ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SERIES WITH SPECIAL COEFFICIENTS

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DEDICATED

TO

MY PARENTS AND GOD

CERTIFICATE

Certified that the dissertation entitled, "**ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SERIES WITH SPECIAL COEFFICIENTS**", which is being submitted by **Miss Raj Rani** (Roll No. 301003017), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of Dr. S.S. Bhatia during the period from January 2012 to June 2012.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University / Institute for the award of any degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

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ABSTRACT

The present dissertation entitled, **"ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SERIES WITH SPECIAL COEFFICIENTS",** contains a brief account of investigations carried out by various authors and by me on L^1-convergence of trigonometric series under the supervision of Dr. S.S. Bhatia, Professor, School of Mathematics and Computer Applications, Thapar University, Patiala.

The work presented in this dissertation has been divided into four chapters. The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in sequel, I have presented some known results interrelated to our results along with a brief plan of our results presented in the subsequent chapters. The purpose of chapter II is to study the L^1-convergence of modified cosine sums introduced by Kumari and Ram in 1988 with class $S(\delta)$ of coefficient sequences. In chapter III, I have studied the L^1-convergence of modified cosine sums introduced by Kumari and Ram under a new class $S_r(\delta)$ of coefficient sequences.

In chapter IV, I have studied the L^1-convergence of modified cosine sums introduced by Rees-Stanojevic with class S_r^{**}. I have also studied the L^1-convergence of the r^{th} derivative of the cosine sums under a class S_r^{**}.

Towards the end, references of various publications cited in the present dissertation have been reported.

CONTENTS

CHAPTER I

INTRODUCTION

1.1 The present dissertation contains certain results studied out by the author **"ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SERIES WITH SPECIAL COEFFICIENTS".** It is known that if a trigonometric series converges in L^1-metric to a function $f \in L^1(T)$, then it is the Fourier series of the function f . Riesz {[21], Vol.II, Ch.VIII article 22} gave a counter example to show that converse of above result does not hold good in L^1-metric. This has encouraged various researchers to carry the research on the topic "On L^1-Convergence of Trigonometric Series".

Integrability and L^1-convergence of Trigonometric series have been studied by number of authors. The work on this topic was initiated by W.H. Young [40] and that of A.N. Kolmogorov [2] by taking into considerations the Classes of Convex sequences $(\Delta^2 a_n \geq 0)$ and quasi-convex sequences $\left(\sum_{n=1}^{\infty} n\left|\Delta^2 a_n\right| < \infty\right)$ respectively. S.A. Teljakovskii [33] studied another class S which was introduced by Sidon [31], for L^1-Covergence of Trigonometric series. The results obtained by these authors were futher generalized and extended by G.H. Hardy and J.E. Littlewood [12], T. Kano [39], J.W. Garrett and C.V. Stanojevic ([14], [15]), B. Ram ([3], [5]), N. Singh and K.M. Sharma ([23], [24], [25]), R. Bojanic and C.V. Stanojevic [29], C.P. Chen [7], R. Bala and B. Ram [28], F. Moricz [10], S.S. Bhatia and B. Ram [36], Z. Tomovski ([41], [42], [43], [44]), N. Hooda and B. Ram [22], K. Kaur, S.S. Bhatia and B. Ram [19], J. Kaur and S.S. Bhatia ([16], [17], [18]) and others by considering various generalizations of classes of sequences mentioned above for one-dimensional trigonometric series.

During investigations on this topic of L^1-convergence of Trigonometric series, various authors introduced number of modified trigonometric sums as these sums approximate in limits better than the classical trigonometric series, since these sums converge in L^1-metric to the sum of trigonometric series where as the classical series itself may not. In this concern, various authors like C.S. Rees and C.V. Stanojevic [9], C.P. Chen [8], S. Kumari and B. Ram [30] and B. Ram and S. Kumari [6], N. Honda and B. Ram [22], K. Kaur, S.S. Bhatia and B. Ram [20] and J. Kaur and S.S. Bhatia ([16], [17], [18]) have introduced various new modified trigonometric sums and have studied their L^1-convergence under various classes of coefficient sequences.

In the present dissertation, some of the results have been studied by the author, most of which are directly associated with the works of above mentioned authors.

To provide sufficient background for later chapters, a summary of basic concepts, techniques and a brief chapter wise resume of the results contained in the dissertation has been given in this introductory chapter. However, some of the definitions and notations will be repeated occasionally in chapters for the sake of convenience.

1.2 DEFINITIONS AND NOTATIONS

Let $\{a_n\}$ be a sequence. Then we write

$$\Delta a_n = a_n - a_{n+1}$$
$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Abel's transformation ([21], Vol.I, p.1) . If $a_0, a_1, ..., a_n, ... v_0, v_1, ..., v_n, ...$ are any real numbers, let us assume that

$$V_n = v_0 + v_1 + ... + v_n$$

Then,

$$\sum_{k=m}^{n} a_k v_k = \sum_{k=m}^{n-1} \Delta a_k V_k + a_n V_n - a_m V_{m-1}$$

where $\Delta a_k = a_k - a_{k+1}$, is called Abel's transformation.

Under the condition that if $m = 0$ and $V_{-1} = 0,$ Abel's transformation reduces to

$$\sum_{k=0}^{n} a_k v_k = \sum_{k=0}^{n-1} \Delta a_k V_k + a_n V_n$$

Null sequence . The sequence $\{a_n\}$ is null sequence if $\{a_n\} \to 0$ as $n \to \infty$.

Trigonometric series: A series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

is called trigonometric series, where $a_0, a_n's, b_n's$ are the coefficients. These coefficients may be real or complex**.**

Fourier Series . A Fourier series may be defined as an expansion of a periodic and integrable function $f(x)$ over interval $(-\pi, \pi)$ in a series of Sines and Cosines such as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

and are called Fourier Coefficients.

Convex sequence: A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$.

