HEAT TRANSFER OF HYBRID NANOFLUID OVER A NON LINEAR STRETCHING SHEET IN THE PRESENCE OF MAGNETIC FIELD

A Thesis

submitted in partial fulfillment of the requirement for the award of the degree of

Masters of Science
(Mathematics and Computing)

Submitted by

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Under the Supervision

of

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Determination is
Nothing without Dedication and
Hard work.

This Dissertation is Dedicated to my

Family, Friends, Future Researches, Supervisor

and Above all....

TO THE ALMIGHTY GOD!

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"Heat Transfer of Hybrid Nanofluid Over a Non Linear Stretching Sheet in the Presence of Magnetic Field"** in partial fulfillment of the requirements for the award of degree of Masters of Science in Mathematics and Computing and submitted to the **School of Mathematics** (SOM), Thapar Institute of Engineering and Technology, Patiala is an authentic record of my own carried out under the supervision of **Dr. Sapna Sharma, Associate Professor** and other research work which is duly listed in the reference section.

The matter presented in this thesis has not been submitted elsewhere for the award of any other degree or diploma from this or any other institution.

Date : 3 July, 19

Nishtha
(Reg. No. 301703022)

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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(Nishtha)

Abstract

Laminar boundary layer flow over nonlinear stretching sheet is a significant type of flow having considerable practical applications in the field of engineering and sciences. The present study includes the study of steady natural convection of boundary layer flow of incompressible hybrid nanofluid over a non-linear stretching sheet in the presence of magnetic field, heat source and Brownian motion. Hybrid nanofluid is taken as a new working fluid as it exhibits better thermophysical and heat transfer properties than conventional nanofluids. The governing system of nonlinear partial differential equations are transformed to the system of ordinary differential equation using similar non dimensional parameters. These equations are solved numerically using finite element method. The chapter wise details of the thesis is summarized below :

Chapter 1: This chapter includes the basic definitions of fluid, fluid dynamics, and computational fluid dynamics. The various basic laws and governing equations (Navier Stoke's equations) in theory of fluid dynamics is stated. This further includes a brief introduction about heat transfer and modes of heat transfer.

Chapter 2: In this chapter, the study of two-phase fluid flow is outlined including classification based on states of matter and flow regime, applications in various fields. Different modelling approaches are given, out of which one basic model (Homogeneous model) is explained. As nanotechnology has enticed good attention in boosting base fluid, the new class of nanofluid called hybrid nanofluid is invented. The review, thermophysical properties, applications, fluid flow characteristics of hybrid nanofluid is incorporated.

Chapter 3: This chapter explains the finite element method used to find the solution to the present problem. History, basic features, applications, advantages of the method is stated. The basic methodology used to solve the system of nonlinear equations is illustrated.

Chapter 4: In this chapter, the mathematical model of the present problem is described. The study of laminar boundary flow of hybrid nanofluid over a nonlinear stretching sheet in the presence of magnetic field is analyzed. The finite element solution of the problem is calculated. The impact of various pertinent parameters like Hartmann number, volume fraction, Heat source parameter, Brownian motion is analyzed on velocity and temperature.

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List of Notations

(u, v)	velocity components in (x, y) coordinates
g	gravitational accerlation
T	temperature
T_∞	reference temperature of fluid
T_w	sheet temperature
D	diffusion coefficient
b	nonlinear parameter
K_{hnf}	thermal conductivity of hybrid nanofluid
K_f	thermal conductivity of base fluid
K_{np1}, K_{np2}	thermal conductivity of nanoparticles (1,2)
K_B	thermal conductivity due to Brownian motion
$C_{p,hnf}$	heat capacity of hybrid nanofluid
$C_{p,f}$	heat capacity of base fluid
$C_{p,np1}, C_{p,np2}$	heat capacity of nanoparticles (1,2)
Q	heat source
B	magnetic field
d_{hp}	diameter of nanoparticles
Ha	dimensionless Hartmann number
Hs	dimensionless Heat source parameter
P_r	dimensionless Prandtl number
G	dimensionless Grashof number
W_i	test or weight functions

List of Greek symbols

κ_b	boltzmann constant
θ	dimensionless temperature
η	dimensionless parameters
ψ_i	shape functions
μ_{hnf}	dynamic viscosity of hybrid nanofluid
μ_f	dynamic viscosity of base fluid
μ_B	dynamic viscosity due to Brownian motion
ν_f	kinematic viscosity of base fluid

ρ_{hnf}	density of hybrid nanofluid
ρ_f	density of base fluid
ρ_{np1}, ρ_{np2}	density of nanoparticles (1,2)
α_{hnf}	thermal diffusivity of hybrid nanofluid
ϕ_{np1}, ϕ_{np2}	volume fraction of nanoparticles (1,2)
ϕ_{hp}	total volume fraction of hybrid nanoparticles (1,2)
σ_{hnf}	electrical conductivity of hybrid nanofluid
β_{hnf}	thermal expansion coefficient of hybrid nanofluid
β_f	thermal expansion coefficient of base fluid
β_{np1}, β_{np2}	thermal expansion coefficient of nanoparticles (1,2)

Subscripts

hnf	hybrid nanofluid
f	base fluid
$np1, np2$	nanoparticles (1,2)
hp	hybrid nanoparticles
eff	effective
∞	condition corresponding to large values of y where fluid in quiescent
w	condition at the surface of sheet
e	element

Chapter 1

Introduction and Preliminaries

In Nature, whatever we are seeing, exists in two states, either solid or fluid. All materials display deformation under the action of forces. If under the effect of shearing forces deformation in the material continuously increases without restriction, however small, the material is called *fluid*. A fluid was described as a substance that alters its shape in response to any force however small [1]. This constant deformity is manifested in the tendency of fluids to flow.. Fluids are a subset of the material phases, which comprise liquids, gases and plasmas.

Fluids are usually classified as liquid and gases. *Liquid* is a substance that flows freely but of a constant volume and has no definite shape. For most of the purpose, it is sufficient to regard liquids as incompressible fluids. According to the old phrase, "Water takes the shape of the vessel containing it", it switches its shape with an upper free surface depending on the form of the container. *Gas* on the other hand is an air like fluid that freely expands and fills any available space, regardless of the quantity. Gas has no specific shape and volume. The gas can fill any space in which it is placed that's why gas is called as compressible fluid.

As, the air is around us and rivers and oceans are close to us. The flow of a river never ends, nevertheless it is not the same water as before. Bubbles that float along the stagnant water disappear and then develop again, but have never remained the same. In this way, the air and water of rivers and seas move constantly. This type of movement of gas or liquid is known as *flow*, and the analysis of this motion is Fluid mechanics.

Fluid mechanics is concerned with the study of motion of fluids and the forces acting on them. Fluid mechanics serves as a basis for the study of various fields in engineering such as irrigation engineering, environmental engineering, hydraulic machinery, mechanical engineering, chemical engineering, lubrication aeronautics, etc. Fluid mechanics has two branches:

1. **Fluid Statics (hydrostatics):** It is the analysis of fluids at rest. Newton's second law for non-accelerating bodies is the primary equation needed for this branch, i.e. $\sum \overrightarrow{F} = 0$

2. **Fluid Dynamics:** It is the analysis of fluids in motion. Newton's second law for accelerating bodies is the basic equation required in this mechanics, i.e. $\sum \overrightarrow{F} = ma$

In the study of fluid flow, it turns out that it is easier to work in terms of stress rather than force. *Stress* is defined as the ratio of magnitude of the force and the area of the surface upon which it is applied. When a normal force is applied, we have normal stresses. The most important and distinctive characteristic of a fluid is that it continually deforms under the influence of shear stress.

1.1 Law of Fluid Mechanics

The basic law of fluid mechanics is well-known as *"Newton's law of Viscosity"*. According to this law "the shear stress between adjacent liquid layers is proportional to the velocity gradients between the two layers". It is given as:

$$\tau \propto \tfrac{du}{dy}$$

$$\tau = \mu \frac{du}{dy} \tag{1.1}$$

where μ is the Coefficient of Viscosity, τ is Shear stress and $\frac{du}{dy}$ is a velocity gradient.

1.2 Types of Fluid

1. **Newtonian Fluid:** The fluid that does not alter its viscosity with the rate of deformation or shear stain is called Newtonian Fluid. Newtonian fluid follows "Newton's law of viscosity". Examples: water, oil, gasoline, glycerin, etc.

2. **Non-Newtonian Fluid:** The fluid whose viscosity alters with the rate of deformation or shear stain, is called Non-Newtonian fluid. This fluid does not comply with the "Newton's law of viscosity". Examples: Blood, Cream, the longer we whip it thicker it gets, Casson fluid, Micro-polar fluid, Bhingam plastic, etc.

1.3 Governing Equations

In the theory of fluid mechanics, some of the fluid flow characteristics are usually anticipated without actually evaluating them. When the initial values of a certain minimum amount of quantities are known, values can be obtained at certain locations using certain fundamental equations or laws. Some equations are widely relevant in a particular flow field, depending on the conservation law of fluid's physical properties.

1. **Conservation of Mass:** The Law states that "mass can be neither created nor be destroyed". It is also known as Continuity equation.

2. **Conservation of Momentum:** It is the fundamental law which states that "momentum of a system is constant if there are no external forces acting on the system". The equation of motion is derived using Newton's Second Law.

3. **Conservation of Energy:** It is based on First Law of Thermodynamics or Energy Equation, i.e. "the total energy of an isolated system remains constant".

1.4 Navier-Stokes Equations

Navier-Stokes equations are the Mass, Momentum and Energy conservation expressions [2]. These equations are used to solve incompressible or compressible, low or high speed, inviscous or viscous flows.

History

In 18 century famous *"Swiss Mathematician Euler"* devised the equation which is generalization of the equation to describe the flow of incompressible and frictionless fluids. The equation is given as:

$$\frac{\partial u}{\partial t} + u.\nabla u = -\frac{\nabla P}{\rho} \tag{1.2}$$

where u is fluid velocity vector, P is pressure and ρ is density of fluid.

In 1822 *"French Engineer Claude-Louis Marie Henry Navier"* introduced the element of viscosity (μ) for the more authentic and far more complicated problems of viscous fluids. In 1845 this work was enhanced by the *"British physicist and mathematician Sir George Gabriel Stokes"*, although complete solutions were only achieved for easy two-dimensional flows. Gabriel gives the famous stoke's equation. The Navier Stoke's equation was titled from the name of *"Claude-Louis Navier and Gabriel Stokes"*.

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0 \qquad (for\ Compressible\ fluids) \tag{1.3}$$

$$\nabla.(\rho u) = 0 \qquad (for\ Incompressible\ fluids) \tag{1.4}$$

In the case of Incompressible fluids:

$$\frac{\partial u}{\partial t} + u.\nabla u = -\frac{\nabla P}{\rho} + \frac{\mu}{\rho}\ \nabla^2 u \tag{1.5}$$

1.4.1 Computational Fluid Dynamics

CFD is a fluid mechanics branch that apply data structures and numerical analysis to solve and analyze problems. The Computational fluid dynamics is generally used tool to generate the solutions for fluid flow with and without solid interaction, with the growth of computers and ever-growing computational power. CFD is implemented in many areas of study and industry to a broad spectrum of research and engineering issues. Computational fluid dynamics are based on the popular Navier-Stokes equations. The application of fundamental mechanical laws to a fluid provides the governing equations for a fluid. The Conservation of Mass equation, Momentum equation and Energy equation form a set of nonlinear, coupled partial differential equations. There are many engineering problems where, such equations can not be analytically solved. However, for a wide range of engineering issues it is feasible to get approximate computerized solutions to governing equations. Computational Fluid Dynamics (CFD) is concerned with this issue.

1.4.2 Applications of CFD

- **Electronics:** For cooling system's design and analysis.
- **Turbo machinery:** For designing and testing of pumps, compressors, fans, blowers, turbines nozzle and diffusers.
- **Power and Energy:** For designing and analyzing of heat, nuclear and hydropower plants. It is also used to model accident situations.
- **Construction:** For the designing and evaluation of building dams, spillways, canals, HVAC systems.
- **Automotive, Aerospace and Marine:** Vehicle aerodynamic design, combustion modeling, component performance such as turbochargers, propellers and cooling fans, etc.
- **Medical Science:** Design of medical equipment like stunts, blood flow through veins and arteries, pathology.
- **Sports:** Evaluating the performance of athletes, design of high performance gear like swimsuit and helmets.
- **Chemical Industry:** Injection of streams or separation mixing.

