Exact and Numerical Solutions of Some Nonlinear Partial Differential Equations

Thesis

Submitted in partial fulfilment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

VIKAS KUMAR

to the

SCHOOL OF MATHEMATICS & COMPUTER APPLICATIONS
THAPAR UNIVERSITY, PATIALA - 147004 (PUNJAB), INDIA
JULY - 2014

Certificate

This is to certify that the thesis entitled **"Exact and Numerical Solutions of Some Nonlinear Partial Differential Equations"**, submitted by Vikas Kumar in the partial fulfilment of the requirements for the award of the degree of Doctoral of Philosophy in the School of Mathematics and Computer Application, Thapar University, Patiala, is a record of the candidate's own work carried out by him under our supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

Attestation by supervisors

Rajesh Gupta

Dr Rajesh Kumar Gupta
Assistant Professor
School of Mathematics and Computer
Applications, Thapar University
Patiala-147004, INDIA

Dr Ramjiwari
Assistant Professor
Department of Mathematics,
Indian Institute of Technology
Roorkee-247667, INDIA

Declaration

It is certified that the thesis is entirely my own and that the idea and references cited herein have been duly acknowledged.

Vikas Kumar

(Regn. No. 951011007)

Attestation by supervisors

Rajesh Gupta

Dr Rajesh Kumar Gupta
Assistant Professor
School of Mathematics and Computer Applications, Thapar University
Patiala-147004, INDIA

Dr Ramjiwari
Assistant Professor
Department of Mathematics,
Indian Institute of Technology
Roorkee-247667, INDIA

Acknowledgement

I begin with the name of God, the creator of the universe, who bestowed his blessing on me in completing this thesis.

I would express sincere and genuine thanks to my research supervisors **Dr. Rajesh Kumar Gupta,** and **Dr. Ramjiwari**, School of Mathematics and Computer Applications, Thapar University, Patiala, for their cooperation, inspiration, suggestions, understanding and patience which added considerably to my research experience. They kept faith through adversity, always determined to produce useful results. I appreciate their vast knowledge and skill in many research areas. I am also grateful to Dr. O. P. Pandey, (Dean, Research and Sponsored Projects), Dr. Rajesh Kumar (Head, School of Mathematics and Computer Applications, Thapar University, Patiala), Prof. S. S. Bhatia, Dr. Amit Kumar, Dr. Deepak Gumber, Dr Satish Kumar Sharma, Dr. V. P. Aggarwal (Mechanical Engineering Department) and the other faculty members for their helpful and valuable suggestions.

I would also like to thank my family members for the support they provided me through out my entire life. Specially, I must acknowledge my parents, my father R.L. Kajal, my mother Urmila Devi, my wife Kavita, my sweet daughter Vinni and my brother Mukesh Kajal. Without their love, encouragement, support and assistance, I would not have finished this thesis.

I recognize that this research would not be possible without the moral support of Dr Pawan Diwan. I would also like to thank my friend Mr Anshu Mli Gaur for his help and support. I acknowledge Lakhveer for helping me in various matters.

Patiala

July, 2014

(Vikas Kumar)

Abstract

In this thesis, we established the range of applicability of Lie's classical symmetry method and differential quadrature method (DQM), with various of its generalizations, in constructing new exact and numerical solutions to the topical some nonlinear partial differential equations, including variable-coefficient Benjamin-Bona-Mahony-Burger (BBMB) equation, Coupled Short Pulse (CSP) equation with constant and variable coefficients, variable coefficients (2+1)-dimensional Diffusion-Advection (DA) equation, variable coefficients coupled KdV-Burgers' equation, generalized Fitzhugh-Nagumo equation with time-dependent coefficients and two-space-dimensional Quasilinear Hyperbolic partial differential equation.

Our thesis comprises of seven chapters. In the introductory part some important features of Lie group of transformations and differential quadrature method (DQM) are demonstrated and the mathematical fundamentals of continuous group theory and weighting coefficients of DQM are reviewed which are of great importance to the work dealt in Chapters 2-7

Chapter 2 is concerned with variable-coefficients Benjamin-Bona-Mahony-Burger (BBMB) equation arising as mathematical model of propagation of small-amplitude long waves in nonlinear dispersive media is investigated. The integrability of such an equation is studied with Painlevé analysis. The Lie symmetry method is performed for the BBMB equation and then similarity reductions and exact solutions are obtained based on the optimal system method. Furthermore different types of solitary, periodic and kink waves can be seen with the change of variable coefficients.

Chapter 3 deals with comparative study of travelling wave and numerical solutions for the Coupled Short Pulse (CSP) equation. The Lie symmetry analysis is performed

for coupled short plus equation. We derive the infinitesimals that admit the classical symmetry group. Five types arise depending on the nature of the Lie symmetry generator. In all types, we find reductions in terms of system of ordinary differential equations and exact solutions of the CSP equation are derived which are compared with numerical solutions using classical fourth order Runge-Kutta (RK4) scheme.

Using the Lie symmetry approach, we have examined herein the variant Coupled Short Pulse (VCSP) equation. The method reduced the system of partial differential equations to a system of ordinary differential equations according to the Lie symmetry admitted. In particular, we found the relevant system of ordinary differential equations in all optimal subgroups. The system of ordinary differential equations is further solved in general to obtain exact and numerical solutions. Several new physically important families of exact and numerical solutions are derived.

Chapter 4 is concerned with (2+1)-dimensional variable coefficients Diffusion-Advection (DA) equation, which arises in modeling of various physical phenomena, is studied by Lie symmetry approach. The similarity reductions are derived by determining the complete sets of point symmetries of this equation and then exact and numerical solutions are reported for the reduced second order nonlinear ordinary differential equations. Further, an extended (G'/G)-expansion method is applied to DA equation for constructing some new non-traveling wave solutions.

In **chapter 5,** we apply Lie symmetry approach, modified (G'/G)-expansion method and classical RK4 method for seeking the exact and numerical solutions of variable coefficients coupled KdV-Burgers' equation. Using suitable similarity transformations, the given system of partial differential equations reduced to a system of ordinary differential equations (ODEs). Moreover, the system of ODEs are solved by the modified (G'/G)-expansion method to obtain exact solutions. Further, RK4 method is applied to system of ODEs for constructing numerical solutions of variable coefficients coupled KdV-Burgers' equation.

Chapter 6 is devoted to derive the numerical solutions of the generalized Fitzhugh-Nagumo equation with time-dependent coefficients in one space dimension are considered using polynomial differential quadrature method (PDQM). The PDQM reduced the problem into a system of second order linear differential equation. Then, the obtained system is solved by optimal four-stage, order three strong stability-preserving time-stepping Runge-Kutta (SSP-RK43) scheme. The accuracy of the proposed method is demonstrated by three test examples. The numerical results are shown in max absolute errors (L_∞), root mean square errors (RMS) and relative errors (L_2) form. Numerical solutions obtained by this method when compared with the exact solutions reveal that the obtained solutions produce high accurate results. Numerical results show that the proposed method is of high accuracy and is efficient for solving the generalized Fitzhugh-Nagumo equation.

In **Chapter 7,** we have proposed a numerical technique based on PDQM to find the numerical solutions of two-space-dimensional Quasilinear Hyperbolic partial differential equations subject to appropriate dirichlet and neumann boundary conditions. The second-order hyperbolic partial differential equations have great impotence in fluid dynamics and aerodynamics, theory of elasticity, optics, electromagnetic etc. The PDQM reduced the equations into a system of second order linear differential equation. The obtained system is solved by RK4 method by converting into a system of first order ordinary differential equations. The accuracy of the proposed method is demonstrated by several test examples. The numerical results are found to be in good agreement with the exact solutions. The proposed technique can be applied easily for multidimensional problems.

List of Research Papers

1. Vikas Kumar, R.K. Gupta and Ram Jiwari, Painlevé Analysis, Lie Symmetries and Exact Solutions for Variable Coefficients Benjamin-Bona-Mahony-Burger (BBMB) Equation, *Communication in Theoretical Physics*, 60 (2013) 175-182. **(IOP Science, Impact Factor 0.954) (SCI)**.
2. Vikas Kumar, R.K. Gupta and Ram Jiwari, Comparative Study of Travelling-Wave and Numerical Solutions for the Coupled Short Pulse (CSP) Equation, *Chinese Physics B*, 22 (2013) 050201 (7 pp.). **(IOP Science, Impact Factor 1.148) (SCI)**.
3. Vikas Kumar, Ram Jiwari and R.K. Gupta, Numerical Simulation of Two Dimensional Quasilinear Hyperbolic Equations by Polynomial Differentia Quadrature Method, *Engineering Computations*, 30 (2013) 892-909. **(Emerald, Impact Factor 1.214) (SCI)**.
4. Vikas Kumar, R.K. Gupta and Ram Jiwari, Lie Group Analysis, Numerical and Non-Traveling Wave Solutions for the (2+1)-Dimensional Diffusion-Advection Equation with Variable Coefficient, *Chinese Physics B*, 23 (2014) 030201 (6 pp.). **(IOP Science, Impact Factor 1.148) (SCI)**.
5. Ram Jiwari, R.K. Gupta and Vikas Kumar, Polynomial Differential Quadrature Method for Numerical Solutions of Generalized Fitzhugh-Nagumo Equation with Time-Dependent Coefficients, *Ain Sham Journal of Engineering* (2014) **DOI: 10.1016/j.asej.2014.06.005 (Elsevier).**
6. R.K. Gupta, Vikas Kumar and Ram Jiwari, Exact and Numerical Solutions of Coupled Short Pulse Equation with Time Dependent Coefficients, *Nonlinear Dynamics*, (2014) **DOI: 10.1007/s11071-014-1678 (Springer, Impact Factor 2.54)**

7. R.K. Gupta, Vikas Kumar and Ram Jiwari, On the Solutions of Coupled KdV-Burgers Equation with Time-Dependent Coefficients, *Acta Applicandae Mathematicae* **(Communicated).**

List of Figures

1.1 An equilateral triangle and its symmetries 5

1.2 A symmetry transformation mapping a solution $y = C$ to the solution $y = C + \varepsilon$ 6

1.3 A manifold M and two charts (ϕ_1, U_1) and (ϕ_2, U_2). 8

1.4 Rectangular domain 32

2.1 The inverted two solitary solution (2.3.11) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $k_1 = k_2 = k_3 = C_1 = 1$ and $e(t) = \sec^2(t)$ 47

2.2 The inverted three Solitary solution (2.3.11) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $k_1 = k_2 = k_3 = C_1 = 1$ and $e(t) = \cos^2(t)$. 48

2.3 The distribution of some singularities of the solution (2.3.13) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_1 = C_2 = C_3 = -1$; $k_2 = 0.5$ and $e(t) = \sec^2(t)$. 49

2.4 The damped oscillatory kink wave solution (2.3.13) in form of 3D plot (left) and contour plot (Projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_1 = C_2 = C_3 = -1$; $k_2 = 0.5$ and $e(t) = \cos^2(t)$. 49

2.5	The two Solitary solution (2.3.15) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_1 = 1$; $k_2 = -0.5$ and $e(t) = \sec^2(t)$.	50
2.6	The three Solitary solution (2.3.15) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 where $C_1 = 1$; $k_2 = -0.5$ and $e(t) = \cos^2(t)$.	50
2.7	Singularity at $t = 0$ of solution (2.3.20) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_6 = C_7 = C_9 = 1$; $k_1 = -0.5$ and $h(t) = \cosh(t)$.	53
2.8	The periodic solution (2.3.20) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $k_1 = -0.5$, $C_6 = C_7 = C_9 = 1$ and $h(t) = \sec^2(t)$.	53
2.9	The kink wave solution (2.3.29) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_1 = C_2 = C_4 = C_5 = C_7 = 1$.	55
3.1	Comparison between exact and numerical solutions for the reduced ODEs system (3.3.2) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.3) at $h = 0.1$. . . .	66
3.2	Comparison between exact and numerical solutions for the reduced ODEs system (3.3.2) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.5) at $h = 0.1$. . . .	68
3.3	Comparison between exact and numerical solutions for the reduced ODEs system (3.3.8) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.9) at $h = 0.1$	69
3.4	Comparison between exact and numerical solutions for the reduced ODEs system (3.3.8) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.11) at $h = 0.1$	71
3.5	Comparison between exact and numerical solutions for the reduced ODEs system (3.3.14) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.15) at $h = 0.1$	72

3.6 The periodic solution (3.5.13) in form of 3D plot with $C_4 = -1$; $C_1 = C_2 = C_3 = 1$; and $\alpha_1(t) = \sin(t)$. 79

3.7 The periodic solution (3.5.17) in form of 3D plot with $C_1 = C_2 = 1$; $C_3 = C_4 = -1$ and $\alpha_1(t) = \cos(t)$. 81

3.8 Numerical solutions for the reduced IVP (3.5.22-3.5.23) for VCSP equation when $\gamma = K_1 = K_2 = K_4 = K_5 = 1$ and $K_3 = 0$, with initial value $z = 1$ at $h = 0.01$ as $z \to 100$ and $z \to 150$ 82

3.9 Numerical solutions for the reduced IVP (3.5.22-3.5.23) for VCSP equation when $\gamma = K_1 = K_2 = K_3 = K_4 = K_5 = 1$, with initial value $z = 1$ at $h = 0.01$ as $z \to 100$ and $z \to 150$. 83

3.10 The periodic solution (3.5.27) in form of 3D plot with $C_2 = 0$ $C_1 = C_3 = C_4 = 1$, $K_1 = K_3 = 1$, $K_4 = -1$ and $\alpha_1(t) = \cos(t)$. 84

4.1 An obstacle of length L parallel to the y-axis centred on the origin and oriented perpendicular to the drift. The direction of the drift is parallel to the x-axis from left to right 88

4.2 Numerical solutions of the reduced IVP (4.2.13-4.2.15) for DA equation at $h = 0.01$ as $z \to 25$ 94

4.3 Numerical solutions of the reduced IVP (4.2.13-4.2.15) for DA equation at $h = 0.1$ as $z \to 50$. 94

4.4 (a) The soliton-like solution given by equation (4.3.11) with the settings $\lambda = 3$, $\mu = 1$, $f(x, t) = \sin(x) + t$, $g(y) = \exp(y)$, $\alpha(t) = \beta(t) = \cos(t)$ and $x, y \in [-2, 2]$ at $t = 0$ (b) The projection of (a) on xoy plane 98

4.5 (a) The soliton-like solution given by equation (4.3.13) with the settings $\lambda = \mu = 2$, $f(x, t) = \sin(x) + t$, $g(y) = \exp(y)$, $\alpha(t) = \beta(t) = \cos(t)$ and $x, y \in [-1, 1]$ at $t = 0$ (b) The projection of (a) on xoy plane 99

4.6 (a) The soliton-like solution given by equation (4.3.15) with the settings $f(x, t) = \sin(x) + t$, $g(y) = \exp(y)$, $\alpha(t) = \beta(t) = \cos(t)$ and $x, y \in [-1, 1]$ at $t = 0$ (b) The projection of (a) on xoy plane 100

5.1 Numerical solutions for the reduced IVP (5.3.10-5.3.11) for equation (5.1.2) when $K_1=1,\ K_2=0.04,\ K_3=5,\ K_4=-0.5,\ K_5=0.5$ and $\lambda_6=10$, with initial value $z=1$ at $h=0.01$ 107

5.2 Periodic solution (left) and soliton-like solution (right) given by equation (5.3.22) with $A=1,\ B=1,\ a_0=c_0=d_1=1$ and $\alpha_1(t)=\exp(t)$ $+\sec(t)\tan(t)+t$ 111

5.3 Numerical solutions for the reduced IVP (5.3.27-5.3.28) for equation (5.1.2) when $K_1=4,\ K_2=-0.22,\ K_3=1,\ K_4=0.5,\ K_5=0.04$ and $\lambda_5=0$, with initial value $z=1$ at $h=0.01$ 112

5.4 U-shaped soliton solution (left) periodic solution (right) given by equation (5.3.37) with $K_3=-1,\ A=B=1,\ a_2=c_0=\lambda_6=1$ and $\alpha_1(t)=\cos(t)+\sin(t)$ 116

5.5 Soliton-like solutions given by equation (5.3.48) with $a_1=1,\ c_1=-1,$ $K_4=-1,\ A=B=1$ and $\alpha_1(t)=\cos ech^2(t)$. 119

6.1 Numerical (left) and exact (right) solutions of example 1with time step length $\Delta t=0.001,\ \rho=-2.0$ upto time $t=5$ 128

6.2 Numerical (left) and exact (right) solutions of example 1 with time step length $\Delta t=0.001,\ \rho=0.75$ upto time $t=5$ 128

6.3 Numerical (left) and exact (right) solutions of example 3 with time step length $\Delta t=0.001,\ \rho=0.5$ upto time $t=3.0$ 130

6.4 Numerical (left) and exact (right) solutions of example 3 with time step length $\Delta t=0.001,\ \rho=0.75$ upto time $t=3.0$ 130

6.5 Numerical (left) and exact (right) solutions of example 3 with time step length $\Delta t=0.001,\ \rho=0.5$ upto time $t=1.0$ 131

6.6 Numerical (left) and exact (right) solutions of example 3 with time step length $\Delta t=0.001,\ \rho=0.75$ upto time $t=1.0$ 132

7.1 Exact (left) and numerical (right) solutions of example 1 with time step length $\Delta t=0.001,\gamma=1$ at $t=1.0$ 140

7.2 Exact (left) and numerical (right) solutions of example 1 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 3.0$ 140

7.3 Contours plots of example 1 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 1.0,\ 3.0$ respectively 140

7.4 Exact (left) and numerical (right) solutions of example 2 with time step length $\Delta t = 0.001$ at $t = 1.0$ 142

7.5 Exact (left) and numerical (right) solutions of example 2 with time step length $\Delta t = 0.001$ at $t = 3.0$ 142

7.6 Contours plots of example 2 with time step length $\Delta t = 0.001$ at $t = 1.0,\ 3.0$ respectively. 142

7.7 Exact (left) and numerical (right) solutions of example 3 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 1.0$ 144

7.8 Exact (left) and numerical (right) solutions of example 3 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 3.0$ 144

7.9 Contours plots of example 3 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 1.0,\ 3.0$ respectively. 144

7.10 Exact (left) and numerical (right) solutions of example 4 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 1.0$ 146

7.11 Exact (left) and numerical (right) solutions of example 4 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 3.0$ 146

7.12 Contours plots of example 4 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 1.0,\ 3.0$ respectively. 146

7.13 Exact (left) and numerical (right) solutions of example 5 with time step length $\Delta t = 0.001,\ \alpha = 0,\ \beta = 5$ at $t = 1.0$ 148

7.14 Exact (left) and numerical (right) solutions of example 5 with time step length $\Delta t = 0.001,\ \alpha = 0,\ \beta = 5$ at $t = 3.0$ 148

7.15 Contours plots of example 5 with time step length $\Delta t = 0.001,\ \alpha = 0,$ $\beta = 5$ at $t = 1.0,\ 3.0$ respectively. 148

List of Tables

3.1 Commutation relations Table 60

3.2 Adjoint Representation Table 61

3.3 Absolute error, relative error, and percentage error for example 1 of subalgebra V_1 67

3.4 Absolute error, relative error, and percentage error for example 2 of subalgebra V_1 68

3.5 Absolute error, relative error, and percentage error for example 1 of subalgebra $V_2 + V_3$ 70

3.6 Absolute error, relative error, and percentage error for example 2 of subalgebra $V_2 + V_3$ 71

3.7 Absolute error, relative error, and percentage error for example of subalgebra $V_2 - V_3$ 73

4.1 The Infinitesimals corresponding to different form of $K(u)$. 90

4.2 The relations between $\alpha(t)$ and $\beta(t)$ for the different form of $K(u)$ 90

4.3 Similarity variables, similarity functions, and the coefficient functions of DA equation when $K(u) = u^m$ for $m \neq 0,1,2$ and $K(u) = \log(u)$. 92

4.4 Form of the coefficient functions of DA equation for different form of $K(u)$ 92

6.1 Max absolute errors (L_∞), RMS errors, L_2 errors and CPU time of example-1 at different time with $\rho = 0.75$. 128

6.2 Max absolute errors (L_∞), RMS errors, L_2 errors and CPU time of example 2 at different time with $\rho=0.75$. 129

6.3 Max absolute errors (L_∞), RMS errors, L_2 errors and CPU time of example 3 at different time with $\rho=0.75$. 131

7.1 RMS, relative, max absolute errors and CPU time of example 1 at different times for $N=M=21, \gamma=1$. 139

7.2 RMS, relative, max absolute errors and CPU time of example 2 at different times for $N=M=21$. 141

7.3 RMS, relative, max absolute errors and CPU time of example 3 at different times for $N=M=21, \gamma=1$. 143

7.4 RMS, relative, max absolute errors and CPU time of example 4 at different times for $N=M=21$. 144

7.5 RMS, relative, max absolute errors and CPU time of example 5 at different times for $N=M=21, \alpha=10, \beta=5$. 147

Contents

Certificate **i**

Declaration **ii**

Acknowledgement **iii**

Abstract **iv**

List of Research Papers **vii**

List of Figures **ix**

List of Tables **xiv**

1 Introduction **1**

1.1 Exact Solution of Partial Differential Equations. 1

1.2 Basic Theory of Lie Groups of Transformations. 4

1.2.1 Differential Equations. 4

1.2.2 Symmetry. 5

1.2.3 Point Transformation. 6

1.2.4 The Group Structure of Symmetries. 7

1.2.5 The Connection Between Groups and Geometry. 7

1.2.6 One-Parameter Lie Group of Point Transformations 9
1.2.7 Infinitesimal Generators 10
1.2.8 Invariants. 11
1.2.9 Prolongation Formulas. 12
1.2.10 Determining Equations 13
1.2.11 Multi Parameter Lie Groups of Transformations. 14
1.2.12 Lie Algebras 15
1.2.13 Lie's Algorithm 16
1.3 Painlevé Analysis. 16
1.3.1 The Algorithm. 17
1.4 The (G'/G)-Expansion Method 18
1.4.1 Extended (G'/G)-Expansion Method 19
1.4.2 Modified (G'/G)-Expansion Method 20
1.5 Numerical Solution of Partial Differential Equations 22
1.6 Basic Approaches for Weighting Coefficients of DQM 23
1.6.1 Weighting Coefficients for the First Order Derivative 23
1.6.2 Bellman's Approach 24
1.6.3 Quan and Chang's Approach 25
1.6.4 Shu's General Approach 26
1.6.5 Weighting Coefficients for the Second Order Derivative 28
1.6.6 Quan and Chang's Approach. 28
1.6.7 Shu's General Approach 28
1.6.8 Shu's Recurrence Formulation for Higher Order Derivatives 29
1.6.9 Weighting Coefficients for the Multi-dimension Case 31
1.6.10 Methodology for Solving a PDE by DQM 34
1.6.10.1 Time dependent PDEs. 35
1.6.10.2 Time independent PDEs 35
1.6.10.3 Solution of a System of ODEs 36
1.7 Outline of Thesis 37

2 Variable Coefficients Benjamin-Bona-Mahony-Burger (BBMB) Equation

2.1 Introduction 41

2.2 Painlevé Analysis. 43

2.3 Lie Symmetric Analysis for BBMB Equation. 44

2.3.1 Reduced Systems and Exact Solutions. 46

2.3.1.1 Subalgebra $\langle V_1 + \alpha V_3 \rangle$ 46

2.3.1.2 Subalgebra $\langle V_2 + \beta V_3 \rangle$ 51

2.3.1.3 Subalgebra $\langle V_3 \rangle$ 54

2.3.1.4 Subalgebra $\langle V_4 \rangle$ 55

2.4 Discussion and Concluding Remarks. 55

3 Coupled Short Pulse (CSP) Equation with its Variable Coefficients Form

3.1 Introduction 57

3.2. Coupled Short Pulse (CSP) Equation 59

3.2.1. Lie Symmetric Analysis for CSP Equation 59

3.3 Reduced Systems and Exact Solutions 61

3.3.1 Subalgebra V_1 61

3.3.2 Subalgebra $V_2 + V_3$ 63

3.3.3 Subalgebra $V_2 - V_3$ 64

3.3.4 Subalgebra V_2 and V_3 64

3.4 Comparative Study of Travelling Wave and Numerical Solutions for CSP - Equation 64

3.4.1 Classical Fourth Order Runge-Kutta (RK4) Method 65

3.4.2 Numerical Experiments 65

3.5 Coupled Short Pulse Equation with Time Dependent Coefficients 73

3.5.1 Lie Symmetric Reduction, Exact and Numerical Solutions 73

3.5.1.1 Subalgebra $V_1 + \mu V_2$ 77

3.5.1.2 Subalgebra $V_2 + \lambda V_3 + V_4$ 79

3.5.1.3 Subalgebra $V_2 + \gamma V_3$ 81

3.5.1.4 Subalgebra $V_3 + V_4$ . . . 83

3.5.1.5 Subalgebra V_3 . . . 84

3.5.1.6 Subalgebra V_4 . . . 85

3.6 Discussion and Concluding Remarks . . . 85

4 (2+1)-Dimensional Diffusion-Advection Equation with Variable Coefficients

4.1 Introduction . . . 87

4.2 Symmetry Analysis for DA Equation . . . 89

4.2.1 Similarity Reductions, Exact and Numerical Solutions When $K(u) = \log(u)$ . . . 92

4.2.1.1 Subalgebra $\langle V_1 + \lambda V_2 + \mu V_6 \rangle$ . . . 93

4.2.1.2 Subalgebra $\langle V_2 + \rho V_6 \rangle$ . . . 93

4.2.1.3 Subalgebra $\langle V_3 + \sigma V_6 \rangle$ . . . 95

4.2.1.4 Subalgebra $\langle V_4 \rangle$, $\langle V_5 \rangle$ and $\langle V_6 \rangle$ . . . 95

4.3 Application of Extended (G'/G)-Expansion Method to DA Equation . . . 96

4.4 Discussion and Concluding Remarks . . . 100

5 Variable Coefficients Coupled KdV-Burgers' Equation

5.1 Introduction . . . 101

5.2 Lie Symmetric Reduction . . . 102

5.3 Exact and Numerical Solutions . . . 105

5.3.1 Subalgebra $V_1 + \lambda_1 V_2 + \lambda_2 V_3$ . . . 105

5.3.2 Subalgebra $V_2 + \lambda_3 V_3 + V_5$ . . . 106

5.3.3 Subalgebra $V_2 + \lambda_4 V_3$ . . . 108

5.3.4 Subalgebra $V_3 + \lambda_5 V_4 + V_5$ . . . 111

5.3.5 Subalgebra $V_3 + \lambda_6 V_4$ . . . 113

5.3.6 Subalgebra $V_4 + V_5$ . . . 116

5.3.7 Subalgebra V_4 . . . 120

5.3.8 Subalgebra V_5 120

5.4. Discussion and Concluding Remarks. 120

6 Generalized Fitzhugh-Nagumo Equation with Time-Dependent Coefficients

6.1 Introduction 123

6.2 Numerical Solution of Generalized Fitzhugh-Nagumo Equation. 125

6.2.1 Implementation of Dirichlet Conditions. 125

6.3 Selections of Grid Points and Stability. 126

6.4 Application of the Proposed Method. 126

6.5 Conclusion and Discussion 132

7 Numerical Simulation of Two Dimensional Quasilinear Hyperbolic Equations

7.1 Introduction 133

7.2 Numerical Scheme for Two Dimensional Quasilinear Hyperbolic Equation . .135

7.2.1 Implementation of Dirichlet Boundary Conditions. 136

7.2.2 Implementation of Neumann Boundary Conditions 136

7.3 Stability of the Differential Quadrature Method. 138

7.4 Numerical Experiments 138

7.5 Discussion 149

Summary **151**

Bibliography **155**

DEDICATED

TO

MY FAMILY

Chapter 1

Introduction

The field of nonlinear partial differential equations (PDEs) is important, mainly because it is used as models to describe complex physical phenomena in various fields of engineering and sciences, especially in fluid mechanics, solid-state physics, plasma physics, plasma wave and chemical physics. It is difficult to imagine any area of applications where its impact is not felt. In order to better understand these phenomena as well as further apply them in practical scientific research, it is important to seek their exact and numerical solutions. Generally, the nonlinear PDEs are still difficult to solve theoretically and numerically. A great deal of effort has been expended over the last 100 years or so in attempting to find robust and stable numerical and analytical methods for solving nonlinear PDEs of physical interest.

1.1 Exact Solutions of Partial Differential Equations

The effort in finding exact solutions to nonlinear PDEs are important for examine the sensitivity of physical phenomena with several important physical parameters described by constant and variable coefficients. These solutions of nonlinear PDEs are of much interest, both from mathematical and application points of view. This is why symmetry analysis using Lie group transformations is a subject which is becoming more and more important. It is the only universal and effective method for solving nonlinear differential equations analytically.

The concept of symmetry fascinated through the centuries many artists and scientists, from the ancient Greeks to Kepler, to Newton, who embodied in the laws of mechanics as a symmetry principle the equivalence of motion in different inertial frames, to Einstein, who generalized the Galileos principle of relativity from mechanics to all the laws of physics. Presently, symmetries play an important role in mathematics, chemistry, engineering and in almost all branches of physics, including classical mechanics, quantum mechanics and general relativity etc. One reason for the overall prominence of the concept of symmetry is its nativeness and its simplicity. Intuitively speaking, symmetry is a transformation of an object leaving that object invariant. This is clearly such a general property that it can be recovered almost everywhere in nature and, correspondingly, innumerous areas of science and art.

Today when students are introduced to differential equations they are presented with a variety of adhoc methods that often seem to be the result of a lucky guess. This was also the state of affairs in the late 19th century when the Norwegian mathematician Sophus Lie made the profound and far-reaching discovery, that most of these solution methods were just special cases of a more general integration theory based on symmetry transformations.

The study of symmetry provides one of the most appealing practical applications of group theory. This was extensively shown by Sophus Lie in the nineteenth century 1872-1899 [178, 179]. Lie formally defined and initiated the mathematical study of continuous group of transformations, now called Lie group. He showed how these led to a symmetries theory of differential equation, in the same spirit of Galois Theory for polynomial equation. This theory enables to derive solutions of differential equations in a completely algorithmic way without appealing to special lucky guesses. Sophus Lie pointed out the main problem he faced with his theory in a foreword to his lecture Differentialgleichungen given in Leipzig (1891). He investigated the continuous groups of transformations leaving differential equations invariant, creating what is now called the symmetry analysis of differential equations. His aim was to solve nonlinear differential equations, which to some extent are may be cumbersome to solve.

Lie had a closed friendship with Klein and led to a long fruitful collaboration. Klein, who published his Erlangen program in 1872 [50], developed the thesis that geometry was the study of invariants of group actions on geometric objects. Lie's starting point had been in modern geometry, after coming across the work of Poncelet and Plucker, [82] and soon turned his attention to the geometry of differential equation with all his genius.

Despite this feature, the Lie's approach to differential equations was not exploited for half a century and only the abstract theory of Lie groups grew. The original Lie group theory was extended to a global theory in the 1930s by Cartan [42] and Weyl [193] (the term Lie group has been coined in 1928 by Hermann Weyl). It was in the Forties of last century, with the work of Birkhoff [53] and Sedov [98] on dimensional analysis, that the theory gave relevant results in concrete applied problems. The first systematic investigation of the problem of group classification was done by Ovsiannikov [103] in 1960 for differential equations in the explicit construction of solutions of any sort of problems, even complicated, of mathematical physics. Among these generalizations in the late 1960s and 1970s there is the non–classical method first proposed by Bluman and Cole [57] which itself have been generalized by Olver and Rosenau [141, 142]. In the 1980s the advent of symbolic manipulation computer program [194], witch takes away the often tedious computations associated with the Lie's algorithm, has also helped in drawing researcher into this area. The basic theory can be found in Ovsiannikov [102], Olver [143, 144], Bluman et al. [58-62], Hydon [140], Stephani [65, 66], Ibragimov [133, 134], Cantwell [24] and in many other text and research papers that have followed since.

Today, symmetry analysis constitutes the most important widely applicable method for finding the exact solutions of the nonlinear problems. Lie's classical theory is a source for various generalizations in this direction, some recent contributions are from Gandarias et al. [121-123, 127], Nucci [107-110], Biswas et al. [3, 9, 54, 69], Singh et al. [93, 94], Gupta et al. [99, 100, 131, 132, 167, 168, 177], Sharma et al. [126, 188-191], Gupta et al. [199, 200], Kara [10], Hill [78, 79], Lakshmanan et al. [119, 128], Oliveri [51, 52], Bhutani et al. [135, 136] and Clarkson et al. [137, 138].

The key idea of Lie's theory of symmetry analysis of differential equations relies on the invariance of the latter under a transformation of independent and dependent variables. The number of independent variables in a PDE is reduced by one by making use of appropriate combinations of the original independent variables as new independent variables, called "similarity variables." The similarity variables can themselves be identified by using the invariance properties of PDEs when subjected to finite or infinitesimal transformations. Thus with the applications of this method one can reduce PDEs to ordinary differential equations (ODEs) or at least reduce the number of independent variables. The reduced system is easier to analyze both numerically and analytically than the original system. Therefore, whenever the need arises, the symmetries are used to construct the invariant solutions; the invariant solutions in turn are used to reduce the underlying PDEs to the systems of ODEs which could be solved or approximated numerically.

There also exist alternative methods, which are not based on the applications of group theory, such as Direct method [40, 96, 196], Bäcklund transformation [28], Painlevé analysis [25, 26], Inverse scattering transformation [116]. Recently, a variety of powerful methods, such as the tanh-sech method [15], extended tanh method [16], sine-cosine method [17], Hirota method [18], homogeneous balance method [44], Jacobi elliptic function method [27], F-expansion method [76], homotopy perturbation method [33], variational iteration method [75], extended $\left(G'/G\right)$-expansion method [176], modified $\left(G'/G\right)$-expansion method [197] were developed.

In the following section, the relevant concepts of the Lie group of transformations are introduced and then we have provided the algorithmic descriptions of the techniques which are applied in the later chapters to derive the symmetry group of the systems under investigation. For more details on Lie groups and various theorems, their proofs and other concepts, we refer our reader to Bluman and Cole [57], Kumei [58] and Olver [143].

1.2 Basic Theory of Lie Groups of Transformations

In this section, some basic definitions and fundamentals of Lie group analysis of differential equations are given; a more detailed exposition can be found in [19, 51, 80, 125, 148].

1.2.1 Differential Equations

$$\text{Let} \quad \Xi^{\sigma}\left(x, u, u_{(1)}, \ldots, u_{(k)}\right) = 0, \qquad \sigma = 1, \ldots, s \tag{1.2.1}$$

be a k^{th}-order $(k \geq 1)$ system Ξ of s differential equations, with independent variables $x \equiv \left(x^1, x^2, \ldots, x^n\right) \in IR^n$ and dependent variables $u \equiv \left(u^1, u^2, \ldots, u^m\right) \in IR^m$

and $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ are respectively the collection of all first, second, up to k^{th}-order partial derivatives. If $n = 1$ the system (1.2.1) becomes a system of ODEs otherwise it is a system of PDEs

1.2.2 Symmetry

A symmetry transformation, or symmetry, is a transformation of an object such that the object is unchanged (invariant). More precisely, the properties are

(i) The transformation preserves the structure of the object.

(ii) The transformation and its inverse are both smooth, i.e. they are infinitely differentiable.

(iii) The transformation maps the object to itself.

In the following, we are presenting some examples and figures from [80]

Example 1.2.1 (Equilateral Triangle). Consider an equilateral triangle like the one illustrated in Figure 1.1. This geometric figure is symmetric with respect to 2π/3 rotations and mirroring along its three axes, in this case geometric object, admitting six discrete symmetries.

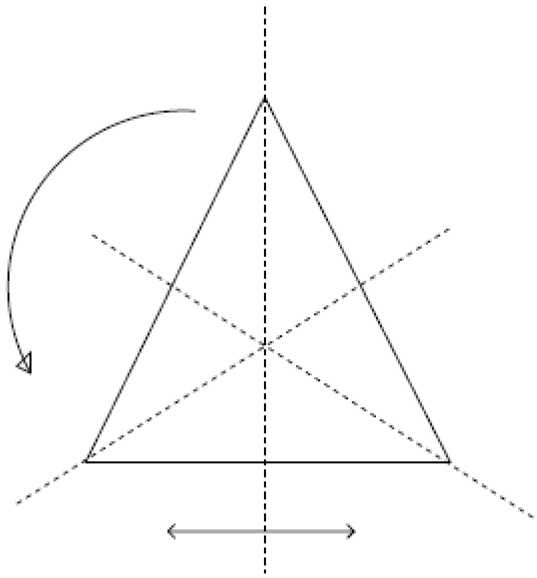


Figure 1.1. An equilateral triangle and its symmetries.

Example 1.2.2 The easiest differential equation is

$$\frac{dy}{dx} = 0, \tag{1.2.2}$$

The set of solutions to this ODE is given by

$$y = C, \qquad C \in R. \tag{1.2.3}$$

The solution curves $y = C$ are independent of x and y. Therefore it admits many symmetry transformations, including

$$T_\varepsilon : \begin{cases} \bar{y} = y + \varepsilon \\ \bar{x} = x \end{cases} \tag{1.2.4}$$

So we can slide any solution curve in the y-direction into any of the other solution curves by means of the correspondence $(x, y) \to (\bar{x}, \bar{y})$. This is illustrated in Figure 1.2.

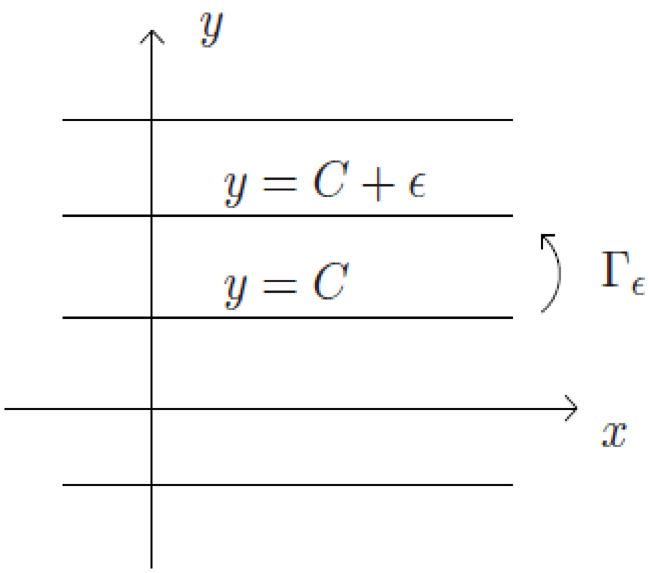


Figure 1.2. A symmetry transformation mapping a solution $y = C$ to the solution $y = C + \varepsilon$

1.2.3 Point Transformation

Transformations of the form

$$T_a : \begin{cases} \bar{x}^i = g^i(x, u, a) \\ \bar{u}^i = \phi^\alpha(x, u, a) \end{cases} \qquad i = 1, \ldots, n; \quad \alpha = 1, \ldots, m, \tag{1.2.5}$$

are called point transformations. Symmetry transformations of this form are called point symmetries.

1.2.4 The Group Structure of Symmetries

To be more specific, in the context of symmetries, we are interested in transformations with a group structure, i.e. they satisfy the axioms of a group.

Definition 1.2.1 A non empty set G together with a composition rule (denoted here by a dot) is said to be a Group if it satisfies the following four postulates

(i) If $g_1 \in G$ and $g_2 \in G$, then $g_1 . g_2 \in G.$ (1.2.6)

(ii) The composition rule is associative, that is for $g_1,\ g_2,\ g_3 \in G$, then

$$(g_1 . g_2).g_3 = g_1 . (g_2 . g_3). \tag{1.2.7}$$

(iii) There is an element $e \in G$, called the identity, such that

$$g = e . g = g . e \quad \forall\ g \in G. \tag{1.2.8}$$

(iv) For every $g \in G$, there is an inverse $g^{-1} \in G$, such that

$$g . g^{-1} = g^{-1} . g = e \tag{1.2.9}$$

1.2.5 The Connection Between Groups and Geometry

When formulating the general theory of symmetry transformations, it is convenient to do so in a coordinate free way. Hence it is natural to do so using differentiable manifolds. Here we shall give a brief formal introduction, for a more rigorous treatment see [80, 129, 184].

Definition 1.2.2 An n-dimensional differentiable manifold is a set M, such that for every $x \in M$, there exists a one-to-one and onto (bijective) map

$$\phi^i = U_i \to V_i \tag{1.2.10}$$

taking x from the subset $U_i \subset M$ to a subset $V_i \subset IR^n$ of Euclidean space. The subsets U_i must be countable, and are together with the mappings ϕ^i called charts or coordinate maps $(\phi^i,\ U_i)$ of the manifold. They must also satisfy the conditions

(i) The charts must cover the entire M,

$$M = \bigcup_i U_i\ . \tag{1.2.11}$$

(ii) On every overlap $U_i \cap V_i$, the composite map

$$\phi^j \, o \, (\phi^i)^{-1} : \ \phi^i (U_i \cap U_j) \to \phi^j (U_i \cap U_j), \tag{1.2.12}$$

between the different charts must be smooth (infinitely differentiable). In other words, we must be able to convert between different charts where there is overlap.

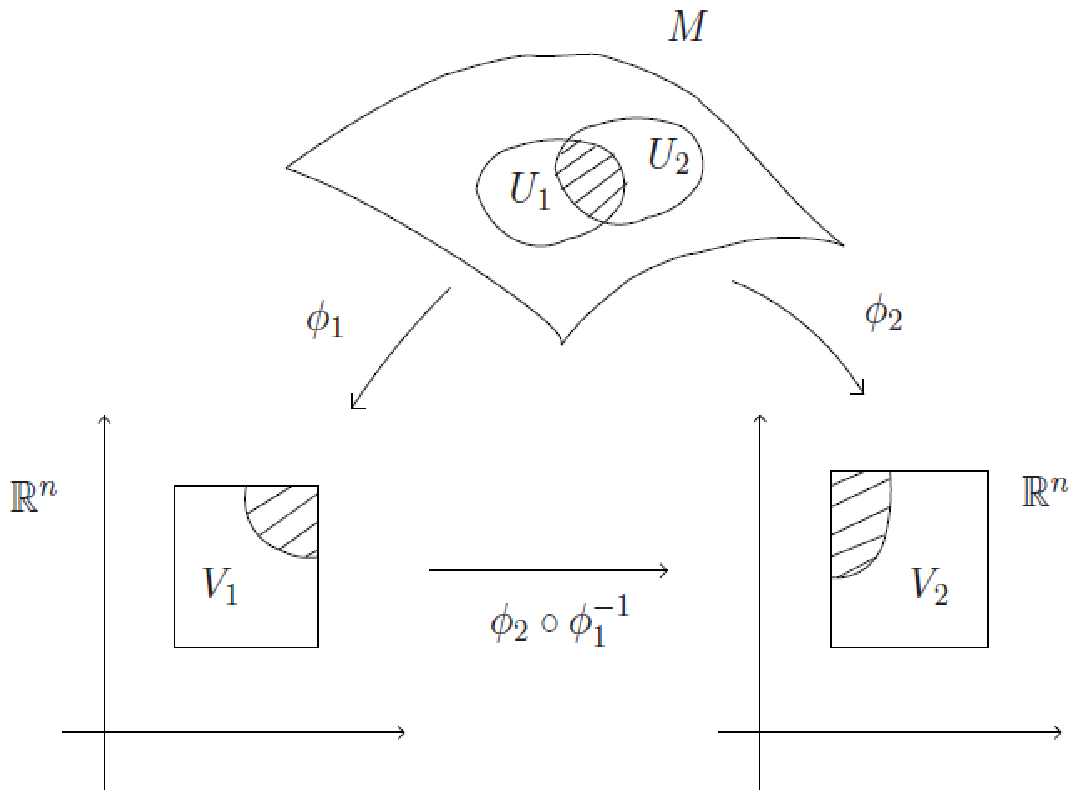


Figure 1.3. A manifold M and two charts (ϕ_1, U_1) and (ϕ_2, U_2).

(iii) If $x_1 \in U_i$ and $x_2 \in U_j$ are distinct points on M, then there exists subsets $W_i \subset U_i$ and $W_j \subset U_j$, with $\phi^i(x_1) \in W_i$ and $\phi^j(x_2) \in W_j$, such that

$$(\phi^i)^{-1}(W_i) \cap (\phi^j)^{-1}(W_j) = \Phi. \tag{1.2.13}$$

When dealing with maps $f : M \to N$ between different manifolds, we define f to be smooth if

$$\tilde{\phi}^j \circ f \circ (\phi^i)^{-1}: \ IR^m \to IR^n, \tag{1.2.14}$$

is smooth for every chart ϕ^i on M and $\tilde{\phi}^j$ on N, where there is overlap such that (1.2.14) is defined.

Now we connect these geometric constructions with our earlier discussion about groups. This marriage of group theory and analysis gives powerful tools to analyze symmetry.

Definition 1.2.3 An r-parameter Lie group is group G which also has the structure of an r-dimensional differentiable manifold, such that both the group composition and inversion

$$(x,\ y) \mapsto x\,.\,y,$$
$$x \mapsto x^{-1}, \tag{1.2.15}$$

are smooth maps between manifolds.

1.2.6 One-Parameter Lie Group of Point Transformations

A set G of transformation

$$T_a: \quad \bar{x}^i = g^i(x,\ u,\ a),\ \bar{u}^\alpha = \phi^\alpha(x,\ u,\ a) \qquad i = 1,\ldots,n; \quad \alpha = 1,\ldots,m, \tag{1.2.16}$$

where a is a real parameter which continuously takes values in a neighbourhood $S \subseteq IR$ of $a = 0$ and g^i, ϕ^α are infinitely differentiable with respect to the real variables x and an analytic function of the real continuous parameter a, which lies in an open interval S. The transformation T_a is a one-parameter Lie group of transformation in IR^{n+m} with respect to the binary operation of composition if and only if

(i) **Closure.** $T_a,\ T_b \in G$ and $a,\ b \in S' \subset S$ then

$$T_a T_b = T_c \in G, \tag{1.2.17}$$

where $c = \varphi(a,\ b) \in S$ The function φ defining the law of composition of T_a is an analytic function of $a,\ b \in S$ and is commutative, i.e., $\varphi(a,\ b) = \varphi(b,\ a)$; thus, Lie groups are abelian.

(ii) **Identity.** There exists an identity element $T_0 \in G$ such that

$$T_0 T_a = T_a T_0 = T_a \tag{1.2.18}$$

(iii) **Inverse.** There is an inverse element $T_a^{-1} = T_{a^{-1}} \in G,\ a^{-1} \in S$ such that

$$T_a^{-1} T_a = T_a T_a^{-1} = T_0 \ \text{ for every } \ T_a \in G,\ a \in S' \subset S. \tag{1.2.19}$$

(iv) **Associative.** The group is associative, i.e.

$$\left(T_a\ T_b\right)T_c = T_a\ \left(T_b\ T_c\right) \quad \text{for } a, b, c \in S \tag{1.2.20}$$

1.2.7 Infinitesimal Generators

Consider a one-parameter Lie group of transformations

$$T_a: \quad \bar{x}^i = g^i\left(x, u, a\right),\ \bar{u}^\alpha = \phi^\alpha\left(x, u, a\right) \tag{1.2.21}$$

where a is the group parameter and assumed to be defined in such a way that the identity element is $a_0 = 0$. Now, expanding (1.2.21) in a Taylor expansion up to the first order terms about $a = 0$, with the initial conditions

$$\xi^i\left(x, u\right) = \left.\frac{\partial g^i\ \left(x, u, a\right)}{\partial a}\right|_{a=0}, \qquad \eta^\alpha\left(x, u\right) = \left.\frac{\partial \phi^\alpha\left(x, u, a\right)}{\partial a}\right|_{a=0}, \tag{1.2.22}$$

we get

$$\bar{x}^i \approx x^i + a\xi^i\left(x, u\right), \qquad \bar{u}^\alpha \approx u^\alpha + a\eta^\alpha\left(x, u\right). \tag{1.2.23}$$

The derivatives of g^i, ϕ^i with respect to the group parameter a evaluated at $a = 0$ are called the infinitesimals of the group and are denoted by $\xi^i\left(x, u\right)$ and $\eta^\alpha\left(x, u\right)$. The first-order approximations (1.2.21) can be written as

$$\bar{x}^i \approx \left(1 + a\ X\right)x^i, \qquad \bar{u}^\alpha \approx \left(1 + a\ X\right)u^\alpha, \tag{1.2.24}$$

where

$$X = \xi^i\left(x, u\right)\frac{\partial}{\partial x^i} + \eta^\alpha\left(x, u\right)\frac{\partial}{\partial u^\alpha}. \tag{1.2.25}$$

The differential operator (1.2.25) is called the infinitesimal generator or vector field of the group G. It is also known as the symmetry generator (operator).

Theorem 1.2.1 (**First Fundamental Theorem of Lie**) The Lie group of transformations (1.2.16) is equivalent to the solution of the initial value problem for the system of first order differential equations

$$\frac{d\,\bar{x}^i}{d\,a} = \xi^i\left(\bar{x},\ \bar{u}\right), \qquad \frac{d\,\bar{u}^\alpha}{d\,a} = \eta^\alpha\left(\bar{x},\ \bar{u}\right), \tag{1.2.26}$$

subject to the initial conditions

$$\bar{x}^i(0) = x^i, \qquad \bar{u}^\alpha(0) = u^\alpha \tag{1.2.27}$$

A one-parameter Lie group of transformations, which by Theorem 1.2.1 is equivalent to its infinitesimal transformation, is also equivalent to its infinitesimal operator. The latter allows to representing the solution of Lie's equations (1.2.26) in terms of a Taylor series (exponential map)

$$\bar{x}^i = \exp(aX)x^i, \qquad \bar{u}^\alpha = \exp(aX)u^\alpha, \tag{1.2.28}$$

where

$$\exp(aX)x^i = 1 + aX + \frac{a^2}{2}X^2 + \ldots = \sum_{r=0}^{\infty} \frac{a^r}{r!} X^r \tag{1.2.29}$$

1.2.8 Invariants

Now, we can introduce the concept of invariants of a function with respect to a Lie group of transformations.

Theorem 1.2.2 A point $(x, u) \in IR^{n+m}$ is an invariant point of a group G with generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{1.2.30}$$

if and only if

$$\xi^i(x, u) = \eta^\alpha(x, u) = 0. \tag{1.2.31}$$

Definition 1.2.4 An infinitely differentiable function $F(x, u)$ is invariant function of the Lie group of transformation (1.2.16) if and only if for any group transformation (1.2.16).

$$F(\bar{x}, \bar{u}) = F(x, u) \qquad \forall\, x, u, a \in S' \subset S. \tag{1.2.32}$$

Theorem 1.2.3 A function $F(x, u)$ is an invariant of a group G with the generator X if and only if

$$X(F) \equiv 0. \tag{1.2.33}$$

1.2.9 Prolongation Formulas

The transformations (1.2.16) determine suitable transformations for the derivatives of system (1.2.1). Let $u_{(1)}$ denote the set of all $m \cdot n$ first order partial derivatives of dependent variable u with respect to independent variable x,

$$u_{(1)} = \left(u_1^1(x), u_1^1(x), \ldots, u_n^1(x), \ldots, u_1^m(x), \ldots, u_n^m(x)\right), \tag{1.2.34}$$

and in general, let $u_{(k)}$ denote

$$u_{(k)} = \left\{ \frac{\partial^k u^\alpha(x)}{\partial x^{i_1}, \ldots, \partial x^{i_k}} \mid \alpha = 1, 2, \ldots, m;\, i_1, \ldots, i_k = 1, \ldots, n \right\} \tag{1.2.35}$$

the set of all partial derivatives of order k.

The transformations of the derivatives of the dependent variables lead to natural extensions (prolongations) of the one-parameter Lie group of transformations (1.2.16). While the one-parameter Lie group of transformations (1.2.16) acts on the space (x, u), the extended group acts on the space $(x, u, u_{(1)})$, and, more in general, on the jet space $(x, u, u_{(1)}, \ldots, u_{(k)})$. Since all the information about a Lie group of transformations is contained in its infinitesimal generator, we need to compute its prolongations,
the first order prolongation

$$X^{[1]} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha\left(x, u, u_{(1)}\right)\frac{\partial}{\partial u_i^\alpha}, \quad u_i^\alpha = D_j\left(u^\alpha\right) \tag{1.2.36}$$

with

$$\zeta_i^\alpha = D_i\left(\eta^\alpha\right) - u_j^\alpha D_i\left(\xi^j\right) \tag{1.2.37}$$

The k^{th} extended infinitesimal generator (k^{th} prolongation of (1.2.25)) is given by

$$\begin{aligned} X^{[k]} = {} & \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha\left(x, u, u_{(1)}\right)\frac{\partial}{\partial u_i^\alpha} + \ldots \\ & + \zeta_{i_1 \ldots i_k}^\alpha\left(x, u, \ldots, u_{(k)}\right)\frac{\partial}{\partial u_{i_1 \ldots i_k}^k}, \end{aligned} \tag{1.2.38}$$

where

$$u_{i_1 \ldots l_k}^\alpha = D_{i_1} D_{i_2} \ldots D_{i_k}\left(u^\alpha\right) \tag{1.2.39}$$

and $\zeta_{i_1 \ldots i_k}^\alpha$ defined by the relation

$$\zeta^{\alpha}_{i_1 \ldots i_k} = D_{ik}\left(\zeta^{\alpha}_{i_1 \ldots i_{k-1}}\right) - u^{\alpha}_{i_1 \ldots i_{k-1} l}\, D_{i_k}\left(\xi^l\right) \tag{1.2.40}$$

In (1.2.37), (1.2.39) and (1.2.40) the total derivative operators

$$D_j = \frac{\partial}{\partial x^j} + u^{\alpha}_j \frac{\partial}{\partial u^{\alpha}} + u^{\alpha}_{jk} \frac{\partial}{\partial u^{\alpha}_k} + \ldots \tag{1.2.41}$$

The converse is also true in that, given the prolonged operators (1.2.38) or their corresponding infinitesimal transformations (1.2.21), we can find the respective one-parameter prolonged groups. This is achieved by using Theorem 1.2.1

1.2.10 Determining Equations

Definition 1.2.5 An invertible transformation acting on the space (x, u) of differential equation Ξ is a point symmetry of Ξ provided every solution h of Ξ is mapped onto another solution $\bar{h}$ of Ξ.

Theorem 1.2.4 Let

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^{\alpha}(x, u) \frac{\partial}{\partial u^{\alpha}} \tag{1.2.42}$$

be the infinitesimal generator corresponding to (1.2.21), and $X^{[k]}$ the k^{th} extended infinitesimal generator. The group of transformations (1.2.21) is admitted by the system (1.2.1) if and only if

$$X^{[k]}\left(E^{\sigma}\left(x, u, u_{(1)}, \ldots, u_{(k)}\right)\right) \equiv 0, \quad \sigma = 1, \ldots, s \tag{1.2.43}$$

The symmetry conditions (1.2.43) can be written compactly as

$$X^{[k]}\left(E^{\sigma}\left(x, u, u_{(1)}, \ldots, u_{(k)}\right)\right)\Big|_{E^{\lambda}\left(x, u_{(1)}, \ldots, u_{(k)}\right) = 0^s_{\lambda = 1}} = 0, \quad \sigma = 1, \ldots, s \tag{1.2.44}$$

Equations (1.2.44) are the so-called determining equations. In general the determining equations comprise an over-determined system of linear homogeneous PDEs for the unknown coordinates ξ^i and η^{α} of the symmetry generator X. The solutions of the determining system form a vector space, that is, any finite linear combination of symmetries is again symmetry.