Quasi-Convex sequence ([21], Vol. II, p. 202). A sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{n=1}^{\infty} n\left|\Delta^2 a_n\right| < \infty.$$

Semi-Convex sequence ([39]). A null sequence $\{a_n\}$ is said to semi-convex if

$$\sum_{n=1}^{\infty} n\left|\Delta^2 a_{n-1} + \Delta^2 a_n\right| < \infty, \quad (a_0 = 0)$$

δ **-Quasi monotone sequence ([27]).** A sequence $\{a_n\}$ of positive numbers is said to be δ -quasi monotone if $a_k \to 0$, $a_k > 0$ and $\Delta a_k \geq -\delta_k$, where δ_k is sequence of positive numbers.

Quasi monotone sequence ([26], [32]). A sequence $\{a_n\}$ of non-negative numbers is said to be a quasi monotone if $a_{n+1} \leq a_n(1+\frac{\alpha}{n})$ for some $\alpha > 0$ and all $n > n_0(\alpha)$. An equivalent definition is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

Monotone decreasing sequence. A sequence $\{a_n\}$ is said to be monotone decreasing sequence if $a_n \geq a_{n+1}$ for all values of n.

O-o Notation: Let u_n and v_n be sequence of real numbers. Then $u_n = o(v_n)$

if $\frac{u_n}{v_n} \to 0$ as $n \to \infty$

and

if $\frac{u_n}{v_n}$ is bounded,

then $u_n = O(v_n)$

Dirichlet kernel ([21], Vol. I, p.85). Let the Dirichlet Kernel be defined as:

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + ... + \cos nx$$

Moreover,

$$2\sin\frac{x}{2} D_n(x) = \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + 2\sin\frac{x}{2}\cos 2x + ... + 2\sin\frac{x}{2}\cos nx$$

$$= \sin(n+\frac{1}{2})x$$

Hence,
$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$$

This expression is known as Dirichlet's kernel.

Let us define Conjugate Dirichlet kernel as:

$$\tilde{D}_n(x) = \sin x + \sin 2x + ... + \sin nx$$

$$2\sin\frac{x}{2}\tilde{D}_n(x) = 2\sin\frac{x}{2}\sin x + 2\sin\frac{x}{2}\sin 2x + ... + 2\sin\frac{x}{2}\sin nx$$

$$= \cos\left(\frac{x}{2}\right) - \cos\left(n + \frac{1}{2}\right)x$$

We get,

$$\tilde{D}_n(x) = \frac{\cos(\frac{x}{2}) - \cos(n + \frac{1}{2})x}{2\sin(\frac{x}{2})}$$

This expression is called the kernel conjugate to the Dirichlet kernel.

If $x \neq 0(\bmod 2\pi)$, then

$$|D_n(x)| \leq \frac{\pi}{2x}, \quad \text{for } 0 < |x| \leq \pi$$

and

$$|\tilde{D}_n(x)| \leq \frac{\pi}{x}, \quad \text{for } 0 < |x| \leq \pi$$

Also, we shall use the uniform estimate

$$|D_n(x)| \leq n + \frac{1}{2}, \quad \text{for any } x$$

and $$|\tilde{D}_n(x)| < n + 1, \quad \text{for any } x .$$

Fejer kernel ([1],[21]): The Fejer kernel $K_n(x)$ is defined as

$$K_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(x)$$

$$= \frac{1}{n+1}\sum_{j=0}^{n}\frac{\sin\left(j + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

Using $\left|D_n(x)\right| \le n+1$, it follows that $K_n(x) \le n+1$

It has the properties

$$(i) \quad K_n(x) \ge 0,$$

$$(ii) \quad \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1.$$

The Conjugate Fejer kernel is defined as

$$\tilde{K}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} \tilde{D}_j(x)$$

We have $\tilde{K}_n(x) > 0 \quad \text{for} \quad 0 < x < \pi, \quad n = 1,2,3,...$

and $\left|\tilde{K}_n(x)\right| < \frac{1}{2}n.$

The class S ([31], [33]): A null sequence $\{a_n\}$ belongs to class S, if there exists a sequence $\{A_n\}$ such that

(1) $A_n \downarrow 0$ as $n \to \infty$,

(2) $\sum_{n=1}^{\infty} A_n < \infty$,

(3) $|\Delta a_n| \le A_n \quad \forall n$

This was given by S.SIDON [31]in 1939

The Class $S(\delta)$ ([34]): A null sequence $\{a_n\}$ belongs to class $S(\delta)$, if there exists a sequence $\{A_n\}$ such that

(1) $\{A_n\}$ is δ-quasi monotone and $\sum n\delta_n < \infty$

(2) $\sum_{n=1}^{\infty} A_n < \infty$,

(3) $|\Delta a_n| \le A_n \quad \forall n$

The Class $S_r(\delta)$ **:** A null sequence $\{a_n\}$ belongs to class $S_r(\delta)$, $r = 0,1,2,...$, if there exists a sequence $\{A_n\}$ such that

(1) $\{A_n\}$ is δ-quasi monotone and $\sum n^{r+1}\delta_n < \infty$

(2) $\sum_{n=1}^{\infty} n^r A_n < \infty,$

(3) $|\Delta a_n| \le A_n \quad \forall n$

For $r = 0$, the class $S_r(\delta)$ reduces to the class $S(\delta)$.

The class S^* **([24]):** A null sequence $\{a_n\}$ belongs to class S^*, if there exists a sequence $\{A_n\}$ such that

(1) $\{A_n\}$ is quasi monotone

(2) $\sum_{n=1}^{\infty} A_n < \infty,$

(3) $|\Delta a_n| \le A_n \quad \forall n$

The class S^{**} **([35]):** A null sequence $\{a_n\}$ of numbers is said to belong to class

S^{**} if $\quad n\Delta a_n = o(1), \quad (n \to \infty).$

The class S_r^{**} **([16]):** A null sequence $\{a_n\}$ of numbers is said to belong to class

S_r^{**}, $r = 0,1,2,...$ if $\quad n^{r+1}\Delta a_n = o(1), \quad (n \to \infty).$

For $r = 0$, this class reduces to the class S^{**}.

