Best Approximation and Best Co-approximation in Metric Linear Space and Normed Linear Space

Thesis Submitted in partial fulfillment of the requirement for

The award of the degree of

Masters of Science

In

Mathematics and Computing

Submitted by

Monika Singla

Reg. No. 301503015

Under the guidance of

Dr. Sumit Chandok

School of Mathematics

Thapar University

Patiala-147004 (PUNJAB)

INDIA

July, 2017

Certificate

This is to certify that the thesis entitled ***"Best Approximation and Best Co-approximation in Metric Linear Space and Normed Linear Space"****, being presented in partial fulfillment of the requirements for the award of the degree of Master of Science in the School of Mathematics, Thapar University, Patiala, is a bonafide work carried out under the supervision of* ***Sumit Chandok.***

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

(Monika Singla)

Reg. No. 301503015

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Dr. Sumit Chandok

Assistant Professor, SOM

Thapar University

Patiala

Acknowledgement

It is my genuine pleasure to express my deep sense of thanks and gratitude to my teachers and supervisor **Sumit Chandok**, Assistant Professor, School of Mathematics, Thapar University, Patiala. His immense interest and support helped me to learn and work in a more practical way. It was really a fortunate experience to work under him and enrich from his vast knowledge and analysis power and affection during the course of this project. Finally, I would like to thank all those who knowingly and unknowingly helped me all throughout this period. I would like to express my sincere thanks to **A.K. Lal**, Head, SOM and to the entire faculty and staff members of School of Mathematics for their direct or indirect help, cooperation, love and affection. My sincere heartfelt gratitude to my family whose prayers, best wishes and encouragement has been a constant source of inspiration. I would like to express my deep and sincere gratitude to all other research fellows for their sincere efforts, keen interest and caring nature. Nevertheless, I will always be grateful to my friends and batch mates for their unconditional love and care.

Date: July 17, 2017 **Monika Singla**
Patiala **Roll No.301503015**

Abstract

Approximation theory is an old and rich branch of analysis and a large number of researchers have studied this subject. The theory has many applications in mathematical analysis, nonlinear problems arising in physical sciences, engineering and social sciences. Since the particular examples of approximation often arise from the problems of Science and Technology, they provide proper motivation for the subject of Approximation Theory. In this work, we consider the problem of characterization, existence and uniqueness of best approximation and best co-approximation in the setting of metric linear space and normed linear space.

Contents

Symbols

$B(x_0; r)$	Open ball with radius r and center x_0
$B[x_0; r]$	Closed ball with radius r and center x_0
$\mathbb{C}$	Field of complex numbers
$\mathbb{C}^n$	Unitary n-space
$C[a, b]$	Space of continuous functions
$d(x, y)$	Distance between two points x and y
$D(T)$	Domain of an operator T
$Im(z)$	Imaginary part of z
$Int(G)$	Interior of the set G
inf	Infimum (greatest lower bound)
l_p	A sequence space
L_p	A function space
$\emptyset$	Emptyset
$\mathbb{R}$	Real line or the field of real numbers
$\mathbb{R}^n$	Euclidean n-space
$Re(z)$	Real part of z
sign	Signum function
sup	Supremum (least upper bound)
$\|x\|$	Norm of x
$\langle x, y \rangle$	Inner product of x and y
$x \perp y$	x orthogonal to y
$X \setminus G$	Set X minus the set G

Chapter 1

Introduction and Preliminaries

1.1 Introduction

Approximation theory is an old and rich branch of mathematics. Approximation means "a value or quantity that is nearly but not exactly correct ". The problem of approximative determination of a given quantity is one of the oldest challenges of mathematics, for instance, finding the formula for approximating the square root of a number, finding the formula to approximate the value of transcendental functions at some given point etc. In mathematics, approximation theory is concerned with how functions can best be approximated and with quantitatively characterizing the error introduced thereby. The historical evolution of the methods and results of approximation theory, starting from the work of Euler in 1777 on minimizing distance errors in maps of Russia, and of Laplace in 1843 on determining the ellipsoid best approximating the surface of the earth, and ending with the work of Bernstein. The maps of the atlas are drawn in the cone projection which preserves distances and is attributed to J. Delisle. Because of the enormous size of the Russian empire all known projections had very large errors near the borders of the map, therefore Euler's approach proved helpful. A problem encountered by Laplace was similar in character, where he deal with the question of determining the ellipsoid best approximating the surface of the earth. In 1820 Fourier generalized Laplace's results in his work, where he approximatively solved linear equational systems with more equations than parametres by minimizing the maximum error of every equation. In 1853 Pafnuti Lvovich Chbeyshev was the first to consolidate these considerations into the ' Theory of functions deviating the least possible from zero'. Starting out from the problem of determining the parameters of the driving mechanism of steam-engines-also called Watts

Parallelogram- in such a manner that the conversion of straight into circular movement becomes as exact as possible everywhere, he was led to the general problem of the uniform approximation of a real analytic function by polynomials of a given degree(see [18]). The theory of best approximation is an important topic in functional analysis and has various applications. The problem of best approximation is defined as:
Let X be a normed linear space, and G be its non empty subset. An element $g_0 \in G$ is called best approximation to $x \in X$ from G if g_0 is closest to x among all the elements of G, that is, $\|x - g_0\| \leq \|x - g\|$ for all $g \in G$.
The set of all best approximations of $x \in X$ from set G is denoted by $P_G(x)$. If $P_G(x) \neq \emptyset$ for each $x \in X$, then the set G is called **proximinal set**. If $P_G(x)$ contains only single element for each x, then G is called **Chebyshev set**, after the name of the Russian Mathematician Pafnuti Lavovich Chebyshev.
This thesis consists of mainly three chapters. In the first chapter, we discuss some basic definitions and concepts related to different spaces. In the next chapter, we discuss about best approximation in metric linear space and normed linear space. In the last chapter, we discuss about another tool of approximation theory, called **"Best co-approximation"** in both spaces. In normed linear space, an element $g_0 \in G$ is said to be the best co-approximation to $x \in X$ from G, if $\|g - g_0\| \leq \|x - g_0\|$. In 1972, Franchetti and Furi introduced best co-approximation in normed linear space(see [3]). Subsequently, this theory has been developed to a large extent in normed linear spaces, metric linear spaces and Hilbert spaces by C. Franchetti and Furi (see [3]), H. Mazaheri (see [8], [9]), T. D. Narang (see [13], [12]), P. L. Papini and Ivan Singer (see [14]), Geeta S. Rao (see [15]) and by many other researchers. In the present work, we concentrate on the following three fundamental questions of best approximation (best co-approximation):(see [4], [13])

1. **Characterization of best approximation (best co-approximation)**: How to recognize whether a given element is best approximation (best co-approximation) to $x \in X$ or not?

2. **Existence of best approximation (best co-approximation)**: Which subsets are proximinal?

3. **Uniqueness of best approximation (best co-approximation)**: Which subsets are Chebyshev?

Now we discuss some basic definitions, important concepts and notations which we require in the sequel.

1.2 Metric Space.

Definition 1.1. [7] A metric space is a pair (X, d) where X is a non-empty set and d is a metric defined on X (or distance on X), that is a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

1. d is real-valued,finite and non-negative.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$ (symmetry).
4. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

Example 1.1. *[7] Let $C[a, b]$ be the set of all real valued functions which are defined and continuous on $[a, b]$. Take $x, y \in C[a, b]$, which are the functions of an independent real variable t. Here metric is defined as*

$$d_{\infty}(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|.$$

Definition 1.2. [7] Let X be a non-empty set and define the metric on it as

$$d(x, y) = \begin{cases} 0 \text{ when } x = y \\ 1 \text{ when } x \neq y \end{cases}$$

which is known as **discrete metric**. The space (X, d) is known as discrete metric space.

Definition 1.3. [19] A metric space (X, d) is said to be **convex** if for every pair $x, y \in X$ and for every λ where $0 \leq \lambda \leq 1$, there exists at least one point $z \in X$ such that $d(x, z) = \lambda d(x, y)$ and $d(z, y) = (1 - \lambda) d(x, y)$.

Example 1.2. *A circle is a convex metric space, if the distance between two points is defined as the length of the shortest arc on the circle connecting them.*

Definition 1.4. [19] A metric space is said to be **strictly convex** if $d(x, 0) \leq t$, $d(y, 0) \leq t$ imply $d(\frac{x+y}{2}, 0) < t$ unless $x = y$, where $x, y \in X$ and t is any positive real number.

Definition 1.5. [7]**(Distance from a point to a set and distance between two sets).**
Let (X, d) be a metric space. The distance from a point 'a' to the non-empty set A of X is defined as

$$d(a, A) = \inf_{b \in A} d(a, b)$$

whereas the distance between two non-empty sets A and B is defined as

$$d(A,B) = \inf_{a\in A, b\in B} d(a,b).$$

Definition 1.6. [7] In a metric space (X,d), a sequence $\{x_m\}$ is said to be **convergent** if there exist an element $x \in X$ such that

$$\lim_{m\to\infty} d(x_m, x) = 0.$$

Definition 1.7. [7] Let (X,d) be a metric space, then a sequence $\{x_n\}$ in X is said to be **Cauchy** if for every $\epsilon > 0$ there exists an integer N such that

$$d(x_m, x_n) < \epsilon \qquad \text{for every } m,n > N.$$

If every Cauchy sequence in X converges (that is, limit of the sequence is a point of X) then X is said to be **complete**.

Definition 1.8. [7] A mapping T of a metric space (X, d_X) into a metric space (Y, d_Y) is **continuous** at a point x_0 if and only if $x_n \longrightarrow x_0$ in (X, d_X) implies $T(x_n) \longrightarrow T(x_0)$ in (Y, d_Y).

1.3 Normed Linear Space.

In this section, we define vector space, normed linear space and some of their properties.

Definition 1.9. [7] A vector space (or linear space) over a *field* K($\mathbb{R}$ or $\mathbb{C}$) is a nonempty set X of vectors $x, y, \dots$ together with two algebraic operations. These operations are called *vector addition* and *multiplication of vectors* by *scalars*, that is, by elements of K.

Vector addition associates with every ordered pair (x,y) of vectors in such a way that the following properties hold:

$$x + y = y + x$$

$$x + (y + z) = (x + y) + z;$$

where a vector $x + y$ called the **sum** of x and y. Furthermore, there exists a vector 0, called the **zero vector**, such that

$$x + 0 = x$$

$$x + (-x) = 0.$$

Multiplication by scalars associates with every vector x and scalar α a vector αx (also written xα), called the *product* of α and x, in such a way that for all vectors x, y and scalars α, β we have

$$\alpha(\beta x) = (\alpha\beta)x$$

$$1x = x$$

and the distribution laws

$$\alpha(x + y) = \alpha x + \alpha y$$

$$(\alpha + \beta)x = \alpha x + \beta x$$

From the definition we see that vector addition is a mapping $X \times X \longrightarrow X$, whereas multiplication by scalars is a mapping $K \times X \longrightarrow X$. If $K = \mathbb{R}$, then X is called a **real vector space** and if $K = \mathbb{C}$, then X is called a **complex vector space**.

Example 1.3. ***Space C[a, b]:*** *The set of real-valued continuous functions on $[a, b]$ forms a real vector space with algebraic operations which are defined as:*

$$(x + y)(t) = x(t) + y(t)$$

$$(\alpha x)(t) = \alpha x(t)$$

where x and y are continuous functions.

Definition 1.10. [7] A **normed space** $(X, \|\cdot\|)$ is a vector space X with a norm $\|\cdot\|$ defined on it.
A **norm** is a real-valued function defined on a vector space X (real or complex) which satisfies the following properties:

1. $\|x\| \geq 0$
2. $\|x\| \Leftrightarrow x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

where $x, y \in X$ are arbitrary vectors and α is a scalar. On vector space X, a metric d is defined by norm as

$$d(x, y) = \|x - y\| \text{ where } x, y \in X$$

and is called *metric induced by the norm.*

If the normed space is complete with respect to the metric induced by the norm, then space is called **Banach space.**

Example 1.4. *[7]**Euclidean space** $\mathbb{R}^n$ **and unitary space** $\mathbb{C}^n$.*

These spaces are Banach spaces with norm defined by

$$\|x\| = (\sum_{j=1}^{n} |\xi_j|^2)^{\frac{1}{2}} = \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2}.$$

and yields the metric

$$d(x, y) = \|x - y\| = \sqrt{|\xi_1 - \eta_1|^2 + \cdots + |\xi_n - \eta_n|^2}.$$

We note in particular that in $\mathbb{R}^3$ *we have*

$$\|x\| = |x| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.$$

This confirms that the norm generalizes the elementary notion of length $|x|$ *of a vector.*

Note 1.3.1. *The norm is a continuous mapping i.e. the mapping* $x \longmapsto \|x\|$ *is continuous.*

Now we discuss *Translation invariance*, which we use widely in the next chapters.

Lemma 1.11. *[7]**(Translation invariance).** Let* $(X, \|\cdot\|)$ *be a normed linear space. A metric* d *induced by a norm satisfies*

(a) $d(x + a, y + a) = d(x, y)$

(b) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$

for all $x, y, a \in X$ *and every scalar* α.

Theorem 1.12. *[1] Let* X *be a normed linear space and* G *be its non-empty subspace. Then,*

(i) $d(x + g, G) = d(x, G)$ *for every* $g \in G$ *and* $x \in X$.

(ii) $d(\alpha x, G) = |\alpha| d(x, G)$ *for any scalar* $\alpha \in \mathbb{R}$ *and* $x \in X$.

(iii) $d(x + y, G) \leq d(x, G) + d(y, G)$ *for every* $x, y \in X$.

A linear space X together with a translation invariant metric d (i.e. $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in X$) such that addition and scalar multiplication are continuous in (X, d) is called a **metric linear space**.
Lemma 1.11 and Theorem 1.12 help in proving many results in metric linear space.

Note 1.3.2. *[13] Every normed linear space is a metric linear space but converse need not be true.*

Definition 1.13. [4] A subset M of a vector space X is said to be **convex** if for each $x, y \in M, z = \lambda x + (1-\lambda)y \in M$ where $0 \leq \lambda \leq 1$.

Remark 1.14. [7] The *closed unit ball* $B[0;1] = \{x \in X \mid \|x\| \leq 1\}$ in a normed space is convex.
Proof: Let $x, y \in B[0;1]$ and $z = \lambda x + (1-\lambda)y$, where $0 \leq \lambda \leq 1$. Then,

$$\begin{aligned} \|z\| &= \|\lambda x + (1-\lambda)y\| \\ &\leq \|\lambda x\| + \|(1-\lambda)y\| \\ &= |\lambda|\|x\| + |1-\lambda|\|y\| \\ &\leq \lambda + (1-\lambda) \\ &= 1 \end{aligned}$$

which implies $\|z\| \leq 1$ and hence, $z \in B[0;1]$. This completes the proof.

Definition 1.15. [7] A normed space $(X, \|\cdot\|)$ is said to be a **strictly convex normed space (or rotund)** if for all x, y $(x \neq y)$ of norm 1

$$\|x+y\| < 2$$

Equality holds only when $x = y$.

