2^* = b_1 + \omega\left(\frac{b_2-b_1}{\omega_B}\right)$ and $b_3^* = b_4 - \omega\left(\frac{b_4-b_3}{\omega_B}\right)$. Thus, above f_C^L and f_C^R becomes

$$\begin{aligned} f_C^L(x) &= \frac{x-a_1-b_1}{\frac{a_2-a_1}{\omega_A}+\frac{b_2^*-b_1}{\omega}} \\ &= \omega\left(\frac{x-a_1-b_1}{a_2-a_1-b_1+b_2^*}\right) \\ &= \omega\left(\frac{x-(a_1+b_1)}{(a_2+b_1+\omega(b_2-b_1)/\omega_B)-(a_1+b_1)}\right) \\ &\quad \text{for } a_1+b_1 \le x \le a_2+b_1+\omega(b_2-b_1)/\omega_B \end{aligned}$$

Similarly,

$$\begin{aligned} f_C^R(x) &= \omega\left(\frac{(a_4+b_4)-x}{(b_4+a_3-\omega(b_4-b_3)/\omega_B)-(a_4+b_4)}\right) \\ &\quad \text{for } b_4+a_3-\omega(b_4-b_3)/\omega_B \le x \le a_4+b_4 \end{aligned}$$

Hence, the addition of two generalized fuzzy number is again a generalized fuzzy number whose memebership function is defined as follow,

$$\mu_C(x) = \begin{cases} \omega\left(\frac{x-(a_1+b_1)}{(a_2+b_1+\omega(b_2-b_1)/\omega_B)-(a_1+b_1)}\right) & ; a_1+b_1 \le x \le a_2+b_1+\omega(b_2-b_1)/\omega_B \\ \omega & ; a_2+b_1+\omega(b_2-b_1)/\omega_B \le x \le b_4+a_3-\omega(b_4-b_3)/\omega_B \\ \omega\left(\frac{(a_4+b_4)-x}{(b_4+a_3-\omega(b_4-b_3)/\omega_B)-(a_4+b_4)}\right) & ; b_4+a_3-\omega(b_4-b_3)/\omega_B \le x \le a_4+b_4 \end{cases}$$

In other words, addition of two generalized fuzzy number $C = A + B = (c_1, c_2, c_3, c_4; \omega = \min(\omega_A, \omega_B)$ is a generalized fuzzy number where

$$\begin{aligned} c_1 &= a_1+b_1 \\ c_2 &= b_1+a_2+\omega(b_2-b_1)/\omega_B \\ c_3 &= b_4+a_3-\omega(b_4-b_3)/\omega_B \\ c_4 &= a_4+b_4 \end{aligned}$$

□

Theorem 3.2.2. *(Scalar multiplication of fuzzy number:) If $A = (a_1, a_2, a_3, a_4; \omega$ be a generalized trapezoidal fuzzy number then kA is again a generalized fuzzy number given by*

$$kA = \begin{cases} (ka_1, ka_2, ka_3, ka_4; \omega) & \text{if } k > 0 \\ (ka_4, ka_3, ka_2, ka_1; \omega) & \text{if } k < 0 \end{cases}$$

Proof. For $k > 0$, the $\alpha-$ cut for membership function of A is

$$A_\alpha = \left[a_1 + \alpha\left(\frac{a_2 - a_1}{\omega_A}\right), a_4 - \alpha\left(\frac{a_4 - a_3}{\omega_A}\right)\right] \quad \forall\alpha \in [0, \omega_A] \quad 0 \leq \omega_A \leq 1$$

that is

$$x \in \left[a_1 + \alpha\left(\frac{a_2 - a_1}{\omega_A}\right), a_4 - \alpha\left(\frac{a_4 - a_3}{\omega_A}\right)\right]$$

Therefore,

$$y = kx \in \left[ka_1 + \alpha\left(\frac{ka_2 - ka_1}{\omega_A}\right), ka_4 - \alpha\left(\frac{ka_4 - ka_3}{\omega_A}\right)\right]$$

Thus, membership function of kA is given by

$$\mu_{kA}(x) = \begin{cases} \omega_A\left(\dfrac{x - ka_1}{ka_2 - ka_1}\right) & ; \; ka_1 \leq x \leq ka_2 \\ \omega_A & ; \; ka_2 \leq x \leq ka_3 \\ \omega_A\left(\dfrac{ka_4 - x}{ka_4 - ka_3}\right) & ; \; ka_3 \leq x \leq ka_4 \end{cases}$$

□

Theorem 3.2.3. *(Subtraction of two numbers:) Consider two generalized fuzzy numbers $A = (a_1, a_2, a_3, a_4; \omega_A)$ and $B = (b_1, b_2, b_3, b_4; \omega_B)$ with different confidence levels then they generates a trapezoidal fuzzy numbers $C = A - B = (c_1, c_2, c_3, c_4; \omega)$ where*

$$\begin{aligned} c_1 &= a_1 - b_4 \\ c_2 &= a_2 - b_4 + \omega(b_4 - b_3)/\omega_B \\ c_3 &= a_3 - b_1 - \omega(b_2 - b_1)/\omega_B \\ c_4 &= a_4 - b_1 \end{aligned}$$

Proof. Follows from Theorem 3.2.1 and Theorem 3.2.2, so we omit here. □

Theorem 3.2.4. *(Multiplication of two numbers:) For two generalized fuzzy number $A = (a_1, a_2, a_3, a_4; \omega_A)$ and $B = (b_1, b_2, b_3, b_4; \omega_B)$ with different confidence level such that $\omega_A \leq \omega_B$ then $C = A \times B = (c_1, c_2, c_3, c_4, \omega = \min(\omega_A, \omega_B)$ generate a fuzzy number where*

$$\begin{aligned} c_1 &= a_1 b_1 \\ c_2 &= \omega(a_2 b_2 - a_2 b_1)/\omega_B + a_2 b_1 \\ c_3 &= \omega(a_3 b_3 - a_3 b_4)/\omega_B + a_3 b_4 \\ c_4 &= a_4 b_4 \end{aligned}$$

Proof. Since there are two different fuzzy number having confidence levels are ω_A and ω_B such that $\omega_A \leq \omega_B$ so we firstly transform the fuzzy number B into $B^* = (b_1, b_2^*, b_3^*, b_4; \omega)$ where $b_2^* = b_1 + \omega\left(\dfrac{b_2 - b_1}{\omega_B}\right)$ and $b_3^* = b_4 - \omega\left(\dfrac{b_4 - b_3}{\omega_B}\right)$. Now, the $\alpha-$ cuts corresponding to A and B^* are

$$\begin{aligned} A_\alpha &= \left[a_1+\alpha\left(\frac{a_2-a_1}{\omega_A}\right), a_4-\alpha\left(\frac{a_4-a_3}{\omega_A}\right)\right] \quad \forall\alpha\in[0,\omega_A] \quad 0\le\omega_A\le 1 \\ B^*_\alpha &= \left[b_1+\alpha\left(\frac{b^*_2-b_1}{\omega}\right), b_4-\alpha\left(\frac{b_4-b^*_3}{\omega}\right)\right] \quad \forall\alpha\in[0,\omega] \quad 0\le\omega\le 1 \end{aligned}$$

Suppose $C = A\times B = \{x \mid x\in C_\alpha\}$ for all $\alpha\in[0,\omega]$. Here $C_\alpha = [C^L_\alpha, C^U_\alpha]$ be its $\alpha-$ cuts such that $C^L_\alpha = A^L_\alpha B^{*L}_\alpha$ and $C^U_\alpha = A^U_\alpha B^{*U}_\alpha$ i.e.