The class C **of Garrett and Stanojevic ([15]).** A null sequence $\{a_n\}$ belongs to class C if for every $\in > 0$, there exists a $\delta(\in) > 0$, independent of n, and such that

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \in, \qquad \text{for all } n \ge 0.$$

(1.3) The following results about the behaviour of cosine and sine series are known:

Theorem I ([2],[21], [40]). If $\{a_n\}$ is a quasi-convex null sequence, then

$$f(x)=\sum_{k=1}^{\infty}a_k\cos kx\in L^1[0,\pi] \tag{1.3.1}$$

Theorem II ([21], [38]). If $\{a_n\}$ is a quasi-convex null sequence, then

$$\sum_{k=1}^{\infty}a_k\sin kx \tag{1.3.2}$$

is a Fourier series if and only if $\sum_{k=1}^{\infty}\frac{|a_k|}{k}<\infty$.

Theorem III ([39]). If $\{a_n\}$ is a null sequence such that

$$\sum_{n=1}^{\infty}n^2\left|\Delta^2\left(\frac{a_n}{n}\right)\right|<\infty \tag{1.3.3}$$

then (1.3.1) and (1.3.2) are the Fourier series, or equivalently they represent integrable functions.

Concerning the integrability of trigonometric series belonging to the class S (introduced already in article 1.2), Teljakovaskii [33] established the following results:

Theorem IV. Let the cosine series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

belongs to the class S. Then, this is a Fouier series and the following relation holds:

$$\int_0^{\pi}\left|\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx\right|dx\le C\sum_{k=0}^{\infty}A_k$$

where C is an absolute constant.

Theorem V. Let the sine series

$$\sum_{n=1}^{\infty}a_n\sin nx$$

belongs to the class S. Then the following relation holds for $p=0,1,2,.......$

$$\int_{\pi/(p+1)}^{\pi}\left|\sum_{n=1}^{\infty}a_n\sin nx\right|dx=\sum_{k=1}^{p}\frac{|a_k|}{k}+O\left(\sum_{k=1}^{\infty}A_k\right)$$

We observe that Theorem I and Theorem III provide just only the sufficient conditions for the integrability of cosine series. Rees and Stanojevic [9] showed that $\sum_{k=1}^{\infty}\frac{a_k}{k}<\infty$ is a necessary and sufficient condition for $L^1[0,\pi]$ integrability but for a different type of cosine sums. They proved the following results.

Theorem VI. Let $b_k=\frac{a_k}{k}\downarrow 0$. Then

$$g(x)=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{b_k}{2}+\left(\sum_{j=k}^{n}b_j\right)\cos kx\right]$$

exists for $x\in(0,\pi]$ and $g\in L^1[0,\pi]$ if and only if $\sum_{k=1}^{\infty}b_k<\infty$.

Theorem VII. Let $b_k=\frac{a_k}{k}\downarrow 0$. Then

$$\frac{1}{x}\sum_{k=1}^{\infty}\frac{b_k}{2}\sin\left(k+\frac{1}{2}\right)x=\frac{h(x)}{x},$$

converges for $x\neq 0$ and $\frac{h(x)}{x}\in L^1[0,\pi]$ if and only if $\sum_{k=1}^{\infty}b_k<\infty$.

Theorem VIII. Let $(k+1)\left|\Delta^2 a_k\right|\downarrow 0$.Then

$$h(x)=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{2}(k+1)\left|\Delta^2 a_k\right|+\left(\sum_{j=k}^{n}(j+1)\left|\Delta^2 a_j\right|\right)\cos kx\right]$$

exists for $x\in(0,\pi)$ and $h\in L^1[0,\pi]$ if and only if $\{a_k\}$ is quasi-convex.

Ram [4] showed that the condition S is sufficient for the integrability of Rees-Stanojevic sums [9]

$$g_n(x)=\frac{1}{2}\sum_{k=0}^{n}\Delta a_k+\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_j)\cos kx \tag{1.3.4}$$

Ram proved the following theorems:

Theorem IX. Let the sequence $\{a_k\}$ satisfy the condition S . Then $g(x)=\lim_{n\to\infty}g_n(x)$ exists for $x\in(0,\pi]$, and

$$\int_0^{\pi}\left|g(x)\right|dx\leq C\sum_{k=0}^{\infty}A_k$$

Theorem X. Let $\{a_k\}$ be a sequence satisfying the condition S . Then

$$\frac{1}{x}\sum_{k=1}^{\infty}\Delta a_k \sin\left(k+\frac{1}{2}\right)x = \frac{h(x)}{x},$$

converges for $x \in (0,\pi]$, and $\frac{h(x)}{x} \in L^1[0,\pi]$.

The above theorems were further studied by Ram [5], under a condition where the monotonicity of the sequence in the definition of the class S is replaced by quasi-monotonicity.

Consider the cosine series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n \cos nx \tag{1.3.5}$$

Let the partial sums of (1.3.5) is denoted by $S_n(x)$ and $f(x)=\lim_{n\to\infty}S_n(x)$. Denote the class of sequence of Fourier coefficients $\{a_k\}$ by F . There are subclasses of F for which $a_n \log n = o(1), \quad n \to \infty$ is a necessary and sufficient condition for $\|S_n - f\|_{L^1} = o(1), \quad n \to \infty.$

A subclass G of F is called a class of L^1-convergence if $\|S_n - f\| = o(1), \quad n \to \infty$ if and only if $a_n \log n = o(1), \quad n \to \infty$.

Concerning the L^1-convergence of the cosine series, we have the following classical result of Kolmogorov [2].

Theorem XI. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1.3.5) in the metric space L, it is necessary and sufficient that $a_k \log k = o(1),\ k \to \infty.$

The case, in which the sequence $\{a_k\}$ is convex, of this theorem was established by Young [40].