1.5 Heat Transfer

In physics, heat is described as the transmission of thermal energy across a well-defined boundary throughout the thermodynamic system. Heat transfer is a phenomenon through which internal energy is transferred from one high temperature material to another low temperature material. Wherever temperature difference is there, heat transfer takes place from a hotter body to a colder body. The flow of heat is all pervasive. The main impact of heat transfer is that particles of one substance collide with the particles of another material. Typically, the more energetic material loses internal energy, i.e. cooling down while the less energetic substance gains internal energy, i.e. heating up [3]. In thermodynamic systems, heat transfer is defined as the movement of heat across the boundary of the system due to a difference in temperature between the system and its surroundings. The most blatant effect of this in our daily lives is a transition stage, in which a material shifts from one state of matter to a different one. Like, ice cube melts at room temperature and shifts from solid state to liquid state.

Heat transfer is very essential in our everyday life. For example, We use warm water in baseboard heaters to heat our homes in the winter, we boil water routinely for cooking purposes. Heat transfer has wide applications in fluid mechanics, architecture, climate engineering, cooling effect and has broad range of implementation in the operation of various devices and systems. When we look inside a contemporary personal computer, we'll see a fan used to cool the electrical circuitry, which becomes hot due to the electrical current flow through resistances. A heat exchanger's size and cost are also determined by considering the heat transfer between the fluid streams of the exchanger.

We often interchangeably use the terms *"heat and temperature"* in heat transfer problems. We usually think that temperatures and heat are identical, but that's not the case. These are interrelated, but are non-identical concepts. Heat refers to the complete energy due to motion of molecules in a substance, while temperature is the estimation of the average molecular energy in a substance. The key distinction is that heat is concerned with thermal energy, whereas temperature deals with the kinetic energy of molecule. Heat is a measure of transition, not a property owned by an object or system. It is therefore categorized as an Process Variable. In contrast, the temperature is a physical property measurable for an object also known as a State Variable.

1.5.1 Thermodynamics and Heat Transfer

Thermodynamics is one of the fundamental physics' branch that focuses on temperature and heat, and their relationship with the energy, work, radiation, and properties of matter bodies. Regardless of the structure of that material or system, the behavior of these measurable quantities is governed by the four laws of thermodynamics. But the rate of heat transfer is the measure of how much heat is transferring and it is main subject of concern. The region of interest is in heat and heat transfer in this writing. Heat transfer obey the laws of thermodynamics.

1.5.2 Four Laws of Thermodynamics

1. **Zeroth law:** Law states that "when two systems are in thermal equilibrium with a third system then, the first two systems are also in thermal equilibrium with each other". This law makes the use of thermometers significant as the third system, and it becomes meaningful to define a temperature scale.

2. **First law:** Law states that "the change in the internal energy is equivalent to the difference between heat supplied to the system from its environment and the work performed by the system on its environment". This is also known as Energy Principle, i.e. "Energy can neither be created nor be destroyed".

3. **Second law:** Law states that "an isolated non-equilibrium system entropy will tend to increase over time, approaching a peak equilibrium value". It expresses the universal principal of decay observable in nature.

4. **Third law:** Law states that "the entropy of a system approaches a constant minimum, as soon as temperature approaches absolute zero". It is an Entropy Statistical Law, which can not achieve zero of absolute temperature.

Thermodynamics covers a range of subjects, particularly physics, chemical engineering and mechanical engineering in the field of science and engineering.

1.5.3 Modes of Heat Transfer

Heat can travel from one place to another in several ways. Heat transfer predominantly takes place in three modes. These are:

1. *Conduction*
2. *Convection*
3. *Radiation*

1. **Conduction:** It is the heat transfer mode in which heat exchange takes place within the solid or between solid objects in thermal contact with each other. The heat transfer takes place from the body of high temperature to the low temperature object. This occurs because the temperature gradient is present between the bodies. Conduction is the movement of heat through a substance by the collision of molecules. For example, on an ironing board a shirt is put for ironing. Heat from the iron is brought to the shirt so that all these nasty wrinkles are easily ironed and the shirt looks smooth. The science of heat conduction mainly focuses on determination of temperature distribution.

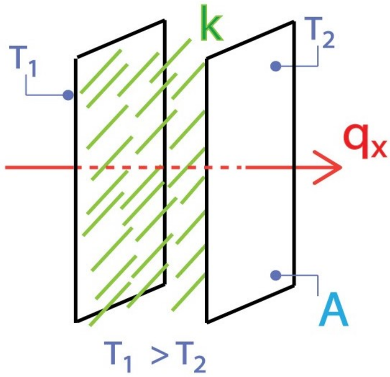


Figure 1.1: Conduction

The fundamental law giving the connection between thermal flow and temperature gradient is entitled by the name of famous *"French Mathematical Physicist, Joseph Fourier"* [4]. In his heat analysis theory, he used this law. The *"Fourier Law"* states that "the time rate of heat transfer through a material is proportional to the negative gradient in the temperature and to the area".

$$Q(r,t) = -kA\ \nabla T(r,t) \tag{1.6}$$

where temperature gradient $\nabla T(r,t)$ is the vector orthogonal to the surface, heat flux $Q(r,t)$ represents heat flow per unit time, in the direction of decreasing temperature, k is the thermal conductivity and A is the area of the surface.

2. **Convection:** When a fluid comes into contact with the solid surface at a distinct temperature, the resulting exchange of thermal energy is called convection. In solids and liquids, convection is generally the dominant heat transfer mode. Convection heat transfer takes place both due to diffusion, the random Brownian motion of individual particles present in the liquid and through advection. There are a lot of everyday examples of convection like boiling water, steaming cup of hot tea, etc. In atmospheres, oceans, planetary mantles, convection takes place on a large scale.

   The rate of convective heat transfer is governed by the *"Newton's Law of Cooling"* which states that "the rate of heat loss of a body is directly proportional to the difference in its own temperature and the ambient temperature".

$$\frac{dQ}{dt} = -h\ A\ (T_{body} - T_{env}) \tag{1.7}$$

$$\frac{dQ}{dt} = -h\ A\ \Delta T \tag{1.8}$$

   where Q is the thermal energy, A is the heat transfer surface area, T_{body} is the temperature of the objects surface and interior, T_{env} is the temperature of the environment. The heat transfer coefficient h relies on the physical characteristics of the fluid and the physical condition in which convection happens. There are two major kinds of Convection:

- *Forced Convection:* It is a unique form of heat transfer where liquids are forced to move in order to improve the transfer of heat. A roof ventilator, a pump, an extracting device or any other means for this forcing can be used. It is basically used to improve the heat exchange. In everyday life, this mechanism is discovered very frequently, including central heating, air conditioning, steam turbines.
- *Natural Convection:* It is a process or form of heat transport in which the fluid movement happens due to variations in the fluid density due to temperature gradients. In natural convection, the fluid surrounding a heat source gets heat and becomes less dense and increases as a result of thermal expansion. Due to its existence in both nature and engineering applications, it has gathered a lot of attention from scientists.

3. **Radiation:** It means the emission and transfer of energy through the space or material medium in the form of electromagnetic waves or particles. Electromagnetic radiations such as radio waves, microwaves, alpha radiation, beta radiation, acoustically radiated materials, etc. are included. Thermal radiation between two bodies depends on the difference between the absolute temperature of bodies. Radiation and radioactive substances are used for diagnosis, cancer treatment and to determine age of material that were once part of a living organism.

   Heat transfer via radiation takes place primarily in the infrared region in the form of electromagnetic waves. Radiation emitted by a body is the outcome of thermal agitation of composing molecules. The transmission of radiation can be explained with reference to black body. The energy of radiation for black bodies can be obtained by using the *"Stefan-Boltzmann Law"* which states that "the radiation energy emitted per unit time from a black body is proportional to the fourth power of the absolute temperature".

$$Q = \sigma T^4 A \tag{1.9}$$

where σ is Stefan-Boltzmann constant. This constant has the value $5.67036710^{-8} W/m^2/K^4$.

We can not directly measure the heat flow, but the notion has a real physical significance. This is because a measurable scalar quantity known as temperature is associated with it. Therefore the thermal transfer in the body can be easily estimated using laws based on heat flow and temperature once the temperature distributed within the body has been determined.

Chapter 2

Two Phase Fluid Flow and Hybrid Nanofluid

2.1 Introduction

Multi-phase fluid flow is the concurrent flow of materials with different states or phases (i.e. gas, liquid or solid) or materials with distinct chemical characteristics, but in the same state or phase i.e. liquid liquid structures (e.g. oil droplets in water). With a two-phase flow, we mean a unique flow problem in which we must concurrently consider the mechanics of two phases of matter. In a wide spectrum of engineering systems, the subject of two or multi-phase flow has become progressively essential for their optimum design and safe operations.

Examples

- **In Nature:** Water wave, Dew on leaves, waterfall, Spitting Cobra etc.
- **In Daily Life:** Sweating, Mixing Tea, Filling Glass, Mixing Butter Milk etc.
- **Sports:** Swimming, Ice Skating, Rafting, Water Polo etc.

2.2 Applications of Two-Phase Fluid

- **Heat Transfer Systems:** Heat exchangers, evaporators, direct thermal exchangers, condensers, cooling spray towers, coolers and electronic cooling systems, heat pipes, dryers, etc.

- **Transportation Systems:** Air-lift pump, ejectors, pipeline transport of gas and oil mixtures; of fibers, of wheat, and pulverized solid particles, pumps with cavitations, management of traffic flow of highway, etc.
- **Lubrication Systems:** Cryogenic cooling, two-phase flow lubrication, etc.
- **Environmental Control:** Air conditioning, cooling and refrigerators, dust collectors, treatment plants, air pollution control separator, space life support systems, etc.
- **Biological Systems:** Cardiovascular system, respiratory system, gastrointestinal tract, flow of blood, flow of bronchus and nasal cavity, conveyance of capillaries, control of body temperature, etc.

2.3 Classification of Two-Phase fluid

Depending on the combination of different phases or states and the interface structures, there are wide range of two-phase flows. Two-phase mixtures at the interface are characterized by existence of one or more interfaces and discontinuities. Two-phase mixtures can be easily classified according to the two-phase combinations, as we have only three material states in standard conditions, solid, liquid and gas (possibly plasma).

- *Solid-Gas Mixture*
- *Liquid-Gas Mixture*
- *Solid-Liquid Mixture*

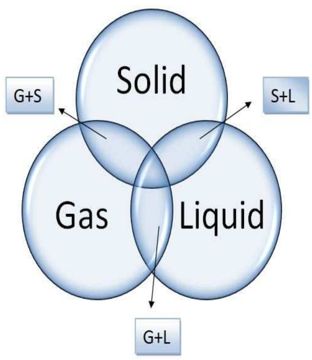


Figure 2.1: Classification based on states of matter

Based on the interface systems, the second classification is much more complex to create as this interface structure continually changes. In accordance with the geometry of the interfaces, two-phase flow can be categorized into three primary groups: Separated flow, temporary or mixed flow and distributed flow as shown in fig. 2.2. Here we follow the standard flow regimes proposed by Ishii [5]. The class of *separated flow* may be split into three systems depending on the type of interface, i.e. film flow, annular, and the jet-flow regimes.