1.2.11 Multi Parameter Lie Groups of Transformations

In this section we summarize some key results pertaining to multi parameter Lie groups of transformations. The definition of one-parameter local groups introduced earlier in sub section 1.2.1 also applies to r-parameter local groups by defining the vector parameter $a = (a_1, a_2, \ldots, a_r)$.

Now we can generalize Theorem 1.2.1 for an r-parameter group of transformation G as follows

Form the set of r-associated infinitesimal generators as

$$X_l = \xi_l^i(x, u)\frac{\partial}{\partial x^i} + \eta_l^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \quad l = 1, \ldots, r \tag{1.2.45}$$

where the infinitesimal $\xi_l^i(x, u)$ and $\eta_l^\alpha(x, u)$ are given by the Lie equations (1.2.26) subject to the initial conditions (1.2.27)

Each transformation $T_{a_l} = T_{a_r} \ldots T_{a_1} \in G$ can be realized as either

$$\exp(\lambda_1 X)\exp(\lambda_2 X)\exp(\lambda_3 X)\ldots \exp(\lambda_r X) \tag{1.2.46}$$

or

$$\exp(\delta X) \tag{1.2.47}$$

where

$$X = \mu_1 X_1 + \mu_2 X_2 + \mu_3 X_3 + \ldots + \mu_r X_r \tag{1.2.48}$$

where λ_i, δ, μ_l are arbitrary real constants with λ_i, δ sufficiently small. Hence, the r infinitesimal generators X_l contain all the information needed to reconstruct the r-parameter Lie group of transformation G.

It is natural to consider the r-dimensional vector space L with basis given by the infinitesimal generator $\{X_1, X_2, \ldots, X_r\}$ over the real number since for any $X \in L$, given by (1.2.48), the corresponding transformation (1.2.47) lies in G for sufficiently small δ. It turns out that L is endowed with an additional algebraic structure which we shall now investigate.

1.2.12 Lie Algebras

Definition 1.2.6 A Lie Algebra is a vector space L over the field IF together with the binary operation

$$[\ ,\ ]:\ L \times L \to L \tag{1.2.49}$$

called the Lie Bracket, such that for all $a, b \in IF$ and $X_1, X_2, X_3 \in L$, if the following axioms are satisfied

(i) **Closure.** $[X_1,\ X_2] \in L;$ (1.2.50)

(ii) **Bilinearity.** $[a X_1 + b X_2,\ X_3] = a[X_1,\ X_3] + b[X_2,\ X_3];$ (1.2.51)

(iii) **Skew- Symmetry.** $[X_1,\ X_2] = -[X_2,\ X_1];$ (1.2.52)

(iv) **Jacobi Identity.** $[X_1,\ [X_2,\ X_3]] + [X_2,\ [X_3,\ X_1]] + [X_3,\ [X_1,\ X_2]] = 0.$ (1.2.53)

The infinitesimal generators of an r-parameter Lie group, by introducing an operation, commutator (Lie bracket) between two infinitesimal generators $X_\alpha,\ X_\beta$ defined by

$$[X_\alpha,\ X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha, \tag{1.2.54}$$

span an r-dimensional vector space over the field IF of real numbers. Clearly (1.2.54) satisfies the bilinear, antisymmetric, and Jacobi identity, say

$$[X_\alpha,\ [X_\beta,\ X_\gamma]] + [X_\beta,\ [X_\gamma,\ X_\alpha]] + [X_\gamma,\ [X_\alpha,\ X_\beta]] = 0 \tag{1.2.55}$$

where $\alpha,\ \beta,\ \gamma = 1, \ldots, r$

i.e. vector space of infinitesimal generators gains the structure of a Lie algebra.

Given an r-dimensional Lie algebra L of infinitesimal generators closed under the operation of commutator with basis $\{X_1,\ X_2, \ldots, X_r\}$, leads to

$$[X_\alpha,\ X_\beta] = \sum_{\gamma=1}^{r} C^{\gamma}_{\alpha\beta} X_\gamma, \quad \text{where} \quad \alpha,\ \beta,\ \gamma = 1, \ldots, r, \tag{1.2.56}$$

for some constants $C^{\gamma}_{\alpha\beta}$, called the structural constants.

Definition 1.2.7 A Subalgebra of a r-dimensional Lie algebra L is a subspace S of L such that $[X_\alpha,\ X_\beta] \in S$ for all $X_\alpha,\ X_\beta \in S$. (1.2.57)

1.2.13 Lie's Algorithm

1. Let the one-parameter Lie group of point transformations (1.2.16) leaves invariant the system of PDEs (1.2.1).
2. Write the generator of symmetry

$$X = \xi^i (x, u) \frac{\partial}{\partial x^i} + \eta^\alpha (x, u) \frac{\partial}{\partial u^\alpha} \tag{1.2.58}$$

3. Apply the prolonged operator $X^{[k]}$ given by (1.2.38) to each equation of the system (1.2.1) and require that

$$X^{[k]} \left(E^\sigma \left(x, u, u_{(1)}, \ldots, u_{(k)}\right)\right)\Big|_{E^\lambda \left(x, u_{(1)}, \ldots, u_{(k)}\right) = 0_{\lambda=1}^s} = 0, \quad \sigma = 1, \ldots, s \tag{1.2.59}$$

4. Separate the expanded expression with respect to the derivatives of the dependent variables and their powers resulting in an over-determined system of linear homogeneous PDEs in terms of ξ^i and η^α.
5. The solutions of the over-determining equations will lead to the explicit forms of ξ^i and η^α.
6. Introducing the corresponding characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \ldots = \frac{dx^n}{\xi^n (x, u)} = \frac{du^1}{\eta^1 (x, u)} = \ldots = \frac{du^m}{\eta^m (x, u)} \tag{1.2.60}$$

   and obtain u in terms of $n-1$ new independent variables.
7. Rewrite the system (1.2.1) in these new coordinates to get the reduced form of the system of PDEs (1.2.1).

1.3 Painlevé Analysis

The Painlevé analysis [34, 35] is a widely applied and quite successful technique to investigate the integrability of nonlinear ODEs and PDEs by analyzing the singularity structure of the solutions. The test is named after the French mathematician Paul Painlevé (1863-1933), who classified second order differential equations that are solvable in terms of known elementary functions or new transcendental functions [45].

Ablowitz et al. [117], stated that when all the ODEs obtained by exact similarity transforms from a given PDE have the Painlevé property, then the PDE is "integrable.

Weiss et al. [87, 88] have defined the Painlevé property for PDE directly, without recourse to the reduction(s) to ODEs. A PDE is said to have the Painlevé property if its solutions in the complex plane are single-valued in the neighbourhood of all its movable singularities. This Painlevé test has proved to be a useful criterion for the identification of completely integrable PDE.

Definition 1.3.1 A differential equation has Painlevé property if their general solutions should have no movable singularities other than poles in the complex plane. A singularity is movable if it depends on the constants of integration of the ODE. For instance, the Riccati equation,

$$w'(z) + w^2(z) = 0, \tag{1.3.1}$$

has the general solution $w(z) = 1/(z-c)$, where c is the constant of integration. Hence, (1.3.1) has a movable simple pole at $z = c$ because it depends on the constant of integration. In the following section, for details of the WTC-method for testing PDEs for the Painlevé property, we are presenting some outlines from [195]

1.3.1 The Algorithm

Consider partial differential equations $\Xi(x, t, Y(x, t)) = 0$ in two independent variables x and t.

Following [34, 35, 195], the Laurent expansion of the solution $Y(x, t)$, about a singular manifold $u(x,t)$ is

$$Y(x, t) = (u(x, t))^{\alpha} \sum_{k=0}^{\infty} Y_k(x, t)\,(u^k(x, t)), \tag{1.3.2}$$

In (1.3.2), $Y_0(x, t) \neq 0$, α is a negative integer, and $Y_k(x, t)$ are analytic functions in a neighborhood of $u(x, t)$. The algorithm for the Painlevé test is composed of the following three steps.

Step 1. Leading order analysis

Determine the (negative) integer α and $Y_0(x, t)$ by balancing the highest derivative with the nonlinear terms after substitution of

$$Y(x,\,t) = Y_0\,(x,\,t)u^{\alpha}\,(x,\,t)\,, \tag{1.3.3}$$

into the given PDE. There may be several branches for Y_0, and for each the next two steps must be performed. If α is non-integer, or $Y_0\,(x,\,t) = 0$, then the algorithm terminates.

Step 2. Determination of the resonances

For a selected α and Y_0, calculate the non-negative integers r, called the resonances, at which arbitrary functions Y_r enter the series (1.3.2). For this, substitute

$$Y = Y_0\, u^r(x,t) + Y_r\, u^{r+\alpha}, \tag{1.3.4}$$

into the equation, only retaining its most singular terms. Require that the coefficient Y_r is arbitrary by equating its coefficient to zero. Compute the integer roots of the resulting polynomial. For (1.3.2) to represent the general solution, the number of roots (including $r = -1$) must match the order of the given equation. The root $r = -1$ corresponds to the arbitrariness of the manifold $u(x,t)$is often called the universal resonance.

Step 3. Verification of the compatibility conditions

Verify that a solution of the form (1.3.2) is indeed admissible, and that it has the necessary number of free coefficients Y_r. Substitute (1.3.2), truncated a largest resonance, into the PDE. Determine Y_k at non-resonance levels k. At resonance levels, Y_r should be arbitrary, and since we are dealing with a nonlinear equation, a compatibility condition must be unconditionally satisfied. An equation which carried out consistently and unambiguously these three steps can be passes the Painlevé test. Details and an abundance of worked examples can be found in [20, 21, 39].

1.4 The (G'/G)-Expansion Method

Wang et al. [124] introduced the (G'/G)-expansion method [181] to look for traveling wave solutions of nonlinear PDEs. This method is based on the assumptions that the traveling wave solutions can be expressed by a polynomial in(G'/G), where G satisfies the second order linear ODE. The degree of the polynomial can be determined by the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given PDEs. The coefficients of the polynomial can be obtained by solving a set of algebraic equations. Being concise and straightforward, this method can be applied to various nonlinear partial differential equations with variable coefficients. As a result, hyperbolic function solution, trigonometric function solution and rational solution with various parameters are obtained.

1.4.1 Extended (G'/G)-Expansion Method

Recently, Bangqing et al. [95] establish a connection between the (G'/G)-expansion method and the soliton structure excitation. By extending the linear traveling wave transformation to the nonlinear transformation with the method, the non-traveling wave solutions with variable separation can be constructed for PDEs. For the non-traveling wave solutions include arbitrary functions of independent variables, it is convenient to excite abundant soliton structures.

Now we describe the extended (G'/G)-expansion method for a given variable coefficient $(2+1)$-dimensional PDEs. Suppose that nonlinear PDEs equation with independent variables $x,\ y,\ t$ and dependent variable u is given by

$$\Xi\left(u,\, u_t,\, u_x,\, u_y,\, u_{tt},\, u_{tx},\, u_{ty},\, u_{xy},\, u_{xx},\, u_{yy},\ \ldots\right), \tag{1.4.1}$$

where $u = u\,(x,\, t)$ is unknown functions and Ξ is a polynomial in $u\,(x,\, t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps [95] for solving equation (1.4.1) using an extended (G'/G)- expansion method.

Step 1. The solution of equation (1.4.1) can be expressed by a polynomial in (G'/G) as follows

$$u = a_0 + \sum_{i=0}^{m} a_i \left[\frac{G'(\theta)}{G(\theta)}\right]^i \tag{1.4.2}$$

where $\theta = s\,x + l\,y - V\,t$ is a linear traveling wave transformation, and $s,\, l,\, V,\, a_i$ $(i = 1,\, 2,\, \ldots,\, m)$ are arbitrary functions of variable $x,\, y,\, t$ to be determined, and $G(\theta)$ satisfies the following equation

$$G''(\theta)+\lambda\, G'(\theta)+\mu\, G(\theta)=0 \tag{1.4.3}$$

We extend $\theta = s\,x + l\,y - V\,t$ into $\theta = \theta(x,\ y,\ t)$, where θ is an arbitrary function of the independent variables $x,\ y,\ t$. The solutions involved $\theta(x,\ y,\ t)$ are called the non-traveling wave solutions. The non-traveling wave solutions allow us to select $\theta(x,\ y,\ t)$ and excite abundant soliton structures.

However, it is extremely arduous to calculate the solutions with $\theta(x,\ y,\ t)$. In general, we may suppose that $\theta(x,\ y,\ t)$ is in variable separation forms, such as $\theta(x,\ y,\ t)=f(x,t)+g(y)$ or $\theta(x,\ y,\ t)=f(x,t)+g(y,\ t)$ where $f(x,t)$ and $g(y,\ t)$ are arbitrary functions of the indicated variables.

Step 2. In order to determine u explicitly, we firstly find the value of integer m by balancing the highest order nonlinear terms and the highest order partial derivative of u in equation (1.4.1).

Step 3. Substituting equation (1.4.2) into equation (1.4.1) and using equation (1.4.3), collecting all terms with the same order of together, the left-hand side of equation (1.4.1) is converted into polynomial in $(G'/G)^j$, $(j = 0, 1, 2, 3)$. Setting each coefficient of this polynomial to zero, we derive a set of overdetermined differential equations for $s,\ l,\ V$ and a_i.

Step 4. Solve the system of over-determined partial differential equations obtained in Step 3 for $s,\ l,\ V$ and a_i.

Step 5. Use the results obtained in above steps to derive the solutions of equation (1.4.1) depending on (G'/G), since the solutions of equation (1.4.3) have been well known to us depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, then the non-traveling wave solutions of equation (1.4.1) are obtained.

1.4.2 Modified (G'/G)-Expansion Method

Suppose that a nonlinear PDE is given by

$$\Xi\left(u,\ u_t,\ u_x,\ u_{tt},\ u_{tx},\ u_{xx},\ .\ .\ .\right) \tag{1.4.4}$$

where $u = u\,(x, t)$ is unknown functions and Ξ is a polynomial in $u\,(x, t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps [197] for solving equation (1.4.4) using modified $(G^{'}/G)$- expansion method.

Step 1. The traveling wave variable

$$u\,(x, t) = u(\xi), \qquad \xi = k(x - ct), \tag{1.4.5}$$

where k and c are constants, permits us to reduce equation (1.4.4) into the following ODE:

$$P(u, u', u'', u''', \ldots). \tag{1.4.6}$$

Step 2. Suppose that the solution of equation (1.4.6) can be expressed by a polynomial (G'/G) as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{m} \left\{ a_i \left(\frac{G'(\xi)}{G(\xi)} \right)^{i} + b_i \left(\frac{G'(\xi)}{G(\xi)} \right)^{-i} \right\}, \tag{1.4.7}$$

where $G = G(\xi)$ satisfies the second order linear ODE

$$G''(\xi) + \mu G(\xi) = 0, \tag{1.4.8}$$

where a_0, a_i, b_i and μ are constants to be determined later.

Step 3. The parameter m in (1.4.7) can be found by balancing the highest order derivative term and the highest nonlinear term in (1.4.4) or (1.4.6) and conclude the following:

(i) If m is a positive integer then go to step 4,

(ii) If m is not positive integer, we put $u = v^m$ and then return to step 1.

Step 4. Substituting (1.4.7) into (1.4.6) and using (1.4.8), collecting all terms with the same powers of (G'/G) together and then equating each coefficient of the resulted polynomial to zero, yield a system of algebraic equations for a_0, a_i, b_i and μ.

Step 5. Since the general solutions of (1.4.8) are well known to us, then substituting $a_0, a_i, b_i,\ \mu$ and the general solutions of (1.4.8) into (1.4.7) we have the traveling wave solutions of equation (1.4.4).

1.5 Numerical Solutions of Partial Differential Equations

The vast majority of PDEs model cannot be solved analytically. So, to investigate the predictions of PDE models of such phenomena it is often necessary to approximate their solutions numerically. In most cases, the approximate solution is represented by functional values at certain discrete points (grid points or mesh points). There seems a bridge between the derivatives in the PDE and the functional values at the grid points. The numerical technique is such a bridge, and the corresponding approximate solution is termed the numerical solution. Currently, there are many numerical techniques available in the literature. Among them, the finite difference (FD) [25, 90], finite element (FE) [5, 6], and finite volume (FV) [25] methods fall under the category of low order methods, whereas spectral and pseudo spectral methods are considered global methods. The differential quadrature method (DQM) is a higher order numerical technique for solving partial differential equations. In the nineteen century, most of the numerical simulations of engineering problems can be carried out by the low order FD, FE, and FV methods using a large number of grid points. In some practical applications, however, numerical solutions of PDEs are required at only a few specified points in the physical domain. To achieve an acceptable degree of accuracy, low order methods still require the use of a large number of grid points to obtain accurate solutions at these specified points. In seeking an efficient discretization technique to obtain accurate numerical solutions using a considerably small number of grid points, Richard Bellman and his associates [151] introduced the method of differential quadrature in the early 1970s. The DQM, akin to the conventional integral quadrature method, approximates the partial derivative of a function at any location by a linear summation of all the function values along a mesh line. The key procedure in the differential quadrature application lies in the determination of the weighting coefficients. Initially, Bellman and his associates proposed two methods to compute the weighting coefficients for the first order derivative. The first method is based on an ill-conditioned algebraic equation system. The second method uses a simple algebraic formulation, but the coordinates of the grid points are fixed by the roots of the shifted Legendre polynomial. In earlier applications of the DQM, Bellman's first method was usually used because it allows the use of an arbitrary grid point distribution. However, since the algebraic equation system of this method is ill-conditioned, the number of the grid points usually used is less than 13. This drawback limits the application of the DQM.

The DQM and its applications were rapidly developed after the late 1980s, thanks to the innovative work in the computation of the weighting coefficients by researchers [29, 32, 145, 146, 192]. As a result, the DQM has emerged as a powerful numerical discretization tool in the past decade. As compared to the conventional low order finite difference and finite element methods, the DQM can obtain very accurate numerical results using a considerably smaller number of grid points and hence requiring relatively little computational effort. So far, the DQM has been efficiently employed in a variety of problems in engineering and physical sciences. The basic theory of this method can be found in Shu [29]. Recently, Mittal et al. [152-160] and Jiwari et al. [162-16] have used polynomial based differential quadrature method for numerical solutions of some nonlinear partial differential equations.

1.6 Basic Approaches for Weighting Coefficients of DQM

In this section, some basic definitions and fundamentals for weighting coefficients of differential quadrature method are given; a more detailed exposition can be found in [29, 30, 166].

1.6.1 Weighting Coefficients for the First Order Derivative

Consider a one dimensional problem over a closed interval [a, b]. It is supposed that there are N grid points with coordinates as $a = x_1, x_2, ..., x_N = b.$ Bellman et al. [151] assumed that a function $f(x)$ is sufficiently smooth over the interval [a, b] so that its first order derivative $f^{(1)}(x)$ at any grid point can be approximated by the following formulation

$$f_x^{(1)}(x_i) = \sum_{j=1}^{N} a_{ij}\; f(x_j), \qquad i = 1,2,..., N\ , \tag{1.6.1}$$

where $f(x_j)$ represents the functional value at a grid point x_j, $f_x^{(1)}(x_i)$ indicates the first order derivative of $f(x)$ at x_i, and a_{ij} is the weighting coefficient of the first order derivative. The determination of weighting coefficients a_{ij} in equations (1.6.1) is a key procedure in the differential quadrature approximation. Once the weighting coefficients are determined the bridge to the link the derivatives in the governing differential equation and the functional value at the mesh points is established. In the following, we shall show that the weighting coefficients can be efficiently computed by employing some explicit formulations.

1.6.2 Bellman's Approach

Bellman et al. [151] proposed two approaches to compute the weighting coefficients a_{ij} in equation (1.6.1). The two approaches are based on the use of two different test functions.

Bellman's first approach: In this approach, the test functions are chosen as

$$g_k(x) = x^k,\ k = 0,1,\ldots,N-1. \tag{1.6.2}$$

Obviously, equations (1.6.2) give *N* test functions. For the weighting coefficients a_{ij} in equation (1.6.1), i and j are taken from I to *N*. Thus, the total number of weighting coefficients is $N \times N$. To obtain these weighting coefficients, the *N* test functions should be applied at *N* grid points $x_1, x_2, \ldots, x_N$. As a consequence, the following $N \times N$ algebraic equations for a_{ij} are obtained

$$\begin{cases} \sum_{j=1}^{N} a_{ij} = 0 \\ \sum_{j=1}^{N} a_{ij} . x_j = 1 \\ \sum_{j=1}^{N} a_{ij} . x_j^k = k . x_i^{k-1}, \quad k = 2,3,\ldots,N-1 \end{cases} , \quad \text{for} \quad i = 1,\ 2,\ \ldots,N \tag{1.6.3}$$

System of equations (1.6.3) has a unique solution because its matrix is of Vandermonde form. Unfortunately, when *N* is large, the matrix is ill-conditioned and its inversion is difficult. In the practical application of this approach, *N* is usually chosen to be less than 13.

Bellman's second approach: This approach is similar to the first approach, but uses different test functions

$$g_k(x) = \frac{L_N(x)}{(x - x_k).\ L_N^{(1)}(x_k)}, \qquad k = 1,2,\ldots,N \tag{1.6.4}$$

where $L_N(x)$ is the Legendre polynomial of degree N and $L_N^{(1)}(x)$ is the first order derivative of $L_N(x)$. By choosing x_k to be the roots of the shifted Legendre polynomial and applying equation (1.6.4) at N grid points $x_1, x_2, ..., x_N$, Bellman et al. [151] obtained a simple algebraic formulation to compute a_{ij}

$$a_{ij} = \frac{L_N^{(1)}(x_i)}{(x_i - x_j).\ L_N^{(1)}(x_j)}, \qquad \text{for} \quad j \neq i \qquad (1.6.5a)$$

$$a_{ii} = \frac{1 - 2x_i}{2x_i(x_i - 1)}, \qquad \text{for} \quad j = i \qquad (1.6.5b)$$

Using equations (1.6.5), the computation of the weighting coefficients is a simple task. However, this approach is not as flexible as the first approach because the coordinates of the grid points in this approach cannot be chosen arbitrary. Instead, they should be chosen as the roots of the Legendre polynomial of degree N. So equation (1.6.5) only reflects a special case. Due to the inflexibility associated with the second approach in selecting the grid points, the first approach is usually adopted in practical applications.

1.6.3 Quan and Chang's Approach

To improve Bellman's approaches in computing the weighting coefficients, many attempts have been made by researchers. One of the most useful approach is the one introduced by Quan and Chang [84, 85]. Quan and Chang used the following Lagrange interpolation polynomials as the test functions,

$$g_k(x) = \frac{M(x)}{(x - x_k).\ M^{(1)}(x_k)},\ k = 1,2,...,N\,, \qquad (1.6.6)$$

where

$$M(x) = (x - x_1)(x - x_2)...(x - x_N), \qquad (1.6.7)$$

$$M^{(1)}(x_i) = \prod_{k=1,k\neq i}^{N}(x_i - x_k). \qquad (1.6.8)$$

Subsequently, by applying equations (1.6.6) at N grid points, they obtained the following algebraic formulations to compute the weighting coefficients a_{ij},

$$a_{ij} = \frac{1}{\left(x_j - x_i\right)} \prod_{k=1, k\neq i, j}^{N} \frac{x_i - x_k}{x_j - x_k}, \quad \text{for} \quad j \neq i \tag{1.6.9a}$$

$$a_{ii} = \sum_{k=1, k\neq i}^{N} \frac{1}{x_i - x_k}. \tag{1.6.9b}$$

It may be noted that there is no restriction on the choice of grid points in the equation (1.6.9).

1.6.4 Shu's General Approach [29]

Shu's general approach was inspired from Bellman's approaches. It covers all the approaches, including Quan and Chang's approach. Shu explains that computation of weighting coefficients by all approaches is same due to polynomial approximation and linear vector space analysis. Shu uses the following two typical sets of the base polynomials listed as follows:

$$g_k(x) = \frac{M(x)}{(x - x_k).\ M^{(1)}(x_k)},\ k = 1,2,...,N\ , \tag{1.6.10a}$$

$$g_k(x) = x^{k-1},\ k = 1,2,...,N\ , \tag{1.6.10b}$$

where $M(x)$ and $M^{(1)}(x_i)$ are the same as in equation (1.6.7) and (1.6.8), but the computation of weighting coefficients is different. For simplicity, we set

$$M(x) = N(x, x_k).(x - x_k), \quad k = 1,2,...,N\ , \tag{1.6.11}$$

with $N(x_i, x_j) = M^{(1)}(x_i).\delta_{ij}$, where δ_{ij} is the Kronecker operator. Using equations (1.6.11), equations (1.6.10) can be simplified to

$$g_k(x) = \frac{N(x, x_k)}{M^{(1)}(x_k)},\ k = 1,2,...,N\ . \tag{1.6.12}$$

Substituting equations (1.6.12) into (1.6.1) gives

$$a_{ij} = \frac{N^{(1)}(x_i, x_j)}{M^{(1)}(x_j)}. \tag{1.6.13}$$

In equations (1.6.13), $M^{(1)}(x_j)$ can be easily computed from the equation (1.6.8). To evaluate $N^{(1)}(x_i, x_j)$, we successively differentiate equations (1.6.11) with respect to x and obtain the following recurrence formulations

$$M^{(m)}(x) = N^{(m)}(x, x_k).(x - x_k) + m.N^{(m-1)}(x, x_k), \qquad (1.6.14)$$

for $m = 1,2,..., N-1;\ \ k = 1,2,..., N$,

where $M^{(m)}(x)$ and $N^{(m)}(x, x_k)$ indicate the mth order derivative of $M(x)$ and $N(x, x_k)$. From equations (1.6.14), we can obtain the expression for $N^{(1)}(x_i, x_j)$ as

$$N^{(1)}(x_i, x_j) = \frac{M^{(1)}(x_i)}{x_i - x_j}, \text{ for } \quad j \neq i \qquad (1.6.15a)$$

$$N^{(1)}(x_i, x_i) = \frac{M^{(2)}(x_i)}{2} \qquad (1.6.15b)$$

Substituting equations (1.6.15) into equations (1.6.13), we obtain

$$a_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j).M^{(1)}(x_j)}, \quad \text{for} \quad j \neq i \qquad (1.6.16a)$$

$$a_{ii} = \frac{M^{(2)}(x_i)}{2.M^{(1)}(x_i)}. \qquad (1.6.16b)$$

It is observed from equations (1.6.16) that, if x_i is given, it is easy to compute $M^{(1)}(x_i)$ from equation (1.6.8), and hence a_{ij} for $i \neq j$. However, the calculation of a_{ii} is based on the computation of the second order derivative $M^{(2)}(x_i)$ which is not easy task. This difficulty can be eliminated by using the second set of base polynomials (1.6.10*b*). According to the property of a linear vector space, if one set of base polynomials satisfies a linear operator, say equation (1.6.1), so does another set of base polynomials. As a consequence, the system of equations for determination of a_{ij} derived from the Lagrange interpolation polynomials equation (1.6.10*a*) should be equivalent to that derived from another set of base polynomials $x^{k-1},\ k = 1,2,..., N$ equations (1.6.10*b*). Thus a_{ij} satisfies the following equations which are obtained by the base polynomial x^{k-1} when $k = 1$

$$\sum_{j=1}^{N} a_{ij} = 0 \quad \text{or} \quad a_{ii} = -\sum_{j=1,\, j\neq i}^{N} a_{ij}\ . \qquad (1.6.17)$$

Equations (1.6.16*a*) and (1.6.17) are two formulations to compute the weighting coefficients a_{ij}. It is noted that in the development of these two formulations, two sets of base polynomials were used in the linear polynomial vector space V_N.

1.6.5 Weighting Coefficients for the Second Order Derivative

For the discretization of the second order derivative, we introduced a similar approximation form given by

$$f_x^{(2)}(x_i) = \sum_{j=1}^{N} b_{ij}\, f(x_j), \qquad i = 1,2,\ldots,N \tag{1.6.18}$$

where $f_x^{(2)}(x_i)$ indicates the second order derivative of $f(x)$ at x_i, and b_{ij} is the weighting coefficient of the second order derivative. Obviously, equations (1.6.18) are a linear operator. The equations (1.6.18) have the same form as equations (1.6.1). The only difference between these equations is the use of different weighting coefficients. The following two approaches can be used to determine b_{ij} as follows

1.6.6 Quan and Chang's Approach

In this approach, Quan and Chang [84, 85] used the Lagrange interpolation polynomials as the test functions and then derived

$$b_{ij} = \frac{2}{(x_j - x_i)}\left(\prod_{k=1,\,k\neq i,\,j}^{N} \frac{x_i - x_k}{x_j - x_k}\right)\left(\sum_{l=1,\,l\neq i,\,j}^{N} \frac{1}{x_i - x_l}\right), \quad \text{for } i \neq j \tag{1.6.19a}$$

$$b_{ii} = 2\sum_{k=1,\,k\neq i}^{N-1}\left[\frac{1}{x_i - x_k}\left(\sum_{l=k+1,\,l\neq i}^{N} \frac{1}{x_i - x_l}\right)\right]. \tag{1.6.19b}$$

1.6.7 Shu's General Approach

Similar to the first order derivative, Shu's general approach is also based on polynomial approximation and linear vector space analyses. Two sets of base polynomials (1.6.10) are used. Substituting equations (1.6.12) into equations (1.6.18) gives

$$b_{ij} = \frac{N^{(2)}(x_i, x_j)}{M^{(1)}(x_j)}. \tag{1.6.20}$$

On the other hand, from equations (1.6.14), we obtain

$$N^{(2)}(x_i, x_j) = \frac{M^{(2)}(x_i) - 2N^{(1)}(x_i, x_j)}{x_i - x_j}, \quad \text{for } i \neq j \tag{1.6.21a}$$

$$N^{(2)}(x_i, x_i) = \frac{M^{(3)}(x_i)}{3} \qquad (1.6.21b)$$

Substituting equation (1.6.21) into equation (1.6.20) yields

$$b_{ij} = \frac{M^{(2)}(x_i) - 2N^{(1)}(x_i, x_j)}{(x_i - x_j).M^{(1)}(x_j)}, \qquad \text{for } i \neq j \qquad (1.6.22a)$$

$$b_{ii} = \frac{M^{(3)}(x_i)}{3M^{(1)}(x_i)}. \qquad (1.6.22b)$$

Finally, by substituting equation (1.6.16) into equation (1.6.22*a*), we obtain

$$b_{ij} = 2a_{ij}\left(a_{ii} - \frac{1}{x_i - x_j}\right), \qquad \text{for } i \neq j \qquad (1.6.23)$$

When $i \neq j$, b_{ij} can be easily computed from equation (1.6.23). However, from equation (1.6.22*a*), it can be seen that the computation of b_{ii} involves the third order derivative $M^{(3)}(x_i)$ which cannot be computed easily. This difficulty can be eliminated by employing the properties of a linear vector space. Similar to the analysis for the case of first order derivative, the equation system for b_{ij} derived from the Lagrange interpolation polynomials (1.6.10*a*) should be equivalent to that derived from another set of base polynomials x^{k-1}, $k = 1,2,...,N$ (1.6.10*b*). Thus b_{ij} satisfies the following equation which is obtained by the base polynomial x^{k-1} when $k = 1$

$$\sum_{j=1}^{N} b_{ij} = 0 \text{ or } \quad b_{ii} = -\sum_{j=1,\, j\neq i}^{N} b_{ij}\ . \qquad (1.6.24)$$

For the application of Shu's general approach, b_{ij} is firstly computed from equation (1.6.23) when $i \neq j$. Subsequently, b_{ii} is computed from equation (1.6.24).

1.6.8 Shu's Recurrence Formulation for Higher Order Derivatives [29]

For the discretization of higher order derivatives, the following two linear operators are applied

$$f_x^{(m-1)}(x_i) = \sum_{j=1}^{N} w_{ij}^{(m-1)}\ f(x_j), \qquad (1.6.25)$$

$$f_x^{(m)}(x_i) = \sum_{j=1}^{N} w_{ij}^{(m)}\ f(x_j), \qquad (1.6.26)$$

for $i = 1,2,..., N;\ m = 2,3,..., N-1$

where $f_x^{(m-1)}(x_i), f_x^{(m)}(x_i)$ indicate the $(m-1)^{th}$ and m^{th} order derivatives of $f(x)$ with respect to x at x_i. $w_{ij}^{(m-1)}, w_{ij}^{(m)}$ are the weighting coefficients related to $f_x^{(m-1)}(x_i)$ and $f_x^{(m)}(x_i)$. Two sets of base polynomials will also be used to derive explicit formulation for $w_{ij}^{(m)}$. The first set of base polynomials is given by equations (1.6.12). Substituting equations (1.6.12) into equations (1.6.25) and (1.6.26) gives

$$w_{ij}^{(m-1)} = \frac{N^{(m-1)}(x_i, x_j)}{M^{(1)}(x_j)}, \tag{1.6.27}$$

$$w_{ij}^{(m)} = \frac{N^{(m)}(x_i, x_j)}{M^{(1)}(x_j)}. \tag{1.6.28}$$

By rewriting equation (1.6.27), we obtain

$$N^{(m-1)}(x_i, x_j) = w_{ij}^{(m-1)}.M^{(1)}(x_j) \tag{1.6.29}$$

Equation (1.6.29) is valid for any i and j. On the other hand, from the recurrence formulation (1.6.14), we have

$$N^{(m-1)}(x_i, x_i) = \frac{M^{(m)}(x_i)}{m}, \tag{1.6.30}$$

$$N^{(m)}(x_i, x_j) = \frac{M^{(m)}(x_i) - m\,N^{(m-1)}(x_i, x_j)}{x_i - x_j}, \quad \text{for} \quad i \neq j \tag{1.6.31}$$

$$N^{(m)}(x_i, x_i) = \frac{M^{(m+1)}(x_i)}{m+1}. \tag{1.6.32}$$

Substituting equation (1.6.30) into equation (1.6.31) leads to,

$$N^{(m)}(x_i, x_j) = \frac{m\left(N^{(m-1)}(x_i, x_i) - N^{(m-1)}(x_i, x_j)\right)}{x_i - x_j}, \quad \text{for} \quad i \neq j \tag{1.6.33}$$

Equation (1.6.33) can be further simplified using equation (1.6.29),

$$N^{(m)}(x_i, x_j) = \frac{m\left(w_{ii}^{(m-1)} M^{(1)}(x_i) - w_{ij}^{(m-1)} M^{(1)}(x_j)\right)}{x_i - x_j}, \quad \text{for} \quad i \neq j \tag{1.6.34}$$

Substituting equation (1.6.34) into equation (1.6.28) and using equation (1.6.16*a*), a recurrence formulation is obtained as follows

$$w_{ij}^{(m)} = m\left(a_{ij}\, w_{ii}^{(m-1)} - \frac{w_{ij}^{(m-1)}}{x_i - x_j}\right), \text{ for } i, j = 1,2,\dots,N;\ m = 2,3,\dots,N-1 \qquad (1.6.35)$$

where a_{ij} is the weighting coefficient of the first order derivative described above. The formulation for $w_{ii}^{(m)}$ can be obtained by substituting equation (1.6.32) into equation (1.6.28), which gives

$$w_{ii}^{(m)} = \frac{M^{(m+1)}(x_i)}{(m+1).M^{(1)}(x_i)}, \qquad \text{for} \quad i, j = 1,2,\dots,N;\ m = 2,3,\dots,N-1 \qquad (1.6.36)$$

Obviously, equations (1.6.35) offer an easy way to compute the weighting coefficients $w_{ij}^{(m)}$ for $i \neq j$. However, it is very difficult to apply equations (1.6.36) to compute the weighting coefficients $w_{ii}^{(m)}$. Again, this difficulty can be overcome by the properties of a linear vector space. In terms of the analysis of the *N*-dimensional linear vector space, the equation system for $w_{ij}^{(m)}$ derived from the Lagrange interpolation polynomials should be equivalent to that derived from the base polynomials $x^{k-1},\ k = 1,2,\dots,N$. Thus $w_{ij}^{(m)}$ should satisfy the following equation obtained from the base polynomial x^{k-1} when $k = 1$

$$\sum_{j=1}^{N} w_{ij}^{(m)} = 0 \quad \text{or} \quad w_{ii}^{(m)} = -\sum_{j=1,\, j\neq i}^{N} w_{ij}^{(m)} \qquad (1.6.37)$$

From this formulation, $w_{ii}^{(m)}$ can be determine from $w_{ij}^{(m)}(i \neq j)$.

1.6.9 Weighting Coefficients for the Multi-dimension Case

In practice, most problems are two-dimensional or three-dimensional. Thus, it is necessary to extend the DQ approximation from the one-dimensional case to the multi-dimensional cases. In this section, we will only demonstrate the extension of the one-dimensional case to the two-dimensional case for regular domain. Shu [29] has shown that the one-dimensional polynomial differential quadrature method (PDQM) can be directly extended to the multi-dimensional case if the discretization domain is regular. The regular domain can be a rectangle or other shapes such as circle. Here, for simplicity, we only consider a rectangular domain for demonstration.

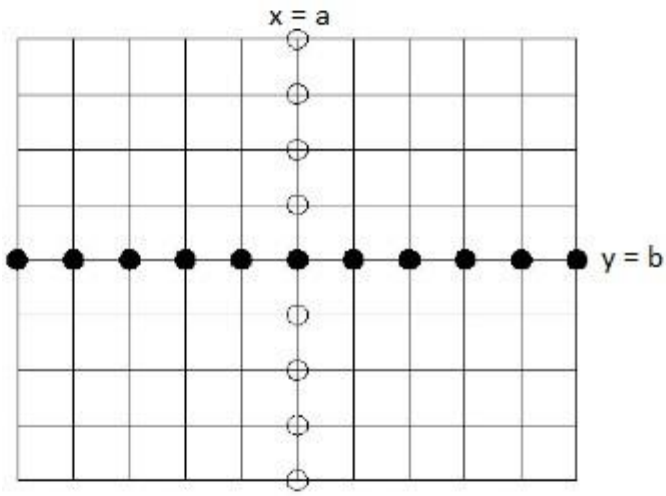


Figure 1.4 Rectangular domain

Consider a two-dimensional function $f(x,y)$ defined on a rectangular domain, as shown in Figure 1.4 It is clearly seen from the figure that along each horizontal line, the *x* interval is same, and along each vertical line, the *y* interval is the same. Thus, we can use the same *x* coordinate distribution for each horizontal line, and the same *y* coordinate distribution for each vertical line. It is supposed that along a $y = b$, where *b* is a constant, the values of $f(x,b)$, denoted by the filled circles, can be approximated by a polynomial of degree $(N-1)$, $P_N(x)$, which constitutes an *N*-dimensional linear vector space V_N with *N* base vectors (polynomials) $r_i(x)$, $i = 1,2,...,N$. Along a vertical line $x = a$, where *a* is a constant, the values of $f(a,y)$, denoted by the open circles, can be approximated by a polynomial of degree $(M-1)$, $P_M(y)$, which constitutes an *M*-dimensional linear vector space V_M with *M* base vectors (polynomials) $s_j(y)$, $j = 1,2,...,M$. The value of the function $f(x,y)$ at any location in the domain can then be approximated by a polynomial $P_{N\times M}(x,y)$ of the form

$$f(x,y) = P_{N\times M}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{M}\overline{c_{ij}}.x^{i-1}.y^{j-1}, \tag{1.6.38}$$

where $\overline{c_{ij}}$ are coefficients. It is clear that $P_{N\times M}(x,y)$ constitutes an $N \times M$ dimensional linear polynomial vector space $V_{N\times M}$ with respect to the operation of vector addition and scalar multiplication. It is easy to show that $\phi_{ij}(x,y) = r_i(x).s_j(y)$ constitutes the base vectors (polynomials) in the linear vector space $V_{N\times M}$.

Assumed that the derivatives of the function $f(x,y)$ at any grid points (x_i, y_j) can be approximated as follows

$$f_x^{(1)}(x_i, y_j) = \sum_{k=1}^{N} a_{ik}^x . f(x_k, y_j), \tag{1.6.39a}$$

$$f_y^{(1)}(x_i, y_j) = \sum_{k=1}^{M} a_{jk}^y . f(x_i, y_k), \tag{1.6.39b}$$

for $i = 1,2,...,N;\ j = 1,2,...,M$

where $f_x^{(1)}, f_y^{(1)}$ are the first order derivatives of the function $f(x,y)$ with respect to x and y respectively, and a_{ik}^x, a_{jk}^y are the corresponding weighting coefficients. From the properties of a linear vector space, we know that if all the base polynomials $\phi_{ij}(x,y)$ satisfy the linear equations (1.6.39), so does any polynomial in $V_{N\times M}$. Substituting $\phi_{ij}(x,y)$ into equations (1.6.39) leads to

$$\sum_{k=1}^{N} a_{ik}^x . r_j(x_k) = r_j^{(1)}(x_i), \quad i, j = 1,2,...,N \tag{1.6.40a}$$

$$\sum_{k=1}^{M} a_{ik}^y . s_j(y_k) = s_j^{(1)}(y_i), \quad i, j = 1,2,...,M \tag{1.6.40b}$$

where $r_j^{(1)}(x_i)$ represents the first order derivative of $r_j(x)$ at x_i and $s_j^{(1)}(y_i)$ represents the first order derivative of $s_j(y)$ at y_i. From equations (1.6.40), it is obvious that $a_{ik}^x\ or\ a_{jk}^y$ is only related to $r_j(x)$ or $s_j(y)$. Hence the formulation of the one dimensional case can be directly extended to the two dimensional case, that is

$$a_{ij}^x = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, \qquad \text{for} \quad i \neq j \tag{1.6.41a}$$

$$a_{ii}^x = -\sum_{j=1,\, j\neq i}^{N} a_{ij}^x, \qquad \text{for} \quad i, j = 1,2,...,N \tag{1.6.41b}$$

$$a_{ij}^y = \frac{P^{(1)}(y_i)}{(y_i - y_j)P^{(1)}(y_j)}, \qquad \text{for} \quad i \neq j \tag{1.6.42a}$$

$$a_{ii}^y = -\sum_{j=1,\, j\neq i}^{N} a_{ij}^y, \qquad \text{for} \quad i, j = 1,2,...,M \tag{1.6.42b}$$

where

$$M^{(1)}(x_i)=\prod_{j=1,\, j\neq i}^{N}(x_i-x_j), \qquad P^{(1)}(y_i)=\prod_{j=1,\, j\neq i}^{M}(y_i-y_j).$$

Similarly, for the second and higher order derivatives, the recurrence relations of the weighting coefficients are obtained as follows

$$w_{ij}^{(n)}=n\left(a_{ij}^{x}.w_{ii}^{(n-1)}-\frac{w_{ij}^{(n-1)}}{x_i-x_j}\right), \text{ for } \quad i\neq j \tag{1.6.43a}$$

$$w_{ii}^{(n)}=-\sum_{j=1,\, j\neq i}^{N}w_{ij}^{(n)}, \quad \text{for} \quad n=2,3,\ldots,N-1;\ i,j=1,2,\ldots,N \tag{1.6.43b}$$

$$\overline{w}_{ij}^{(m)}=m\left(a_{ij}^{y}.\overline{w}_{ii}^{(m-1)}-\frac{\overline{w}_{ij}^{(m-1)}}{y_i-y_j}\right), \text{ for } \quad i\neq j \tag{1.6.44a}$$

$$\overline{w}_{ii}^{(m)}=-\sum_{j=1,\, j\neq i}^{M}\overline{w}_{ij}^{(m)}, \text{ for } m=2,3,\ldots,M-1;\ i,j=1,2,\ldots,M \tag{1.6.44b}$$

where $w_{ij}^{(n)}$ are the weighting coefficients of the nth order derivative of $f(x,y)$ with respect to *x* at (x_i,y_j), namely $f_x^{(n)}(x_i,y_j)$ and $\overline{w}_{ij}^{(m)}$ are the weighting coefficients of the mth order derivative of $f(x,y)$ with respect to *y* at (x_i,y_j), namely $f_y^{(m)}(x_i,y_j)$. They satisfy

$$f_x^{(n)}(x_i,y_j)=\sum_{k=1}^{N}w_{ik}^{(n)}.\,f(x_k,y_j), \tag{1.6.45a}$$

$$f_y^{(m)}(x_i,y_j)=\sum_{k=1}^{M}\overline{w}_{jk}^{(m)}.\,f(x_i,y_k), \tag{1.6.45b}$$

$$\text{for} \quad i=1,2,\ldots,M;\quad j=1,2,\ldots,M;\quad n=1,2,\ldots,N;\quad m=1,2,\ldots,M-1$$

1.6.10 Methodology for Solving a PDE by DQM

The partial differential equations (PDEs) are divided into two parts. First, time dependent (unsteady state) and second, time independent (steady state). We will discuss the methodology of DQM for both parts.

1.6.10.1 Time dependent PDEs

Most of the engineering problems are governed by time-dependent partial differential equations. The spatial derivatives are discretized by the DQM whereas the time derivatives are discretized by low order finite difference schemes.

For the general case, we consider a time-dependent PDE as follows

$$\frac{\partial w}{\partial t}+\ell(w)=g, \tag{1.6.46}$$

where $\ell(w)$ is a differential operator containing all the spatial derivatives and g is a given function. Equation (1.6.46) should be specified with proper initial and boundary conditions for the solution to a specific problem. By using DQM to discretize the spatial derivatives in the differential operator $\ell(w)$ and applying equation at all the interior grid points, we can obtain a set of ordinary differential equations (ODEs)

$$\frac{d\{W\}}{dt}+L\{W\}=\{G\}, \tag{1.6.47}$$

where $\{W\}$ is a vector representing a set of unknown functional values at all the interior points, $L\{W\}$ is a vector resulting from the differential quadrature discretization, $\{G\}$ is a known vector arising from the function g and the given initial and boundary conditions.

For a time-dependent problem, equation (1.6.47) constitutes a standard form of ODEs. The time derivative in equation (1.6.47) can be discretized explicitly or implicitly using low order finite difference schemes.

1.6.10.2 Time independent PDEs

To demonstrate the solution techniques for the time independent PDEs, we consider the solution of a two-dimensional Poison's equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(x,y) \tag{1.6.48}$$

on a rectangle domain, $0\le x\le L, 0\le y\le H$, where $f(x,y)$ is a given function. Suppose that the equation (1.6.48) is subjected to a Dirichlet boundary condition, that is, w is given at the boundary points. Let N and M be the number of grid points in the x and y direction respectively, $b_{i,j}$ and $\bar{b}_{i,j}$ be the differential quadrature weighting coefficients of the second order derivatives of w with respect to x and y respectively. At any point (x_i, y_j), equation (1.6.48) can be discretized as

$$\sum_{k=1}^{N} b_{i,k} w_{k,j} + \sum_{k=1}^{M} \bar{b}_{j,k} w_{i,k} = f_{i,j}, \tag{1.6.49}$$

where $w_{i,j}$ and $f_{i,j}$ represent the values of w and f at the grid point (x_i, y_j) respectively. By shifting the functional values at the boundary points within the summation sign to the right hand side, equation (1.6.49) can be written as

$$\sum_{k=2}^{N-1} b_{i,k} w_{k,j} + \sum_{k=2}^{M-1} \bar{b}_{j,k} w_{i,k} = s_{i,j}, \tag{1.6.50}$$

$$\text{for} \quad 2 \le i \le N-1, \; 2 \le j \le M-1$$

where $\quad s_{i,j} = f_{i,j} - \left(b_{i,1} w_{1,j} + b_{i,N} w_{N,j} + \bar{b}_{j,1} w_{i,1} + \bar{b}_{j,M} w_{i,M} \right).$

Equations (1.6.42) is a set of differential quadrature algebraic equations, which can be written in matrix form

$$[A]\{w\} = \{s\}, \tag{1.6.51}$$

where $\{w\}$ is a vector of unknown functional values at all the interior points given by

$$\{w\} = \left(w_{2,2}, w_{2,3}, \ldots, w_{2,M-1}, w_{3,2}, \ldots., w_{3,M-1}, \ldots, w_{N-1,2}, \ldots, w_{N-1,M-1} \right)^T,$$

and $\{s\}$ is known vector given by

$$\{s\} = \left(s_{2,2}, s_{2,3}, \ldots, s_{2,M-1}, s_{3,2}, \ldots., s_{3,M-1}, \ldots, s_{N-1,2}, \ldots, s_{N-1,M-1} \right)^T.$$

The dimension of the matrix $[A]$ is $(N-2)(M-2) \times (N-2)(M-2)$. Equation (1.6.51) can be solved by direct methods or iterative methods.

1.6.10.3 Solution of a System of ODEs

If we consider the ordinary differential equation of the form

$$\frac{dY}{dt} = f(Y), \tag{1.6.52}$$

with given initial condition $Y(0)$.

Pike and Roe [81] have given the following four-stage RK4 scheme when f is independent of t.

$$\begin{cases} Y = Y_n \\ G = f(Y) \end{cases} \tag{1.6.53}$$

$$\begin{cases} Y = Y + \dfrac{\Delta t}{4}.G \\ G = f(Y) \end{cases} \tag{1.6.54}$$

$$\begin{cases} Y = Y + \dfrac{\Delta t}{3}.G \\ G = f(Y) \end{cases} \tag{1.6.55}$$

$$\begin{cases} Y = Y + \dfrac{\Delta t}{2}.G \\ G = f(Y) \end{cases} \tag{1.6.56}$$

$$Y_{n+1} = Y + \Delta t.G \tag{1.6.57}$$

1.7 Outline of Thesis

In the thesis, an attempt has been made to analyse the exact and numerical solutions of some nonlinear partial differential equations. To this effect we used the Lie group analysis techniques for the systematic search of exact solutions, and DQM method for the numerical solutions. The following six physically important partial differential equations are studied in this thesis.

1. Variable Coefficients Benjamin-Bona-Mahony-Burger (BBMB) Equation.
2. Coupled Short Pulse (CSP) Equation with its Variable Coefficients Form.
3. (2+1)-Dimensional Diffusion-Advection (DA) Equation with Variable Coefficients.
4. Variable Coefficients Coupled KdV-Burgers' Equation
5. Generalized Fitzhugh-Nagumo Equation with Time-Dependent Coefficients.
6. Two Dimensional Quasilinear Hyperbolic Equation.

In **Chapter 2**, a Painlevé analysis of variable-coefficients Benjamin-Bona-Mahony-Burger (BBMB) equation

$$Y_t - e(t)Y_{xxt} - f(t)Y_{xx} + g(t)Y\,Y_x + h(t)Y_x = 0, \tag{1.7.1}$$

where $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are arbitrary time-dependent coefficients has been performed to study the Painlevé properties. We derived the infinitesimals that admit the classical symmetry group. The reduced ordinary differential equation is further studied and many new families of exact solutions are successfully obtained.

Furthermore three-dimensional profile plots of some of the solutions like periodic, kink–antikink solitons etc. can easily be derived from the general results.

In **Chapter 3,** deals with comparative study of travelling wave and numerical solutions for the Coupled Short Pulse (CSP) equation

$$u_{xt} = \alpha_1 u + \beta_1 u \left(u^2\right)_x + \delta_1 \left(u^2 + w^2\right) u_{xx}$$
$$w_{xt} = \alpha_2 w + \beta_2 w \left(w^2\right)_x + \delta_2 \left(u^2 + w^2\right) w_{xx}, \tag{1.7.2}$$

where α_i, β_i and δ_i $i=1, 2$ are arbitrary constant coefficients. Using the invariance group properties of the governing system of partial differential equations, admitting Lie group of point transformations with commuting infinitesimal generators, some appropriate canonical variables are characterized that transform the equations at hand to an equivalent system of ordinary differential equations and some physically important travelling wave solutions of CSP equations are constructed which are compared with numerical solutions using classical fourth order Runge-Kutta scheme.

To understand more about physical phenomena of CSP equation, we consider the variable variant Coupled Short Pulse (VCSP) equation

$$u_{xt} = \alpha_1(t)\, u + \beta_1(t)\, u \left(u^2\right)_x + \delta_1(t) \left(u^2 + w^2\right) u_{xx},$$
$$w_{xt} = \alpha_2(t) w + \beta_2(t)\, w \left(w^2\right)_x + \delta_2(t) \left(u^2 + w^2\right) w_{xx}, \tag{1.7.3}$$

where $\alpha_i(t)$, $\beta_i(t)$ and $\delta_i(t)$ $i=1, 2$ are arbitrary time-dependent coefficients. The method of Lie symmetry of differential equations is utilized to obtain new exact and numerical solutions of VCSP equations. Six cases arise depending on the nature of the Lie symmetry generator. In all cases, we find reductions in terms of system of ODEs. The system of ordinary differential equations is further solved in general to obtain exact solutions. Now with respect to the some reduced system of ODEs we are unable to obtain the nontrivial exact solutions for the VCSP equation so we present the numerical solutions of main system by solving reduced ODEs with the help of RK4 method.

In **Chapter 4,** we study the following (2+1)-dimensional Diffusion-Advection (DA) equation

$$u_t = \alpha(t)\, u_{yy} - \beta(t) \left[K(u)\right]_x, \tag{1.7.4}$$

where $K(u)$ is an arbitrary functions of *u* and $\alpha(t),\ \beta(t)$ are arbitrary functions of *t*. On carrying over the Lie group method to DA equation, we have derived the groups of transformations admitted by the equation under consideration. Consequently, by using the symmetries involving arbitrary functions, the equation has been reduced to (1+1)-dimensional PDE with variable coefficients which is again studied for its symmetries, reductions, exact and numerical solutions. Then, with the use of extended (G'/G)-expansion method, more explicit non traveling wave solutions involving arbitrary parameters are found out, which are expressed in terms of hyperbolic functions, the trigonometric functions and rational functions.

In **Chapter 5,** with the help of classical Lie group method, we study a variable coefficient coupled KdV-Burgers' equation,

$$\begin{aligned} &u_t + \alpha_1(t)\, u u_x + \beta_1(t)\, w_{xx} + \delta_1(t)\, u_{xxx} = 0, \\ &w_t + \alpha_2(t) w w_x + \beta_2(t)\, w_{xx} + \delta_2(t)\, u_{xxx} = 0, \end{aligned} \tag{1.7.5}$$

where $\alpha_i(t), \beta_i(t)$ and $\delta_i(t)$ $i=1, 2$ are arbitrary time-dependent coefficients and get its infinitesimal generator, commutation table of Lie algebra. An optimal list of inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra is constructed and used for Lie reductions. This method reduces the number of independent variables by one. Therefore, for a system of PDEs with two independent variables we applied the method once to yield a system represented by ODEs. Further, modified (G'/G)-expansion and RK4 methods are successfully used to obtain exact and numerical solutions. The results are illustrated graphically for different parameters.