Most common examples of strictly convex normed spaces are L_p and l_p spaces for any $1 < p < \infty$. The spaces L_p with norm $\|f\| = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}}$ are strictly convex when $1 < p < \infty$(see [2]).

Definition 1.16. [7] Let (X, d) be a metric space and G be its subset. Then the **boundary of a set** G is defined as closure minus the interior of G and is denoted by $bd(G)$. Also **Closure of a set** is defined as the union of the given set and its limit points that is, $cl(G) = G \cup G'$ (where G' denotes set of limit points and $cl(G)$ denotes the closure of G).

Next two theorems are related to convergence and completeness that needed in the sequel.

Theorem 1.17. *[7] Let G be a non-empty subset of a metric space (X, d). Then*

(i) $x \in cl(G)$ *if and only if there is a sequence* $\{x_n\}$ *in G such that* $x_n \longrightarrow x$.

(ii) *G is closed if and only if the sequence* $\{x_n\} \in G$ *and* $x_n \longrightarrow x$ *implies that* $x \in G$.

Theorem 1.18. *[7] Let X be a Banach space and G be its subspace. Then G is complete if and only if G is closed.*

Definition 1.19. [4] Let X be a vector space and G be its subset. Then for $x \in X$ we define $G + x = \{g + x : g \in G\}$ and $\alpha G = \{\alpha g : g \in G\}$ for a scalar α.

Definition 1.20. [7] A vector space X is said to be the **direct sum** of two subspaces Y and Z of X, if each $x \in X$ has a unique representation $x = y + z$, where $y \in Y$ and $z \in Z$ and it is represented as $X = Y \oplus Z$. Here Z is called an **algebraic complement** of Y in X and vice versa.

Definition 1.21. [7] Let V be a subspace of a linear space X. The coset of an element $x \in X$ with respect to V, denoted to the set $x + V = \{x + V : x \in X\}$. Linear operations on V/X are defined by $(x + V) + (y + V) = (x + y) + V$, $\alpha(x + V) = \alpha x + V$, where $x, y \in X$ and $\alpha \in \mathbb{R}$. V/X is a linear space over $\mathbb{R}$. The space is called the **quotient space** of X by V.

Remark 1.22. [7]Let X and Y be normed spaces and $T : D(T) \longrightarrow Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be **bounded** if there is a real number k such that for all $x \in D(T)$,

$$\|Tx\| \leq k\|x\|.$$

Definition 1.23. [7] A **bounded linear functional** f is a bounded linear operator. The range of f lies in the the scalar field of normed linear space X in which the $D(f)$ lies. Thus there exists a real number k such that for all $x \in D(f)$,

$$|f(x)| \leq k\|x\|.$$

Notation 1.3.1. *[7] Let X be a normed space. Then the set of all bounded linear functionals on X constitutes normed space with norm defined by*

$$\|f\| = \sup_{x \in X, x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x \in X, \|x\| = 1} |f(x)|,$$

which is called ***dual space*** *of X and is denoted by* X'.

Remark 1.24. Let q is a real-valued functional defined on normed space X. If q is *subadditive*, that is,

$$q(x+y) \leq q(x) + q(y)$$

for all $x, y \in X$, and *positive-homogeneous*, that is,

$$q(\alpha x) = \alpha q(x)$$

where $\alpha \geq 0$ and belongs to $\mathbb{R}$, then q is said to be **sublinear functional** on X.

Theorem 1.25. *[7] Let q be a sublinear functional on a real vector space X. Also let f be a functional which is defined on a subspace Y of X and satisfies*

$$f(x) \leq q(x) \tag{1}$$

for all $x \in Y$. Then f has a linear extension $\hat{f}$ from Y to X satisfying

$$\hat{f}(x) \leq q(x) \tag{2}$$

for all $x \in X$, that is, $\hat{f}$ is a linear functional on X, satisfies (2) on X and $\hat{f}(x) = f(x)$ for every $x \in Y$.

Now, we discuss *Hahn-Banach Theorem*, which is used in some of the results.

Theorem 1.26. *[7] Let f be a bounded linear functional on a subspace Y of a normed linear space X. Then there exists a bounded linear functional $\hat{f}$ on X which is an extension of f to X and has the same norm,*

$$\|\hat{f}\|_X = \|f\|_Y \tag{3}$$

where

$$\|\hat{f}\|_X = \sup_{x \in X}\{|\hat{f}(x)| : \|x\| = 1\} \qquad \|f\|_Y = \sup_{x \in Y}\{|f(x)| : \|x\| = 1\}.$$

Proof. If $Y = \{0\}$, then $f = 0$ and the extension is $\hat{f} = 0$. So, let $Y \neq \{0\}$. We need Theorem 1.25 to prove this result. Hence, we have to first find a suitable q. Now, we have

$$|f(x)| \leq \|f\|_Y \|x\|$$

for all $x \in Y$. This is of the form (1), where $q(x) = \|f\|_Y \|x\|$ and q is defined on all X. Also,

$$q(x+y) = \|f\|_Y \|x+y\| \leq \|f\|_Y (\|x\| + \|y\|) = q(x) + q(y)$$

and

$$q(\alpha x) = \|f\|_Y \alpha \|x\| = |\alpha| \|f\|_Y \|x\| |\alpha| q(x).$$

Hence q is a sublinear functional on X. Now, by Theorem 1.25, there exists a linear functional $\hat{f}$ on X which is an extension of f and satisfies

$$|\hat{f}(x)| \leq q(x) = \|f\|_Y \|x\|$$

where $x \in X$. Taking supremum over all $x \in X$ of norm 1, we get the following inequality

$$\|\hat{f}\|_X = \sup_{x \in X, \|x\|=1} |\hat{f}(x)| \leq \|f\|_Y.$$

Also, $\|\hat{f}\|_X \geq \|f\|_Y$ as norm cannot decrease, under an extension. Hence, we obtain (3), which completes the proof. □

Theorem 1.27. *[7] Let X be a normed space and let $x_0 \neq 0$ be any element of X. Then there exists a bounded linear functional $\hat{f}$ on X such that*

$$\|\hat{f}\| = 1 \qquad \hat{f}(x_0) = \|x_0\|.$$

Proof. Let Y be the subspace of X consisting of elements of type $x = \alpha x_0$, where α is a scalar. Now, define a linear functional f on Y by

$$f(x) = f(\alpha x_0) = \alpha \|x_0\| \tag{4}$$

which implies f is bounded. Also,

$$|f(x)| = |f(\alpha x_0)| = |\alpha| \|x_0\| = \|\alpha x_0\| = \|x\|$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} = 1.$$

Hence, by Theorem 1.26, f has a linear extension $\hat{f}$ from Y to X, of norm $\|f\| = \|\hat{f}\| = 1$. From (4), we see that $\hat{f}(x_0) = f(x_0) = \|x_0\|$. □

Definition 1.28. [4] Let X be a normed linear space. By Hahn Banach Theorem, for each $x \in X$ we have $f \in X'$ such that $\|f\| = 1$ and $f(x) = \|x\|$.
X is said to be **smooth normed linear space** if for each $x \in X$ such f is unique .

Theorem 1.29. *[7] Let X be a normed space. For every x in a normed space X, we have*

$$\|x\| = \sup_{f \in X'} \left\{ \frac{|f(x)|}{\|f\|} | f \neq 0 \right\}.$$

*If x_0 is such that $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$ and x_0 is called the **zero vector**.*

Proof. From Theorem 1.27 we have, writing x for x_0,

$$\sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|} \geq \frac{|\hat{f}(x)|}{\|\hat{f}\|} = \frac{\|x\|}{1} = \|x\|,$$

and from $|f(x)| \leq \|f\|\|x\|$ we get

$$\sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|} \leq \|x\|.$$

Hence,

$$\sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|} = \|x\|.$$

□

1.4 Inner Product Space.

Definition 1.30. [7] An **inner product space** (or pre Hilbert space) is a vector space with an inner product defined on X. Here an **inner product** on X is a mapping of $X \times X$ into the scalar field K of X, that is, with every pair of vectors x and y there is associated a scalar which is written as $\langle x, y \rangle$ and satisfies the following properties:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in X$ and scalar α.

A complete inner product space (complete in the metric defined by the inner product) is called ***Hilbert space***. Norm on X defined by an inner product on X is given by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

and a metric on X given by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}.$$

Example 1.5. *[7]* ***Euclidean space*** $\mathbb{R}^n$**.** *The space* $\mathbb{R}^n$ *is a Hilbert space with inner product defined by*

$$\langle x,y\rangle = \xi_1\eta_1 + \cdots + \xi_n\eta_n$$

where $x=(\xi_j)=(\xi_1,\cdots,\xi_n)$ *and* $y=(\eta_j)=(\eta_1,\cdots,\eta_n)$*. Also norm on X is given as*

$$\|x\| = \langle x,x\rangle^{\frac{1}{2}} = (\xi_1^2 + \cdots + \xi_n)^{\frac{1}{2}}$$

and from this the Euclidean metric defined by

$$d(x,y) = \|x-y\| = \langle x-y, x-y\rangle^{\frac{1}{2}} = [(\xi_1-\eta_1)^2 + \cdots + (\xi_n-\eta_n)^2]^{\frac{1}{2}}.$$

Note 1.4.1. *An inner product space is a continuous function.*

Definition 1.31. [7]An element $x \in X$ is said to be **orthogonal** to an element $y \in X$ if

$$\langle x,y\rangle = 0.$$

It is denoted by $x \perp y$. Similarly, for subsets $A, B \subset X$ we write $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and $b \in B$.

Definition 1.32. [7]**(Pythagorean theorem).** If $x \perp y$ in an inner product space X, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

There are many different ways to express the orthogonality of two vectors in real inner product space. We discuss one of them, that is **Birkhoff orthogonality** which is of our interest.

Definition 1.33. [4] Let X be a normed linear space. Two vectors $x, y \in X$ are said to be **Birkhoff orthogonal** if and only if $\|x+\alpha y\| \geq \|x\|$ for all scalar α. Symbolically, it is represented as $x \perp_B y$.
In general, two sets G_1 and G_2 are said to be **Birkhoff orthogonal** if and only if $g_1 \perp_B g_2$ for all $g_1 \in G_1$ and $g_2 \in G_2$.

Proposition 1.34. *[4] Birkhoff orthogonality is homogeneous, that is, if* $x \perp_B y$*, then* $x \perp_B \lambda y$*, for all* $\lambda \in \mathbb{R}$.

Proof. Let $x \perp_B y$, which implies $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{R}$.

$$\begin{aligned} \|x + \mu(\lambda y)\| &= \|x + (\mu\lambda)y\| \quad where(\mu \in \mathbb{R}) \\ &= \|x + \alpha y\| \quad (\mu\lambda = \alpha) \\ &\geq \|x\| \end{aligned}$$

Hence, Birkhoff orthogonality is homogeneous. □

Chapter 2

Best Approximation in Metric Linear Space and Normed Linear Space

In this chapter, we discuss about *best approximation* in metric linear space and normed linear space. First, we discuss about best approximation and then define some terms like proximinal set, Chebyshev set etc. We also concerned with the *characterization*, *existence* and *uniqueness* of best approximation in metric linear space and normed linear space.

2.1 Best approximation in metric linear space.

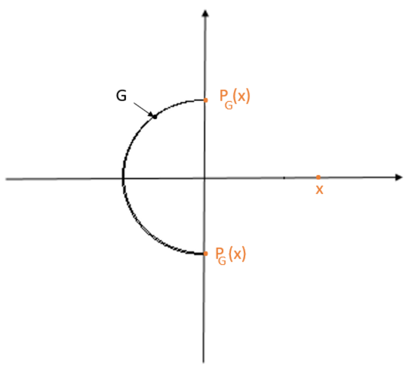
Definition 2.1. [13] Let (X, d) be a metric space and G be its closed subset. An element $g_0 \in G$ is said to be best approximation to $x \in X$ if

$$d(x, g_0) \leq d(x, g) \qquad (\forall g \in G\ ).$$

The set of all such elements that are best approximation to $x \in X$ is denoted by $P_G(x)$. If $P_G(x) \neq \emptyset$, then the set is called a **proximinal set.**

Example 2.1. *[13] Let $X = \mathbb{R}^2$ be a metric space with metric $d(x, y) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ where $x = (x_1, y_1)$ and $y = (x_2, y_2)$. Then $G = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$ is a proximinal subset of X.*

Example 2.2. *Let $\mathbb{R}^2$ be a metric space with metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $G = \{x = -\sqrt{1 - y^2}, -1 \leq y \leq 1\}$ be its subset.*

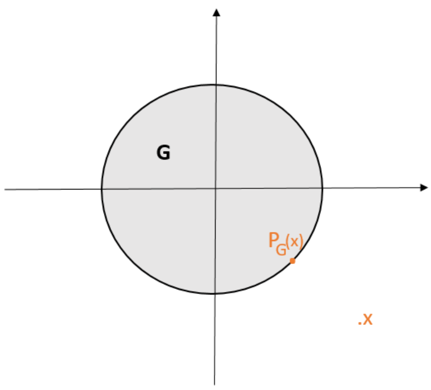


Since every point of $\mathbb{R}^2$ *has atleast one best approximation in G, therefore set G is a* proximinal set.

Definition 2.2. [13] If $P_G(x)$ contains exactly one element for each $x \in X$, then the set G is called **Chebyshev set.**

Example 2.3. *[13] Let* $X = \mathbb{R}$ *be a metric space with metric* $d(x, y) = |x - y|$ *and* $G = [1, 2]$ *be its subset. Then G is a Chebyshev subset of X.*

Example 2.4. *[2] Consider* $G = B[0; 1] \subseteq \mathbb{R}^2$ *equipped with metric* $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ *where* $x = (x_1, y_1)$ *and* $y = (x_2, y_2)$.



It is easy to check that $P_G(x) = \{\frac{x}{d(x,0)}\}$, *for any* $x \in \mathbb{R}^2 \setminus G$*. Therefore, G is a Chebyshev set.*

In metric linear space, **metric projection** for a proximinal set G is defined as the mapping from the set X to the power set of G, i.e. $P_G(x) : X \to 2^G$, whereas for the chebyshev set G it is defined as the mapping from the set X to the set G, i.e. $P_G(x) : X \to G$.

Definition 2.3. [12] Let (X, d) be a metric linear space. An element $x \in X$ is said to be **orthogonal** to an element $y \in X$ i.e. $x \perp y$ if $d(x, 0) \leq d(x, \alpha y)$ for each scalar α. Let G

be a subset of X, then G is said to be orthogonal to some element $x \in X$ i.e. $G \perp x$ if $d(g,0) \leq d(g, \alpha x)$ for all $g \in G$ and scalar α.

Notation 2.1.1. *[13] Let (X,d) be a metric space and G be its non-empty set. Then we define the set $P_G^{-1}(0)$ as*

$$\breve{G} = P_G^{-1}(0) = \{x \in X : d(x,0) = d(x,G)\}$$

2.1.1 Characterization of best approximation in metric linear space.

In this section, we recognize whether the given element $g \in G$ is the best approximtaion of $x \in X$ or not.