Generalizing the above classical result, Teljakovskii [34] proved the following result:

Theorem XII. If the coefficient sequence $\{a_k\}$ of the cosine series (1.3.5) belongs to the class S, Then a necessary and sufficient condition for L^1-convergence of (1.3.5) is

$$a_k \log k = o(1),\ k \to \infty$$

Rees and Stanojevic [9] introduced modified cosine sums (1.3.4) and obtained an analogue of theorem XI for these sums. These modified cosine sums approximate their limits better than the classical cosine series as they converge in L^1-metric to the sum of the cosine series whereas the classical cosine series itself may not. They proved the following result:

Theorem XIII. Let f be the sum of the cosine series (1.3.5). Then $g_n(x)$ converges to f in L^1-metric if and only if $\{a_k\}$ belongs to the class C.

Ram [3] proved the following result on L^1-convergence of Rees-Stanojevic sums (1.3.4).

Theorem XIV. If (1.3.5) belongs to class S. Then

$$\|f - g_n\|_{L^1} = o(1), \quad n \to \infty$$

Theorem XII of Teljakovskii [33] follows as corollary of this theorem.

Singh and Sharma [24] proved the above theorem by replacing the monotonicity of sequence $\{A_n\}$ in the definition of class S by quasi-monotonocity of $\{A_n\}$. Their result reads as:

Theorem XV. Let $a_n \in S^*$, then $f_n(x)$ converges to $f(x)$ in L^1- metric.

Further, Ram and Kumari ([6], [30]) introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \tag{1.3.6}$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \sin kx \tag{1.3.7}$$

and studied their L^1-convergence under the condition that the cosine series and sine series belong to the classes R and S. They also deduced the results about L^1-convergence of cosine and sine series. Their results state as below:

Theorem XVI. Let $\{a_n\}$ belongs to the class S . If $\lim_{n\to\infty}|a_{n+1}|\log n = 0$, then

$$\| f - f_n\|_{L^1} = o(1), \qquad n \to \infty$$

Theorem XVII. Let $\{a_n\}$ belongs to the class R . If $t_n(x)$ represents $f_n(x)$ and $g_n(x)$, then $\| f - t_n\|_{L^1} = o(1), \quad n \to \infty.$

Theorem XVII. Let the sequence $\{a_n\}$ belongs to class $S(\delta)$ and $\lim_{n\to\infty}|a_{n+1}|\log n = 0,$

then $\| f - g_n\| = o(1), \quad n \to \infty.$

In chapter II, Theorem XII, XVI have been studied for the cosine series with class $S(\delta)$ by using the modified cosine sum (1.3.6) of Kumari and Ram [30].

Whereas, in chapter III, Theorem XVI have been proved for the cosine series with class $S_r(\delta)$ by using the modified cosine sum (1.3.6) of Kumari and Ram [30].

The objective of chapter IV is to generalize Theorem XIV for the cosine series with class S_r^{**}, $\quad r = 0,1,2,....$ of the coefficient sequences using the modified cosine sum (1.3.4) of Rees–Stanojevic [9] and also obtain the L^1-convergence of the r^{th} derivative of the cosine series.

CHAPTER II

ON L^1-CONVERGENCE OF A MODIFIED COSINE SUM

2.1 Introduction. In this chapter, we shall study the L^1-convergence of cosine series by using the modified cosine sums of Kumari and Ram [30] and the class $S(\delta)$ of coefficient sequence introduced by Zahid and Hasan [34].

Consider the cosine series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \tag{2.1.1}$$

Let the partial sum of (2.1.1) be denoted by $S_n(x)$ and $f(x)=\lim_{n\to\infty} S_n(x)$.

Regarding L^1-convergence of (2.1.1), the following theorem of Teljakovskii [33] is well known:

Theorem A. If the coefficient sequence $\{a_n\}$ of the cosine series (2.1.1) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (2.1.1) is $|a_{n+1}|\log n = o(1), \quad n\to\infty$.

Huseyin Bor [13] studied the L^1-convergence of cosine trigonometric series by using the modified cosine sum of Ram and Kumari [30] as

$$g_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx \tag{2.1.2}$$

They have studied these results under a new class $S(\delta)$ of coefficient sequences defined as:

Class $S(\delta)$ of coefficient Sequence ([34]): A null sequence $\{a_n\}$ belongs to class $S(\delta)$, if there exists a sequence $\{A_n\}$ such that

(i) $\{A_n\}$ is δ-quasi monotone and $\sum n\delta_n < \infty$

(ii) $\sum_{n=1}^{\infty} A_n < \infty,$

(iii) $|\Delta a_n| \le A_n \quad \forall n$

Theorem A. Let $\{a_n\}$ belongs to the class S. If $\lim\limits_{n\to\infty}|a_{n+1}|\log n = 0$, then

$$\|f - g_n\| = o(1), \quad n \to \infty$$

The aim of this chapter is to study the L^1-convergence of Kumari and Ram cosine sums (2.1.2) to a cosine trigonometric series belonging to the class $S(\delta)$ defined by Zaini and Hasan [34].

2.2 Lemmas: The proof of our results are based upon the following two lemmas:

Lemma 1 ([11]): If $|c_k| \le 1$, then

$$\int_0^{\pi}\left|\sum_{k=0}^{n} c_k \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| dx \le C(n+1)$$

where C is a positive absolute constant**.**

Proof: We know that

$$D_k(x) = \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}$$

Since Dirichlet Kernel is bounded.

$$\therefore \qquad \left|\frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| \le B$$

$$\Rightarrow \int_0^{\pi}\left|\sum_{k=0}^{n} c_k \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| dx \le \int_0^{\pi}\left| B\sum_{k=0}^{n} 1\right| dx$$

$$\le B(n+1)\pi = C(n+1)$$

$$\Rightarrow \int_0^{\pi}\left|\sum_{k=0}^{n} c_k \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}\right| dx \le C(n+1)$$

Lemma 2. ([34]). Let $\{a_n\}$ be a δ-quasi monotone sequence with $\sum n\delta_n < \infty$. If

$\sum a_n < \infty$, then $\sum (n+1)\Delta a_n < \infty$.