In **Chapter 6,** we have developed a numerical method based on polynomial differential quadrature method to solve time dependent Generalized Fitzhugh-Nagumo equation

$$u_t + \nu(t)\, u_x - \mu(t)\, u_{xx} - \eta(t) u\, (1-u)\, (\rho - u) = 0; \quad (x, t) \in [A, B]\times[0, T], \tag{1.7.6}$$

subject to the boundary conditions

$$u\,(A, t) = g_1(t),\ u\,(B, t) = g_2(t), \qquad t \in [0,\ T] \tag{1.7.7}$$

and the initial condition

$$u\,(x,0) = f\,(x), \qquad x \in [A,\ B] \qquad (1.7.8)$$

where $\nu(t)$, $\mu(t)$, and $\eta(t)$ are arbitrary real-valued functions of t. The equation has various applications in the fields of flame propagation, logistic population growth, neurophysiology, branching brownian motion process, autocatalytic chemical reaction and nuclear reactor theory. In the construction of numerical method, spatial derivatives are discretized by polynomial differential quadrature method (PDQM) which leads to a system of first order initial value problem. For total discretization, we discretize the system of first order initial value problem by optimal four-stage, order three strong stability-preserving time-stepping Runge-Kutta (SSP-RK43) scheme. Finally, the efficiency of the method is shown with the numerical solutions of three test examples.

Chapter 7 is devoted to the numerical simulation of two dimensional Quasilinear Hyperbolic equations

$$\frac{\partial^2 u}{\partial t^2} + A(x,y,t,u)\frac{\partial^2 u}{\partial x^2} + B(x,y,t,u)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,t,u,u_x,u_y,u_t\right) = 0, \qquad (x,y,t) \in [0,1]\times[0,1]\times[t>0] \qquad (1.7.9)$$

with initial conditions

$$u(x,y,0) = g_1(x,y), \quad (x,y) \in [0,1]\times[0,1],$$
$$u_t(x,y,0) = g_2(x,y), \quad (x,y) \in [0,1]\times[0,1], \qquad (1.7.10)$$

subject to dirichlet and neumann boundary conditions by polynomial differential quadrature method. The equation (1.7.9) is assumed to be hyperbolic if $A(x,y,t,u) > 0$ and $B(x,y,t,u) > 0$ in the solutions region $\Omega \equiv \{(x,y,t) : (x,y,t) \in [0,1]\times[0,1]\times[t>0]\}$. To check the efficiency and accuracy of the method, five examples of quasilinear hyperbolic equation are included with their numerical solutions, L_∞, RMS and L_2 errors and comparisons are done with the exact solutions. The main advantage of the present scheme is that the scheme gives very accurate and similar results to the exact solutions by choosing less number of grid points and the problem can be solved up to big time. The proposed method produces better results as compared to the results obtained by almost all the schemes available in the literature, and approaching to the exact solutions.

Chapter 2

Variable Coefficients Benjamin-Bona-Mahony-Burger (BBMB) Equation

2.1 Introduction

In recent years, nonlinear PDEs have been used to model many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics, optical fiber, and geochemistry. Thus, it is important to investigate the exact explicit solutions of nonlinear PDEs.

One such nonlinear PDE is the celebrated Benjamin-Bona-Mahony-Burger (BBMB) equation

$$Y_t - Y_{xxt} - \alpha\, Y_{xx} + Y\, Y_x + Y_x = 0, \tag{2.1.1}$$

where $Y(x,t)$ represents the fluid velocity in the horizontal direction x, α is a positive constant. It has been proposed in [182], as a model to study the unidirectional long waves of small amplitudes in water, which is an alternative to the Korteweg-de Vries equation. For more details on both the mathematical theory and the physical significance of equation (2.1.1), we refer the reader to the references [37, 182]. We call equation (2.1.1) the BBMB equations, but it is proposed neither by Benjamin, Bona and

A part of this chapter has been published in **Communication in Theoretical Physics**, 60 (2013) 175-182

Mahony nor by Burgers. Many authors studied equation (2.1.1) using so many aspects [55, 67, 89, 91].

The physical phenomena in which many nonlinear integrable equations with constant coefficients arise tend to highly idealized due the presence of constant coefficients; therefore, equations with variable coefficients may provide various models for real physical phenomena, such as, in the propagation of small-amplitude surface waves, which runs on large channels of slowly varying depth variable coefficients of nonlinear integrable evolution equations [93, 94]. As there are choices for parameters, the variable-coefficients nonlinear equations can be considered as generalization of the constant coefficients equations.

In this paper, we study variable coefficients version of Benjamin-Bona-Mahony-Burger equation

$$Y_t - e(t)Y_{xxt} - f(t)Y_{xx} + g(t)Y\,Y_x + h(t)Y_x = 0, \tag{2.1.2}$$

where $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are arbitrary time-dependent coefficients. When $f(t)=0$, $h(t)=1$ and $e(t)=1$ equation (2.1.2) is the alternative regularized long wave equation proposed by Peregrine [37] and Benjamin [182].

In the physical case, the dispersive effect of equation (2.1.2) is the same as the variable coefficients Benjamin-Bona-Mahony (BBM) equation

$$Y_t - e(t)Y_{xxt} + g(t)Y\,Y_x + h(t)Y_x = 0, \tag{2.1.3}$$

while the dissipative effect is the same as the variable coefficient Burgers equation

$$Y_t - f(t)Y_{xx} + g(t)Y\,Y_x + h(t)Y_x = 0. \tag{2.1.4}$$

The study of nonlinear models for integrable properties is one of the important works in nonlinear science. Therefore, in this paper, we check the Painlevé property of the variable coefficient Benjamin-Bona-Mahony-Burger (BBMB) equation (2.1.2) firstly, and then the symmetries of equation (2.1.2) are considered. The exact solutions generated from the symmetries are presented.

The outline of this chapter is as follows, In section 2.2, we study the Painlevé property of BBMB equation. In section 2.3, the classical Lie method is utilized to obtain the optimal system of sub-algebras for equation (2.1.2), and group invariant solutions of the nonlinear equation (2.1.2) are sought. This analysis generates a rich class of solutions. A number of cases arise depending on the nature of the Lie symmetry generator. This analysis demonstrates the value of Lie analysis of differential equations for this application. In section 2.4, we derive some explicit exact solutions of the nonlinear equation (2.1.2). In the last section, some conclusions are drawn.

2.2 Painlevé Analysis

Ablowitz et al. [117] stated that when all the ODEs obtained by exact similarity transforms from a given PDE have the Painlevé property, then the PDE is "integrable". The definition of "Painlevé property" of the ODE (WTC) was extended to the case of PDE by Weiss et al. [87]. Briefly, a partial differential equation has the Painlevé property [20, 39] when the solutions of the PDE are "single-valued" in the neighbourhood of all its movable singularities. Before finding some exact solutions of equation (2.1.2), we briefly discuss its Painlevé property by means of the standard WTC approach. The Laurent expansion of the function $Y = Y(x, t)$ about a singular manifold $u(x,t) = 0$ is

$$Y(x,t) = (u(x,t))^{\alpha} \sum_{k=0}^{\infty} Y_k(x,t)\,(u(x,t))^{k}, \tag{2.2.1}$$

whereby one makes the ansätz

$$Y(x,t) = Y_0(x,t)\,[u(x,t)]^{\alpha}. \tag{2.2.2}$$

Substitution of (2.2.2) into (2.1.2) and balancing the highest derivative with the nonlinear term gives

$$\alpha = -2 \quad \text{and} \quad Y_0(x,t) = 12\,\frac{e(t)}{g(t)}\,u_x\,u_y\,. \tag{2.2.3}$$

The resonances are determined by setting

$$Y(x,t) = Y_0(x,t)\,u^{-2}(x,t) + Y_r(x,t)u^{r-2}(x,t), \tag{2.2.4}$$

and balancing the most singular term in equation (2.1.2), giving recursion relation

$$Y_r\left[2g(t)Y_0(x,t)u_x(x,t) - g(x)Y_0(x,t)(r-2)u_x(x,t)\right] +$$
$$Y_r\left[e(t)(r-2)(r-3)(r-3)(r-4)u_t(x,t)u_x^2(x,t)\right] = H(Y_O, Y_1, \ldots, Y_{r-1}, u_t, u_x, \ldots). \tag{2.2.5}$$

Using (2.2.3) into (2.2.5), we find that the resonances for this branch are $r = -1$, 4 and 6. The resonance at $J = -1$ corresponds to the arbitrary function defining the singularity manifold for the (BBMB) equation. After detailed calculation, we find that compatibility conditions at J = 4 and 6 are satisfied identically. According to WTC approach, equation (2.1.2) possesses "Painlevé property".

2.3 Lie Symmetric Analysis for BBMB Equation

Let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned} x^* &= x + \varepsilon\, \eta(x,t,Y) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\, \psi(x,t,Y) + O(\varepsilon^2), \\ Y^* &= Y + \varepsilon\, \phi(x,t,Y) + O(\varepsilon^2), \end{aligned} \tag{2.3.1}$$

with a small parameter ε.

The BBMB equation (2.1.2) is said to admit a symmetry generated by the vector field

$$V = \eta(x,t,Y)\frac{\partial}{\partial x} + \psi(x,t,Y)\frac{\partial}{\partial t} + \phi(x,t,Y)\frac{\partial}{\partial Y}, \tag{2.3.2}$$

if it is left invariant by the transformation $(x,\, t,\, Y) \to (x^*,\, t^*,\, Y^*)$.

Applying the third prolongation $\mathrm{pr}^{(3)}V$ [202] to (2.1.2) results in an overdetermined system of linear partial differential equations. After some straightforward and lengthy calculations, the resultant overdetermined system gives the following infinitesimal elements

$$\begin{aligned} \eta &= a\,x + b \\ \phi &= cY + d \\ \psi &= 2a\,\frac{e(t)}{e'(t)}, \end{aligned} \tag{2.3.3}$$

where a, b, c, and d are arbitrary constants and the function $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are governed by the following conditions

$$\frac{d}{dt}[h(t)\psi] - a\,h(t) + g(t)d = 0$$

$$\frac{d}{dt}[g(t)\psi] + (c-a)g(t) = 0$$

$$\frac{d}{dt}\left[f(t)\psi\right] - 2a\, f(t) = 0. \tag{2.3.4}$$

The Lie algebra associated with equation (2.1.2) consists of following four vector fields

$$V_1 = x\frac{\partial}{\partial x} + 2\frac{e(t)}{e'(t)}\frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x},$$

$$V_3 = Y\frac{\partial}{\partial Y}, \qquad V_4 = \frac{\partial}{\partial Y}. \tag{2.3.5}$$

The commutator relations between these vector fields are given by

$\left[V_i, V_j\right] = V_i V_j - V_j V_i$, where $i, j = 1, \ldots, 4$.

It immediately follows that

$$[V_1, V_1] = [V_2, V_2] = [V_3, V_3] = [V_4, V_4] = 0$$

$$[V_1, V_3] = [V_1, V_4] = [V_2, V_3] = [V_2, V_4] = [V_3, V_1] = [V_3, V_2] = [V_4, V_1] = [V_4, V_2] = 0$$

$$[V_1, V_2] = -[V_2, V_1] = -V_2$$

$$[V_3, V_4] = -[V_4, V_3] = -V_4.$$

The adjoint representation of a Lie group on the Lie algebra is given as the Lie series

$$\mathrm{Ad}\left(\exp(\varepsilon V_i)\right)V_j = V_j - \varepsilon\left[V_i, V_j\right] + \frac{\varepsilon^2}{2}\left[V_i, \left[V_i, V_j\right]\right] - \cdots,$$

where ε is a parameter and $\left[V_i, V_j\right]$ is the commutator of the Lie algebra.

It follows that the adjoint representations of the vector fields for the BBMB equation is

$\mathrm{Ad}\left(\exp(\varepsilon V_i)\right)V_i = V_i$, where $i = 1, \ldots, 4$

and

$$\mathrm{Ad}(\exp(\varepsilon V_1))V_2 = e^{\varepsilon} V_2, \quad \mathrm{Ad}(\exp(\varepsilon V_1))V_3 = V_3, \quad \mathrm{Ad}(\exp(\varepsilon V_1))V_4 = V_4,$$

$$\mathrm{Ad}(\exp(\varepsilon V_2))V_1 = V_i - \varepsilon V_2, \quad \mathrm{Ad}(\exp(\varepsilon V_2))V_3 = V_3, \quad \mathrm{Ad}(\exp(\varepsilon V_2))V_4 = V_4,$$

$$\mathrm{Ad}(\exp(\varepsilon V_3))V_1 = V_1, \quad \mathrm{Ad}(\exp(\varepsilon V_3))V_2 = V_2, \quad \mathrm{Ad}(\exp(\varepsilon V_3))V_4 = e^{\varepsilon} V_4,$$

$$\mathrm{Ad}(\exp(\varepsilon V_4))V_1 = V_1, \quad \mathrm{Ad}(\exp(\varepsilon V_4))V_2 = V_2, \quad \mathrm{Ad}(\exp(\varepsilon V_4))V_3 = V_3 - \varepsilon V_4.$$

The classification of one-dimensional subalgebras of the whole symmetry algebra (2.3.5) is done by an inductive approach [143]. If $V = \sum_{i=1}^{4} a_i V_i$ is non zero vector field, we will simplify as many of the coefficients a_i as possible by acting it to various adjoint transformations to obtain the subalgebra.

The types of one-dimensional subalgebras of equation (2.1.2) are as follows

(i) $\langle V_1 + \alpha V_3 \rangle$,

(ii) $\langle V_2 + \beta V_3 \rangle$,

(iii) $\langle V_3 \rangle$,

(iv) $\langle V_4 \rangle$, (2.3.6)

where α and β are arbitrary constants.

2.3.1 Reduced Systems and Exact Solutions

In the following, some exact solutions for the reduced systems are studied for each type of subalgebra. These solutions are obtained by solving the associated Lagrange equations [26, 100, 132]

$$\frac{dx}{\eta} = \frac{dt}{\psi} = \frac{dY}{\phi}. \tag{2.3.7}$$

2.3.1.1 Subalgebra $\langle V_1 + \alpha V_3 \rangle$

The invariants associated with the symmetry operators, can be obtained by integrating the characteristic equation (2.3.7). In this type, the form of invariant solution and coefficient functions are as follows

$$Y(x,t) = x^{\alpha} F(\xi),$$

$$g(t) = k_2 \frac{[e(t)]^{\frac{-1-\alpha}{2}} e'(t)}{2}, \quad h(t) = k_1 \frac{[e(t)]^{\frac{-1}{2}} e'(t)}{2} \quad \text{and} \quad f(t) = k_3 \frac{e'(t)}{2}, \tag{2.3.8}$$

where $\xi = \dfrac{x^2}{e(t)}$ is an invariant of the symmetry. Substitution of (2.3.8) into (2.1.2) gives the following ODE

$$\left[\left(-\alpha^2 + \alpha\right)k_3 + \alpha \xi^{\frac{1}{2}} k_1\right] F + \left[-2\xi^2 + \left(2\alpha^2 + 6\alpha - 4\alpha k_3 - 2k_3 + 4\right)\xi + 2k_1 \xi^{\frac{3}{2}}\right] F'$$

$$+ \left[(4k_3 + 8\alpha + 20)\right]\xi^2 F'' + \alpha k_3 \xi^{\frac{\alpha+1}{2}} F^2 + 2k_2 \xi^{\frac{\alpha+3}{2}} F F' + 8\xi^3 F''' = 0. \tag{2.3.9}$$

To solve equation (2.3.9), we consider two cases

Case (i) $\alpha = 0,$ the reduced ODE (2.3.9) is as follows

$$\left[-2\xi^2+4\xi-2k_3\xi+2k_1\xi^{\frac{3}{2}}\right]F'+\left[20+4k_3\right]\xi^2F''+2k_2\xi^{\frac{3}{2}}FF'+8\xi^3F'''=0. \qquad (2.3.10)$$

Thus, by solving equation (2.3.10) and reverting back to the original variables, we obtain the following group-invariant solutions of the BBMB equation (2.1.2) as follows

Solution 1:-

$$\begin{aligned} Y(x,t)=&\frac{1}{25}\frac{25+55k_3+25\left(\frac{x^2}{e(t)}\right)+3k_3^{\ 2}-25k_1\left(\frac{x^2}{e(t)}\right)^{\frac{1}{2}}}{k_2\left(\frac{x^2}{e(t)}\right)^{\frac{1}{2}}} \\ &+\frac{6}{25}\frac{\left(25+10k_3+k_3^{\ 2}\right)\tanh\left(-C_1+\frac{1}{4}+\frac{1}{20}k_3\right)}{k_2\left(\frac{x^2}{e(t)}\right)^{\frac{1}{2}}} \\ &-\frac{3}{25}\frac{\left(25+10k_3+k_3^{\ 2}\right)\tanh\left(-C_1+\frac{1}{4}+\frac{1}{20}k_3\right)^2}{k_2\left(\frac{x^2}{e(t)}\right)^{\frac{1}{2}}}, \end{aligned} \qquad (2.3.11)$$

where k_1, k_2, k_3 and C_1 are arbitrary constant and $e(t)$ is an arbitrary function of t.

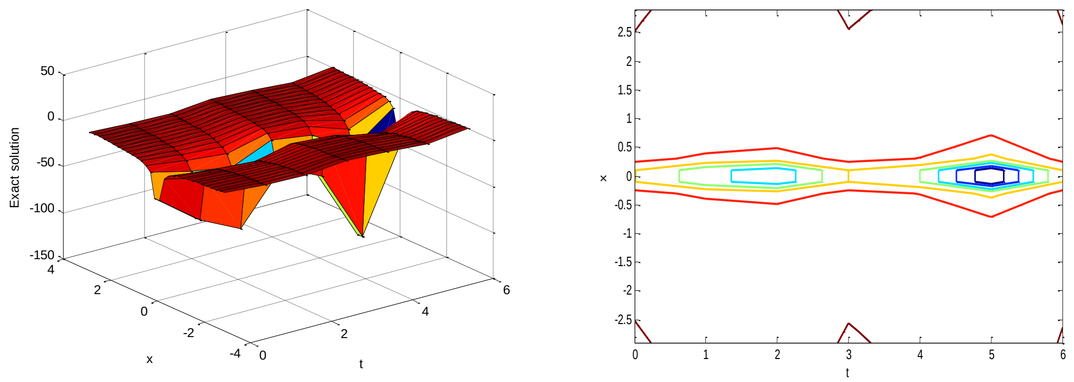


Figure 2.1: The inverted two solitary solution (2.3.11) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $k_1 = k_2 = k_3 = C_1 = 1$ and $e(t) = \sec^2(t)$.

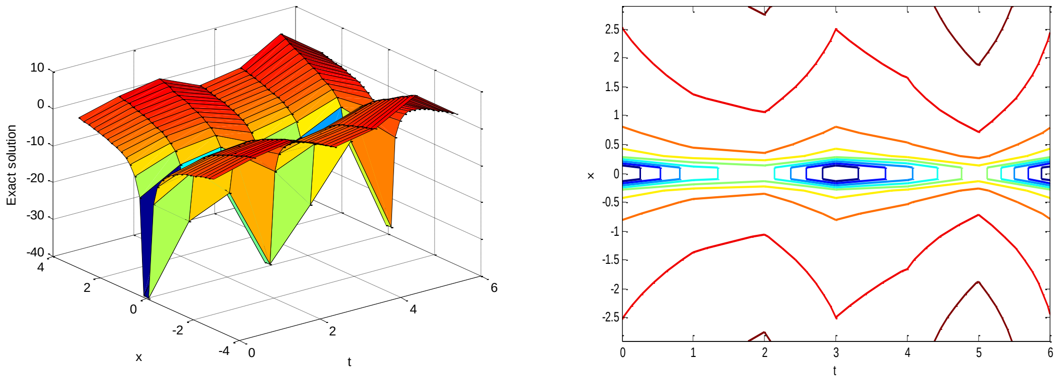


Figure 2.2: The inverted three Solitary solution (2.3.11) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $k_1 = k_2 = k_3 = C_1 = 1$ and $e(t) = \cos^2(t)$.

Solution 2:-

$$Y(x,t)=C_3-\frac{96\left(\frac{x^2}{e(t)}\right)^{\frac{3}{2}} C_2^{\,2}\tanh\left(C_1+C_2\left(\frac{x^2}{e(t)}\right)\right)}{k_2}-\frac{48\left(\frac{x^2}{e(t)}\right)^{\frac{3}{2}} C_2^{\,2}\tanh\left(C_1+C_2\left(\frac{x^2}{e(t)}\right)\right)^2}{k_2}, \tag{2.3.12}$$

where $k_1=-\dfrac{C_3\,k_2^{\,2}\left(\frac{x^2}{e(t)}\right)^{\frac{1}{2}}+7+20\,C_2\frac{x^2}{e(t)}-\frac{x^2}{e(t)}-48\left(\frac{x^2}{e(t)}\right)^2 C_2^{\,2}}{\left(\frac{x^2}{e(t)}\right)^{\frac{1}{2}}}$, $k_3=-20\,C_2\dfrac{x^2}{e(t)}-5$

and k_2, C_1, C_2, C_3 are arbitrary constants and $e(t)$ be an arbitrary function of t.

Solution 3:-

$$Y(x,t)=C_3-\frac{48\left(\frac{x^2}{e(t)}\right)^{\frac{3}{2}} C_2^{\,2}\tanh\left(C_1+C_2\frac{x^2}{e(t)}\right)^2}{k_2}, \tag{2.3.13}$$

where $k_1=-\dfrac{C_3\,k_2\left(\frac{x^2}{e(t)}\right)^{\frac{1}{2}}-32\left(\frac{x^2}{e(t)}\right)^2 C_2^{\,2}-\frac{x^2}{e(t)}+7}{\left(\frac{x^2}{e(t)}\right)^{\frac{1}{2}}}$, $k_3=-5$ and k_2, C_1, C_2 are arbitrary constants; also $e(t)$ is an arbitrary function of t.

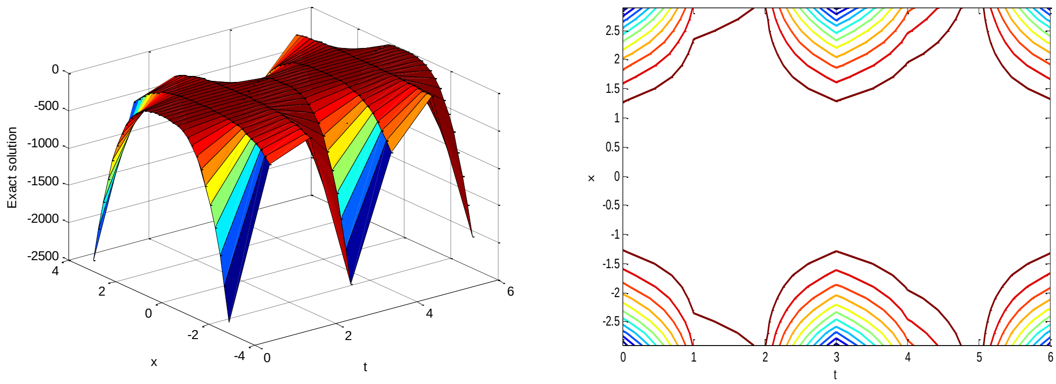


Figure 2.3: The distribution of some singularities of the solution (2.3.13) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_1 = C_2 = C_3 = -1$; $k_2 = 0.5$ and $e(t) = \sec^2(t)$.

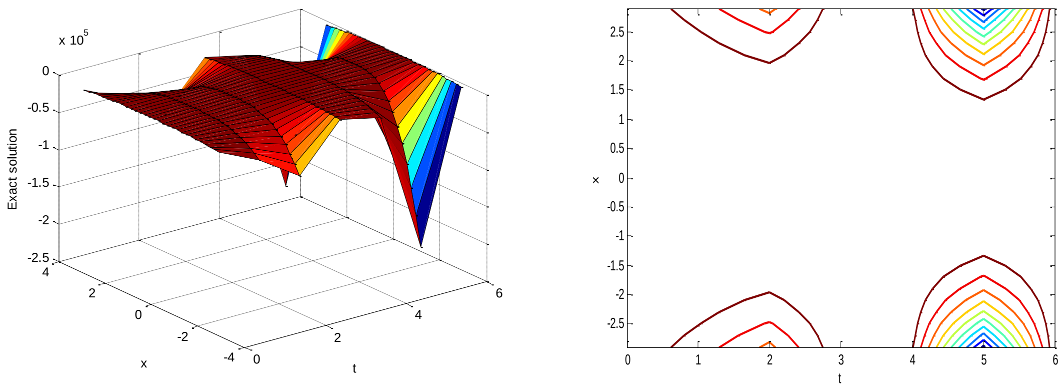


Figure 2.4: The damped oscillatory kink wave solution (2.3.13) in form of 3D plot (left) and contour plot (Projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_1 = C_2 = C_3 = -1$; $k_2 = 0.5$ and $e(t) = \cos^2(t)$.

Case (ii) $\alpha = 1$

The reduced ODE is as follows

$$k_1\xi^{\frac{1}{2}}F + \left[(12-6k_3)\xi - 2\xi^2 + 2k_1\xi^{\frac{3}{2}}\right]F' + [28+4k_3]\xi^2 F'' + k_3\xi F^2 + 2k_2\xi^2 F F'$$

$$+8\xi^3 F''' = 0. \qquad (2.3.14)$$

Solving this equation (2.3.14) and reverting back to the original variables, we obtain the following group-invariant solutions of the BBMB equation (2.1.2) as follows

Solution 1:-

$$Y(x,t)=\frac{1}{25}\frac{-3+25\dfrac{x^2}{e(t)}}{k_2}-\frac{294}{25}\frac{\tanh\left(C_1-\dfrac{7}{20}\right)}{k_2\dfrac{x^2}{e(t)}}-\frac{147}{25}\frac{\tanh\left(C_1-\dfrac{7}{20}\right)^2}{k_2\dfrac{x^2}{e(t)}}, \qquad (2.3.15)$$

where $k_1=0, k_3=0$ and k_2, C_1 are arbitrary constants and $e(t)$ is arbitrary function of *t*.

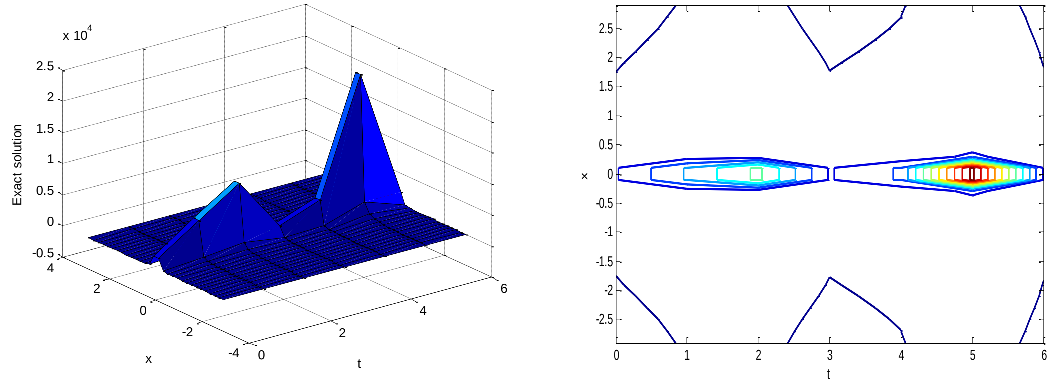


Figure 2.5: The two Solitary solution (2.3.15) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time *t* = 0, 1, 2, 3, 4, 5, 6 with $C_1=1;\ k_2=-0.5$ and $e(t)=\sec^2(t)$.

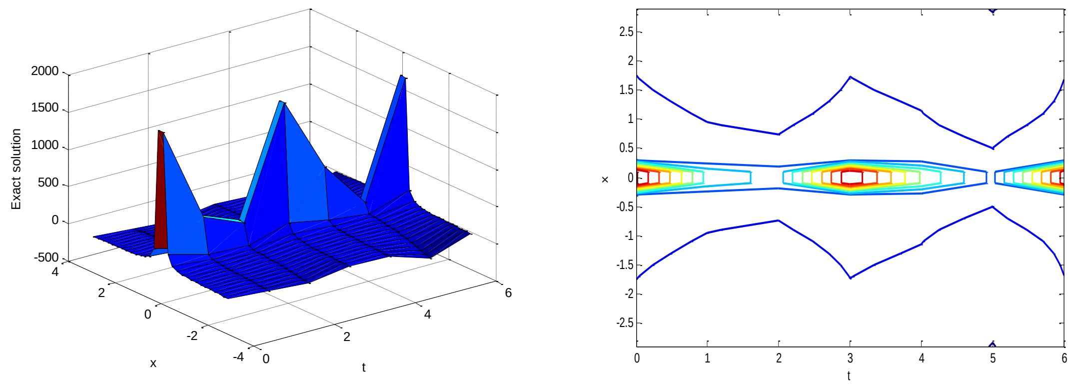


Figure 2.6: The three Solitary solution (2.3.15) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time *t* = 0, 1, 2, 3, 4, 5, 6 where $C_1=1;\ k_2=-0.5$ and $e(t)=\cos^2(t)$.

Solution 2:-

$$Y(x,t)=-\frac{147}{50\dfrac{x^2}{e(t)}}-\frac{147}{25}\frac{\tanh\left(C_1-\dfrac{7}{20}\right)}{\dfrac{x^2}{e(t)}}-\frac{147}{50}\frac{\tanh\left(C_1-\dfrac{7}{20}\right)^2}{\dfrac{x^2}{e(t)}}, \qquad (2.3.16)$$

where $k_1 = \dfrac{147}{550}\dfrac{25\dfrac{x^2}{e(t)}+144}{\left(\dfrac{x^2}{e(t)}\right)^{\frac{1}{2}}}$, $k_3 = \dfrac{25}{44}\dfrac{x^2}{e(t)}+\dfrac{36}{11}$, $k_2 = 2$ and C_1 is an arbitrary constant and $e(t)$ is an arbitrary function of t.

Solution 3:-

$$Y(x,t) = \frac{441}{50\dfrac{x^2}{e(t)}} - \frac{147}{25}\frac{\tanh\left(C_1 - \dfrac{7}{20}\right)}{\dfrac{x^2}{e(t)}} - \frac{147}{50}\frac{\tanh\left(C_1 - \dfrac{7}{20}\right)^2}{\dfrac{x^2}{e(t)}}, \tag{2.3.17}$$

where $k_1 = \dfrac{444}{6800}\dfrac{25\dfrac{x^2}{e(t)}+144}{\left(\dfrac{x^2}{e(t)}\right)^{\frac{1}{2}}}$, $k_3 = -\dfrac{25}{544}\dfrac{x^2}{e(t)}+\dfrac{111}{136}$, $k_2 = 2$ and C_1 is an arbitrary constant and $e(t)$ is an arbitrary function of t.

2.3.1.2 Subalgebra $\langle V_2 + \beta V_3 \rangle$

To solve the equation (2.1.2) we consider the two cases under this type of subalgebra.

Case (i) $e(t) =$ constant and $\psi =$ arbitrary

The symmetry in this case gives the group invariant solution and form of the coefficient functions as follows

$$Y(x,t) = \exp\left(\frac{\beta}{k_1}\int h(t)dt\right)F(\xi),$$

$$g(t) = k_2 h(t)\exp\left(-\frac{\beta}{k_1}\int h(t)dt\right),\ h(t) = \text{arbitrary and } f(t) = k_3\, h(t), \tag{2.3.18}$$

where $\xi = x - \dfrac{1}{k_1}\int h(t)dt$ is an invariant of the symmetry. Substitution of (2.3.18) into (2.1.2) gives the ODE where 'F' satisfies

$$C_1 F + C_2 F' - C_3 F'' + C_4 F F' + C_5 F''' = 0, \tag{2.3.19}$$

where $C_1 = \dfrac{\beta}{k_1}$, $C_2 = 1 - \dfrac{1}{k_1}$, $C_3 = \dfrac{e\beta}{k_1} + k_3$, $C_4 = k_2$, $C_5 = \dfrac{e}{k_1}$, $k_1 \neq 0$ and k_2, k_3, β, e are arbitrary constants.

The solutions of the main equation (2.1.2) corresponding to the solutions of equation (2.3.19) are

Solution 1:-

$$Y(x,t) = C_9 \tanh\left(C_6 + C_7\left(x - \frac{1}{k_1}\int h(t)dt\right)\right), \tag{2.3.20}$$

where $C_1 = 0$, $C_2 = 0$, $C_3 = -\dfrac{1}{2}\dfrac{C_9\,C_4}{C_7}$, $C_5 = 0$ and C_4, C_6, C_7, C_9 are arbitrary constants and $h(t)$ is an arbitrary function of t.

Solution 2:-

$$Y(x,t) = C_8 + C_9 \tanh\left(C_6 + C_7\left(x - \frac{1}{k_1}\int h(t)dt\right)\right), \tag{2.3.21}$$

where $C_1 = 0$, $C_2 = -C_4\,C_8$, $C_3 = -\dfrac{1}{2}\dfrac{C_9\,C_4}{C_7}$, $C_5 = 0$ and C_4, C_6, C_7, C_8, C_9 are arbitrary constants and $h(t)$ is an arbitrary function of t.

Solution 3:-

$$Y(x,t) = C_8 + C_9 \tanh\left(C_6 + C_7\left(x - \frac{1}{k_1}\int h(t)dt\right)\right) + C_{10} \tanh\left(C_6 + C_7\left(x - \frac{1}{k_1}\int h(t)dt\right)\right) \tag{2.3.22}$$

where $C_1 = 0$, $C_2 = 0$, $C_3 = 0, C_4 = 0$, $C_5 = 0$ and $C_6, C_7, C_8, C_9, C_{10}$ are arbitrary constants and $h(t)$ is an arbitrary function of t.

Solution 4:-

$$Y(x,t) = C_8 + C_{10} \tanh\left(C_6 + C_7\left(x - \frac{1}{k_1}\int h(t)dt\right)\right)^2, \tag{2.3.23}$$

where $C_1 = 0$, $C_2 = -C_4\,C_8 - \dfrac{2}{3}C_4 C_{10}$, $C_3 = 0, C_5 = -\dfrac{1}{12}\dfrac{C_{10}\,C_4}{{C_7}^2}$ and C_4, C_6, C_7, C_8 C_{10} are arbitrary constants and $h(t)$ is an arbitrary function of t.

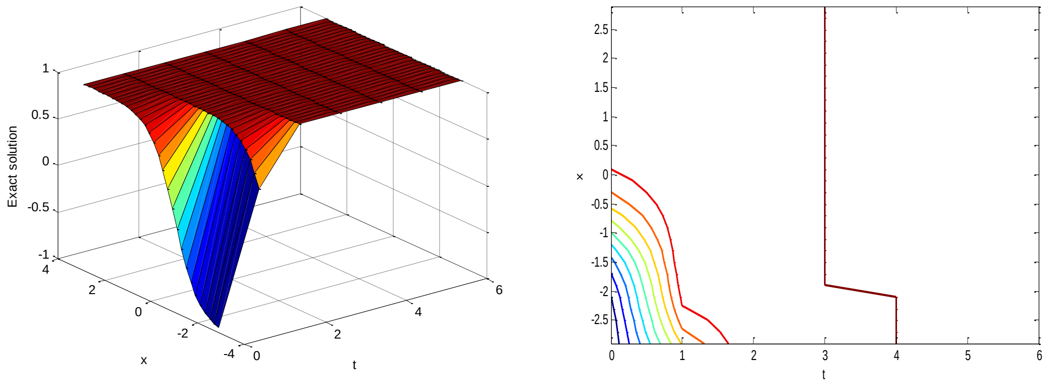


Figure 2.7: Singularity at $t = 0$ of solution (2.3.20) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_6 = C_7 = C_9 = 1;\ k_1 = -0.5$ and $h(t) = \cosh(t)$.

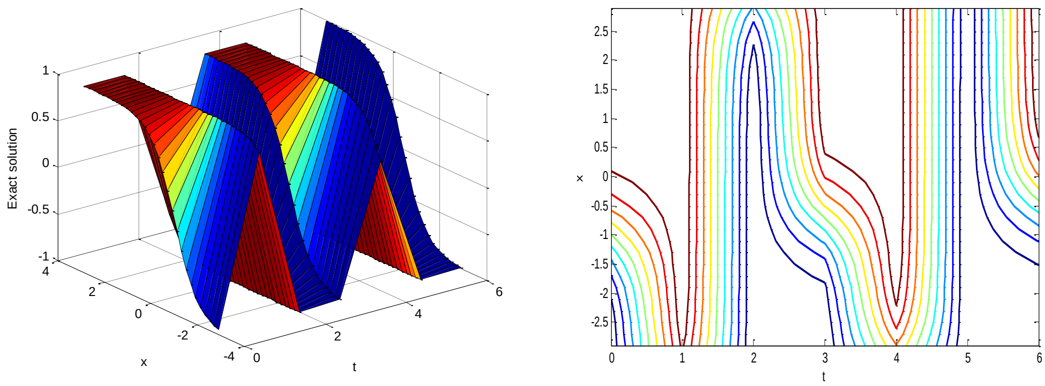


Figure 2.8: The periodic solution (2.3.20) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_6 = C_7 = C_9 = 1;\ k_1 = -0.5$ and $h(t) = \sec^2(t)$.

Solution 5:-

$$Y(x,t) = C_8 - \frac{6}{25}\,\frac{C_3^{\,2}\tanh\left(-C_6 + \dfrac{1}{10}\,\dfrac{C_3\left(x - \dfrac{1}{k1}\displaystyle\int h(t)dt\right)}{C_5}\right)}{C_5\,C_4} - \frac{3}{25}\,\frac{C_3^{\,2}\tanh\left(-C_6 + \dfrac{1}{10}\,\dfrac{C_3\left(x - \dfrac{1}{k_1}\displaystyle\int h(t)dt\right)}{C_5}\right)^2}{C_5\,C_4}, \tag{2.3.24}$$

where $C_1 = 0,\, C_2 = \frac{1}{25}\frac{-25C_8\,C_5\,C_4 + 3C_3{}^2}{C_5}$ and $C_3,\, C_4, C_5, C_6, C_8,$ are arbitrary constants and $h(t)$ is an arbitrary function of t.

Case (ii) $e(t) =$ arbitrary and $\psi\ =\ 0$

Suitable invariants and form of the coefficient functions of this case for reduction are as follows

$$Y(x,t) = F(\xi),$$

$$g(t) = \text{ arbitrary, } \ h(t) = \text{ arbitrary } \ \text{ and } \ g(t) = \text{ arbitrary,} \tag{2.3.25}$$

where $\xi = t$ represents the invariant of the symmetry which reduce equation (2.1.2) to

$$F' = 0\,. \tag{2.3.26}$$

The obvious solution of equation (2.1.2), in this case, is $Y =$ constant.

2.3.1.3 Subalgebra $\langle V_3 \rangle$

Under this type to find the solutions of equation we consider the following cases

Case (i) $e(t) =$ constant and $\psi =$ arbitrary. In this case we derive the following expression of Y

$$Y(x,t) = \exp\left(\frac{1}{k_1}\int h(t)dt\right)F(\xi),$$

$$g(t) = k_2 h(t)\exp\left(-\frac{1}{k_1}\int h(t)dt\right),\ h(t) = \text{ arbitrary } \ \text{ and } \ f(t) = k_3\,h(t), \tag{2.3.27}$$

where $\xi = x$ and F must satisfy

$$C_1\,F + F' + C_2\,F\,F' - C_3\,F'' \ = 0 \tag{2.3.28}$$

where $C_1 = \frac{1}{k_1},\, C_2 = k_2,\, C_3 = \frac{e}{k_1} + k_3,\ k_1 \neq 0$ and $k_2, k_3,\, e$ are arbitrary constants.

The solution of the main equation (2.1.2) is given by

$$Y(x,t) = -\frac{1}{C_2} + C_7\,\tanh(C_4 + C_5\,x), \tag{2.3.29}$$

where $C_1 = 0,\, C_3 = -\frac{1}{2}\frac{C_7}{C_5},\, C_5 \neq 0$ and $C_2,\, C_4$ are arbitrary constants.

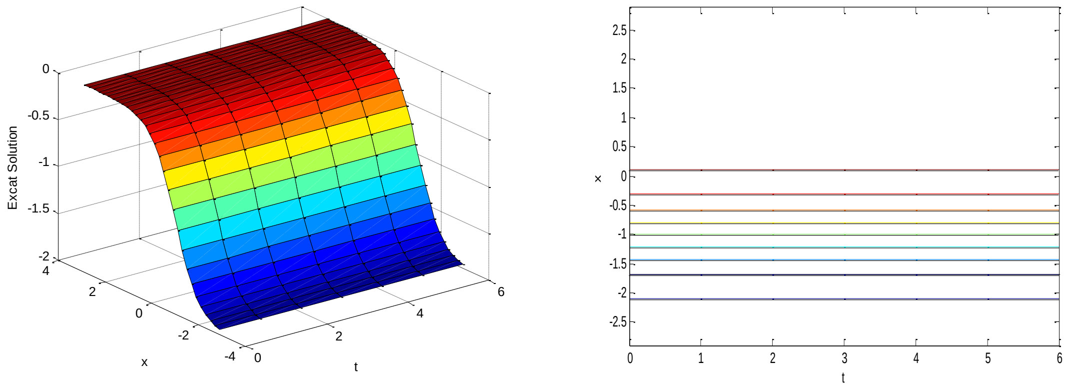


Figure 2.9: The kink wave solution (2.3.29) in form of 3D plot (left) and contour plot (projection of some of its level curves, right) at different time t = 0, 1, 2, 3, 4, 5, 6 with $C_2 = C_4 = C_5 = C_7 = 1$.

Case (ii) $e(t) =$ arbitrary and $\psi = 0$

This gives a constant solution.

2.3.1.4 Subalgebra $\langle V_4 \rangle$

It leads to a constant solution.

2.4 Discussion and Concluding Remarks

We have performed Painlevé analysis to cheek the integrability of the nonlinear variable coefficient BBMB equation. The Lie symmetry method is utilized to investigate the symmetries and invariant solutions of the BBMB equations. The vector fields of the optimal system lead to reduction of the nonlinear PDE to system of ODEs. The infinitesimal generators in the optimal system are used for reductions and exact solutions. By using these vector fields, we have found group-invariant solutions. In almost all the cases, one can choose the arbitrary function $e(t)$ or $h(t)$ along with various other arbitrary parameters, in a suitable manner, to simulate physical situations with the help of figures (2.1-2.9) governed by the equation (2.1.2).

Chapter 3

Coupled Short Pulse (CSP) Equation with its Variable Coefficients Form

3.1 Introduction

The short pulse (SP) equation, which has the form

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx} \tag{3.1.1}$$

was derived by Schäfer and Wayne [186, 198] as a model equation describing the propagation of ultra-short light pulses in silica optical fibres. Here $u = u(x, t)$ represents the magnitude of the electric field, while the subscripts t and x denote partial differentiations. The SP equation represents an alternative approach in contrast with the slowly varying envelope approximation which leads to the nonlinear Schrödinger (NLS) equation.

The form of coupled short pulse (CSP) equation [22] is

$$\begin{aligned} u_{xt} &= u + \frac{1}{6}\left(u^3\right)_{xx} + \frac{1}{2}w^2 u_{xx} \\ w_{xt} &= w + \frac{1}{6}\left(w^3\right)_{xx} + \frac{1}{2}u^2 w_{xx}. \end{aligned} \tag{3.1.2}$$

A part of this chapter has been published in **Chinese Physics B 22 (2013) 050201 (7 pp.)** and some part has been accepted in **Nonlinear Dynamics DOI 10.1007/s11071-014-1678-5.**

A CSP equation is proposed for the propagation of ultra-short pulses in optical fibers. Based on two sets of bilinear equations to a two-dimensional Toda lattice linked by a Bäcklund transformation, and an appropriate hodograph transformation, the CSP equation is derived. In particular, system (3.1.2) can be written as

$$
\begin{aligned}
u_{xt} &= u + u\left(u^2\right)_x + \frac{1}{2}\left(u^2 + w^2\right)u_{xx} \\
w_{xt} &= w + w\left(w^2\right)_x + \frac{1}{2}\left(u^2 + w^2\right)w_{xx}.
\end{aligned} \tag{3.1.3}
$$

If $u = 0$ or $w = 0$ the system is reduced to the SP equation (3.1.1).

The system (3.1.3), with constant coefficients, has been proposed by Feng in [22] for the propagation of ultra-short pulses in optical fibers. Based on two sets of bilinear equations to a two-dimensional Toda lattice linked by a Bäcklund transformation, and an appropriate hodograph transformation, the CSP equation is derived. The determinant *N*-soliton solution of the CSP equation in the parameter form and the properties of the one and two-soliton solutions were discussed in [22] in detail. The hamiltonian integrability of two-Component SP equations was given in [73]. In this chapter, we study the general form of the CSP equation

$$
\begin{aligned}
u_{xt} &= \alpha_1 u + \beta_1 u\left(u^2\right)_x + \delta_1\left(u^2 + w^2\right)u_{xx} \\
w_{xt} &= \alpha_2 w + \beta_2 w\left(w^2\right)_x + \delta_2\left(u^2 + w^2\right)w_{xx},
\end{aligned} \tag{3.1.4}
$$

where α_i, β_i and δ_i $i=1, 2$ are arbitrary constant coefficients. For more details on both the mathematical theory and the physical significance of equation (3.1.4), we refer the readers to [23, 64, 74]

With the consideration of inhomogeneous media and non-uniform boundaries, the variable-coefficient PDEs can be used to describe some complex and realistic phenomena [4]. Due to this, much attention has been paid on study of nonlinear equations with variable coefficients [93, 94]. Motivated by the reasons above in this chapter, we also study the variable coefficients version of the Coupled Short Pulse (VCSP) equation [22]

$$
\begin{aligned}
u_{xt} &= \alpha_1(t)\, u + \beta_1(t)\, u\left(u^2\right)_x + \delta_1(t)\left(u^2 + w^2\right)u_{xx}, \\
w_{xt} &= \alpha_2(t) w + \beta_2(t)\, w\left(w^2\right)_x + \delta_2(t)\left(u^2 + w^2\right)w_{xx},
\end{aligned} \tag{3.1.5}
$$

where $\alpha_i(t)$, $\beta_i(t)$ and $\delta_i(t)$ $i=1, 2$ are arbitrary time-dependent coefficients, for exact and numerical solutions.

The study in this chapter has been structured as follows. In section 3.2, we study the Lie symmetries of the equation (3.1.4), its Lie algebra, and also the corresponding optimal system. In section 3.3, we present the reduced ODEs systems and exact solutions. Section 3.4, devoted to the comparative study of travelling wave and numerical solutions for CSP equation. In Section 3.5, we generate various symmetries of the VCSP equation, the admissible forms of the coefficients that admit the classical symmetry group and PDEs corresponding to each member of the optimal system of subalgebras and it also contains the reductions of reduced PDEs to corresponding ODEs along with their exact and numerical solutions. Some concluding remarks are given in Section 3.6.

3.2. Coupled Short Pulse (CSP) Equation

In this section, the Lie symmetry analysis is performed for coupled Short Pulse equation. On considering one point group transformations of point-like transformations acting on the space of independent variables (x, t) and dependent variables u, the associated infinitesimal generator is obtained. We derive the infinitesimals that admit the classical symmetry group. Five types arise depending on the nature of the Lie symmetry generator. In all types, we find reductions in terms of system of ordinary differential equations and exact solutions of the CSP equation are derived which are compared with numerical solutions using classical fourth order RK4 scheme.

3.2.1. Lie Symmetric Analysis for CSP Equation

In this section, Lie's method [178] of infinitesimal transformation groups is applied on system of Einstein equations for perfect fluids. On considering a one point group transformation of point-like transformations acting on the space of independent variables (x, t) and dependent variables u and w, the associated infinitesimal generator is given by

$$V = \xi(x,t,u,w)\frac{\partial}{\partial x} + \tau(x,t,u,w)\frac{\partial}{\partial t} + \phi(x,t,u,w)\frac{\partial}{\partial u} + \psi(x,t,u,w)\frac{\partial}{\partial w}, \quad (3.2.2)$$

Then, it is required that this transformation leave the set of solutions of system (3.1.4) invariant. This yields an over determined linear system of equations for the infinitesimals $\xi(x,t,u,w), \phi(x,t,u,w), \tau(x,t,u,w)$ and $\psi(x,t,u,w)$.

The structure of the determining equations prompts the selection of the following forms of infinitesimals

$$\begin{aligned}
\xi &= a\,x + c,\\
\tau &= -a\,t + b,\\
\phi &= a\,u,\\
\psi &= a\,w,
\end{aligned} \tag{3.2.3}$$

where a, b, and c are arbitrary constants.

The Lie algebra associated with system (3.1.4) consists of following three vector fields

$$\begin{aligned}
V_1 &= x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w},\\
V_2 &= \frac{\partial}{\partial t},\\
V_3 &= \frac{\partial}{\partial x}.
\end{aligned} \tag{3.2.4}$$

The commutation relations between these vector fields are given in the Table 3.1.

Table 3.1

Commutation relations between vector fields

$[V_i,\ V_j]$	V_1	V_2	V_3
V_1	0	V_2	$-V_3$
V_2	$-V_2$	0	0
V_3	V_3	0	0

To compute the adjoint representation, we use the Lie series.

$$\mathrm{Ad}(\exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i,\ V_j] + \frac{\varepsilon^2}{2}[V_i,[V_i,\ V_j]] - \cdot\cdot\cdot,$$

where ε is a parameter, $[V_i,\ V_j]$ is the commutator of the Lie algebra and $i,\ j = 1, 2, 3.$ Then we have the Table 3.2.

Table 3.2 Adjoint Representation Table

Ad	V_1	V_2	V_3
V_1	V_1	$V_2\, e^{-\varepsilon}$	$V_3\, e^{-\varepsilon}$
V_2	$V_1 + \varepsilon V_2$	V_2	V_3
V_3	$V_1 - \varepsilon V_3$	V_2	V_3

The classification of one-dimensional subalgebras of the whole symmetry algebra (3.2.4) is done by an inductive approach [125]. If $V = \sum_{i=1}^{3} a_i V_i$ is a non zero vector field we will simplify as many of the coefficients a_i as possible by acting it to various adjoint transformations to obtain the subalgebra.

The types of one-dimensional subalgebras of system (3.1.4) are as follows.

(i) V_1,

(ii) $V_2 + V_3$,

(iii) $V_2 - V_3$,

(iv) V_2,

(v) V_3. (3.2.5)

3.3 Reduced Systems and Exact Solutions

In the following section, some exact solutions for the reduced systems are studied for each type of subalgebra. For this first we solve the associated Lagrange equations for subalgebra (3.2.5) to obtain the group invariant solutions and then we obtained reduced system of ODEs [13, 46, 63] which are further studied for the exact and numerical solutions

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi} = \frac{dw}{\psi}. \qquad (3.2.6)$$

3.3.1 Subalgebra V_1

The symmetry in this type gives the invariant of the symmetry and group invariant solution as

$$z = xt$$

$$u(x,t) = \frac{F(z)}{t}$$

$$w(x,t) = \frac{G(z)}{t}. \tag{3.3.1}$$

Using this similarity variable and the forms of the similarity solution, the system of PDEs (3.1.4) reduces to the following system of ODEs

$$F'' z - \alpha_1 F - \beta_1 F (F')^2 - \delta_1 (F^2 + G^2) F'' = 0$$

$$G'' z - \alpha_2 G - \beta_2 G (G')^2 - \delta_2 (F^2 + G^2) G'' = 0. \tag{3.3.2}$$

The solutions of system (3.1.4) are discussed under the following cases.

Case (i)

If $\alpha_1 = C_4^{\,2}\delta_1 C_3^{\,2} + \delta_1 C_3^{\,2} C_1^{\,2} - z C_3^{\,2}$, $\alpha_2 = -C_3^{\,2}\beta_2 C_4^{\,2} - z C_3^{\,2}$, $\beta_1 = \dfrac{-\delta_1\left(C_1^{\,2} + C_4^{\,2}\right)}{C_1^{\,2}}$,

$\delta_2 = -\dfrac{\beta_2 C_4^{\,2}}{C_1^{\,2} + C_4^{\,2}}$ and β_2, δ_1 are arbitrary constants. Then the solution of the system (3.3.2) is given by

$$F(z) = C_1 \sin(C_2 + C_3 z) \qquad G(z) = C_4 \sin(C_2 + C_3 z), \tag{3.3.3}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Corresponding solution of the system (3.1.4) can be expressed as.

$$u(x,t) = C_1 \sin(C_2 + C_3 xt) \qquad w(x,t) = C_4 \sin(C_2 + C_3 xt), \tag{3.3.4}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Case (ii)

When constants are under the restriction $\alpha_1 = -C_4^{\,2}\beta_1 C_3^{\,2} - \beta_1 C_3^{\,2} C_1^{\,2} - z C_3^{\,2}$, $\alpha_2 = 0$, $\delta_1 = -\beta_1$ and $\beta_1, \beta_2, \delta_2$ are arbitrary constants then solution of system (3.1.2) is as follows

$$F(z) = C_1 \sin(C_2 + C_3 z) \qquad G(z) = C_4, \tag{3.3.5}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

The solution of the main system (3.1.4) corresponding to the solution (3.3.5) is

$$u(x,t) = C_1 \sin(C_2 + C_3 xt) \qquad w(x,t) = C_4, \tag{3.3.6}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

3.3.2 Subalgebra $V_2 + V_3$

We derive the following expression of z, u and w by solving Lagrange equation

$$z = x + t$$
$$u(x,t) = F(z)$$
$$w(x,t) = G(z). \tag{3.3.7}$$

Using the similarity variable and the forms of the similarity solution (3.3.7), the system of PDEs (3.1.4) reduces to the following system of ODEs

$$F'' + \alpha_1 F + \beta_1 F\,(F')^2 + \delta_1 \left(F^2 + G^2\right)F'' = 0$$
$$G'' + \alpha_2 G + \beta_2 G\,(G')^2 + \delta_2 \left(F^2 + G^2\right)G'' = 0. \tag{3.3.8}$$

The solutions of system (3.1.4) are as follows

Case (i)

If $\alpha_1 = C_4{}^2\delta_1 C_3{}^2 + \delta_1 C_3{}^2 C_1{}^2 + C_3{}^2,\ \alpha_2 = -C_3{}^2\,\beta_2\,C_4{}^2 + C_3{}^2,\ \beta_1 = \dfrac{-\delta_1\left(C_1{}^2 + C_4{}^2\right)}{C_1{}^2}$,

$\delta_2 = -\dfrac{\beta_2\,C_4{}^2}{C_1{}^2 + C_4{}^2}$ and β_2, δ_1 are arbitrary constants. Then the solution of the system (3.3.8) is

$$F(z) = C_1 \sin(C_2 + C_3\,z) \qquad G(z) = C_4 \sin(C_2 + C_3\,z), \tag{3.3.9}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Corresponding solutions of the system (3.1.4) can be expressed as follows

$$u(x,t) = C_1 \sin(C_2 + C_3\,xt) \qquad w(x,t) = C_4 \sin(C_2 + C_3\,xt), \tag{3.3.10}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Case (ii)

When constants satisfy the following restriction $\alpha_2 = -C_1{}^2\beta_2\,C_4{}^2 - \beta_2\,C_4{}^2\,C_2{}^2 + C_4{}^2$, $\alpha_1 = 0,\ \delta_2 = -\beta_2$ and $\beta_1, \beta_2, \delta_1$ are arbitrary constants then solution of the system (3.3.8) is

$$F(z) = C_1 \qquad G(z) = C_2 \sin(C_3 + C_4\,z), \tag{3.3.11}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Thus on using the back transformation we get the solution of the system (3.1.4) as

$$u(x,t) = C_1 \qquad w(x,t) = C_2 \sin(C_3 + C_4\, x\, t), \tag{3.3.12}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

3.3.3 Subalgebra $V_2 - V_3$

Suitable invariant of the symmetry and group invariant solution of this type for reduction are as follows

$$z = x - t$$
$$u(x,t) = F(z)$$
$$w(x,t) = G(z). \tag{3.3.13}$$

Using the similarity variable and the forms of the similarity solution (3.3.13), the system of PDEs (3.1.4) reduces to the following system of ODEs

$$F'' - \alpha_1 F - \beta_1 F\,(F')^2 - \delta_1 \left(F^2 + G^2\right)F'' = 0$$
$$G'' - \alpha_2 G - \beta_2 G\,(G')^2 - \delta_2 \left(F^2 + G^2\right)G'' = 0. \tag{3.3.14}$$

The solution of system (3.1.4) is as follows

If $\alpha_2 = -C_3^{\,2} \beta_2\, C_4^{\,2} - C_3^{\,2}$, $\delta_2 = -\beta_2$ then solution of the system (3.3.14) corresponds to the arbitrary constants α_1, β_1, β_2, δ_1 is

$$F(z) = 0 \qquad G(z) = C_1 \sin(C_2 + C_3\, z), \tag{3.3.15}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Corresponding solution of the system (3.1.4) can be expressed as

$$u(x,t) = 0 \qquad w(x,t) = C_1 \sin(C_2 + C_3\,(x-t)), \tag{3.3.16}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

3.3.4 Subalgebra V_2 and V_3

Corresponding to these subalgebras we get only constant solutions of system (3.1.4).