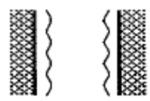
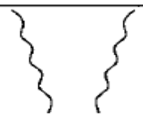
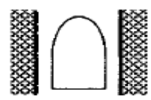
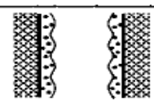
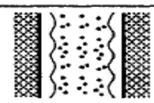
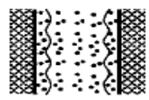
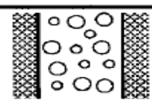
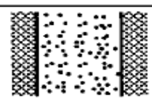
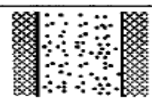
Class	Typical regimes	Geometry	Configuration	Examples
Separated flows	Film flow		Liquid film in gas Gas film in liquid	Film condensation Film boiling
	Annular flow		Liquid core and gas film Gas core and liquid film	Film boiling Boilers
	Jet flow		Liquid jet in gas Gas jet in liquid	Atomization Jet condenser
Mixed or Transitional flows	Cap, Slug or Churn-turbulent flow		Gas pocket in liquid	Sodium boiling in forced convection
	Bubbly annular flow		Gas bubbles in liquid film with gas core	Evaporators with wall nucleation
	Droplet annular flow		Gas core with droplets and liquid film	Steam generator
	Bubbly droplet annular flow		Gas core with droplets and liquid film with gas bubbles	Boiling nuclear reactor channel
Dispersed flows	Bubbly flow		Gas bubbles in liquid	Chemical reactors
	Droplet flow		Liquid droplets in gas	Spray cooling
	Particulate flow		Solid particles in gas or liquid	Transportation of powder

Figure 2.2: Classification based on Flow regimes

The *dispersed flow* class can also be split into several kinds i.e. spherical, elliptical, granular particles, etc. can be taken into consideration depending on the interface geometry. However, by considering the dispersion phase, it is more convenient to divide the class. We can therefore distinguish between three regimes, i.e. bubbly, droplet or mist, and particulate flow. The geometry of dispersion can be spherical, spheroidal, distorted, etc. in each regime.

We have the third class which is characterized by the existence of both separate and dispersed flow. Subdividing the category of *mixed flow* according to the dispersion phase was more convenient. We can therefore distinguish five flow regimes, i.e. cap, slug or chum-turbulent flow, bubbly-annular flow, bubbly annular-droplet flow and film flow.

2.4 Modelling Approach for Two-Phase Flow

Mathematical modeling is about mathematically expressing the problem with proper differential equations and appropriate boundary conditions. A broad variety of models for two-phase flow schemes have been developed. Some of the models are listed below:

- **Homogeneous Model:** In the homogenous model, both phases of the channel are considered to be moving at the same rate and the flow is considered to be analogous to a single phase flow.
- **Drift-Flux Model:** Here, drift velocity difference between the phases is considered and overall conservation equations are obtained for mixture.
- **Separated Flow Model:** In this case, the two liquids are deemed to be traveling at distinct velocities and conservation equations are written for phases individually are taken into consideration.
- **Multi-Fluid Model:** Here, for each phase, distinct conservation equations are written. These equations containing terms describes the interaction between the phases.
- **CFD Model:** In contrast to the above models, CFD models generally involve two or three dimensions and try to portray the complete flow field.

The selection of the modelling approach rely on the accessibility and exactness of the data (the more complicated the models are, the more comprehensive information is needed to be fed into them).

2.4.1 Homogeneous Model

This flow model considers the two fluids to be mixed intimately such that they can be approximated as a pseudo fluid with suitable average properties. This model is suitable for bubbly flow, annular flow. Thus, the single phase equations for Continuity, Momentum and Energy can be applied to the two phase mixture. The assumptions of the homogeneous model are as follows:

- Two fluids are uniformly mixed and moving as a pseudo fluid at the homogeneous velocity (u_h), i.e. $u_g = u_f = u_h$
- There is a thermodynamic equilibrium between the phases.

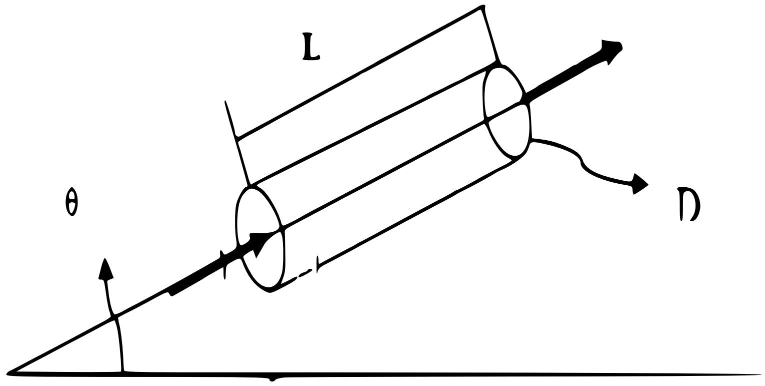


Figure 2.3: Uniform Cross-section of pipe (Homogeneous Model)

Here, we had considered a pipe of uniform cross-sectional area A and diameter D. The pipe is inclined at an angle θ with horizontal as depicted in figure 2.3. Liquid phase (water) is signifying 'f' and gaseous gas phase (air) is denoted by 'g'.

Accordingly, the mass (Continuity equation), momentum equations for two phase homogeneous flow inclined at an angle θ from the horizontal can be written as:
Continuity equation:

$$A\rho_h u_h = constant$$

Momentum equation: Pressure (P) drop in pipeline is given as

$$-\frac{dP}{dz} = -\left(\frac{dP}{dz}\right)_F - \left(\frac{dP}{dz}\right)_z - \left(\frac{dP}{dz}\right)_a \tag{2.1}$$

where dz is the length of pipeline, $(\frac{dP}{dz})_F$ is pressure drop due friction between phases, $(\frac{dP}{dz})_z$ is potential heat, $(\frac{dP}{dz})_a$ is pressure drop due to accerlation.

For tube of diameter (D):

$$-\left(\frac{dP}{dz}\right)_F = \frac{2f_h G^2(v_f + xv_{fg})}{D} \tag{2.2}$$

Accerlation Component:

$$-\left(\frac{dP}{dz}\right)_a = G^2\left\{v_{fg}\frac{dx}{dz} + \frac{dp}{dz}\left[x\frac{dv_g}{dp} + (1-x)\frac{dv_f}{dp}\right] - \frac{v_f + xv_{fg}}{A}\frac{dA}{dz}\right\} \tag{2.3}$$

Pressure drop due to potential heat:

$$-\left(\frac{dP}{dz}\right)_z = \rho_h g\sin\theta = \frac{g\sin\theta}{v_f + xv_{fg}} \tag{2.4}$$

where f_h is homogeneous friction factor, ρ_h is homogeneous density, G is mass flux, v_f is volume of fluid, x is Mass-quality, $p = \Pi D$ is perimeter of pipe.

Substituting (2.2), (2.3), (2.4) in equation (2.1) and for uniform cross-section ($\frac{dA}{dz} = 0$) of pipe. We get,

$$-\frac{dP}{dz} = 2f_h\frac{G^2}{D}\left(v_f + xv_{fg}\right) + G^2 v_{fg}\frac{dx}{dz} - \frac{g\sin\theta}{v_f + xv_{fg}} \tag{2.5}$$

In order to find pressure gradient from the equation (4.3), the only unknown is f_h, the equivalent friction factor during two phase flow under homogeneous equilibrium and it depends on the type of flow i.e.

If flow is *Laminar*: $f_h = \frac{64}{R_e}$
If flow is *Turbulent*: $f_h = 0.079R_e^{-1/4}$

where R_e is Reynold's Number.

Further, making the use of above mentioned model we will define and solve the problem. The problem is based on incompressible two phase fluid (Hybrid Nanofluid), which is explained in next section.

2.5 Hybrid Nanofluid

Non-Newtonian two-phase fluid experiments have accomplished a excellent position in pioneer research in latest years, mostly because of their broad range of utilization in different problems. Later, advancement in nanotechnology resulted to the new innovation called *Nanofluid*. Nanofluids which were first invented by *"Choi"* [6] in 1995 are engineered colloidal suspensions of nanosized particles (smaller than 100nm) in a base fluid. Nanosized particles which are used in the base fluid for dispersion are nanotubes, nanoparticles, nanofibres and nanowires. Materials usually used as nanoparticles includes metal oxides (e.g. Al_2O_3, CuO), chemically stable metals (e.g. gold, silver, copper, iron, aluminium) and multiple types of carbon (e.g. nano-diamond, graphite), single-walled carbon nanotubes, double-wall and multi-wall composites, fullerenes and shell composites. The base fluids used are mainly water, organic liquids (ethylene glycol), oils and lubricants, bio fluids and other conventional fluids.

The implementation of nanoparticles improves the efficiency of heat transfer of base fluids. In many industrial and technological applications, nanofluids are widely used, such as polymer melting, biological solutions, paints, tars. The thermal conductivity of nanoparticles generally exceeds in order of magnitude when compared to those with base fluids. Nanofluids are mainly used for their enhanced thermal properties as coolants in heat transmission equipment such as heat exchangers, electronic cold systems and radiators.

Later, an experimental research on nanofluid with two kinds of nanoparticles called as

Hybrid Nanofluid was conducted [7]. Hybrid nanofluid is a unique class of nanofluid engineered by dispersion of two entirely different kind of nanoparticles into the standard heat transfer fluid. The term *hybrid* can be regarded as a combination of the physical and chemical properties of nanoparticles to form a homogeneous mixture. If particle materials are properly selected, they may improve each other's positive characteristics and cover one single material's disadvantage. The preparation of hybrid nanofluid is highly critical and crucial to the stability of the nanofluid. In order to use hybrid nanofluid for reliable applications, stability, durability and chemical inertness of suspended nanoparticles are crucial. In 2017, [8] author gives a review on hybrid nanofluid. This review brings together present research on the synthesis, thermophysical properties, thermal transfer features, hydrodynamic behavior, and the fluid flow features of the different hybrid nanofluids reported by the various scientists. Some of hybrid nanofluids are Silver-Graphene with base fluid ethylene glycol, $Al_2O_3 - Cu$ with base fluid water, $SiC - TiO_2$ with base fluid oil, $Fe - CuO$ with base fluid water, $MWCNT - Fe_3O_4$ (multiwall-carbon nanotube) with base fluid water, etc.

2.5.1 Thermophysical Properties of Hybrid Nanofluid

Hybrid nano-fluids are prospective liquids that perform better than traditional heat transfer liquids and nanofluids in terms of thermophysical and heat transfer capability. The main purpose of synthesizing the hybrid nanofluid is to improve the properties of single materials. It has great improvements in thermal or rheological properties that are much better than standard nanofluids. To investigate the performance of any thermal system functioned with hybrid nanofluid, the accurate knowledge of thermophysical properties of working fluid is essential. The nanoparticles is supposed to be dispersed uniformly in base fluids, for deriving the thermophysical properties. Many parameters must be taken into consideration like, type of nanoparticles, nanoparticles shape and size, working temperature, base fluid, etc. These characteristics depend heavily on the amount of nanoparticles being dispersed to the base liquid. The empirical correlations that are accessible to obtain thermophysical properties are very limited in the open literature [8, 9]. This section contains correlations that are consistent with experimental outcomes for the evaluation of density, specific heat, viscosity and thermal conductivity of hybrid nanofluids [10].

Table 2.1: Thermophysical properties of base fluid and Nanoparticles

Physical properties	Fluid phase (water)	Cu	Al_2O_3
C_p	4179	531.8	765
ρ	997.1	6320	3970
K	0.613	76.5	40.0
$\beta \times 10^{-5}$	21.0	1.8	0.85
$d_p(nm)$	–	29	47

2.5.2 Density of hybrid nanofluid

The density of the nanofluid is predicted using the principle of mixture rule. The density of hybrid nanofluid is estimated by *"Takabi"* by expanding the same mixture rule as given below:

$$\rho_{hnf} = \phi_{np1}\rho_{np1} + \phi_{np2}\rho_{np2} + (1 - \phi_{hp})\rho_f \tag{2.6}$$

ϕ_{hp} is the total volume fraction of two individual nanoparticles suspended uniformly in hybrid nanofluid is given as:

$$\phi_{hp} = \phi_{np1} + \phi_{np2} \tag{2.7}$$

2.5.3 Heat Capacity of hybrid nanofluid

Mono nanofluid's heat capacity was also estimated using the mixture rule by *"Pak et al."* with the supposition that there is the thermal equilibrium between nanoparticles and host fluid. By using the same assumption and rule, the heat capacity of hybrid nanofluid is estimated as given below:

$$(\rho C_p)_{hnf} = \phi_{np1}(\rho C_p)_{np1} + \phi_{np1}(\rho C_p)_{np1} + (1 - \phi_{hp})(\rho C_p)_f \tag{2.8}$$

The prediction of density and heat capacity for hybrid nanofluid using mixture rule at different volume fractions are found to be in good agreement with experimental outcomes by [9].

