STATISTICAL CONVERGENCE OF SEQUENCES, SERIES, AND MEASURABLE FUNCTIONS WITH APPLICATION IN FOURIER SERIES

A
Dissertaton In Accomplishment Of Requirements
For The Award Of The Degree Of

Master Of Science
In
Mathematics And Computing

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July, 2018

CERTIFICATE

This is to certify that the dissertation titled STATISTICAL CONVERGENCE OF SEQUENCES, SERIES, AND MEASURABLE FUNCTIONS WITH APPLICATION IN FOURIER SERIES is a bonafide record of the work done in the accomplishment of the requirements for the award of the degree of Master of Science in MATHEMATICS AND COMPUTING from the THAPAR INSTITUTE OF ENGINEERING AND TECHNOLOGY, during the year 2016-2018. The matter embodied in this report is of candidate's own record and not governed by any other university in part or full form for the award of such a degree.

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ACKNOWLEDGEMENT

The matter of fact is that the only elements *i.e.*, concentration, dedication and hard work does not plays a vital role in achieving appropriate goals. However, it is essential to have assistance and co-operation, guidance and support of people.

A honest and cordial appreciation is exhibited to my supervisor, **Dr.Jatinderdeep Kaur**, Assistant Professor, School of Mathematics for being an inspiration and also for untiring support with constant motivation throughout this dissertation.

The entire faculty and staff members of School of Mathematics pleases with help, cooperation, love and affection for comfortable stasy at Thapar Institute of Engineering and Technology and most importantly unforgettable.

A token of thanks to my friends who are always there in the hour of need.

The unconditional support of my family gave me the courage to achieve my goals throughout life. Their encouragement plays vital role in completion of my work and without this support it would have been difficult to tackle.

Apart from this, I pay my admiration to the almighty God.

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Abstract

The dissertation entitled as "**Statistical Convergence of Sequences, Series and Measurable functions with applications in Fourier Series**", encompasses a concise description of inquisition prosecute by numerous researchers. Besides this certain results are evinced on statistical convergence under the enlightenment of **Dr. Jatinderdeep Kaur**, Assistant Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

Currently, the dissertation presents four chapters along with the conclusion. Chapter I is the introductory which includes certain well known results, examples and assertions and comparison of Statistical convergence with classical convergence.The objective of Chapter II and III is to study the convergence of single and multiple sequences and series statistically already explained by Ferenc Móricz and evaluating certain results and remarks. In Chapter IV, the statistical limit of measurable function at ∞ is explained with hypothesis and assertions with application to Fourier Transform.

Towards the end, references of various publications cited in the current dissertation have been reported.

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Chapter 1

1 Introduction

1.1 Origination Of Statistical Convergence

Statistical Convergence,was published almost fifty years ago, has flatter the domain of recent research. Unlike mathematicians studied characteristics of statistical convergence and applied this notion in numerous extent such as measure theory, trigonometric series, approximation theory, locally compact spaces, and Banach spaces,etc.

The present thesis emphasis on certain results studied by Ferenc Mórícz in his two research papers *i.e.*, "*Statistical Convergence of Sequences and Series of Complex Numbers with applications in Fourier Analysis and summability*" and in "*Statistical Limit of Lebesgue Measurable functions with* ∞ *with applications in Fourier Analysis and summability*".

The perception of conjunction has been generalized in various ways through different methods such as summability and also a method in which one moves from a sequence to functions. In 1932 earlies, Banach coined the first generalization of it and named as "*almost convergence*". Later it was studied by Lorentz [6] in 1948.

The most recent generalization of the classical convergence *i.e.,*a new type of conjunction named as Statistical Convergence had been originated first via Henry Fast[3] in 1951. He characterizes this hypothesis to Hugo Steinhaus[19]. Actually, it was Antoni Zygmund[20] who evince the results, prepositions and assertion on Statistical Convergence in a Monograph in 1935. Antoni Zygmund in 1935 demonstrated in his book "*Trigonometric Series*" where instead of *Statistical convergence* he proposes the term "*almost convergence*" which was later proved by Steinhaus and Fast([19] and [3]).

Then, Henry Fast[3] in 1951 developed the notion analogous to Statistical Convergence, Lacunary Statistical Convergence and λ Statistical Convergence and it was reintroduced by Schoenberg[18] in 1959.Since then the several research paper related to the concept have been published explaining the notion of convergence and is applications.The objective of the study is to discuss the fundamentals and results along with various extensions which have been subsequently formulated.

1.2 Basic Terminology

In this subsection, basic concepts are presented as follows :

Borgohain and Savas introduced this term in [1] as listed below :

Definition 1.2.1. Asymptotic Density *The theory of Convergent statistically depends on the perception of asymptotic density of subsets of $\mathcal{N}$ of natural numbers. $\mathcal{P} \subseteq \mathcal{N}$ is termed as natural density $\delta(\mathcal{P})$ i.e.,*

$$\delta(\mathcal{P}) = \lim_{m\to\infty} \frac{1}{m} \sum_{p=1}^{m} \chi_{\mathcal{P}}(p)$$

where $\chi_{\mathcal{P}}$ is the characteristic function of $\mathcal{P}$.

Móricz defined certain terms such as Statistical Convergent, Statistically Bounded, Statistically Cauchy, Summable explained in [9] are listed as follows :-

Definition 1.2.2. Statistically Convergent - *A sequence $(t_p) := (t_p : p = 0, 1, 2, ...)$ or a series. Consider partial sums (t_p) of series, where the (t_p) are real or complex numbers, is called convergent statistically to limit (or sum) t, in symbols :-*

$$st - \lim_{p\to\infty} t_p = t$$

if for all $\epsilon > 0$,

$$\lim_{m\to\infty} ((m+1)^{-1})|\{p \leq m : |t_p - t| > \epsilon\}| = 0$$

where $p \leq m$ where $p = 0, 1, 2, ..., m$; and where $|\mathcal{T}|$ is cardinality of finite set $\mathcal{T} = |\{p \leq m : |t_p - t| > \epsilon\}| \subseteq \mathcal{N} := 0, 1, 2, ...$

Definition 1.2.3. Bounded Statistically - *A sequence (t_p) is called bounded statistically, if some constant $D > 0$ exists so that*

$$\lim_{m\to\infty} \frac{1}{m+1} |\{p \leq m : |t_p| > D\}| = 0,$$

where D is a bound .

If definition of Statistical Convergence validates for some $\epsilon > 0$. Moreover, the inequality validates for $D = |t| + \epsilon$; t – statistical limit of sequence, say.

Definition 1.2.4. Statistically Cauchy *A sequence (t_p) is termed as Statistically Cauchy, if for all $\epsilon > 0 \ \exists \ v = v(\epsilon)$ there exists $\mathcal{N}$ (natural numbers) so that*

$$\lim_{m\to\infty} \frac{1}{m+1} |\{p \leq m : |t_p - t_v| > \epsilon\}| = 0$$

Definition 1.2.5. SUMMABLE *A sequence* (t_p) *is coined as Summable* S_q *to limit (or sum)* t *for certain real* $q > 0$ *if*

$$\lim_{m\to\infty} (m+1)^{-1} \sum_{p=0}^{m} |t_p - t|^q = 0$$

In year, 2014 Móricz defines in [10] the terms coined as Statistical limit, Approximate limit. These are listed below :-

Definition 1.2.6. STATISTICAL LIMIT OF MEASURABLE FUNCTION AT ∞
A function $g : [s, \infty) \to \mathcal{C}$*, measurable in Lebesgue, where* $s \geq 0$*, is said to have statistical limit* r *at* ∞*, in symbols:*

$$st - \lim_{v\to\infty} g(v) = r$$

if for every $\epsilon > 0$

$$\lim_{d\to\infty} (d - s)^{-1} |\{v \in (s, d) : |g(v) - r| > \epsilon\}| = 0$$

where

$$|\{v \in (s, d) : |g(v) - r| > \epsilon\}|$$

denotes the measure of the set in Lebesgue's sense

$$\{v \in (s, d) : |g(v) - r| > \epsilon\}$$

Definition 1.2.7. APPROXIMATE LIMIT *A. Zygmund commenced its conceptualization. A function* g *is given on a set* $A \subset \mathcal{R}_+$ *which is measurable and there exists* $v_0 > 0$*, then* g *has an approximate limit* r *as* $v \to v_0$*, in symbols:*

$$\lim_{v\to v_0} ap(g(v)) = r$$

1.3 LITERATURE REVIEW

1.3.1 THEOREMS

Some of the well known theorems are stated as follows :

