Some Aspects of Duality in Mathematical Programming Problems

A Thesis

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by

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Certificate

I hereby certify that the work, which is being presented in the thesis, entitled **Some Aspects of Duality in Mathematical Programming Problems**, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work carried out during the period **January 19, 2016** to **June 16, 2020** under the supervision of **Dr. Vikas Sharma**, Assistant Professor, School of Mathematics and **Dr. Navdeep Kailey**, Assistant Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

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.........dedicated to my parents

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Sonali

Abstract

The work exhibited in this thesis is an endeavor to achieve various duality results for minimax fractional programming and multiobjective programming problems. The proposed work encapsulates these results which are weaved into six chapters. The present thesis is assembled into chapters as described below:

Chapter 1 is introductory and consists of definitions, notations and prerequisites of the present work. A brief account of the related work studied by various authors in the field and a summary of the thesis are also presented.

Chapter 2 presents a parametric dual model for nondifferentiable minimax fractional programming (NMFP) problems. Optimality conditions and duality relations are acquired using (p, r)-ρ-(η, θ)-invex suppositions. Two types of second-order dual models are proposed for NMFP problem and usual duality results are developed under second-order B-(p, r)-invex functions.

In **Chapter** 3, we present a novel concept of higher-order B-(p, r)-invex functions. we construct a higher-order dual for NMFP problem and achieve duality results under higher-order B-(p, r)-invexity. A numerical example is solved for finding optimal solution of NMFP problem.

In Chapter 4, we develop second-order duality results for nondifferentiable multiobjective fractional variational problem under second-order (F, α, ρ, d)-pseudoconvexity suppositions. An illustration showing the existence of second-order (F, α, ρ, d)-pseudoconvex functions is provided. An example is obtained to validate the theoretical results of weak duality.

Chapter 5 presents a new pair of higher-order symmetric dual for multiobjective programming problems involving support functions over arbitrary cones. We construct an example of a non trivial function that shows the existence of higher-order K-η-convex functions. Various duality relations are explored under aforesaid assumptions. Some special cases are also examined to show that this work extends known results of the literature.

In **Chapter** 6, we propose a mixed type higher-order symmetric dual model for multiobjective programming problems. Weak, strong and converse duality theorems are established under higher-order K-(F, α, ρ, d)-convexity assumptions.

List of Published/Communicated Papers

1. Sonali, Kailey N. and Sharma V., *On second order duality of minimax fractional programming with square root term involving generalized* B*-*(p,r)*-invex functions*, Annals of Operations Research, 244 (2016) 603-617. (**Impact Factor: 2.284**)
2. Kailey N. and Sonali, *Higher-order symmetric duality in nondifferentiable multiobjective optimization over cones*, Filomat, 33(3) (2019) 711-724. (**Impact Factor: 0.789**)
3. Sonali, Sharma V. and Kailey N., *Higher-order non-symmetric duality for nondifferentiable minimax fractional programs with square root terms*, Acta Mathematica Scientia, 40B(1) (2020) 127-140. (**Impact Factor: 0.992**)
4. Sonali, Kailey N. and Sharma V., *Duality in nondifferentiable minimax fractional programming with generalized invexity.* (**Communicated**)
5. Sonali, Kailey N. and Sharma V., *Parametric approach for a class of fractional variational programs involving support functions.* (**Communicated**)
6. Sonali, Kailey N. and Sharma V., *Higher-order mixed symmetric duality in multiobjective programming involving support functions.* (**Communicated**)

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List of Abbreviations

ES	Efficient Solution
FP	Fractional Programming
FVP	Fractional Variational Programming
LI	Linearly Independent
LP	Linear Programming
MFP	Minimax Fractional Programming
MFVP	Multiobjective Fractional Variational Programming
MP	Multiobjective Programming
MVP	Multiobjective Variational Programming
NLFP	Nonlinear Fractional Programming
NLP	Nonlinear Programming
NMFP	Nondifferentiable Minimax Fractional Programming
NMFVP	Nondifferentiable Multiobjective Fractional Variational Programming
NMP	Nondifferentiable Minimax Programming
PES	Properly Efficient Solution
QFP	Quadratic Fractional Programming
VP	Variational Programming
WES	Weak Efficient Solution

Chapter 1

Introduction

Optimization techniques deal with selecting the best solution to a problem under a given set of constraints. Various practical problems arising in engineering, economics, science and finance often lead to minimize or maximize a function of several variables. Such problems are known as optimization problems. Optimization problems occur frequently in day to day life in order to take the best possible decision from the given set of possible alternatives. The decision making situation could be something like buying a certain product from market, deciding on means of transport to reach a particular destination etc. In such situations a fundamental question arises in one's mind is, "What decision should I take now ?" Interestingly, finding an answer to such situations is equivalent of solving an optimization problem. A general form of optimization problem can be described as:

$$\begin{array}{ll}\textbf{(P)} & \text{Minimize } \vartheta(s) \\ & \text{subject to } \zeta(s) \leqq 0, \\ & s \in V,\end{array}$$

where ϑ is a real valued function on $V(V \subseteq R^n)$ and ζ is a mapping from V to R^m. The function ϑ is called the objective function. If either the objective function or at least one of the constraint of above optimization problem or both are nonlinear then, the problem is called nonlinear programming (NLP) problem. One of the applications of NLP is to maximize or minimize ratio of two functions. Ratio optimization is commonly called fractional programming (FP). FP can be used in various fields of study. The main reason for interest in FP stems from the fact that programming models could better fit the real problems if we consider the maximization of return on investment, maximization of return/risk, minimization of cost/time etc. Many applications of FP problem are given by Schaible [115], Schaible and Ibaraki [116] and Stancu-Minasian [121].

In many real-world applications there are often a number of conflicting objective functions that are all important to optimize. Let us consider an example of a shipping company which is interested in minimizing the total duration of its routes to improve customer service. On the other hand, the company also wants to minimize the number of trucks used in order to reduce operating costs. Clearly, these objectives are in conflict since adding more trucks reduces the duration of the routes, but increases operation costs. In this way there exist a number of problems where a decision maker is interested in maximizing or minimizing various criteria function simultaneously. Such class of problems is known as multiobjective programming (MP) problems where two or more functions are optimized simultaneously. The purpose of MP is to give an understanding of how these functions are conflicting and to give the user the possibility to choose an appropriate trade-off between them.

There exist some real life problems where we have to maximize or minimize functionals e.g. Brachistochrone problem, Isoperimetric problem etc. Such problems can be solved by various techniques found in Calculus of variations. Calculus of variations plays a significant role in many problems arising in mechanics, analysis, geometry etc. It has its origin in the generalization of the elementary theory of minima and maxima of function of one or more than one variables. Its aim is to explore analytical techniques for finding extrema or stationary values of functionals. A functional is a kind of function where domain is itself a set of functions and it associates a definite (real) number to each function belonging to some class. Fermat's principle in optics, the principle of conservation of linear momentum, the principle of least action are some examples of variational principles. In general, we want to find the curve or function $y = y(s)$ where $y(s_0) = y_0$ and $y(s_1) = y_1$ such that for some known function $\phi(s, y(s), \dot{y}(s))$ of variables $s, y, \dot{y}$, the integral

$$\int_{s_0}^{s_1} \phi(s, y(s), \dot{y}(s))ds$$

has an extremum. The curve that satisfy this property is known as extremal and the problem for finding such extremal for above program is called variational programming (VP) problem. In 1992, Bector and Husain [23] obtained dual model and acquired duality results for multiobjective VP problems using convexity assumptions. Convexity suppositions makes the solution of a problem relatively easy. It assures global optimal results. But many problems exist which includes nonconvex functions and functionals. Therefore, it is essential to generalize convexity to larger classes.

In the last three decades, the field of FP and VP has grown rapidly due to the pioneering contributions of the active researchers. But duality theory in the field of fractional variational programming (FVP) and multiobjective variational programming (MVP) is yet in its emerging phase. Moreover, acquiring duality results for nondifferentiable minimax fractional programming (NMFP) is also a tedious task. In this thesis, we achieved second and higher-order duality relations for NMFP problems. Although there is wide research available on FVP, we have made an attempt to obtain duality results for nondifferentiable

FVP problems. We have provided an illustration to corroborate the theoretical results acquired for nondifferentiable FVP problems.

1.1 Preliminaries

1.1.1 Notations and definitions

The following notations are used throughout the work. R^n represents n-dimensional Euclidean space, $R^n_+ = \{s \in R^n : s_i \geqq 0, i = 1, 2, \ldots, n\}$ and R_+ represents the set of nonnegative real numbers. We will apply superscript T to represent transpose of a vector. The vectors will be studied as column vectors. The term w.r.t. represents with respect to. The index sets will be $K = \{1, 2, \ldots, k\}$, $M = \{1, 2, \ldots, m\}$ and $Q = \{1, 2, \ldots, q\}$. Small letters are used to denote vectors or vector functions. For $s, v \in R^n$,

$$
\begin{aligned}
& s \geqq v \Leftrightarrow s_i \geqq v_i,\ i = 1, 2, ..., n, \\
& s \geq v \Leftrightarrow s \geqq v \text{ and } s \neq v, \\
& s > v \Leftrightarrow s_i > v_i,\ i = 1, 2, ..., n. \\
& s \nleq v \text{ means negation of } s \leq v.
\end{aligned}
$$

The vector $\nabla\psi(\bar{s})$ represents the gradient of differentiable function $\psi : R^n \to R$ at $\bar{s}$. It is described as

$$
\nabla\psi(\bar{s}) = \left[\frac{\partial}{\partial s_1}\psi(\bar{s}),\ \frac{\partial}{\partial s_2}\psi(\bar{s}), \ldots, \frac{\partial}{\partial s_n}\psi(\bar{s})\right]^T.
$$

If the function ψ is twice differentiable at $\bar{s}$ then there exists an $n \times n$ symmetric matrix $\nabla_{ss}\psi$ or $\nabla^2\psi$, called Hessian matrix of ψ at $\bar{s}$. For any function $\phi : R^n \to R^k$, $\nabla\phi(\bar{s})$ represents $k \times n$ jacobian matrix of ϕ at $\bar{s}$, whose *ith* row is the vector $\nabla\phi_i(\bar{s})^T$. A vector valued function is differentiable if its every component is differentiable.

We need the following definitions in our work.

Lemma 1.1.1 *(Generalized Schwartz inequality) Let G be a positive semidefinite matrix of order n. Then, $\forall\ s, w \in R^n$,*

$$
s^T G w \leq (s^T G s)^{1/2} (w^T G w)^{1/2}. \tag{1.1.1}
$$

Definition 1.1.1 *A convex set Q of R^n is called a **convex cone** if $\forall\ s \in Q$ and $\beta \geq 0$,*

$\beta s \in Q$.

Definition 1.1.2 *$Q^* = \{z \in R^n : s^T z \geq 0, \forall\ s \in Q\}$ is called **positive polar cone** of Q.*

Definition 1.1.3 *Let E be a compact convex set in R^n. The support function of E is defined by*

$$S(s|E) = \max\{s^T u : u \in E\}.$$

A support function which is convex and everywhere finite, has a subdifferential, that is, $\exists$ $z \in R^n$ such that

$$S(u|E) \geqq S(s|E) + z^T(u - s)\ \forall\ u \in E.$$

The subdifferential of $S(s|E)$ is given by

$$\partial S(s|E) = \{z \in E : z^T s = S(s|E)\}.$$

For any convex set $S \subset R^n$, the normal cone to S at $s \in S$ is described as

$$N_S(s) = \{u \in R^n : u^T(z - s) \leqq 0\ \forall\ z \in S\}.$$

Now for compact convex set E, $u \in N_E(s)$ iff $S(u|E) = s^T u$, or equivalently, s is in $\partial S(u|E)$.

1.1.2 Convexity and its generalization

Most of the real world applications lead to optimization problems which are inherently nonlinear and therefore are void of linearity structure. Fortunately most often this nonlinearity is leading to the convexity structure. Let V be an open convex subset of R^n and the function $\vartheta : V \to R$. Then at $a \in V$, ϑ is said to be

(i) ***convex*** if $\forall\ s \in V$ and $0 \leqq \beta \leqq 1$,

$$\vartheta(\beta s + (1 - \beta)a) \leqq \beta\vartheta(s) + (1 - \beta)\vartheta(a).$$

If ϑ is differentiable at a, then we have

$$\vartheta(s) - \vartheta(a) \geqq \nabla\vartheta(a)^T(s - a)\ \ \forall\ s \in V.$$

The function ϑ is defined as strictly convex if above inequality holds strictly for $s \neq a,\ 0 < \beta < 1$.

The important property of convex functions is that its level sets are convex sets. But it was noted that its converse is not true i.e. if level sets are convex then the function need not be convex. For this a new class of functions was obtained termed as quasiconvex functions which generalize the convexity. Finetti [43] in 1949 was the first one to introduce the generalized convex functions with the idea of quasiconvex functions(named by Fenchel [41]). A function will be quasiconvex iff its level sets are convex sets.

(ii) ***quasiconvex*** if $\forall\ s \in V$ and $0 \leqq \beta \leqq 1$,

$$\vartheta(\beta s + (1-\beta)a) \leqq \max\{\vartheta(s), \vartheta(a)\}$$

or equivalently, if

$$\vartheta(s) \leqq \vartheta(a) \Rightarrow \vartheta(\beta s + (1-\beta)a) \leqq \vartheta(a).$$

Also if ϑ is differentiable at a, then we have

$$\vartheta(s) \leqq \vartheta(a) \Rightarrow \nabla\vartheta(a)^T(s-a) \leqq 0 \quad \forall\ s \in V.$$

Remark: Every convex function is quasiconvex but its converse is not true. For instance, $\vartheta(s) = s^3$ is quasiconvex but not convex.

Later on it was found that quasiconvex functions do not share the property of convex functions, which says that if $\nabla\vartheta(a) = 0$ at some point a, then a is a global minimum of ϑ. This motivated the definition of pseudoconvex functions which share this important property of convex functions. The concept of pseudoconvexity was

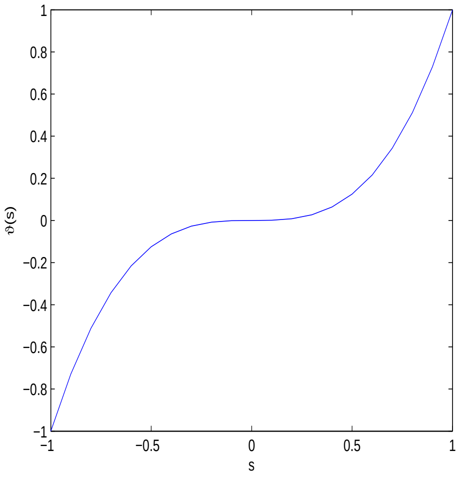

Figure 1.1: Quasiconvex function $\vartheta(s) = s^3$

introduced by Mangasarian [90].

(iii) ***pseudoconvex*** if ϑ is differentiable at a and $\forall\ s \in V$,

$$\nabla\vartheta(a)^T(s-a) \geqq 0 \Rightarrow \vartheta(s) \geqq \vartheta(a),$$

or equivalently, if

$$\vartheta(s) < \vartheta(a) \Rightarrow \nabla\vartheta(a)^T(s-a) < 0.$$

For example, $\vartheta(s) = s^3 + s$ is pseudoconvex.

Hanson [60] introduced the concept of invexity. By choosing $\eta(s,a) = s - a$, we can deduce the definition of convex function.

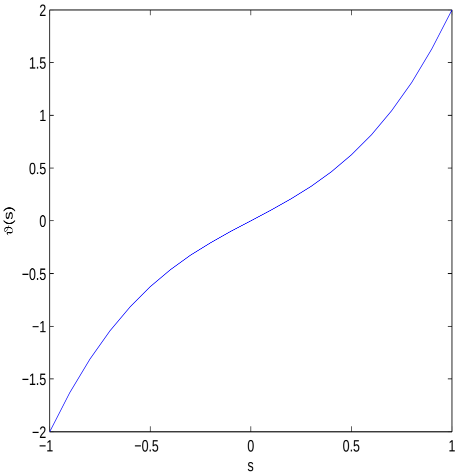

Figure 1.2: Pseudoconvex function $\vartheta(s) = s^3 + s$

(iv) ***invex*** if there exists $\eta : V \times V \to R^n$ such that $\forall\ s \in V$,

$$\vartheta(s) - \vartheta(a) \geqq \eta(s,a)^T\nabla\vartheta(a).$$

(v) ***pseudoinvex*** if there exists $\eta : V \times V \to R^n$ such that $\forall\ s \in V$,

$$\eta(s,a)^T\nabla\vartheta(a) \geqq 0 \Rightarrow \vartheta(s) \geqq \vartheta(a).$$

Hanson and Mond [61] introduced F-convex function using sublinear functional.

(vi) A functional $F : V \times V \times R^n \to R$ is sublinear in the third variable, if $\forall\ s, a \in V$,

(a) $F(s,a;\mu_1+\mu_2) \leqq F(s,a;\mu_1) + F(s,a;\mu_2)$, for all $\mu_1, \mu_2 \in R^n$

(b) $F(s,a;\alpha\mu) = \alpha F(s,a;\mu)$ for all $\alpha \in R_+$and $\mu \in R^n$.

(vii) F***-convex*** if $\forall\ s \in V$,

$$\vartheta(s) - \vartheta(a) \geqq F(s,a;\nabla\vartheta(a)).$$

(viii) F***-pseudoconvex*** if $\forall\ s \in V$,

$$F(s,a;\nabla\vartheta(a)) \geqq 0 \Rightarrow \vartheta(s) \geqq \vartheta(a).$$

Preda [113] extended F-convexity to (F,ρ)-convexity. Liang et al. [86] defined (F,α,ρ,d)-convexity where F was a sublinear functional and proved optimality and duality results.

(ix) (F,ρ)***-convex*** if there exists $d : V \times V \to R$ and $\rho \in R$ such that $\forall\ s \in V$,

$$\vartheta(s) - \vartheta(a) \geqq F(s,a;\nabla\vartheta(a)) + \rho d^2(s,a).$$

(x) (F,ρ)***-pseudoconvex*** if there exists $d : V \times V \to R$ and $\rho \in R$ such that $\forall\ s \in V$,

$$F(s,a;\nabla\vartheta(a)) \geqq -\rho d^2(s,a) \Rightarrow \vartheta(s) \geqq \vartheta(a),$$

or equivalently

$$\vartheta(s) < \vartheta(a) \Rightarrow F(s,a;\nabla\vartheta(a)) < -\rho d^2(s,a).$$

(xi) (F,α,ρ,d)***-convex*** if there exists $\alpha : V \times V \to R_+ \setminus \{0\}, d : V \times V \to R$ and $\rho \in R$ such that $\forall\ s \in V$,

$$\vartheta(s) - \vartheta(a) \geqq F(s,a;\alpha(s,a)\nabla\vartheta(a)) + \rho d^2(s,a).$$

(xii) (F,α,ρ,d)***-pseudoconvex*** if there exists $\alpha : V \times V \to R_+ \setminus \{0\}, d : V \times V \to R$ and $\rho \in R$ such that $\forall\ s \in V$,

$$F(s,a;\alpha(s,a)\nabla\vartheta(a)) \geqq -\rho d^2(s,a) \Rightarrow \vartheta(s) \geqq \vartheta(a),$$

or equivalently

$$\vartheta(s) < \vartheta(a) \Rightarrow F(s,a;\alpha(s,a)\nabla\vartheta(a)) < -\rho d^2(s,a).$$

The class of (p,r)-invex functions contains the class of invex functions defined by Hanson [60]. These functions possess a result that is, any local minimum of a function is global minimum.

(xiii) (p,r)***-invex (strictly)*** if for all $s \in V$,

$\frac{1}{r}e^{r\vartheta(s)} \geqq \frac{1}{r}e^{r\vartheta(a)}\left[1+\frac{r}{p}\left(\nabla\vartheta(a)^T(e^{p\eta(s,a)}-\mathbf{1})\right)\right] (> \text{ if } s \neq a)$ for $p \neq 0,\ r \neq 0$,
$e^{r\vartheta(s)} - e^{r\vartheta(a)} \geqq re^{r\vartheta(a)}\big(\nabla\vartheta(a)^T\eta(s,a)\big) \quad (> \text{ if } s \neq a)$ for $p = 0,\ r \neq 0$,
$\vartheta(s) - \vartheta(a) \geqq \frac{1}{p}\big(\nabla\vartheta(a)^T(e^{p\eta(s,a)}-\mathbf{1})\big) \quad (> \text{ if } s \neq a)$ for $p \neq 0,\ r = 0$,
$\vartheta(s) - \vartheta(a) \geqq \nabla\vartheta(a)^T\eta(s,a) \qquad (> \text{ if } s \neq a)$ for $p = 0,\ r = 0$.

The class of (p,r)-invex functions lies in the class of B-(p,r)-invex functions.

(xiv) B-(p,r)***-invex (strictly)*** w.r.t. η and b at $a \in V$ if there exists $\eta : V \times V \to R^n$ and $b : V \times V \to R_+$ such that $\forall\ s \in V$,

$$\frac{1}{r}b(s,a)(e^{r(\vartheta(s)-\vartheta(a))}-1) \geqq \frac{1}{p}\big(\nabla\vartheta(a)^T(e^{p\eta(s,a)}-\mathbf{1})\big) \ (> \text{ if } s \neq a) \ \text{ for } \ p \neq 0,\ r \neq 0,$$
$$\frac{1}{r}b(s,a)(e^{r(\vartheta(s)-\vartheta(a))}-1) \geqq \nabla\vartheta(a)^T\eta(s,a) \ (> \text{ if } s \neq a) \ \text{ for } \ p = 0,\ r \neq 0,$$
$$b(s,a)(\vartheta(s)-\vartheta(a)) \geqq \frac{1}{p}\big(\nabla\vartheta(a)^T(e^{p\eta(s,a)}-\mathbf{1})\big) \ (> \text{ if } s \neq a) \ \text{ for } \ p \neq 0,\ r = 0,$$
$$b(s,a)(\vartheta(s)-\vartheta(a)) \geqq \nabla\vartheta(a)^T\eta(s,a) \ (> \text{ if } s \neq a) \ \text{ for } \ p = 0,\ r = 0.$$

Here $\mathbf{1} = (1,1,\ldots,1) \in R^n$, $(e^{p\eta(s,a)}-\mathbf{1})$ represents $(e^{p\eta_1(s,a)}-1,\ e^{p\eta_2(s,a)}-1, \ldots,\ e^{p\eta_n(s,a)}-1)$.

1.1.3 Extensions of convexity

Bector and Chandra [19] defined bonvex functions which were later on extended to η-bonvex by Pandey [111]. If ϑ is twice differentiable at a, then, at $a \in V$, ϑ is said to be

(xv) ***second-order convex (bonvex)*** if $\forall\ s \in V,\ l \in R^n$,

$$\vartheta(s) - \vartheta(a) \geqq \ (\nabla\vartheta(a) + \nabla^2\vartheta(a)l)^T(s-a) - \frac{1}{2}l^T\nabla^2\vartheta(a)l.$$

(xvi) ***second-order pseudoconvex (pseudobonvex)*** if $\forall\ s \in V,\ l \in R^n$,

$$(\nabla\vartheta(a) + \nabla^2\vartheta(a)l)^T(s-a) \geqq 0 \Rightarrow \vartheta(s) \geqq \ \vartheta(a) - \frac{1}{2}l^T\nabla^2\vartheta(a)l.$$

(xvii) η-***bonvex*** if $\forall\ s \in V,\ l \in R^n$,

$$\vartheta(s) - \vartheta(a) \geqq\ \eta(s,a)^T[\nabla\vartheta(a) + \nabla^2\vartheta(a)l] - \frac{1}{2}l^T\nabla^2\vartheta(a)l.$$

(xviii) η-***pseudobonvex*** if $\forall\ s \in V,\ l \in R^n$,

$$\eta(s,a)^T(\nabla\vartheta(a) + \nabla^2\vartheta(a)l) \geqq 0 \Rightarrow \vartheta(s) \geqq\ \vartheta(a) - \frac{1}{2}l^T\nabla^2\vartheta(a)l.$$

(xix) ***second-order*** F***-convex*** if $\forall\ s \in V,\ l \in R^n$,

$$\vartheta(s) - \vartheta(a) \geqq\ F(s,a;\nabla\vartheta(a) + \nabla^2\vartheta(a)l) - \frac{1}{2}l^T\nabla^2\vartheta(a)l.$$

(xx) ***second-order*** F***-pseudoconvex*** if $\forall\ s \in V,\ l \in R^n$,

$$F(s,a;\nabla\vartheta(a) + \nabla^2\vartheta(a)l) \geqq 0 \Rightarrow \vartheta(s) \geqq \vartheta(a) - \frac{1}{2}l^T\nabla^2\vartheta(a)l.$$

(xxi) ***second-order*** (F,ρ)***-convex*** if there exists $d : V \times V \to R$ and $\rho \in R$ then $\forall$ $s \in V,\ l \in R^n$

$$\vartheta(s) - \vartheta(a) + \frac{1}{2}l^T\nabla^2\vartheta(a)l \geqq F(s,a;\nabla\vartheta(a) + \nabla^2\vartheta(a)l) + \rho d^2(s,a).$$

(xxii) ***second-order*** (F,ρ)***-pseudoconvex*** if there exists $d : V \times V \to R$ and $\rho \in R$ then $\forall\ s \in V,\ l \in R^n$

$$F(s,a;\nabla\vartheta(a) + \nabla^2\vartheta(a)l) \geqq -\rho d^2(s,a) \Rightarrow \vartheta(s) \geqq \vartheta(a) - \frac{1}{2}l^T\nabla^2\vartheta(a)l.$$

(xxiii) ***second-order*** (F,α,ρ,d)***-convex*** if there exists $\alpha : V \times V \to R_+\backslash\{0\}, d : V \times V \to R$ and $\rho \in R$ then $\forall\ s \in V,\ l \in R^n$

$$\vartheta(s) - \vartheta(a) + \frac{1}{2}l^T\nabla^2\vartheta(a)l \geqq F(s,a;\alpha(s,a)(\nabla\vartheta(a) + \nabla^2\vartheta(a)l)) + \rho d^2(s,a).$$

(xxiv) ***second-order*** (F,α,ρ,d)***-pseudoconvex*** if $\forall\ s \in V,\ l \in R^n$

$$F(s,a;\alpha(s,a)(\nabla\vartheta(a) + \nabla^2\vartheta(a)l)) \geqq -\rho d^2(s,a) \Rightarrow \vartheta(s) \geqq \vartheta(a) - \frac{1}{2}l^2\nabla^2\vartheta(a)l.$$

1.2 Literature Review

1.2.1 Optimality conditions for single objective programming

The problems of optimizing a function under some constraints are known as mathematical programming or constrained optimization problems. Optimality conditions plays a significant role for finding solutions of mathematical programming problems. Initially John obtained necessary optimality conditions for **(P)**. Fritz John conditions have not only shed new light on the notion of Lagrangian multipliers but also gave an easier approach to develop the Lagrangian multiplier rule for equality constrained optimization. These conditions with an added requirement on the lagrange multiplier for the objective function were independently derived by Karush [75] and Kuhn and Tucker [82]. Thus these conditions are called Karush-Kuhn-Tucker (KKT) conditions.