3.4 Comparative Study of Travelling Wave and Numerical Solutions for CSP Equation

In this section, we drive numerical solutions of the main system (3.1.4). For this purpose the reduced systems of ODEs obtained in section, 3.3 are solved numerically. Also in this section we compare the exact and numerical solutions of system (3.1.4).

3.4.1 Classical Fourth Order Runge-Kutta (RK4) Method

The classical RK4 method applied to the nth order differential equation

$$y^{(n)} = f\left(t, y, y^{'}, y^{''}, \ . \ . \ . \ , y^{(n-1)}\right), \tag{3.4.1}$$

with initial conditions $y(t_0) = y_0, \quad y'(t_0) = y_0, \quad y''(t_0) = y_0, \ . \ . \ . \ , y^{(n-1)}(t_0) = y_0,$ evaluates the function $f\left(t, y, y', y'', . \ . \ . \ , y^{(n-1)}\right)$ four times per step and can be derived by transforming the problem to a system of first order differential equations. To solve the reduced ODEs the RK4 method is given by the following equations.

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \qquad t_{n+1} = t_n + h, \tag{3.4.2}$$

where y_{n+1} is the RK4 approximation of $y\left(t_{n+1}\right)$ and

$$k_1 = h\, f\left(t_n, y_n\right), \qquad k_2 = h\, f\left(t_n + \frac{1}{2}h, \quad y_n + \frac{1}{2}k_1\right),$$

$$k_3 = h\, f\left(t_n + \frac{1}{2}h, \quad y_n + \frac{1}{2}k_2\right), \quad k_4 = h\, f\left(t_n + h, \quad y_n + k_3\right). \tag{3.4.3}$$

3.4.2 Numerical Experiments

In this sub-section, the proposed scheme based on RK4 method applied on reduced ODEs of the system (3.1.4) to show the efficiency and accuracy of the numerical method. The exact solutions of these reduced ODEs are obtained in section 3.3 by using Lie classical method. The software Dev C++, Matlab and Maple are used for whole computational work. For each reduced ODEs absolute errors E_a, relative error E_r and percentage error E_p are computed by the following formulas

Absolute Error $E_a = \left| \ u - \bar{u} \ \right|,$

Relative Error $E_r = \left| \ \dfrac{u - \bar{u}}{u} \ \right|,$

Percentage Error $E_r = \left| \ \dfrac{u - \bar{u}}{u} \ \right| \times 100,$

where u and $\bar{u}$ denote the exact and numerical solution of the problem.

3.4.3 Subalgebra V_1

For the purpose of numerical solutions of the main system (3.1.4) we solve numerically the reduced system of ODEs (3.3.2) under the following restrictions with the help of RK4 method.

Example 1

For numerical solutions we start with initial values at $z = 5$ and the initial approximation by solution (3.3.3). If constants are under the restrictions $\beta_2 = \delta_1 = 0$,

$$\alpha_1 = C_4^{\,2}\,\delta_1\,C_3^{\,2} + \delta_1\,C_3^{\,2}\,C_1^{\,2} - z\,C_3^{\,2},\quad \alpha_2 = -C_3^{\,2}\,\beta_2\,C_4^{\,2} - z\,C_3^{\,2},\quad \delta_2 = -\frac{\beta_2\,C_4^{\,2}}{C_1^{\,2} + C_4^{\,2}},$$

$\beta_1 = \dfrac{-\delta_1\left(C_1^{\,2} + C_4^{\,2}\right)}{C_1^{\,2}}$ and $C_1 = 1.5$, $C_2 = 0$, $C_3 = \dfrac{1}{5}, C_4 = -1$ then reduced ODEs system (3.3.2) for CSP equation transforms into the following system of first order differential equation

$$\frac{dy_1}{dz} = y_2, \qquad \frac{dy_2}{dz} = -0.04\,y_1,$$

$$\frac{dy_3}{dz} = y_4, \qquad \frac{dy_4}{dz} = -0.04\,y_3, \tag{3.4.4}$$

with initial conditions $y_1 = 1.5\sin(1), y_2 = 0.3\cos(1),\ y_3 = -1\sin(1),$ and $y_4 = -0.2\cos(1)$.

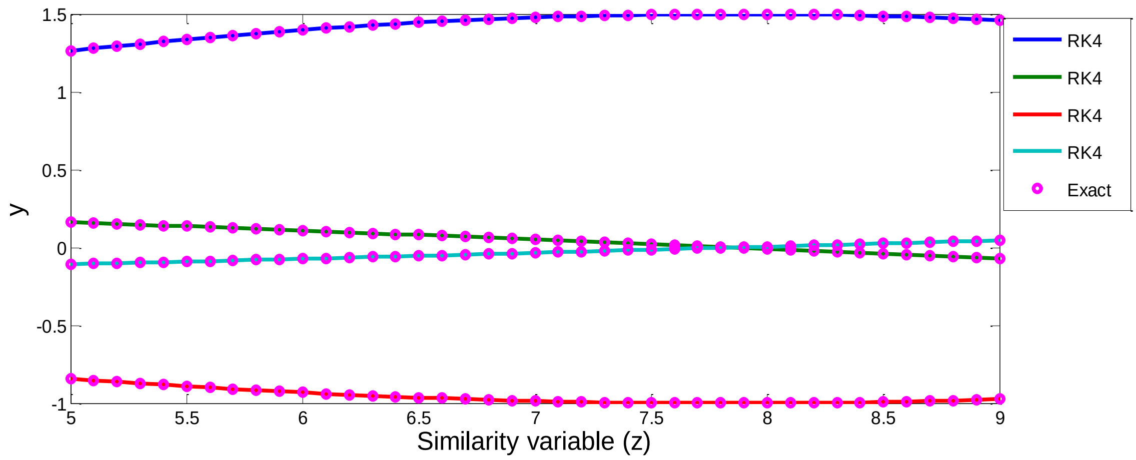


Figure 3.1: Comparison between exact and numerical solutions for the reduced ODEs system (3.3.2) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.3) at $h = 0.1$.

Table – 3.3

Absolute error, relative error, and percentage error for example 1 of subalgebra V_1

	z	Absolute error E_a	Relative error E_r	Percentage error E_p
y_1	6	$2.4513724383\times10^{-13}$	$1.7534117579\times10^{-13}$	$1.7534117579\times10^{-11}$
	7	$1.4983125851\times10^{-11}$	$1.0136235531\times10^{-11}$	$1.0136235531\times10^{-9}$
	8	$1.7643597693\times10^{-10}$	$1.1767416053\times10^{-10}$	$1.1767416053\times10^{-8}$
	9	$1.0003580186\times10^{-9}$	$6.8481487721\times10^{-10}$	$6.8481487721\times10^{-8}$
	10	$3.7432299482\times10^{-9}$	$2.7444118486\times10^{-9}$	$2.7444118486\times10^{-7}$
y_2	6	$2.0997092953\times10^{-14}$	$1.9315251013\times10^{-13}$	$1.9315251013\times10^{-11}$
	7	$7.3100553388\times10^{-13}$	$1.4336212701\times10^{-13}$	$1.4336212701\times10^{-9}$
	8	$2.5900011302\times10^{-12}$	$2.9566706645\times10^{-10}$	$2.9566706645\times10^{-8}$
	9	$1.8625004316\times10^{-11}$	$2.7325165204\times10^{-10}$	$2.7325165204\times10^{-8}$
	10	$1.9830399822\times10^{-10}$	$1.5884136784\times10^{-9}$	$1.5884136784\times10^{-7}$
y_3	6	$1.6298074001\times10^{-13}$	$1.7486470521\times10^{-13}$	$1.7486470521\times10^{-11}$
	7	$9.9880104187\times10^{-12}$	$1.0135484454\times10^{-11}$	$1.0135484454\times10^{-9}$
	8	$1.1762302243\times10^{-10}$	$1.1767319792\times10^{-10}$	$1.1767319792\times10^{-8}$
	9	$6.6690508671\times10^{-10}$	$6.8481461120\times10^{-10}$	$6.8481461120\times10^{-8}$
	10	$2.4954870392\times10^{-9}$	$2.7444122963\times10^{-9}$	$2.7444122963\times10^{-7}$
y_4	6	$1.4002687898\times10^{-14}$	$1.9321634110\times10^{-13}$	$1.9321634110\times10^{-11}$
	7	$4.8699932975\times10^{-13}$	$1.4326278639\times10^{-11}$	$1.4326278639\times10^{-9}$
	8	$1.7269996197\times10^{-12}$	$2.9572395087\times10^{-10}$	$2.9572395087\times10^{-8}$
	9	$1.2416997985\times10^{-11}$	$2.7325887998\times10^{-10}$	$2.7325887998\times10^{-8}$
	10	$1.3220199934\times10^{-10}$	$1.5884056748\times10^{-9}$	$1.5884056748\times10^{-7}$

Example 2

In this example to solve the system (3.3.2) numerically we start with initial values at $z = 5$ and the initial approximation by solutions (3.3.5).

If $\alpha_1 = -C_4{}^2\beta_1 C_3{}^2 - \beta_1 C_3{}^2 C_1{}^2 - zC_3{}^2$, $\alpha_2 = 0$, $\delta_1 = -\beta_1$, $\beta_1 = 0$ $\beta_2 = 1$, $\delta_2 = \dfrac{1}{2}$,

$C_1 = C_2 = C_4 = 1$ and $C_3 = -\dfrac{1}{5}$ then reduced ODEs system (3.3.2) for CSP equation transforms into the following system of first order differential equation

$$\frac{dy_1}{dz} = y_2, \qquad \frac{dy_2}{dz} = -0.04\,y_1$$

$$\frac{dy_3}{dz} = y_4, \qquad \frac{dy_4}{dz} = \frac{y_3\,(y_4)^2}{z - \frac{1}{2}\left(y_1^{\,2} + y_3^{\,2}\right)}, \qquad (3.4.5)$$

with initial conditions $y_1 = \sin(0),\ y_2 = -0.2\cos(0),\ y_3 = 1,$ and $y_4 = 0$.

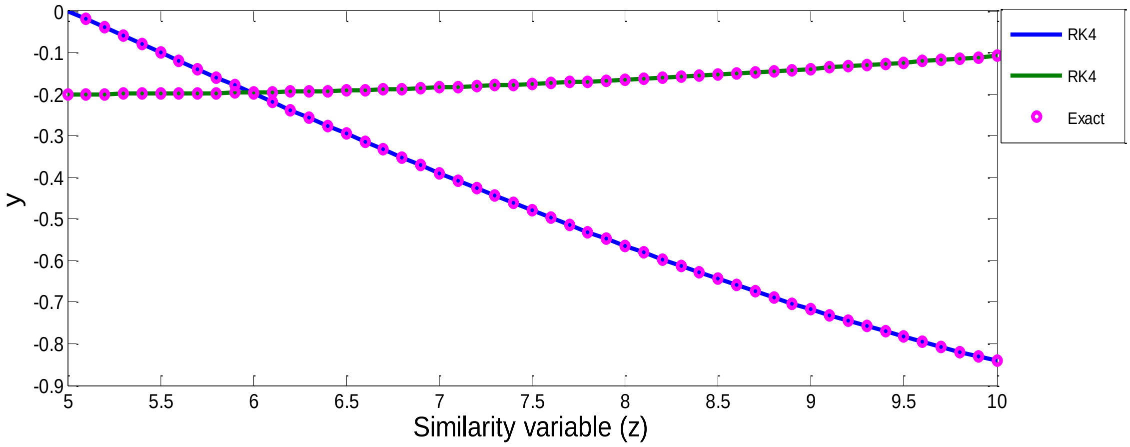


Figure 3.2: Comparison between exact and numerical solutions for the reduced ODEs system (3.3.2) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.5) at $h = 0.1$.

Table 3.4

Absolute error, relative error, and percentage error for example 2 of subalgebra V_1

	z	Absolute error E_a	Relative error E_r	Percentage error E_p
y_1	6	$2.6006974351\times10^{-14}$	$1.3090583356\times10^{-13}$	$1.3090583356\times10^{-11}$
	7	$3.3590352721\times10^{-12}$	$8.6257756946\times10^{-12}$	$8.6257756946\times10^{-10}$
	8	$5.5817017674\times10^{-11}$	$9.8853735424\times10^{-11}$	$9.8853735424\times10^{-9}$
	9	$4.0669900780\times10^{-10}$	$5.6694159701\times10^{-10}$	$5.6694159701\times10^{-8}$
	10	$1.8716309613\times10^{-9}$	$2.2242370742\times10^{-9}$	$2.2242370742\times10^{-7}$
y_2	6	$3.2002178684\times10^{-14}$	$1.6326533017\times10^{-13}$	$1.6326533017\times10^{-11}$
	7	$1.9509949211\times10^{-12}$	$1.0591019128\times10^{-11}$	$1.0591019128\times10^{-9}$
	8	$2.0566992553\times10^{-11}$	$1.2459775261\times10^{-10}$	$1.2459775261\times10^{-8}$
	9	$1.0410100559\times10^{-10}$	$7.4709346269\times10^{-10}$	$7.4709346269\times10^{-8}$
	10	$3.4276399774\times10^{-10}$	$3.1719649723\times10^{-9}$	$3.1719649723\times10^{-7}$

3.4.4 Subalgebra $V_2 + V_3$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (3.1.4) reduces to the system (3.3.8) of ODEs. For numerical solutions of the system (3.1.4) we solve the reduced ODEs (3.3.8) with the help of fourth order RK4 method under the two examples given below.

Example 1

In this example we start with initial values at $z=5$ and the initial approximation by equation (3.3.9). When $\alpha_1 = C_4^{\,2}\delta_1 C_3^{\,2} + \delta_1 C_3^{\,2} C_1^{\,2} + C_3^{\,2}$, $\alpha_2 = -C_3^{\,2}\beta_2 C_4^{\,2} + C_3^{\,2}$, $\beta_1 = \dfrac{-\delta_1\left(C_1^{\,2} + C_4^{\,2}\right)}{C_1^{\,2}}$, $\delta_2 = -\dfrac{\beta_2 C_4^{\,2}}{C_1^{\,2} + C_4^{\,2}}$, $\beta_2 = \delta_1 = 0$, $C_1 = 5$, $C_2 = 0$, $C_3 = 0.1$ and $C_4 = -6$ then system (3.3.8) for CSP equation becomes as

$$\frac{dy_1}{dz} = y_2, \qquad \frac{dy_2}{dz} = -0.01 y_1,$$

$$\frac{dy_3}{dz} = y_4, \qquad \frac{dy_4}{dz} = -0.01 y_3, \tag{3.4.6}$$

with initial conditions

$$y_1 = 5\sin(0.5), \; y_2 = 0.5\cos(0.5), \; y_3 = -6\sin(0.5) \text{ and } y_4 = -0.6\cos(0.5).$$

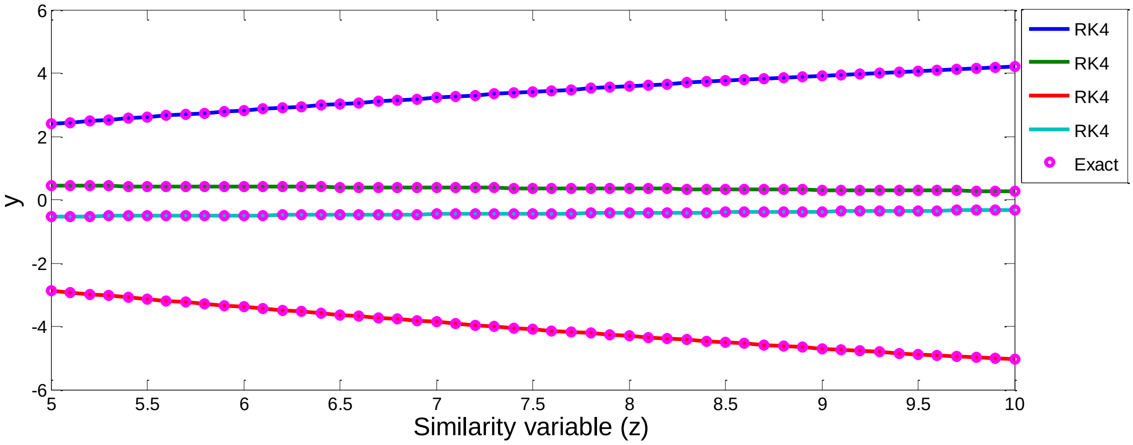


Figure 3.3: Comparison between exact and numerical solutions for the reduced ODEs system (3.3.8) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.9) at $h = 0.1$.

Table – 3.5

Absolute error, relative error, and percentage error for example 1 of subalgebra $V_2 + V_3$

	z	Absolute error E_a	Relative error E_r	Percentage error E_p
y_1	6	$9.7699626167\times10^{-15}$	$3.4605836709\times10^{-15}$	$3.4605836709\times10^{-13}$
	7	$5.0004445029\times10^{-13}$	$1.5524083246\times10^{-13}$	$1.5524083246\times10^{-11}$
	8	$6.2598815020\times10^{-12}$	$1.7452647524\times10^{-12}$	$1.7452647524\times10^{-10}$
	9	$3.8620218134\times10^{-11}$	$9.8605620871\times10^{-12}$	$9.8605620871\times10^{-10}$
	10	$1.5941026276\times10^{-10}$	$3.7888475216\times10^{-11}$	$3.7888475216\times10^{-9}$
y_2	6	$9.9920072216\times10^{-16}$	$2.4213197737\times10^{-15}$	$2.4213197737\times10^{-13}$
	7	$6.3005156647\times10^{-14}$	$1.6475335093\times10^{-13}$	$1.6475335093\times10^{-11}$
	8	$6.7501559897\times10^{-13}$	$1.9377324487\times10^{-12}$	$1.9377324487\times10^{-10}$
	9	$3.5179636981\times10^{-12}$	$1.1318878002\times10^{-11}$	$1.1318878002\times10^{-9}$
	10	$1.2271017535\times10^{-11}$	$4.5422784253\times10^{-11}$	$4.5422784253\times10^{-9}$
y_3	6	$9.7699626167\times10^{-15}$	$2.8838197257\times10^{-15}$	$2.8838197257\times10^{-13}$
	7	$5.9996452250\times10^{-13}$	$1.5521785425\times10^{-13}$	$1.5521785425\times10^{-11}$
	8	$7.5193185011\times10^{-12}$	$1.7469981311\times10^{-12}$	$1.7469981311\times10^{-10}$
	9	$4.6339820869\times10^{-11}$	$9.8596172087\times10^{-12}$	$9.8596172087\times10^{-10}$
	10	$1.9130030892\times10^{-10}$	$3.7890058477\times10^{-11}$	$3.7890058477\times10^{-9}$
y_4	6	$1.9984014443\times10^{-15}$	$4.0355329561\times10^{-15}$	$4.0355329561\times10^{-13}$
	7	$7.5994766035\times10^{-14}$	$1.6560010090\times10^{-13}$	$1.6560010090\times10^{-11}$
	8	$8.0896400689\times10^{-13}$	$1.9352093595\times10^{-12}$	$1.9352093595\times10^{-10}$
	9	$4.2230108299\times10^{-12}$	$1.1322777533\times10^{-11}$	$1.1322777533\times10^{-9}$
	10	$1.4724998997\times10^{-11}$	$4.5422099313\times10^{-11}$	$4.5422099313\times10^{-9}$

Example 2

Here to fulfill the purpose of numerical solutions we start with initial values at $z = 5$ and the initial approximation by equation (3.3.11).

When $\alpha_1 = 0$, $\alpha_2 = -C_1^{\,2}\beta_2 C_4^{\,2} - \beta_2 C_4^{\,2} C_2^{\,2} + C_4^{\,2}$, $\delta_2 = -\beta_2$, $\beta_2 = 0$, $\beta_1 = 1$, $\delta_1 = \dfrac{1}{2}$, $C_1 = C_2 = C_3 = 1$ and $C_4 = 0.1$ then the system of ODEs (3.3.8) for CSP equation can be expressed into the following system of first order differential equations

$$\frac{dy_1}{dz} = y_2, \qquad \frac{dy_2}{dz} = \frac{-y_1(y_2)^2}{z + \frac{1}{2}\left(y_1^{\ 2} + y_3^{\ 2}\right)},$$

$$\frac{dy_3}{dz} = y_4, \qquad \frac{dy_4}{dz} = -0.01 y_3, \tag{3.4.7}$$

with initial conditions $y_1 = 1, y_2 = 0, y_3 = \sin(1.5)$, and $y_4 = 0.1\cos(1.5)$.

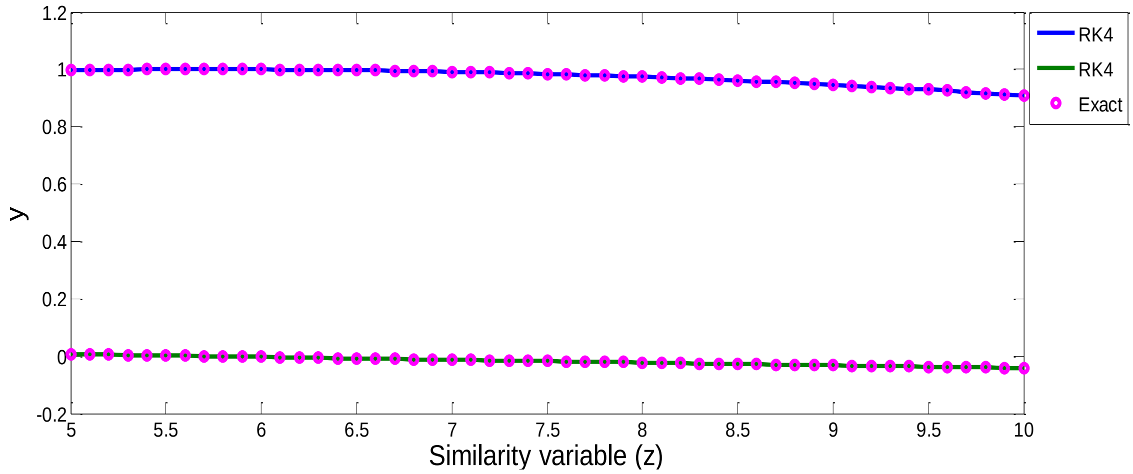


Figure 3.4 Comparison between exact and numerical solutions for the reduced ODEs system (3.3.8) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by Eq. (3.3.11) at $h = 0.1$.

Table- 3.6

Absolute error, relative error, and percentage error for example 2 of subalgebra $V_2 + V_3$

	z	Absolute error E_a	Relative error E_r	Percentage error E_p
y_3	6	$2.9976021664\times10^{-15}$	$2.9988808801\times10^{-15}$	$2.9988808801\times10^{-13}$
	7	$1.5998313784\times10^{-13}$	$1.6132783594\times10^{-13}$	$1.6132783594\times10^{-11}$
	8	$1.8119949984\times10^{-12}$	$1.8606555492\times10^{-12}$	$1.8606555492\times10^{-10}$
	9	$1.0095035918\times10^{-11}$	$1.0667901281\times10^{-11}$	$1.0667901281\times10^{-9}$
	10	$3.7877034841\times10^{-11}$	$4.1655275517\times10^{-11}$	$4.1655275517\times10^{-9}$
y_4	6	$1.9949319973\times10^{-17}$	$6.8320706646\times10^{-15}$	$6.8320706646\times10^{-13}$
	7	$1.6011497683\times10^{-15}$	$1.2426994083\times10^{-13}$	$1.2426994083\times10^{-11}$
	8	$3.2602393007\times10^{-14}$	$1.4349512512\times10^{-12}$	$1.4349512512\times10^{-11}$
	9	$2.6960378374\times10^{-13}$	$8.3393901745\times10^{-12}$	$8.3393901745\times10^{-10}$
	10	$1.3569007029\times10^{-12}$	$3.2606296233\times10^{-11}$	$3.2606296233\times10^{-9}$

3.4.5 Subalgebra $V_2 - V_3$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (3.1.4) reduces to the system (3.3.14) of ODEs. For the numerical solutions of CSP Equation we solve the system (3.3.14) under the restrictions given below.

Example

Herein, we start with initial values at $z = 5$ and the initial approximation by equation (3.3.15) to construct the numerical solutions of system (3.1.4). When $\delta_2 = -\beta_2$, $\alpha_2 = -C_3^{\,2}\,\beta_2\, C_4^{\,2} - C_3^{\,2}$, and $\beta_2 = 0,\ \alpha_1 = \beta_1 = C_1 = C_2 = C_3 = 1,\ \delta_1 = \frac{1}{2}$ then reduced first order system of ODEs corresponding to the systems (3.3.14) for CSP equation is as follows

$$\frac{dy_1}{dz} = y_2, \qquad \frac{dy_2}{dz} = \frac{y_1 + y_1(y_2)^2}{z - \frac{1}{2}\left(y_1^{\,2} + y_3^{\,2}\right)},$$

$$\frac{dy_3}{dz} = y_4, \qquad \frac{dy_4}{dz} = -y_3, \tag{3.3.8}$$

with initial conditions $y_1 = 0,\ y_2 = 0,\ y_3 = \sin(6)$, and $y_4 = \cos(6)$.

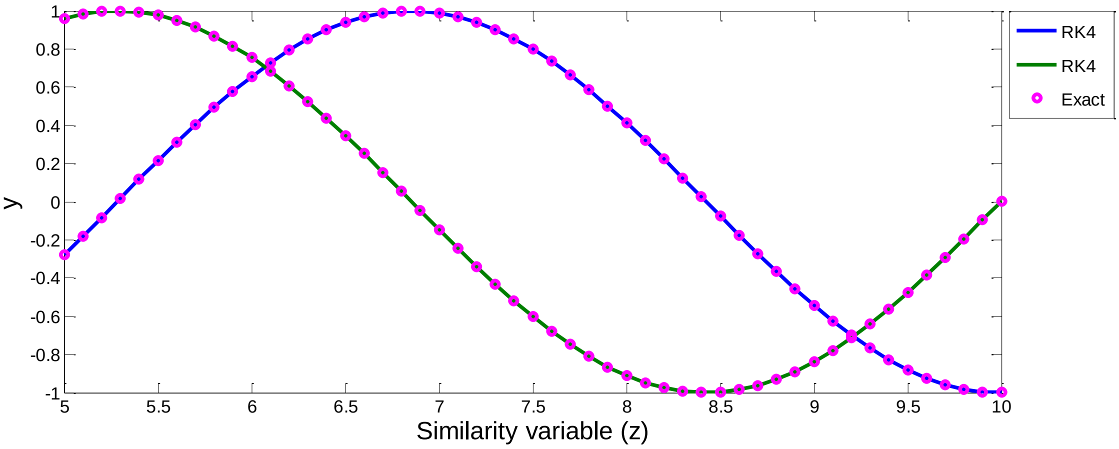


Figure 3.5: Comparison between exact and numerical solutions for the reduced ODEs system (3.3.14) for CSP equation when we start with initial values at $z = 5$ and the initial approximation by equation (3.3.15) at $h = 0.1$.

Table- 3.7

Absolute error, relative error, and percentage error for example of subalgebra $V_2 - V_3$

	z	Absolute error E_a	Relative error E_r	Percentage error E_p
y_3	6	$1.3090309947\times10^{-9}$	$1.9924774679\times10^{-9}$	$1.9924774679\times10^{-7}$
	7	$1.6135250902\times10^{-7}$	$1.6308805184\times10^{-7}$	$1.6308805184\times10^{-5}$
	8	$1.5353387229\times10^{-6}$	$3.7254789046\times10^{-6}$	$3.7254789046\times10^{-4}$
	9	$1.0498759309\times10^{-6}$	$1.9298440997\times10^{-6}$	$1.9298440997\times10^{-4}$
	10	$2.8677316509\times10^{-5}$	$2.8677597361\times10^{-5}$	$2.8677597361\times10^{-3}$
y_4	6	$2.1546260331\times10^{-9}$	$2.8579647039\times10^{-9}$	$2.8579647039\times10^{-7}$
	7	$3.1689822987\times10^{-8}$	$2.1779942009\times10^{-7}$	$2.1779942009\times10^{-5}$
	8	$1.1455247350\times10^{-6}$	$1.2572568193\times10^{-6}$	$1.2572568193\times10^{-4}$
	9	$1.0913914822\times10^{-5}$	$1.3007132818\times10^{-5}$	$1.3007132818\times10^{-3}$
	10	$3.1303523470\times10^{-5}$	$7.0731268953\times10^{-3}$	$7.0731268953\times10^{-1}$

3.5 Coupled Short Pulse Equation with Time Dependent Coefficients

In this section, using the Lie symmetry approach, we have examined herein the variant Coupled Short Pulse (VCSP) equation (3.1.5). The method used reduces the system of PDEs to a system of ODEs according to the Lie symmetry admitted. In particular, we found the relevant system of ODEs is all optimal subgroups. The system of ODEs is further solved in general to obtain exact and numerical solutions. Several new physically important families of exact and numerical solutions are derived

3.5.1 Lie Symmetric Reduction, Exact and Numerical Solutions

The application of the classical Lie group method requires considering a one parameter Lie group of infinitesimal transformations as follows

$$
\begin{aligned}
x^* &= x + \varepsilon X(x,t,u,w) + O(\varepsilon^2),\\
t^* &= t + \varepsilon T(x,t,u,w) + O(\varepsilon^2),\\
u^* &= u + \varepsilon U(x,t,u,w) + O(\varepsilon^2),\\
w^* &= w + \varepsilon W(x,t,u,w) + O(\varepsilon^2),
\end{aligned} \tag{3.5.1}
$$

with a small parameter ε. The vector field associated with the above group of transformations can be written as

$$V = X\left(x,t,u,w\right)\frac{\partial}{\partial x}+T\left(x,t,u,w\right)\frac{\partial}{\partial t}+U\left(x,t,u,w\right)\frac{\partial}{\partial u}+W\left(x,t,u,w\right)\frac{\partial}{\partial w}. \quad (3.5.2)$$

The method for determining the symmetry group of (3.1.5) consists of finding the infinitesimals X, T, U and W, which are functions of x, t, u and w. Assuming that system (3.1.5) is invariant under the transformations (3.5.1). In other words, the transformations are such that if u is a solution of equation (3.1.5), then u^{*} is also a solution. Herein, on invoking the invariance criterion as mentioned in [57], we obtain the following relation from the coefficients of the first order of ε:

$$U_{xt}-\alpha_1(t)U-\alpha_1'(t)Tu-2\beta_1(t)uu_xU_x-\left(\beta_1(t)U+\beta_1'(t)uT\right)u_x^{\,2}-\delta_1(t)\left(u^2+w^2\right)U_{xx}$$

$$-\left(2\delta_1(t)uU+2\delta_1(t)\,wW+\delta_1'(t)u^2T+\delta_1'(t)w^2T\right)u_{xx}=0, \quad (3.5.3)$$

$$W_{xt}-\alpha_2(t)W-\alpha_2'(t)wT-2\beta_2(t)ww_xW_x-\left(\beta_2(t)W+\beta_2'(t)wT\right)w_x^{\,2}-\delta_2(t)\left(u^2+w^2\right)$$

$$W_{xx}-\left(2\delta_2(t)uU+2\delta_2(t)\,wW+\delta_2'(t)u^2T+\delta_2'(t)w^2T\right)w_{xx}=0, \quad (3.5.4)$$

where the prime (') denotes the derivatives with respect to t and $U_x, U_{xt}, U_{xx}, W_x, W_{xt}, W_{xx}$ are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables $u_x, u_{xt}, u_{xx}, w_x, w_{xt}$, and w_{xx} (for more details, see [57, 102, 143]). The infinitesimals are determined from invariance conditions (3.5.3) and (3.5.4) by setting the coefficients of different differentials equal to zero. We obtain a large number of the following PDEs in X, T, U and W that need to be satisfied:

(i) $T_x = 0,\quad T_u = 0,\quad T_w = 0,$

(ii) $X_t = 0,\quad X_u = 0,\quad X_t = 0,$

(iii) $U_w = 0,\quad U_x = 0,\quad U_t = 0,\quad U_{uu} = 0,$

(iv) $W_u = 0,\quad W_x = 0,\quad W_t = 0,\quad W_{ww} = 0,$

(v) $-\alpha_1(t)\left(X_x+T_t\right)-\alpha_1'(t)\,T=0,$

(vi) $-2\beta_1(t)U_u+\beta_1(t)X_x-\beta_1(t)T_t-\beta_1'(t)\,T=0,$

(vii) $-2\delta_1(t)U_u+\delta_1(t)X_x-\delta_1(t)T_t-\delta_1'(t)\,T=0,$

(viii) $U_u = W_w,$

(ix) $-\alpha_2(t)\left(X_x + T_t\right) - \alpha_2'(t)\, T = 0,$

(x) $-2\beta_2(t)U_u + \beta_2(t)\, X_x - \beta_2(t)T_t - \beta_2'(t)\, T = 0,$

(xi) $-2\delta_2(t)U_u + \delta_2(t)\, X_x - \delta_2(t)T_t - \delta_2'(t)\, T = 0,$ (3.5.5)

The general solution of this large system (3.5.5) leads to the expressions for the functions $X(x,t,u,w)$, $T(x,t,u,w)$, $U(x,t,u,w)$ and $W(x,t,u,w)$ of the form

$$X = a\,x + c, \qquad T = \frac{-a\int \alpha_1(t)\,dt + b}{\alpha_1(t)},$$

$$U = d\,u, \qquad W = d\,w. \tag{3.5.6}$$

The functions $\alpha_i(t)$, $\beta_i(t)$ and $\delta_i(t)$, $i = 1, 2$ are connected by the relations

$$\frac{d}{dt}\left[\alpha_i(t)\,T\right] = -a\,\alpha_i(t), \quad \frac{d}{dt}\left[\beta_i(t)\,T\right] = (a - 2d)\,\beta_i(t),$$

$$\frac{d}{dt}\left[\delta_i(t)\,T\right] = (a - 2d)\,\delta_i(t), \tag{3.5.7}$$

where a, b, c and d are arbitrary constants.

Hence, the Lie algebra of infinitesimal symmetries of the equations (3.1.5) is spanned by the following four linearly independent vector fields

$$V_1 = x\frac{\partial}{\partial x} - \frac{1}{\alpha_1(t)}\int \alpha_1(t)\,dt\,\frac{\partial}{\partial t}, \qquad V_2 = u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w},$$

$$V_3 = \frac{1}{\alpha_1(t)}\frac{\partial}{\partial t}, \qquad V_4 = \frac{\partial}{\partial x}. \tag{3.5.8}$$

In general, one may obtain the reduced system of ODEs from any linear combination of vector fields V_j, $j = 1, 2, 3, 4$. Since there exist infinite possibilities for such combinations, a systematic procedure to classify these reductions is based on the property that the transformations of the symmetry group will transform solutions of equations (3.1.5) into other solutions. Therefore, we classify the symmetry algebra of system (3.1.5) into conjugacy inequivalent subalgebra under the adjoint action of the symmetry group. We will work out first an optimal system and then embark upon the various reductions associated with generators in the optimal system [143]. The adjoint action is given by the Lie series

$$\mathrm{Ad}\left(\exp\left(\varepsilon V_i\right)\right)V_j \;=\; V_j - \varepsilon\left[V_i,\ V_j\right] + \frac{\varepsilon^2}{2}\left[V_i,\left[V_i,\ V_j\right]\right] - \cdots, \qquad (3.5.9)$$

where $\left[V_i, V_j\right] \;=\; V_i V_j - V_j V_i$ is the commutator for the Lie algebra, ε is a parameter and Ad denotes adjoint action.

The commutation relations of Lie algebra (3.5.8) are as follows:

$$\left[V_1,\ V_1\right] = \left[V_2,\ V_2\right] = \left[V_3,\ V_3\right] = \left[V_4,\ V_4\right] = 0,$$

$$\left[V_1,\ V_2\right] = \left[V_2,\ V_1\right] = \left[V_2, V_3\right] = \left[V_2,\ V_4\right] = \left[V_3,\ V_2\right] = \left[V_3, V_4\right] = \left[V_4,\ V_2\right] = \left[V_4,\ V_3\right] = 0,$$

$$\left[V_1,\ V_3\right] = -\left[V_3,\ V_1\right] = V_3,$$

$$\left[V_1,\ V_4\right] = -\left[V_4,\ V_1\right] = -V_4. \qquad (3.5.10)$$

Using these commutation relations, we can easily construct following adjoint table from (3.5.9)

$\mathrm{Ad}\left(\exp\left(\varepsilon V_i\right)\right)V_i \;=\; V_i,$ where $i = 1, 2, 3, 4$

and

$$\mathrm{Ad}\left(\exp\left(\varepsilon V_1\right)\right)V_2 = V_2, \quad \mathrm{Ad}\left(\exp\left(\varepsilon V_1\right)\right)V_3 = e^{-\varepsilon} V_3, \quad \mathrm{Ad}\left(\exp\left(\varepsilon V_1\right)\right)V_4 = e^{\varepsilon} V_4,$$

$$\mathrm{Ad}\left(\exp\left(\varepsilon V_2\right)\right)V_1 = V_2, \quad \mathrm{Ad}\left(\exp\left(\varepsilon V_2\right)\right)V_3 = V_3, \quad \mathrm{Ad}\left(\exp\left(\varepsilon V_2\right)\right)V_4 = V_4,$$

$$\mathrm{Ad}\left(\exp\left(\varepsilon V_3\right)\right)V_1 = V_1 + \varepsilon V_3, \quad \mathrm{Ad}\left(\exp\left(\varepsilon V_3\right)\right)V_2 = V_2, \quad \mathrm{Ad}\left(\exp\left(\varepsilon V_3\right)\right)V_4 = V_4,$$

$$\mathrm{Ad}\left(\exp\left(\varepsilon V_4\right)\right)V_1 = V_1 - \varepsilon V_4, \quad \mathrm{Ad}\left(\exp\left(\varepsilon V_4\right)\right)V_2 = V_2, \quad \mathrm{Ad}\left(\exp\left(\varepsilon V_4\right)\right)V_3 = V_3.$$

(3.5.11)

The classification of one-dimensional subalgebras of the whole symmetry algebra (3.5.8) is done by an inductive approach (for details see [102] and [143]). We begin by considering a general element $V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4$ of symmetry algebra and subject it to various adjoint transformations to simplify it as much as possible. The types of one-dimensional subalgebras of system (3.1.5) are as follows:

(i) $V_1 + \mu V_2$,

(ii_a) $V_2 + \lambda V_3 + V_4$,

(ii_b) $V_2 + \lambda V_3 - V_4$,

(iii) $V_2 + \gamma V_3$,

(iv_a) $V_3 + V_4$,

(iv_b) $V_3 - V_4$,

(v) V_3,

(vi) V_4, (3.5.12)

where μ, λ and γ are arbitrary constants. Because the discrete symmetry $(x,\, t,\, u) \to (-x,\, t,\, u)$ will map (ii_a), (iv_a) to (ii_b), (iv_b) respectively, and therefore, in the optimal system, we confine ourselves to six essential vector fields of the optimal system, while neglecting the other two (ii_b), (iv_b).

Once the subalgebra of the equation (3.1.5) have been determined, the symmetry variables and similarity solutions for the associated reduction can be found by solving the characteristic equation

$$\frac{dx}{X(x,t,u,w)} = \frac{dt}{T(x,t,u,w)} = \frac{du}{U(x,t,u,w)} = \frac{dw}{W(x,t,u,w)}. \tag{3.5.13}$$

The general solution of these equations involves three constants; one becomes the new independent variable z (similarity variable) and the others, say F (similarity solution) and G (similarity solution), play the role of new dependent variables. On substituting non-equivalent Lie ansätze (similarity variable, similarity solutions and forms of the coefficient functions) for the function $u(x,\, t)$ and $w(x,\, t)$ in system (3.1.5), one gets the reduced system of ordinary differential equations.

3.5.1.1 Subalgebra $V_1 + \mu V_2$

In this subalgebra we obtain the following similarity variable and similarity solutions for the reduction

$$z = x \int \alpha_1(t)dt,$$

$$u(x,t) = \left(\int \alpha_1(t)dt\right)^{-\mu} F(z),$$

$$w(x,t) = \left(\int \alpha_1(t)dt\right)^{-\mu} G(z), \tag{3.5.14}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = -K_1\left(\int \alpha_1(t)dt\right)^{2\mu-2}\alpha_1(t), \qquad \delta_1(t) = -K_2\left(\int \alpha_1(t)dt\right)^{2\mu-2}\alpha_1(t),$$

$$\alpha_2(t) = -K_3\,\alpha_1(t), \qquad \beta_2(t) = -K_4\left(\int \alpha_1(t)dt\right)^{2\mu-2}\alpha_1(t),$$

$$\delta_2(t) = -K_5\left(\int \alpha_1(t)dt\right)^{2\mu-2}\alpha_1(t). \tag{3.5.15}$$

Using (3.5.14) and (3.5.15), the system of PDEs (3.1.5) reduces to the following system of ODEs:

$$(-\mu+1)F' + F''z - F + K_1 F\left(F'\right)^2 + K_2\left(F^2+G^2\right)F'' = 0,$$

$$(-\mu+1)G' + G''z + K_3 G + K_4 G\left(G'\right)^2 + K_5\left(F^2+G^2\right)G'' = 0. \tag{3.5.16}$$

Thus, by solving equation (3.5.16) for $\mu=1$ and reverting back to the original variables, we obtain the group-invariant solutions of the equation (3.1.5) as follows

(i)
$$u(x,t) = \left(\int \alpha_1(t)dt\right)^{-1}\left(C_1 - \frac{C_2 + C_3\left(x\int \alpha_1(t)dt\right)}{\sqrt{K_1}\,C_3}\right),$$

$$w(x,t) = \left(\int \alpha_1(t)dt\right)^{-1}\left(C_4 - \frac{\sqrt{-K_4K_3}\left(C_2 + C_3\left(x\int \alpha_1(t)dt\right)\right)}{K_4C_3}\right), \tag{3.5.17}$$

where $C_1,\ C_2,\ C_3,\ C_4, K_1,\ K_2,\ K_3$ and K_4 are arbitrary constants.

(ii)
$$u(x,t) = C_1\left(\int \alpha_1(t)dt\right)^{-1}\sec\left(-C_3 + \frac{x\int \alpha_1(t)dt}{\sqrt{-x\left(\int \alpha_1(t)dt\right)}}\right),$$

$$w(x,t) = C_2\left(\int \alpha_1(t)dt\right)^{-1}\sec\left(-C_2 + \frac{x\int \alpha_1(t)dt}{\sqrt{-x\left(\int \alpha_1(t)dt\right)}}\right), \tag{3.5.18}$$

where $K_1 = \dfrac{4\left(x\int \alpha_1(t)dt\right)}{C_1^{\,2}},\ K_2 = K_5 = \dfrac{-2\left(x\int \alpha_1(t)dt\right)}{C_1^{\,2} + C_2^{\,2}},\ K_3 = -1,\ K_4 = \dfrac{4\left(x\int \alpha_1(t)dt\right)}{C_2^{\,2}}$

and $C_1,\ C_2,\ C_3$ are arbitrary constants.

(iii)
$$u(x,t) = C_1\left(\int \alpha_1(t)dt\right)^{-1}\sin\left(C_2 + C_3\left(x\int \alpha_1(t)dt\right)\right),$$

$$w(x,t) = C_4\left(\int \alpha_1(t)dt\right)^{-1}\sin\left(C_2 + C_3\left(x\int \alpha_1(t)dt\right)\right), \tag{3.5.19}$$

where $K_1 = \dfrac{C_3^{\,2}\left(x\int \alpha_1(t)dt\right)+1}{C_1^{\,2}\,C_3^{\,2}}$, $K_2 = \dfrac{-C_3^{\,2}\left(x\int \alpha_1(t)dt\right)+1}{C_3^{\,2}\left(C_1^{\,2}+C_4^{\,2}\right)}$, $K_4 = K_4$, $K_5 = -\dfrac{K_4\,C_4^{\,2}}{C_1^{\,2}+C_4^{\,2}}$,

$K_3 = -\left(-x\left(\int \alpha_1(t)dt\right)+K_4\,C_4^{\,2}\right)C_3^{\,2}$ and $C_1,\ C_2, C_3,\ C_4$ are arbitrary constants.

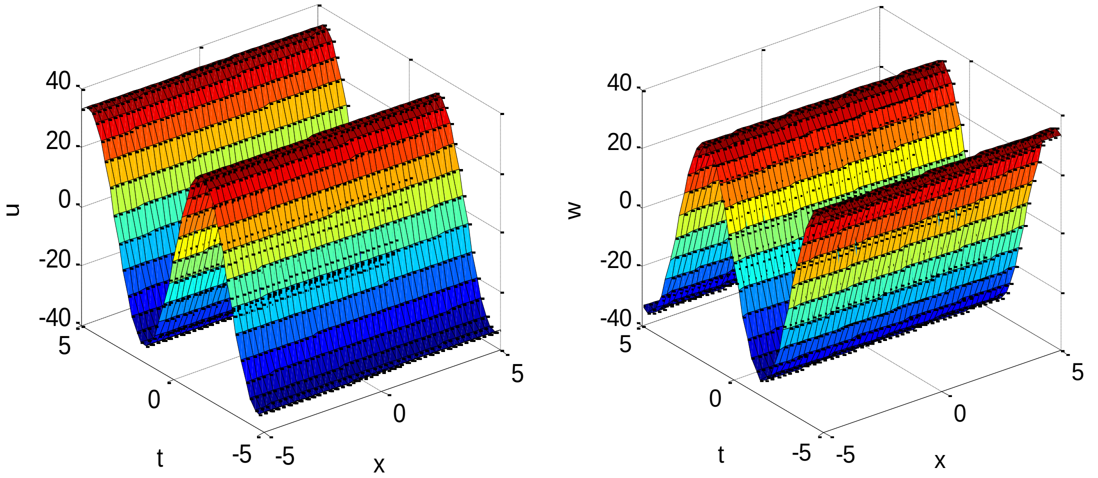


Figure 3.6: The periodic solution (3.5.19) in form of 3D plot with $C_1 = C_2 = C_3 = 1$; $C_4 = -1$ and $\alpha_1(t) = \sin(t)$.

3.5.1.2 Subalgebra $V_2 + \lambda V_3 + V_4$

Under this subalgebra, the optimal system defines the similarity variable and similarity solution as follows

$$z = \lambda x - \int \alpha_1(t)dt,$$

$$u(x,t) = \exp\left(\frac{1}{\lambda}\int \alpha_1(t)\,dt\right)F(z),$$

$$w(x,t) = \exp\left(\frac{1}{\lambda}\int \alpha_1(t)\,dt\right)G(z). \tag{3.5.20}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = \frac{K_1}{\lambda}\exp\left(\frac{-2}{\lambda}\int \alpha_1(t)dt\right)\alpha_1(t),\quad \delta_1(t) = \frac{K_2}{\lambda}\exp\left(\frac{-2}{\lambda}\int \alpha_1(t)dt\right)\alpha_1(t),$$

$$\alpha_2(t) = \frac{K_3}{\lambda}\alpha_1(t),\quad \beta_2(t) = \frac{K_4}{\lambda}\exp\left(\frac{-2}{\lambda}\int \alpha_1(t)dt\right)\alpha_1(t),$$

$$\delta_2(t) = \frac{K_5}{\lambda}\exp\left(\frac{-2}{\lambda}\int \alpha_1(t)dt\right)\alpha_1(t). \tag{3.5.21}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (3.1.5) reduces to the following system of ODEs

$$F' - F'' - F - K_1 \lambda\, F\left(F'\right)^2 - K_2 \lambda \left(F^2 + G^2\right)F'' = 0,$$

$$\lambda G' - \lambda G'' - K_3\, G - K_4 \lambda^2\, G\left(G'\right)^2 - K_5 \lambda^2 \left(F^2 + G^2\right)G'' = 0. \quad (3.5.22)$$

(*i*) $$u(x,t) = \exp\left(\frac{1}{\lambda}\int \alpha_1(t)dt + \frac{1}{2}\left(\lambda x - \int \alpha_1(t)dt\right)\right)\times \left(C_1 \sin\left(\frac{1}{2}\sqrt{3}\left(\lambda x - \int \alpha_1(t)dt\right)\right) + C_2 \cos\left(\frac{1}{2}\sqrt{3}\left(\lambda x - \int \alpha_1(t)dt\right)\right)\right),$$

$$w(x,t) = \exp\left(\frac{1}{\lambda}\int \alpha_1(t)dt + \frac{1}{2}\left(\lambda x - \int \alpha_1(t)dt\right)\right)\times \left(C_3 \sin\left(\frac{1}{2}\sqrt{3}\left(\lambda x - \int \alpha_1(t)dt\right)\right) + C_4 \cos\left(\frac{1}{2}\sqrt{3}\left(\lambda x - \int \alpha_1(t)dt\right)\right)\right), \quad (3.5.23)$$

where $\lambda = K_3 \neq 0$, $K_1 = K_2 = K_4 = K_5 = 0$ and C_1, C_2, C_3, C_4 are arbitrary constants.

$$u(x,t) = C_1 \exp\left(C_2 + \frac{1}{\lambda}\int \alpha_1(t)dt + \left(\frac{1}{2} - \frac{1}{2}\sqrt{-3}\right)\left(\lambda x - \int \alpha_1(t)dt\right)\right),$$

$$w(x,t) = C_3 \exp\left(C_2 + \frac{1}{\lambda}\int \alpha_1(t)dt + \left(\frac{1}{2} - \frac{1}{2}\sqrt{-3}\right)\left(\lambda x - \int \alpha_1(t)dt\right)\right), \quad (3.5.24)$$

where $\lambda = K_3 \neq 0$, $K_1 = K_2 = K_4 = K_5 = 0$ and C_1, C_2, C_3 are arbitrary constants.

(ii) Herein, we seek the solution of equation (3.5.22) in the following form

$$F(z) = a_1 \exp(mz)$$

$$G(z) = a_2 \exp(mz) \quad (3.5.25)$$

where a_1 and m are constants to be found out. On substitution of the form of $F(z)$ and $G(z)$ in equation (3.5.22), and performing the algebraic calculations on the relations obtained among various parameters, we get the corresponding solution of equation (3.1.5) as follows:

$$u(x,t) = a_1 \exp\left(\frac{1}{\lambda}\int \alpha_1(t)\, dt\right)\left(\exp\left(m\lambda x - m\int \alpha_1(t)\, dt\right)\right),$$

$$w(x,t) = a_2 \exp\left(\frac{1}{\lambda}\int \alpha_1(t)\, dt\right)\left(\exp\left(m\lambda x - m\int \alpha_1(t)\, dt\right)\right). \quad (3.5.26)$$

where $K_1 = -\frac{K_2(a_1^2 + a_2^2)}{a_1^2}$, $K_3 = (1-m)\lambda m$, $K_4 = -\frac{K_5(a_1^2 + a_2^2)}{a_1^2}$, $m = \frac{1 \pm \sqrt{-3}}{2}$

and $K_2, K_5, a_1, a_2, \lambda$ are arbitrary constants.

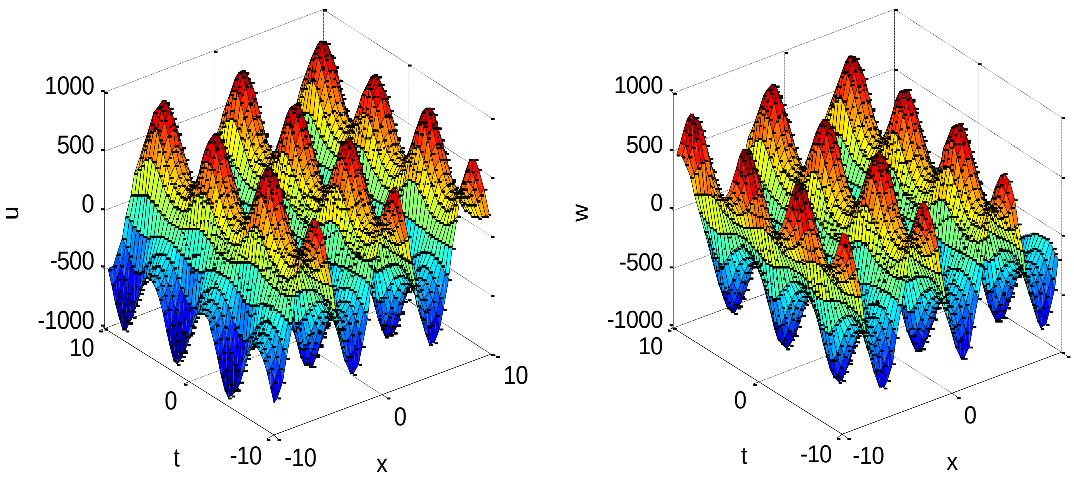


Figure 3.7: The periodic solution (3.5.23) in form of 3D plot with $C_1 = C_2 = 1$; $C_3 = C_4 = -1$ and $\alpha_1(t) = \cos(t)$.