Theorem 1.3.1. [9] *In Convergence of sequence if the limit exists for any sequence* (t_p) *then it is uniquely determined and same is followed by statistical convergence also.*

Theorem 1.3.2. [9] *Any Sequence that converges is Statistically Convergent however the implication is not valid for converse.*

Theorem 1.3.3. [9] *Statistical limit follows the algebra of limits.*
i.e.
Let (t_p) *and* (b_p) *be two sequences such that*

$$st - \lim_{p\to\infty} t_p = t$$

and,

$$st - \lim_{p\to\infty} b_p = b$$

then,

1. $t_p + b_p$ *converge statistically to t + b.*
2. $t_p - b_p$ *converge statistically to t - b.*
3. $t_p * b_p$ *converge statistically to t * b.*
4. t_p/b_p *converge statistically to* t/b*; provided* $b_p \neq 0$ *and* $b \neq 0$*.*

Theorem 1.3.4. [9] *If a sequence is convergent, then the sequence is also bounded.*

Theorem 1.3.5. [9] *Every Statistically Convergent Sequence is Statistically Bounded.*

Theorem 1.3.6. [9] *All the sequences that converges are statistically convergent with common limit however, the sequences which are Statistically Convergent may neither be convergent nor bounded.*

Theorem 1.3.7. [9] (t_p) *is a sequence converge to a limit t statistically if and only if there exists two sequences* (u_p) *and* (w_p) *such that*

1. $t_p = u_p + w_p; p = 0, 1, 2, ...$
2. $\lim_{p\to\infty} u_p = t$
3. $\lim_{m\to\infty}(\frac{1}{m+1})|\{p \leq m : w_p \neq 0\}| = 0$

Moreover, if sequence (t_p) *is bounded, then both* (u_p) *and* (w_p) *are also bounded.*

Theorem 1.3.8. [9] *Convergence of sequence statistically is equivalent to almost convergence of sequences.*

Theorem 1.3.9. [5] *A function h with periodicity is continuous i.e., $h \in C(\mathcal{S})$, subsequently for certain $q > 0$ Fourier Series is S_q summable to $h(a)$ uniformly in $a \in \mathcal{S}$*

Theorem 1.3.10. [5] *If $h \in \mathcal{L}^1(\mathcal{S})$, afterwards for any $q > 0$ Fourier Series is S_q summable to h(a) at almost every $a \in \mathcal{S}$.*

Theorem 1.3.11. [10] *Any integrable function of Fourier series is statistically convergent at almost all point.*

Theorem 1.3.12. [10] *Any continuous function of Fourier series is uniformly statistically convergent.*

The above hypothesis also hold for the Lebesgue measurable function at ∞ along with the statistical limit and there are some more hypothesis for Lebesgue measurable function at ∞ along with the Statistical limit.

Theorem 1.3.13. [10] *If a function g has limit at ∞ statistically then it is bounded statistically.*

Theorem 1.3.14. [10] *If $h \in \mathcal{L}^1(\mathcal{R})$ then at every $a \in \mathcal{R}$,*

$$st - \lim_{v \to \infty} t_v(h; a) = h(a)$$

$$st - \lim_{v \to \infty} \tilde{t}_v(h; a) = \tilde{h}(a)$$

Theorem 1.3.15. [10] *If $h \in \mathcal{L}^1(\mathcal{R})$, h has a Lebesgue point a, and the Hilbert transform $\tilde{h}$ exists at a, then Drichlet integral and conjugate Drichlet integral i.e., $t_v(h; a)$ and $\tilde{t}_v(h; a)$ respectively are S_1 summable then,*

$$\lim_{d \to \infty} d^{-1} \int_o^d |t_v(h; a) - h(a)| dv = 0$$

and,

$$\lim_{d \to \infty} d^{-1} \int_o^d |\tilde{t}_v(h; a) - \tilde{h}(a)| dv = 0$$

Theorem 1.3.16. [10] *If a function $g \in \mathcal{L}^1(\mathcal{R}) \cap \mathcal{L}^\infty_{loc}(\mathcal{R})$ on bounded interval U is continuous, then the limit in theorem 1.3.15 holds locally uniformly on U.*

Corollary 1.3.16.1. [10] *Based on the assumptions of theorem 1.3.16, the statistical limit in theorem 1.3.14 is locally uniform in $a \in U$*

Theorem 1.3.17. [10] *If $g \in \mathcal{L}^1(\mathcal{R}) \cup C_0(\mathcal{R})$, then limit in theorem 1.3.15 holds uniformly on the whole $\mathcal{R}$.*

Corollary 1.3.17.1. [10] *Based on the assumptions of theorem 1.3.17, the statistical limit in theorem 1.3.14 is uniform in $a \in \mathcal{R}$.*

1.3.2 Examples

Some common examples which are used in every concept and satisfy certain properties:-

1. Consider a sequence

$$(t_p) := \begin{cases} t; \ t = l^2 \\ 0; \ otherwise \end{cases}$$

   be a statistical convergent sequence where the limit $t = 0$ *for every* $\epsilon > 0$. But (t_p) is not bounded.

   $\implies$ it may or may not be convergent.

2. If the sequence

$$(a_n) := \begin{cases} t; \ if \ t = l^2, \ l = 0, 1, 2, .. \\ 1/t; \ otherwise \end{cases}$$

   Then, the given sequence statistically converges to 0 . *i.e.*

$$st - \lim_{p \to \infty} t_p = 0$$

   but the sequence is divergent in ordinary sense.

3. Statistical Convergence based on intervals [1,p] and Lacunary Convergence based on $(t_{p-1}, t_p]$. Therefore, the sequence that converges is statistically convergent however converse is not valid.

4. A function denoted by g is coined as

$$g(v) := \begin{cases} t; \ v \in (t^2, t^2 + 1), \ t = 1, 2, ...; \\ 0; \ otherwise \ on \ (0, \infty) \end{cases}$$

   Then, its statistical limit exists in Lebesgue measure at ∞ *for every* $\epsilon > 0$ with r=0 but g is not bounded, and therefore its ordinary limit cannot exist at ∞

1.3.3 Remarks

Some remarks concluded from the well known results are stated below:-

- Limit of a sequence classically exists *imply that* statistical limit of sequence also exists.
  Also, two limits correlate. In general, the converse is not valid.

- The abstraction of limit statistically also boasts the laws of limit which validates inquest of limit classically in the terms of Cauchy *1.e.,* additivity, homogeneity, etc.

  Both assertions stated above are valid for Lebesgue measurable functions at ∞ with the statistical limit.

- Holder's Inequality provides that

$$\lim_{m\to\infty}(m+1)^{-1}\sum_{p=0}^{m}|t_p-t|^q=0,$$

  is valid for certain $q > 0$. It validates *for all* smaller exponent r_0, $0< r_0 < q$. Thus, Summability S_1 is termed as *strong summability* (C,1) *implies* Summability by first arithmetic means *i.e.*,

$$\lim_{m\to\infty}(m+1)^{-1}\sum_{p=0}^{m}t_p=t$$

- Holder's Inequality provides that if

$$\lim_{d\to\infty}(d-s)^{-1}\int_s^d|g(v)-r|^q dv=0$$

  is valid for certain $q > 0$. It validates for any smaller exponent $\tilde{q}$, $0<\tilde{q}<q$. If the above equality is satisfied for q = 1, then it is trivial that

$$|(d-s)^{-1}\int_s^d g(v)dv-r|\le(d-s)^{-1}\int_s^d|g(v)-r|dv\to 0$$

$$as\ d\to\infty$$

  So, the summability S_1 of g to r at ∞ imply the classical summability of g at ∞ by *Cesáro summability* (C,1) of g at ∞. Thus, the summability S_q is termed as strong *Cesáro* summability (C,q), where $q > 0$.

Both, the assertions are valid for Lebesgue measurable functions at ∞ with the statistical limit.

1.4 Comparison Of Statistical Convergence And Classical Convergence

The idea is used to define statistical convergence was following a sequence may be infinite which are not included in ϵ neighborhoods of limit point but the index of terms have density zero, which is not possible in ordinary sense.