2.5.4 Viscosity of hybrid nanofluid

The flow properties play a very significant role in many heat transfer applications. That's why fluid's viscosity has great predominance. Several inter-relations have been established for dynamic viscosity of nanofluid. In 1906, *"Einstein"* elaborated the first theoretical formula for the assessment of viscosity values for composite and mixture. The model had been developed for linear viscous fluid. Einsteins model is valid for very meager volume fraction ($\phi < 0.02$).

$$\mu_{nf} = \mu_f(1 + 2.5\phi) \tag{2.9}$$

where nf denotes nanofluid and ϕ is the volume fraction of nanofluid.
"Brinkman" further modified Einsteins model which is valid up to 4% of particle volume fraction.

$$\mu_n f = \frac{\mu_f}{(1 - \phi)^{2.5}} \tag{2.10}$$

By introducing the overall effect of particle size and temperature on viscosity *"Nguyen et al"* developed numerous correlations. With particle concentration, the viscosity of the nanofluid increases consistently. While, it reduces as temperature rises because the cohesive force between the particles reduces quickly as the temperature goes up, resulting a decrease in viscosity. The correlations that are provided for dynamic viscosity of nanofluid in literature can be implement to estimate the viscosity for hybrid nanofluid. Using "Brinkman" model, viscosity of hybrid nanofluid can be estimated as:

$$\mu_{hnf} = \mu_f (1 + 2.5\phi_h) \tag{2.11}$$

2.5.5 Thermal Conductivity of hybrid nanofluid

It is one of the eminent thermophysical property, which controls heat transfer properties of fluids. The net thermal conductance of the nanofluid increases rapidly due to convection current between base fluid and nanoparticles by dispersing high thermal conductive nanoparticles in base fluid. The above mentioned reason can also be applied to hybrid nanofluids. Many scientists suggested correlations by anticipating multiple parameters to estimate and illustrate improvement in thermal conductivity. *"Maxwell"* [11] was a innovator in the field, developed and verified the conceptual model for thermal conductivity. For determining the thermal conductivity of nanofluids, it is the basic correlation available till now. Thermal conductivity of hybrid nanofluid can also be calculated using the modified Maxwell model [12] given as:

$$K_{hnf} = K_f \left[\frac{\left(\frac{\phi_{np1}K_{np1} + \phi_{np2}K_{np2}}{\phi_{hp}}\right) + 2K_f + 2(\phi_{np1}K_{np1} + \phi_{np2}K_{np2}) - 2\phi_{hp}K_f}{\left(\frac{\phi_{np1}K_{np1} + \phi_{np2}K_{np2}}{\phi_{hp}}\right) + 2K_f - 2(\phi_{np1}K_{np1} + \phi_{np2}K_{np2}) + \phi_{hp}K_f} \right] \tag{2.12}$$

A vast literature is available on the thermal conductivity of hybrid nanofluid. There are various parameters which affect the thermal conductivity of hybrid nanofluid. These are:

1. **Particle Material:** The size of nanocluster formed around nanoparticles depends on the type of nanoparticle material. It is believed that particles extracted from the material with large thermal conductance, promotes greater effective thermal conductivity to the nanofluid.

2. **Base fluid:** The thickness of an electronic dual layer, generated around the nanoparticles depends on the nature of host fluid used.

3. **Particle Size:** One of the major factor for improved heat conductivity is the Brownian motion of nanoparticles. Also with reduction in particle size, speed of particles increases. Hence, transportation of energy in liquid medium enhances due to increase in colloidal motion.

4. **Particle Shape:** Cylindrical particles exhibit more increased thermal conductivity as compared to particles in a spherical shape. The long length permits the promotion of high heat transfer rates in cylindrical particles.

5. **Volume fraction:** It's relationship with thermal conductivity is linear.

6. **Temperature:** With the increase in temperature, the Brownian motion of nanoparticles improves and thus, results in enhanced thermal conductivity. However, extreme rise in temperature would affect nanocluster size and layer thickness.

2.5.6 Applications of Hybrid Nanofluid

A handful of meaningful experiments have been conducted in the recent many years on various applications of hybrid nanofluids and several review articles covering residential, commercial, industrial and transport applications had also been published recently. Hybrid nanofluids has been used in various heat transfer applications such as heat sink, thermal storage, plate heat exchanger, air conditioning system, tabular heat exchanger, micro-channel, sheet and tube heat exchanger, helical coil heat exchanger, heat pipe, refrigeration. In multiple systems, the extensive use of hybrid nanofluids had been indicated. This fluid is also used as a coolant in machining process like generator cooling, electronic cooling, nuclear system cooling, cooling and heating in buildings, transformer cooling etc. These liquids are reiterated to be appropriate for environmental and economic gains. The efficiency of hybrid nanofluid has demonstrated remarkable progress in electronics, cutting processes and lubrication. In addition, the use of hybrid liquids in the solar power systems had yielded remarkable outcomes. Due to the synergistic impact by which they provide beneficial characteristics of all their components, their practical uses can take place nearly in all branches of heat transfer.

Hybrid nanofluids had a tremendous contribution in field of biology especially in cancer detection and treatment. The aim of this sort of studies is to construct a nanostructure which has a higher therapeutic effect that could be realized from any simple mixture of the individual components. The Scientists have seen numerous instances of nanoparticles that can travel through blood stream and can image the tumors at their earliest stage of development. Imaging of tumors is an outstanding illustration that how nanoparticles can combine their characteristics to improve the surgical therapy of cancer victims. A foremost important application of hybrid nanofluid is the concurrent detection and treatment of diseases, especially cancer. They had demonstrated the ability to perform multiple functions in biological systems.

Chapter 3

Finite Element Method (FEM)

Every natural phenomenon can be outlined in the form of algebraic, differential or integral equations with the help of laws of physics. Majority of scientists and engineers, probing the physical phenomenon are involved with two primary tasks:

1. **Formulation of Mathematical model of the physical process**
2. **Numerical study of the model**

Mathematical formulation of a problem is an art which requires a good vision of the related subject and mathematical tools. The formulation of results in the form of mathematical statements, often includes differential equations and some appropriate quantities of interest for comprehensive knowledge of the process design. It is achieved through some assumptions concerned with the working of the process. Further, in the numerical simulation we use certain statistical methods and computer to evaluate the developed model and estimate the peculiarity and significance of the process. The solution of governing equations using precise techniques is an onerous challenge. Then, approximate methods such as variational methods (like, Rayleigh-Ritz method, Galerkin method, Least square method), and finite difference method gives an excellent way of finding solutions. In the literature, these techniques are often applied.

In *finite difference* approximation, the differential equation involving derivatives are substituted by difference quotients that include the solution values at discrete points of the domain. Thereafter, imposing the boundary conditions, the resulting algebraic equations are solved. In *Variational methods*, the equations are placed in an equivalent weighted integral form. Then, the estimated solution over the domain is supposed to be linear combination of approximation functions (ϕ_i) and undetermined coefficients (c_i), i.e. $\sum_i \phi_i ci$. The coefficients are determined so that integral statement equivalent to original differential equation is satisfied. Large number of variational methods vary greatly in the selection of integral form, weight functions, and approximation functions. The solution thus obtained are continuous.

The main drawback of the variational method that prohibits them from competing with the finite difference method is the inconvenience caused while choosing the approximation functions. If the domain is geometrically complicated or boundary conditions are complex, the selection method becomes more pathetic or even impossible. Most real-world problems are defined on geometrically complex regions, and therefore it is challenging to generate approximation functions that satisfy different types of boundary conditions. This leads to the invention of the new method, which could overcome the drawback of above methods. *Finite element method* destroys all the barriers of long-established methods, by offering an well organized way for finding the approximate functions.

3.1 Historical Background

This method can be retrieved back to work by "Alexander Hrennikoff" (1941) and "Richard Courant" (1942), while solving the problem of torsion of non-circular shafts in late 1940. He divided the shaft cross-section into different triangular elements and assumed a linear combination of the primary variable. The term *"finite element"* was first used by "Ray W. Clough" in his paper in the proceeding and ASCE conference on Electronic computation in 1960 [13]. Clough extended the matrix method to two dimensional continuum domains by dividing domains into triangular segments. "Argyris" around the same time developed the technique in Germany. The process obtained a real impetus in the period from 1960 to 1975 by the Argyris with the co-workers at the University of Stuttgart, Germany. It was extended from static, small deformation, elastic problems to dynamic, non-structural problems like fluid flow and heat transfer problems. In 1965, First conference on "finite elements" was held. In 1967, the first book on the "Finite Element Method" was published by "Zienkiewicz and Chung". The large deformation structural problems, where the domains changes significantly were solved using FEM around 1976 using Lagrangian formulation.

NASA financed the initial version of NASTARA, UC Berkely created the commonly accessible finite element program SAP. Some new packages for analyzing significant deformation problems was incorporated in existing packages like LS DYNA, DEFORM, ANSYS, ABAQUS were developed. With the evolution of the technique of finite elements, along with phenomenal rise in computing capacity and comfort, it is now viable to comprehend structural functioning with a high level of perfection. In fact, this was beyond imagination before the era of the computer.

3.2 Introduction

The finite element method (abbreviated as FEM) is a strong computing technique to solve differential and integral equations that occur in different areas of Engineering and Sciences. The variational and weighted residue methods are heart of FEM. The most exclusive trait of the finite element method that distinguishes it from all other methods is the division of specified geometrically complex domain into the collection of simple sub-domains called *finite elements*. Thus, the given domain can be treated as an assemblage of simple geometric shapes for which the approximation functions necessary to the solution can be generated systematically. The approximation function can be represented over each element by a linear combination of algebraic polynomials. The undetermined coefficients are obtained by satisfying the governing equations. They denotes the solution value at the fixed number of pre-selected nodes on the boundary and interior of the component. The approximation function depend not only on the geometry but also on the nodes. This makes the technique a useful practical gear for the solving the boundary value and initial value problems. The finite element method is element wise implementation of the weighted residual method. The major steps involved in the finite element formulation and evaluations of the problem are:

1. Discretization of the domain into a group of preselected finite elements. The finite element is not just a geometric shape, but it is endowed with certain geometric and physical features. Build the mesh and number the nodes and components of the finite element.

2. Derivation of the element equations for all elements in consideration. Development of the finite element model using the weighted-integral statement or weak form over an element using methods like Galerkin, least-squares, and so on.

3. Assembly of finite elements to attain the global system of algebraic equations.

4. The imposition of boundary conditions of the problem. They must be altered to account for the boundary conditions of the problem, before the system equations are prepared to be solved.

5. The solution of assembled equations.

6. Post computation of the solution or desired quantities.

3.2.1 Discretization of domain

The domain Ω of any given problem is divided into a set of smaller subdomains called finite elements, which are connected at points common to two or more nodes as shown in fig. 3.1. The subdivision process is a vital task. The reason for dividing the domain is twofold: first, to represent the geometry of domain, second, to approximate the solution over each of the element in order to represent the solution in a better way over the entire domain. The number, shape, size and configuration of elements must be pronounced in this mechanism in order to simulate the actual structure in the closest possible way. The number of elements usually depends on the element type and accuracy desired. The order of an element refers to the degree of the polynomial used to represent the solution over the element. To connect the element and impose the continuity of the solution at nodes common to elements, we identify the end points of each line element as the element nodes. The discretization is to be done in such an efficient way so that it converges to the actual solution. Fig. 3.2 demonstrates a finite element mesh of continuum with triangular and quadrilateral components. This situation demonstrates a better representation of the continuum by the assembly of triangular components.

