DEA-MCDM APPROACH FOR RANKING DECISION MAKING UNITS USING OWA AGGREGATION OPERATORS

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CERTIFICATE

I hereby certify that the dissertation entitled, **"DEA-MCDM Approach for Ranking Decision Making Units using OWA Aggregation Operators"** which is being submitted by **Ms. Meenu Verma** (Roll No. 301503014), in the partial fulfillment of the requirement for the award of the degree of Master of Science in the School of Mathematics, Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of Dr. Jolly Puri during the period from January 2017 to July 2017. The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/Institute for the award of any degree.

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Abstract

Data Envelopment Analysis (DEA) (Charnes et al., 1978) is a linear programming based non-parametric technique to measure the relative efficiencies of homogeneous decision-making units (DMUs) in the presence of a production process consisting of multiple inputs and outputs. DMU is an entity who is responsible to convert inputs to outputs and whose performance is to be evaluated. DEA is a popular management tool which is commonly used to evaluate the efficiency of a number of similar DMUs. It identifies the production frontier which envelopes the whole input-output data. The DMUs that lie on the efficient frontier are called efficient DMUs with efficiency score equals to one. On the other hand, all the DMUs which are enveloped by the efficient frontier are termed as inefficient DMUs which take the efficiency score less than one.

DEA provides an opportunity to a DMU to self-evaluate its efficiency relative to the other DMUs. The efficiencies in self-evaluation are then compared and ranked. Self-evaluation allows each DMU to rate its efficiency with the most favorable weights to itself. Despite of the fact that DEA has proven an effective approach in identifying best practice frontiers, its flexibility in weighting multiple inputs and outputs and its nature of self-evaluation have been criticized. Self-evaluation lacks in discriminating among DMUs possessing efficiency score equals to one. It allows each DMU to be evaluated with its most favorable weights. The inputs and outputs favorable to a particular DMU will be heavily weighted, whereas those not favorable to the DMU will be less weighted/ignored. Hence, more than one DMU take an efficiency score equals to one. To overcome the drawback of self-evaluation, to make the weights more realistic and to increase the discrimination power, cross efficiency evaluation has been introduced by Sexton et al.(1986) which also allows peer-evaluation. The main point of attraction in using cross-efficiency approach is that it provides an ordering among DMUs and also eliminates unrealistic weight schemes without requiring the elicitation of weight restrictions from application area expert. However, the usefulness of cross efficiency is reduced by the non-uniqueness of the DEA optimal weights. Thus, depending on which of the alternate optimal solutions to the DEA linear programs is used, it is possible to improve the performance rating of a DMU, but generally only by worsening the ratings of others. With this drawback, Sexton et al. (1986) and Doyle and Green (1994) proposed the method to deal with the non-unique DEA solutions. They developed aggressive (benevolent) model formulations to identify optimal weights that not only maximize the efficiency of a particular DMU under evaluation, but also minimize (maximize) the average efficiency of the other DMUs. Later on, Wang and Chin (2010) proposed a neutral DEA model that determines one set of input and output weights for every DMU without being aggressive or benevolent to the others. In real applications, sometimes decision maker finds it difficult to make a choice among aggressive, benevolent and neutral cross-efficiency formulations. It can be treated as a decision making problem having the following major issues:

i. How to choose among aggressive, benevolent and neutral cross-efficiency formulations?

ii. How to aggregate the cross-efficiencies based on selected formulation so that the DMUs are ranked realistically?

iii. Whether the selection is appropriate?

To overcome the above discussed issues, the present study is mainly focused on developing an algorithmic approach based on DEA and multi-criteria decision making (MCDM) so that all the three cross-efficiency formulations can be included along with the subjectivity and preference of the decision maker while measuring the final ranking of the DMUs. Therefore, in the present work, we have interpreted the aforementioned decision problem as a MCDM problem and proposed a DEA-MCDM algorithmic approach

for ranking DMUs. The proposed approach has the following characteristics: // i. The proposed DEA-MCDM approach includes the subjectivity and preference of the decision maker in more realistic way due to the use of orness constraint while computing results.

ii. Aggregation of cross-efficiencies in each formulation is done by using ordered weighted averaging (OWA) / induced OWA (IOWA) operators instead of the traditional approach of taking average.

iii. Each of the aggressive, benevolent and neutral cross-efficiency formulations contribute to select the best alternative among the DMUs and to achieve complete ranking of the DMUs.

The outline of the thesis is summarized below:

Chapter 1 is based on the introduction to the performance evaluations problems and objective of the thesis. It includes the literature review on efficiency and cross-efficiency in DEA. It also presents the existing literature on OWA/IOWA aggregation operators using degree of orness and the approaches to determine OWA operator weights.

Chapter 2 includes the mathematical formulation of CCR efficiency model. In this chapter, different cross-efficiency formulations (Wang and Chin, 2010): aggressive, benevolent and neutral, and their mathematical models are discussed in detail.

Chapter 3 is related to the cross-efficiency aggregation by OWA operator. It includes properties and characteristics of OWA operator weights. It presents minimax disparity approach(Wang and Parkan,2005) and other approaches to determine OWA operator weights.

Chapter 4 includes the introduction to decision making process and its various elements. It demonstrates about MCDM technique(Alias et al.,2008) and steps for utilizing this technique. It also includes the proposed DEA-MCDM algorithmic approach for ranking DMUs in real situations. The proposed approach is further illustrated by an application to the educational institution in order to prove its validity in real situations.

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INTRODUCTION AND LITERATURE SURVEY

1.1 Introduction

A variety of techniques has been available in existing literature to study performance evaluation of profit/non- profit organisations also known as Decision Making Units (DMUs). The efficiency analysis is an important tool to analyse the performance in the presence of the set of inputs and outputs. It is found that efficiency evaluation is sensitive to the choice of technique. In existing literature, two approaches have been extensively used to measure the efficiency of homogeneous DMUs, namely, traditional ratio approach and frontier approach. In ratio approach, the researchers employ different ratios to examine different aspects of DMU's performance. As an example in the banking industry, a ratio of return on assets to return on equity has been considered as a key ratio for analysing the financial performance of banks (Avkiran, 2006). However, it has been observed that the ratio approach comprises of certain problems which are summarized as under:

(i) the choice of ratios.

(ii) the choice of benchmark against which the efficiency of a DMU has been assessed.

On the other hand, the frontier approach overcomes the shortcomings of ratio approach and does not possess the aforementioned problems. The following five frontier techniques has been widely used in analysing performance of various organizations(Bauer et al., 1998):

1. Stochastic Frontier Analysis (SFA)
2. Distribution Free Approach (DFA)
3. Thick Frontier Approach (TFA)
4. Data Envelopment Analysis (DEA)
5. Free Disposal Hull(FDH)

The three techniques, namely, SFA, DFA and TFA are known as parametric techniques which require explicit specification of production frontier. The remaining techniques, DEA and FDH are the non-parametric techniques which do not need any prior information regarding production frontier. These are mathematical programming techniques. The most popular technique among these non-parametric technique is DEA which has been extensively employed to analyse the performance of DMUs like banks, hospitals, universities, etc. Here DMU is an entity who is responsible to convert inputs to outputs and whose performance is to be evaluated.

DEA is a non-parametric technique for measuring the relative efficiencies of homogeneous DMUs in terms of multiple inputs and multiple outputs. It identifies the production frontier which envelopes the whole input-output data. All the DMUs which lie on the efficient frontier are called efficient DMUs and these DMUs takes the efficient value equal to one. On the other hand, all the DMUs which are enveloped by the efficient frontier are called inefficient DMUs which take the efficiency value less than one.It formulates the choice of input and output weights as a linear program that further allows each DMU to appear in the best possibility (maximization of its own measured efficiency relative to the other DMUs)given a minimal set of constraints on the weights. It has been used for non- profit organizations to study their performances in the real situations. It is applied for the financial ratios, product analysis, software productivity estimation. It is a popular management tool which is commonly used to evaluate the efficiency of a number of DMUs. It is a linear programming based technique. It compares each DMU with only the best DMU(s). The efficiency of a DMU in DEA is defined as the ratio of the weighted sum of its outputs to the weighted sum of its inputs. This is also known as ′engineering ratio′ and is popularly used to measure the efficiency of a number of alternatives.

DEA provided opportunity to a DMU to self-evaluate its efficiency relative to the other DMUs. The efficiencies in self-evaluation are then compared and ranked. Self-evaluation allows each DMU to rate its efficiency with the most favorable weights to itself. Various theoretical and methodlogical extensions in DEA have been reported in Cooper et al.(2007). Despite of various advantages, DEA possess some drawbacks. Self-evaluation lacks in discriminating among DMUs possessing efficiency score equal to one. It allows each DMU to be evaluated with its most favorable weights. The inputs and outputs favorable to a particular DMU will be heavily weighted, whereas those not favorable to the DMU will be less weighted/ignored. Hence, more than one DMU take an efficiency score equal to one. To overcome the drawback of self-evaluation, to make the weights more realistic and to increase the discrimination power, cross efficiency evaluation which allows peer-evaluation has been introduced by Sexton et al.(1986). While DEA has been proven an effective approach in identifying best practice frontiers, its flexibility in weighting multiple inputs and outputs and its nature of self-evaluation have been criticized.

The cross efficiency evaluation was developed as a DEA extension to rank DMUs. The concept of cross-efficiency was firstly proposed by Sexton et al.(1986) has been further investigated by Doyle and Green(1994). In this method, the best set of weights has been choosen for the targeted DMU and this set is further used to weight the input and output of all other DMUs. In this way, the cross-efficiencies are calculated for all other DMUs using the weights of the targeted DMU. This procedure can be repeated for all DMUs, so filling out a matrix of cross-efficiencies, row-by-row. The leading diagonal of this matrix shows the usual efficiency measurements for each DMUs. There are mainly two advantages of the cross-evaluation method.
(i) It provides an ordering among DMUs.
(ii)It eliminates unrealistic weight schemes without requiring the elicitation of weight restrictions from application area expert.
In real world, cross - efficiency model is used in efficiency evaluations of nursing homes, R and D project selection and preference voting etc. The usefulness of cross efficiency is reduced by the non-uniqueness of the DEA optimal weights. Thus, depending on which of the alternate optimal solutions to the DEA linear programs is used, it is possible to improve the preformance rating of a DMU, but generally only by worsening the ratings of others. With this drawback, Sexton et al.(1986) and Doyle and Green(1994) proposed the method to deal with the non-unique DEA solutions. They developed aggressive (benevolent) model formulations to identify optimal weights that not only maximize the efficiency of a particular DMU under evaluation, but also minimize (maximize) the average efficiency of other DMUs.
Wang and Chin(2010) proposed neutral DEA model that determines one set of input and output weights for every DMU without being aggressive or benevolent to the others.
In present work, we demonstrate the aggressive, benevolent and neutral formulations of cross-efficiency in DEA.

In real applications, sometimes decision maker finds it difficult to make a choice among aggressive, benevolent and neutral cross-efficiency formulations. It can be treated as a decision making problem having the following major issues:

i. How to choose among aggressive, benevolent and neutral cross-efficiency formulations?

ii. How to aggregate the cross-efficiencies based on selected formulation so that the DMUs are ranked realistically?

iii. Whether the selection is appropriate?

To overcome the above discussed issues, the present thesis has the following objectives:

i. To develop an algorithmic approach based on DEA and multi-criteria decision making (MCDM) so that all the three cross-efficiency formulations can be included along with the subjectivity and preference of the decision maker while measuring the final ranking of the DMUs.

ii. To interpret the aforementioned issues as a decision making problem, in particular, MCDM problem in which alternatives are the set of DMUs and criteria as the set of different cross-efficiency formulations.

iii. To propose a DEA-MCDM algorithmic approach for identifying the best DMU in a set of alternatives as DMUs ranking DMUs.

1.2 Literature Survey

1.2.1 Literature Survey on DEA

Charnes et al.(1978) were the first to developed the DEA as a traditional programming model to analyse the performance/efficiency of the organisation in terms of multiple inputs and multiple outputs. Banker et al.(1984) extended their work on DEA and introduced it as a linear programming technique of inputs and outputs for a frontier analysis. Further, Golany(1988) presented a method for restricting weights flexibility in DEA. Charnes et al.(1990) introduced new models named as Polyhedral Cone-Ratio DEA models using CCR ratio models to improve the CCR- efficiencies. Seiford and Thrall(1990) discussed the effect of DEA on the efficient frontier and the effect of convexity rquirements on returns to scale. They also proposed the methodological extensions and alternate model for efficiency evaluation. The various theoretical and methodological extensions in DEA can be seen in Cooper et al.(2007).

Anderson and Peterson(1993) presented a technique for ranking efficient DMUs in which they worked on the ideology to compare the units evaluation with linear combination of all other units in the sample. Sueyoshi(1999) introduced a new ranking approach that combines efficiency analysis with the index measurement using DEA sensitivity analysis. Later, Adler and Golany(2001)presented new method named as Principal Component Analysis(PCA) to improve the discrimination power of DEA. Jerkins and Anderson(2003) suggested the use of partial covariance analysis to choose a subset of variables that provides the information which should be retained.

Beasley (2003) used a non-linear program to maximize the average efficiency scores of the DMUs to simultaneously allocate fixed costs, input resources, and output targets. Further, Amirteimoori and Kordrostami (2005) modified the constraints of Beasley's (2003) model to minimize cases of infeasibility. Asmild et al.(2006) proposed new models to measure effectiveness when value measures are represented by separable or linked cones, where the latter can be used to analyze profit-maximizing effectiveness. Wei and Chang (2011) introduced the optimal system design DEA model to optimally implement a DMU′s resource allocation. Their model helps DMUs discover an optimal design or configuration given some cost or effort constraints.