Thus, a convergence defined in this manner gives us the statistical convergence different from classical convergence. Researchers focuses on convergence obtained from different density functions. But all density function are based on class of intervals different from each other. Also, recall the finite subsets of Natural Numbers having natural density as zero, and if the fact is combined with that of classical convergence of a sequence to a real number k,

$$\implies t : |t_p - k| \geq \epsilon$$

also a finite set which says that classical convergence imply statistical convergence. But, the property of boundedness does not hold by Statistical convergence.

Therefore, in Classical convergence, convergent sequences are all bounded *i.e.*, the result stated above shows that Statistical Convergence differs from that of Classical convergence.

1.5 Summary Of Chapters

In Chapter II, the Convergence of single sequence and series statistically with application in single Fourier series have been studied.

In Chapter III, the Convergence of multiple sequence and series statistically with application in multiple Fourier series have been studied.

The limit of Lebesgue Measurable function at ∞ statistically has been studied with application in Fourier Analysis have been presented in Chapter IV.

Moreover, some concepts, hypothesis, examples, and remarks have been used repeatedly in the further study of this dissertation. Apart from these some new theorems will be proved in the further chapters and new concepts have been explained.Some of the examples and concepts are developed from [2], [13], [15], [16] and [17].

Chapter II

2 Statistical Convergence Of Single Sequence And Series With Application In Fourier Analysis

2.1 Single Sequence And Series - Introduction

In this chapter, Convergence of Single Sequence and Series statistically have been studied. Here, few terms related to Convergence of single sequence and series statistically has been already defined in Chapter I.

The concept defined below is the main notion used in this chapter to formulate the main result. *i.e.,*

Definition 2.1.1. **Statistical Convergent** - *A sequence* $(t_p) := (t_p : p = 0, 1, 2, ...)$ *or a series. Consider partial sums* (t_p) *of series, where the* (t_p) *are real or complex numbers, is called convergent statistically to limit (or sum) t, in symbols :-*

$$st - \lim_{p\to\infty} t_p = t$$

if for all $\epsilon > 0$*,*

$$\lim_{m\to\infty}((m+1)^{-1})|\{p \leq m : |t_p - t| > \epsilon\}| = 0 \tag{2.1.1}$$

where $p \leq m$ *where* $p = 0, 1, 2, ..., m$*; and where* $|\mathcal{T}|$ *is cardinality of finite set* $\mathcal{T} = |\{p \leq m : |t_p - t| > \epsilon\}| \subseteq \mathcal{N} := 0, 1, 2, ...$

Definition 2.1.2. **Bounded Statistically** - *A sequence* (t_p) *is called bounded statistically, if some constant D > 0 exists so that*

$$\lim_{m\to\infty} \frac{1}{m+1}|\{p \leq m : |t_p| > D\}| = 0, \tag{2.1.2}$$

where D is a bound .

If definition of Statistical Convergence validates for some $\epsilon > 0$*. Moreover, the inequality validates for* $D = |t| + \epsilon$*; t – statistical limit of sequence, say.*

2.2 Main Result

Some generalization of notion discussed in Chapter I with the some well known hypothesis, examples and assertions evolved new main results for single sequence and series are stated below along with the proofs as:-

Theorem 2.2.1. *Statistical Limit is Uniquely determined, if exists.*

Proof. Suppose (t_p) is a sequence and, Also assume that (t_p) statistically converges to t_1 and t_2

$$\implies st - \lim_{p\to\infty} t_p = t_1$$

and,

$$\implies st - \lim_{p\to\infty} t_p = t_2$$

$$\implies t_1 - t_2 = 0$$

$$\implies t_1 = t_2$$

Hence, limit is unique.

□

Theorem 2.2.2. *Sequence (t_p) Statistically converges if and only if sequence (t_p) is Statistically Cauchy.*

Proof. Recall [4], and [9] as it defines
$\mathcal{P} \subseteq \mathcal{N}$ is termed as natural density denoted by $\delta(\mathcal{P})$ if

$$\delta(\mathcal{P}) = \lim_{n\to\infty} \frac{1}{n+1} |\{t \leq n : t \in \mathcal{P}\}| \tag{2.2.1}$$

and limit exists. Thus, if the sequence is statistically convergent and say, it converges to t *then*,

$$\delta(\{t \in \mathcal{N} : |t_p - t| > \epsilon\}) = 0$$

is given from (2.1.1) and (2.1.2) along with (2.2.1).
Thus, by the concept defined in earlier Chapter I it's proved to be Statistically Cauchy.

Conversely,if it is Statistically Cauchy
for each $\epsilon > 0$ there exists $\nu(\epsilon) \in \mathcal{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} |\{t < n : |t_p - t_\nu| > \epsilon\}| = 0$$

i.e., subset has natural density as 0.

$$\implies \lim_{n\to\infty} \frac{1}{n+1} |\{t \leq n : t \in \mathcal{P}\}| = \delta(P)$$

and limit exists.

Therefore, the sequence converges statistically. ☐

Theorem 2.2.3. *If a sequence (t_p) is S_q summable to a limit t for certain q > 0 subsequently, (t_p) is almost convergent to t.*

Proof. Assume that t = 0. Then, firstly to show the condition necessary and sufficient for (t_p) to be almost convergent to 0 *i.e., for all* $\epsilon > 0$, the p's *so that*

$$|t_p| \leq \epsilon$$

has density 1.

Thus, it is suffices to prove only the sufficient condition.
Let$\mathcal{T}$ be the set of p indices *so that*

$$|t_p| \leq \frac{1}{k}, (k = 1, 2, ..)$$

Then, $\mathcal{T}_1 \supset \mathcal{T}_2 \supset ... \supset \mathcal{T}_k \supset ...$ and each of $\mathcal{T}_k$ has density 1.
Then, defining a sequence

$$B_1 < B_2 < ... < B_k < ...$$

so that B_m has density $> 1 - \frac{1}{k}$ in (0,B) is not less than density of $\mathcal{T}_k$ in (0,B), and so exceeds $1 - \frac{1}{k}$. Hence ,$\mathcal{T}$ is of density 1, and also as $p \to \infty$ *implies* $t_p \to 0$ in $\mathcal{T}$ ☐

Theorem 2.2.4. *Conversely, if (t_p) is almost convergent to t and bounded, afterwards for certain q > 0, (t_p) is S_q summable to t.*

Proof. Now, for this using the proof done for above theorem 2.2.3
Then, fix an $\epsilon > 0$. Let $v(B)$ be the number of $s \leq B$ with $|t_p| > \epsilon$. Recalling the summability defined by Hardy and Littlewood in [5]

$$\frac{|t_0 - t|^q + |t_1 - t|^q + ... + |t_p - t|^q}{p + 1} \to 0 \text{ } as \text{ } p \to \infty$$

with $t = 0$ is not less than $\epsilon^q \frac{v(p)}{p+1}$. Thus, this is an application of the theorem (2.2.3) yields above. ☐

2.3 Application To Single Fourier Series

A special type of series has been considered as an application to Single Sequence and Series Statistically.

The Fourier Analysis endlessly work on time domain and move to frequency domain which are considered to be symmetric and real valued. Sometimes these are complex valued when perform under some conditions and give rise to the pair of functions.

2.3.1 Conversion Of Classical Fourier Series

The Fourier series classically is of the form

$$a(s) = \sum_{l=-\infty}^{\infty} \eta_l \; e^{jz_l s} \longleftrightarrow \eta_l = \frac{1}{H} \int_0^H a(s) \; e^{-jz_l s} ds \tag{2.3.1}$$

The classical Fourier Series demonstrates a function with continuity, piecewise continuity and periodicity

i.e., $a(s) = a(s + H)$

represented as a sum of functions of sin and cos. These functions are with increasing frequencies harmonically

i.e., $z_l = \frac{2\pi l}{H}$; $l = 1, 2, ...$

such that

$$\begin{aligned} a(s) &= \beta_0 + \sum_{l=1}^{\infty} \beta_l \cos(z_l s) + \sum_{l=1}^{\infty} \beta_l \sin(z_l s) \\ &= \beta_0 + \sum_{l=1}^{\infty} \phi_l \cos(z_l s - \psi_l) \end{aligned} \tag{2.3.2}$$

Now, $z_1 = \frac{2\pi}{H}$ *i.e.,* the angular velocity or fundamental frequency.

This further proportionates to trigonometric function which finishes one cycle in [0,H] interval or in some interval of length H such as $[-\frac{H}{2}, \frac{H}{2}]$.