$$\begin{aligned} C_\alpha &= \left[A^L_\alpha B^{*L}_\alpha, A^U_\alpha B^{*U}_\alpha\right] \\ &= \left[\left\{a_1+\alpha\left(\frac{a_2-a_1}{\omega_A}\right)\right\}\left\{b_1+\alpha\left(\frac{b^*_2-b_1}{\omega}\right)\right\}, \left\{a_4-\alpha\left(\frac{a_4-a_3}{\omega_A}\right)\right\}\left\{b_4-\alpha\left(\frac{b_4-b^*_3}{\omega}\right)\right\}\right] \\ &= \left[a_1b_1+\alpha\left(\frac{a_1(b^*_2-b_1)+b_1(a_2-a_1)}{\omega}\right)+\frac{\alpha^2}{\omega^2}(a_2-a_1)(b^*_2-b_1),\right. \\ &\qquad \left. a_4b_4-\alpha\left(\frac{a_4(b_4-b^*_3)(a_4-a_3)}{\omega}\right)+\frac{\alpha^2}{\omega^2}(a_4-a_3)(b_4-b^*_3)\right] \end{aligned}$$

Therefore,

$$\begin{aligned} a_1b_1+\alpha\left(\frac{a_1(b^*_2-b_1)+b_1(a_2-a_1)}{\omega}\right)+\frac{\alpha^2}{\omega^2}(a_2-a_1)(b^*_2-b_1)-x &= 0 \quad \text{and} \\ a_4b_4-\alpha\left(\frac{a_4(b_4-b^*_3)(a_4-a_3)}{\omega}\right)+\frac{\alpha^2}{\omega^2}(a_4-a_3)(b_4-b^*_3)-x &= 0 \end{aligned}$$

which is quadratic in α and hence its roots give the left and right membership functions for C as

$$\begin{aligned} f^L_C(x) &= \frac{-H_1+\sqrt{H_1^2+4G_1(x-P_1)}}{2G_1} \quad ; \quad c_1\le x\le c_2 \\ f^U_C(x) &= \frac{-H_2-\sqrt{H_2^2+4G_2(x-P_2)}}{2G_2} \quad ; \quad c_3\le x\le c_4 \end{aligned}$$

where $H_1 = \frac{a_1(b^*_2-b_1)+b_1(a_2-a_1)}{\omega}$, $H_2 = \frac{a_4(b_4-b^*_3)(a_4-a_3)}{\omega}$, $G_1 = \frac{(a_2-a_1)(b^*_2-b_1}{\omega^2}$, $G_2 = \frac{(a_4-a_3)(b_4-b^*_3)}{\omega^2}$, $P_1 = a_1b_1$ and $P_2 = a_4b_4$. Substitute the value of b^*_2 and b^*_3, we get

$$\begin{aligned} H_1 &= \frac{(a_2-a_1)(b_2-b_1)}{\omega\omega_B} \quad ; \quad H_2 = \frac{(a_4-a_3)(b_4-b_3)}{\omega\omega_B} \\ G_1 &= \frac{a_1(b_2-b_1)}{\omega_B}+\frac{b_1(a_2-a_1)}{\omega} \quad ; \quad G_2 = \frac{a_4(b_4-b_3)}{\omega_B}+\frac{b_4(a_4-a_3)}{\omega} \end{aligned}$$

Thus, the multiplication operations between two generalized fuzzy numbers is a fuzzy number whose membership function is defined as

$$\mu_C(x) = \begin{cases} \frac{-H_1+\sqrt{H_1^2+4G_1(x-P_1)}}{2G_1} & ; \quad c_1\le x\le c_2 \\ \omega & ; \quad c_2\le x\le c_3 \\ \frac{-H_2-\sqrt{H_2^2+4G_2(x-P_2)}}{2G_2} & ; \quad c_3\le x\le c_4 \end{cases}$$

where

$$\begin{aligned} c_1 &= a_1b_1 \\ c_2 &= \omega(a_2b_2 - a_2b_1)/\omega_B + a_2b_1 \\ c_3 &= \omega(a_3b_3 - a_3b_4)/\omega_B + a_3b_4 \\ c_4 &= a_4b_4 \end{aligned}$$

□

Theorem 3.2.5. *For two generalized fuzzy number* $A = (a_1, a_2, a_3, a_4; \omega_A)$ *and* $B = (b_1, b_2, b_3, b_4; \omega_B)$ *with different confidence level such that* $\omega_A \le \omega_B$ *then* $C = A/B = (c_1, c_2, c_3, c_4, \omega = \min(\omega_A, \omega_B))$ *generate a fuzzy number where*

$$\begin{aligned} c_1 &= a_1/b_4 \\ c_2 &= \omega(a_2/b_3 - a_2/b_4)/\omega_B + a_2/b_4 \\ c_3 &= \omega(a_3/b_2 - a_3/b_1)/\omega_B + a_3/b_1 \\ c_4 &= a_4/b_1 \end{aligned}$$

Proof. As $B = (b_1, b_2, b_3, b_4; \omega_B)$. Thus $1/B = (1/b_4, 1/b_3, 1/b_2, 1/b_1; \omega)$ and $A/B = A \times (1/B)$. Therefore the proof of this theorem follows from Theorem 3.2.4, so we omit here. □

Example 3.2.1.

If we apply the proposed arithmetic operations between fuzzy numbers A and B defined in Example 3.1.1 then we get $A + B = (1.1, 1.256, 1.344, 1.5; 0.5)$, $A - B = (-0.3, -0.144, -0.056, 0.1; 0.5)$, $A(\cdot)B = (0.3, 0.393, 0.447, 0.56; 0.5)$, $A/B = (0.625,\ 0.81,\ 1.01,\ 1.167; 0.5)$.

Example 3.2.2.

If we apply the proposed arithmetic operations in Example 3.1.2 then we get $A_1 + A_2 = (0.7, 1.043, 1.157, 1.5; 0.5)$ and $A_1 + A_3 = (0.7, 1.011, 1.189, 1.5; 0.5)$. Thus, it has been clearly seen that $A_1 + A_2 \in A_1 + A_3$. Therefore these improved arithmetic operations can overcome the shortcomings of the inconsistency of Chen [5] approach in addition between the generalized fuzzy numbers.

3.3 Numerical examples

In this section, some numerical example has been used for showing the validity of these operations.

3.3.1 Length of the Rod

Let length of the rod is a triangular fuzzy number $A = (12cm, 13.5cm, 15cm; 0.8)$. If the length $B = (5cm, 6.5cm, 8cm; 0.7)$, a triangular fuzzy number, is cut off from this rod then the remaining length of the rod is $C = A(-)B$.