3.5.1.3 Subalgebra $V_2 + \gamma V_3$

For this subalgebra, on solving the equations (3.5.7) and (3.5.13) we obtain the following similarity variable and similarity solution for the reduction

$$z = x,$$

$$u(x,t) = \exp\left(\frac{1}{\gamma}\int \alpha_1(t)\,dt\right) F(z),$$

$$w(x,t) = \exp\left(\frac{1}{\gamma}\int \alpha_1(t)\,dt\right) G(z), \tag{3.5.27}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = \frac{K_1}{\gamma}\exp\left(\frac{-2}{\gamma}\int \alpha_1(t)dt\right)\alpha_1(t), \quad \delta_1(t) = \frac{K_2}{\gamma}\exp\left(\frac{-2}{\gamma}\int \alpha_1(t)dt\right)\alpha_1(t),$$

$$\alpha_2(t) = \frac{K_3}{\gamma}\alpha_1(t), \quad \beta_2(t) = \frac{K_4}{\gamma}\exp\left(\frac{-2}{\gamma}\int \alpha_1(t)dt\right)\alpha_1(t),$$

$$\delta_2(t) = \frac{K_5}{\gamma}\exp\left(\frac{-2}{\gamma}\int \alpha_1(t)dt\right)\alpha_1(t). \tag{3.5.28}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (3.1.5) reduces to the following system of ODEs

$$
\begin{aligned}
&F' - \gamma F - K_1 F \left(F'\right)^2 - K_2 \left(F^2 + G^2\right) F'' = 0, \\
&G' - K_3 G - K_4 G \left(G'\right)^2 - K_5 \left(F^2 + G^2\right) G'' = 0. \qquad (3.5.29)
\end{aligned}
$$

Now with respect to the ODEs (3.5.29), we are unable to obtain the nontrivial exact solutions for the VCSP equation so under this subalgebra we present the numerical solutions of main system (3.1.5) by solving reduced ODEs (3.5.29) with the help of classical RK4 method. For numerical solution we pose the following initial value problem (IVP) for the ODEs (3.5.29) after reducing it first order ODEs.

$$
\frac{dy_1}{dz} = y_2, \qquad \frac{dy_3}{dz} = \frac{y_2 - \gamma y_1 - 2\,K_1 y_1 \left(y_2\right)^2}{K_2 \left(y_1{}^2 + y_3{}^2\right)},
$$

$$
\frac{dy_3}{dz} = y_4, \qquad \frac{dy_4}{dz} = \frac{y_4 - K_3\, y_3 - 2\,K_4 y_3 \left(y_4\right)^2}{K_5 \left(y_1{}^2 + y_3{}^2\right)}, \qquad (3.5.30)
$$

with $y_1(1) = 0.01$, $y_2(1) = 0$, $y_3(1) = 3.9$ and $y_4(1) = 0$. (3.5.31)

The numerical solution to the IVP (3.5.30)–(3.5.31) are depicted below

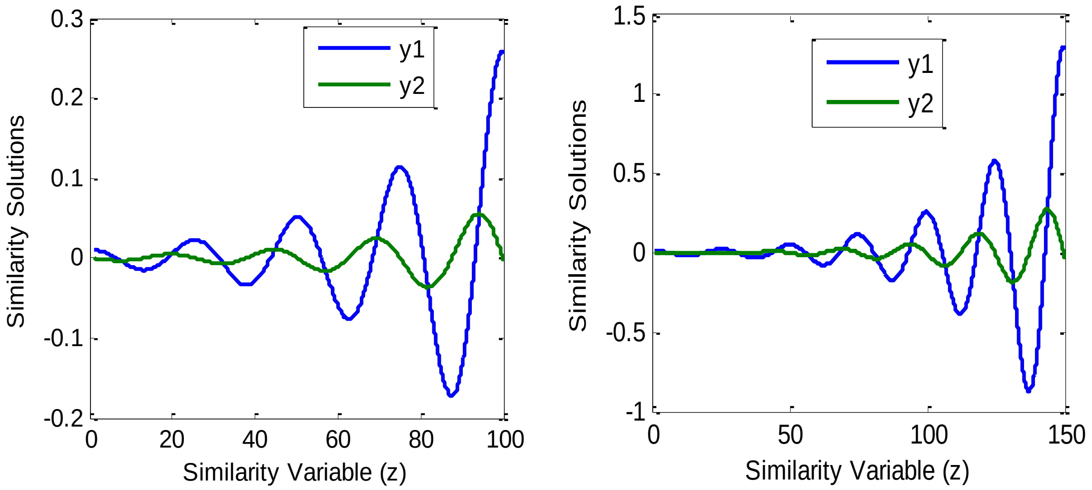
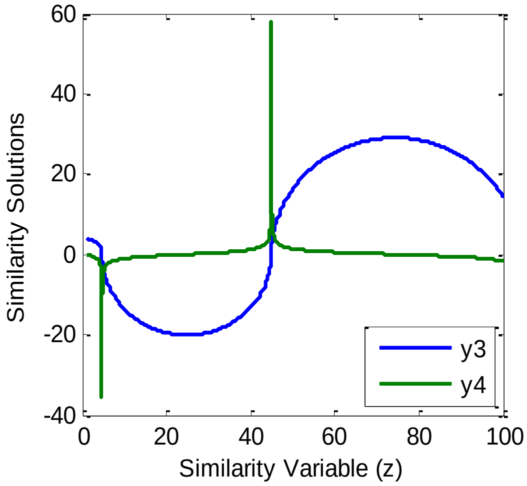


Figure 3.8: Numerical solutions for the reduced IVP (3.5.30-3.5.31) for VCSP equation when $\gamma = K_1 = K_2 = K_4 = K_5 = 1$ and $K_3 = 0$, with initial value $z = 1$ at $h = 0.01$ as $z \to 100$ and $z \to 150$.



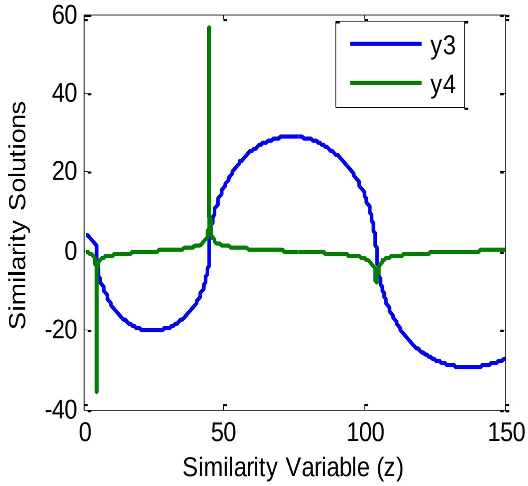


Figure 3.9: Numerical solutions for the reduced IVP (3.5.30-3.5.31) for VCSP equation when $\gamma = K_1 = K_2 = K_3 = K_4 = K_5 = 1$, with initial value $z = 1$ at $h = 0.01$ as $z \to 100$ and $z \to 150$.

3.5.1.4 Subalgebra $V_3 + V_4$

For this subalgebra, on solving the equation (3.5.7) and (3.5.13) we obtain

$$z = x - \int \alpha_1(t)dt,$$

$$u(x,t) = F(z),$$

$$w(x,t) = G(z), \tag{3.5.32}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = K_1\ \alpha_1(t),\ \ \delta_1(t) = K_2\ \alpha_1(t),\ \ \alpha_2(t) = K_3\ \alpha_1(t),$$

$$\beta_2(t) = K_4\,\alpha_1(t),\ \ \delta_2(t) = K_5\ \alpha_1(t). \tag{3.5.33}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (3.1.5) reduces to the following system of ODEs

$$F'' + F + K_1 F\left(F'\right)^2 + K_2\left(F^2 + G^2\right)F'' = 0,$$

$$G'' + K_3 G + K_4 G\left(G'\right)^2 + K_5\left(F^2 + G^2\right)G'' = 0. \tag{3.5.34}$$

Solving this equation (3.5.34) and using the back transformation we get the solutions of the main system (3.1.5) as

(i)
$$u(x,t)=C_1-\frac{C_2+C_3\left(x-\int\alpha_1(t)dt\right)}{\sqrt{K_1}\,C_3},$$
$$w(x,t)=C_4-\frac{\sqrt{-K_4K_3}\left(C_2+C_3\left(x-\int\alpha_1(t)dt\right)\right)}{K_4C_3},\qquad(3.5.35)$$

where C_1, C_2, C_3, C_4, K_1, K_2, K_3 and K_4 are arbitrary constants.

(ii)
$$u(x,t)=C_1\sec\left(\pm C_2+x-\int\alpha_1(t)dt\right),$$
$$w(x,t)=C_3\sec\left(\pm C_2+x-\int\alpha_1(t)dt\right),\qquad(3.5.36)$$

where $K_1=\frac{4}{C_1^{\,2}}$, $K_2=K_5=-\frac{2}{C_1^{\,2}+C_3^{\,2}}$, $K_3=1$, and $K_4=\frac{4}{C_3^{\,2}}$,

and C_1, C_2, C_3 are arbitrary constants.

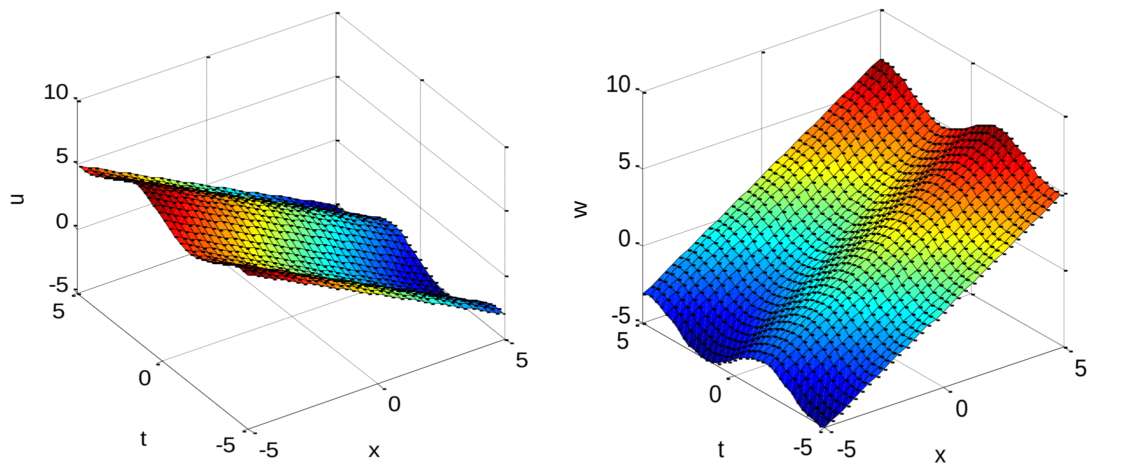


Figure 3.10: The periodic solution (3.5.35) in form of 3D plot with $C_1=C_3=C_4=1$, $C_2=0$, $K_1=K_3=1$, $K_4=-1$ and $\alpha_1(t)=\cos(t)$.

3.5.1.5 Subalgebra V_3

Following the same way as above, we get

$$\xi=x,$$
$$u(x,t)=F(\xi),$$
$$w(x,t)=G(\xi).\qquad(3.5.37)$$

and the coefficient functions are given by following relations

$$\beta_1(t) = K_1\ \alpha_1(t),\ \ \delta_1(t) = K_2\ \alpha_1(t),\ \ \alpha_2(t) = K_3\ \alpha_1(t),$$
$$\beta_2(t) = K_4\,\alpha_1(t),\ \ \delta_2(t) = K_5\ \alpha_1(t). \qquad (3.5.38)$$

Substituting (3.5.37) into equation (3.1.5), we have the functions $F(\xi)$ and $G(\xi)$ which must satisfy the following ODE:

$$F + K_1 F\left(F^{'}\right)^2 + K_2\left(F^2 + G^2\right)F^{''} = 0,$$
$$K_3 G + K_4 G\left(G^{'}\right)^2 + K_5\left(F^2 + G^2\right)G^{''} = 0. \qquad (3.5.39)$$

As one can see from equation (3.5.37) that $u(x,\,t)$ and $w(x,\,t)$ both degenerate into functions of variable x alone, which may not really correspond to a physically interesting case, we, therefore, prefer not to study the reduced system (3.5.39) further.

3.5.1.6 Subalgebra V_4

The reduced ODEs, in this case are

$$F' = 0, \qquad G' = 0. \qquad (3.5.40)$$

The obvious solution of (1.4) is as follows

$$u(x,\,t) = \text{constant}, \qquad w(x,\,t) = \text{constant}. \qquad (3.5.41)$$

3.6 Discussion and Concluding Remarks

In this chapter, we have investigated the symmetries, exact and numerical solutions of CSP and VCSP equation. Firstly, The Lie symmetry method is utilized to investigate the symmetries and invariant solutions of the CSP equations. By determining the transformation group under which a given system is invariant, we obtained information about the invariants and symmetries of that equation. This information, in turn, was used to determine similarity variables that reduced the number of independent variables. The vector fields of the optimal system lead to reduction of the nonlinear system of PDEs to ODEs. The infinitesimal generators in the optimal system are used for reductions and exact solutions. By using these vector fields, we have found group-invariant solutions. Here, we found new exact solutions which are compared with numerical solutions using classical RK4 scheme. The numerical solutions are presented in Tables 3.3-3.7and Figures 3.1-3.5. From these results, we found that the numerical solution is in a good agreement with the analytical solution. The results reported here provide further evidence of the usefulness of RK4 method of fourth order for solving nonlinear system of PDEs.

In this chapter, we have also studied the CSP equation with time dependent variable coefficients using the Lie symmetry group method. The Lie symmetry classification with respect to the special form of the time dependent variable coefficients equation was presented. We used this classification of optimal system of one-dimensional subalgebras of the Lie symmetry algebras to construct symmetry reductions and exact group-invariant solutions with physical behavior given in Figure 3.6 and Figure 3.7 for the subalgebra 3.5.1.1 and 3.5.1.2 Then under the subalgebra 3.5.1.3 initial value problems are posed to (3.5.21) and the numerical solutions are shown to decay for a fairly finite value of z in the Figure 3.8 and Figure 3.9. Further, for subalgebra 3.5.1.4 some exact solutions with physical behavior given in Figure 3.10 to the equation (3.1.5) are also obtained.

Chapter 4

(2+1)-Dimensional Diffusion-Advection Equation with Variable Coefficients

---

4.1 Introduction

The continuum problem of the driven diffusive flow of particles past an impenetrable strip of length L which is parallel to the y-axis ; the strip is centered on the origin and normal to the incident flow which taking the direction parallel to the x-axis from left to right (as shown in Figure 4.1). The flux of particles in the fluid is composed of row tends: a linear diffusive term $-Du_y$ perpendicular to the nonlinear drift tends $vLk(u)$, where $u(x, y, t)$ is the particle concentration, D is the diffusion coefficient, v is the drift velocity, $k(u)$ is a nonlinear function of the particle concentration. We further impose that there can be no flux through the strip. The velocity v may result from the motion of the fluid (advection), or from gravitational field acting on the particles (drift). Conservations of particles implies that the divergence of the flux equals to $-u_t$. Therefore, when D and v are constants, $u(x, y, t)$ satisfies the following nonlinear diffusion-advection (DA) equation in (2+1) dimensions

---

A part of this chapter has been published in **Chinese Physics B 22 (2013) 050201 (7 pp.)**

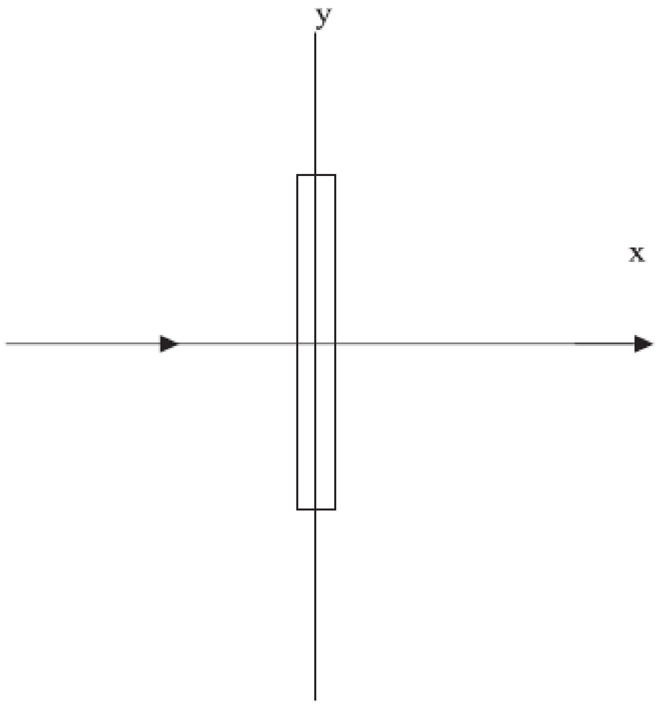


Figure 4.1: An obstacle of length *L* parallel to the *y*-axis, centred on the origin and oriented perpendicular to the drift. The direction of the drift is parallel to the *x*-axis from left to right.

$$u_t = D\, u_{yy} - [K(u)]_x, \tag{4.1.1}$$

where $K(u) = v\,L\,k(u)$. A number of many special cases of this class of equations have been used to model successfully problems in mathematical physics, chemistry and biology. In fact, Alexander and Lebowitz [48] investigated when $K(u) = u(1-u)$ equation (4.1.1) represents the flow of particles in a lattice fluid past an impenetrable obstacle [48, 49], where $u(x,\, y,\, t)$ is the particle concentration moving by diffusion and advection or drift, D and v are constants that they represent the diffusion coefficient and the drift velocity, respectively.

In [41, 175], the classical Lie group method was used for studying another model of the driven diffusive flow whose behavior is quite similar to that of a rod in a lattice fluid given in [48, 49]. Following [41, 43, 175] and using the classical symmetry reduction technique, RK4 and extended (G'/G)-expansion methods, in this chapter we study the variable-coefficient version of the DA equation

$$u_t = \alpha(t)\, u_{yy} - \beta(t)\,[K(u)]_x, \tag{4.1.2}$$

where $K(u)$ is an arbitrary functions of u and $\alpha(t)$, $\beta(t)$ are arbitrary functions of t, which arises in modeling of various physical phenomena, is studied by Lie symmetry approach. The similarity reductions are derived by determining the complete sets of point symmetries of this equation and then exact and numerical solutions are reported for the reduced second order nonlinear ODEs. Further, an extended (G'/G)-expansion method is applied to DA equation (4.1.2) for constructing some new non-traveling wave solutions.

The outline of this chapter is as follows, In section 4.2, the classical Lie method is utilized to obtain the optimal system of sub-algebras for equation (4.1.2). In sub-section 4.2.1, we derive some explicit exact and numerical solutions of the nonlinear equation (4.1.2). In section 4.3, we apply extended (G'/G)-expansion method to DA equation for constructing some new non-traveling wave solutions. In the last section, some conclusions are drawn.

4.2 Symmetry Analysis for DA Equation

Firstly, let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned} x^* &= x + \varepsilon X(x, y, t, u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon T(x, y, t, u) + O(\varepsilon^2), \\ y^* &= y + \varepsilon Y(x, y, t, u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon U(x, y, t, u) + O(\varepsilon^2), \end{aligned} \tag{4.2.1}$$

with a small parameter ε. The vector field associated with the above group of transformations can be written as

$$V = X(x, y, t, u)\frac{\partial}{\partial x} + T(x, y, t, u)\frac{\partial}{\partial t} + Y(x, y, t, u)\frac{\partial}{\partial y} + U(x, y, t, u)\frac{\partial}{\partial u}. \tag{4.2.2}$$

The condition that the above-mentioned transformation (4.2.1) leaves invariant the equation (4.1.2) under consideration yields an overdetermined system of linear equations for the infinitesimals $X(x,y,t,u)$, $T(x,y,t,u)$, $Y(x,y,t,u)$ and $U(x,y,t,u)$. The infinitesimals for the different form of $K(u)$ are given in the Table 4.1.

Table 4.1

The Infinitesimals corresponding to different form of $K(u)$

$K(u)$	X	Y	T	U
$u^m,\ m \neq 0, 1, 2$	$k_2\, x + k_3$	$k_4\, y + k_5$	$\dfrac{2k_4 \int \alpha(t)dt + k_6}{\alpha(t)}$	$k_1\, u$
e^{mu}	$k_2\, x + k_3$	$k_4\, y + k_5$	$\dfrac{2k_4 \int \alpha(t)dt + k_6}{\alpha(t)}$	k_1
$\log(u)$	$k_2\, x + k_3$	$k_4\, y + k_5$	$\dfrac{2k_4 \int \alpha(t)dt + k_6}{\alpha(t)}$	$k_1\, u$

The relations between $\alpha(t)$ and $\beta(t)$ for the different form of $K(u)$ are given in the Table 4.2

Table 4.2

Relation between $\alpha(t)$ and $\beta(t)$

$K(u)$	Relation between $\alpha(t)$ and $\beta(t)$
$u^m,\ m \neq 0, 1, 2$	$2k_4 - T\dfrac{\alpha'(t)}{\alpha(t)} + (m-1)k_1 - k_2 + \dfrac{\beta'(t)}{\beta(t)}T = 0$
e^{mu}	$2k_4 - T\dfrac{\alpha'(t)}{\alpha(t)} + m\, k_1 - k_2 + \dfrac{\beta'(t)}{\beta(t)}T = 0$
$\log(u)$	$2k_4 - T\dfrac{\alpha'(t)}{\alpha(t)} - k_1 - k_2 + \dfrac{\beta'(t)}{\beta(t)}T = 0$

The Lie algebra of infinitesimal symmetries of equations (4.1.2) when $K(u) = u^m$, $m \neq 0, 1, 2$ and $\log(u)$ is spanned by the following six linearly independent infinitesimal generators

$$V_1 = u\frac{\partial}{\partial u},$$

$$V_2 = x\frac{\partial}{\partial x},$$

$$V_3 = \frac{\partial}{\partial x},$$

$$V_4 = \frac{\partial}{\partial y},$$

$$V_5 = \frac{1}{\alpha(t)}\frac{\partial}{\partial t},$$

$$V_6 = y\frac{\partial}{\partial y} + \frac{2}{\alpha(t)}\int \alpha(t)\, dt\ \frac{\partial}{\partial t}. \tag{4.2.3}$$

It is easy to verify that $V_j \ \ j = 1, 2, 3, 4, 5, 6$ is closed under the Lie bracket.

However, if the two lie algebras are similar, i.e. they are connected to each other by a transformation from the symmetry group, then their corresponding invariant solutions are connected to each other by the same transformation. Therefore, it is sufficient to put all similar subalgebras into one class and select a representative from each class. The set of all these representatives is called an optimal system [143].

For brevity we omit the details and just state the result that an optimal system of one-dimensional subalgebras of equation (4.1.2) is as follows

(i) $\langle V_1 + \lambda V_2 + \mu V_6 \rangle$,

(ii) $\langle V_2 + \rho V_6 \rangle$,

(iii) $\langle V_3 + \sigma V_6 \rangle$,

(iv) $\langle V_4 \rangle$,

(v) $\langle V_5 \rangle$,

(vi) $\langle V_6 \rangle$, (4.2.4)

where λ, μ, ρ and σ are arbitrary constants. Once the subalgebra of the equation (4.1.2) have been determined, the symmetry variables for the associated reduction can be found by solving the characteristic equation

$$\frac{dx}{X(x,y,t,u)} = \frac{dy}{Y(x,y,t,u)} = \frac{dt}{T(x,y,t,u)} = \frac{du}{U(x,y,t,u)}. \tag{4.2.5}$$

In Table 4.3, we have listed the Lie ansätze for all the essential fields comprising the optimal system and forms of the coefficient functions of equation (4.1.2) are listed in Table 4.4

Table 4.3

Similarity variables, Similarity functions, and the Coefficient functions of DA equation when $K(u) = u^m$ for $m \neq 0,1,2$ and $K(u)=\log(u)$

Essential vector fields	Similarity variables $(f,\ g)$	Similarity solution (u)
$\langle V_1 + \lambda V_2 + \mu V_6 \rangle$	$x\left(\int \alpha(t)dt\right)^{\frac{-\lambda}{2\mu}},\ \ y\left(\int \alpha(t)dt\right)^{\frac{-1}{2}}$	$\left(\int \alpha(t)dt\right)^{\frac{1}{2\mu}} H(f,\ g)$
$\langle V_2 + \rho V_6 \rangle$	$x\left(\int \alpha(t)dt\right)^{\frac{-1}{2\rho}},\ y\left(\int \alpha(t)dt\right)^{\frac{-1}{2}}$	$H(f,\ g)$
$\langle V_3 + \sigma V_6 \rangle$	$x\left(\int \alpha(t)dt\right)^{\frac{-1}{2\sigma}},\ y\left(\int \alpha(t)dt\right)^{\frac{-1}{2}}$	$H(f,\ g)$
$\langle V_6 \rangle$	$x,\ \ y\left(\int \alpha(t)dt\right)^{\frac{-1}{2}}$	$H(f,\ g)$

Table 4.4

Form of the Coefficient functions of DA equation for different form of $K(u)$

Essential vector fields	Form of coefficient function when $K(u) = u^m$ for $m \neq 0,1,2$	Form of coefficient function when $K(u)=\log(u)$
$\langle V_1 + \lambda V_2 + \mu V_6 \rangle$	$k\left(\int \alpha(t)dt\right)^{\frac{\lambda-(m-1)-2\mu}{2\mu}} \alpha(t)$	$k\left(\int \alpha(t)dt\right)^{\frac{\lambda+1-2\mu}{2\mu}} \alpha(t)$
$\langle V_2 + \rho V_6 \rangle$	$k\left(\int \alpha(t)dt\right)^{\frac{1}{2\rho}-1} \alpha(t)$	$k\left(\int \alpha(t)dt\right)^{\frac{1}{2\rho}-1} \alpha(t)$
$\langle V_3 + \sigma V_6 \rangle$	$k\left(\int \alpha(t)dt\right)^{-1} \alpha(t)$	$k\left(\int \alpha(t)dt\right)^{-1} \alpha(t)$
$\langle V_6 \rangle$	$k\left(\int \alpha(t)dt\right)^{-1} \alpha(t)$	$k\left(\int \alpha(t)dt\right)^{-1} \alpha(t)$

4.2.1 Similarity Reductions, Exact and Numerical Solutions when $K(u) = \log(u)$

Now, we deal with the symmetry reductions and exact solutions to equation (4.1.2). We will consider the similarity reductions and group-invariant solutions. From an optimal system of group-invariant solutions to equation (4.1.2), every other such solution to the equation can be derived. Using the ansätze mentioned in Table 4.3, for the form of $K(u) = \log(u)$ we can reduce the nonlinear equation (4.1.2) to ODE.

4.2.1.1 Subalgebra $\langle V_1 + \lambda V_2 + \mu V_6 \rangle$

Substituting the forms of the similarity solution and the coefficient functions corresponding to this vector field, equation (4.1.2) yields the following PDE

$$H - \lambda\, f\, H_f - \mu\, g\, H_g - 2\mu H_{gg} + 2\,\mu\, k\, H^{-1} H_f = 0. \tag{4.2.6}$$

To reduce the PDE, let us assume transformation of the form

$$H(f, g) = f^{-1}\, J(z), \qquad z = g, \tag{4.2.7}$$

which reduces equation (4.2.6) to the ODE of the form

$$(1+\lambda) J - \mu\, z\, J' - 2\,\mu\, J'' - 2\mu\, k = 0. \tag{4.2.8}$$

Solving this equation (4.2.8) and reverting back to the original variables, we obtain the following group-invariant solutions of the DA equation (4.1.2)

$$\begin{aligned} u(x, y, t) = {} & (x)^{-1}\left(\int \alpha(t)dt\right)^{\frac{1+\lambda}{2\mu}} \exp\left(\frac{-1}{4} y^2 \left(\int \alpha(t)dt\right)^{-1}\right) \times \\ & KummerM\left(\frac{1}{2}\,\frac{2\mu+1+\lambda}{\mu},\ \frac{3}{2},\ \frac{1}{4} y^2 \left(\int \alpha(t)dt\right)^{-1}\right) y \left(\int \alpha(t)dt\right)^{\frac{-1}{2}} C_1 \\ & + (x)^{-1}\left(\int \alpha(t)dt\right)^{\frac{1+\lambda}{2\mu}} \exp\left(\frac{-1}{4} y^2 \left(\int \alpha(t)dt\right)^{-1}\right) \times \\ & KummerU\left(\frac{1}{2}\,\frac{2\mu+1+\lambda}{\mu},\ \frac{3}{2},\ \frac{1}{4} y^2 \left(\int \alpha(t)dt\right)^{-1}\right) y \left(\int \alpha(t)dt\right)^{\frac{-1}{2}} C_2 + \frac{2\mu k}{1+\lambda}, \end{aligned} \tag{4.2.9}$$

where λ, μ, k, C_1 and C_2 are arbitrary constants.

4.2.1.2 Subalgebra $\langle V_2 + \rho V_6 \rangle$

For the case under consideration, the reduced PDE is as follows

$$f\, H_f + \rho\, g\, H_g + 2\,\rho\, H_{gg} - 2\,\rho\, k\, H^{-1} H_f = 0. \tag{4.2.10}$$

Perform the transformation as follows

$$H(f, g) = f^{-1}\, J(z), \qquad z = g, \tag{4.2.11}$$

Substituting (4.2.11) into equation (4.2.10) one obtains an ODE of the form

$$J - \rho\, z\, J' - 2\,\rho\, J'' - 2\rho\, k \;=\; 0. \tag{4.2.12}$$

Now with respect to the ODEs (4.2.12) we are unable to obtain the nontrivial exact solutions for the DA equation so under this subalgebra we present the numerical solutions of main equation (4.1.2) by solving reduced ODEs (4.2.12) with the help of Classical RK4 method. For numerical solutions we start with initial values at $z = 0$. If constants are under the restrictions ρ=100, $k = 0.5$. We pose the following initial value problem (IVP) for the ODE (4.2.12) after reducing it first order ODEs.

$$\frac{dF(z)}{dz} = G(z), \tag{4.2.13}$$

$$\frac{d\,G(z)}{dz} = \frac{F(z)}{2\rho} - \frac{z}{2}G(z) - k\,, \tag{4.2.14}$$

$$F(0) = 100\,, \quad G\,(0) = -1.019607920\times 10^{-9}. \tag{4.2.15}$$

The numerical solutions to the IVP (4.2.13)–(4.2.15) are depicted below

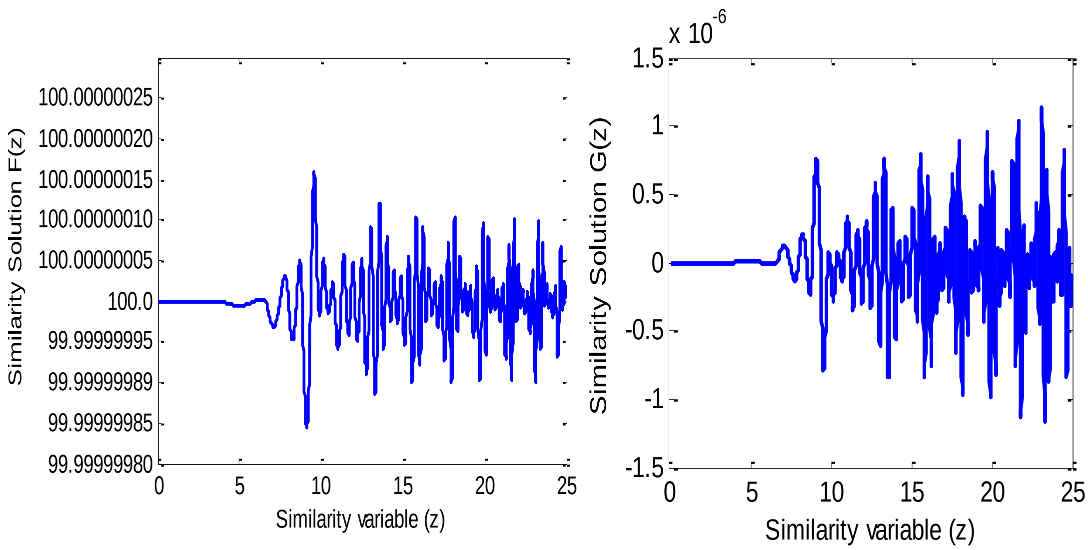


Figure 4.2: Numerical solutions of the reduced IVP (4.2.13-4.2.15) for DA equation at $h = 0.01$ as $z \to 25$

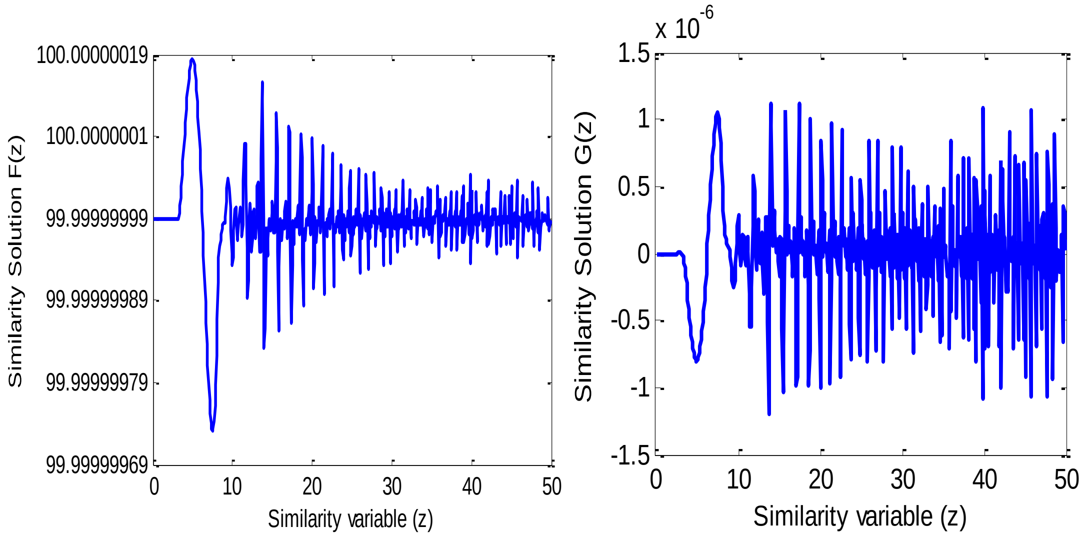


Figure 4.3: Numerical solutions of the reduced IVP (4.2.13-4.2.15) for DA equation at $h = 0.1$ as $z \to 50$.

4.2.1.3 Subalgebra $\langle V_3 + \sigma V_6 \rangle$

For this vector field, equation (4.1.2) is transformed into the following second order nonlinear PDE

$$H_f + \sigma g H_g + 2\sigma H_{gg} - 2\sigma k H^{-1} H_f = 0. \tag{4.2.16}$$

We further transform (4.2.16) via $H(f, g) = J(z)$, $z = g$ to ODE

$$z J' + 2 J'' = 0. \tag{4.2.17}$$

Then the solution of the above ODE leads to the following trivial group-invariant solution of the equation (4.1.2)

$$u(x, y, t) = C_1 + erf\left(\frac{1}{2}\, y \left(\int \alpha(t) dt\right)^{\frac{-1}{2}}\right) C_2, \tag{4.2.18}$$

where C_1 and C_2 are arbitrary constants.

4.2.1.4 Subalgebra $\langle V_4 \rangle, \langle V_5 \rangle$ and $\langle V_6 \rangle$

Under these subalgebras, we obtain the solutions which may not really correspond to a physically interesting case. Thus, we omit the details here.

4.3 Application of Extended (G'/G)-Expansion Method to DA Equation

when $K(u) = u(1-u)$ then the equation (4.1.2) can be written as

$$u_t = \alpha(t)\, u_{yy} - \beta(t)\, [u\,(1-u)]_x . \tag{4.3.1}$$

In this section, we will apply the extended (G'/G)-expansion method to DA equation (4.1) as a result, some non-traveling wave solutions with parameters are obtained.

The solution of equation (4.3.1) can be expressed by a polynomial in (G'/G)as follows

$$u = a_0 + \sum_{i=0}^{m} a_i \left[\frac{G'(\theta)}{G(\theta)}\right]^i , \tag{4.3.2}$$

where a_i $(i = 1, 2, \ldots, m)$ are arbitrary functions of variable *x, y, t* to be determined, and $G(\theta)$ satisfies the following equation

$$G''(\theta) + \lambda\, G'(\theta) + \mu\, G(\theta) = 0 , \tag{4.3.3}$$

where $\theta(x, y, t) = f(x,t) + g(y)$ and $f(x,t),\ g(y)$ are arbitrary functions of the indicated variables.

By balancing the highest order nonlinear term and the highest order partial derivative term of *u* in equation (4.3.1), we obtain $m = 1$. In order to search for explicit solutions, we suppose that equation (4.3.1) has the following formal solution

$$u = a_0 + a_1\left(\frac{G'}{G}\right), \qquad a_1 \neq 0. \tag{4.3.4}$$

Substituting equation (4.3.4) into equation (4.3.1) and using equation (4.3.3), collecting all terms with the same order of together, the left-hand side of equation (4.3.1) is converted into polynomial in $\left(\frac{G'}{G}\right)^j$, $(j = 0, 1, 2, 3)$. Setting each coefficient of this polynomial to zero, we derive a set of overdetermined differential equations for a_0 and a_1.

$$\left(\frac{G'}{G}\right)^3 : \; -2\, a_1 (g')^2\, \alpha(t) + 2\, \beta(t) a_1^{\,2}\, f_x = 0 , \tag{4.3.5}$$

$$\left(\frac{G'}{G}\right)^{2}:-a_1 f_t-g'\left[3\lambda a_1 g'-2(a_1)_y\right]\alpha(t)-a_1 f_x \beta(t)+2\lambda\beta(t)a_1^{2} f_x+2\beta(t)a_0 a_1 f_x$$
$$+a_1 g''\alpha(t)=0, \tag{4.3.6}$$

$$\left(\frac{G'}{G}\right):(a_1)_t-\lambda a_1 f_t-g'\left[(\lambda^2+2\mu)a_1 g'-2\lambda(a_1)_y\right]\alpha(t)+\left[(a_1)_x-\lambda a_1 f_x\right]\beta(t)$$
$$-2a_1\left[(a_0)_x-\mu a_1 f_x\right]\beta(t)-2a_0\left[(a_1)_x-\lambda a_1 f_x\right]\beta(t)-\left[(a_1)_{yy}-\lambda a_1 g''\right]\alpha(t)=0, \tag{4.3.7}$$

$$\left(\frac{G'}{G}\right)^{0}:\ (a_0)_t-\mu a_1 f_t+g'\left[\lambda\mu a_1-2\mu(a_1)_y\right]\alpha(t)+\left[(a_0)_x-\mu a_1 f_x\right]\beta(t)$$
$$-2a_0\left[(a_0)_x-\mu a_1 f_x\right]\beta(t)-\left[(a_0)_{yy}-\mu a_1 g''\right]\alpha(t)=0, \tag{4.3.8}$$

Solving this set of equations, we have obtained the following results

$$a_0=\frac{f_t+\lambda(g')^2\alpha(t)+f_x\beta(t)-5g''\alpha(t)}{2\beta(t)f_x},\ \ a_1=\frac{\alpha(t)(g')^2}{\beta(t)f_x},\quad f_x\neq 0. \tag{4.3.9}$$

Substituting the general solutions of equation (4.3.3) into equation (4.3.4) and using equations (4.3.5-4.3.9), we have three types of exact solutions of equation (4.3.1) as follows

Type 1. When $\lambda^2-4\mu>0$ we have

$$\frac{G'(\theta)}{G(\theta)}=-\frac{\lambda}{2}+\Lambda_1\frac{\mathrm{A}_1\sinh\Lambda_1(f+g)+\mathrm{A}_2\cosh\Lambda_1(f+g)}{\mathrm{A}_1\cosh\Lambda_1(f+g)+\mathrm{A}_2\sinh\Lambda_1(f+g)}, \tag{4.3.10}$$

where $\Lambda_1=\dfrac{\sqrt{\lambda^2-4\mu}}{2}$ and $\mathrm{A}_1,\ \mathrm{A}_2$ are arbitrary constants.

We can obtain the non-traveling wave solutions with hyperbolic functions for the DA equation (4.3.1) in the form

$$u(x,y,t)=\frac{f_t+f_x\beta(t)-5g''\alpha(t)}{2\beta(t)f_x}$$
$$+\left(\frac{\alpha(t)(g')^2}{\beta(t)f_x}\right)\Lambda_1\frac{\mathrm{A}_1\sinh\Lambda_1(f+g)+\mathrm{A}_2\cosh\Lambda_1(f+g)}{\mathrm{A}_1\cosh\Lambda_1(f+g)+\mathrm{A}_2\sinh\Lambda_1(f+g)}, \tag{4.3.15}$$

where the arbitrary function $f(x,t)$ satisfies the conditions (i) $f_x \neq 0$; (ii) f_x exists and $g(y)$ satisfies (i) $g' \neq 0$; (ii) g' exists; (iii) g'' exists.

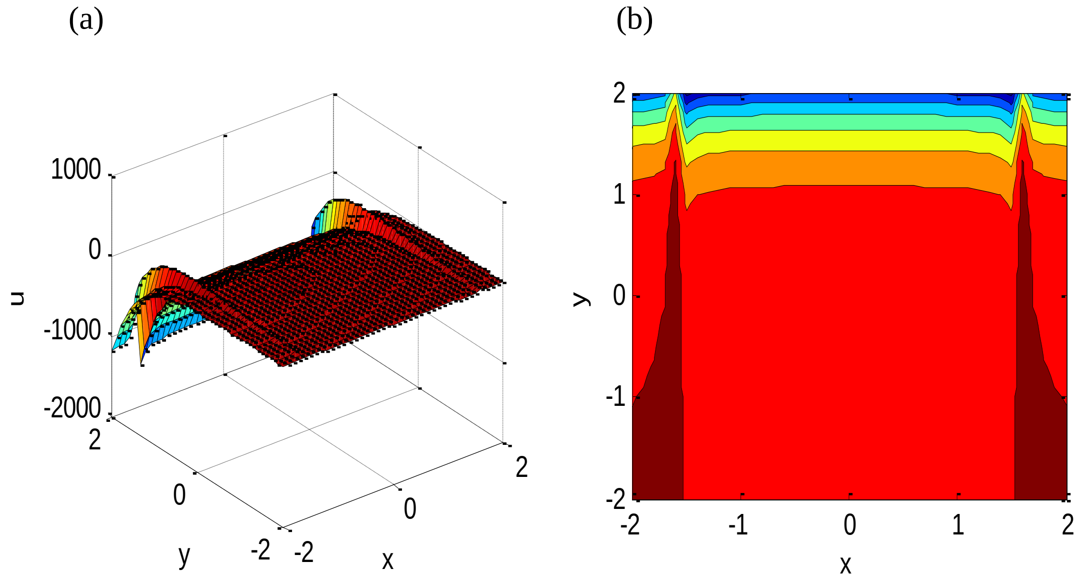


Figure 4.4: (a) The soliton-like solution given by equation (4.3.11) with the settings $\lambda=3$, $\mu=1$, $f(x,t)=\sin(x)+t$, $g(y)=\exp(y)$, $\alpha(t)=\beta(t)=\cos(t)$ and $x, y\in[-2,2]$ at $t = 0$ **(b)** The projection of (a) on *xoy* plane.

Type 2. When $\lambda^2 - 4\mu < 0$ we have

$$\frac{G'(\theta)}{G(\theta)} = -\frac{\lambda}{2} + \Lambda_2 \frac{-A_1 \sin\Lambda_2(f+g) + A_2\cos\Lambda_2(f+g)}{A_1\cos\Lambda_2(f+g) + A_2\sin\Lambda_2(f+g)}, \tag{4.3.12}$$

where $\Lambda_2 = \dfrac{\sqrt{4\mu-\lambda^2}}{2}$ and A_1, A_2 are arbitrary constants.

We can obtain the non-traveling wave solutions with trigonometric functions for the DA equation (4.1) in the form

$$u(x,y,t) = \frac{f_t + f_x\beta(t) - 5g''\alpha(t)}{2\beta(t)f_x} + \left(\frac{\alpha(t)(g')^2}{\beta(t)f_x}\right)\Lambda_2 \frac{-A_1 \sin\Lambda_2(f+g) + A_2\cos\Lambda_2(f+g)}{A_1\cos\Lambda_2(f+g) + A_2\sin\Lambda_2(f+g)}, \tag{4.3.13}$$

where the arbitrary function $f(x, t)$ satisfies the conditions (i) $f_x \neq 0$; (ii) f_x exists and $g(y)$ satisfies (i) $g' \neq 0$; (ii) g' exists; (iii) g'' exists.

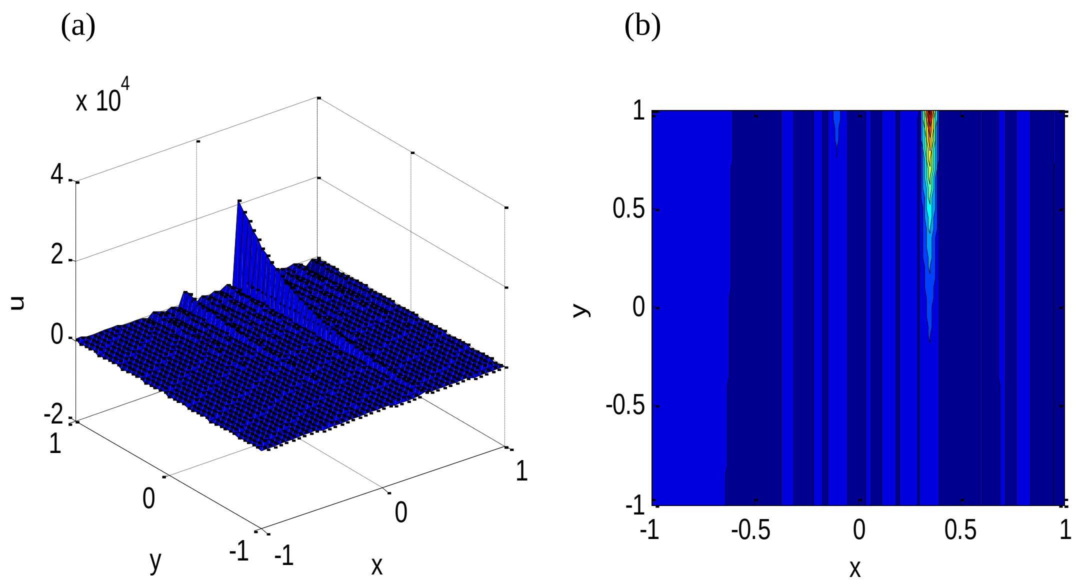


Figure 4.5: (a) The soliton-like solution given by equation (4.3.13) with the settings $\lambda = \mu = 2$, $f(x, t) = \sin(x) + t$, $g(y) = \exp(y)$, $\alpha(t) = \beta(t) = \cos(t)$ and $x, y \in [-1, 1]$ at $t = 0$ **(b)** The projection of (a) on *xoy* plane.

Type 3. When $\lambda^2 - 4\mu = 0$ we have

$$\frac{G'(\theta)}{G(\theta)} = -\frac{\lambda}{2} + \frac{A_2}{A_1 + A_2(f+g)}, \tag{4.3.14}$$

where A_1, A_2 are arbitrary constants.

We can obtain the non-traveling wave solutions with rational functions for the DA equation (4.3.1) in the form

$$u(x, y, t) = \frac{f_t + f_x \beta(t) - 5g''\alpha(t)}{2\beta(t) f_x} + \left(\frac{\alpha(t)(g')^2}{\beta(t) f_x}\right) \frac{A_2}{A_1 + A_2(f+g)}, \tag{4.3.15}$$

where the arbitrary function $f(x, t)$ satisfies the conditions (i) $f_x \neq 0$; (ii) f_x exists and $g(y)$ satisfies (i) $g' \neq 0$; (ii) g' exists; (iii) g'' exists.

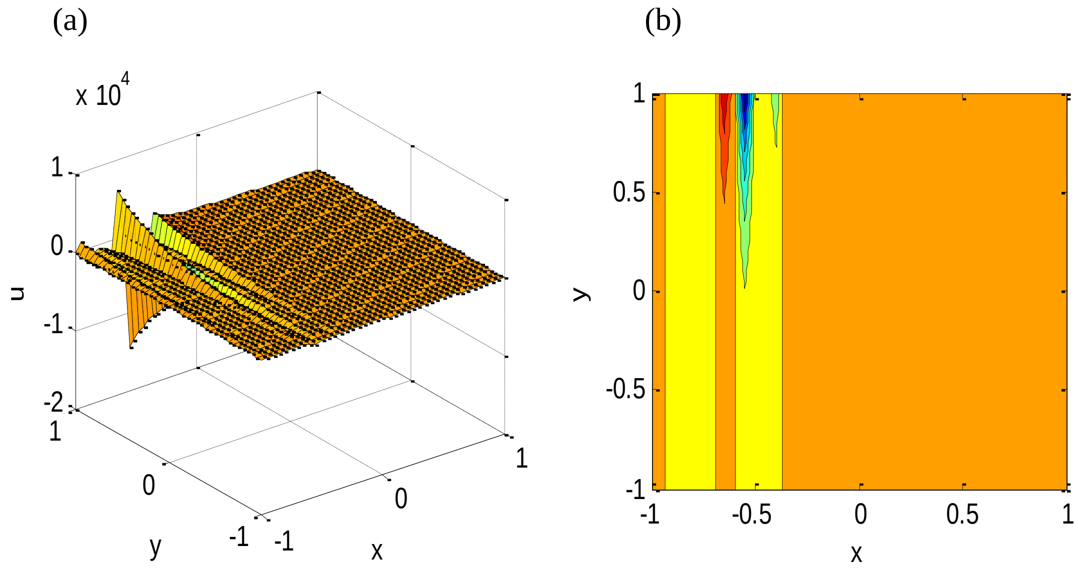


Figure 4.6: (a) The soliton-like solution given by equation (4.3.15) with the settings $f(x, t) = \sin(x) + t$, $g(y) = \exp(y)$, $\alpha(t) = \beta(t) = \cos(t)$ and $x, y \in [-1, 1]$ at $t = 0$ **(b)** The projection of (a) on *xoy* plane.

4.4 Discussion and Concluding Remarks

The Lie symmetry method is utilized to investigate the symmetries and invariant solutions of the DA equations. By determining the transformation group under which a given system is invariant, we obtained information about the invariants and symmetries of that equation. This information, in turn, was used to determine similarity variables that reduced the number of independent variables. The vector fields of the optimal system lead to reduction of the nonlinear PDEs to ODEs. The infinitesimal generators in the optimal system are used for reductions and exact solution. By using these vector fields, we have found group-invariant solution under the subalgebra 4.2.1.1 Then under the subalgebra 4.2.1.2 IVP are posed to (4.2.12) and the numerical solutions are shown with the help of Figures (4.2, 4.3) to decay for a fairly finite value of z. Further, we successfully obtained non-traveling solutions with arbitrary parameters to DA equation (4.3.1) via extended (G'/G)-expansion method and in all the cases, we choose the arbitrary function $f(x, t)$ or $g(y)$ along with various other arbitrary parameters, in a suitable manner, to simulate physical behavior of the solutions with the help of Figures (4.4-4.6). Thus, we found some new exact, numerical and non-traveling wave solutions that might prove to be potentially useful for applications in mathematical physics and applied mathematics.

Chapter 5

Variable Coefficients Coupled KdV-Burgers' Equation

5.1 Introduction

One of the famous nonlinear PDE is Korteweg-de-Vries-Burgers' equation [31]

$$u_t + u u_x + \alpha u_{xx} + \beta u_{xxx} = 0, \tag{5.1.1}$$

where α, β are dissipation and dispersion coefficients respectively. The equation describes the mathematical modeling of liquid flow in bubble and the liquid flow in the elastic pipeline [104]. Also, it was derived in fluid mechanics to describe shallow water waves in a rectangular channel [173] and plays an important role in plasma physics [147]. The Korteweg-de-Vries-Burgers' equation(5.1.1) for $\alpha = 0$, becomes KdV equation derived by Korteweg and de Vries [38] to describe the theory of water waves in shallow channels, such as a canal. Further, for $\beta = 0$, the equation (5.1.1) reduces to the well known Burgers' equation [77] which encountered in the theory of shock waves, mathematical modeling of turbulent fluid and in continuous stochastic processes.

In this chapter, we consider the variable coefficients version of the coupled KdVBurgers' equation [101, 139, 201]

A part of this chapter has been communicated in **Acta Applicandae Mathematicae**

$$u_t + \alpha_1(t)\, u u_x + \beta_1(t)\, w_{xx} + \delta_1(t)\, u_{xxx} = 0,$$
$$w_t + \alpha_2(t) w w_x + \beta_2(t)\, w_{xx} + \delta_2(t)\, u_{xxx} = 0, \tag{5.1.2}$$

where $\alpha_i(t), \beta_i(t)$ and $\delta_i(t)$ $i = 1, 2$ are arbitrary time-dependent coefficients. We have studied the equation (5.1.2) for exact and numerical solutions using Lie symmetry analysis, modified (G'/G)-expansion method and RK4 method.

The layout of this chapter is as follows. Section 5.2 is devoted to an outline of the Lie classical method for generating various symmetries of the equation (5.1.2) and the admissible forms of the coefficients that admit the classical symmetry group corresponding to each member of the optimal system of subalgebras. In Section 5.3, with the use of the modified (G'/G)-expansion method, more explicit exact solutions involving arbitrary parameters of equation (5.1.2) are found out, which are expressed by the hyperbolic functions, the trigonometric functions, and rational functions. Further, in this section with the help of RK4 method numerical solutions of equation (5.1.2) are also found out. Some concluding remarks are given in section 5.4.

5.2 Lie Symmetric Reduction

Let us consider the Lie group of point transformations

$$\begin{aligned} x^* &= x + \varepsilon X(x,t,u,w) + O(\varepsilon^2), \\ t^* &= t + \varepsilon T(x,t,u,w) + O(\varepsilon^2), \\ u^* &= u + \varepsilon U(x,t,u,w) + O(\varepsilon^2), \\ w^* &= w + \varepsilon W(x,t,u,w) + O(\varepsilon^2), \end{aligned} \tag{5.2.1}$$

which leaves the system (5.1.2) invariant. The method for determining the symmetry group of (5.1.2) consists of finding the infinitesimals X, T, U and W, which are functions of x, t, u and w.

Assuming that system (5.1.2) is invariant under the transformations (5.2.1), the infinitesimals X, T, U and W must satisfy the symmetry conditions

$$\begin{aligned} &U_t + \alpha_1(t) U u_x + \alpha_1(t) U_x u + \alpha_1'(t) T u u_x + \beta_1(t) W_{xx} + \beta_1'(t) T w_{xx} \\ &+ \delta_1(t) U_{xxx} + \delta_1'(t) T u_{xxx} = 0, \end{aligned} \tag{5.2.2a}$$

$$\begin{aligned}&W_t+\alpha_2(t)Ww_x+\alpha_2(t)W_x w+\alpha_2{}'(t)Tww_x+\beta_2(t)W_{xx}+\beta_2{}'(t)Tw_{xx}\\&+\delta_2(t)U_{xxx}+\delta_2{}'(t)Tu_{xxx}=0,\end{aligned}\tag{5.2.2b}$$

where $U_t, U_x, W_x, W_t, W_{xx}$ and U_{xxx} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables (for more details, see [58]). Substituting expressions for $U_t, U_x, W_x, W_t, W_{xx}$ and U_{xxx}, into symmetry conditions (5.2.2), then equating the coefficients of the various monomials in the first, second and the other order partial derivatives of u, w and their powers, we can find the determining equations for the symmetry group of the equations (5.1.2). Solving these equations, we get the following forms of infinitesimals:

$$\begin{aligned}&X = a_1\, x + a_5, \quad T = \frac{1}{\alpha_1(t)}\left[(a_1-a_2)\int\alpha_1(t)dt + a_4\right],\\&U = a_2\, u, \qquad W = a_3\, w.\end{aligned}\tag{5.2.3}$$

The functions $\alpha_i(t)$, $\beta_i(t)$ and $\delta_i(t)$, $i=1, 2$ are connected by the relations

$$\begin{aligned}&\beta_1{}'(t)\,T+\beta_1(t)\,T_t+(a_3-2a_1-a_2)\beta_1(t)=0,\\&\delta_1{}'(t)\,T+\delta_1(t)\,T_t-3a_1\,\delta_1(t)=0,\\&\alpha_2{}'(t)\,T+\alpha_2(t)\,T_t+(a_3-a_1)\alpha_2(t)=0,\\&\beta_2{}'(t)\,T+\beta_2(t)\,T_t-2a_1\,\beta_2(t)=0,\\&\delta_2{}'(t)\,T+\delta_2(t)\,T_t-(a_3-a_2+3a_1)\delta_2(t)=0.\end{aligned}\tag{5.2.4}$$

where a_1, a_2, a_3, a_4 and a_5 are arbitrary constants. The Lie algebra associated with system (5.1.2) consists of following five vector fields

$$\begin{aligned}&V_1 = x\frac{\partial}{\partial x} + \frac{1}{\alpha_1(t)}\int\alpha_1(t)\,dt\,\frac{\partial}{\partial t}, \quad V_2 = u\frac{\partial}{\partial u} - \frac{1}{\alpha_1(t)}\int\alpha_1(t)\,dt\,\frac{\partial}{\partial t},\\&V_3 = w\frac{\partial}{\partial w}, \quad V_4 = \frac{1}{\alpha_1(t)}\frac{\partial}{\partial t}, \quad V_5 = \frac{\partial}{\partial x}.\end{aligned}\tag{5.2.5}$$

It is easy to verify that V_j, $j=1, 2, 3, 4, 5$ is closed under the Lie bracket.