The expression of a(s) depends on a fact when written again *i.e.,*

$$\phi_l^2 = \beta_l^2 + \alpha_l^2$$

and,

$$\psi_l = \tan^{-1}\left(\frac{\alpha_l}{\beta_l}\right) \tag{2.3.3}$$

and the identity of trigonometry is followed by equality *i.e.,*

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B) \tag{2.3.4}$$

Therefore, the periodicity of a(s) is illustrated by considering a map from [0,H] an interval onto the circumference of circle on which a function is defined where the end points of the interval coincide.

Moreover, continuous cycles will be generated by the laps or circuits of circle successively.

These are according to Euler's equation :

$$\cos(z_l s) = \frac{1}{2}(e^{jz_{[l]}s} + e^{-jz_l s})$$

and,

$$\sin(z_l s) = -\frac{\iota}{2}(e^{jz_l s} - e^{-jz_l s}) \tag{2.3.5}$$

So, equation (2.3.2) is evaluated as

$$a(s) = \beta_0 + \sum_{l=1}^{\infty} \frac{\beta_l + \iota\alpha_l}{2} e^{-jz_l s} + \sum_{l=1}^{\infty} \frac{\beta_l - \iota\alpha_l}{2} e^{jz_l s} \tag{2.3.6}$$

that can also be expressed as :-

$$a(s) = \sum_{l=-\infty}^{\infty} \eta_l e^{jz_l s} \tag{2.3.7}$$

where,

$$\begin{aligned} \eta_0 = \beta_0; \eta_l &= \frac{\beta_l - \iota\alpha_l}{2} \\ \eta_{-l} = \eta_l^* &= \frac{\beta_l + \iota\alpha_l}{2} \end{aligned} \tag{2.3.8}$$

Now, the inverse of the classical Fourier series transform *i.e.,* equation (2.3.7) is written as,

$$\begin{aligned} \eta_l &= \frac{1}{H}\int_0^H a(s)e^{-jz_l s}ds \\ &= \frac{1}{H}\int_0^H \sum_{p=-\infty}^{\infty} \eta_p e^{jz_l s} e^{jz_l s} ds \\ &= \frac{1}{H}\sum_{p=-\infty}^{\infty} \eta_p \int_0^H e^{j(z_p - z_l)s} ds \end{aligned} \tag{2.3.9}$$

where the final equality is followed from a condition of orthogonality *i.e.,* of form

$$\int_0^H e^{j(z_p - z_l)s} ds = \begin{cases} 0, \ if \ l \neq p \\ H, \ if l = p \end{cases} \tag{2.3.10}$$

Thus, the relation between the periodic function which is continuous and its Fourier Transform can be summarized as

$$a(s) = \sum_{l=-\infty}^{\infty} \eta_l \ e^{jz_l s} \longleftrightarrow \eta_l = \frac{1}{H}\int_0^H a(s) \ e^{-jz_l s} ds \tag{2.3.11}$$

2.3.2 Conditions For Convergence Of Classical Fourier Series

The visualization of Fourier series is considered along the conditions established under which convergent to function for its partial sums.
Sufficiency for convergence is that a(s) should be continuous and in the interval [0,H] is bounded.

Although, the theory of classical convergence of Fourier series is concerned with the relationship existence in the case that a(s) is bounded and is allowed with finite maxima, minima and jump discontinuities.

Further, we see that if the terms which are successive when added, the Fourier series of equation (2.4.2.2) converges to $\frac{1}{2}a(s+0)+a(s-0)$
where, $a(s+0)$ is the value approached from right of s and $a(s-0)$ is the value approached from the left of s.

Thus, if a(s) attains continuity at a point, then Fourier series converges to a(s) only.

2.3.3 Convergence of Fourier Series

Consider, Mó́ricz describes the special type of series in his paper [9] , recalling the term

FOURIER SERIES - A function h:$\mathcal{S} \rightarrow \mathcal{C}$ in Lebesgue's measure is integrable on the core $\mathcal{S} := [-\phi, \phi)$, its represented as : $h \in \mathcal{L}^1(\mathcal{S})$ .The Fourier Series is termed by

$$h(a) \sim \sum_{l \in \mathcal{Z}} \hat{h}(l) e^{jla}, a \in \mathcal{S}, \tag{2.3.12}$$

where the Fourier coefficients $\hat{h}(l)$ are defined by

$$\hat{h}(l) := \frac{1}{2\pi} \int_{\mathcal{S}} h(s) \exp^{-jla} ds, i \in \mathcal{Z} := ..., -2, -1, 0, 1, 2, ... \tag{2.3.13}$$

Therefore, the series with symmetric partial sums in (2.3.12) is coined by

$$P_p(h; a) := \sum_{|l| \leq p} \hat{h}(l) e^{jla}, p \in \mathcal{N} \ and \ a \in \mathcal{S} \tag{2.3.14}$$

Now, an example is quoted i.e., the Conjugate Series to Fourier Series defined by :-

$$\sum_{l\in\mathcal{Z}}(-j\ sign\ l)\hat{h}(l)e^{jla}$$

$$where\ sign\ l = \begin{cases} \frac{l}{|l|}; l \neq 0 \\ 0; l = 0 \end{cases} \tag{2.3.15}$$

Clearly from (2.3.12) and (2.3.15) it follows that

$$\sum_{l\in\mathcal{Z}}\hat{h}(l)e^{jla} + j\sum_{l\in\mathcal{Z}}(-j\ sign\ l)\hat{h}(l)e^{jla} = 1 + 2\sum_{l=1}^{\infty}\hat{h}(l)e^{jla} \tag{2.3.16}$$

Thus, the power series is considered from (2.3.16) we get,

$$1 + 2\sum_{l=1}^{\infty}\hat{h}(l)z^{l}, where\ z\ := Re^{ja}, 0 \leq R < 1$$

on open unit disk *i.e.* $|z| < 1$ is analytic.

because, $|\hat{h}(l)| \leq \frac{1}{2\pi}\int_{\pi}|h(s)|ds,\ l \in \mathcal{Z}$ which vindicates the term *"Conjugate series"* in (2.3.16). Thus, the conjugate function $\tilde{h}$ of a function $h \in \mathcal{L}^1(\mathcal{S})$ is termed by

$$\begin{aligned} \tilde{h}(a) &= -\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{\epsilon\leq|s|\leq\pi}\frac{h(a+s)}{2\tan(\frac{s}{2})}ds \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{\epsilon}^{\pi}\frac{h(a-s)-h(a+s)}{2\tan(\frac{s}{2})}ds \end{aligned} \tag{2.3.17}$$

Thus, it is defined in the sense of principle value and that $\tilde{h}(a)$ exists at almost every $a \in \mathcal{S}$. Then by the results stated in Chapter I, conclusion for Statistical convergence single Fourier series of function $h \in \mathcal{L}^1(\mathcal{S})$ diverges everywhere.

2.4 Remarks

- The conjugate series written above is summable S_q *for certain* $q > 0$ to the conjugate function $\tilde{h}(a)$ explained above at each $a \in \mathcal{S}$.
- If $h \in C(\mathcal{S})$, then Fourier Series converges statistically to h(a) uniformly in $a \in \mathcal{S}$.
- If by the virtue of above results in Chapter I, the divergence of a function obtained from the Fourier series i.e. $h \in C(\mathcal{S})$ at one point $a \in \mathcal{S}$ is due to existence of a subsequence $0 < p_1 < p_2 < ... < p_o < ...$ with natural density as 0 of integers *so that* the subsequences $(P_{p_o}(h; a))$ of the partial sums of Fourier series of h diverges at the point a.

Chapter III

3 Statistical Convergence Of Multiple Sequence And Series With Application

3.1 Multiple Sequence And Series - Basic Concepts

In this chapter, Convergence of Multiple Sequence and Series statistically have been studied. Here, few terms related to Statistical Convergence of double sequence and series are explained below which are discussed by *Šalát* in [14].

The concept of Statistical convergence can be expanded to k - multiple sequences and series, where a fixed integer is $k \geq 2$ Notation for the k-tuples $p = (p_1, p_2, p_3, ..., p_k)$ with the coordinates p_l all are non-negative integers.

The main concepts used to explain the convergence of k - multiple sequences and series statistically discussed by M$\acute{o}$ricz in [8] are listed below :-

Definition 3.1.1. Convergence Of k - Multiple Sequence Statistically - *A k - multiple sequence $(t_p) = (t_p : p \in \mathcal{N}^k)$ of real or complex numbers is said to converge statistically to limit t, in symbols :*

$$st - \lim_{p \to \infty} t_p = t,$$

if for every $\epsilon > 0$,

$$\lim_{m \to \infty} (\prod_{i=1}^{k} (m_i + 1)^{-1}) |p \leq m : |t_p - t| > \epsilon| = 0$$

where by $m \to \infty$, we mean

$$min(m_1, m_2, ..., m_k) \to \infty$$

.