The membership function corresponding to generalized triangular number A and B are defined as below

$$\mu_A(x) = \begin{cases} 0.8\left(\dfrac{x-12}{1.5}\right) & ; \quad 12 \le x \le 13.5 \\ 0.8 & ; \quad x = 13.5 \\ 0.8\left(\dfrac{15-x}{1.5}\right) & ; \quad 13.5 \le x \le 15 \\ 0 & ; \quad \text{otherwise} \end{cases} \quad ; \quad \mu_B(y) = \begin{cases} 0.7\left(\dfrac{y-5}{1.5}\right) & ; \quad 5 \le y \le 6.5 \\ 0.7 & ; \quad y = 6.5 \\ 0.7\left(\dfrac{8-y}{1.5}\right) & ; \quad 6.5 \le y \le 8 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Now $B = (-8cm, -6.5cm, -5cm; 0.7)$ be the negative of the fuzzy number B, then their corresponding membership functions is given as

$$\mu_{-B}(x) = \begin{cases} 0.7\left(\dfrac{x+8}{1.5}\right) & ; \quad -8 \le x \le -6.5 \\ 0.7 & ; \quad x = -6.5 \\ 0.7\left(\dfrac{-5-x}{1.5}\right) & ; \quad -6.5 \le x \le -5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Since $\omega_A = 0.8$ and $\omega_B = 0.7$ so take $\omega = 0.7$ which is $\min(\omega_A, \omega_B)$. As $\omega = \omega_B$ so there is a need to preserve the flatness of fuzzy number A. So for this, the triangular fuzzy number A is converted into the generalized trapezoidal fuzzy number $A^* = (12cm, a_2^*, a_3^*, 15cm; 0.7)$ where $a_2^* = a_1 + \omega(a_2 - a_1)./\omega_A = 12 + (0.7)(1.5)/(0.8) = 13.3125$ and $a_3^* = a_3 - \omega(a_3 - a_2)./\omega_A = 15 - (0.7)(1.5)/(0.8) = 13.6875$. Thus $A^* = (12cm, 13.3125cm, 13.6875cm, 15cm; 0.7)$ and their corresponding membership function becomes

$$\mu_{A^*}(x) = \begin{cases} 0.7\left(\dfrac{x-12}{1.3125}\right) & ; \quad 12 \le x \le 13.3125 \\ 0.7 & ; \quad 13.3125 \le x \le 13.6875 \\ 0.7\left(\dfrac{15-x}{1.3125}\right) & ; \quad 13.6875 \le x \le 15 \end{cases}$$

Hence, by using the property of the addition of two generalized fuzzy numbers, the membership functions of the remaining length of the rod is trapezoidal fuzzy number and is given as

$$\mu_C(x) = \begin{cases} 0.7\left(\dfrac{x-4}{2.8125}\right) & ; \quad 4 \le x \le 6.8125 \\ 0.7 & ; \quad 6.8125 \le x \le 7.6875 \\ 0.7\left(\dfrac{10-x}{2.3125}\right) & ; \quad 7.6875 \le x \le 10 \end{cases}$$

i.e., $C = (4cm, 6.8125cm, 7.1875cm, 10cm; 0.7)$. Hence, we conclude that the remaining length of the rod lies between 4cm and 10cm while there is a 70% probability that the length takes the values between 6.8125cm an 7.6875cm. On the other hand, if we utilize the standard approach for computing the fuzzy number of C then we get $C = (4cm, 7cm, 10cm; 0.7)$. The variation of the membership function for the remaining length of rod by the proposed approach are depicted graphically in Fig. 3.3. On the other hand, the membership degrees at different levels of significance by the existing and proposed arithmetic operations are summarized in Table 3.1. From this table, it has been observed that

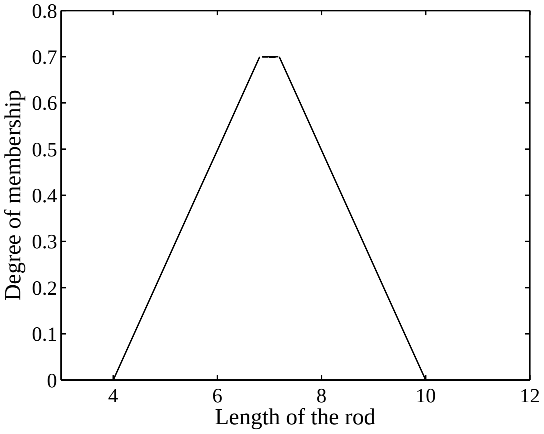


Figure 3.3: Membership function of subtraction of two numbers

- The results computed by the traditional or crisp methodology with ‘crisp’ label shows that their results are independent of the level of confidence levels. Hence, their corresponding results are not so useful for the practical purposes.

- The results computed by taking the existing arithmetic operations have triangular fuzzy number with confidence level 0.7. Also, it has been observed that there is 70% probability that the length of rod takes the value 7cm.

- The result computed by the proposed arithmetic operations have a trapezoidal fuzzy number and hence conclude that the obtained values by the proposed approach is more reasonable than the outcome obtained by the existing approach.

Table 3.1: Comparison of results for Subtraction of two numbers

$\alpha-$ cut	Crisp [22]		Banerjee and Roy [2]		Ansha [1]		Kaur [20]		Proposed approach	
	L	R	L	R	L	R	L	R	L	R
0	7.0000	7.0000	4.0000	10.0000	4.0000	10.000	4.0000	10.000	4.0000	10.0000
0.1	7.0000	7.0000	4.4286	9.5714	5.1339	8.8661	4.9740	9.0260	4.4018	9.5982
0.2	7.0000	7.0000	4.8571	9.1429	5.6036	8.3964	5.2293	8.7707	4.8036	9.1964
0.3	7.0000	7.0000	5.2857	8.7143	5.9640	8.0360	5.4149	8.5851	5.2054	8.7946
0.4	7.0000	7.0000	5.7143	8.2857	6.2678	7.7322	5.5851	8.4149	5.6071	8.3929
0.5	7.0000	7.0000	6.1429	7.8571	6.5355	7.4645	5.7707	8.2293	6.0089	7.9911
0.6	7.0000	7.0000	6.5714	7.4286	6.7775	7.2225	6.0260	7.9740	6.4107	7.5893
0.7	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	6.8125	7.1875

L: left membership values; R: right membership values

3.3.2 Area of the rectangle

Let length and breath of a rectangle are two sigmoidal fuzzy numbers given by $A = (1cm, 2cm, 4cm; 0.75)$ and $B = (3cm, 5cm, 6cm; 0.85)$, then area C of the rectangle is $C = A \cdot B$.

Since $\omega_A < \omega_B$ so, $\omega = \min(0.75, 0.85) = 0.75 = \omega_A$ and hence the fuzzy number B is converted into B^* by preserving the flatness of the numbers and hence we get $B^* = (3cm, 4.7647cm, 5.1176cm, 6cm; 0.75)$. Now, area of rectangle has obtained as $C = (3cm^2, 9.5294cm^2, 10.2352cm^2, 24cm^2; 0.75)$ and hence its membership function becomes

$$\mu_C(x) = \begin{cases} 0.75\left(\dfrac{x-3}{6.5294}\right) & ; \quad 3 \leq x \leq 9.5294 \\ 0.75 & ; \quad 9.5294 \leq x \leq 10.2352 \\ 0.75\left(\dfrac{24-x}{13.7648}\right) & ; \quad 10.2352 \leq x \leq 24 \end{cases}$$

The variation of the membership functions of the area of rectangle by existing as well as by proposed approach are depicted graphically in Fig. 3.4 while the complete range of their membership values are arranged in Table 3.2.

