Linear Programming Problems with Generalized Fuzzy Sets

A

Thesis

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CANDIDATE'S DECLARATION

I hereby certify that the work being presented in the thesis entitled **"Linear Programming Problems with Generalized Fuzzy Sets"** in fulfillment of the requirements for the award of degree of Doctor of Philosophy, submitted in the School of Mathematics and Computer Applications of Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Amit Kumar, Assistant Professor, School of Mathematics and Computer Applications, Thapar University, Patiala.

The matter embodied in this thesis has not been submitted by me to any other university or institute for the award of any other degree.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Dr. Amit Kumar

(Supervisor)

DEDICATED

TO

MY PARENTS

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Abstract

In this thesis, the limitations and shortcomings of existing methods for solving linear programming problems with fuzzy sets are pointed out. To overcome the limitations and shortcomings of existing methods, some new ranking approaches are proposed for comparing generalized fuzzy sets and vague sets. On the basis of proposed ranking approaches, some new methods are proposed to find the appropriate solution of such linear programming problems with generalized fuzzy sets and vague sets in which only the parameters cost (or profit) are represented by fuzzy sets or vague sets.

The chapter wise summary of the thesis is as follows:

In **Chapter 1**, a brief review of the work done in the area of linear programming problems with fuzzy sets is presented.

In **Chapter 2**, it is shown that the existing method for solving linear programming problems with fuzzy sets can be used only for solving such linear programming problems in which either the parameters are represented by normal fuzzy sets or generalized fuzzy sets having equal height. Although, the limitations of the existing method can be removed by replacing an appropriate ranking approach instead of already used existing ranking approach for comparing fuzzy sets but there are some shortcomings in all the existing ranking approaches so none of the existing ranking

approach can be used to overcome the limitations of existing method. To overcome the limitations of existing method, a new ranking approach is proposed for comparing generalized trapezoidal fuzzy sets and on the basis of proposed ranking approach a new method is proposed for solving linear programming problems with generalized fuzzy sets.

In **Chapter 3**, the limitations of some existing results for comparing generalized p-norm fuzzy sets are pointed out and with the help of the ranking approach, proposed in Chapter 2, some new results are proposed by modifying the existing results to overcome the limitations of existing results. It is shown that the existing results are the particular cases of the proposed results. Also, the method for solving linear programming problems with generalized fuzzy sets, proposed in Chapter 2, is used to solve a linear programming problem with generalized p-norm trapezoidal fuzzy sets.

In **Chapter 4**, the shortcomings of the ranking approach, proposed in Chapter 2, are pointed out and to overcome these shortcomings a new ranking approach, named as RM ranking approach, is proposed for comparing generalized trapezoidal fuzzy sets. Also, with the help of RM ranking approach, a new method is proposed for solving linear programming problems with generalized trapezoidal fuzzy sets.

In **Chapter 5**, the limitations of some existing results related to comparison of intuitionistic fuzzy sets and shortcomings of an existing ranking approach for comparing triangular intuitionistic fuzzy sets are pointed out. To overcome the limitations of existing results, some new results are proposed by modifying the existing results and to overcome the shortcomings of existing ranking approach, a new ranking approach is proposed for comparing trapezoidal vague sets. Also, the method for solving linear programming problems with generalized fuzzy sets, proposed in Chapter 4, with proposed ranking approach is used to solve a linear programming problem with trapezoidal vague sets.

In **Chapter 6**, it is shown that the results of the linear programming problems with fuzzy sets obtained by using the existing and proposed methods are not appropriate. It is pointed out that all the shortcomings in the results are occurring due to used ranking approaches. To overcome the shortcomings of existing and proposed ranking approaches, a new ranking approach, named as RMDS ranking approach, is proposed for comparing trapezoidal vague sets. On the basis of proposed RMDS ranking approach, a new method is proposed for solving linear programming problems with trapezoidal vague sets. To show the advantage of the proposed method the linear programming problems with fuzzy and vague sets, for which the results obtained by using the existing and other proposed methods are not appropriate, are solved by using the proposed method and it is shown that the obtained results are appropriate.

Finally, in **Chapter 7**, based on the present study, conclusions are drawn and future work have been suggested.

List of Research Papers

1. A. Kumar, **P. Singh**, J. Kaur, Generalized simplex algorithm to solve fuzzy linear programming problems with ranking of generalized fuzzy numbers, Turkish Journal of Fuzzy Systems 1 (2010) 80-103.

2. A. Kumar, **P. Singh** and J. Kaur, Two phase method for solving fuzzy linear programming problems using ranking of generalized fuzzy numbers. International Journal of Applied Science and Engineering 8 (2010) 127-147.

3. A. Kumar, **P. Singh**, A. Kaur, P. Kaur, RM approach for ranking of generalized trapezoidal fuzzy numbers, Fuzzy Information and Engineering 2 (2010) 37-47.

4. A. Kumar, **P. Singh**, P. Kaur and A. Kaur, RM approach for ranking of L-R type generalized fuzzy numbers, Soft Computing 15 (2011) 1373-1381.

5. A. Kumar, **P. Singh**, A. Kaur, P. Kaur, Ranking of generalized trapezoidal fuzzy numbers based on rank, mode, divergence and spread, Turkish Journal of Fuzzy Systems 1 (2010) 141-152.

6. A. Kumar, **P. Singh**, A. Kaur, Ranking of generalized exponential fuzzy numbers using integral value approach, International Journal of Advances in Soft Computing and Its Applications 2 (2010) 221-230.

7. A. Kumar, **P. Singh**, A. Kaur, P. Kaur, Equality of generalized triangular fuzzy numbers, International Journal of Physical and Mathematical Sciences 1 (2010) 43-48.
8. A. Kumar, **P. Singh**, P. Kaur and A. Kaur, A new approach for ranking of generalized trapezoidal fuzzy numbers, International Journal of Engineering and Physical Sciences 4 (2010) 102-105.
9. A. Kumar, **P. Singh**, A. Kaur and P. Kaur, A new approach for ranking nonnormal p-norm trapezoidal fuzzy numbers, Computers and Mathematics with Applications 61 (2011) 881-887.
10. A. Kumar, **P. Singh**, A. Kaur, P. Kaur, A new approach for ranking of L-R type generalized fuzzy numbers, Tamsui Oxford Journal of Mathematical Sciences 27(2011) 197-211..
11. A. Kumar, **P. Singh**, Ranking of generalized fuzzy numbers with generalized fuzzy simplex algorithm, Journal of Mathematics and Applications (Accepted)
12. A. Kumar, **P. Singh**, A new approach for the ranking of trapezoidal vague sets, Applied Soft Computing (Communicated).
13. A. Kumar, **P. Singh**, RMDS ranking approach for solving linear programming problems with trapezoidal vague sets, Expert Systems with Applications (Communicated)
14. A. Kumar, **P. Singh**, Ranking of generalized exponential fuzzy numbers using integral value approach, Eighteen International Conference on Interdisciplinary and Statistical Techniques, Jaypee University of Information and

Technology, Waknaghat, Himachal Pradesh, India, August 02-04, 2009.

15. A. Kumar, **P. Singh**, Big-M method for solving fuzzy linear programming problems using ranking of generalized fuzzy numbers, International Congress on Productivity, Quality, Reliability, Optimization and Modeling, ISI Delhi, India, Feb. 07-08, 2011.

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Chapter 1

INTRODUCTION

Linear programming is one of the most frequently applied operations research techniques. The classical tool for solving the linear programming problem in practice is the class of simplex algorithm which was proposed and developed by Dantzig [47]. A lot of real world decision problems are described by linear programming models and sometimes it is necessary to formulate them with elements of imprecision or uncertainty. This imprecise nature has long been studied with the help of the probability theory. However, the probability theory might not provide the correct interpretation to solve some practical decision making problems. Stochastic programming is built on the assumption that some of the parameters used to describe the problem are random variables, which are difficult to determine exactly and stochastic programming models are often very complex in structure and therefore rarely used in practice. In these cases, the fuzzy set theory [189] might be more helpful.

1.1 Literature review

The concept of decision making in fuzzy environment was proposed by Bellman and Zadeh [15]. Tanaka et al. [163] adopted this concept for solving the problems of mathematical programming. Rodder and Zimmermann [141] studied the first duality results using fuzzy parameters. Zimmermann [193] proposed the first formulation of fuzzy linear programming. Chanas [28] proposed the possibility of the identification of a complete fuzzy decision in fuzzy linear programming by use of the parametric programming technique. Verdegay [169] defined the fuzzy dual problem with the help of parametric programming and proved that fuzzy primal and fuzzy dual have the same fuzzy solution. Tanaka and Asai [161] formulated a fuzzy linear programming problem to obtain a reasonable solution under consideration of ambiguity of parameters. Carlsson and Korhonen [27] proposed a method to deal with decision problems having uncertainity about the parameters. Werners [178] introduced an interactive system which supports a decision maker in solving programming models with crisp or fuzzy constraints and crisp or fuzzy goals.

Luhandjula [102] proposed some ways to deal with linear programming problems when the coefficients of the objective function are subject to possibilistic imprecision. Campos and Verdegay [26] considered linear programming problems with fuzzy constraints and fuzzy coefficients in both matrix and right hand of the constraint set. Delgado et al. [49] described more important problems in fuzzy linear programming and proposed a method for solving them. Rommelfanger et al. [145] presented a new method for solving linear programming problems with fuzzy parameters in the objective function. Inuiguchi et al. [71] dealt with the fuzzy linear programming problems with continuous piecewise linear membership function. Zhao and Govind [192] represented fuzzy equality constraints by fuzzy intervals and proposed an expression from which the complete fuzzy solution set can be conveniently studied. Lai and Hwang [89] investigated an interactive fuzzy linear programming approach to improve the flexibility and robustness of linear programming technique.

Fuller and Zimmermann [61] interpreted fuzzy linear programming problems with fuzzy coefficients and fuzzy inequality relations as multiple fuzzy reasoning schemes. Shaocheng [149] focused on two kinds of linear programming: interval number and fuzzy number linear programming. Herrera and Verdegay [70] studied some models for dealing with fuzzy integer linear programming problems. Cadenas and Verdegay [24] studied a linear programming problem in which all its elements are defined as fuzzy sets. Wang [172] proposed an inexact approach to solve objective/resource type of fuzzy linear programming problems. Fang et al. [59] presented a method for solving linear programming with fuzzy coefficients in constraints. Guu and Wu [67] proposed a two phase approach to solve the fuzzy linear programming problems.

Buckley and Feuring [22] introduced a method to find solution of the fully fuzzified linear programming problems with all the parameters and variables as fuzzy numbers by changing the objective function into a multiobjective fuzzy linear programming problem. Jiuping [78] proposed a kind of fuzzy linear programming problems based on interval-valued fuzzy sets by converting interval valued fuzzy linear programming into parametric linear programming. Inuiguchi and Ramik [72] proposed some fuzzy linear programming methods and techniques from a practical point of view. Maleki et al. [111] solved the linear programming problems in which all decision parameters are fuzzy numbers by the comparison of fuzzy numbers. Liu [97] proposed a method for solving fuzzy linear programming problems based on the satisfaction degree of the constraints.

Bector and Chandra [13] discussed duality in fuzzy linear programming based on modification of dual formulation stated by Rodder and Zimmermann [141]. Maleki [110] introduced a new method for solving linear programming with vagueness in constraints by using ranking function. Zhang et al. [191] proposed a method for solving fuzzy linear programming problems which involve fuzzy numbers in coefficients of objective functions. Wu [179] proposed fuzzy duality of linear programming problems with fuzzy coefficients and formulated it by using the fuzzy scalar product. Nehi et al. [126] defined the concept of optimality for linear programming problem with fuzzy parameters by transforming fuzzy linear programming problem into multiobjective linear programming problem. Ramik [132] introduced fuzzy linear programming problems based on fuzzy relations. Nasseri et al. [120] presented a new method for solving fuzzy number linear programming problems.

Mahdavi-Amiri and Nasseri [108] proposed a method to obtain the dual of a fuzzy linear programming problem. Ganesan and Veeramani [62] proposed an approach to solve a fuzzy linear programming problem involving symmetric trapezoidal fuzzy numbers without converting it to crisp linear programming problem. Hashemi et al. [69] proposed a two phase approach to find the optimal solutions of class of fuzzy linear programming problems called fully fuzzified linear p rogramming, where all decision parameters and variables are fuzzy numbers. Rommelfanger [142] proposed a new method for solving stochastic linear programming problems with fuzzy parameters. Lin and Lee [95] presented a genetic algorithm for solving linear programming problem having fuzzy constraints. Jimenez et al. [77] proposed a method for solving linear programming problems where all the coefficients are fuzzy numbers and used a fuzzy ranking method to rank the fuzzy objective values and to deal with the inequality relation on constraints.

Mahdavi-Amiri and Nasseri [107] established some duality results for linear programming problems with trapezoidal fuzzy variables. Safi et al. [146] improved the Zimmermann method for solving fuzzy linear programming problems. Allahviranloo et al. [5] proposed a new method for solving fully fuzzy linear programming problems by the use of ranking function. Nasseri [118] proposed a method for solving fuzzy linear programming problems by solving the classical linear programming. Mahdavi-Amiri and Nasseri [109] introduced some duality results on linear programming problems with symmetric fuzzy numbers. Lotfi et al. [101] discussed fully fuzzy linear programming problems by representing all parameters and variables as triangular fuzzy numbers. Ebrahimnejad and Nasseri [55] used the complementary slackness property to solve fuzzy linear programming problem with fuzzy parameters without the need of a simplex tableau. Stanojevic and Stanojevic [156] proposed an algorithm for solving fuzzy linear programming problems with trapezoidal fuzzy numbers using a penalty method. Ebrahimnejad et al. [57] proposed a new primal-dual algorithm for solving linear programming problems with fuzzy variables by using duality results.

In this thesis, the limitations and shortcomings of existing methods for solving linear programming problems with fuzzy sets are pointed out. To overcome the limitations and shortcomings of existing methods, some new ranking approaches are proposed for comparing generalized fuzzy sets and vague sets. On the basis of proposed ranking approaches, some new methods are proposed for solving such linear programming problems with generalized fuzzy sets and vague sets in which only the parameters cost (or profit) are represented by fuzzy sets or vague sets.

1.2 Organization of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

Several researchers [6, 55, 56, 57, 58, 107, 108, 109, 119, 120] have used the existing method [111] for solving different types of linear programming problems with fuzzy sets. In this chapter, it is shown that due to limitations of existing ranking approach [96], used in the existing method [111] for comparing fuzzy sets, the existing method [111] can be used for solving such linear programming problems in which either the parameters are represented by normal fuzzy sets or generalized fuzzy sets having equal heights but the existing method can't be used for solving such linear programming problems in which some or all the parameters are represented by generalized fuzzy sets having different heights.

Since, the limitations in the existing method are occurring due to limitations of existing ranking approach so, the limitations of the existing method can be removed by using an appropriate alternative ranking approach instead of existing ranking approach.

Recently, Chen and Chen [37], Chen and Sanguansat [41] have pointed out the limitations and shortcomings of several existing ranking approaches for comparing generalized fuzzy sets and proposed new ranking approaches for comparing generalized trapezoidal fuzzy sets but there are also some shortcomings in these existing ranking approaches [37, 41] due to which these ranking approaches can also not be used for comparing fuzzy sets in the existing methods.

In this chapter, the shortcomings of the existing ranking approaches [37, 41] are pointed out and to overcome the shortcomings of the existing ranking approaches

[37, 41] a new ranking approach is proposed for comparing generalized trapezoidal fuzzy sets. Also, a new method, on the basis of proposed ranking approach, is proposed for solving linear programming problems with generalized trapezoidal fuzzy sets by modifying the existing method [111].

Chapter 3

Chen and Tang [42] introduced the definition of generalized p-norm fuzzy sets and proposed some results for comparing generalized p-norm fuzzy sets. In this chapter, with the help of several counter examples it is proved that the existing results [42] are applicable only for the generalized p-norm fuzzy sets having equal height. To overcome the limitations of existing results, on the basis of ranking approach, proposed in Chapter 2, some new results are proposed for comparing generalized p-norm fuzzy sets having different height. It is shown that the existing results, [42] are particular cases of the proposed results. Also, the method for solving linear programming problems with generalized fuzzy sets, proposed in Chapter 2, is used for solving a linear programming problem with generalized p-norm fuzzy sets.

Chapter 4

In Chapter 2, a new ranking approach is proposed for comparing generalized fuzzy sets in which only rank is used for comparing generalized fuzzy sets. In this chapter, it is shown that only rank is not sufficient for comparing generalized fuzzy sets and to overcome this limitation, a new ranking approach, named as RM ranking approach, is proposed by modifying the ranking approach proposed in Chapter 2. Also, on the basis of the proposed RM ranking approach, a new method is proposed for solving linear programming problems with generalized trapezoidal fuzzy sets. The proposed method is illustrated by solving a numerical example.

Chapter 5

In real life, a person may assume that an object belongs to a set, but it is possible that he is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use intuitionistic fuzzy set [9] or vague set [63]. Bustince and Burillo [23] pointed out that the notion of vague set is the same as that of intuitionistic fuzzy set.

Li [94] pointed out the shortcomings of all the existing ranking approaches for comparing intuitionistic fuzzy sets and proposed some results and a new approach for comparing intuitionistic fuzzy sets.

In this chapter, the limitations of the existing results [94] and the shortcomings of the existing ranking approach [94] are pointed out. To overcome the limitations of the existing results, some new results are proposed by modifying the existing results. Also, to overcome the shortcomings of the existing ranking approach [94], a new ranking approach is proposed for comparing trapezoidal vague sets. To show the application of proposed ranking approach a linear programming problem with trapezoidal vague sets is solved.

Chapter 6

Several researchers [6, 55, 57, 56, 58, 107, 108, 109, 119, 120] have used the existing method [111] for solving different types of linear programming problems with fuzzy sets. In this chapter, it is shown that the results of the linear programming problems, obtained by the researchers by using the existing method [111] and the results of the linear programming problems with generalized and vague sets, obtained by using the proposed methods, are not appropriate. Also, it is shown that the shortcomings in the results are occurring due to existing and proposed ranking approaches which are used in the existing and proposed methods for comparing fuzzy and vague sets. To overcome the shortcomings of the existing and proposed methods, a new ranking approach, named as RMDS ranking approach, is proposed for comparing trapezoidal vague sets having equal degree of membership and non membership. On the basis of proposed RMDS ranking approach, a new method is proposed for solving linear programming problems with trapezoidal vagues sets. To show the advantages of the proposed RMDS ranking approach, the chosen linear programming problems with fuzzy and vague sets, for which the results obtained by using the existing and proposed methods are not appropriate, are solved by the proposed method and it is shown that the obtained results are appropriate.

Chapter 7

Finally, in this chapter, based on the present study, conclusions are drawn and future work have been suggested.

Chapter 2

A NEW RANKING APPROACH FOR SOLVING LINEAR PROGRAMMING PROBLEMS WITH GENERALIZED TRAPEZOIDAL FUZZY SETS [1]

Several researchers [6, 55, 56, 57, 58, 107, 108, 109, 119, 120] have used the existing method [111] for solving different types of linear programming problems with fuzzy sets. In this chapter, it is shown that due to limitations of existing ranking approach [96], used in the existing method [111] for comparing fuzzy sets, the existing method [111] can be used for solving such linear programming problems in which either the parameters are represented by normal fuzzy sets or generalized fuzzy sets having equal heights but the existing method can't be used for solving such linear programming problems in which some or all the parameters are represented by generalized fuzzy sets having different heights.

Since, the limitations in the existing method are occurring due to limitations of existing ranking approach so, the limitations of the existing method can be removed by using an appropriate alternative ranking approach instead of existing ranking

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[1] Some part of this chapter is published in the *International Journal of Engineering and Physical Sciences* 4 (2010) 102-105 and remaining part is published in *International Journal of Applied Science and Engineering* 8 (2010) 127-147.

approach.

Recently, Chen and Chen [37], Chen and Sanguansat [41] have pointed out the limitations and shortcomings of several existing ranking approaches for comparing generalized fuzzy sets and proposed new ranking approaches for comparing generalized trapezoidal fuzzy sets but there are also some shortcomings in these existing ranking approaches [37, 41] due to which these ranking approaches can also not be used for comparing fuzzy sets in the existing methods.

In this chapter, the shortcomings of the existing ranking approaches [37, 41] are pointed out and to overcome the shortcomings of the existing ranking approaches [37, 41], a new ranking approach is proposed for comparing generalized trapezoidal fuzzy sets. Also, a new method, on the basis of proposed ranking approach, is proposed for solving linear programming problems with generalized trapezoidal fuzzy sets by modifying the existing method [111].

2.1 Preliminaries

In this section, some basic definitions and arithmetic operations are presented [37].

2.1.1 Basic definitions

In this section, some basic definitions are presented .

Definition 2.1 The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\tilde{A}} : X \to [0, 1]$. The assigned value indicates the membership grade of the element in the set A.

The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ defined by $\mu_{\tilde{A}}(x)$ for each $x \in X$ is called a fuzzy set.

Definition 2.2 A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be a generalized fuzzy set if its membership function has the following characteristics:

1. $\mu_{\tilde{A}} : R \longrightarrow [0, w]$ is continuous, where $0 < w \leq 1$ and w is said to be height of generalized fuzzy set.
2. $\mu_{\tilde{A}}(x) = 0$, for all $x \in (-\infty, a] \bigcup [d, \infty)$.
3. $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a, b]$ and is strictly decreasing on $[c, d]$.
4. $\mu_{\tilde{A}}(x) = w$, for all $x \in [b, c]$.

Definition 2.3 Let $\tilde{A}$ be a generalized fuzzy set then $A^{\alpha} = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha, 0 \leq \alpha \leq w\}$ is said to be an α-cut of $\tilde{A}$.

Definition 2.4 A generalized fuzzy set $\tilde{A}$ defined on the universal set of real numbers R, denoted as $\tilde{A} = (a, b, c; w)$, shown in Fig. 2.1, is said to be a generalized triangular fuzzy set if its membership function, $\mu_{\tilde{A}}(x)$, is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{w(x-a)}{(b-a)} & , \quad a \leq x < b \\ w & , \quad x = b \\ \frac{w(x-c)}{(b-c)} & , \quad b < x \leq c \\ 0 & , \quad \text{otherwise} \end{cases}$$

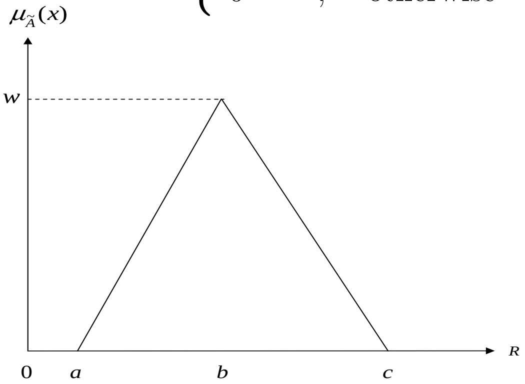

Fig. 2.1 A Generalized Triangular Fuzzy Set

Definition 2.5 A generalized fuzzy set $\tilde{A}$ defined on the universal set of real numbers R, denoted as $\tilde{A} = (a, b, c, d; w)$, shown in Fig. 2.2, is said to be a generalized trapezoidal fuzzy set if its membership function, $\mu_{\tilde{A}}(x)$, is given by

$$\mu_{\tilde{A}}(x)=\begin{cases} \frac{w(x-a)}{(b-a)} & , \quad a \leq x < b \\ w & , \quad b \leq x \leq c \\ \frac{w(x-d)}{(c-d)} & , \quad c < x \leq d \\ 0 & , \quad \text{otherwise} \end{cases}$$

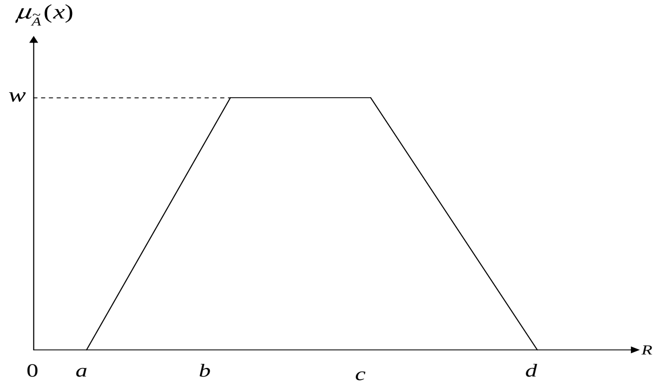

Fig. 2.2 A Generalized Trapezoidal Fuzzy Set

Definition 2.6 Let $\tilde{A} = (a_1, a_2, a_3; w)$ and $\tilde{B} = (b_1, b_2, b_3, b_4; w)$ be the generalized triangular and generalized trapezoidal fuzzy sets respectively. Then the α-cuts A^α and B^α for these generalized fuzzy sets can be defined as follows:

$$A^\alpha = [a_1 + (a_2 - a_1)\frac{\alpha}{w}, a_3 - (a_3 - a_2)\frac{\alpha}{w}], 0 \leq \alpha \leq w$$

$$B^\alpha = [b_1 + (b_2 - b_1)\frac{\alpha}{w}, b_4 - (b_4 - b_3)\frac{\alpha}{w}], 0 \leq \alpha \leq w$$

2.1.2 Arithmetic operations

In this section, some arithmetic operations between two generalized triangular fuzzy sets and between two generalized trapezoidal fuzzy sets, defined on universal set of real numbers R, are presented.

2.1.2.1 Arithmetic operations between generalized triangular fuzzy sets

Let $\tilde{A}_1 = (a_1, b_1, c_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2; w_2)$ be two generalized triangular fuzzy sets. Then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2; min\ (w_1, w_2))$

(ii) $\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - c_2, b_1 - b_2, c_1 - a_2; min\ (w_1, w_2))$

(iii) $\gamma \tilde{A}_1 = \begin{cases} (\gamma a_1, \gamma b_1, \gamma c_1, w_1) & \gamma \geq 0 \\ (\gamma c_1, \gamma b_1, \gamma a_1, w_1) & \gamma \leq 0 \end{cases}$

2.1.2.2 Arithmetic operations between generalized trapezoidal fuzzy sets

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy sets. Then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; min\ (w_1, w_2))$

(ii) $\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; min\ (w_1, w_2))$

(iii) $\gamma \tilde{A}_1 = \begin{cases} (\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1, w_1) & \gamma \geq 0 \\ (\gamma d_1, \gamma c_1, \gamma b_1, \gamma a_1, w_1) & \gamma \leq 0 \end{cases}$

Remark 2.1 For $w = 1$ generalized triangular fuzzy sets and generalized trapezoidal fuzzy sets are said to be normal triangular fuzzy sets and normal trapezoidal fuzzy sets respectively.

Remark 2.2 Assuming $w_1 = w_2 = 1$, the arithmetic operations of generalized fuzzy sets, presented in Section 2.1.2, can also be used for normal fuzzy sets.

2.2 Linear programming problems with fuzzy sets

Linear programming is one of the most frequently applied operations research techniques. The classical tool for solving the linear programming problem in practice is the class of simplex algorithm which was proposed and developed by Dantzig [47]. A lot of real world decision problems are described by linear programming models and sometimes it is necessary to formulate them with elements of imprecision or uncertainty. This imprecise nature has long been studied with the help of the probability theory. However, the probability theory might not provide the correct interpretation to solve some practical decision making problems. Stochastic programming is built on the assumption that some of the parameters used to describe the problem are random variables, which are difficult to determine exactly and stochastic programming models are often very complex in structure and therefore rarely used in practice. In these cases, the fuzzy set theory [189] might be more helpful.

2.2.1 Linear programming problems with normal fuzzy sets

Any linear programming problem having m constraints and n decision variables in which the profit (or cost) are represented by normal fuzzy sets can be formulated as follows [57, 58, 107, 108, 109, 111]:

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n} \tilde{c}_j x_j)\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij} x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad\qquad (P_1)\\
&\qquad x_j \geq 0
\end{aligned}$$

where, $\tilde{c}_j$ is a normal fuzzy set and $a_{ij}, b_i, x_j \in R$.

2.2.2 Linear programming problems with generalized fuzzy sets

If in the existing linear programming problem with fuzzy sets (P_1) the parameters $\tilde{c}_j$ are represented by generalized fuzzy sets instead of normal fuzzy sets then the problem (P_1) is converted into a linear programming problem with generalized fuzzy sets (P_2).

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n}\tilde{c}_j x_j)\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i=1,2,...,m \qquad\qquad (P_2)\\
&\qquad x_j \geq 0
\end{aligned}$$

where, $\tilde{c}_j$ is a generalized fuzzy set.

2.2.3 Comparison of fuzzy sets

Fuzzy sets must be ranked before an action is taken by a decision maker. Real numbers can be linearly ordered by the relation ≥ or ≤, however this type of inequality does not exist in fuzzy sets. Since fuzzy sets are represented by possibility distribution, they can overlap with each other and it is difficult to determine clearly whether one fuzzy set is larger or smaller than other. An efficient method for ordering the fuzzy sets is by the use of a ranking function, which maps each fuzzy set into the real line, where a natural order exists. Jain [74] proposed the concept of ranking function for comparing normal fuzzy sets. Chen [32] pointed out that in many cases it is not possible to restrict the membership function to the normal form and proposed the concept of generalized fuzzy sets. Since then, tremendous efforts are spent and significant advances are made on the development of numerous methodologies [1, 36, 37, 41, 44, 46, 88, 96, 113, 135, 150, 176, 177] for the ranking of generalized fuzzy sets.

Although there are several ranking approaches for comparing fuzzy sets but several authors [6, 55, 57, 56, 58, 107, 108, 109, 111, 119, 120] have used the existing ranking approach [96] for solving linear programming problem with normal fuzzy sets.

2.2.3.1 Existing ranking approach

In this section, the existing ranking approach [96] for comparing generalized fuzzy sets is presented.

Two generalized trapezoidal fuzzy sets $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ can be compared by using the following steps of the existing ranking approach [96]:

Step 1 Calculate

$$I_L(\tilde{A}) = \int_0^{w_1} (\nu^L_{\tilde{A}}(\alpha))d\alpha = \int_0^{w_1} (a_1 + \frac{(b_1 - a_1)}{w_1}\alpha)d\alpha = \frac{w_1}{2}(a_1 + b_1) \tag{2.1}$$

$$I_R(\tilde{A}) = \int_0^{w_1} (\nu^R_{\tilde{A}}(\alpha))d\alpha = \int_0^{w_1} (d_1 + \frac{(d_1 - c_1)}{w_1}\alpha)d\alpha = \frac{w_1}{2}(c_1 + d_1) \tag{2.2}$$

and

$$I_L(\tilde{B}) = \int_0^{w_2} (\nu^L_{\tilde{B}}(\alpha))d\alpha = \int_0^{w_2} (a_2 + \frac{(b_2 - a_2)}{w_2}\alpha)d\alpha = \frac{w_2}{2}(a_2 + b_2) \tag{2.3}$$

$$I_R(\tilde{B}) = \int_0^{w_2} (\nu^R_{\tilde{A}}(\alpha))d\alpha = \int_0^{w_2} (d_2 + \frac{(d_2 - c_2)}{w_2}\alpha)d\alpha = \frac{w_2}{2}(c_2 + d_2) \tag{2.4}$$

where, $\nu^L_{\tilde{A}}(\alpha) = infimum(A^\alpha)$, $\nu^L_{\tilde{B}}(\alpha) = infimum(B^\alpha)$ and $\nu^R_{\tilde{A}}(\alpha) = supremum(A^\alpha)$, $\nu^R_{\tilde{B}}(\alpha) = supremum(B^\alpha)$.

Step 2 Calculate

$$I^\lambda_T(\tilde{A}) = \lambda I_L(\tilde{A}) + (1 - \lambda)I_R(\tilde{A}) = \frac{w_1}{2}\{(\lambda(a_1 + b_1) + (1 - \lambda)(c_1 + d_1))\} \tag{2.5}$$

and

$$I_T^\lambda(\tilde{B}) = \lambda I_L(\tilde{B}) + (1-\lambda)I_R(\tilde{B}) = \frac{w_2}{2}\{(\lambda(a_2+b_2)+(1-\lambda)(c_2+d_2))\} \tag{2.6}$$

where, λ represents the degree of optimism of a decision maker. A larger value of λ indicates a higher degree of optimism. For $\lambda = 0$ and $\lambda = 1$ value of $I_T^\lambda(\tilde{A})$ and $I_T^\lambda(\tilde{B})$ represents the view points of a pessimistic and optimistic decision maker respectively while for $\lambda = 0.5$ values of $I_T^\lambda(\tilde{A})$ and $I_T^\lambda(\tilde{B})$ represents the view points of a moderate decision maker.