Theorem 1.2.1 *(Necessary conditions) Let* ϑ *and* ζ *be differentiable at* $\bar{s} \in V$ *and let the set of gradients of active constraints at* $\bar{s}$ *be linearly independent (LI). If* $\bar{s}$ *solves the problem* ***(P)****, there exists* $\bar{\mu} \in R^n$ *such that*

$$\nabla\vartheta(\bar{s}) + \bar{\mu}^T \nabla\zeta(\bar{s}) = 0,$$

$$\zeta(\bar{s}) \leqq 0$$

$$\bar{\mu}^T \zeta(\bar{s}) = 0, \ \bar{\mu} \geqq 0.$$

Theorem 1.2.2 *(Sufficient conditions) Let there exists* $\bar{\mu} \in R^n$ *satisfying the Karush-Kuhn-Tucker conditions*

$$\nabla\vartheta(\bar{s}) + \bar{\mu}^T \nabla\zeta(\bar{s}) = 0,$$

$$\bar{\mu}^T \zeta(\bar{s}) = 0,$$

$$\zeta(\bar{s}) \leqq 0, \ \bar{\mu} \geqq 0.$$

If ϑ *is pseudoconvex and* ζ *is quasiconvex and differentiable at* $\bar{s}$*, then* $\bar{s}$ *is a global optimal solution of the problem* ***(P)****.*

1.2.2 Duality in mathematical programming

Duality in linear programming (LP) was first introduced by Neumann [109] in association with Dantzig in 1947. Wolfe [130] gave duality results for convex primal program. For nonlinear problems, several dual programs were defined in the literature. But in nonlinear case, unlike the linear case, the dual of a dual not always be primal.

Wolfe gave this dual for NLP problem **(P)**:
(D1)

$$\begin{aligned} \text{Maximize} \quad & \vartheta(a) + \mu^T \zeta(a) \\ \text{subject to} \quad & \nabla \vartheta(a) + \nabla \mu^T \zeta(a) = 0, \\ & a \in V, \mu \geqq 0 \end{aligned}$$

and developed various duality results between **(P)** and **(D1)** under convexity assumptions. Mond and Weir [106], in order to relax condition of convexity which is needed for Wolfe duality to hold, associated a different type of dual to the primal problem **(P)**. This dual is known as Mond-Weir dual presented as:
(D2)

$$\begin{aligned} \text{Maximize} \quad & \vartheta(a) \\ \text{subject to} \quad & \nabla \vartheta(a) + \nabla \mu^T \zeta(a) = 0, \\ & \mu^T \zeta(a) \geqq 0, \\ & a \in V, \ \mu \geqq 0. \end{aligned}$$

The duality theorems between problem **(P)** and problem **(D2)** had been worked out under the pseudoconvexity of objective function ϑ and quasiconvexity of $\mu^T\zeta$ by Mond and Weir [106]. This was the first attempt to move away from convexity in obtaining the duality results. Various duality results for nonconvex optimization problems have been studied by Floudas and Pardalos [42].

1.2.2.1 Second and higher-order duality

Second-order duality attains great attention due to its computational advantage. It gives better bounds for the value of objective function. Initially, second-order dual for NLP problem **(P)** was proposed by Mangasarian [91].
(D3)

$$\begin{aligned} \text{Maximize} \quad & \vartheta(a) + u^T \zeta(a) - \frac{1}{2} l^T \left[\nabla^2 \vartheta(a) + \nabla^2 u^T \zeta(a)\right] l \\ \text{subject to} \quad & \nabla \vartheta(a) + u^T \nabla \zeta(a) + l^T \left[\nabla^2 \vartheta(a) + \nabla^2 u^T \zeta(a)\right] = 0, \\ & u \geqq 0. \end{aligned}$$

By introducing differentiable functions $h : R^n \times R^n \to R$ and $k : R^n \times R^n \to R^m$, Mangasarian [91] constructed a higher-order dual for NLP problem **(P)**:

(D4)

$$\begin{array}{rl} \text{Maximize} & \vartheta(a) + h(a,l) + u^T\zeta(a) + u^Tk(a,l) \\ \text{subject to} & \nabla_l h(a,l) + \nabla_l u^T k(a,l) = 0, \\ & u \geqq 0. \end{array}$$

1.2.3 Multiobjective programming

We now take a step further and enter in the area where the programming problems require optimization of more than one objective function. MP is used for solving these type of problems. A general MP problem is stated as:

(MP1)

$$\begin{array}{rl} \text{Minimize} & \phi(s) = (\phi_1(s), \dots, \phi_k(s)) \\ \text{subject to} & s \in V^1 = \{s \in V : \zeta(s) \leqq 0\}, \end{array}$$

Here we assume that V is non-empty subset of R^n. $\phi : V \to R^k$ is a given vector function comprising of k objective criteria to be minimized and $\zeta : V \to R^m$.

For k=1, problem **(MP1)** reduces to a single objective programming problem. So, for the multiobjective case, we take $k \geq 2$. Moreover it is not necessary that all the k objective criteria are to be minimized, some criteria may involve maximization process. For instance, in a car buying problem, we would like to maximize comfort and minimize cost of a car. Actually in context of modeling the problem it does not matter whether we investigate minimization or maximization problem. One can convert all the maximization criteria into the minimization form by using Max $\phi_i(s)$ =-Min$(-\phi_i(s))$.

We introduce some concepts related to the problem **(MP1)**:

Weak Efficient Solution:- A point $\bar{s} \in V^1$ is called a weak efficient solution (WES) of **(MP1)** if $\nexists$ $s \in V^1$ such that

$$\phi(s) < \phi(\bar{s}).$$

Efficient Solution:- A point $\bar{s} \in V^1$ is called an efficient solution (ES) of **(MP1)** if $\nexists$ $s \in V^1$ such that

$$\phi(s) \leq \phi(\bar{s}).$$

Example of weak efficient and efficient solutions:- Consider $V = [0,1] \times [0,1]$ and $\phi : V \to R^2$ is the identity map given by $\phi(s_1, s_2) = (s_1, s_2)$.
Here all the points in the set $\{(s_1, 0) : 0 \leq s_1 \leq 1\} \bigcup \{(0, s_2) : 0 \leq s_2 \leq 1\}$ are WESs and $(0, 0)$ is the only ES.
Efficiency in turn means decrease in value of one objective and raise in value of some other. However, it may happen that the decrease is relatively too large as compared to the increase in the other objective. Sometimes this is not a favorable scenario. To avoid such pathological efficient point, Geoffrion [46] proposed the concept of properly efficient solution defined as:
A point $\bar{s} \in V^1$ is called a *properly effcient solution* (PES) of **(MP1)** if $\bar{s}$ is an ES of problem **(MP1)** and $\exists$ a real number $M > 0$ such that for every index i $(i = 1, \ldots, k)$ and every $s \in V^1$ with $\phi_i(s) < \phi_i(\bar{s})$, $\exists$ at least one index j $(j = 1, \ldots, k)$ such that $\phi_j(\bar{s}) < \phi_j(s)$ and

$$\frac{\phi_i(\bar{s}) - \phi_i(s)}{\phi_j(s) - \phi_j(\bar{s})} \leq M.$$

An ES that is not PES is called improperly efficient.

1.2.3.1 Duality in multiobjective programming

Some interesting results for the problem **(MP1)** were discussed by Kuhn and Tucker [82] in his classical work in 1951. Since then, the researchers are paying more attention in this field. In primitive times, optimality conditions for efficiency were attained by Kuhn and Tucker [82] and Arrow et al. [16]. Geoffrion [46] introduced scalar parametric problem:
(EP)

$$\begin{aligned} \text{Minimize} \quad & \beta^T \phi(s) = \sum_{j \in K} \beta_j \phi_j(s) \\ \text{subject to} \quad & s \in V^1, \end{aligned}$$

where $\beta_j > 0 \ \forall \ j, \ \sum\limits_{j \in K} \beta_j = 1$ and associated its results with PES of **(MP1)**.

Theorem 1.2.3 *(Necessary conditions) Let $\bar{s} \in V^1$ be PES of **(MP1)**. Let ζ satisfy a constraint qualification at $\bar{s}$. Then $\exists$ $\bar{\beta} \in R^k$ and $\bar{v} \in R^m$ such that*

$$\nabla \bar{\beta}^T \phi(\bar{s}) + \bar{v}^T \nabla \zeta(\bar{s}) = 0,$$

$$\bar{v}^T \zeta(\bar{s}) = 0,$$

$$\bar{\beta} > 0, \bar{v} \geqq 0, \ \sum_{j=1}^{k} \bar{\beta}_j = 1.$$

Theorem 1.2.4 *(Sufficient conditions) Let ϕ and ζ be convex at $\bar{s} \in V^1$. If $\exists\ \bar{\beta} \in R^k$ and $\bar{v} \in R^m$ such that*

$$\nabla \bar{\beta}^T \phi(\bar{s}) + \bar{v}^T \nabla \zeta(\bar{s}) = 0,$$
$$\bar{v}^T \zeta(\bar{s}) = 0,$$
$$\bar{\beta} > 0, \bar{v} \geqq 0,\ \sum_{j=1}^{k} \bar{\beta}_j = 1,$$

then $\bar{s}$ is PES of ***(MP1)***.

Weir and Mond [129] constructed two symmetric dual models for MP problem. Motivated by Weir and Mond [129], several researchers, such as the ones of (Chen [30]; Yang et al. [136]), developed second and higher-order symmetric dual for MP problems. Yang et al. [134] acquired higher-order duality results for MP problems. Usual higher-order duality results for Mond-Weir type dual are achieved by Ahmad et al. [11] using higher-order (F, α, ρ, d)-type I functions. Batatorescu et al. [17] constructed higher-order dual for MP problem with generalized invexity suppositions. Two dual models for fractional MP problem are proposed by Suneja et al. [124]. Mishra and Giorgi [93] explored various types of invexity and duality relations. Kim and Lee [81] introduced the nondifferentiable MP problem involving support function with cone constraints and for this problem, they proposed Wolfe and Mond-Weir type higher-order duals. Symmetric higher-order duality results are proved by Gulati and Gupta [53] with cone constraints. Jayswal et al. [72] developed higher-order Wolfe and Mond-Weir dual for MP problems. They explored duality relations with higher-order (F, α, ρ, d)-V-type-I functions. Various methods for solving nonconvex multiobjective optimization problems have been presented by Pardalos et al. [112]. Dubey and Mishra [39] constructed second-order dual for MP problems. They also introduced the novel concept of G_f-bonvex/G_f-pseudobonvex functions.

1.2.4 Symmetric and self duality

A pair of primal and dual problem is known as symmetric if dual of dual becomes primal. Duality in LP is always symmetric. However, all dual models in NLP are not always symmetric. Symmetric duality in quadratic programming was initiated by Dorn [35]. Dantzig et al. [32] formulated the following symmetric dual:

(PM1)

$$\begin{aligned} \text{Minimize} \quad & F(a, b) = H(a, b) - b^T \nabla_b H(a, b) \\ \text{subject to} \quad & \nabla_b H(a, b) \leqq 0, \end{aligned}$$

$$(a,b) \geqq 0.$$

(DM1)

$$\begin{aligned} \text{Maximize} \quad & G(u,w) = H(u,w) - u^T \nabla_a H(u,w) \\ \text{subject to} \quad & \nabla_a H(u,w) \geqq 0, \\ & (u,w) \geqq 0. \end{aligned}$$

where $H : R^n \times R^m \to R$ is a continuously differentiable function. For the weak duality theorem, Dantzig et al. [32] required $H(\cdot, b)$ to be convex in a and $H(a, \cdot)$ to be concave in b. Mond and Weir [107] weakened the convexity-concavity supposition on $H(a,b)$ to pseudoconvexity-pseudoconcavity. They constructed the following pair:

(PM2)

$$\begin{aligned} \text{Minimize} \quad & H(a,b) \\ \text{subject to} \quad & \nabla_b H(a,b) \leqq 0, \\ & b^T \nabla_b H(a,b) \geqq 0, \\ & a \geqq 0. \end{aligned}$$

(DM2)

$$\begin{aligned} \text{Maximize} \quad & H(u,w) \\ \text{subject to} \quad & \nabla_a H(u,w) \geqq 0, \\ & u^T \nabla_a H(u,w) \leqq 0, \\ & w \geqq 0. \end{aligned}$$

and discussed duality results using pseudoconvexity-pseudoconcavity suppositions.

A problem is called self dual [36] if its dual is similar to problem itself. Mond and Cottle [100] noticed that the symmetric duals given by Dantzig et al. [32] are self duals by taking $H(a,b)$ as skew symmetric. A nondifferentiable symmetric dual was proposed by Chandra and Husain [25] with convexity/concavity suppositions of $H(a,b)$:

(PM3)

$$\begin{aligned} \text{Minimize} \quad & H(a,b) - b^T \nabla_b H(a,b) + (a^T B a)^{\frac{1}{2}} \\ \text{subject to} \quad & -\nabla_b H(a,b) + Cu \geqq 0, \\ & u^T C u \leqq 1, \end{aligned}$$

$$a \geqq 0,$$
$$b \geqq 0$$

(DM3)

$$\begin{aligned} \text{Maximize} \quad & H(a,b) - a^T \nabla_a H(a,b) - (b^T C b)^{\frac{1}{2}} \\ \text{subject to} \quad & -\nabla_a H(a,b) - Bz \leqq 0, \\ & z^T B z \leqq 1, \\ & a \geqq 0, \\ & b \geqq 0 \end{aligned}$$

Subsequently, Chandra et al. [24] explored nondifferentiable symmetric dual programs inspired by Mond and Weir [107] and acquired results with pseudoconvexity suppositions. Later on, Mond and Schechter [105] studied symmetric nondifferentiable Wolfe and Mond-Weir dual. They established duality relations with convexity/concavity suppositions for Wolfe type model and pseudoconvexity/pseudoconcavity suppositions for Mond-Weir type model respectively.

1.2.5 Symmetric duality with cone constraints

Several researchers have explored symmetric duality results with cone constraints. Bazaraa and Goode [18] generalized the results provided by Dantzig et al. [32] to include the case where constraints involve cones. They studied the following dual pair over arbitrary cones:
(PM4)

$$\begin{aligned} \text{Minimize} \quad & F(a,b) = H(a,b) - b^T \nabla_b H(a,b) \\ \text{subject to} \quad & \nabla_b H(a,b) \in C_2^*, \\ & (a,b) \in C_1 \times C_2. \end{aligned}$$

(DM4)

$$\begin{aligned} \text{Maximize} \quad & G(u,w) = H(u,w) - u^T \nabla_a H(u,w) \\ \text{subject to} \quad & -\nabla_a H(u,w) \in C_1^*, \\ & (u,w) \in C_1 \times C_2. \end{aligned}$$

where

1. C_1 and C_2 are closed convex cones having non-empty interiors in R^n and R^m, respectively.
2. For $j = 1, 2$, C_j^* is the polar of C_j.

Chandra and Kumar [27] constructed the following pair:
(PM5)

$$\begin{aligned} \text{Minimize} \quad & H(a,b) \\ \text{subject to} \quad & \nabla_b H(a,b) \in C_2^*, \\ & b^T \nabla_b H(a,b) \geqq 0, \\ & a \in C_1. \end{aligned}$$

(DM5)

$$\begin{aligned} \text{Maximize} \quad & H(u,w) \\ \text{subject to} \quad & -\nabla_a H(u,w) \in C_1^*, \\ & u^T \nabla_a H(u,w) \leqq 0, \\ & w \in C_2. \end{aligned}$$

and proved usual duality theorems under pseudoinvexity type assumptions.

Later on various problems have been studied where the objective function is maximized or minimized w.r.t. a cone. Consider the following problem:
(PP1)

$$\begin{aligned} & K\text{-minimize } \vartheta(s) \\ & \text{subject to } s \in V^0 = \{s \in V : -\zeta(s) \in C\}, \end{aligned}$$

where $V \subseteq R^n$, $\vartheta : V \to R^k$, $\zeta : V \to R^m$, C is a closed convex cone in R^m and K is closed convex pointed cone in R^k with nonempty interior.

Definition 1.2.1 *A point $\bar{s} \in V^0$ is said to be WES of* ***(PP1)*** *if there exists no $s \in V^0$ such that $\vartheta(\bar{s}) - \vartheta(s) \in int\ K$.*

Definition 1.2.2 *A point $\bar{s} \in V^0$ is said to be an ES of* ***(PP1)*** *if there exists no $s \in V^0$ such that $\vartheta(\bar{s}) - \vartheta(s) \in K \setminus \{0\}$.*

Suneja et al. [122] formulated this dual program:

(PM6)

$$\begin{aligned} K\text{-minimize} \quad & \psi(a,b) - [b^T \nabla_b (\lambda^T \psi)(a,b)]e \\ \text{subject to} \quad & -\nabla_b (\lambda^T \psi)(a,b) \in C_2^*, \ \lambda^T e = 1, \\ & \lambda \in K^*, \ (a,b) \in C_1 \times C_2, \end{aligned}$$

(DM6)

$$\begin{aligned} K\text{-maximize} \quad & \psi(u,w) - [u^T \nabla_a (\lambda^T \psi)(u,w)]e \\ \text{subject to} \quad & \nabla_a (\lambda^T \psi)(u,w) \in C_1^*, \ \lambda^T e = 1, \\ & \lambda \in K^*, \ (u,w) \in C_1 \times C_2, \end{aligned}$$

where $e \in \text{int } K$ and acquired usual relations using K-convexity suppositions.

Later, Khurana [77] discussed the following pair:

(PM7)

$$\begin{aligned} K\text{-minimize} \quad & \psi(a,b) \\ \text{subject to} \quad & -\nabla_b (\lambda^T \psi)(a,b) \in C_2^*, \\ & b^T \nabla_b (\lambda^T \psi)(a,b) \geqq 0, \\ & \lambda \in K^*, \ a \in C_1, \end{aligned}$$

(DM7)

$$\begin{aligned} K\text{-maximize} \quad & \psi(u,w) \\ \text{subject to} \quad & \nabla_a (\lambda^T \psi)(u,w) \in C_1^*, \\ & u^T \nabla_a (\lambda^T \psi)(u,w) \leqq 0, \\ & \lambda \in K^*, \ w \in C_2, \end{aligned}$$

and proved the duality theorems under K-pseudoinvexity/strongly K-pseudoinvexity assumptions. Kim and Kim [78] extended the above two results to nondifferentiable multiobjective symmetric dual programs containing support functions. Ahmad and Husain [9] constructed the mixed type symmetric dual problems which unifies the above two dual formulations.

1.2.6 Second and higher-order symmetric duality

Mond [99] established second-order symmetric dual pair:
(PM8)

$$\begin{aligned} \text{Minimize} \quad & H(a,b) - b^T\nabla_b H(a,b) - b^T\nabla_{bb}H(a,b)l - \frac{1}{2}l^T\nabla_{bb}H(a,b)l \\ \text{subject to} \quad & \nabla_b H(a,b) + \nabla_{bb}H(a,b)l \leqq 0 \\ & a \geqq 0. \end{aligned}$$

(DM8)

$$\begin{aligned} \text{Maximize} \quad & H(a,b) - a^T\nabla_a H(a,b) - a^T\nabla_{aa}H(a,b)r - \frac{1}{2}r^T\nabla_{aa}H(a,b)r \\ \text{subject to} \quad & \nabla_a H(a,b) + \nabla_{aa}H(a,b)r \geqq 0 \\ & b \geqq 0. \end{aligned}$$

and achieved second-order duality results using simpler assumptions. Gulati et al. [47] formulated the pairs of second-order nonlinear symmetric dual with η_1- pseudoconvexity/η_2-pseudoconcavity suppositions. Yang et al. [135] achieved duality relations for second-order symmetric pair with second-order F-convexity suppositions. Suneja et al. [123] constructed second-order symmetric pair for MP. This pair was generalized to nondifferentiable case by Yang et al. [136]. Chen [30] studied multiobjective higher-order dual involving nondifferentiable functions and achieved duality results with higher-order F-convexity:
(PM9)

$$\begin{aligned} \text{Minimize} \quad & (\phi_1(a,b) + S(a|C_1) - b^T z_1 + h_1(a,b,l_1) - l_1^T[\nabla_{l_1}h_1(a,b,l_1)], \dots, \\ & \phi_k(a,b) + S(a|C_k) - b^T z_k + h_k(a,b,l_k) - l_k^T[\nabla_{l_k}h_k(a,b,l_k)]) \\ \text{subject to} \quad & \sum_{i=1}^{k} \lambda_i[\nabla_b\phi_i(a,b) - z_i + \nabla_{l_i}h_i(a,b,l_i)] \leqq 0, \\ & b^T\sum_{i=1}^{k} \lambda_i[\nabla_b\phi_i(a,b) - z_i + \nabla_{l_i}h_i(a,b,l_i)] \geqq 0, \\ & z_i \in D_i,\ i = 1,2,\dots,k,\ \lambda > 0,\ \lambda^T e = 1. \end{aligned}$$

(DM9)

$$\begin{aligned} \text{Maximize} \quad & (\phi_1(u,w) - S(w|D_1) + u^T v_1 + g_1(u,w,r_1) - r_1^T[\nabla_{r_1}g_1(u,w,r_1)], \dots, \\ & \phi_k(u,w) - S(w|D_k) + u^T v_k + g_k(u,w,r_k) - r_k^T[\nabla_{r_k}g_k(u,w,r_k)]) \end{aligned}$$

$$\begin{aligned}
\text{subject to}\quad & \sum_{i=1}^{k}\lambda_i[\nabla_a\phi_i(u,w)+v_i+\nabla_{r_i}g_i(u,w,r_i)] \geqq 0,\\
& u^T\sum_{i=1}^{k}\lambda_i[\nabla_a\phi_i(u,w)+v_i+\nabla_{r_i}g_i(u,w,r_i)] \leqq 0,\\
& v_i\in C_i,\ i=1,2,\dots,k,\ \lambda>0,\ \lambda^Te=1.
\end{aligned}$$

Gulati and Gupta [51] proposed second-order Wolfe type dual with η_1-bonvexity suppositions. Srivastava and Bhatia [120] studied second-order duality for MP with second-order $(F,\,\rho)$-convexity suppositions. Later on, Gulati and Gupta [52] achieved results for higher-order Wolfe dual. Ahmad et al. [11] presented higher-order Mond-Weir type dual for MP. They acquired results with higher-order (F,α,ρ,d)-type I functions. Second-order Wolfe dual pair for MP was constructed by Ahmad and Husain [10]. They also noticed that the results given in Mishra [92] and Mishra and Wang [95] are erroneous. Later Agarwal et al. [3] found some gaps in Chen [30]. They provided a corrected version of strong duality. Gupta and Jayswal [54] obtained symmetric higher-order Mond-Weir dual for MP. Gulati et al. [58] considered the following pair:

(PM10)

$$\begin{aligned}
K\text{-minimize}\quad & \phi(a,b)+S(a|D)e-b^T\nabla_b(\lambda^T\phi)(a,b)e-b^T(\nabla_{bb}(\lambda^T\phi)(a,b)l)e\\
& -\frac{1}{2}l^T(\nabla_{bb}(\lambda^T\phi)(a,b)l)e\\
\text{subject to}\quad & -(\nabla_b(\lambda^T\phi)(a,b)-z+\nabla_{bb}(\lambda^T\phi)(a,b)l)\in C_2^*,\\
& \lambda^Te=1,\\
& a\in C_1,\ \lambda\in \text{int } K^*,\ z\in E.
\end{aligned}$$

(DM10)

$$\begin{aligned}
K\text{-maximize}\quad & \phi(u,w)-S(w|E)e-u^T\nabla_a(\lambda^T\phi)(u,w)e-u^T(\nabla_{aa}(\lambda^T\phi)(u,w)r)e\\
& -\frac{1}{2}r^T(\nabla_{aa}(\lambda^T\phi)(u,w)r)e\\
\text{subject to}\quad & (\nabla_a(\lambda^T\phi)(u,w)+v+\nabla_{aa}(\lambda^T\phi)(u,w)r)\in C_1^*,\\
& \lambda^Te=1,\\
& w\in C_2,\ \lambda\in \text{int } K^*,\ v\in D.
\end{aligned}$$

where $\phi: S_1\times S_2\to R^k$($S_1$ and S_2 are open subsets of R^n and R^m), $l\in R^m,\ r\in R^n, \lambda\in R^k$ and $e=(1,1,\dots,1)^T\in R^k$. Duality results are acquired using K-η-bonvex assumptions. After that Debnath et al. [33] constructed the following higher-order pair:

(PM11)

$$\begin{aligned}
K\text{-minimize} \quad & \phi(a,b)+h(a,b,l)+S(a|D)e-l^T\nabla_l h(a,b,l) \\
& -b^T(\sum_{i=1}^{k}\lambda_i\{\nabla_b\phi_i(a,b)+\nabla_{l_i}h_i(a,b,l_i)\}e) \\
\text{subject to} \quad & -(\sum_{i=1}^{k}\lambda_i\{\nabla_b\phi_i(a,b)+\nabla_{l_i}h_i(a,b,l_i)\}-z)\in C_2^* \\
& \lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)\in \text{int } K^*,\ \lambda^Te=1,\ a\in C_1,\ z\in E,
\end{aligned}$$

(DM11)

$$\begin{aligned}
K\text{-maximize} \quad & \phi(u,w)+g(u,w,r)-S(w|E)e-r^T\nabla_r g(u,w,r) \\
& -u^T(\sum_{i=1}^{k}\lambda_i\{\nabla_u\phi_i(u,w)+\nabla_{r_i}g_i(u,w,r_i)\}e) \\
\text{subject to} \quad & (\sum_{i=1}^{k}\lambda_i\{\nabla_u\phi_i(u,w)+\nabla_{r_i}g_i(u,w,r_i)\}+u)\in C_1^* \\
& \lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)\in \text{int } K^*,\ \lambda^Te=1,\ w\in C_2,\ v\in D,
\end{aligned}$$

where $\phi_i : S_1 \times S_2 \to R(S_1$ and S_2 are open subsets of R^n and $R^m),\ l_i \in R^m$ and $r_i \in R^n$. Gao [44] proposed Mond-Weir type higher-order symmetrical pair for MP with generalized invexity. Recently, Dubey and Mishra [39] constructed second-order Mond-Weir dual for nondifferentiable MP problem.