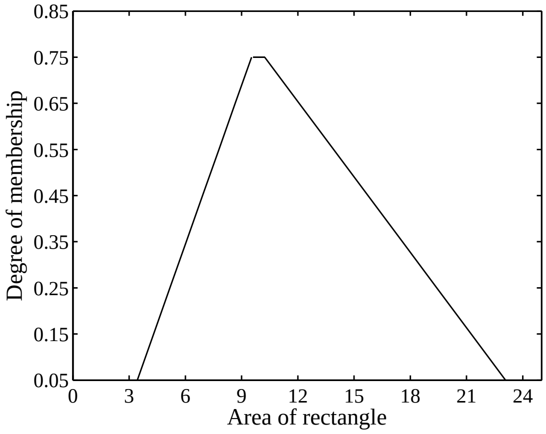


Figure 3.4: Membership function of Area of the rectangle

Table 3.2: Comparison of results for Area of the rectangle

$\alpha-$ cut	Kumar and Aggarwal [22]		Banerjee and Roy [2]		Ansha [1]		Kaur [20]		Proposed approach	
	L	R	L	R	L	R	L	R	L	R
0	10.000	10.000	3.0000	24.0000	3.0000	24.000	3.0000	24.0000	3.0000	24.0000
0.15	10.000	10.000	4.4000	21.2000	6.1305	17.7390	5.0841	18.4430	4.3059	21.2470
0.25	10.000	10.000	5.3333	19.3333	7.0415	15.9171	5.5211	17.4657	5.1765	19.4117
0.35	10.000	10.000	6.2667	17.4667	7.7819	14.4362	5.9081	16.6848	6.0471	17.5764
0.45	10.000	10.000	7.2000	15.6000	8.4222	13.1556	6.2831	15.9434	6.9176	15.7411
0.55	10.000	10.000	8.1333	13.7333	8.9944	12.0111	6.7168	15.1259	7.7882	13.9058
0.65	10.000	10.000	9.0667	11.8667	9.5166	10.9667	7.3480	14.0050	8.6588	12.0705
0.75	10.000	10.000	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000	9.5294	10.2352

L: left membership values; R: right membership values

3.3.3 Length of the rectangle

Let area and breadth of the rectangle be given as a sigmoidal fuzzy numbers $A = (1cm^2, 2cm^2, 4cm^2; 0.75)$ and $B = (3cm, 5cm, 6cm; 0.85)$ respectively, the length of the rectangle is given by $A(\div)B$ or $A(\cdot)B^{-1}$. Now the fuzzy number B is transformed into a new generalized fuzzy number as $B^* = (3cm, 4.7647cm, 5.1176cm, 6cm; 0.75)$. Thus, the length of the rod is computed which is a generalized trapezoidal fuzzy number and given by $C = (0.1667cm, 0.3908cm, 0.4197cm, 1.3333cm; 0.75)$. On the other hand, if we compute the length of rectangle by the existing CHEN approach then we get $C = (0.1667cm, 0.4cm, 1.3333cm; 0.75)$ which is clearly not reasonable than the proposed result. The variations of it at different level of confidence is shown in Fig. 3.5 by the different approaches. The complete variety of these membership functions in the degree of satisfaction α are given in Table 3.3 along with the values by taking the existing approaches results.

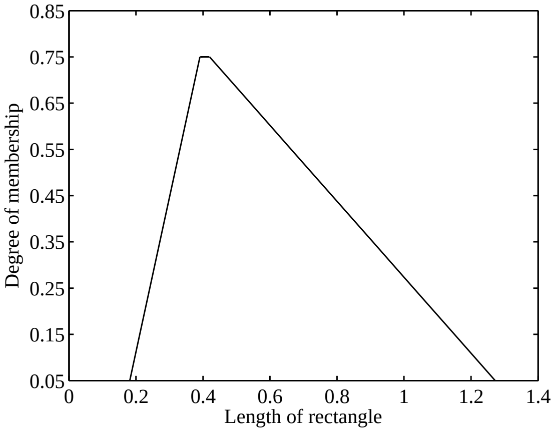


Figure 3.5: Membership function of Length of the rectangle

Table 3.3: Comparison of results for length of the rectangle

$\alpha-$ cut	Kumar and Aggarwal [22]		Banerjee and Roy [2]		Ansha [1]		Kaur [20]		Proposed approach	
	L	R	L	R	L	R	L	R	L	R
0	0.4000	0.4000	0.1667	1.3333	0.1667	1.3333	0.1667	1.3333	0.1667	1.3333
0.15	0.4000	0.4000	0.2133	1.1467	0.2710	0.9159	0.2439	0.9320	0.2115	1.1506
0.25	0.4000	0.4000	0.2444	1.0222	0.3014	0.7945	0.2592	0.8650	0.2414	1.0288
0.35	0.4000	0.4000	0.2756	0.8978	0.3261	0.6957	0.2719	0.8123	0.2713	0.9070
0.45	0.4000	0.4000	0.3067	0.7733	0.3474	0.6104	0.2844	0.7631	0.3012	0.7851
0.55	0.4000	0.4000	0.3378	0.6489	0.3665	0.5341	0.2986	0.7097	0.3310	0.6633
0.65	0.4000	0.4000	0.3689	0.5244	0.3839	0.4644	0.3188	0.6381	0.3609	0.5415
0.75	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.3908	0.4197

L: left membership values; R: right membership values

3.3.4 Perimeter of the rectangle

Consider the length and breadth of the rectangle as a sigmoidal fuzzy numbers $A = (12cm, 13.5cm, 14cm; 0.9)$ and $B = (6cm, 7.5cm, 9cm; 0.8)$, respectively, then perimeter C of the rectangle is given by $2(A + B)$. The membership function corresponding to fuzzy number of A and B are

$$\mu_A(x) = \begin{cases} 0.9\left(\dfrac{x-12}{1.5}\right) & ; \quad 12 \le x < 13.5 \\ 0.9 & ; \quad x = 13.5 \\ 0.9\left(\dfrac{14-x}{0.5}\right) & ; \quad 13.5 < x \le 14 \end{cases} \quad ; \quad \mu_B(y) = \begin{cases} 0.8\left(\dfrac{y-6}{1.5}\right) & ; \quad 6 \le y < 7.5 \\ 0.8 & ; \quad y = 7.5 \\ 0.8\left(\dfrac{9-y}{1.5}\right) & ; \quad 7.5 < y \le 9 \end{cases}$$

In order the preserve the flatness of fuzzy number A, we convert it into the new generalized trapezoidal number as $A^* = (12cm, 13.3333cm, 13.5555cm, 14cm; 0.8)$. Therefore by using the addition and scalar multiplication property of the two generalized fuzzy number, we get the perimeter of the rectangle as $C = (36cm, 41.6666cm, 42.1110, 46cm; 0.8)$ and hence membership function of C becomes

$$\mu_C(x) = \begin{cases} 0.8\left(\dfrac{x-36}{5.6666}\right) & ; \quad 36 \leq x \leq 41.66666 \\ 0.8 & ; \quad 41.6666 \leq x \leq 42.1110 \\ 0.8\left(\dfrac{46-x}{3.889}\right) & ; \quad 42.1110 \leq x \leq 46 \end{cases}$$

The results corresponding to this membership function have been summarized in Fig. 3.6 for the different level of significance. From these results it has been seen that the 80% of the probability to get the perimeter of the rectangle lies between [41.6666cm, 42.1110cm]. The corresponding values by taking existing approaches are summarized in Table 3.4.