Step 3 Chose a particular value of λ and check that $I_T^\lambda(\tilde{A}) > I_T^\lambda(\tilde{B})$ or $I_T^\lambda(\tilde{A}) < I_T^\lambda(\tilde{B})$ or $I_T^\lambda(\tilde{A}) = I_T^\lambda(\tilde{B})$

Case (i) If $I_T^\lambda(\tilde{A}) > I_T^\lambda(\tilde{B})$ for chosen value of λ, then $\tilde{A} \succ \tilde{B}$ for same value of λ.

Case (ii) If $I_T^\lambda(\tilde{A}) < I_T^\lambda(\tilde{B})$ for chosen value of λ, then $\tilde{A} \prec \tilde{B}$ for same value of λ.

Case (iii) If $I_T^\lambda(\tilde{A}) = I_T^\lambda(\tilde{B})$ for chosen value of λ, then $\tilde{A} \sim \tilde{B}$ for same value of λ.

2.2.4 Feasible and optimal solutions

In this section, the definitions of feasible and optimal solution of linear programming problem with fuzzy sets (P_1) are presented [107, 108].

Definition 2.7 A set of real numbers $\{x_j\}$ is said to be feasible solution of the linear programming problem with fuzzy sets if

(i) $x_j \geq 0,\ j = 1, 2..., n$

(ii) $\sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i,\ i = 1, 2, ..., m$

Definition 2.8 A feasible solution $\{x_j\}$ of the linear programming problem with fuzzy sets is said to be an optimal solution if:

(i) $\sum_{j=1}^{n} \tilde{c}_j x_j \succ \sum_{j=1}^{n} \tilde{c}_j y_j,\ \forall\ y_j \in S_F$ (in case of maximization)

(ii) $\sum_{j=1}^{n} \tilde{c}_j x_j \prec \sum_{j=1}^{n} \tilde{c}_j y_j,\ \forall\ y_j \in S_F$ (in case of minimization) where, S_F is the set of all feasible solutions.

Definition 2.9 Let $\{x_j\}$ be an optimal solution of the linear programming problem with fuzzy sets (P_1) and $\{x'_j\}$ be a feasible solution of the same problem such that $\sum\limits_{j=1}^{n} \tilde{c}_j x_j \sim \sum\limits_{j=1}^{n} \tilde{c}_j x'_j$ then $\{x'_j\}$ is said to be an alternative optimal solution.

2.3 Existing method for solving linear programming problems with normal trapezoidal fuzzy sets

Several researchers [6, 55, 56, 57, 58, 107, 108, 109, 119, 120] have used the existing method [111] for solving different types of linear programming problems with fuzzy sets.

The solution of a linear programming problem with normal fuzzy sets (P_1) can be obtained by using the following steps of the existing method.

Step 1 Assuming $\tilde{c}_j = (c_{j1}, c_{j2}, c_{j3}, c_{j4}; 1)$ the linear programming problem with normal fuzzy sets (P_1) can be converted into linear programming problem with normal trapezoidal fuzzy sets (P_3):

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n}(c_{j1}, c_{j2}, c_{j3}, c_{j4}; 1)x_j)\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad (P_3)\\
&\qquad x_j \geq 0
\end{aligned}$$

Step 2 Using Definition 2.8, the optimal solution of the linear programming problem with normal trapezoidal fuzzy sets (P_3) can be obtained by solving the crisp linear programming problem (P_4)

$$\begin{aligned}
&Maximize\ \ (or\ \ Minimize)\ \ (I_T^{\lambda}(\sum_{j=1}^{n}((c_{j1}, c_{j2}, c_{j3}, c_{j4}; 1)x_j)))\\
&subject\ to
\end{aligned}$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad (P_4)$$

$$x_j \geq 0$$

Step 3 Using linearity property $I_T^{\lambda}\sum_{j=1}^{n}((c_{j1}, c_{j2}, c_{j3}, c_{j4}; 1)x_j) = \sum_{j=1}^{n} I_T^{\lambda}((c_{j1}, c_{j2}, c_{j3}, c_{j4}; 1)x_j)$, the crisp linear programming problem (P_4) can be converted into crisp linear programming problem (P_5)

$$Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n} I_T^{\lambda}((c_{j1}, c_{j2}, c_{j3}, c_{j4}; 1)x_j))$$

$$subject\ to$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad (P_5)$$

$$x_j \geq 0$$

Step 4 Using $I_T^{0.5}(a, b, c, d; 1) = \frac{1}{4}(a+b+c+d)$, the crisp linear programming problem (P_5) can be converted into crisp linear programming problem (P_6)

$$Maximize\ (or\ Minimize)\ (\frac{1}{4}\sum_{j=1}^{n}(c_{j1} + c_{j2} + c_{j3} + c_{j4})x_j)$$

$$subject\ to$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad (P_6)$$

$$x_j \geq 0$$

Step 5 Use an appropriate existing method such as simplex method or Big-M method or two phase method or dual simplex method etc. to find the optimal solution $\{x_j\}$ of the the crisp linear programming problem (P_6).

Step 6 Find the fuzzy optimal value of the linear programming problem with trapezoidal fuzzy sets (P_3) by putting the obtained values of x_j in $\sum_{j=1}^{n}(c_{j1}, c_{j2}, c_{j3}, c_{j4}; 1)x_j$.

2.3.1 Illustrative example

In this section, the existing method is illustrated by solving a numerical example.

Example 2.1 Solve

$$\begin{aligned}&Maximize\ \ ((2,3,5,6;1)x_1 \oplus (1,2,4,5;1)x_2)\\&subject\ to\\&\qquad 8x_1+6x_2 \leq 25\\&\qquad 3x_1+4x_2 \leq 15\\&\qquad x_1,x_2 \geq 0\end{aligned}$$

Solution:- Using the existing method the solution of linear programming problem with normal trapezoidal fuzzy sets, chosen in Example 2.1, can be obtained as follows:

Step 1 Using Step 2, of the existing method the optimal solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$$\begin{aligned}&Maximize\ \ (I_T^{0.5}((2,3,5,6;1)x_1 \oplus (1,2,4,5;1)x_2))\\&subject\ to\\&\qquad 8x_1+6x_2 \leq 25\\&\qquad 3x_1+4x_2 \leq 15\\&\qquad x_1,x_2 \geq 0\end{aligned}$$

Step 2 Using Step 3 of the existing method, the crisp linear programming problem, obtained in Step 1, can be written as:

$$\begin{aligned}&Maximize\ \ (I_T^{0.5}((2,3,5,6;1)x_1) + I_T^{0.5}((1,2,4,5;1)x_2))\\&subject\ to\\&\qquad 8x_1+6x_2 \leq 25\\&\qquad 3x_1+4x_2 \leq 15\\&\qquad x_1,x_2 \geq 0\end{aligned}$$

Step 3 Using $I_T^{0.5}(a,b,c,d;w) = \frac{w}{4}(a+b+c+d)$ the crisp linear programming problem, obtained in Step 2, can be written as:

$$\begin{aligned}
&Maximize \ \ (4x_1 + 3x_2)\\
&subject\ to\\
&\qquad 8x_1 + 6x_2 \leq 25\\
&\qquad 3x_1 + 4x_2 \leq 15\\
&\qquad x_1, x_2 \geq 0
\end{aligned}$$

Step 4 On solving the crisp linear programming problem, obtained in Step 3, the obtained optimal solution is $x_1 = \frac{25}{8}$ and $x_2 = 0$

Step 5 Putting $x_1 = \frac{25}{8}$ and $x_2 = 0$ in $((2,3,5,6;1)x_1 \oplus (1,2,4,5;1)x_2)$ the fuzzy optimal value of the chosen problem is $(\frac{50}{8}, \frac{75}{8}, \frac{125}{8}, \frac{150}{8}; 1)$

2.4 Limitations of existing method

In Step 3, the existing method the linearity property $I_T^{\lambda}(\tilde{c}_1 x_1 \oplus \tilde{c}_2 x_2 \oplus ... \oplus \tilde{c}_n x_n) = I_T^{\lambda}(\tilde{c}_1 x_1) + I_T^{\lambda}(\tilde{c}_2 x_2) + ... + I_T^{\lambda}(\tilde{c}_n x_n)$ is used to convert the crisp linear programming problem (P_4) into the crisp linear programming problem (P_5). But the linearity property $I_T^{\lambda}(\tilde{c}_1 x_1 \oplus \tilde{c}_2 x_2 \oplus ... \oplus \tilde{c}_n x_n) = I_T^{\lambda}(\tilde{c}_1 x_1) + I_T^{\lambda}(\tilde{c}_2 x_2) + ... + I_T^{\lambda}(\tilde{c}_n x_n)$ is satisfied only if $\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n$ are either normal fuzzy sets or generalized fuzzy sets having equal heights but if neither $\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n$ are normal fuzzy sets nor generalized fuzzy sets having equal heights then the linearity property is not satisfied i.e. $I_T^{\lambda}(\tilde{c}_1 x_1 \oplus \tilde{c}_2 x_2 \oplus ... \oplus \tilde{c}_n x_n) \neq I_T^{\lambda}(\tilde{c}_1 x_1) + I_T^{\lambda}(\tilde{c}_2 x_2) + ... + I_T^{\lambda}(\tilde{c}_n x_n)$.

So the existing method [111] can't be used for solving linear programming problems with generalized fuzzy sets having different heights.

Example 2.2 Let $\tilde{C}_1 = (1, 2, 4, 5; 1)$ and $\tilde{C}_2 = (3, 5, 7, 9; 1)$ be two normal trapezoidal fuzzy sets then using equation (2.5), $I_T^{0.5}(\tilde{C}_1) = 4$, $I_T^{0.5}(\tilde{C}_2) = 6$ and $I_T^{0.5}(\tilde{C}_1 \oplus \tilde{C}_2) = 9$ i.e. $I_T^{0.5}(\tilde{C}_1 \oplus \tilde{C}_2) = I_T^{0.5}(\tilde{C}_1) + I_T^{0.5}(\tilde{C}_2)$.

Example 2.3 Let $\tilde{C}_1 = (1, 2, 4, 5; 0.3)$ and $\tilde{C}_2 = (3, 5, 7, 9; 0.3)$ be two generalized trapezoidal fuzzy sets then using equation (2.5), $I_T^{0.5}(\tilde{C}_1) = 0.9$, $I_T^{0.5}(\tilde{C}_2) = 1.8$ and $I_T^{0.5}(\tilde{C}_1 \oplus \tilde{C}_2) = 2.7$ i.e. $I_T^{0.5}(\tilde{C}_1 \oplus \tilde{C}_2) = I_T^{0.5}(\tilde{C}_1) + I_T^{0.5}(\tilde{C}_2)$.

Example 2.4 Let $\tilde{C}_1 = (1, 2, 4, 5; 0.3)$ and $\tilde{C}_2 = (3, 5, 7, 9; 0.7)$ be two generalized trapezoidal fuzzy sets then using equation (2.5), $I_T^{0.5}(\tilde{C}_1) = 0.9$, $I_T^{0.5}(\tilde{C}_2) = 4.2$ and $I_T^{0.5}(\tilde{C}_1 \oplus \tilde{C}_2) = 2.7$ i.e. $I_T^{0.5}(\tilde{C}_1 \oplus \tilde{C}_2) \neq I_T^{0.5}(\tilde{C}_1) + I_T^{0.5}(\tilde{C}_2)$.

2.5 Shortcomings of existing ranking approaches

In Section 2.4, it is pointed out that the existing method can be used for solving such linear programming problems in which either the parameters are represented by normal fuzzy sets or generalized fuzzy sets having equal heights but the existing method can't be used for solving such linear programming problems in which some or all the parameters are represented by generalized fuzzy sets having different heights.

If in the existing method instead of existing ranking approach, any such alternative ranking approach for comparing generalized fuzzy sets is used for the which the linearity property, $I_T^{\lambda}(\tilde{c}_1 x_1 \oplus \tilde{c}_2 x_2 \oplus ... \oplus \tilde{c}_n x_n) = I_T^{\lambda}(\tilde{c}_1 x_1) + I_T^{\lambda}(\tilde{c}_2 x_2) + ... + I_T^{\lambda}(\tilde{c}_n x_n)$, is satisfied then the existing method can be used for solving programming problems with generalized fuzzy sets having different heights.

Recently, Chen and Chen [37], Chen and Sanguansat [41] have pointed out the limitations and shortcomings of several existing ranking approaches for comparing generalized fuzzy sets and proposed a new ranking approach for comparing generalized trapezoidal fuzzy sets.

In this section, it is shown that there are some shortcomings in the existing ranking approaches [37, 41] so neither these existing ranking approaches should be used for comparing normal fuzzy sets nor should be used for comparing generalized fuzzy sets.

2.5.1 Chen and Chen ranking approach

In this section, the existing ranking approach [37] for comparing generalized trapezoidal fuzzy sets is presented.

If $\tilde{A} = (a_1, b_1, c_1, d_1; w_{\tilde{A}})$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_{\tilde{B}})$ are two generalized trapezoidal fuzzy sets then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Standardize each generalized trapezoidal fuzzy set $\tilde{A}$ and $\tilde{B}$ into $\tilde{A}^*$ and $\tilde{B}^*$ as follows:

$$\tilde{A}^* = \left(\frac{a_1}{k_1}, \frac{b_1}{k_1}, \frac{c_1}{k_1}, \frac{d_1}{k_1}; w_{\tilde{A}} \right) = (a_1^*, b_1^*, c_1^*, d_1^*; w_{\tilde{A}}), k_1 = max(\lceil |a_1| \rceil, \lceil |b_1| \rceil, \lceil |c_1| \rceil, \lceil |d_1| \rceil, 1), \tag{2.7}$$

where, $|a_1|, |b_1|, |c_1|, |d_1|$ denotes the absolute value of a_1, b_1, c_1, d_1 respectively and $\lceil |a_1| \rceil, \lceil |b_1| \rceil, \lceil |c_1| \rceil, \lceil |d_1| \rceil$ denotes the upper bound of $|a_1|, |b_1|, |c_1|, |d_1|$ respectively.

$$\tilde{B}^* = \left(\frac{a_2}{k_2}, \frac{b_2}{k_2}, \frac{c_2}{k_2}, \frac{d_2}{k_2}; w_{\tilde{B}} \right) = (a_2^*, b_2^*, c_2^*, d_2^*; w_{\tilde{B}}), k_2 = max(\lceil |a_2| \rceil, \lceil |b_2| \rceil, \lceil |c_2| \rceil, \lceil |d_2| \rceil, 1) \tag{2.8}$$

where, $|a_2|, |b_2|, |c_2|, |d_2|$ denotes the absolute value of a_2, b_2, c_2, d_2 respectively and $\lceil |a_2| \rceil, \lceil |b_2| \rceil, \lceil |c_2| \rceil, \lceil |d_2| \rceil$ denotes the upper bound of $|a_2|, |b_2|, |c_2|, |d_2|$ respectively.

Step 2 Calculate the defuzzified values $x_{\tilde{A}^*}$ and $x_{\tilde{B}^*}$ for the standardized generalized trapezoidal fuzzy sets $\tilde{A}^*$ and $\tilde{B}^*$ as follows:

$$x_{\tilde{A}^*} = \frac{a_1^* + b_1^* + c_1^* + d_1^*}{4} \tag{2.9}$$

and

$$x_{\tilde{B}^*} = \frac{a_2^* + b_2^* + c_2^* + d_2^*}{4} \tag{2.10}$$

Step 3 Calculate the standard deviations $STD_{\tilde{A}^*}$ and $STD_{\tilde{B}^*}$ for the standardized generalized trapezoidal fuzzy sets $\tilde{A}^*$ and $\tilde{B}^*$ as follows:

$$STD_{\tilde{A}^*} = \sqrt{\frac{(a_1^* - x_{\tilde{A}^*})^2 + (b_1^* - x_{\tilde{A}^*})^2 + (c_1^* - x_{\tilde{A}^*})^2 + (d_1^* - x_{\tilde{A}^*})^2}{4-1}} \tag{2.11}$$

and

$$STD_{\tilde{B}^*} = \sqrt{\frac{(a_2^* - x_{\tilde{B}^*})^2 + (b_2^* - x_{\tilde{B}^*})^2 + (c_2^* - x_{\tilde{B}^*})^2 + (d_2^* - x_{\tilde{B}^*})^2}{4-1}} \tag{2.12}$$

Step 4 Calculate the ranking scores $Score(\tilde{A}^*)$ and $Score(\tilde{B}^*)$ for the standardized generalized trapezoidal fuzzy sets $\tilde{A}^*$ and $\tilde{B}^*$ as follows:

$$Score(\tilde{A}^*) = \frac{x_{\tilde{A}^*} \times w_{\tilde{A}}}{1 + STD_{\tilde{A}^*}} \tag{2.13}$$

and

$$Score(\tilde{B}^*) = \frac{x_{\tilde{B}^*} \times w_{\tilde{B}}}{1 + STD_{\tilde{B}^*}} \tag{2.14}$$

where, $Score(\tilde{A}^*)$ and $Score(\tilde{B}^*) \in [-1, 1]$

Step 5 Check $Score(\tilde{A}^*) > Score(\tilde{B}^*)$ or $Score(\tilde{A}^*) < Score(\tilde{B}^*)$ or $Score(\tilde{A}^*) = Score(\tilde{B}^*)$

Case (i) If $Score(\tilde{A}^*) > Score(\tilde{B}^*)$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If $Score(\tilde{A}^*) < Score(\tilde{B}^*)$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If $Score(\tilde{A}^*) = Score(\tilde{B}^*)$ then $\tilde{A} \sim \tilde{B}$

2.5.1.1 Illustrative example

To illustrate the existing ranking approach [37] two generalized trapezoidal fuzzy sets, $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 0.3)$ and $\tilde{B} = (0.2, 0.3, 0.3, 0.4; 0.2)$ are compared by using the existing ranking approach [37]:

Step 1 Using equation (2.7) and (2.8), $\tilde{A}^* = (0.1, 0.3, 0.3, 0.5; 0.3)$ and $\tilde{B}^* = (0.2, 0.3, 0.3, 0.3; 0.2)$

Step 2 Using equation (2.9) and (2.10), $x_{\tilde{A}^*} = 0.3$ and $x_{\tilde{B}^*} = 0.3$.

Step 3 Using equation (2.11) and (2.12), $STD_{\tilde{A}^*} = 0.1633$ and $STD_{\tilde{B}^*} = 0.08165$

Step 4 Using equation (2.13) and (2.14), $Score(\tilde{A}^*) = 0.0.07737$ and $Score(\tilde{B}^*) = 0.0555$

Step 5 Since $Score(\tilde{A}^*) > Score(\tilde{B}^*)$, so $\tilde{A} \succ \tilde{B}$.

2.5.2 Chen and Sanguansat ranking approach

In this section, the existing ranking approach [41] for comparing generalized trapezoidal fuzzy sets is presented.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_{\tilde{A}})$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_{\tilde{B}})$ be two generalized trapezoidal fuzzy sets then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Standardize each generalized trapezoidal fuzzy set $\tilde{A}$ and $\tilde{B}$ into $\tilde{A}^*$ and $\tilde{B}^*$ as follows:

$$\tilde{A}^* = \left(\tfrac{a_1}{k_1}, \tfrac{b_1}{k_1}, \tfrac{c_1}{k_1}, \tfrac{d_1}{k_1}; w_{\tilde{A}} \right) = (a_1^*, b_1^*, c_1^*, d_1^*; w_{\tilde{A}}), k_1 = max(\lceil |a_1| \rceil, \lceil |b_1| \rceil, \lceil |c_1| \rceil, \lceil |d_1| \rceil, 1) \tag{2.15}$$

where, $|a_1|, |b_1|, |c_1|, |d_1|$ denotes the absolute value of a_1, b_1, c_1, d_1 respectively and $\lceil |a_1| \rceil, \lceil |b_1| \rceil, \lceil |c_1| \rceil, \lceil |d_1| \rceil$ denotes the upper bound of $|a_1|, |b_1|, |c_1|, |d_1|$ respectively.

$$\tilde{B}^* = \left(\tfrac{a_2}{k_2}, \tfrac{b_2}{k_2}, \tfrac{c_2}{k_2}, \tfrac{d_2}{k_2}; w_{\tilde{B}} \right) = (a_2^*, b_2^*, c_2^*, d_2^*; w_{\tilde{B}}), k_2 = max(\lceil |a_2| \rceil, \lceil |b_2| \rceil, \lceil |c_2| \rceil, \lceil |d_2| \rceil, 1) \tag{2.16}$$

where, $|a_2|, |b_2|, |c_2|, |d_2|$ denotes the absolute value of a_2, b_2, c_2, d_2 respectively and $\lceil |a_2| \rceil, \lceil |b_2| \rceil, \lceil |c_2| \rceil, \lceil |d_2| \rceil$ denotes the upper bound of $|a_2|, |b_2|, |c_2|, |d_2|$ respectively.

Step 2 Calculate the areas $AreaA_L^{*-}$ and $AreaA_R^{*-}$ on the negative side, respectively, which denote the trapezoidal areas from the membership function curves of $\mu_{\tilde{A}^*}^L$ and $\mu_{\tilde{A}^*}^R$, respectively, to the membership function curve of the generalized trapezoidal fuzzy set $(-1, -1, -1, -1; w_{\tilde{A}})$ respectively, where

$$\mu_{\tilde{A}^*}^L = w_{\tilde{A}} \frac{(x - a_1^*)}{(b_1^* - a_1^*)}, a_1^* \leq x < b_1^* \tag{2.17}$$

$$\mu_{\tilde{A}^*}^R = w_{\tilde{A}} \frac{(x - d_1^*)}{(c_1^* - d_1^*)}, c_1^* < x \leq d_1^* \tag{2.18}$$

and

$$AreaA_L^{*-} = w_{\tilde{A}^*} \frac{(a_1^* + 1) + (b_1^* + 1)}{2} \tag{2.19}$$

$$AreaA_R^{*-} = w_{\tilde{A}^*} \frac{(c_1^* + 1) + (d_1^* + 1)}{2} \tag{2.20}$$

Then, calculate the areas $AreaA_L^{*+}$ and $AreaA_R^{*+}$ on the positive side, respectively, which denote the trapezoidal areas from the membership function curves of $\mu_{\tilde{A}^*}^L$ and $\mu_{\tilde{A}^*}^R$, defined in equation (2.17) and (2.18), respectively, to the membership function curve of the generalized trapezoidal fuzzy set $(1, 1, 1, 1; w_{\tilde{A}})$, where

$$AreaA_L^{*+} = w_{\tilde{A}} \frac{(1 - a_1^*) + (1 - b_1^*)}{2} \tag{2.21}$$

$$AreaA_R^{*+} = w_{\tilde{A}} \frac{(1 - c_1^*) + (1 - d_1^*)}{2} \tag{2.22}$$

Similarly, calculate $AreaB_L^{*-}$, $AreaB_R^{*-}$, $AreaB_L^{*+}$ and $AreaB_L^{*+}$ for generalized trapezoidal fuzzy set $\tilde{B}^*$ as follows:

$$AreaB_L^{*-} = w_{\tilde{B}} \frac{(a_2^* + 1) + (b_2^* + 1)}{2} \tag{2.23}$$

$$AreaB_R^{*-} = w_{\tilde{B}} \frac{(c_2^* + 1) + (d_2^* + 1)}{2} \tag{2.24}$$

$$AreaB_L^{*+} = w_{\tilde{B}}\frac{(1-a_2^*)+(1-b_2^*)}{2} \tag{2.25}$$

$$AreaB_R^{*+} = w_{\tilde{B}}\frac{(1-c_2^*)+(1-d_2^*)}{2} \tag{2.26}$$

Step 3 Calculate the values $XI_{\tilde{A}^*}, XD_{\tilde{A}^*}, XI_{\tilde{B}^*}$ and $XD_{\tilde{B}^*}$ as follows:

$$XI_{\tilde{A}^*} = AreaA_L^{*-} + AreaA_R^{*-} \tag{2.27}$$

$$XD_{\tilde{A}^*} = AreaA_L^{*+} + AreaA_R^{*+} \tag{2.28}$$

and

$$XI_{\tilde{B}^*} = AreaB_L^{*-} + AreaB_R^{*-} \tag{2.29}$$

$$XD_{\tilde{B}^*} = AreaB_L^{*+} + AreaB_R^{*+} \tag{2.30}$$

Step 4 Calculate the ranking scores $Score(\tilde{A}^*)$ and $Score(\tilde{B}^*)$ for each generalized trapezoidal fuzzy set $\tilde{A}^*$ and $\tilde{B}^*$ as follows:

$$Score(\tilde{A}^*) = \frac{XI_{\tilde{A}^*} - XD_{\tilde{A}^*}}{XI_{\tilde{A}^*} + XD_{\tilde{A}^*} + (1-w_{\tilde{A}})} \tag{2.31}$$

and

$$Score(\tilde{B}^*) = \frac{XI_{\tilde{B}^*} - XD_{\tilde{B}^*}}{XI_{\tilde{B}^*} + XD_{\tilde{B}^*} + (1-w_{\tilde{B}})} \tag{2.32}$$

Step 5 Check $Score(\tilde{A}^*) > Score(\tilde{B}^*)$ or $Score(\tilde{A}^*) < Score(\tilde{B}^*)$ or $Score(\tilde{A}^*) = Score(\tilde{B}^*)$

Case (i) If $Score(\tilde{A}^*) > Score(\tilde{B}^*)$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If $Score(\tilde{A}^*) < Score(\tilde{B}^*)$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If $Score(\tilde{A}^*) = Score(\tilde{B}^*)$ then $\tilde{A} \sim \tilde{B}$

2.5.2.1 Illustrative example

To illustrate the existing ranking approach [41] two generalized trapezoidal fuzzy sets, $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 0.8)$ and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$ are compared by using the existing ranking approach [41].

Step 1 Using equation (2.15) and (2.16), $\tilde{A}^* = (0.1, 0.3, 0.3, 0.5; 0.8)$ and

$\tilde{B}^* = (0.1, 0.3, 0.3, 0.5; 1)$

Step 2 Using equation (2.19) and (2.20), $AreaA_L^{*-} = 0.96$ and $AreaA_R^{*-} = 1.12$.

Using equation (2.21) and (2.22), $AreaA_L^{*+} = 0.64$ and $AreaA_R^{*+} = 0.48$.

Using equation (2.23) and (2.24), $AreaB_L^{*-} = 1.2$ and $AreaB_R^{*-} = 1.4$.

Using equation (2.25) and (2.26), $AreaB_L^{*+} = 0.8$ and $AreaB_R^{*+} = 0.6$

Step 3 Using equation (2.27) and (2.28), $XI_{\tilde{A}^*} = AreaA_L^{*-} + AreaA_R^{*-} = 2.08$, $XD_{\tilde{A}^*} = AreaA_L^{*+} + AreaA_R^{*+} = 1.12$.

Using equation (2.29) and (2.30), $XI_{\tilde{B}^*} = AreaB_L^{*-} + AreaB_R^{*-} = 2.6$, $XD_{\tilde{B}^*} = AreaB_L^{*+} + AreaB_R^{*+} = 1.4$

Step 4 Using equation (2.31) and (2.32) $Score(\tilde{A}^*) = 0.2824$ and $Score(\tilde{B}^*) = 0.3$

Step 5 Since $Score(\tilde{A}^*) < Score(\tilde{B}^*)$ so $\tilde{A} \prec \tilde{B}$.

2.5.3 Shortcomings on the basis of reasonable properties of fuzzy quantities

In this section, the shortcomings of existing ranking approaches [37, 41] on the basis of reasonable properties of fuzzy quantities are pointed out.

Wang and Keere [174] proposed the following reasonable properties for the validation of any ranking function.

If $\tilde{A}$ and $\tilde{B}$ are normal fuzzy sets then

(i) $\tilde{A} \succ \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \succ (\tilde{B} \oplus \tilde{C})$

(ii) $\tilde{A} \prec \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \prec (\tilde{B} \oplus \tilde{C})$

(iii) $\tilde{A} \sim \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \sim (\tilde{B} \oplus \tilde{C})$

where, $\tilde{C}$ is normal fuzzy set.

For the generalized trapezoidal fuzzy sets the same property can be written as:

If $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ are two generalized trapezoidal fuzzy sets then

(i) $\tilde{A} \succ \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \succ (\tilde{B} \oplus \tilde{C})$

(ii) $\tilde{A} \prec \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \prec (\tilde{B} \oplus \tilde{C})$

(iii) $\tilde{A} \sim \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \sim (\tilde{B} \oplus \tilde{C})$

where, $\tilde{C} = (a, b, c, d; w_3)$ is a generalized trapezoidal fuzzy set and $w_3 \leq min(w_1, w_2)$.

There can exist several fuzzy sets for which the existing ranking functions [37, 41] does not satisfies the reasonable property $\tilde{A} \succ \tilde{B} \Rightarrow \tilde{A} \oplus \tilde{C} \succ \tilde{B} \oplus \tilde{C}$ i.e., according to existing ranking approaches $\tilde{A} \succ \tilde{B} \not\Rightarrow \tilde{A} \oplus \tilde{C} \succ \tilde{B} \oplus \tilde{C}$ which is a contradiction according to Wang and Keere [174].

Example 2.5 Let $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 1)$, $\tilde{B} = (0.2, 0.3, 0.3, 0.4; 1)$ and $\tilde{C} = (-0.5, -0.3, -0.3, -0.1; 1)$ be three generalized trapezoidal fuzzy sets. Then according to existing ranking approaches [37, 41], $\tilde{B} \succ \tilde{A}$ but $\tilde{B} \oplus \tilde{C} \prec \tilde{A} \oplus \tilde{C}$ i.e., $\tilde{B} \succ \tilde{A} \not\Rightarrow \tilde{B} \oplus \tilde{C} \succ \tilde{A} \oplus \tilde{C}$.

Example 2.6 Let $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 0.8)$, $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$ and $\tilde{C} = (-0.5, -0.3, -0.3, -0.1; 0.8)$ be three generalized trapezoidal fuzzy sets. Then according to existing ranking approaches [37, 41], $\tilde{B} \succ \tilde{A}$ but $\tilde{B} \oplus \tilde{C} \prec \tilde{A} \oplus \tilde{C}$ i.e., $\tilde{B} \succ \tilde{A} \not\Rightarrow \tilde{B} \oplus \tilde{C} \succ \tilde{A} \oplus \tilde{C}$.

Example 2.7 Let $\tilde{A} = (-0.8, -0.6, -0.4, -0.2; 0.35)$, $\tilde{B} = (-0.4, -0.3, -0.2, -0.1; 0.7)$ and $\tilde{C} = (0.2, 0.4, 0.6, 0.8; 0.35)$ be three generalized trapezoidal fuzzy sets. Then according to existing ranking approaches [37, 41], $\tilde{A} \succ \tilde{B}$ but $\tilde{A} \oplus \tilde{C} \prec \tilde{B} \oplus \tilde{C}$ i.e., $\tilde{A} \succ \tilde{B} \not\Rightarrow \tilde{A} \oplus \tilde{C} \succ \tilde{B} \oplus \tilde{C}$.

Example 2.8 Let $\tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$, $\tilde{B} = (0.1, 0.2, 0.3, 0.4; 0.7)$ and $\tilde{C} = (-0.8, -0.6, -0.4, -0.2; 0.35)$ be three generalized trapezoidal fuzzy sets. Then according to existing ranking approaches [37, 41], $\tilde{B} \succ \tilde{A}$ but $\tilde{B} \oplus \tilde{C} \prec \tilde{A} \oplus \tilde{C}$ i.e., $\tilde{B} \succ \tilde{A} \not\Rightarrow \tilde{B} \oplus \tilde{C} \succ \tilde{A} \oplus \tilde{C}$.

2.5.4 Shortcomings on the basis of height of fuzzy sets

In this section, it is shown that in some cases the ranking results, obtained by using existing ranking approaches [37, 41], depends upon height of fuzzy sets while in several cases the ranking results does not depend upon the height of fuzzy sets.