Proof: Consider the series $\sum_{n=0}^{m} a_n$

Applying Abel's transformation, we get

$$\sum_{n=0}^{m} a_n = \sum_{n=0}^{m-1} (n+1)\Delta a_n - m a_m \tag{2.2.1}$$

and

$$-ma_m = -m\sum_{n=m}^{\infty} \Delta a_n$$

$$\leq m\sum_{n=m}^{\infty} \delta_n$$

$$= \sum_{n=m}^{\infty} n\delta_n = o(1)$$

$-ma_m$ is zero and $\sum a_n < \infty$. from (2.2.1), we get

$$\sum (n+1)\Delta a_n < \infty$$

Hence the proof of lemma.

2.3 Main Result. The main results of this chapter are given as under:

Theorem ([13]). Let $\{a_n\}$ belongs to the class $S(\delta)$. If $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then

$$\| f - g_n \| = o(1), \quad n \to \infty.$$

Proof: We have

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx (\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + ...\Delta\left(\frac{a_n}{n}\right))$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx \left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx$$

$$= S_n(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)$$

where $\tilde{D}_n'(x)$ is the derivative of conjugate Dirichlet kernel.

Now, making use of Abel's transformation and lemma 1, we have

$$\int_0^{\pi} |f(x) - g_n(x)| dx = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) + \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$$

$$\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$$

$$\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$$

$$= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{l=0}^{k} \frac{\Delta a_l}{A_l} D_l(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$$

$$\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{l=0}^{k} \frac{\Delta a_l}{A_l} D_l(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$$

$$\leq C \sum_{k=n+1}^{\infty} (k+1) \Delta A_k + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx \tag{2.3.1}$$

Under the assumed hypothesis, $\sum (k+1)\Delta A_k$ converges and therefore the first term in (2.3.1) tends to zero as $n \to \infty$

Moreover, by Zygmund's Theorem ([1], Vol II, p.458) that $\int_{-\pi}^{\pi} \left| \tilde{D}_n'(x) \right| dx$ behaves like $\log n$.

Therefore, we get

$$\int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right| dx \leq \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$$

$$= \frac{|a_{n+1}|}{n+1} \int_{-\pi}^{\pi} \left| \tilde{D}_n'(x) \right| dx$$

$$= C |a_{n+1}| \int_{-\pi}^{\pi} \left| \tilde{D}_n'(x) \right| dx$$

$$\sim |a_{n+1}| \log n \tag{2.3.2}$$

The conclusion of the theorem now follows from (2.3.1) and (2.3.2).

Corollary ([13]). If (2.1.1) belongs to the class $S(\delta)$ and $\lim_{n\to\infty}|a_{n+1}|\log n = 0$,then

$$\| f - S_n \| = o(1), \quad n \to \infty.$$

Proof: We notice that

$$\int_{-\pi}^{\pi} | f(x) - S_n(x)| dx = \int_{-\pi}^{\pi} | f(x) - g_n(x) + g_n(x) - S_n(x)| dx$$

$$\leq \int_{-\pi}^{\pi} | f(x) - g_n(x)| dx + \int_{-\pi}^{\pi} |g_n(x) - S_n(x)| dx$$

$$\leq \int_{-\pi}^{\pi} | f(x) - g_n(x)| dx + \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$$

Since $\lim_{n\to\infty} \int_{-\pi}^{\pi} | f(x) - g_n(x)| dx = 0,$ by our theorem and $\int_{0}^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n'(x) \right| dx$ behaves like $|a_{n+1}|\log n$. By Zygmund's Theorem [1], for large of n , the conclusion of the corollary follows.

CHAPTER III

ON L^1-CONVERGENCE OF r^{th} DIFFERENTIAL OF RAM KUMARI'S MODIFIED TRIGONOMETRIC SUMS

3.1 Introduction: Consider the cosine series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \tag{3.1.1}$$

Let the partial sum of (3.1.1) be denoted by $S_n(x)$ and $f(x)=\lim_{n\to\infty} S_n(x)$. Further, let $f^r(x)=\lim_{n\to\infty} S_n^r(x),\ \ r\in\{0,1,2,.....\}$ where $f^r(x)$ represents r^{th} derivative of $f(x)$ and $S_n^r(x)$ represents r^{th} derivative of $S_n(x)$.

Ram and Kumari introduced a modified cosine sums [30] as

$$g_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx \tag{3.1.2}$$

and Huseyin Bor [13] studied the L^1-convergence of cosine series by using above modified cosine sum.

Further, Zaini and Hasan [34] generalized the Sidon class S to a new class $S(\delta)$ defined as follows:

Class $S(\delta)$ of coefficient Sequence [34] : A null sequence $\{a_n\}$ belongs to class $S(\delta)$, if there exists a sequence $\{A_n\}$ such that

(i) $\{A_n\}$ is δ-quasi monotone and $\sum n\delta_n<\infty$,

(ii) $\sum_{n=1}^{\infty} A_n<\infty$,

(iii) $|\Delta a_n|\le A_n \quad \forall n$

Now, we introduce a new extended Class $S_r(\delta),\ r=0,1,2,.....$ of coefficient sequences defined as:

Class $S_r(\delta)$ of coefficient Sequence: A null sequence $\{a_n\}$ belongs to class $S_r(\delta),\ r = 0,1,2,.....$, if there exists a sequence $\{A_n\}$ such that

(i) $\{A_n\}$ is δ-quasi monotone and $\sum n^{r+1}\delta_n < \infty$

(ii) $\sum_{n=1}^{\infty} n^r A_n < \infty,$

(iii) $|\Delta a_n| \le A_n \quad \forall n$

For $r = 0,$ the class $S_r(\delta)$ reduces to class $S(\delta)$.

The aim of this chapter is to study the L^1-convergence of the r^{th} derivative of modified cosine trigonometric sums (3.1.2) under a new class $S_r(\delta)$ of coefficient sequences.