The classification of one-dimensional subalgebras of the whole symmetry algebra (5.2.5) is done by an inductive approach [143]. We will work out first an optimal system and then embark upon the various reductions associated with generators in the optimal system. We begin by considering a general element $V = a_1 V_1 + a_2 V_2 + a_3 V_3 + a_4 V_4 + a_5 V_5$ of symmetry algebra and subject it to various adjoint transformations to simplify it as much as possible. The adjoint action is given by the Lie series

$$\mathrm{Ad}\left(\exp\left(\varepsilon V_i\right)\right)V_j = V_j - \varepsilon\left[V_i, V_j\right] + \frac{\varepsilon^2}{2}\left[V_i, \left[V_i, V_j\right]\right] - \cdots, \tag{5.2.6}$$

where $\left[V_i, V_j\right] = V_i V_j - V_j V_i$ is the commutator for the Lie algebra, and ε is a parameter.

The types of one-dimensional subalgebras of system (5.1.2) are as follows

(i) $V_1 + \lambda_1 V_2 + \lambda_2 V_3$,

(ii_a) $V_2 + \lambda_3 V_3 + V_5$,

(ii_b) $V_2 + \lambda_3 V_3 - V_5$,

(iii) $V_2 + \lambda_4 V_3$,

(iv_a) $V_3 + \lambda_5 V_4 + V_5$,

(iv_b) $V_3 + \lambda_5 V_4 - V_5$,

(v) $V_3 + \lambda_6 V_4$,

(vi_a) $V_4 + V_5$,

(vi_b) $V_4 - V_5$,

(vii) V_4,

(viii) V_5, (5.2.7)

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 are arbitrary constants. Because the discrete symmetry $(x, t, u) \to (-x, t, u)$ will map (ii_a), (iv_a) and (vi_a) to (ii_b), (iv_b) and (vi_b) respectively, and therefore, in the optimal system, we confine ourselves to eight essential vector fields of the optimal system, while neglecting the other three (ii_b), (iv_b) and (vi_b).

Once the subalgebra of the equation (5.1.2) have been determined, the symmetry variables for the associated reduction can be found by solving the characteristic equation

$$\frac{dx}{X(x,t,u,w)} = \frac{dt}{T(x,t,u,w)} = \frac{du}{U(x,t,u,w)} = \frac{dw}{W(x,t,u,w)}. \tag{5.2.8}$$

On solving the equation (5.2.8) for the various operators in the optimal system, we obtain a set of non-equivalent Lie ansätze for the function $u(x, t)$ and $w(x, t)$. Using these ansätze, we can reduce the nonlinear system (5.1.2) to system of ODEs.

5.3 Exact and Numerical Solutions

In this section, the primary focus is on the reductions associated with the subalgebras in the optimal system. Corresponding, to every essential subalgebras in the optimal system, the reduced ODEs are obtained. Each reduced system of ODEs can be further studied for exact and numerical solutions

5.3.1 Subalgebra $V_1 + \lambda_1 V_2 + \lambda_2 V_3$

In this subalgebra we obtain the following similarity variables and similarity solution for the reduction as

$$\xi = x\left(\int \alpha_1(t)\right)^{\frac{-1}{1-\lambda_1}},$$

$$u(x,t) = \left(\int \alpha_1(t)dt\right)^{\frac{\lambda_1}{1-\lambda_1}} F(\xi),$$

$$w(x,t) = \left(\int \alpha_1(t)dt\right)^{\frac{\lambda_2}{1-\lambda_1}} G(\xi), \tag{5.3.1}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = \frac{K_1}{1-\lambda_1}\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\frac{2\lambda_1-\lambda_2+1}{1-\lambda_1}}, \qquad \delta_1(t) = \frac{K_2}{1-\lambda_1}\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\frac{\lambda_1+2}{1-\lambda_1}},$$

$$\alpha_2(t) = \frac{K_3}{1-\lambda_1}\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\frac{\lambda_1-\lambda_2}{1-\lambda_1}}, \qquad \beta_2(t) = \frac{K_4}{1-\lambda_1}\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\frac{\lambda_1+1}{1-\lambda_1}},$$

$$\delta_2(t) = \frac{K_5}{1-\lambda_1}\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\frac{\lambda_2+2}{1-\lambda_1}}. \tag{5.3.2}$$

Using the similarity variable and the forms of the similarity solution (5.3.1), the system of PDEs (5.1.2) reduces to the following system of ODEs:

$$\lambda_1 F - F'\,\xi + (1-\lambda_1)\,F\,F' + K_1\,G'' + K_2\,F''' = 0,$$

$$\lambda_2 G - G'\,\xi + K_3\,G\,G' + K_4 G'' + K_5\,F''' = 0. \tag{5.3.3}$$

Solving the above ODEs (5.3.3) main system (5.1.2) possesses the following solutions: Herein, we seek a special solution for (5.3.3) in the following form:

$$F = a_1\,\xi^{-2} + b_1\,\xi$$

$$G = a_2\,\xi^{-2} + b_2\,\xi \tag{5.3.4}$$

By substituting these expressions for F and G in (5.3.3), we get the following relations among the various parameters:

$$b_1 = 1,\ K_1 = 0,\ K_2 = -\frac{a_1}{8},\ K_3 = \frac{3}{2b_2},\ K_4 = 0,\ K_5 = -\frac{a_2{}^2}{8b_2\,a_1},\ \lambda_1 = \lambda_2 = -\frac{1}{2} \tag{5.3.5}$$

where a_1, a_2 and b_2 are arbitrary constants.

Thus, on using the back transformation, we get the solution of the system (5.1.2) as

$$u(x,t) = \left(\int \alpha_1(t)dt\right)^{-1}\left(a_1\left(x\left(\int \alpha_1(t)\,dt\right)^{-2}\right)^{-2} + x\left(\int \alpha_1(t)\,dt\right)^{-2}\right),$$

$$w(x,t) = \left(\int \alpha_1(t)dt\right)^{-1}\left(a_2\left(x\left(\int \alpha_1(t)\,dt\right)^{-2}\right)^{-2} + b_2\,x\left(\int \alpha_1(t)\,dt\right)^{-2}\right) \tag{5.3.6}$$

5.3.2 Subalgebra $V_2 + \lambda_3\,V_3 + V_5$

Under this subalgebra the optimal system defines the similarity variable and similarity solution as follows

$$\xi = x + \log\int \alpha_1(t)dt,$$

$$u(x,t) = \left(\int \alpha_1(t)\,dt\right)^{-\lambda_3} F(\xi),$$

$$w(x,t) = \left(\int \alpha_1(t)\,dt\right)^{-1} G(\xi). \tag{5.3.7}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = K_1\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\lambda_3 - 2}, \qquad \delta_1(t) = K_2\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{-1},$$

$$\alpha_2(t) = K_3\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\lambda_3 - 1}, \qquad \beta_2(t) = K_4\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{-1},$$

$$\delta_2(t) = K_5\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{-\lambda_3}. \tag{5.3.8}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (5.1.2) reduces to the following system of ODEs

$$-F + F' + F F' + K_1 G'' + K_2 F''' = 0,$$
$$-\lambda_3 G + G' + K_3 G G' + K_4 G'' + K_5 F''' = 0. \tag{5.3.9}$$

Now with respect to the ODEs (5.3.9) we are unable to obtain the nontrivial exact solutions for the coupled KdV-Burgers' equation (5.1.2) so under this subalgebra we present the numerical solutions of main system (5.1.2) by solving reduced ODEs (5.3.9) with the help of RK4 method. For numerical solution we pose the following initial value problem (IVP) for the ODEs (5.3.9) after reducing it first order ODEs.

$$\frac{dy_1}{dz} = y_2, \qquad \frac{dy_2}{dz} = y_3, \qquad \frac{dy_3}{dz} = \frac{y_1 - y_2 - y_1 y_2 - K_1 y_6}{K_2},$$

$$\frac{dy_4}{dz} = y_5, \qquad \frac{dy_5}{dz} = y_6 \qquad \frac{dy_3}{dz} = \frac{\lambda_6 y_4 - y_5 - K_3 y_4 y_5 - K_4 y_6}{K_5}, \tag{5.3.10}$$

with $y_1(1) = -1.5$, $y_2(1) = 1$, $y_3(1) = 0$, $y_4(1) = 0$, $y_5(1) = -0.5$ and $y_6(1) = -1$.
(5.3.11)

The numerical solution to the IVP (5.3.10)–(5.3.11) are depicted below

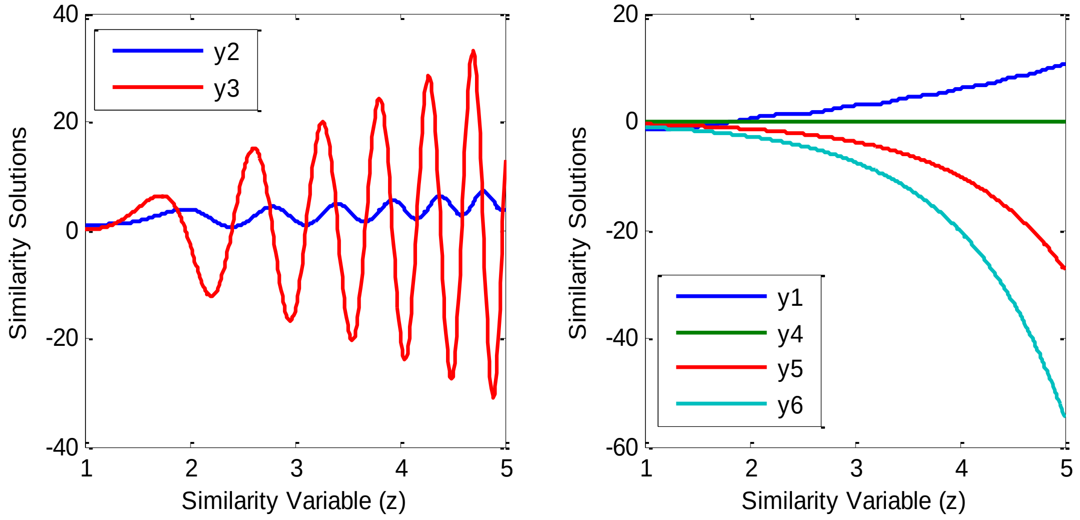


Figure 5.1: Numerical solutions for the reduced IVP (5.3.10-5.3.11) for equation (5.1.2) when $K_1 = 1$, $K_2 = 0.04$, $K_3 = 5$, $K_4 = -0.5$, $K_5 = 0.5$ and $\lambda_6 = 10$, with initial value $z = 1$ at $h = 0.01$.

5.3.3 Subalgebra $V_2 + \lambda_4 V_3$

For this subalgebra, on solving the equations (5.2.4) and (5.2.8) we obtain the following similarity variables and similarity solution for the reduction

$$\xi = x,$$
$$u(x,t) = \left(\int \alpha_1(t)\,dt\right)^{-1} F(\xi),$$
$$w(x,t) = \left(\int \alpha_1(t)\,dt\right)^{-\lambda_4} G(z), \tag{5.3.12}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = K_1\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\lambda_4-2}, \qquad \delta_1(t) = K_2\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{-1},$$
$$\alpha_2(t) = K_3\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{\lambda_4-1}, \qquad \beta_2(t) = K_4\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{-1},$$
$$\delta_2(t) = K_5\,\alpha_1(t)\left(\int \alpha_1(t)dt\right)^{-\lambda_4}. \tag{5.3.13}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (5.1.2) reduces to the following system of ODEs

$$-F + F\,F' + K_1\,G'' + K_2\,F''' = 0,$$
$$-\lambda_4 G + K_3\,G\,G' + K_4\,G'' + K_5\,F''' = 0. \tag{5.3.14}$$

Herein, we exploit the idea used in [197] to construct solutions of equation (5.3.14) by modified (G'/G)-expansion method with the restriction $\lambda_4 = 0$.

Let us assume that the system (5.3.14) admits a solution in the (G'/G) form as follows

$$F(\xi) = a_0 + \sum_{i=1}^{m}\left\{a_i\left(\frac{G'(\xi)}{G(\xi)}\right)^{i} + b_i\left(\frac{G'(\xi)}{G(\xi)}\right)^{-i}\right\},$$
$$G(\xi) = c_0 + \sum_{j=1}^{n}\left\{c_j\left(\frac{G'(\xi)}{G(\xi)}\right)^{j} + d_j\left(\frac{G'(\xi)}{G(\xi)}\right)^{-j}\right\}. \tag{5.3.15}$$

where the function $G(\xi)$ is the solution of the linear ODE

$$G''(\xi) + \mu G(\xi) = 0, \tag{5.3.16}$$

While $a_0, a_i, b_i, c_0, c_j, d_j$ and μ are constants to be determined later, the positive integer m, n can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODEs (5.3.14).

Now, balancing the highest-order derivatives and nonlinear terms appearing in equation (5.3.14), we get $m = 2$, $n = 2$ and thus we write

$$F(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + b_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} + b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2},$$

$$G(\xi) = c_0 + c_1\left(\frac{G'(\xi)}{G(\xi)}\right) + c_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + d_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} + d_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2}. \quad (5.3.17)$$

By substituting (5.3.17) into the first equation of system (5.3.14) and using linear ODE (5.3.16), and equating the coefficients of same powers of (G'/G) to zero, yields a set of simultaneous algebraic equations among $a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, d_1, d_2$ and μ as follows:

(i) $24b_2 K_2 \mu^3 + 2b_2^2 \mu = 0$,

(ii) $6d_2 K_1 \mu^2 + 6b_1 \mu^3 K_2 + 3\mu b_1 b_2 = 0$,

(iii) $2d_1 K_1 \mu^2 + 40b_2 \mu^2 K_2 + 2a_0 \mu b_2 + b_1^2 \mu + 2b_2^2 = 0$,

(iv) $8d_2 K_1 \mu + 8b_1 \mu^2 K_2 + a_0 \mu b_1 + b_2 a_1 \mu + 3b_1 b_2 - b_2 = 0$,

(v) $2d_1 K_1 \mu + 16b_2 \mu K_2 + 2a_0 b_2 - 2b_2 a_2 \mu + b_1^2 + 2a_2 b_2 \mu - b_1 = 0$,

(vi) $2d_2 K_2 - 2a_1 \mu^2 K_2 + 2b_1 \mu K_2 - a_0 a_1 \mu + a_0 b_1 + a_1 b_2 - b_1 a_2 \mu + 2c_2 K_1 \mu^2 - a_0 = 0$,

(vii) $2c_1 K_1 \mu - 16a_2 \mu^2 K_2 - 2a_0 a_2 \mu - a_1^2 \mu - a_1 = 0$,

(viii) $8c_2 K_1 \mu - 8a_1 \mu K_2 - 3a_1 a_2 \mu - a_0 a_1 - a_2 b_1 - a_2 = 0$,

(ix) $2c_1 K_1 - 40a_2 \mu K_2 - 2a_0 a_2 - a_1^2 - 2a_2^2 \mu = 0$,

(x) $-6a_1 K_2 - 3a_1 a_2 + 6K_1 c_2 = 0$,

(xi) $-24a_2 K_2 - 2a_2^2 = 0$, (5.3.18)

Substituting (5.3.17) into the second equation of ODEs (5.3.14) yields the following system of algebraic equations:

(i) $24b_2 K_5 \mu^3 + 2K_3 d_2^2 \mu = 0$,

(ii) $6d_2 K_4 \mu^2 + 6b_1 \mu^3 K_5 + 3K_3 \mu d_1 d_2 = 0$,

(iii) $2d_1 K_4 \mu^2 + 40b_2 \mu^2 K_5 + 2c_0 \mu d_2 K_3 + K_3 d_1^2 \mu + 2K_3 d_2^2 = 0$,

(iv) $8d_2 K_4 \mu + 8b_1 \mu^2 K_5 + K_3 c_0 \mu d_1 + K_3 d_2 c_1 \mu + 3K_3 d_1 d_2 = 0,,$

(v) $2d_1 K_4 \mu + 16b_2 \mu K_5 + 2K_3 c_0 d_2 - 2K_3 d_2 c_2 \mu + K_3 d_1^2 + 2K_3 c_2 d_2 \mu = 0,$

(vi) $2d_2K_4 - 2a_1\mu^2K_5 + 2b_1\mu K_5 - K_3c_0 c_1\mu + K_3c_0d_1 + K_3c_1d_2 - K_3d_1c_2\mu + 2c_2K_4\mu^2 = 0,$

(vii) $2c_1 K_4 \mu - 16a_2 \mu^2 K_5 - 2K_3 c_0 c_2 \mu - K_3 c_1^2 \mu = 0,$

(viii) $8c_2 K_4 \mu - 8a_1 \mu K_5 - 3K_3 c_1 c_2 \mu - K_3 c_0 c_1 - K_3 c_2 d_1 = 0,$

(ix) $2c_1 K_4 - 40a_2 \mu K_5 - 2K_3 c_0 c_2 - K_3 c_1^2 - 2K_3 c_2^2 \mu = 0,$

(x) $-6a_1 K_5 - 3K_3 c_1 c_2 + 6K_4 c_2 = 0,$

(xi) $-24a_2 K_5 - 2K_3 c_2^2 = 0.$ (5.3.19)

Solving these algebraic equations we obtain the following results:

Case1:

$$a_0 = a_0,\ b_1 = 1,\ c_0 = c_0,\ d_1 = d_1 \text{ and } a_1 = a_2 = b_2 = c_1 = c_2 = d_2 = K_3 = \mu = 0, \quad (5.3.20)$$

Case 2:

$$a_0 = a_0,\ b_1 = 1,\ c_1 = \frac{2K_4}{K_3} \text{ and } a_1 = a_2 = b_2 = c_0 = c_2 = d_2 = d_2 = K_1 = \mu = 0, \quad (5.3.21)$$

From (5.3.16) and the general solution of (5.3.18) and (5.3.19), we deduce the rational solutions of (5.1.2) as follows:

Case 1, gives the rational solution of equation (5.1.2) as follows:

$$u(x,t) = \left(\int \alpha_1(t)\,dt\right)^{-1}\left[a_0 + \left(\frac{B}{A + Bx}\right)^{-1}\right],$$

$$w(x,t) = c_0 + d_1\left(\frac{B}{A + Bx}\right)^{-1}, \quad (5.3.22)$$

where A and B are arbitrary constants.

Similarly, **case 2**, gives the rational solution as follows:

$$u(x,t) = \left(\int \alpha_1(t)\,dt\right)^{-1}\left[a_0 + \left(\frac{B}{A + Bx}\right)^{-1}\right],$$

$$w(x,t) = \frac{2K_4}{K_3}\left(\frac{B}{A + Bx}\right)^{2}, \quad (5.3.23)$$

where A, B, K_3 and K_4 are arbitrary constants.

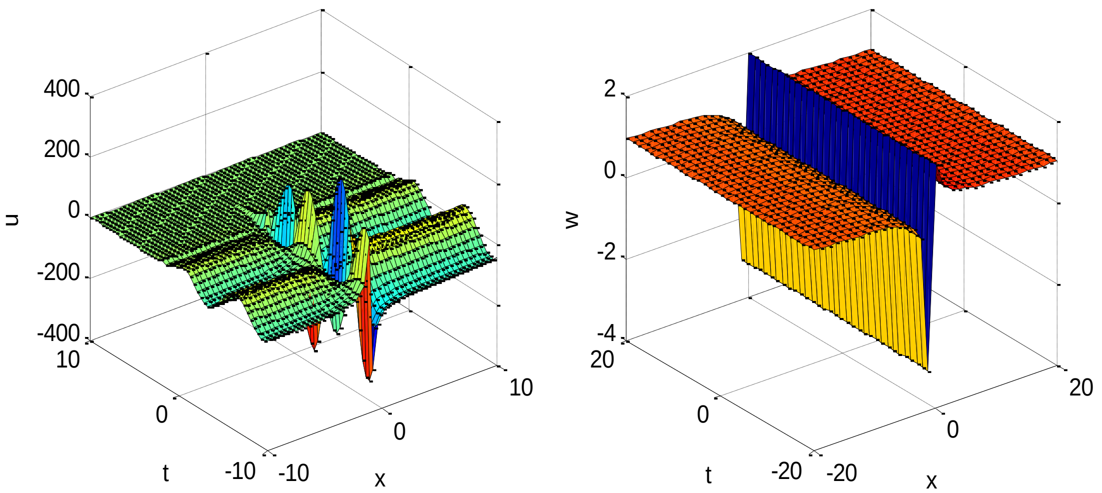


Figure 5.2: Periodic solution (left) and soliton-like solution (right) given by equation (5.3.22) with $A = 1,\ \ B = 1,\ a_0 = c_0 = d_1 = 1$ and $\alpha_1(t) = \exp(t) + \sec(t)\,\tan(t) + t$.

5.3.4 Subalgebra $V_3 + \lambda_5 V_4 + V_5$

Following the same way as above, we get similarity variable and similarity solution as follows

$$\xi = x - \frac{1}{\lambda_5}\int \alpha_1(t)dt,$$

$$u(x,t) = F(\xi),$$

$$w(x,t) = \exp\left(\frac{1}{\lambda_5}\int \alpha_1(t)dt\right)G(\xi), \tag{5.3.24}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = \frac{K_1}{\lambda_5}\,\alpha_1(t)\exp\left(-\frac{1}{\lambda_5}\int \alpha_1(t)dt\right), \qquad \delta_1(t) = \frac{K_2}{\lambda_5}\,\alpha_1(t),$$

$$\alpha_2(t) = \frac{K_3}{\lambda_5}\,\alpha_1(t)\exp\left(-\frac{1}{\lambda_5}\int \alpha_1(t)dt\right), \quad \beta_2(t) = \frac{K_4}{\lambda_5}\,\alpha_1(t),$$

$$\delta_2(t) = \frac{K_5}{\lambda_5}\,\alpha_1(t)\exp\left(\frac{1}{\lambda_5}\int \alpha_1(t)dt\right). \tag{5.3.25}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (5.1.2) reduces to the following system of ODEs

$$-\ F' + \lambda_5\ F\ F' + K_1\ G'' + K_2\ F''' = 0,$$

$$G -\ G' + K_3 G\, G' + K_4\ G'' + K_5\ F''' = 0. \tag{5.3.26}$$

With respect to the ODEs (5.3.26), we are also unable to obtain the nontrivial exact solutions for the system (5.1.2). So, under this subalgebra, we present the numerical solutions of main system by solving reduced ODEs (5.3.26) with the help of RK4 method. For numerical solution, we pose the following initial value problem (IVP) for the ODEs (5.3.26) after reducing it first order ODEs.

$$\frac{dy_1}{dz} = y_2, \qquad \frac{dy_2}{dz} = y_3, \qquad \frac{dy_3}{dz} = \frac{y_1 - \lambda_5\, y_1\, y_2 - K_1\, y_6}{K_2},$$

$$\frac{dy_4}{dz} = y_5, \qquad \frac{dy_5}{dz} = y_6, \qquad \frac{dy_3}{dz} = \frac{-\,y_4 + y_5 - K_3 y_4\, y_5 - K_4\, y_6}{K_5}, \tag{5.3.27}$$

with $y_1(1) = 3.5$, $y_2(1) = 6.9$, $y_3(1) = 0$, $y_4(1) = 0$, $y_5(1) = 0.43$ and $y_6(1) = 0.87$.

(5.3.28)

The numerical solution to the IVP (5.3.27)–(5.3.28) are depicted as follow

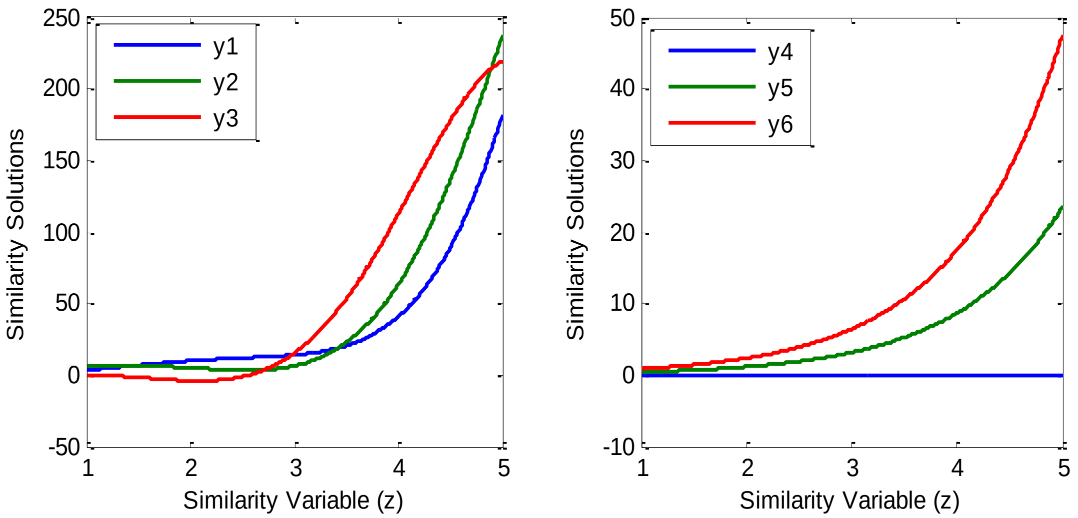


Figure 5.3: Numerical solutions for the reduced IVP (5.3.27-5.3.28) for equation (5.1.2) when $K_1 = 4$, $K_2 = -0.22$, $K_3 = 1$, $K_4 = 0.5$, $K_5 = 0.04$ and $\lambda_5 = 0$, with initial value $z = 1$ at $h = 0.01$.

5.3.5 Subalgebra $V_3 + \lambda_6 \ V_4$

For this subalgebra the associated similarity variable and similarity solution are as follows:

$$\xi = x,$$
$$u(x,t) = F(\xi),$$
$$w(x,t) = \exp\left(\frac{1}{\lambda_6}\int \alpha_1(t)dt\right)G(\xi), \tag{5.3.29}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = \frac{K_1}{\lambda_6}\,\alpha_1(t)\exp\left(-\frac{1}{\lambda_6}\int \alpha_1(t)dt\right), \qquad \delta_1(t) = \frac{K_2}{\lambda_6}\,\alpha_1(t),$$
$$\alpha_2(t) = \frac{K_3}{\lambda_6}\,\alpha_1(t)\exp\left(-\frac{1}{\lambda_6}\int \alpha_1(t)dt\right), \quad \beta_2(t) = \frac{K_4}{\lambda_6}\,\alpha_1(t)\ ,$$
$$\delta_2(t) = \frac{K_5}{\lambda_6}\,\alpha_1(t)\exp\left(\frac{1}{\lambda_6}\int \alpha_1(t)dt\right). \tag{5.3.30}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (5.1.2) reduces to the following system of ODEs

$$\lambda_6\ F\ F' + K_1\ G'' + K_2\,F''' = 0,$$
$$G\ + K_3\,G\,G' + K_4\ G'' + K_5\,F''' = 0. \tag{5.3.31}$$

In this subalgebra, we seek solutions of equation (5.3.31) by modified (G'/G)-expansion method. Following the procedure adopted in subalgebra 5.3.3, suppose the solution of (5.3.31) can be expressed in (G'/G) as in (5.3.15), where $G(\xi)$ satisfies the linear ODE (5.3.16). Considering the homogeneous balance between highest-order derivatives with the nonlinear terms appearing in (5.3.31), we get $m = 2,\ n = 2$. So we can suppose that the solution of ODEs (5.3.31) is of the form

$$F(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + b_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} + b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2},$$
$$G(\xi) = c_0 + c_1\left(\frac{G'(\xi)}{G(\xi)}\right) + c_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + d_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} + d_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2}. \tag{5.3.32}$$

By substituting (5.3.32) into the first equation of system (5.3.31) and using linear ODE (5.3.16), and equating the coefficients of same powers of (G'/G) to zero, yields a set of simultaneous algebraic equations among $a_0,\ a_1,\ a_2, b_1,\ b_2,\ c_0,\ c_1,\ c_2,\ d_1,\ d_2$ and μ as follows:

(i) $24b_2 K_2 \mu^3 + 2\lambda_6 b_2^2 \mu = 0,$

(ii) $6d_2 K_1 \mu^2 + 6b_1 \mu^3 K_2 + 3\lambda_6 \mu b_1 b_2 = 0,$

(iii) $2d_1 K_1 \mu^2 + 40b_2 \mu^2 K_2 + 2\lambda_6 a_0 \mu b_2 + \lambda_6 b_1^2 \mu + 2\lambda_6 b_2^2 = 0,$

(iv) $8d_2 K_1 \mu + 8b_1 \mu^2 K_2 + a_0 \lambda_6 \mu b_1 + b_2 \lambda_6 a_1 \mu + 3\lambda_6 b_1 b_2 = 0,$

(v) $2d_1 K_1 \mu + 16b_2 \mu K_2 + 2\lambda_6 a_0 b_2 - 2\lambda_6 b_2 a_2 \mu + \lambda_6 b_1^2 + 2\lambda_6 a_2 b_2 \mu = 0,$

(vi) $$2d_2K_2 - 2a_1\mu^2K_2 + 2b_1\mu K_2 - \lambda_6 a_0 a_1\mu + \lambda_6 a_0 b_1 + \lambda_6 a_1 b_2 - \lambda_6 b_1 a_2 \mu + 2c_2 K_1 \mu^2 = 0,$$

(vii) $2c_1 K_1 \mu - 16a_2 \mu^2 K_2 - 2\lambda_6 a_0 a_2 \mu - a_1^2 \lambda_6 \mu = 0,$

(viii) $8c_2 K_1 \mu - 8a_1 \mu K_2 - 3\lambda_6 a_1 a_2 \mu - \lambda_6 a_0 a_1 - \lambda_6 a_2 b_1 = 0,$

(ix) $2c_1 K_1 - 40a_2 \mu K_2 - 2\lambda_6 a_0 a_2 - \lambda_6 a_1^2 - 2\lambda_6 a_2^2 \mu = 0,$

(x) $-6a_1 K_2 - 3\lambda_6 a_1 a_2 + 6K_1 c_2 = 0,$

(xi) $-24a_2 K_2 - 2\lambda_6 a_2^2 = 0.$ (5.3.33)

Substituting (5.3.32) into the second equation of ODEs (5.3.31) yields the following system of algebraic equations:

(i) $24b_2 K_5 \mu^3 + 2K_3 d_2^2 \mu = 0,$

(ii) $6d_2 K_4 \mu^2 + 6b_1 \mu^3 K_5 + 3K_3 \mu d_1 d_2 = 0,$

(iii) $2d_1 K_4 \mu^2 + 40b_2 \mu^2 K_5 + 2c_0 \mu d_2 K_3 + K_3 d_1^2 \mu + 2K_3 d_2^2 = 0,$

(iv) $8d_2 K_4 \mu + 8b_1 \mu^2 K_5 + K_3 c_0 \mu d_1 + K_3 d_2 c_1 \mu + 3K_3 d_1 d_2 + d_2 = 0,,$

(v) $2d_1 K_4 \mu + 16b_2 \mu K_5 + 2K_3 c_0 d_2 - 2K_3 d_2 c_2 \mu + K_3 d_1^2 + 2K_3 c_2 d_2 \mu + d_1 = 0,$

(vi) $$2d_2K_4 - 2a_1\mu^2K_5 + 2b_1\mu K_5 - K_3c_0c_1\mu + K_3c_0d_1 + K_3c_1d_2 - K_3d_1c_2\mu + 2c_2K_4\mu^2 + c_0 = 0$$

(vii) $2c_1 K_4 \mu - 16a_2 \mu^2 K_5 - 2K_3 c_0 c_2 \mu - K_3 c_1^2 \mu + c_1 = 0,$

(viii) $8c_2 K_4 \mu - 8a_1 \mu K_5 - 3K_3 c_1 c_2 \mu - K_3 c_0 c_1 - K_3 c_2 d_1 + c_2 = 0,$

(ix) $2c_1 K_4 - 40 a_2 \mu K_5 - 2K_3 c_0 c_2 - K_3 c_1^2 - 2K_3 c_2^2 \mu = 0,$

(x) $-6a_1 K_5 - 3K_3 c_1 c_2 + 6K_4 c_2 = 0,$

(xi) $-24a_2 K_5 - 2K_3 c_2^2 = 0.$ (5.3.34)

Solving these algebraic equations we obtain the following results:

Case1:

$$a_2 = a_2,\ c_0 = c_0,\ d_1 = -\frac{1}{K_3},\ K_2 = -\frac{1}{12}\lambda_6 a_2,\ a_0 = a_1 = b_1 = b_2 = c_1 = c_2 = d_2 = K_5 = \mu = 0 \tag{5.3.35}$$

Case 2:

$$a_2 = -\frac{12K_2}{\lambda_6},\ c_0 = c_0,\ d_1 = -\frac{1}{K_3},\ \text{and}\ a_0 = a_1 = b_1 = b_2 = c_1 = c_2 = d_2 = K_5 = \mu = 0, \tag{5.3.36}$$

From (5.3.16) and the general solution of (5.3.33) and (5.3.34), we deduce the rational solutions of (5.1.2) as follows:

Case 1, gives the rational solution of equation (5.1.2) as follows:

$$u(x,t) = a_2\left(\frac{B}{A + Bx}\right)^2,$$

$$w(x,t) = \exp\left(\frac{1}{\lambda_6}\int \alpha_1(t)\,dt\right)\left[c_0 - \frac{1}{K_3}\left(\frac{B}{A + Bx}\right)^{-1}\right], \tag{5.3.37}$$

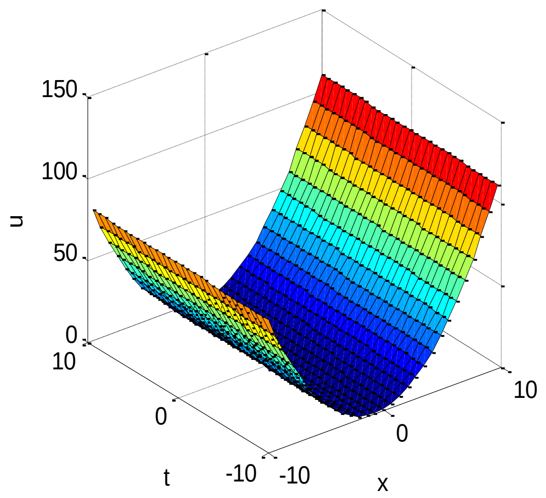
where A, B, K_3 and λ_6 are arbitrary constants.

Similarly, **case 2**, gives the rational solution as follows:

$$u(x,t) = -\frac{12K_2}{\lambda_6}\left(\frac{A_2}{A_1 + A_2 x}\right)^2,$$

$$w(x,t) = \exp\left(\frac{1}{\lambda_6}\int \alpha_1(t)\,dt\right)\left[c_0 - \frac{1}{K_3}\left(\frac{A_2}{A_1 + A_2 x}\right)^{-1}\right], \tag{5.3.38}$$

where A, B, K_2, K_3 and λ_6 are arbitrary constants.



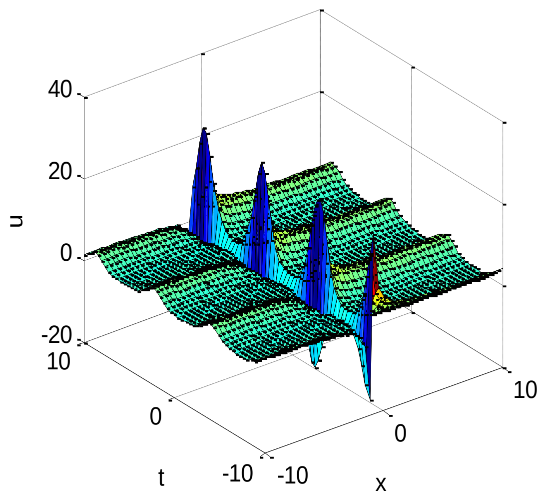


Figure 5.4: U-shaped soliton solution (left) periodic solution (right) given by equation (5.3.37) with $K_3 = -1,\ \ A = B = 1,\ a_2 = c_0 = \lambda_6 = 1$ and $\alpha_1(t) = \cos(t) + \sin(t)$.

5.3.6 Subalgebra $V_4 + V_5$

On solving the equations (5.2.4) and (5.2.8), we get

$$\xi = x - \int \alpha_1(t)dt,$$
$$u(x,t) = F(\xi),$$
$$w(x,t) = G(\xi). \tag{5.3.39}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = K_1\ \alpha_1(t),\ \ \delta_1(t) = K_2\ \alpha_1(t),\ \ \alpha_2(t) = K_3\ \alpha_1(t),$$
$$\beta_2(t) = K_4\,\alpha_1(t),\ \ \delta_2(t) = K_5\ \alpha_1(t). \tag{5.3.40}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (5.1.2) reduces to the following system of ODEs

$$-F' + F\,F' + K_1\ G'' + K_2\,F''' = 0,$$
$$-G' + K_3\,G\,G' + K_4\ G'' + K_5\,F''' = 0. \tag{5.3.41}$$

In this subalgebra, follow the procedure in a manner similar to the preceding subalgebra to solve the equations (5.3.41) by modified (G'/G)-expansion method we get solution of ODEs (5.3.41) as follows:

$$F(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + b_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} + b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2},$$

$$G(\xi) = c_0 + c_1\left(\frac{G'(\xi)}{G(\xi)}\right) + c_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + d_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} + d_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2}. \quad (5.3.42)$$

By substituting (5.3.42) into the first ODE of (5.3.41) and using linear ODE (5.3.16), and equating the coefficients of same powers of (G'/G) to zero, yields a set of simultaneous algebraic equations among $a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, d_1, d_2$ and μ as follows:

(i) $24b_2 K_2 \mu^3 + 2b_2^2 \mu = 0$,

(ii) $6d_2 K_1 \mu^2 + 6b_1 \mu^3 K_2 + 3\mu b_1 b_2 = 0$,

(iii) $2d_1 K_1 \mu^2 + 40b_2 \mu^2 K_2 + 2a_0 \mu b_2 + b_1^2 \mu + 2b_2^2 - 2b_2 \mu = 0$,

(iv) $8d_2 K_1 \mu + 8b_1 \mu^2 K_2 + a_0 \mu b_1 + b_2 a_1 \mu + 3b_1 b_2 - b_1 \mu = 0$,

(v) $2d_1 K_1 \mu + 16b_2 \mu K_2 + 2a_0 b_2 - 2b_2 a_2 \mu + b_1^2 + 2a_2 b_2 \mu - 2b_2 = 0$,

(vi) $2d_2K_2 - 2a_1\mu^2K_2 + 2b_1\mu K_2 - a_0a_1\mu + a_0b_1 + a_1b_2 - b_1a_2\mu + 2c_2K_1\mu^2 + a_1\mu - b_1 = 0$,

(vii) $2c_1 K_1 \mu - 16a_2 \mu^2 K_2 - 2a_0 a_2 \mu - a_1^2 \mu + 2a_2 \mu = 0$,

(viii) $8c_2 K_1 \mu - 8a_1 \mu K_2 - 3a_1 a_2 \mu - a_0 a_1 - a_2 b_1 + a_1 = 0$,

(ix) $2c_1 K_1 - 40a_2 \mu K_2 - 2a_0 a_2 - a_1^2 - 2a_2^2 \mu + 2a_2 = 0$,

(x) $-6a_1 K_2 - 3a_1 a_2 + 6K_1 c_2 = 0$,

(xi) $-24a_2 K_2 - 2a_2^2 = 0$, (5.3.43)

By substituting (5.3.42) into the second equation of ODEs (5.3.41) yields the following system of algebraic equations:

(i) $24b_2 K_5 \mu^3 + 2K_3 d_2^2 \mu = 0$,

(ii) $6d_2 K_4 \mu^2 + 6b_1 \mu^3 K_5 + 3K_3 \mu d_1 d_2 = 0$,

(iii) $2d_1 K_4 \mu^2 + 40b_2 \mu^2 K_5 + 2c_0 \mu d_2 K_3 + K_3 d_1^2 \mu + 2K_3 d_2^2 - 2d_2 \mu = 0$,

(iv) $8d_2 K_4 \mu + 8b_1 \mu^2 K_5 + K_3 c_0 \mu d_1 + K_3 d_2 c_1 \mu + 3K_3 d_1 d_2 - d_1 \mu = 0$,

(v) $2d_1 K_4 \mu + 16b_2 \mu K_5 + 2K_3 c_0 d_2 - 2K_3 d_2 c_2 \mu + K_3 d_1^2$
$+ 2K_3 c_2 d_2 \mu - 2d_2 = 0,$

(vi) $2d_2 K_4 - 2a_1\mu^2 K_5 + 2b_1\mu K_5 - K_3 c_0 c_1 \mu + K_3 c_0 d_1 + K_3 c_1 d_2 - K_3 d_1 c_2 \mu + 2c_2 K_4 \mu^2$
$+ c_1\mu - d_1 = 0,$

(vii) $2c_1 K_4 \mu - 16a_2 \mu^2 K_5 - 2K_3 c_0 c_2 \mu - K_3 c_1^2 \mu + 2c_2 \mu = 0,$

(viii) $8c_2 K_4 \mu - 8a_1 \mu K_5 - 3K_3 c_1 c_2 \mu - K_3 c_0 c_1 - K_3 c_2 d_1 + c_1 = 0,$

(ix) $2c_1 K_4 - 40a_2 \mu K_5 - 2K_3 c_0 c_2 - K_3 c_1^2 - 2K_3 c_2^2 \mu + 2c_2 = 0,$

(x) $-6a_1 K_5 - 3K_3 c_1 c_2 + 6K_4 c_2 = 0,$

(xi) $-24a_2 K_5 - 2K_3 c_2^2 = 0.$ (5.3.44)

Solving these algebraic equations, yields

$$a_0 = 1, \quad a_1 = a_1, \quad a_2 = 0, \quad b_1 = b_2 = 0,$$

$$c_0 = \frac{c_1}{2K_4}, \quad c_1 = c_1, \quad c_2 = 0, \quad d_1 = d_2 = 0,$$

$$K_1 = \frac{a_1^2}{2c_1}, \quad K_2 = 0, \quad K_3 = \frac{2K_4}{c_1}, \quad K_5 = 0, \quad K_4 = K_4, \quad \mu = \mu. \quad (5.3.45)$$

Substituting the general solutions of equation (5.3.16) into equation (5.3.42) and using equation (5.3.45), we have three types of exact solutions of equation (5.1.2) as follows:

If $\mu > 0$, we have trigonometric function solution

$$u(x,t) = 1 + a_1\left[\sqrt{\mu}\,\frac{A\sin\left(\sqrt{\mu}\left(x-\int\alpha_1(t)dt\right)\right) - B\sin\left(\sqrt{\mu}\left(x-\int\alpha_1(t)dt\right)\right)}{A\sin\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right) + B\sin\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right)}\right],$$

$$w(x,t) = \frac{c_1}{2K_4} + c_1\left[\sqrt{\mu}\,\frac{A\sin\left(\sqrt{\mu}\left(x-\int\alpha_1(t)dt\right)\right) - B\sin\left(\sqrt{\mu}\left(x-\int\alpha_1(t)dt\right)\right)}{A\sin\left(\sqrt{\mu}\left(x-\int\alpha_1(t)dt\right)\right) + B\sin\left(\sqrt{\mu}\left(x-\int\alpha_1(t)dt\right)\right)}\right],$$

(5.3.46)

where A, B and K_4 are arbitrary constants.

If $\mu < 0$, then we obtain hyperbolic function solution in the form

$$u(x,\, t) = 1 + a_1\left[\sqrt{-\mu}\,\frac{A\sinh\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right) - B\sinh\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right)}{A\sin\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right) + B\sin\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right)}\right],$$

$$w(x,\, t) = \frac{c_1}{2\,K_4} + c_1\left[\sqrt{-\mu}\,\frac{A\sinh\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right) - B\sinh\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right)}{A\sinh\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right) + B\sinh\left(\sqrt{-\mu}\left(x-\int\alpha_1(t)dt\right)\right)}\right],$$

(5.3.47)

where A, B and K_4 are arbitrary constants.

When $\mu = 0$, we obtain rational solution as follows:

$$u(x,\, t) = 1 + a_1\left[\frac{B}{A + B\left(x - \int\alpha_1(t)dt\right)}\right],$$

$$w(x,\, t) = \frac{c_1}{2K_4} + c_1\left[\frac{B}{A + B\left(x - \int\alpha_1(t)dt\right)}\right], \qquad (5.3.48)$$

where A, B and K_4 are arbitrary constants.

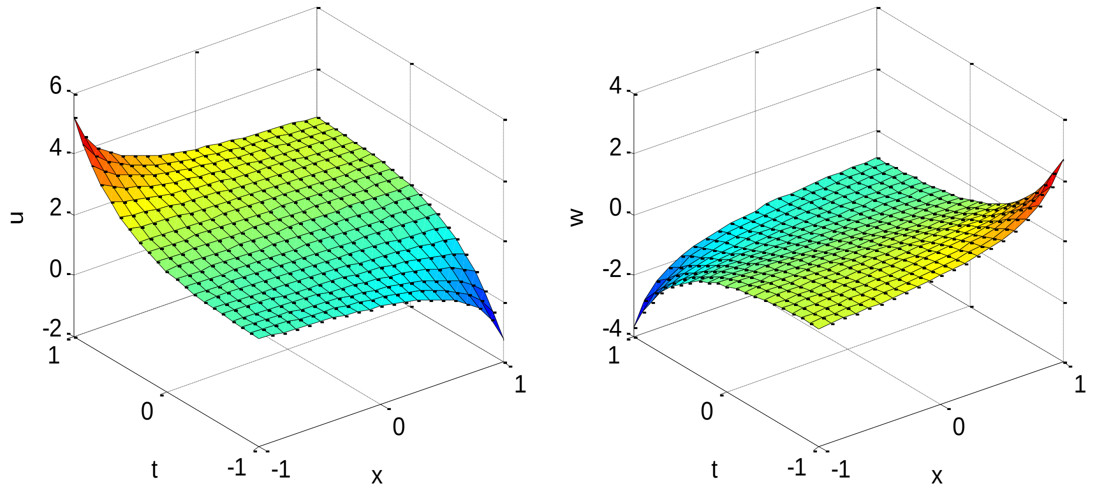


Figure 5.5: Soliton-like solutions given by equation (5.3.48) with $a_1 = 1,\ c_1 = -1,$ $K_4 = -1,\ \ A = B = 1$ and $\alpha_1(t) = \cos ech^2(t)$.

5.3.7 Subalgebra V_4

On solving the equations (5.2.4) and (5.2.8), we get

$$\xi = x,$$
$$u(x,t) = F(\xi),$$
$$w(x,t) = G(\xi). \tag{5.3.49}$$

and the coefficient functions are given by following relations

$$\beta_1(t) = K_1\ \alpha_1(t),\ \ \delta_1(t) = K_2\ \alpha_1(t),\ \ \alpha_2(t) = K_3\ \alpha_1(t),$$
$$\beta_2(t) = K_4\,\alpha_1(t),\ \ \delta_2(t) = K_5\ \alpha_1(t). \tag{5.3.50}$$

Using the similarity variable and the forms of the similarity solution, the system of PDEs (5.1.2) reduces to the following system of ODEs

$$F\ F' + K_1\ G'' + K_2\,F''' = 0,$$
$$K_3\,G\,G' + K_4\ G'' + K_5\,F''' = 0. \tag{5.3.51}$$

As one can see from equation (5.3.51) that $u(x,\,t)$ and $w(x,\,t)$ both degenerate into functions of variable x alone, which may not really correspond to a physically interesting case, we, therefore, prefer not to study the reduced system (5.3.51) further.

5.3.8 Subalgebra V_5

The reduced ODEs, in this case are

$$F' = 0, \qquad G' = 0. \tag{5.3.52}$$

The obvious solution of (5.1.2) is

$$u(x,\,t) = \text{constant}, \qquad w(x,\,t) = \text{constant}. \tag{5.3.53}$$

5.4. Discussion and Concluding Remarks

In this chapter, the authors have studied solution of coupled KdV-Burgers' equation with variable coefficients by using Lie symmetry group method, modified (G'/G)-expansion method and RK4 method. The Lie symmetry classification with respect to the special form of the time dependent variable coefficients equation was presented. We used this classification of optimal system of one-dimensional subalgebras of the Lie symmetry algebras to construct symmetry reductions, exact and numerical solutions. Herein, with the aid of the modified (G'/G)-expansion method some exact solutions are found out under the subalgebra 5.3.3, 5.3.5 and 5.3.6. Then under the subalgebra 5.3.2 and 5.3.4 initial value problems are posed to the reduced ODEs and the numerical solutions are shown to decay for a fairly finite value of z in the Figure 5.1 and Figure 5.3.

Chapter 6

Generalized Fitzhugh-Nagumo Equation with Time-Dependent Coefficients

6.1 Introduction

The classical Fitzhugh-Nagumo (FN) equation

$$u_t = u_{xx} - u(1-u)(\rho - u), \tag{6.1.1}$$

where ρ is constant and $u(x,t)$ is the unknown function depending on the temporal variable t and the spatial variable x has been derived by both FitzHugh [161] and Nagumo et al. [86]. When $\rho = -1$, equation (6.1.1) reduces to the real Newell-Whitehead equation describes the dynamical behavior near the bifurcation point for the Rayleigh-Bénard convection of binary fluid mixtures [72]. The Fitzhugh-Nagumo equation has various applications in the fields of flame propagation, logistic population growth, neurophysiology, branching Brownian motion process, autocatalytic chemical reaction and nuclear reactor theory. This equation combines diffusion, and nonlinearity which is controlled by the term $u(1-u)(\rho - u)$.

In [130], Shih et al. proposed the approximate conditional symmetry method to determine approximate solutions admitted by a perturbation of equation (6.1.1), where the perturbation is proportional to u. Abbasbandy [174] applied the homotopy analysis

A part of this chapter has been **accepted** in **Ain Sham Journal of Engineering** approach to obtain the soliton solution of the Fitzhugh-Nagumo equation. The Haar wavelet method has been presented by Hariharan and Kannan [56], for solving the standard Fitzhugh-Nagumo equation (6.1.1). Meanwhile, Van Gorder and Vajravelu [150] investigated the variational method to obtain analytical solutions for both the Nagumo telegraph and the Nagumo reaction-diffusion PDEs. Moreover, the homotopy analysis method has been successfully applied by Van Gorder [149], to present the approximate solution of the standard Fitzhugh-Nagumo equation. By using Hirota method, Kawahara and Tanaka [185] have found new exact solutions of equation (6.1.1), by applying the nonclassical symmetry reductions approach, Nucci and Clarkson [107] have obtained some new exact solutions with Jacobi elliptic function. Huaying Li and Yucui Guo [70] have obtained the new exact solutions of the FN equation by using first integral method. Recently, Hajipour et al [8] proposed exp-function method for Fitzhugh-Nagumo equation.

In this chapter, we are interested to exhibit the numerical behavior of the generalized Fitzhugh-Nagumo equation with time dependent coefficients and linear dispersion term given by

$$u_t + \nu(t)\, u_x - \mu(t)\, u_{xx} - \eta(t) u\, (1-u)\, (\rho - u) = 0\,; \;\; (x,t) \;\in\; [A, B] \times [0, T], \tag{6.1.2}$$

subject to the boundary conditions

$$u\,(A,t) = g_1(t), \;\; u\,(B,t) \;= g_2(t), \qquad\qquad t \;\in\; [0,\; T] \tag{6.1.3}$$

and the initial condition

$$u\,(x,0) = f\,(x), \qquad\qquad x \;\in\; [A,\; B] \tag{6.1.4}$$

where $\nu(t)$, $\mu(t)$, and $\eta(t)$ are arbitrary real-valued functions of t.

Recently, the generalized Fitzhugh-Nagumo equation with time dependent coefficients has been considered by Bhrawy [7] and Triki et al. [68]. In [7], the author used Jacobi-Gauss-Lobatto collocation method to find the numerical solutions of equation (6.1.2), while in [57] the authors considered a generalized Fitzhugh-Nagumo equation exhibiting time-varying coefficients and linear dispersion term. By means of specific solitary wave ansatz and the tanh method, new variety of soliton solutions are derived in [68] and the conditions of existence and uniqueness of solitons are also presented.

In this chapter, we propose a numerical scheme based on polynomial differential quadrature method to find the numerical solutions of the equation (6.1.2) with initial conditions and Dirichlet boundary conditions (6.1.3)-(6.1.4). The PDQM reduced the problem into a system of first order non-linear ordinary differential equations. Then, the obtained system is solved by optimal four-stage, order three strong stability-preserving time-stepping Runge-Kutta (SSP-RK43) scheme [83]. The accuracy and efficiency of the proposed method is demonstrated by three test examples. The numerical results are discussed in L_∞, RMS and L_2 errors and figures form. The obtained numerical solutions are very similar to the exact solutions.

6.2 Numerical Solutions of Generalized Fitzhugh-Nagumo Equation

The space derivatives in the generalized Fitzhugh-Nagumo with time dependent coefficients equation (6.1.2) are approximated by the differential quadrature methods discussed in chapter 1. The equation (6.1.2) changed into a system of first order nonlinear ODEs in time dependent coefficients.

$$\frac{du(x_i,t)}{dt} = v(t)\sum_{j=1}^{N} w_{ij}^{(1)} u(x_j,t) + \mu(t)\sum_{j=1}^{N} w_{ij}^{(2)} u(x_j,t) - \eta(t)u(x_j,t)(-u(x_j,t))(\rho - u(x_j,t)),$$

$$x_i \in \Omega,\ t \geq 0 \qquad (6.2.1)$$

with initial condition

$$u(x_i,0) = f(x_i), \qquad (6.2.2)$$

and Dirichlet boundary conditions (6.1.3).