Lotfi et al.(2013) recently proposed an allocation mechanism using a common dual weights approach for allocating the fixed resources to the units and equitably setting the expected common increase of the targets to the DMUs. Zhang et al.(2014) implemented feature selection as an efficiency evaluation process with multiple evaluation indices, and proposed a novel feature selection framework based on (DEA). Hashemi et al.(2015) proposed an alternative DEA model for centrally imposed resource or output reduction across the reference set. They determined the amount of input and output reduction needed for each DMU to increase the efficiency values of all the DMUs. Al-Refaie et al.(2017)proposed an approach that integrates the desirability function and DEA to enhance process performance with dynamic multi-responses.

1.2.2 Literature Survey on Cross-Efficiency

Sexton et al.(1986) was the first to introduced the cross-efficiency evaluation that is an effective way of ranking DMUs. Self and peer-evaluation were used to evaluate the overall efficiencies of the DMUs. Doyle and Green(1991) developed new concepts on cross-efficiency. They presented of aggressive and benevolent cross-efficiencies formulations. Doyle and Green(1995) showed that cross-evaluation is more powerful than the reference-set count. They also described four variants of cross-evaluation and presented their implementations as secondary goals to the usual DEA efficiency-maximizing primary goal. Later,they compared the performance of the four variants on a dozen data sets that have appeared in the DEA literature, paying particular attention to the effect of the different input-output structures among the data sets.

Anderson et al.(2002) demonstrated the fixed weighting nature of cross-efficiency evaluation for single input and multiple outputs. Karsak and Ashika(2007) presented an Multi-Criteria Decision Making(MCDM) technique with improved discrimination power. The coming results indicated that the proposed methodology enables the ranking of DEA−DMUs with a notable saving in computations compared with cross-efficiency analysis. Bao et al.(2008) offered an alternative interpretation to the cross-efficiency evaluation from the slack analysis in DEA. Liang et al.(2008a) extended the model of cross-efficiency evaluation introduced by Doyle and Green(1994). They presented a number of alternative goals for cross-efficiency evaluation. Later, Liang et al.(2008b) developed the cross-efficiency concept to game cross-efficiency by viewing each DMU as a player that seeks to maximize its own efficiency under the condition that the cross-efficiency of each of the other DMUs doesnot deteriorate and the cross-efficiencies as payoffs.

Wu et al.(2009) suggested an extended model for the game cross-efficiency model by appending an extra constraint to avoid producing negative cross-efficiencies under variable returns to scale(VRS). Wu et al.(2009) presented a revised benevolent cross-efficiency model and used the cross-efficiencies obtained to construct a fuzzy preference relation. Further, Wu et al.(2009) suggested a bargaining game model and a mixed integer programming model for the cross-efficiency evaluation. Wang and Chin(2010) proposed a new secondary goal for cross-efficiency named as neutral DEA model that determines one set of input and output weights for each DMU from its own point of view. Later, Wang and Chin (2011) suggested the use of ordered weighting averaging(OWA) operator weights for the cross-efficiency evaluation.

Lim et al.(2014) proposed a way of using DEA cross-efficiency evaluation in portfolio selection and improved its use in portfolio selection. In addition to (average) cross-efficiency scores, they suggested to examine the variations of cross-efficiencies, and to incorporate two statistics of cross-efficiencies into the mean-variance formulation of portfolio selection. Wu et al.(2016) presented a cross-efficiency evaluation approach based on Pareto improvement, which contains two models (Pareto optimality estimation model and cross-efficiency Pareto improvement model) and an algorithm. There models overcome the drawback that the cross-efficiency scores generated may not be Pareto optimal, which has reduced the effectiveness of cross-efficiency method. Liu et al.(2017) presented an equitable model for efficiency evaluation of DMUs with undesirable outputs and introduced a technique for cross-efficiency evaluation considering undesirable outputs. Then, a ranking priority model is proposed considering the DMUs' intentions of pursuing the best ranking positions. They presented an aggressive model to guarantee the uniqueness of the optimal solution.

1.2.3 Literature survey on OWA Operators

Yager(1988) presented an operator for aggregation known as Ordered Weighted Aggregation (OWA) operator, and its various the properties. O'Hagan (1988) introduced maximum entropy approach using the degree of orness to determine OWA operator weights. In 1993, Yager introduced the families of OWA operators like maximum entropy, S-OWA, step, window and lot more. He presented the evaluation of quantified proposition using these operators and also introduced the idea of aggregate dependent weights.

Fodor et al.(1995) presented the classification of aggregation operators as monotonic and neutral(MN). They showed that the OWA aggregator can be expressed as a Choquet integral. Mitchell and Estrakh(1996)proposed a modified OWA operator in which the input arguments are not re-arranged according to their actual relative values but rather according to their estimated relative values. They presented an unusual application of this operator to loss less image compression. Yager and Filev(1999) presented a new class of OWA operator weights known as Induced OWA operators(IOWA). Further, IOWA operator are used to present different types of aggregation models.

Fuller and Majlender(2001) presented a technique for weight determination. They introduced a method of Lagrange multipliers that helps to solve the constrained optimization problem analytically and derive a polynomial equation which was then solved to obtain the optimal weighting vector. Xu and Da(2002) investigated the uncertain OWA operator in which the associated weighting parameters cannot be specified, but value ranges can be obtained and each input argument is given in the form of an interval of numerical values. The problem of ranking a set of interval numbers and obtaining the weights associated with the uncertain OWA operator is studied.

Pereira and Ribeiro(2003) presented weighted aggregation operators in multiple attribute decision making and its main goal is to investigate ways in which weights can depend on the satisfaction degrees of the various attributes (criteria). Xu(2006) proposed new aggregation operators named as the uncertain linguistic geometric mean (ULGM) operator, uncertain linguistic weighted geometric mean (ULWGM) operator, uncertain linguistic ordered weighted geometric (ULOWG) operator, and induced uncertain linguistic ordered weighted geometric (IULOWG) operator. The IULOWG operator is a more general type of aggregation operator, based on the ULGM and ULOWG operators. Moreover, based on the ULOWG and IULOWG operators and the formula for the comparison between two uncertain multiplicative linguistic variables, he developed an approach to group decision making with uncertain multiplicative linguistic preference relations, and, finally, an application of the approach to group decision-making problem with uncertain multiplicative linguistic preference relations is pointed out.

Merig and Lafuente(2009), presented the induced generalized ordered weighted averaging (IGOWA) operator. It included the main characteristics of both the generalized

OWA and the induced OWA operator. This operator uses generalized means and order-inducing variables in the reordering process. They further introduced the Quasi-IOWA operator. Merig and Casanovas(2011) introduced a new aggregation operator named as the induced ordered weighted averaging distance (IOWAD) operator that extended the OWA operator by using distance measures and a reordering of arguments that depends on order-inducing variables. The advantage of IOWAD is that it provides a parameterized family of distance aggregation operators between the maximum and the minimum distance based on a complex reordering process that reflects the complex attitudinal character of the decision-maker.

Merigo(2011) investigated a new model that uses the weighted average (WA) and the induced ordered weighted averaging (IOWA) operator in the same formulation named as induced ordered weighted averaging weighted average (IOWAWA) operator. Liu(2013) firstly developed score function and accuracy function of intuitionistic linguistic numbers and then an intuitionistic linguistic generalized dependent ordered weighted average (ILGDOWA) operator and an intuitionistic linguistic generalized dependent hybrid weighted aggregation (ILGDHWA) operator were introduced. He at al.(2015) proposed a kind of ordered weighted averaging (OWA) operator based link prediction ensemble algorithm (LPEOWA) for social network by assigning the aggregation weights for nine local information-based link prediction algorithms with three different OWA operators.

Zhang et al.(2017) firstly defined a new score function and a new accuracy function of intuitionistic linguistic number (ILN) and presented a simple method for the comparison between two ILNs. Then, based on the intuitionistic linguistic weighted geometric averaging (ILWGA) operator, they proposed the intuitionistic linguistic ordered weighted geometric (ILOWG) operator and intuitionistic linguistic hybrid geometric (ILHG) operator, and established various properties of these operators. Further, they applied the ILHG and ILWGA operators to solve multi-criteria group decision making problems, in which the criterion values take the form of ILNs and the criterion weight information is known completely.

1.2.3.1 OWA operator using Orness

Yager(1988) was the first to introduced the degree of orness considered as a family of mean operators. Further, Yager(1992) introduced characteristics of orness by proofing many theorems. Yager et al.(1994) introduced a concept of immediate probabilities for use by decision makers in selecting alternatives. They also discussed the role of the decision maker's level of orness(optimism) in transforming probabilities into immediate probabilities to be used for a decision.

Yager(1996) introduced fuzzy representation of linguistic quantifier and also discussed the problems regarding the measure of orness and determination of weights in OWA aggregation. Wang and Parkan(2005) proposed a minimax disparity approach for obtaining OWA operator weights. This involves the formulation and solution of a linear programming (LP) model for a given degree of orness. The proposed approach generates the OWA operator weights by minimizing the maximum difference between any two adjacent weights. Seok Ahn(2009) revisited the least-squared OWA method, which intends to produce spread-out weights as much as possible while strictly satisfying a predefined value of orness, and showed that it is an equivalent of the minimax disparity approach.

Hong(2011) worked on the problem that there would be an interval for the degree of orness for having an even number of arguments, such that the optimal solution of the extended model could be obtained by a compact mathematical form. He completely proved this open problem mathematically for all orness levels whatever the number of arguments. Belles-Sampera et al.(2013) showed that (OWA) and Weighted Ordered Weighted Averaging (WOWA) operators can be derived from the Choquet integral, and then the mathematical relationship between distortion risk measures and the OWA and WOWA operators for discrete and finite random variables is presented. Also the degree of orness indicator characterizing distortion risk measures was introduced. Wang et al.(2017) introduces the OWA in the mean-variance model. The main idea is to replace the classical mean and variance by the OWA operator. Then the new model is able to study different degrees of orness(optimism) in the analysis being able to develop an approach that considers the decision makers attitude in the selection process.

1.2.3.2 OWA operator using Minimax Disparity Approach

O'Hagan(1988) was the first to introduced the maximum entropy approach that contains a non-linear optimization problem with a particular degree of orness. Yager and Filev(1995) further worked on maximum entropy approach and developed its analytic properties. Fuller and Majlender(2001) transformed the maximum entropy model into polynomial equation that can be solved analytically. Later,Fuller and Majlender(2003) suggested minimax variance approach that helps to obtain minimal variability OWA operator weights. Liu and Chen(2004) proposed a parametric geometric approach to obtain maximum entropy weights.

Wang and Parkan(2005) developed a simple and effective approach named as minimax disparity approach for generating the OWA operator weights. Amin and Emrouznejad (2006) introduced the minimax disparity approach between any distinct pairs of the weights and uses the duality of linear programming to prove the feasibility of the extended OWA operator weights model. Further, Amin(2007) worked on the problem created in the research of Amin and Emrouznejad(2006) regarding the OWA weights determination. Emrouznejad and Amin (2010) extended their work and suggested mathematical model by minimizing the sum of the deviation between two distinct OWA weights in existing disparity models.

Hong(2011) put forward the work done by Amin and Emrouznejad(2006) and solved the problem of having an interval for the degree of orness such that the optimal solution of the extended model could be obtained by a compact mathematical form, for an even number of arguments. They completely proved this problem mathematically for all orness levels whatever the number of arguments. Ouyang(2015) pointed out that the improved minimax disparity model, which was suggested by Emrouznejad and Amin (2010) having has infinite solutions. Tohidi and Khodadadi(2015) worked on removing some of the constraints of the improved minimax disparity (MD) model and obtains its optimal simplex tableau in the general case. The study also presented the weights of the preference ranking aggregation system without solving any model and suggested a secondary goal model for selecting a unique preference ranking aggregation weights through the alternative optimal weights of the improved MD model. Gong et al.(2016) provided two new disparity models to obtain the associated weights, which is determined by considering the absolute deviation and relative deviation of any distinct pairs of weights. The proposed mathematical models improve the existing minimax disparity approach and chi-square method, which is suggested by Amin and Emrouzenjad (2006, 2010).

2

CROSS-EFFICIENCY IN DATA ENVELOPMENT ANALYSIS

2.1 Efficiency in DEA

Data Envelopment Analysis(DEA) is a linear programming that evaluates the efficiency of the transformation of a DMU′s inputs into its outputs. It computes the relative efficiencies of all DMUs by finding a set of the best weights for every DMU by maximizing its efficiency. DMU is an entity whose performance is to be evaluated and it is also responsible for converting all inputs into outputs. DEA provides a categorical classification of the units into efficient and inefficient ones. In DEA, it is not necessary to know the production function in advance, however, it includes an LP problem which is solved for each DMU to find its relative efficiency as a linear combination of corresponding optimal weights. In the theory of microeconomic production, production function is used to present the combination of inputs and outputs. Using this production function, construction of production technology frontier is possible that maximize outputs subject to any possible combination of inputs. The term technical efficiency was presented by Farrell(1957) that was based on the principle of maximizing outputs for the given set of inputs. Later, Charnes et al.(1978) introduced a technique named as DEA that was the mathematical extension of Farrell's technical efficiency. In DEA, a linear programming is applied to estimate a empirical production technology frontier for measuring the technical efficiency of DMUs. Seiford et al.(1990) made a great development in DEA. Now DEA became capable of handling multiple inputs and outputs, and also capable of being used with any input-output measurement. It became possible to identify the sourse of inefficiency. Results are sensitive to the selection of input and output. One must choose relative criteria for the selection of input and output variables. With the increase in the number of inputs and outputs variables, the number of efficient DMUs on the frontier also tends to increase. DEA identifies the most efficient units and is a very powerful benchmarking technique. It estimates the input and output targets to make inefficient DMUs as efficient. It is a non-parametric technique that do not make any assumption regarding the distribution of inefficiencies. By using linear programming techniques, DEA uses the input and output data themselves to compute the production possibility frontier. The efficiency of each DMUs are measured as the ratio of the weighted sum of outputs to the weighted sum of inputs where the weights used are evaluated by the technique so as to reflect the unit at its most efficient relative to all others in the data set.