1.2.7 Fractional programming

The fractional programs were firstly discussed by Neumann [109]. In 1962, Charnes and Cooper [28] constructed an algorithm for solving a linear FP problem. Later on Dinkelbach's algorithm was given by Dinkelbach [34] for nonlinear fractional programming (NLFP) programs. Dinkelbach has taken the following two problems

I

$$\max\{\vartheta(s)/\theta(s)|\ s\in S_1\}$$

and

II

$$\max\{\vartheta(s)-q\theta(s)|\ s\in S_1\}\ \text{for}\ q\in E^1$$

where S_1 is compact and connected subset of E^n, $\theta(s) > 0$ and shown the relationship between the problems **I** and **II**. He proved that $q_0 = \vartheta(s_0)/\theta(s_0) = \max\{\vartheta(s)/\theta(s)|\ s \in S_1\}$ if and only if $F(q_0) = F(q_0, s_0) = \max\{\vartheta(s) - q_0\theta(s)|s \in S_1\} = 0$.

Duality in FP is a prominent class of duality theory and several researchers are engaged in its development. Jagannathan [65] discussed the problems

(P1)

$$\min_{s\in S}\ \vartheta(s)/\theta(s)$$

and

(P2)

$$\min_{s\in S}\ \vartheta(s) - \lambda\theta(s)$$

where $S = \{s|\ \zeta_i(s) \leq 0,\ i = 1, \ldots, m\}$, $\vartheta(s)$ and $\theta(s)$ are convex and concave functions respectively. Moreover these functions are differentiable also. He gave the following duals **(DP1)** and **(DP2)** for the problems **(P1)** and **(P2)** respectively.

(DP1)

$$\begin{aligned}
&\max \qquad \lambda \\
&\nabla\vartheta(s) + \sum_{i=1}^{m} u_i \nabla\zeta_i(s) = \lambda\nabla\theta(s) \\
&\vartheta(s) - \lambda\theta(s) + \sum_{i=1}^{m} u_i\zeta_i(s) \geq 0 \\
&u_i \geq 0, \quad i = 1, \ldots, m \\
&\lambda \geq 0.
\end{aligned}$$

(DP2)

$$\max_{(s,u)\in T_p} \vartheta(s) - \lambda\theta(s) + \sum_{i=1}^{m} u_i\zeta_i(s)$$

$$\text{where} \qquad T_p = \Big\{(s,u)|\nabla\vartheta(s) - \lambda\nabla\theta(s) + \sum_{i=1}^{m} u_i\nabla\zeta_i(s) = 0,\ u_i \geq 0\Big\}.$$

Bector and Chandra [20] studied the duality for the following FP problem:

(PM12)

$$\begin{aligned}
&\text{Minimize } \frac{\vartheta(s)}{\theta(s)} \\
&\text{subject to } s \in V,\ \zeta(s) \leq 0.
\end{aligned}$$

(DM12)

$$\begin{aligned}
&\text{Maximize } \frac{\vartheta(a)+\mu^T\zeta(a)}{\theta(a)}\\
&\text{subject to } \nabla\left(\frac{\vartheta(a)+\mu^T\zeta(a)}{\theta(a)}\right)=0,\\
&\quad a\in V, \mu\geq 0.
\end{aligned}$$

where V is an open subset of R^n. $\vartheta,\ \theta : V \to R,\ \zeta : V \to R^m$ and for all $s,\ \theta(s) > 0$ and $\vartheta(s) \geq 0$ (if θ is nonlinear). Ahmad and Husain [8] developed a Mond-Weir type dual for NMFP problems with generalized convexity.

1.2.8 Minimax programming

The problems, in which both minimization and maximization are evaluated, are called minimax programming problems. In [117], Schmitendorf derived optimality conditions for generalized minimax programming using convexity. Schmitendorf [117] studied the following problem:
(PP)

$$\begin{aligned}
&\text{Minimize } \sup_{a\in A}\psi(s,a)\\
&\text{subject to } s\in X=\{s|C(s)\leqq 0\},
\end{aligned}$$

where $\psi(\cdot,\cdot) : R^n \times R^m \to R$ is C^1 on $R^n \times R^m$, X contains feasible solutions of **(PP)**, A is a specified subset of R^m and $C(\cdot) : R^n \to R^p$ is C^1 on R^n.

For $s \in X$, the defined sets are

$$J(s)=\{j : C_j(s)=0\},$$

$$\hat{A}(s)=\{a\in A : \psi(s,a)=\sup_{z\in A}\psi(s,z)\}.$$

and following are the necessary and sufficient optimality conditions.

Theorem 1.2.5 *(Necessary Conditions) Let s^* be a solution of **(PP)**. Then $\exists$ a positive integer β, $\lambda_i \geqq 0,\ a_i \in A(s^*),\ i = 1, 2, \dots, \beta$ and $\mu_j \geqq 0,\ j = 1, 2, \dots, p$ such that*

$$\sum_{i=1}^{\beta}\lambda_i\psi_s(s^*,a_i)+\sum_{j=1}^{p}\mu_jC_{js}(s^*)=0$$

$$\mu_jC_j(s^*)=0,\quad j=1,2,...,p,$$

$$\sum_{i=1}^{\beta} \lambda_i + \sum_{j=1}^{p} \mu_j \neq 0.$$

If the vectors $C_{js}(s^*),\ j \in J(s^*)$ *are LI, then* $\sum_{i=1}^{\beta} \lambda_i \neq 0$ *can replace* $\sum_{i=1}^{\beta} \lambda_i + \sum_{j=1}^{p} \mu_j \neq 0.$

Theorem 1.2.6 *(Sufficient Conditions) Let* $s^* \in S$. *Let* $C(\cdot)$ *be convex function of* s *and for each* $a \in A$, *let* $\psi(\cdot, a)$ *be convex function of* s. *If* $\exists$ *a positive integer* $\beta, 1 \leqq \beta \leqq n+1$, $\lambda_i \geqq 0,\ i = 1, 2, \dots, \beta,\ \sum_{i=1}^{\beta} \lambda_i \neq 0,\ a_i \in \hat{A}(s^*),\ i = 1, 2, \dots, \beta$ *and* $\mu_j \geqq 0,\ j = 1, 2, \dots, p$ *such that*

$$\sum_{i=1}^{\beta} \lambda_i \psi_s(s^*, a_i) + \sum_{j=1}^{p} \mu_j C_{js}(s^*) = 0$$

$$\mu_j C_j(s^*) = 0,\ j = 1, 2, \dots, p,$$

then s^* *is minimax solution.*

Later on Tanimoto [125] used these conditions to construct a dual and derived duality theorems. Crouzeix et al. [31] have given a variety of applications of generalized FP and examined that minimax fractional programming (MFP) can be solved by finding solution of minimax parametric program. Moreover, numerous applications of minimax programming problems has been discussed by Du and Pardalos [37].

1.2.8.1 Duality in minimax fractional programming

Duality for MFP is getting more attention with the passage of time. Lai et al. [84] derived optimality conditions for NMFP. They used these conditions to obtain a parametric dual. Liang et al. [86] acquired duality results for NLFP with (F, α, ρ, d)-convexity. Ho and Lai [62] defined exponential (p, r)-invexity whereas differentiable (p, r)-invexity was defined by Antczak [14]. Later on Yang and Hou [132] gave optimality conditions for generalized MFP using generalized convexity. Jayswal and Stancu-Minasian [70] introduced higher-order duality for nondifferentiable minimax programming (NMP). Further Ahmad et al. [5] acquired various duality results for NMFP with B-(p, r)-invexity. Ahmad [4] also achieved higher-order duality relations for NMFP problems. Second-order dual for NMFP was achieved by Gupta et al. [48]. Sonali et al. [119] constructed second-order dual models for NMFP problem using generalized invexity. Recently, Jayswal et al. [69] proposed higher-order dual model for NMFP problem. They achieved results using generalized convexity suppositions.

1.2.9 Variational programming

Calculus of variation deals with maximizing or minimizing functionals. It gives an analytical approach to examine that curve joining two given points which either minimizes or maximizes some known integral. For instance, to find a curve which will generate the minimum surface area when revolved about the x-axis. The curve which is to be determined is called an extremal and the problem of finding such extremal is known as VP problem.

1.2.9.1 Duality in variational programming

Mond and Hanson [101] introduced duality in VP problems. They studied the following pair under convexity assumptions:
(PM13)

$$\begin{aligned} &\text{Minimize} && \int_b^c \phi(t,s,\dot{s})dt \\ &\text{subject to} && Q(t,s,\dot{s}) \geq 0,\ s(b)=s_0,\ s(c)=s_1, \end{aligned}$$

(DM13)

$$\begin{aligned} &\text{Maximize} && \int_b^c \phi(t,s,\dot{s}) - \lambda(t)Q(t,s,\dot{s})dt \\ &\text{subject to} && \phi_s(t,s,\dot{s}) - \lambda(t)Q_s(t,s,\dot{s}) = \frac{d}{dt}[\phi_{\dot{s}}(t,s,\dot{s}) - \lambda(t)Q_{\dot{s}}(t,s,\dot{s})], \\ &&& \lambda(t) \geq 0,\ s(b)=s_0,\ s(c)=s_1, \end{aligned}$$

where $B=[b,c]$, $\phi : B \times R^n \times R^n \to R$ and $Q : B \times R^n \times R^n \to R^m$. $s(t)$ is piecewise smooth function of t. Mond and Hanson [102] studied the following dual programs:
(PM14)

$$\begin{aligned} &\text{Minimize} && \int_b^c [\phi(t,s,\dot{s},r,\dot{r}) - r(t)^T\phi_r(t,s,\dot{s},r,\dot{r}) + r(t)^T\frac{d}{dt}\phi_{\dot{r}}(t,s,\dot{s},r,\dot{r})]dt \\ &\text{subject to} && s(b)=\alpha,\ s(c)=\beta \\ &&& r(b)=\gamma,\ r(c)=\delta \\ &&& \frac{d}{dt}\phi_{\dot{r}}(t,s,\dot{s},r,\dot{r}) \geq \phi_r(t,s,\dot{s},r,\dot{r}) \\ &&& s(t) \geq 0. \end{aligned}$$

(DM14)

$$\begin{aligned}
\text{Maximize} \quad & \int_b^c [\phi(t,u,\dot{u},w,\dot{w}) - u(t)^T \phi_s(t,u,\dot{u},w,\dot{w}) + u(t)^T \frac{d}{dt}\phi_{\dot{s}}(t,u,\dot{u},w,\dot{w})]dt \\
\text{subject to} \quad & u(b) = \alpha,\ u(c) = \beta \\
& w(b) = \gamma,\ w(c) = \delta \\
& \frac{d}{dt}\phi_{\dot{s}}(t,u,\dot{u},w,\dot{w}) \leq \phi_s(t,u,\dot{u},w,\dot{w}) \\
& w(t) \geq 0.
\end{aligned}$$

where $s : B \to R^n$, $r : B \to R^m$ and $\phi(t,s,\dot{s},r,\dot{r})$ is continuously differentiable scalar function. Here ϕ is considered as convex in s and $\dot{s}$ for every r and $\dot{r}$ and concave in r and $\dot{r}$ for every s and $\dot{s}$ to establish duality results. If we eliminate $s(t) \geqq 0$ and $w(t) \geqq 0$, we acquire the problem presented in Smart and Mond [118] in which weak duality theorem is obtained with invexity suppositions. Bector et al. [22] proposed symmetric dual for VP problems. Chandra and Husain [26] discussed self duality for FVP. Later, Mond and Husain [104] obtained optimality conditions using weaker invexity assumptions and proved results for Mond-Weir dual. Bector and Husain [23] generalized Wolfe and Mond-Weir duals to their multiobjective analogue. Kim and Lee [79] formulated symmetric dual for VP problem using pseudoinvexity suppositions. A symmetric dual for multiobjective VP problems were constructed by Kim and Lee [80] which unifies the Wolfe and Mond-Weir models. They developed duality relations with the notion of efficiency. Husain et al. [63] proposed second-order Mond-Weir dual by introducing second-order invexity. After that Gulati and Mehndiratta [57] found some gaps in Husain et al. [63] and gave modified results for multiobjective VP problems. Recently Jayswal and Jha [67] established second-order dual for FVP problems using generalized invexity.

1.2.10 Mixed duality

Mixed duality theory has been widely studied by researchers. Two types of mixed type duals were constructed for MP and multiobjective FP problems by Xu [131] and usual duality relations were established using generalized convexity. Aghezzaf [1] defined second-order (F,ρ)-convexity suppositions and obtained second-order mixed dual model for MP. Husain et al. [64] constructed a mixed type vector dual for MP problem containing support functions and acquired duality theorems with generalized invexity. Ahmad and Husain [9] developed a mixed symmetric dual for MP problem. They achieved results with K-preinvexity/K-pseudoinvexity suppositions. Gupta and Kailey [55] constructed a second-order mixed symmetric dual for nondifferentiable MP problem. They proved relations with second-order F-convexity/pseudo-convexity suppositions. Tripathy and Devi [127] introduced second-order (ϕ, ρ)-univexity and obtained second-order mixed symmetric dual for MP problem. A second-order mixed dual was established by Gupta et al. [49] for nondifferentiable MP problem.

1.3 Summary of the thesis

The aim of the present thesis is to acquire duality results for mathematical programming problems with generalized convexity suppositions. The results obtained are discussed in Chapter 2 to 6. Chapter 1 is introductory.
Chapterwise summary is as follows:

In Chapter 2, we consider the following NMFP problem:

(NFP1)
$$\text{Minimize } \psi(s) = \sup_{a \in A} \frac{d(s,a) + (s^T L s)^{1/2}}{e(s,a) - (s^T N s)^{1/2}},$$
$$\text{subject to } h(s) \leq 0.$$

Here A is a compact subset of $R^{l'}$. $d(\cdot,\cdot) : R^n \times R^{l'} \to R$ and $e(\cdot,\cdot) : R^n \times R^{l'} \to R$ are C^1 functions on $R^n \times R^{l'}$. $h(\cdot) : R^n \to R^m$ is C^1 function on R^n. L and N are $n \times n$ positive semidefinite matrices.

We established the following dual model for **(NFP1)**

(NFD)
$$\max_{(q,\xi,\bar{a}) \in K(c)} \quad \sup_{(c,\zeta,k,w,v) \in H_1(q,\xi,\bar{a})} \quad k,$$

where $H_1(q,\xi,\bar{a})$ contains $(c,\zeta,k,w,v) \in R^n \times R^m_+ \times R_+ \times R^n \times R^n$ that satisfy

$$\sum_{j=1}^{q} \xi_j \{\nabla d(c,\bar{a}_j) + Lw - k(\nabla e(c,\bar{a}_j) - Nv)\} + \nabla \sum_{i=1}^{m} \zeta_i h_i(c) = 0,$$

$$\sum_{j=1}^{q} \xi_j \{d(c,\bar{a}_j) + c^T Lw - k(e(c,\bar{a}_j) - c^T Nv)\} \geq 0,$$

$$\sum_{i=1}^{m} \zeta_i h_i(c) \geq 0,$$

$$(q,\xi,\bar{a}) \in K(c),$$

$$w^T Lw \leq 1, \quad v^T Nv \leq 1.$$

and proved duality results using (p,r)-ρ-(η,θ) invexity. Then we proposed two second-order duality models for **(NFP1)** and achieved duality results with second-order B-(p,r) invexity.

Model-I

(M1) $$\max_{(q,\xi,\bar{a})\in K(c)} \quad \sup_{(c,\zeta,w,v,l)\in H_2(q,\xi,\bar{a})} \quad F(c),$$

where $F(c)=\sup\limits_{a\in A}\dfrac{d(c,a)+(c^TLc)^{1/2}}{e(c,a)-(c^TNc)^{1/2}}$ and $H_2(q,\xi,\bar{a})$ contains $(c,\zeta,w,v,l)\in R^n\times R^m_+\times R^n\times R^n\times R^n$ that satisfy

$$\begin{gathered}\nabla\psi_1(c)+\nabla^2\psi_1(c)l=0,\\ \sum_{i=1}^m\zeta_ih_i(c)-\frac{1}{2}l^T\nabla^2\psi_1(c)l\geq 0,\\ w^TLw\leq 1,\quad v^TNv\leq 1,\\ (c^TLc)^{1/2}=c^TLw,\quad (c^TNc)^{1/2}=c^TNv.\end{gathered}$$

Model-II

(M2) $$\max_{(q,\xi,\bar{a})\in K(c)} \sup_{(c,\zeta,w,v,l)\in H_3(q,\xi,\bar{a})} \frac{\sum\limits_{j=1}^q\xi_j\big(d(c,\bar{a}_j)+(c^TLc)^{1/2}\big)+\sum\limits_{i=1}^m\zeta_ih_i(c)}{\sum\limits_{j=1}^q\xi_j\big(e(c,\bar{a}_j)-(c^TNc)^{1/2}\big)},$$

where $H_3(q,\xi,\bar{a})$ represents the set of $(c,\zeta,w,v,l)\in R^n\times R^m_+\times R^n\times R^n\times R^n$ that satisfy

$$\begin{gathered}\nabla\psi_2(c)+\nabla^2\psi_2(c)l=0,\\ l^T\nabla^2\psi_2(c)l\leq 0,\\ w^TLw\leq 1,\quad v^TNv\leq 1,\quad (c^TLc)^{1/2}=c^TLw,\quad (c^TNc)^{1/2}=c^TNv.\end{gathered}$$

We have also given an illustration of second-order B-(p,r) invexity.

In Chapter 3, We proposed the following higher-order dual for NMFP problem:

(FPD) $$\max_{(q,\xi,\bar{a})\in K(c)} \quad \sup_{(c,\zeta,w,v,l)\in H_1(q,\xi,\bar{a})} \quad F(c),$$

where $F(c)=\sup\limits_{a\in A}\dfrac{d(c,a)+(c^TLc)^{1/2}}{e(c,a)-(c^TNc)^{1/2}}$ and $H_1(q,\xi,\bar{a})$ contains $(c,\zeta,w,v,l)\in R^n\times R^m_+\times R^n\times R^n\times R^n$ that satisfy

$$\nabla\psi_1(c)+\nabla_l\phi_1(c,l)=0,$$

$$\sum_{i=1}^{m} \zeta_i h_i(c) + \phi_1(c,l) - l^T \nabla_l \phi_1(c,l) \geq 0,$$

$$w^T L w \leq 1, \quad v^T N v \leq 1,$$

$$(c^T L c)^{1/2} = c^T L w, \quad (c^T N c)^{1/2} = c^T N v.$$

Various duality relations have been acquired using higher-order B-(p,r) invexity. An illustration has been given which shows that there exists some functions that are higher-order B-(p,r) invex but not second-order B-(p,r) invex. Also we solved an example of NMFP using optimality conditions.

In Chapter 4, We focused on exploring second-order duality for nondifferentiable multiobjective fractional variational programming (NMFVP) problem under second-order (F,α,ρ,d)-pseudoconvexity assumptions. An illustration showing the existence of second-order (F,α,ρ,d)-pseudoconvex functions has been given. Symmetric second-order duality for NMFVP and results are discussed with aforesaid suppositions in Section 4.4. An illustration which corroborate the weak duality has been provided.

In Chapter 5, Duality relations have been constructed for this higher-order symmetric dual program involving support functions:

(SPP) K-minimize

$$\begin{aligned} S(a,b,\delta,p) = \quad & \Big(\phi(a,b) + S(a|E)e_k - b^T \sum_{i=1}^{k} \delta_i(\nabla_b \phi_i(a,b) + \nabla_{p_i} h_i(a,b,p_i))e_k \\ & + \sum_{i=1}^{k} \delta_i h_i(a,b,p_i)e_k - \sum_{i=1}^{k} \delta_i(p_i^T \nabla_{p_i} h_i(a,b,p_i))e_k\Big) \\ \text{subject to} \quad & -\sum_{i=1}^{k} \delta_i(\nabla_b \phi_i(a,b) - z + \nabla_{p_i} h_i(a,b,p_i)) \in C_2^*, \\ & z \in D \\ & \delta^T e_k = 1 \\ & \delta \in \text{int } K^*, \ a \in C_1 \end{aligned}$$

(SDP) K-maximize

$$\begin{aligned} T(u,w,\delta,q) = \quad & \phi(u,w) - S(w|D)e_k - u^T \sum_{i=1}^{k} \delta_i(\nabla_b \phi_i(u,w) + \nabla_{q_i} g_i(u,w,q_i))e_k \\ & + \sum_{i=1}^{k} \delta_i g_i(u,w,q_i)e_k - \sum_{i=1}^{k} \delta_i (q_i^T \nabla_{q_i} g_i(u,w,q_i))e_k \\ \text{subject to} \quad & \sum_{i=1}^{k} \delta_i(\nabla_a \phi_i(u,w) + v + \nabla_{q_i} g_i(u,w,q_i)) \in C_1^*, \\ & v \in E \\ & \delta^T e_k = 1 \\ & \delta \in \text{int } K^*, \ \ w \in C_2 \end{aligned}$$

where

(i) $\phi_i : S_1 \times S_2 \to R, \ h_i : S_1 \times S_2 \times R^m \to R$ and $g_i : S_1 \times S_2 \times R^n \to R, \ i = 1, 2, \ldots, k$ are differentiable functions, where $h(a,b,p)$ denotes $(h_1(a,b,p_1), h_2(a,b,p_2), \ldots, h_k(a,b,p_k))$ and $g(u,w,q)$ denotes $(g_1(u,w,q_1), g(u,w,q_2), \ldots, g(u,w,q_k))$, $e_k = (1,\ldots,1)^T \in R^k$, $\delta = (\delta_1, \delta_2, \ldots, \delta_k)$,

(ii) C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively,

(iii) q_i and p_i are vectors in R^n and R^m, respectively for $i = 1, 2, \ldots, k$.

(iv) E and D are compact convex sets in R^n and R^m, respectively, and

(v) $S(a|E)$ and $S(w|D)$ are the support functions of E and D, respectively.

A non trivial example of higher-order K-η-convex functions is provided.

In Chapter 6, the duality relations for higher-order multiobjective mixed symmetric dual are acquired.
Primal Problem **(NMSP)**
K-minimize
$P(a^1, b^1, a^2, b^2, \lambda, r, s) = (P_1(a^1, b^1, a^2, b^2, \lambda, r_1, s_1), P_2(a^1, b^1, a^2, b^2, \lambda, r_2, s_2), \ldots,$
$P_l(a^1, b^1, a^2, b^2, \lambda, r_k, s_k))$
subject to

$$-\sum_{j=1}^{k} \lambda_j (\nabla_{b^1} \varphi_j(a^1, b^1) - q_j^1 + \nabla_{r_j} h_j^1(a^1, b^1, r_j)) \in C_3^*,$$

$$
\begin{aligned}
&-\sum_{j=1}^{k}\lambda_j(\nabla_{b^2}\phi_j(a^2,b^2)-q_j^2+\nabla_{s_j}g_j^1(a^2,b^2,s_j))\in C_4^*,\\
&(b^2)^T\sum_{j=1}^{k}\lambda_j(\nabla_{b^2}\phi_j(a^2,b^2)-q_j^2+\nabla_{s_j}g_j^1(a^2,b^2,s_j))\geqq 0,\\
&\lambda^T e_k=1,\\
&\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)\in \text{int } K^*,\quad a^1\in C_1,\quad a^2\in C_2,\\
&q_j^1\in Q_j^1,\ q_j^2\in Q_j^2,\ j=1,2,\ldots,k.
\end{aligned}
$$

Dual Problem **(NMSD)**
K-maximize
$T(u^1,w^1,u^2,w^2,\lambda,c,d)=(T_1(u^1,w^1,u^2,w^2,\lambda,c_1,d_1),T_2(u^1,w^1,u^2,w^2,\lambda,c_2,d_2),\ldots,$
$T_l(u^1,w^1,u^2,w^2,\lambda,c_k,d_k))$
subject to

$$
\begin{aligned}
&\sum_{j=1}^{k}\lambda_j(\nabla_{a^1}\varphi_j(u^1,w^1)+v_j^1+\nabla_{c_j}h_j^2(u^1,w^1,c_j))\in C_1^*,\\
&\sum_{j=1}^{k}\lambda_j(\nabla_{a^2}\phi_j(u^2,w^2)+v_j^2+\nabla_{d_j}g_j^2(u^2,w^2,d_j))\in C_2^*,\\
&(u^2)^T\sum_{j=1}^{k}\lambda_j(\nabla_{a^2}\phi_j(u^2,w^2)+v_j^2+\nabla_{d_j}g_j^2(u^2,w^2,d_j))\leqq 0,\\
&\lambda^T e_k=1,\\
&\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)\in \text{int } K^*,\quad w^1\in C_3,\quad w^2\in C_4,\\
&v_j^1\in V_j^1,\ v_j^2\in V_j^2,\ j=1,2,\ldots,k,
\end{aligned}
$$

where
$P_j(a^1,b^1,a^2,b^2,\lambda,r_j,s_j)=\varphi_j(a^1,b^1)+S(a^1|V_j^1)+\phi_j(a^2,b^2)+S(a^2|V_j^2)-(b^2)^Tq_j^2$
$-(b^1)^T\sum\limits_{j=1}^{k}\lambda_j(\nabla_{b^1}\varphi_j(a^1,b^1)+\nabla_{r_j}h_j^1(a^1,b^1,r_j))+h_j^1(a^1,b^1,r_j)-r_j^T\nabla_{r_j}h_j^1(a^1,b^1,r_j)+g_j^1(a^2,b^2,s_j)-$
$s_j^T\nabla_{s_j}g_j^1(a^2,b^2,s_j)$.
$T_j(u^1,w^1,u^2,w^2,\lambda,c_j,d_j)=\varphi_j(u^1,w^1)-S(w^1|Q_j^1)+\phi_j(u^2,w^2)-S(w^2|Q_j^2)+(u^2)^Tv_j^2$
$-\ (u^1)^T\sum\limits_{j=1}^{k}\lambda_j(\nabla_{u^1}\varphi_j(u^1,w^1)+\nabla_{c_j}h_j^2(u^1,w^1,c_j))+h_j^2(u^1,w^1,c_j)-c_j^T\nabla_{c_j}h_j^2(u^1,w^1,c_j)+$
$g_j^2(u^2,w^2,d_j)-d_j^T\nabla_{d_j}g_j^2(u^2,w^2,d_j)$.
and

(i) $\varphi_j:R^{|N_1|}\times R^{|M_1|}\to R,\ \phi_j:R^{|N_2|}\times R^{|M_2|}\to R,\ h_j^1:R^{|N_1|}\times R^{|M_1|}\times R^{|M_1|}\to R,\ g_j^1:$

$R^{|N_2|} \times R^{|M_2|} \times R^{|M_2|} \to R,\ h_j^2 : R^{|N_1|} \times R^{|M_1|} \times R^{|N_1|} \to R$ and $g_j^1 : R^{|N_2|} \times R^{|M_2|} \times R^{|N_2|} \to R$ are differentiable functions, respectively.