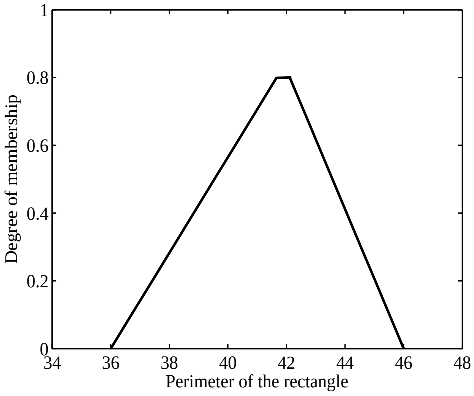


Figure 3.6: Membership function of Perimeter of the rectangle

Table 3.4: Comparison of results for perimeter of the rectangle

$\alpha-$ cut	Kumar and Aggarwal [22]		Banerjee and Roy [2]		Ansha [1]		Kaur [20]		Proposed approach	
	L	R	L	R	L	R	L	R	L	R
0	42.0000	42.0000	36.0000	46.0000	36.0000	46.0000	36.0000	46.0000	36.0000	46.0000
0.1	42.0000	42.0000	36.7500	45.5000	38.1213	44.5858	37.8595	44.7603	36.7083	45.5139
0.2	42.0000	42.0000	37.5000	45.0000	39.0000	44.0000	38.3515	44.4323	37.4167	45.0277
0.3	42.0000	42.0000	38.2500	44.5000	39.6742	43.5505	38.6978	44.2015	38.1250	44.5416
0.4	42.0000	42.0000	39.0000	44.0000	40.2426	43.1716	39.0000	44.0000	38.8333	44.0555
0.5	42.0000	42.0000	39.7500	43.5000	40.7434	42.8377	39.3022	43.7985	39.5416	43.5694
0.6	42.0000	42.0000	40.5000	43.0000	41.1962	42.5359	39.6485	43.5677	40.2500	43.0833
0.7	42.0000	42.0000	41.2500	42.5000	41.6125	42.2583	40.1405	43.2397	40.9583	42.5971
0.8	42.0000	42.0000	42.0000	42.0000	42.0000	42.0000	42.0000	42.0000	41.6666	42.1110

L: left membership values; R: right membership values

3.4 Conclusion

The objective of this chapter is to address some improved arithmetic operations between the generalized fuzzy numbers to overcome the shortcoming of the existing operations. As the existing work on the arithmetic operations consider the same degree of confidence level for different fuzzy numbers and hence it observed that it loss the information which cause the inexact results. So in order to avoid this and to preserve the flatness of the fuzzy numbers we derived an improved arithmetic operators such as addition, scalar multiplication, subtraction, multiplication for generalized trapezoidal (triangular) fuzzy numbers. Several examples demonstrating the usage and advantages of the proposed operations than the existing ones. From the computed results it has been observed that the system analyst may use their results in increasing the performance of the system and may change their target goals with the proposed results rather than existing results.

Chapter 4

Reliability analysis of repairable industrial systems using improved arithmetic operations

In this chapter, reliability analysis of repairable industrial systems has been analyzed by using the improved arithmetic operations for a generalized fuzzy numbers as described in Chapter 3 by considering the different degree of confidence levels. A case study from the washing unit of a paper mill, a complex repairable industrial system, has been taken for demonstrating the approach. Finally, the computed results are compared with the existing methodologies results.

4.1 Introduction

Today with growing complexity of the repairable industrial systems along with advances in technology the job of the system analysts becomes more tedious and challenging as they have to study, characterize, measure and analyze the behavior of the systems in both a qualitative and a quantitative manner using various traditional and non-traditional techniques. But unfortunately, failure is an unavoidable phenomenon associated with technological advancement of the equipments used in these industries. These failures may be the result of human error, poor maintenance or inadequate testing/inspection. The traditional analytical techniques (mathematical and statistical models) need large amounts of data, which are difficult to obtain because of constraints (i.e. rare events of components, human errors and economic considerations) for estimation of the failure/repair characteristics of the system. If data are collected or available from resources and are used as such for analysis then they have a large amount of uncertainty because historical records can only represent the past behavior but may be unable to predict the future behavior of the equipment. However, the authors [4–7, 9–19, 21, 23, 24] have analyzed the behavior of the industrial systems using traditional arithmetic operations. Based on it, they have analyzed the behavior of the system in terms of several reliability parameters which depicts the performance and behavior of the system. But all the above existing studies do not consider the level of significance during an analysis and hence they do not conserve the flatness of the data which results the lose of significance. Therefore, there is a need for developing such type of arithmetic operations which consider the effect of different levels of significance during the analysis and will reduce the uncertainties, for each reliability index, up to a desired degree of accuracy.

Therefore, in this light, the main objective of this chapter is to analyze the behavior of repairable system using an improved arithmetic operations as described in Chapter 3. The presented technique utilizes uncertain data of the system and analyze its behavior with reduced level of uncertainty which makes the decision more realistic and generic for further application.

4.2 Methodology

An approach has been presented by taking the different fuzzy numbers for analyzing the behavior of the system under the following restrictions.

(i) component parameters are independent.

(ii) maintenance facility is separate for each component.

(iii) after repairs, component is assumed to be as new.

(iv) standby components are of same nature as of active units.

Based on these assumptions, the methodology for conducting the behavior analysis are summarized in the following five steps as follows:

(Step 1:) The approach has been started from the information collection phase in which the data related to the various components' of the system are extracted in the form of failure rates (λ_i's) and repair times (τ_i's).

(Step 2:) Since the obtained information from step 1 are generally out of date or imprecise due to lack of proper update or by human errors. So in order to quantify the uncertainties in the data, this data is converted into the different form of the fuzzy numbers with some known spread as suggested by the decision makers towards the data. For instance, if the decision-maker gives ±15% spread towards the data, then their corresponding generalized fuzzy number becomes $(\lambda_{i1}, \lambda_{i2}, \lambda_{i3}; \omega_i) = (0.85\lambda_{i2}, \lambda_{i2}, 1.15\lambda_{i2}; \omega_i)$ corresponding to i^{th} component of the failure rate λ_i where ω_i is the level of satisfaction towards the data. Similarly for the repair times.

(Step 3:) Since there are different levels of satisfaction towards the data i.e. ω_i's are different so take $\omega = \min_i\{\omega_i\} = \omega_k$(say) and hence the generalized triangular fuzzy numbers corresponding to λ_i's and τ_i's $(i \neq k)$ are transformed into the generalized trapezoidal fuzzy number $\lambda_k = (0.85\lambda_{k2}, \lambda^*_{k2}, \lambda^*_{k3}, 1.15\lambda_{k2}; \omega_k)$.

(Step 4:) Using the transformed fuzzy numbers and their corresponding $\alpha-$ cuts obtained from the above step have been used for finding the overall system fuzzy numbers by using an improved fuzzy arithmetic operations on conventional AND/OR expressions as listed in Table 4.1. Based on these operations, various reliability parameters listed in Table 4.2 are measured in terms of membership functions.