The production process of a DMU in DEA can be represented by Figure 2.1.

Figure 2.1: Production process of DMU$_k$

$$(x_{1k}, x_{2k}, x_{3k}, \ldots\ldots\ldots, x_{mk}) \rightarrow DMU_k \rightarrow (y_{1k}, y_{2k}, y_{3k}, \ldots\ldots\ldots, y_{sk})$$

NOMENCLATURE

- n : Number of DMUs to be evaluated.
- m : Number of inputs consumed.
- s : Number of outputs produced.
- x_{ij} : i^{th} input consumed by the j^{th} DMU, (i=1,2,3,...,m).
- y_{rj} :r^{th} output produced by the s^{th} DMU, (r=1,2,3...,s).
- v_i : weight associated with the i^{th} input, (i=1,2,3,...,m).
- u_r : weight associated with the r^{th} output, (r=1,2,3...,.s).
- E_k: efficiency of the k^{th} DMU.

The efficiency of DMU$_j$ in DEA is defined by:

$$E_j = \frac{weighted\ \ sum\ \ of\ \ all\ \ outputs}{weighted\ \ sum\ \ of\ \ all\ \ inputs} = \frac{Virtual\ \ output}{Virtual\ \ input} = \frac{\sum\limits_{r=1}^{s} u_r\, y_{rj}}{\sum\limits_{i=1}^{m} v_i\, x_{ij}}, \forall\ j = 1, 2, 3, ...n.$$

2.1.1 Mathematical formulation of CCR model

CCR model was developed by Chames et al.(1978) and is the first and fundamental DEA model that built on the notion of efficiency with multiple inputs and multiple outputs. CCR ratio model aims to maximize the efficiency of the targeted DMU subject to the condition that the efficiencies of all DMUs should be less than or equal to one. It calculates an efficiency for the DMU in which both its pure technical efficiency and scale efficiency are aggregated into a single value. The obtained efficiency of a targeted unit/DMU is never absolute as it is always measured relative to the other units/DMUs.

Consider a DMU, say, DMU$_k$; $k \in \{$1,2,3, . . . ,$n\}$ whose efficiency is to be evaluated. The efficiency E$_k$ of DMU$_k$ relative to the other DMUs is evaluated by using the following fractional programming model:

Model-2.1

$$\text{Maximize} \;\; E_k = \frac{\sum\limits_{r=1}^{s} u_r\, y_{rk}}{\sum\limits_{i=1}^{m} v_i\, x_{ik}}$$

$$\text{subject to} \quad \mathrm{E_j} = \frac{\sum\limits_{r=1}^{s} u_r\, y_{rj}}{\sum\limits_{i=1}^{m} v_i\, x_{ij}} \leq 1 \quad \forall\; j = 1, 2, 3...., n,$$

$$u_r \geq \epsilon \;\; \forall\; r = 1, 2, 3, ..., s,$$

$$v_i \geq \epsilon \;\; \forall\; i = 1, 2, 3, ..., m,$$

where $\epsilon > 0$ is an Archimedean infinitesimal.

Model-2.1 aims to find the set of input and output weights that is more favourable to DMU_k.

Charnes and Cooper transformations

The transformation was introduced by Charnes and Cooper (1962) which provides the equivalence about programming with linear fractional functional. Model-2.1 is a fractional programming problem that can be reduced to a linear format in the usual manner of Charnes and Cooper transformation. Specifically, use the transformation $(\sum\limits_{i=1}^{m} v_i\, x_{ik})^{-1}$ $= t$ and let $t\, v_i = v_i \;\; \forall i\,,\; t\, u_r = u_r \;\; \forall r$. It is noted that due to scaling, we may restrict the virtual input $(i.e. \frac{1}{t})$ to be bounded above by some constant C. Without loss of generality, we may choose C=1. Proceeding in the manner of Charnes and Cooper transformation, Model-2.1 can be expressed in the following linear form:

Model-2.2

$$\text{maximize} \;\; E_k = \sum_{r=1}^{s} u_r y_{rk}$$

$$\text{subject to} \quad \sum_{i=1}^{m} u_i x_{ik} = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} u_i x_{ij} \leq 0 \quad \forall \ j = 1, 2, 3...., n,$$

$$u_r \geq \epsilon \ \ \forall \ r = 1, 2, 3, ...s,$$

$$v_i \geq \epsilon \ \ \forall \ i = 1, 2, 3, ...m,$$

The weighted sum of inputs is constrained to be unity in this linear program. The objective function is to maximize the weighted sum of outputs. It is the primal form of the CCR model. The output CCR efficiency lies between 0 and 1.
Let the optimal solution of Model-2.2 be $\left(u_1^*, u_2^*, u_3^*, ..., u_s^*, v_1^*, v_2^*, v_3^*, ..., v_m^*\right)$ .Then,

$$\mathrm{E_k^*} = \sum_{r=1}^{s} u_r^* y_{rk}$$

is referred to as the CCR-efficiency of the DMU$_k$. It reflects the self evaluated efficiency of DMU$_k$.
Definition 2.1 A DMU$_k$ is said to be efficient if $E_k^* = 1$.

2.2 Cross-Efficiency Evaluation in DEA

NOMENCLATURE

- n : Number of DMUs to be evaluated.
- m : Number of inputs consumed by each DMU.
- s : Number of outputs produced by each DMU.
- x_{ij}: i^{th} input value of the j^{th} DMU, (i=1,2,3,...,m).
- y_{rj} :r^{th} output value of the j^{th} DMU, (r=1,2,3...,s).

- $V_j = (v_{1j}, v_{2j}, v_{3j}, ..., v_{mj})^T$:Input Weight vector obtained for DMU$_j$ using Model-2.2, $j = 1, 2, 3......, n$.

- $U_j = (u_{1j}, u_{2j}, u_{3j}, ..., u_{sj})^T$:Output Weight vector obtained for DMU$_j$ using Model-2.2, $j = 1, 2, 3......, n$.

- E_{jk} is the cross efficiency of jth DMU using weights of the k^{th} DMU.

Model-2.2 is solved for each DMU which results into n different sets of optimal input and output weights, one for each DMU.

Defination 2.2 The cross- efficiency of DMU$_j$ (j=1,2,3....n; $j \neq k$) which reflects the peer-evaluation of DMU$_k$ to DMU$_j$ is defined by:

$$E_{jk} = \frac{\sum_{r=1}^{s} u_{rk}^{*} y_{rj}}{\sum_{i=1}^{m} v_{ik}^{*} x_{ij}}$$

Each DMU have (n-1) cross efficiency values E_{jk} and one CCR-efficiency value which further makes a cross-efficiency value E_{kk} as depicted in Figure 2.2

Figure 2.2: Cross-efficiency matrix

$$\begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{bmatrix}$$

The weights obtained in Model-2.2, which are used to determine the cross- efficiencies of a DMU may not be unique. To overcome this failure, Sexton et.al.(1986) introduced a secondary objective model. The secondary objective contains two formulations named as aggressive formulation and benevolent formulation. Further, Wang and Chin(2010) introduced neutral formulation to make the cross-efficiency evaluation technique more practical.

Aggressive formulation not only provides the choice of weights to obtain the maximum efficiency of the j^{th} DMU as a primary goal but it also minimizes the cross- efficiencies of other DMUs. On the other hand, benevolent formulation provides the primary goal in which the choice of weights are made to get maximum efficiency for the j^{th} DMU and also the secondary goal provides the maximization of cross-efficiencies of the other DMUs. Neutral formulation is neither aggressive nor benevolent formulation. In neutral formulation, each DMU determines the weights only from its own point of view without considering their impact on the other DMUs.The cross-efficiencies determined by the neutral DEA model are more neutral. The mathematical formulations for all three are discussed as under.

2.2.1 Aggressive Formulation

Sexton et al.(1986) presented the following aggressive formulation for DMU$_k$:

Model-2.3

$$\text{Minimize } \; E_k^{(a)} = \frac{\sum_{r=1}^{s} u_{rk}\left(\sum_{j=1 j\neq k}^{n} y_{rj}\right)}{\sum_{i=1}^{m} v_{ik}\left(\sum_{j=1 j\neq k}^{n} x_{ij}\right)}$$

$$\text{subject to } \; \frac{\sum_{r=1}^{s} u_{rk} y_{rk}}{\sum_{i=1}^{m} v_{ik} x_{ik}} = E_{kk}^{*},$$

$$\frac{\sum_{r=1}^{s} u_{rk} y_{rj}}{\sum_{i=1}^{m} v_{ik} x_{ij}} \leq 1 \quad \forall\ j = 1, 2, 3.....n; j \neq k$$

$$u_{rk} \geq \epsilon \;\; \forall\ r = 1, 2, 3..., s,$$

$$v_{ik} \geq \epsilon \;\; \forall\ \ i = 1, 2, 3, ..., m,$$

where $\epsilon > 0$ is an Archimedean infinitesimal.

By using Charnes and Cooper transformation, (Cooper et al.,2007), Model-2.4 can be expressed in the following linear form:

Model-2.4

$$\begin{aligned}
\text{Minimize} \quad E_k^{(a)} &= \sum_{r=1}^{s} u_{rk}\Bigg(\sum_{j=1,j\neq k}^{n} y_{rj}\Bigg) \\
\text{subject to} \quad & \sum_{i=1}^{m} v_{ik}\Bigg(\sum_{j=1,j\neq k}^{n} x_{ij}\Bigg) = 1, \\
& \sum_{r=1}^{s} u_{rk}y_{rk} - E_{kk}^{*}\sum_{i=1}^{m} v_{ik}x_{ik} = 0, \\
& \sum_{r=1}^{s} u_{rk}y_{rj} - \sum_{i=1}^{m} v_{ik}x_{ij} \leq 0 \ \ \forall\, j = 1,2,3....n; j \neq k, \\
& u_{rk} \geq \epsilon \ \ \forall\ r = 1,2,3, ...s, \\
& v_{ik} \geq \epsilon \ i = 1,2,3, ...m.
\end{aligned}$$

2.2.2 Benevolent Formulation

Sexton et al.(1986) presented a benevolent formulation for DMU_k which is given by the following mathematical model.

Model-2.5

$$\begin{aligned}
\text{Maximize} \quad E_k^{(b)} &= \frac{\sum_{r=1}^{s} u_{rk}\Big(\sum_{j=1 j\neq k}^{n} y_{rj}\Big)}{\sum_{i=1}^{m} v_{ik}\Big(\sum_{j=1 j\neq k}^{n} x_{ij}\Big)} \\
\text{subject to} \quad & \frac{\sum_{r=1}^{s} u_{rk}y_{rk}}{\sum_{i=1}^{m} v_{ik}x_{ik}} = E_{kk}^{*}, \\
& \frac{\sum_{r=1}^{s} u_{rk}y_{rj}}{\sum_{i=1}^{m} v_{ik}x_{ij}} \leq 1 \ \ \forall\ j = 1,2,3.....n; j \neq k \\
& u_{rk} \geq \epsilon \ \ \forall\ \ r = 1,2,3..., s, \\
& v_{ik} \geq \epsilon \ \ \forall\ \ i = 1,2,3..., m,
\end{aligned}$$

where $\epsilon > 0$ is an Archimedean infinitesimal.

By using Charnes and Cooper transformation, Model-2.5 can also be reduced to the following linear form:

Model-2.6

$$\begin{aligned}
\text{Maximize} \quad E_k^{(b)} &= \sum_{r=1}^{s} u_{rk}\Bigg(\sum_{j=1,j\neq k}^{n} y_{rj}\Bigg) \\
\text{subject to} \quad & \sum_{i=1}^{m} v_{ik}\Bigg(\sum_{j=1,j\neq k}^{n} x_{ij}\Bigg) = 1, \\
& \sum_{r=1}^{s} u_{rk}y_{rk} - E_{kk}^{*}\sum_{i=1}^{m} v_i x_{ik} = 0, \\
& \sum_{r=1}^{s} u_{rk}y_{rj} - \sum_{i=1}^{m} v_{ik}x_{ij} \leq 0 \ \ \forall\ j = 1,2,3....n; j \neq k, \\
& u_{rk} \geq \epsilon\ \forall\ r = 1,2,3,.....s, \\
& v_{ik} \geq \epsilon\ \forall\ i = 1,2,3,.....m.
\end{aligned}$$

2.2.3 Neutral Formulation

In existing literature, all cross-efficiencies are computed either aggressively or benevolently. Both formulations can lead to different efficiency rankings for the DMUs. To overcome this difficulty and to avoid a choice between the two different formulations, Wang and Chin(2010) proposed a neutral DEA model for cross-efficiency evaluation. The neutral DEA model determines one set of input and output weights for each DMU without being aggressive or benevolent.