(ii) $V_j^1,\ V_j^2,\ Q_j^1$ and Q_j^2 are compact convex sets in $R^{|N_1|},\ R^{|N_2|},\ R^{|M_1|}$ and $R^{|M_2|}$, respectively.

(iii) $C_1,\ C_2,\ C_3$ and C_4 are closed convex cones in $R^{|N_1|},\ R^{|N_2|},\ R^{|M_1|}$ and $R^{|M_2|}$, respectively and $C_1^*,\ C_2^*,\ C_3^*$ and C_4^* are their respective positive polar cones.

Here $N_1 \subset N,\ M_1 \subset M,\ N_2 = N \setminus N_1$ and $M_2 = M \setminus M_1$.
Duality relations are acquired for aforesaid pair with higher-order K-(F, α, ρ, d)-convex suppositions.
The overall concluding observations of this study and few significant directions for the future scope are given in the end of chapters.

Chapter 2

Duality in nondifferentiable minimax fractional programming with generalized invexity[1]

2.1 Introduction

There are numerous decision making problems which directly or indirectly lead to FP problems. Some examples are efficiency measure for systems, portfolio selection problem, agricultural planning, information transfer, stochastic processes numerical analysis, resource allocation problems and cargo loading problems etc. In physics, maximization of signal-to-noise ratio of a spectral filter gives rise to concave quadratic fractional programming (QFP) which was studied by Falk [40].

In the primitive years, much attention has been given to MFP problems. Schmitendorf [117] obtained optimality conditions for static minimax problems. A parametric dual for MFP problem was studied by Ahmad et al. [5]. Khan and Al-Solamy [76] obtained sufficient conditions and duality relations for NMFP using (H_p, r)-invexity.

In this chapter, we explore the following problem:

(NFP1)
$$\text{Minimize } \psi(s) = \sup_{a \in A} \frac{d(s,a) + (s^T L s)^{1/2}}{e(s,a) - (s^T N s)^{1/2}},$$
$$\text{subject to } h(s) \leq 0.$$

Here A is a compact subset of $R^{l'}$. $d(\cdot,\cdot) : R^n \times R^{l'} \to R$, $e(\cdot,\cdot) : R^n \times R^{l'} \to R$ are C^1 functions on $R^n \times R^{l'}$ and $h(\cdot) : R^n \to R^m$ is C^1 function on R^n. L and N are $n \times n$ positive semidefinite matrices. We presume that $e(s,a) - (s^T N s)^{1/2} > 0$ and $d(s,a) + (s^T L s)^{1/2} \geq 0$ for every $(s,a) \in S \times A$, where $S = \{s \in R^n : \ h(s) \leq 0\}$ contains feasible solutions of **(NFP1)**.

Antczak [14] introduced p-invex sets and (p,r)-invexity. He established sufficient conditions for NLP problem under (p,r)-invexity assumptions. Further generalization of (p,r)-invexity was studied by Mandal and Nahak [89] as (p,r)-ρ-(η,θ)-invexity. In this chapter,

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[1] A part of this chapter is published as " *On second order duality of minimax fractional programming with square root term involving generalized B-(p,r)-invex functions*", Annals of Operations Research, 244 (2016) 603-617.

we derive duality theorems to relate NMFP problem and parametric dual with (p,r)-ρ-(η,θ)-invexity suppositions. Section 2.2 reviews definitions. Optimality conditions for finding optimal solution of NMFP are established in Section 2.3. Further, a parametric dual and its duality results are discussed with (p,r)-ρ-(η,θ)-invexity suppositions in Section 2.4. In Section 2.5, second-order duality relations are explored with second-order B-(p,r)-invexity.

2.2 Notations and preliminaries

For each $(s,a)\in S\times A$ and $M=\{1,2,\dots,m\}$, we define
$I(s)=\{i\in M : h_i(s)=0\}$,
$A(s)=\left\{a\in A : \dfrac{d(s,a)+(s^TLs)^{1/2}}{e(s,a)-(s^TNs)^{1/2}}=\sup\limits_{u\in A}\dfrac{d(s,u)+(s^TLs)^{1/2}}{e(s,u)-(s^TNs)^{1/2}}\right\}$,
$K(s)=\left\{(q,\xi,\bar{a})\in N\times R^q_+\times R^{l'q} : 1\le q\le n+1,\ \xi=(\xi_1,\xi_2,\dots,\xi_q)\in R^q_+,\ \sum\limits_{j=1}^{q}\xi_j=1,\ \bar{a}=(\bar{a}_1,\bar{a}_2,\dots,\bar{a}_q),\ \bar{a}_j\in A(s), j=1,2,\dots,q\right\}$.
Since d and e are of class C^1 and A is compact in $R^{l'}$, therefore, for every $s^*\in S,\ A(s^*)\neq\phi$ and for any $\bar{a}_j\in A(s^*)$, we have a positive constant

$$k_0=\psi(s^*)=\frac{d(s^*,\bar{a}_j)+(s^{*T}Ls^*)^{1/2}}{e(s^*,\bar{a}_j)-(s^{*T}Ns^*)^{1/2}}.$$

Let $\varphi:\hat{E}\to R$ (where $\hat{E}\subset R^n$) be a differentiable function. Let p,r be arbitrary real numbers.

Definition 2.2.1 *[89] φ is called (p,r)-ρ-(η,θ)-invex (strictly (p,r)-ρ-(η,θ)-invex) w.r.t. η and θ at $\nu\in\hat{E}$ on $\hat{E}$ if $\exists\ \eta,\theta:\hat{E}\times\hat{E}\to R^n$ and $\rho\in R$ such that $\forall\ s\in\hat{E}$,*

$$\left[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(\nu))}-1)\right]\ge\tfrac{1}{p}(\nabla\varphi(\nu))^T(e^{p\eta(s,\nu)}-\boldsymbol{1})+\rho||\theta(s,\nu)||^2\quad(>\ \textit{if}\ s\neq\nu)\ \textit{for}\ p\neq0,r\neq0,$$

$$\left[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(\nu))}-1)\right]\ge\left[(\nabla\varphi(\nu))^T\eta(s,\nu)\right]+\rho||\theta(s,\nu)||^2\qquad(>\ \textit{if}\ s\neq\nu)\ \textit{for}\ p=0,r\neq0,$$

$$(\varphi(s)-\varphi(\nu))\ge\tfrac{1}{p}[(\nabla\varphi(\nu))^T(e^{p\eta(s,\nu)}-\boldsymbol{1})]+\rho||\theta(s,\nu)||^2\qquad(>\ \textit{if}\ s\neq\nu)\ \textit{for}\ p\neq0,r=0,$$

$$(\varphi(s)-\varphi(\nu))\ge(\nabla\varphi(\nu))^T\eta(s,\nu)+\rho||\theta(s,\nu)||^2\qquad(>\ \textit{if}\ s\neq\nu)\ \textit{for}\ p=0,r=0.$$

Definition 2.2.2 *φ is called second-order B-(p,r)-invex (strictly second-order B-(p,r)-invex) w.r.t. η and b at $\nu\in\hat{E}$ on $\hat{E}$ if $\exists\ \eta:\hat{E}\times\hat{E}\to R^n$ and $b:\hat{E}\times\hat{E}\to R_+$ such that $\forall\ s\in\hat{E}$ and $l\in R^n$,*

$$b(s,\nu)\left[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(\nu))}-1)+\tfrac{1}{2}l^T\nabla^2\varphi(\nu)l\right]\ge\tfrac{1}{p}(\nabla\varphi(\nu)+\nabla^2\varphi(\nu)l)^T(e^{p\eta(s,\nu)}-\boldsymbol{1})$$
$$(>\ \textit{if}\ s\neq\nu)\ \textit{for}\ p\neq0,r\neq0.$$

$$b(s,\nu)\big[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(\nu))}-1)+\tfrac{1}{2}l^T\nabla^2\varphi(\nu)l\big]\geq\big[(\nabla\varphi(\nu)+\nabla^2\varphi(\nu)l)^T\eta(s,\nu)\big]$$

$$(> \textit{ if } s\neq\nu) \textit{ for } p=0, r\neq 0,$$

$$b(s,\nu)(\varphi(s)-\varphi(\nu)+\tfrac{1}{2}l^T\nabla^2\varphi(\nu)l)\geq\tfrac{1}{p}[(\nabla\varphi(\nu)+\nabla^2\varphi(\nu)l)^T(e^{p\eta(s,\nu)}-\mathbf{1})]$$

$$(> \textit{ if } s\neq\nu) \textit{ for } p\neq 0, r=0,$$

$$b(s,\nu)(\varphi(s)-\varphi(\nu)+\tfrac{1}{2}l^T\nabla^2\varphi(\nu)l)\geq[(\eta(s,\nu))^T(\nabla\varphi(\nu)+\nabla^2\varphi(\nu)l)]$$

$$(> \textit{ if } s\neq\nu) \textit{ for } p=0, r=0.$$

Here $\mathbf{1}=(1,1,\ldots,1)\in R^n$, $(e^{p\eta(s,\nu)}-\mathbf{1})$ *represents* $(e^{p\eta_1(s,\nu)}-1,\ e^{p\eta_2(s,\nu)}-1,\ldots,\ e^{p\eta_n(s,\nu)}-1)$.

Example 2.2.1 *Let* $\hat{E}=[-0.8,0]\subset R$. *Take* $\varphi:\hat{E}\to R$ *as* $\varphi(s)=\log(\sec s)$. *Consider* $\eta:\hat{E}\times\hat{E}\to R$ *as*

$$\eta(s,\nu)=\log\left(\frac{1}{1+\nu}\right)$$

and

$$b(s,\nu)=-8\sin\nu.$$

Here φ *is second-order* B-$(1,1)$-*invex as*

$$\begin{aligned}L&=b(s,\nu)\big[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(\nu))}-1)+\tfrac{1}{2}l^T\nabla^2\varphi(\nu)l\big]-\tfrac{1}{p}(\nabla\varphi(\nu)+\nabla^2\varphi(\nu)l)^T(e^{p\eta(s,\nu)}-\mathbf{1})\\&=-8(\sin\nu)\left[\left(\frac{\sec s}{\sec\nu}-1\right)+\tfrac{1}{2}l^2\sec^2\nu\right]-\big(\tan\nu+l\sec^2\nu\big)\left(\frac{1}{1+\nu}-1\right)\\&=\left[-8(\sin\nu)\left(\frac{\sec s}{\sec\nu}-1\right)-(\tan\nu)\left(\frac{1}{1+\nu}-1\right)\right]+\left[-8(\sin\nu)\left(\tfrac{1}{2}l^2\sec^2\nu\right)\right.\\&\qquad\left.-(l\sec^2\nu)\left(\frac{1}{1+\nu}-1\right)\right]\\&=L_1+L_2\end{aligned}$$

where

$$\begin{aligned}L_1&=-8(\sin\nu)\left(\frac{\sec s}{\sec\nu}-1\right)-(\tan\nu)\left(\frac{1}{1+\nu}-1\right)\\&\geq 0 \textit{ for all } s,\nu\in\hat{E} \textit{ (see Figure 2.1)}\end{aligned}$$

and

$$\begin{aligned}L_2&=-8(\sin\nu)\left(\tfrac{1}{2}l^2\sec^2\nu\right)-(l\sec^2\nu)\left(\frac{1}{1+\nu}-1\right)\\&\geq 0 \textit{ for all } \nu\in\hat{E} \textit{ and } l\in(-10^{18},10^{18}) \textit{ (see Figure 2.2).}\end{aligned}$$

Therefore, $L\geq 0$.

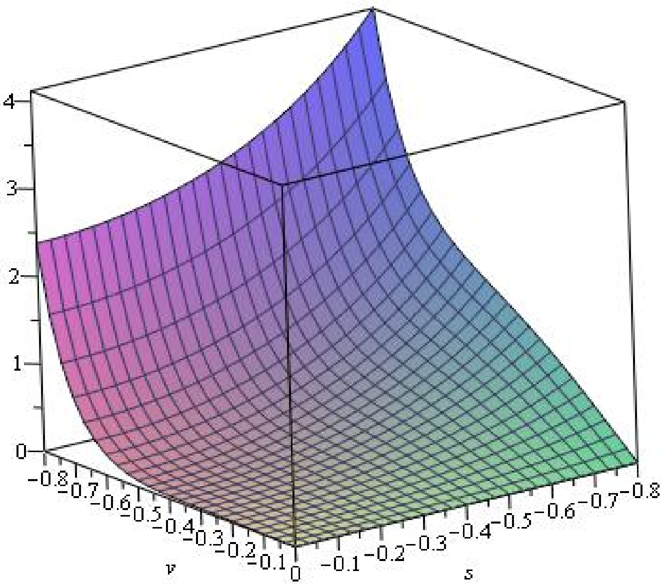

Figure 2.1: graph of L_1 against s and ν

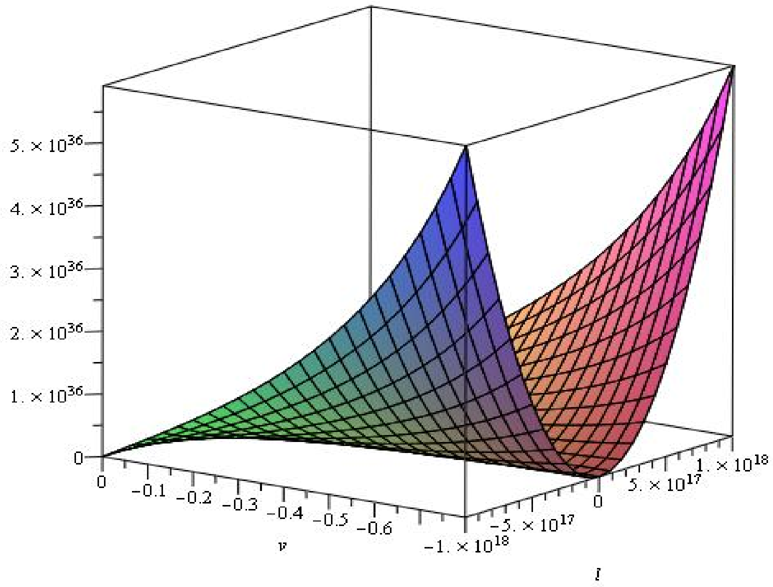

Figure 2.2: graph of L_2 against ν and l

But φ *is not* B-$(-1,-1)$-*invex as*

$$
\begin{aligned}
&b(s,\nu)\left[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(\nu))}-1)\right]-\tfrac{1}{p}(\nabla\varphi(\nu))^T(e^{p\eta(s,\nu)}-\boldsymbol{1})\\
&\quad=8(\sin(-0.8))\left(\frac{\sec(-0.8)}{\sec(-0.1)}-1\right)+(-0.8)\tan(-0.8)\\
&\quad=-1.6338<0 \ \textit{for}\ s=-0.1,\ \nu=-0.8.
\end{aligned}
$$

Hence φ *is second-order* B*-*$(1,1)$*-invex but not* B*-*$(-1,-1)$*-invex.*

2.3 Optimality conditions

Theorem 2.3.1 *[84] (Necessary conditions) If* s^* *is an optimal solution of* ***(NFP1)*** *that satisfies* $s^{*T}Ls^*>0,\ s^{*T}Ns^*>0$ *and* $\nabla h_i(s^*),\ i\in I(s^*)$ *are LI, then* $\exists\ (q,\xi^*,\bar{a})\in K(s^*)$*,* $k_0\in R_+, w, v\in R^n$ *and* $\zeta^*\in R^m_+$ *s.t.*

$$\sum_{j=1}^{q}\xi_j^*\{\nabla d(s^*,\bar{a}_j)+Lw-k_0(\nabla e(s^*,\bar{a}_j)-Nv)\}+\nabla\sum_{i=1}^{m}\zeta_j^*h_i(s^*)=0, \tag{2.3.1}$$

$$d(s^*,\bar{a}_j)+(s^{*T}Ls^*)^{1/2}-k_0(e(s^*,\bar{a}_j)-(s^{*T}Ns^*)^{1/2})=0,\quad j=1,2,\ldots,q, \tag{2.3.2}$$

$$\sum_{i=1}^{m}\zeta_i^*h_i(s^*)=0, \tag{2.3.3}$$

$$\xi_j^*\geq 0\ (j=1,2,\ldots,q),\quad \sum_{j=1}^{q}\xi_j^*=1, \tag{2.3.4}$$

$$w^TLw\leq 1,\quad v^TNv\leq 1,\quad (s^{*T}Ls^*)^{1/2}=s^{*T}Lw,\quad (s^{*T}Ns^*)^{1/2}=s^{*T}Nv. \tag{2.3.5}$$

In the above theorem, L and N are positive definite. If either $s^{*T}Ls^*$ or $s^{*T}Ns^*$ is zero or both of them are zero, then, functions considered in objective of **(NFP1)** are not differentiable. To establish necessary conditions for such cases, for $(q,\xi^*,\bar{a})\in K(s^*)$, we take

$$
\begin{aligned}
Z_{\bar{a}}(s^*)=\big\{&z\in R^n: z^T\nabla h_i(s^*)\leq 0, i\in I(s^*),\\
&\text{with any one of the next conditions (i)-(iii) holds}\big\}.
\end{aligned}
$$

(i) $s^{*T}Ls^*>0,\quad s^{*T}Ns^*=0$

$$\Rightarrow z^T\left(\sum_{j=1}^{q}\xi_j^*\left\{\nabla d(s^*,\bar{a}_j)+\frac{Ls^*}{(s^{*T}Ls^*)^{1/2}}-k_0\nabla e(s^*,\bar{a}_j)\right\}\right)$$

$$+ (z^T(k_0^2N)z)^{1/2} < 0,$$

(ii) $s^{*T}Ls^* = 0, \quad s^{*T}Ns^* > 0$

$$\Rightarrow z^T\Bigg(\sum_{j=1}^{q}\xi_j^*\Bigg\{\nabla d(s^*,\bar{a}_j) - k_0\Bigg(\nabla e(s^*,\bar{a}_j) - \frac{Ns^*}{(s^{*T}Ns^*)^{1/2}}\Bigg)\Bigg\}\Bigg) + (z^TLz)^{1/2} < 0,$$

(iii) $s^{*T}Ls^* = 0, \quad s^{*T}Ns^* = 0$

$$\Rightarrow z^T\Bigg(\sum_{j=1}^{q}\xi_j^*\Bigg\{\nabla d(s^*,\bar{a}_j) - k_0\nabla e(s^*,\bar{a}_j)\Bigg\}\Bigg) + (z^T(k_0^2N)z)^{1/2} + (z^TLz)^{1/2} < 0,$$

If we add the condition $Z_{\bar{a}}(s^*) = \phi$, then Theorem 2.3.1 still satisfies.

Remark 2.3.1 *The theoretical results has been given for $p \neq 0$ and $r \neq 0$. For rest conditions, theorems can be obtained easily than this one. Also we will assume that $r > 0$.*

Theorem 2.3.2 *(Sufficient conditions) Suppose s^* is a feasible solution of* ***(NFP1)*** *and $\exists$ a positive integer q, $1 \le q \le n+1$, $\xi^* \in R_+^q$, $\bar{a}_j \in A(s^*)(j = 1,2,\ldots,q)$, $k_0 \in R_+$, w, $v \in R^n$ and $\zeta^* \in R_+^m$ satisfying (2.3.1)-(2.3.5). Suppose*

(i) $\sum_{j=1}^{q}\xi_j^*(d(\cdot,\bar{a}_j) + (\cdot)^TLw - k_0(e(\cdot,\bar{a}_j) - (\cdot)^TNv))$ *is (p,r)-ρ^1-(η,θ)-invex function at s^* w.r.t. η and θ for all $s \in S$,*

(ii) $\sum_{i=1}^{m}\zeta_i^*h_i(\cdot)$ *is (p,r)-ρ^2-(η,θ)-invex function at s^* w.r.t. same function η and θ,*

(iii) $\rho^1 + \rho^2 \ge 0$.

Then, s^* is an optimal solution of ***(NFP1)****.*

Proof Consider s^* is not an optimal solution of **(NFP1)**. Therefore, $\exists\ \bar{s} \in S$ s.t.

$$\sup_{\bar{a}\in A}\frac{d(\bar{s},\bar{a}) + (\bar{s}^TL\bar{s})^{1/2}}{e(\bar{s},\bar{a}) - (\bar{s}^TN\bar{s})^{1/2}} < \sup_{\bar{a}\in A}\frac{d(s^*,\bar{a}) + (s^{*T}Ls)^{1/2}}{e(s^*,\bar{a}) - (s^{*T}Ns^*)^{1/2}}.$$

Since

$$\sup_{\bar{a}\in A}\frac{d(s^*,\bar{a}) + (s^{*T}Ls^*)^{1/2}}{e(s^*,\bar{a}) - (s^{*T}Ns^*)^{1/2}} = \frac{d(s^*,\bar{a}_j) + (s^{*T}Ls^*)^{1/2}}{e(s^*,\bar{a}_j) - (s^{*T}Ns^*)^{1/2}} = k_0,$$

for any $\bar{a}_j \in A(s^*)$, $j = 1,2,\ldots,q$ and

$$\frac{d(\bar{s},\bar{a}_j) + (\bar{s}^TL\bar{s})^{1/2}}{e(\bar{s},\bar{a}_j) - (\bar{s}^TN\bar{s})^{1/2}} \le \sup_{\bar{a}\in A}\frac{d(\bar{s},\bar{a}) + (\bar{s}^TL\bar{s})^{1/2}}{e(\bar{s},\bar{a}) - (\bar{s}^TN\bar{s})^{1/2}}.$$

Therefore, we get

$$\frac{d(\bar{s},\bar{a}_j) + (\bar{s}^TL\bar{s})^{1/2}}{e(\bar{s},\bar{a}_j) - (\bar{s}^TN\bar{s})^{1/2}} < k_0.$$

Also from $\xi_j^* \geq 0, j = 1, 2, \ldots, q, \xi^* \neq 0$ and $\bar{a}_j \in A(s^*)$, we get

$$\sum_{j=1}^{q} \xi_j^* \big[d(\bar{s}, \bar{a}_j) + (\bar{s}^T L \bar{s})^{1/2} - k_0\big(e(\bar{s}, \bar{a}_j) - (\bar{s}^T N \bar{s})^{1/2}\big)\big] < 0. \tag{2.3.6}$$

Using Lemma 1.1.1, (2.3.2), (2.3.5), and (2.3.6), we have

$$\begin{aligned}
&\sum_{j=1}^{q} \xi_j^* \big[d(\bar{s}, \bar{a}_j) + \bar{s}^T Lw - k_0\big(e(\bar{s}, \bar{a}_j) - \bar{s}^T Nv\big)\big] \\
&\qquad \leq \sum_{j=1}^{q} \xi_j^* \big[d(\bar{s}, \bar{a}_j) + (\bar{s}^T L \bar{s})^{1/2} - k_0\big(e(\bar{s}, \bar{a}_j) - (\bar{s}^T N \bar{s})^{1/2}\big)\big] \\
&\qquad < 0 = \sum_{j=1}^{q} \xi_j^* \big[d(s^*, \bar{a}_j) + (s^{*T} L s^*)^{1/2} - k_0\big(e(s^*, \bar{a}_j) - (s^{*T} N s^*)^{1/2}\big)\big], \\
&\qquad = \sum_{j=1}^{q} \xi_j^* \big[d(s^*, \bar{a}_j) + s^{*T} Lw - k_0\big(e(s^*, \bar{a}_j) - s^{*T} Nv\big)\big].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{j=1}^{q} \xi_j^* \big[d(\bar{s}, \bar{a}_j) + \bar{s}^T Lw - k_0\big(e(\bar{s}, \bar{a}_j) - \bar{s}^T Nv\big)\big] \\
&\qquad < \sum_{j=1}^{q} \xi_j^* \big[d(s^*, \bar{a}_j) + s^{*T} Lw - k_0\big(e(s^*, \bar{a}_j) - s^{*T} Nv\big)\big].
\end{aligned} \tag{2.3.7}$$

As $\sum_{j=1}^{q} \xi_j^* \big[d(\cdot, \bar{a}_j) + (\cdot)^T Lw - k_0\big(e(\cdot, \bar{a}_j) - (\cdot)^T Nv\big)\big]$ is (p, r)-ρ^1-(η, θ)-invex at s^* on S w.r.t. η and θ, we have

$$\begin{aligned}
&\left[\tfrac{1}{r}\Big(e^{r\big[\sum_{j=1}^{q} \xi_j^* \big(d(s,\bar{a}_j)+s^T Lw-k_0\big(e(s,\bar{a}_j)-s^T Nv\big)\big)-\sum_{j=1}^{q} \xi_j^* \big(d(s^*,\bar{a}_j)+s^{*T} Lw-k_0\big(e(s^*,\bar{a}_j)-s^{*T} Nv\big)\big)\big]} - 1\Big)\right] \\
&\qquad \geq \tfrac{1}{p}\Big(\sum_{j=1}^{q} \xi_j^* \big(\nabla d(s^*, \bar{a}_j) + Lw - k_0\big(\nabla e(s^*, \bar{a}_j) - Nv\big)\big)\Big)^T \big(e^{p\eta(s,s^*)} - \mathbf{1}\big) + \rho^1 ||\theta(s, s^*)||^2
\end{aligned}$$

holds $\forall\ s \in S$, and also for $s = \bar{s}$. From (2.3.7), together with the inequality above, we get

$$\frac{1}{p}\left[\Big(\sum_{j=1}^{q} \xi_j^* \big[\nabla d(s^*, \bar{a}_j) + Lw - k_0\big(\nabla e(s^*, \bar{a}_j) - Nv\big)\big]\Big)^T \big(e^{p\eta(\bar{s},s^*)} - \mathbf{1}\big)\right] + \rho^1 ||\theta(\bar{s}, s^*)||^2 < 0. \tag{2.3.8}$$

From the feasibility of $\bar{s}$ along with $\zeta_i^* \geq 0,\ i \in I$, we acquire

$$\sum_{i=1}^{m} \zeta_i^* h_i(\bar{s}) \leq 0. \tag{2.3.9}$$

Using (ii), we conclude

$$\tfrac{1}{r}(e^{r\left(\sum\limits_{i=1}^{m}\zeta_i^*h_i(\bar{s})-\sum\limits_{i=1}^{m}\zeta_i^*h_i(s^*)\right)}-1)\geq \tfrac{1}{p}\left[(\sum_{i=1}^{m}\nabla\zeta_i^*h_i(s^*))^T\big(e^{p\eta(\bar{s},s^*)}-\mathbf{1}\big)\right]+\rho^2||\theta(\bar{s},s^*)||^2$$

Using (2.3.3), (2.3.9) and above inequality, we get

$$\frac{1}{p}\left[(\sum_{i=1}^{m}\nabla\zeta_i^*h_i(s^*))^T\big(e^{p\eta(\bar{s},s^*)}-\mathbf{1}\big)\right]+\rho^2||\theta(\bar{s},s^*)||^2\leq 0. \tag{2.3.10}$$

By summing up (2.3.8) and (2.3.10), we have
$\frac{1}{p}\left[\big(\sum\limits_{j=1}^{q}\xi_j^*\big[\nabla d(s^*,\bar{a}_j)+Lw-k_0\big(\nabla e(s^*,\bar{a}_j)-Nv\big)\big]+\sum\limits_{i=1}^{m}\nabla\zeta_i^*h_i(s^*)\big)^T\big(e^{p\eta(\bar{s},s^*)}-\mathbf{1}\big)\right]+(\rho^1+\rho^2)||\theta(\bar{s},s^*)||^2<0,$
Using (2.3.1), we get

$$(\rho^1+\rho^2)||\theta(\bar{s},s^*)||^2<0,$$

which contradicts hypothesis (iii). Hence proved.