Theorem 2.4. *Let (X,d) be a metric space and G be its non-empty subspace. Then,*

(i) If $x \in G$, then $P_G(x) = \{x\}$.

(ii) If $x \in cl(G) \setminus G$, then $P_G(x) = \emptyset$.

Proof. (i) Let $x \in G$, so $d(x,G) = 0$. Also, $P_G(x) = \{g \in G : d(x,g) = d(x,G)\} = \{g \in G : d(x,g) = 0\}$. Hence $P_G(x) = \{x\}$.

(ii) Let $x \in cl(G) \setminus G$. Then, by Theorem 1.17, there exists a sequence $\{x_n\}$ of elements in G such that $x_n \longrightarrow x$, that is $d(x_n, x) \to 0$ as $n \to \infty$. Hence, $d(x,G) = 0$. Therefore,

$$\begin{aligned} P_G(x) &= \{g \in G : d(x,g) = d(x,G)\} \\ &= \{g \in G : d(x,g) = 0\} \\ &= \{g \in G : x = g\} \\ &= \emptyset. \end{aligned}$$

□

Theorem 2.5. *[19] Let (X,d) be a real metric linear space and G be its closed linear subspace. Let $x_0 \in X \setminus G$, then $g_0 \in P_G(x)$ iff there exists $h \in X^{\varphi}$[1], with the following properties:*

(i) $|h(x) - h(y)| \leq d(x,y)$ where $x, y \in X$

[1] $X^{\varphi} = \left\{ f : X \to \mathbb{R} \mid \sup_{x \in X - \{0\}} \frac{|f(x)|}{d(x,0)} < \infty, f(0) = 0, \text{ f is subadditive.} \right\}$

(ii) $h(x+g) = h(x)$, *for each* $x \in X, g \in G$ *or* $h|_G = 0$

(iii) $h(x_0 - g_0) = h(x_0) = d(x_0, g_0)$.

Proof. Let $g_0 \in P_G(x)$. Now define a function h as follows: $h : X \to \mathbb{R}$ such that $h(x) = d(x + g_0, G)$ for all $x \in X$. Clearly, $h(0) = d(g_0, G) = 0$ and

$$\begin{aligned} \sup_{x\in X, x\neq 0} \frac{|h(x)|}{d(x,0)} &= \sup_{x\in X, x\neq 0} \frac{|d(x+g_0,G)|}{d(x,0)} \\ &\leq \sup_{x\in X, x\neq 0} \frac{|d(x,G)|}{d(x,0)} \\ &< \infty. \end{aligned}$$

Also,

$$\begin{aligned} h(x+y) &= d(x+y+g_0, G) \\ &= d(x+y+2g_0, G+g_0) \\ &\leq d(x+g_0, G+g_0) + d(y+g_0, G+g_0) \quad \text{(by using Theorem 1.12)} \\ &= h(x) + h(y) \end{aligned}$$

which implies h is subadditive. Hence $h \in X^{\varphi}$. Now,

$$\begin{aligned} h(x+g) &= d(x+g+g_0, G) \\ &= d(x+g_0, G-g) \\ &= d(x+g_0, G) \\ &= h(x). \end{aligned}$$

Thus $d(x_0, g_0) = d(x_0, G) = h(x_0 - g_0) = h(x_0)$. To prove the first property, consider

$$\begin{aligned} h(y) &= d(y+g_0, G) \\ &= \inf_{g\in G} d(y+g_0, g) \\ &\leq \inf_{g\in G} d(y+g_0, x+g_0) + \inf_{g\in G} d(x+g_0, g) \text{ (by using triangle inequality)} \\ &= d(y,x) + \inf_{g\in G} d(x+g_0, g) \\ &= d(y,x) + h(x). \end{aligned}$$

So we have $h(y) - h(x) \leq d(y,x)$. Now interchange the role of x and y, and we have $|h(x) - h(y)| \leq d(x,y)$ where $x, y \in X$.

Conversely, let there exists an $h \in X^\varphi$ satisfying all the three properties. Then $d(x_0, g_0) = h(x_0 - g_0) = h(x_0) = h(x_0) - h(g) \leq d(x_0, g)$ for all $g \in G$. Hence $g_0 \in P_G(x_0)$, which completes the proof. □

Lemma 2.6. *[19] Let G be a closed linear subspace of be a real metric linear space (X, d). Let $x \in X \setminus G$ and $M \subset G$, then $M \subset P_G(x)$ if and only if there exists an $h \in X^\varphi$ satisfying*

(i) $|h(x) - h(y)| \leq d(x, y)$ $x, y \in X$

(ii) $h|_G = 0$

(iii) $h(x - g) = d(x, g)$ *for all* $g \in M$.

Proof. Suppose $M \subset P_G(x)$ and let $g_0 \in M$. Then $g_0 \in P_G(x)$ and by Theorem 2.5, there exists $h \in X^\varphi$ such that

$$|h(x) - h(y)| \leq d(x, y) \text{ for all } x, y \in X, \quad h|_G = 0$$

and

$$h(x - g_0) = d(x, g_0), \quad g_0 \in M$$

Now let $g \in M$, then $g \in P_G(x)$. Consider

$$\begin{aligned} h(x - g) &= h(x - g_0) \\ &= d(x, g_0) \\ &= d(x, g) \quad (\text{as } d(x, g_0) = d(x, g) = d(x, G)). \end{aligned}$$

Hence there exists an $h \in X^\varphi$ which satisfies all given three properties. Conversely, let there exists an $h \in X^\varphi$ satisfying $(i), (ii)$ and (iii). Now we show that $M \subset P_G(x)$. Let $g_0 \in M$. Then $g_0 \in P_G(x)$ by Theorem 2.5. Hence $M \subset P_G(x)$. □

Lemma 2.7. *[13] Let G be a subspace of a metric linear space (X, d). Then $g_0 \in P_G(x)$ if and only if $x - g_0 \in \breve{G}$ for all $x \in X$.*

Proof. Let $g_0 \in P_G(x)$ if and only if $d(x, g_0) \leq d(x, g)$ for all $g \in G$ if and only if $d(x - g_0, 0) \leq d(x - g_0, g - g_0)$ if and only if $d(x - g_0, 0) \leq d(x - g_0, g_1)$ for all $g_1 \in G$ if and only if $x - g_0 \in \breve{G}$. □

Theorem 2.8. *[13] Let (X, d) be a metric linear space and G be a linear subspace of X. If $P_G(x)$ is the set of all best approximation of $x \in X$, then $P_G(x) = G \cap (x - \breve{G})$.*

Proof. Let $g_0 \in G \cap (x - \breve{G})$ if and only if $g_0 \in G$, and $g_0 \in (x - \breve{G})$ if and only if $g_0 \in G$ and $g_0 = x - \breve{x}$ for some $\breve{x} \in \breve{G}$ if and only if $g_o \in G$ and $\breve{x} = x - g_0 \in \breve{G}$ if and only if $g_0 \in P_G(x)$, (by Lemma 2.7). Therefore, $P_G(x) = G \cap (x - \breve{G})$. □

Theorem 2.9. *Let (X, d) be a metric linear space and G be a non-empty subset of X. Then $P_{G+u}(x + u) = P_G(x) + u$ for every $u \in X$.*

Proof. Let $u_0 \in P_{G+u}(x + u)$, then $d(x + u, u_0) \leq d(x + u, g + u)$ for every $g + u \in G + u$. Now,

$$\begin{aligned} d(x, u_0 - u) &= d(x + u, u_0) \\ &\leq d(x + u, g + u) \\ &= d(x, g) \end{aligned}$$

which implies $d(x, u_0 - u) \leq d(x, g)$ and we have, $u_0 - u \in P_G(x)$. Thus $u_0 \in P_G(x) + u$ and $P_{G+u}(x + u) \subseteq P_G(x) + u$.
Conversely let, $g_0 \in P_G(x) + u$. This implies $g_0 - u \in P_G(x)$, then $d(x, g_0 - u) \leq d(x, g)$ for all $g \in G$. Now,

$$\begin{aligned} d(x + u, g_0) &= d(x, g_0 - u) \\ &\leq d(x, g) \\ &= d(x + u, g + u) \end{aligned}$$

for every $g + u \in G + u$. This implies $g_0 \in P_{G+u}(x + u)$, and thus $P_G(x) + u \subseteq P_{G+u}(x + u)$. Hence $P_{G+u}(x + u) = P_G(x) + u$ for every $u \in X$. □

Our next result is related to the characterization of semi-Chebyshev subspaces in the framework of real metric linear space.
Let G be a linear subspace of metric linear space (X, d). Then G is said to be **semi-Chebyshev** if for every $x \in X$, the set $P_G(x)$ contains at most one element.

Theorem 2.10. *[19] Let G be a linear subspace of real metric linear space (X, d). Then the following statements are equivalent:*

(i) G is semi-Chebyshev

(ii) There exists an $h \in X^{\varphi}$ and $x_1, x_2 \in X$ with $x_1 - x_2 \in G \setminus \{0\}$ such that

$$|h(x)| \leq d(x, 0), \quad x \in X \tag{2.1}$$

$$h|_G = 0 \tag{2.2}$$

$$h(x_1) = d(x_1, 0), \;\; h(x_2) = d(x_2, 0) \tag{2.3}$$

(iii) There exists no $h \in X^\varphi$, $x \in X$ and $g_0 \in G \setminus \{0\}$ with the properties (2.1), (2.2) and

$$h(x) = d(x, 0) = d(x, g_0). \tag{2.4}$$

Proof. First, consider G is a semi-Chebyshev subspace. Assume that there exist $h \in X^\varphi$ and x_1, $x_2 \in X$ with $x_1 - x_2 \in G \setminus \{0\}$, such that we have (2.1), (2.2) and (2.3). Put $g_0 = x_1 - x_2$. Then,

$$h(x_1 - g_0) = h(x_2) = d(x_2, 0) = d(x_1 - g_0, 0). \tag{2.5}$$

Now, $x_1 \in X \setminus cl(G)$ as if $x_1 \in cl(G)$, then from (2.2), (2.3) and (2.5) we have $0 = h(x_1) = d(x_1, 0)$, $0 = h(x_1 - g_0) = d(x_2, 0)$, which contradicts $x_2 - x_1 \neq 0$. From, Theorem 2.5 we have, $g_0 \in P_G(x)$ and $0 \in P_G(x)$. Since, $g_0 \neq 0$, it follows that G is not a semi-Chebyshev subspace, which is a contradiction. Hence, there do not exist $h \in X^\varphi$ and x_1, $x_2 \in X$ with $x_1 - x_2 \in G \setminus \{0\}$, such that we have (2.1), (2.2) and (2.3). Thus $(i) \Rightarrow (ii)$.

Now assume that there exist $h \in X^\varphi$, $x \in X$ and $g_0 \in G \setminus \{0\}$, such that we have (2.1), (2.2) and (2.4). By putting $x_1 = x$ and $x_2 = x - g_0$ in (2.4), we get $h(x_1) = d(x_1, 0)$ and $h(x) = d(x_2, 0)$. Now by (2.2), we have $h(x_2) = h(x - g_0) = h(x) = d(x_2, 0)$. Thus we have (2.1), (2.2) and (2.3) and also $x_1 - x_2 = g_0 \in G \setminus \{0\}$. Hence $(ii) \Rightarrow (iii)$.

Now suppose that G is not a semi-Chebyshev subspace. For a suitable $y \in X \setminus cl(G)$ there exist $g_1, g_2 \in P_G(x)$, such that $g_1 \neq g_2$. By putting

$$x = y - g_1, \;\; g_0 = g_2 - g_1,$$

we have $x \in X \setminus cl(G)$, $g_0 \in G \setminus \{0\}$ and $0, g_0 \in P_G(x)$. By Lemma 2.6, there exist an $h \in X^\varphi$ such that we have (2.1), (2.2) and (2.4), which is a contradiction. Thus $(iii) \Rightarrow (i)$. □

2.1.2 Existence of best approximation in metric linear space.

In this section, we discuss some conditions under which the given set is proximinal.

Theorem 2.11. *Let X be a metric linear space and G be its non-empty subset. Then $G + u$ is proximinal if and only if G is proximinal for any given $u \in X$.*

Proof. First assume that G is proximinal. Then G is proximinal if and only if $P_G(x) \neq \emptyset$ if and only if $P_G(x) + u \neq \emptyset$ if and only if $P_{G+u}(x+u) \neq \emptyset$ (by Theorem 2.9) if and only if $G + u$ is proximinal.

□

Theorem 2.12. *[12] Let (X, d) be a metric linear space and G be its subspace. Then following are equivalent:*

(i) G is proximinal

(ii) $X = G + \breve{G}$.

Proof. Let G is proximinal, $x \in X$ and $g_0 \in P_G(x)$. Then $(x - g_0) \in \breve{G}$ and $g_0 \in G$, (by Lemma 2.7). Now, $x = g_0 + (x - g_0) \in G + \breve{G}$. Hence $X = G + \breve{G}$.
Conversly, let $X = G + \breve{G}$ and let $x \in X$. Then $x = g_0 + \breve{g}$ where $g_0 \in G$ and $\breve{g} \in \breve{G}$. Now $\breve{g} \in \breve{G}$, and so, $0 \in P_G(\breve{g})$. But $\breve{g} = x - g_0$, so $P_G(\breve{g}) = P_G(x - g_0)$. Hence $0 \in P_G(x - g_0)$. Then for all $g \in G$, $d(x - g_0, 0) \leq d(x - g_0, g)$, and so, $d(x, g_0) \leq d(x, g + g_0)$. Since G is a subspace therefore, $g + g_0 \in G$. Let $g_1 = g + g_0$ and so $d(x, g_0) \leq d(x, g_1)$ for all $g_1 \in G$. But this implies that $g_0 \in P_G(x)$. Hence, G is proximinal. □

Theorem 2.13. *Let G be a subspace of a metric linear space (X, d) and $x \in X$. Then $P_G(x)$ is a bounded set.*

Proof. Let $g_1, g_2 \in P_G(x)$, then $d(x, g_1) \leq d(x, g)$ for all $g \in G$ and $d(x, g_2) \leq d(x, g)$ for all $g \in G$. Now,

$$\begin{aligned} d(g_1, g_2) &\leq d(g_1, x) + d(x, g_2) \text{ where } x \in G \\ &\leq d(g_0, x) + d(x, g_0) \text{ for some fix } g_0 \in G \\ &= 2d(x, g_0) \\ &= t \end{aligned}$$

where t is a constant and ≥ 0. Hence, the proof is complete. □

Now we discuss some results related to the proximinality of quotient spaces.
For a closed linear subspace G of a metric linear space (X, d), **canonical mapping** $\pi : X \to X/G$ is defined as $\pi(x) = x + G$, where $x \in X$. This mapping π is linear[13].