3.2 Lemmas. The proof of our results are based on the following lemmas:

Lemma 1 ([37]). Let r be a non-negative integer and $x \in \left[\frac{\pi}{n}, \pi\right]$, where $n \ge 1$. Then

$$D_n^{(r)}(x) = \sum_{k=0}^{r-1} \frac{(n+\frac{1}{2})^k \sin\left[(n+\frac{1}{2})x + \frac{k\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}} \psi(x) + \frac{(n+\frac{1}{2})^r \sin\left[(n+\frac{1}{2})x + \frac{r\pi}{2}\right]}{2\sin\frac{x}{2}} \tag{3.2.1}$$

where the same ψ denotes various analytical functions of x independent of n and $D_n^r(x)$ is the r^{th} derivative of Dirichlet kernel.

Proof. The proof is straightforward in the case of $r = 0$. Assuming that Equation (3.2.1) holds, and taking the derivative of both sides in this equation, we have

$$D_n^{(r+1)}(x) = \sum_{k=0}^{r-1} \left(n+\frac{1}{2}\right)^{k+1} \sin\left[\left(n+\frac{1}{2}\right)x + \frac{(k+1)\pi}{2}\right]\left(\sin\left(\frac{x}{2}\right)\right)^{k-r-1} \psi(x)$$
$$+ \left(n+\frac{1}{2}\right)^{k} \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k\pi}{2}\right]\left(\sin\left(\frac{x}{2}\right)\right)^{k-r-2} \psi(x)$$

$$+\frac{\left(n+\frac{1}{2}\right)^{r+1}\sin\left[\left(n+\frac{1}{2}\right)x+\frac{(r+1)\pi}{2}\right]}{2\sin\frac{x}{2}}$$

$$+\frac{\left(n+\frac{1}{2}\right)^{r}\sin\left[\left(n+\frac{1}{2}\right)x+\frac{r\pi}{2}\right]}{\sin^{2}\left(\frac{x}{2}\right)}\psi(x)$$

Therefore

$$D_n^{(r+1)}(x)=\sum_{k=0}^{r}\left(n+\frac{1}{2}\right)^{k}\sin\left[\left(n+\frac{1}{2}\right)x+\frac{k\pi}{2}\right]\left(\sin\left(\frac{x}{2}\right)\right)^{k-r-2}\psi(x)$$

$$+\frac{\left(n+\frac{1}{2}\right)^{r+1}\sin\left[\left(n+\frac{1}{2}\right)x+\frac{(r+1)\pi}{2}\right]}{2\sin\frac{x}{2}}$$

The proof follows by induction.

Moreover, we can also write it as follow

$$D_n^{(r)}(x)=\sum_{k=0}^{r}\frac{(n+\frac{1}{2})^{k}\sin\left[(n+\frac{1}{2})x+\frac{k\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}}\psi(x)$$

Lemma 2.([11]). If $|a_k|\leq 1$, then

$$\int_0^{\pi}\left|\sum_{k=0}^{n}a_k\frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}\right|dx\leq C(n+1)$$

where C is a positive absolute constant**.** Moreover, by Bernstein's inequality, for $r=0,1,2,.....$

$$\int_{\frac{\pi}{n+1}}^{\pi}\left|\sum_{k=0}^{n}a_kD_k^{r}(x)\right|dx\leq C(n+1)^{r+1}$$

where $D_n^{r}(x)$ represents the r^{th} derivative of Dirichlet kernel.

Lemma 3 [37]. $\left\|\tilde{D}_n^r(x)\right\|_{L^1} = O(n^r \log n), \quad r = 0,1,2,...$ where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet kernel.

Proof. Let r be any non-negative integer.

Now with the Bernstein inequality in L^p -space, we have

On the other hand,

$$\int_0^\pi \left|\tilde{D}_n^r(x)\right| dx \le n^r \int_0^\pi \left|\tilde{D}_n(x)\right| dx$$

$$\lim_{n\to\infty} \int_{\frac{\pi}{n}}^{\pi} \left|\tilde{D}_n(x)\right| dx \le \lim_{n\to\infty} \int_{\frac{\pi}{n}}^{\pi} \frac{\pi}{x} dx$$

$$= \lim_{n\to\infty} \pi \left[\log x\right]_{\frac{\pi}{n}}^{\pi} = \pi\left[\log \pi - \log \frac{\pi}{n}\right]$$

$$= \pi\left[\log \pi - \log \pi + \log n\right]$$

Therefore

$$= \pi \log n = O(\log n)$$

$$\left\|\tilde{D}_n^r(x)\right\|_{L^1} = O\left(n^r \log n\right), \qquad r = 0,1,2,....$$

Lemma 4. Let $\{a_n\}$ be a δ -quasi monotone sequence with $\sum n^{r+1}\delta_n < \infty$. If $\sum n^r a_n < \infty$, then $\sum (n+1)^{r+1} \Delta a_n < \infty$.

Proof: We prove it by induction,

Consider

$$\sum_{n=0}^{m} n^r a_n \le \sum_{n=0}^{m-1} (n+1)^{r+1} \Delta a_n - m^{r+1} a_m \qquad (3.2.2)$$

for $r = 0$, this is hold.

Assume that (3.2.2) hold for $r = k$

$$\sum_{n=0}^{m} n^k a_n \le \sum_{n=0}^{m-1} (n+1)^{k+1} \Delta a_n - m^{k+1} a_m \qquad (3.2.3)$$

Now, for $r = k+1$, Multiply (3.2.3) by n,

$$\sum_{n=0}^{m} n^{k+1}a_n \le \sum_{n=0}^{m-1} n(n+1)^{k+1}\Delta a_n - m^{k+1}na_m$$

$$\le \sum_{n=0}^{m-1}(n+1)^{k+2}\Delta a_n - m^{k+2}a_m$$

and $$-m^{k+1}a_m = -m\sum_{n=m}^{\infty}\Delta a_n$$

$$\le m\sum_{n=m}^{\infty}\delta_n$$

$$= \sum_{n=m}^{\infty} n\delta_n = o(1)$$

$-m^{k+1}a_m$ is zero and $\sum n^r a_n < \infty$. from (3.2.2), we get

$$\sum (n+1)^{r+1}\Delta a_n = o(1) \quad \text{as} \quad n \to \infty.$$

This is proved by induction.