Let $\tilde{A} = (a_1, a_2, a_3, a_4; w_1)$ and $\tilde{B} = (a_1, a_2, a_3, a_4; w_2)$ be two generalized trapezoidal fuzzy sets. Then according to existing ranking approaches [37, 41] there can be two cases:

Case (i) If $(a_1 + a_2 + a_3 + a_4) \neq 0$, then $\begin{cases} \tilde{A} \prec \tilde{B}, & \text{if } w_1 < w_2 \\ \tilde{A} \succ \tilde{B}, & \text{if } w_1 > w_2 \\ \tilde{A} \sim \tilde{B}, & \text{if } w_1 = w_2. \end{cases}$

Case (ii) If $(a_1 + a_2 + a_3 + a_4) = 0$, then $\tilde{A} \sim \tilde{B}$ for all values of w_1 and w_2.

Example 2.9 Let $\tilde{A} = (1, 1, 1, 1; w_1)$ and $\tilde{B} = (1, 1, 1, 1; w_2)$ be two generalized trapezoidal fuzzy sets. Then according to existing ranking approaches $\tilde{A} \prec \tilde{B}$ if $w_1 < w_2$, $\tilde{A} \succ \tilde{B}$ if $w_1 > w_2$ and $\tilde{A} \sim \tilde{B}$ if $w_1 = w_2$.

Example 2.10 Let $\tilde{A} = (-.4, -.2, -.1, .7; w_1)$ and $\tilde{B} = (-.4, -.2, -.1, .7; w_2)$, be two generalized trapezoidal fuzzy sets. Then $\tilde{A} \sim \tilde{B}$ for all values of w_1 and w_2.

According to existing ranking approaches [37, 41], in first case comparison of fuzzy sets depends upon heights of fuzzy sets and in second case comparison of fuzzy sets does not depend upon the heights of fuzzy sets which is a contradiction.

2.6 Proposed ranking approach

In this section, to overcome the shortcomings of the existing approaches [37, 41], a new ranking approach is proposed for comparing generalized trapezoidal fuzzy sets.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy sets then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Transform $\tilde{A}$, $\tilde{B}$ into $\tilde{A}^*$, $\tilde{B}^*$ as follows:

$$\tilde{A}^* = (a_1, b_1, c_1, d_1; w), \tilde{B}^* = (a_2, b_2, c_2, d_2; w), \text{where } w = min(w_1, w_2) \tag{2.33}$$

Step 2 Calculate

$$\Re_L(\tilde{A}^*) = \int_0^w (\nu^L_{\tilde{A}^*}(\alpha))d\alpha = \int_0^w (a_1 + \frac{(b_1 - a_1)}{w}\alpha)d\alpha = \frac{w}{2}(a_1 + b_1) \tag{2.34}$$

$$\Re_R(\tilde{A}^*) = \int_0^w (\nu^R_{\tilde{A}^*}(\alpha))d\alpha = \int_0^w (d_1 + \frac{(d_1 - c_1)}{w}\alpha)d\alpha = \frac{w}{2}(c_1 + d_1) \tag{2.35}$$

and

$$\Re_L(\tilde{B}^*) = \int_0^w (\nu^L_{\tilde{B}^*}(\alpha))d\alpha = \int_0^w (a_2 + \frac{(b_2 - a_2)}{w}\alpha)d\alpha = \frac{w}{2}(a_2 + b_2) \tag{2.36}$$

$$\Re_R(\tilde{B}^*) = \int_0^w (\nu^R_{\tilde{B}^*}(\alpha))d\alpha = \int_0^w (d_2 + \frac{(d_2 - c_2)}{w}\alpha)d\alpha = \frac{w}{2}(c_2 + d_2) \tag{2.37}$$

where, $\nu^L_{\tilde{A}^*}(\alpha) = infimum(A^{*\alpha})$, $\nu^L_{\tilde{B}^*}(\alpha) = infimum(B^{*\alpha})$, $\nu^R_{\tilde{A}^*}(\alpha) = supremum(A^{*\alpha})$ and $\nu^R_{\tilde{B}^*}(\alpha) = supremum(B^{*\alpha})$.

Step 3 Calculate

$$\Re^\lambda_T(\tilde{A}^*) = \lambda\Re_L(\tilde{A}^*) + (1-\lambda)\Re_R(\tilde{A}^*) = \frac{w}{2}\{(\lambda(a_1 + b_1) + (1-\lambda)(c_1 + d_1))\} \tag{2.38}$$

and

$$\Re^\lambda_T(\tilde{B}^*) = \lambda\Re_L(\tilde{B}^*) + (1-\lambda)\Re_R(\tilde{B}^*) = \frac{w}{2}\{(\lambda(a_2 + b_2) + (1-\lambda)(c_2 + d_2))\} \tag{2.39}$$

Step 4 Choose a particular value of λ and check that $\Re_T^{\lambda}(\tilde{A}^*) > \Re_T^{\lambda}(\tilde{B}^*)$ or $\Re_T^{\lambda}(\tilde{A}^*) < \Re_T^{\lambda}(\tilde{B}^*)$ or $\Re_T^{\lambda}(\tilde{A}^*) = \Re_T^{\lambda}(\tilde{B}^*)$

Case (i) If $\Re_T^{\lambda}(\tilde{A}^*) > \Re_T^{\lambda}(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \succ \tilde{B}$ for same value of λ.

Case (ii) If $\Re_T^{\lambda}(\tilde{A}^*) < \Re_T^{\lambda}(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \prec \tilde{B}$ for same value of λ.

Case (iii) If $\Re_T^{\lambda}(\tilde{A}^*) = \Re_T^{\lambda}(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \sim \tilde{B}$ for same value of λ

2.6.1 Illustrative examples

In this section, the proposed ranking approach is illustrated by solving some numerical examples.

Example 2.11 Using the proposed ranking approach the normal trapezoidal fuzzy sets $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 1)$ and $\tilde{B} = (0.2, 0.3, 0.3, 0.4; 1)$ can be compared as follows:

Step 1 Using equation (2.33), $\tilde{A}^* = (0.1, 0.3, 0.3, 0.5; 1)$ and $\tilde{B}^* = (0.2, 0.3, 0.3, 0.4; 1)$

Step 2 Using equation (2.34), (2.35), (2.36) and (2.37), $\Re_L(\tilde{A}^*) = 0.2$, $\Re_R(\tilde{A}^*) = 0.4$, $\Re_L(\tilde{B}^*) = 0.25$ and $\Re_R(\tilde{B}^*) = 0.35$

Step 3 Using equation (2.38) and (2.39), $\Re_T^{\lambda}(\tilde{A}^*) = (0.4 - 0.2\lambda)$ and $\Re_T^{\lambda}(\tilde{B}^*) = (0.35 - 0.1\lambda)$

Step 4 For a pessimistic decision maker i.e. for $\lambda = 0$, $\Re_T^{0}(\tilde{A}^*) = 0.4$, $\Re_T^{0}(\tilde{B}^*) = 0.35$ Since $\Re_T^{0}(\tilde{A}^*) > \Re_T^{0}(\tilde{B}^*)$ so $\tilde{A} \succ \tilde{B}$.

For optimistic decision maker i.e. for $\lambda = 1$, $\Re_T^{1}(\tilde{A}^*) = 0.2$, $\Re_T^{1}(\tilde{B}^*) = 0.25$. Since $\Re_T^{1}(\tilde{A}^*) < \Re_T^{1}(\tilde{B}^*)$ so $\tilde{A} \prec \tilde{B}$.

For moderate decision maker i.e for $\lambda = 0.5$, $\Re_T^{0.5}(\tilde{A}^*) = 0.3$ and $\Re_T^{0.5}(\tilde{B}^*) = 0.3$

Since $\Re_T^{0.5}(\tilde{A}^*) = \Re_T^{0.5}(\tilde{B}^*)$ so $\tilde{A} \sim \tilde{B}$.

Example 2.12 Using the proposed ranking approach the generalized trapezoidal fuzzy sets $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 0.8)$, $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 0.9)$ can be compared as follows:

Step 1 Using equation (2.33), $\tilde{A}^* = (0.1, 0.3, 0.3, 0.5; 0.8)$, $\tilde{B}^* = (0.1, 0.3, 0.3, 0.5; 0.8)$

Step 2 Using equation (2.34), (2.35), (2.36) and (2.37), $\Re_L(\tilde{A}^*) = 0.16$, $\Re_R(\tilde{A}^*) = 0.32$, $\Re_L(\tilde{B}^*) = 0.16$, $\Re_R(\tilde{B}^*) = 0.32$.

Step 3 Using equation (2.38) and (2.39), $\Re_T^{\lambda}(\tilde{A}^*) = 0.4(0.8 - 0.4\lambda)$ and $\Re_T^{\lambda}(\tilde{B}^*) = 0.4(0.8 - 0.4\lambda)$

Step 4 Since $\Re_T^{\lambda}(\tilde{A}^*) = \Re_T^{\lambda}(\tilde{B}^*)\ \ \forall\ \lambda \in [0, 0.8]$ so $\tilde{A} \sim \tilde{B}$

Example 2.13 Using the proposed ranking approach the generalized trapezoidal fuzzy sets $\tilde{A} = (-0.8, -0.6, -0.4, -0.2; 0.35)$, $\tilde{B} = (-0.4, -0.3, -0.2, -0.1; 0.7)$ can be compared as follows:

Step 1 Using equation (2.33),

$\tilde{A}^* = (-0.8, -0.6, -0.4, -0.2; 0.35)$, $\tilde{B}^* = (-0.4, -0.3, -0.2, -0.1; 0.35)$

Step 2 Using equation (2.34), (2.35), (2.36) and (2.37), $\Re_L(\tilde{A}^*) = -0.245$, $\Re_R(\tilde{A}^*) = -0.105$, $\Re_L(\tilde{B}^*) = -0.1225$, $\Re_R(\tilde{B}^*) = -0.0525$

Step 3 Using equation (2.38) and (2.39), $\Re_T^{\lambda}(\tilde{A}^*) = 0.35(-0.3 - 0.4\lambda)$, $\Re_T^{\lambda}(\tilde{B}^*) = 0.175(-0.3 - 0.4\lambda)$

Step 4 For a pessimistic decision maker i.e. for $\lambda = 0$, $\Re_T^{0}(\tilde{A}^*) = -0.105$, $\Re_T^{0}(\tilde{B}^*) = -0.0525$

Since $\Re_T^{0}(\tilde{A}^*) < \Re_T^{0}(\tilde{B}^*)$ so $\tilde{A} \prec \tilde{B}$.

For optimistic decision maker i.e. for $\lambda = 1$, $\Re_T^{1}(\tilde{A}^*) = -2.45$, $\Re_T^{1}(\tilde{B}^*) =$

0.1225.

Since so $\tilde{A} \prec \tilde{B}$

For moderate decision maker i.e. for $\lambda = 0.5$, $\Re_T^{0.5}(\tilde{A}^*) = -0.175$, $\Re_T^{0.5}(\tilde{B}^*) = 0.0875$.

Since $\Re_T^{0.5}(\tilde{A}^*) < \Re_T^{0.5}(\tilde{B}^*)$ so $\tilde{A} \prec \tilde{B}$.

2.7 Validity of the proposed ranking approach

In Section 2.5.3, it is shown that if $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are three fuzzy sets then by using the existing ranking approaches [37, 41] the property $\tilde{A} \succ \tilde{B} \Rightarrow \tilde{A} \oplus \tilde{C} \succ \tilde{B} \oplus \tilde{C}$ is not always satisfied. In this section, it is proved that by using the proposed approach the property $\tilde{A} \succ \tilde{B} \Rightarrow \tilde{A} \oplus \tilde{C} \succ \tilde{B} \oplus \tilde{C}$ is always satisfied.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy sets then according proposed ranking approach for the generalized trapezoidal fuzzy set $\tilde{C} = (a_3, b_3, c_3, d_3; w_3)$, where, $w_3 \leq min(w_1, w_2)$,

$(\tilde{A} \oplus \tilde{C})^* = (a_1 + a_3, b_1 + b_3, c_1 + c_3, d_1 + d_3; w_3)$ and $(\tilde{B} \oplus \tilde{C})^* = (a_2 + a_3, b_2 + b_3, c_2 + c_3, d_2 + d_3; w_3)$

Now, $\Re_T^{\lambda}(\tilde{A} \oplus \tilde{C})^* - \Re_T^{\lambda}(\tilde{B} \oplus \tilde{C})^*$

$= \frac{w_3}{2}[\lambda\{(a_1 + a_3) + (b_1 + b_3)\} + (1 - \lambda)\{(c_1 + c_3) + (d_1 + d_3)\}] - \frac{w_3}{2}[\lambda\{(a_2 + a_3) + (b_2 + b_3)\} + (1 - \lambda)\{(c_2 + c_3) + (d_2 + d_3)\}]$

$= \frac{w_3}{2}[\lambda\{(a_1 - a_2) + (b_1 - b_2)\} + (1 - \lambda)\{(c_1 - c_2) + (d_1 - d_2)\}]$

$= \frac{w_3}{2}[\lambda\{(a_1 + b_1) + (1 - \lambda)(c_1 + d_1)\}] - \frac{w_3}{2}[\lambda\{(a_2 + b_2) + (1 - \lambda)(c_2 + d_2)\}]$

$$= \frac{w_3}{min(w_1, w_2)}(\Re_T^{\lambda}(\tilde{A}^*) - \Re_T^{\lambda}(\tilde{B}^*)) \tag{2.40}$$

From (2.40) it can be easily seen that

(i) If $\Re_T^{\lambda}(\tilde{A}^*) - \Re_T^{\lambda}(\tilde{B}^*) > 0$ then $\Re_T^{\lambda}(\tilde{A} \oplus \tilde{C})^* - \Re_T^{\lambda}(\tilde{B} \oplus \tilde{C})^* > 0$ i.e. $\tilde{A} \succ \tilde{B} \Rightarrow$

$(\tilde{A} \oplus \tilde{C}) \succ (\tilde{B} \oplus \tilde{C})$

(ii) If $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) < 0$ then $\Re_T^\lambda(\tilde{A} \oplus \tilde{C})^* - \Re_T^\lambda(\tilde{B} \oplus \tilde{C})^* < 0$ i.e. $\tilde{A} \prec \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \prec (\tilde{B} \oplus \tilde{C})$

(iii) If $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) = 0$ then $\Re_T^\lambda(\tilde{A} \oplus \tilde{C})^* - \Re_T^\lambda(\tilde{B} \oplus \tilde{C})^* = 0$ i.e. $\tilde{A} \sim \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \sim (\tilde{B} \oplus \tilde{C})$

2.8 Proposed method for solving linear programming problems with generalized trapezoidal fuzzy sets

In this section, to overcome the limitations of the existing method, a new method is proposed to find the solution of linear programming problems with generalized trapezoidal fuzzy sets. The same method can also be used for solving linear programming problems with normal trapezoidal fuzzy sets.

The solution of linear programming problem with generalized trapezoidal fuzzy sets can be obtained by using the following steps:

Step 1 Assuming $\tilde{c}_j = (c_{j1}, c_{j2}, c_{j3}, c_{j4}; w_j)$ the linear programming problem with generalized fuzzy sets (P_2) can be converted into linear programming problem with generalized trapezoidal fuzzy sets (P_7):

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n}(c_{j1}, c_{j2}, c_{j3}, c_{j4}; w_j)x_j)\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad (P_7)\\
&\qquad x_j \geq 0
\end{aligned}$$

Step 2 Using Definition 2.8 and on the basis of the proposed ranking approach, the optimal solution of the linear programming problem with generalized trapezoidal fuzzy sets (P_7) can be obtained by solving the crisp linear programming problem

(P_8)

$$Maximize\ \ (or\ \ Minimize)\ \ (\Re_T^{\lambda}(\sum_{j=1}^{n}((c_{j1},c_{j2},c_{j3},c_{j4};min(w_1,w_2,...,w_n))x_j)))$$

$$subject\ to$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq,=,\leq b_i, \quad i=1,2,...,m \qquad (P_8)$$

$$x_j \geq 0$$

Step 3 Since the proposed ranking function $\Re_T^{\lambda}$ satisfies the linearity property i.e. $\Re_T^{\lambda}(\sum_{j=1}^{n}((a_j,b_j,c_j,d_j;w)x_j)) = \sum_{j=1}^{n}\Re_T^{\lambda}((a_j,b_j,c_j,d_j;w)x_j)$, so the crisp linear programming problem (P_8) can be converted into crisp linear programming problem (P_9)

$$Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n}\Re_T^{\lambda}((c_{j1},c_{j2},c_{j3},c_{j4};min(w_1,w_2,...,w_n))x_j))$$

$$subject\ to$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq,=,\leq b_i, \quad i=1,2,...,m \qquad (P_9)$$

$$x_j \geq 0$$

Step 4 Using $\Re_T^{0.5}(a,b,c,d;w) = \frac{w}{4}(a+b+c+d)$, the crisp linear programming problem (P_9) can be converted into crisp linear programming problem (P_{10})

$$Maximize\ (or\ Minimize)\ \sum_{j=1}^{n}(\tfrac{min(w_1,w_2,...,w_n)}{4}(c_{j1}+c_{j2}+c_{j3}+c_{j4})x_j)$$

$$subject\ to$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq,=,\leq b_i, \quad i=1,2,...,m \qquad (P_{10})$$

$$x_j \geq 0$$

Step 5 Use an appropriate existing method such as simplex method or Big-M method or two phase method or dual simplex method etc. to find the optimal solution $\{x_j\}$ of the the crisp linear programming problem (P_{10}).

Step 6 Find the fuzzy optimal value of the linear programming problem with trapezoidal fuzzy sets (P_7) by putting the obtained values of x_j in $\sum_{j=1}^{n}(c_{j1},c_{j2},c_{j3},c_{j4};w_j)x_j$.

2.8.1 Advantage of proposed method over existing method

The existing method can be used to solve such linear programming problems in which either the parameters $\tilde{c}_j$ are represented by normal fuzzy sets or generalized fuzzy sets having equal heights but the existing method can't be used for solving such linear programming problems in which some or all the parameters $\tilde{c}_j$ are represented by generalized fuzzy sets having different heights while the proposed method can be used for solving both types of linear programming problems with fuzzy sets.

To show the advantage of the proposed method over existing method a linear programming problem with generalized fuzzy sets, chosen in Example 2.17, which can't be solved by using existing method is solved by the proposed method.

Example 2.17 Solve

$$Minimize \quad ((2,5,8,10;0.8)x_1 \oplus (5,10,15,30;0.6)x_2 \oplus (1,5,6,8;0.9)x_3)$$

$$subject\ to$$

$$x_1 + 2x_2 + x_3 \leq 5$$

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

Solution:-The linear programming problem, chosen in Example 2.17, can be solved by using the following steps of the proposed method:

Step 1 Using Step 2 of the proposed method the chosen linear programming problem can be written as

$$Minimize \quad (\Re_T^{0.5}((2,5,8,10;0.6)x_1 \oplus (5,10,15,30;0.6)x_2 \oplus (1,5,6,8;0.6)x_3))$$

$$subject\ to$$

$$x_1 + 2x_2 + x_3 \leq 5$$

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

Step 2 Using Step 3 of the proposed method the chosen linear programming problem can be written as

$$Minimize\ \ (\Re_T^{0.5}((2,5,8,10;0.6)x_1 + \Re_T^{0.5}(5,10,15,30;0.6)x_2 + \Re_T^{0.5}(1,5,6,8;0.6)x_3))$$

subject to

$$x_1 + 2x_2 + x_3 \leq 5$$

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

Step 3 Using $\Re_T^{0.5}(a,b,c,d;w) = \frac{w}{4}(a+b+c+d)$, the crisp linear programming problem, obtained in Step 2, can be written as

$$Maximize\ \ (5x_1 + 12x_2 + 4x_3)$$

subject to

$$x_1 + 2x_2 + x_3 \leq 5$$

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

Step 4 On solving the crisp linear programming problem, obtained in Step 2, the obtained optimal solution is $x_1 = \frac{9}{5}$, $x_2 = \frac{8}{5}$ and $x_3 = 0$. Putting $x_1 = \frac{9}{5}$, $x_2 = \frac{8}{5}$ and $x_3 = 0$ in $(2,5,8,10;0.8)x_1 \oplus (5,10,15,30;0.6)x_2 \oplus (1,5,6,8;0.9)x_3$ the fuzzy optimal value of the chosen problem is $(\frac{58}{5}, 25, \frac{192}{5}, 66; 0.6)$.

2.9 Conclusions

In this chapter limitations of existing method for solving linear programming problems with fuzzy sets and the shortcomings of existing ranking approaches [37, 41] for comparing generalized trapezoidal fuzzy sets are pointed out. To overcome the shortcomings of existing ranking approaches, a new ranking approach is proposed for comparing generalized trapezoidal fuzzy sets and also on the basis of proposed ranking approach, a new method is proposed for solving linear programming problems with generalized fuzzy sets.

Chapter 3

LINEAR PROGRAMMING PROBLEMS WITH GENERALIZED P-NORM TRAPEZOIDAL FUZZY SETS [1]

Chen and Tang [42] introduced the definition of generalized p-norm fuzzy sets and proposed some results for comparing generalized p-norm fuzzy sets. In this chapter, with the help of several counter examples it is proved that the existing results [42] are applicable only for the generalized p-norm fuzzy sets having equal height. To overcome the limitations of existing results, on the basis of ranking approach, proposed in Chapter 2, some new results are proposed for comparing generalized p-norm fuzzy sets having different height. It is shown that the existing results [42] are particular cases of the proposed results. Also, the method for solving linear programming problems with generalized fuzzy sets, proposed in Chapter 2, is used for solving a linear programming problem with generalized 2-norm trapezoidal fuzzy sets.

3.1 Preliminaries

In this section some basic definitions and arithmetic operations are presented.

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3.1.1 Basic definitions

In this section some basic definitions are presented [42].

Definition 3.1 A generalized fuzzy set defined on the universal set of real numbers R, denoted as $\tilde{A} = (a, b, c; w)_p$, is said to be a generalized p-norm triangular fuzzy set if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} w(1-(\frac{x-b}{a-b})^p)^{\frac{1}{p}}, & a \leq x < b \\ w \qquad\qquad\qquad , & x = b \\ w(1-(\frac{x-b}{c-b})^p)^{\frac{1}{p}}, & b < x \leq c \\ 0 \qquad\qquad\qquad , & \text{otherwise} \end{cases}$$

where, p is a positive integer.

Definition 3.2 A generalized fuzzy set defined on the universal set of real numbers R, denoted as $\tilde{A} = (a, b, c, d; w)_p$, is said to be a generalized p-norm trapezoidal fuzzy set if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} w(1-(\frac{x-b}{a-b})^p)^{\frac{1}{p}}, & a \leq x < b \\ w \qquad\qquad\qquad , & b \leq x \leq c \\ w(1-(\frac{x-c}{d-c})^p)^{\frac{1}{p}}, & c < x \leq d \\ 0 \qquad\qquad\qquad , & \text{otherwise} \end{cases}$$

where, p is a positive integer.

Definition 3.3 Let $\tilde{A} = (a_1, a_2, a_3; w)_p$ and $\tilde{B} = (b_1, b_2, b_3, b_4; w)_p$ be generalized p-norm triangular and generalized p-norm trapezoidal fuzzy sets respectively. Then the α-cuts A^α and B^α for these generalized fuzzy sets can be defined as follows:

$$A^\alpha = [a_2 + (a_1 - a_2)(1 - (\frac{\alpha}{w})^2)^2, a_2 + (a_3 - a_2)(1 - (\frac{\alpha}{w})^2)^2]$$

$$B^\alpha = [b_2 + (b_1 - b_2)(1 - (\frac{\alpha}{w})^2)^2, b_3 + (b_4 - b_3)(1 - (\frac{\alpha}{w})^2)^2]$$

3.1.2 Arithmetic operations

In this section, some arithmetic operations between two generalized p-norm triangular fuzzy sets and between two generalized p-norm trapezoidal fuzzy sets, defined on universal set of real numbers R, are presented.

3.1.2.1 Arithmetic operations between generalized p-norm triangular fuzzy sets

Let $\tilde{A}_1 = (a_1, b_1, c_1; w_1)_p$ and $\tilde{A}_2 = (a_2, b_2, c_2; w_2)_p$ be two generalized p-norm triangular fuzzy sets. Then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2; min\ (w_1, w_2))_p$

(ii) $\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - c_2, b_1 - b_2, c_1 - a_2; min\ (w_1, w_2))_p$

(iii) $\gamma\tilde{A}_1 = \begin{cases} (\gamma a_1, \gamma b_1, \gamma c_1, w_1)_p & \gamma \geq 0 \\ (\gamma c_1, \gamma b_1, \gamma a_1, w_1)_p & \gamma \leq 0. \end{cases}$

3.1.2.2 Arithmetic operations between generalized p-norm trapezoidal fuzzy sets

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1; w_1)_p$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2; w_2)_p$ be two generalized p-norm trapezoidal fuzzy sets. Then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; min\ (w_1, w_2))_p$

(ii) $\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; min\ (w_1, w_2))_p$

(iii) $\gamma\tilde{A}_1 = \begin{cases} (\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1, w_1)_p & \gamma \geq 0 \\ (\gamma d_1, \gamma c_1, \gamma b_1, \gamma a_1, w_1)_p & \gamma \leq 0. \end{cases}$

Remark 3.1 For $p = 1$ generalized p-norm triangular and generalized p-norm trapezoidal fuzzy sets are converted into generalized triangular and generalized trapezoidal fuzzy sets respectively. For $p = 1$ and $w = 1$ generalized p-norm triangular and generalized p-norm trapezoidal fuzzy sets are converted into normal triangular and normal trapezoidal fuzzy sets respectively.

Remark 3.2 For $p = 1$, the arithmetic operations between generalized p-norm fuzzy sets, presented in Section 3.1.2, are converted into arithmetic operations between generalized fuzzy sets. For $p = 1$ and $w = 1$, the arithmetic operations between generalized p-norm fuzzy sets, presented in Section 3.1.2, are converted into arithmetic operations between normal fuzzy sets.

3.2 Limitations of existing ranking results

Chen and Tang [42] proposed the following results for comparing generalized p-norm fuzzy sets having equal height.

Proposition 3.1 [42] Let $\tilde{A} = (a, b, c, d; w)$ and $\tilde{B} = (a, e, d; w)$ be generalized trapezoidal and generalized triangular fuzzy sets, respectively and $-\infty < a \leq b \leq e \leq c \leq d < \infty$. Then

(1) $I_L(\tilde{B}) \geq I_L(\tilde{A})$

(2) $I_R(\tilde{A}) \geq I_R(\tilde{B})$

(3) $I_T^{\lambda}(\tilde{A}) > I_T^{\lambda}(\tilde{B})$ if $e < c\lambda + (1 - \lambda)b$

(4) $I_T^{\lambda}(\tilde{A}) = I_T^{\lambda}(\tilde{B})$ if $e = c\lambda + (1 - \lambda)b$

(5) $I_T^{\lambda}(\tilde{A}) < I_T^{\lambda}(\tilde{B})$ if $e > c\lambda + (1 - \lambda)b$.

where, $I_L(\tilde{A}) = \frac{w}{2}(a + b)$, $I_L(\tilde{B}) = \frac{w}{2}(a + e)$, $I_R(\tilde{A}) = \frac{w}{2}(c + d)$, $I_R(\tilde{B}) = \frac{w}{2}(e + d)$, $I_T^{\lambda}(\tilde{A}) = \frac{w}{2}(\lambda(c + d) + (1 - \lambda)(a + b))$ and $I_T^{\lambda}(\tilde{B}) = \frac{w}{2}(\lambda d + e + (1 - \lambda)a)$

The above results are valid only if either both $\tilde{A}$ and $\tilde{B}$ are normal fuzzy sets or generalized fuzzy sets having equal height.

But the results of Proposition 3.1 are invalid if the height of $\tilde{A}$ and $\tilde{B}$ are different.

Let $\tilde{A} = (a, b, c, d; w_1)$ and $\tilde{B} = (a, e, d; w_2)$ be generalized trapezoidal and triangular fuzzy sets respectively and $-\infty < a \leq b \leq e \leq c \leq d < \infty$ then according to existing approach [42], the values of $I_L(\tilde{A})$, $I_L(\tilde{B})$, $I_R(\tilde{A})$, $I_R(\tilde{B})$ etc. will be

$$I_L(\tilde{A}) = \frac{w_1}{2}(a + b) \tag{3.1}$$

$$I_L(\tilde{B}) = \frac{w_2}{2}(a + e) \tag{3.2}$$

$$I_R(\tilde{A}) = \frac{w_1}{2}(c + d) \tag{3.3}$$

$$I_R(\tilde{B}) = \frac{w_2}{2}(e + d) \tag{3.4}$$

$$I_T^{\lambda}(\tilde{A}) = \frac{w_1}{2}(\lambda(c + d) + (1 - \lambda)(a + b)) \tag{3.5}$$

and

$$I_T^{\lambda}(\tilde{B}) = \frac{w_2}{2}(\lambda d + e + (1 - \lambda)a) \tag{3.6}$$

To illustrate the limitations of the existing results, presented in Proposition 3.1, the following counter examples are chosen:

Example 3.1 Let $\tilde{A} = (5, 7, 9, 10; 0.4)$ and $\tilde{B} = (5, 8, 10; 0.2)$ be generalized trapezoidal and generalized triangular fuzzy sets. Then using equation (3.1) and equation (3.2), it can be easily seen that $I_L(\tilde{A}) > I_L(\tilde{B})$, which contradicts the existing result $I_L(\tilde{A}) < I_L(\tilde{B})$.

Example 3.2 Let $\tilde{A} = (5, 7, 9, 10; 0.2)$ and $\tilde{B} = (5, 8, 10; 0.8)$ be generalized trapezoidal and generalized triangular fuzzy sets. Then using equation (3.3) and equation (3.4), it can be easily seen that $I_R(\tilde{A}) < I_R(\tilde{B})$, which contradicts the existing result $I_R(\tilde{A}) > I_R(\tilde{B})$.

Example 3.3 Let $\tilde{A} = (0, 1, 4, 5; 0.1)$ and $\tilde{B} = (0, 3, 5; 0.8)$ be generalized trapezoidal and generalized triangular fuzzy sets. For $\lambda = 0.9$, $e < c\lambda + (1 - \lambda)b$ but using equation (3.5) and equation (3.6), it can be easily seen that $I_T^{0.9}(\tilde{A}) < I_T^{0.9}(\tilde{B})$, which contradicts the existing result $I_T^{0.9}(\tilde{A}) > I_T^{0.9}(\tilde{B})$

Example 3.4 Let $\tilde{A} = (0, 1, 4, 5; 0.1)$ and $\tilde{B} = (0, 3, 5; 0.2)$ be generalized trapezoidal and generalized triangular fuzzy sets. For $\lambda = \frac{2}{3}$, $e = c\lambda + (1-\lambda)b$ but using equation (3.5) and equation (3.6), it can be easily seen that $I_T^{\frac{2}{3}}(\tilde{A}) \neq I_T^{\frac{2}{3}}(\tilde{B})$, which contradicts the existing result $I_T^{\frac{2}{3}}(\tilde{A}) = I_T^{\frac{2}{3}}(\tilde{B})$

Example 3.5 Let $\tilde{A} = (0, 1, 4, 5; 0.9)$ and $\tilde{B} = (0, 3, 5; 0.1)$ be generalized trapezoidal and generalized triangular fuzzy sets. For $\lambda = 0.3$, $e > c\lambda + (1-\lambda)b$ but using equation (3.5) and equation (3.6), it can be easily seen that $I_T^{0.3}(\tilde{A}) > I_T^{0.3}(\tilde{B})$, which contradicts the existing result $I_T^{0.3}(\tilde{A}) < I_T^{0.3}(\tilde{B})$

Proposition 3.2 [**42**] Let $\tilde{A} = (a, b, c, d; w)$ and $\tilde{C}_2 = (a, b, c, d; w)_2$ be generalized trapezoidal and generalized 2-norm trapezoidal fuzzy sets respectively. Then

(1) $I_L(\tilde{A}) \geq I_L(\tilde{C}_2)$

(2) $I_R(\tilde{A}) \leq I_R(\tilde{C}_2)$

(3) $I_T^{\lambda}(\tilde{A}) < I_T^{\lambda}(\tilde{C}_2)$ if $\lambda(d-c) + (1-\lambda)(a-b) > 0$

(4) $I_T^{\lambda}(\tilde{A}) = I_T^{\lambda}(\tilde{C}_2)$ if $\lambda(d-c) + (1-\lambda)(a-b) = 0$

(5) $I_T^{\lambda}(\tilde{A}) > I_T^{\lambda}(\tilde{C}_2)$ if $\lambda(d-c) + (1-\lambda)(a-b) < 0$

where, $I_L(\tilde{A}) = \frac{w}{2}(a+b)$, $I_R(\tilde{A}) = \frac{w}{2}(c+d)$, $I_T^{\lambda}(\tilde{A}) = \frac{w}{2}(\lambda(c+d) + (1-\lambda)(a+b))$, $I_L(\tilde{C}_2) = bw + (\frac{a-b}{4})\pi w$, $I_R(\tilde{C}_2) = cw + (\frac{d-c}{4})\pi w$ and $I_T^{\lambda}(\tilde{C}_2) = w\{\frac{\pi}{4}[\lambda(d-c) + (1-\lambda)(a-b)] + \lambda c + (1-\lambda)b\}$

The above results are valid only if either both $\tilde{A}$ and $\tilde{C}_2$ are normal fuzzy sets or generalized fuzzy sets having equal height.