Theorem 2.14. *[13] Let G be a closed linear subspace of a metric linear space (X, d) and M be a proximinal subspace of X containing G. Then M/G is proximinal in X/G.*

Proof. Let $x + G \in X/G$, $x \in X$ and m be a best approximation to x. We show that $m + G$ is best approximation to $x + G$. To the contrary, suppose that there exists $m' + G \in M/G$ such that $d(x + G, m + G) > d(x + G, m' + G)$, that is, $\inf_{g \in G} d(x - m', g) < d(x - m, G)$. Then there exist some $g_0 \in G$ such that

$$d(x - m', g_0) < d(x - m, G) \leq d(x - m, 0)$$

that is, $d(x, m' + g_0) < d(x, m)$. Thus m is not a best approximation to x from M, which is a contradiction. Hence $m + G$ is a best approximation to $x + G$ and consequently, M/G is a proximinal in X/G. □

Theorem 2.15. *[13] Let (X, d) be a metric linear space and let M be a subspace of X containing G where G is a proximinal subspace of X. If M/G is proximinal in X/G, then M is proximinal in X.*

Proof. Let $x \in X$ be arbitrary, then $x + G \in X/G$. Since M/G is proximinal in X/G, there is some $m + G \in P_{M/G}(x + G)$, that is, $d(m + G, x + G) \leq d(x + G, m' + G)$ for every $m' + G \in M/G$. Also, since G is proximinal, there exist some $g_0 \in G$ such that $d(m - x, g_0) \leq d(x - m', G) \leq d(x - m', 0)$ for every $m' \in M$. This implies that $m - g_0 \in P_M(x)$. Hence M is proximinal in X. □

2.1.3 Uniqueness of best approximation in metric linear space.

In this section, we discuss the cases in which the given set G is Chebyshev.

Theorem 2.16. *[13] Let (X, d) be a metric linear space and G be its linear subspace. Then G is Chebyshev if and only if $X = G \oplus \breve{G}$.*

Proof. Let G be a subspace of metric linear space X and $x \in X$. Then by Theorem 2.12 we have, $x = g + \breve{g}$ for some $g \in G$ and $\breve{g} \in \breve{G}$. Let $x = g_1 + \breve{g}_1 = g_2 + \breve{g}_2$, where $g_1, g_2 \in G$ and $\breve{g}_1, \breve{g}_2 \in \breve{G}$. Now we will show that $g_1 = g_2$ and $\breve{g}_1 = \breve{g}_2$. Since $x = g_1 + \breve{g}_1$ and $x = g_2 + \breve{g}_2$, then $x - g_1 = \breve{g}_1$ and $x - g_2 = \breve{g}_2$. This implies that $g_1, g_2 \in P_G(x)$ by Lemma 2.7. Now, since G is Chebyshev, therefore $g_1 = g_2$, and it follows that $\breve{g}_1 = \breve{g}_2$. Hence $X = G \oplus \breve{G}$.

Conversely, let $X = G \oplus \breve{G}$, and $x \in X$. On contrary, assume that $g_1, g_2 \in P_G(x)$. Then by Lemma 2.7, we have $x - g_1, x - g_2 \in \breve{G}$ and so $x = g_1 + \breve{g}_1 = g_2 + \breve{g}_2$ for some $\breve{g}_1, \breve{g}_2 \in \breve{G}$. But $X = G \oplus \breve{G}$, therefore $g_1 = g_2$ and $\breve{g}_1 = \breve{g}_2$. Hence, G is Chebyshev. □

Lemma 2.17. *[13] Let (X, d) be a metric linear space and G be a non-empty subspace of X. Then $G \cap \breve{G} = \{0\}$.*

Proof. Let $g \in G \cap \breve{G}$ if and only if $g \in G$ and $g \in \breve{G}$ if and only if $g \in G$ and $d(g, 0) \leq d(g, h)$ for all $h \in G$ if and only if $g \in G$ and $d(g, 0) \leq d(g, g)$ when $h = g$, if and only if $g \in G$ and $d(g, 0) \leq 0$ if and only if $g \in G$ and $d(g, 0) = 0$ as d can't be negative, if and only if $g \in G$ and $g = 0$ if and only if $G \cap \breve{G} = \{0\}$. □

The following theorem give condition under which proximinal set is Chebyshev and metric projection P_G is linear.

Theorem 2.18. *[13] Let (X, d) be a metric linear space and G be a proximinal subspace of X, then the following are equivalent:*

(i) P_G is one-valued and linear

(ii) $\breve{G}$ is a linear subspace of X.

Proof. First assume, P_G is one-valued and linear. Let $x, y \in \breve{G}$ and α, β be scalars. Then $P_G(x) = \{0\}$ and $P_G(y) = \{0\}$. Since P_G is linear, we have $P_G(\alpha x + \beta y) = \alpha P_G(x) + \beta P_G(y) = \{0\}$. This implies that $\alpha x + \beta y \in \breve{G}$ and hence $\breve{G}$ is a linear subspace of X.
Conversely let, $\breve{G}$ is a linear subspace of X, and also let $g_1, g_2 \in P_G(x)$. This gives $x - g_1, x - g_2 \in \breve{G}$ by Theorem 2.7. Now $(x - g_1) - (x - g_2) \in \breve{G}$, that is, $g_2 - g_1 \in \tilde{G}$ as $\breve{G}$ is a subspace . This implies that $g_2 - g_1 \in \breve{G} \cap G = \{0\}$ and so $g_1 = g_2$. Hence P_G is one valued. Now let $x, y \in X$ and α, β are scalars. Suppose $g_1 \in P_G(x)$ and $g_2 \in P_G(y)$. This gives $x - g_1$, $y - g_2 \in \breve{G}$. Since $\tilde{G}$ is a subspace, therefore we have $\alpha(x - g_1) + \beta(y - g_2) \in \breve{G}$, that is, $(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) \in \breve{G}$. Now $P_G(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) = P_G(\alpha x + \beta y - (\alpha g_1 + \beta g_2)) = \{0\}$, as P_G is one-valued. Hence $P_G(\alpha x + \beta y) = \alpha g_1 + \beta g_2 = \alpha P_G(x) + \beta P_G(y)$ and this completes the proof. □

Theorem 2.19. *[13] Let (X, d) be a metric linear space and G be a Chebyshev subspace of X. Then the following are equivalent:*

(i) P_G is linear

(ii) $\breve{G}$ *is a subspace*

(iii) $\breve{G}$ *contains a subspace* H *for which* $X = G \oplus H$.

Proof. Let P_G is linear and it is given that G is Chebyshev, therefore $\breve{G}$ is a subspace of X by Theorem 2.18. Hence $(i) \Rightarrow (ii)$.

Now by Theorem 2.16, it is obvious that $(ii) \Rightarrow (iii)$. Now we show that $(iii) \Rightarrow (i)$. Let $x, y \in X$ and α, β be scalars. Then $x = g_1 + h_1$ and $y = g_2 + h_2$ for some $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Thus $x - g_1, y - g_2 \in H$. Since H is a subspace, therefore we have $\alpha(x - g_1) + \beta(y - g_2) \in H \subseteq \breve{G}$ for all scalars α, β. This implies that $(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) \in \breve{G} = P_G^{-1}(0)$, that is, $P_G(\alpha x + \beta y - (\alpha g_1 + \beta y_2)) = \{0\}$ as G is Chebyshev. Hence $P_G(\alpha x + \beta y) = \alpha g_1 + \beta g_2 = \alpha P_G(x) + \beta P_G(y)$. □

Theorem 2.20. *[13] Let G be a proximinal subspace of a metric linear space X and* $\breve{G}$ *is convex. Then*

(i) G is Chebyshev

(ii) M is a proximinal subspaces of X containing G, then the quotient space M/G is a chebyshev subspace of X/G.

Proof. (i) Suppose $g_1, g_2 \in P_G(x)$. Then $\breve{g}_1 = x - g_1, \breve{g}_2 = x - g_2 \in \breve{G}$ (by Lemma 2.7.) Since $\breve{G}$ is convex, therefore $\frac{1}{2}(\breve{g}_2 - \breve{g}_1) = \frac{1}{2}(g_1 - g_2) \in \breve{G}$. Also, $\frac{1}{2}(g_1 - g_2) \in G$ as G is a subspace of X. Then $\frac{1}{2}(g_1 - g_2) \in G \cap \breve{G} = \{0\}$(by Lemma 2.17), and this implies $g_1 = g_2$. Hence G is Chebyshev.

(ii) As described earlier, $\pi : X \to X/G$ is the canonical map defined by $\pi(x) = x + G$. Also M/G is proximinal with X/G, by Theorem 2.14. Thus, $\pi(P_M(x)) \subseteq P_{M/G}(x + G)$. Now, let $m + G \in P_{M/G}(x + G)$. Then by Theorem 2.15 there exists $g_0 \in G$ such that $m - g_0 \in P_M(x)$. This implies that $m + G \in \pi(P_M(x))$. Therefore, $P_{M/G}(x + G) \subseteq \pi(P_M(x))$ and hence $\pi(P_M(x)) = P_{M/G}(x + G)$.
Now we show that $P_{M/G}^{-1}(G) = \widehat{M/G}$ is convex. Suppose $x + G, y + G \in \widehat{M/G}$ and $0 < \lambda < 1$. Since $G \in P_{M/G}(x + G)$, then $G \in \pi(P_M(x))$, it means there exists $m_1 \in P_M(x)$ such that $\pi(m_1) = G$. Also, since $G \in P_{M/G}(y + G)$, then $G \in \pi(P_M(y))$, it means there exists $m_2 \in P_M(y)$ such that $\pi(m_2) = G$, so we have $\pi(m_1) = G = \pi(m_2)$. Now since $m_1 \in P_M(x)$ and $m_2 \in P_M(y)$, therefore $x - m_1, y - m_2 \in \breve{M}$ and $\breve{M}$ is convex, so $\lambda(x - m_1) + (1 - \lambda)(y - m_2) \in \breve{M}$, then $(\lambda x + (1 - \lambda)y) - (\lambda m_1 + (1 - \lambda)m_2) \in \breve{M}$. It follows that $\lambda m_1 + (1 - \lambda)m_2 \in P_M(\lambda x + (1 - \lambda)y)$, that is

$\pi(\lambda m_1 + (1-\lambda)m_2) = \lambda m_1 + (1-\lambda)m_2 + G = G$. Therefore $G \in P_{M/G}(\lambda x + (1-\lambda)y + G)$, and $\lambda(x+G) + (1-\lambda)(y+G) \in \widehat{M/G}$. This implies that $\widehat{M/G}$ is convex. Then (by part (1)) we have, M/G is Chebyshev subspace of X/G.

□

Theorem 2.21. *[19] Let (X,d) be a strictly convex metric linear space and G be its convex proximinal subset, then G is Chebyshev.*

Proof. Let G be a convex proximinal set in a strictly convex metric linear space X. For a given pair $x_1, x_2 \in X$, if possible, let $g_1^*, g_2^* \in G$ be such that

$$\max\{d(g_1^*, x_1), d(g_1^*, x_2)\} = \max\{d(g_2^*, x_1), d(g_2^*, x_2)\} = t$$

where

$$t = \inf\{max\{d(g, x_1), d(g, x_2)\} : g \in G\}.$$

Then $d(g_1^*, x_1), d(g_2^*, x_1) \leq t$ and $d(g_1^*, x_2), d(g_2^*, x_2) \leq t$. Since X is a strictly convex, therefore

$$d\left(\frac{g_1^* + g_2^*}{2}, x_1\right) < t \text{ and } d\left(\frac{g_1^* + g_2^*}{2}, x_2\right) < t$$

unless $g_1^* = g_2^*$. But this contradicts the definition of t, as $\frac{1}{2}(g_1^* + g_2^*) \in G$. Hence $g_1^* = g_2^*$. Thus G is a Chebyshev set. □

2.2 Best approximation in normed linear space.

In this section, we discuss some results related to the *characterization*, *existence* and *uniqueness* of best approximation in normed linear space. To start with, we give a following notation to be used in the sequel.

Notation 2.2.1. *[11] Let X be a normed linear space and G be its subspace. Define,*

$$\breve{G} = P_G^{-1}(0) = \{x \in X | 0 \in P_G(x)\} = \{x \in X | \|x\| \leq \|x - g\| \ \ \forall g \in G\} = \{x \in X | x \perp_B G\}.$$

2.2.1 Characterization of best approximation in normed linear space.

Theorem 2.22. *[16] Let X be a normed space and G be its non-empty subspace. Then,* *(i) If $x \in G$, then $P_G(x) = \{x\}$.*

(ii) If $x \in cl(G) \setminus G$, then $P_G(x) = \emptyset$ ($cl(G)$ is the closure of G).

Proof. The result follows from Theorem 2.4. □

As a consequence of this result, we will characterize the best approximation of only those elements which belong to the set $X \setminus cl(G)$.

Theorem 2.23. *[6] Let G be a subspace of a normed linear space X over the field $\mathbb{K}$. Let $x_0 \in X$ be such that $d(x_0, G) = d > 0$. Then, there exists a $f \in X'$ such that*

(i) $f(x_0) = 1$

(ii) $f(G) = 0$, (that is, $f(g) = 0\ \forall g \in G$)

(iii) $\|f\| = \frac{1}{d}$.

Proof. Let

$$G_0 = G + [x_0]$$

be the space spanned by G and x_0. This $x_0 \notin G$ as $d > 0$. Each point of $x \in G_0$ can be expressed uniquely in the form $x = g + \alpha x_0$, where $g \in G$ and $\alpha \in \mathbb{K}$. Consider the functional $h : G_0 \longrightarrow \mathbb{K}$ defined as

$$h(g + \alpha x_0) = \alpha.$$

Note that h is a linear functional on G_0 satisfying $h(G) = 0$ and $h(x_0) = 1$. Also,

$$|h(x)| = |h(g + \alpha x_0)| = |\alpha|, \text{ for all } x \in G_0.$$

Now if $\alpha \neq 0$, then

$$\begin{aligned} \|x\| &= \|g + \alpha x_0\| \\ &= |\alpha| \|\frac{g}{\alpha} + x_0\| \\ &\geq |\alpha| d \\ &= d|h(x)|, \quad \textit{where} \quad x_0 \in G_0 \end{aligned}$$

If $\alpha = 0$, then

$$\|x\| = \|g\| \geq 0 = d|h(x)|.$$

Hence, in either case, it follows that

$$|h(x)| \leq \frac{\|x\|}{d} \text{ for all } x \in G_0.$$

which implies, h is bounded on G_0 and $\|h\| \leq \frac{1}{d}$. To prove $\|h\| \geq \frac{1}{d}$, consider a sequence $\{g_k\} \subset G$ such that $\|x_0 - g_k\| \longrightarrow d$. Then

$$1 = h(x_0 - g_k) \leq \|h\| \|x_0 - g_k\| \longrightarrow \|h\| d$$

which implies $\|h\| \geq \frac{1}{d}$.
This means that there exists an $h \in G_0'$ such that $h(G) = 0, h(x_0) = 1$ and $\|h\| = \frac{1}{d}$. By Theorem 1.26 we obtain a functional $f \in X'$ such that

$$f|_{G_0} = h \text{ and } \|f\| = \|h\| = \frac{1}{d}$$

But

$$f|_{G_0} = h \Rightarrow \begin{cases} f(G) = h(G) = 0 \\ f(x_0) = h(x_0) = 1. \end{cases}$$

Hence, the proof is complete. □

Corollary 2.24. *[6] Let G be a closed linear subspace of a normed linear space X over the field $\mathbb{K}$. Let $x_0 \in X \setminus G$. If $d(x_0, G) = d$, then there exists a $f \in X'$ such that*

(i) $f(x_0) = 1$

(ii) $f(G) = 0$, *(that is, $f(g) = 0 \; \forall g \in G$)*

(iii) $\|f\| = \frac{1}{d}$.