2.4 Duality results

We construct the following dual for **(NFP1)**:
(NFD) $\max\limits_{(q,\xi,\bar{a})\in K(c)}\ \sup\limits_{(c,\zeta,k,w,v)\in H_1(q,\xi,\bar{a})} k,$
where $H_1(q,\xi,\bar{a})$ signifies the set of $(c,\zeta,k,w,v)\in R^n\times R^m_+\times R_+\times R^n\times R^n$ satisfying

$$\sum_{j=1}^{q}\xi_j\{\nabla d(c,\bar{a}_j)+Lw-k(\nabla e(c,\bar{a}_j)-Nv)\}+\nabla\sum_{i=1}^{m}\zeta_ih_i(c)=0, \tag{2.4.1}$$

$$\sum_{j=1}^{q}\xi_j\{d(c,\bar{a}_j)+c^TLw-k(e(c,\bar{a}_j)-c^TNv)\}\geq 0, \tag{2.4.2}$$

$$\sum_{i=1}^{m}\zeta_ih_i(c)\geq 0, \tag{2.4.3}$$

$$(q,\xi,\bar{a})\in K(c), \tag{2.4.4}$$

$$w^TLw\leq 1,\quad v^TNv\leq 1. \tag{2.4.5}$$

If for $(q,\xi,\bar{a})\in K(c)$, $H_1(q,\xi,\bar{a})=\phi$, then we take supremum as $-\infty$.

Now we acquire the following results.

Theorem 2.4.1 *(Weak duality) Let s be a feasible solution of **(NFP1)** and $(c,\zeta,k,w,v,q,\xi,\bar{a})$*

be a feasible solution of ***(NFD)****. Suppose*

(i) $\sum\limits_{j=1}^{q}\xi_j(d(\cdot,\bar{a}_j)+(\cdot)^T Lw-k(e(\cdot,\bar{a}_j)-(\cdot)^T Nv))$ *is* (p,r)-ρ^1-(η,θ)*-invex at* c *w.r.t.* $\eta,\ \theta,$

(ii) $\sum\limits_{i=1}^{m}\zeta_i h_i(\cdot)$ *is* (p,r)-ρ^2-(η,θ)*-invex at* c *w.r.t. same* $\eta,\ \theta,$

(iii) $\rho^1+\rho^2\geq 0.$

Then,

$$\sup_{a\in A}\frac{d(s,a)+(s^T Ls)^{1/2}}{e(s,a)-(s^T Ns)^{1/2}}\geq k. \tag{2.4.6}$$

Proof Let

$$\sup_{a\in A}\frac{d(s,a)+(s^T Ls)^{1/2}}{e(s,a)-(s^T Ns)^{1/2}}< k.$$

From above, we have

$$d(s,\bar{a}_j)+(s^T Ls)^{1/2}-k(e(s,\bar{a}_j)-(s^T Ns)^{1/2})<0, \text{ for all } \bar{a}_j\in A.$$

Using (2.3.4), we get

$$\xi_j(d(s,\bar{a}_j)+(s^T Ls)^{1/2}-k(e(s,\bar{a}_j)-(s^T Ns)^{1/2}))<0, \tag{2.4.7}$$

as $\xi=(\xi_1,\xi_2,\ldots,\xi_q)\neq 0.$

From Lemma 1.1.1, (2.4.2), (2.4.5) and (2.4.7), we have

$$\begin{aligned}&\sum_{j=1}^{q}\xi_j\big[d(s,\bar{a}_j)+s^T Lw-k\big(e(s,\bar{a}_j)-s^T Nv\big)\big]\\&\quad\leq\sum_{j=1}^{q}\xi_j\big[d(s,\bar{a}_j)+(s^T Ls)^{1/2}-k\big(e(s,\bar{a}_j)-(s^T Ns)^{1/2}\big)\big]\\&\quad<0\leq\sum_{j=1}^{q}\xi_j\big[d(c,\bar{a}_j)+c^T Lw-k\big(e(c,\bar{a}_j)-c^T Nv\big)\big]\end{aligned}$$

Hence $\sum\limits_{j=1}^{q}\xi_j\big[d(s,\bar{a}_j)+s^T Lw-k\big(e(s,\bar{a}_j)-s^T Nv\big)\big]$

$$<\sum_{j=1}^{q}\xi_j\big[d(c,\bar{a}_j)+c^T Lw-k\big(e(c,\bar{a}_j)-c^T Nv\big)\big]. \tag{2.4.8}$$

From hypothesis (i), we acquire

$$\begin{aligned}&\big[\tfrac{1}{r}\big(e^{r\big[\sum_{j=1}^{q}\xi_j\big(d(s,\bar{a}_j)+s^T Lw-k\big(e(s,\bar{a}_j)-s^T Nv\big)\big)-\sum_{j=1}^{q}\xi_j\big(d(c,\bar{a}_j)+c^T Lw-k\big(e(c,\bar{a}_j)-c^T Nv\big)\big)\big]}-1\big)\big]\\&\quad\geq\tfrac{1}{p}\Bigg[\Big(\sum_{j=1}^{q}\xi_j\big[\nabla d(c,\bar{a}_j)+Lw-k\big(\nabla e(c,\bar{a}_j)-Nv\big)\big]\Big)^T\big(e^{p\eta(s,c)}-\mathbf{1}\big)\Bigg]+\rho^1||\theta(s,c)||^2.\end{aligned}$$

Using (2.4.8) together with above, we get

$$\frac{1}{p}\Bigg[\Big(\sum_{j=1}^{q}\xi_j\big[\nabla d(c,\bar{a}_j)+Lw-k\big(\nabla e(c,\bar{a}_j)-Nv\big)\big]\Big)^T\big(e^{p\eta(s,c)}-\mathbf{1}\big)\Bigg]+\rho^1||\theta(s,c)||^2<0. \quad (2.4.9)$$

From the feasibility of s along with $\zeta_i \geq 0,\ i \in I$, we have

$$\sum_{i=1}^{m}\zeta_i h_i(s) \leq 0. \quad (2.4.10)$$

Using (ii), we acquire

$$\tfrac{1}{r}\big(e^{r\left(\sum\limits_{i=1}^{m}\zeta_i h_i(s)-\sum\limits_{i=1}^{m}\zeta_i h_i(c)\right)}-1\big) \geq \tfrac{1}{p}\big(\sum_{i=1}^{m}\nabla\zeta_i h_i(c)\big)^T\big(e^{p\eta(s,c)}-\mathbf{1}\big)+\rho^2||\theta(s,c)||^2.$$

Now from (2.4.3), (2.4.10) and using above, we get

$$\frac{1}{p}\big(\sum_{i=1}^{m}\nabla\zeta_i h_i(c)\big)^T\big(e^{p\eta(s,c)}-\mathbf{1}\big)+\rho^2||\theta(s,c)||^2 \leq 0. \quad (2.4.11)$$

By adding (2.4.9) and (2.4.11), we obtain

$$\tfrac{1}{p}\Bigg[\Bigg(\sum_{j=1}^{q}\xi_j\big[\nabla d(c,\bar{a}_j)+Lw-k\big(\nabla e(c,\bar{a}_j)-Nv\big)\big]+\sum_{i=1}^{m}\nabla\zeta_i h_i(c)\Bigg)^T\big(e^{p\eta(s,c)}-\mathbf{1}\big)\Bigg]+(\rho^1+\rho^2)||\theta(s,c)||^2<0.$$

Consequently (2.4.1) and above inequality yield

$$(\rho^1+\rho^2)||\theta(s,c)||^2<0,$$

which contradicts to the fact that $\rho^1+\rho^2 \geq 0$.

Theorem 2.4.2 *(Strong duality) Let s^* be an optimal solution for **(NFP1)** and let $\nabla h_i(s^*)$, $i \in I(s^*)$ be LI. Then $\exists$ $(q^*,\xi^*,\bar{a}^*) \in K(s^*)$ and $(s^*,\zeta^*,k^*,w^*,v^*) \in H_1(q^*,\xi^*,\bar{a}^*)$, such that $(s^*,\zeta^*,k^*,w^*,v^*,q^*,\xi^*,\bar{a}^*)$ is feasible solution of **(NFD)**. If suppositions of Theorem 2.4.1 hold $\forall$ feasible solutions $(c,\zeta,k,w,v,q,\xi,\bar{a})$ of **(NFD)**, then $(s^*,\zeta^*,k^*,w^*,v^*,q^*,\xi^*,\bar{a}^*)$ is an optimal solution of **(NFD)**. Also both objectives have equal optimal values.*

Proof Given that s^* is an optimal solution of **(NFP1)**. $\nabla h_i(s^*)$, $i \in I(s^*)$ are LI, therefore by Theorem 2.3.1, $\exists$ $(q^*,\xi^*,\bar{a}^*) \in K(s^*)$ and $(s^*,\zeta^*,k^*,w^*,v^*) \in H_1(q^*,\xi^*,\bar{a}^*)$ such that $(s^*,\zeta^*,k^*,w^*,v^*,q^*,\xi^*,\bar{a}^*)$ is feasible solution of **(NFD)**. Also objectives gives equal values as

$$k^*=\frac{d(s^*,\bar{a}_j^*)+(s^*Ls^*)^{1/2}}{e(s^*,\bar{a}_j^*)-(s^*Ns^*)^{1/2}}$$

Optimality of $(s^*,\zeta^*,k^*,w^*,v^*,q^*,\xi^*,\bar{a}^*)$ for **(NFD)**, thus follows from Theorem 2.4.1.

Theorem 2.4.3 *(Strict converse duality) Let* s^* *and* $(\bar{c}, \zeta^*, k^*, w^*, v^*, q^*, \xi^*, \bar{a}^*)$ *be optimal solutions of **(NFP1)** and **(NFD)** respectively, and let* $\nabla h_i(s^*), i \in I(s^*)$ *be LI. Assume that*

(i) $\sum\limits_{j=1}^{q^*} \xi_j^*(d(\cdot, \bar{a}_j^*) + (\cdot)^T Lw^* - k^*(e(\cdot, \bar{a}_j^*) - (\cdot)^T Nv^*))$ *is strictly* (p, r)*-*ρ^1*-*(η, θ)*-invex at* $\bar{c}$ *w.r.t. to* η*,* θ*,*

(ii) $\sum\limits_{i=1}^{m} \zeta_i^* h_i(\cdot)$ *is* (p, r)*-*ρ^2*-*(η, θ)*-invex at* $\bar{c}$ *w.r.t. same* η*,* θ*,*

(iii) $\rho^1 + \rho^2 \geq 0$.

Then, $s^* = \bar{c}$*, that is,* $\bar{c}$ *is an optimal solution in **(NFP1)** and*

$$\sup_{\bar{a}^* \in A} \frac{d(\bar{c}, \bar{a}^*) + (\bar{c}^T L\bar{c})^{1/2}}{e(\bar{c}, \bar{a}^*) - (\bar{c}^T N\bar{c})^{1/2}} = k^*.$$

Proof We will suppose that $s^* \neq \bar{c}$ and comes up with a contradiction. From Theorem 2.4.2, we have

$$\sup_{\bar{a}^* \in A} \frac{d(s^*, \bar{a}^*) + (s^{*T} Ls^*)^{1/2}}{e(s^*, \bar{a}^*) - (s^{*T} Ns^*)^{1/2}} = k^*. \tag{2.4.12}$$

By the feasibility of s^* along with $\zeta_i^* \geq 0,\ i \in I$, we get

$$\sum_{i=1}^{m} \zeta_i^* h_i(s^*) \leq 0. \tag{2.4.13}$$

Now, (2.4.3) and (2.4.13) gives

$$\tfrac{1}{r}\big(e^{r\big(\sum\limits_{i=1}^{m} \zeta_i^* h_i(s^*) - \sum\limits_{i=1}^{m} \zeta_i^* h_i(\bar{c})\big)} - 1\big) \leq 0.$$

Using (ii) and above, we obtain

$$\tfrac{1}{p}\Bigg[\big(\sum_{i=1}^{m} \nabla \zeta_i^* h_i(\bar{c})\big)^T \big(e^{p\eta(s^*, \bar{c})} - \mathbf{1}\big)\Bigg] + \rho^2 ||\theta(s^*, \bar{c})||^2 \leq 0.$$

that is

$$\frac{1}{p}\Bigg[\big(\sum_{i=1}^{m} \nabla \zeta_i^* h_i(\bar{c})\big)^T \big(e^{p\eta(s^*, \bar{c})} - \mathbf{1}\big)\Bigg] \leq -\rho^2 ||\theta(s^*, \bar{c})||^2. \tag{2.4.14}$$

Now using (2.4.1), (2.4.14) and assumption $\rho^1 + \rho^2 \geq 0$, we get

$$\frac{1}{p}\Bigg[\big(\sum_{j=1}^{q^*} \xi_j^* \big[\nabla d(\bar{c}, \bar{a}_j^*) + Lw^* - k^*\big(\nabla e(\bar{c}, \bar{a}_j^*) - Nv^*\big)\big]\big)^T \big(e^{p\eta(s^*, \bar{c})} - \mathbf{1}\big)\Bigg] \geq -\rho^1 ||\theta(s^*, \bar{c})||^2. \tag{2.4.15}$$

From (i) and using (2.4.15), we acquire

$$\big[\tfrac{1}{r}\big(e^{r\big[\sum\limits_{j=1}^{q^*} \xi_j^* \big(d(s^*, \bar{a}_j^*) + s^{*T} Lw^* - k^*\big(e(s^*, \bar{a}_j^*) - s^{*T} Nv^*\big)\big) - \sum\limits_{j=1}^{q^*} \xi_j^* \big(d(\bar{c}, \bar{a}_j^*) + \bar{c}^T Lw^* - k^*\big(e(\bar{c}, \bar{a}_j^*) - \bar{c}^T Nv^*\big)\big)\big]} - 1\big)\big] > 0.$$

This further gives
$$\sum_{j=1}^{q^*}\xi_j^*\big[d(s^*,\bar{a}_j^*)+s^{*T}Lw^*-k^*\big(e(s^*,\bar{a}_j^*)-s^{*T}Nv^*\big)\big]-\sum_{j=1}^{q^*}\xi_j^*\big[d(\bar{c},\bar{a}_j^*)+\bar{c}^TLw^*-k^*\big(e(\bar{c},\bar{a}_j^*)-\bar{c}^TNv^*\big)\big]>0.$$
Therefore, from (2.4.2),
$$\sum_{j=1}^{q^*}\xi_j^*\big[d(s^*,\bar{a}_j^*)+s^{*T}Lw^*-k^*\big(e(s^*,\bar{a}_j^*)-s^{*T}Nv^*\big)\big]>0.$$
Since $\xi_j^*\geq 0$ and $\xi^*\neq 0$, therefore $\exists\ j$ such that
$$d(s^*,\bar{a}_j^*)+s^{*T}Lw^*-k^*\big(e(s^*,\bar{a}_j^*)-s^{*T}Nv^*\big)>0.$$

Hence, we get
$$\frac{d(s^*,\bar{a}_j^*)+s^{*T}Lw^*}{e(s^*,\bar{a}_j^*)-s^{*T}Nv^*}>k^*,$$

which contradicts (2.4.12). Hence proved.

2.5 Second-order duality results

Second-order duality plays a vital role because of computational importance. It gives tighter bounds whenever an approximation is used. For the sake of convenience, let

$$\psi_1(\cdot)=\beta_1(\cdot)+\sum_{i=1}^{m}\zeta_i(h_i(\cdot)-h_i(c)) \tag{2.5.1}$$

and
$$\psi_2(\cdot)=\left[\sum_{j=1}^{q}\xi_j\big(e(c,\bar{a}_j)-c^TNv\big)\right]\left[\sum_{j=1}^{q}\xi_j(d(\cdot,\bar{a}_j)+(\cdot)^TLw)+\sum_{i=1}^{m}\zeta_ih_i(\cdot)\right]$$
$$-\left[\sum_{j=1}^{q}\xi_j(d(c,\bar{a}_j)+c^TLw)+\sum_{i=1}^{m}\zeta_ih_i(c)\right]\left[\sum_{j=1}^{q}\xi_j\big(e(\cdot,\bar{a}_j)-(\cdot)^TNv\big)\right],$$
where
$$\beta_1(\cdot)=\sum_{j=1}^{q}\xi_j[(e(c,\bar{a}_j)-c^TNv)(d(\cdot,\bar{a}_j)+(\cdot)^TLw)-(d(c,\bar{a}_j)+c^TLw)(e(\cdot,\bar{a}_j)-(\cdot)^TNv)].$$

Now we propose second-order duals for **(NFP1)**:

Model-I

$$\max_{(q,\xi,\bar{a})\in K(c)}\ \sup_{(c,\zeta,w,v,l)\in H_2(q,\xi,\bar{a})} F(c), \tag*{\textbf{(M1)}}$$

where $F(c)=\sup\limits_{a\in A}\dfrac{d(c,a)+(c^TLc)^{1/2}}{e(c,a)-(c^TNc)^{1/2}}$ and $H_2(q,\xi,\bar{a})$ contains $(c,\zeta,w,v,l)\in R^n\times R_+^m\times$

$R^n \times R^n \times R^n$ that satisfy

$$\nabla\psi_1(c) + \nabla^2\psi_1(c)l = 0, \tag{2.5.2}$$

$$\sum_{i=1}^{m} \zeta_i h_i(c) + \frac{1}{2} l^T \nabla^2 \psi_1(c) l \geq 0, \tag{2.5.3}$$

$$w^T L w \leq 1, \quad v^T N v \leq 1,$$

$$(c^T L c)^{1/2} = c^T L w, \quad (c^T N c)^{1/2} = c^T N v. \tag{2.5.4}$$

If the set $H_2(q, \xi, \bar{a}) = \phi$, we take the supremum as $-\infty$.

Theorem 2.5.1 *(Weak duality) Let s and $(c, \zeta, w, v, q, \xi, \bar{a}, l)$ be feasible solutions of **(NFP1)** and **(M1)**, respectively. Suppose*

(i) $\psi_1(\cdot)$ is second-order B-(p, r)-invex at c,

(ii) $\nabla^2\psi_1(c)$ is negative semidefinite.

Then

$$\sup_{\bar{a} \in A} \frac{d(s, \bar{a}) + (s^T L s)^{1/2}}{e(s, \bar{a}) - (s^T N s)^{1/2}} \geq F(c).$$

Proof Since hypothesis (i) gives

$$b(s, c)\left[\tfrac{1}{r}(e^{r(\psi_1(s) - \psi_1(c))} - 1) + \tfrac{1}{2} l^T \nabla^2 \psi_1(c) l\right] \geq \tfrac{1}{p}\left[\left(\nabla\psi_1(c) + \nabla^2\psi_1(c)l\right)^T \left(e^{p\eta(s,c)} - \mathbf{1}\right)\right],$$

from above and (2.5.2), we have

$$b(s, c)\left[\tfrac{1}{r}(e^{r(\psi_1(s) - \psi_1(c))} - 1) + \tfrac{1}{2} l^T \nabla^2 \psi_1(c) l\right] \geq 0,$$

that is

$$\left[\frac{1}{r}(e^{r(\psi_1(s) - \psi_1(c))} - 1) + \frac{1}{2} l^T \nabla^2 \psi_1(c) l\right] \geq 0,$$

which gives

$$e^{r(\psi_1(s) - \psi_1(c))} \geq 1 - \frac{r}{2} l^T \nabla^2 \psi_1(c) l, \tag{2.5.5}$$

Further from hypothesis (ii), we have

$$l^T \nabla^2 \psi_1(c) l \leq 0 \ \forall \ l, \tag{2.5.6}$$

Using (2.5.5) and (2.5.6), we get

$$e^{r(\psi_1(s) - \psi_1(c))} \geq 1,$$

which implies

$$\psi_1(s) \geq \psi_1(c).$$

From (2.5.1), (2.5.3), (2.5.6) and the feasibility of s implies

$$\beta_1(s) \geq -\sum_{i=1}^{m} \zeta_i h_i(s) \geq 0 = \beta_1(c). \tag{2.5.7}$$

Suppose

$$\sup_{\bar{a}\in A} \frac{d(s,\bar{a}) + (s^T L s)^{1/2}}{e(s,\bar{a}) - (s^T N s)^{1/2}} < F(c). \tag{2.5.8}$$

Since $\bar{a}_j \in A(c), j = 1, 2, \ldots, q$, we have

$$F(c) = \frac{d(c,\bar{a}_j) + (c^T L c)^{1/2}}{e(c,\bar{a}_j) - (c^T N c)^{1/2}}. \tag{2.5.9}$$

Using (2.5.8) and (2.5.9), for $j = 1, 2, \ldots, q$, we have

$$\frac{d(s,\bar{a}_j) + (s^T L s)^{1/2}}{e(s,\bar{a}_j) - (s^T N s)^{1/2}} \leq \sup_{\bar{a}\in A} \frac{d(s,\bar{a}) + (s^T L s)^{1/2}}{e(s,\bar{a}) - (s^T N s)^{1/2}} < \frac{d(c,\bar{a}_j) + (c^T L c)^{1/2}}{e(c,\bar{a}_j) - (c^T N c)^{1/2}}.$$

Also from $\xi_j \geq 0, j = 1, 2, \ldots, q, \xi \neq 0$ and $\bar{a}_j \in A(c)$, we get

$$\sum_{j=1}^{q} \xi_j \Big[\big(e(c,\bar{a}_j) - (c^T N c)^{1/2}\big)\big(d(s,\bar{a}_j) + (s^T L s)^{1/2}\big) - \big(d(c,\bar{a}_j) + (c^T L c)^{1/2}\big)$$

$$\times\big(e(s,\bar{a}_j) - (s^T N s)^{1/2}\big)\Big] < 0. \tag{2.5.10}$$

Now,

$$\begin{aligned}
\beta_1(s) &= \sum_{j=1}^{q} \xi_j \Big[\big(e(c,\bar{a}_j) - c^T N v\big)\big(d(s,\bar{a}_j) + s^T L w\big) \\
&\quad - \big(d(c,\bar{a}_j) + c^T L w\big)\big(e(s,\bar{a}_j) - s^T N v\big)\Big] \\
&\leq \sum_{j=1}^{q} \xi_j \Big[\big(e(c,\bar{a}_j) - (c^T N c)^{1/2}\big)\big(d(s,\bar{a}_j) + (s^T L s)^{1/2}\big) \\
&\quad - \big(d(c,\bar{a}_j) + (c^T L c)^{1/2}\big)\big(e(s,\bar{a}_j) - (s^T N s)^{1/2}\big)\Big] \\
&\quad \text{(using Lemma 1.1.1 and (2.5.4))} \\
&< 0 \qquad \text{(from (2.5.10))}.
\end{aligned}$$

Therefore,

$$\beta_1(s) < 0 = \beta_1(c).$$

This contradicts (2.5.7), hence proved.