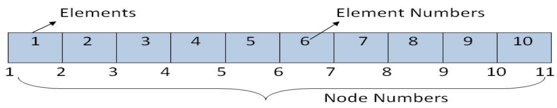
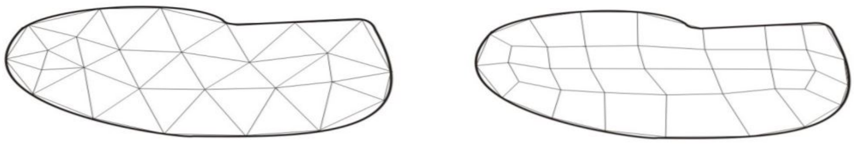


Figure 3.1: Discretization into elements

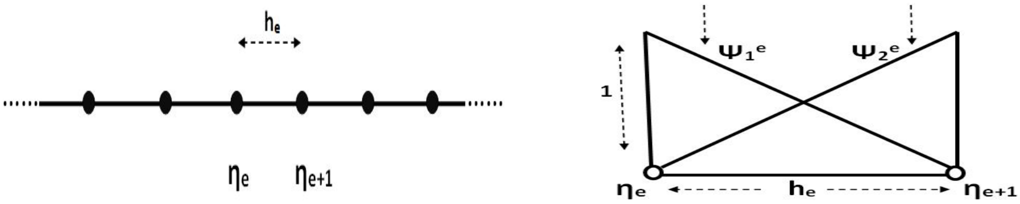


(a) Triangular mesh (b) Quadrilateral mesh

Figure 3.2: Discretization of Continuum

One-dimensional domain

If the domain Ω of the problem is one-dimensional, then it will be partitioned into set of line elements each being of length h_e as shown in fig. 3.3. The group of such elements is called finite element mesh of domain. A typical element $\Omega^e = (\eta_e, \eta_{e+1})$ whose end points are $\eta = \eta_e$ and $\eta = \eta_{e+1}$ are isolated from the mesh.



(a) (b)

Figure 3.3: One-dimensional domain

The shape (approximation) function over a specific line element is given by:

$$\varphi_1^e = \frac{\eta_{e+1} - \eta}{\eta_{e+1} - \eta_e}\,, \qquad \varphi_2^e = \frac{\eta - \eta_e}{\eta_{e+1} - \eta_e} \qquad \text{where, } \eta_e < \eta < \eta_{e+1} \tag{3.1}$$

Approximation of domain, in this case is not required, as it is a straight line. If the domain is a curve, the approximation by set of the straight or curved line is necessary. Hence, approximation of solution over each element is more effortless than approximation over the entire domain.

3.2.2 Derivation of element equation

The derivation of algebraic equations, i.e. element equations that link the primary variables to the secondary variables at the nodes of the element involves three steps:

1. Establish the weighted-residual or weak form of the differential equation.
2. Suppose the approximate solution form over a particular component.
3. Obtain the equations of the finite element by putting the approximate functions into weak form.

Let, us suppose any general problem of finding $u(x)$ that satisfies the differential equation:

$$a\frac{d^2u}{dx^2} + b = 0 \ , \qquad 0 < x < l \ , \qquad \Omega = (0, l) \tag{3.2}$$

We seek the approximate solution to the governing equations over the element using any method like Rayley-Ritz method. In general, any method that permits the deduction of the necessary relations among the nodal values of the dependent variable can be used. The equations arising from the application of the variational method are an alliance between *primary* variables and the *secondary* variables. The polynomial approximation of the solution within a specific element Ω^e is of the form:

$$U^e = \sum_{j=1}^{n} u_j^e \ \varphi_j^e(x) \tag{3.3}$$

where u_j^e are the values of solution at the nodes and φ_j^e are the approximation functions over the element. The necessary and sufficient number of algebraic relations among the u_j^e can be obtained by revising the differential equation (3.2) in weighted residual form:

$$0 = \int_{\eta_e}^{\eta_e+1} w \left[a\frac{d^2u}{dx^2} + b = 0 \right] dx \tag{3.4}$$

where w designates the weight function and $\Omega^e = (\eta_e, \ \eta_{e+1})$ is the domain of the element. For $u \approx U^e$ and for each independent choice of w we obtain an independent algebraic equations connecting all u_j^e. A total of n independent equations are required to solve for n values u_j^e. When w is selected to be φ_j^e and equation (3.4) is used to obtain the jth equation of the required n equations, we will get resulting finite element model.

3.2.3 Interpolation functions

The approximation solution U^e are generally represented in the form of algebraic polynomials because the numerical evaluation of integrals of algebraic polynomials are easy. The approximation functions are often established using concepts from theory of interpolation like Lagrange's interpolation method. Hence, they are called *interpolation functions*. U^e is an interpolant of $u(x)$ over the element Ω^e and φ_j^e are called interpolation functions. The approximation solution U^e must attain specific requirements so that it be convergent to the actual solution (u) as the number of elements increases. These are:

1. The approximate solution should be continuous, differentiable over the element.
2. It should be a complete polynomial.
3. It should be an interpolant of the primary variables at the nodes.
4. All terms in the polynomial should be linearly independent.

The reason for the first requirement ensures a non-zero coefficient matrix. The second requirement is necessary to capture all possible states i.e., constant, linear and so on of the actual solution. The third requirement is required to satisfy the essential boundary conditions of the element and to enforce continuity of primary variables at nodes [14].

For variational statement, the minimum polynomial order is linear. A complete linear polynomial is of the form:

$$U^e = a + bx \tag{3.5}$$

where a and b are constants. This expression meets the first two requirements in 3.2.3. To satisfy the third condition, we express the constants a and b in terms of u_1^e and u_2^e such that

$$U^e(\eta_e) = u_1^e = a + b\ \eta_e \tag{3.6}$$

$$U^e(\eta_{e+1}) = u_2^e = a + b\ \eta_{e+1} \tag{3.7}$$

U^e can be illustrated in terms of interpolation functions as:

$$U^e(x) = \varphi_1^e(x)\ u_1^e + \varphi_2^e(x)\ u_2^e = \sum_{j=1}^{2} \varphi_j^e(x)\ u_j^e \tag{3.8}$$

φ_j^e are evaluated meticulously, beginning with an pre-assumed degree of a polynomial for the dependent unknown and determining the coefficients of polynomials in the expression of primary degrees of freedom. The dependent variable can be expressed as linear combination of approximation functions and the primary nodal variables. It is necessary to select the location and number of nodes in the element, so that geometry is uniquely determined. The number of nodes should be sufficient so that the assumed degree of interpolation can be expressed in terms of primary variables.

Like for linear polynomial approximation (3.5) two nodes with one primary unknown per node is sufficient to define the geometry of element, provided two nodes are end points of the element. In quadratic polynomial, a total of three nodal points must be present to depict the geometry uniquely.

$$U^e(x) = a + bx + cx^2 \tag{3.9}$$

Thus, two of the nodes must be the end points, and third nodal can be inside the element. In theory, the third node can be placed anywhere in the interior, however midpoint of element, being equidistant from end points is a good option. Thus, we identify three nodes in the element and re-represent $U^e(x)$ in terms of the terms of three nodal values (u_1^e, u_2^e, u_3^e) as:

$$U^e(\eta^e) = u_1^e = a + b\eta^e + c(\eta^e)^2 \tag{3.10}$$

$$U^e(\eta^{e-1}) = u_2^e = a + b\eta^{e-1} + c(\eta^{e-1})^2 \tag{3.11}$$

$$U^e(\eta^{e+1}) = u_3^e = a + b\eta^{e+1} + c(\eta^{e+1})^2 \tag{3.12}$$

Further, the number of linearly independent terms in the representation of U^e tells the shape and number of degree of freedom. Like if we have a polynomial

$$U^e(x, y) = c_1 + c_2x + c_3y$$

containing three linearly independent terms. It is linear in x, y and to write three constants in terms of the nodal values of U^e, we require an element with three nodes. These three nodes will uniquely define the geometry and obviously, it will be a triangle in two-dimensional domain as shown in fig. 3.4. On the other hand, for a polynomial with four linearly independent terms, it requires an element with four nodes. There are two possible geometries i.e., a triangle with fourth node at the center and rectangle. A triangle does not provide single value variation at inter-element boundaries and is, therefore not compatible. The linear rectangular element is compatible. Thus, depending on the number of nodes in the component and the order of differential equation to be solved, the degree of interpolation functions can be found.

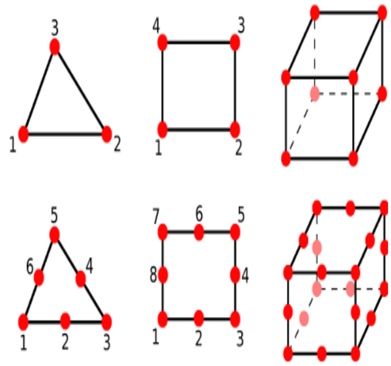


Figure 3.4: Some simple element shapes and standard node placement

3.2.4 Assembly of Equations

While deriving the element equations, we separated a typical component from the mesh and formulated the weak form and derives the model. To solve the total problem, we should substitute the elements back to their original positions. Thus, assembly of equations are required to be done by imposing two conditions:

1. Continuity of primary variables at connecting nodes, i.e. $u_n^e = u_1^{e+1}$. The last nodal value of the element Ω^e should be the same as the first nodal value of the adjacent element Ω^{e+1}.
2. Equilibrium of secondary variables at the connecting nodes.

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & & & & \\ K_{21}^1 & K_{22}^1+K_{11}^2 & K_{12}^2 & & 0 & \\ & K_{21}^2 & K_{22}^2+K_{11}^3 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ & 0 & & & K_{22}^{e-1}+K_{11}^e & K_{12}^e \\ & & & & K_{21}^e & K_{22}^e \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_e \\ U_{e+1} \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1+f_1^2 \\ f_2^2+f_1^3 \\ \vdots \\ f_2^{e-1}+f_1^e \\ f_2^e \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1+Q_1^2 \\ Q_2^2+Q_1^3 \\ \vdots \\ Q_2^{e-1}+Q_1^e \\ Q_2^e \end{Bmatrix} \quad (3.13)$$

where K_{ij}^e are global node numbers, $[K^e]$ is called aa stiffness matrix. The column vector $\{f_n^e\}$ is the source vector. $(U_1, U_2, U_3, ...U_e)$ and $\{Q_n^e\}$ are the global primary and secondary variables respectively.

3.2.5 Post-processing of Solution

When, the finite element model had been developed and studied by imposing the given boundary conditions, then it is time to evaluate the result of the analysis. This exercise is referred as post-processing phase of FEM. Post-processing starts with a complete inspection of problems that may happen during solutions. Post processing of finite element data predominantly seeks additional software to assemble the output so that it is clearly understandable whether the construction is allowed or not. Post-processing of result comprises of the following one or more aspects:

1. Computation of any secondary variables.
2. Interpretation of the results.
3. Tabular and or graphical representation of the results.

3.3 Advantages of FEM

- In the engineering community, FEM is basic, compact and result-oriented and is therefore commonly used tool.
- This technique can analyze the physical characteristics that are ungovernable and complicated for any closed bounded solution.
- In this method, approximations are restricted to small sub domains.
- Modeling of complex geometries like 2D or 3D domains and irregular shapes are easier in FEM.
- Boundary conditions can be easily be assimilated in FEM. The weak form automatically includes “natural boundary conditions”.
- It can handle bodies comprised of non-homogeneous and non-isotropic materials without any difficulty.
- FEM is superior for multi physics problems where there is a high degree of coupling and non-linearity.
- There are plenty of computer software packages available to build FEM a flexible and robust numerical method.

3.4 Applications of FEM

- To discover the distribution of displacement and stress for mechanical or thermal loading in solid mechanics
- Aerospace
- Civil or Automotive Engineering
- Structural and thermal analysis of systems.

- Fluid Flow
- To determine velocity, pressure, temperature, and density distributions of equilibrium problems in fluid mechanics.
- Heat Transfer
- Electromagnetic Fields
- Acoustics
- Allows the construction, refining and optimization of entire models before the design is produced.

Chapter 4

Heat transfer and Brownian motion of Hybrid Nanofluid over non-linear stretching sheet using MHD

4.1 Introduction

Humankind is living in the arena of modern technologies, and over the previous few decades there has been a definite affirmation of the rapid growth in diverse sectors like electronics, power generation with heat transfer being an inherently important component. This revolutionary development demands the establishment of an effective medium of heat transfer. Several methods for heat transfer improvement like extended surfaces and mini-channels were used to improve the heat transfer rate. The single phase heat transfer fluids like oils, water, organic liquids (ethylene glycol) are broadly used in various process industries, chemical and thermal power plants. Single phase fluid's heat transfer performance is usually very poor due to their lower thermal conductivity values. The heat transfer escalation is the foremost necessity to achieve significant outcomes. The new strategy for improving the thermal conductivity of the heat transfer medium is to add solid particles of large thermal conductivity in inherently poor thermal conductive conventional fluids. The dispersing of solid particles in liquid medium was first experimented by *"Maxwell"* [11]. He gave the proper adequate base to calculate thermal conductivities and observed the enhanced thermal conductive values. In extension of this, magnificent research was carried out by "Hamilton-Crosser" on particles suspended in heterogeneous two component mixtures. The sheer dispersion of solid particles in single-phase liquids, however, results in their sedimentation and causes the fluid flow to be obstructed. This further originates erosion on the walls of the flow passage, while increasing pressure drop through the installations. The key objective to enhance thermal conductivity gets diminished. Later on, micro-sized solid particles were dispersed in single phase fluids and observed thermal conductivity enhancement. But also faced particle sedimentation in the base fluid, wall erosion and clogging of the flow passage in the flow field which reduces the enhancement in thermal conductivity. Therefore, the use of micro-sized particles in base liquids is a huge constrain.