Proof. Let $d(x_0, G) = 0$, which implies $x_0 \in cl(G)$. Since G is closed, therefore $G = cl(G)$. Also $x_0 \notin G$, so

$$d = d(x_0, G) > 0.$$

Hence, the result follows from Theorem 2.23. □

Theorem 2.25. *[16] Let G be a non-empty linear subspace of a normed space X such that $x \in X \setminus cl(G)$ and $g_0 \in G$. We have $g_0 \in P_G(x)$ if and only if there exists an $f \in X'$ with the following properties:*

$$\|f\| = 1 \tag{2.6}$$

$$f(g) = 0 \quad (\forall g \in G) \tag{2.7}$$

$$f(x - g_0) = \|x - g_0\|. \tag{2.8}$$

Proof. Assume that $g_0 \in P_G(x)$. Also, since $x \in X \setminus cl(G)$ therefore, we have $d(x, G) = \|x - g_0\| > 0$. Now, by Corollary 2.24, there exists an $f_0 \in X'$ such that $\|f_0\| = \frac{1}{d(x,G)}$, $f_0(g) = 0$ $(\forall g \in G)$ and $f_0(x) = 1$. Take $f = \|x - g_0\| f_0 \in X'$ which satisfies (2.6), (2.7) and (2.8).

Conversely, assume that there exists a $f \in X'$ which satisfies (2.6), (2.7) and (2.8). Then for any $g \in G$ we have

$$\|x - g_0\| = |f(x - g)| = |f(x - g)| \leq \|f\| \|x - g\| = \|x - g\|.$$

Hence $g_0 \in P_G(x)$, which completes the proof. □

Lemma 2.26. *[16] Let X be a normed linear subspace and G be its linear subspace. If $x \in X \setminus cl(G), g_0 \in G$ and $f \in X'$, then*

(i) f satisfies (2.6) and (2.8) if and only if satisfies (2.6) and

$$Ref(x - g_0) = \|x - g_0\|. \tag{2.9}$$

(ii) f satisfies (2.7) if and only if

$$Ref(g) = 0 \quad (\forall g \in G). \tag{2.10}$$

(iii) f satisfies (2.6), (2.7) and

$$|f(x - g_0)| = \|x - g_0\| \tag{2.11}$$

if and only if either $f_1 = [signf(x - g_0)]f$ satisfies (2.6), (2.7) and (2.8).

(iv) f satisfies (2.6), (2.7) and

$$|Ref(x - g_0)| = \|x - g_0\|, \tag{2.12}$$

if and only if either $f_1 = f$, or $f_2 = -f$ satisfies (2.6), (2.7) and (2.8).

Proof. (i) Clearly, (2.9) is directly obtained from (2.8). Hence (2.6) and (2.8) $\Longrightarrow$ (2.6) and (2.9). Conversely, if f satisfies (2.6) and (2.9), we have

$$\|x - g_0\| = Ref(x - g_0) \leq |f(x - g_0)| \leq \|x - g_0\|,$$

hence $Ref(x-g_0) = |f(x-g_0)|$, and as a result, $f(x-g_0)$ is real and positive. Hence (2.6) and (2.9) $\Longrightarrow$ (2.6) and (2.8).

(ii) Obviously, (2.7) $\Longrightarrow$ (2.10). Conversely, let f satisfies (2.10). Consider that $ig \in G$ and the relation $Imf(g) = -Ref(ig)$, we have

$$f(g) = Ref(g) - iRef(ig) = 0 \quad \text{where}(g \in G).$$

(iii) If $f_1 = [\text{sign} f(x-g_0)]f$ satisfies (2.6), (2.7) and (2.8) then obviously $f = e^{iargf(x-g_0)f_1}$ satisfies (2.6), (2.7) and (2.11). Conversely, if f satisfies (2.6), (2.7) and (2.11), then f_1 satisfies (2.6), (2.7) and

$$f_1(x-g_0) = |f(x-g_0)| = \|x-g_0\|.$$

(iv) If either $f_1 = f$, or $f_2 = -f$ satisfies (2.6), (2.7) and (2.8) then f satisfies (2.6), (2.7) and (2.12). Conversely, if f satisfies (2.6), (2.7) and (2.12), then we have

$$\|x-g_0\| = |Ref(x-g_0)| \leq |f(x-g_0)| \leq \|x-g_0\|,$$

hence $|Ref(x-g_0)| = |f(x-g_0)|$. As a result, $f(x-g_0)$ is real, whence either $f_1 = f$ or $f_2 = f$ satisfies (2.6), (2.7) and (2.8), which completes the proof.

□

Lemma 2.27. *(see[4], [10]) Let G be a linear subspace of normed linear space X. Then $g_0 \in P_G(x)$ if and only if $(x-g_0) \perp_B G$.*

Proof. Let $g_0 \in P_G(x)$. Put $g_1 = g_0 - \alpha g$ for any fixed $g \in G$ and $\alpha \in \mathbb{R}$. Since $g_0 \in P_G(x)$, therefore $\|x-g_0\| \leq \|x-g\|$ for all $g \in G$. In particular, $\|x-g_0\| \leq \|x-g_1\|$, which implies $\|x-g_0\| \leq \|x-(g_0-\alpha g)\|$. Then, $\|x-g_0\| \leq \|(x-g_0)+\alpha g\|$. Hence, $(x-g_0) \perp_B G$.
Conversely, let $(x-g_0) \perp_B G$. Then $\|x-g_0\| \leq \|x-g_0+\alpha g_1\|$. Since G is a subspace, therefore $g_0 - \alpha g_1 \in G$. Put $g = g_0 - \alpha g_1$ and we get $\|x-g_0\| \leq \|x-g\|$. Hence, $g_0 \in P_G(x)$. □

The following two results, follows from the Lemma 2.7 and Theorem 2.8.

Lemma 2.28. *(see[4], [11]) Let G be a subspace of a normed linear space X. Then $g_0 \in P_G(x)$ if and only if $x - g_0 \in \breve{G}$ for all $x \in X$.*

Theorem 2.29. *(see[4], [10]) Let X be a normed linear space and G be a linear subspace of X. If $P_G(x)$ is the set of all best approximation of $x \in X$.Then, $P_G(x) = G \cap (x - \breve{G})$.*

Lemma 2.30. *(see[4],[11]) Let G be a subspace of normed linear space X. Then $G \cap \breve{G} = \{0\}$.*

Proof. Let $g \in G \cap \breve{G}$, then $g \in G$ and $g \in \breve{G}$. Now $g \perp_B G$ as $g \in \breve{G}$. This implies that $\|g\| \leq \|g + \alpha h\|$ for all $g \in G$ and all scalar α. Now choose $h = g$ and $\alpha = -\frac{1}{2}$, then $\|g\| \leq \|\frac{1}{2}g\|$, and hence $g = 0$. Therefore, $G \cap \breve{G} \subseteq \{0\}$. But $\{0\} \subseteq G \cap \breve{G}$, hence together we have $G \cap \breve{G} = \{0\}$. □

Theorem 2.31. *[4] Let X be a normed linear space and G be its non-empty subset. Then*

(i) $P_{G+u}(x + u) = P_G(x) + u$ for every $x, u \in X$.

(ii) $P_{\beta G}(\beta x) = \beta P_G(x)$ where $x \in X$ and $\beta \in \mathbb{R}$.

Proof. (i) Let $g_0 \in P_{G+u}(x + u)$ if and only if $g_0 \in G + u$ and $\|x + u - g_0\| \leq \|x + u - (g + u)\|$ for all $g + u \in G + u$ if and only if $g_0 - u \in G$ and $\|x - (g_0 - u)\| \leq \|x - g\|$ for all $g \in G$ if and only if $(g_0 - u) \in P_G(x)$ if and only if $g_0 \in P_G(x) + u$, which completes the proof.

(ii) If $\beta = 0$ then the result trivially holds, as $P_{\beta G}(\beta x) = P_0(0) = 0$ (because $0 \in \{0\}$). Also $\beta P_G(x) = 0P_G(x) = 0$.
So, let $\beta \neq 0$. Now, assume $g_0 \in P_{\beta G}(\beta x)$ if and only if $g_0 \in \beta G$ and $\|\beta x - g_0\| \leq \|\beta x - \beta g\|$ for all $g \in G$ if and only if $|\beta|\|x - \frac{g_0}{\beta}\| \leq |\beta|\|x - g\|$ for all $g \in G$ if and only if $\frac{g_0}{\beta} \in G$ and $\|x - \frac{g_0}{\beta}\| \leq \|x - g\|$ if and only if $\frac{g_0}{\beta} \in P_G(x)$ if and only if $g_0 \in \beta P_G(x)$. Hence $P_{\beta G}(\beta x) = \beta P_G(x)$.

□

Theorem 2.32. *[16] Let G be a linear subspace of a normed linear space X. Then following are equivalent:*

1. *G is a semi-Chebyshev subspace*
2. *There do not exist $f \in X'$ and $x_1, x_2 \in X$ with $x_1 - x_2 \in G \setminus \{0\}$, such that*

$$\|f\| = 1 \tag{2.13}$$

$$f(g) = 0 \quad (\forall g \in G) \tag{2.14}$$

$$f(x_1) = \|x_1\|, f(x_2) = \|x_2\| \tag{2.15}$$

3. *There do not exist $f \in X'$, $x \in X$ and $g_0 \in G \setminus \{0\}$ with the properties (2.13), (2.14) and*

$$f(x) = \|x\| = \|x - g_0\|.$$

Proof. Proceeding on the similar lines of Theorem 2.10, we obtain the result. □

2.2.2 Existence of best approximation in normed linear space.

Theorem 2.33. *[4] Let X be a normed linear space and G be its non-empty subset. Then*

(i) G is proximinal if and only if $G + u$ is proximinal for any given $u \in X$.

(ii) G is proximinal if and only if βG is proximinal where $\beta \in \mathbb{R} \setminus \{0\}$.

Proof. (i) G is proximinal if and only if $P_G(x) \neq \emptyset$ if and only if $P_G(x) \neq \emptyset$ if and only if $P_{G+u}(x+u) \neq \emptyset$ (by theorem 2.31) if and only if $G + u$ is proximinal.

(ii) G is proximinal if and only if $P_G(x) \neq \emptyset$ if and only if $\beta P_G(x) \neq \emptyset$ if and only if $P_{\beta G}(\beta x) \neq \emptyset$ (by Theorem 2.31) if and only if βG is proximinal.

□

Theorem 2.34. *(see[4], [17]) Let X be a normed linear space and G be a linear subspace of X. Then following statements are equivalent:*

(i) G is Proximinal,

(ii) $X = G + \check{G} = \{g + y | g \in G, y \in \check{G}\}$.

Proof. Proof follows from Theorem 2.12. □

Theorem 2.35. *(see[2], [4]) Let X be a normed linear space and G be a proximinal subset of X. Then G is non-empty and closed.*

Proof. Since $\emptyset \neq P_G(0) \subset G$, therefore, G is a non-empty set. To show that G is closed, take any sequence $\{x_n\} \in G$ which converges to x. If we show $x \in G$ then we have done. Since G is proximinal, therefore $P_G(x) \neq \emptyset$. Let $g_0 \in P_G(x)$ such that $\|x - g_0\| \leq \|x - x_n\|$ for all n. But $\lim\limits_{n\to\infty} x_n = x$ implies $\|x_n - x\| = 0$. Then $x = g_0 \in G$, and hence G is closed. □

Corollary 2.36. *(see[4], [9]) Let G be a subspace of a normed linear space X and $x \in X$, then*

1. *If G is a convex subset of X, then $P_G(x)$ is a convex subset of X for every $x \in X$*
2. *$P_G(x)$ is a bounded set.*

Proof. Let $g_1, g_2 \in P_G(x)$, then

$$\|x - g_1\| \leq \|x - g\| \quad and \quad \|x - g_2\| \leq \|x - g\| \forall g \in G. \tag{1}$$

1. Now, for $0 \leq \lambda \leq 1$ and any $g \in G$, we have

$$\begin{aligned} \|x - (\lambda g_1 + (1-\lambda) g_2)\| &= \|x - \lambda g_1 - g_2 + \lambda g_2\| \\ &= \|x - \lambda g_1 - g_2 + \lambda g_2 - \lambda x + \lambda x\| \\ &= \|\lambda(x - g_1) + (1-\lambda)(x - g_2)\| \\ &\leq \lambda\|x - g_1\| + (1-\lambda)\|x - g_2\| \\ &\leq \lambda\|x - g\| + (1-\lambda)\|x - g\|, \end{aligned}$$

   so we have, $\|x - (\lambda g_1 + (1-\lambda) g_2)\| \leq \|x - g\|$. which implies $\lambda g_1 + (1-\lambda) g_2 \in P_G(x)$, and hence $P_G(x)$ is a convex set.

2. Now, we show $P_G(x)$ is a bounded set. Fix $g_0 \in G$. Then

$$\begin{aligned} \|g_1 - g_2\| &= \|g_1 - x + x - g_2\| \\ &\leq \|g_1 - x\| + \|x - g_2\| \qquad (by \;\; using \;\; (1)) \\ &\leq 2\|x - g_0\| = t(say), \end{aligned}$$

   where t is a constant and ≥ 0.