Theorem 2.5.2 *(Strong duality) Suppose s^* is an optimal solution of **(NFP1)** and $\nabla h_i(s^*)$, $i \in I(s^*)$ are LI. Then there exist $(q^*, \xi^*, \bar{a}^*) \in K(s^*)$ and $(s^*, \zeta^*, w^*, v^*, l^* = 0) \in H_2(q^*, \xi^*, \bar{a}^*)$, such that $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ is feasible solution of **(M1)**. Also* *objectives have equal values. If suppositions of Theorem 2.5.1 satisfy for all feasible solutions* $(s, \zeta, w, v, q, \xi, \bar{a}, l)$ *of **(M1)**, then* $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ *is an optimal solution of **(M1)**.*

Proof Given that s^* is an optimal solution of **(NFP1)**. $\nabla h_i(s^*), i \in I(s^*)$ are LI. From Theorem 2.3.1, there exist $(q^*, \xi^*, \bar{a}^*) \in K(s^*)$ and $(s^*, \zeta^*, w^*, v^*, l^* = 0) \in H_2(q^*, \xi^*, \bar{a}^*)$ such that $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ is feasible solution of **(M1)**. Both objectives gives equal values. Optimality of $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ for **(M1)** follows from Theorem 2.5.1.

Theorem 2.5.3 *(Strict converse duality) Let* s^* *be an optimal solution of **(NFP1)** and* $(c^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^*)$ *be an optimal solution of **(M1)**. Assume that*

(i) $\psi_1(\cdot)$ *is strictly second-order* B*-*(p, r)*-invex at* c^**,*

(ii) $\{\nabla h_i(s^*), i \in I(s^*)\}$*, are LI,*

(iii) $\nabla^2 \psi_1(c^*)$ *is negative semidefinite.*

Then $c^* = s^*$.

Proof Using (i), we acquire

$$b(s^*, c^*)\left[\tfrac{1}{r}(e^{r(\psi_1(s^*)-\psi_1(c^*))} - 1) + \tfrac{1}{2} l^{*T} \nabla^2 \psi_1(c^*) l^*\right]$$
$$> \tfrac{1}{p}\left[\left(\nabla \psi_1(c^*) + \nabla^2 \psi_1(c^*) l^*\right)^T \left(e^{p\eta(s^*, c^*)} - \mathbf{1}\right)\right],$$

which using (2.5.2) give

$$b(s^*, c^*)\left[\tfrac{1}{r}(e^{r(\psi_1(s^*)-\psi_1(c^*))} - 1) + \tfrac{1}{2} l^{*T} \nabla^2 \psi_1(c^*) l^*\right] > 0,$$

that is

$$\left[\frac{1}{r}(e^{r(\psi_1(s^*)-\psi_1(c^*))} - 1) + \frac{1}{2} l^{*T} \nabla^2 \psi_1(c^*) l^*\right] > 0,$$

which implies

$$e^{r(\psi_1(s^*)-\psi_1(c^*))} > 1 - \frac{r}{2} l^{*T} \nabla^2 \psi_1(c^*) l^*, \tag{2.5.11}$$

Now from hypothesis (ii), we acquire

$$l^{*T} \nabla^2 \psi_1(c^*) l^* \leq 0 \ \forall \ l^*, \tag{2.5.12}$$

Using (2.5.11) and (2.5.12), we obtain

$$e^{r(\psi_1(s^*)-\psi_1(c^*))} > 1,$$

which implies

$$\psi_1(s^*) > \psi_1(c^*).$$

This further from (2.5.1), (2.5.3), (2.5.12) and the feasibility of s^* gives

$$\beta_1(s^*) > -\sum_{i=1}^{m} \zeta_i h_i(s^*) \geq 0 = \beta_1(c^*). \tag{2.5.13}$$

Let $c^* \neq s^*$. Given that s^* and $(c^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^*)$ are optimal solutions to **(NFP1)** and **(M1)**, respectively. $\{\nabla h_i(s^*), i \in I(s^*)\}$, are LI, by Theorem 2.5.2, we get

$$\sup_{\bar{a}^* \in A} \frac{d(s^*, \bar{a}^*) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{a}^*) - (s^{*T} N s^*)^{1/2}} = F(c^*). \tag{2.5.14}$$

Since $\bar{a}_j^* \in A(c^*), j = 1, 2, \ldots, q^*$, we have

$$F(c^*) = \frac{d(c^*, \bar{a}_j^*) + (c^{*T} L c^*)^{1/2}}{e(c^*, \bar{a}_j^*) - (c^{*T} N c^*)^{1/2}}. \tag{2.5.15}$$

By (2.5.14) and (2.5.15), we get

$$\begin{aligned}&\big[\big(e(c^*, \bar{a}_j^*) - (c^{*T} N c^*)^{1/2}\big)\big(d(s^*, \bar{a}_j^*) + (s^{*T} L s^*)^{1/2}\big)\\ &\qquad - \big(d(c^*, \bar{a}_j^*) + (c^{*T} L c^*)^{1/2}\big)\big(e(s^*, \bar{a}_j^*) - (s^{*T} N s^*)^{1/2}\big)\big] \leq 0\end{aligned}$$

for all $j = 1, 2, \ldots, q^*$ and $\bar{a}_j^* \in A$. From $\bar{a}_j^* \in A(c^*) \subset A$ and $\xi^* \in R_+^{q^*}$, with $\sum_{j=1}^{q^*} \xi_j^* = 1$, we obtain

$$\begin{aligned}&\sum_{j=1}^{q^*} \xi_j^* \big[\big(e(c^*, \bar{a}_j^*) - (c^{*T} N c^*)^{1/2}\big)\big(d(s^*, \bar{a}_j^*) + (s^{*T} L s^*)^{1/2}\big)\\ &\qquad\qquad - \big(d(c^*, \bar{a}_j^*) + (c^{*T} L c^*)^{1/2}\big)\big(e(s^*, \bar{a}_j^*) - (s^{*T} N s^*)^{1/2}\big)\big] \leq 0. \end{aligned} \tag{2.5.16}$$

From Lemma 1.1.1, (2.5.4), and (2.5.16), we have

$$\begin{aligned}\beta_1(s^*) &= \sum_{j=1}^{q^*} \xi_j^* \big[\big(e(c^*, \bar{a}_j^*) - c^{*T} N v^*\big)\big(d(s^*, \bar{a}_j^*) + s^{*T} L w^*\big)\\ &\quad - \big(d(c^*, \bar{a}_j^*) + c^* L w^*\big)\big(e(s^*, \bar{a}_j^*) - s^{*T} N v^*\big)\big]\\ &\leq \sum_{j=1}^{q^*} \xi_j^* \big[\big(e(c^*, \bar{a}_j^*) - (c^{*T} N c^*)^{1/2}\big)\big(d(s^*, \bar{a}_j^*) + (s^{*T} L s^*)^{1/2}\big)\\ &\quad - \big(d(c^*, \bar{a}_j^*) + (c^{*T} L c^*)^{1/2}\big)\big(e(s^*, \bar{a}_j^*) - (s^{*T} N s^*)^{1/2}\big)\big]\\ &\leq 0 = \beta_1(c^*),\end{aligned}$$

which contradicts (2.5.13), hence the result.

Model-II

$$\max_{(q,\xi,\bar{a})\in K(c)} \sup_{(c,\zeta,w,v,l)\in H_3(q,\xi,\bar{a})} \frac{\sum_{j=1}^{q}\xi_j\big(d(c,\bar{a}_j)+(c^TLc)^{1/2}\big)+\sum_{i=1}^{m}\zeta_i h_i(c)}{\sum_{j=1}^{q}\xi_j\big(e(c,\bar{a}_j)-(c^TNc)^{1/2}\big)}, \qquad \textbf{(M2)}$$

where $H_3(q,\xi,\bar{a})$ contains $(c,\zeta,w,v,l)\in R^n\times R_+^m\times R^n\times R^n\times R^n$ that satisfy

$$\nabla\psi_2(c)+\nabla^2\psi_2(c)l=0, \tag{2.5.17}$$

$$l^T\nabla^2\psi_2(c)l\le 0, \tag{2.5.18}$$

$$w^TLw\le 1,\quad v^TNv\le 1,\quad (c^TLc)^{1/2}=c^TLw,\quad (c^TNc)^{1/2}=c^TNv. \tag{2.5.19}$$

If $H_3(q,\xi,\bar{a})$ is empty, the supremum will be $-\infty$.

Theorem 2.5.4 *(Weak duality) Let s and $(c,\zeta,w,v,q,\xi,\bar{a},l)$ be feasible solutions of **(NFP1)** and **(M2)**, respectively. Suppose that $\psi_2(.)$ is second-order B-(p,r)-invex at c. Then*

$$\sup_{\bar{a}\in A}\frac{d(s,\bar{a})+(s^TLs)^{1/2}}{e(s,\bar{a})-(s^TNs)^{1/2}}\ge\frac{\sum_{j=1}^{q}\xi_j\big(d(c,\bar{a}_j)+(c^TLc)^{1/2}\big)+\sum_{i=1}^{m}\zeta_i h_i(c)}{\sum_{j=1}^{q}\xi_j\big(e(c,\bar{a}_j)-(c^TNc)^{1/2}\big)}.$$

Proof Since $\psi_2(.)$ is second-order B-(p,r)-invex at c, we acquire

$$b(s,c)\big[\tfrac{1}{r}(e^{r(\psi_2(s)-\psi_2(c))}-1)+\tfrac{1}{2}l^T\nabla^2\psi_2(c)l\big]\ge\tfrac{1}{p}\Big[\big(\nabla\psi_2(c)+\nabla^2\psi_2(c)l\big)^T\big(e^{p\eta(s,c)}-\mathbf{1}\big)\Big].$$

Using above and (2.5.17), we get

$$b(s,c)\big[\tfrac{1}{r}(e^{r(\psi_2(s)-\psi_2(c))}-1)+\tfrac{1}{2}l^T\nabla^2\psi_2(c)l\big]\ge 0.$$

This implies

$$\big[\tfrac{1}{r}(e^{r(\psi_2(s)-\psi_2(c))}-1)+\tfrac{1}{2}l^T\nabla^2\psi_2(c)l\big]\ge 0.$$

From (2.5.18), we have

$$\tfrac{1}{r}(e^{r(\psi_2(s)-\psi_2(c))}-1)\ge 0,$$

that is

$$\psi_2(s)\ge\psi_2(c). \tag{2.5.20}$$

Suppose

$$\sup_{\bar{a}\in A}\frac{d(s,\bar{a})+(s^TLs)^{1/2}}{e(s,\bar{a})-(s^TNs)^{1/2}}<\frac{\sum_{j=1}^{q}\xi_j\big(d(c,\bar{a}_j)+(c^TLc)^{1/2}\big)+\sum_{i=1}^{m}\zeta_i h_i(c)}{\sum_{j=1}^{q}\xi_j\big(e(c,\bar{a}_j)-(c^TNc)^{1/2}\big)}.$$

or

$$\big(d(s,\bar{a}_j)+(s^TLs)^{1/2}\big)\Big[\sum_{j=1}^{q}\xi_j\big(e(c,\bar{a}_j)-(c^TNc)^{1/2}\big)\Big]$$

$$< \big(e(s,\bar{a}_j) - (s^T N s)^{1/2}\big)\Bigg[\sum_{j=1}^{q} \xi_j\big(d(c,\bar{a}_j) + (c^T L c)^{1/2}\big) + \sum_{i=1}^{m} \zeta_i h_i(c)\Bigg],$$

for all $\bar{a}_j \in A(c),\ j = 1,2,\ldots,q$.
From $\xi_j \geq 0, j = 1,2,\ldots,q$ and (2.5.19) in above, we have

$$\sum_{j=1}^{q} \xi_j\big(d(s,\bar{a}_j) + (s^T L s)^{1/2}\big)\Bigg[\sum_{j=1}^{q} \xi_j\big(e(c,\bar{a}_j) - c^T N v\big)\Bigg]$$
$$< \sum_{j=1}^{q} \xi_j\big(e(s,\bar{a}_j) - (s^T N s)^{1/2}\big)\Bigg[\sum_{j=1}^{q} \xi_j\big(d(c,\bar{a}_j) + c^T L w\big) + \sum_{i=1}^{m} \zeta_i h_i(c)\Bigg] \quad (2.5.21)$$

Now, $\psi_2(s) = \Bigg[\sum\limits_{j=1}^{q} \xi_j\big(d(s,\bar{a}_j) + s^T L w\big) + \sum\limits_{i=1}^{m} \zeta_i h_i(s)\Bigg]\Bigg[\sum\limits_{j=1}^{q} \xi_j\big(e(c,\bar{a}_j) - c^T N v\big)\Bigg]$

$$- \Bigg[\sum_{j=1}^{q} \xi_j\big(e(s,\bar{a}_j) - s^T N v\big)\Bigg]\Bigg[\sum_{j=1}^{q} \xi_j\big(d(c,\bar{a}_j) + c^T L w\big) + \sum_{i=1}^{m} \zeta_i h_i(c)\Bigg]$$
$$\leq \Bigg[\sum_{j=1}^{q} \xi_j\big(d(s,\bar{a}_j) + (s^T L s)^{1/2}\big) + \sum_{i=1}^{m} \zeta_i h_i(s)\Bigg]\Bigg[\sum_{j=1}^{q} \xi_j\big(e(c,\bar{a}_j) - c^T N v\big)\Bigg]$$
$$- \Bigg[\sum_{j=1}^{q} \xi_j\big(e(s,\bar{a}_j) - (s^T N s)^{1/2}\big)\Bigg]\Bigg[\sum_{j=1}^{q} \xi_j\big(d(c,\bar{a}_j) + c^T L w\big) + \sum_{i=1}^{m} \zeta_i h_i(c)\Bigg]$$

(from Lemma 1.1.1 and (2.5.19))

$$< \sum_{j=1}^{q} \xi_j\big(e(c,\bar{a}_j) - c^T N v\big)\sum_{i=1}^{m} \zeta_i h_i(s) \quad \text{(using (2.5.21))}$$

$$\leq 0 \qquad \Bigg(\text{Since } \sum_{j=1}^{q} \xi_j\big(e(c,\bar{a}_j) - c^T N v\big) > 0 \text{ and } \sum_{i=1}^{m} \zeta_i h_i(s) \leq 0\Bigg).$$

Hence,

$$\psi_2(s) < 0 = \psi_2(c).$$

which contradicts (2.5.20). Hence proved.

Theorem 2.5.5 *(Strong duality) If s^* is an optimal solution for **(NFP1)** and $\nabla h_i(s^*)$, $i \in I(s^*)$ are LI. Then there exist $(q^*, \xi^*, \bar{a}^*) \in K(s^*)$ and $(s^*, \zeta^*, w^*, v^*, l^* = 0) \in H_3(q^*, \xi^*, \bar{a}^*)$, such that $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ is feasible solution of **(M2)**. Also objectives gives equal values. If suppositions of weak duality hold for all feasible solutions $(s, \zeta, w, v, q, \xi, \bar{a}, l)$ of **(M2)**, then $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ is an optimal solution of **(M2)**.*

Theorem 2.5.6 *(Strict converse duality) Let s^* and $(c^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^*)$ be optimal solutions of **(NFP1)** and **(M2)**, respectively. Suppose*

(i) $\psi_2(\cdot)$ is strictly second-order B-(p,r)-invex at c^*,

(ii) $\{\nabla h_i(s^*), i \in I(s^*)\}$, are LI,

Then $c^*=s^*$.

Proof From (i), we acquire

$$b(s^*,c^*)\left[\tfrac{1}{r}\left(e^{r(\psi_2(s^*)-\psi_2(c^*))}-1\right)+\tfrac{1}{2}l^{*T}\nabla^2\psi_2(c^*)l^*\right]$$
$$>\tfrac{1}{p}\left[\left(\nabla\psi_2(c^*)+\nabla^2\psi_2(c^*)l^*\right)^T\left(e^{p\eta(s^*,c^*)}-\mathbf{1}\right)\right],$$

which using (2.5.17) give

$$b(s^*,c^*)\left[\tfrac{1}{r}\left(e^{r(\psi_2(s^*)-\psi_2(c^*))}-1\right)+\tfrac{1}{2}l^{*T}\nabla^2\psi_2(c^*)l^*\right]>0,$$

that is

$$\left[\tfrac{1}{r}\left(e^{r(\psi_2(s^*)-\psi_2(c^*))}-1\right)+\tfrac{1}{2}l^{*T}\nabla^2\psi_2(c^*)l^*\right]>0.$$

From (2.5.18) and above we have,

$$\tfrac{1}{r}\left(e^{r(\psi_2(s^*)-\psi_2(c^*))}-1\right)>0.$$

From this, we get

$$\psi_2(s^*)>\psi_2(c^*). \tag{2.5.22}$$

Let $c^*\neq s^*$. Given that s^* and $(c^*,\zeta^*,w^*,v^*,q^*,\xi^*,\bar{a}^*,l^*)$ are optimal solutions to **(NFP1)** and **(M2)**, respectively. $\{\nabla h_i(s^*), i\in I(s^*)\}$, are LI, by Theorem 2.5.5, we get

$$\sup_{\bar{a}^*\in A}\frac{d(s^*,\bar{a}^*)+(s^{*T}Ls^*)^{1/2}}{e(s^*,\bar{a}^*)-(s^{*T}Ns^*)^{1/2}}=\frac{\sum\limits_{j=1}^{q^*}\xi_j^*\left(d(c^*,\bar{a}_j^*)+(c^{*T}Lc^*)^{1/2}\right)+\sum\limits_{i=1}^{m}\zeta_ih_i(c^*)}{\sum\limits_{j=1}^{q^*}\xi_j^*\left(e(c^*,\bar{a}_j^*)-(c^{*T}Nc^*)^{1/2}\right)}$$

or

$$\left(d(s^*,\bar{a}_j^*)+(s^{*T}Ls^*)^{1/2}\right)\left[\sum_{j=1}^{q^*}\xi_j^*\left(e(c^*,\bar{a}_j^*)-(c^{*T}Nc^*)^{1/2}\right)\right]$$
$$\leq\left(e(s^*,\bar{a}_j^*)-(s^{*T}Ns^*)^{1/2}\right)\left[\sum_{j=1}^{q^*}\xi_j^*\left(d(c^*,\bar{a}_j^*)+(c^{*T}Lc^*)^{1/2}\right)+\sum_{i=1}^{m}\zeta_ih_i(c^*)\right],$$

for all $\bar{a}_j^*\in A(c), j=1,2,\ldots,q^*$.
Using $\xi_j^*\geq 0, j=1,2,\ldots,q^*$ and (2.5.19) in above, we have

$$\sum_{j=1}^{q^*}\xi_j^*\left(d(s^*,\bar{a}_j^*)+(s^{*T}Ls^*)^{1/2}\right)\left[\sum_{j=1}^{q^*}\xi_j^*\left(e(c^*,\bar{a}_j^*)-c^{*T}Nv^*\right)\right]$$
$$\leq\sum_{j=1}^{q^*}\xi_j^*(e(s^*,\bar{a}_j^*)-(s^{*T}Ns^*)^{1/2})\left[\sum_{j=1}^{q^*}\xi_j^*\left(d(c^*,\bar{a}_j^*)+c^{*T}Lw^*\right)+\sum_{i=1}^{m}\zeta_ih_i(c^*)\right]. \tag{2.5.23}$$

From Lemma 1.1.1,(2.5.19) and (2.5.23), we have

$$\psi_2(s^*)=\left[\sum_{j=1}^{q^*}\xi_j^*\left(d(s^*,\bar{a}_j^*)+s^{*T}Lw^*\right)+\sum_{i=1}^{m}\zeta_ih_i(s^*)\right]\left[\sum_{j=1}^{q^*}\xi_j^*\left(e(c^*,\bar{a}_j^*)-c^{*T}Nv^*\right)\right]$$
$$-\left[\sum_{j=1}^{q^*}\xi_j^*\left(e(s^*,\bar{a}_j^*)-s^{*T}Nv^*\right)\right]\left[\sum_{j=1}^{q^*}\xi_j^*\left(d(c^*,\bar{a}_j^*)+c^{*T}Lw^*\right)+\sum_{i=1}^{m}\zeta_ih_i(c^*)\right]$$

$$\leq \left[\sum_{j=1}^{q^*} \xi_j^* \big(d(s^*, \bar{a}_j^*) + (s^{*T} L s^*)^{1/2}\big) + \sum_{i=1}^{m} \zeta_i h_i(s^*)\right] \left[\sum_{j=1}^{q^*} \xi_j^* \big(e(c^*, \bar{a}_j^*) - c^{*T} N v^*\big)\right]$$

$$- \left[\sum_{j=1}^{q^*} \xi_j^* \big(e(s^*, \bar{a}_j^*) - (s^{*T} N s^*)^{1/2}\big)\right] \left[\sum_{j=1}^{q^*} \xi_j^* \big(d(c^*, \bar{a}_j^*) + c^{*T} L w^*\big) + \sum_{i=1}^{m} \zeta_i h_i(c^*)\right]$$

(from Lemma 1.1.1 and (2.5.19))

$$\leq \sum_{j=1}^{q^*} \xi_j^* \big(e(c^*, \bar{a}_j^*) - c^{*T} N v^*\big) \sum_{i=1}^{m} \zeta_i h_i(s^*) \quad \text{(using (2.5.23))}$$

$$\leq 0 \qquad \left(\text{Since } \sum_{j=1}^{q^*} \xi_j^* \big(e(c^*, \bar{a}_j^*) - c^{*T} N v^*\big) > 0 \text{ and } \sum_{i=1}^{m} \zeta_i h_i(s^*) \leq 0\right).$$

Hence,

$$\psi_2(s^*) < 0 = \psi_2(c^*).$$

which contradicts (2.5.22), hence the result.

2.6 Conclusion

We studied NMFP problem and acquired duality results with (p, r)-ρ-(η, θ)-invexity. We have also constructed two second-order duals and achieved results using generalized invexity.

Chapter 3

Higher-order non-symmetric duality for non-differentiable minimax fractional programs with square root terms[1]

3.1 Introduction

The theory of higher-order duality is most significant. It gives better bounds to the infimum of primal problem when it is hard to examine feasible solution for first and second-order dual. Initially, higher-order duality was introduced by Mangasarian [91]. Suneja et al. [124] considered Mond-Weir and Schaible type nondifferentiable higher-order dual. They achieved results using higher-order (F, ρ, σ)- type I suppositions. Various higher-order duality models for minimax problems were derived by Jayswal and Stancu-Minasian [70]. Later on Jayswal et al. [68] has taken NMFP problem and established duality theorems using generalized convexity. Duality relations for fractional MP were discussed by Dubey et al. [38]. Sonali et al. [119] studied second-order duality for NMFP with second-order B-(p, r)-invexity. Jayswal et al. [69] corroborated a higher-order duality model for NMFP with (C, α, ρ, d)-convexity. Recently Li et al. [85] acquired optimality conditions and duality results for MFP with data uncertainty.

This chapter is organized as follows. In Section 3.2, a novel concept of higher-order B-(p, r)-invex functions has been introduced. An illustration of higher-order B-(p, r)-invex functions has been given. Optimality conditions for NMFP problems are also discussed. Section 3.4 presents a higher-order dual for NMFP problem. A numerical example of NMFP program has been solved in Section 3.5. The paper closes with conclusions in Section 3.7.

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[1] The content of this chapter is published as "*Higher-order non-symmetric duality for nondifferentiable minimax fractional programs with square root terms*", Acta Mathematica Scientia, 40B(1) (2020) 127-140.

3.2 Preliminaries

We focus on the following problem:

$$\text{Minimize } \psi(s) = \sup_{a\in A} \frac{d(s,a) + (s^T Ls)^{1/2}}{e(s,a) - (s^T Ns)^{1/2}}, \qquad \textbf{(NMFPP)}$$

subject to $h(s) \leq 0$.

where A is a compact subset of $R^{l'}$, $d(\cdot,\cdot) : R^n \times R^{l'} \to R$, $e(\cdot,\cdot) : R^n \times R^{l'} \to R$ are C^2(twice continuously differentiable) functions on $R^n \times R^{l'}$ and $h(\cdot) : R^n \to R^m$ is C^2 function on R^n. L and N are $n \times n$ positive semidefinite matrices. We assume that $e(s,a) - (s^T Ns)^{1/2} > 0$ and $d(s,a) + (s^T Ls)^{1/2} \geq 0$ for each $(s,a) \in S \times A$, where $S = \{s \in R^n : \ h(s) \leq 0\}$ contains feasible solutions of **(NMFPP)**.