Evolution in material technology, however, provides heat transfer fluids a uniqueness by dispersing nano-sized particles in parent fluid, referred as *Nanofluids*. This was first originated by *"Choi"* [6] in the year 1995. Choi gives the glimpse of the procedure of adding nanoparticles in the base fluid. The most significant attribute of nanofluids is that coagulation can be overpower to a great extent. Agglomeration can be further vanquished by analogous (homogeneous) dispersion of particles and favorable surfactant. Variety of nanofluids are analyzed for multiple heat transfer applications. Some commonly used nanoparticles have already been listed in section 2.5. In review articles published over the previous few years, the experimental information on amplification of heat transfer with various nanofluids are well summarized. Researchers have witnessed a higher heat transfer rate under laminar and turbulent flow situations. The increase in heat transfer rate depends on particle concentration, thermal conductivity, mass flow. Despite of various applications in material sciences, the root issue with monokind nanofluids is that they either own great thermal properties or better rheological properties. Nanofluids does not hold all favorable characteristics required for a particular application. But, real time applications demand several characteristics of nanofluids. Like Al_2O_3 possess lower thermal conductivity but, demonstrate useful chemical inertness and stability. On the other hand, the fluid comprising metallic nanoparticles (like Aluminium, Copper, Silver) retains higher thermal conductivity, but are chemically more reactive and unstable.

By hybridizing metallic nanoparticles with metal or ceramic oxides, the arising fluid shows more superior thermophysical properties and rheological attributes along with the improved heat transfer characteristics. This opened the door for advanced arena of *Hybrid Nanofluid*, defined as a mixture of two or more different nanosized materials in some base fluid. The advancement in the nanotechnology and increased growth in hybrid nanofluids is experienced due to its potential influence in material sciences and engineering. The heat transfer enhancement with different hybrid nanofluids and their thermophysical properties is reviewed by various authors worldwide [9, 10, 15]. The hybrid nanofluids are a fairly new group of nanofluids, with a wide range of applications in almost every field of heat transfer.

The boundary layer flow over a stretching sheet had induced considerable interest because of its practical utilization in engineering, metallurgical, industrial processes, manufacturing of materials by extrusion, drawing and paper production, crystal growing, aerodynamic extrusion of plastic sheets and fiber, tinning of copper wire, etc. The manufacturing of sheeting material has been created in several industrial procedures and involves sheets of metal and polymer. During the process of formation of sheets, the melt which comes out of slit is stretched to get the required thickness. The rate of cooling and stretching also effect the quality of the final product. Several studies have been undertaken by considering the flow of viscous and the non-newtonian fluids over linear and nonlinear stretching flat sheets. *"Crane"* [16] was the first known person who investigated the flow over a stretching sheet. Later, many researchers like *"Gupta"* [17] extended the work of crane by incorporating the impact of heat and mass transfer under various physical conditions. After this work, fluid flow over a stretching sheet has received extensive recognition. The fluid flow by curved nonlinear stretching sheet was initiated by *"Sajid"* [18] in 2010. *"Rosca and Pop"* [19] solved the time-dependent boundary layer flow of nanofluid over nonlinear stretching sheet, using several numerical methods. Thus, the study of fluid flow past stretching sheet had resulted in comprehensive research as mentioned in open literature [20–22] so far.

Magnetohydrodynamics also called magneto-fluid dynamics (MHD) is a study of hydrodynamics in the presence of a magnetic field. It is the physical-mathematical framework which helps to analyze the dynamics of magnetic fields in electrically conducting fluids like plasma. The Swedish physicist *"Hannes Alfvén"* was the first person who used MHD and received Nobel prize for his fundamental work and discoveries in MHD with fruitful applications in plasma physics. The fluid flow under the influence of magnetic field has several industrial applications such as metal casting, crystal growth, astrophysical flows, solar power technology, electrical power generation, etc. MHD is a macroscopic theory. The critical point of MHD theory is based on the fact that conductive fluids can support magnetic field and Lorentz force interacts with the buoyancy force. The application of magnetic field to the convection process will act as a controlling factor by damping both the flow and temperature oscillations. Recently, the problem of MHD natural convection for hybrid nanofluid is studied in an open wavy cavity [23]. In engineering applications, it is extremely important as it controls the flow of fluids without any physical contact [20].

There are various modelling mechanisms to model relative velocity of the nanoparticles and the base fluid, such as the drag forces, inertia, Brownian motion, thermophoresis, diffusiophoresis, the Magnus effect, etc. Out of all these, Brownian diffusion (random zig-zag motion of particles in a fluid due to the collision) and thermophoresis (phenomenon in mixture where different particle type exhibit different responses to temperature gradient) were found to be more relevant. The Brownian force plays vital importance for particle deposition when particles with smaller size are used. It is a crucial nanoscale mechanism for nanoparticles governing their thermal behavior. For engineers and researchers the analysis of heat transfer with thermophoresis particle deposition is of great practical importance these days.

It is evident from the citations listed above, no numerical research has been carried out so far on hybrid nanofluid for heat transfer analysis on the stretching sheet. In the present problem, we intend to use hybrid nanofluid to numerically study the enhancement in natural convection flow and heat transfer over the stretching sheet in the presence of magnetic field and heat source. Here, a combined effect of Brownian diffusion and thermophoresis is considered on the thermal conductivity. The governing set of partial differential equations describing the problem are simplified to the system of nonlinear ordinary differential equations. These are further elucidated by Finite Element Method. The effect of various decisive parameters such as Hartmann number (Ha) representing the magnetic field, Heat source parameter (Hs), volume fraction, thermal conductivity due to Brownian motion K_B on the flow variables are discussed and corresponding plot are obtained by considering variation in different parameter values.

4.2 Mathematical Analysis

We consider a two-dimensional steady, isothermal, laminar boundary layer flow of incompressible hybrid nanofluid over a nonlinear stretching sheet. The coordinate system under consideration is such that x-axis is drawn in the direction of motion along the stretching surface and y measure the distance into the fluid. The flow is supposed to be limited to $y > 0$ region. The origin is kept fixed, and the surface is stretched by imposing two equal forces but in opposite direction along the x-axis. Additionally, the size and shape of nanoparticles are considered to follow a uniform distribution. Moreover, for the stability of fluid, the fluid phase and nanoparticles are assumed to be in thermal equilibrium with each other. The fluid is electrically conductive due to an applied magnetic field (B). The heat source (Q) is also added normal to the stretching sheet. The problem's physical model is given below in fig. 4.1.

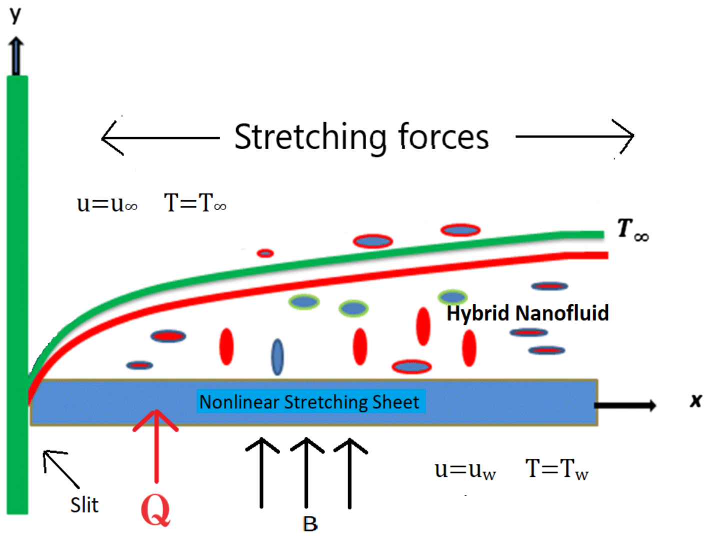


Figure 4.1: Geometry of the problem

Under the boundary conditions, the basic Conservation of Mass, Momentum and Energy equations governing the flow and heat transfer characteristics of hybrid nanofluid, nanoparticles has been stated below:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \frac{\mu_{hnf}}{\rho_{hnf}}\frac{\partial^2 u}{\partial y^2} + \beta_{hnf}g(T - T_\infty) - \frac{\sigma_{hnf}}{\rho_{hnf}}B^2 u \tag{4.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{hnf}\frac{\partial^2 T}{\partial y^2} + \frac{Q}{(\rho c_p)_{hnf}}(T - T_\infty) \tag{4.3}$$

The appropriate boundary conditions are:

$$u(x,0) = Dx^b\ , \qquad v(x,0) = 0\ , \qquad T(x,0) = T_w\ ,$$
$$\text{as}\ \ y \to \infty: \quad u \to 0\ , \qquad T \to T_\infty \tag{4.4}$$

Here $u,\ v$ represents the velocity components in $x,\ y$ direction respectively and T denotes the temperature. $T_w,\ T_\infty$ represents the temperature of sheet and ambient fluid respectively, D denotes the combined diffusion constant for brownian and thermophoresis, b is positive nonlinear parameter and σ_{hnf} is the electrical conductivity of hybrid nanofluid. The properties of the hybrid nanofluid [23, 24] are listed as:

The effective density of hybrid nanofluid is given as:

$$\rho_{hnf} = \phi_{np1}\rho_{np1} + \phi_{np2}\rho_{np2} + (1-\phi_{hp})\rho_f \tag{4.5}$$

$$\rho_{hnf} = \phi_{hp}\rho_{hp} + (1-\phi_{hp})\rho_f \tag{4.6}$$

$$\phi_{hp} = \phi_{np1} + \phi_{np2} \tag{4.7}$$

$$\rho_{hp} = \frac{\phi_{np1}\rho_{np1} + \phi_{np2}\rho_{np2}}{\phi_{hp}} \tag{4.8}$$

The effective heat capacity of hybrid nanofluid is given as:

$$(\rho C_p)_{hnf} = \phi_{hp}(\rho C_p)_{hp} + (1-\phi_{hp})(\rho C_p)_f \tag{4.9}$$

$$(C_p)_{hp} = \frac{\phi_{np1}(C_p)_{np1} + \phi_{np2}(C_p)_{np2}}{\phi_{hp}} \tag{4.10}$$

Thermal expansion coefficient of hybrid nanofluid is given as:

$$(\rho\beta)_{hnf} = (1-\phi_{hp})(\rho\beta)_f + \phi_{hp}(\rho\beta)_{hp} \tag{4.11}$$

$$\beta_{hp} = \frac{\phi_{np1}\beta_{np1} + \phi_{np2}\beta_{np2}}{\phi_{hp}} \tag{4.12}$$

The effective viscosity is proposed to be sum of static part and Brownian part given as:

$$\mu_{eff} = \mu_{hnf} + \mu_B \tag{4.13}$$

$$\mu_B = \frac{K_B}{K_f} \times \frac{\mu_f}{P_r} \tag{4.14}$$

The dynamic viscosity of hybrid nanofluid given by Brinkman [25] is:

$$\mu_{hnf} = \frac{\mu_f}{(1-\phi_{hp})^{2.5}} \tag{4.15}$$

The effective thermal diffusivity of hybrid nanofluid is given as:

$$\alpha_{hnf} = \frac{K_{eff}}{(\rho C_p)_{hnf}} \tag{4.16}$$

As discussed earlier, the consideration of Brownian motion significantly impact the thermal conductivity in the hybrid nanofluids. By taking into account both static effect and Brownian motion impact, "Koo and Kleinstreuer" established a correlation. The Maxwell [11] model is regarded as a static portion while the movement of nanoparticles in the surrounded fluid is included in Brownian part.

$$K_{eff} = K_{hnf} + K_B \tag{4.17}$$

$$K_B = 5 \times 10^4 \beta \ \phi_{hp} \ \rho_f \ C_{p,f} \sqrt{\frac{\kappa_b T}{\rho_{hp} d_{hp}}} \ g'(T, \phi_{hp}) \tag{4.18}$$

where d_{hp} represents diameter of hybrid nanoparticles. β and g' are empirical functions which are used to include the interaction between different nanoparticles in addition to the temperature effect. The function g' captures the influence of both the temperature and volume fraction. It is continuous function of particle volume fraction ϕ_{hp} and can be approximated using Taylor series as

$$g'(T, \phi_{hp}) = (-6.04 \ \phi_{hp} + 0.4705)T + (1722.3 \ \phi_{hp} - 134.63)$$

Due to the relationship with the particle motion, the parameter β is supposed to depend solely not only on volume fraction but also on the shape of particle, temperature and material properties of the hybrid nanofluid [8].

$$\beta = \begin{cases} 0.0011(100\phi_{hp})^{-0.7272} & \text{for} \quad \phi_h < 1\% \\ 0.0017(100\phi_{hp})^{-0.0841} & \text{for} \quad \phi_h > 1\% \end{cases}$$

For simplicity, we assumed the parameter β to be dependent only on the volume fraction as all other dependencies mentioned is considered in the function g'.