Let $\tilde{A} = (a, b, c, d; w_1)$ and $\tilde{C}_2 = (a, b, c, d; w_2)_2$ be generalized trapezoidal and generalized 2-norm trapezoidal fuzzy sets respectively and then according to existing approach [42], the values of $I_L(\tilde{A})$, $I_L(\tilde{C_2})$, $I_R(\tilde{A})$, $I_R(\tilde{C_2})$ etc. will be

$$I_L(\tilde{A}) = \frac{w_1}{2}(a+b) \tag{3.7}$$

$$I_R(\tilde{A}) = \frac{w_1}{2}(c+d) \tag{3.8}$$

$$I_T^{\lambda}(\tilde{A}) = \frac{w_1}{2}(\lambda(c+d) + (1-\lambda)(a+b)) \tag{3.9}$$

$$I_L(\tilde{C_2}) = bw_2 + (\frac{a-b}{4})\pi w_2 \tag{3.10}$$

$$I_R(\tilde{C_2}) = cw_2 + (\frac{d-c}{4})\pi w_2 \tag{3.11}$$

and

$$I_T^{\lambda}(\tilde{C_2}) = w_2\{\frac{\pi}{4}[\lambda(d-c) + (1-\lambda)(a-b)] + \lambda c + (1-\lambda)b\} \tag{3.12}$$

To illustrate the limitations of the results, proposed in Proposition 3.2, the following counter examples are chosen:

Example 3.6 Let $\tilde{A} = (5, 7, 8, 9; 0.4)$ and $\tilde{C_2} = (5, 7, 8, 9; 0.6)_2$ be generalized trapezoidal fuzzy set and generalized 2-norm trapezoidal fuzzy sets respectively. Then using equation (3.7) and equation (3.10), it can be easily seen that $I_L(\tilde{A}) < I_L(\tilde{C_2})$, which contradicts the existing result $I_L(\tilde{A}) > I_L(\tilde{C_2})$.

Example 3.7 Let $\tilde{A} = (5, 7, 8, 9; 0.4)$ and $\tilde{C_2} = (5, 7, 8, 9; 0.1)_2$ be generalized trapezoidal fuzzy set and generalized 2-norm trapezoidal fuzzy sets respectively. Then using equation (3.8) and equation (3.11), it can be easily seen that $I_R(\tilde{A}) > I_R(\tilde{C_2})$, which contradicts the existing result $I_R(\tilde{A}) < I_R(\tilde{C_2})$.

Example 3.8 Let $\tilde{A} = (5, 7, 8, 9; 0.4)$ and $\tilde{C_2} = (5, 7, 8, 9; 0.1)_2$ be generalized trapezoidal fuzzy set and generalized 2-norm trapezoidal fuzzy sets respectively. For $\lambda > \frac{1}{2}$, $\lambda(d-c) + (1-\lambda)(a-b) > 0$ but using equation (3.9) and equation (3.12), it can be easily seen that $I_T^{\frac{1}{2}}(\tilde{A}) > I_T^{\frac{1}{2}}(\tilde{C_2})$, which contradicts the existing result

$I_T^{\frac{1}{2}}(\tilde{A}) < I_T^{\frac{1}{2}}(\tilde{C}_2)$

Example 3.9 Let $\tilde{A} = (1, 1, 1, 1; 0.4)$ and $\tilde{C}_2 = (1, 1, 1, 1; 0.1)_2$ be generalized trapezoidal fuzzy set and generalized 2-norm trapezoidal fuzzy sets respectively. For $\lambda = \frac{1}{2}$, $\lambda(d - c) + (1 - \lambda)(a - b) = 0$ but using equation (3.9) and equation (3.12), it can be easily seen that $I_T^{\frac{1}{2}}(\tilde{A}) > I_T^{\frac{1}{2}}(\tilde{C}_2)$, which contradicts the existing result $I_T^{\frac{1}{2}}(\tilde{A}) < I_T^{\frac{1}{2}}(\tilde{C}_2)$

Example 3.10 Let $\tilde{A} = (5, 7, 8, 9; 0.1)$ and $\tilde{C}_2 = (5, 7, 8, 9; 0.4)_2$ be generalized trapezoidal fuzzy set and generalized 2-norm trapezoidal fuzzy sets respectively. For $\lambda < \frac{1}{2}$, $\lambda(d - c) + (1 - \lambda)(a - b) < 0$ but using equation (3.9) and equation (3.12), it can be easily seen that $I_T^{\frac{1}{2}}(\tilde{A}) < I_T^{\frac{1}{2}}(\tilde{C}_2)$, which contradicts the existing result $I_T^{\frac{1}{2}}(\tilde{A}) > I_T^{\frac{1}{2}}(\tilde{C}_2)$.

3.3 Generalization of existing results

In Section 3.2, using counter examples, the shortcomings in the existing results are pointed out.

In this section, on the basis of the ranking approach, proposed in Chapter 2, new results are proposed in Proposition 3.3 and Proposition 3.4 which are the generalization of results of Proposition 3.1 and Proposition 3.2, discussed in Section 3.3 respectively. In the proposed propositions it is assumed that the heights of fuzzy sets are different.

The results, proposed by Chen and Tang [42], are the particular cases of proposed results and can be obtained by assuming that the height of all fuzzy sets are equal.

Proposition 3.3 Let $\tilde{A} = (a, b, c, d; w_1)$ and $\tilde{B} = (a, e, d; w_2)$ be generalized trapezoidal and generalized triangular fuzzy sets respectively and $-\infty < a \leq b \leq e \leq c \leq d < \infty$. Then

(1) $\Re_L(\tilde{B}^*) \geq \Re_L(\tilde{A}^*)$

(2) $\Re_R(\tilde{A}^*) \geq \Re_R(\tilde{B}^*)$

(3) $\Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{B}^*)$ if $e < c\lambda + (1-\lambda)b$

(4) $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{B}^*)$ if $e = c\lambda + (1-\lambda)b$

(5) $\Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{B}^*)$ if $e > c\lambda + (1-\lambda)b$

where, $\tilde{A}^* = (a, b, c, d; w)$, $\tilde{B}^* = (a, e, d; w)$ and $w = min(w_1, w_2)$.

Proof:- The proposed results can be proved as follows:

(1) Using equation (2.28), $\Re_L(\tilde{A}^*) = \frac{w}{2}(a+b)$ and $\Re_L(\tilde{B}^*) = \frac{w}{2}(a+e)$

Now,

$$\Re_L(\tilde{B}^*) - \Re_L(\tilde{A}^*) = \frac{w}{2}(e-b) \tag{3.13}$$

Since $b \leq e \leq c$, so using equation (3.13), $\Re_L(\tilde{B}^*) - \Re_L(\tilde{A}^*) \geq 0 \Rightarrow \Re_L(\tilde{B}^*) \geq \Re_L(\tilde{A}^*)$

(2) Using equation (2.29), $\Re_R(\tilde{A}^*) = \frac{w}{2}(c+d)$ and $\Re_R(\tilde{B}^*) = \frac{w}{2}(e+d)$

Now,

$$\Re_R(\tilde{B}^*) - \Re_R(\tilde{A}^*) = \frac{w}{2}(e-c) \tag{3.14}$$

Since $e \leq c \leq d$, so using equation (3.14), $\Re_R(\tilde{B}^*) - \Re_R(\tilde{A}^*) \leq 0 \Rightarrow \Re_R(\tilde{B}^*) \leq \Re_R(\tilde{A}^*)$

(3) Using equation (2.32), $\Re_T^\lambda(\tilde{A}^*) = \lambda\Re_L(\tilde{A}^*) + (1-\lambda)\Re_R(\tilde{A}^*) = \frac{w}{2}(\lambda(c+d) + (1-\lambda)(a+b))$ and using equation (2.33), $\Re_T^\lambda(\tilde{B}^*) = \lambda\Re_L(\tilde{B}^*) + (1-\lambda)\Re_R(\tilde{B}^*) =$

$\frac{w}{2}(\lambda d + e + (1-\lambda)a)$

Now,

$$\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) = \frac{w}{2}(c\lambda - e + (1-\lambda)b) \tag{3.15}$$

Since $e < c\lambda + (1-\lambda)b$, so using equation (3.15), $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) > 0 \Rightarrow \Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{B}^*)$

(4) Since $e > c\lambda + (1-\lambda)b$, so using equation (3.15), $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) < 0 \Rightarrow \Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{B}^*)$

(5) Since $e = c\lambda + (1-\lambda)b$, so using equation (3.15), $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) = 0 \Rightarrow \Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{B}^*)$.

Proposition 3.4 Let $\tilde{A} = (a,b,c,d;w_1)$ and $\tilde{C}_2 = (a,b,c,d;w_2)_2$ be generalized trapezoidal and generalized 2-norm trapezoidal fuzzy sets respectively. Then

(1) $\Re_L(\tilde{A}^*) \geq \Re_L(\tilde{C}_2^*)$

(2) $\Re_R(\tilde{A}^*) \leq \Re_R(\tilde{C}_2^*)$

(3) $\Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{C}_2^*)$ if $\lambda(d-c) + (1-\lambda)(a-b) > 0$

(4) $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{C}_2^*)$ if $\lambda(d-c) + (1-\lambda)(a-b) = 0$

(5) $\Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{C}_2^*)$ if $\lambda(d-c) + (1-\lambda)(a-b) < 0$

where $\tilde{A}^* = (a,b,c,d;w)$, $\tilde{C}_2^* = (a,b,c,d;w)_2$ and $w = min(w_1, w_2)$.

Proof:- The proposed results can be proved as follows:

(1) Using equation (2.28),

$\Re_L(\tilde{C}_2^*) = \int_0^w (b + (a-b)(1-(\frac{\alpha}{w})^2)^{\frac{1}{2}})d\alpha = bw + \frac{(a-b)}{4}w\pi$ and $\Re_L(\tilde{A}^*) = \frac{w}{2}(a+b)$

Now,

$$\Re_L(\tilde{C}_2^*) - \Re_L(\tilde{A}^*) = \frac{(\pi-2)}{4}(a-b)w \tag{3.16}$$

Since $a \leq b$, so using equation (3.16), $\Re_L(\tilde{C_2^*}) - \Re_L(\tilde{A}^*) \leq 0 \Rightarrow \Re_L(\tilde{C_2^*}) \leq \Re_L(\tilde{A}^*)$

(2) Using equation (2.29),

$\Re_R(\tilde{C_2^*}) = \int\limits_0^w (c + (d-c)(1-(\frac{\alpha}{w})^2)^{\frac{1}{2}})d\alpha = cw + \frac{(a-b)}{4}w\pi$ and $\Re_R(\tilde{A}^*) = \frac{w}{2}(c+d)$

Now,

$$\Re_R(\tilde{C_2^*}) - \Re_R(\tilde{A}^*) = \frac{(\pi-2)}{4}(d-c)w \tag{3.17}$$

Since $d \geq c$, so using equation (3.17), $\Re_R(\tilde{C_2^*}) - \Re_R(\tilde{A}^*) \geq 0 \Rightarrow \Re_R(\tilde{C_2^*}) \geq \Re_R(\tilde{A}^*)$

(3) Using equation (2.32),

$\Re_T^\lambda(\tilde{C_2^*}) = (1-\lambda)\Re_L(\tilde{C_2^*}) + \lambda\Re_R(\tilde{C_2^*}) = w\{\frac{\pi}{4}[\lambda(d-c) + (1-\lambda)(a-b)] + \lambda c + (1-\lambda)b\}$ and $\Re_T^\lambda(\tilde{A}^*) = \frac{w}{2}(\lambda(c+d) + (1-\lambda)(a+b))$

Now,

$$\Re_T^\lambda(\tilde{C_2^*}) - \Re_T^\lambda(\tilde{A}^*) = \frac{(\pi-2)}{4}w(\lambda(d-c) + (1-\lambda)(a-b)) \tag{3.18}$$

Since $\lambda(d-c)+(1-\lambda)(a-b) > 0$, so using equation (3.18), $\Re_T^\lambda(\tilde{C_2^*}) - \Re_T^\lambda(\tilde{A}^*) > 0 \Rightarrow \Re_T^\lambda(\tilde{C_2^*}) > \Re_T^\lambda(\tilde{A}^*)$

(4) Since $\lambda(d-c)+(1-\lambda)(a-b) < 0$, so using equation (3.18), $\Re_T^\lambda(\tilde{C_2^*}) - \Re_T^\lambda(\tilde{A}^*) < 0 \Rightarrow \Re_T^\lambda(\tilde{C_2^*}) < \Re_T^\lambda(\tilde{A}^*)$

(5) Since $\lambda(d-c)+(1-\lambda)(a-b) = 0$, so using equation (3.18), $\Re_T^\lambda(\tilde{C_2^*}) - \Re_T^\lambda(\tilde{A}^*) = 0 \Rightarrow \Re_T^\lambda(\tilde{C_2^*}) = \Re_T^\lambda(\tilde{A}^*)$

3.3.1 Particular cases

The existing results, presented in Proposition 3.1 and Proposition 3.2 can be easily obtained by using the results, proposed in Proposition 3.3 and Proposition

3.4 respectively, by using the following assumption:

(i) Assuming $w_1 = w_2 = w$ the results, proposed in Proposition 3.3 and Proposition 3.4, will be converted into the existing results, presented in Proposition 3.1 and Proposition 3.2 respectively.

3.4 Limitations of proposed results

In the Section 3.2, limitations of existing results [42] are pointed out and to overcome these limitations new results are proposed in Section 3.3. But the results, proposed in Section 3.3, are valid only for special type of generalized fuzzy sets i.e, results, proposed in Proposition 3.3, are valid only for such generalized fuzzy sets $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, e_2, d_2; w_2)$ in which the conditions $a_1 = a_2$, $d_1 = d_2$ and $-\infty < a_1 = a_2 \leq b_1 \leq e_2 \leq c_1 \leq d_1 = d_2 < \infty$ are satisfied and the results, proposed in Proposition 3.4, are valid only for such generalized fuzzy sets $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{C}_2 = (a_2, b_2, c_2, d_2; w_2)_2$ in which the conditions $a_1 = a_2, b_1 = b_2, c_1 = c_2$ and $d_1 = d_2$ are satisfied.

But the results proposed in Section 3.4, can not be used for such generalized fuzzy sets in which these conditions are not satisfied.

To illustrate the limitations of the proposed results, proposed in Proposition 3.3 and Proposition 3.4, the following counter examples are chosen:

Example 3.11 Let $\tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\tilde{B} = (0.1, 0.2, 0.3; .7)$ be generalized trapezoidal and generalized triangular fuzzy sets respectively. Then using equation (3.1) and equation (3.2), it can be easily seen that then $\Re_L(\tilde{B}^*) < \Re_L(\tilde{A}^*)$, which contradicts the proposed result $\Re_L(\tilde{B}^*) > \Re_L(\tilde{A}^*)$.

Example 3.12 Let $\tilde{A} = (0.1, 0.2, 0.2, 0.5; 1)$ and $\tilde{B} = (0.1, 0.3, 0.5; 1)$ be generalized trapezoidal and generalized triangular fuzzy sets respectively. Then using equation (3.3) and equation (3.4), it can be easily seen that $\Re_R(\tilde{B}^*) > \Re_R(\tilde{A}^*)$, which contradicts the proposed result $\Re_R(\tilde{B}^*) < \Re_R(\tilde{A}^*)$.

Example 3.13 Let $\tilde{A} = (5, 7, 8, 9; 0.4)$ and $\tilde{C}_2 = (6, 7, 8, 12; 0.6)_2$ be generalized trapezoidal and generalized 2-norm trapezoidal fuzzy sets respectively. Then using equation (3.7) and equation (3.10), it can be easily seen that $\Re_L(\tilde{A}^*) < \Re_L(\tilde{C}_2^*)$, which contradicts the proposed result $\Re_L(\tilde{A}^*) > \Re_L(\tilde{C}_2^*)$.

Example 3.14 Let $\tilde{A} = (6, 7, 10, 12; 0.4)$ and $\tilde{C}_2 = (5, 7, 8, 9; 0.1)_2$ be generalized trapezoidal and generalized 2-norm trapezoidal fuzzy sets respectively. Then using equation (3.8) and equation (3.11), it can be easily seen that $\Re_R(\tilde{A}^*) > \Re_R(\tilde{C}_2^*)$, which contradicts the proposed result $\Re_R(\tilde{A}^*) < \Re_R(\tilde{C}_2^*)$

3.5 Generalization of the proposed results

In this section, new results are proposed in Proposition 3.5 and Proposition 3.6 which are the generalization of the results of Proposition 3.3 and Proposition 3.4 respectively.

Proposition 3.5 Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, e_2, d_2; w_2)$ be generalized trapezoidal and generalized triangular fuzzy sets respectively. Then

(1) $\Re_L(\tilde{B}^*) \geq \Re_L(\tilde{A}^*)$ if $(a_2 + e_2) \geq (a_1 + b_1)$

(2) $\Re_R(\tilde{A}^*) \geq \Re_R(\tilde{B}^*)$ if $(c_1 + d_1) \geq (e_2 + d_2)$

(3) $\Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{B}^*)$ if $\lambda(c_1 + d_1) + (1 - \lambda)(a_1 + b_1) > (1 - \lambda)a_2 + e_2 + \lambda d_2$

(4) $\Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{B}^*)$ if $\lambda(c_1 + d_1) + (1 - \lambda)(a_1 + b_1) < (1 - \lambda)a_2 + e_2 + \lambda d_2$

(5) $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{B}^*)$ if $\lambda(c_1 + d_1) + (1 - \lambda)(a_1 + b_1) = (1 - \lambda)a_2 + e_2 + \lambda d_2$

where, $\tilde{A}^* = (a_1, b_1, c_1, d_1; w)$, $\tilde{B}^* = (a_2, e_2, d_2; w_2)$ and $w = min\ (w_1, w_2)$

Proof:- The proposed results can be proved as follows:

(1) Using equation (2.28), $\Re_L(\tilde{A}^*) = \frac{w}{2}(a_1 + b_1)$ and $\Re_L(\tilde{B}^*) = \frac{w}{2}(a_2 + e_2)$

Now,

$$\Re_L(\tilde{B}^*) - \Re_L(\tilde{A}^*) = \frac{w}{2}(a_2 + e_2 - a_1 - b_1) \tag{3.19}$$

Since $(a_2 + e_2) \geq (a_1 + b_1)$, so using equation (3.19), $\Re_L(\tilde{B}^*) - \Re_L(\tilde{A}^*) \geq 0$

$\Rightarrow \Re_L(\tilde{B}^*) \geq \Re_L(\tilde{A}^*)$

(2) Using equation (2.29), $\Re_R(\tilde{A}^*) = \frac{w}{2}(c_1 + d_1)$ and $\Re_R(\tilde{B}^*) = \frac{w}{2}(e_2 + d_2)$

Now,

$$\Re_R(\tilde{A}^*) - \Re_R(\tilde{B}^*) = \frac{w}{2}(c_1 + d_1 - e_2 - d_2) \tag{3.20}$$

Since $(c_1 + d_1) \geq (e_2 + d_2)$, so using equation (3.20), $\Re_R(\tilde{A}^*) - \Re_R(\tilde{B}^*) \geq 0$

$\Rightarrow \Re_R(\tilde{A}^*) \geq \Re_R(\tilde{B}^*)$

(3) Using equation (2.32), $\Re_T^\lambda(\tilde{A}^*) = \frac{w}{2}(\lambda(c_1+d_1)+(1-\lambda)(a_1+b_1))$ and $\Re_T^\lambda(\tilde{B}^*) = \frac{w}{2}(\lambda d_2 + e_2 + (1-\lambda)a_2)$

Now,

$$\Re_T^\lambda(\tilde{A}^*)-\Re_T^\lambda(\tilde{B}^*) = \frac{w}{2}(\lambda(c_1+d_1)+(1-\lambda)(a_1+b_1)-\lambda d_2-e_2-(1-\lambda)a_2) \tag{3.21}$$

Since $\lambda(c_1 + d_1) + (1-\lambda)(a_1 + b_1) \geq (1-\lambda)a_2 + e_2 + \lambda d_2$, so using equation (3.21), $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) \geq 0 \Rightarrow \Re_T^\lambda(\tilde{A}^*) \geq \Re_T^\lambda(\tilde{B}^*)$

(4) Since $\lambda(c_1 + d_1) + (1-\lambda)(a_1 + b_1) \leq (1-\lambda)a_2 + e_2 + \lambda d_2$, so using equation (3.21), $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) \leq 0 \Rightarrow \Re_T^\lambda(\tilde{A}^*) \leq \Re_T^\lambda(\tilde{B}^*)$

(5) Since $\lambda(c_1 + d_1) + (1-\lambda)(a_1 + b_1) = (1-\lambda)a_2 + e_2 + \lambda d_2$, so using equation (3.21), $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{B}^*) = 0 \Rightarrow \Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{B}^*)$

Proposition 3.6 Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{C}_2 = (a_2, b_2, c_2, d_2; w_2)_2$ be generalized trapezoidal and 2-norm trapezoidal fuzzy sets respectively. Then

(1) $\Re_L(\tilde{A}^*) \geq \Re_L(\tilde{C}_2^*)$ if $(a_1 + b_1) \geq \frac{1}{2}(\pi a_2 + (4 - \pi)b_2)$

(2) $\Re_R(\tilde{A}^*) \leq \Re_R(\tilde{C}_2^*)$ if $(c_1 + d_1) \leq \frac{1}{2}(\pi d_2 + (4 - \pi)c_2)$

(3) $\Re_T^\lambda(\tilde{A}) < \Re_T^\lambda(\tilde{C}_2)$ if $\frac{1}{2}(\lambda(c_1 + d_1) + (1 - \lambda)(a_1 + b_1)) < \frac{\pi}{4}(\lambda(d_2 - c_2) + (1 - \lambda)(a_2 - b_2)) + \lambda c_2 + (1 - \lambda)b_2$

(4) $\Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{C}_2^*)$ if $\frac{1}{2}(\lambda(c_1 + d_1) + (1 - \lambda)(a_1 + b_1)) > \frac{\pi}{4}(\lambda(d_2 - c_2) + (1 - \lambda)(a_2 - b_2)) + \lambda c_2 + (1 - \lambda)b_2$

(5) $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{C}_2^*)$ if $\frac{1}{2}(\lambda(c_1 + d_1) + (1 - \lambda)(a_1 + b_1)) = \frac{\pi}{4}(\lambda(d_2 - c_2) + (1 - \lambda)(a_2 - b_2)) + \lambda c_2 + (1 - \lambda)b_2$

where, $\tilde{A}^* = (a_1, b_1, c_1, d_1; w)$, $\tilde{C}_2^* = (a_2, b_2, c_2, d_2; w)_2$ and $w = min\ (w_1, w_2)$

Proof:- The proposed results can be proved as follows:

(1) Using equation (2.28), $\Re_L(\tilde{A}^*) = \frac{w}{2}(a_1 + b_1)$ and $\Re_L(\tilde{C}_2^*) = b_2 w + (\frac{a_2 - b_2}{4})\pi w$

Now,

$$\Re_L(\tilde{A}^*) - \Re_L(\tilde{C}_2^*) = \frac{w}{2}(a_1 + b_1) - (b_2 w + (\frac{a_2 - b_2}{4})\pi w) \tag{3.22}$$

Since $(a_1 + b_1) \geq \frac{1}{2}(\pi a_2 + (4 - \pi)b_2)$, so using equation (3.22), $\Re_L(\tilde{A}^*) - \Re_L(\tilde{C}_2^*) \geq 0 \Rightarrow \Re_L(\tilde{A}^*) \geq \Re_L(\tilde{C}_2^*)$

(2) Using equation (2.29), $\Re_R(\tilde{A}^*) = \frac{w}{2}(c_1 + d_1)$ and $\Re_R(\tilde{C}_2^*) = c_2 w + (\frac{d_2 - c_2}{4})\pi w$

Now,

$$\Re_R(\tilde{A}^*) - \Re_R(\tilde{C}_2^*) = \frac{w}{2}(c_1 + d_1) - (c_2 w + (\frac{d_2 - c_2}{4})\pi w) \tag{3.23}$$

Since $c_1 + d_1 \leq \frac{1}{2}(d_2\pi + (4 - \pi)c_2)$, so using equation (3.23), $\Re_R(\tilde{A}^*) - \Re_R(\tilde{C}_2^*) \leq 0 \Rightarrow \Re_R(\tilde{A}^*) \leq \Re_R(\tilde{C}_2^*)$

(3) Using equation (2.32), $\Re_T^\lambda(\tilde{C}_2^*) = w\{\frac{\pi}{4}[\lambda(d_2-c_2)+(1-\lambda)(a_2-b_2)]+\lambda c_2+(1-\lambda)b_2$ and $\Re_T^\lambda(\tilde{A}^*) = \frac{w}{2}(\lambda(c_1+d_1)+(1-\lambda)(a_1+b_1))$

Now, $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{C}_2^*)$

$$= (\frac{w}{2}(\lambda(c_1+d_1)+(1-\lambda)(a_1+b_1)))-(w\{\frac{\pi}{4}[\lambda(d_2-c_2)+(1-\lambda)(a_2-b_2)]+\lambda c_2+(1-\lambda)b_2) \tag{3.24}$$

Since $\frac{1}{2}(\lambda(c_1+d_1)+(1-\lambda)(a_1+b_1)) < \frac{\pi}{4}(\lambda(d_2-c_2)+(1-\lambda)(a_2-b_2))+\lambda c_2+(1-\lambda)b_2$, so using equation (3.24), $\Re_T^\lambda(\tilde{A}^*)-\Re_T^\lambda(\tilde{C}_2^*) < 0 \Rightarrow \Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{C}_2^*)$

(4) Since $\frac{1}{2}(\lambda(c_1+d_1)+(1-\lambda)(a_1+b_1)) < \frac{\pi}{4}(\lambda(d_2-c_2)+(1-\lambda)(a_2-b_2))+\lambda c_2+(1-\lambda)b_2$, so using equation (3.24), $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{C}_2^*) > 0$

$\Rightarrow \Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{C}_2^*)$

(5) Since $\frac{1}{2}(\lambda(c_1+d_1)+(1-\lambda)(a_1+b_1)) = \frac{\pi}{4}(\lambda(d_2-c_2)+(1-\lambda)(a_2-b_2))+\lambda c_2+(1-\lambda)b_2$, so using equation (3.24), $\Re_T^\lambda(\tilde{A}^*) - \Re_T^\lambda(\tilde{C}_2^*) = 0$

$\Rightarrow \Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{C}_2^*)$

3.5.1 Particular cases

The existing results, presented in Section 3.3, and the results, proposed in the Section 3.4 can be easily obtained by using the results, proposed in Section 3.6, by using the following assumptions:

(i) Assuming $a_1 = a_2 = a, b_1 = b, c_1 = c, d_1 = d_2 = d, e_2 = e$ and $w_1 = w_2 = w$ the results, proposed in Proposition 3.5 and Proposition 3.6 are converted into the existing results, presented in Proposition 3.1 and Proposition 3.2 respectively.

(ii) Assuming $a_1 = a_2 = a, b_1 = b, c_1 = c, d_1 = d_2 = d$ and $e_2 = e$ the results, proposed in Proposition 3.5 and Proposition 3.6 are converted into results proposed in Proposition 3.3 and Proposition 3.4 respectively.

3.6 Linear programming problems with generalized p-norm trapezoidal fuzzy sets

If in the linear programming problem with generalized fuzzy sets (P_2), defined in Section 2.2 of Chapter 2, the parameters $\tilde{c}_j$ are represented by generalized p-norm trapezoidal fuzzy sets $(c_{j1}, c_{j2}, c_{j3}, c_{j4}; w_j)_p$ then linear programming problem (P_2) can be converted into the linear programming problem with generalized p-norm trapezoidal fuzzy sets (P_{11}):

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n}(c_{j1}, c_{j2}, c_{j3}, c_{j4}; w_j)_p x_j)\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \;\; i = 1, 2, ..., m \qquad\qquad (P_{11})\\
&\qquad x_j \geq 0
\end{aligned}$$

Example 3.15 Solve

$$\begin{aligned}
&Minimize\ \ ((2, 4, 8, 10; 0.8)_2 x_1 \oplus (1, 3, 5, 7; 0.7)_2 x_2)\\
&subject\ to\\
&\qquad x_1 + x_2 \leq 3\\
&\qquad x_1 + x_2 \geq 2\\
&\qquad x_1 \geq 3\\
&\qquad x_1, x_2 \geq 0
\end{aligned}$$

Solution:-Using the method, proposed in Chapter 2, with proposed comparing results the linear programming problem with generalized 2-norm trapezoidal fuzzy sets, chosen in Example 3.15, can be solved by using the following steps:

Step 1 Using Step 2 of the method, proposed in Section 2.8, the chosen linear programming problem can be written as

$$Minimize\ \ (\Re_T^{0.5}((2, 4, 8, 10; 0.7)_2 x_1 \oplus (1, 3, 5, 7; 0.7)_2 x_2))$$

subject to

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$x_1 \geq 3$$

$$x_1, x_2 \geq 0$$

Step 2 Using Step 3 of the method, proposed in Section 2.8, the chosen linear programming problem can be written as

$$Minimize \ \ (\Re_T^{0.5}((2,4,8,10;0.7)_2 x_1) + \Re_T^{0.5}((1,3,5,7;0.7)_2 x_2))$$

subject to

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$x_1 \geq 3$$

$$x_1, x_2 \geq 0$$

Step 3 Using $\Re_T^{0.5}(a,b,c,d;w)_2 = w((\frac{a-b-c+d}{2})\frac{\pi}{4} + \frac{b+c}{2})$, the crisp linear programming problem, obtained in Step 2, can be written as

$$Maximize \ \ (\tfrac{21}{5}x_1 + \tfrac{14}{5}x_2)$$

subject to

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$x_1 \geq 3$$

$$x_1, x_2 \geq 0$$

Step 4 On solving the crisp linear programming problem, obtained in Step 3, the obtained optimal solution is $x_1 = 3$ and $x_2 = 0$. Putting $x_1 = 3$ and $x_2 = 0$ in $((2,4,8,10;0.8)_2 x_1 \oplus (1,3,5,7;0.7)_2 x_2)$ the fuzzy optimal value of the chosen problem is $(6, 12, 24, 30; 0.7)_2$.

3.7 Conclusions

In this chapter, the limitations of the existing results [42] are pointed out and some new results are proposed. It is shown that the existing results are particular cases of the proposed results. Also, a linear programming problem with generalized 2-norm trapezoidal fuzzy sets is solved.

Chapter 4

RM Ranking Approach For Solving Linear Programming Problems With Generalized Trapezoidal Fuzzy Sets[1]

In Chapter 2, a new ranking approach is proposed for comparing generalized fuzzy sets in which only rank is used for comparing generalized fuzzy sets. In this chapter, it is shown that only rank is not sufficient for comparing generalized fuzzy sets and to overcome this limitation, a new ranking approach, named as RM ranking approach, is proposed by modifying the ranking approach proposed in Chapter 2. Also, on the basis of the proposed RM ranking approach, a new method is proposed for solving linear programming problems with generalized trapezoidal fuzzy sets. The proposed method is illustrated by solving a numerical example.