3.3 Main Result. The main results of this chapter are given as under:

Theorem. If the $\{a_k\}$ belongs to class $S_r(\delta)$ and $n^r|a_{n+1}|\log n = o(1), \quad n \to \infty,$ then

$$\|f^r - g_n^r\| = o(1), \quad n \to \infty$$

Proof. We have

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx$$

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}k\cos kx(\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \ldots\ldots\ldots\Delta\left(\frac{a_n}{n}\right))$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n}k\cos kx\left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n}a_k\cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n}k\cos kx$$

$$= S_n(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)$$

we have then,

$$g_n^r(x) = S_n^r(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) \tag{3.3.1}$$

where $g_n^r(x)$ represents the r^{th} derivative of $g_n(x)$ and $\tilde{D}_n^r(x)$ represents the r^{th} derivative of Conjugate Dirichlet kernel.

Since $\{a_n\} \in S_r(\delta), \; r \in \{0,1,2,.....\}$ is a null sequence and $\tilde{D}_n^r(x)$ is bounded in $(0,\pi]$, therefore

$$\lim_{n\to\infty} g_n^r(x) = \lim_{n\to\infty} S_n^r(x) = f^r(x)$$

Now, it follows from (3.3.1) that

$$f^r(x) - g_n^r(x) = \sum_{k=1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) - \sum_{k=1}^{n} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x)$$

$$= \sum_{k=n+1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x)$$

Making use of Abel's transformation, we have

$$f^r(x) - g_n^r(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) - a_{n+1} D_n^r(x) + \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x)$$

where $D_n^r(x)$ denotes the r^{th} derivative of Dirichlet kernel.

$$\left\| f^r(x) - g_n^r(x) \right\| = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) - a_{n+1} D_n^r(x) + \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$$

$$\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) \right| dx + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$$

$$= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k^r(x) \right| dx + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$$

Applying Abel's transformation, we have

$$\left\| f^r - g_n^r \right\| = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j^r(x) \right| dx + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$$

$$\leq \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i^r(x) \right| dx + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$$

Moreover, by use of lemma 2, 3 and 4, we have

$$\left\| f^r - g_n^r \right\| \le C\int_0^\pi \left| \sum_{k=n=1}^{\infty} (k+1)^{r+1} \Delta A_k \right| dx + O\left(n^r |a_{n+1}| \log n\right) + O\left(n^r |a_{n+1}| \log n\right)$$

Therefore $\left\| f^r - g_n^r \right\| = o(1) + O\left(n^r |a_{n+1}| \log n\right)$

Hence

$$\left\| f^r - g_n^r \right\| = o(1),\ n \to \infty \quad \text{if only if} \quad n^r |a_{n+1}| \log n = o(1),\ n \to \infty$$

Corollary 1. If $\{a_n\}$ belongs to class $S_r(\delta)$, $r = 0,1,2,......$ then

$\left\| f^r - S_n^r \right\|_{L^1} = o(1),\ n \to \infty$ if and only if $|a_{n+1}| n^r \log n = o(1),\ n \to \infty$.

Proof. Consider

$$\begin{aligned}\left\| f^r - S_n^r \right\| &= \left\| f^r - g_n^r + g_n^r - S_n^r \right\| \\ &\le \left\| f^r - g_n^r \right\| + \left\| g_n^r - S_n^r \right\| \\ &= \left\| f^r - g_n^r \right\| + \left\| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right\| \\ &= \left\| f^r - g_n^r \right\| + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx\end{aligned}$$

Further, $\left\| f^r - g_n^r \right\|_{L^1} = o(1),\quad n \to \infty$ (by theorem) and also we know that

$\int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| dx$ behaves like $n^r |a_{n+1}| \log n$ for large value of n by lemma 3.

Thus $\left\| f^r - S_n^r \right\|_{L^1} = o(1),\quad n \to \infty$ if and only if $n^r |a_{n+1}| \log n = o(1),\quad n \to \infty$

CHAPTER IV

INTEGRABILITY AND L^1-CONVERGENCE OF r^{th} DIFFERENTIAL OF REES MODIFIED TRIGONOMETRIC SUMS

4.1 Introduction. Consider the cosine series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \tag{4.1.1}$$

Let the partial sum of (4.1.1) be denoted by $S_n(x)$ and $f(x)=\lim_{n\to\infty}S_n(x)$.

Further, let $f^r(x)=\lim_{n\to\infty}S_n^r(x),\ \ r\in\{0,1,2,.....\}$ where $f^r(x)$ represents r^{th} derivative of $f(x)$ and $S_n^r(x)$ represents r^{th} derivative of $S_n(x)$.

Rees and Stanojevic [9] introduced the modified cosine sums as

$$g_n(x)=\frac{1}{2}\sum_{k=0}^{n}\Delta a_k+\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a_j\cos kx \tag{4.1.2}$$

and studied the L^1-convergence of trigonometric cosine series under class S .

In 1985, Zahid and Hasan [35] studied the L^1-convergence of trigonometric cosine series using the above modified cosine sum (4.1.2) under the class S^{**} of coefficient sequence defined as below:

The class S^{**} ([35]): A null sequence $\{a_n\}$ of numbers is said to belongs to class S^{**} if $n\Delta a_n=o(1)\quad(n\to\infty)$.

Theorem ([35]). Let the sequence $\{a_n\}$ satisfy condition S^{**} . Then

$$g(x)=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\left(\frac{1}{2}\right)\Delta a_k+\sum_{j=k}^{n}\Delta a_j\cos kx\right]$$

exists for $x\in(0,\pi]$ and $g(x)\in L(0,\pi]$.

The aim of this chapter is to study the integrability and L^1 -convergence of r^{th} derivative of Rees and Stanojevic modified cosine sum under a class S_r^{**}, $r = 0,1,2,......$ introduced by J.Kaur and S.S. Bhatia [16] defined as :

Class S_r^{**} of coefficient Sequence ([16]): A null sequence $\{a_n\}$ belongs to class S_r^{**}, $r = 0,1,2,......$ if $n^{r+1}\Delta a_n = o(1),\ \ n \to .\infty$

Remark. For $r = 0,$ the class S_r^{**} reduces to the class S^{**} of Zahid and Hasan [34]. Clearly, $S_{r+1}^{**} \subset S_r^{**}$, but converse is not true.