Definition 3.1.2. Convergence Of k - Multiple Series Statistically - *An k - multiple series*

$$\sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} ... \sum_{l_k=0}^{\infty} b_{l_1, l_2, ..., l_k}$$

of real or complex numbers, considering its rectangular partial sums as (t_p) *defined by*

$$t_p := \sum_{l_1=0}^{p_1} \sum_{l_2=0}^{p_2} ... \sum_{l_k=0}^{p_k} b_{l_1,l_2,...,l_k}, \ p \in \mathcal{N}^k.$$

The k - multiple series converges to sum t statistically if the k - multiple sequence (t_p) *of its rectangular partial sums converges to t statistically.*

Definition 3.1.3. STATISTICALLY CAUCHY - *A multiple sequence* $(t_p : p \in \mathcal{N}^k)$ *is Statistically Cauchy if for every* $\epsilon > 0$ *there exists* $v \in \mathcal{N}^k$ *such that*

$$\lim_{m\to\infty} (\prod_{i=1}^{k} (m_i + 1)^(-1)|p \leq m : |t_p - t_v| > \epsilon| = 0$$

Definition 3.1.4. ASYMPTOTIC DENSITY - *The natural density of a set* $\mathcal{P} \subset \mathcal{N}^k$ *is represented as*

$$\delta(\mathcal{P}) := \lim_{m\to\infty} (\prod_{i=1}^{k} (m_l + 1)^{-1})|p \leq m : p \in \mathcal{P}|,$$

provided limit above exists.

Definition 3.1.5. SUMMABLE - *An k - multiple sequence* (t_p) *or series with rectangular partial sums* (t_p) *is said to be* S_q *summable to limit (or sum) t for certain real q > 0 if*

$$\lim_{m\to\infty} (\prod_{i=1}^{k} (m_i + 1)^{-1}) \sum_{p_1=0}^{m_1} \sum_{p_2=0}^{m_2} ... \sum_{p_n=0}^{m_n} |t_p - t|^q = 0.$$

3.2 MAIN RESULTS

Now, here are some **well known results** based on the generalization of multiple sequences and series converges statistically.

The result stated below are for double sequence and series can be expanded to multiple sequence and series.
Following are the results discussed in [8] as:-

Theorem 3.2.1. *A double sequence to a number t converges Statistically if and only if there exists a subset* $A = (l, p) \subseteq \mathcal{N} \times \mathcal{N}; l, p = 1, 2, ...,$ *such that*

$$\delta_2(A) = 1$$

and,

$$\lim_{l,p \to \infty \ (l,p)\in A} a_{lp} = t$$

Proof. Let (a) be statistically convergent to t. But,

$$A_{r_0} = \{(l,p) \in \mathcal{N} \times \mathcal{N} : |a_{lp} - t| \geq \frac{1}{r_0}\}$$

and,

$$\mathcal{M}_{r_0} = \{(l,p) \in \mathcal{N} \times \mathcal{N} : |a_{lp} - t| \geq \frac{1}{r_0}\}$$

where $r_0 = 1, 2, ...$ Then, $\delta_2(A_{r_0}) = 0$

1. $M_1 \supset M_2 \supset M_3 ... \supset M_i \supset M_{i+1} \supset ...$ and
2. $\delta_2(M_{r_0}) = 1, r_0 = 1, 2, ...$
   Now, to show that for $(l,p) \in M_{r_0}$, (a_{lp}) converges to t.
   Suppose that (a_{lp}) is not convergent to t.

   Therefore, there exists $\epsilon > 0$, *so that* $|a_{lp} - t| \geq \epsilon$ for infinitely many values.
   Let
   $$M_{r_0} = \{(l,p) : |a_{lp} - t| < \epsilon\}$$
   and, $\epsilon > \frac{1}{r_0}$ (r=1,2,...) Then,
3. $\delta_2(M_\epsilon) = 0$

and by (1) $M_{r_0} \subset M_\epsilon$. Thus, $\delta_2(M_{r_0}) = 0$ contradicts (2). So, (a_{lp}) is convergent to t.

Conversely,
Suppose that *there exists* a subset $A = (l,p) \subseteq \mathcal{N} \times \mathcal{N}$ *such that*

$$\delta_2(A) = 1 \ and \ \lim_{l,p} a_{lp} = t$$

i.e., there exits $B \in \mathcal{N}$ *such that* for all $\epsilon > 0$, $|a_{lp} - t| < \epsilon, \forall \ l,p \geq B$
Now,

$$A_\epsilon = \{(l,p) : |a_{lp} - t| \geq \epsilon \ \subseteq \mathcal{N} \times \mathcal{N} - \{(l_{B+1}, p_{B+1}), (l_{B+2}, p_{B+2}), ...\}$$

$$\delta_2(A_\epsilon) \geq 1 - 1 = 0$$

Hence, (a) is statistically convergent to t. □

Theorem 3.2.2. *A double sequence (a_{lp}) is Convergent Statistically if and only if (a_{lp}) is Statistically Cauchy.*

Proof. Let (a) converges a number t statistically.
Then, *for all* $\epsilon > 0$, the set

$$\{(l,p) : l \leq m,\ p \leq k : |a_{lp} - t| \geq \epsilon\}$$

has natural density 0.

Choosing Number B and $\mathcal{M}$ *such that* $|a_{B\mathcal{M}} - t| \geq \epsilon$.
Now, let

$$C_\epsilon = \{(l,p) : l \leq m,\ p \leq k : |a_{lp} - a_{B\mathcal{M}}| \geq \epsilon\}$$

$$E_\epsilon = \{(l,p) : l \leq m,\ p \leq k : |a_{lp} - t| \geq \epsilon\}$$

$$F_\epsilon = \{(l,p) : l = B \leq m,\ p = \mathcal{M} \leq k : |a_{B\mathcal{M}} - t| \geq \epsilon\}$$

Then,

$$C_\epsilon \subseteq E_\epsilon \cup F_\epsilon$$

Therefore, $\delta_2(C_\epsilon) \leq \delta_2(E_\epsilon) + \delta_2(F_\epsilon) = 0$.

Conversely, Let (a) be statistically Cauchy but not convergent statistically.
Then, *there exists* B and $\mathcal{M}$ *such that* the set C_ϵ has natural density 0.
Hence, the set

$$G_\epsilon = (l,p) : l \leq m,\ p \leq k : |a_{lp} - a_{B\mathcal{M}}| < \epsilon$$

has natural density 1. In particular, it can also be written as:-

$$|a_{lp} - a_{B\mathcal{M}}| \leq 2|a_{lp} - t| < \epsilon \tag{3.2.1}$$

if $|a_{lp} - t| < \frac{\epsilon}{2}$
Since, (a) is not statistically Convergent, the set E_ϵ has natural density 1 i.e. the set

$$\{(l,p) : l \leq m, p \leq k : |a_{lp} - t| < \epsilon\}$$

has natural density 0.

Therefore, by equation (3.1.1),
the set

$$\{(l,p) : l \leq m, p \leq k : |a_{lp} - a_{B\mathcal{M}}| < \epsilon\}$$

has natural density 0 , *i.e.,* the set C_ϵ has natural density 1 which is a contradiction.
Hence,(a) Converges statistically. □

Theorem 3.2.3. *A double sequence (a_{lp}) to some number η converges Statistically if and only if there exist two sequences (u_lp) and (w_lp) such that*

$$a_{lp} = u_{lp} + w_{lp},\ \ l, p = 0, 1, 2, ..$$

$$\lim_{l,p\to\infty} u_{lp} = \eta$$

and,

$$\lim_{k,m\to\infty} \frac{1}{(k+1)(m+1)} |\{l \leq k, p \leq m : w_{lp} \neq 0\}| = 0$$

Moreover, if (a_{lp}) is bounded, then (u_{lp}) and (w_{lp}) are also bounded.