6.2.1 Implementation of Dirichlet Boundary Conditions

The Dirichlet boundary conditions are directly applied in the PDQM and gives numerical solutions on boundary in the following way

$$u_1 = g_1(t),\ u_N = g_2(t),\ t > 0. \qquad (6.2.3)$$

The above system (6.2.3) can be written as follows

$$\frac{d\,\{U\}}{d\,t} = [A]\,\{U\} + \{s\}, \qquad (6.2.4)$$

with initial conditions

$$U(0) = \{F\}, \qquad (6.2.5)$$

where $\{U\}=\{u_2, u_3,..., u_{N-1}\}$, is unknown vector, $[A]$ is full matrix of dimension $N-2\times N-2$ and $\{F\}=\{f(x_2), f(x_3),..., f(x_{N-1})\}$, is known vector from initial conditions. Keeping in mind the stability criteria, we preferred the optimal SSP-RK43 scheme [83] to solve the system of ODE (6.2.4) with initial conditions (6.2.5).

6.3 Selections of Grid Points and Stability

The accuracy, stability and rate of convergence of the numerical solutions depend on the choice of grid points selected. It is well known that uniformly distributed grid points are not desirable [32]. Therefore it is suggested that non-uniformly spaced grid points may give better solutions. The zeros of some orthogonal polynomials are commonly used as the grid points. Infact Bellman et al. [151] had proposed to use the zeros of the Legendre polynomials as the grid points in one of his papers.

The stability of the DQM applied depends on the eigenvalues of differential quadrature discretization matrices. These eigenvalues in turn vary much depend on the distribution of grid points. It has been shown by Shu [29] in his book that the uniform grid points distribution does not give stable solution which we have also noticed in our numerical experiments. According to Shu the stable solution can be obtained when Chebyshev-Gauss- Lobatto grid points are chosen. The Chebyshev-Gauss-Lobatto grid points are given by

$$x_i = a+\frac{1}{2}\left(1-\cos\frac{((i-1)\pi)}{N-1}\right) L_x, \qquad i=1,2,..., N \tag{6.3.1}$$

where L_x is the length of the interval.

6.4 Application of the Proposed Method

In this section, the numerical solutions by the proposed method are evaluated for three examples of generalized Fitzhugh-Nagumo equation. Existence of analytical solutions helps to measure the accuracy of numerical method. The numerical computations have been done with the help of software DEV C++ and MATLAB. In numerical experiments L_2 , L_∞ and RMS errors are calculated by the formulas

$$L_2 = \left(\sum_{i=1}^{N} e_i^2 \right)^{1/2},$$

$$L_\infty = \max_{1 \le i \le N} |e_i|,$$

$$\text{RMS} = \left(\sum_{i=1}^{N} \frac{e_i^2}{N} \right)^{1/2}, \tag{6.4.1}$$

where $e_i = \left(u_i - \bar{u}_i\right)$, u_i are exact solutions and $\bar{u}_i$ are approximate solutions.

Example 1

In this example, we consider the nonlinear standard Fitzhugh-Nagumo PDE [7]

$$u_t = u_{xx} - u\,(1-u)\,(\rho - u) = 0, \qquad (x,\, t) \in [A,\, B] \times [0,\, T], \tag{6.4.2}$$

subject to the boundary conditions

$$u\,(A,\, t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(A - \frac{2\,\rho - 1}{\sqrt{2}}\,t\right)\right), \tag{6.4.3}$$

$$u\,(B,\, t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(B - \frac{2\,\rho - 1}{\sqrt{2}}\,t\right)\right), \tag{6.4.4}$$

and initial conditions

$$u\,(x,\, 0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right). \qquad x \in [A,\, B] \tag{6.4.5}$$

The exact solution of equation (6.4.2) is

$$u\,(x,\, t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(x - \frac{2\,\rho - 1}{\sqrt{2}}\,t\right)\right). \tag{6.4.6}$$

The numerical results of example 1 are shown in table 6.1 and Figures 6.1-6.2 with $A = -10$, $B = 10$. Table shows L_∞, RMS and L_2 errors of the PDQM for $\rho = 0.75$ at different times $t = 0.2, 0.5, 1.0, 1.5, 2.0, 3.0, 5.0$. The Figure 6.1 and 6.2 compares the numerical results with the exact one for $\rho = -2$ and $\rho = 0.75$ respectively in 3D form upto time $t = 5$. From numerical results of this example, we conclude that the obtained results are quite agreed with the exact one.

Table 6.1 Max absolute errors L_∞, RMS errors, L_2 errors and CPU time of example 1 at different time with $\rho =$ 0.75.

T	L_∞	RMS	L_2	CPU Time(s)
0.2	4.7416×10^{-5}	1.5880×10^{-5}	2.3012×10^{-6}	0.10
0.5	1.2312×10^{-4}	3.8433×10^{-5}	5.5695×10^{-6}	0.10
1.0	2.6261×10^{-4}	8.1870×10^{-5}	1.1864×10^{-5}	0.18
1.5	4.2096×10^{-4}	1.3387×10^{-4}	1.9400×10^{-5}	0.24
2.0	5.9999×10^{-4}	1.9433×10^{-4}	2.8162×10^{-5}	0.31
3.0	1.0324×10^{-3}	3.4320×10^{-4}	4.9735×10^{-5}	0.35
5.0	2.3050×10^{-3}	7.8638×10^{-4}	1.1395×10^{-4}	0.55

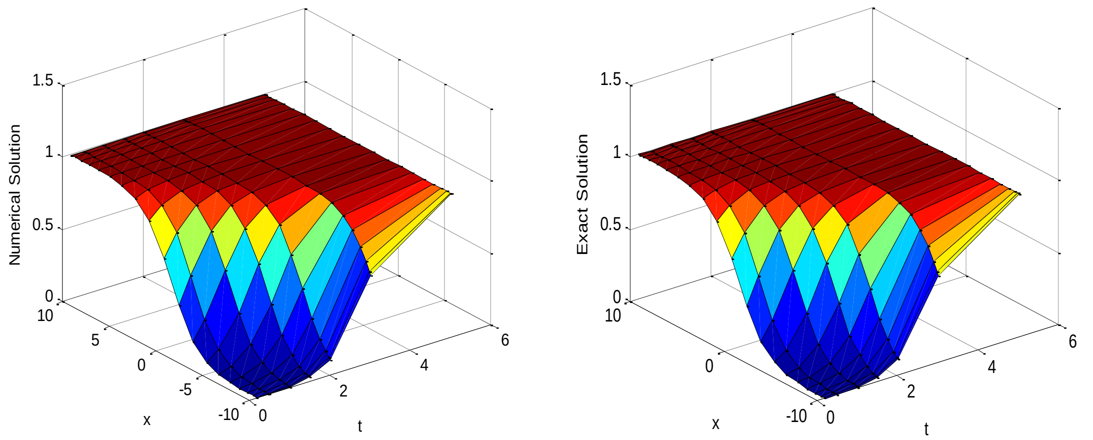


Figure 6.1: Numerical (left) and exact (right) solutions of example 1 with time step length $\Delta t = 0.001, \rho = -2.0$ upto time $t = 5$.

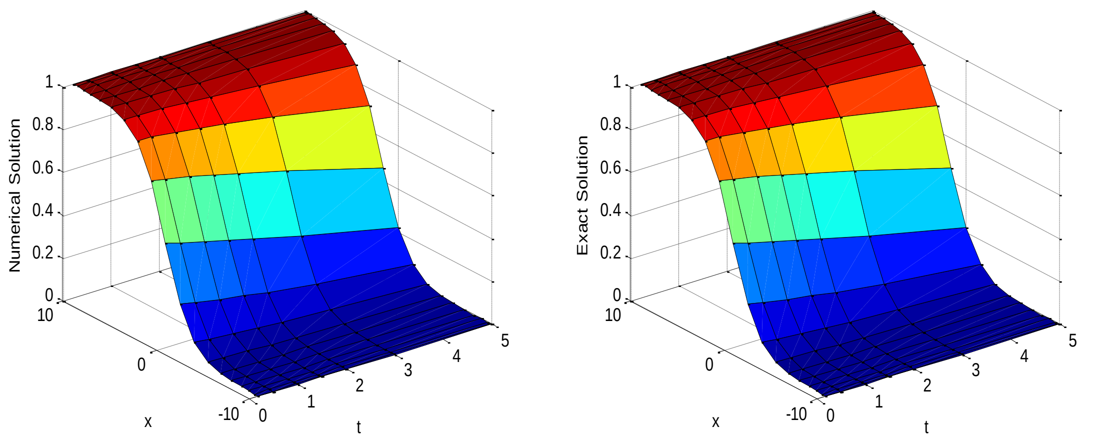


Figure 6.2: Numerical (left) and exact (right) solutions of example 1 with time step length $\Delta t = 0.001,\ \rho = 0.75$ upto time $t = 5$.

Example 2

In this example we examine the Fisher type standard Fitzhugh-Nagumo PDE [14, 185]

$$u_t = u_{xx} - u(1-u)(\rho - u), \qquad 0 < \rho < 1, \tag{6.4.7}$$

subject to the initial condition

$$u(x, 0) = \frac{1}{1 + \exp\left(-\frac{1}{\sqrt{2}}x\right)}, \tag{6.4.8}$$

by taking boundary conditions from the exact solution

$$u(x, t) = \frac{1}{2}(1+\rho) + \left(\frac{1}{2} - \frac{1}{2}\rho\right)\tanh\left(\pm\sqrt{2}(1-\rho)\frac{x}{4} + \frac{(1-\rho^2)}{4}t\right), \tag{6.4.9}$$

of equation (6.4.7) as presented in [185].

The numerical results of example 2 are shown in table 6.2 and Figures 6.3-6.4 with $A = 0$, $B = 1$. Table shows L_∞, RMS and L_2 errors of the PDQM for $\rho = 0.75$ at different times $t = 0.2, 0.5, 1.0, 1.5, 2.0, 3.0, 5.0$. The Figure 6.3 and 6.4 compares the numerical results with the exact one for $\rho = 0.5$ and $\rho = 0.75$ respectively in 3D form upto time $t = 3$. From numerical results of this example, we conclude that the obtained results are quite agreed with the exact one.

Table 6.2 Max absolute errors L_∞, RMS errors, L_2 errors and CPU time of example 2 at different time with $\rho = 0.75$.

T	L_∞	RMS	L_2	CPU Time(s)
0.2	7.0244×10^{-5}	2.3380×10^{-5}	9.6401×10^{-6}	0.11
0.5	7.3492×10^{-4}	2.3469×10^{-5}	9.6764×10^{-6}	0.12
1.0	7.9036×10^{-4}	2.3778×10^{-5}	9.8004×10^{-5}	0.21
1.5	8.4706×10^{-4}	2.4276×10^{-4}	1.0009×10^{-5}	0.27
2.0	9.0446×10^{-4}	2.4937×10^{-4}	1.0282×10^{-5}	0.32
3.0	1.0187×10^{-3}	2.6623×10^{-4}	1.0977×10^{-5}	0.36
5.0	1.2229×10^{-3}	3.0427×10^{-4}	1.2545×10^{-4}	0.57

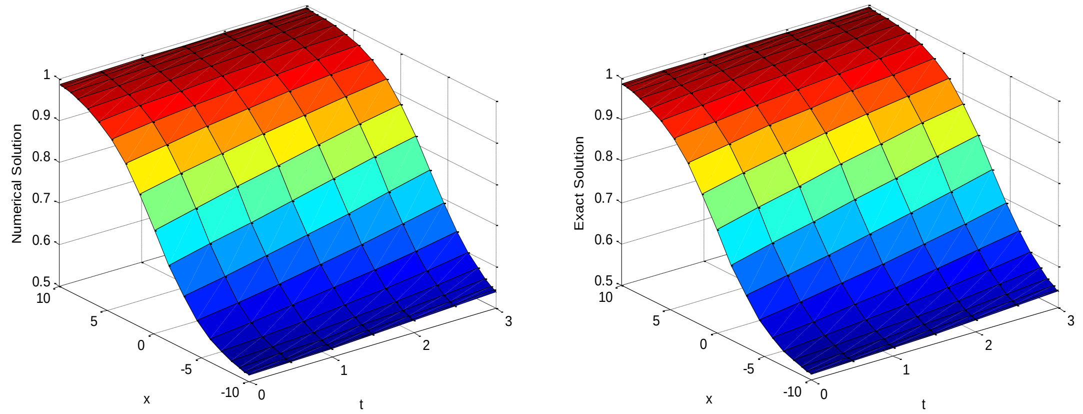


Figure 6.3: Numerical (left) and exact (right) solutions of example 2 with time step length $\Delta t = 0.001,\ \rho = 0.5$ upto time $t = 3.0$.

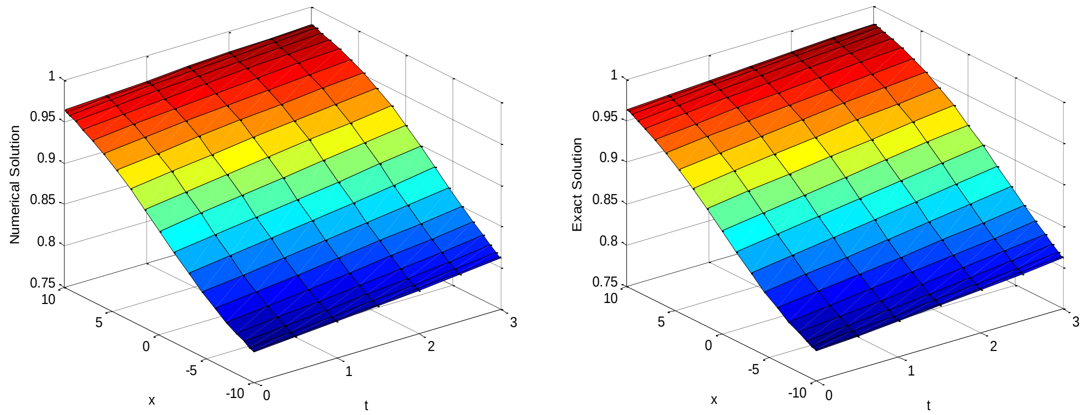


Figure 6.4: Numerical (left) and exact (right) solutions of example 2 with time step length $\Delta t = 0.001,\ \rho = 0.75$ upto time $t = 3.0$.

Example 3

Consider the nonlinear time-dependent generalized Fitzhugh-Nagumo equation with time-dependent coefficients [7, 68]

$$u_t + \cos(t)\, u_x - \cos(t)\, u_{xx} - 2\cos(t)(u\,(1-u)\,(\rho - u)) = 0;$$

$$(x, t) \in [A, B] \times [0, T], \tag{6.4.10}$$

subject to the boundary conditions

$$u\,(A, t) = \frac{\rho}{2} + \frac{\rho}{2}\tanh\left(\frac{\rho}{2}(A - (3-\rho)\sin(t))\right), \tag{6.4.11}$$

$$u\,(B, t) = \frac{\rho}{2} + \frac{\rho}{2}\tanh\left(\frac{\rho}{2}(B - (3-\rho)\sin(t))\right), \tag{6.4.12}$$

And initial conditions

$$u\ (x,\,0) = \frac{\rho}{2} + \frac{\rho}{2}\tanh\left(\frac{\rho\ x}{2}\right), \qquad x \in [A,\ B]. \tag{6.4.13}$$

The exact solution of equation (6.10) is

$$u\ (x,\,t) = \frac{\rho}{2} + \frac{\rho}{2}\tanh\left(\frac{\rho}{2}(x - (3 - \rho)\sin(t))\right). \tag{6.4.14}$$

In this example, the results are computed with $A=-10,\,B=10$, $\rho = 0.5,\,0.75$ and shown in table 6.3 and Figures 6.5-6.6. Table 6.3 shows the L_∞, RMS and L_2 errors of PDQM at different time t with $\rho =\ 0.75$. Figures 6.5-6.6 depict a comparison of numerical and exact solutions with $\rho =\ 0.5,\ 0.75$ and time upto $t = 1.0$.

Table 6.3 Max absolute errors L_∞, RMS errors, L_2 errors and CPU time of example 3 at different time with $\rho =\ 0.75$.

T	L_∞	RMS	L_2	CPU Time(s)
0.2	1.235×10^{-5}	4.5670×10^{-5}	1.3122×10^{-6}	0.12
0.5	5.1986×10^{-4}	5.6423×10^{-5}	6.0995×10^{-6}	0.15
1.0	6.3283×10^{-4}	8.1671×10^{-5}	2.1213×10^{-5}	0.23
1.5	8.5383×10^{-4}	2.3681×10^{-4}	3.2340×10^{-5}	0.26
2.0	9.9123×10^{-4}	3.0123×10^{-4}	4.8342×10^{-5}	0.37
3.0	1.7904×10^{-3}	5.3420×10^{-4}	6.1098×10^{-5}	0.39
5.0	3.3551×10^{-3}	7.8900×10^{-4}	1.0015×10^{-4}	0.61

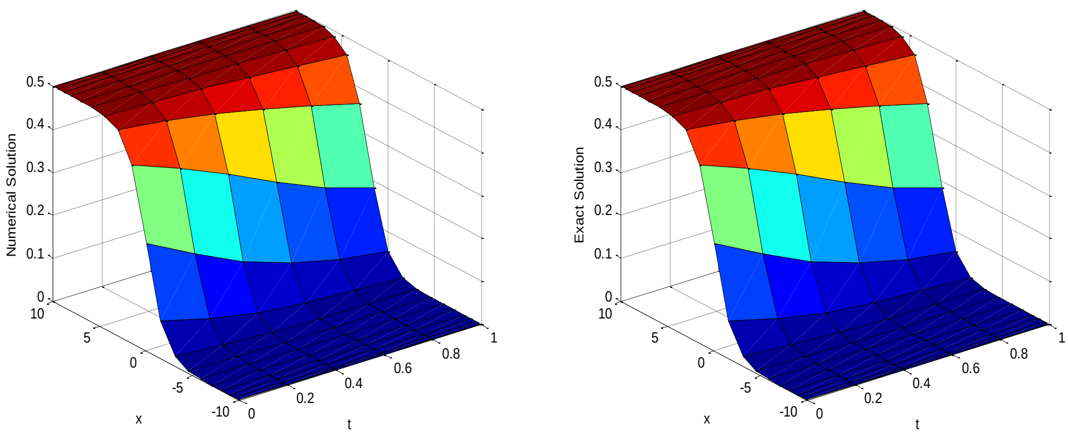


Figure 6.5: Numerical (left) and exact (right) solutions of example 3 with time step length $\Delta t = 0.001, \rho = 0.5$ upto time $t = 1.0$.

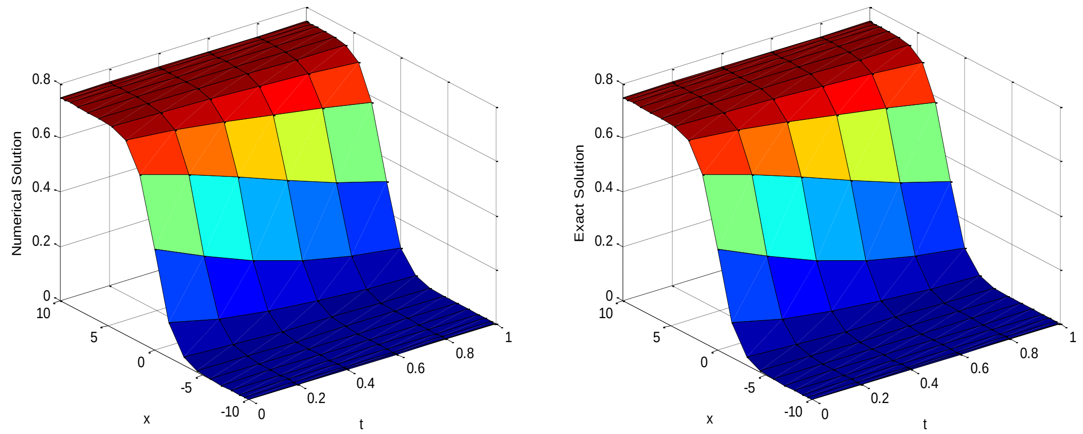


Figure 6.6: Numerical (left) and exact (right) solutions of example 3 with time step length $\Delta t = 0.001, \rho = 0.75$ upto time $t = 1.0$.

6.5. Conclusion and Discussion

In this chapter, we proposed PDQM based on Lagrange interpolation to find the approximate solution of well known generalized Fitzhugh-Nagumo equation. To check the accuracy of the method three test problems available in the literature are solved. The L_∞, RMS and L_2 errors are presented in tables with time step length $\Delta t = 0.001$. It is suggested that the Chebyshev-Gauss-Lobatto grid points produce accurate solutions. Finally, the authors conclude that the proposed methods give very accurate and similar results to the exact solutions by choosing less number of grid points. Also, the methods can be applicable for two and multidimensional problems that arise in biological, physical and chemical phenomena.

Chapter 7

Numerical Simulation of Two Dimensional Quasilinear Hyperbolic Equations

7.1 Introduction

In this chapter, we have considered the two dimensional quasilinear hyperbolic PDE:

$$\frac{\partial^2 u}{\partial t^2} + A(x,y,t,u)\frac{\partial^2 u}{\partial x^2} + B(x,y,t,u)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,t,u,u_x,u_y,u_t\right) = 0, \quad (x,y,t) \in [0,1]\times[0,1]\times[t>0] \tag{7.1.1}$$

with initial conditions

$$u(x,y,0) = g_1(x,y), \quad (x,y) \in [0,1]\times[0,1],$$
$$u_t(x,y,0) = g_2(x,y), \quad (x,y) \in [0,1]\times[0,1], \tag{7.1.2}$$

subject to Dirichlet boundary conditions given by

$$u(0,y,t) = h_1(y,t), \quad (y,t) \in [0,1]\times[t>0], \ u(1,y,t) = h_2(y,t), \ (y,t) \in [0,1]\times[t>0],$$
$$u(x,0,t) = h_3(x,t), \quad (x,t) \in [0,1]\times[t>0], \ u(x,1,t) = h_4(x,t), \quad (x,t) \in [0,1]\times[t>0], \tag{7.1.3}$$

A part of this chapter has been published in **Engineering Computations**, 30 (2013) 892-909

and Neumann boundary conditions given by

$$u_x(0,y,t)= f_1(y,t),\quad (y,t)\in[0,1]\times[t>0],\ u_x(1,y,t)= f_2(y,t),\quad (y,t)\in[0,1]\times[t>0],$$
$$u_y(x,0,t)= f_3(x,t),\quad (x,t)\in[0,1]\times[t>0],\ u_y(x,1,t)= f_4(x,t),\quad (x,t)\in[0,1]\times[t>0],$$
$$(7.1.4)$$

The equation (7.1.1) is assumed to be hyperbolic if $A(x,y,t,u)>0$ and $B(x,y,t,u)>0$ in the solution region $\Omega \equiv \{(x,y,t):(x,y,t)\in[0,1]\times[0,1]\times[t>0]\}$. Further, it is assumed that $g_1(x,y)$, $g_2(x,y)$ are sufficiently differentiable functions as required and $u(x,y,t)\in C^2$, $A(x,y,t,u)$, $B(x,y,t,u)\in C^2$.

The second-order quasilinear hyperbolic PDE (7.1.1) with appropriate initial and boundary conditions serves as models in many branches of physics, fluid dynamics and aerodynamics, theory of elasticity, optics, electromagnetic etc. The wave equation with nonlinear first derivative terms has been studied by Lees [121] and Greenspan [36]. Unconditionally stable ADI schemes of $O(h^2+k^2)$ and $O(h^4+k^2)$ for the wave equation with constant coefficients and variable coefficients were proposed by Jain et al. [118] and Iyengar and Mittal [180]. Using operator compact implicit method, Ciment and Leventhal [105, 106] have developed fourth order difference schemes for linear two-dimensional wave equation. After that, Mohanty et al. [171] have proposed a high order conditionally stable numerical method for the solution of two-space dimensional second-order nonlinear hyperbolic equation. Later, Mohanty et al. [172] have extended their technique to solve second order quasilinear hyperbolic equations by using seven evaluations of the function and 19-grid points. Mohanty and Jain [169] proposed an unconditionally stable ADI scheme for the two space dimensional linear hyperbolic equation. Meanwhile, Koksal et al. [1, 2] have proposed the difference scheme for hyperbolic equations in a Hilbert space and for the hyperbolic equation with self-adjoint operator. Schwartzkopff et al. [187] have proposed and analyzed a higher order ADER finite volume schemes for linear hyperbolic equations using Cartesian meshes while Dumbser and Kaser [115] used arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems. Dehghan and Mehebbi [111] proposed a high order implicit collocation method for solving the two dimensional hyperbolic equation. Dehghan and Shokri [113] solved the one and two dimensional hyperbolic equations using collocation points and used the thin platespline radial basis functions to approximate the solution. In [112, 114], Dehghan and Mohebbi proposed some numerical techniques for two dimensional wave equation and Schrödinger equation. Recently, Mohanty and Singh [170] proposed a new high-order approximation for the solution of two-space-dimensional quasilinear hyperbolic PDE of the form (7.1.1).

In the following sections, we proposed a numerical scheme based on PDQM to find the numerical solutions of two-space-dimensional quasilinear hyperbolic PDEs with Dirichlet and Neumann boundary conditions. The PDQM reduced the problems into a system of second order quasilinear differential equations. Then, the obtained system is changed into coupled differential equations and lastly, RK4 method is used to solve the coupled system. The accuracy and efficiency of the proposed method are demonstrated by several test examples of nonlinear wave equations with constants coefficients and nonlinear wave equations with variable coefficients.

7.2 Numerical Scheme for Two Dimensional Quasilinear Hyperbolic Equation

The space derivatives in the two-dimensional quasilinear hyperbolic equation (7.1.1) are approximated by the polynomial differential quadrature method discussed in chapter 1. The equation (7.1.1) changed into the following form

$$\frac{d^2u_{i,j}}{dt^2} + A\left(x_i, y_j, t, u_{i,j}\right)\sum_{k=1}^{N} w_{i,k}^{(2)} u_{k,j} + B\left(x_i, y_j, t, u_{i,j}\right)\sum_{k=1}^{M} \overline{w}_{j,k}^{(2)} u_{i,k}$$

$$+ f\left(x_i, y_j, t, u_{i,j}, \sum_{k=1}^{N} w_{i,k}^{(1)} u_{k,j}, \sum_{k=1}^{M} \overline{w}_{j,k}^{(1)} u_{i,k}, \frac{du_{i,j}}{dt}\right) = 0, \ \left(x_i, y_j, t\right) \in [0,1]\times[0,1]\times[t>0],$$
$$i = 1,2,\ldots,N, \ \ j = 1,2,\ldots,M \qquad (7.2.1)$$

with initial conditions

$$u\left(x_i, y_j, 0\right) = g_1\left(x_i, y_j\right), \qquad (7.2.2)$$

$$\frac{du}{dt}\left(x_i, y_j, 0\right) = g_2\left(x_i, y_j\right), \qquad (7.2.3)$$

where $u_{i,j} = u\left(x_i, y_j, t\right)$.

Equation (7.2.1) represents a system of second order nonlinear differential equations. Now let

$$\frac{du}{dt}(x_i, y_j, t) = v(x_i, y_j, t), \quad \text{than} \quad \frac{d^2 u}{dt^2}(x_i, y_j, t) = \frac{dv}{dt}(x_i, y_j, t). \tag{7.2.4}$$

Using (7.2.4) into the equation (7.2.1), the equation (7.2.1) can be written as the following system

$$\frac{du}{dt}(x_i, y_j, t) = v(x_i, y_j, t), \tag{7.2.5}$$

$$\frac{dv_{i,j}}{dt} + A(x_i, y_j, t, u_{i,j})\sum_{k=1}^{N} w_{i,k}^{(2)} u_{k,j} + B(x_i, y_j, t, u_{i,j})\sum_{k=1}^{M} \overline{w}_{j,k}^{(2)} u_{i,k}$$

$$+ f\left(x_i, y_j, t, u_{i,j}, \sum_{k=1}^{N} w_{i,k}^{(1)} u_{k,j}, \sum_{k=1}^{M} \overline{w}_{j,k}^{(1)} u_{i,k}, v_{i,j}\right) = 0, \quad (x_i, y_j, t) \in [0,1]\times[0,1]\times[t>0],$$

$$i = 1,2,\ldots,N, \quad j = 1,2,\ldots,M \tag{7.2.6}$$

with initial conditions

$$u(x_i, y_j, 0) = g_1(x_i, y_j), \tag{7.2.7}$$

$$v(x_i, y_j, 0) = g_2(x_i, y_j), \tag{7.2.8}$$

7.2.1 Implementation of Dirichlet Boundary Conditions

The Dirichlet boundary conditions are directly applied in the PDQM like [163] and we can get numerical solutions at the rectangular or square boundary in the following way

$$u_{N,j} = h_2(y_j, t), \quad (y_j, t) \in [0,1]\times[t>0], \quad \text{for} \quad j = 1,2,\ldots,M \tag{7.2.9}$$

$$u_{i,1} = h_3(x_i, t), \quad (x_i, t) \in [0,1]\times[t>0],$$

$$u_{i,M} = h_4(x_i, t), \quad (x_i, t) \in [0,1]\times[t>0], \quad \text{for} \quad i = 1,2,\ldots,N \tag{7.2.10}$$

7.2.2 Implementation of Neumann Boundary Conditions

The Neumann boundary conditions (7.1.4) at $x = 0$ and $x = 1$ can be approximated as

$$\sum_{k=1}^{N} w_{1,k}^{(1)} u_{k,j} = f_1(0, y, t), \tag{7.2.11}$$

$$\sum_{k=1}^{N} w_{N,k}^{(1)} u_{k,j} = f_2(1,y,t), \qquad j=1,2,...,M \tag{7.2.12}$$

Equations (7.2.11) and (7.2.12) are simplifying as in [163] and finally, we get solutions at boundaries

$$u_{1,j} = \frac{w_{1,N}^{(1)}\left(f_2 - S_2\right) - w_{N,N}^{(1)}\left(f_1 - S_1\right)}{\left(w_{1,N}^{(1)} w_{N,1}^{(1)} - w_{1,1}^{(1)} w_{N,N}^{(1)}\right)}, \qquad j=1,2,...,M \tag{7.2.13}$$

$$u_{N,j} = \frac{w_{N,1}^{(1)}\left(f_1 - S_1\right) - w_{1,1}^{(1)}\left(f_2 - S_2\right)}{\left(w_{1,N}^{(1)} w_{N,1}^{(1)} - w_{1,1}^{(1)} w_{N,N}^{(1)}\right)}, \qquad j=1,2,...,M \tag{7.2.14}$$

where $S_1 = \sum_{k=2}^{N-1} w_{1,k}^{(1)} u_{k,j}$, $S_2 = \sum_{k=2}^{N-1} w_{N,k}^{(1)} u$, $f_1 = f_1(0,y,t)$ and $f_2 = f_2(1,y,t)$.

In a similar way, the Neumann boundary conditions (7.1.4) at $y=0$ and $y=1$ can be approximated as

$$\sum_{k=1}^{M} \overline{w}_{1,k}^{(1)} u_{i,k} = f_3(x,0,t), \tag{7.2.15}$$

$$\sum_{k=1}^{M} \overline{w}_{M,k}^{(1)} u_{i,k} = f_4(x,1,t), \qquad i=1,2,...,N \tag{7.2.16}$$

Equations (7.2.15) and (7.2.16) are simplifying as in [163] and we get solutions at boundaries as follows

$$u_{i,1} = \frac{\overline{w}_{1,M}^{(1)}\left(f_4 - S_4\right) - \overline{w}_{M,M}^{(1)}\left(f_3 - S_3\right)}{\left(\overline{w}_{1,M}^{(1)} \overline{w}_{M,1}^{(1)} - \overline{w}_{1,1}^{(1)} \overline{w}_{M,M}^{(1)}\right)}, \qquad j=1,2,...,M \tag{7.2.17}$$

$$u_{i,M} = \frac{\overline{w}_{M,1}^{(1)}\left(f_3 - S_3\right) - \overline{w}_{1,1}^{(1)}\left(f_4 - S_4\right)}{\left(\overline{w}_{1,M}^{(1)} \overline{w}_{M,1}^{(1)} - \overline{w}_{1,1}^{(1)} \overline{w}_{M,M}^{(1)}\right)}, \qquad j=1,2,...,M \tag{7.2.18}$$

where $S_3 = \sum_{k=2}^{M-1} \overline{w}_{1,k}^{(1)} u_{i,k}$, $S_4 = \sum_{k=2}^{M-1} \overline{w}_{M,k}^{(1)} u_{i,k}$, $f_3 = f_3(x,0,t)$ and $f_4 = f_4(x,1,t)$.

The system of ODEs (7.2.5)-(7.2.6) with initial and boundary conditions (7.2.7)-(7.2.8), discussed in subsection 7.1 and 7.2, cannot be solved directly by standard RK4 method since RK4 method is used to solve only initial value problems (IVP). So, after applying the boundary conditions on the system of equations (7.2.5)-(7.2.6), the system (7.2.5)-(7.2.6) is changed into a system of IVPs. Then, the obtained system is solved by RK4 method.

7.3 Stability of the Differential Quadrature Method

The stability of the PDQM depends on the eigen-values of differential quadrature discretization matrices. These eigen-values in turn very much depend on the distribution of grid points. It has been shown by Shu [29] in his book that the uniform grid points distribution does not give stable solutions. So, in the whole numerical experiments, we have used the Chebyshev-Gauss-Lobatto grid points to find the numerical solutions of the two dimensional quasilinear hyperbolic partial differential equation (7.1.1). The Chebyshev-Gauss-Lobatto grid points are

$$x_i = a + \frac{1}{2}\left(1 - \cos\frac{((i-1)\pi)}{N-1}\right)(b-a), \qquad i = 1,2,\ldots,N \tag{7.3.1}$$

7.4 Numerical Experiments

In this section, the proposed scheme based on PDQM is applied on several test examples of nonlinear wave equations with constants and variable coefficients to show the efficiency and accuracy of the numerical method. The examples are chosen such that their exact solutions are known and already discussed in literature. The software Dev C++ and Matlab are used for whole computational work in the paper. In each example, relative errors, root mean square (RMS) errors and absolute errors are computed by the following formulas

$$\text{Absolute error} = \left| u_{ij} - \bar{u}_{ij} \right|$$

$$\text{Relative error} = \sqrt{\left[\frac{\sum_{i,j=1}^{N}\left(u_{ij} - \bar{u}_{ij}\right)^2}{\sum_{i,j=1}^{N} u_{ij}^2}\right]}$$

$$\text{RMS error} = \sqrt{\frac{\sum_{i,j=1}^{N}\left(u_{ij} - \bar{u}_{ij}\right)^2}{N \times N}}$$

where u_{ij} and $\bar{u}_{ij}$ denote the exact and numerical solution of the problem.

Example 1

In this example, Van der Pol type nonlinear wave equation is considered with domain $0 \le x, y \le 1,\ t > 0$,

$$u_{tt} = u_{xx} + u_{yy} + \gamma\left(u^2 - 1\right)u_t + \left(2\pi^2 + \gamma^2 e^{-2\gamma t} \sin^2(\pi x)\sin^2(\pi y)\right)e^{-\gamma t} \sin(\pi x)\sin(\pi y) \tag{7.4.1}$$

with initial and Dirichlet boundary conditions

$$\begin{cases} u(x, y, 0) = \sin(\pi x)\sin(\pi y), \\ u_t(x, y, 0) = -\gamma \sin(\pi x)\sin(\pi y), \end{cases}$$

$$\begin{cases} u(0, y, t) = u(1, y, t) = 0, \\ u(x, 0, t) = u(x, 1, t) = 0 \end{cases} \tag{7.4.2}$$

The exact solution is given by

$$u(x, y, t) = e^{-\gamma t} \sin(\pi x)\sin(\pi y), \quad t \ge 0. \tag{7.4.3}$$

Table 7.1 presents RMS errors, relative errors, max absolute errors and CPU time in seconds of example 1. The table shows that the errors are small and negligible. The figures 7.1 and 7.2 show the numerical and exact solutions at time $t = 1.0$ and $t = 3.0$ respectively. The Figures show that the numerical results are in good agreements with the exact solutions. The Figure 7.3 shows physical behavior of the Van der Pol type nonlinear wave equation (7.4.1) in contour form at $t = 1.0$.

Table-7.1

RMS, relative, max absolute errors and CPU time of example 1 at different times for $N = M = 21,\ \gamma = 1.$

T	PDQM RMS	PDQM Relative Error	PDQM Max Absolute Error	CPU Time(s)
0.5	2.688×10^{-6}	1.337×10^{-5}	8.114×10^{-6}	9
1.0	1.121×10^{-6}	9.194×10^{-5}	3.454×10^{-6}	19
2.0	1.169×10^{-6}	2.609×10^{-5}	3.488×10^{-6}	38
3.0	3.018×10^{-6}	1.829×10^{-5}	8.989×10^{-6}	56
5.0	2.282×10^{-6}	1.023×10^{-5}	6.818×10^{-6}	94
7.0	7.133×10^{-7}	2.360×10^{-4}	2.199×10^{-7}	133
10.0	1.696×10^{-7}	1.127×10^{-4}	5.139×10^{-7}	186

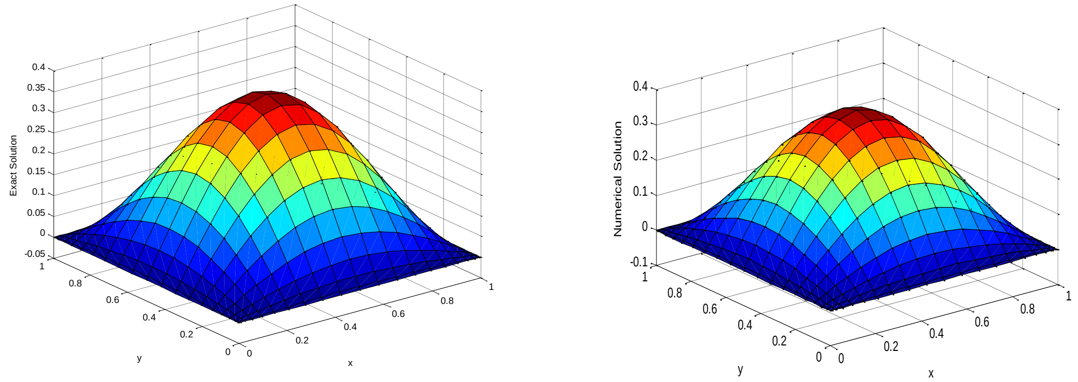


Figure 7.1: Exact (left) and numerical (right) solutions of example 1 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 1.0$.

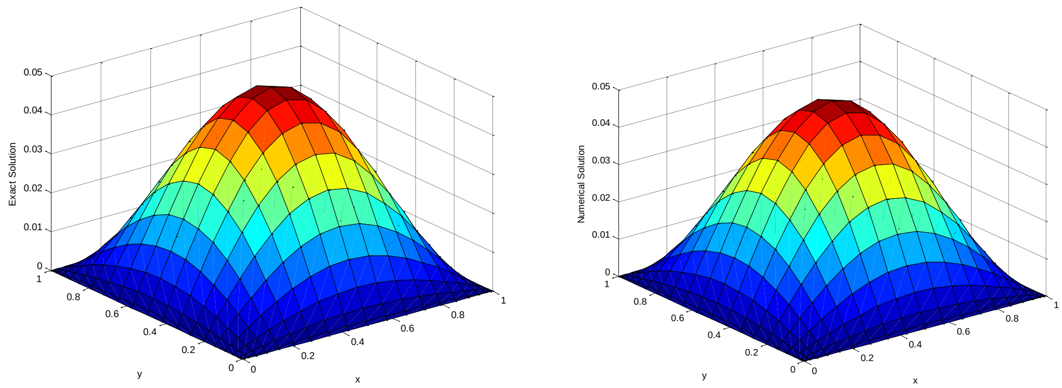


Figure: 7.2: Exact (left) and numerical (right) solutions of example 1 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 3.0$.

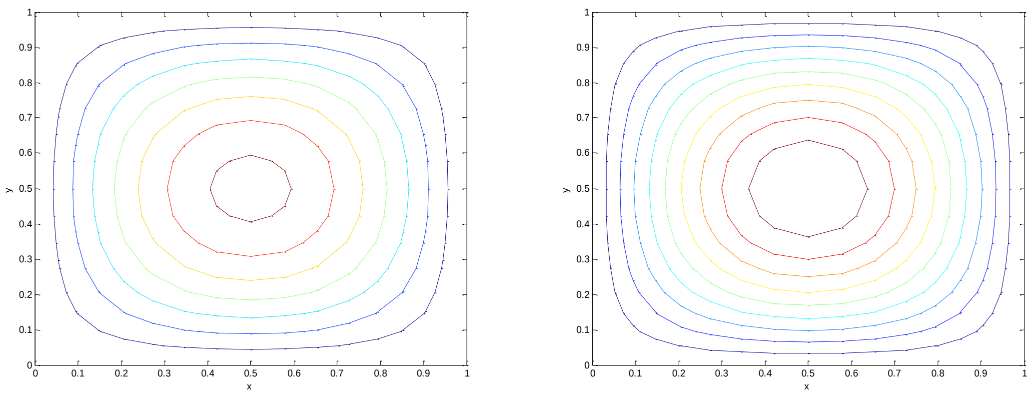


Figure 7.3: Contours plots of example 1 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 1.0$, 3.0 respectively.

Example 2

Consider the dissipative nonlinear hyperbolic wave equation in the region $0 \le x, y \le 1,\ t > 0$;

$$u_{tt} = u_{xx} + u_{yy} - 2u\,u_t + \left(2\pi^2 - 1 + 2\sin(\pi x)\sin(\pi y)\cos(t)\right)\sin(\pi x)\sin(\pi y)\sin(t) \quad (7.4.4)$$

with initial and Dirichlet boundary conditions

$$\begin{cases} u(x,y,0) = 0, \\ u_t(x,y,0) = \sin(\pi x)\sin(\pi y), \end{cases}$$

$$\begin{cases} u(0,y,t) = u(1,y,t) = 0, \\ u(x,0,t) = u(x,1,t) = 0 \end{cases} \quad (7.4.5)$$

The exact solution is given by

$$u(x,y,t) = \sin(\pi x)\sin(\pi y)\sin(t), \quad t \ge 0, \quad (7.4.6)$$

The RMS errors, relative errors, max absolute errors and CPU time in seconds of example 2 are shown in table 7.2. The table shows that the errors are small and negligible. The numerical and exact solutions at time $t = 1.0$ and $t = 3.0$ respectively are shown in figures 7.4 and 7.5. The figures show that the numerical results are in good agreements with the exact solutions. The figure 7.6 shows physical behavior of the equation (7.4.4) in contour form at $t = 1.0$.

Table-7.2

RMS, relative, max absolute errors and CPU time of example 2 at different times for $N = M = 21$.

T	PDQM RMS	PDQM Relative Error	PDQM Max Absolute Error	CPU Time(s)
0.5	3.066×10^{-6}	1.930×10^{-5}	9.098×10^{-6}	8
1.0	1.399×10^{-6}	5.018×10^{-6}	4.402×10^{-6}	15
2.0	2.862×10^{-7}	9.500×10^{-7}	9.305×10^{-7}	31
3.0	2.439×10^{-6}	5.218×10^{-7}	7.379×10^{-6}	45
5.0	1.805×10^{-6}	5.684×10^{-5}	5.403×10^{-6}	76
7.0	8.099×10^{-7}	3.720×10^{-7}	2.305×10^{-7}	106
10.0	2.348×10^{-6}	1.302×10^{-5}	7.119×10^{-6}	153

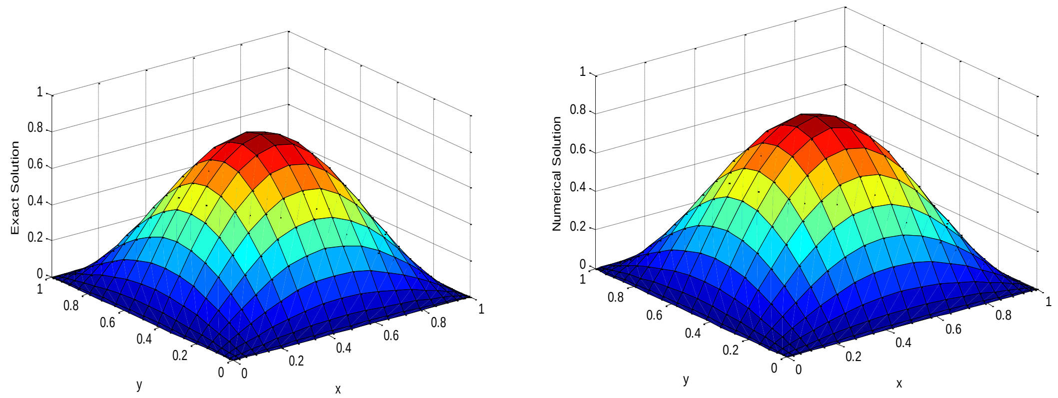


Figure 7.4: Exact (Left) and Numerical (Right) solutions of Example 2 with time step length $\Delta t = 0.001$ at $t = 1.0$.

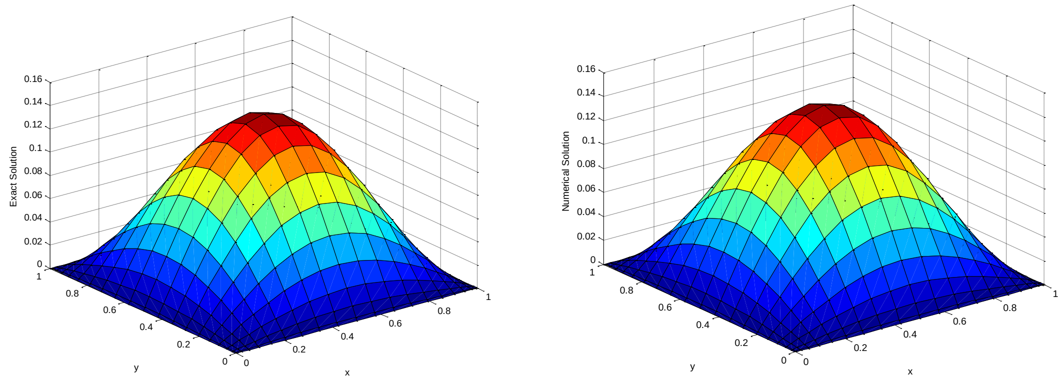


Figure 7.5: Exact (left) and numerical (right) solutions of example 2 with time step length $\Delta t = 0.001$ at $t = 3.0$.

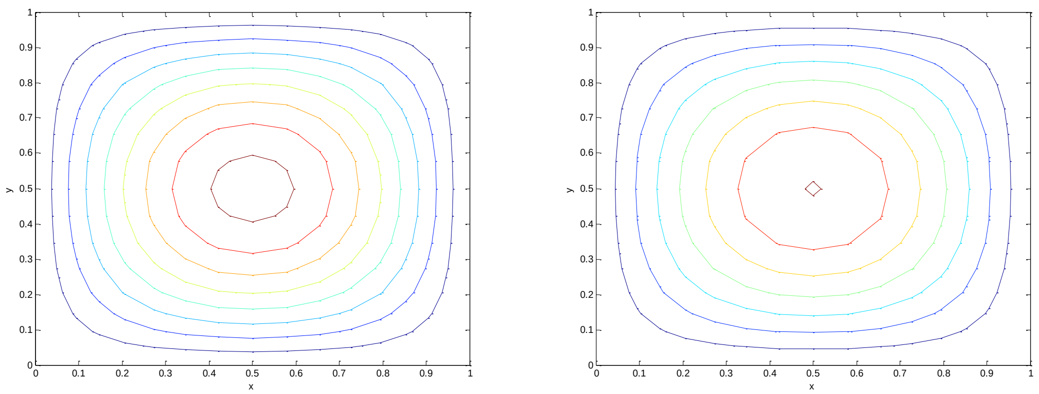


Figure 7.6: Contours plots of example 2 with time step length $\Delta t = 0.001$ at $t = 1.0,\ 3.0$ respectively.

Example 3

In this example, we have considered a nonlinear hyperbolic wave equation with variable coefficients

$$u_{tt} = (1+x^2)u_{xx} + (1+y^2)u_{yy} + \gamma u(u_x + u_y + u_t)$$
$$+ \left(\gamma e^{-t}(\cosh(x)\cosh(y) - \sinh(x+y)) - 1 - x^2 - y^2\right)e^{-t}\cosh(x)\cosh(y),$$
$$0 \le x, y \le 1,\ t > 0 \quad (7.4.7)$$

The initial and Dirichlet boundary conditions of (7.4.7) are given by

$$\begin{cases} u(x,y,0) = \cosh(x)\cosh(y), \\ u_t(x,y,0) = -\cosh(x)\cosh(y), \end{cases} \begin{cases} u(0,y,t) = e^{-t}\cosh(y), & 0 \le y \le 1, \quad x = 0, \\ u(1,y,t) = e^{-t}\cosh(1)\cosh(y), & 0 \le y \le 1, \quad x = 1, \\ u(x,0,t) = e^{-t}\cosh(x), & 0 \le x \le 1, \quad y = 0, \\ u(x,1,t) = e^{-t}\cosh(x)\cosh(1), & 0 \le x \le 1, \quad y = 1. \end{cases}$$

(7.4.8)

The exact solution of the nonlinear hyperbolic wave equation (7.4.7) is given

$$u(x,y,t) = e^{-t}\cosh(x)\cosh(y). \quad (7.4.9)$$

The numerical results of the example 3 are presented in table 7.3 in form of RMS errors, relative errors, max absolute errors and CPU time in seconds. The figures 7.7 and 7.8 show the numerical and exact solutions at time $t = 1.0$ and $t = 3.0$ respectively and show that the numerical results are in good agreements with the exact solution. The figure 7.9 show the contour plot of the considered equation (7.4.7) at $t = 1.0$.

Table-7.3

RMS, relative, max absolute errors and CPU time of example 3 at different times for $N = M = 21,\ \gamma = 1.$

T	PDQM RMS	PDQM Relative Error	PDQM Max Absolute Error	CPU Time(s)
0.5	1.917×10^{-5}	2.126×10^{-5}	6.643×10^{-5}	13
1.0	1.056×10^{-5}	1.932×10^{-5}	2.643×10^{-5}	26
2.0	2.465×10^{-5}	1.225×10^{-5}	7.615×10^{-5}	52
3.0	2.236×10^{-5}	3.020×10^{-4}	5.430×10^{-5}	86
5.0	1.430×10^{-5}	1.427×10^{-4}	4.942×10^{-5}	130
7.0	2.609×10^{-5}	1.920×10^{-4}	6.067×10^{-5}	193
10.0	1.082×10^{-5}	1.604×10^{-3}	3.130×10^{-5}	270

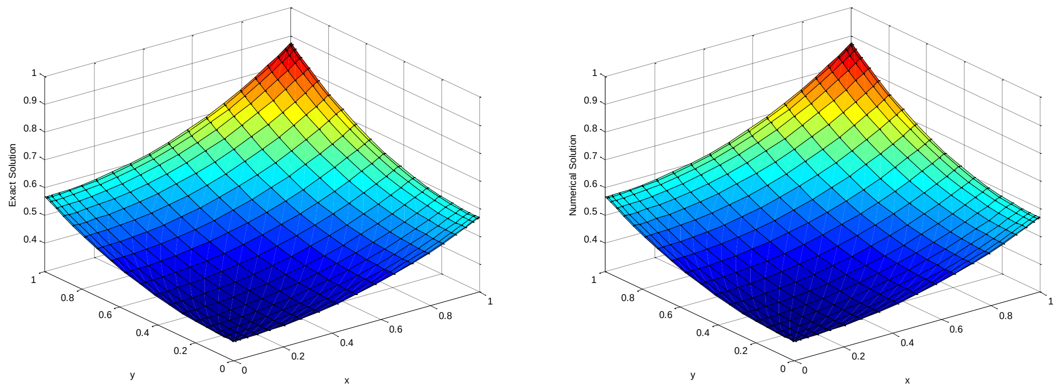


Figure 7.7: Exact (left) and numerical (right) solutions of example 3 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 1.0$.

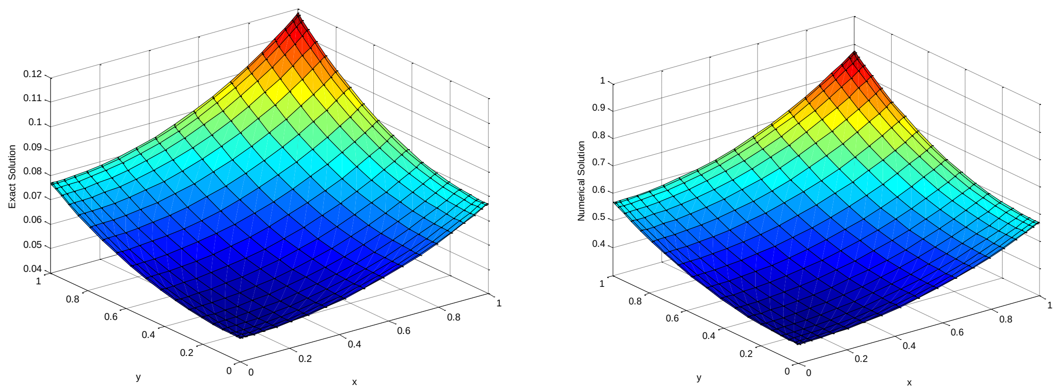


Figure 7.8: Exact (left) and numerical (right) solutions of example 3 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 3.0$.

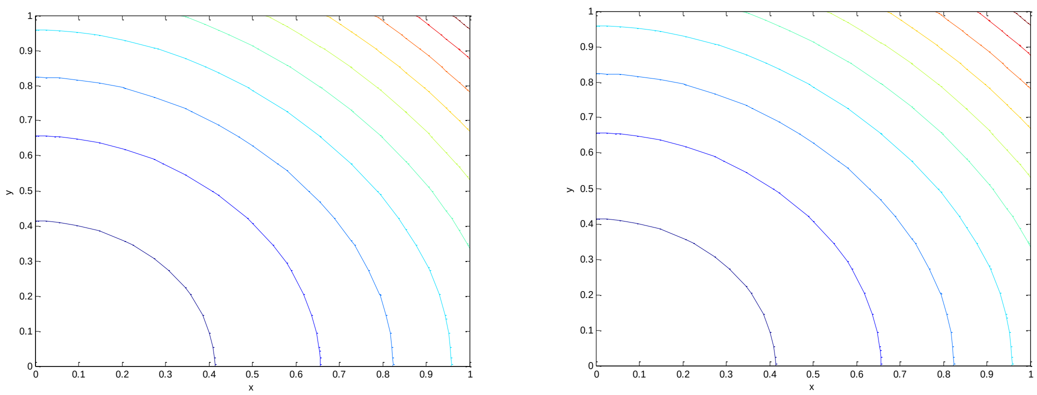


Figure 7.9: Contours plots of example 3 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 1.0,\ 3.0$ respectively.