In realistic situation, when a DMU is given a chance to decide its set of input and output weights, DMU needs to determine weights that are as favorable as possible to itself. In such case, a DMU do not care about how to be aggressive or benevolent to the other DMUs. Based on the aforementioned viewpoint, Wang and Chin(2010) proposed the following neutral DEA model for cross-efficiency evaluation:

Model-2.7

$$\text{Maximize} \ \ \delta = \text{Minimum}_{r\in 1,2,3,....s} \left\{ \frac{u_{rk}\, y_{rk}}{\sum_{i=1}^{m} v_{ik}\, x_{ik}} \right\}$$

$$\text{subject to} \quad E^*_{kk} = \frac{\sum_{r=1}^{s} u^*_{rk} y_{rk}}{\sum_{i=1}^{m} v^*_{ik} x_{ik}},$$

$$E_{jk} = \frac{\sum_{r=1}^{s} u_{rk} y_{rj}}{\sum_{i=1}^{m} v_{ik} x_{ij}} \leq 1, \quad j = 1, 2, 3....n; j \neq k,$$

$$u_{rk} \geq 0; \; r = 1, 2, 3, ...s,$$

$$v_{ik} \geq 0; \; i = 1, 2, 3, ...m,$$

where $u_{rk} y_{rj} / \sum_{i=1}^{m} v_{ik} x_{ij}$, is the efficiency of the r^{th} output of the DMU$_k$. The economic interpretation of Model-2.7 is that DMU$_k$ searches for a set of input and output weights that not only maximize its efficiency as a whole but at the same time it makes its each output as efficient as possible to obtain sufficient efficiency as an individual. Therefore, the objective of Model-2.7 is to determine input and output weights just from the point of view of DMU$_k$ and has nothing to do with the other DMUs. In this way, the decision maker do not need to make any choice between aggressive and benevolent formulations. Further, Model-2.7 can be reduced to the following linear form by using Charnes and Chooper transformation:

Model-2.8

$$\text{Maximize} \quad \delta$$

$$\text{subject to} \quad \sum_{i=1}^{m} v_{ik} x_{ik} = 1,$$

$$\sum_{r=1}^{s} u_{rk} y_{rk} = E^*_{kk},$$

$$\sum_{r=1}^{s} u_{rk} y_{rj} - \sum_{i=1}^{m} v_{ik} x_{ij} \leq 0 \quad j = 1, 2, 3....n; j \neq k,$$

$$u_{rk} y_{rk} - \delta \geq 0 \quad r = 1, 2, 3......s,$$

$$v_{ik} \geq 0 \quad i = 1, 2, 3.....m,$$

$$\delta \geq 0,$$

where $u_{rk}\,(r = 1, 2, 3...., s)$, $v_{ik}\,(i = 1, 2, 3...., m)$ and δ are the decision variables. The neutral formulation can effectively reduce the number of zero weights for outputs.

3

CROSS-EFFICIENCY AGGREGATION BY OWA OPERATOR

3.1 Introduction

In cross-efficiency evaluation, self and peer-evaluation techniques are used to evaluate overall efficiencies of the DMUs. In self-evaluation technique, the efficiencies of the DMUs are evaluated with the most favourable weights to get the best relative efficiency, whereas in peer-evaluation, the efficiency of each DMU should be evaluated with the weights determined by the other DMUs. The idea of using equal weights for cross-efficiency evaluation has a drawback that peer-evaluation efficiencies are much more weighted than self-evaluation efficiencies. This is due to the reason that each DMU has only one self-evaluated efficiency value whereas peer-evaluation has multiple efficiency values. So, self-evaluation lacks in the final overall assessment and ranking of the DMUs. Wu et al.(2009) proposed an approach to give different and fixed weights to each DMU but it also suffers by a drawback. That is, the self-efficiencies lies on the leading diagonal of the cross-efficiency matrix and also have different weights which results that the self-evaluation plays a distinct role in the final overall assessment and ranking of the DMUs. Therefore, it has been observed that the problem of aggregation criteria functions (in our case, cross-efficiencies under different formulations) to form overall decision function is of considerable importance in many areas/disciplines. The primary factor that involves in determining the structure of aggregation functions is the relationship between the given criteria. The two extreme situations involve

(i) all the criteria to be satisfied, and

(ii) the satisfaction of any of the criteria is desired.

These two extreme cases lead to the use of "and" and "or" operators to aggregate the criteria functions. In that first extreme situation where we require that all criteria are to be satisfied can be inferred as the "anding" of the criteria values. On the other hand, the second extreme situation where we require that atleast one of the criteria is to be satisfied can be inferred as the "oring" of the criteria values in the formulation of the decision function. In many situation, the interrelationship between the given criteria lies between these two extreme situations of "all" or "atleast one". That is, the desirable situations can be "most" or "many" or "atleast half" etc. The present work is to discuss aggregation operators that can handle the aforementioned situations. OWA operators, developed by Yager(1988) is one of the aggregation operators that allow to easily adjust the degree of "anding" and "oring" implicit in the aggregation. This operator is also known as "orand" operator.

In present study, our objective is to aggregate self and peer-evaluated cross-efficiencies for ranking DMUs in different cross-efficiency formulations. In OWA, weights are allocated between self and peer-evaluated efficiencies in terms of the decision maker's optimum level which is characterized by an orness degree. Also, the coefficients are not associated directly with a particular attribute but rather to an ordered position. It allows to adjust the degree of "anding" and "oring" implicit in aggregation.

Definition 3.1 An OWA operator of dimension n is a mapping

$$\mathrm{F} : R^n \to R$$

with an associated weight vector $W = (w_1, w_2, w_3, ...w_n)^T$ such that

$$\sum_{i=1}^{n} w_i = 1,\ \ 0 \leq w_i \leq 1,\ \ i = 1, 2, 3, ..., n.$$

and

$$F(a_1, a_2, a_3, ..., a_n) \ = \ \sum_{i=1}^{n} w_i b_i$$

where b_i is the ith largest of $a_1, a_2, a_3, ..., a_n$.

Example (Yager,1988) Assume F is an ordered weighting averaging operator of size $n = 4$ with the weighting vector, $W = (0.2, 0.3, 0.1, 0.4)^T$. Then the aggregation of $(0.6, 1, 0.3, 0.5)^T$ is done in the following manner:

Ordered argument vector B for the given argument vector

$(0.6, 1, 0.3, 0.5)^T$ is B=$(1,0.6,0.5,0.3)^T$ = F(B) = W^TB=[0.2,0.3,0.1,0.4]

=(0.2)(1)+(0.3)(0.6)+(0.1)(5)+(0.3)(0.4)=0.55

Definition 3.2 (Merigo and Gil-Lafuente,2009) An induced OWA(IOWA) operator of dimension n is a mapping $F_I : R^n \to R$ defined by an associated weighting vector $W = (w_1, w_2, w_3, ..., w_n)^T$ of dimension n such that $w_i \geq 0$and $\sum_{i=1}^{n} w_i = 1$ and a set of order-inducing variables u_i by a formula of the following form:

$$\mathrm{F}_I\Big(\Big\langle \mathrm{u}_1,\mathrm{a}_1 \Big\rangle, \Big\langle \mathrm{u}_1,\mathrm{a}_2 \Big\rangle, \Big\langle \mathrm{u}_3,\mathrm{a}_3 \Big\rangle,......, \Big\langle \mathrm{u}_n,\mathrm{a}_n \Big\rangle \Big) = \textstyle\sum_{i=1}^{n} \mathrm{w}_i\ \mathrm{b}_i$$

where $(b_1, b_2, b_3, ..., b_n)$ is simply $(a_1, a_2, a_3, ..., a_n)$ reordered in decreasing order of the values of u_i, u_i is the order-inducing variable and a_i is the argument variable.

3.2 Properties of OWA aggregation operator

The OWA aggregation operator possesses the properties like commutativity, monotonicity, idempotency and boundedness which are discussed in the following theorems (Yager,1988):

Theorem 3.1 Assume that F is an OWA operator. Let $A = [a_1, a_2, a_3, ..., a_n]^T$ be an ordered argument vector. Let $B = [b_1, b_2, b_3, ..., b_n]^T$ be a second ordered argument vector such that for each j, $a_j \geq b_j$ then $F(A) \geq F(B)$.

Proof Since, $F(A) = W^T A$ and $F(B) = W^T B$. The result follows directly from the property that $a_j \geq b_j$.

Theorem 3.2 Assume F is an OWA operator.

Then $F(a_1, a_2, a_3, ..., a_n) = F(a_1^{'}, a_2^{'}, a_3^{'}, ..., a_n^{'})$ where $(a_1^{'}, a_2^{'}, a_3^{'}, ..., a_n^{'})$ is any permutation of the element in $(a_1, a_2, a_3, ..., a_n)$.

Proof If B and $B^{'}$ are the ordered argument vectors of $(a_1, a_2, a_3, ..., a_n)$ and $(a_1^{'}, a_2^{'}, a_3^{'}, ..., a_n^{'})$ respectively then $B = B^{'}$. Therefore, $F(B) = F(B^{'})$.

Theorem 3.3 All OWA operators are idemponent in the sense that if $a_j = a$, for all j=1,2,3....,n, then $F(a_1, a_2, a_3, ..., a_n)$=$a$.

Proof From the definition,

$$F(a_1, a_2, a_3, ..., a_n) = \sum_{j=1}^{n} w_j b_j$$

as $a_j = a$ then also $b_j = a$

$$\Rightarrow \; F(a_1, a_2, a_3, ..., a_n) = \sum_{j=1}^{n} w_j a$$

We know that

$$\sum w_i = 1,$$

$$\Rightarrow \; F(a_1, a_2, a_3, ..., a_n) = a.$$

Theorem 3.4 Assume that $(a_1, a_2, a_3, ..., a_n)$ is the collection of numbers each lying in the unit interval then

(a)$F_*(a_1, a_2, a_3, ..., a_n)$= $\text{Min}_j(a_j)$

(b)$F^*(a_1, a_2, a_3, ..., a_n)$= $\text{Max}_j(a_j)$

Proof (a) The weighting vector W_* defined such that

$$w_n = 1 \text{ and } \ w_i = 0 \quad \text{for all i } \neq n$$

gives us the aggregation $F_*(a_1, a_2, a_3, ..., a_n) = \text{Max}_i[a_i]$. Therefore W_* provides the smallest possible aggregation.

(b) The weighting vector W^* defined such that

$$w_1 = 1 \text{ and } \ w_j = 0 \quad \text{for all j } \neq 1$$

gives us the aggregation $F^*(a_1, a_2, a_3, ..., a_n) = \text{Max}_i[a_i]$. Therefore W^* provides the largest aggregation of the arguments.

Theorem 3.5 Assume B is an arbitrary ordered input vector. Then for any weighted vector W,

$(W_*)^T B \leq W^T B \leq (W^*)^T B$

Proof $(W_*)^T B \leq W^T B$

$(W_*)^T B = b_n$

$$W^T B = \sum_n w_j b_j \ = b_n w_n \ + \sum_{j=1}^{n-1} w_j bj.$$

Since B is an ordered input vector then $b_j \geq b_k$ for $k > j$. In particular, $b_n < b_j$ for $j = 1, 2, 3, ..., n-1$, hence

$$W^T B \geq b_n w_n + b_n \sum_{j=2} w_j,$$

however,

$$\sum_{j=1}^{n-1} w_j = 1 - w_n,$$

thus

$$W^T B \geq b_n w_n + (1 - w_n) b_n \geq b_n \geq (W_*)^T B$$

(b) $(W^*)^T B \geq W^T B$

$(W^*)^T B = b_1$

$$W^T B = \sum_{n} w_j b_j \ = b_1 w_1 \ + \ \sum_{j=2}^{n} w_j bj.$$

Since, $b_i \geq b_j$ then

$$W^T B \leq b_1 w_1 + b_1 \sum_{j=2}^{n} w_j \leq b_1$$

3.3 Weight determination for OWA aggregation operators

In this section, we define measure of orness of an aggregation operator measures in which degree the method behaves as the maximum operator or as the minimum. This can be understood as being optimistic or not, or being compensative or not. This measure reflects the or-like level of the aggregation operator. The measure of orness was first introduced by Dujmovic(1974) for the power means under the name of disjunction degree. Yager(1988) defined the orness concept independently in the case of OWA operator. It can be proved that Yager's orness measure for OWA operator coincides with Dujmovic's defination as a special case (Salido and Murakami,2003). Marichal (2000) proposed the orness definition of discrete Choquet integral for multicriteria decision problems. Marichal(1998) also proposed that the degree of orness can be defined for any compensative aggregation operator.

3.3.1 Measure of orness

Let F be a OWA operator with weighting function $W = (w_1, w_2, w_3, ...w_n)^T$.We know that if $W = W^*$, then F is a pure "or" operator. We are here introducing a measure of orness associated with a weighting function. The orness is the value that lies in an interval [0,1].

Definition 3.3 (Yager,1988) Assume F is a OWA operator with weighting function $W = (w_1, w_2, , ...w_n)^T$. The degree of orness, denoted by α, associated with this operator is defined as

$$\alpha = \frac{1}{n-1}\sum_{i=1}^{n}((n-i)w_i) \tag{3.1}$$

This measure of "orness" is also defined as

$$\alpha = \sum_{i=1}^{n}\big(h_n(i) * w_i\big) \tag{3.2}$$

where $h_n(i)$ is a linear type function. That is
$h_n(i) = (n/n-1) - (i/n-1) = (n-i)/(n-1), \quad i = 1, 2, ...n$. Since $h_n(i) > h_n(j)$ for $j < i$, then we see that h_n is a really "linear argument vector". Therefore the measure of "orness" of a OWA operator is its aggregated value under a linear argument vector.

It is observed that the weights near the top will be an "$orlike''$ operator $orness(W) \geq 0.5$ while the weights near the bottom will be an "$andlike''$ operator$orness(W) \leq 0.5$.

The following theorems express the above mentioned characteristic.