Table 4.1: Basic Expressions of Lambda Tau Methodology

Gate	λ_{AND}	τ_{AND}	λ_{OR}	τ_{OR}
Expression	$\prod_{j=1}^{n} \lambda_j \left[\sum_{i=1}^{n} \prod_{\substack{j=1 \\ i\neq j}}^{n} \tau_j\right]$	$\frac{\prod_{i=1}^{n} \tau_i}{\sum_{j=1}^{n} \left[\prod_{\substack{i=1 \\ i\neq j}}^{n} \tau_i\right]}$	$\sum_{i=1}^{n} \lambda_i$	$\frac{\sum_{i=1}^{n} \lambda_i \tau_i}{\sum_{i=1}^{n} \lambda_i}$

(Step 5:) After fuzzifying the data, it is necessary to decode it in the form of crisp number so as to implement their results in the real systems. For this, center of gravity

Table 4.2: Some Reliability Parameters

Parameters	Expressions
Failure rate	$\text{MTTF}_s = \frac{1}{\lambda_s}$
Repair time	$\text{MTTR}_s = \frac{1}{\mu_s} = \tau_s$
Mean Time Between Failures	$\text{MTBF}_s = \text{MTTF}_s + \text{MTTR}_s$
Reliability	$\text{R}_s = e^{-\lambda_s t}$
Availability	$\text{A}_s = \frac{\mu_s}{\lambda_s+\mu_s} + \frac{\lambda_s}{\lambda_s+\mu_s} e^{-(\lambda_s+\mu_s)t}$
Expected numbers of failures	$\text{W}_s(0,t) = \frac{\lambda_s \mu_s t}{\lambda_s+\mu_s} + \frac{\lambda_s^2}{(\lambda_s+\mu_s)^2}[1 - e^{-(\lambda_s+\mu_s)t}]$

(COG) method has been used as it gives the mean of the membership function in the interval $[x_1, x_2]$. Mathematically, it is represented as

$$COG = \frac{\int_{x_1}^{x_2} x \cdot \mu_A(x)dx}{\int_{x_1}^{x_2} \mu_A(x)dx} \tag{4.2.1}$$

4.3 Case study

In this section, a brief description of the system i.e. a paper mill situated in the northern part of India and producing approximately 200 tons of paper per day have been presented. The paper mills are large capital oriented engineering systems, comprising of units/subsystems namely, feeding, pulping, washing, screening, bleaching, forming, dryer and press, arranged in predefined configuration. Out of it, the washing unit is one of the most important functioning unit of the mill. The brief description of the system is summarized as below.

4.3.1 System description

The Washing of prepared pulp is done in three to four stages, to get it free from blackness and to prepare the fine fibers of the pulp. The system consists of four main subsystems, defined as:

- **Filter (A):** It consists of single unit which is used to drain black liqueur from the cooked pulp.
- **Cleaners (B):** In this subsystem three units of cleaners are arranged in parallel configuration. Each unit may be used to clean the pulp by centrifugal action. Failure of anyone will reduce the efficiency of the system as well as quality of paper.

- **Screeners (C):** Herein two units of screeners are arranged in series. These are used to remove oversized, uncooked and odd shaped fibers from pulp through straining action. Failure of any one will cause the complete failure of the system.
- **Deckers (D):** Two units of deckers are arranged in parallel configuration. The function of deckers is to reduce the blackness of pulp. Complete failure of decker occurs when both the components will fail.

The systematic diagram of the system are shown in Fig. 4.1.

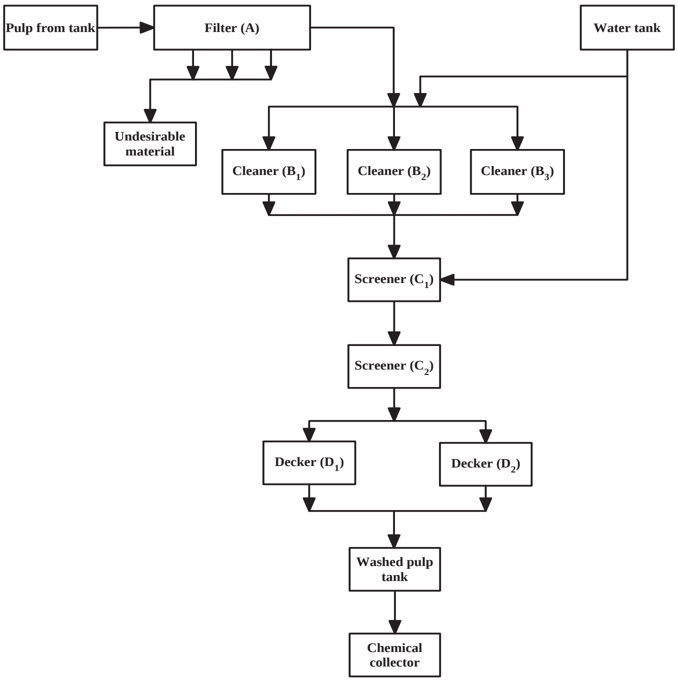


Figure 4.1: Systematic diagram of Washing system

4.3.2 Behavior analysis

The procedure steps used for conducting the analysis by using proposed methodology are given as below.

(Step 1:) Under the information extraction phase, the data related to main components' of the system are extracted in the form of failure rates (λ_i's) and repair times (τ_i's) and are summarized in Table 4.3 along with their level of significance.

Table 4.3: Input data for the main components of the Washing system

Components→	Filter	Cleaners	Screeners	Deckers
	$(i=1)$	$(i=2,3,4)$	$(i=5,6)$	$(i=7,8)$
Failure rate λ_i (hrs^{-1})	1×10^{-3}	3×10^{-3}	5×10^{-3}	5×10^{-3}
Repair time τ_i (hrs)	3	2	3	3

(Step 2:) To handle the uncertainties, the obtained(crisp) data are fuzzified into TFNs with some known spread or support, ±15% (also at ±25%, ±50%) on both sides of data, as suggested by system analyst are arranged in Table 4.4.

Table 4.4: Generalized Triangular fuzzified data for the main components system

Components ↓	Failure rate λ_i (hrs^{-1})	Repair time τ_i (hrs)
Filter $(i=1)$	(0.0085, 0.001, 0.00115; 0.65)	(2.55, 3, 3.45; 0.65)
Cleaners $(i=2,3,4)$	(0.00255, 0.003, 0.00345; 0.75)	(1.70, 2, 2.30; 0.75)
Screeners $(i=5,6)$	(0.00425, 0.005, 0.00575; 0.80)	(2.55, 3, 3.45; 0.80)
Deckers $(i=7,8)$	(0.00425, 0.005, 0.00575; 0.90)	(2.55, 3, 3.45; 0.90)

(Step 3:) Since there are different level of significance and hence take $\omega=\min(0.65, 0.75,$ $0.80,\ 0.90)=0.65$, i.e. $\omega=\omega_1=0.65$. So the given collected data have been transformed into the new generalized trapezoidal fuzzy number corresponding to λ_i's and τ_i's for $i=2,\ldots,8$ and their values are arranged in Table 4.5.

Table 4.5: Generalized data for the main components system

Components ↓	Failure rate λ_i^* (hrs^{-1})	Repair time τ_i^* (hrs)
Filter $(i=1)$	(0.00085, 0.001, 0.00115; 0.65)	(2.55, 3, 3.45; 0.65)
Cleaners $(i=2,3,4)$	(0.00255, 0.00294, 0.00306, 0.00345; 0.65)	(1.70, 1.96, 2.04, 2.30; 0.65)
Screeners $(i=5,6)$	(0.00425, 0.004859, 0.005141, 0.00575; 0.65)	(2.55, 2.9156, 3.0843, 3.45; 0.65)
Deckers $(i=7,8)$	(0.00425, 0.004791, 0.005208, 0.00575; 0.65)	(2.55, 2.875, 3.125, 3.45; 0.65)

(Step 4:) Based on these transformed data and the minimal cut set of the system obtained as $\{A\}$, $\{B_1B_2B_3\}$, $\{C_1\}$, $\{C_2\}$ and $\{D_1D_2\}$, the system expression for systems' failure rate (λ_s) and repair time (τ_s) are obtained using the results given in Table 4.1, as below

$$\begin{aligned}\lambda_s &= \lambda_1+\lambda_2^*\lambda_3^*\lambda_4^*(\tau_2^*\tau_3^*+\tau_3^*\tau_4^*+\tau_4^*\tau_2^*)+\lambda_5^*+\lambda_6^*+\lambda_7^*\lambda_8^*(\tau_7^*+\tau_8^*)\\ \tau_s &= \frac{\lambda_1\tau_1+\lambda_2^*\lambda_3^*\lambda_4^*\tau_2^*\tau_3^*\tau_4^*+\lambda_5^*\tau_5^*+\lambda_6^*\tau_6^*+\lambda_7^*\lambda_8^*\tau_7^*\tau_8^*}{\lambda_s}\end{aligned}$$