Hence, the proof is complete. □

Theorem 2.37. *[9] Let G be a subspace of a normed linear space X. If G is closed and $g_0 \in G$ is a best approximation for $x \in X \setminus G$, then $g_0 \in bd(G)$.*

Proof. On the contrary , suppose that $g_0 \in Int(G)$, then there exists $r > 0$ such that $B(g_0; r) \subset G$. Since $g_0 \in G$ is a best approximation for x, then $d(x, G) = \|x_0 - g_0\| = t(say) > 0$. Let $g_1 = \frac{t}{t+r} g_0 + \frac{r}{t+r} x_0$. Then $\|g_1 - g_0\| = \frac{rt}{t+r} < r$ and so $g_1 \in B(g_0; r)$. Also, $\|g_1 - x_0\| = \frac{t^2}{t+r} < t = \|x_0 - g_0\|$, which implies that g_0 is not a best approximation of x_0 and hence, a contradiction. Therefore, $g_0 \in bd(G)$. □

Corollary 2.38. *[9] Let G be a closed subspace of a normed linear space X. If $x_0 \in X \setminus G$ has a best approximation in G, then $d(x_0, G) = d(x_0, bd(G))$.*

Proof. $bd(G) \subset G$ as G is a closed set. Now,

$$\begin{aligned} d(x_0, bd(G)) &= \inf\{\|x_0 - b\| : b \in bd(G)\} \\ &\geq \inf\{\|x_0 - b\| : b \in G\} = d(x_0, G). \end{aligned}$$

Hence,

$$d(x_0, bd(G)) \geq d(x_0, G) \tag{1}$$

Now, suppose that $g_0 \in G$ is the best approximation for x_0. Then $d(x_0, G) = \|x_0 - g_0\|$. Hence, $g_0 \in bd(G)$ (by Theorem 2.37). Also, $d(x_0, bd(G)) = \inf\{\|x_0 - b\| : b \in bd(G)\} \leq \|x_0 - g_0\| = d(x_0, G)$. So, we have

$$d(x_0, G) \geq d(x_0, bd(G)) \tag{2}$$

Hence, from (1) and (2), we have $d(x_0, G) = d(x_0, bd(G))$. □

2.2.3 Uniqueness of best approximation in normed linear space.

Theorem 2.39. *[16] Let G be a linear subspace of a normed linear space X. Let $x \in X \setminus cl(G)$ and $g_0 \in G$. Then $P_G(x)$ is a Chebyshev set, if and only if $g_0 \in P_G(x)$ and there do not exist $g \in G \setminus \{g_0\}$ and $f \in X'$ such that*

$$\|f\| = 1 \tag{2.16}$$

$$f(g) = f(g_0) \tag{2.17}$$

$$f(x - g) = \|x - g\|. \tag{2.18}$$

Proof. Let $P_G(x)$ be a Chebyshev set such that $P_G(x) = \{g_0\}$. On the contrary, assume that there exist $g \in G \setminus \{g_0\}$ and $f \in X'$ satisfying (2.16), (2.17) and (2.18). Then,

$$\|x - g\| = |f(x - g)| = |f(x - g_0) + f(g_0 - g)| = |f(x - g_0)| \leq \|x - g_0\|.$$

Since $g_0 \in P_G(x)$, therefore $g \in P_G(x)$. Hence, $P_G(x)$ cannot be a Chebyshev set which is a contradiction. Therefore, there do not exist $g \in G \setminus \{g_0\}$ and $f \in X$ satisfying (2.16), (2.17) and (2.18).

Conversely let, $g_0 \in P_G(x)$ and there do not exist $g \in G \setminus \{g_0\}$ and $f \in X'$ satisfying (2.16), (2.17) and (2.18). Assume that some $g_1 \in P_G(x)$. Then by Theorem 2.25, there exist $h \in X'$ such that

$$\|h\| = 1$$

$$h(g) = 0 \quad (\forall g \in G)$$

$$h(x - g_1) = \|x - g_1\|.$$

which implies $h(g) = h(g_0) = 0$. But this contradicts to our assumption that there do not exist $g \in G \setminus \{g_0\}$ and $f \in X'$ satisfying (2.16), (2.17) and (2.18). Therefore, $g_1 \notin P_G(x)$ and thus, $P_G(x)$ is a Chebyshev set. Hence, the proof is complete. □

Theorem 2.40. *[2] Let X be a strictly convex normed linear space and G be a non-empty and closed subset of X. Then for each $x \in X$, $P_G(x)$ contains at most one element.*

Proof. The result is obviously true if $x \in G$. So, assume that $x \in X \setminus G$. On the contrary, suppose that there exist two distinct g_1 and $g_2 \in P_G(x)$. Then,

$$\|x - g_1\| = \|x - g_2\| = d(x, G) > 0.$$

Now, $\frac{g_1+g_2}{2} \in G$ as G is convex, and so

$$\begin{aligned} d(x,G) &\leq \|x - (\frac{g_1+g_2}{2})\| \\ &= \left\| \frac{(x-g_1)+(x-g_2)}{2} \right\| \\ &< \frac{\|x-g_1\| + \|x-g_2\|}{2} \text{ (since } \|\cdot\| \text{ is strictly convex)} \\ &= d(x,G), \end{aligned}$$

which is not possible. Hence, $g_1 = g_2$, and so $P_G(x)$ contains at most one element. □

Corollary 2.41. *[2] Every convex proximinal set in a strictly convex normed linear space $(X, \|\cdot\|)$ is a Chebyshev set.*

Proof. By Theorem 2.35, proximinal set is closed. Hence by Theorem 2.40, every convex proximinal set in a srictly convex normed linear space is Chebyshev set. □

The following example shows that if normed linear space X is not strictly convex, then there exists a convex proximinal set G which is not Chebyshev.

Example 2.5. *Let $X = \mathbb{R}^2$ equipped with the norm $\|(x,y)\| = |x| + |y|$ be a normed linear space and $G = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1$ where $0 \leq x \leq 1$ and $-1 \leq y \leq 1\}$. Then, G is convex proximinal set of X but not Chebyshev.*

Proof. First we show that X is not a strictly convex. Let $x = (\frac{1}{3}, \frac{2}{3})$, $y = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$. Clearly, both x and $y \in S_X$ as they have norm 1. Also,

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|_1 &= \frac{1}{2}\left\|\left(\frac{1}{3}, \frac{2}{3}\right) + \left(\frac{1}{2}, \frac{1}{2}\right)\right\|_1 \\ &= \frac{1}{2}\left\|\left(\frac{5}{6}, \frac{7}{6}\right)\right\|_1 \\ &= 1 \end{aligned}$$

which implies that $\frac{x+y}{2} \in S_X$. Hence, X is not strictly convex as $x \neq y$. Now, we show this is a convex set. Let $(x_1, y_1), (x_2, y_2) \in G$, which implies

$$\begin{aligned} &x_1^2 + y_1^2 \leq 1; 0 \leq x_1 \leq 1 \text{ and } -1 \leq y_1 \leq 1 \\ &x_2^2 + y_2^2 \leq 1; 0 \leq x_2 \leq 1 \text{ and } -1 \leq y_2 \leq 1. \end{aligned}$$

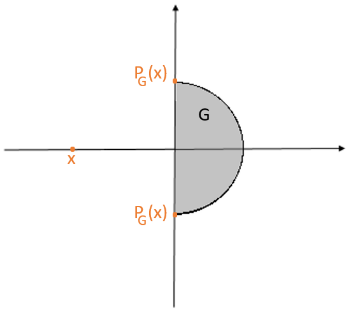
Now,

$$\begin{aligned} \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) &= (\lambda x_1, \lambda y_1) + ((1-\lambda)x_2, (1-\lambda)y_2) \\ &= (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \end{aligned}$$

such that $0 \leq \lambda x_1 + (1-\lambda)x_2 \leq 1$ and $-1 \leq \lambda y_1 + (1-\lambda)y_2 \leq 1$. Also,

$$\begin{aligned} [\lambda x_1 + (1-\lambda)x_2]^2 + [\lambda y_1 + (1-\lambda)y_2]^2 &= \lambda^2(x_1^2 + y_1^2) + (1-\lambda)^2(x_2^2 + y_2^2) + \\ &\quad 2\lambda(1-\lambda)(x_1x_2 + y_1y_2) \\ &\leq \lambda^2 + (1-\lambda)^2 + \\ &\quad 2\lambda(1-\lambda)(x_1x_2 + y_1y_2) \\ &\leq \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)(1) \\ &= 1. \end{aligned}$$

Hence, this is a convex set. Now, we show this is a proximinal set but not Chebyshev.



From this figure, it is clear that point x has two best approximations in G. Therefore, it is a proximinal set, not Chebyshev. Hence, the proof is complete. □

Theorem 2.42. *[2] Let X be a normed linear space and G be its non-empty set. Suppose that $x \in X \setminus G$ and $y \in P_G(x)$, then for any $u \in \lambda x + (1-\lambda)y$, where $0 \le \lambda < 1$, $y \in P_G(u)$.*

Proof. Since $u \in \lambda x + (1-\lambda)y$, therefore, $\|x-u\| + \|u-y\| = \|x-y\|$. Now, $y \in G \cap B[u; \|u-y\|]$ and so $d(u,G) \le \|u-y\|$. On the other hand, by triangle inequality,

$$B(u; \|u-y\|) \subseteq B(x; \|x-u\| + \|u-y\|) = B(x; \|x-y\|) \subseteq X \setminus G$$

and hence, we have $\|u-y\| \le d(u,G)$. Thus, $\|u-y\| = d(u,G)$, which implies $y \in P_G(u)$. □

The proof of following two theorems follows from Theorem 2.16 and Theorem 2.20.

Theorem 2.43. *[4] Let G be a subspace of a normed linear space X. Then following are equivalent:*

(i) G is Chebyshev

(ii) $X = G \oplus \breve{G}$.

Theorem 2.44. *(see[10], [4]) Let G be a proximinal subspace of a normed linear space X and $\breve{G}$ is convex. Then*

(i) G is Chebyshev

(ii) M is a proximinal subspaces of X containing G, then the quotient space M/G is a Chebyshev subspace of X/G.

Chapter 3

Best Co-approximation in Metric Linear Space and Normed linear space

In this chapter, we discuss about *best co-approximation* in metric linear space and normed linear space. First we define best co-approximation in metric linear space, and then define some terms and definitions like co-proximinal set, co-Chebyshev set etc. We also discuss some results related to the *characterization*, *existence* and *uniqueness* of best co-approximation in metric linear space and normed linear space.

3.1 Best co-approximation in metric linear space.

Let (X, d) be a metric linear space and G be its non-empty subset. Then element $g_0 \in G$ is called **best co-approximation** to $x \in X$ from set G if

$$d(g_0, g) \leq d(x, g)$$

for every $g \in G$. This set is denoted by $R_G(x)$.

If $R_G(x) \neq \emptyset$ for every $x \in X$, then G is called **co-proximinal set.** If every $x \in X$ has unique best co-approximation in G, then G is called **co-Chebyshev set**. **Metric projection** for co-proximinal set G of metric linear space X is the set valued mapping $R_G(x) : X \to 2^G$, which is defined as $R_G(x) = \{g_0 \in G : d(g_0, g) \leq d(x, g)$ for every $g \in G\}$.

Example 3.1. *[13] Let $X = R \setminus \{1\}$ be a metric space with metric $d(x,y) = |x-y|$ and $G = (1,2]$ be its subset. Then G is co-proximinal subset of X.*

Example 3.2. *[13] Let $X = \mathbb{R}^2$ be a metric space with the metric $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$ and $G = \{(x,y) \in \mathbb{R}^2 : x = y\}$ be its subset. Then $R_G(x,y) = \{(\frac{x+y}{2}, \frac{x+y}{2})\}$, that is, G is co-Chebychev.*

Notation 3.1.1. *[13] Let G be a subspace of a metric linear space X. Then the set $R_G^{-1}(0)$ is defined as*

$$\tilde{G} = R_G^{-1}(0) = \{x \in X : d(g,0) \leq d(x,g) \text{ for all } g \in G\}.$$

Theorem 3.1. *[12] Let (X,d) be a metric linear space and G be its linear subspace. If $G \perp (x - g_0)$, then $g_0 \in R_G(x)$.*

Proof. Since $G \perp (x - g_0)$, so $g \perp (x - g_0)$ for all $g \in G$. Then by the definition of orthogonality we have, $d(g,0) \leq d(g, \alpha(x - g_0))$ for every scalar α. Now take $\alpha = 1$, we get $d(g,0) \leq d(g, x - g_0)$ for every $g \in G$. This implies $d(g_0, g + g_0) \leq d(x, g + g_0)$ for every $g \in G$. Hence $g_0 \in R_G(x)$. □

Lemma 3.2. *[12] Let (X,d) be a metric linear space and G be its subspace. Then, for all $x \in X$, $g_0 \in R_G(x)$ if and only if $(x - g_0) \in \tilde{G}$.*

Proof. $g_0 \in R_G(x)$ if and only if $d(g_0, g) \leq d(x, g)$ for all $g \in G$ if and only if $d(0, g - g_0) \leq d(x - g_0, g - g_0)$ for all $(g - g_0) \in G$ if and only if $x - g_0 \in \tilde{G}$. Hence, the proof is complete. □

Theorem 3.3. *Let (X,d) be a metric linear space and G be its subspace. Then, $R_G(x) = G \cap (x - \tilde{G})$.*

Proof. $g_0 \in G \cap (x - \tilde{G})$, if and only if $g_0 \in G$ and $g_0 \in (x - \tilde{G})$, if and only if $g_0 \in G$ and $g_0 = x - \tilde{g}$ where $\tilde{g} \in \tilde{G}$ if and only if $g_0 \in G$ and $\tilde{g} = x - g_0 \in \tilde{G}$ if and only if $g_0 \in R_G(x)$ by using [Lemma 3.2]. Hence, $R_G(x) = G \cap (x - \tilde{G})$. □

Theorem 3.4. *[12] If G be a subspace of a metric linear space (X,d), then $G \cap \tilde{G} = \{0\}$.*

Proof. Let $g \in G \cap \tilde{G}$ if and only if $g \in G$ and $g \in \tilde{G}$ if and only if $g \in G$ and $d(h,0) \leq d(g,h)$ for all $h \in G$ if and only if $g \in G$ and $d(g,0) \leq d(g,g)$ when $h = g$, if and only if $g \in G$ and $d(g,0) \leq 0$ if and only if $g \in G$ and $d(g,0) = 0$ as d can't be negative, if and only if $g \in G$ and $g = 0$ if and only if $G \cap \breve{G} = \{0\}$. □

Theorem 3.5. *[11] Let (X, d) be a metric linear space and G be its subspace. Then G is a co-proximinal subspace if and only if $X = G + \tilde{G}$.*

Proof. First let, G is a co-proximinal subspace of X. Also, let $x \in X$ and $g_0 \in R_G(x)$. Then $x - g_0 \in \tilde{G}$ (by Lemma 3.2). Now $x = g_0 + (x - g_0) \in G + \tilde{G}$ as $g_0 \in G$. Hence, $X = G + \tilde{G}$.
Conversely, let $X = G + \tilde{G}$ and $x \in X$. Then $x = g + \tilde{g}$ where $g \in G$ and $\tilde{g} \in \tilde{G}$. Since $\tilde{g} \in \tilde{G} = R_G^{-1}(0)$, then $0 \in R_G(\tilde{g})$. Now $\tilde{g} = x - g_0$ as $x = g_0 + \tilde{g}$ so, $R_G(\tilde{g}) = R_G(x - g_0)$ which implies $0 \in R_G(\tilde{g}) = R_G(x - g_0)$. Then $d(0, g) \leq d(x - g_0, g)$ for all $g \in G$, so $d(g_0, g + g_0) \leq d(x, g + g_0)$ where $g + g_0 \in G$ as G is a subspace. Hence $g_0 \in R_G(x)$. Therefore, G is co-proximinal. □

Theorem 3.6. *Let (X, d) be a metric linear space and G be a subspace of X. Then $R_{G+u}(x + u) = R_G(x) + u$ for every $u \in X$.*

Proof. Let $u_0 \in R_{G+u}(x + u)$, then $d(g + u, u_0) \leq d(x + u, g + u)$ for every $g + u \in G + u$. Now,

$$\begin{aligned} d(g, u_0 - u) &= d(g + u, u_0) \\ &\leq d(x + u, g + u) \\ &= d(x, g) \end{aligned}$$

which implies $d(g, u_0 - u) \leq d(x, g)$ and we have, $u_0 - u \in R_G(x)$. Thus $u_0 \in R_G(x) + u$ and $R_{G+u}(x + u) \subseteq R_G(x) + u$.
Conversely, let $g_0 \in R_G(x) + u$. This implies $g_0 - u \in R_G(x)$, then $d(g, g_0 - u) \leq d(x, g)$ for all $g \in G$. Now,

$$\begin{aligned} d(g + u, g_0) &= d(g, g_0 - u) \\ &\leq d(x, g) \\ &= d(x + u, g + u) \end{aligned}$$

for every $g + u \in G + u$. This implies $g_0 \in R_{G+u}(x + u)$, and thus $R_G(x) + u \subseteq R_{G+u}(x + u)$. Hence $R_{G+u}(x + u) = R_G(x) + u$ for every $u \in X$. □

Theorem 3.7. *[13] Let (X, d) be a metric linear space and G be a co-proximinal subspace of X, then the following are equivalent:*

(i) R_G is one-valued and linear.