4.1 Shortcomings of existing and proposed ranking approaches

In all the existing ranking approaches and ranking approach, proposed in Chapter 2, only rank is used for comparing fuzzy sets. In this section, it is shown that

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only rank is not sufficient for comparing fuzzy sets.

Let $\tilde{A} = (a_1, a_2, a_3, a_4; w)$ be a generalized trapezoidal fuzzy set then $\tilde{A} \ominus \tilde{A} = (a_1 - a_4, a_2 - a_3, a_3 - a_2, a_4 - a_1; w)$ i.e., $\tilde{A} \ominus \tilde{A} \neq (0, 0, 0, 0; w)$. On the basis of this property of fuzzy sets it can be concluded that two generalized trapezoidal fuzzy sets $\tilde{A} = (a_1, a_2, a_3, a_4; w_1)$ and $\tilde{B} = (b_1, b_2, b_3, b_4; w_2)$ will be equivalent if for $\tilde{A} \ominus \tilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1; \min(w_1, w_2))$ the following two properties are satisfied:

(i) $a_1 - b_4 = -(a_4 - b_1)$

(ii) $a_2 - b_3 = -(a_3 - b_2)$

For example, if $\tilde{A}$ and $\tilde{B}$ are any two generalized trapezoidal fuzzy sets such that $\tilde{A} \ominus \tilde{B} = (-4, -2, 2, 4; w)$ then $\tilde{A} \sim \tilde{B}$.

Similarly two generalized triangular fuzzy sets $\tilde{A} = (a_1, a_2, a_3; w_1)$ and $\tilde{B} = (b_1, b_2, b_3; w_2)$ will be equivalent if for $\tilde{A} \ominus \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1; \min(w_1, w_2))$ the following properties are satisfied:

(i) $a_2 - b_2 = 0$

(ii) $a_1 - b_3 = -(a_3 - b_1)$

According to existing ranking approaches and ranking approach, proposed in Chapter 2, if $\tilde{A}$ and $\tilde{B}$ are any two generalized fuzzy sets such that $\Re_T^{\lambda}(\tilde{A}) = \Re_T^{\lambda}(\tilde{B})$ then $\tilde{A} \sim \tilde{B}$ but there may exist several generalized fuzzy sets for which $\Re_T^{\lambda}(\tilde{A}) = \Re_T^{\lambda}(\tilde{B})$ but for $\tilde{A} \ominus \tilde{B}$ the above two properties are not satisfied, e.g, if $\tilde{A} = (0, 2, 4; 1)$ and $\tilde{B} = (0, 1, 6; 1)$ are two generalized triangular fuzzy sets then according to ranking approach, proposed in Chapter 2, $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B}) = 2$ but $\tilde{A} \ominus \tilde{B} = (-6, 1, 4; 1)$ i.e., $\tilde{A} \ominus \tilde{B} \neq (-a, 0, a; w)$.

Similarly if $\tilde{A} = (1, 2, 3, 6; 1)$ and $\tilde{B} = (1, 2, 4, 5; 1)$ are two generalized trapezoidal fuzzy sets then according to ranking approach, proposed in Chapter 2, $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ but $\tilde{A} \ominus \tilde{B} = (-4, -2, 1, 5; 1)$ i.e., $\tilde{A} \ominus \tilde{B} \neq (-a, -b, b, a; w)$.

On the basis of above discussion the following conclusions can be drawn :

If $\tilde{A}$ and $\tilde{B}$ are two generalized triangular fuzzy sets then

(i) $\Re_T^{\lambda}(\tilde{A}) = \Re_T^{\lambda}(\tilde{B}) \not\Rightarrow \tilde{A} \ominus \tilde{B} = (-a, 0, a; w)$

(ii) $\tilde{A} \ominus \tilde{B} = (-a, 0, a; w) \Rightarrow \Re_T^{\lambda}(\tilde{A}) = \Re_T^{\lambda}(\tilde{B})$.

Similarly, if $\tilde{A}$ and $\tilde{B}$ are two generalized trapezoidal fuzzy sets then

(i) $\Re_T^{\lambda}(\tilde{A}) = \Re_T^{\lambda}(\tilde{B}) \not\Rightarrow \tilde{A} \ominus \tilde{B} = (-a, -b, b, a; w)$

(ii) $\tilde{A} \ominus \tilde{B} = (-a, -b, b, a; w) \Rightarrow \Re_T^{\lambda}(\tilde{A}) = \Re_T^{\lambda}(\tilde{B})$

i.e., it is not correct to check the equality relation on the basis of rank only. Hence, to overcome these shortcomings, in the next section, a new ranking approach, named as RM ranking approach, is proposed by modifying the ranking approach, proposed in Chapter 2.

4.2 RM ranking approach

In this section, a new ranking approach, named as RM ranking approach, is proposed for comparing generalized trapezoidal fuzzy sets.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy sets then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Transform $\tilde{A}$, $\tilde{B}$ into $\tilde{A}^*$, $\tilde{B}^*$ as follows:

$$\tilde{A}^* = (a_1, b_1, c_1, d_1; w), \tilde{B}^* = (a_2, b_2, c_2, d_2; w) \text{ where, } w = min(w_1, w_2) \tag{4.1}$$

Step 2 Calculate

$$\Re_T^{\lambda}(\tilde{A}^*) = \frac{w}{2}\{(\lambda(a_1 + b_1) + (1 - \lambda)(c_1 + d_1))\} \tag{4.2}$$

and

$$\Re_T^\lambda(\tilde{B}^*) = \frac{w}{2}\{(\lambda(a_2 + b_2) + (1-\lambda)(c_2 + d_2))\} \tag{4.3}$$

Step 3 Chose a particular value of λ, and check that $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{B}^*)$ or $\Re_T^\lambda(\tilde{A}^*) \neq \Re_T^\lambda(\tilde{B}^*)$

Case (i) If $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{B}^*)$ then Go to Step 4.

Case (ii) If $\Re_T^\lambda(\tilde{A}^*) \neq \Re_T^\lambda(\tilde{B}^*)$ then check that $\Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{B}^*)$ or $\Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{B}^*)$.

Case (a) If $\Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{B}^*)$ then $\tilde{A} \succ \tilde{B}$

Case (b) If $\Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{B}^*)$ then $\tilde{A} \prec \tilde{B}$

Step 4 Calculate

$$\mathrm{RM}^\lambda(\tilde{A}^*) = \mathrm{Mode}^\lambda(\tilde{A}^*) = w(\lambda b_1 + (1-\lambda)c_1) \tag{4.4}$$

and

$$\mathrm{RM}^\lambda(\tilde{B}^*) = \mathrm{Mode}^\lambda(\tilde{B}^*) = w(\lambda b_2 + (1-\lambda)c_2) \tag{4.5}$$

Step 5 Check that for the chosen value of λ, $\mathrm{RM}^\lambda(\tilde{A}^*) > \mathrm{RM}^\lambda(\tilde{B}^*)$ or $\mathrm{RM}^\lambda(\tilde{A}^*) < \mathrm{RM}^\lambda(\tilde{B}^*)$ or $\mathrm{RM}^\lambda(\tilde{A}^*) = \mathrm{RM}^\lambda(\tilde{B}^*)$

Case (i) If $\mathrm{RM}^\lambda(\tilde{A}^*) > \mathrm{RM}^\lambda(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \succ \tilde{B}$ for same value of λ.

Case (ii) If $\mathrm{RM}^\lambda(\tilde{A}^*) < \mathrm{RM}^\lambda(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \prec \tilde{B}$ for same value of λ.

Case (iii) If $\mathrm{RM}^\lambda(\tilde{A}^*) = \mathrm{RM}^\lambda(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \sim \tilde{B}$ for same value of λ.

4.2.1 Illustrative examples

In this section, to illustrate the proposed RM ranking approach, some normal and generalized trapezoidal fuzzy sets are compared by using the proposed RM ranking approach.

Example 4.1 Let $\tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\tilde{B} = (0.1, 0.2, 0.3, 0.4; 0.7)$ be two generalized trapezoidal fuzzy sets. Then $\tilde{A}$ and $\tilde{B}$ can be compared by using the following steps:

Step 1 Using equation (4.1),

$\tilde{A}^* = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\tilde{B}^* = (0.1, 0.2, 0.3, 0.4; 0.35)$

Step 2 Using equation (4.2) and (4.3),

$\Re_T^\lambda(\tilde{A}^*) = 0.35(0.3\lambda + (1-\lambda)0.7)$ and $\Re_T^\lambda(\tilde{B}^*) = 0.175(0.3\lambda + (1-\lambda)0.7)$

Step 3 For a pessimistic decision maker i.e. for $\lambda = 0$,

$\Re_T^0(\tilde{A}^*) = 0.245$, $\Re_T^0(\tilde{B}^*) = 0.1225$. Since $\Re_T^0(\tilde{A}^*) > \Re_T^0(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

For optimistic decision maker i.e. for $\lambda = 1$,

$\Re_T^1(\tilde{A}^*) = 0.105$, $\Re_T^1(\tilde{B}^*) = 0.0525$. Since $\Re_T^1(\tilde{A}^*) > \Re_T^1(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

For moderate decision maker i.e. for $\lambda = 0.5$,

$\Re_T^{0.5}(\tilde{A}^*) = 0.175$, $\Re_T^{0.5}(\tilde{B}^*) = 0.0875$. Since $\Re_T^{0.5}(\tilde{A}^*) > \Re_T^{0.5}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

Example 4.2

Let $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B} = (0.1, 0.2, 0.3, 0.6; 1)$ be two generalized trapezoidal fuzzy sets. Then $\tilde{A}$ and $\tilde{B}$ can be compared by using the following steps:

Step 1 Using equation (4.1),

$\tilde{A}^* = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B}^* = (0.1, 0.2, 0.3, 0.5; 1)$

Step 2 Using equation (4.2) and (4.3),

$\Re_T^\lambda(\tilde{A}^*) = (0.15\lambda + (1-\lambda)0.45)$ and $\Re_T^\lambda(\tilde{B}^*) = (0.15\lambda + (1-\lambda)0.45)$

Step 3 Since $\Re_T^{\lambda}(\tilde{A}^*) = \Re_T^{\lambda}(\tilde{B}^*) \quad \forall\ \lambda$ so go to Step 4

Step 4 Using equation (4.4) and (4.5),

$\mathrm{RM}^{\lambda}(\tilde{A}^*) = 0.2\lambda + (1-\lambda)0.4$ and $\mathrm{RM}^{\lambda}(\tilde{B}^*) = 0.2\lambda + (1-\lambda)0.3$

Step 5 For a pessimistic decision maker i.e. for $\lambda = 0$,

$\mathrm{RM}^0(\tilde{A}^*) = 0.4$ and $\mathrm{RM}^0(\tilde{B}^*) = 0.3$. Since $\mathrm{RM}^0(\tilde{A}^*) > \mathrm{RM}^0(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

For optimistic decision maker i.e. for $\lambda = 1$,

$\mathrm{RM}^1(\tilde{A}^*) = 0.2$ and $\mathrm{RM}^1(\tilde{B}^*) = 0.2$. Since $\mathrm{RM}^1(\tilde{A}^*) = \mathrm{RM}^1(\tilde{A}^*) \Rightarrow \tilde{A} \sim \tilde{B}$

For moderate decision maker i.e. for $\lambda = 0.5$,

$\mathrm{RM}^{0.5}(\tilde{A}^*) = 0.3$ and $\mathrm{RM}^0(\tilde{B}^*) = 0.25$. Since $\mathrm{RM}^{0.5}(\tilde{A}^*) > \mathrm{RM}^{0.5}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

Example 4.3

Let $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 0.1)$ and $\tilde{B} = (1, 1, 1, 1; 0.1)$ be two generalized fuzzy sets. Then $\tilde{A}$ and $\tilde{B}$ can be compared by using the following steps:

Step 1 Using equation (4.1),

$\tilde{A}^* = (0.1, 0.2, 0.4, 0.5; 0.1)$ and $\tilde{B}^* = (1, 1, 1, 1; 0.1)$

Step 2 Using equation (4.2) and (4.3),

$\Re_T^{\lambda}(\tilde{A}^*) = (0.03\lambda + (1-\lambda)0.09)$ and $\Re_T^{\lambda}(\tilde{B}^*) = (0.02\lambda + (1-\lambda)0.02)$.

Step 3 For a pessimistic decision maker i.e. for $\lambda = 0$,

$\Re_T^0(\tilde{A}^*) = 0.09$ and $\Re_T^0(\tilde{B}^*) = 0.02$. Since $\Re_T^0(\tilde{A}^*) > \Re_T^0(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

For optimistic decision maker i.e. for $\lambda = 1$,

$\Re_T^1(\tilde{A}^*) = 0.03$ and $\Re_T^1(\tilde{B}^*) = 0.02$

Since $\Re_T^1(\tilde{A}^*) > \Re_T^1(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

For moderate decision maker i.e. for $\lambda = 0.5$,

$\Re_T^{0.5}(\tilde{A}^*) = 0.06$ and $\Re_T^{0.5}(\tilde{B}^*) = 0.02$. Since $\Re_T^{0.5}(\tilde{A}^*) > \Re_T^{0.5}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

4.3 Proposed method based on RM ranking approach

In this section by modifying the method, proposed in Chapter 2, a new method, based on proposed RM ranking approach, is proposed for solving linear programming problems with generalized trapezoidal fuzzy sets. The steps of the proposed method are as follows:

Step 1 Use Step 1 to Step 6 of the method, proposed in Chapter 2, to solve the chosen linear programming problem with generalized fuzzy sets (P_2) and check that alternative optimal solution exist or not.

Case (i) If there does not exist any alternative optimal solution then obtained solution $\{x_j\}$ is the optimal solution and find the fuzzy optimal value by putting the values of x_j in $\sum\limits_{j=1}^{n}(\tilde{c}_j x_j)$.

Case (ii) If alternative optimal solution exist then Go to Step 2.

Step 2 Solve the crisp linear programming problem (P_{12})

$$
\begin{aligned}
&\mathit{Maximize}\ (\mathit{or}\ \mathit{Minimize})(\ (\sum_{j=1}^{n} \mathrm{Mode}^{0.5}(\tilde{c}_j x_j)))\\
&\mathit{subject\ to}\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i,\ \ i = 1, 2, ..., m \qquad\qquad (P_{12})\\
&\qquad (\sum_{j=1}^{n} \Re_T^{0.5}(\tilde{c}_j x_j)) = a\\
&\qquad x_j \geq 0
\end{aligned}
$$

where, a is the optimal value of the crisp linear programming problem (P_{12}).

The obtained solution is optimal solution and fuzzy optimal value can be obtained by putting the values of x_j in $\sum\limits_{j=1}^{n}(\tilde{c}_j x_j)$.

4.3.1 Illustrative examples

To illustrate the proposed method based on RM ranking approach, a linear programming problem with generalized trapezoidal fuzzy sets, chosen in Example 4.6, is solved by using the proposed method based on RM ranking approach.

Example 4.4 Solve

$$Maximize \ \ ((2,4,5,6;0.9)x_1 \oplus (2,3,5,7;0.8)x_2)$$

$$subject\ to$$

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Solution:-The linear programming problem with trapezoidal fuzzy sets, chosen in Example 4.4, can be solved by using the following steps of the proposed method:

Step 1 Since on solving the chosen linear programming problem with generalized trapezoidal fuzzy sets by using the method, proposed in Section 2.8, alternative optimal solution is obtained and the optimal value of the crisp linear programming problem corresponding to crisp linear programming problem (P_{10}) is $\frac{17}{5}$ so according to Case (ii) of Step 1, of the proposed method the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$$Maximize \ \ (\text{Mode}^{0.5}((2,4,5,6;0.8)x_1 + \text{Mode}^{0.5}(2,3,5,7;0.8)x_2))$$

$$subject\ to$$

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$\tfrac{17}{5}x_1 + \tfrac{17}{5}x_2 = \tfrac{17}{5}$$

$$x_1, x_2 \geq 0$$

Step 2 Using $\text{Mode}^{0.5}(a,b,c,d;w) = \frac{(b+c)}{2}$, the crisp linear programming problem obtained in Step 1, can be written as

$$Maximize\ \ (\tfrac{18}{5}x_1 + \tfrac{16}{5}x_2)$$

$$subject\ to$$

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$\tfrac{17}{5}x_1 + \tfrac{17}{5}x_2 = \tfrac{17}{5}$$

$$x_1, x_2 \geq 0$$

Step 3 On solving the crisp linear programming problem, obtained in Step 2, the obtained optimal solution is $x_1 = 3$ and $x_2 = 0$. Putting $x_1 = 3$ and $x_2 = 0$ in $((2,4,5,6;0.9)x_1 \oplus (2,3,5,7;0.8)x_2)$ the fuzzy optimal value of the chosen problem is $(6,12,15,18;0.8)$.

4.4 Conclusions

In this chapter, the shortcomings of existing ranking approaches and the ranking approach, proposed in Chapter 2, are pointed out and a new ranking approach, named as RM ranking approach is proposed for comparing generalized trapezoidal fuzzy sets. Also, a new method, on the basis of RM ranking approach, is proposed for solving linear programming problems with generalized trapezoidal fuzzy sets. To illustrate the proposed method a linear programming problem with generalized trapezoidal fuzzy set is solved.

Chapter 5

A NEW RANKING APPROACH FOR SOLVING LINEAR PROGRAMMING PROBLEMS WITH TRAPEZOIDAL VAGUE SETS [1]

In real life, a person may assume that an object belongs to a set, but it is possible that he is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use intuitionistic fuzzy set [9] or vague set [63]. Bustince and Burillo [23] pointed out that the notion of vague set is the same as that of intuitionistic fuzzy set.

Li [94] pointed out the shortcomings of all the existing ranking approaches for comparing intuitionistic fuzzy sets and proposed some results and a new approach for comparing intuitionistic fuzzy sets.

In this chapter, the limitations of the existing results [94] and the shortcomings of the existing ranking approach [94] are pointed out. To overcome the limitations of the existing results, some new results are proposed by modifying the existing

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[1] A part of this chapter has appeared in *Journal of Advances in Soft Computing and Its Applications* 2 (2010) 221-230 and some part of this chapter has been communicated in *Applied Soft Computing.*

results and to overcome the shortcomings of the existing ranking approach [94]. Also, a new approach is proposed for comparing trapezoidal vague sets. To show the application of proposed ranking approach, a linear programming problem with trapezoidal vague sets is solved.

5.1 Preliminaries

In this section some basic definitions and arithmetic operations are presented.

5.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 5.1 [9] An intuitionistic fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) | x \in X\}$ on the universal set X is characterized by a truth membership function $\mu_{\tilde{A}}, \mu_{\tilde{A}} : X \to [0,1]$ and a false membership function $\nu_{\tilde{A}}, \nu_{\tilde{A}} : X \to [0,1]$. The values $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ represents the degree of membership and degree of non-membership for $x \in X$ and always satisfies the condition $\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1 \ \forall \ x \in X$. The value $(1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x))$ represents the degree of hesitation for $x \in X$.

Definition 5.2 [9] Let $\tilde{A}$ be an intuitionistic fuzzy set then $A^{(\alpha,\beta)} = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha, \nu_{\tilde{A}}(x) \leq \beta, \alpha, \beta \in [0,1]\}$ is said to be an (α, β)-cut of $\tilde{A}$.

Definition 5.3 [9] An intuitionistic fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, denoted as $\tilde{A} =< [(a, b, c); w, u] >$, is said to be triangular intuitionistic fuzzy set, if degree of membership $\mu_{\tilde{A}}(x)$ and degree of non-membership $\nu_{\tilde{A}}(x)$ are given by:

$$\mu_{\tilde{A}}(x)=\begin{cases} \frac{w(x-a)}{(b-a)}, & a \le x < b \\ w \quad , & x=b \\ \frac{w(c-x)}{(c-b)}, & b < x \le c \\ 0 \quad , & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{\tilde{A}}(x)=\begin{cases} \frac{(b-x+u(x-a))}{(b-a)}, & a \le x < b \\ u \quad , & x=b \\ \frac{(x-b+u(c-x))}{(c-b)}, & b < x \le c \\ 1 \quad , & \text{otherwise} \end{cases}$$

respectively, where, $w = supremum\{\mu_{\tilde{A}}(x) : x \in R\}$ and $u = infimum\{\nu_{\tilde{A}}(x) : x \in R\}$.

Definition 5.4 [9] Let $\tilde{A} =< [(a_1, a_2, a_3); w, u] >$ be a triangular intuitionistic fuzzy set then the (α, β)-cut $A^{(\alpha,\beta)}$ for the triangular intuitionistic fuzzy set $\tilde{A}$ can be defined as follows:

$$A^{(\alpha,\beta)} = \{[a_1 + (a_2 - a_1)\frac{\alpha}{w}, a_3 - (a_3 - a_2)\frac{\alpha}{w}]; [a_2 - (a_2 - a_1)\frac{\beta}{u}, a_2 + (a_3 - a_2)\frac{\beta}{u}]\}$$

Definition 5.5 [63] A vague set $\tilde{A} =< [x; \mu_{\tilde{A}}(x), (1 - \nu_{\tilde{A}}(x))|x \in X] >$, defined on the universal set X, is characterized by a truth membership function $\mu_{\tilde{A}}$, $\mu_{\tilde{A}} : X \to [0, 1]$ and complement of a false membership function $(1 - \nu_{\tilde{A}})$, $(1 - \nu_{\tilde{A}}) : X \to [0, 1]$. The values $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ represents the degree of membership and degree of non-membership for $x \in X$ and always satisfies the condition $\mu_{\tilde{A}}(x) \le (1 - \nu_{\tilde{A}}(x))\ \forall\ x \in X$. The value $(1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x))$ represents the degree of hesitation for $x \in X$.

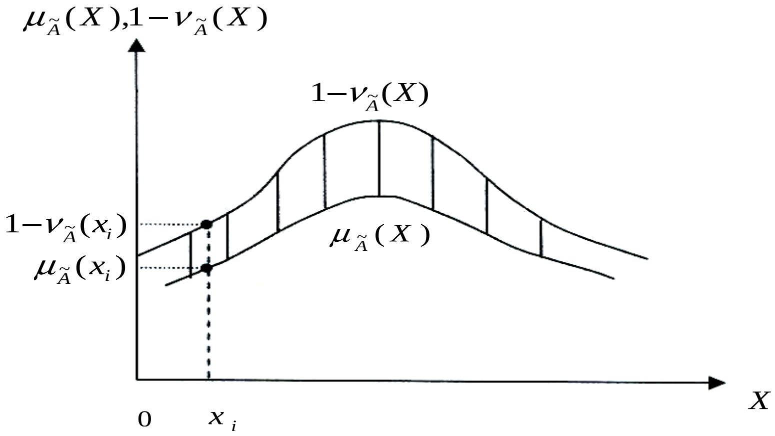

Fig 5.1 Vague Set

Definition 5.6 [63] Let $\tilde{A}$ be a vague set then $A^{(\alpha,\beta)} = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha, (1-\nu_{\tilde{A}}(x)) \geq \beta, \alpha, \beta \in [0,1]\}$ is said to be an (α,β)-cut of $\tilde{A}$.

Definition 5.7 [30] A vague set $\tilde{A}$, defined on the universal set of real numbers R, denoted as $\tilde{A} = < [(a,b,c);\delta,\rho] >$, where $a \leq b \leq c$ and $\delta \leq \rho$, is said to be a triangular vague set if degree of membership, $\mu_{\tilde{A}}(x)$, and complement of the degree of non-membership, $(1-\nu_{\tilde{A}}(x))$, are given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{\delta(x-a)}{(b-a)} & , \quad a \leq x < b \\ \delta & , \quad x = b \\ \frac{\delta(x-c)}{(b-c)} & , \quad b < x \leq c \\ 0 & , \quad \text{otherwise} \end{cases} \quad \text{and} \quad (1-\nu_{\tilde{A}}(x)) = \begin{cases} \frac{\rho(x-a)}{(b-a)} & , \quad a \leq x < b \\ \rho & , \quad x = b \\ \frac{\rho(x-c)}{(b-c)} & , \quad b < x \leq c \\ 0 & , \quad \text{otherwise} \end{cases}$$

where, $\delta = supremum\{\mu_{\tilde{A}}(x) : x \in R\}$ and $\rho = supremum\{(1-\nu_{\tilde{A}}(x) : x \in R\}$

Definition 5.8 [30] Let $\tilde{A} =< [(a_1,a_2,a_3);\delta,\rho] >$ be a triangular vague set then the (α,β)-cut $A^{(\alpha,\beta)}$ for the triangular vague set can be defined as follows:

$$A^{(\alpha,\beta)} = \{[a_1 + (a_2 - a_1)(\frac{\alpha}{\delta}), a_3 - (a_3 - a_2)(\frac{\alpha}{\delta})]; [a_1 + (a_2 - a_1)(\frac{\beta}{\rho}), a_3 - (a_3 - a_2)(\frac{\beta}{\rho})]\}$$

Definition 5.9 [30] A vague set $\tilde{A}$, defined on the universal set of real numbers R, denoted as $\tilde{A} = < [(a,b,c,d);\delta,\rho] >$, where $a \leq b \leq c \leq d$ and $\delta \leq \rho$, is said to be a trapezoidal vague set if degree of membership, $\mu_{\tilde{A}}(x)$, and complement of the degree of non-membership, $(1-\nu_{\tilde{A}}(x))$, are given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{\delta(x-a)}{(b-a)} & , \quad a \leq x < b \\ \delta & , \quad b \leq x \leq c \\ \frac{\delta(x-d)}{(c-d)} & , \quad c < x \leq d \\ 0 & , \quad \text{otherwise} \end{cases} \quad \text{and} \quad (1-\nu_{\tilde{A}}(x)) = \begin{cases} \frac{\rho(x-a)}{(b-a)} & , \quad a \leq x < b \\ \rho & , \quad b \leq x \leq c \\ \frac{\rho(x-d)}{(c-d)} & , \quad c < x \leq d \\ 0 & , \quad otherwise \end{cases}$$

where, $\delta = supremum\{\mu_{\tilde{A}}(x) : x \in R\}$ and $\rho = supremum\{(1-\nu_{\tilde{A}}(x)) : x \in R\}$

Definition 5.10 [30] Let $\tilde{A} =< [(a_1,a_2,a_3,a_4);\delta,\rho] >$ be a trapezoidal vague set then the (α, β)-cut $A^{(\alpha,\beta)}$ for the trapezoidal vague set $\tilde{A}$ can be defined as follows:

$$A^{(\alpha,\beta)} = \{[a_1 + (a_2 - a_1)(\frac{\alpha}{\delta}), a_4 - (a_4 - a_3)(\frac{\alpha}{\delta})]; [a_1 + (a_2 - a_1)(\frac{\beta}{\rho}), a_4 - (a_4 - a_3)(\frac{\beta}{\rho})]\}$$

5.1.2 Arithmetic operations

In this section, some arithmetic operations between triangular intuitionistic fuzzy sets, triangular vague sets and trapezoidal vague sets, defined on universal set of real numbers R, are presented.

5.1.2.1 Arithmetic operations between triangular intuitionistic fuzzy sets

In this section, some arithmetic operations between triangular intuitionistic fuzzy sets, defined on universal set of real numbers R, are presented 87-96.

Let $\tilde{A} = < [(a_1, b_1, c_1); w_1; u_1] >$ and $\tilde{B} = < [(a_2, b_2, c_2); w_2, u_2] >$ be two triangular intuitionistic fuzzy set. Then

(i) $\tilde{A} \oplus \tilde{B} =< [(a_1 + a_2, b_1 + b_2, c_1 + c_2); min\ (w_1, w_2), max(u_1, u_2)] >$

(ii) $\tilde{A} \ominus \tilde{B} =< [(a_1 - c_2, b_1 - b_2, c_1 - a_2); min\ (w_1, w_2), max(u_1, u_2)] >$

(iii) $\gamma\tilde{A} = \begin{cases} < [(\gamma a_1, \gamma b_1, \gamma c_1); w_1, u_1] >, & \gamma \geq 0 \\ < [(\gamma c_1, \gamma b_1, \gamma a_1); w_1, u_1] >, & \gamma \leq 0 \end{cases}$

5.1.2.2 Arithmetic operations between triangular vague sets

In this section, some arithmetic operations between triangular vague sets, defined on universal set of real numbers R, are presented [30].

Let $\tilde{A} = < [(a_1, b_1, c_1); \delta_1, \rho_1] >$ and $\tilde{B} = < [(a_2, b_2, c_2); \delta_2, \rho_2] >$ be two triangular vague sets. Then

(i) $\tilde{A} \oplus \tilde{B} =< [(a_1 + a_2, b_1 + b_2, c_1 + c_2); min\ (\delta_1, \delta_2), min(\rho_1, \rho_2)] >$

(ii) $\tilde{A} \ominus \tilde{B} =< [(a_1 - c_2, b_1 - b_2, c_1 - a_2); min\ (\delta_1, \delta_2), min(\rho_1, \rho_2)] >$

(iii) $\gamma\tilde{A} = \begin{cases} < [(\gamma a_1, \gamma b_1, \gamma c_1); \delta_1, \rho_1] >, & \gamma \geq 0 \\ < [(\gamma c_1, \gamma b_1, \gamma a_1); \delta_1, \rho_1] >, & \gamma \leq 0 \end{cases}$

5.1.2.3 Arithmetic operations between trapezoidal vague sets

In this section, some arithmetic operations between trapezoidal vague sets, defined on universal set of real numbers R, are presented [30].

Let $\tilde{A} = < [(a_1, b_1, c_1, d_1); \delta_1, \rho_1] >$ and $\tilde{B} = < [(a_2, b_2, c_2, d_2); \delta_2, \rho_2] >$ be two trapezoidal vague sets. Then

(i) $\tilde{A} \oplus \tilde{B} =< [(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); min\ (\delta_1, \delta_2), min(\rho_1, \rho_2)] >$

(ii) $\tilde{A} \ominus \tilde{B} =< [(a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2); min\ (\delta_1, \delta_2), min(\rho_1, \rho_2)] >$

(iii) $\gamma\tilde{A} = \begin{cases} < [(\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1); \delta_1, \rho_1] >, & \gamma \geq 0 \\ < [(\gamma d_1, \gamma c_1, \gamma b_1, \gamma a_1); \delta_1, \rho_1] >, & \gamma \leq 0 \end{cases}$

5.2 Li ranking approach

In this section, the existing ranking approach [94] for comparing triangular intuitionistic fuzzy sets is presented.

Let $\tilde{A} =< [(a_1, b_1, c_1); w_1, u_1] >$ and $\tilde{B} =< [(a_2, b_2, c_2); w_2, u_2] >$ be two triangular intuitionistic fuzzy sets. Then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Calculate

$$V^{\lambda}(\tilde{A}) = V_{\mu}(\tilde{A}) + \lambda(V_{\nu}(\tilde{A}) - V_{\mu}(\tilde{A})) \tag{5.1}$$

$$A^{\lambda}(\tilde{A}) = A_{\mu}(\tilde{A}) - \lambda(A_{\nu}(\tilde{A}) - A_{\mu}(\tilde{A})) \tag{5.2}$$

$$V^{\lambda}(\tilde{B}) = V_{\mu}(\tilde{B}) + \lambda(V_{\nu}(\tilde{B}) - V_{\mu}(\tilde{B})) \tag{5.3}$$

$$A^{\lambda}(\tilde{B}) = A_{\mu}(\tilde{B}) - \lambda(A_{\nu}(\tilde{B}) - A_{\mu}(\tilde{B})) \tag{5.4}$$

where,

$$V_{\mu}(\tilde{A}) = \frac{w_1}{6}(a_1 + 4b_1 + c_1) \tag{5.5}$$

$$V_{\nu}(\tilde{A}) = \frac{(1-u_1)}{6}(a_1 + 4b_1 + c_1) \tag{5.6}$$

$$A_{\mu}(\tilde{A}) = \frac{w_1}{3}(c_1 - a_1) \tag{5.7}$$

$$A_{\nu}(\tilde{A}) = \frac{(1-u_1)}{3}(c_1 - a_1) \tag{5.8}$$

$$V_{\mu}(\tilde{B}) = \frac{w_2}{6}(a_2 + 4b_2 + c_2) \tag{5.9}$$

$$V_{\nu}(\tilde{B}) = \frac{(1-u_2)}{6}(a_2 + 4b_2 + c_2) \tag{5.10}$$

$$A_{\mu}(\tilde{B}) = \frac{w_2}{3}(c_2 - a_2) \tag{5.11}$$

$$A_{\nu}(\tilde{B}) = \frac{(1-u_2)}{3}(c_2 - a_2) \tag{5.12}$$

and $\lambda \in [0,1]$.