Theorem 3.2.4. *A double sequence (a_{lp}) converges to some number η statistically if and only if there exists a $\mathcal{T} \subseteq \mathcal{N}^2$ such that the natural density of $\mathcal{T}$ is 1 and*

$$\lim_{l,p\to\infty\ \ and\ \ (l,p)\in\mathcal{T}} a_{lp} = \eta$$

i.e., for each $\epsilon > 0$, there exists B such that

$$|a_{lp} - \eta| \leq \epsilon\ , if\ \ l, p \geq B\ and\ (j, k) \in \mathcal{T}$$

3.3 Application To k - Multiple Fourier Series

A special type of series has been considered as an application to multiple Sequence and Series Statistically.
The Fourier Series repeatedly move from time to frequency domain an expressed in symmetric and real valued. Under certain conditions it is expressed in complex value and give rise to pair of functions.

3.3.1 Convergence of k- multiple Fourier Series

Let $h : \mathcal{S}^k \to \mathcal{C}$ in Lebesgue's measure is integrable on the k- dimensional core $\mathcal{S}^k := [-\phi, \phi)^k$ where k is an integer ≤ 2.
Recalling the special type of Fourier Series discussed by Móricz in [9]

k- MULTIPLE FOURIER SERIES - A function h:$\mathcal{S}^k \to \mathcal{C}$ in Lebesgue's measure is integrable on the k - dimensional core $\mathcal{S}^k := [-\phi, \phi)$. The Fourier Series is termed by

$$h(a) \sim \sum_{l_1\in\mathcal{Z}} \sum_{l_2\in\mathcal{Z}} ... \sum_{l_k\in\mathcal{Z}} \hat{h}(l) e^{jla}, a \in \mathcal{S}^k, \tag{3.3.1}$$

where the Fourier coefficients $\hat{h}(l)$ are defined by

$$\hat{h}(l) := \frac{1}{2\pi^k} \int_{\mathcal{S}} \int_{\mathcal{S}} ... \int_{\mathcal{S}} h(s) \exp^{jla} ds_1 ds_2 ... ds_k$$

$$l.s = \sum_{s_i}^{k} l_i s_i ; l \in \mathcal{Z}^k and\ s \in \mathcal{S}^k \tag{3.3.2}$$

Therefore, the symmetric rectangular partial sums of the series in (3.3.1) is coined by

$$P_p(h; a) := \sum_{|l_1| \leq p_1} \sum_{|l_2| \leq p_2} ... \sum_{|l_k| \leq p_k} \hat{h}(l) e^{jla}, p \in \mathcal{N}^k \ and \ a \in \mathcal{S}^k \tag{3.3.3}$$

So, the convergence of $P_p(h; a)$ meaning in *Pringsheim's sense, i.e.,*
the finite limit $\lim P_p(h; a)$ *exists as* $min(p_1, p_2, ..., p_k) \to \infty$
An example is now quoted i.e., the *Conjugate Series* for the sake of impermanence to Fourier Series for the case when $k = 2$ defined by :-

$$\sum_{l_1 \in \mathcal{Z}} \sum_{l_2 \in \mathcal{Z}} (-j \ sign \ l_1) \hat{h}(l) e^{jla},$$

$$\sum_{l_1 \in \mathcal{Z}} \sum_{l_2 \in \mathcal{Z}} (-j \ sign \ l_2) \hat{h}(l) e^{jla},$$

$$\sum_{l_1 \in \mathcal{Z}} \sum_{l_2 \in \mathcal{Z}} (-j \ sign \ l_1)(-j \ sign \ l_2) \hat{h}(l) e^{mla}, a \in \mathcal{S}^2 \tag{3.3.4}$$

The corresponding conjugate functions analogous to one dimensional case are :-

$$\tilde{h}^{(1,0)}(a) := -\lim_{\epsilon_1 \downarrow 0} \frac{1}{\pi} \int_{\epsilon_1 \leq |s_1| \leq \pi} \frac{h(a_1 + s_1, a_2)}{2 \tan \frac{s_1}{2}} ds_1,$$

$$\tilde{h}^{(0,1)}(a) := -\lim_{\epsilon_2 \downarrow 0} \frac{1}{\pi} \int_{\epsilon_2 \leq |s_2| \leq \pi} \frac{h(a_1, a_2 + s_2,)}{2 \tan \frac{s_2}{2}} ds_2,$$

$$\tilde{h}^{(1,1)}(a) := \lim_{\epsilon_1 \to 0} \lim_{\epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1 \leq |s_1| \leq \pi} \int_{\epsilon_2 \leq |s_2| \leq \pi} \frac{h(a_1 + s_1, a_2 + s_2)}{4 \tan \frac{s_1}{2} \tan \frac{s_2}{2}} ds_1 ds_2,$$

all the integrals are in the sense of principle value. These functions exists at all $a \in \mathcal{S}^2$ provided

$$\int_{\mathcal{S}} \int_{\mathcal{S}} |h(s_1, s_2)| log^+ |h(s_1, s_2)| ds_1 ds_2 < \infty,$$

represented as:
$h \in \mathcal{L}^1 log^+ \mathcal{L}(\mathcal{S}^2)$, where

$$log^+|e| := max\{0, log|e|\}, e \in \mathcal{C}$$

In general, take case when $k \geq 3$, then,

$$\varphi = (\varphi_1, \varphi_2, ..., \varphi_k) \in 0, 1^k$$

is such that one of the component is 1 at least.

Precisely,

$$\varphi_{i_1} = ... = \varphi_{i_t} = 1, \; where \; 1 \leq i_1 < ... < i_t \leq k \; and \; 1 \leq t < k;$$

while $\varphi_t = 0$ and for the other remaining indicies t is between 1 and k . Thus, the k- multiple series

$$\sum_{l_1 \in \mathcal{Z}} \sum_{l_2 \in \mathcal{Z}} ... \sum_{l_t \in \mathcal{Z}} (-jsign\; l_{i_1})(-jsign\; l_{i_2})...(-jsign\; l_{i_t}) \hat{h}(l)\; e^{jla}$$

is the conjugate series of Fourier Series which corresponds to $\varphi \in 0,1^k$ There are $2^k - 1$ conjugate series to Fourier series altogether .
Also, the symmetric partial sums of the k - multiple series is denoted by $\tilde{t}_p^{\varphi}$(h;a). all these are defined in sense of principle value and that $\tilde{h}(a)$ esists at almost every $a \in \mathcal{S}^k$.

Then by the result stated in Chapter I, the conclusion of Statistical convergence of k - multiple Fourier series of function $h \in \mathcal{L}^1(\mathcal{S})$ diverges everywhere.

3.4 Remarks

- Some results proved above for double sequences can be generalized for the conceptualization of the k - multiple sequences and series.
- If $f \in \mathcal{L}^1(log^+\mathcal{L})^{k-1}(\mathcal{S}^k)$,then the k- multiple sequence $(t_p(h; a))$ of the symmetric rectangular partial sums of the Fourier series is summable S_q to h(a) at every point $a \in \mathcal{S}^k$ for certain $0 < q < \infty$. Moreover, for each $\varphi \in 0,1^k$ with anyone component $\varphi_i = 1$, the k- multiple sequence $(t_p^{\varphi}(h; a))$ of the conjugate series is S_q summable at ever point $a \in \mathcal{S}^k$ to the conjugate function $\tilde{h}^{\varphi}(a)$ *for certain* $0 < q < \infty$.
- If $f \in \mathcal{L}^1(log^+\mathcal{L})^{k-1}(\mathcal{S}^k)$,then the k- multiple Fourier Series to h(a) at every point $a \in \mathcal{S}^k$. Besides this, each of its conjugate series converges statistically to the corresponding conjugate function $\tilde{h}^{\varphi}(a)$ at almost every $a \in \mathcal{S}^k$.
- For certain $0 < q < \infty$, *exists* a constant C_q which depends only on q in such a way that if $h \in C(\mathcal{S}^2)$, then for every $(m_1, m_2) \in \mathcal{N}^2$ and $(a_1, a_2) \in \mathcal{S}^2$ then

$$(m_1+1)^{-1}(m_2+1)^{-1} \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} |t_{l_1,l_2}(h; a_1, a_2) - h(a_1, a_2)|^q$$

$$\leq C_q(m_1+1)^{-1}(m_2+1)^{-1} \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} (G_{l_1,l_2}(h))^q,$$

  where $G_{l_1,l_2}(h)$ is the most suitable uniform approximation by two- dimensional trigonometric polynomials to the function h which is continuous .
- If $h \in C(\mathcal{S}^k)$, then the k - multiple Fourier series of h statistically converges to h(a) uniformly in $a \in \mathcal{S}^k$.