Thermal conductivity of hybrid nanofluid as predicted from modified classical Maxwell model [12] is given as

$$K_{hnf} = K_f \left[\frac{(K_{hp} + 2K_f) - 2\phi_{hp}(K_f - K_{hp})}{(K_{hp} + 2K_f) + \phi_{hp}(K_f - K_{hp})} \right] \tag{4.19}$$

$$K_{hp} = \frac{\phi_{np1} K_{np1} + \phi_{np2} K_{np2}}{\phi_{hp}} \tag{4.20}$$

Now, for equation simplification we define the similar non-dimensional parameters to examine flow regime:

$$\eta = y\sqrt{\frac{D(b+1)}{2\nu_f}}x^{\frac{(b-1)}{2}}, \qquad \theta(\eta) = \frac{T - T_w}{T_w - T_\infty}$$

$$u = Dx^b f'(\eta), \qquad v = -\sqrt{\frac{D(b+1)\nu_f}{2}}x^{\frac{(b-1)}{2}}\left[f + \left(\frac{b-1}{b+1}\right)\eta f'\right] \tag{4.21}$$

Using the above transformations, the equation of Continuity (4.1) is satisfied automatically and the equations (4.2) and (4.3) are simplified into system of non-linear differential equations:

$$A_1 A_2 f''' + A_3 A_2 \theta - A_4 (f')^2 + f f'' - A_5 f' = 0 \tag{4.22}$$

$$A_6 \theta'' + A_7 f \theta' + \frac{2}{(b+1)}\theta = 0 \tag{4.23}$$

The coefficients $A_i (i = 1, 2..., 7)$ are dimensionless constants given in the table 4.1. The boundary conditions (4.4) are also transformed to:

$$f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1 \quad \text{and} \quad f'(\infty) = 0, \quad \theta(\infty) = 0 \tag{4.24}$$

The parameter G is the modified Grashof number, P_r is the Prandtl number, Hs is the modified heat source parameter, Ha is the modified Hartmann number.

4.3 Numerical Modelling

The dimensionless equations obtained (4.22),(4.23) are nonlinear in nature, therefore it is impossible to solve the above equations analytically. To achieve a numerical solution, the finite element method is used. To solve the governing differential equations, we suppose

$$f' = h \tag{4.25}$$

The system of equations (4.22),(4.23) reduces to:

$$A_1 A_2 h'' + A_3 A_2 \theta - A_4 \bar{h} h + \bar{f} h' - A_5 h = 0 \tag{4.26}$$

$$A_6 \theta'' + A_7 \bar{f} \theta' + \frac{2}{(b+1)}\theta = 0 \tag{4.27}$$

Table 4.1: List of coefficients of the model

A_1	A_2	A_3	A_4
$\left(\frac{1}{(1-\phi_{hp})^{2.5}}+\frac{K_B}{K_f}\times\frac{1}{P_r}\right)$	$\left(\frac{1}{\frac{(\phi\rho)_{hp}}{\rho_f}+(1-\phi_{hp})}\right)$	$\frac{2G}{b+1}\left(\frac{\phi_{hp}(\rho\beta)_{hp}}{\beta_f\rho_f}+(1-\phi_{hp})\right)$	$\frac{2b}{(b+1)}$
A_5	A_6	A_7	
$\frac{2Ha^2}{b+1}$	$\frac{K_{hnf}}{K_f}+\frac{K_B}{K_f}$	$P_r\left(\frac{\phi_{hp}(\rho C_p)_{hp}}{(\rho C_p)_f}+(1-\phi_{hp})\right)$	
G	P_r	Hs	Ha^2
$\frac{g(T_w-T_\infty)\beta_f}{D^2x^{2b-1}}$	$\frac{\nu_f(\rho C_p)_f}{K_f}$	$\frac{Q\nu_f}{K_f D x^{b-1}}$	$(\frac{B^2}{Dx^{b-1}})\frac{\sigma_{hnf}}{\rho_{hnf}}$

The respective boundary conditions get revised as:

$$f(0)=0,\quad h(0)=1,\quad \theta(0)=1 \quad \text{and} \quad h(\infty)=0,\quad \theta(\infty)=0 \tag{4.28}$$

These are linearized by incorporating the known functions $\bar{f}$ and $\bar{h}$. The complete domain is divided into set the of 150 components.

4.3.1 Variational formulation

The variation formulation of equations (4.26), (4.27) over the element (η_e, η_{e+1}) is given by:

$$\int_{\eta_e}^{\eta_{e+1}} W_1(f'-h)d\eta=0 \tag{4.29}$$

$$\int_{\eta_e}^{\eta_{e+1}} W_2(A_1A_2h''+A_3A_2\theta-A_4\bar{h}h+\bar{f}h'-A_5h)d\eta=0 \tag{4.30}$$

$$\int_{\eta_e}^{\eta_{e+1}} W_3(A_6\theta''+A_7\bar{f}\theta'+\frac{2}{(b+1)}\theta)d\eta=0 \tag{4.31}$$

where $W_1,\ W_2,\ W_3$ are arbitrary weight functions and considered as the deviation in $f,\ h,\ \theta$ respectively. These functions are chosen in such a way that it satisfy the homogeneous boundary conditions.

4.3.2 Finite element formulation

Let the finite element approximation for two-noded element as:

$$f = \sum_{i=1}^{2} f_i \psi_i \ , \qquad h = \sum_{i=1}^{2} h_i \psi_i \ , \qquad \theta = \sum_{i=1}^{2} \theta_i \psi_i \tag{4.32}$$

where $W_j = \psi_1$ is for first node and $W_j = \psi_2$ is for second node with $j = 1, 2, 3$. Here, ψ_i are the shape functions for the line element (η_e, η_{e+1}) defined as:

$$\varphi_1^e = \frac{\eta_{e+1} - \eta}{\eta_{e+1} - \eta_e} \ , \qquad \varphi_2^e = \frac{\eta - \eta_e}{\eta_{e+1} - \eta_e} \qquad \text{where, } \ \eta_e \leq \eta \leq \eta_{e+1} \tag{4.33}$$

The finite element model of the equations (4.29), (4.30), (4.31) is given below:

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix} \begin{bmatrix} \{f\} \\ \{h\} \\ \{\theta\} \end{bmatrix} = \begin{bmatrix} \{r^1\} \\ \{r^2\} \\ \{r^3\} \end{bmatrix} \tag{4.34}$$

Here, each $[K^{pq}]$ is of the order (2×2) and $\{r^p\}$ is of order (2×1) $(p, q = 1, 2, 3)$. The system of coupled ordinary differential equations are nonlinear. These matrices are described as:

$$K_{ij}^{11} = \int_{\eta_e}^{\eta_{e+1}} \psi_i \frac{\partial \psi_i}{\partial \eta} d\eta \ , \qquad K_{ij}^{12} = -\int_{\eta_e}^{\eta_{e+1}} \psi_i \psi_j d\eta \ , \qquad K_{ij}^{13} = 0 \tag{4.35}$$

$$K_{ij}^{21} = 0 \ , \qquad K_{ij}^{23} = A_3 A_2 \int_{\eta_e}^{\eta_{e+1}} \psi_i \psi_j d\eta \ , \tag{4.36}$$

$$\begin{aligned} K_{ij}^{22} = & -A_1 A_2 \int_{\eta_e}^{\eta_{e+1}} \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_j}{\partial \eta} d\eta + \bar{f} \int_{\eta_e}^{\eta_{e+1}} \psi_i \frac{\partial \psi_i}{\partial \eta} d\eta - A_4 \bar{h} \int_{\eta_e}^{\eta_{e+1}} \psi_i \psi_j d\eta \\ & - A_5 \int_{\eta_e}^{\eta_{e+1}} \psi_i \psi_j d\eta \end{aligned} \tag{4.37}$$

$$K_{ij}^{31} = 0 \,, \qquad K_{ij}^{32} = 0 \,, \tag{4.38}$$

$$K_{ij}^{33} = -A_6 \int_{\eta_e}^{\eta_{e+1}} \frac{\partial \psi_i}{\partial \eta}\frac{\partial \psi_j}{\partial \eta} d\eta \;+\; A_7 \bar{f} \int_{\eta_e}^{\eta_{e+1}} \psi_i \frac{\partial \psi_i}{\partial \eta} d\eta \;+\; \frac{2Hs}{b+1} \int_{\eta_e}^{\eta_{e+1}} \psi_i \psi_j d\eta \tag{4.39}$$

$$r_i^1 = 0 \,, \qquad r_i^2 = -A_1 A_2 \left(\psi_i \frac{\partial h}{\partial \eta} \right)_{\eta_e}^{\eta_{e+1}} , \qquad r_i^3 = A_6 \left(\psi_i \frac{\partial \theta}{\partial \eta} \right)_{\eta_e}^{\eta_{e+1}} \tag{4.40}$$

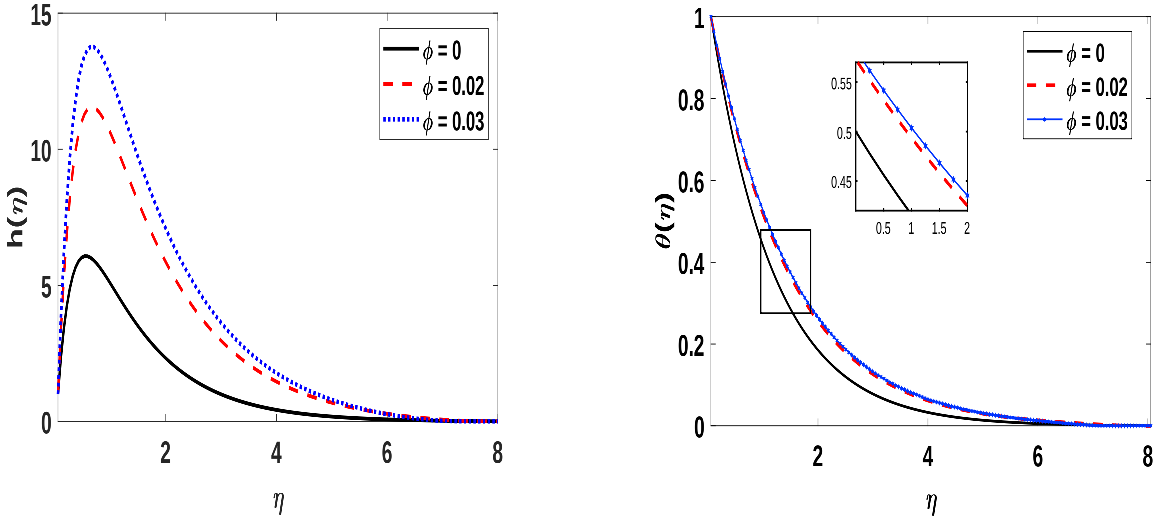
The system of equations obtained after equations have been assembled is nonlinear, therefore iterative scheme is used to elucidate it. Finite element method is used to solve the governing nonlinear system of equations (4.26), (4.27) with boundary conditions (4.28).