Based on these values, the various reliability parameters have been computed at different confidence levels ranging from 0 to 0.65 at the mission time t = 10(hrs) with left and right spread. These computed results are depicted graphically in Fig. 4.2 for ±15% spreads along with the existing techniques results. From this figure, it has been concluded that

- The results computed by FLT methodology, shown with 'FLT' legend in the figures, do not consider the degree of satisfaction levels and hence their results are impractical in real-life situation.
- The results computed by existing methodology have consider the minimum level of satisfaction levels and hence their corresponding results becomes triangular in shape. Thus, it has been observed that they lose its significance and hence their results are limited to the system analysts.
- On the other hand, the results computed by the proposed approach have trapezoidal in nature which clearly signifies the level of satisfaction during the analysis. Further, there is a increase in the crisp level which suggests that DM has smaller and more sensitive region to make more sound and effective decision in lesser time.

(Step 5:) As for implementing these results in real-life situation then it is necessary that these fuzzified valued should be converted to crisp values as most of the actions implemented by the human or machines are binary or crisp in nature. Thus for this, COG method is used and the corresponding crisp and defuzzified values of reliability indices at ±15%, ±25% and ±50% spreads are computed and compared with the existing results are shown in Table 4.6. It shows that when uncertainty level in the form of spread increases from ±15% to ±25% and further ±50%, the variation in defuzzified values for almost all the reliability indices are not so much as shown by results of other techniques.

Table 4.6: Crisp and Defuzzified Values of Reliability Indices for Washing System

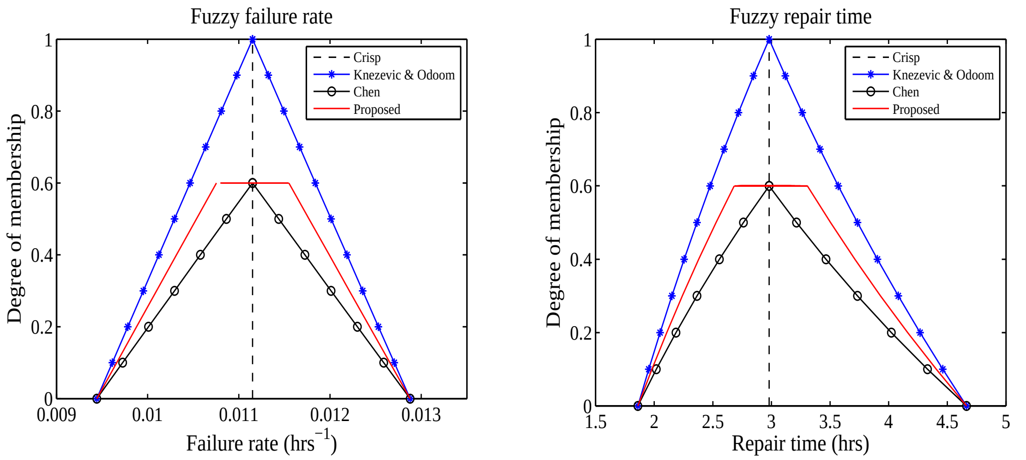
		Computed spread for reliability indices					
Spread	Technique	Failure rate	Repair time	MTBF	ENOF	Reliability	Availability
±0%	Crisp	0.01115032	2.979763	92.66325	0.108919	0.894488	0.968846
±15%	FLT [21]	0.01115544	3.121068	93.85087	0.109248	0.894508	0.967489
	Chen [5]	0.01115547	3.121943	93.85823	0.109251	0.894509	0.967482
	proposed	0.01115904	3.206889	94.57983	0.109458	0.894518	0.966693
±25%	FLT [21]	0.01116453	3.387354	96.080550	0.109969	0.894545	0.965238
	Chen [5]	0.01116462	3.390104	96.103664	0.109980	0.894545	0.965225
	proposed	0.01117456	3.625623	98.119361	0.110633	0.894571	0.963330
±50%	FLT [21]	0.01120723	4.969436	109.158853	0.117150	0.894718	0.958872
	Chen [5]	0.01120757	4.990784	109.336319	0.117373	0.894719	0.959000
	proposed	0.01124728	5.934077	117.680954	0.121917	0.894823	0.955575

4.3.3 Sensitivity analysis

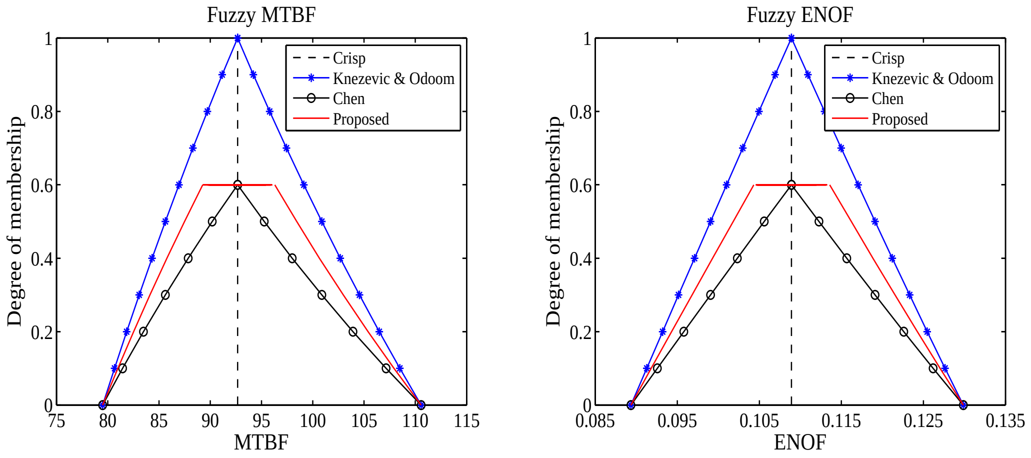
In order to sustain the approach, an effect of the various component's parameters on the system MTBF has been investigated. For this, repair time and ENOF have fixed along the ranges taken from Fig. 4.2(b) and Fig. 4.2(d) at cut level $\alpha = 0$ corresponding to all distribution functions respectively and hence the impact of the reliability, availability and failure rates on MTBF are summarized in Table 4.7. From this table, it has been seen that different combinations have been made in which failure rate are changes from 0.0083 to 0.0111 and further to 0.0155 and then their simultaneously ranges of MTBF are varies from, for the first combination, 154.1984 to 225.7793 and 153.7955 to 225.8555. A similar conclusion has been observed for other combinations too. The complete variations of these have been summarized through the surface plot in Fig. 4.3. These results will be highly beneficial for the plant personnel to depict the effect of each component and hence change their strategy/target goals accordingly.