THEOREM 3.6 (Yager,1993)Assume that $W = (w_1, w_2, w_3, ..., w_n)^T$ and $W = (w_1^{'}, w_2^{'}, w_3^{'}, ..., w_n^{'})^T$ are the two n-dimensional OWA weight vectors such that

(i) $w_i = w_i^{'},\ i \neq j\ or\ k,$

(ii) $w_j = w_j^{'} + \Delta$

(iii) $w_k = w_k^{'} - \Delta$

where $\Delta > 0, j > k$. Then $orness\,(W) > orness\,(W^{'})$

Proof From the definition of the orness ,

$$orness(W) = \frac{1}{n-1}\sum_{i}^{n}(n-i)w_i$$

$$= \frac{1}{n-1}\left[\sum_{i}^{n}(n-i)w_i^{'} + (n-j)\Delta - (n-k)\Delta\right]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i^{'} + \frac{(k-j)}{n-1}\Delta$$

$$= orness(W^{'}) + \frac{(k-j)}{n-1}\Delta$$

Since $k > j$, $orness(W) > orness(W^{'})$

If F is an OWA operator with weights w_i , i=1,2,3,.....n, then the OWA operator with the weights $\hat{w}_i = w_{n-i+1}, i = 1,2,3....n$ is called the **dual of F**, denoted by$\hat{F}$.

Remark: It is easy to prove that if OWA operators F with the weight vector W and $\hat{F}$ with the weight vector $\hat{W}$ are mutually dual, then

$$orness(\hat{W}) = 1 - orness(W)$$

Definition 3.4 (Yager,1993) Assume that F is any OWA operator of dimension n with weights w_i. F is said to be *buoyancy measure*, if the weights satisfy the additional condition that

$$w_i \geq w_j, \text{for } \; i < j.$$

Definition 3.5 (Yager,1993) An OWA operator is called a *strong buoyancy measure* if

$$w_i \geq \sum_{j=i+1}^{n} w_i, \text{for} \qquad i = 1,2,3,.....,n-1.$$

Theorem 3.7 (Yager,1993) If F is a buoyancy measure then orness (F) ≥ 0.5.

Proof Let $W = (w_1, w_2, w_3, ..., w_n)^T$ be the n-dimensional OWA weight vector.Using Definition of orness,

$$orness(F) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i \tag{3.3}$$

Add and subtract the value $\frac{1}{2}$ on the both side of the equation(3.3) as given below:

$$orness(F) = \frac{1}{2} + \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i - \frac{1}{2} \tag{3.4}$$

As we know

$$\sum_{i=1}^{n} w_i = 1,$$

hence

$$\frac{1}{2}\sum_{i=1}^{n} w_i = \frac{1}{2},$$

Putting the value of $\frac{1}{2}$ in its second occurrence in the equation (3.4),

$$orness(F) = \frac{1}{2} + \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i - \frac{1}{2}\sum_{i=1}^{n} w_i$$

$$= \frac{1}{2} + \sum_{i=1}^{n}\left(\frac{(n-i)}{n-1} - \frac{1}{2}\right)w_i$$

$$= \frac{1}{2} + \sum_{i=1}^{n}\frac{(n-2i+1)}{2(n-1)}w_i$$

$$orness(F) = \frac{1}{2} + \sum_{i=1}^{n} q_i w_i,$$

where

$$q_i = \frac{n-2i+1}{2(n-1)}$$

Case 1: When n is even, $n = 2m$.

Note for $i = a$, where $a \leq m$ and $i = n+1-a$, we get

$$q_a = \frac{n-2a+1}{2(n-1)}$$

and

$$q_{n+1-a} = \frac{n-2n-2+2a+1}{n-1}$$

$$= \frac{-n+2a-1}{2(n-1)} = -q_a$$

Thus, $orness(F) = \frac{1}{2} + \sum_{a=1}^{m} q_a(w_a - w_{n+1-a})$ Now, for $a \geq m$, we have

$$q_a = \frac{n-2a+1}{2(n-1)} \geq \frac{2m-2m+1}{2(n-1)} \geq 0$$

Since, $w_i > w_j$, if $i < j$ and,$q_a \geq 0$, we get

$$\sum_{a=1}^{m} q_a(w_a - w_{n+1-a}) \geq 0$$

Hence, $orness(F) \geq \frac{1}{2}$

Case 2: When n is odd, $n = 2m + 1$,we have

$$orness(F) = \frac{1}{2} + \sum_{a=1}^{m} q_a(w_a - w_{n+1-a}) + q_{m+1}w_{m+1}$$

but,

$$q_{m+1} = \frac{2m + 1 - 2(m + 1) + 1}{2(n - 1)} = 0$$

Hence, we get, $orness(F) \geq \frac{1}{2}$.

Remark: If the weights are $w_i \geq w_{n-i+1}$ then orness (F)$\geq \frac{1}{2}$. If the weights are $w_i \leq w_{n-i+1}$ then orness (F)$\leq \frac{1}{2}$.

3.3.2 Characteristics of OWA operator weights

Let W=$(w_1, w_2, w_3, ...w_n)^T$ be the weighting vector and b_i be the i^{th} largest of $a_1, a_2, a_3, ..., a_n$. Then, the characteristics of OWA operator weights (Wang and Chin,2011) determined by orness are summarized below:

1. **Ordered Weight**: If the orness degree $\alpha > 0.5$ then $w_1 \geq w_2 \geq ...... \geq w_n \geq 0$ and if $\alpha \leq 0.5$ then $0 \leq w_1 \leq w_2 \leq ..... \leq w_n$.

2. The weights have nothing to do with the magnitudes of the aggregates $b_1 \sim b_n$, but depend upon their ranking orders and the decision maker's optimum level.

3. **Optimistic**: If $\alpha = 1$, then we have $w_1 = 1$ and $w_j = 0(j \neq 1)$. This shows that the decision maker is purely optimistic and consider only the biggest value $b_1 = \max_i(a_i)$ in decision analysis.

4. **Pessimistic**: If $\alpha = 0$, then we have $w_n = 1$ and $w_j = 0(j \neq n)$. This shows that the decision maker is purely pessimistic and is concerned with the most conservative $b_n = \min_i(a_i)$ when making decision.

5. **Neutral**: If $\alpha = 0.5$, then we have $w_1 = ... = w_n = (1/n)$. This implies that the decision maker is neutral and make use of all aggregates $b_1 \sim b_n$ equally in decision making.

3.4 Cross-efficiency aggregation approaches

Tradionally, the ranking of the DMUs can be done using average cross-efficiencies defined by

$$C_k = \frac{1}{n}\sum_{j=1}^{n} E_{kj}, \qquad \forall\ k = 1, 2, 3, ..., n.$$

However, the traditional approach of evaluating average cross-efficiencies possess some drawbacks which are summarized as below:

(i) The weight assigned to the self-evaluated efficiency of each DMU is fixed and has no way of incorporating the decision maker's subjective preferences into the aggregation.

(ii) In aggregation process, it has been observed that the contribution of self-evaluated efficiency is less as compared to peer-evaluated efficiencies.

To overcome the aforementioned drawbacks and to reflect the decision maker's subjective preferences, the use of OWA operator weights for cross-efficiency aggregation has been proposed by eminent researchers. In order to apply OWA operator weights, the reordering of both self and peer-evaluated efficiencies is required as shown in Figure 3.1. In Figure 3.1, $b_{ij} (\forall i, j = 1, 2, 3, ..., n)$ are reordered cross-efficiencies of each DMU from the largest to the smallest. In this way, the self-evaluated efficiencies are always ranked in the first place, i.e, $b_{i1} = E^*_{ii}$.

The next goal is to determine the weights of the OWA operators. In existing literature on OWA operator weights, many researchers presented different approaches to determine weights. O'Hagan(1988) suggested maximum entropy approach in which a non-linear optimization problem was presented with a pre defined degree of orness as a constraint. The resultant weights are termed as maximum entropy weights which are further used in OWA aggregation . Further, Fuller and Majlender(2001) transformed the maximum entropy model into a polynomial equation that can easily be solved analytically. A quantifier guided aggregation approach was suggested by Yager(1996). Further, an exponential smoothing method was presented by Yager and Filev(1998) to obtain exponential OWA operator weights. Fuller and Majlender(2003) proposed a minimum variance approach to determine the minimal variability OWA operator weights. However, these approaches are quite complex due to the presence of non-linear constraints and thus, are difficult to solve. To overcome this difficulty, Wang and Parkan(2005) suggested minimal disparity approach to determine OWA operator weights which is simple and effective approach. This approach involves a linear programming problem that can easily be solved by using existing LP softwares such as LINGO, LINDO, MATLAB,etc.

In this study, we utilize minimax disparity approach suggested by Wang and Parkan (2005) for obtaining OWA operator weights which is discussed as below:

Figure 3.1: Re-ordered Cross-efficiency matrix

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

3.5 Minimax disparity approach

Major problem in the theory of (OWA) operators is to determine its weights. The Maximun Entropy approach (O'Hagan,1988) and Minimum Variance approach(Fuller and Majlender,2003) could be interepted as making the weights close to each other when the degree of orness is taken into consideration. The minimax disparity approach(Wang and Parkan,2005) is also based upon the fact that the disparities between two adjacent weights are made as small as possible. They suggested the following mathematical model for minimizing the maximum disparity:

Model-3.1

$$\text{Minimize}\left\{ Max_{i\in\left\{1,2,3....,n-1\right\}} |w_i - w_{i+1}| \right\}$$

$$\text{subject to orness(W)} = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, 0 \le \alpha \le 1,$$

$$\sum_{i=1}^{n} w_i = 1, 0 \le w_i \le 1, i = 1, 2, 3....n.$$

Let the maximum disparity between adjacent OWA weights is given by

$$\delta = \text{Max}_{i\in\left\{1,2,3,....n-1\right\}} |w_i - w_{i+1}|$$

This implies

$$|w_i - w_{i+1}| \le \delta, i = 1, 2, 3....n-1.$$

$$\Rightarrow -\delta \le w_i - w_{i+1} \le \delta \quad i = 1, 2, 3....n-1,$$

$$\Rightarrow w_i - w_{i+1} - \delta \le 0 \quad i = 1, 2, 3....n-1,$$

and

$$w_i - w_{i+1} + \delta \ge 0, \quad i = 1, 2, 3....n-1.$$

Therefore, the minimax disparity Model-3.1 is reduced to the following LP form:

Model-3.2

$$\text{Minimize } \delta$$

$$\text{subject to orness(W)} = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)\ w_i, \quad 0 \le \alpha \le 1,$$

$$\sum_{i=1}^{n} w_i = 1,$$

$$w_i \ - w_{i+1} \ - \ \delta \ \leq \ 0, \quad i = 1, 2, 3....n-1,$$

$$w_i \ - w_{i+1} \ + \ \delta \ \geq \ 0, \quad i = 1, 2, 3....n-1,$$

$$w_i \ \geq \ 0, \quad i = 1, 2, 3....n.$$

The main characteristics of using are stated in the following theorems.

Theorem 3.8 For an OWA operator weight vector $W = (w_1, w_2, ..., w_n)^T$ determined in Model-3.2,

(i) if orness(W)= 1, then $W = (1, 0, 0, ...0)^T$

(ii) if orness(W)= 0, then $W = (0, 0, 0, ...1)^T$

(iii) if orness(W)= 0.5, then $W = (1/n, 1/n, ...1/n)^T$

Proof

(i) If orness(W)= 1, $W = (1, 0, 0, ..., 0)^T$ is the only feasible and hence optimal solution of Model-3.2. We have

$$\text{orness(W)} \ = \ \alpha \ = \frac{1}{n-1}\sum_{i=1}^{n}(n-i) \ w_i \ = 1$$

(ii) If orness(W)= 0, $W = (0, 0, 0, ..., 1)^T$ is the only feasible and hence optimal solution Model-3.2. We have

$$\text{orness(W)} \ = \ \alpha \ = \frac{1}{n-1}\sum_{i=1}^{n}(n-i) \ w_i \ = 0$$

(iii) If orness(W)= 0.5 then $W = (1/n, , 1/n, ..., 1/n)^T$ is the feasible solution of Model-3.2. For $W = (1/n, , 1/n, ..., 1/n)^T$, we have $\delta = 0$ which is the lower bound of the objective function value. Therefore, $W = (1/n, , 1/n, ...1/n)^T$ is the optimal solution of Model-3.2.

Theorem 3.9 If $W^* = (w_1^*, w_2^*, ..., w_n^*)^T$ is an optimal solution of Model-3.2 for the given degree of orness(W) = α, then $\hat{W}^* = (\hat{w}_1^*, \hat{w}_2^*, ..., \hat{w}_n^*)^T$, where $\hat{w}_i^* = w_{n-i+1}^*$, $i = 1, 2, 3, ...n,$ is the optimal solution of Model-3.2 for the degree of orness(W)=1 - α and vice-versa.