Step 2 Calculate

$L_T^{\lambda}(\tilde{A}) = \frac{V^{\lambda}(\tilde{A})}{1+A^{\lambda}(\tilde{A})}$ and $L_T^{\lambda}(\tilde{B}) = \frac{V^{\lambda}(\tilde{B})}{1+A^{\lambda}(\tilde{B})}$

Step 3 Chose a particular value of λ and check that $L_T^{\lambda}(\tilde{A}) > L_T^{\lambda}(\tilde{B})$ or $L_T^{\lambda}(\tilde{A}) < L_T^{\lambda}(\tilde{B})$ or $L_T^{\lambda}(\tilde{A}) = L_T^{\lambda}(\tilde{B})$

Case (i) If $L_T^{\lambda}(\tilde{A}) > L_T^{\lambda}(\tilde{B})$ for chosen value of λ, then $\tilde{A} \succ \tilde{B}$ for same value of λ.

Case (ii) If $L_T^{\lambda}(\tilde{A}) < L_T^{\lambda}(\tilde{B})$ for chosen value of λ, then $\tilde{A} \prec \tilde{B}$ for same value of λ.

Case (iii) If $L_T^{\lambda}(\tilde{A}) = L_T^{\lambda}(\tilde{B})$ for chosen value of λ, then $\tilde{A} \sim \tilde{B}$ for same value of λ.

5.3 Limitations of existing results

Li [94] pointed out the shortcomings of all the existing ranking approaches for comparing triangular intuitionistic fuzzy sets and proposed some results and a new ranking approach for comparing triangular intuitionistic fuzzy sets. In this section, the limitations of existing results [94] are pointed out.

Li [94] proposed the following results [94], (Theorem 5.1-Theorem 5.6), for the the triangular intuitionistic fuzzy sets $\tilde{A} =< [(a_1, a_2, a_3); w_a, u_a] >$ and $\tilde{B} =< [(b_1, b_2, b_3); w_b, u_b] >$:

(i) $V_\mu(\tilde{A} \oplus \tilde{B}) = V_\mu(\tilde{A}) + V_\mu(\tilde{B})$

(ii) $V_\nu(\tilde{A} \oplus \tilde{B}) = V_\nu(\tilde{A}) + V_\nu(\tilde{B})$

(iii) $A_\mu(\tilde{A} \oplus \tilde{B}) = A_\mu(\tilde{A}) + A_\mu(\tilde{B})$

(iv) $A_\nu(\tilde{A} \oplus \tilde{B}) = A_\nu(\tilde{A}) + A_\nu(\tilde{B})$

(v) $V^\lambda(\tilde{A} \oplus \tilde{B}) = V^\lambda(\tilde{A}) + V^\lambda(\tilde{B})$

(vi) $A^\lambda(\tilde{A} \oplus \tilde{B}) = A^\lambda(\tilde{A}) + A^\lambda(\tilde{B})$

where, $\lambda \in [0, 1]$

The existing results are valid only if $w_a = w_b$ and $u_a = u_b$.

But if $w_a \neq w_b$ and $u_a \neq u_b$ then the existing results are not valid.

Example 5.1 Let $\tilde{A} =< [(1, 3, 5); 0.3, 0.2] >$ and $\tilde{B} =< [(4, 8, 9); 0.4, 0.1] >$ be two triangular intuitionistic fuzzy sets. Then using equation (5.5), it can be easily seen that

$V_\mu(\tilde{A} \oplus \tilde{B}) \neq V_\mu(\tilde{A}) + V_\mu(\tilde{B})$, which contradicts the existing result $V_\mu(\tilde{A} \oplus \tilde{B}) = V_\mu(\tilde{A}) + V_\mu(\tilde{B})$

Example 5.2 Let $\tilde{A} =< [(0, 2, 5); 0.2, 0.1] >$ and $\tilde{B} =< [(2, 3, 7); 0.3, 0.2] >$ be two triangular intuitionistic fuzzy sets. Then using equation (5.6), it can be easily seen that

$V_\nu(\tilde{A} \oplus \tilde{B}) \neq V_\nu(\tilde{A}) + V_\nu(\tilde{B})$, which contradicts the existing result $V_\nu(\tilde{A} \oplus \tilde{B}) = V_\nu(\tilde{A}) + V_\nu(\tilde{B})$

Example 5.3 Let $\tilde{A} =< [(1, 2, 7); 0.4, 0.3] >$ and $\tilde{B} =< [(0, 3, 7); 0.3, 0.2] >$ be two triangular intuitionistic fuzzy sets. Then using equation (5.7), it can be easily seen that

$A_\mu(\tilde{A} \oplus \tilde{B}) \neq A_\mu(\tilde{A}) + A_\mu(\tilde{B})$, which contradicts the existing result $A_\mu(\tilde{A} \oplus \tilde{B}) = A_\mu(\tilde{A}) + A_\mu(\tilde{B})$

Example 5.4 Let $\tilde{A} =< [(-2, 0, 3); 0.3, 0.1] >$ and $\tilde{B} =< [(0, 3, 8); 0.2, 0.1] >$ be two triangular intuitionistic fuzzy sets. Then using equation (5.8), it can be easily seen that

$A_\nu(\tilde{A} \oplus \tilde{B}) \neq A_\nu(\tilde{A}) + A_\nu(\tilde{B})$, which contradicts the existing result $A_\nu(\tilde{A} \oplus \tilde{B}) = A_\nu(\tilde{A}) + A_\nu(\tilde{B})$

Example 5.5 Let $\tilde{A} =< [(1, 3, 9); 0.6, 0.4] >$ and $\tilde{B} =< [(2, 3, 8); 0.5, 0.3] >$ be two triangular intuitionistic fuzzy sets. Then using equation (5.1), it can be easily seen that

$V^\lambda(\tilde{A} \oplus \tilde{B}) \neq V^\lambda(\tilde{A}) + V^\lambda(\tilde{B})$, which contradicts the existing result $V^\lambda(\tilde{A} \oplus \tilde{B}) = V^\lambda(\tilde{A}) + V^\lambda(\tilde{B})$

Example 5.6 Let $\tilde{A} =< [(2, 3, 5); 0.5, 0.4] >$ and $\tilde{B} =< [(2, 3, 8); 0.5, 0.3] >$ be two triangular intuitionistic fuzzy sets. Then using equation (5.2), it can be easily seen that

$A^\lambda(\tilde{A} \oplus \tilde{B}) \neq A^\lambda(\tilde{A}) + A^\lambda(\tilde{B})$, which contradicts the existing result $A^\lambda(\tilde{A} \oplus \tilde{B}) = A^\lambda(\tilde{A}) + A^\lambda(\tilde{B})$

5.4 Proposed results

In this section, to overcome the limitation of the existing results, pointed out in Section 5.3, new results are proposed.

Proposition 5.1 Let $\tilde{A} =< [(a_1, a_2, a_3); w_a, u_a] >$ and $\tilde{B} =< [(b_1, b_2, b_3); w_b, u_b] >$ be two triangular intuitionistic fuzzy sets. Then

(i) $V_\mu(\tilde{A}^* \oplus \tilde{B}^*) = V_\mu(\tilde{A}^*) + V_\mu(\tilde{B}^*)$

(ii) $V_\nu(\tilde{A}^* \oplus \tilde{B}^*) = V_\nu(\tilde{A}^*) + V_\nu(\tilde{B}^*)$

(iii) $A_\mu(\tilde{A}^* \oplus \tilde{B}^*) = A_\mu(\tilde{A}^*) + A_\mu(\tilde{B}^*)$

(iv) $A_\nu(\tilde{A}^* \oplus \tilde{B}^*) = A_\nu(\tilde{A}^*) + A_\nu(\tilde{B}^*)$

(v) $V^\lambda(\tilde{A}^* \oplus \tilde{B}^*) = V^\lambda(\tilde{A}^*) + V^\lambda(\tilde{B}^*)$

(vi) $A^\lambda(\tilde{A}^* \oplus \tilde{B}^*) = A^\lambda(\tilde{A}^*) + A^\lambda(\tilde{B}^*)$

where, $\tilde{A}^* =< [(a_1, a_2, a_3); w, u] >$, $\tilde{B}^* =< [(b_1, b_2, b_3); w, u] >$, $\tilde{A}^* \oplus \tilde{B}^* =< [(a_1 + b_1, a_2 + b_2, a_3 + b_3; w, u)] >$, $w = min(w_a, w_b)$ and $u = max(u_a, u_b)$.

Proof:- The proposed results can be proved as follows:

(i) Using equation (5.5),

$$V_\mu(\tilde{A}^* \oplus \tilde{B}^*) = \tfrac{w}{6}(a_1+b_1+4(a_2+b_2)+a_3+b_3) = \tfrac{w}{6}(a_1+4a_2+a_3)+\tfrac{w}{6}(b_1+4b_2+b_3)$$
$$= V_\mu(\tilde{A}^*) + V_\mu(\tilde{B}^*)$$

(ii) Using equation (5.6),

$$V_\nu(\tilde{A}^* \oplus \tilde{B}^*) = \tfrac{(1-u)}{6}(a_1 + b_1 + 4(a_2 + b_2) + a_3 + b_3) = \tfrac{(1-u)}{6}(a_1 + 4a_2 + a_3) + \tfrac{(1-u)}{6}(b_1 + 4b_2 + b_3)$$
$$= V_\nu(\tilde{A}^*) + V_\mu(\tilde{B}^*)$$

(iii) Using equation (5.7),

$$A_\mu(\tilde{A}^* \oplus \tilde{B}^*) = \tfrac{w}{3}((a_1 + b_3) - (a_1 + b_1)) = \tfrac{w}{3}(a_3 - a_1) + \tfrac{w}{3}(b_3 + b_1)$$

$= A_\mu(\tilde{A}^*) + A_\mu(\tilde{B}^*)$

(iv) Using equation (5.8),

$A_\nu(\tilde{A}^* \oplus \tilde{B}^*) = \frac{(1-u)}{3}((a_1+b_3)-(a_1+b_1)) = \frac{(1-u)}{3}(a_3-a_1) + \frac{(1-u)}{3}(b_3+b_1)$

$= A_\nu(\tilde{A}^*) + A_\nu(\tilde{B}^*)$

(v) Using equation (5.1),

$V^\lambda(\tilde{A}^* \oplus \tilde{B}^*) = V_\mu(\tilde{A}^* \oplus \tilde{B}^*) + \lambda(V_\nu(\tilde{A}^* \oplus \tilde{B}^*) - V_\mu(\tilde{A}^* \oplus \tilde{B}^*))$

Using (i) and (ii),

$V^\lambda(\tilde{A}^* \oplus \tilde{B}^*) = V_\mu(\tilde{A}^*) + V_\mu(\tilde{B}^*) + \lambda(V_\nu(\tilde{A}^*) + V_\nu(\tilde{B}^*) - (V_\mu(\tilde{A}^*) + V_\mu(\tilde{B}^*)))$

$= V_\mu(\tilde{A}^*) + \lambda(V_\nu(\tilde{A}^*) - V_\mu(\tilde{A}^*)) + V_\mu(\tilde{B}^*) + \lambda(V_\nu(\tilde{B}^*) - V_\mu(\tilde{B}^*))$

$= V^\lambda(\tilde{A}^*) + V^\lambda(\tilde{B}^*)$

(vi) Using equation (5.2),

$A_\mu^\lambda(\tilde{A}^* \oplus \tilde{B}^*) = A_\mu(\tilde{A}^* \oplus \tilde{B}^*) + \lambda(A_\nu(\tilde{A}^* \oplus \tilde{B}^*) - A_\mu(\tilde{A}^* \oplus \tilde{B}^*))$

Using (iii) and (iv),

$A^\lambda(\tilde{A}^* \oplus \tilde{B}^*) = A_\mu(\tilde{A}^*) + A_\mu(\tilde{B}^*) + \lambda(A_\nu(\tilde{A}^*) + A_\nu(\tilde{B}^*) - (A_\mu(\tilde{A}^*) + A_\mu(\tilde{B}^*)))$

$= A_\mu(\tilde{A}^*) + \lambda(A_\nu(\tilde{A}^*) - A_\mu(\tilde{A}^*)) + A_\mu(\tilde{B}^*) + \lambda(A_\nu(\tilde{B}^*) - A_\mu(\tilde{B}^*))$

$= A^\lambda(\tilde{A}^*) + A^\lambda(\tilde{B}^*)$

5.5 Shortcomings of existing ranking approach

Li [94] pointed out the shortcomings of all the existing ranking approaches for comparing intuitionistic fuzzy sets and proposed a new ranking approach for comparing intuitionistic fuzzy sets. In this section, the shortcomings of existing ranking approach [94] are pointed out.

5.5.1 Shortcomings on the basis of reasonable properties of fuzzy quantities

In Section 2.5.3, on the basis of reasonable properties of fuzzy quantities, the shortcomings of existing ranking approaches [37, 41] are pointed out. In this section, it is shown that the ranking function used in the existing ranking approach [94] for comparing intuitionistic fuzzy sets also does not satisfies the same properties so the ranking results, obtained by using the existing ranking approach [94] are not valid.

Example 5.7 Let $\tilde{A} =< [(-a, 0, a); w, u] >$, $\tilde{B} =< [(-b, 0, b); w, u] >$ and $\tilde{C} =< [(c_1, c_2, c_3); w, u] >$ be three triangular intuitionistic fuzzy sets. Then λ $L_T^{\lambda}(\tilde{A}) = L_T^{\lambda}(\tilde{B})\ \forall\ \lambda \in [0, 1]$ but $L_T^{\lambda}(\tilde{A} \oplus \tilde{C}) \neq L_T^{\lambda}(\tilde{B} \oplus \tilde{C})\ \forall\ \lambda \in [0, 1]$ i.e. $\tilde{A} \sim \tilde{B} \not\Rightarrow (\tilde{A} \oplus \tilde{C}) \sim (\tilde{B} \oplus \tilde{C})$, which contradicts the reasonable property, $\tilde{A} \sim \tilde{B} \Rightarrow (\tilde{A} \oplus \tilde{C}) \sim (\tilde{B} \oplus \tilde{C})$.

Example 5.8 Let $\tilde{A} =< (1, 3, 5; 0.3, 0.2) >$, $\tilde{B} =< (4, 8, 9; 0.4, 0.1) >$ and $\tilde{C} =< (-9, -8, -4; 0.4, 0.1) >$ be two triangular intuitionistic fuzzy sets. According to existing ranking approach [94] the values of $L_T^{0.5}(\tilde{A})$ and $L_T^{0.5}(\tilde{B})$ are 0.6347 and 0.822 respectively so $\tilde{A} \prec \tilde{B}$ but the values of $L_T^{0.5}(\tilde{A} \oplus \tilde{C})$ and $L_T^{0.5}(\tilde{B} \oplus \tilde{C})$ are -10.2 and 0 respectively so $L_T^{0.5}(\tilde{A} \oplus \tilde{C}) > L_T^{0.5}(\tilde{B} \oplus \tilde{C})$, which contradicts the resealable property $\tilde{A} \prec \tilde{B} \Rightarrow \tilde{A} \oplus \tilde{C} \prec \tilde{B} \oplus \tilde{C}$

5.5.2 Shortcomings on the basis of height of fuzzy sets

In this section, it is shown that in some cases the ranking results, obtained by using existing ranking approach [94], depends upon degree of membership and degree of non-membership of intuitionistic fuzzy sets while in several cases the ranking results does not depend upon degree of membership and degree of non-membership of intuitionistic fuzzy sets.

Let $\tilde{A} =< [(a_1, a_2, a_3); w_1, u_1] >$ and $\tilde{B} =< [(a_1, a_2, a_3); w_2, u_2] >$ be two triangular intuitionistic fuzzy sets. Then according to existing ranking approach [94] there can be two cases:

Case (i) If $(a_1 + 4a_2 + a_3) \neq 0$ then the comparison of $\tilde{A}$ and $\tilde{B}$ will depend upon the values of w_1, u_1, w_2 and u_2

Case (ii) If $(a_1 + 4a_2 + a_3) = 0$ then $\tilde{A} \sim \tilde{B}$ all values of w_1, u_1, w_2 and u_2 i.e. the comparison of $\tilde{A}$ and $\tilde{B}$ does not depend upon the values of w_1, u_1, w_2 and u_2.

Example 5.9 Let $\tilde{A} =< [(1, 1, 1); w_1, u_1] >$ and $\tilde{B} =< [(1, 1, 1); w_2, u_2] >$ be two triangular intuitionitsic fuzzy sets. Then according to existing ranking approach [94]

(i) $\tilde{A} \prec \tilde{B}$ if $L_T^{\lambda}(\tilde{A}) < L_T^{\lambda}(\tilde{B})$

(ii) $\tilde{A} \succ \tilde{B}$ if $L_T^{\lambda}(\tilde{A}) > L_T^{\lambda}(\tilde{B})$

(iii) $\tilde{A} \sim \tilde{B}$ $L_T^{\lambda}(\tilde{A}) = L_T^{\lambda}(\tilde{B})$.

Example 5.10 Let $\tilde{A} =< [(-2, 0, 2); w_1, u_1] >$ and $\tilde{B} =< [(-2, 0, 2); w_2, u_2] >$ be two triangular intuitionitsic fuzzy sets. Then according to existing ranking approach [94] $\tilde{A} \sim \tilde{B}$ for all values of w_1, u_1, w_2 and u_2.

Example 5.11 Let $\tilde{A} =< [(-8, 1, 4); w_1, u_1] >$ and $\tilde{B} =< [(-8, 1, 4); w_2, u_2] >$ be two triangular intuitionitsic fuzzy sets. Then according to existing ranking approach [94] $\tilde{A} \sim \tilde{B}$ for all values of w_1, u_1, w_2 and u_2.

According to existing approach [94], in first case comparison of triangular intuitionistic fuzzy sets depends upon degree of membership and degree of non-membership of intuitionistic fuzzy sets while in second case comparison of triangular intuitionistic fuzzy sets does not depend upon degree of membership and degree of non-membership of intuitionistic fuzzy sets which is a contradiction.

5.6 Proposed ranking approach

In this section, a new ranking approach is proposed for comparing trapezoidal vague sets.

Let $\tilde{A} = < [(a_1, b_1, c_1, d_1); \delta_1, \rho_1] >$ and $\tilde{B} =< [(a_2, b_2, c_2, d_2); \delta_2, \rho_2] >$ be two triangular vague sets, where $a_1 \leq b_1 \leq c_1 \leq d_1$, $\delta_1 \leq \rho_1$ and $a_2 \leq b_2 \leq c_2 \leq d_2$, $\delta_2 \leq \rho_2$. Then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Transform $\tilde{A}$, $\tilde{B}$ into $\tilde{A}^*$, $\tilde{B}^*$ as follows:

$$\tilde{A}^* =< [(a_1, b_1, c_1, d_1); \delta, \rho] >, \tilde{B}^* =< [(a_2, b_2, c_2, d_2); \delta, \rho] > \tag{5.13}$$

where, $\delta = min(\delta_1, \delta_2)$ and $\rho = min(\rho_1, \rho_2)$

Step 2 Calculate $\Re_T^\lambda(\tilde{A}^*_\delta) = \int\limits_0^\delta \lambda(a_1 + \frac{(b_1-a_1)\alpha}{\delta})d\alpha + \int\limits_0^\delta (1-\lambda)(d_1 + \frac{(c_1-d_1)\alpha}{\delta})d\alpha$

$$\Rightarrow \Re_T^\lambda(\tilde{A}^*_\delta) = \lambda\delta(\frac{a_1+b_1}{2}) + (1-\lambda)\delta(\frac{c_1+d_1}{2}) \tag{5.14}$$

$\Re_T^\lambda(\tilde{A}^*_{\rho-\delta}) = \int\limits_\delta^\rho (\lambda(b_1 - (\frac{\alpha-\delta}{\rho-\delta})(a_1 + (b_1-a_1)\frac{\delta}{\rho} - b_1)))d\alpha + \int\limits_\delta^\rho ((1-\lambda)(c_1 - (\frac{\alpha-\rho}{\rho-\delta})(d_1 + (c_1-d_1)\frac{\delta}{\rho} - c_1)))d\alpha,$

$$\Rightarrow \Re_T^\lambda(\tilde{A}^*_{\rho-\delta}) \Rightarrow (\rho-\delta)\{\lambda(\frac{(b_1+a_1)}{2} + \frac{\delta}{\rho}\frac{(b_1-a_1)}{2}) + (1-\lambda)(\frac{(c_1+d_1)}{2} + \frac{\delta}{\rho}\frac{(d_1-c_1)}{2})\} \tag{5.15}$$

Now $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{A}^*_\delta) + \Re_T^\lambda(\tilde{A}^*_{\rho-\delta})$

$$= \lambda\{\delta\frac{(a_1+b_1)}{2} + \frac{(\rho-\delta)}{2}(b_1+a_1+\frac{\delta}{\rho}(b_1-a_1))\} + (1-\lambda)\{\delta\frac{(c_1+d_1)}{2} + \frac{(\rho-\delta)}{2}(c_1+d_1+\frac{\delta}{\rho}(d_1-c_1))\} \tag{5.16}$$

where, $\lambda \in [0, 1]$

Similarly, $\Re_T^\lambda(\tilde{B}^*) = \Re_T^\lambda(\tilde{B}^*_\delta) + \Re_T^\lambda(\tilde{B}^*_{\rho-\delta})$

$$= \lambda\{\delta\frac{(a_2+b_2)}{2} + \frac{(\rho-\delta)}{2}(b_2+a_2+\frac{\delta}{\rho}(b_2-a_2))\} + (1-\lambda)\{\delta\frac{(c_2+d_2)}{2} + \frac{(\rho-\delta)}{2}(c_2+d_2+\frac{\delta}{\rho}(d_2-c_2))\} \tag{5.17}$$

where, $\lambda \in [0, 1]$

Step 3 Check that for particular value of λ, $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{B}^*)$ or $\Re_T^\lambda(\tilde{A}^*) \neq \Re_T^\lambda(\tilde{B}^*)$

Case (i) If $\Re_T^\lambda(\tilde{A}^*) = \Re_T^\lambda(\tilde{B}^*)$ then Go to Step 4.

Case (ii) If $\Re_T^\lambda(\tilde{A}^*) \neq \Re_T^\lambda(\tilde{B}^*)$ then check that $\Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{B}^*)$ or $\Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{B}^*)$.

Case (a) If $\Re_T^\lambda(\tilde{A}^*) > \Re_T^\lambda(\tilde{B}^*)$ then $\tilde{A} \succ \tilde{B}$.

Case (b) If $\Re_T^\lambda(\tilde{A}^*) < \Re_T^\lambda(\tilde{B}^*)$ then $\tilde{A} \prec \tilde{B}$.

Step 4 Calculate

$$\mathrm{RM}^\lambda(\tilde{A}^*) = \mathrm{Mode}^\lambda(\tilde{A}^*) = \lambda b_1 + (1-\lambda)c_1 \tag{5.18}$$

and

$$\mathrm{RM}^\lambda(\tilde{B}^*) = \mathrm{Mode}^\lambda(\tilde{B}^*) = \lambda b_2 + (1-\lambda)c_2 \tag{5.19}$$

Step 5 Check that for the chosen value of λ, $\mathrm{RM}^\lambda(\tilde{A}^*) > \mathrm{RM}^\lambda(\tilde{B}^*)$ or $\mathrm{RM}^\lambda(\tilde{A}^*) < \mathrm{RM}^\lambda(\tilde{B}^*)$ or $\mathrm{RM}^\lambda(\tilde{A}^*) = \mathrm{RM}^\lambda(\tilde{B}^*)$

Case (i) If $\mathrm{RM}^\lambda(\tilde{A}^*) > \mathrm{RM}^\lambda(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \succ \tilde{B}$ for same value of λ.

Case (ii) If $\mathrm{RM}^\lambda(\tilde{A}^*) < \mathrm{RM}^\lambda(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \prec \tilde{B}$ for same value of λ.

Case (iii) If $\mathrm{RM}^\lambda(\tilde{A}^*) = \mathrm{RM}^\lambda(\tilde{B}^*)$ for chosen value of λ, then $\tilde{A} \sim \tilde{B}$ for same value of λ.

5.6.1 Particular cases

Since trapezoidal vague sets are the generalization of normal triangular fuzzy sets, trapezoidal fuzzy sets, generalized triangular fuzzy sets, generalized trapezoidal fuzzy sets and triangular vague sets so the proposed ranking approach can also be used for comparing these sets by considering the following assumptions:

(i) Assuming $b_1 = c_1 = b_1'$ and $b_2 = c_2 = b_2'$ the proposed ranking approach can be used for comparing triangular vague sets $\tilde{A} =< [(a_1, b_1', d_1); \delta, \rho] >$ and $\tilde{B} =< [(a_2, b_2', d_2); \delta, \rho] >$.

(ii) Assuming $\delta = \rho = \delta_1'$ the proposed ranking approach can be used for comparing generalized trapezoidal fuzzy sets $\tilde{A} = (a_1, b_1, c_1, d_1; \delta_1')$ and $\tilde{B} = (a_2, b_2, c_2, d_2; \delta_1')$.

(iii) Assuming $b_1 = c_1 = b_1'$, $\delta = \rho = \delta_1'$ and $b_2 = c_2 = b_2'$ the proposed ranking approach can be used for comparing generalized triangular fuzzy sets $\tilde{A} =< [(a_1, b_1', d_1; \delta_1')] >$ and $\tilde{B} =< [(a_2, b_2', d_2; \delta_1')] >$.

(iv) Assuming $\delta = \rho = 1$ the proposed ranking approach can be used for comparing trapezoidal fuzzy numbers $\tilde{A} =< [(a_1, b_1, c_1, d_1; 1)] >$ and $\tilde{B} =< [(a_2, b_2, c_2, d_2; 1)] >$.

(v) Assuming $b_1 = c_1 = b_1'$, $\delta = \rho = 1$ and $b_2 = c_2 = b_2'$ the proposed ranking approach , can be used for comparing triangular fuzzy sets $\tilde{A} =< [(a_1, b_1', d_1; 1)] >$ and $\tilde{B} =< [(a_2, b_2', d_2; 1)] >$.

5.6.2 Illustrative examples

In this section, to illustrate the proposed ranking approach some triangular and trapezoidal vague sets are compared.

Example 5.12 Let $\tilde{A} = < [(5, 7, 9); 0.4, 0.8] >$ and $\tilde{B} = < [(2, 4, 8); 0.2, 0.9] >$ be two triangular vague sets. Then $\tilde{A}$ and $\tilde{B}$ can be compared by using the following steps:

Step 1 Using equation (5.13),

$\tilde{A}^* =< [(5, 7, 9); min(0.2, 0.4), min(0.8, 0.9)] >$ and $\tilde{B}^* =< [(2, 4, 8); min(0.2, 0.4), min(0.8, 0.9)] >$

Step 2 Using equation (5.16), $\Re_T^{\lambda}(\tilde{A}^*) = 4.95\lambda + 3.25(1-\lambda)$ and $\Re_T^{\lambda}(\tilde{B}^*) = 2.7\lambda + 2.45(1-\lambda)$

Step 3 For a pessimistic decision maker i.e. for $\lambda = 0$,

$\Re_T^{0}(\tilde{A}^*) = 3.25$ and $\Re_T^{0}(\tilde{B}^*) = 2.45$. Since $\Re_T^{0}(\tilde{A}^*) > \Re_T^{0}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

For optimistic decision maker i.e. for $\lambda = 1$,

$\Re_T^{1}(\tilde{A}^*) = 4.95$ and $\Re_T^{1}(\tilde{B}^*) = 2.7$. Since $\Re_T^{1}(\tilde{A}^*) > \Re_T^{1}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$.

For moderate decision maker i.e. for $\lambda = 0.5$,

$\Re_T^{0.5}(\tilde{A}^*) = 4.1$, $\Re_T^{0.5}(\tilde{B}^*) = 2.6$. Since $\Re_T^{0.5}(\tilde{A}^*) > \Re_T^{0.5}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$.

Example 5.13 Let $\tilde{A} = < [(3,7,14); 0.2, 0.7] >$ and $\tilde{B} = < [(1,6,9); 0.3, 0.4] >$ be two triangular vague sets. Then $\tilde{A}$ and $\tilde{B}$ can be compared by using the following steps:

Step 1 Using equation (5.13),

$\tilde{A}^* =< [(3,7,14); 0.2, 0.4)] >$ and $\tilde{B}^* =< [(1,6,9); 0.2, 0.4)] >$

Step 2 Using equation (5.16), $\Re_T^{\lambda}(\tilde{A}^*) = 2.25\lambda + 4.35(1-\lambda)$ and $\Re_T^{\lambda}(\tilde{B}^*) = 1.65\lambda + 3.15(1-\lambda)$

Step 3 For a pessimistic decision maker i.e. for $\lambda = 0$,

$\Re_T^{0}(\tilde{A}^*) = 4.35$ and $\Re_T^{0}(\tilde{B}^*) = 3.15$

Since $\Re_T^{0}(\tilde{A}^*) > \Re_T^{0}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$

For optimistic decision maker i.e. for $\lambda = 1$,

$\Re_T^{1}(\tilde{A}^*) = 2.25$ and $\Re_T^{1}(\tilde{B}^*) = 1.65$. Since $\Re_T^{1}(\tilde{A}^*) > \Re_T^{1}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$.

For moderate decision maker i.e. for $\lambda = 0.5$,

$\Re_T^{0.5}(\tilde{A}^*) = 3.3$ and $\Re_T^{0.5}(\tilde{B}^*) = 2.4$. Since $\Re_T^{0.5}(\tilde{A}^*) > \Re_T^{0.5}(\tilde{B}^*) \Rightarrow \tilde{A} \succ \tilde{B}$.

Example 5.14

Let $\tilde{A} = < [(1,2,3,4); 0.5, 0.6] >$ and $\tilde{B} = < [(2,4,5,6,1); 0.6, 0.8] >$ be two vague sets. Then $\tilde{A}$ and $\tilde{B}$ can be compared by using the following steps:

Step 1 Using equation (5.13),

$\tilde{A}^* =< [(1,2,3,4);0.5,0.6] >$ and $\tilde{B}^* =< [(2,4,5,6,1);0.5,0.6] >$

Step 2 Using equation (5.13), $\Re_T^\lambda(\tilde{A}^*) = 0.942\lambda + 2.14(1-\lambda)$ and $\Re_T^\lambda(\tilde{B}^*) = 1.88\lambda + 3.34(1-\lambda)$

Step 3 For a pessimistic decision maker i.e. for $\lambda = 0$,

$\Re_T^0(\tilde{A}^*) = 2.14$ and $\Re_T^0(\tilde{B}^*) = 3.34$. Since $\Re_T^0(\tilde{A}^*) < \Re_T^0(\tilde{B}^*) \Rightarrow \tilde{A} \prec \tilde{B}$

For optimistic decision maker i.e. for $\lambda = 1$,

$\Re_T^1(\tilde{A}^*) = 0.942$ and $\Re_T^1(\tilde{B}^*) = 1.88$. Since $\Re_T^1(\tilde{A}^*) < \Re_T^1(\tilde{B}^*) \Rightarrow \tilde{A} \prec \tilde{B}$.

For moderate decision maker i.e for $\lambda = 0.5$,

$\Re_T^{0.5}(\tilde{A}^*) = 1.54$ and $\Re_T^{0.5}(\tilde{B}^*) = 2.6$. Since $\Re_T^{0.5}(\tilde{A}^*) < \Re_T^{0.5}(\tilde{B}^*) \Rightarrow \tilde{A} \prec \tilde{B}$.