For $(s,a) \in S \times A$ and $M = \{1, 2, \ldots, m\}$, Consider

$I(s) = \{i \in M : h_i(s) = 0\}$,

$$A(s) = \left\{ a \in A : \frac{d(s,a) + (s^T Ls)^{1/2}}{e(s,a) - (s^T Ns)^{1/2}} = \sup_{u\in A} \frac{d(s,u) + (s^T Ls)^{1/2}}{e(s,u) - (s^T Ns)^{1/2}} \right\},$$

$$K(s) = \left\{ (q, \xi, \bar{a}) \in N \times R^q_+ \times R^{l'q} : 1 \leq q \leq n+1, \ \xi = (\xi_1, \xi_2, \ldots, \xi_q) \in R^q_+, \ \sum_{j=1}^{q} \xi_j = 1, \ \bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_q), \ \bar{a}_j \in A(s), j = 1, 2, \ldots, q \right\}.$$

Since d and e are C^2 and A is compact in $R^{l'}$, for each $s^* \in S$, $A(s^*) \neq \phi$, and for any $\bar{a}_j \in A(s^*)$, we have a positive constant

$$k_0 = \psi(s^*) = \frac{d(s^*, \bar{a}_j) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{a}_j) - (s^{*T} N s^*)^{1/2}}.$$

Definition 3.2.1 *Let $\varphi : \hat{E} \to R$ (where $\hat{E} \subset R^n$) be a differentiable function and let p, r be arbitrary real numbers. Then φ is called higher-order B-(p,r)-invex (strictly higher-order B-(p,r)-invex) w.r.t. η, b and a differentiable function $k : \hat{E} \times R^n \to R$ ($\hat{E} \subset R^n$) at $z \in \hat{E}$ if $\exists$ a function $\eta : \hat{E} \times \hat{E} \to R^n$ and a function $b : \hat{E} \times \hat{E} \to R_+$ such that $\forall$ $s \in \hat{E}$ and $l \in R^n$,*

$$b(s,z)\Big[\frac{1}{r}(e^{r(\varphi(s)-\varphi(z))} - 1) - k(z,l) + l^T \nabla_l k(z,l)\Big] \geq \frac{1}{p}(\nabla_s \varphi(z) + \nabla_l k(z,l))^T (e^{p\eta(s,z)} - \boldsymbol{1})$$
$$(> \ \textit{if } s \neq z) \ \textit{for } p \neq 0, r \neq 0. \qquad (3.2.1)$$

Now

(i) when $r \neq 0, p = 0$ in (3.2.1), we acquire

$$b(s,z)\big[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(z))} - 1) - k(z,l) + l^T \nabla_l k(z,l)\big] \geq \big[(\nabla_s \varphi(z) + \nabla_l k(z,l))^T \eta(s,z)\big]$$
$$(> \textit{if } s \neq z)$$

(ii) when $p \neq 0, r = 0$ in (3.2.1), we acquire

$$b(s,z)(\varphi(s)-\varphi(z)-k(z,l)+l^T\nabla_l k(z,l)) \geq \tfrac{1}{p}[(\nabla_s\varphi(z)+\nabla_l k(z,l))^T(e^{p\eta(s,z)}-\boldsymbol{1})]$$
$$(> \text{ if } s \neq z)$$

(iii) when $r=0, p=0$ in (3.2.1), we acquire

$$b(s,z)(\varphi(s)-\varphi(z)-k(z,l)+l^T\nabla_l k(z,l)) \geq [\eta(s,z)^T(\nabla_s\varphi(z)+\nabla_l k(z,l))]$$
$$(> \text{ if } s \neq z)$$

Here $\boldsymbol{1}=(1,1,\ldots,1)\in R^n$, $(e^{p\eta(s,z)}-\boldsymbol{1})$ *represents* $(e^{p\eta_1(s,z)}-1,\ e^{p\eta_2(s,z)}-1,\ldots,\ e^{p\eta_n(s,z)}-1)$.

Example 3.2.1 *Let $\hat{E}=[0.1,1]\subset R$. Take $\varphi:\hat{E}\to R$ as $\varphi(s)=\log(\sin s)$.*
Consider $\eta:\hat{E}\times\hat{E}\to R$ as

$$\eta(s,z)=\log\left(\frac{1}{1+z}\right)$$

and

$$b(s,z)=\cos^2(z).$$

Let $k(z,l)=(z+1)l$ where $k:\hat{E}\times R\to R$.

Now φ is higher order B-$(1,1)$-invex as

$$\begin{aligned}
&b(s,z)\left[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(z))}-1)-k(z,l)+l^T\nabla_l k(z,l)\right]-\tfrac{1}{p}(\nabla\varphi(z)+\nabla_l k(z,l))^T(e^{p\eta(s,z)}-\boldsymbol{1})\\
&=\cos^2(z)\left(\frac{\sin(s)}{\sin(z)}-1\right)+z\left(\frac{\cot(z)}{z+1}+1\right)\\
&\geq 0 \text{ for all } s,z\in\hat{E} \text{ (see Figure 3.1).}
\end{aligned}$$

But it is not second-order B-$(-1,-1)$-invex as

$$\begin{aligned}
&b(s,z)\left[\tfrac{1}{r}(e^{r(\varphi(s)-\varphi(z))}-1)+\tfrac{1}{2}l^T\nabla^2\varphi(z)l\right]-\tfrac{1}{p}(\nabla\varphi(z)+\nabla^2\varphi(z)l)^T(e^{p\eta(s,z)}-\boldsymbol{1})\\
&=-\cos^2(z)\left(\frac{\sin(z)}{\sin(s)}-1\right)+z\cot(z)-\tfrac{1}{2}l^2\cos^2(z)\csc^2(z)\\
&\quad -lz\csc^2(z)\\
&=-\cos^2(0.8)\left(\frac{\sin(0.8)}{\sin(0.1)}-1\right)+(0.8)\cot(0.8)-\tfrac{1}{2}l^2\cos^2(0.8)\csc^2(0.8)\\
&\quad -l(0.8)\csc^2(0.8)\\
&=-0.47163l^2-1.55461l-2.22503<0 \text{ for } s=0.1,\ z=0.8 \text{ and } l\in R.
\end{aligned}$$

Therefore the function φ is higher-order B-$(1,1)$-invex function but not second-order B-$(-1,-1)$-invex.

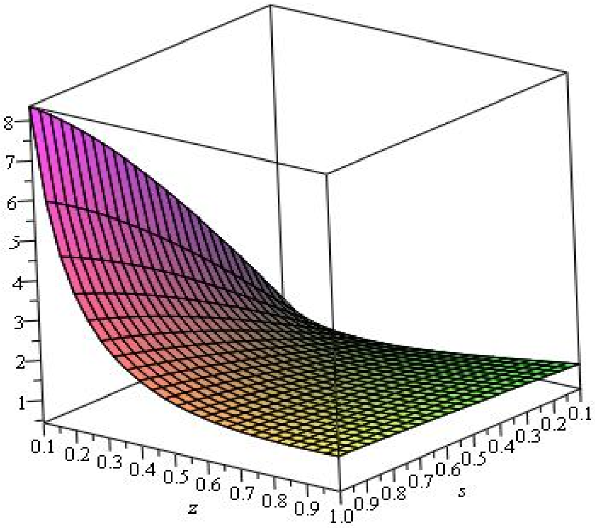

Figure 3.1: graph of function

Remark 3.2.1 *The theoretical results has been given for* $p \neq 0$ *and* $r \neq 0$*. Moreover we will suppose that* $r > 0$*.*

3.3 Optimality conditions

Theorem 3.3.1 *[84] (Necessary conditions) If* s^* *is an optimal solution of* ***(NMFPP)*** *that satisfies* $s^{*T}Ls^* > 0$, $s^{*T}Ns^* > 0$, *and* $\nabla h_i(s^*)$, $i \in I(s^*)$ *are LI, then* $\exists\ (q, \xi^*, \bar{a}) \in K(s^*)$, $k_0 \in R_+, w, v \in R^n$ *and* $\zeta^* \in R^m_+$ *s.t.*

$$\sum_{j=1}^{q} \xi_j^* \{\nabla d(s^*, \bar{a}_j) + Lw - k_0(\nabla e(s^*, \bar{a}_j) - Nv)\} + \sum_{i=1}^{m} \zeta_j^* \nabla h_i(s^*) = 0, \tag{3.3.1}$$

$$d(s^*, \bar{a}_j) + (s^{*T}Ls^*)^{1/2} - k_0(e(s^*, \bar{a}_j) - (s^{*T}Ns^*)^{1/2}) = 0, \quad j = 1, 2, \ldots, q, \tag{3.3.2}$$

$$\sum_{i=1}^{m} \zeta_i^* h_i(s^*) = 0, \tag{3.3.3}$$

$$\xi_j^* \geq 0\ (j = 1, 2, \ldots, q), \quad \sum_{j=1}^{q} \xi_j^* = 1, \tag{3.3.4}$$

$$w^T Lw \leq 1, \quad v^T Nv \leq 1, \quad (s^{*T}Ls^*)^{1/2} = s^{*T}Lw, \quad (s^{*T}Ns^*)^{1/2} = s^{*T}Nv. \tag{3.3.5}$$

Theorem 3.3.2 *(Sufficient Condition) Let* s^* *be a feasible solution of* ***(NMFPP)*** *and* $\exists$ *a positive integer* q, $1 \leq q \leq n+1$, $\xi^* \in R_+^q$, $\bar{a}_j \in A(s^*)(j = 1, 2, \ldots, q)$, $k_0 \in R_+$, w, $v \in R^n$ *and* $\zeta^* \in R_+^m$ *satisfying the relations (3.3.1-3.3.5). Assume that*

(i) $\vartheta(\cdot) = \vartheta_1(\cdot) + \sum\limits_{i=1}^{m} \zeta_i^* h_i(\cdot)$ *is higher-order* B*-*(p, r)*-invex at* s^* *w.r.t.* η *and* b *satisfying* $b(s, s^*) > 0 \ \forall \ s \in S$ *where* $\vartheta_1(\cdot) = \sum\limits_{j=1}^{q} \xi_j^*[d(\cdot, \bar{a}_j) + (\cdot)^T Lw - k_0(e(\cdot, \bar{a}_j) - (\cdot)^T Nv)]$,

(ii) $\nu(s^*, l^*) \geq 0$ *and* $\nabla_l \nu(s^*, l^*) = 0$.

Then, s^* *is an optimal solution of problem* ***(NMFPP)****.*

Proof Suppose s^* is not an optimal solution of **(NMFPP)**. Then there exists an $\bar{s} \in S$ such that

$$\sup_{\bar{a} \in A} \frac{d(\bar{s}, \bar{a}) + (\bar{s}^T L \bar{s})^{1/2}}{e(\bar{s}, \bar{a}) - (\bar{s}^T N \bar{s})^{1/2}} < \sup_{\bar{a} \in A} \frac{d(s^*, \bar{a}) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{a}) - (s^{*T} N s^*)^{1/2}}.$$

Since

$$\sup_{\bar{a} \in A} \frac{d(s^*, \bar{a}) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{a}) - (s^{*T} N s^*)^{1/2}} = \frac{d(s^*, \bar{a}_j) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{a}_j) - (s^{*T} N s^*)^{1/2}} = k_0,$$

for $\bar{a}_j \in A(s^*)$, $j = 1, 2, \ldots, q$ and

$$\frac{d(\bar{s}, \bar{a}_j) + (\bar{s}^T L \bar{s})^{1/2}}{e(\bar{s}, \bar{a}_j) - (\bar{s}^T N \bar{s})^{1/2}} \leq \sup_{\bar{a} \in A} \frac{d(\bar{s}, \bar{a}) + (\bar{s}^T L \bar{s})^{1/2}}{e(\bar{s}, \bar{a}) - (\bar{s}^T N \bar{s})^{1/2}},$$

therefore, we get

$$\frac{d(\bar{s}, \bar{a}_j) + (\bar{s}^T L \bar{s})^{1/2}}{e(\bar{s}, \bar{a}_j) - (\bar{s}^T N \bar{s})^{1/2}} < k_0.$$

Also $\xi_j^* \geq 0, j = 1, 2, \ldots, q, \xi^* \neq 0$ and $\bar{a}_j \in A(s^*)$ implies

$$\sum_{j=1}^{q} \xi_j^* \big[d(\bar{s}, \bar{a}_j) + (\bar{s}^T L \bar{s})^{1/2} - k_0 \big(e(\bar{s}, \bar{a}_j) - (\bar{s}^T N \bar{s})^{1/2}\big)\big] < 0. \tag{3.3.6}$$

Using Lemma 1.1.1, (3.3.5) and (3.3.6), we obtain

$$\begin{aligned} \vartheta_1(\bar{s}) \quad & = \sum_{j=1}^{q} \xi_j^* \big[d(\bar{s}, \bar{a}_j) + \bar{s}^T Lw - k_0 \big(e(\bar{s}, \bar{a}_j) - \bar{s}^T Nv\big)\big] \\ & \leq \sum_{j=1}^{q} \xi_j^* \big[d(\bar{s}, \bar{a}_j) + (\bar{s}^T L \bar{s})^{1/2} - k_0 \big(e(\bar{s}, \bar{a}_j) - (\bar{s}^T N \bar{s})^{1/2}\big)\big] \\ & < 0 = \vartheta_1(s^*), \end{aligned} \tag{3.3.7}$$

From hypothesis (i), we acquire

$$b(s, s^*)\big[\tfrac{1}{r}(e^{r(\vartheta(s) - \vartheta(s^*))} - 1) - \nu(s^*, l^*) + l^{*T} \nabla_l \nu(s^*, l^*)\big] \geq \tfrac{1}{p}\Big[\big(\nabla \vartheta(s^*) + \nabla_l \nu(s^*, l^*)\big)^T \big(e^{p\eta(s, s^*)} -$$

$\mathbf{1)}\Big]$

holds for all $s \in S$, and so for $s = \bar{s}$. Using (3.3.1), (ii) and $b(\bar{s}, s^*) > 0$ along with the inequality above, we have

$$\tfrac{1}{r}(e^{r(\vartheta(\bar{s})-\vartheta(s^*))} - 1) - \nu(s^*, l^*) + l^{*T}\nabla_l \nu(s^*, l^*) \geq 0,$$

$$\Rightarrow \tfrac{1}{r}(e^{r(\vartheta(\bar{s})-\vartheta(s^*))} - 1) \geq \nu(s^*, l^*) - l^{*T}\nabla_l \nu(s^*, l^*) \geq 0, \text{ (From hypothesis } (ii))$$

$$\Rightarrow \tfrac{1}{r}(e^{r(\vartheta(\bar{s})-\vartheta(s^*))} - 1) \geq 0,$$

$$\Rightarrow \vartheta(\bar{s}) \geq \vartheta(s^*).$$

Now,

$$\vartheta_1(\bar{s}) = \vartheta(\bar{s}) - \sum_{i=1}^{m} \zeta_i^* h_i(\bar{s}),$$

$$\geq \vartheta(s^*), \text{ (From above and } \sum_{i=1}^{m} \zeta_i^* h_i(\bar{s}) \leq 0)$$

$$= \vartheta_1(s^*),$$

that is $\vartheta_1(\bar{s}) \geq \vartheta_1(s^*)$, which contradicts (3.3.7). Hence the result.

For our convenience, Assume that

$$\psi_1(\cdot) = \beta_1(\cdot) + \sum_{i=1}^{m} \zeta_i(h_i(\cdot) - h_i(c)) \tag{3.3.8}$$

where

$$\beta_1(\cdot) = \sum_{j=1}^{q} \xi_j[(e(c, \bar{a}_j) - c^T Nv)(d(\cdot, \bar{a}_j) + (\cdot)^T Lw) - (d(c, \bar{a}_j) + c^T Lw)(e(\cdot, \bar{a}_j) - (\cdot)^T Nv)].$$

3.4 Higher-order duality results

Duality theory is well-developed for nonlinear programming problems. It is exciting to enhance the second order results to higher-order as it provides tighter bounds when approximations are used. A unified higher-order dual for NMP was constructed by Ahmad et al. [12] which was the extension of second-order dual proposed by Mishra and Rueda [94]. Later Ahmad [4] discussed higher-order duality for NMFP problem using generalized convexity assumptions. We have considered a higher-order dual for **(NMFPP)** problem:

Model-I

$$\max_{(q,\xi,\bar{a}) \in K(c)} \quad \sup_{(c,\zeta,w,v,l) \in H_1(q,\xi,\bar{a})} F(c), \tag{FPD}$$

where $F(c) = \sup\limits_{a \in A} \dfrac{d(c,a) + (c^T Lc)^{1/2}}{e(c,a) - (c^T Nc)^{1/2}}$ and $H_1(q, \xi, \bar{a})$ contains $(c, \zeta, w, v, l) \in R^n \times R^m_+ \times R^n \times R^n \times R^n$ that satisfy

$$\nabla \psi_1(c) + \nabla_l \phi_1(c, l) = 0, \tag{3.4.1}$$

$$\sum_{i=1}^{m} \zeta_i h_i(c) + \phi_1(c, l) - l^T \nabla_l \phi_1(c, l) \geq 0, \tag{3.4.2}$$

$$w^T Lw \leq 1, \quad v^T Nv \leq 1,$$

$$(c^T Lc)^{1/2} = c^T Lw, \quad (c^T Nc)^{1/2} = c^T Nv. \tag{3.4.3}$$

If $H_1(q, \xi, \bar{a}) = \phi$, then the supremum of $F(c)$ will be $-\infty$.

Now, we acquire the following duality results.

Theorem 3.4.1 *(Weak duality) Let s and $(c, \zeta, w, v, q, \xi, \bar{a}, l)$ be feasible solutions of* ***(NMFPP)*** *and* ***(FPD)****, respectively. Suppose*

(i) $\psi_1(\cdot)$ is higher-order B-(p, r)-invex at c w.r.t. η and b that satisfy $b(s, c) > 0 \; \forall \; s \in S$,

(ii) $\phi_1(c, l) = l^T \nabla_l \phi_1(c, l)$.

Then

$$\sup_{\bar{a} \in A} \frac{d(s, \bar{a}) + (s^T Ls)^{1/2}}{e(s, \bar{a}) - (s^T Ns)^{1/2}} \geq F(c).$$

Proof Assume that

$$\sup_{\bar{a} \in A} \frac{d(s, \bar{a}) + (s^T Ls)^{1/2}}{e(s, \bar{a}) - (s^T Ns)^{1/2}} < F(c). \tag{3.4.4}$$

Since $\bar{a}_j \in A(c), j = 1, 2, \ldots, q$, we have

$$F(c) = \frac{d(c, \bar{a}_j) + (c^T Lc)^{1/2}}{e(c, \bar{a}_j) - (c^T Nc)^{1/2}}. \tag{3.4.5}$$

From (3.4.4) and (3.4.5), for $j = 1, 2, \ldots, q$, we get

$$\frac{d(s, \bar{a}_j) + (s^T Ls)^{1/2}}{e(s, \bar{a}_j) - (s^T Ns)^{1/2}} \leq \sup_{\bar{a} \in A} \frac{d(s, \bar{a}) + (s^T Ls)^{1/2}}{e(s, \bar{a}) - (s^T Ns)^{1/2}} < \frac{d(c, \bar{a}_j) + (c^T Lc)^{1/2}}{e(c, \bar{a}_j) - (c^T Nc)^{1/2}}.$$

Also from $\xi_j \geq 0, j = 1, 2, \ldots, q, \xi \neq 0$ and $\bar{a}_j \in A(c)$, we get

$$\sum_{j=1}^{q} \xi_j \big[\big(e(c, \bar{a}_j) - (c^T Nc)^{1/2}\big)\big(d(s, \bar{a}_j) + (s^T Ls)^{1/2}\big) - \big(d(c, \bar{a}_j) + (c^T Lc)^{1/2}\big)$$
$$\times \big(e(s, \bar{a}_j) - (s^T Ns)^{1/2}\big)\big] < 0. \tag{3.4.6}$$

Now,

$$\begin{aligned}\beta_1(s) &= \sum_{j=1}^{q} \xi_j \big[\big(e(c,\bar{a}_j) - c^T Nv\big)\big(d(s,\bar{a}_j) + s^T Lw\big) \\ &\quad - \big(d(c,\bar{a}_j) + c^T Lw\big)\big(e(s,\bar{a}_j) - s^T Nv\big)\big] \\ &\leq \sum_{j=1}^{q} \xi_j \big[\big(e(c,\bar{a}_j) - (c^T Nc)^{1/2}\big)\big(d(s,\bar{a}_j) + (s^T Ls)^{1/2}\big) \\ &\quad - \big(d(c,\bar{a}_j) + (c^T Lc)^{1/2}\big)\big(e(s,\bar{a}_j) - (s^T Ns)^{1/2}\big)\big] \\ &\qquad \text{(using Lemma 1.1.1 and (3.4.3))} \\ &< 0 \qquad \text{(from (3.4.6))}.\end{aligned}$$

Therefore,

$$\beta_1(s) < 0. \tag{3.4.7}$$

Using (i), we achieve

$b(s,c)\big[\frac{1}{r}(e^{r(\psi_1(s)-\psi_1(c))} - 1) - \phi_1(c,l) + l^T \nabla_l \phi_1(c,l)\big] \geq \frac{1}{p}\Bigg[\big(\nabla\psi_1(c) + \nabla_l \phi_1(c,l)\big)^T \big(e^{p\eta(s,c)} - \mathbf{1}\big)\Bigg].$

From above, $b(s,c) > 0$ and (3.4.1), we have

$$\begin{aligned}& b(s,c)\Bigg[\frac{1}{r}(e^{r(\psi_1(s)-\psi_1(c))} - 1) - \phi_1(c,l) + l^T \nabla_l \phi_1(c,l)\Bigg] \geq 0, \\ &\Rightarrow \Bigg[\frac{1}{r}(e^{r(\psi_1(s)-\psi_1(c))} - 1) - \phi_1(c,l) + l^T \nabla_l \phi_1(c,l)\Bigg] \geq 0, \\ &\Rightarrow \frac{1}{r}(e^{r(\psi_1(s)-\psi_1(c))} - 1) \geq \phi_1(c,l) - l^T \nabla_l \phi_1(c,l), \\ &\Rightarrow \frac{1}{r}(e^{r(\psi_1(s)-\psi_1(c))} - 1) \geq 0, \text{ (From } (ii))\end{aligned}$$

which gives

$$\psi_1(s) \geq \psi_1(c). \tag{3.4.8}$$

Now,

$$\begin{aligned}\beta_1(s) &= \psi_1(s) - \sum_{i=1}^{m} \zeta_i (h_i(s) - h_i(c)), \\ &\geq \psi_1(c) + \sum_{i=1}^{m} \zeta_i h_i(c), \text{ (From (3.4.8) and } \sum_{i=1}^{m} \zeta_i h_i(s) \leq 0) \\ &\geq \psi_1(c) = 0. \text{ (Using hypothesis } (ii)\text{, (3.3.8) and (3.4.2))}\end{aligned}$$

Therefore, $\beta_1(s) \geq 0$. This contradicts (3.4.7).

Theorem 3.4.2 *(Strong duality) Suppose s^* is an optimal solution for* ***(NMFPP)****. Suppose $\nabla h_i(s^*), i \in I(s^*)$ are LI. Assume that*

$$\phi_1(s^*,0) = 0 \qquad\qquad \nabla_l \phi_1(s^*,0) = \nabla\psi_1(s^*)$$

Then there exist $(q^*, \xi^*, \bar{a}^*) \in K(s^*)$ and $(s^*, \zeta^*, w^*, v^*, l^* = 0) \in H_1(q^*, \xi^*, \bar{a}^*)$, such that

$(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ is feasible solution of ***(FPD)***. *Also objectives have equal values. If suppositions of Theorem 3.4.1 satisfy for all feasible solutions $(s, \zeta, w, v, q, \xi, \bar{a}, l)$ of* ***(FPD)***, *then $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ is optimal solution of* ***(FPD)***.

Proof Given that s^* is optimal solution of **(NMFPP)**. $\nabla h_i(s^*), i \in I(s^*)$ are LI, then by Theorem 3.3.1, there exist $(q^*, \xi^*, \bar{a}^*) \in K(s^*)$ and $(s^*, \zeta^*, w^*, v^*, l^* = 0) \in H_1(q^*, \xi^*, \bar{a}^*)$ such that $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ is feasible solution of **(FPD)** and the objectives gives equal values.

$$
\begin{aligned}
F(s^*) &= \sup_{\bar{a}^* \in A} \frac{d(s^*, \bar{a}^*) + (s^* L s^*)^{1/2}}{e(s^*, \bar{a}^*) - (s^* N s^*)^{1/2}} \\
&= \frac{d(s^*, \bar{a}_j^*) + (s^* L s^*)^{1/2}}{e(s^*, \bar{a}_j^*) - (s^* N s^*)^{1/2}} \\
&= k_0 \\
&= \psi(s^*).
\end{aligned}
$$

Optimality of $(s^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^* = 0)$ for **(FPD)** follows from Theorem 3.4.1.

Theorem 3.4.3 *(Strict converse duality) Let s^* be an optimal solution of* ***(NMFPP)*** *and $(c^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^*)$ be an optimal solution of* ***(FPD)***. *Suppose*

(i) *$\psi_1(\cdot)$ is strictly higher-order B-(p, r)-invex at c^* w.r.t. η and b that satisfy $b(s, c^*) > 0$ $\forall\ s \in S$,*

(ii) *$\{\nabla h_i(s^*), i \in I(s^*)\}$, are LI,*

(iii) *$\phi_1(c^*, l^*) = l^{*T} \nabla_l \phi_1(c^*, l^*)$.*

Then $c^* = s^*$.