Proof

$$\text{For } \hat{w}_i^* = w_{n-i+1}^*, \ \forall\, i = 1, 2, 3, ...n. \text{ we have}$$

$$w_i^* = \hat{w}_{n-i+1}^*, \ \forall\, i = 1, 2, 3, ...n. \tag{3.5}$$

Now,

$$|w_i^* - w_{i+1}^*| = |\hat{w}_{n-i+1}^* - \hat{w_{n-i}^*}| = |\hat{w}_{n-i}^* - \hat{w_{n-i+1}^*}|$$

$$= |\hat{w}_j^* - \hat{w}_{j+1}^*|, \ j = n - i, \ , i = 1, 2, 3......n - 1. \tag{3.6}$$

Also

$$\sum_{i=1}^{n} w_i^* = \sum_{j=1}^{n} \hat{w}_j^* = 1 \tag{3.7}$$

Since $W^* = (w_1^*, w_2^*, w_3^*, ..., w_n^*)^T$ is an optimal solution of Model-3.2 for orness(W)=α, the Model-3.2 becomes

Model-3.3

$$\text{Min } \delta = \delta^*$$

$$\text{subject to orness}(W^*) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)\ w_i^*, \quad 0 \le \alpha \le 1$$

$$\sum_{i=1}^{n} w_i^* = 1$$

$$w_i^* - w_{i+1}^* - \delta^* \le 0, \quad i = 1, 2, 3....n - 1.$$

$$w_i^* - w_{i+1}^* + \delta^* \ge 0, \quad i = 1, 2, 3....n - 1.$$

$$w_i^* \ge 0, \quad i = 1, 2, 3....n.$$

where δ^* is the optimum value of the objective function. By using equations(3.5),(3.6) and(3.7), Model-3.3 reduces to the following form:

Model-3.4

$$\text{Min}\,\delta \;=\; \delta^*$$

$$\text{subject to}\;\; \text{orness}(\hat{\text{W}}^*) \quad = \frac{1}{n-1}\sum_{j=1}^{n}\,(n-j)\,\hat{w}_j^* \;=\; 1\;-\;\alpha,\;\;0\le\alpha\le 1$$

$$\sum_{j=1}^{n}\hat{w}_j^* \;=\; 1$$

$$\hat{w}_j^* \;-\hat{w}_{j+1}^* \;-\;\delta^* \;\le\; 0, \quad j=1,2,3....n-1.$$

$$\hat{w}_j^* \;-\hat{w}_{j+1}^* \;+\;\delta^* \;\ge\; 0, \quad j=1,2,3....n-1.$$

$$\hat{w}_j^* \;\ge\; 0, \quad j=1,2,3....n.$$

It is observed that $\hat{W}^* \;=\; (\hat{w}_1^*, \hat{w}_2^*, ..., \hat{w}_n^*)^T$ and δ^* is an optimum solution of Model-3.2 for $orness(W) \;=\; 1 \;-\; \alpha.$

In the similar way, we can show that Model-3.4 can be reduced to Model-3.3 by using some equations(3.5),(3.6) and (3.7). This completes the proof.

Theorem 3.10 (Wang et al., 2011) For the given w_1, there exists an integer $k \;\le\; n$ such that $w_i \;=\; w_1 - (i-1)\,d \ge\;$ 0for $i \;=\; 1,2,3....k$and $w_i \;=\; 0 \;for\; i \;=\; k+1, k+2, ..., n,$ where k and d are determined by

$$k = min\left(n, INT\left[\frac{2}{w_1}\right]\right) \text{ and } d = \frac{2(kw_1-1)}{k(k-1)}$$

where$INT[x]$ is the function that rounds $x$down to the nearest integer.

Proof It is derived that $kw_1 - (1+2+.....k-1)d = 1$ from

$$\sum_{i=1}^{n} w_i \;=\; 1$$

. That is,

$kw_1 - (k-1)kd/2 \;=\; 1 \;or\; d \;=\; 2(kw_1-1)/k(k-1)$. Since,$w_k \;=\; w_1 \;-\; (k-1)d \ge 0,$ which follows that $d \;\le w_1/(k-1)$. Therefore, gives $kw_1 \le 2$ from

$$2(kw_1-1)/k(k-1) \le w_1(k-1)$$

Using the fact that K is an integer and is $\leq$ n. Therefore,

$$k \leq min\left(n, INT\left[\frac{2}{w_1}\right]\right)$$

As already know, the minimax disparity approach requires minimizing the weight difference between the two adjacent weights that minimizes

$$d = 2(kw_1 - 1)/k(k-1)$$

K must take its maximum value, that is

$$k = min\left(n, INT\left[\frac{2}{w_1}\right]\right).$$

Theorem 3.11 For a given degree $\alpha \in (0.5, 1)$, there exists an integer $k \leq n$ such that $w_i = w_1 - (i-1)\, d \geq 0$ for$i = 1, 2, 3....k$ and$w_i = 0$ for $i = k+1, k+2....., n$ where k,w_1 and d are determined by $k = Min\Big(n, INT[3n - 1 - 3\alpha(n-1)]\Big)$.

$$w_1 = \frac{4(k+1) - 6n + 6\alpha(n-1)}{k(k+1)} \text{ and } d = \frac{2k(w_1 - 1)}{k(k-1)}$$

Proof From Theorem 3.8, $d = 2k(kw_1 - 1)/k(k-1)$. By the orness degree $\alpha = (1/(n-1)) \sum_{i=1}^{n}(n-i)w_i$, it is derived that

$$\alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \frac{1}{n-1}\sum_{i=1}^{k}(n-i)[w_i - (i-1)d]$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{k}(n-i)w_i - d\sum_{i=1}^{k}(n-i)(i-1)\right)$$

$$= \frac{1}{n-1}\left(k\left(n - \frac{k+1}{2}\right)w_1 - d\left(\frac{nk(k+1)}{2} - nk - \frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2}\right)\right)$$

$$= \frac{1}{n-1}\left(k\left(n - \frac{k+1}{2}\right)w_1 - \frac{k(k-1)(3n-2(k+1))}{6}d\right)$$

$$= \frac{1}{n-1}\left(k\left(n - \frac{k+1}{2}\right)w_1 - \frac{k(k-1)(3n-2(k+1))}{6} \times \frac{2(kw_1-1)}{k(k-1)}\right)$$

$$= \frac{1}{n-1}\left(k\left(n - \frac{k+1}{2}\right)w_1 - \frac{(3n-2(k+1))(kw_1-1)}{3}\right)$$

$$= \frac{1}{n-1}\left(\frac{k(k+1)}{6}w_1 + \frac{3n-2(k+1)}{3}\right)$$

As a result,

$$w_1 = \frac{6\alpha(n-1) - 6n + 4(k+1)}{k(k+1)}$$

It is known that $k \leq \frac{2}{w_1}$, k is an integer $\leq n$ and able to minimize the weight difference between two adjacent weights, it is therefore concluded that $k = min(n, INT[3n-1-3\alpha(n-1)])$.

Theorem 3.12 (Wang et al.,2011) For a given integer$P < n$, if it is desired that $w_{n-P+1} = w_{n-P+2} = ... = w_n = 0$, then the weights $w_i = w_1 - (i-1)\,d > 0$ $(i = 1, 2, 3....n-P)$ can be determined by$d = 2[(n-P)w_1 - 1]/((n-P)(n-P-1))$, where

$$w_1 \in (2/(n+P+1), 2/(n-P))$$

is a weight parameter specified by the deicision maker.

Proof From theorem 3.10, it is known that $k = n - P$ and

$$d = \frac{2(kw_1 - 1)}{k(k_1)} = \frac{2[(n-P)w_1 - 1}{(n-P)(n-P-1)}$$

From,

$$w_{n-P} = w_1 - (n-P-1)d = w_1 - \frac{2[(n-P)w_1 - 1]}{(n-P)} > 0$$

It is derived that $w_1 < 2/(n-P)$, since $w_n - P$ is the last non zero weight, after that resultant value should be positive. That is,

$$w_1 - (n-P)d = w_1 - \frac{2[(n-P)w_1 - 1]}{n-P-1} < 0$$

from which $w_1 > 2/(n-P+1)$ is derived.Therefore, $w_1 \in (2/(n-P+1), 2/(n-P))$.

Theorem 3.13 (Wang et al.,2011) For a given orness degree α,if it is desired that$w_1 > w_2 > .... > w_n > 0$, then α has to satisfy $0.5 < \alpha <\ (2n-1)/3(n-1))$.

Proof From theorem 3.11, when $k = n$ there exist,

$$w_1 = \frac{4(k+1) - 6n + 6\alpha(n-1)}{k(k+1)} = \frac{4(n+1) - 6n + 6\alpha(n-1)}{n(n+1)}$$

$$= \frac{6\alpha(n-1) - 2n + 4}{n(n+1)}$$

$$\text{and} d = \frac{2k(w_1 - 1)}{k(k-1)} = \frac{2(nw_1 - 1)}{n(n-1)} = \frac{6(2\alpha - 1)}{n(n+1)}$$

Accordingly,

$$w_n = w_1 - (n-1)d = \frac{6\alpha(n-1) - 2n + 4}{n(n+1)} - \frac{6(2\alpha - 1)(n-1)}{n(n+1)}$$

$$= \frac{4n - 2 - 6\alpha(n-1)}{n(n+1)}$$

by $d > 0$ and $w_n > 0$, it is easily derived that $0.5 < \alpha < (2n-1)/(3(n-1))$.

4

A NEW DEA-MCDM ALGORITHM FOR RANKING DECISION MAKING UNITS

4.1 Introduction

Decisions are very important part of our life. We make decisions at every stage. Decision making is an essential part of planning. However, studies have shown that most people are much poorer at decision making than they think. An understanding of what decision making involves, together with a few elective techniques, will help you make better decisions. Elective and successful decisions make profit to the company and unsuccessful ones make losses. Therefore, corporate decision making process is the most critical process in any organization. In the decision making process, we choose one course of action from a few possible alternatives. In the process of decision making, we may use many tools, techniques and perceptions. Usually, decision making is hard. Majority of corporate decisions involve some level of dissatisfaction or conflict with another party.

Decision-making is an integral part of modern management. Essentially, rational or sound decision making is taken as primary function of management. Every manager takes hundreds and hundreds of decisions subconsciously or consciously making it as the key component in the role of a manager. Decisions play important roles as they determine both organizational and managerial activities. A decision can be defined as a course of action purposely chosen from a set of alternatives to achieve organizational or managerial objectives or goals. Decision making process is continuous and indispensable component of managing any organization or business activities. Decisions are made to sustain the activities of all business activities and organizational functioning.
The following are some examples which involve decision making:

1. Buying a property in good price.

2. Choosing a University for higher studies.

3. Vote a candidate.

Operations Research Agarwal et al.(2010) is a discipline that deals with the application of advanced techniques for making better decisions in practical situations. Its approach is particularly useful in balancing conflicting objectives (goals or interests) where there are many alternative courses of action available to the decision makers. It attempts to resolve the conflicts of interest among various sections of the organization and seeks the optimal solution which may not be acceptable to one department but is in the interest of the organization as a whole. It facilitates the decision-maker with decision aids (or rules) derived from:

1. Total system orientation

2. Scientific methods of investigation

3. Models of reality, generally based on quantitative measurement and techniques.

4.2 Decision making

Decision theory (or the theory of choice) is the study of identifying and choosing alternatives (where alternatives are possibilities or choices) based on the values and preferences of the decision maker. Making a decision implies that there are alternative choices to be considered, to choose the one that best fits with our goals, objectives and desires.

Decision making deals with methods for determining the optimal course of action when a number of alternatives are available.

Application of decision making

The broad area where we need decision making are summarized as follows (Alias et al.,2008):

1. **MANUFACTURING:** It includes the following decision making problem:
   (a) Supplier selection
   (b) IT department performance evaluation
   (c) Machine-tool selection
   (d) Decision making in equipment selection
2. **MANAGEMENT**: It involves following decision making problem:
   (a) Data mining software comparison and scenario analysis.
   (b) Software development allocation.
   (c) Agroenvironmental policies.
   (d) Assessment of ecological criteria and indicators.
3. **EDUCATION**: It involves the following decision making problems:
   (a) Basic research evaluation
   (b) Educational project evaluation

(c) Environmental education

4. Decision making is used in **MEDICAL SCIENCE** for Cancer screening option

5. Decision making is an important process in **CONSTRUCTION** for Selection of location.

6. Decision making is used in **TRANSPORTATION** for Public transport line that facilitate conservation

4.2.1 Classification of decisions

Decision theory divides the decisions into three classes:

1. **Decisions under certainty**

   Conditions under certainty are that the decision maker should have full and needed information to make a decision. Decision is made under condition of certainty. The decision maker knows exactly what the outcome will be, as he/she has enough clarity about the situation and knows the resources, time available for decision making,the nature of the problem itself, possible alternatives to resolve the problem and undoubtedly clarify or certain with result of alternatives. In most situations, the solutions are already available from the past experience and are appropriate for the problem at hand.

   For example, if the optimization criterion is least cost and you are considering two different brands of a product, which appear to be equal in value to you,one costing 20percent less than the other, all other things being equal, you will choose the less expensive brand.

2. **Decisions under risk**

   Conditions under risk provides the probability regarding expected results for decision making alternatives. It is due to the nature of future conditions that are not always known in advance. Although some good information may be available, it is not enough to answer all questions about outcomes. Decision makers could define the nature of the problem, possible alternatives and the probability of each alternative leading to desired results, but could not guarantee how each alternative may work. Decision has clear-cut goals, but future outcomes associated with each alternative are subject to chance.

For example, testing of nuclear leakage in Japan after Tsunami hit in year 2011 is a risky decision made by Japanese Govt., as the government does not know how wide the range of effecting area and also nuclear substance itself is a life threatening factor.

3. **Decisions under uncertainty**

   Conditions under uncertainty provide no or incomplete information, many unknowns and possibilities to predict expected results for decision-making alternatives. An assumption is often made; decision maker has no information or intuitive judgment to use as a basis for assigning the probabilities to each state of nature. One should have to come up with creative approaches and alternatives to solve the problem.

   For example, flood may cause panic and environment of uncertainty among the victims, which leads to uncertain decision making of victims, some may flee from home and take only important documents with them, some who lived at higher ground, may wait and observe if the flood worsen then decide the next approach.

4.2.2 Elements of decision making

1. Decision Makers: the one who takes decisions.
2. Goals/Criterion to be served: the standard that can be fulfilled to test
3. Relevant Alternatives: various option relative to the given criteria.