Chapter IV

4 Statistical Convergence Of Lebesgue Measurable functions at ∞ With Application

This chapter has been presented with the Statistical convergence of Measurable functions in Lebesgue sense at ∞. Some of the abstractions related to this have already been materialized in the introductory chapter. This is the major flattering part of the domain of research .

4.1 Basic Concepts

Some the conceptions are determined and certain results are demonstrated below for the visualization of conjunction. All the hypothesis of this scope are analogous to that of the notion of convergence of single and multiple sequences and series statistically.

Definition 4.1.1. Statistical Limit Of Measurable Function At ∞ - *A function $g : [s, \infty) \to \mathcal{C}$, measurable in Lebesgue, where $s \geq 0$, is said to have statistical limit r at ∞ , in symbols:*

$$st - \lim_{v \to \infty} g(v) = r$$

if for every $\epsilon > 0$ we have

$$\lim_{d \to \infty} (d - s)^{-1} |\{v \in (s, d) : |g(v) - r| > \epsilon\}| = 0$$

where

$$|\{v \in (s, d) : |g(v) - r| > \epsilon\}|$$

denotes the measure of the set in Lebesgue's sense

$$\{v \in (s, d) : |g(v) - r| > \epsilon\}$$

Definition 4.1.2. Bounded Statistically - *A function g is called bounded statistically, if some constant $D_0 > 0$ exists so that*

$$\lim_{d \to \infty} (d - s)^{-1} |\{\nu \in (s, d)| > D_0\}| = 0,$$

where D_0 is a bound .

If definition of Statistical Convergence validates for some $\epsilon > 0$. Moreover, the inequality validates for $D_0 = |r| + \epsilon$; r – statistical limit of measurable functions, say.

Definition 4.1.3. Asymptotic Density - *A subset $K \subset (s, \infty)$ is measurable called to have the (non-discrete) natural density zero if*

$$\delta(K) := \lim_{d\to\infty} (d - s)^{-1} |\{v \in (s, d) \cap K\}| = 0$$

Definition 4.1.4. Statistically Cauchy - *A function g is measurable if for all $\epsilon > 0$, there exists certain $v_0 = v_0(\epsilon) > s$* such that

$$\lim_{d\to\infty} (d - s)^{-1} |\{v \in (s, d) : |g(v) - g(v_0)| > \epsilon\}| = 0$$

Definition 4.1.5. Summable - *A function g is called to be summable S_q to the limit r at ∞ for certain exponent $q > 0$ if*

$$\lim_{d\to\infty} (d - s)^{-1} \int_s^d |g(v) - r|^q dv = 0$$

Definition 4.1.6. Approximate Limit - *A. Zygmund commenced its conceptualization. A function g is given on a set $A \subset \mathcal{R}_+$ which is measurable and there exists $v_0 > 0$, then g has an approximate limit r as $v \to v_0$, in symbols:*

$$\lim_{v\to v_0} ap(g(v)) = r$$

These basic concepts are required to formulate certain results. Some of the results have already been discussed in introductory Chapter.

4.2 Main Result

Apart from those hypothesis, examples and assertions some main results are evolved for measurable functions along with the proofs are stated below:-

Theorem 4.2.1. *The statement follows are equivalent pairwise : -*

1. *g has limit at ∞ statistically.*
2. *g is Cauchy statistically.*
3. *g can be represented as sum of two measurable functions $g_1 : (s, \infty) \to \mathcal{C}$ and $g_2 : (s, \infty) \to \mathcal{C}$ so that*

$$\lim_{v\to\infty} g_1(v) = st - \lim_{v\to\infty} g_1(v)$$

and,

$$\lim_{d\to\infty} (d - s)^{-1} |\{s < v < d : g_2 \neq 0\}| = 0$$

Moreover, if g is bounded, then both g_1 and g_2 are also bounded, in the classical sense.

Proof. The proof is schemed as (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).
(1) $\implies$ (2)
A sequence which is convergent is always Cauchy. Given $\epsilon > 0$, then

$$|h(a)^{-}r| < \frac{\epsilon}{2}$$

for all $x \in (s, \infty)$ and choose a_0. Then, *for all* $s < a_0 < d$,

$$|h(a) - h(a_0)| \leq |h(a) - r| + |r - h(a_0)|$$
$$< |h(a) - r| + \frac{\epsilon}{2}$$

Consequently, we have

$$\{s < a < d : |h(a) - h(a_0)| > \epsilon\}$$
$$\subseteq \{s < a < d : |h(a) - r| > \frac{\epsilon}{2}\},$$

hence it follows. (2) $\implies$ (3) Suppose a real-valued function is considered and denoted as h. If U and V are intervals *such that* both includes the value of a function h(a) for every $a \in (s, \infty)$, also their intersection $U \cap V$.

Then, $\epsilon_1 := \frac{1}{2}$. So, $U := [h(a_o) - 1/2, h(a_o) + 1/2]$ contains the value of function h(a) *for all* $a \in (s, \infty)$ and $\epsilon_2 := \frac{1}{4}$ to get certain $a_1 > s$ *so that* the interval $V := [h(a_1) - \frac{1}{4}, h(a_1) + \frac{1}{4}]$ comprises the value of function h(a) *for all* $a \in (s, \infty)$. The interval $U_1 := U \cap V$ also includes the value of function $h(a)\ \forall\ a$. Clearly, U_1 is a closed interval whose length $|U_1| < \frac{1}{2}$.

Then, use $\epsilon = \frac{1}{8}$ to get some $a_2 > s$ so that $V_1 = [h(a) - \frac{1}{8}, h(a) + \frac{1}{8}]$ involves the value of function $h(a)\ \forall\ a\ \in (s, \infty)$.

Hence, $U_2 = U_1 \cap V_1$ that comprises the value of function $h(a)\ \forall\ a\ \in (s, \infty)$. Then, U_2 is closed interval and $|U_2| \leq \frac{1}{4}$.

Then, by method of induction

A sequence $U_k \supseteq U_{k+1}$, $|U_k| \leq 2^{-k}$ and U_k contains the value of function $h(a)\ \forall\ a\ \in (s, \infty)$.

Therefore, by Nested Interval Theorem $\exists$ a unique no. r so that $r \in U_k\ \forall\ k \geq 1$.

On the other hand, select the number $(s <)d_1 < d_2 < d_3 < ...$ *so that* $d_k \to \infty$ as $k \to \infty$ and,

$$\frac{1}{d-s}|\{s < a < d : h(a) \notin U_k\}| < \frac{1}{k}, if\ d > d_k$$

Then, for $s < a \leq d$, $\exists$ two functions f_1 and f_2 *such that*

$$f_1(a) := g(a)\ and\ f_2(a) := 0$$

If, $a > d_1$, then $d_k < a \leq d_{k+1}$ for some $k \geq 1$.

and, $f_1(a) = \begin{cases} g(a), if \ g(a) \in U_k \\ r, otherwise \end{cases}$

and, $f_2 = g - f_1$

As, f_1 and f_2 are measurable function and $g = f_1 + f_2$

Claiming that the classical limit of $f_1(a)$ exists as $a \to \infty$ and equals r.

Let $\epsilon > 0$ be given.

Assume $\epsilon < 1$

Choosing p_0 so that $2^{-p_0} < \epsilon$.

$if \ a > d_{p_0} \ then \ d_k < a \leq d_{k+1} \ \forall \ k \geq p_0$.

$if \ h(a) \in U_k \ then \ f_1(a) = g(a)$

and $|f_1(a) - r| \leq |U_k| \leq 2^{-k} \leq 2^{-p_0} < \epsilon$

while if $a \notin U_k \ then \ f_1(a) = r$.

The above inequality holds $\forall \ a > d_{p_0}$ which proves the result.