5.7 Linear programming problems with trapezoidal vague sets

If in the existing linear programming problem with fuzzy sets (P_2) the parameters $\tilde{c}_j$ are represented by trapezoidal vague sets $< [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta_j, \rho_j] >$ then linear programming problem with fuzzy set (P_2) can be converted into linear programming problem with trapezoidal vague set (P_{13}):

$$\begin{aligned}
&\mathit{Maximize\ (or\ Minimize)}\ \sum_{j=1}^{n}(< [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta_j, \rho_j] > x_j)\\
&\mathit{subject\ to}\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \ \ i = 1,2,...,m \qquad\qquad (P_{13})\\
&\qquad (\sum_{j=1}^{n} \Re_T^{0.5}(\tilde{c}_j x_j)) = a\\
&\qquad x_j \geq 0
\end{aligned}$$

5.7.1 Application of proposed ranking approach

To show the application of proposed ranking approach a linear programming problem with vague sets, chosen in Example 5.15, is solved by using the method, proposed in Chapter 4, with the proposed ranking approach.

Example 5.15 Solve

$$Maximize\ \ (<[(1,2,3,4);0.5,0.6]>\oplus<[(2,4,5,6);0.7,0.7]>x_2)$$

$$subject\ to$$

$$x_1+x_2\geq 2$$

$$x_1+x_2\leq 5$$

$$x_1\geq -1$$

$$x_2\leq 7$$

$$x_1,x_2\geq 0$$

Solution:- Using the method, proposed in Section 4.4, with proposed proposed ranking approach the linear programming problem with trapezoidal vague sets, chosen in Example 5.15, can be solved by using the following steps:

Step 1 Using Definition 2.8 and on the basis of the proposed ranking approach, the optimal solution of the chosen linear programming problem with trapezoidal vague sets can be obtained by solving the crisp linear programming problem

$$Maximize\ \ (\Re_T^{0.5}(<[(1,2,3,4);0.5,0.6]>\oplus<[(2,4,5,6);0.7,0.9]>x_2))$$

$$subject\ to$$

$$x_1+x_2\geq 2$$

$$x_1+x_2\leq 5$$

$$x_1 \geq -1$$

$$x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

Step 2 Using linearity property

$\Re_T^{0.5} \sum\limits_{j=1}^{n} (< [(a_j, b_j, c_j, d_j); \delta, \rho] > x_j) = \sum\limits_{j=1}^{n} \Re_T^{0.5} (< [(a_j, b_j, c_j, d_j); \delta, \rho] > x_j)$, the crisp linear programming problem, obtained in Step 1, can be written as

$$Maximize \ \ (\Re_T^{0.5}(< [(1,2,3,4); 0.5, 0.6] >) + \Re_T^{0.5}(< [(2,4,5,6); 0.5, 0.6] > x_2))$$

$$subject \ to$$

$$x_1 + x_2 \geq 2$$

$$x_1 + x_2 \leq 5$$

$$x_1 \geq -1$$

$$x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

Step 3 Using $\Re_T^{0.5} < [(a,b,c,d); \delta, \rho] >= \frac{1}{2}\left\{\frac{\delta}{2}(a+b+c+d) + \frac{(\rho-\delta)}{2}((a+b+c+d) + \frac{\delta}{\rho}(-a+b+d-c))\right\}$, the crisp linear programming problem, obtained in Step 2, can be written as

$$Maximize \ \ (\tfrac{771}{50}x_1 + \tfrac{653}{250}x_2)$$

$$subject \ to$$

$$x_1 + x_2 \geq 2$$

$$x_1 + x_2 \leq 5$$

$$x_1 \geq -1$$

$$x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

Step 4 On solving the crisp linear programming problem, obtained in Step 3, the obtained optimal solution is $x_1 = 5$, $x_2 = 0$. Putting $x_1 = 5$ and $x_2 = 0$ in $(< [(1, 2, 3, 4); 0.5, 0.6] > \oplus < [(2, 4, 5, 6); 0.7, 0.7] > x_2)$ the vague optimal value of the chosen problem is $< [(5, 10, 15, 20); 0.5, 0.6] >$.

5.8 Conclusions

In this chapter, the shortcomings of existing ranking approaches [94] for comparing triangular intuitionistic fuzzy sets are pointed out and a new ranking approach is proposed for comparing trapezoidal vague sets. To show the application of proposed ranking approach a linear programming problem with trapezoidal vague set is solved by using the method, proposed in Chapter 4, with proposed ranking approach.

Chapter 6

RMDS Ranking Approach For Solving Linear Programming Problems With Trapezoidal Vague Sets [1]

Several researchers [6, 55, 57, 56, 58, 107, 108, 109, 119, 120] have used the existing method [111] for solving different types of linear programming problems with fuzzy sets. In this chapter, it is shown that the results of the linear programming problems, obtained by the researchers by using the existing method [111] and the results of the linear programming problems with generalized and vague sets obtained by using the methods, proposed in previous chapters, are not appropriate. Also, it is shown that the shortcomings in the results are occurring due to existing and proposed ranking approaches which are used in the existing and proposed methods for comparing fuzzy and vague sets.

To overcome the shortcomings of the existing and proposed methods, a new ranking approach, named as RMDS ranking approach, is proposed for comparing trapezoidal vague sets having equal degree of membership and non membership. On the basis of proposed RMDS ranking approach, a new method is proposed for solving

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[1] Some part of this chapter has been communicated in *Expert Systems with Applications*

linear programming problems with trapezoidal vagues sets. To show the advantages of the proposed RMDS ranking approach, the chosen linear programming problems with fuzzy and vague sets, for which the results obtained by using the existing and proposed methods are not appropriate, are solved by the proposed method and it is shown that the obtained results are appropriate.

6.1 Shortcomings of existing methods

Let $\{x_j\}$ and A be the optimal solution and optimal value of a linear programming problem respectively. If there exist any feasible solution $\{y_j\}$ of the same linear programming problem such that the value of the objective function of the linear programming problem corresponding to $\{y_j\}$ is also A then $\{y_j\}$ is said to be an alternative optimal solution of the same linear programming problem i.e. corresponding to all alternative optimal solutions the values of objective function should be same.

In this section, it is shown that the results obtained by using the existing and proposed methods are contradicting this property of alternative optimal solutions i.e. by using the existing methods the value of the objective function corresponding to all the alternative optimal solution are not same e.g. the results of the linear programming problem with normal triangular fuzzy set, chosen in Example 6.1, by using the existing and proposed methods are contradicting this property of alternative optimal solution.

Example 6.1: Solve

$$Maximize \quad ((4,8,12;1)x_1 \oplus (6,8,10;1)x_2)$$

$$subject\ to$$

$$x_1 + x_2 \leq 10$$

$$x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Solution:- On solving the problem by using the existing and proposed methods the following alternative optimal solutions are obtained:

(i) $x_1 = 10$ and $x_2 = 0$

(ii) $x_1 = 0$ and $x_2 = 10$

Putting $x_1 = 10$ and $x_2 = 0$ in $((4, 8, 12; 1)x_1 \oplus (6, 8, 10; 1)x_2)$ the obtained fuzzy optimal value is (40,80,120;1).

Similarly, on putting $x_1 = 0$ and $x_2 = 10$ in $((4, 8, 12; 1)x_1 \oplus (6, 8, 10; 1)x_2)$ the obtained fuzzy optimal value is $(60, 80, 100; 1)$.

It is obvious from the results that the normal triangular fuzzy sets, representing the fuzzy optimal values corresponding to two different alternative optimal solutions, are not equal which is contradicting the property of alternative optimal solution.

6.2 Proposed method based on Kaufmann and Gupta ranking approach

Kaufmann and Gupta [80] defined the three parameters, Rank, Mode and Divergence for a normal triangular fuzzy set and proposed a ranking approach, on the basis of these parameters, for comparing normal triangular fuzzy sets.

In this section, the existing ranking approach [80] is presented and to overcome the shortcoming of existing method [111], a new method is proposed for solving linear programming problems with normal triangular fuzzy sets.

6.2.1 Kaufmann and Gupta ranking approach

In this section, the existing ranking approach is presented.

Let $\tilde{A} = (a_1, b_1, c_1; 1)$ and $\tilde{B} = (a_2, b_2, c_2; 1)$ be two normal triangular fuzzy sets. Then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Find $\Re_T^{0.5}(\tilde{A}) = \frac{1}{4}(a_1 + 2b_1 + c_1)$ and $\Re_T^{0.5}(\tilde{B}) = \frac{1}{4}(a_2 + 2b_2 + c_2)$

Case (i) If $\Re_T^{0.5}(\tilde{A}) > \Re_T^{0.5}(\tilde{B})$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If $\Re_T^{0.5}(\tilde{A}) < \Re_T^{0.5}(\tilde{B})$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ then go to Step 2.

Step 2 Find $\text{Mode}^{0.5}(\tilde{A}) = b_1$ and $\text{Mode}^{0.5}(\tilde{B}) = b_2$

Case (i) If $\text{Mode}^{0.5}(\tilde{A}) > \text{Mode}^{0.5}(\tilde{B})$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If $\text{Mode}^{0.5}(\tilde{A}) < \text{Mode}^{0.5}(\tilde{B})$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ then go to Step 3

Step 3 Find $\text{Divergence}(\tilde{A}) = (c_1 - a_1)$ and $\text{Divergence}(\tilde{B}) = (c_2 - a_2)$

Case (i) If $\text{Divergence}(\tilde{A}) > \text{Divergence}(\tilde{B})$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If $\text{Divergence}(\tilde{A}) < \text{Divergence}(\tilde{B})$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$ then $\tilde{A} = \tilde{B}$

Remark 6.1 The Mode and Divergence for a triangular vague set $\tilde{A} =< [(a, b, c); \delta, \rho] >$ can be defined as:

$\text{Mode}^{0.5}(\tilde{A}) = b$ and $\text{Divergence}(\tilde{A}) = c - a$

Remark 6.2 The Mode and Divergence for a trapezoidal vague set $\tilde{A} =< [(a, b, c, d); \delta, \rho] >$ can be defined as:

$\text{Mode}^{0.5}(\tilde{A}) = \frac{(b+c)}{2}$ and $\text{Divergence}(\tilde{A}) = d - a$.

6.2.2 Proposed method

In this section, to overcome the shortcomings of existing method [111], on the basis of Kaufmann and Gupta ranking approach [80], a new method is proposed to find the fuzzy optimal solution of linear programming problems with fuzzy sets.

The steps of the proposed method are as follows:

Step 1 Use Step 1 to Step 6 of the method, proposed in Chapter 2, to solve the chosen linear programming problem with normal fuzzy sets (P_1) and check that alternative optimal solution exist or not.

Case (i) If there does not exist any alternative optimal solution then obtained solution $\{x_j\}$ is the optimal solution and find the fuzzy optimal value by putting the values of x_j in $\sum\limits_{j=1}^{n}(\tilde{c}_j x_j)$.

Case (ii) If alternative optimal solution exist then Go to Step 2.

Step 2 Solve the crisp linear programming problem (P_{14}) and check that alternative optimal solution exist or not.

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\ (\sum_{j=1}^{n}\text{Mode}^{0.5}(\tilde{c}_j x_j)))\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i,\ \ i = 1, 2, ..., m\\
&\qquad \sum_{j=1}^{n} \Re_T^{0.5}(\tilde{c}_j x_j) = a\\
&\qquad x_j \geq 0
\end{aligned} \qquad (P_{14})$$

where, a is the optimal value of the crisp linear programming problem (P_5).

Case (i) If there does not exist any alternative optimal solution then obtained solution $\{x_j\}$ is the optimal solution and find the fuzzy optimal value by putting the values of x_j in $\sum\limits_{j=1}^{n}(\tilde{c}_j x_j)$.

Case (ii) If alternative solution exist then Go to Step 3.

Step 3 Solve the crisp linear programming problem (P_{15}) and check that alternative optimal solution problem exist or not.

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\ (\sum_{j=1}^{n} \text{Divergence}(\tilde{c}_j x_j)))\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij} x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad (P_{15})\\
&\qquad \sum_{j=1}^{n} \Re_T^{0.5}(\tilde{c}_j x_j) = a\\
&\qquad \sum_{j=1}^{n} \text{Mode}^{0.5}(\tilde{c}_j x_j) = b\\
&\qquad x_j \geq 0
\end{aligned}$$

where, b is the optimal value of the crisp linear programming problem (P_{14})

The obtained solution is optimal solution and unique fuzzy optimal value can be obtained by putting the values of x_j in $\sum_{j=1}^{n}(\tilde{c}_j x_j)$.

6.3 Advantage of proposed method based on Kaufmann and Gupta ranking approach

To show the advantage of the proposed method based on Kaufmann and Gupta ranking approach the linear programming problem with normal triangular fuzzy sets chosen in Example 6.1, for which the results obtained by using the existing methods are not appropriate, is solved by using the proposed method based on Kaufmann and Gupta approach and it is shown that all the shortcomings occurring in the results are removed.

6.3.1 Solution of the chosen problem

The linear programming problem, chosen in Example 6.1, can be solved by using the following steps of the proposed method based on Kaufmann and Gupta approach:

Step 1 Since on solving the linear programming problem with normal triangular fuzzy sets, chosen in Example 6.1, by using the existing method [111] and the proposed methods, an alternative optimal solution is obtained and the optimal value of the crisp linear programming problem corresponding to crisp linear programming problem (P_6) is 80 so the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$$\begin{aligned}
&Maximize \quad (\mathrm{Mode}^{0.5}(4,8,12;1)x_1 + \mathrm{Mode}^{0.5}(6,8,10;1)x_2))\\
&subject\ to\\
&\qquad x_1 + x_2 \leq 10\\
&\qquad x_1 + x_2 \geq 3\\
&\qquad \Re_T^{0.5}((4,8,12;1)x_1) + \Re_T^{0.5}((6,8,10;1)x_2) = 80\\
&\qquad x_1, x_2 \geq 0
\end{aligned}$$

Step 2 On solving the crisp linear programming problem, obtained in Step 1, alternative optimal solution is obtained. Since the optimal value of the crisp linear programming problem is 8 so the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$$\begin{aligned}
&Maximize \quad (\mathrm{Divegence}^{0.5}(4,8,12;1)x_1 + \mathrm{Divegence}^{0.5}(6,8,10;1)x_2))\\
&subject\ to\\
&\qquad x_1 + x_2 \leq 10\\
&\qquad x_1 + x_2 \geq 3\\
&\qquad \Re_T^{0.5}((4,8,12;1)x_1) + \Re_T^{0.5}((6,8,10;1)x_2) = 80\\
&\qquad \mathrm{Mode}^{0.5}((4,8,12;1)x_1) + \mathrm{Mode}^{0.5}((6,8,10;1)x_2)) = 8\\
&\qquad x_1, x_2 \geq 0
\end{aligned}$$

Step 3 On solving the crisp linear programming problem, obtained in Step 2, the obtained optimal solution is $x_1 = 10$ and $x_2 = 0$. Putting $x_1 = 10$ and $x_2 = 0$ in $((4, 8, 12; 1)x_1 \oplus (6, 8, 10; 1)x_2)$ the fuzzy optimal value of the chosen problem is $(40, 80, 120; 1)$.

6.4 Limitations of proposed method based on Kaufmann and Gupta ranking approach

In this section, some important results are proved to show that the proposed method based on Kaufmann and Gupta ranking approach can be used to find the appropriate optimal solution of such linear programming problems in which the parameters are represented by normal triangular fuzzy sets or generalized triangular fuzzy sets having same heights or triangular vague sets but the same method can't be used for solving such linear programming problem in which the parameters are represented by normal trapezoidal fuzzy sets or generalized trapezoidal fuzzy sets or triangular vague sets.

The results of the proposed method based on Kaufmann and Gupta approach are appropriate due to following property, proved in Proposition 6.1, of normal triangular fuzzy sets:

If $\tilde{A}$ and $\tilde{B}$ are two normal triangular fuzzy sets or generalized triangular fuzzy sets having equal heights or triangular vague sets such that $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$, $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ and $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$ then $\tilde{A} = \tilde{B}$.

In Proposition 6.2 to Proposition 6.6 it is proved that if all these three conditions are satisfied for normal trapezoidal fuzzy sets or generalized trapezoidal fuzzy sets or trapezoidal vague sets then chosen sets may or may not be equal. Due to the same reason, proposed method based on Kaufmann and Gupta approach can't be used to find the appropriate results of such linear programming problems in which the parameters are represented by normal trapezoidal or generalized trapezoidal or trapezoidal vague sets.

Proposition 6.1 Let $\tilde{A} = (a_1, b_1, c_1; 1)$ and $\tilde{B} = (a_2, b_2, c_2; 1)$ be two normal triangular fuzzy sets such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$.

Then $\tilde{A} = \tilde{B}$

Proof:- Using equation (2.38),

$\Re_T^{0.5}(\tilde{A}) = \frac{1}{4}(a_1 + 2b_1 + c_1)$, $\Re_T^{0.5}(\tilde{B}) = \frac{1}{4}(a_2 + 2b_2 + c_2)$

From (i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$

$$\Rightarrow (a_1 + 2b_1 + c_1) = (a_2 + 2b_2 + c_2) \tag{6.1}$$

Using equation (4.4), $\text{Mode}^{0.5}(\tilde{A}) = b_1$, $\text{Mode}^{0.5}(\tilde{B}) = b_2$

From (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$

$$\Rightarrow b_1 = b_2 \tag{6.2}$$

From (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$

$$\Rightarrow (c_1 - a_1) = (c_2 - a_2) \tag{6.3}$$

Solving (6.1), (6.2) and (6.3),

$a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. Therefore $\tilde{A} = \tilde{B}$.

Proposition 6.2 Let $\tilde{A} = (a_1, b_1, c_1; w)$ and $\tilde{B} = (a_2, b_2, c_2; w)$ be two generalized triangular fuzzy sets such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ (iii) $\text{Divergence}(\tilde{A}) =$

Divergence($\tilde{B}$).

Then $\tilde{A} = \tilde{B}$.

Proof:- Using equation (2.38),

$\Re_T^{0.5}(\tilde{A}) = \frac{w}{4}(a_1 + 2b_1 + c_1)$, $\Re_T^{0.5}(\tilde{B}) = \frac{w}{4}(a_2 + 2b_2 + c_2)$

From (i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$

$$\Rightarrow w(a_1 + 2b_1 + c_1) = w(a_2 + 2b_2 + c_2) \tag{6.4}$$

Using equation (4.4), $\text{Mode}^{0.5}(\tilde{A}) = b_1$, $\text{Mode}^{0.5}(\tilde{B}) = b_2$

From (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$

$$\Rightarrow wb_1 = wb_2 \tag{6.5}$$

From (iii) Divergence($\tilde{A}$) = Divergence($\tilde{B}$)

$$\Rightarrow (c_1 - a_1) = (c_2 - a_2) \tag{6.6}$$

Solving (6.4), (6.5) and (6.6),

$a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. Therefore $\tilde{A} = \tilde{B}$.

Proposition 6.3 Let $\tilde{A} = (a_1, b_1, c_1, d_1; w)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w)$ be two generalized trapezoidal fuzzy sets such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ (iii) Divergence($\tilde{A}$) = Divergence($\tilde{B}$).

Then $\tilde{A} \neq \tilde{B}$

Proof:- Using equation (2.38),

$\Re_T^{0.5}(\tilde{A}) = \frac{1}{4}(a_1 + b_1 + c_1 + d_1)$, $\Re_T^{0.5}(\tilde{B}) = \frac{1}{4}(a_2 + b_2 + c_2 + d_2)$

From (i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$

$$\Rightarrow w(a_1 + b_1 + c_1 + d_1) = w(a_2 + b_2 + c_2 + d_2) \tag{6.7}$$

Using equation (4.4), $\text{Mode}^{0.5}(\tilde{A}) = \frac{1}{2}(b_1 + c_1)$, $\text{Mode}^{0.5}(\tilde{B}) = \frac{1}{2}(b_2 + c_2)$

From (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$

$$\Rightarrow w(b_1 + c_1) = w(b_2 + c_2) \tag{6.8}$$

From (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$

$$\Rightarrow (d_1 - a_1) = (d_2 - a_2) \tag{6.9}$$

Solving (6.7), (6.8) and (6.9),

$a_1 = a_2$ and $d_1 = d_2$. Therefore $(a_1, b_1, c_1, d_1; w) \neq (a_2, b_2, c_2, d_1; w) \Rightarrow \tilde{A} \neq \tilde{B}$

Proposition 6.4 Let $\tilde{A} = (a_1, b_1, c_1, d_1; 1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; 1)$ be two trapezoidal fuzzy set such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$.

Then $\tilde{A} \neq \tilde{B}$.

Proof:- Putting $w = 1$ in the Proposition 6.3 we get

$a_1 = a_2$, and $d_1 = d_2$. Therefore $(a_1, b_1, c_1, d_1; 1) \neq (a_2, b_2, c_2, d_1; 1) \Rightarrow \tilde{A} \neq \tilde{B}$

Proposition 6.5 Let $\tilde{A} =< [(a_1, b_1, c_1); \delta, \rho] >$ and $\tilde{B} =< [(a_2, b_2, c_2); \delta, \rho] >$ be two triangular vague sets such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$.

Then $\tilde{A} = \tilde{B}$

Proof:- Using equation (5.16),

$\Re_T^{0.5}(\tilde{A}) = \frac{1}{2}\big\{\frac{\delta}{2}(a_1 + 2b_1 + c_1) + \frac{(\rho-\delta)}{2}((a_1 + 2b_1 + c_1) + \frac{\delta}{\rho}(-a_1 + c_1))\big\}$,

$\Re_T^{0.5}(\tilde{B}) = \frac{1}{2}\big\{\frac{\delta}{2}(a_2 + 2b_2 + c_2) + \frac{(\rho-\delta)}{2}((a_2 + 2b_2 + c_2) + \frac{\delta}{\rho}(-a_2 + c_2))\big\}$

From (i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$

$$\Rightarrow \frac{1}{2}\Big\{\frac{\delta}{2}(a_1+2b_1+c_1)+\frac{(\rho-\delta)}{2}((a_1+2b_1+c_1)+\frac{\delta}{\rho}(-a_1+c_1))\Big\}$$

$$=\frac{1}{2}\Big\{\frac{\delta}{2}(a_2+2b_2+c_2)+\frac{(\rho-\delta)}{2}((a_2+2b_2+c_2)+\frac{\delta}{\rho}(-a_2+c_2))\Big\} \tag{6.10}$$

Using equation (5.18), $\text{Mode}^{0.5}(\tilde{A}) = b_1$, $\text{Mode}^{0.5}(\tilde{B}) = b_2$

From (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$

$$\Rightarrow b_1 = b_2 \tag{6.11}$$

Since $\text{Divergence}(\tilde{A}) = (c_1 - a_1)$, $\text{Divergence}(\tilde{B}) = (c_2 - a_2)$

From (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$

$$\Rightarrow (c_1 - a_1) = (c_2 - a_2) \tag{6.12}$$

Solving (6.10), (6.11) and (6.12), we get

$a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. Therefore $\tilde{A} = \tilde{B}$.

Proposition 6.6 Let $\tilde{A} =< [(a_1, b_1, c_1, d_1); \delta, \rho] >$ and $\tilde{B} =< [(a_2, b_2, c_2, d_2); \delta, \rho] >$ be two trapezoidal vague sets such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$.

Then $\tilde{A} \neq \tilde{B}$

Proof:- Using equation (5.16),

$\Re_T^{0.5}(\tilde{A}) = \frac{1}{2}\big\{\frac{\delta}{2}(a_1+b_1+c_1+d_1)+\frac{(\rho-\delta)}{2}((a_1+b_1+c_1+d_1)+\frac{\delta}{\rho}(-a_1+d_1))\big\}$,

$\Re_T^{0.5}(\tilde{B}) = \frac{1}{2}\big\{\frac{\delta}{2}(a_2+b_2+c_2+d_2)+\frac{(\rho-\delta)}{2}((a_2+b_2+c_2+d_2)+\frac{\delta}{\rho}(-a_2+d_2))\big\}$

From (i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$

$$\Rightarrow \frac{1}{2}\Big\{\frac{\delta}{2}(a_1+b_1+c_1+d_1)+\frac{(\rho-\delta)}{2}((a_1+b_1+c_1+d_1)+\frac{\delta}{\rho}(-a_1+d_1))\Big\}$$

$$= \frac{1}{2}\{\frac{\delta}{2}(a_2 + b_2 + c_2 + d_2) + \frac{(\rho - \delta)}{2}((a_2 + b_2 + c_2 + d_2) + \frac{\delta}{\rho}(-a_2 + d_2))\} \quad (6.13)$$

Using equation (5.18), $\text{Mode}^{0.5}(\tilde{A}) = b_1 + c_1$, $\text{Mode}^{0.5}(\tilde{A}) = b_2 + c_2$

From (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$

$$\Rightarrow (b_1 + c_1) = (b_2 + c_2) \quad (6.14)$$

Since $\text{Divergence}(\tilde{A}) = (d_1 - a_1)$, $\text{Divergence}(\tilde{B}) = (d_2 - a_2)$

From (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$

$$\Rightarrow (d_1 - a_1) = (d_2 - a_2) \quad (6.15)$$

Solving (6.13), (6.14) and (6.15) we get

$a_1 = a_2$ and $d_1 = d_2$. Therefore $< [(a_1, b_1, c_1, d_1); \delta, \rho] > \neq < [(a_2, b_2, c_2, d_1); \delta, \rho] > \Rightarrow$ $\tilde{A} \neq \tilde{B}$

6.5 Some important results

In Section 6.4, it is shown that Rank, Mode and Divergence are not sufficient for the equality of fuzzy sets and vague sets. To overcome this shortcoming in this section, one more parameter Spread, is taken in the account. Also, some important results that are proved.

Let $\tilde{A} =< [(a, b, c, d); \delta, \rho] >$ be a trapezoidal vague set then $(b - a)$ and $(d - c)$ may be be called Left spread and Right spread respectively.

Proposition 6.7 Let $\tilde{A} =< [(a_1, b_1, c_1, d_1); \delta, \rho] >$ and $\tilde{B} =< [(a_2, b_2, c_2, d_2); \delta, \rho] >$ be two trapezoidal vague sets such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$.

Then

(a) Left spread $(\tilde{A}) >$ Left spread $(\tilde{B})$ iff $b_1 > b_2$

(b) Left spread $(\tilde{A}) <$ Left spread $(\tilde{B})$ iff $b_1 < b_2$

(c) Left spread $(\tilde{A}) =$ Left spread $(\tilde{B})$ iff $b_1 = b_2$

Proof: Since $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$, Mode$^{0.5}(\tilde{A})$ = Mode$^{0.5}(\tilde{B})$ and Divergence$(\tilde{A})$ = Divergence$(\tilde{B})$ so, from Proposition 6.6, we have

$$a_1 = a_2 \tag{6.16}$$

$$d_1 = d_2 \tag{6.17}$$

$$(b_1 + c_1) = (b_2 + c_2) \tag{6.18}$$

(a) Left spread $(\tilde{A}) >$ Left spread $(\tilde{B})$

iff $(b_1 - a_1) > (b_2 - a_2)$

iff $b_1 > b_2 \qquad (\because a_1 = a_2)$

Hence, Left spread $(\tilde{A}) >$ Left spread $(\tilde{B})$ iff $b_1 > b_2$.

(b) Left spread $(\tilde{A}) <$ Left spread $(\tilde{B})$

iff $(b_1 - a_1) < (b_2 - a_2)$

iff $b_1 < b_2 \qquad (\because a_1 = a_2)$

Hence, Left spread $(\tilde{A}) <$ Left spread $(\tilde{B})$ iff $b_1 < b_2$.

(c) Left spread $(\tilde{A}) =$ Left spread $(\tilde{B})$

iff $(b_1 - a_1) = (b_2 - a_2)$

iff $b_1 = b_2 \qquad (\because a_1 = a_2)$

Hence, Left spread $(\tilde{A}) =$ Left spread $(\tilde{B})$ iff $b_1 = b_2$.

Proposition 6.8 Let $\tilde{A} =< [(a_1, b_1, c_1, d_1); \delta, \rho] >$ and $\tilde{B} =< [(a_2, b_2, c_2, d_2); \delta, \rho] >$ be two trapezoidal vague sets such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) Mode$^{0.5}(\tilde{A})$ = Mode$^{0.5}(\tilde{B})$ (iii) Divergence$(\tilde{A})$ =

Divergence($\tilde{B}$).

Then

(a) Right spread $(\tilde{A}) >$ Right spread $(\tilde{B})$ iff $c_1 < c_2$

(b) Right spread $(\tilde{A}) <$ Right spread $(\tilde{B})$ iff $c_1 > c_2$

(c) Right spread $(\tilde{A}) =$ Right spread $(\tilde{B})$ iff $c_1 = c_2$

Proof: Since $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$, Mode$^{0.5}(\tilde{A}) =$ Mode$^{0.5}(\tilde{B})$ and Divergence($\tilde{A}$) = Divergence($\tilde{B}$) so, from Proposition 6.6, we have

$$a_1 = a_2 \tag{6.19}$$

$$d_1 = d_2 \tag{6.20}$$

$$(b_1 + c_1) = (b_2 + c_2) \tag{6.21}$$

(a) Right spread $(\tilde{A}) >$ Right spread $(\tilde{B})$

iff $(d_1 - c_1) > (d_2 - c_2)$

iff $c_1 < c_2 \qquad (\because a_1 = a_2)$

Hence, Right spread $(\tilde{A}) >$ Right spread $(\tilde{B})$ iff $c_1 < c_2$.

(b) Right spread $(\tilde{A}) <$ Right spread $(\tilde{B})$

iff $(d_1 - c_1) < (d_2 - c_2)$

iff $c_1 > c_2 \qquad (\because a_1 = a_2)$

Hence, Right spread $(\tilde{A}) <$ Right spread $(\tilde{B})$ iff $c_1 > c_2$.

(c) Right spread $(\tilde{A}) =$ Right spread $(\tilde{B})$

iff $(d_1 - c_1)=(d_2 - c_2)$

iff $c_1 = c_2 \qquad (\because a_1 = a_2)$

Hence, Right spread $(\tilde{A}) =$ Right spread $(\tilde{B})$ iff $c_1 = c_2$.

Proposition 6.9 Let $\tilde{A} = <[(a_1, b_1, c_1, d_1); \delta, \rho]>$ and $\tilde{B} = <[(a_2, b_2, c_2, d_2); \delta, \rho]>$

be two trapezoidal vague sets such that

(i) $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$ (ii) $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ (iii) $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$.

Then

(a) Left spread $(\tilde{A}) >$ Left spread $(\tilde{B})$ iff Right spread $(\tilde{A}) >$ Right spread $(\tilde{B})$

(b) Left spread $(\tilde{A}) <$ Left spread $(\tilde{B})$ iff Right spread $(\tilde{A}) <$ Right spread $(\tilde{B})$

(c) Left spread $(\tilde{A}) =$ Left spread $(\tilde{B})$ iff Right spread $(\tilde{A}) =$ Right spread $(\tilde{B})$

Proof: Since $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$, $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$ and $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$ so, from Proposition 6.7, we have

$$a_1 = a_2$$

$$d_1 = d_2$$

$$(b_1 + c_1) = (b_2 + c_2)$$

(a) Left spread $(\tilde{A}) >$Left spread $(\tilde{B})$

iff $b_1 > b_2$ (from Proposition 6.7)

iff $c_1 < c_2$ $(\because (b_1 + c_1) = (b_2 + c_2))$

iff $-c_1 > -c_2$

iff $(d_1 - c_1) > (d_2 - c_2)$ $(\because d_1 = d_2)$

iff Right spread $(\tilde{A}) >$ Right spread $(\tilde{B})$

(b) Left spread $(\tilde{A}) <$ Left spread $(\tilde{B})$

iff $b_1 < b_2$ (from Proposition 6.7)

iff $c_1 > c_2$ $(\because (b_1 + c_1) = (b_2 + c_2))$

iff $-c_1 < -c_2$

iff $(d_1 - c_1) < (d_2 - c_2)$ $(\because d_1 = d_2)$

iff Right spread $(\tilde{A}) <$ Right spread $(\tilde{B})$

(c) Left spread $(\tilde{A}) =$ Left spread $(\tilde{B})$

iff $b_1 = b_2$ (from Proposition 6.7)

iff $c_1 = c_2$ $(\because (b_1 + c_1) = (b_2 + c_2))$

iff $-c_1 = -c_2$

iff $(d_1 - c_1) = (d_2 - c_2)$ $(\because d_1 = d_2)$

iff Right spread $(\tilde{A}) =$ Right spread $(\tilde{B})$

6.6 Proposed RMDS ranking approach

In this section, to overcome the limitations of Kaufmann and Gupta approach a new ranking approach, named as RMDS ranking approach, is proposed for comparing trapezoidal vague sets having equal degree of membership and non-membership.