4. Ordering of Alternatives: alternatives are ordered according to their preferences.

5. Choice among Alternatives: choose the best alternative among the various alternatives to make the good decision.

4.2.3 Decisison making process

Seven most essential steps involved in decision making process are(Forman and Selly,2001):

1. **Define the problem**

   Firstly, realize the situation which involves decision. Try to clearly define the statement of the problem and nature of decision to be made.

2. **Gather relevant information**

   Collect some pertinent information before one should make their decision. It involves the knowledge about what information is needed, the best sources of information and how to get it.

3. **Identify the alternatives**

   As the information get collected, several paths of action should be identified. In this step, one must list all possible and desirable alternatives.

4. **Weight the evidence**

   Evidence may be given full weight, partial weight, more or less weight than other evidence, or no weight at all. Evidence is weighed against other evidence to determine which evidence is more reliable.

5. **Choose among alternatives**

   Once, weighed all the evidence, select the alternative that seems to be the best one. Combination of alternatives may also prefered.

6. **Take action**

   After the choice of alternatives in step 5, take some positive action by beginning to implement that alternative.

7. **Review your decision and its consequences**
   In this final step, consider the results of your decision and evaluate whether or not it has resolved the need you identified in Step 1. If the decision has not met to the identified need, you may want to repeat certain steps of the process to make a new decision. In such situation, gather more detailed or somewhat different information or explore additional alternatives.

4.3 Multi-criteria decision making (MCDM)

"Multi-Criteria Decision Making (MCDM) is the study of methods and procedures by which concerns about multiple conflicting criteria can be formally incorporated into the management planning process", as defined by the **International Society on Multiple Criteria Decision Making**. It is a branch of a general class of Operations Research models which deal with decision problems under the presence of a number of decision criteria. It is a rich and well-studied problem solving approach usually aimed at **ranking of the alternatives**. It has some unique characteristics such as the presence of multiple non-commensurable and conflicting criteria, different units of measurement among the criteria, and the presence of quite different alternatives. It is an important branch of decision making theory. The problems related to MCDM are generally classified into two classes with respect to the solution space of the given problem: continuous and discrete. The continuous problems are handled by multi-objective decision making (MODM) methods. On the other hand, discrete problems are solved by multi-attribute decision making (MADM) methods. In existing literature, MCDM is popularly used to describe discrete MCDM. The present work is focused on the discrete decision making problem and therefore, we use MCDM to state discrete problem corresponding to performance evaluation.

The objective is to identify the best/most desirable alternative, i.e., to select an alternative with the best overall value.

Table 4.1: Decision matrix

Alternative	C_1	C_2	C_3	.	.	.	C_m
A_1	a_{11}	a_{12}	a_{13}	.	.	.	a_{1m}
A_2	a_{21}	a_{22}	a_{23}	.	.	.	a_{2m}
A_3	a_{31}	a_{32}	a_{33}	.	.	.	a_{3m}
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
A_n	a_{n1}	a_{n2}	a_{n3}	.	.	.	a_{nm}

4.3.1 Preliminaries

1. CRITERIA: It is refered to as “goals”. It represents the different dimensions from which alternatives can be viewed.

2. MULTIPLE CRITERIAS: Each MCDM problem is associated with multiple criterias.

3. CONFLICT AMONG CRITERIAS: Since,different criteria represents different dimensions of alternatives they may conflict with each other.

4. DECISION WEIGHTS: Most of MCDM methods require that criteria be assigned weights of importance.

5. INCOMMENSURABLE UNITS: Different criteria may be associated with different units of measure.

6. DECISION MATRIX: An MCDM problem can be easily expressed as a matrix format of order (nXm) matrix in which element a_{ij} indicates the performance of alternativeA_i when it is evaluated in terms of decision criterion C_j.

4.3.2 Steps for utilizing any decision-making technique

1. Determine the relevant criteria and alternatives.

2. Attach numerical measures to the relative importance of the criteria and to the impacts of the alternatives on these criteria.

3. Process the numerical values to determine ranking of each alternative.

General discrete MCDM problem Let $A = A_i, i = 1, 2, 3...., n$ be a (finite) set of decision alternatives and $C = C_j, i = 1, 2, 3...., m$ be a (finite) set of criteria. A general discrete MCDM problem can easily be expressed as a decision matrix that is depicted in Table 4.1. A decision matrix of order $n \times m$ is a matrix in which each entry a_{ij} indicates the performance of alternative A_i when it is evaluated in terms of decision criterion C_j, for$i = 1, 2, 3, ..., n$ and $j = 1, 2, 3, ..., m$. Let $w_{ij} \geq 0$ be the weight associated to the j^{th} criterion C_j, such that $\sum_{j=1}^{m} w_j = 1$.

4.4 Proposed DEA-MCDM approach for ranking DMUs

One of the important issues discussed in previous chapters in DEA is ranking efficient DMUs possessing efficiency score equal to one. Various ranking techniques have been developed and many more are developing to increase the discrimination power of existing DEA models. Cross-efficiency (Sexton et al., 1986) is one among these techniques to achieve complete ranking of the DMUs (Refer to Chapter 2). However, the existence of various cross-efficiency evaluation formulations, namely, aggressive, benevolent and neutral, lead to different rankings of the DMUs. Therefore, in many practical applications of decision making, decision maker finds it difficult to make any choice among these available cross-efficiency formulations. In addition, if at certain time, a decision maker needs to make decision based on "all" or "some" or "most" of the cross-efficiency evaluation formulations, then this cannot be done by using existing ranking approaches. Thus, there is a need to develop an algorithmic approach, where a decision maker can specify his degree of orness for selecting aggressive/benevolent/neutral formulations and measure a complete ranking of the DMUs. To the best of our knowledge, there does not exist any study in DEA literature in which the final decision for selection of the best DMU is made on the basis of all cross-efficiency formulations at a specified degree of orness. In the present work, we consider all the three formulations, namely, aggressive, benevolent and neutral as three criteria and the problem of identifying best DMU among the efficient ones as a multi-criteria decision making problem. The proposed DEA-MCDM approach is discussed as under.

4.4.1 Problem formulation

Consider a performance evaluation problem of n DMUs in terms of m inputs and s outputs.

Nomenclature

- α : Degree of orness
- $W_\alpha = (w_1^\alpha, w_2^\alpha, , w_2^\alpha, ..., , w_n^\alpha)^T$: Weight vector obtained by minimax disparity approach (Model-3.2) at degree of orness .
- $E_{k\alpha}^{(A)}$: OWA aggregated aggressive cross-efficiency of DMU$_k$ at degree of orness α.
- $E_{k\alpha}^{(B)}$: OWA aggregated benevolent cross-efficiency of DMU$_k$ at degree of orness α.
- $E_{k\alpha}^{(N)}$: OWA aggregated neutral cross-efficiency of DMU$_k$ at degree of orness α.

The present problem is to achieve complete ranking of the DMUs as a MCDM problem. Here we have considered the given set of DMUs as alternatives and aggressive, benevolent and neutral cross-efficiency evaluation formulations as three criteria at orness α. Let $A = \left\{DMU_1, DMU_2, DMU_3, ..., DMU_n\right\}$be a set of decision alternatives and C=$\left\{Aggressive, Benevolent, Neutral\right\}$ be a set of criteria at degree of orness α. The proposed DEA-MCDM problem is expressed as a decision matrix as shown in Table 4.2.

Let a decision matrix of order $n \times 3$ is a matrix where each entry indicates the performance of alternative DMU_i when it is evaluated in terms of decision criterion C_j, for $i = 1, 2, 3, ..., n$ and $j = 1, 2, 3$. Let $w_{ij} \geq 0$ be the weight associated to the j^{th} criterion C_j such that $w_1 + w_2 + w_3 = 1$ The objective of the present study is to develop a hybrid

Table 4.2: Decision matrix of proposed MCDM problem

	Criteria		
Alternative	Aggressive	Benevolent	Neutral
A_1	$E_{1\alpha}^{(A)}$	$E_{1\alpha}^{(B)}$	$E_{1\alpha}^{(N)}$
A_2	$E_{2\alpha}^{(A)}$	$E_{2\alpha}^{(B)}$	$E_{2\alpha}^{(N)}$
A_3	$E_{3\alpha}^{(A)}$	$E_{3\alpha}^{(B)}$	$E_{3\alpha}^{(N)}$
.	.	.	.
.	.	.	.
.	.	.	.
A_n	$E_{n\alpha}^{(A)}$	$E_{n\alpha}^{(B)}$	$E_{n\alpha}^{(N)}$

DEA-MCDM algorithmic approach so that a decision maker can select/identify the most desirable DMU from a set of DMUs and rank them by considering aggressive, benevolent and neutral cross-efficiency formulations at a desired orness level. The algorithmic approach is summarized as follows.

4.4.2 A new DEA-MCDM algorithm for ranking DMUs

The steps involved in new DEA-MCDM algorithm are summarized below:

Step 1. Consider a performance evaluation problem.

Step 2. Selecting input and output data variables and define a performance model.

Step 3. Evaluate CCR efficiency E_k for each DMU_k by using Model-2.2.

Step4. Evaluate aggressive, benevolent and neutral cross-efficiency matrices using Model-2.4, Model-2.6 and Model-2.8 respectively.

Step 5. Re-order the aggressive, benevolent and neutral cross-efficiencies row-wise from largest to smallest and obtain re-ordered cross-efficiency matrices.

Step 6. Select a degree of orness α from an interval [0, 1].

Step 7. Obtain a weight vector $W_\alpha = (w_1^\alpha, w_2^\alpha, w_3^\alpha, ..., w_n^\alpha)^T$ for OWA operator by utilizing minimax disparity approach (Model-3.2) at orness level α.

Step 8. Evaluate OWA aggregated aggressive cross-efficiency $E_{k\alpha}^{(A)}$, benevolent cross-efficiency $E_{k\alpha}^{(B)}$ and neutral cross-efficiency $E_{k\alpha}^{(N)}$ for each DMU_k at orness level α by employing OWA operator (Definition 3.1).

Step 9. Define an MCDM problem with A=$\left\{ \mathrm{DMU}_1,\mathrm{DMU}_2,\mathrm{DMU}_3,.....,\mathrm{DMU}_n \right\}$ as a set of decision alternatives and C=$\left\{ Aggressive, Benevolent,\ Neutral \right\}$ as a set of criteria at degree of orness α. By utilizing OWA aggregated cross-efficiencies $E_{k\alpha}^{(A)}$, $E_{k\alpha}^{(B)}$, $E_{k\alpha}^{(N)}$ measured in Step 8 at a particular orness α and make a decision matrix for DEA-MCDM problem as shown in Table 4.2.

Step 10. Obtain a weighting vector $W_\alpha^C = (w_1^\alpha, w_2^\alpha, w_3^\alpha)^T$ associated with criteria using minimax.

Step 11. Select a disparity approach at orness level α, preference order for three criterions, that is, $(C_i, C_j, C_k), i, j,\ k$=1,2,3;$i \neq j \neq k$

Step 12. Apply induced OWA (IOWA) operator (Definition 3.2) with preference order as an inducing variable and weighting vector $W_\alpha^C = (w_1^\alpha, w_2^\alpha, w_3^\alpha)^T$ to aggregate cross-efficiencies $E_{k\alpha}^{(A)}$, $E_{k\alpha}^{(B)}$ and $E_{k\alpha}^{(N)}$ for each alternative DMU_k. Let the final aggregated value be denoted by AV_k^α.

Step 13. Rank the DMUs as the decreasing values of $AV_k^\alpha, k = 1, 2, 3, ...n$.

4.4.3 Characteristics of the proposed DEA-MCDM approach

The proposed DEA-MCDM approach has the following characteristics:

i. The proposed DEA-MCDM approach includes the subjectivity and preference of the decision maker in more realistic way due to the use of orness constraint while computing results.

ii. Aggregation of cross-efficiencies in each formulation is done by using ordered weighted averaging (OWA)/induced OWA (IOWA) operators instead of the traditional approach of taking average.

iii Each of the aggressive, benevolent and neutral cross-efficiency formulations contribute to select the best alternative among the DMUs and to achieve complete ranking of the DMUs.

4.5 Application to Educational Institution

Consider an efficiency evaluation problem of seven departments of a certain University (Wang et al.,1990; Wang and Chin,2010). The performance of seven academic departments in a university is to be measured in terms of three inputs and three outputs. Here, the seven academic departments are the DMUs. The description of input and output data is shown in Table 4.3, and the data is listed in Table 4.4.

Table 4.3: Input and Output Data variables

S.No.	Input	output
1	Number of academic staff (I_1)	Number of under graduates (O_1)
2	Academic staff salaries (in pounds) (I_2)	Number of post graduate students (O_2)
3	Sports Staff Salaries (in pound) (I_3)	Number of research papers (O_3)

The CCR-efficiency for each academic department has been evaluated by using Model-2.2 and the efficiency results are listed in Table 4.4. In Table 4.4, it has been observed that by Definition 2.1, six DMUs among seven are evaluated as efficient DMUs. The only inefficient DMU is DMU_4. Carefully observing Table 4.4, we can depict that CCR-efficiency using Model-2.2 lacks in discriminating among the DMUs. Therefore, in order to rank the DMUs and to increase the discrimination power of the model, we have utilized cross-efficiency evaluation technique as discussed in Section 2.4.

Table 4.4: Input and Output data for seven academic departments in a university

S.No.	I_1	I_2	I_3	O_1	O_2	O_3	CCR-efficiency
1	12	400	20	60	35	17	1.0000
2	19	750	70	139	41	40	1.0000
3	42	1500	70	225	68	75	1.0000
4	15	600	100	90	12	17	0.8135
5	45	2000	250	253	145	130	1.0000
6	19	730	50	132	45	45	1.0000
7	41	2350	600	305	159	97	1.0000

The aggressive, benevolent and neutral cross-efficiencies of the seven academic departments are measured by using Model-2.4, Model-2.6 and Model-2.8 respectively. The cross-efficiency results for aggressive, benevolent and neutral models are listed in Table 4.5, Table 4.6 and Table 4.7 respectively. In these tables, it has been observed that the ranking result of the DMUs using the traditional approach are different and therefore it is quite difficult for the decision maker to make a choice among these three different ranking results.