Example: For $n = 1,2,3,........$ Define $a_n = \dfrac{(-1)^{n+1}}{n^{r+2}},\quad r = 0,1,2,........$ Now $S_{r+1}^{**} \subset S_r^{**}$ but the converse is not true.

$$n^{r+1}\Delta a_n = n^{r+1}(-1)^{n+1}\left[\frac{1}{n^{r+2}} + \frac{1}{(n+1)^{r+2}}\right] = o(1),\quad n \to \infty$$

But

$$n^{r+2}\Delta a_n = n^{r+2}(-1)^{n+1}\left[\frac{1}{n^{r+2}} + \frac{1}{(n+1)^{r+2}}\right]$$

does not tends to zero as n tends to infinity.

3.2 Lemmas. The proofs of our results are based on the following lemmas:

Lemma 1 [37]: $\left\|D_n^{(r)}(t)\right\|_{L^1} = O(n^r \log n),\ \ \mathrm{r} = 0,1,2,....$ where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet kernel.

Proof. Let r be any non-negative integer.

Now with the Bernstein inequality in L^p -space, we have

$$\int_0^\pi \left|D_n^{(r)}(t)\right| dt \le n^r \int_0^\pi \left|D_n(t)\right| dt \tag{4.2.1}$$

On the other hand,

$$\lim_{n\to\infty} \int_{\frac{\pi}{n}}^{\pi} \left|D_n(t)\right| dt \le \lim_{n\to\infty} \int_{\frac{\pi}{n}}^{\pi} \frac{\pi}{x} dx$$

$$= \lim_{n\to\infty} \pi\left[\log x\right]_{\frac{\pi}{n}}^{\pi}$$

$$= \pi\left[\log \pi - \log \frac{\pi}{n}\right]$$

$$= \pi[\log \pi - \log \pi + \log n]$$

$$= \pi \log n = O(\log n)$$

Therefore, by (4.2.1), we have

$$\left\| D_n^{(r)}(t) \right\|_{L^1} = O\left(n^r \log n\right); \qquad r = 0,1,2,.....$$

4.3 Main Result: The main results of this chapter are given as under:

Theorem: Let $\{a_n\}$ be a null sequence and $\{a_n\} \in S_r^{**}, \quad r = \{0,1,2,.......\}$ then

$\left\| g^r - g_n^r \right\| = o(1), \; n \to \infty$

Proof. Consider

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx$$

$$= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} [\Delta a_k + \Delta a_{k+1} + ... + \Delta a_n] \cos kx$$

$$= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} [a_k - a_{k+1} + a_{k+1} - a_{k+2} + ... + a_n - a_{n+1}] \cos kx$$

$$= \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} a_k \cos kx - \sum_{k=1}^{n} a_{n+1} \cos kx$$

$$= \frac{1}{2}[\Delta a_1 + \Delta a_2 + ... + \Delta a_n] + \sum_{k=1}^{n} a_k \cos kx - a_{n+1}\sum_{k=1}^{n} \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x)$$

$$= S_n(x) - a_{n+1} D_n(x)$$

Taking r^{th} derivate on both sides, we get

$$g_n^r(x) = S_n^r(x) - a_{n+1} D_n^r(x) \tag{4.3.1}$$

where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet kernel.

Since $\{a_n\} \in S_r^{**}, \quad r = 0,1,2,....$ is a null sequence and $D_n^r(x)$ is bounded in $(0, \pi]$. Then

$$\lim_{n \to \infty} g_n^r(x) = \lim_{n \to \infty} S_n^r(x) = g^r(x)$$

Now, it follows from (4.3.1), that

$$g^r(x) - g_n^r(x) = \sum_{k=n+1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1} D_n^r(x)$$

Making use of Abel's transformation, we have

$$g^r(x) - g_n^r(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) - a_{n+1} D_n^r(x) + a_{n+1} D_n^r(x)$$

Further, we have

$$\left\| g^r(x) - g_n^r(x) \right\| = \left\| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) \right\|$$

$$\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) \right| dx$$

Moreover, by use of lemma and given hypothesis, we have

$$\left\| g^r(x) - g_n^r(x) \right\| = O\left(\sum_{k=n+1}^{\infty} k^{r+1} \Delta a_k \right)$$

$$= o(1)$$

therefore .

$$\left\| g^r - g_n^r \right\| = o(1), \; n \to \infty$$

Corollary 1. If $\{a_n\}$ belongs to class S_r^{**}, $r = 0,1,2,3,.......$ then $\left\| g^r - S_n^r \right\|_{L^1} = o(1), \; n \to \infty$

if and only if $n^r |a_{n+1}| \log n = o(1), \quad n \to \infty$.

Proof. Consider

$$\left\| g^r - S_n^r \right\| = \left\| g^r - g_n^r + g_n^r - S_n^r \right\|$$

$$\leq \left\| g^r - g_n^r \right\| + \left\| g_n^r - S_n^r \right\|$$

$$= \left\| g^r - g_n^r \right\| + \left\| a_{n+1} D_n^r(x) \right\|$$

$$= \left\| g^r - g_n^r \right\| + \int_0^{\pi} \left| a_{n+1} D_n^r(x) \right| dx$$

By using, lemma (1) , we get,

$$= \left\| g^r - g_n^r \right\| + O(n^r |a_{n+1}| \log n)$$

Further, $\left\| g^{r} - g_{n}^{r} \right\|_{L^{1}} = o(1), \quad n \to \infty$ (by theorem 1).

Thus $\left\| g^{r} - S_{n}^{r} \right\|_{L^{1}} = o(1), \quad n \to \infty$ if and only if $n^{r} \left| a_{n+1} \right| \log n = o(1), \quad n \to \infty$.

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