Table 4.7: Change in MTBF for Various Combination of Reliability Parameters for Washing System

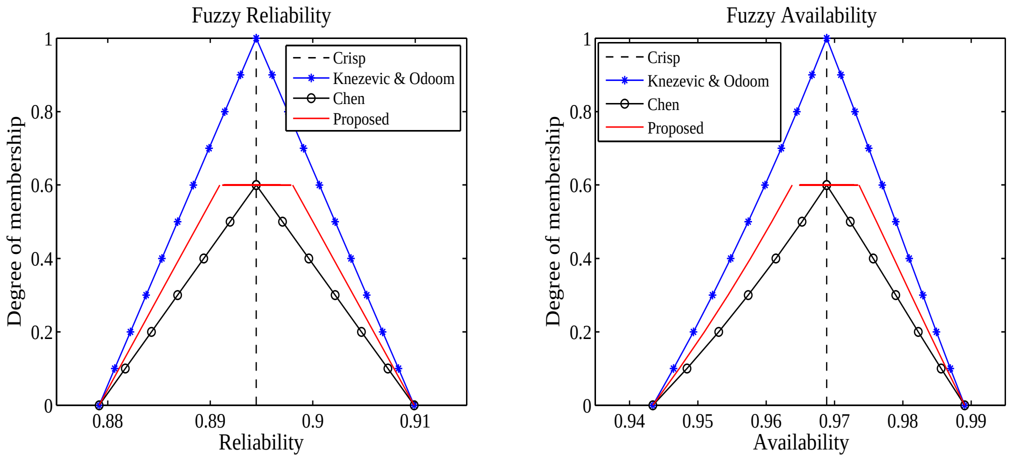
		[Reliability, Failure rate, Availability]								
Methods	Range of MTBF	[0.8482, 0.0083, 0.9269]	[0.8482, 0.0111, 0.9269]	[0.8482, 0.0155, 0.9269]	[0.8945, 0.0083, 0.9612]	[0.8945, 0.0111, 0.9612]	[0.8945, 0.0155, 0.9612]	[0.9629, 0.0083, 0.9806]	[0.9629, 0.0111, 0.9806]	[0.9629, 0.0155, 0.9806]
Knezevic and Odoom [21]	Min:	154.1984	115.5664	83.05852	104.2666	78.10572	56.09198	35.44586	26.57485	19.11013
	Max:	225.7793	169.7882	122.6732	152.3350	114.4190	82.51354	51.98171	39.12459	28.30566
Chen [5]	Min:	153.7955	115.2644	82.84147	103.9941	77.90161	55.94539	35.35896	26.51114	19.06592
	Max:	225.8555	169.8456	122.7147	152.3865	114.4576	82.54142	52.02015	39.15869	28.33610
proposed	Min:	153.7955	115.2644	82.84147	103.9941	77.90161	55.94539	35.35896	26.51114	19.06592
	Max:	225.8555	169.8456	122.7147	152.3865	114.4576	82.54142	52.02015	39.15869	28.33610



(a) Fuzzy Failure Rate (b) Fuzzy Repair Time

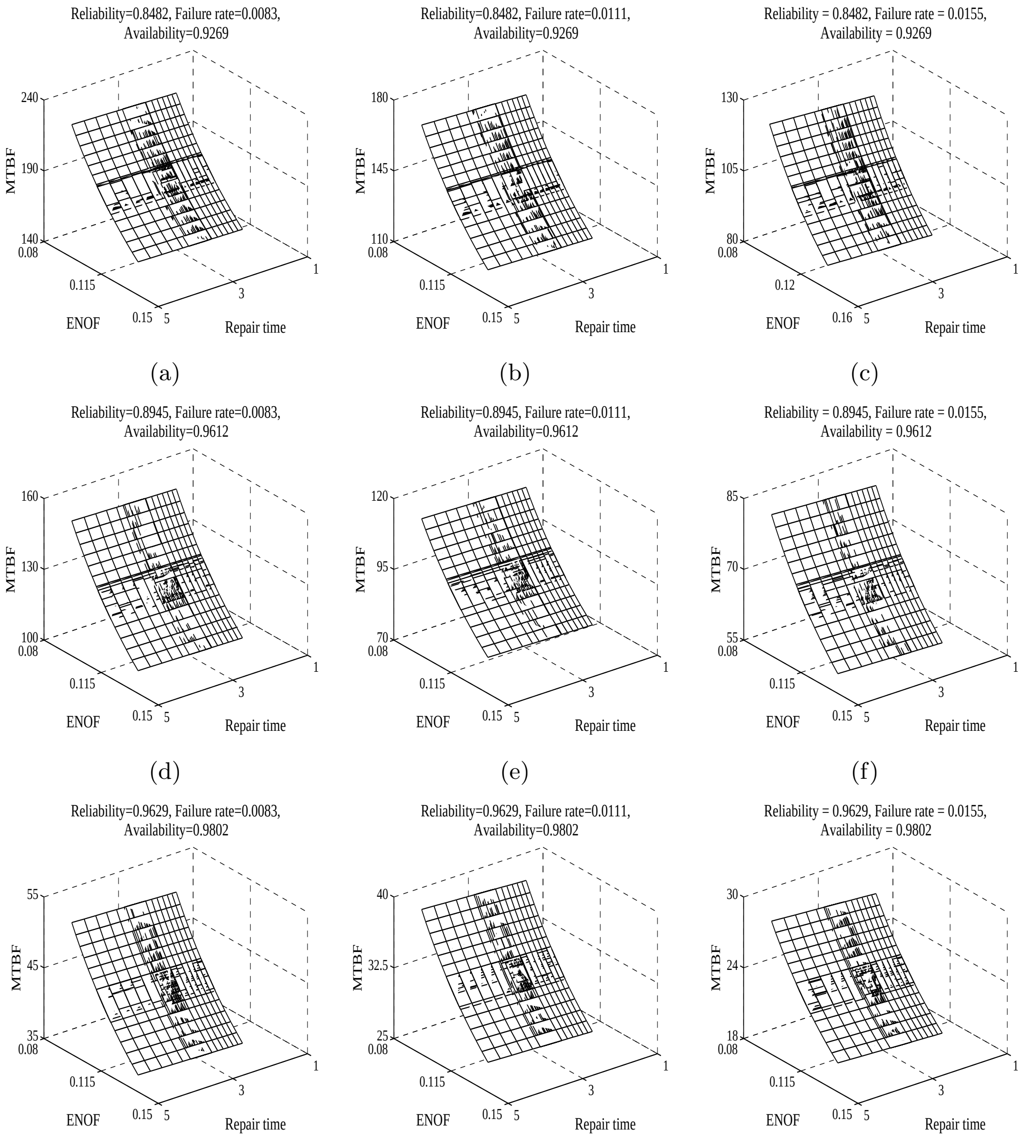


(c) Fuzzy MTBF (d) Fuzzy ENOF



(e) Fuzzy Reliability (f) Fuzzy Availability

Figure 4.2: Fuzzy Reliability Indices Plots for Washing system



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 4.3: Effect of different parameters on MTBF of the system

4.4 Conclusion

In this chapter, an analysis has been conducting for investigating the performance of an industrial systems using an improved arithmetic operation between the generalized fuzzy numbers. For this, a case study of washing unit, from a complex repairable system of paper mill, has been taken for illustration. In it, the different level of significance towards the collected data have been taken in account and hence based on it the fuzzified data are transferred into the generalized trapezoidal fuzzy numbers for preserving the flatness of the number. Then various reliability parameters at different level of spread, suggested by decision makers', have been computed by using an improved arithmetic operations and hence compare their performance with some existing operations. Sensitivity analysis on the effect of system MTBF have also been addressed by varying the other parameters. These results of system will help the concern managers to plan and adapt suitable maintenance practices/strategies for improving system performance and thereby reduce operational and maintenance costs.

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