4.4 Results and Discussions

To analyze the heat transfer and Brownian motion of hybrid nanofluid, various important and interesting parameters such as volume fraction, nonlinear parameter (b), modified Grashof number (G), Prandtl number (P_r), Hartmann number (Ha), Heat source parameter (Hs), Brownian motion thermal conductivity (K_B) are discussed to study their impact on dimensionless velocity $f'(\eta) = h(\eta)$ and dimensionless temperature $\theta(\eta)$. The total volume fraction ϕ_{hp} of hybrid nanofluid ranges from $0\% < \phi_{hp} < 4\%$ in this problem. Here, we had considered Cu and Al_2O_3 as two different types of nanoparticles. The thermophysical properties of these nanoparticles and base fluid are listed in table 2.1. Results for velocity and temperature are plotted by taking different values of the parameters involved.

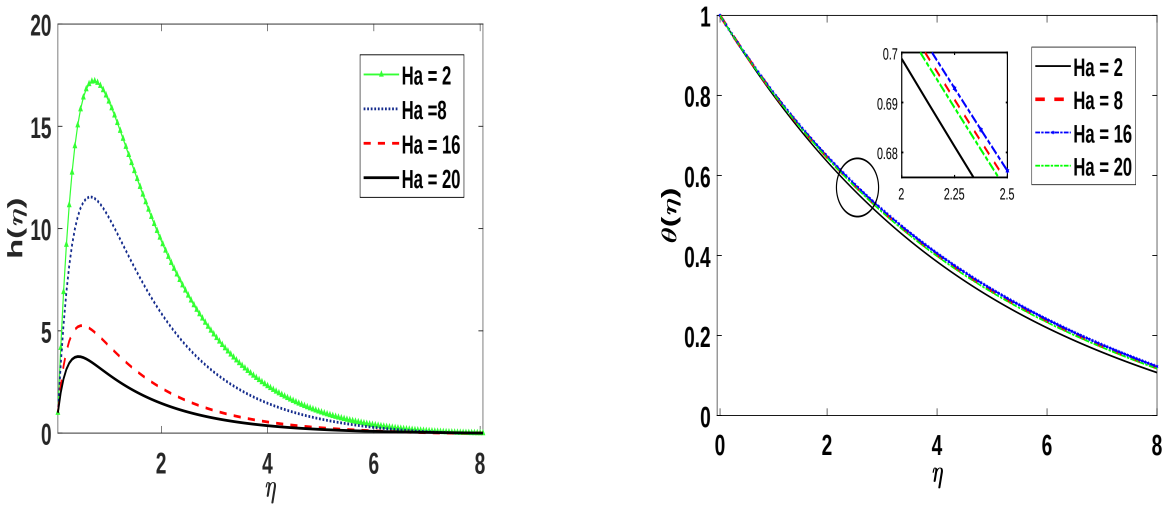
The numerical outcomes of the present analysis depicts that velocity and temperature of hybrid nanofluid increases remarkably with the increasing volume fraction of hybrid nanoparticles as shown in fig. 4.2. From the physics' point of view, it can be justified that the random movement of the particle increases with the rise in volume fraction. As a result, the rate of exchange energy within the liquid increases and enhances the thermal dispersion in the flow of hybrid nanofluids.

The influence of Hartmann number (Ha) on the velocity and temperature is displayed in figure 4.3. The figure 4.3(a) represents that velocity declines within the boundary layer with the rise in Hartmann number. This behavior is observed because retarding body force known as Lorentz force is introduced with rise in the magnetic field, which acts in the transverse direction to the applied magnetic field. Hence, the momentum boundary layer becomes narrower as Ha rises. In opposition to this behavior, fig. 4.3(b) represents that temperature enhances insignificantly because as the nature of Lorentz force is resistive, it opposes the fluid motion. Due to which heat is produced which makes the thermal boundary layer thick in this case.



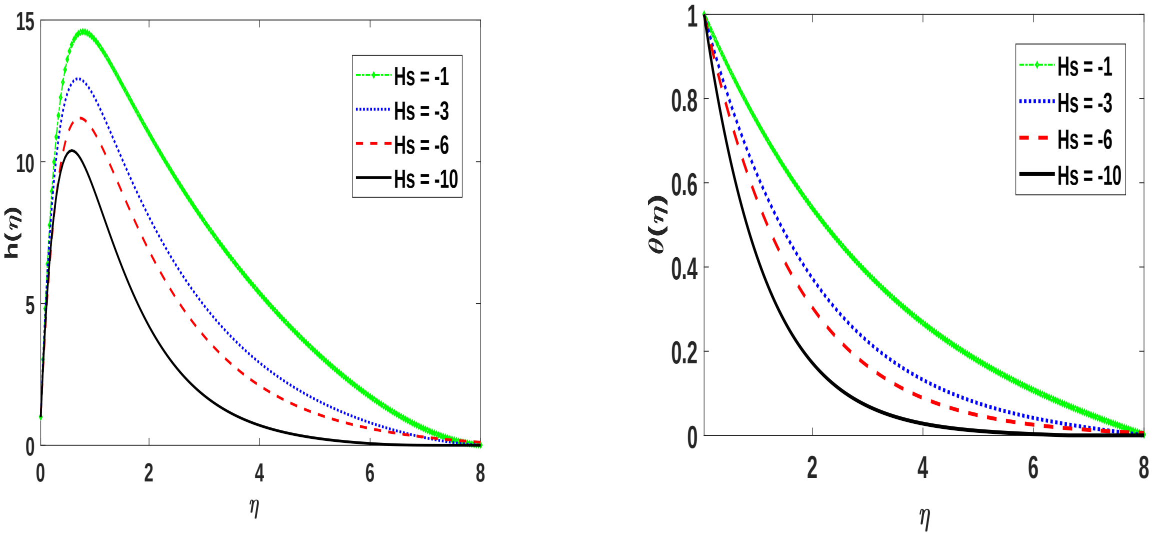
(a) (b)

Figure 4.2: For $T = 315K$, $G = 1000$, $Pr = 0.7$, $Hs = -1$, $b = 3$, $Ha = 8$ (a) Effect of volume fraction ϕ_h on velocity (b) Effect of volume fraction ϕ_h on temperature



(a) (b)

Figure 4.3: For $T = 315K$, $\phi_h = 0.02$, $G = 1000$, $Pr = 0.7$, $b = 3$, $Hs = -1$ (a) Effect of Hartmann number Ha on velocity (b) Effect of Hartmann number Ha on temperature

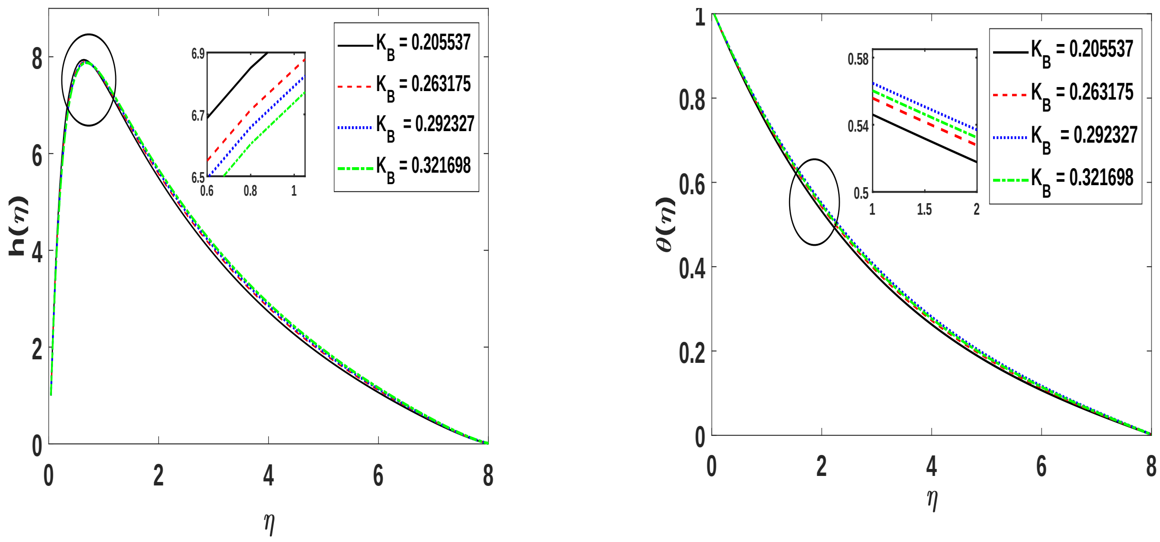


(a) (b)

Figure 4.4: For $T = 315K$, $\phi_h = 0.02$, $G = 1000$, $Pr = 0.7$, $b = 3$, $Hs = -1$, $Ha = 8$ (a) Effect of Heat source parameter Hs on velocity (b) Effect of Heat source parameter Hs on temperature

The impact of Heat source parameter (Hs) on velocity and temperature is picturised in fig. 4.4. It is discovered that when heat source parameter numerically increases, temperature decreases inside the boundary layer, as displayed in fig. 4.4(b). In contrast to temperature profile, fig. 4.4(a) revealed that velocity overshoots in the vicinity of the boundary layer and decreases as Hs goes on increasing numerically.

The thermal conductivity due to Brownian motion K_B is calculated for different values of temperature (T). Fig. 4.5 displays the impact of K_B on velocity and temperature field. Fig. 4.5(a) tells the influence of thermal conductivity due to Brownian motion on velocity. It is analyzed that velocity intensify in the zone of the boundary layer up to some extent and thereafter, it decreases. This is due to the reason that K_B serves to heat the boundary layer and at the same time heighten particle deposition away from the fluid regime or on the surface of fluid system. The fig. 4.5(b) sketch the effect of thermal conductivity due to Brownian motion on temperature. It is noted that K_B imposes a significant temperature enhancing impact. Temperature rises with a rise in K_B.



(a) (b)

Figure 4.5: For $\phi_h = 0.02$, $G = 1000$, $Pr = 0.7$, $b = 3$, $Ha = 8$ (a) Influence of thermal conductivity due to Brownian motion K_B on velocity (b) Influence of thermal conductivity due to Brownian motion K_B on temperature

4.5 Conclusion

The influence of heat source (Q) and thermal conductivity due to Brownian motion on steady MHD (hybrid nanofluid) flow over a nonlinear stretching sheet with two phase mixture model is evaluated. The central surveillance of the present study are as follows:

- The dimensionless velocity rises with the increase in total concentration of hybrid nanoparticles. It is also noted that when Heat source increases in magnitude, velocity enhances in the region of boundary layer and after that, it declines. In contrast, velocity decreases as the Hartmann number (Ha) escalate. The behavior of velocity is somewhat different for Brownian motion thermal conductivity. Firstly, velocity enhances with the rise in K_B up to some extent and then it decreases.

- The temperature amplifies with the rise in the concentration of nanoparticles, Hartmann number and K_B. Temperature profile declines when heat source parameter (Hs) increases numerically.

- The heat transfer rate decreases at the surface with an increase in Hartmann number. It is also observed that the rate of heat transfer rises when heat source parameter (Hs) increases in magnitude.

- Employing hybrid nanofluid is more effective than nanofluid as it offers better heat transfer characteristics. With increasing volume fraction and temperature, the characteristics of the hybrid nanofluid get enhanced.

Hybrid nanofluids are revolutionary unique class of advanced heat transfer fluids and they are at the research and development phase. The use of hybrid nanofluids is limited to very few research-oriented applications. There are some challenges and issues related to hybrid nanofluid identified in the literature. Firstly, there is a lack of agreements between experimental outcomes and theoretical predictions. Secondly, there are shortage of theoretical models that can anticipate the actual behavior of fluids. Stability is one of the main contributing factors to hybrid nanofluid performance. Lack of adequate stability adversely affect the nanofluid. The stability of some hybrid nanofluid has been shown to be deteriorating over a lengthy span of time by the earlier researchers. Scientists are pacing in the direction of investigating hybrid nanofluids' rheological and heat transfer behavior. Further, studies are progressing to find the root source behind the unique modifications in thermophysical properties like thermal conductivity, heat transfer characteristics, density, specific heat capacity and viscosity of hybrid nanofluids.

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