Therefore in order to achieve complete ranking based on all the three cross-efficiency formulation, we have applied the proposed DEA-MCDM algorithm, discussed in section 4.4.2. Firstly we have to select a degree of orness α in [0,1]. Here we have selected eleven different values of α as 0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9 and 1. Further the OWA operator weights for these different orness levels have been evaluated by using minimax disparity approach (Model-3.2) and the weights are listed in Table-4.8. Now we have re-ordered the aggressive, benevolent and neutral cross-efficiencies(see Tables-4.5, 4.6, 4.7) row wise from largest to smallest and then we have evaluated OWA aggregated aggressive cross-efficiencies $E_{k\alpha}^{(A)}$, benevolent cross-efficiency $E_{k\alpha}^{(B)}$ and neutral cross-efficiency $E_{k\alpha}^{(N)}$ for each DMU$_k$ at each orness level α and the results are listed in Table-4.9,Table- 4.10 and Table 4.11 respectively in 0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1. Further, we have defined a decision matrix in Table-4.12 of an MCDM problem for a university selection with A=$\left\{ \text{DMU}_1\text{,DMU}_2\text{,DMU}_3\text{,.....,DMU}_n \right\}$ as a set of decision alternatives and C=$\left\{ Aggressive,Benevolent,\ Neutral \right\}$ as a set of criteria at degree of orness α,in particular $\alpha = 0.8$. The entries in Table-4.12 denote the OWA aggregated cross-efficiencies $E_{k\alpha}^{(A)}$, $E_{k\alpha}^{(B)}$ f each DMU$_k oE_{k\alpha}^{(N)}$ of each DMU$_k$ at $\alpha = 0.8$. For aggregation of each alternative with respect to different criterions, we have obtained a weighting vector $W_{0.8}^C = (w_1^{0.8} = 0.6333, w_2^{0.8} = 0.3333, w_3^{0.8} = 0.0334)^T$ by utilizing minimax disparity approach(Model-3.2) at orness level$\alpha = 0.8$. In the next step, a decision maker has to select a prefernce order for three criterions which is application dependent. Here, for an illustration we have selected perference order as(C_2, C_1, C_3) where$C_2 \equiv Benevolent$, $C_1 \equiv Aggressive$,$C_3 \equiv Neutral$. Lastly, the final aggregated value $AV_k^{0.8}$ for each DMU$_k$ has been measured by using IOWA operator with preference order (C_2, C_1, C_3) as an inducing variable and weighting vector $(0.6333, 0.3333, 0.0334)^T$, an the final results are listed in Table-4.13 along with the final ranking of the DMUs. Table-4.13 depicts that DMU$_6$ is evaluated as the best DMU by using the proposed DEA-MCDM approach and it outperformance the other DMUs. The DMU$_1$ has been measured as the least preferable alternative. The final ranking of the DMUs is given by DMU$_6 > DMU_5 > DMU_2 > DMU_7 > DMU_4 > DMU_3 > DMU_1$

Table 4.5: Aggressive Cross-efficiencies and ranking of DMUs

DMUs	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	Average cross-effficiency	Ranking
DMU_1	1.0000	0.3284	0.5558	0.0705	0.3359	0.5186	0.1542	0.4233	7
DMU_2	0.8464	1.0000	0.0856	0.7562	0.6635	1.0000	0.6060	0.7082	5
DMU_3	0.0010	0.6237	1.0000	0.2839	0.3194	0.8259	1.0000	0.7123	4
DMU_4	0.0010	0.9995	0.7771	0.8134	0.7209	0.9102	0.8192	0.7202	3
DMU_5	0.6456	0.8241	0.8254	0.3680	1.0000	0.9999	0.8906	0.7934	1
DMU_6	0.7949	0.7012	0.9996	0.0246	0.9998	1.0000	0.8906	0.7730	2
DMU_7	0.7629	0.5594	0.4201	0.2078	0.8336	0.9999	1.0000	0.6834	6

Table 4.6: Benevolent Cross-efficiencies and ranking of DMUs

DMUs	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	Average cross-efficiency	Ranking
DMU_1	1.0000	0.9581	0.7667	0.5782	1.0000	1.0000	1.0000	0.9004	2
DMU_2	0.9226	1.0000	0.7026	0.9011	1.0000	1.0000	1.0000	0.8999	3
DMU_3	1.0000	0.5874	1.0000	0.2281	0.7090	1.0000	0.2747	0.6856	7
DMU_4	0.8878	0.1180	0.9292	0.8134	0.2227	1.0000	0.9855	0.7081	6
DMU_5	0.9224	1.0000	0.7727	0.7025	1.0000	1.0000	1.0000	0.9139	1
DMU_6	0.9971	0.8868	0.7625	0.5747	0.9969	1.0000	0.9966	0.8878	5
DMU_7	1.0000	1.0000	0.7661	0.5776	0.9533	1.0000	1.0000	0.8996	4

Table 4.7: Neutral Cross-efficiencies and ranking of DMUs

DMUs	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	Average cross-effficiency	Ranking
DMU_1	1.0000	0.8509	1.0000	0.9033	0.4950	1.0000	0.2941	0.7490	4
DMU_2	0.7368	1.0000	1.0000	0.7904	0.7245	0.9696	0.8029	0.8606	2
DMU_3	0.9363	0.6909	1.0000	0.3279	0.3639	0.8356	0.1936	0.6212	7
DMU_4	0.7186	0.9996	0.7554	0.8134	0.7386	0.9582	0.8848	0.8384	3
DMU_5	0.6626	0.9701	0.7649	0.6684	1.0000	0.9999	1.0000	0.8666	1
DMU_6	0.8065	0.9565	0.8565	0.3589	0.6264	1.0000	0.4909	0.7280	5
DMU_7	0.6754	0.9870	0.0723	0.8093	0.7580	0.4360	1.0000	0.6769	6

Table 4.8: OWA operator weights using Minimax Disparity weights

$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
1.0000	0.0000	0.0000	0.0143	0.0789	0.1429	0.2071	0.2714	0.3600	0.5333	1.0000
0.0000	0.0000	0.0000	0.0571	0.1000	0.1429	0.1857	0.2286	0.2800	0.3333	0.0000
0.0000	0.0000	0.0400	0.1000	0.1214	0.1429	0.1643	0.1857	0.2000	0.1333	0.0000
0.0000	0.0000	0.1200	0.1429	0.1429	0.1429	0.1429	0.1429	0.1200	0.0000	0.0000
0.0000	0.1333	0.2000	0.1857	0.1643	0.1429	0.1214	0.1000	0.0400	0.0000	0.0000
0.0000	0.3333	0.2800	0.2286	0.1857	0.1429	0.1000	0.0571	0.0000	0.0000	0.0000
0.0000	0.5333	0.3600	0.2714	0.2071	0.1429	0.0786	0.0143	0.0000	0.0000	0.0000

Table 4.9: Aggregated Aggressive formulation using OWA operators

	$E^{(A)}_{k(0)}$	$E^{(A)}_{k(0.1)}$	$E^{(A)}_{k(0.2)}$	$E^{(A)}_{k(0.3)}$	$E^{(A)}_{k(0.4)}$	$E^{(A)}_{k(0.5)}$	$E^{(A)}_{k(0.6)}$	$E^{(A)}_{k(0.7)}$	$E^{(A)}_{k(0.8)}$	$E^{(A)}_{k(0.9)}$	$E^{(A)}_{k(1.0)}$
DMU_1	0.0705	0.1897	0.1953	0.2611	0.3423	0.4235	0.5047	0.5852	0.6727	0.7875	1.0000
DMU_2	0.0856	.0.3360	0.4578	0.5491	0.6287	0.7082	0.7878	0.8674	0.9266	0.9795	1.0000
DMU_3	0.0010	0.1372	0.2516	0.3676	0.4734	0.5791	0.6849	0.7907	0.8928	0.9768	1.0000
DMU_4	0.0010	0.3444	0.4880	0.5738	0.6470	0.7202	0.7934	0.8666	0.9072	0.9857	0.9995
DMU_5	0.3680	0.5213	0.6127	0.6789	0.7361	0.7930	0.8506	0.9078	0.9501	0.9854	1.0000
DMU_6	0.0246	0.3528	0.5110	0.6132	0.6931	0.7730	0.8528	0.9327	0.9785	0.9999	1.0000
DMU_7	0.2078	0.3254	0.4292	0.5201	0.6017	0.6834	0.7650	0.8477	0.9206	0.9778	1.0000

Table 4.10: Aggregated Benevolent formulation using OWA operators

	$E^{(B)}_{k(0)}$	$E^{(B)}_{k(0.1)}$	$E^{(B)}_{k(0.2)}$	$E^{(B)}_{k(0.3)}$	$E^{(B)}_{k(0.4)}$	$E^{(B)}_{k(0.5)}$	$E^{(B)}_{k(0.6)}$	$E^{(B)}_{k(0.7)}$	$E^{(B)}_{k(0.8)}$	$E^{(B)}_{k(0.9)}$	$E^{(B)}_{k(1.0)}$
DMU_1	0.5782	0.6917	0.7744	0.8244	0.8624	0.9004	0.9384	0.9765	0.9983	1.0000	1.0000
DMU_2	0.7026	0.7981	0.8498	0.8823	0.9073	0.9323	0.9573	0.9824	0.9969	1.0000	1.0000
DMU_3	0.2281	0.2915	0.4016	0.5065	0.5961	0.6856	0.7751	0.8647	0.9486	1.0000	1.0000
DMU_4	0.1180	0.2456	0.4112	0.5243	0.6162	0.7081	0.80000	0.8918	0.9609	0.9857	1.0000
DMU_5	0.7025	0.6322	0.8137	0.9957	0.8834	0.9139	0.9445	0.9750	0.9969	1.0000	1.0000
DMU_6	0.5747	0.6622	0.7572	0.8083	0.8480	0.8878	0.9276	0.9673	0.9936	0.9986	1.0000
DMU_7	0.5776	0.6905	0.7731	0.8232	0.8914	0.8996	0.9378	0.9759	0.9981	1.0000	1.0000

Table 4.11: Aggregated Neutral formulation using OWA operators

	$E^{(C)}_{k(0)}$	$E^{(C)}_{k(0.1)}$	$E^{(C)}_{k(0.2)}$	$E^{(C)}_{k(0.3)}$	$E^{(C)}_{k(0.4)}$	$E^{(C)}_{k(0.5)}$	$E^{(C)}_{k(0.6)}$	$E^{(C)}_{k(0.7)}$	$E^{(C)}_{k(0.8)}$	$E^{(C)}_{k(0.9)}$	$E^{(C)}_{k(1.0)}$
DMU_1	0.2941	0.4353	0.5631	0.6515	0.7217	0.7919	0.8621	0.9323	0.9824	1.0000	1.0000
DMU_2	0.7245	0.7374	0.7603	0.7949	0.8278	0.8606	0.8934	0.9263	0.9619	1.0000	1.0000
DMU_3	0.1936	0.2611	0.3506	0.4451	0.5331	0.6212	0.7092	0.7972	0.8867	0.9568	1.0000
DMU_4	0.7186	0.7302	0.7496	0.7779	0.8081	0.8384	0.8686	0.8989	0.9329	0.9705	0.8134
DMU_5	0.6626	0.6782	0.7351	0.7847	0.8256	0.8666	0.9075	0.9484	0.9870	1.00000	1.0000
DMU_6	0.3589	0.4386	0.5230	0.5958	0.6619	0.7280	0.7941	0.8602	0.9210	0.9664	1.0000
DMU_7	0.0723	0.2739	0.4065	0.5046	0.5907	0.6769	0.7630	0.8491	0.9162	0.9702	1.0000

Table 4.12: Decision matrix for a university selection

	Criteria		
Alternative	Aggressive	Benevolent	Neutral
A_1	0.6727	0.9983	0.9824
A_2	0.9266	0.9969	0.9619
A_3	0.8928	0.9486	0.8867
A_4	0.9072	0.9609	0.9329
A_5	0.9501	0.9969	0.9870
A_6	0.9785	0.9981	0.9210
A_7	0.9206	0.9981	0.9162

Table 4.13: Cross-efficiency aggregation and Ranking of DMUs

DMU$_k$	$AV_k^{(0.8)}$	Ranking
DMU_1	0.7915	7
DMU_2	0.9511	3
DMU_3	0.9111	6
DMU_4	0.9259	5
DMU_5	0.9668	2
DMU_6	0.9815	1
DMU_7	0.9462	4

Conclusion

In the present work, a new DEA-MCDM algorithmic approach for ranking DMUs has been proposed. The proposed approach includes the subjectivity and preference of the decision maker in more realistic way while computing the ranking results due to a choice of an orness level in [0, 1] by a decision maker. The aggregation of cross-efficiencies in each formulation (Aggressive, Benevolent and Neutral) is done by using ordered weighted averaging (OWA) / induced OWA (IOWA) operators instead of the traditional approach based on simple average. The proposed approach has an advantage that each of the aggressive, benevolent and neutral cross-efficiency formulations contribute to select the best alternative among the DMUs in a MCDM problem and to achieve complete ranking of the DMUs. This new approach can be applied in various real life application areas like R and D projects, flexible manufacturing systems (FMSs), project ranking and preference voting, industrial robot selection, electricity distribution sector, labor assignment, the economic environmental performances and in olympics ranking and benchmarking.

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