Let $\tilde{A} = < [(a_1, b_1, c_1, d_1); \delta, \rho] >$ and $\tilde{B} = < [(a_2, b_2, c_2, d_2); \delta, \rho] >$ be two trapezoidal vague sets. Then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Find $\Re_T^\lambda(\tilde{A})$ and $\Re_T^\lambda(\tilde{B})$ for some particular value of λ.

Case (i) If $\Re_T^\lambda(\tilde{A}) > \Re_T^\lambda(\tilde{B})$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If $\Re_T^\lambda(\tilde{A}) < \Re_T^\lambda(\tilde{B})$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If $\Re_T^\lambda(\tilde{A}) = \Re_T^\lambda(\tilde{B})$ then go to Step 2.

Step 2 Find Mode$^\lambda(\tilde{A})$ and Mode$^\lambda(\tilde{B})$

Case (i) If Mode$^\lambda(\tilde{A}) >$ Mode$^\lambda(\tilde{B})$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If Mode$^\lambda(\tilde{A}) <$ Mode$^\lambda(\tilde{B})$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If Mode$^\lambda(\tilde{A}) =$ Mode$^\lambda(\tilde{B})$ then go to Step 3.

Step 3 Find Divergence$(\tilde{A})$ and Divergence$(\tilde{B})$

Case (i) If Divergence$(\tilde{A}) >$ Divergence$(\tilde{B})$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If Divergence($\tilde{A}$) < Divergence($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$

Case (iii) If Divergence($\tilde{A}$) = Divergence($\tilde{B}$) then go to Step 4.

Step 4 Find Left spread ($\tilde{A}$) and Left spread ($\tilde{B}$)

Case (i) If Left spread ($\tilde{A}$) > Left spread ($\tilde{B}$)

i.e., $b_1 > b_2$ then $\tilde{A} \succ \tilde{B}$. (from Proposition 6.7)

Case (ii) Left spread ($\tilde{A}$) < Left spread ($\tilde{B}$)

i.e., $b_1 < b_2$ then $\tilde{A} \prec \tilde{B}$. (from Proposition 6.7)

Case (iii) Left spread ($\tilde{A}$) = Left spread ($\tilde{B}$)

i.e., $b_1 = b_2$ then $\tilde{A} = \tilde{B}$. (from Proposition 6.7)

6.6.1 Particular cases

Since trapezoidal vague sets are the generalization of normal triangular fuzzy sets, trapezoidal fuzzy sets, generalized triangular fuzzy sets, generalized trapezoidal fuzzy sets and triangular vague sets so the approach RMDS ranking approach can also be used for comparing these sets by considering the following assumptions:

(i) Assuming $b_1 = c_1 = b_1'$ and $b_2 = c_2 = b_2'$ the proposed RMDS ranking approach can be used for comparing triangular vague sets $\tilde{A} =< [(a_1, b_1', d_1); \delta, \rho] >$ and $\tilde{B} =< [(a_2, b_2', d_2); \delta, \rho] >$.

(ii) Assuming $\delta = \rho = \delta_1'$ the proposed RMDS ranking approach can be used for comparing generalized trapezoidal fuzzy sets $\tilde{A} = (a_1, b_1, c_1, d_1; \delta_1')$ and $\tilde{B} = (a_2, b_2, c_2, d_2; \delta_1')$.

(iii) Assuming $b_1 = c_1 = b_1'$, $\delta = \rho = \delta_1'$ and $b_2 = c_2 = b_2'$ the proposed RMDS ranking approach can be used for comparing generalized triangular fuzzy sets

$\tilde{A} =< [(a_1, b'_1, d_1; \delta'_1)] >$ and $\tilde{B} =< [(a_2, b'_2, d_2; \delta'_1)] >$.

(iv) Assuming $\delta = \rho = 1$ the proposed RMDS ranking approach can be used for comparing trapezoidal fuzzy numbers $\tilde{A} =< [(a_1, b_1, c_1, d_1; 1)] >$ and $\tilde{B} =< [(a_2, b_2, c_2, d_2; 1)] >$.

(v) Assuming $b_1 = c_1 = b'_1$, $\delta = \rho = 1$ and $b_2 = c_2 = b'_2$ the proposed RMDS ranking approach , can be used for comparing triangular fuzzy sets $\tilde{A} =< [(a_1, b'_1, d_1; 1)] >$ and $\tilde{B} =< [(a_2, b'_2, d_2; 1)] >$.

6.6.2 Illustrative examples

In this section, the proposed RMDS ranking approach is illustrated by comparing some fuzzy sets and vague sets.

Example 6.2

Using the proposed RMDS ranking approach the generalized trapezoidal fuzzy sets $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 0.8)$ and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 0.8)$ can be compared as follows:

Step 1 $\Re_T^{0.5}(\tilde{A}) = 0.24$ and $\Re_T^{0.5}(\tilde{B}) = 0.24$. Since, $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$, so go to Step 2.

Step 2 $\text{Mode}^{0.5}(\tilde{A}) = 0.3$ and $\text{Mode}^{0.5}(\tilde{B}) = 0.3$. Since, $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$, so go to Step 3.

Step 3 $\text{Divergence}(\tilde{A}) = 0.4$ and $\text{Divergence}(\tilde{B}) = 0.4$. Since, $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$, so go to Step 4.

Step 4 Left spread $(\tilde{A}) = 0.1$ and Left spread $(\tilde{B}) = 0.2$. Since, Left spread $(\tilde{A}) <$ Left spread $(\tilde{B})$, so $\tilde{A} \prec \tilde{B}$

Example 6.3 Using the proposed RMDS ranking approach the normal trapezoidal fuzzy sets $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$ can be compared as follows:

Step 1 $\Re_T^{0.5}(\tilde{A}) = 0.3$ and $\Re_T^{0.5}(\tilde{B}) = 0.3$. Since, $\Re_T^{0.5}(\tilde{B}) = \Re_T^{0.5}(\tilde{B})$, so go to Step 2.

Step 2 $\text{Mode}^{0.5}(\tilde{A}) = 0.3$ and $\text{Mode}^{0.5}(\tilde{B}) = 0.3$. Since, $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$, so go to Step 3.

Step 3 $\text{Divergence}(\tilde{A}) = 0.4$ and $\text{Divergence}(\tilde{B}) = 0.4$. Since, $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$, so go to Step 4.

Step 4 Left spread $(\tilde{A}) = 0.1$ and Left spread $(\tilde{B}) = 0.2$. Left spread $(\tilde{A}) <$ Left spread $(\tilde{B}) \Rightarrow \tilde{A} \prec \tilde{B}$

Example 6.4 Using the proposed RMDS ranking approach the trapezoidal vague sets $\tilde{A} = <[(4, 5, 9, 14); 0.3, 0.4]>$ and $\tilde{B} = <[(4, 6, 8, 14); 0.3, 0.4]>$ can be compared as follows:

Step 1 $\Re_T^{0.5}(\tilde{A}) = 3.3875$ and $\Re_T^{0.5}(\tilde{B}) = 3.3875$. Since, $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$, so go to Step 2.

Step 2 $\text{Mode}^{0.5}(\tilde{A}) = 7$ and $\text{Mode}^{0.5}(\tilde{B}) = 7$. Since, $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$, so go to Step 3.

Step 3 $\text{Divergence}(\tilde{A}) = 10$ and $\text{Divergence}(\tilde{B}) = 10$. Since, $\text{Divergence}(\tilde{A}) = \text{Divergence}(\tilde{B})$, so go to Step 4.

Step 4 Left spread $(\tilde{A}) = 1$ and Left spread $(\tilde{B}) = 2$. Since, Left spread $(\tilde{A}) <$ Left spread $(\tilde{B})$, so $\tilde{A} \prec \tilde{B}$

Example 6.5 Using the proposed RMDS ranking approach the trapezoidal vague sets $\tilde{A} = <[(2, 6, 8); 0.5, 0.8]>$ and $\tilde{B} = <[(3, 6, 7); 0.5, 0.8]>$ can be compared as follows:

Step 1 $\Re_T^{0.5}(\tilde{A}) = 2.425$ and $\Re_T^{0.5}(\tilde{B}) = 2.425$. Since, $\Re_T^{0.5}(\tilde{A}) = \Re_T^{0.5}(\tilde{B})$, so go to

Step 2.

Step 2 $\text{Mode}^{0.5}(\tilde{A}) = 6$ and $\text{Mode}^{0.5}(\tilde{B}) = 6$. Since, $\text{Mode}^{0.5}(\tilde{A}) = \text{Mode}^{0.5}(\tilde{B})$, so go to Step 3.

Step 3 $\text{Divergence}(\tilde{A}) = 6$ and $\text{Divergence}(\tilde{B}) = 4$. Since, $\text{Divergence}(\tilde{A}) > \text{Divergence}(\tilde{B})$, so $\tilde{A} \succ \tilde{B}$.

6.7 Proposed method based on RMDS ranking approach

To overcome the shortcomings of existing method [111] and to overcome the limitations of proposed method based on Kaufmann and Gupta approach, a new method, based on RMDS ranking approach, is proposed for solving such linear programming problems in which the parameters are represented by trapezoidal vague sets. The same method can also be used for solving such linear programming problems in which the parameters are represented by normal triangular fuzzy sets or generalized triangular fuzzy sets or normal trapezoidal fuzzy sets or generalized trapezoidal fuzzy sets or triangular vague sets.

The steps of the proposed method are as follows:

Step 1 Assuming $\tilde{c}_j =< [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] >$ the linear programming problem with fuzzy set (P_1) can be converted into (P_{16}):

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j)\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad\qquad (P_{16})\\
&\qquad x_j \geq 0
\end{aligned}$$

Step 2 Using Definition 2.8 and on the basis of the proposed RMDS ranking approach, the optimal solution of the linear programming problem with trapezoidal vague sets (P_{16}) can be obtained by solving the crisp linear programming problem (P_{17})

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\Re_T^{0.5}(\sum_{j=1}^{n} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j))\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \ \ i = 1, 2, ..., m \qquad (P_{17})\\
&\qquad x_j \geq 0
\end{aligned}$$

Step 3 Using linearity property

$\Re_T^{0.5}(\sum_{j=1}^{n} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j) = \sum_{j=1}^{n}(\Re_T^{0.5} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j)$, the crisp linear programming problem (P_{17}) can be converted into crisp linear programming problem (P_{18})

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n}(\Re_T^{0.5} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j))\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \ \ i = 1, 2, ..., m \qquad (P_{18})\\
&\qquad x_j \geq 0
\end{aligned}$$

Step 4 Using $\Re_T^{0.5}(< [(a, b, c, d); \delta, \rho] >) = \frac{1}{2}\{\frac{\delta}{2}(a_1+b_1+c_1+d_1)+\frac{(\rho-\delta)}{2}((a_1+b_1+c_1+d_1)+\frac{\delta}{\rho}(-a_1+d_1))\}$ the crisp linear programming problem (P_{18}) can be converted into (P_{19})

$$\begin{aligned}
&Maximize\ (or\ Minimize)\ (\tfrac{1}{2}\{\tfrac{\delta}{2}(c_{j1} + c_{j2} + c_{j3} + c_{j4}) + \tfrac{(\rho-\delta)}{2}((c_{j1} + c_{j2} + c_{j3} + c_{j4}) +\\
&\tfrac{\delta}{\rho}(-c_{j1} + c_{j4}))x_j)\})\\
&subject\ to\\
&\qquad \sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \ \ i = 1, 2, ..., m \qquad (P_{19})\\
&\qquad x_j \geq 0
\end{aligned}$$

Step 5 Solve the crisp linear programming problem (P_{19}) and check that alternative optimal solution exist or not.

Case (i) If there does not exist any alternative optimal solution then the obtained solution is optimal solution. Find the fuzzy optimal value by putting the values of x_j in $(\sum\limits_{j=1}^{n} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j)$.

Case (ii) If alternative optimal solution exist then Go to Step 6.

Step 6 Solve the crisp linear programming problem (P_{20}) and check that alternative optimal solution exist or not.

$$Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n}(\text{Mode}^{0.5}(< [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j))$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \\ &\sum_{j=1}^{n}(\Re_T^{0.5} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j) = a \\ &x_j \geq 0 \end{aligned} \qquad (P_{20})$$

where a, is the optimal value of the crisp linear programming problem (P_{19}).

Case (i) If there does not exist any alternative solution obtained then solution is optimal solution. Find the fuzzy optimal value by values of x_j in $(\sum\limits_{j=1}^{n} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j)$.

Case (ii) If alternative solution exist then Go to Step 7.

Step 7 Solve the crisp linear programming problem (P_{21}) and check that alternative optimal solution exist or not.

$$Maximize\ (or\ Minimize)\ (\sum_{j=1}^{n}(\text{Divergence}(< [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j))$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad (P_{21}) \\ &\sum_{j=1}^{n}(\Re_T^{0.5} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j) = a \end{aligned}$$

$$\sum_{j=1}^{n}(\text{Mode}^{0.5} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j) = b$$
$$x_j \geq 0$$

where b, is the optimal value of the crisp linear programming problem (P_{20}).

Case (i) If there does not exist any alternative optimal solution then obtained solution is optimal solution. Find the fuzzy optimal value by values of x_j in $(\sum_{j=1}^{n} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j)$

Case (ii) If alternative solution exist then Go to Step 8.

Step 8 Solve the crisp linear programming problem (P_{22}) and check that alternative optimal solution exist or not.

$$\textit{Maximize (or Minimize)}\ (\sum_{j=1}^{n}(\text{Left Spread}(< [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} a_{ij}x_j \geq, =, \leq b_i, \quad i = 1, 2, ..., m \qquad (P_{22})\\
&\sum_{j=1}^{n}(\Re_T^{0.5} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j) = a\\
&\sum_{j=1}^{n}(\text{Mode}^{0.5} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j) = b\\
&\sum_{j=1}^{n}(\text{Divergence} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j) = c\\
&x_j \geq 0
\end{aligned}$$

where, c is the optimal value of the crisp linear programming problem (P_{22}).

The solution obtained in Step 8 is optimal solution and fuzzy optimal value is obtained by putting the values of x_j in $(\sum_{j=1}^{n} < [(c_{j1}, c_{j2}, c_{j3}, c_{j4}); \delta, \rho] > x_j)$

6.7.1 Illustrative examples

In this section proposed method based on RMDS ranking approach is illustrated by solving some linear programming problems with normal and generalized trapezoidal vague set fuzzy sets. To illustrate the proposed method a linear programming problem with trapezoidal vagues set, chosen in Example 6.6, is solved by using the proposed method based on RMDS ranking approach.

Example 6.6 Solve

$$Maximize \;\; (<[(1,4,5,8);0.6,0.8]> x_1 \oplus <[(1,3,6,8);0.6,0.8]> x_2)$$

$$subject\ to$$

$$x_1 + x_2 \leq 2$$

$$x_1 + x_2 \geq 1$$

$$x_1 \leq 4$$

$$x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Solution:-Using Step 1, the linear programming problem, chosen in Example 6.6, can be solved by using the following steps of the proposed method:

Step 1 Using Step 2 of the proposed method the chosen linear programming problem can be written as

$$Maximize \;\; (\Re_T^{0.5}(<[(1,4,5,8);0.6,0.8]> x_1 \oplus <[(1,3,6,8);0.6,0.8]> x_2))$$

$$subject\ to$$

$$x_1 + x_2 \leq 2$$

$$x_1 + x_2 \geq 1$$

$$x_1 \leq 4$$

$$x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Step 2 Using linearity property

$\Re_T^{0.5}(\sum_{j=1}^{n} <[(c_{j1},c_{j2},c_{j3},c_{j4});\delta,\rho]> x_j) = \sum_{j=1}^{n}(\Re_T^{0.5} <[(c_{j1},c_{j2},c_{j3},c_{j4});\delta,\rho]> x_j)$, the crisp linear programming problem, obtained in Step 1, can be written as:

$Maximize\ \ (\Re_T^{0.5}(<[(1,4,5,8);0.6,0.8]>x_1)+\Re_T^{0.5}(<[(1,3,6,8);0.6,0.8]>x_2))$

subject to

$$x_1+x_2\leq 2$$

$$x_1+x_2\geq 1$$

$$x_1\leq 4$$

$$x_2\geq 3$$

$$x_1,x_2\geq 0$$

Step 3 Using $\Re(<[(a,b,c,d);\delta,\rho]>)=\frac{1}{2}\big\{\frac{\delta}{2}(a+b+c+d)+\frac{(\rho-\delta)}{2}((a+b+c+d)+\frac{\delta}{\rho}(-a+d))\big\}$, the crisp linear programming problem, obtained in Step 2, can be written as

$$Maximize\ \ (\tfrac{309}{80}x_1+\tfrac{309}{80}x_2)$$

subject to

$$x_1+x_2\leq 2$$

$$x_1+x_2\geq 1$$

$$x_1\leq 4$$

$$x_2\geq 3$$

$$x_1,x_2\geq 0$$

Step 4 Since on solving the crisp linear programming problem, obtained in Step 3, alternative optimal solution is obtained and the optimal value is $\frac{309}{40}$ so the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$Maximize\ \ (\text{Mode}^{0.5}(<[(1,4,5,8;)0.6,0.8]>x_1)+\text{Mode}^{0.5}(<[(1,3,6,8);0.6,0.8]>x_2))$

subject to

$x_1 + x_2 \leq 2$

$x_1 + x_2 \geq 1$

$\frac{309}{80}x_1 + \frac{309}{80}x_2 = \frac{309}{40}$

$x_1 \leq 4$

$x_2 \geq 3$

$x_1, x_2 \geq 0$

Step 5 Using Mode($< [(a,b,c,d);\delta,\rho] >$) $= \frac{(b+c)}{2}$, the crisp linear programming problem, obtained in Step 4, can be written as

$Maximize \ \ (\frac{9}{2}x_1 + \frac{9}{2}x_2)$

subject to

$x_1 + x_2 \leq 2$

$x_1 + x_2 \geq 1$

$\frac{309}{80}x_1 + \frac{309}{80}x_2 = \frac{309}{40}$

$x_1 \leq 4$

$x_2 \geq 3$

$x_1, x_2 \geq 0$

Step 6 Since on solving the crisp linear programming problem, obtained in Step 5, alternative optimal solution is obtained and the optimal value is 9 so the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$Maximize$ (Divergence($< [(1,4,5,8);0.6,0.8] > x_1$)+Divergence($< [(1,3,6,8);0.6,0.8] > x_2$))

subject to

$x_1 + x_2 \leq 2$

$x_1 + x_2 \geq 1$

$\frac{309}{80}x_1 + \frac{309}{80}x_2 = \frac{309}{40}$

$\frac{9}{2}x_1 + \frac{9}{2}x_2 = 9$

$x_1 \leq 4$

$x_2 \geq 3$

$x_1, x_2 \geq 0$

Step 7 Using Divergence($< [(a, b, c, d); \delta, \rho] >$) $= d - a$, the crisp linear programming problem, obtained in Step 6, can be written as

$Maximize \;\; (7x_1 + 7x_2)$

subject to

$x_1 + x_2 \leq 2$

$x_1 + x_2 \geq 1$

$\frac{309}{80}x_1 + \frac{309}{80}x_2 = \frac{309}{40}$

$\frac{9}{2}x_1 + \frac{9}{2}x_2 = 9$

$x_1 \leq 4$

$x_2 \geq 3$

$x_1, x_2 \geq 0$

Step 8 Since on solving the crisp linear programming problem, obtained in Step 7, alternative optimal solution is obtained and the optimal value is 14 so the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

$Maximize$ (Left Spread($< [(1, 4, 5, 8); 0.6, 0.8] > x_1$)+Left Spread($< [(1, 3, 6, 8); 0.6, 0.8] > x_2$))

subject to

$$x_1 + x_2 \leq 2$$

$$x_1 + x_2 \geq 1$$

$$\frac{309}{80}x_1 + \frac{309}{80}x_2 = \frac{309}{40}$$

$$\frac{9}{2}x_1 + \frac{9}{2}x_2 = 9$$

$$7x_1 + 7x_2 = 14$$

$$x_1 \leq 4$$

$$x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Step 9 Using Left Spread($< [(a, b, c, d); \delta, \rho] >$) $= b - a$, the crisp linear programming problem, obtained in Step 8, can be written as

$$Maximize \ \ (3x_1 + 2x_2)$$

subject to

$$x_1 + x_2 \leq 2$$

$$x_1 + x_2 \geq 1$$

$$\frac{309}{80}x_1 + \frac{309}{80}x_2 = \frac{309}{40}$$

$$\frac{9}{2}x_1 + \frac{9}{2}x_2 = 9$$

$$7x_1 + 7x_2 = 14$$

$$x_1 \leq 4$$

$$x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Step 10 On solving the crisp linear programming problem, obtained in Step 9, the obtained solution is $x_1 = 2$ and $x_2 = 0$ and putting $x_1 = 2$ and $x_2 = 0$ in ($< [(1, 4, 5, 8); 0.6, 0.8] > x_1 \oplus < [(1, 3, 6, 8); 0.6, 0.8] > x_2$) the fuzzy optimal value of the chosen problem is $< [(2, 8, 10, 16); 0.6, 0.8] >$.

6.8 Advantages of proposed method based on RMDS ranking approach

To show the advantages of the proposed method based on RMDS ranking approach over existing methods, method based on Kaufmann and Gupta approach and methods proposed in previous chapters, the results of the linear programming problems with fuzzy sets and vague sets, chosen in Example 6.7 to Example 6.11 and linear programming problem with trapezoidal vague set, chosen in Example 6.1, obtained by using all these methods and the proposed method based on RMDS approach are compared in Table 1.

Example 6.7 Solve

$$Maximize \ \ ((1, 5, 8; 0.8)x_1 \oplus (2, 5, 7; 0.8)x_2)$$

$$subject\ to$$

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$x_1 \leq 5$$

$$x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Example 6.8 Solve

$$Maximize \ \ ((1, 3, 4, 9; 1)x_1 \oplus (1, 2, 5, 9; 1)x_2)$$

$$subject\ to$$

$$x_1 + x_2 \leq 5$$

$$x_1 + x_2 \geq 1$$

$$x_1 \leq 6$$

$$x_1, x_2 \geq 0$$

Example 6.9 Solve

$$Maximize \ \ ((1,4,7,9;0.6)x_1 \oplus (1,3,8,9;0.6)x_2)$$

subject to

$$x_1 + x_2 \leq 5$$

$$x_1 + x_2 \geq 1$$

$$x_1 \leq 6$$

$$x_1, x_2 \geq 0$$

Example 6.10 Solve

$$Maximize \ \ (< [(1,5,8);0.6,0.8] > x_1 \oplus < [(2,5,7);0.6,0.8] > x_2)$$

subject to

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$x_1 \leq 5$$

$$x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Example 6.11 Solve

$$Maximize \ \ (< [(2,4,7,9);0.6,0.7] > x_1 \oplus < [(2,3,8,9);0.6,0.7] > x_2)$$

subject to

$$x_1 + x_2 \leq 3$$

$$x_1 + x_2 \geq 2$$

$$x_1 \leq 5$$

$$x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Table 6.1 Comparison of results

Example	Fuzzy optimal value					
	Existing method [111]	Proposed method based on ranking approach proposed in Chapter 2	Proposed method based on RM ranking approach proposed in Chapter 4	Proposed method based on ranking approach proposed in Chapter 5	Proposed method based on Kaufmann and Gupta ranking approach	Proposed method based on RMDS ranking approach
6.1	$(40, 80, 120; 1)$ and $(60, 80, 100; 1)$	$(40, 80, 120; 1)$ and $(60, 80, 100; 1)$	$(40, 80, 120; 1)$ and $(60, 80, 100; 1)$	$(40, 80, 120; 1)$ and $(60, 80, 100; 1)$	$(40, 80, 120; 1)$	$(40, 80, 120; 1)$
6.7	$(3, 15, 24; 0.8)$ and $(6, 15, 21; 0.8)$	$(3, 15, 24; 0.8)$ and $(6, 15, 21; 0.8)$	$(3, 15, 24; 0.8)$ and $(6, 15, 21; 0.8)$	$(3, 15, 24; 0.8)$ and $(6, 15, 21; 0.8)$	$(3, 15, 24; 0.8)$	$(3, 15, 24; 0.8)$
6.8	$(5, 15, 20, 45; 1)$ and $(5, 10, 25, 45; 1)$	$(5, 15, 20, 45; 1)$ and $(5, 10, 25, 45; 1)$	$(5, 15, 20, 45; 1)$ and $(5, 10, 25, 45; 1)$	$(5, 15, 20, 45; 1)$ and $(5, 10, 25, 45; 1)$	$(5, 15, 20, 45; 1)$ and $(5, 10, 25, 45; 1)$	$(5, 15, 20, 45; 1)$
6.9	$(5, 20, 35, 45; 0.6)$ and $(5, 15, 40, 45; 0.6)$	$(5, 20, 35, 45; 0.6)$ and $(5, 15, 40, 45; 0.6)$	$(5, 20, 35, 45; 0.6)$ and $(5, 15, 40, 45; 0.6)$	$(5, 20, 35, 45; 0.6)$ and $(5, 15, 40, 45; 0.6)$	$(5, 20, 35, 45; 0.6)$ and $(5, 15, 40, 45; 0.6)$	$(5, 20, 35, 45; 0.6)$
6.10	Not applicable	Not applicable	Not applicable	$< [(3, 15, 24); 0.6, 0.8] >$ and $< [(6, 15, 21); 0.6, 0.8] >$	$< [(3, 15, 24); 0.6, 0.8] >$ and $< [(6, 15, 21); 0.6, 0.8] >$	$< [(3, 15, 24); 0.6, 0.8] >$
6.11	Not applicable	Not applicable	Not applicable	$< [(6, 12, 21, 27); 0.6, 0.7] >$ and $< [(6, 9, 24, 27); 0.6, 0.7] >$	$< [(6, 12, 21, 27); 0.6, 0.7] >$ and $< [(6, 9, 24, 27); 0.6, 0.7] >$	$< [(6, 12, 21, 27); 0.6, 0.7] >$

The results of the chosen linear programming problems with generalized fuzzy sets and vague sets obtained by using different existing and proposed methods, shown in Table 6.1, can be explained as follows:

(i) If the linear programming problem with triangular fuzzy sets, chosen in Example 6.1 and Example 6.7, are solved by existing method [111] or proposed method based on ranking approach proposed in Chapter 2 or proposed method based on RM ranking approach proposed in Chapter 4 or proposed method based on ranking approach proposed in Chapter 5 then two different normal triangular fuzzy sets, representing the fuzzy optimal value of the same problem, are obtained which contradicts the property of alternative optimal solutions while on solving the same problem by using the proposed method based on Kaufmann and Gupta ranking approach and by using the proposed method based RMDS ranking approach a unique triangular fuzzy set, representing the fuzzy optimal value, is obtained which is appropriate.

(ii) If the linear programming problem with trapezoidal fuzzy sets, chosen in Example 6.8 and Example 6.9, are solved by existing method [111] or proposed method based on ranking approach proposed in Chapter 2 or proposed method based on RM ranking approach proposed in Chapter 4 or proposed method based on ranking approach proposed in Chapter 5 or the proposed method based on Kaufmann and Gupta ranking approach then two different trapezoidal fuzzy sets, representing the fuzzy optimal value of the same problem, are obtained which contradicts the property of alternative optimal solutions while on solving the same problem by using the proposed method based RMDS ranking approach a unique trapezoidal fuzzy set, representing the fuzzy optimal value, is obtained which is appropriate.

(iii) Since the existing method, proposed method based on ranking approach proposed in Chapter 2 or proposed method based on RM ranking approach proposed in Chapter 4 can be used only for solving such linear programming problems in which either the parameters are represented by normal fuzzy set or generalized fuzzy set but in the linear programming problems, chosen in Example 6.10 and Example 6.11, the parameters are represented by vague sets so none of these proposed method can be used for solving these chosen problems. On solving the linear programming problems with triangular vague sets, chosen in Example 6.10 by using the proposed method based on Kaufmann and Gupta ranking approach and proposed method based on RMDS ranking approach a unique triangular vague set, representing the fuzzy optimal value, of the chosen problem is obtained. On solving the linear programming problem with trapezoidal vague set, chosen in Example 6.11, by using the proposed method based on RMDS approach a unique trapezoidal vague set, representing the fuzzy optimal value of the chosen problem, is obtained while on solving the same problem by using the proposed method based on Kaufmann and Gupta ranking approach two different trapezoidal vague sets, representing the fuzzy optimal value of the same problem, are obtained which contradicts the properties of alternative optimal solution.

On the basis of above discussion it can be concluded that it is better to use proposed method based on RMDS ranking approach as compared to existing and other proposed methods for solving linear programming problems with fuzzy and vague sets.

6.9 Conclusions

In this chapter, the shortcomings of the existing method for solving linear programming problems with fuzzy and vague sets are pointed out and to overcome these shortcomings a new method with a new ranking approach is proposed for solving linear programming problems with fuzzy and vague sets. Also, it is shown that it is better to use the proposed method as compared to existing and other proposed methods for solving linear programming problems with fuzzy and vague sets.

Chapter 7

Conclusions And Future Scope

On the basis of the work done in this Chapter 6, it can be concluded that it is better to use RMDS ranking approach, proposed in Chapter 6, as compared to other existing ranking approaches for comparing fuzzy sets and vague sets. Also it is better to use method, proposed in Chapter 6 on the basis of RMDS ranking approach, for solving linear programming problems with generalized fuzzy sets and linear programming problems with vague sets.

The RMDS ranking approach, proposed in Chapter 6, can be used for comparing such trapezoidal vague sets ($\tilde{A}_1 =< [(a_1, b_1, c_1, d_1); \delta_1, \rho_1] >$ and ($\tilde{A}_2 =< [(a_2, b_2, c_2, d_2); \delta_2, \rho_2] >$ for which the property $\delta_1 = \delta_2$ and $\rho_1 = \rho_2$ is satisfied but the same approach can't be used for comparing such trapezoidal vague sets for which the property $\delta_1 = \delta_2$ and $\rho_1 = \rho_2$ is not satisfied.

Due to the same reason the method, proposed in Chapter 6, based on RMDS ranking approach can be used to find the appropriate solution of such linear programming problems with trapezoidal vague sets in which all the parameters $\tilde{c}_j$ are represented by such trapezoidal vague sets in which the property $\delta_i = \delta_j$ and $\rho_i = \rho_j$, $i = j = 1, 2..., n$ is satisfied but the same method can't be used for solving such linear programming problems with trapezoidal vague sets in which some or all the parameters $\tilde{c}_j$ are represented by such trapezoidal vague sets for which the property $\delta_i = \delta_j$ and $\rho_i = \rho_j$ is not satisfied.

In future, it may be tried to modify the proposed RMDS ranking approach for comparing such trapezoidal vague sets ($\tilde{A}_1 =< [(a_1, b_1, c_1, d_1); \delta_1, \rho_1] >$ and ($\tilde{A}_2 =< [(a_2, b_2, c_2, d_2); \delta_2, \rho_2] >$ for which the property $\delta_1 = \delta_2$ and $\rho_1 = \rho_2$ is not satisfied and on the basis of proposed approach it may be tried to modify the method for solving linear programming problems with trapezoidal vague sets, proposed in Chapter 6, to solve such linear programming problems with trapezoidal vague sets in which some or all the parameters $\tilde{c}_j$ are represented by such trapezoidal vague sets for which the property $\delta_i = \delta_j$ and $\rho_i = \rho_j$ is not satisfied.

Also, the proposed methods can be used only to find the appropriate solution of such linear programming problems with generalized fuzzy sets and vague sets in which only the parameters cost (or profit) are represented by fuzzy sets or vague sets. In future, it may be tried to modify these methods for solving such linear programming problems with generalized fuzzy sets and vague sets in which all the parameters are represented by fuzzy sets or vague sets.

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