Example 4

In this example, we have considered the quasilinear hyperbolic equation

$$u_{tt} = (1+u^2)(u_{xx}+u_{yy}) + \gamma u(u_x+u_y+u_t)$$
$$+(\gamma e^{-t}(\sinh(x)\cosh(y) - \cosh(x+y)) - 1)e^{-t}\sinh(x)\cosh(y) - 2(e^{-t}\sinh(x)\cosh(y))^3,$$
$$0 \le x, y \le 1,\ t > 0 \qquad (7.4.10)$$

with initial and Neumann boundary conditions are given by

$$\begin{cases} u(x,y,0) = \sinh(x)\cosh(y), \\ u_t(x,y,0) = -\sinh(x)\cosh(y), \end{cases} \begin{cases} \dfrac{\partial u(0,y,t)}{\partial x} =, e^{-t}\cosh(y) & 0 \le y \le 1, \quad x=0, \\ \dfrac{\partial u(1,y,t)}{\partial x} = e^{-t}\cosh(1)\cosh(y), & 0 \le y \le 1, \quad x=1, \\ \dfrac{\partial u(x,0,t)}{\partial y} = 0, & 0 \le x \le 1, \quad y=0, \\ \dfrac{\partial u(x,1,t)}{\partial y} = e^{-t}\sinh(x)\sinh(1), & 0 \le x \le 1, \quad y=1. \end{cases}$$

$$(7.4.11)$$

The exact solution is given by

$$u(x,y,t) = e^{-t}\ \sinh(x)\cosh(y), \qquad (7.4.12)$$

Table 7.4 presents RMS errors, relative errors, max absolute errors and CPU time in seconds of example 4. The table shows that the errors are small and negligible. The figures 7.10 and 7.11 show the numerical and exact solutions at time $t = 1.0$ and $t = 3.0$ respectively. The figures show that the numerical results are in good agreements with the exact solutions. The figure 7.12 shows physical behavior of the equation (7.4.12) in contour form at $t = 1.0$.

Table-7.4

RMS, relative, max absolute errors and CPU time of example 4 at different times for $N = M = 21$.

T	PDQM RMS	PDQM Relative Error	PDQM Max Absolute Error	CPU Time(s)
0.5	9.985×10^{-6}	1.934×10^{-5}	3.846×10^{-6}	17
1.0	7.545×10^{-6}	2.410×10^{-5}	2.277×10^{-6}	36
2.0	1.220×10^{-6}	1.059×10^{-5}	4.086×10^{-6}	70
3.0	8.040×10^{-6}	1.897×10^{-4}	2.106×10^{-6}	103
5.0	9.168×10^{-6}	1.599×10^{-4}	2.669×10^{-6}	179
7.0	5.395×10^{-6}	6.953×10^{-3}	1.499×10^{-6}	248
10.0	5.939×10^{-6}	1.537×10^{-3}	1.619×10^{-6}	339

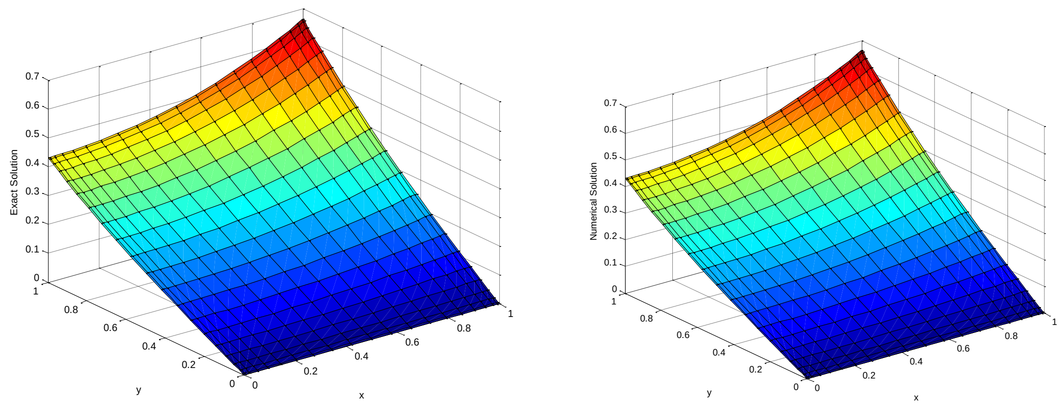


Figure 7.10: Exact (left) and numerical (right) solutions of example 4 with time step length $\Delta t = 0.001$, $\gamma = 1$ at $t = 1.0$.

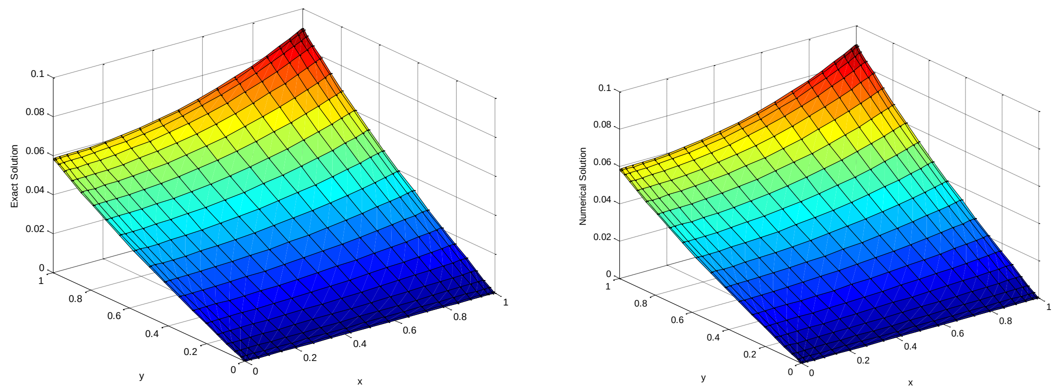


Figure 7.11: Exact (left) and numerical (right) solutions of example 4 with time step length $\Delta t = 0.001$, $\gamma = 1$ at $t = 3.0$.

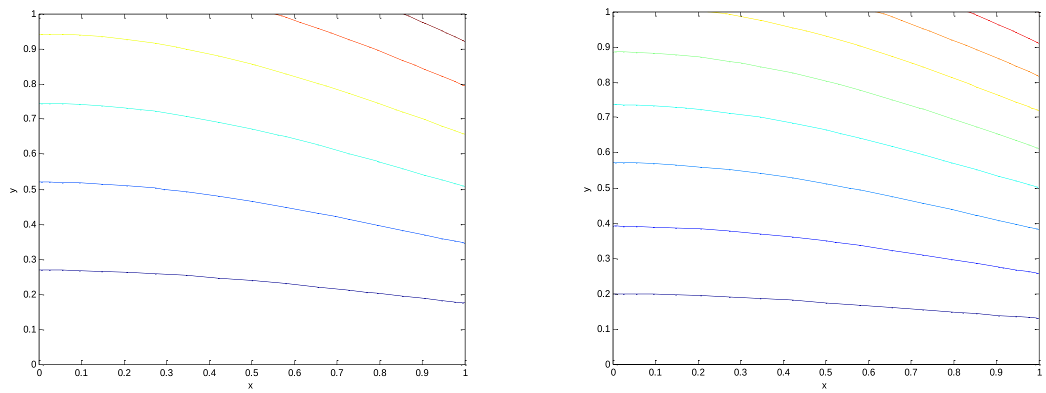


Figure 7.12: Contours plots of example 4 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 1.0$, 3.0 respectively.

Example 5

In this example, we have considered the following two dimensional hyperbolic equation with forcing function

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + u_{yy} + \left(2 - 4\alpha + \beta^2\right)e^{-2t}\sinh(x)\sinh(y), \quad 0 \le x, y \le 1, \;\; t > 0, \tag{7.4.13}$$

where $\alpha > 0$ *and* $\beta \ge 0$ are real parameters and exact solution is given by $u(x, y, t) = e^{-2t}\sinh(x)\sinh(y), \quad t \ge 0.$ The initial and Dirichlet boundary conditions

$$\begin{cases} u(x, y, 0) = \sinh(x)\sinh(y), \\ u_t(x, y, 0) = -2\sinh(x)\sinh(y), \end{cases} \begin{cases} \dfrac{\partial(0, y, t)}{\partial x} = e^{-2t}\sinh(y), & 0 \le y \le 1, \quad x = 0, \\ \dfrac{\partial(1, y, t)}{\partial x} = e^{-2t}\cosh(1)\sinh(y), & 0 \le y \le 1, \quad x = 1, \\ \dfrac{\partial(x, 0, t)}{\partial y} = e^{-2t}\sinh(x), & 0 \le x \le 1, \quad y = 0, \\ \dfrac{\partial(x, 1, t)}{\partial y} = e^{-2t}\cosh(1)\sinh(x), & 0 \le x \le 1, \quad y = 1. \end{cases} \tag{7.4.14}$$

The RMS errors, relative errors, max absolute errors and CPU time in seconds of example 5 are shown in table 7.5 at different time. The table shows that the errors are small and negligible. The numerical and exact solutions at time $t = 1.0$ and $t = 3.0$ respectively are shown in figures 7.13 and 7.14. The figures show that the numerical results are in good agreements with the exact solutions. The figure 7.15 shows physical behavior of the telegraph equation (7.4.14) in contour form at $t = 1.0$.

Table-7.5

RMS, relative, max absolute errors and CPU time of example 5 at different times for $N = M = 21, \alpha = 10, \beta = 5.$

T	PDQM RMS	PDQM Relative Error	PDQM Max Absolute Error	CPU Time(s)
0.5	6.111×10^{-6}	3.411×10^{-5}	2.422×10^{-6}	6
1.0	6.066×10^{-6}	9.206×10^{-5}	2.319×10^{-6}	12
2.0	6.072×10^{-6}	6.809×10^{-4}	2.266×10^{-6}	24
3.0	6.079×10^{-6}	5.035×10^{-4}	2.197×10^{-6}	35
5.0	6.080×10^{-6}	4.984×10^{-3}	2.765×10^{-6}	57

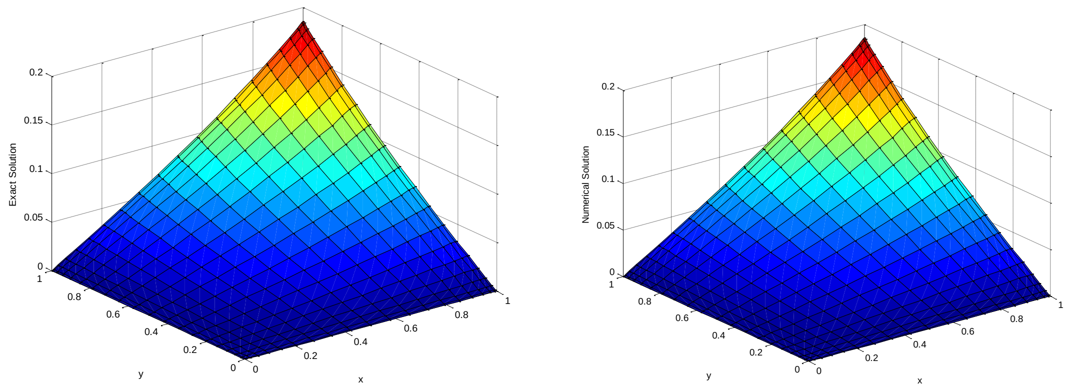


Figure 7.13: Exact (left) and numerical (right) solutions of example 5 with time step length $\Delta t = 0.001$, $\alpha = 0$, $\beta = 5$ at $t = 1.0$.

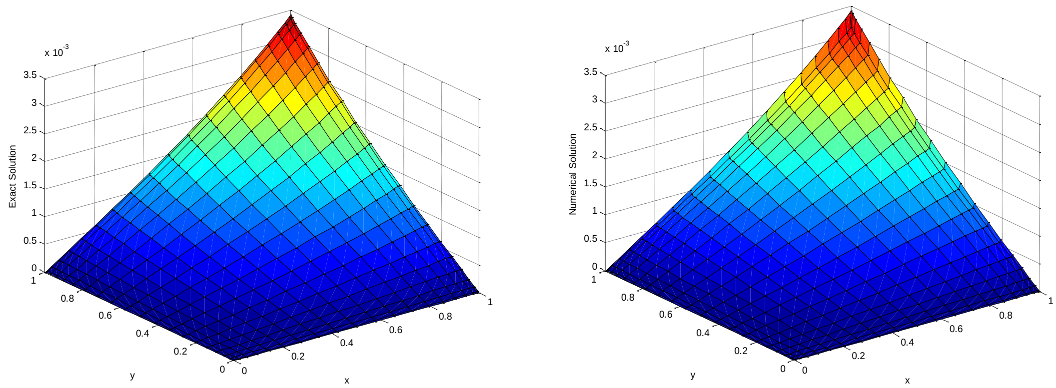


Figure 7.14: Exact (left) and numerical (right) solutions of example 5 with time step length $\Delta t = 0.001$, $\alpha = 0$, $\beta = 5$ at $t = 3.0$.

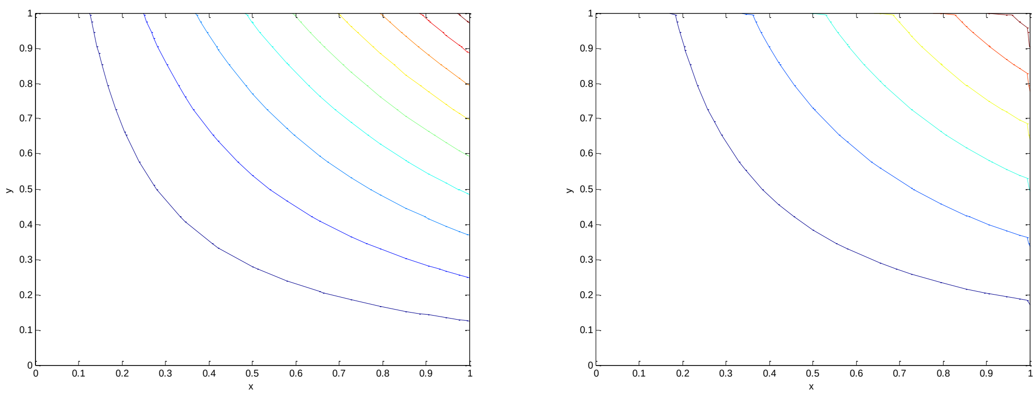


Figure 7.15: Contours plots of example 5 with time step length $\Delta t = 0.001, \alpha = 0$, $\beta = 5$ at $t = 1.0$, 3.0 respectively.

7.5 Discussion

In this chapter, numerical solutions of the second order two dimensional quasilinear hyperbolic equations are discussed by PDQM. Many authors [105, 106, 118, 111-114] have proposed the numerical schemes for the two dimensional quasilinear hyperbolic equations. In this study, numerical results obtained by using Gauss-Lobatto-Chebyshev grid points are presented. In all problems, the RSM errors, relative errors, absolute errors and CPU times are presented in tables 7.1-7.5 with time step length $\Delta t = 0.001$ and $M = N = 21$. The main advantage of the present scheme is that the scheme gives very accurate and similar results to the exact solutions by choosing less number of grid points and the problem can be solved upto big time. The good thing of the present technique is that it is easy to apply and gives us better accuracy in less numbers of grid points as comparison to the other numerical techniques. The multidimensional problems can be easily handled by the present numerical technique. Due to easiness, the present scheme is very useful for engineers and physical scientists.

Summary

The study of nonlinear partial differential equations has an important role in the domains of mathematics, physics, classical mechanics, electromagnetism, relativity theory and quantum mechanics etc. The exact and numerical solutions for nonlinear systems are used for physical investigations. In this thesis entitled "**Exact and Numerical Solutions of some Nonlinear Partial Differential Equations**", we present the applications of Lie classical approach and Differential quadrature method to obtain the exact and numerical solutions of certain nonlinear PDEs representing important physical phenomenon which are the variable coefficients Benjamin-Bona-Mahony-Burger (BBMB) equation, Coupled Short Pulse (CSP) equation along with time dependent coefficients, (2+1)-dimensional Diffusion-Advection equation with variable coefficient, variable coefficient coupled KdV-Burgers's equation, generalized Fitzhugh-Nagumo equation with time-dependent coefficients and two dimensional Quasilinear Hyperbolic equations.

In chapters 2, Lie symmetry method has been applied to investigate the symmetries and invariant solutions of Benjamin-Bona-Mahony-Burger equation. The vector fields of the optimal system lead to reductions of the nonlinear PDE to ODEs and some new exact solutions are derived. Chapter 3 is devoted to utilize the Lie symmetry method to obtain the symmetries admitted by CSP and variable coefficients CSP equations. The symmetries of these equations are exploited to derive some ansätz leading to the reduction of variables, where the traveling wave and exact solutions are easier to obtain by considering the optimal system of conjugacy inequivalent subgroups. The system of ordinary differential equations is further solved in general to obtain numerical solutions with the help of RK4 method.

(2+1)-dimensional variable coefficients Diffusion Advection (DA) equation is studied in chapter 4. The admissible forms of coefficients have been derived for which this nonlinear equation possess the Lie symmetries. Further, we have successfully established exact and explicit non-traveling wave solutions with arbitrary parameters to variable coefficients DA equation via extended (G'/G)-expansion method. These solutions are expressed in terms of the hyperbolic functions, trigonometric functions and rational functions. Most of the solutions obtained involve an arbitrary coefficient function and it may enable one to control and discuss the behavior of solution as governed by the choice of this arbitrary function. In Chapter 5, based on classical Lie group method, with the help of modified (G'/G)-expansion and RK4 methods, we study a variable coefficient coupled KdV-Burgers' equation, and get its infinitesimal generator, commutation table of Lie algebra, symmetry group and similarity reductions. In particular, exact and numerical solutions of the coupled KdV-Burgers equation are derived from the reduction equations. Further, results are illustrated graphically for different parameters.

Another important aspect that deserves special attention is to study numerical solutions of PDEs with the help of PDQM method. Chapter 6, 7 are concerned with numerical solutions of generalized Fitzhugh-Nagumo equation with time-dependent coefficients and two-space-dimensional Quasilinear Hyperbolic partial differential equations with the help of PDQM. In construction of the numerical method, Lagrange's interpolation is used to find the weighting coefficients of differential quadrature method that is why the method is called PDQ method. PDQ method reduced the problem into a system of nonlinear ODEs with initial and boundary conditions. After using the boundary conditions, the obtained ODEs with initial conditions are solved by RK4 method. Finally, numerical examples support to check the accuracy and efficiency of the developed method.

We make the remark that, PDEs and the PDEs systems which arise from differential geometry were not systematically studied numerically by using the symmetry group theory. That is our motivation to apply the modern symmetry groups theory in the case of certain PDEs connected with physical phenomena to obtain the exact as well as numerical solutions. After obtaining the point symmetries of the PDEs under investigation, our attempt is to reduce the number of independent variables of the PDEs corresponding to each element of the optimal system and then reduced ODEs have been studied for exact solution. But with respect to the some reduced ODEs we are unable to obtain the nontrivial exact solutions for the PDEs so for these ODEs we present the numerical solutions of main equation by solving reduced ODEs with the help of classical RK4 method. For numerical solution we pose the initial value problem (IVP) for the ODEs after reducing it to first order ODEs.

Finally, it is worth mentioning that in spite of the focus on the numerical solutions, the author found that the accuracy of the PDQM depends on the number of grid points. As we increase the grid points it gives better results and the method gives better accuracy in less number of grid points while in other numerical methods large number of grid point are used. It is also observed that time dependent problems can be best solved by forming initial valued problems. The multidimensional problems can be easily handled by the present numerical technique. Due to easiness, the PDQM scheme is very useful for engineers and physical scientists.

In short, we can say that in spite of the focus on symmetries and exact solutions, the authors found it really difficult at times to find symmetries for variable coefficient equations as symmetries for constant coefficient equations can be obtained by some mathematical softwares. Keeping in view this limitation, it will be really interesting if such softwares can be developed. In this thesis, it is also a herculean task to extract solutions of PDEs from reduced ODEs and to fulfill this aim, some specific forms of solutions of ODEs have been assumed. Thus, the general solution of reduced ODEs, their physical interpretation and to study the nonclassical symmetries of variable coefficient equations brings forth tremendous scope of future work. Also there is gap in literature to study the numerical analysis and simulation of elliptic PDEs by DQM. In future, numerical analysis and simulation of elliptic PDEs can be done by developing numerical scheme based on B-spline differential quadrature methods, quasilinearization process and some standard numerical methods such as Runge-Kutta methods, Gauss-elimination method etc.

Bibliography

[1] A. Ashyralyeva and M.E. Koksal, On the second order of accuracy difference scheme for hyperbolic equations in a Hilbert space, *Numerical Functional Analysis and Optimization*, 26 (2005) 739-772.

[2] A. Ashyralyeva, M.E. Koksal and R.P. Agarwal, A difference scheme for Cauchy problem for the hyperbolic equation with self-adjoint operator, *Mathematical and Computer Modelling*, 52 (2010) 409-424.

[3] A. Biswas, A. Yildirim, T. Hayat, O.M. Aldossary and R. Sassaman, Soliton perturbation theory of the generalized Klein-Gordon equation with full nonlinearity, *Proceedings of the Romanian Academy*, 13 (2012) 32-41.

[4] A.E. Mowafy, E.K. El-Shewy, W.M. Moslem and M.A. Zahran, Effect of dust charge fluctuation on the propagation of dust-ion acoustic waves in inhomogeneous mesospheric dusty plasma, *Physics of Plasmas*, 15 (2008) 073708 (9 pp.).

[5] A. Gerisch, D.F. Griffiths, R. Weiner and M.A.J Chaplain, A positive splitting method for mixed hyperbolic-parabolic systems, *Numerical Methods for Partial Differential Equations*, 17 (2001) 152-168.

[6] A. Gerisch, J. Lang, H. Podhaisky and R. Weiner, High-order linearly implicit two-step peer-finite element methods for time-dependent PDEs, *Applied Numerical Mathematics*, 59 (2009) 624-638.

[7] A.H. Bhrawy, A Jacobi-Gauss-Lobatto collocation method for solving generalized Fitzhugh-Nagumo equation with time-dependent coefficients, *Applied Mathematics and Computation*, 222 (2013) 255-264.

[8] A. Hajipour and S.M. Mahmoudi, Application of exp-function method to Fitzhugh-Nagumo equation, *World Applied Sciences Journal*, 9 (2010) 113-117.

[9] A.G. Johnpillai, A.H. Kara and A. Biswas, Symmetry reduction, exact group-invariant solutions and conservation laws of the Benjamin-Bona-Mahoney equation, *Applied Mathematics Letters*, 26 (2013) 376-381.

[10] A.H. Kara and F.M. Mahomed, Relationship between symmetries and conservation laws, *International Journal of Theoretical Physics*, 39 (2000) 23-40.

[11] A. Korkmaz and I. Dag, A differential quadrature algorithm for simulations of nonlinear Schrödinger equation, *Computer & Mathematics with Applications*, 56 (2008) 2222-2234.

[12] A. Korkmaz and I. Dag, Crank-Nicolson-differential quadrature algorithms for the Kawahara equation, *Chaos, Solitons and Fractals*, 42 (2009) 65-73.

[13] A.M. Kawala, Comparison study for numerical solution and travelling wave solution for Ito coupled system, *Numerical Methods for Partial Differential Equation*, 33 (2008) 870-884.

[14] A.M. Wazwaz and A. Gorguis, An analytic study of Fisher's equation by using adomian decomposition method, *Applied Mathematics and Computation*, 154 (2004) 609-620.

[15] A.M.Wazwaz, The tanh method for travelling wave solutions of nonlinear equations, *Applied Mathematics and Computation*, 154 (2004) 713-723.

[16] A.M. Wazwaz, The extended tanh method for new soliton solutions for many forms of the fifth-order KdV equations, *Applied Mathematics and Computation*, 184 (2007) 1002-1014.

[17] A.M. Wazwaz, A sine-cosine method for handling nonlinear wave equations, *Mathematical and Computer Modelling*, 40 (2004) 499-508.

[18] A.M. Wazwaz, The Hirota's direct method for multiple-soliton solutions for three model equations of shallow water waves, *Applied Mathematics and Computation*, 201 (2008) 489-503.

[19] Anupma, *Lie Group Applications to Some Nonlinear Systems*, Ph.D. Thesis, School of Mathematics and Computer Application Thapar University, Patiala, India, 2013.

[20] A. Pickering, A new truncation in Painlevé analysis, *Journal of Physics A: Mathematical and General*, 26 (1993) 4395-4405.

[21] A. Ramani, B. Grammaticos and T. Bountis, The Painlevé property and singularity analysis of integrable and non-integrable systems, *Physics Reports*, 180 (1989) 159-245.

[22] B.F. Feng, An integrable coupled short pulse equation, *Journal of Physics A: Mathematical and Theoretical*, 45 (2012) 085202.

[23] B.F. Feng, K. Maruno and Y. Ohta, Integrable discretizations of the short pulse equation, *Journal of Physics A: Mathematical and Theoretical*, 43 (2010) 085203.

[24] B.J. Cantwell, *Introduction to Symmetry Analysis*, Cambridge University Press, Cambridge, UK, 2002.

[25] C. Grossmann and H.G. Roos, *Numerical Treatment of Partial Differential Equations*, Springer-Verlag, Berlin Heidelberg, 2007.

[26] C.M. Khalique and K.R. Adem, Exact solutions of the (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation using Lie group analysis, *Mathematical and Computer Modelling*, 54 (2011) 184-189.

[27] C.Q. Dai and J.F. Zhang, Jacobian elliptic function method for nonlinear differential-difference equations, *Chaos, Solitons and Fractals*, 27 (2006) 1042-1049.

[28] C. Rogers and W.F. Shadwick, *Bäcklund Transformations and Their Applications*, Academic Press, New York, 1982.

[29] C. Shu, *Differential Quadrature and its Application in Engineering*, Springer-Verlag, London, 2000.

[30] C. Shu, *Generalized Differential-Integral Quadrature and Application to the Simulation of Incompressible Viscous Flows Including Parallel Computation*, Ph.D. Thesis, University of Glasgow, UK, 1991.

[31] C.S. Su and C.S. Gardner, Drivation of the Korteweg-de Vries and Burgers' equation, *Journal of Mathematical Physics*, 10 (1969) 536-539.

[32] C.W. Bert and M. Malik, Differential quadrature method in computational mechanics, *Applied Mechanics Reviews*, 49 (1996) 1-28.

[33] D.D. Ganji and A. Sadighi, Application of He’s homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations, *International Journal of Nonlinear Sciences and Numerical Simulation*, 7 (2006) 411-418.

[34] D.E. Baldwin and W. Hereman, Symbolic software for the Painlevé test of nonlinear ordinary and partial differential equations, *Journal of Nonlinear Mathematical Physics*, 13 (2006) 1-21.

[35] D.E. Baldwin, *Symbolic Algorithms and Softwares for the Painlevé Test and Recursion Operators for Nonlinear Partial Differential Equations*, Ph.D. Thesis, Colorado School of Mines, Colorado, 2004.

[36] D. Greenspan, Approximate solution of initial boundary wave equation problems by boundary-value techniques, *Communications of the ACM*, 11 (1968) 760-763.

[37] D.H. Peregrine, Calculations of the development of an undular bore, *Journal of Fluid Mechanics*, 25 (1996) 321-330.

[38] D.J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Philosophical Magazine*, 39 (1895) 422-443.

[39] D. Rollins, Painlevé analysis and Lie group symmetries of the regularized long-wave equation, *Journal of Mathematical Physics*, 32 (1991) 3331–3332.

[40] D.Z. Zhou, C. Yong and W. Ling, Similarity reductions of (2+1)-dimensional multi-component Broer-Kaup system, *Communication in Theoretical Physics*, 50 (2008) 803-808.

[41] E.A. Saied and R.G. Abd El-Rahman, Analytic solutions for asymmetric model of a rod in a lattice fluid, *Journal of Statistical Physics*, 9 (1998) 639-652.

[42] E. Cartan, *Geometry of Riemannian Spaces (Lie Groups; History, Frontiers and Applications Series)*, Math Sci Press, Massachusetts, 1983.

[43] E. Demetriou, M.A. Christou and C. Sophocleous, On the classification of similarity solutions of a two-dimensional diffusion-advection equation, *Applied Mathematics and Computation*, 187 (2007) 1333-1350.

[44] E. Fan and H. Zhang, A note on the homogeneous balance method, *Physics Letters A*, 246 (1998) 403-406.

[45] E. L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[46] E.M.E. Zayed, H.A. Zedan and K.A. Gepreel, Group analysis and modified extended tanh- function to find the invariant solutions and soliton solutions for nonlinear Euler equations, *International Journal of Nonlinear Science and Numerical simulation*, 5 (2004) 221-234.

[47] F. Civan and C.M Sliepcevich, Differential quadrature for multi-dimensional problems, *Journal of Mathematical Analysis and Applications*, 101 (1984) 423-443.

[48] F.J. Alexander and J.L. Lebowitz, Driven diffusive systems with a moving obstacle: a variation on the Brazil nuts problem, *Journal of Physics A: Mathematical and General*, 23 (1990) 375- 382.

[49] F.J. Alexander and J.L. Lebowitz, On the drift and diffusion of a rod in a lattice fluid, *Journal of Physics A: Mathematical and General*, 27 (1994) 683-696.

[50] F. Klein, Vergleichende betrachtungen über neuere geometrische forsuchungen, (A comparative review of recent researches in geometry), *Mathematische Annalen*, 43 (1893) 63-100.

[51] F. Oliveri, Lie Symmetries of differential equations: classical results and recent contributions, *Symmetry Open Access Journal*, 2 (2010) 658-706.

[52] F. Oliveri, *Relie: A Reduce Program for Calculating Lie Symmetries*. Technical Report: Department of Mathematics, University of Messina, Italy, 2001.

[53] G. Birkhoff, *Hydrodynamics-A Study in Logic, Fact and Similitude*, Princeton University Press: Princeton, NJ, USA, 1950.

[54] G. Ebadi, A.H. Kara, M.D. Petkovic, A. Yildirim and A. Biswas, Solitons and conserved quantities of the Ito equation, *Proceedings of the Romanian Academy*, 13 (2012) 215-224.

[55] G. Fakhari, Domairry and Ebrahimpour, Approximate explicit solutions of nonlinear BBMB equations by homotopy analysis method and comparison with the exact solution, *Physics Letter A*, 368 (2007) 64-68.

[56] G. Hariharan and K. Kannan, Haar wavelet method for solving Fitzhugh-Nagumo equation, *World Academy of Science, Engineering and Technology*, 43 (2010) 560-564.

[57] G.W. Bluman and J.D. Cole, The general similarity solution of the heat equation, *Journal of Mathematics and Mechanics*, 18 (1969) 1025–1042.

[58] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations,* Springer-Verlag, New York, 1989.

[59] G.W. Bluman, A.F. Cheviakov and S.C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Springer-Verlag, New York, USA, 2009.

[60] G.W. Bluman, and J.D. Cole, *Similarity Methods for Differential Equations,* Springer-Verlag, New York, USA, 1974.

[61] G.W. Bluman, and S. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer-Verlag, New York, 2002.

[62] G.W. Bluman, *Construction of Solutions to Partial Differential Equations by the Use of Transformation Groups*, Ph.D. Thesis, California Institute of Tecnology, California, 1967.

[63] H.A. Zedan and A.M. Kawala, Classes of solution for a system of one dimensional motion of a gas, *Applied Mathematics and Computation*, 142 (2003) 271–282.

[64] H. Liu and J. Li, Lie symmetry analysis and exact solutions for the short pulse equation, *Nonlinear Analysis*, 71 (2009) 2126-2133.

[65] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, Cambridge, 2003.

[66] H. Stephani, *Differential Equations Their Solutions Using Symmetries*, Cambridge University Press, Cambridge, UK, 1989.

[67] H. Tari and D.D. Ganji, Approximate explicit solutions of nonlinear BBMB equations by He's methods and comparison with the exact solution, *Physics Letter A*, 367 (2007) 95-101.

[68] H. Triki and A.M. Wazwaz, On soliton solutions for the Fitzhugh-Nagumo equation with time-dependent coefficients, *Applied Mathematical Modelling,* 37 (2013) 3821-3828.

[69] H. Triki, A. Yildirim, T. Hayat, O.M. Aldossary and A. Biswas, Topological and non-topological soliton solutions of the Bretherton equation, *Proceedings of the Romanian Academy,* 13 (2012) 103-108.

[70] H. Li and Y. Guo, New exact solutions to the Fitzhugh-Nagumo equation, *Applied Mathematics and Computation*, 180 (2006) 524-528.

[71] I. Dag, A. Korkmaz and B. Saka, Cosine expansion-based differential quadrature algorithm for numerical solution of the RLW equation, *Numerical Methods for Partial Differential Equations*, 26 (2010) 544-560.

[72] J.A. Whitehead and A.C. Newell, Finite bandwidth, finite amplitude convection, *Journal of Fluid Mechanics*, 38 (1969) 279-303.

[73] J.C. Brunelli and S. Sakovich, Hamiltonian integrability of two-component short pulse equation, *Journal of Mathematical Physics*, 54 (2013) 012701.

[74] J.C. Brunelli, The short pulse hierarchy, *Journal of Mathematical Physics*, 46 (2005) 123507.

[75] J.H. He, Some asymptotic methods for strongly nonlinear equations, *International Journal of Modern Physics B*, 20 (2006) 1141-1199.

[76] J.L. Zhang, M.L.Wang, Y.M. Wang and Z.D. Fang, The improved F-expansion method and its applications, *Physics Letters A*, 350 (2006) 103-109.

[77] J.M. Burgers, Mathematical example illustrating relations occurring in the theory of turbulent fluid motion. *Transactions of Royal Netherlands Academy of Arts and Sciences Amsterdam*, 17 (1939) 1-53.

[78] J.M. Hill, *Differential Equations and Group Methods for Scientists and Engineers*, CRC Press, Boca Raton, FL, 1992.

[79] J.M. Hill, Similarity solutions for nonlinear diffusion- a new integration procedure, *Journal of Engineering Mathematics*, 23 (1989) 141-155.

[80] J. Häggblad, *Symmetries and Recursion Operators of Nonlinear Differential Equations*, C/D EXTENDED ESSAY, Department of Mathematics Luleå University of Technology, SE-971 87 Luleå, Sweden, 2006.

[81] J. Pike and P.L. Roe, Accelerated convergence of Jameson's finite volume Euler scheme using Van Der Houwen integrators, *Computers & Fluids*, 13 (1985) 223-236.

[82] J. Plucker. On a new geometry of space, *Philosophical Transactions of the Royal Society of London*, 155 (1865) 725-791.

[83] J.R. Spiteri and S.J. Ruuth, A new class of optimal high-order strong stability-preserving time-stepping schemes, *SIAM Journal on Numerical Analysis,* 40 (2002) 469-491.

[84] J.R. Quan and C.T. Chang, New insights in solving distributed system equations by the quadrature methods-I, *Computers & Chemical Engineering,* 13 (1989) 779-788.

[85] J.R. Quan and C.T. Chang, New insights in solving distributed system equations by the quadrature methods-II, *Computers & Chemical Engineering*, 13 (1989) 1017-1024.

[86] J.S. Nagumo, S. Arimoto and S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proceedings of the IRE,* 50 (1962) 2061-2071.

[87] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, *Journal of Mathematical Physics*, 24 (1983) 522–526.

[88] J. Weiss, The Painlevé property for partial differential equation II: Bäcklund transformation, lax pairs and the Schwarzian derivative, *Journal of Mathematical Physics*, 24 (1983) 1405-1414.

[89] K.Al. Khaled, S. Momani and A. Alawneh, Approximate wave solutions for generalized Benjamin-Bona-Mahony-Burgers equations, *Applied Mathematics and Computation*, 171 (2005) 281-292.

[90] K.M. Morton and D. Mayers, *Numerical Solution of Partial Differential Equations*, Cambridge University Press, New York, 2005.

[91] K. Omrani, and M. Ayadi, Finite difference discretization of the Benjamin-Bona-Mahony-Burgers equation, *Numerical Methods for Partial Differential Equations*, 24 (2008) 239-248.

[92] K.R. Helfrich, W.K. Melville and J.W. Miles, On interfacial solitary waves over variable topography, *Journal of Fluid Mechanics*, 149 (1984) 305-317.

[93] K. Singh and R.K. Gupta, Exact solutions of a variant Boussinesq system, *International Journal of Engineering Science*, 44 (2006) 1256-1268.

[94] K. Singh and R.K. Gupta, Lie symmetries and exact solutions of a new generalized Hirota-Satsuma coupled KdV system with variable coefficients, *International Journal of Engineering Science*, 44 (2006) 241-255.

[95] L. Bangqing and M. Yulan, The non-traveling wave solutions and novel fractal soliton for the (2+1)-dimensional Broer-Kaup equations with variable coefficients, *Communication in Nonlinear Science and Numerical Simulation,* 16 (2011) 144-149.

[96] L.C. Cheng and C. Yong, Symmetry and exact solutions of (2+1)-dimensional generalized Sasa-Satsuma equation via a modified direct method, *Communication in Theoretical Physics*, 51 (2009) 973-978.

[97] L. Han-Ze, L. Ji-Bin, and L. Lei, Lie symmetry analysis, Bäcklund transformations and exact solutions to (2+1)-dimensional Burgers' types of equations, *Communication in Theoretical Physics*, 57 (2012) 737-742.

[98] L.I. Sedov, *Similarity and Dimensional Methods in Mechanics*, Academic Press, New York, USA, 1959.

[99] L. Kaur and R.K. Gupta, Kawahara equation and modified Kawahara equation with time dependent coefficients: symmetry analysis and generalized $\left(G'/G\right)$-expansion method, *Mathematical Methods in Applied Sciences*, 36 (2012) 584-601.

[100] L. Kaur and R.K. Gupta, On symmetries and exact solutions of the Einstein Maxwell field equations via the symmetry approach, *Physica Scripta*, 87 (2013) 035003 (11 pp).

[101] L. Shi-da and L. Shi-kuo, KdV-Burgers equation model of turbulence, *Science China Mathematics*, 34 (1991) 938-946.

[102] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, UK, 1982.

[103] L.V. Ovsiannikov, *Group Properties of Differential Equations*, Nauka, Novosibirsk, Russia, 1962.

[104] L.V. Wijngaarden, On the motion of gas bubbles in a perfect fluid, *Annual Review of Fluid Mechanics*, 4 (1972) 369-373.

[105] M. Ciment and S.H. Leventhal, A note on the operator compact implicit method for the wave equation, *Mathematics of Computation*, 32 (1978) 143-147.

[106] M. Ciment and S.H. Leventhal, Higher order compact implicit schemes for the wave equation, *Mathematics of Computation*, 29 (1975) 985-994.

[107] M.C. Nucci and P.A. Clarkson, The nonclassical method is more general than the direct method for symmetry reductions: An example of the Fitzhugh-Nagumo equation, *Physics Letters A*, 164 (1992) 49-56.

[108] M.C. Nucci and P.G.L. Leach, Singularity and symmetry analyses of mathematical models of epidemics, *South African Journal of Science*, 105 (2009) 136-146.

[109] M.C. Nucci, Nonclassical symmetries as special solutions of Heir equations, *Journal of Mathematical Analysis and Applications*, 279 (2003) 168-179.

[110] M.C. Nucci, The complete Kepler group can be derived by Lie group analysis, *Journal of Mathematical Physics*, 37 (1996) 1772–1775.

[111] M. Dehghan and A. Mehebbi, A high order implicit collocation method for the solution of two-dimensional linear hyperbolic equation, *Numerical Methods for Partial Differential Equations*, 25 (2009) 232-243.

[112] M. Dehghan and A. Mohebbi, The combination of collocation, finite difference, and multigrid methods for solution of the two-dimensional wave equation, *Numerical Methods for Partial Differential Equations*, 24 (2008) 897-910.

[113] M. Dehghan and A. Shokri, A meshless method for numerical solution of a linear hyperbolic equation with variable coefficients in two space dimensions, *Numerical Methods for Partial Differential Equations*, 25 (2009) 494-506.

[114] M. Dehghan, Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices, *Mathematics and Computers in Simulation*, 71 (2006) 16-30.

[115] M. Dumbser and M. Kaser, Arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems, *Journal of Computational Physics*, 221 (2007) 693-723.

[116] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Mathematical Society, Lecture Note Series 149, Cambridge University Press, London, 1991.

[117] M.J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type I, *Journal of Mathematical Physics*, 21 (1980) 715-721.

[118] M.K. Jain, R. Ahuja and S. Bhattacharya, Difference schemes for second order hyperbolic equations, *International Journal for Numerical Methods in Engineering*, 10 (1976) 960-964.

[119] M. Lakshmanan and K.M. Tamizhmani, Lie-Bäcklund symmetries of certain nonlinear evolution equations under perturbation around their solutions, *Journal of Mathematical Physics*, 26 (1985) 1189-1200.

[120] M. Lees, Alternating direction methods for hyperbolic differential equations, *Journal of the Society for Industrial and Applied Mathematics*, 10 (1962) 610-616.

[121] M.L. Gandarias and M.S. Bruzón, Nonclassical potential symmetries for the Burgers equation, *Nonlinear Analysis*, 71 (2009) 1826-1834.

[122] M.L. Gandarias and M.S. Bruzón, Symmetry analysis and exact solutions of some Ostrovsky equations, *Theoretical and Mathematical Physics*, 168 (2011) 898-911.

[123] M.L. Gandarias, M. Torrisi and A. Valenti, Symmetry classification and optimal systems of a non-linear wave equation, *International Journal of Non-Linear Mechanics*, 39 (2004) 389-398.

[124] M.L. Wang, X. Li and J. Zhang, The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Physics Letters A*, 372 (2008) 417-423.

[125] M. Molati, *Group Classification of Coupled Partial Differential Equations with Application to Flow in a Collapsible Channel and Diffusion Processes*, Ph.D. Thesis, University of the Witwatersrand, Johannesburg, 2010.

[126] M. Pandey, B.D. Pandey and V.D. Sharma, Symmetry groups and similarity solutions for the system of equations for a viscous compressible fluid, *Applied Mathematics and Computation*, 215 (2009) 681-685.

[127] M.S. Bruzón and M.L. Gandarias, Classical and nonclassical symmetries for the Krichever-Novikov equation, *Theoretical and Mathematical Physics*, 168 (2011) 875-885.

[128] M. Senthil and M. Lakshmanan, Lie symmetries and infinite-dimensional Lie algebras of certain nonlinear dissipative systems, *Journal of Physics A: Mathematical and General*, 28 (1995) 1929-1942.

[129] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, Houston, Texas, 1999.

[130] M. Shih, E. Momoniat and F.M. Mahomed, Approximate conditional symmetries and approximate solutions of the perturbed Fitzhugh-Nagumo equation, *Journal of Mathematical Physics,* 46 (2005) 023503.

[131] N. Goyal and R.K. Gupta, New exact solutions of the Einstein Maxwell equations for magnetostatic fields, *Chinese Physics B*, 21 (2012) 090401.

[132] N. Goyal and R.K. Gupta, Symmetries and exact solutions of the nondiagonal Einstein-Rosen metrics, *Physica Scripta*, 85 (2012) 015004.

[133] N.H. Ibragimov, *Handbook of Lie Group Analysis of Differential Equations*, CRC Press, Boca Raton, USA, 1996.

[134] N.H Ibragimov, *Transformation Groups Applied to Mathematical Physics,* Reidel Publishing Company, Dordrecht, Netherlands, 1985.

[135] O.P. Bhutani, G. Chandrasekarann and P. Mittal, On the exact solutions of nonlinear differential equations of nonlinear engineering system via invariant variational principles, *International Journal of Engineering Science*, 26 (1988) 243-248.

[136] O.P. Bhutani, P. Mittal and G. Chandrasekaran, On invariant solutions of the generalized Boussinesq equation, *International Journal of Engineering Science*, 26 (1988) 307-310.

[137] P.A. Clarkson and E.L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear equations, *Physica D: Nonlinear Phenomena*, 70 (1993) 250-288.

[138] P.A. Clarkson and M.D. Kruskal, New similarity solutions of the Boussinesq equation, *Journal of Mathematical Physics*, 30 (1989) 2201-2213.

[139] P. Drazin and R. Johnson, *Solitons: An Introduction*, Cambridge Univesity Press, New York, 1989.

[140] P.E. Hydon, *Symmetry Methods for Differential Equations, a Beginner's Guide*, Cambridge University Press, New York, USA, 2000.

[141] P.J. Olver and P. Rosenau, The construction of special solutions to partial differential equations, *Physics Letters A*, 114 (1986) 107-112.

[142] P.J. Olver and P. Rosenau, Group-invariant solutions of differential equations, *SIAM Journal on Applied Mathematics*, 47 (1987) 263–278.

[143] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, USA, 1993.

[144] P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press, Cambridge, UK, 1995.

[145] P. Malekzadeh and G. Karami, A mixed differential quadrature and finite element free vibration and buckling analysis of thick beams on two parameter elastic foundations, *Applied Mathematical Modelling*, 32 (2008) 1381-1394.

[146] P. Malekzadeh and G. Karami, Polynomial and harmonic differential quadrature methods for free vibration of variable thickness thick skew plates, *Engineering Structures*, 27 (2005) 1563-1574.

[147] P.N. Hu, Collisional theory of shock and nonlinear waves in a plasma, *Physics of Fluids*, 15 (1972) 854-864.

[148] P.R. Doran-Wu, *Extension of Lie's algorithm: A potential symmetries classification of PDEs*, Ph.D. Thesis, University of British Columbia, Canada, 1996.

[149] R.A. Van Gorder, Gaussian waves in the Fitzhugh-Nagumo equation demonstrate one role of the auxiliary function $H(x, t)$ in the homotopy analysis method, *Communications in Nonlinear Science and Numerical Simulation*, 17 (2012) 1233-1240.

[150] R.A. Van Gorder and K. Vajravelu, A variational formulation of the Nagumo reaction-diffusion equation and the Nagumo telegraph equation, *Nonlinear Analysis: Real World Applications*, 11 (2010) 2957-2962.

[151] R. Bellman, B.G. Kashef and J. Casti, Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations, *Journal of Computational Physics*, 10 (1972) 40-52.

[152] R.C. Mittal and R. Jiwari, A differential quadrature method for numerical solutions of Burgers'-type equations, *International Journal of Numerical Methods for Heat & Fluid Flow*, 22 (2012) 880-895.

[153] R.C. Mittal, R. Jiwari and K.K. Sharma, A numerical scheme based on differential quadrature method to solve time dependent Burgers' equation, *Engineering Computations*, 30 (2013) 117-131.

[154] R.C. Mittal and R. Jiwari, A numerical scheme for singularly perturbed Burger-Huxley equation, *Journal of Applied Mathematics & Informatics*, 29 (2011) 813-829.

[155] R.C. Mittal and R. Jiwari, Numerical study of Burger-Huxley equation by differential quadrature method, *International Journal of Applied Mathematics and Mechanics*, 5 (2009) 1-9.

[156] R.C. Mittal and R. Jiwari, A numerical scheme for some nonlinear differential equations models in biology, *International Journal for Computational Methods in Engineering Science and Mechanics*, 12 (2011) 134-140.

[157] R.C. Mittal and R. Jiwari, Differential quadrature method for numerical solution of coupled viscous Burgers' equations, *International Journal for Computational Methods in Engineering Science and Mechanics*, 13 (2012), 1-5.

[158] R.C. Mittal and R. Jiwari, Differential quadrature method for two dimensional Burgers' equations, *International Journal for Computational Methods in Engineering Science and Mechanics*, 10 (2009) 450-459.

[159] R.C. Mittal and R. Jiwari, Numerical study of two-dimensional reaction-diffusion brusselator system, *Applied Mathematics and Computation*, 217 (2011) 5404-5415.

[160] R.C. Mittal and R. Jiwari, Numerical study of Fisher's equation by using differential quadrature method, *International Journal of Information and Systems Sciences*, 5 (2008) 143-160.

[161] R. Fitzhugh, Impulse and physiological states in models of nerve membrane, *Biophysical Journal*, 1 (1961) 445-466.

[162] R. Jiwari, R.C. Mittal and K.K. Sharma, A numerical scheme based on weighted average differential quadrature method for the numerical solution of Burgers' equation, *Applied Mathematics and Computation*, 219 (2013) 6680-6691.

[163] R. Jiwari, S. Pandit and R.C. Mittal, A differential quadrature algorithm to solve the two dimensional linear hyperbolic equation with Dirichlet and Neumann

boundary conditions, *Applied Mathematics and Computation*, 218 (2012) 7279-7294.

[164] R. Jiwari, S. Pandit and R.C. Mittal, A differential quadrature algorithm for the numerical solutions of the second-order one dimensional hyperbolic Telegraph equation, *International Journal of Nonlinear Sciences*, 13 (2012) 259-266.

[165] R. Jiwari, S. Pandit and R.C. Mittal, Numerical simulation of two-dimensional sine-Gordon solitons by differential quadrature method, *Computer Physics Communication*, 183 (2012) 600-616.

[166] R. Jiwari, *Numerical Treatment of Some Partial Differential Equations*, Ph.D. Thesis, Department of Mathematics, Indian Institute of Technology, Roorkee, India, 2010.

[167] R.K. Gupta, and A. Bansal, Painlevé analysis, Lie symmetries and invariant solutions of potential Kadomstev-Petviashvili equation with time dependent coefficients, *Applied Mathematics and Computation*, 219 (2013) 5290–5302.

[168] R.K. Gupta and K. Singh, Symmetry analysis and some exact solutions of cylindrically symmetric null fields in general relativity, *Communications in Nonlinear Science and Numerical Simulation*, 16 (2011) 4189-4196.

[169] R.K. Mohanty and M.K. Jain, An unconditionally stable alternating direction implicit scheme for the two space dimensional linear hyperbolic equation, *Numerical Methods for Partial Differential Equations*, 22 (2001) 983-993.

[170] R.K. Mohanty and S. Singh, A new high-order approximation for the solution of two-space-dimensional quasilinear hyperbolic equations, *Advances in Mathematical Physics*, 2011 (2011) 1-22.

[171] R.K. Mohanty, M.K. Jain, and K. George, High order difference schemes for the system of two space second order nonlinear hyperbolic equations with variable coefficients, *Journal of Computational and Applied Mathematics*, 70 (1996) 231-243.

[172] R.K. Mohanty, U. Arora, and M.K. Jain, Linear stability analysis and fourth-order approximations at first time level for the two space dimensional mildly quasi-linear hyperbolic equations, *Numerical Methods for Partial Differential Equations*, 17 (2001) 607-618.

[173] R.S. Johnson, Shallow water waves on a viscous fluid-the undular bore, *Physics of Fluids*, 15 (1972) 1693-1699.

[174] S. Abbasbandy, Soliton solutions for the Fitzhugh-Nagumo equation with the homotopy analysis method, *Applied Mathematical Modelling*, 32 (2008) 2706-2714.

[175] S.A. Elwakil, M.A. Zahran and R. Sabry, Group classification and symmetry reductions of a (2+1)-dimensional diffusion-advection equation, *Zeitschrift für angewandte Mathematik und Physik*, 56 (2005) 986–999.

[176] S. Guo and Y. Zhou, The extended (G'/G)-expansion method and its applications to the Whitham-Broer-Kaup like equations and coupled Hirota-Satsuma KdV equations, *Applied Mathematics and Computation*, 215 (2010) 3214–3221.

[177] S. Kumar, K. Singh and R.K. Gupta, Painlevé analysis, Lie symmetries and exact solutions for (2+1)-dimensional variable coefficients Broer-Kaup equations, *Communication in Nonlinear Sciences and Numerical Simulation*, 17 (2012) 1529-1541.

[178] S. Lie and F. Engel, *Theorie der transformationsgruppen*, Teubner, Leipzig, Germany, 1888.

[179] S. Lie, *Vorlesungena über differentialgleichungen mit bekannten infinitesimalen transformationen*, Teubner, Leipzig, Germany, 1891.

[180] S.R.K. Iyengar and R.C. Mittal, High order difference schemes for the wave equation, *International Journal for Numerical Methods in Engineering*, 12 (1978) 1623-1628.

[181] S. Zhang, L. Dong, J.M. Ba and Y.N. Sun, The (G'/G)-expansion method for nonlinear differential difference equations, *Physics Letters A*, 373 (2009) 905-910.

[182] T. B. Benjamin, J. Bona and J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philosophical Transaction of the Royal Society London A*, 272 (1972) 47-78.

[183] T. Chou, *Lie Group and its Applications in Differential Equations*, Science Press, Beijing, China, 2001.

[184] T. Frankel, *The Geometry of Physics*, Cambridge University Press, UK, 2003.

[185] T. Kawahara and M. Tanaka, Interactions of traveling fronts: an exact solution of a nonlinear diffusion equation, *Physics Letters A*, 97 (1983) 311–314.

[186] T. Schäfer and C.E. Wayne, Propagation of ultra-short optical pulses in cubic nonlinear media, *Physica D: Nonlinear Phenomena*, 196 (2004) 90–105.

[187] T. Schwartzkopff, M. Dumbser and C.D. Munz, Fast high order ADER schemes for linear hyperbolic equations, *Journal of Computational Physics*, 197 (2004) 532-539.

[188] V.D. Sharma and C. Radha, Similarity solutions for converging shocks in a relaxing gas, *International Journal of Engineering Science*, 33 (1995) 535-553.

[189] V.D. Sharma and R. Arora, Similarity solutions for strong shocks in an ideal gas, *Studies in Applied Mathematics*, 114 (2005) 375-394.

[190] V.D. Sharma and R. Radha, Exact solutions of Euler equations of ideal gas dynamics via Lie group analysis, *Zeitschrift für angewandte Mathematik und Physik*, 59 (2008) 1029-1038.

[191] V.D. Sharma, *Quasilinear Hyperbolic Systems, Compressible Flows, and Waves, Monographs and Surveys in Pure and Applied Mathematics*, Chapman & Hall CRC Press, New York, 2010.

[192] W. Chen, T.X. Zhong and C. Shu, A Lyapunov formulation for efficient solution of the Poisson and convection-diffusion equations by differential quadrature method, *Journal of Computational Physics*, 141 (1998) 78-84.

[193] W. Hermann, Cartan on groups and differential geometry, *Bulletin of the American Mathematical Society*, 44 (1938) 598-601.

[194] W. Hereman, Review of symbolic software for the computation of Lie symmetries of differential equation, *Euromath Bulletin*, 1 (1993) 45-79.

[195] W. Hereman, The Painlevé integrability test, *arXiv preprint solv-int/9803006*, (1998) 1-5.

[196] X. Hu, S. Lou and Y. Chen, Explicit solutions from eigen function symmetry of the Korteweg-de Vries equation, *Physical Review E*, 85 (2012) 056607.

[197] X. Miao and Z. Zhang, The modified (G'/G)-expansion method and traveling wave solutions of nonlinear perturbed Schrödinger's equation with Kerr law nonlinearity, *Communications in Nonlinear Sciences and Numerical Simulation*, 16 (2011) 4259-4267.

[198] Y. Chung, C.K.R.T. Jones, T. Schäfer and C.E. Wayne, Ultra-short pulse in linear and nonlinear media, *Nonlinearity*, 18 (2005) 1351-1374.

[199] Y.K. Gupta and J.R. Sharma, On similarity solutions for type D spherical and pseudospherical fluid distributions in 5-flat form in general relativity, *Indian Journal of Pure and Applied Mathematics*, 27 (1996) 723-729.

[200] Y.K. Gupta and J.R. Sharma, Similarity solutions for the type D fluid plates in 5-D flat space, *Journal of Mathematical Physics*, 37 (1996) 531487.

[201] Y. Shaojie and H. Cuncai, Lie symmetry reductions and exact solutions of a coupled KdV-Burgers equation, *Applied Mathematics and Computation*, 234 (2014) 579-583.

[202] Y. Yunqing and C. Yong, Prolongation structure of the equation studied by Qiao, *Communication in Theoretical Physics*, 56 (2011) 463-466.