(ii) $\tilde{G}$ *is a linear subspace of* X.

Proof. First assume, R_G is one-valued and linear. Let $x, y \in \tilde{G}$ and α, β be scalars. Then $R_G(x) = \{0\}$ and $R_G(y) = \{0\}$. Since R_G is linear, we have $R_G(\alpha x + \beta y) = \alpha R_G(x) + \beta R_G(y) = \{0\}$. This implies that $\alpha x + \beta y \in \tilde{G}$ and hence $\tilde{G}$ is a linear subspace of X.
Conversely let, $\tilde{G}$ is a linear subspace of X, and also let $g_1, g_2 \in R_G(x)$. This gives $x - g_1, x - g_2 \in \tilde{G}$ by Lemma 3.2. Now $(x - g_1) - (x - g_2) \in \tilde{G}$, that is, $g_2 - g_1 \in \tilde{G}$ as $\tilde{G}$ is a subspace . This implies that $g_2 - g_1 \in \tilde{G} \cap G = \{0\}$ and so $g_1 = g_2$. Hence R_G is one valued. Now let $x, y \in X$ and α, β are scalars. Suppose $g_1 \in R_G(x)$ and $g_2 \in R_G(y)$. This gives $x - g_1, y - g_2 \in \tilde{G}$. Since $\tilde{G}$ is a subspace, therefore we have $\alpha(x - g_1) + \beta(y - g_2) \in \tilde{G}$, that is, $(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) \in \tilde{G}$. Now $R_G(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) = R_G(\alpha x + \beta y - (\alpha g_1 + \beta g_2)) = \{0\}$, as R_G is one-valued. Hence $R_G(\alpha x + \beta y) = \alpha g_1 + \beta g_2 = \alpha R_G(x) + \beta R_G(y)$ and this completes the proof. □

Theorem 3.8. *[13] Let G be a closed linear subspace of a metric linear space (X, d) and M be a co-proximinal subspace of X containing G. Then M/G is co-proximinal in X/G.*

Proof. Let $x + G \in X/G, x \in X$ and m be a best approximation to x. We show that $m + G$ is best co-approximation to $x + G$. On the contrary, suppose that there exists $m' + G \in M/G$ such that $d(m + G, m' + G) > d(x + G, m' + G)$, that is, $\inf_{g \in G} d(x - m', g) < d(m - m', G)$. Then there exist some $g_0 \in G$ such that

$$d(x - m', g_0) < d(m - m', G) \leq d(m - m', g_0)$$

that is, $d(x, m' + g_0) < d(m, m' + g_0)$. Thus m is not best co-approximation to x from M, which is a contradiction. Hence $m + G$ is a best co-approximation to $x + G$ and consequently, M/G is a co-proximinal in X/G. □

Theorem 3.9. *[13] Let (X, d) be a metric linear space and let M be a subspace of X containing G where G is a proximinal subspace of X. If M/G is co-proximinal in X/G, then M is co-proximinal in X.*

Proof. Let $x \in X$ be arbitrary, then $x + G \in X/G$. Since M/G is co-proximinal in X/G, there is some $m + G \in R_{M/G}(x + G)$, that is, $d(m + G, m' + G) \leq d(x + G, m' + G)$ for every $m' + G \in M/G$. Also, since G is proximinal, there exists some $g_0 \in G$ such that $d(m - m', g_0) \leq d(x - m', G) \leq d(x - m', 0)$ for every $m' \in M$. This implies that $m - g_0 \in R_M(x)$. Hence M is co-proximinal in X. □

Theorem 3.10. *[13] Let G be a subspace of a metric linear space X. Then following are equivalent:*

(i) *G is co-Chebyshev*

(ii) $X = G \oplus \tilde{G}$.

Proof. Result follows from Theorem 2.16. □

Theorem 3.11. *[13] Let (X, d) be a metric linear space and G be a co-Chebyshev subspace of X. Then the following are equivalent:*

(i) R_G *is linear*

(ii) $\tilde{G}$ *is a subspace*

(iii) $\tilde{G}$ *contains a subspace H for which* $X = G \oplus H$.

Proof. Let R_G is linear and it is given that G is co-Chebyshev, therefore $\tilde{G}$ is a subspace of X by Theorem 3.7. Hence $(i) \Rightarrow (ii)$.
Now by Theorem 3.10, it is obvious that $(ii) \Rightarrow (iii)$. Now we show that $(iii) \Rightarrow (i)$. Let $x, y \in X$ and α, β be scalars. Then $x = g_1 + h_1$ and $y = g_2 + h_2$ for some $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Thus $x - g_1, y - g_2 \in H$. Since H is a subspace, therefore we have $\alpha(x - g_1) + \beta(y - g_2) \in H \subseteq \tilde{G}$ for all scalars α, β. This implies that $(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) \in \tilde{G} = R_G^{-1}(0)$, that is, $R_G(\alpha x + \beta y - (\alpha g_1 + \beta y_2)) = \{0\}$ as G is co-Chebyshev. Hence $R_G(\alpha x + \beta y) = \alpha g_1 + \beta g_2 = \alpha R_G(x) + \beta R_G(y)$. □

Theorem 3.12. *[12] Let (X, d) be a metric linear space and G be its co-proximinal subspace. If $\tilde{G}$ is a convex set, then G is co-Chebyshev.*

Proof. Suppose $x \in X$ and $g_1, g_2 \in R_G(x)$, then $x - g_1, x - g_2 \in \tilde{G}$ by Lemma 3.2. Put $x - g_1 = \tilde{g_1}$ and $x - g_2 = \tilde{g_2}$, where $\tilde{g_1}, \tilde{g_2} \in \tilde{G}$. Now we first show that $g_1 - x \in \tilde{G}$. Since $0 \in R_G(x - g_1)$, so we have $d(g, 0) \le d(x - g_1, g)$ for every $g \in G$. This implies that $d(-g, 0) \le d(-g, g_1 - x)$ for every $g \in G$ that is, $d(g', 0) \le d(g_1 - x, g')$ for every $g' \in G$. Therefore, $g_1 - x \in \tilde{G}$. Now, $x - g_2$, $g_1 - x \in \tilde{G}$ and $\tilde{G}$ is convex, we have $\frac{1}{2}[(x - g_2) + (g_1 - x)] \in \breve{G}$. This implies $\frac{1}{2}[\tilde{g_2} - \tilde{g_1}] \in \tilde{G}$ and also, $\frac{1}{2}[\tilde{g_2} - \tilde{g_1}] = \frac{1}{2}[g_1 - g_2] \in G$. So we have $\frac{1}{2}[g_1 - g_2] \in \tilde{G} \cap G = \{0\}$ and this implies $g_1 = g_2$. Hence G is co-Chebyshev. □

Theorem 3.13. *[13] Let G be a proximinal subspace of a metric linear space (X, d) and M be a co-proximinal subspace of X containing G. If $\pi : X \to X/G$ is the canonical map, then $\pi(R_M(x)) = R_{M/G}(x + G)$.*

Proof. Since it is given that G is a proximinal subspace, therefore by Theorem 2.35, G is a closed subspace. Then by using Theorem 3.8, we have M/G is co-proximinal subspace of X/G. Thus, $\pi(R_M(x)) \subseteq R_{M/G}(x + G)$. Now, let $m + G \in R_{M/G}(x + G)$. Then by Theorem 3.9 there exists $g_0 \in G$ such that $m - g_0 \in R_M(x)$. This implies that $m + G \in \pi(R_M(x))$. Therefore, $R_{M/G}(x + G) \subseteq \pi(R_M(x))$ and hence $\pi(R_M(x)) = R_{M/G}(x + G)$. □

Theorem 3.14. *[13] Let G be a proximinal subspace of a metric linear space (X, d) and M be a co-proximinal subspace containing G. If $\tilde{G}$ is a convex set then M/G is a co-Chebyshev subspace of X/G.*

Proof. From Theorem 3.13, we have $\pi(R_M(x)) = R_{M/G}(x + G)$. Also by Theorem 3.8, M/G is co-proximinal in X/G. Now in view of Theorem 3.12 it is sufficient to prove that $R^{-1}_{M/G}(G) = \widetilde{M/G}$ is convex. Let $x + G, y + G \in \widetilde{M/G}$ and $0 < \lambda < 1$. Since $G \in R_{M/G}(x + G)$, then $G \in \pi(R_M(x))$, it means there exists $m_1 \in R_M(x)$ such that $\pi(m_1) = G$. Also , since $G \in R_{M/G}(y + G)$, then $G \in \pi(R_M(y))$, it means there exists $m_2 \in R_M(y)$ such that $\pi(m_2) = G$, so we have $\pi(m_1) = G = \pi(m_2)$. Therefore, $x - m_1, y - m_2 \in \tilde{M}$ (as $m_1 \in R_M(x), m_2 \in R_M(y)$). Since $\tilde{M}$ is a convex set, therefore $\lambda(x - m_1) + (1 - \lambda)(y - m_2) \in \tilde{M}$, that is, $d(0, m) \leq d(\lambda(x - m_1) + (1 - \lambda)(y - m_2), m)$ for all $m \in M$. This implies $d(\lambda m_1 + (1 - \lambda)m_2, \lambda m_1 + (1 - \lambda)m_2 + m) \leq d(\lambda x + (1 - \lambda)y, \lambda m_1 + (1 - \lambda)m_2 + m)$ for all $m \in M$. Therefore, $\lambda m_1 + (1 - \lambda)m_2 \in R_M(\lambda x + (1 - \lambda)y)$. Also $\pi(\lambda m_1 + (1 - \lambda)m_2) = G$. Therefore, $G \in R_{M/G}(\lambda x + (1 - \lambda)y + G)$, that is, $\lambda(x + G) + (1 - \lambda)(y + G) \in \widetilde{M/G}$ and so $\widetilde{M/G}$ is convex. Hence M/G is co-Chebyshev in X/G. □

3.2 Best co-approximation in normed linear space

In this section, we discuss about the *characterization*, *existence* and *uniqueness* of best co-approximation in normed linear space. Before discussing this, we define a notation.

Notation 3.2.1. *[11] Let G be a subspace of a normed linear space X. Then the set $R^{-1}_G(0)$ is defined as*

$$\tilde{G} = R^{-1}_G(0) = \{x \in X : \|g\| \leq \|x - g\| \textit{ for all } g \in G\} = \{x \in X : G \perp_B x\}.$$

Theorem 3.15. *Let G be a non-empty subset of a normed linear space X. If $x \in G$, then $R_G(x) = \{x\}$.*

Proof. It is easy to see that $x \in R_G(x)$. Now we show that x is the only element of G which belongs to the set $R_G(x)$. Assume some $g_0 \in G$, which is not equal to x but belongs to $R_G(x)$. From the definition of $R_G(x)$, we have $\|g - g_0\| \leq \|g - x\|$ for all $g \in G$. This implies $\|x - g_0\| \leq \|x - x\|$ when $g = x$, which in turn gives $\|x - g_0\| \leq 0$. Now $\|x - g_0\| \neq 0$ as $g_0 \neq x$, so it means $\|x - g_0\| < 0$, which is a contradiction. Hence $R_G(x) = \{x\}$. □

Lemma 3.16. *[4] Let X be a normed linear space and G be its subspace. Then for all $x \in X$, $g_0 \in R_G(x)$ if and only if $G \perp_B (x - g_0)$.*

Proof. First suppose that $g_0 \in R_G(x)$. For $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, put $g_1 = g_0 - \frac{1}{\alpha}g$ where $g \in G$. Since $g_0 \in R_G(x)$ so, $\|g_0 - g_1\| \leq \|x - g_1\|$. Therefore,

$$\begin{aligned} \left\|\frac{1}{\alpha}g\right\| &\leq \left\|x - g_0 + \frac{1}{\alpha}g\right\| \\ &= \frac{1}{\alpha}\|\alpha(x - g_0) + g\| \end{aligned}$$

which implies $\|g\| \leq \|g + \alpha(x - g_0)\|$. Hence, $g \perp_B (x - g_0)$. Since g is arbitrary, therefore $G \perp_B (x - g_0)$.
Conversely, let $G \perp_B (x - g_0)$. Then for all $g_1 \in G$ and $\alpha \in \mathbb{R}$ we have,

$$\|g_1\| \leq \|\alpha(x - g_0) + g_1\|.$$

Put $g_1 = g_0 - g$ where $g \in G$ and $\alpha = 1$, then we have

$$\|g_0 - g\| \leq \|x - g\|.$$

Hence $g_0 \in R_G(x)$, which completes the proof. □

By proceeding on similar lines of Lemma 3.2 and Theorem 3.3, we get the following results.

Lemma 3.17. *[10] Let X be a normed space and G be its subspace. Then, for all $x \in X$, $g_0 \in R_G(x)$ if and only if $(x - g_0) \in \tilde{G}$.*

Theorem 3.18. *[4] Let X be a normed space and G be its subspace. Then, $R_G(x) = G \cap (x - \tilde{G})$.*

Proposition 3.19. *[4] Let X be a normed linear space and G be its subspace, then*

(i) $R_{G+u}(x+u) = R_G(x) + u$ *for every* $x, u \in X$,

(ii) $R_{\beta G}(\beta x) = \beta R_G(x)$ *for every* $x \in X$ *and any scalar* β.

Proof. (i) $g_0 \in R_{G+u}(x+u)$, if and only if $\|g_0 - (g+u)\| \leq \|x+u-(g+u)\|\ \forall\ (g+u) \in G+u$ if and only if $\|(g_0-u)-g\| \leq \|x-g\|$ for all $g \in G$, if and only if $(g_0-u) \in R_G(x)$, if and only if $g_0 \in R_G(x)+u$. Hence, $R_{G+u}(x+u) = R_G(x)+u$.