Now, by definition $f_2(a) \neq 0 \ if \ and \ only \ if \ f_1(a) \neq g(a)$

Therefore, if $d_k \leq d \leq d_{k+1}$ for certain $k \geq 1$.
then, by definition

$$\{s < a < d : f_2(a) \neq 0\} = \bigcup_{m=1}^{k-1} \{d_k < a < d_{k+1} : g(a) \notin U_m\}$$

$$\cup \{d_k < a < d : g(a) \notin U_k\} \not\subseteq \{s < a < d : g(a) \notin U_k\}$$

Therefore,

$$\frac{1}{d-s} |\{s < a < d : f_2(a) \neq 0\}| \leq \frac{1}{k}$$

for $d_k < d < d_{k+1}; k = 1, 2, ...$

Since, $k \to \infty$ as $d \to \infty$
in that case, g is bounded say, $|g(a)| \leq D_0 \ \forall \ a$

Therefore, $|f_1(a)| \leq max\{D_0, r\}$

and $|f_2(a)| \leq D_0 + r \ \forall \ a \in (s, \infty)$

lastly, to show : $(3) \implies (1)$
The implication is valid under the certain assumptions that $g = f_1 + f_2$

$$st - \lim_{a\to\infty} f_1(a) = r$$

and,

$$st - \lim_{a\to\infty} f_2(a) = 0$$

due to the additive property of limit at ∞ statistically. The result is proved. □

Theorem 4.2.2. *Suppose $g \in \mathcal{L}^q_{loc}[a, \infty)$ for certain $s \geq 0$ and $q \geq 0$. If g is summable S_q to the limit r at ∞ then*

$$st - \lim_{v\to\infty} g(v) = r$$

Proof. Given $\epsilon > 0$.
So, by Markov's Inequality

$$\epsilon^q |\{s < a < d : |g(a) - r| > \epsilon\}|$$

$$\leq \int_s^d |g(a) - r|^q da$$

$\forall \ 0 < q < \infty$ and $a < d < \infty$.
Thus,

$$\lim_{d\to\infty} \frac{1}{d-s} |\{s < a < d : |g(a) - r| > \epsilon\}| = 0$$

follows from

$$\lim_{d\to\infty} \frac{1}{d-s} \int_s^d |g(a) - r|^q da = 0$$

□

Theorem 4.2.3. *Suppose $g \in \mathcal{L}^q_{loc}[a, \infty)$ for certain $s \geq 0$ and $q \geq 0$. Then after the proof of the above theorem. Conversely, if $st - \lim_{v\to\infty} g(v) = r$ holds and $g \in \mathcal{L}^\infty[s, \infty)$, then g is summable S_q to the same r for each $q > 0$.*

Proof. Assume $|g(a)| \leq D_0 \ \forall \ a$
then,

$$\int_s^d |g(a) - r|^q da = \int_{\{s<a<d:|g(a)-r|\leq\epsilon\}} + \int_{\{s,a<d:|g(a)-r|>\epsilon\}}$$

$$\leq (d - s)\epsilon^q + (D_0 + |r|)^q |\{s < a < d : |g(a) - r| > \epsilon\}|$$

So, by

$$\lim_{d\to\infty} \frac{1}{d-s}|\{s < a < d : |g(a) - r| > \epsilon\}| = 0$$

$\Longrightarrow$

$$\lim_{d\to\infty} \frac{1}{d-s} \int_s^d |g(a) - r|^q da \leq \epsilon^q$$

as, $\epsilon > 0$ is arbitrary. The result is proved. □

4.3 Application to Fourier Transform

A special type of transform has been used here an application of limit of Measurable function at ∞ statistically. Fourier Transforms expressed as a function of time into the frequencies.It is a Complex valued function.

4.3.1 Convergence of Fourier Transform

The Fourier transform $\hat{h}$ of a function $h \in \mathcal{L}^1(\mathcal{R})$ is termed by

$$\hat{h}(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}} h(a)e^{-jsa}da \ , s \ \in \mathcal{R}$$

The above integral exists in Lebesgue's sense. Also, $\hat{h} \in C_0(\mathcal{R})$ *i.e.,* $\hat{h}$ is continuous on $\mathcal{R}$ and, $\lim_{|s|\to\infty} \hat{h}(s) = 0$

Thus, most important part in Fourier Analysis is to recover the function h in terms of Fourier transform $\hat{h}$.
Considering an example for this *i.e.,* if h and $\hat{h}$ both lies in $\mathcal{L}^1(\mathcal{R})$ then, the inversion formula

$$h(a) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}} \hat{h}(s)e^{jsa}ds$$

holds at almost all $a \in \mathcal{R}$. Also, if the above integral exists in Lebesgue's sense then it equals to $\hat{h}^\wedge(-a)$ In general, $\hat{h} \notin \mathcal{L}^1(\mathcal{R})$. The integral existing on right side of Inversion formula does not exist in Lebesgue's sense.To deal with this situation, the integral is assumed to be improper integral and as a result of it consider Drichlet integral $t_v(h; a)$ of function $h \in \mathcal{L}^1(\mathcal{R})$ is defined by

$$t_v(h; a) := \frac{1}{\sqrt{2\pi}} \int_v^v \hat{h}(s)e^{jsa}ds \ ; v \in \mathcal{R}_+ anda \in \mathcal{R}$$

It follows from the above that

$$t_v(h; a) := \frac{1}{\pi} \int_{\mathcal{R}} h(a - s)\frac{\sin vs}{s}ds \ ; v \in \mathcal{R}_+ anda \in \mathcal{R}$$

which is analogous to the classical Drichlet formula of a periodic function for the partial sum of the Fourier series. Now on successive integration using Fubini's Theorem. The hope of saving inversion formula by replacing the integral fails.

Furthermore, recalling the (Fourier) integral conjugate on the right side of inversion formula it is defined by

$$\frac{1}{\sqrt{2\pi}}\int_{\mathcal{R}}(-j\ sign\ s)\hat{h}(s)e^{jsa}ds\ ,a\in\mathcal{R}$$

Therefore, the conjugate Drichlet integral analogous to that of Drichlet integral defined $t_v(h;a)$ of a function $h\in\mathcal{L}^1(\mathcal{R})$ is defined by

$$\tilde{t}_v(h;a):=\frac{1}{\sqrt{2\pi}}\int_{-v}^{v}(-jsign\ s)\hat{h}(s)e^{jsa}ds$$

$$=\frac{1}{\pi}\int_{\mathcal{R}}h(a-s)\frac{1-\cos vs}{v}ds\ ;v\in\mathcal{R}_{+}and\ a\ \in\mathcal{R}$$

Again on successive integration using Fubini's Theorem it follows that

$$t_v(h;a)+it_v(h;a)=\sqrt{\frac{2}{\pi}}\int_{0}^{v}h(s)e^{jsa}ds\ ,v\in\mathcal{R}_{+}and\ a\ \in\mathcal{R},$$

and hence justify the term used known as "Conjugate".

Also, the Hilbert Transform of a function $h\in\mathcal{L}^1(\mathcal{R})$ is defined by

$$\tilde{h}(a):=\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{|s|\geq\epsilon}\frac{h(a+s)}{s}ds$$

$$=\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{\epsilon}^{\infty}\frac{h(a-s)-h(a+s)}{s}ds$$

then, the classical limit called the principle value in terms of Cauchy exists at almost every $a\in\mathcal{R}$.
Thus, in general case the claim is that the inversion formula can be saved when $h\in\mathcal{L}^1(\mathcal{R})$ and $\hat{h}\notin\mathcal{L}^1(\mathcal{R})$ by considering the statistical limit of the Drichlet integral. Moreover by the results stated in Chapter I, conclusion for Statistical limit of Fourier Transform $h\in\mathcal{L}^1(\mathcal{R})$ diverges everywhere and satisfies Continuity characteristics under certain conditions defined earlier.

4.4 Remarks

- Statistical limit of a function is determined uniquely, if exists.
- The statistical limit existing as per the definition of statistical limit of measurable functions at ∞ is equivalently defined by the necessity for all $\epsilon>0$, the set

$$R_\epsilon:=\{v>s:|g(v)-r|>\epsilon\}$$

has natural density zero.

- The measurable function g is such that $|g|^q$ is locally integrable on some interval $[a, \infty)$, and represented as: $g \in \mathcal{L}^q_{loc}[a, \infty)$, where $s \geq 0$ and $q > 0$.
- If two sets having a point v_0 as density then they must have points in common in every neighborhood of v_0 and *there exists* at most one approximate limit.
- Fourier series of a periodic function $h \in \mathcal{L}^1(-\phi, \phi)$ and its conjugate series are summable S_q at almost every point $a \in (-\phi, \phi)$ for certain $q > 0$.
- The notion of limit of a measurable function g at ∞ can be generalized as a nondiscrete version of the notion of convergence of sequences of numbers statistically.

5 Conclusion

The present dissertation reveals certain results for *Statistical convergence of Sequences, Series and Measurable functions with application in Fourier series* explains that certain conditions to be needed for special series such as Fourier Series to converge statistically in Lebesgue Measure. One of these conditions are termed as *Tauberian Conditions for Statistical summability*. This is research are carried out later on in future.

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