(ii) $g_0 \in R_{\beta G}(\beta x)$, if and only if $\|g_0 - g\| \leq \|\beta x - g\|\ \forall g \in G$, if and only if $\left\|\frac{1}{\beta}g_0 - \frac{1}{\beta}g\right\| \leq \|x - \frac{g}{\beta}\|$ for all $\frac{1}{\beta}g \in G$ if and only if $\left\|\frac{1}{\beta}g_0 - g_1\right\| \leq \|x-g_1\|$ for all $g_1 = \frac{1}{\beta}g \in G$ if and only if $\frac{1}{\beta}g_0 \in R_G(x)$, if and only if $g_0 \in \beta R_G(x)$. Therefore, $R_{\beta G}(\beta x) = \beta R_G(x)$ for every $x \in X$ and any scalar β.

□

Theorem 3.20. *[4] Let G be a subset of a normed linear space X. If $g_0 \in R_G(x)$ and $(1-\lambda)x + \lambda g_0 \in G$, for some scalar λ, then $(1-\lambda)x + \lambda g_0 \in R_G(x)$.*

Proof. Since $g_0 \in R_G(x)$, then by definition we have,

$$\|g - g_0\| \leq \|x - g\| \text{ for all } g \in G. \tag{1}$$

Now,

$$\begin{aligned}
\|g-(1-\lambda)x+\lambda g_0\| &= \|g-(1-\lambda)x+\lambda g-\lambda g-\lambda g_0\| \\
&= \|(1-\lambda)g-(1-\lambda)x+\lambda(g-g_0)\| \\
&= \|(1-\lambda)(g-x)+\lambda(g-g_0)\| \\
&\leq (1-\lambda)\|g-x\|+\lambda|g-g_0\| \\
&\leq (1-\lambda)\|g-x\|+\lambda\|x-g\| \text{ (by using (1))} \\
&\leq \|g-x\| \text{ for all } g \in G.
\end{aligned}$$

Hence, $(1-\lambda)x+\lambda g_0 \in R_G(x)$. □

Theorem 3.21. *[4] Let G be a subspace of normed linear space X. If $x \in X \setminus G$ and $g_0 \in R_G(x)$, then $g_0 \in bd(G)$.*

Proof. Assume that $g_0 \notin bd(G)$. Since $g_0 \in R_G(x)$, therefore we have

$$\|g - g_0\| \leq \|x - g\|$$

for all $g \in G$. Now,

$$\begin{aligned}
\|g - [(1-\lambda)x + \lambda g_0]\| &= \|g - (1-\lambda)x - \lambda g + \lambda g - \lambda g_0\| \\
&= \|(1-\lambda)g - (1-\lambda)x + \lambda(g - g_0)\| \\
&\leq \|(1-\lambda)(g - x) + \lambda(g - g_0)\| \\
&\leq (1-\lambda)\|g - x\| + \lambda\|g - g_0\| \\
&\leq (1-\lambda)\|g - x\| + \lambda\|x - g\| \\
&= \|g - x\| \ \text{ for all } g \in G.
\end{aligned}$$

This implies $(1-\lambda)x + \lambda g_0 \in R_G(x)$, and hence $(1-\lambda)x + \lambda g_0 \in G$. It means $x \in G$, as G is a subspace which is a contradiction. Hence $g_0 \in bd(G)$. □

Theorem 3.22. *[4] Let X be a normed linear space and G be its subspace. Then,*

(i) G is co-proximinal if and only if $G + u$ is co-proximinal for every element $u \in X$.

(ii) G is co-proximinal if and only if βG is co-proximinal for any scalar β.

Proof. (i) G is co-proximinal if and only if $R_G(x) \neq \emptyset$ if and only if $R_G(x) + u \neq \emptyset$ if and only if $R_{G+u}(x + u) \neq \emptyset$ (by using Proposition 3.19) if and only if $G + u$ is co-proximinal.

(ii) G is co-proximinal if and only if $R_G(x) \neq \emptyset$ if and only if $\beta R_G(x) \neq \emptyset$ if and only if $R_{\beta G}(\beta x) \neq \emptyset$ (by using Proposition 3.19) if and only if βG is co-proximinal.

□

Theorem 3.23. *[11] Let X be a normed linear space and G be its subspace. Then G is a co-proximinal subspace if and only if $X = G + \tilde{G}$.*

Proof. The proof follows the same steps written in the Theorem 3.5. □

Theorem 3.24. *(see[11], [4]) If G be a subspace of a normed linear space X, then $G \cap \tilde{G} = \{0\}$.*

Proof. Let $u \in G + \tilde{G}$ and we show that $u = 0$. To show this, we have $u \in \tilde{G}$, as $u \in G + \tilde{G}$. Then, $G \perp_B u$ for all $g \in G$. This implies that $g \perp_B u$ for all $g \in G$. Therefore, $\|g\| \leq \|g + \beta u\|$ for all $g \in G$ and scalar β. Now choose $\beta = -\frac{1}{2}$ and $g = u$, then $\|g\| \leq \|g - \frac{1}{2}g\|$, and so, $\|g\| \leq \|\frac{1}{2}g\|$ and we have, $g = 0$. This implies $G \cap \tilde{G} \subseteq \{0\}$. Also, $\{0\} \subseteq G \cap \tilde{G}$. Therefore, we get $G \cap \tilde{G} = \{0\}$. □

Lemma 3.25. *(see[11], [4]) Let X be a normed linear space and G be its subset. Then $d(x, \tilde{G}) = \|g\|$ for every $g \in G$.*

Proof. Let $s \in \tilde{G}$, then $G \perp_B s$, that is, $g \perp_B s$ for every $g \in G$ and scalar α. Now put $\alpha = -1$, then $\|g\| \leq \|g - s\|$ for every $g \in G$. For fixed $g \in G$, we have $\|g\| \leq \|g - s\|$ for every $s \in \tilde{G}$. Now this implies

$$\|g\| \leq \inf_{s \in \tilde{G}} \|g - s\| = d(g, \tilde{G}) \leq \|g - 0\| = \|g\|, \text{ where } g \in \tilde{G}.$$

Therefore, $d(g, \tilde{G}) = \|g\|$. □

Theorem 3.26. *(see[11], [4]) Let X be a smooth Banach space and G be its co-proximinal subspace. Then, $\tilde{G}$ is a proximinal subspace of X.*

Proof. Suppose G is co-proximinal, then by Theorem 3.23 we have $X = G + \tilde{G}$. Therefore, $x = g + \tilde{G}$ where $g \in G$ and $\tilde{g} \in \tilde{G}$. Then, $x - \tilde{g} = g$, and so, $\|x - \tilde{g}\| = \|x - (x - g)\| = \|g\| = d(g, \tilde{G})$. Now since $\|x - \tilde{g}\| = d(g, \tilde{G})$, then $\tilde{g} \in R_{\tilde{G}}(x)$ for all $x \in X$. Therefore, $R_{\tilde{G}}(x) \neq \emptyset$, that is, G is co-proximinal subspace of X. □

Theorem 3.27. *[11] Let G be a subspace of a normed linear space X. Then following are equivalent:*

(i) G is co-Chebyshev

(ii) $X = G \oplus \tilde{G}$.

Proof. The proof runs on similar lines of Theorem 2.16. □

Theorem 3.28. *[4] Let X be a smooth Banach space and G be its co-Chebyshev subspace. Then, $\tilde{G}$ is a Chebyshev subspace of X.*

Proof. Suppose G is co-Chebyshev, then by Theorem 3.27 we have $X = G \oplus \tilde{G}$. So $\tilde{G}$ is a proximinal subspace of X, by Theorem 3.26. Now we have to only show that for every $x \in X$, the set $P_{\tilde{G}}(x)$ is singleton. To show this, assume that $g_1, g_2 \in P_{\tilde{G}}(x)$, then $x - g_1 = \tilde{g_1} \in \tilde{G}$ and $x - g_2 = \tilde{g_2} \in \tilde{G}$. Therefore, $x = g_1 + \tilde{g_1} = g_2 + \tilde{g_2}$, so $\tilde{g_1} = \tilde{g_2}$ and $g_1 = g_2$. Thus $P_{\tilde{G}}$ is a singleton set, that is, $\tilde{G}$ is co-Chebyshev subspace of X. □

Definition 3.29. [8] Let X be a normed linear space and G be its subspace. For $x \in X$, let $d(x, G)$ denotes the distance between x and G, that is, $d(x, G) = \inf\{\|x - g\| : g \in G\}$. Then the quotient space X/G is equipped with the norm

$$\|x + G\| = d(x, G).$$

If X is a normed space or a Banach space, and G is a closed subspace of X, then X/G is a normed space or a Banach space.

Theorem 3.30. *[15] Let X be a normed linear space and let G and M be its subspaces such that $G \subset M$. Let $x \in X \setminus M$ and $m \in M$. If m is a best approximation to x from M, then $m + G$ is a best co-approximation to $x + G$ from the quotient space M/G.*

Proof. On the contrary, assume that $m + G$ is not a best co-approximation to $x + G$ from M/G. Then there exists $m' + G \in M/G$ such that

$$|\|m' + G - (m + G)\|| > |\|x + G - (m' + G)\||,$$

that is,

$$|\|m' - m + G\|| > |\|x - m' + G\||,$$

that is,

$$d(x - m', G) < d(m' - m, G).$$

This implies that there exists $g \in G$ such that

$$\begin{aligned} \|x - m' - g\| &< d(m' - m, G) \\ &< \|m' - m + g\|. \end{aligned}$$

That is,

$$\|(g + m') - m\| > \|x - (g + m')\|.$$

Thus m is not a best approximation to x from M, which is a contradiction. Hence $m + G$ is a best co-approximation to $x + G$ from the quotient space M/G. □

Theorem 3.31. *[8] Let G be a closed linear subspace of normed linear space X and M be a co-proximinal subspace of X containing G. Then, M/G is co-proximinal to X/G.*

Proof. Proof follows from Theorem 3.8. □

Theorem 3.32. *[13] Let (X, d) be a metric linear space and let M be a subspace of X containing G where G is a proximinal subspace of X. If M/G is co-proximinal in X/G, then M is co-proximinal in X.*

Proof. Proof follows from Theorem 3.9. □

Theorem 3.33. *[8] Let G be a proximinal subspace of a normed space X and let M be co-proximinal subspace of X containing G. If $\pi : X \to X/G$ is the canonical map, then $\pi(R_M(x)) = R_{M/G}(x + G)$.*

Proof. As G is proximinal subspace therefore, G is closed subspace by Theorem 2.35. Also, by Theorem 3.31 M/G is co-proximinal with X/M. Now suppose $m \in R_M(x)$, and let $u \in M$, then for every $g \in G$ we have

$$\begin{aligned} \|(m+G)-(u+G)\| &= d(m-u, G) \\ &\leq \|m-(u+g)\| \\ &\leq \|x-(u+g)\| \\ &= \|(x-u)+g\| \\ &= \|(x+G)-(u+G)\|. \end{aligned}$$

Therefore, $\|(m+G)-(u+G)\| \leq |(x+G)-(u+G)\|$. This implies that $\pi(m) = m + G \in R_{M/G}(x+G)$ and thus $\pi(R_M(x)) \subseteq R_{M/G}(x+G)$. Now, let $m + G \in R_{M/G}(x+G)$. Then by Theorem 3.32, there exists $g_0 \in G$ such that $m - g_0 \in R_M(x)$. This implies that $m + G \in \pi(R_M(x))$. Therefore, $R_{M/G}(x+G) \subseteq \pi(R_M(x))$ and hence $\pi(R_M(x)) = R_{M/G}(x+G)$. □

Theorem 3.34. *[8] Let G be a proximinal subspace of a normed space X and let M be a co-proximinal subspace of X containing G. If $\tilde{M}$ is convex, then M/G is co-Chebyshev with X/G.*

Proof. Proof is similar of Theorem 3.14. □

Future Work

Best approximation is a vast topic of the approximation theory. Many researchers have done work in this field. But still there are many areas in which work can be done. For instance:

(i) Error of approximation:
How to compute the error of approximation $d(x, G)$?

(ii) Computation of best approximation:
Which algorithms are useful for computing best approximation?

(iii) Continuity of best approximation:
How does the set of all best approximation vary as a function of x or G?

Also, there is an open problem which is referred to as the 'Chebyshev set problem':

"Is every Chebyshev set in a Hilbert space convex"?

Bibliography

[1] F. Deutsch. *Best Approximation in Inner Product Spaces*. Springer-Verlag, 2000.

[2] J. Fletcher and W. B. Moors. Chebyshev sets. *Journal of the Australian Mathematical Society*, 98(2):161–231, 2015.

[3] C. Franchetti and M. Furi. Some characteristic properties of real hilbert spaces. *Revue Roumaine des Mathematiques Pures et Appliquees*, 17:1045–1048, 1972.

[4] A. M. A. Ghazal. Best approximation and best co-approximation in normed spaces. M.sc. thesis, The Islamic University of Gaza, January 2010.

[5] P. K. Jain and K. Ahmad. *Metric Spaces*. Nrosa Publishing House, 1996.

[6] P. K. Jain and O. P. Ahuja. *Functional Analysis*. New Age International(P) Limited, Publishers, 1995.

[7] E. Kreyszig. *Introductory Functional Analysis with Applications*. John Wiley & Sons, 1978.

[8] H. Mazaheri. Best coapproximation in quotient spaces. *Nonlinear Analysis*, 68(10):3122–3126, 2008.

[9] H. Mazaheri. The nearest points in normed linear spaces. *Applied mathematical science*, 2(16):763–766, 2008.

[10] H. Mazaheri and F. M. Maleek Ghaini. Quasi-orthagonality of the best approximant sets. *Nonlinear Analysis*, 65(3):534–537, 2006.

[11] H. Mazaheri and S. M. S. Modarees. Some results concerning proximinality and co-proximinality. *Nonlinear Analysis*, 62(6):1123–1126, 2005.

[12] T. D. Narang and S. Gupta. Proximinality and co-proximinality in metric linear spaces. *Annales Universitatis Mariae Curie-Sklodowska Section A*, 69(1):83–90, 2015.

[13] T. D. Narang and S. Gupta. Best approximation and best coapproximation in metric linear spaces. *Journal of Nonlinear Analysis and Optimizations*, 7:137–143, 2016.

[14] P. L. Papini and I. Singer. Best coapproximation in normed linear space. *Michigan Mathematical Journal*, 88:27–44, 1979.

[15] G. S. Rao and R. Saravanan. Some results concerning best uniform coapproximation. *Journal of Inequalities in Pure and Applied Mathematics*, 3(2):1–13, 2002.

[16] I. Singer. *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*. Springer-Verlag, 1970.

[17] I. Singer. *The Theory of Best Approximation and Functional Analysis*. Society for Industrial and Applied Mathematics, Pennsylvania, 1974.

[18] K. G. Steffens. *The History of Approximation Theory*. Birkhauser, 2006.

[19] H. M. Tehrani, T. D. Narang, and H. R. Khademzadeh. *A first course in nearest and farthest points*. Iran, 2015.