Proof From (i), we acquire

$$
\begin{aligned}
b(s^*, c^*)\Big[\frac{1}{r}(e^{r(\psi_1(s^*) - \psi_1(c^*))} - 1) - \phi_1(c^*, l^*) + l^{*T} \nabla_l \phi_1(c^*, l^*)\Big] \\
> \frac{1}{p}\Big[\big(\nabla \psi_1(c^*) + \nabla_l \phi_1(c^*, l^*)\big)^T \big(e^{p\eta(s^*, c^*)} - \mathbf{1}\big)\Big]. \qquad (3.4.9)
\end{aligned}
$$

Using (3.4.1) and hypothesis *(iii)* in (3.4.9), we get

$$\tfrac{1}{r}(e^{r(\psi_1(s^*) - \psi_1(c^*))} - 1) > 0,$$

that is

$$\psi_1(s^*) > \psi_1(c^*). \qquad (3.4.10)$$

Now,

$$\beta_1(s^*) = \psi_1(s^*) - \sum_{i=1}^{m} \zeta_i^*(h_i(s^*) - h_i(c^*)),$$

$$\begin{aligned}&> \psi_1(c^*) - \sum_{i=1}^{m} \zeta_i^*(h_i(s^*) - h_i(c^*)), \text{ (From (3.4.10))}\\
&\geq \psi_1(c^*) + \sum_{i=1}^{m} \zeta_i^* h_i(c^*), \text{ (As } \sum_{i=1}^{m} \zeta_i^* h_i(s^*) \leq 0)\\
&\geq \psi_1(c^*), \text{ (Using hypothesis } (iii) \text{ and (3.4.2)}\\
&= \beta_1(c^*),\end{aligned}$$

that is

$$\beta_1(s^*) > \beta_1(c^*). \tag{3.4.11}$$

Now, we suppose that $c^* \neq s^*$. As s^* and $(c^*, \zeta^*, w^*, v^*, q^*, \xi^*, \bar{a}^*, l^*)$ are optimal solutions to **(NMFPP)** and **(FPD)**, respectively, and $\{\nabla h_i(s^*), i \in I(s^*)\}$, are LI, by Theorem 3.4.2, we get

$$\sup_{\bar{a}^* \in A} \frac{d(s^*, \bar{a}^*) + (s^{*T} L s^*)^{1/2}}{e(s^*, \bar{a}^*) - (s^{*T} N s^*)^{1/2}} = F(c^*). \tag{3.4.12}$$

Since $\bar{a}_j^* \in A(c^*), j = 1, 2, \ldots, q^*$, we have

$$F(c^*) = \frac{d(c^*, \bar{a}_j^*) + (c^{*T} L c^*)^{1/2}}{e(c^*, \bar{a}_j^*) - (c^{*T} N c^*)^{1/2}}. \tag{3.4.13}$$

By (3.4.12) and (3.4.13), we get

$$\begin{aligned}&\big[\big(e(c^*, \bar{a}_j^*) - (c^{*T} N c^*)^{1/2}\big)\big(d(s^*, \bar{a}_j^*) + (s^{*T} L s^*)^{1/2}\big)\\
&\qquad - \big(d(c^*, \bar{a}_j^*) + (c^{*T} L c^*)^{1/2}\big)\big(e(s^*, \bar{a}_j^*) - (s^{*T} N s^*)^{1/2}\big)\big] \leq 0\end{aligned}$$

for all $j = 1, 2, \ldots, q^*$ and $\bar{a}_j^* \in A$. From $\bar{a}_j^* \in A(c^*) \subset A$ and $\xi^* \in R_+^{q^*}$, with $\sum_{j=1}^{q^*} \xi_j^* = 1$, we obtain

$$\begin{aligned}&\sum_{j=1}^{q^*} \xi_j^* \big[\big(e(c^*, \bar{a}_j^*) - (c^{*T} N c^*)^{1/2}\big)\big(d(s^*, \bar{a}_j^*) + (s^{*T} L s^*)^{1/2}\big)\\
&\qquad -\big(d(c^*, \bar{a}_j^*) + (c^{*T} L c^*)^{1/2}\big)\big(e(s^*, \bar{a}_j^*) - (s^{*T} N s^*)^{1/2}\big)\big] \leq 0.\end{aligned} \tag{3.4.14}$$

From Lemma 1.1.1, (3.4.3) and (3.4.14), we have

$$\begin{aligned}\beta_1(s^*) &= \sum_{j=1}^{q^*} \xi_j^* \big[\big(e(c^*, \bar{a}_j^*) - c^{*T} N v^*\big)\big(d(s^*, \bar{a}_j^*) + s^{*T} L w^*\big)\\
&\quad - \big(d(c^*, \bar{a}_j^*) + c^{*T} L w^*\big)\big(e(s^*, \bar{a}_j^*) - s^{*T} N v^*\big)\big]\\
&\leq \sum_{j=1}^{q^*} \xi_j^* \big[\big(e(c^*, \bar{a}_j^*) - (c^{*T} N c^*)^{1/2}\big)\big(d(s^*, \bar{a}_j^*) + (s^{*T} L s^*)^{1/2}\big)\\
&\quad - \big(d(c^*, \bar{a}_j^*) + (c^{*T} L c^*)^{1/2}\big)\big(e(s^*, \bar{a}_j^*) - (s^{*T} N s^*)^{1/2}\big)\big]\\
&\leq 0 = \beta_1(c^*),\end{aligned}$$

which contradicts (3.4.11), hence the result.

3.5 Example of a minimax problem

Let $n = l' = 1,\ m = 2$ and $A = [1, 2]$.
Let $d(s, a) = s^4 + 2s^2a + a,\ e(s, a) = s^2 + 4,\ L = N = 1,\ h_1(s) = s - 3$ and $h_2(s) = -s + 1$.
Also $h(s) \leq 0\ \Rightarrow s - 3 \leq 0$ and $-s + 1 \leq 0$,

$$\Rightarrow 1 \leq s \leq 3.$$

Therefore $S = \{s \in R|\ 1 \leq s \leq 3\}$.

Now $d(s, a) + (s^T Ls)^{\frac{1}{2}} = s^4 + 2s^2a + a + |s| > 0$
and $e(s, a) - (s^T Ns)^{\frac{1}{2}} = s^2 + 4 - |s| > 0\ \forall (s, a) \in S \times A$
where $S \times A = \{(s, a)|1 \leq s \leq 3,\ 1 \leq a \leq 2\}$.
Now the problem **(NMFPP)** becomes

$$\text{Minimize } \psi(s) = \sup_{a\in A} \frac{s^4 + 2s^2a + a + |s|}{s^2 + 4 - |s|},$$

$$\text{subject to } s - 3 \leq 0,$$

$$-s + 1 \leq 0,$$

$$\text{and } 1 \leq a \leq 2.$$

Here $A(s) = \{2\}$ and $K(s) = \{(1, 1, 2)\}$.

In order to calculate minimax solution of **(NMFPP)** for $s^* \in [1, 3]$, we have considered the following cases.
Case I: Take $s^* = 3$
Using (3.3.2), (3.3.4) and (3.3.5), we get $k_0 = 12.2,\ w = v = 1$ and $\xi_1^* = 1$.

Since $h_1(s^*) = 0$ for $s^* = 3$, therefore from (3.3.3), we have $\zeta_2^* h_2(s^*) = 0$ which implies $\zeta_2^* = 0$. Hence from (3.3.1), $\zeta_1^* = -72$ does not satisfy the condition that $\zeta_1^* \in R_+$.

Case II: Take $1 < s^* < 3$
Using (3.3.3), we obtain $\zeta_1^*(s^* - 3) + \zeta_2^*(-s^* + 1) = 0$ which implies $\zeta_2^* = \zeta_1^* \frac{(s^* - 3)}{(s^* - 1)}$. From (3.3.4), $\xi_1^* = 1$ and from (3.3.5), we get $w^2 \leq 1,\ v^2 \leq 1,\ |s^*| = s^*w$ and $|s^*| = s^*v$ which implies $w = v = 1$. By using values of $\zeta_2^*,\ \xi_1^*,\ w$ and v in (3.3.1) and (3.3.2), we get

$$(4s^{*3} + 8s^* + 1) - \frac{(s^{*4} + 4s^{*2} + 2 + |s^*|)}{(s^{*2} + 4 - |s^*|)}(2s^* - 1) + \frac{2\zeta_1^*}{(s^* - 1)} = 0$$

$$\Rightarrow (4s^{*3} + 8s^* + 1) - \frac{(s^{*4} + 4s^{*2} + 2 + s^*)}{(s^{*2} + 4 - s^*)}(2s^* - 1) = \frac{-2\zeta_1^*}{(s^* - 1)}$$

Since $\frac{-2\zeta_1^*}{(s^* - 1)} \leq 0$ for $s^* \in (1, 3)$

therefore, $(4s^{*3}+8s^*+1)-\dfrac{(s^{*4}+4s^{*2}+2+s^*)}{(s^{*2}+4-s^*)}(2s^*-1)\leq 0$,
that is $(s^{*2}+4-s^*)(4s^{*3}+8s^*+1)-(s^{*4}+4s^{*2}+2+s^*)(2s^*-1)\leq 0$
that is $2s^{*5}-3s^{*4}+16s^{*3}-5s^{*2}+28s^*+6\leq 0$, which is not possible for any $s^*\in(1,3)$. Therefore $s^*\in(1,3)$ cannot be a minimax solution.

Case III: Take $s^*=1$
Since $h_2(s^*)=0$ for $s^*=1$, therefore from (3.3.3), we have $\zeta_1^* h_1(s^*)=0$ implies $\zeta_1^*=0$ and using (3.3.4), we get $\xi_1^*=1$. By solving (3.3.2), we have $k_0=2$ and using (3.3.5), we obtain $w=v=1$. Putting these values in (3.3.1), we get $\zeta_2^*=11$. Now using $w=v=1$ along with $\xi_1^*=1,\ k_0=2,\ \zeta_1^*=0$ and $\zeta_2^*=11$, we can check that all these values satisfy the necessary conditions for a minimax solution. Therefore $s^*=1$ is the only candidate for a minimax solution.

Now we will justify that $s^*=1$ is an optimal solution of **(NMFPP)**. For this we will show that $\vartheta(\cdot)$ is higher-order B-(p,r)-invex function at s^*.
Here $\vartheta(\cdot)=\sum\limits_{j=1}^{q}\xi_j^*[d(\cdot,\bar{a}_j)+(\cdot)^TLw-k_0(e(\cdot,\bar{a}_j)-(\cdot)^TNv)+\sum\limits_{i=1}^{m}\zeta_i^*(h_i(\cdot))$,

$$=[((\cdot)^4+4(\cdot)^2+2+(\cdot)w)-k_0((\cdot)^2+4-(\cdot)v)+\zeta_1^*((\cdot)-3)+\zeta_2^*(-(\cdot)+1)],$$

Using $w=v=1,\ k_0=2$ and $\zeta_1^*=0$ and $\zeta_2^*=11$, we get

$$\vartheta(\cdot)=[((\cdot)^4+4(\cdot)^2+2+(\cdot))-2((\cdot)^2+4-(\cdot))+11(-(\cdot)+1)],$$

$$=(\cdot)^4+2(\cdot)^2-8(\cdot)+5,$$

which is convex and hence higher-order B-(p,r)-invex at $s^*=1$. By taking $q=1,\ \xi_1^*=1,\ w=v=1,\ k_0=2,\ \zeta_1^*=0$ and $\zeta_2^*=11$, the sufficient conditions of the theorem are easily verified and $s^*=1$ is a minimax solution.

Now we will formulate dual model **(FPD)** for **(NMFPP)** problem.
Let $\phi_1(c,l)=(c+1)l$. From **(FPD)**, we have

$$\max_{(q,\xi,\bar{a})\in K(c)}\ \sup_{(c,\zeta,w,v,l)\in H_1(q,\xi,\bar{a})} F(c),$$

$$\text{where } F(c)=\sup_{a\in A}\frac{c^4+2c^2a+a+|c|}{c^2+4-|c|}$$

and $H_1(q,\xi,\bar{a})$ contains $(c,\zeta,w,v,l)\in R\times R_+^2\times R\times R\times R$ that satisfy

$$2c^5-3c^4v+16c^3-(4v+w)c^2+29c+4w+2v+\zeta_1-\zeta_2+1=0,$$

$$\zeta_1(c-3)+\zeta_2(-c+1)\geq 0,$$

$$w^2\leq 1,\ v^2\leq 1,$$

$$|c|=cw,\ |c|=cv.$$

3.6 Special cases

(*i*) If we take $\phi_1(c, l) = \frac{1}{2} l^T \nabla^2 \psi_1(c) l$ in **(FPD)** then, our dual will be reduced to the second-order dual given by Gupta et al. [50].

(*ii*) If $l = 0$, then the dual model **(FPD)** reduces to the problems studied in [7, 66, 83].

(*iii*) If $l = 0$ and L and N are zero matrices of order n, then **(FPD)** becomes the dual problem given in [87, 88].

3.7 Conclusion

We proposed a higher-order dual for **(NMFPP)** and proved duality results with higher-order B-(p, r)-invexity. An illustration of higher-order B-(p, r)-invex function has been given. Also we have solved a NMFP problem using optimality conditions.

Chapter 4

Parametric approach for a class of fractional variational programs involving support functions

4.1 Introduction

Duality results have been applied to several problems of calculus of variations that arises in physics, filtering and optimal control theory. The purpose of calculus of variations is to provide analytical methods to find maxima or minima of functionals. Several physical laws can be derived from mathematical principles to the effect that a certain functional acquires a maximum and minimum. The connection between mathematical programming and the classical calculus of variations was examined by [59]. Mond and Hanson [101] discussed the results of duality for VP using convexity. A symmetric dual pair for NMFVP problems was considered by Mishra et al. [96]. Mititelu and Stancu-Minasian [98] studied duality for MFVP problems and used a parametric approach in order to relate ESs of MFVP problem and a non-fractional problem. Kailey and Gupta [73] achieved appropriate duality relations for symmetric NMFVP with arbitrary cones using generalized (F, α, ρ, d)-convexity. Mond-Weir and Wolfe duals for MVP were studied by Antczak [15].

Second-order duality models for VP problems were formulated by Chen [29]. Second-order Mond-Weir dual for VP was explored by Husain et al. [63]. Later Gulati and Mehndiratta [57] provided a modified version of converse duality which was studied by Husain et al. [63]. Symmetric second-order duals for VP were constructed by Padhan et al. [110] and relations between primal and dual were proved under generalized invexity. Second-order $(F, \alpha, \rho, \theta)$-convexity for VP was introduced by Jayswal et al. [71] and various duality results were derived for VP and its second-order dual. Jayswal and Jha [67] constructed symmetric second-order dual for FVP involving cone constraints.

In this chapter, we study second-order duality for NMFVP problem with second-order (F, α, ρ, d)-pseudoconvexity suppositions. Section 4.2 reviews the definitions. An illustration showing the existence of second-order (F, α, ρ, d)-pseudoconvexity has been provided. Symmetric second-order duality for NMFVP and results are discussed with aforesaid suppositions in Section 4.4. Also an illustration which corroborate weak duality is obtained. The chapter closes with conclusions in Section 4.6.

4.2 Notations and preliminaries

Let $B = [c, d]$ be a real interval. Let $C_1 \subset R^n$, $C_2 \subset R^m$ be closed convex cones with nonempty interiors and their respective positive polar cones are C_1^* and C_2^*. Let $h^i(t, w(t), \dot{w}(t), s(t), \dot{s}(t))$ and $e^i(t, w(t), \dot{w}(t), s(t), \dot{s}(t))$ are C^2 functions $\forall\, i \in \hat{L} = \{1, 2, \ldots, l\}$, where $w : B \to R^n$ and $s : B \to R^m$ with derivatives $\dot{w}$ and $\dot{s}$. For $i \in \hat{L}$, the symbols $h^i_w, h^i_{\dot{w}}, h^i_s, h^i_{\dot{s}}$ signifies gradient vectors of $h^i(t, w, \dot{w}, s, \dot{s})$ with respect to $w, \dot{w}, s$ and $\dot{s}$, respectively. We have

$$h^i_w = \left(\frac{\partial h^i}{\partial w^1}, \ldots, \frac{\partial h^i}{\partial w^n}\right)^T, \qquad h^i_{\dot{w}} = \left(\frac{\partial h^i}{\partial \dot{w}^1}, \ldots, \frac{\partial h^i}{\partial \dot{w}^n}\right)^T$$

Similarly, $e^i_w, e^i_{\dot{w}}, e^i_s$ and $e^i_{\dot{s}}$ signifies the gradient vectors of $e^i(t, w, \dot{w}, s, \dot{s})$ with respect to $w, \dot{w}, s$ and $\dot{s}$, respectively.

These observations are considered to establish Theorem 4.4.2:

$$Dh^i_{\dot{s}} = h^i_{\dot{s}t} + h^i_{\dot{s}s}\dot{s} + h^i_{\dot{s}\dot{s}}\ddot{s} + h^i_{\dot{s}w}\dot{w} + h^i_{\dot{s}\dot{w}}\ddot{w}.$$

Consequently,

$$\tfrac{\partial}{\partial s} Dh^i_{\dot{s}} = Dh^i_{\dot{s}s}, \quad \tfrac{\partial}{\partial \dot{s}} Dh^i_{\dot{s}} = Dh^i_{\dot{s}\dot{s}} + h^i_{\dot{s}s}, \quad \tfrac{\partial}{\partial \ddot{s}} Dh^i_{\dot{s}} = h^i_{\dot{s}\dot{s}},$$

$$\tfrac{\partial}{\partial w} Dh^i_{\dot{s}} = Dh^i_{\dot{s}w}, \quad \tfrac{\partial}{\partial \dot{w}} Dh^i_{\dot{s}} = Dh^i_{\dot{s}\dot{w}} + h^i_{\dot{s}w}, \quad \tfrac{\partial}{\partial \ddot{w}} Dh^i_{\dot{s}} = h^i_{\dot{s}\dot{w}}, i \in \hat{L}.$$

Similarly, $De^i_{\dot{s}}$ can be defined.

Let $W(B, R^n)$ denotes the space of piecewise smooth functions w with the norm

$$||w|| = ||w||_\infty + ||Dw||_\infty.$$

Here the differentiation operator D is defined as

$$\vartheta = Dw \Leftrightarrow w(t) = \mu_1 + \int_c^t \vartheta(g)dg,$$

where μ_1 is a given boundary value. So, $\frac{d}{dt} \equiv D$ except at discontinuities. The space of piecewise smooth functions $s : B \to R^m$ is denoted with $\check{S}(B, R^m)$ with the same norm as defined for $W(B, R^n)$.

Consider the following MVP problem:

(MVP1) Minimize $\int_c^d \varphi(t, w(t), \dot{w}(t))dt = \big(\int_c^d \varphi^1(t, w(t), \dot{w}(t))dt, \ldots, \int_c^d \varphi^l(t, w(t), \dot{w}(t))dt\big)$

subject to $w(c) = 0 = w(d)$,

$\dot{w}(c) = 0 = \dot{w}(d)$,

$$\zeta(t, w(t), \dot{w}(t)) \leqq 0,\ t \in B.$$

Here $\varphi : B \times R^n \times R^n \to R^l$ and $\zeta : B \times R^n \times R^n \to R^m$ are differentiable functions. Let $\hat{S}$ contains feasible solutions of **(MVP1)**,i.e.,
$\hat{S} = \{w \in W(B, R^n)|\ w(c) = 0 = w(d),\ \dot{w}(c) = 0 = \dot{w}(d),\ \zeta(t, w(t), \dot{w}(t)) \leqq 0,\ t \in B\}.$

Definition 4.2.1 *[73] A point $\bar{w} \in \hat{S}$ is an ES of **(MVP1)** if $\nexists\ w \in \hat{S}$ s.t.*
$\int_c^d \varphi(t, w(t), \dot{w}(t))dt \leq \int_c^d \varphi(t, \bar{w}(t), \dot{\bar{w}}(t))dt.$

Definition 4.2.2 *[73] A point $\bar{w} \in \hat{S}$ is a WES of **(MVP1)** if $\nexists\ w \in \hat{S}$ s.t.*
$\int_c^d \varphi(t, w(t), \dot{w}(t))dt < \int_c^d \varphi(t, \bar{w}(t), \dot{\bar{w}}(t))dt.$

Definition 4.2.3 *[73] A functional $F : B \times W \times W \times W \times W \times R^n \to R$ is called sublinear in sixth argument, if $\forall\ w, \dot{w}, z, \dot{z} \in W$,*

(i) $F(t, w, \dot{w}, z, \dot{z}; \nu_1 + \nu_2) \leqq F(t, w, \dot{w}, z, \dot{z}; \nu_1) + F(t, w, \dot{w}, z, \dot{z}; \nu_2), \forall\ \nu_1, \nu_2 \in R^n$

(ii) $F(t, w, \dot{w}, z, \dot{z}; a_1\nu) = a_1 F(t, w, \dot{w}, z, \dot{z}; \nu), \forall\ a_1 \in R_+$ *and* $\nu \in R^n$.

Let F and G be functionals sublinear in sixth argument. Let $d = (d^1, d^2)$, where $d^1 = (d_1^1, d_2^1, \ldots, d_l^1) : B \times W \times W \times W \times W \to R^l$, $d^2 = (d_1^2, d_2^2, \ldots, d_l^2) : B \times \check{S} \times \check{S} \times \check{S} \times \check{S} \to R^l$. Let $h = (h^1, h^2, \ldots, h^l) : B \times W \times W \times \check{S} \times \check{S} \to R^l$ be a differentiable function, $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_1 : W \times W \to R_+ \setminus \{0\}$, $\alpha_2 : \check{S} \times \check{S} \to R_+ \setminus \{0\}$ and $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho_1^1, \rho_2^1, \ldots, \rho_l^1)$, $\rho^2 = (\rho_1^2, \rho_2^2, \ldots, \rho_l^2) \in R^l$.

Definition 4.2.4 *For every $i \in \hat{L}$, $\int_c^d h^i(t, w, \dot{w}, s, \dot{s})dt$ is called second-order $(F, \alpha_1, \rho_i^1, d_i^1)$-pseudoconvex at z for fixed s and $\dot{s}$, if $\forall\ w, \dot{w} \in W$ and $q^i \in R^n$,*
$\int_c^d F(t, w, \dot{w}, z, \dot{z}; \alpha_1(w, z)(h_w^i(t, z, \dot{z}, s, \dot{s}) - Dh_{\dot{w}}^i(t, z, \dot{z}, s, \dot{s}) + M^i q^i(t)))dt \geqq 0$
$\Rightarrow \int_c^d h^i(t, w, \dot{w}, s, \dot{s})dt \geqq \int_c^d h^i(t, z, \dot{z}, s, \dot{s})dt - \frac{1}{2}\int_c^d q^i(t)^T M^i q^i(t)dt + \rho_i^1 \int_c^d (d_i^1(t, w, \dot{w}, z, \dot{z}))^2 dt$
where $M^i = h_{ww}^i(t, z, \dot{z}, s, \dot{s}) - Dh_{w\dot{w}}^i(t, z, \dot{z}, s, \dot{s})$, $t \in B$.

Remark 4.2.1 *(i) If we take $q^i(t) = 0$ in above definition, then it will be reduced to the definition of $(F, \alpha_1, \rho_i^1, d_i^1)$-pseudoconvex functions given by [73].*

(ii) With (i), consider $F(t, w, \dot{w}, z, \dot{z}; a) = \eta(t, w, \dot{w}, z, \dot{z})^T a$, $\alpha_1(w, z) = 1$ and $\rho_i^1 = 0$ then it will become the definition of pseudoinvex functions studied by [79].

4.3 Example

Many optimization problems exist in the literature which includes nonconvex functions and functionals. Therefore it is essential to generalize the concept of convexity. In this work, we considered second-order (F,α,ρ,d)-pseudoconvexity. Now we construct an illustration which shows the existence of aforementioned functions.

Example 4.3.1 *Let* $B=[0,1]$*. Let* W*,* $\check{S}$ *be the space of piecewise smooth functions* $w: B\to[0,\infty)$ *and* $s: B\to[0,\infty)$ *respectively. For* $i=1$*, define* $h^1(t,w,\dot w,s,\dot s): B\times W\times W\times\check{S}\times\check{S}\to R$ *as* $h^1(t,w,\dot w,s,\dot s)=w^4-w+\sin\left(\frac{w}{2}\right)+s$*. Consider the functional* $F: B\times W\times W\times W\times W\times R\to R$ *be defined as* $F(t,w,\dot w,z,\dot z;\nu)=-\nu(w+z+1)$ *and* $\alpha_1: W\times W\to R_+\setminus\{0\}$ *be defined as* $\alpha_1(w,z)=\dfrac{1}{w+z+1}$*.*
Take $d_1^1(t,w,\dot w,z,\dot z)=(w+z)^{\frac{1}{2}}$ *and* $\rho_1^1=-6$*.*
Since $\int\limits_0^1 F(t,w,\dot w,z,\dot z;\alpha_1(w,z)(h^1_w(t,z,\dot z,s,\dot s)-Dh^1_{\dot w}(t,z,\dot z,s,\dot s)+M^1q^1(t)))dt$*,*

$$=\int\limits_0^1\left(-4z^3+1-\frac{1}{2}\cos\left(\frac{z}{2}\right)-\left(12z^2-\frac{1}{4}\sin\left(\frac{z}{2}\right)\right)q^1(t)\right)dt\geqq 0 \text{ for } w(t)\in W,\ z(t)=0$$

and $q^1(t)\in R$

and $\int\limits_0^1 h^1(t,w,\dot w,s,\dot s)dt-\int\limits_0^1 h^1(t,z,\dot z,s,\dot s)dt+\frac{1}{2}\int\limits_0^1 q^1(t)^TM^1q^1(t)dt-\rho_1^1\int\limits_0^1(d_1^1(t,w,\dot w,z,\dot z))^2dt$

$$=\int\limits_0^1\left(w^4-w+\sin\left(\frac{w}{2}\right)+s-z^4+z-\sin\left(\frac{z}{2}\right)-s+\frac{1}{2}\left(12z^2-\frac{1}{4}\sin\left(\frac{z}{2}\right)\right)(q^1(t))^2+6(w+z)\right)dt$$

$$=\int\limits_0^1\left(w^4-w+\sin\left(\frac{w}{2}\right)+6w\right)dt\geqq 0 \text{ for } w(t)\in W,\ z(t)=0 \text{ and } q^1(t)\in R$$

where $M^1=h^1_{ww}(t,z,\dot z,s,\dot s)-Dh^1_{w\dot w}(t,z,\dot z,s,\dot s)$*,*
Therefore, the functional $\int\limits_0^1 h^1(t,w,\dot w,s,\dot s)dt$ *is second-order* $(F,\alpha_1,\rho_1^1,d_1^1)$*-pseudoconvex at* $z(t)=0$*, But it is not second-order* F*-pseudoconvex at* $z(t)=0$ *as*
$\int\limits_0^1 F(t,w,\dot w,z,\dot z;h^1_w(t,z,\dot z,s,\dot s)-Dh^1_{\dot w}(t,z,\dot z,s,\dot s)+M^1q^1(t))dt\geqq 0$*,*
but $\int\limits_0^1 h^1(t,w,\dot w,s,\dot s)dt-\int\limits_0^1 h^1(t,z,\dot z,s,\dot s)dt+\frac{1}{2}\int\limits_0^1 q^1(t)^TM^1q^1(t)dt$

$$=\int\limits_0^1\left(w^4-w+\sin\left(\frac{w}{2}\right)-z^4+z-\sin\left(\frac{z}{2}\right)+\frac{1}{2}(12z^2-\frac{1}{4}\sin\left(\frac{z}{2}\right))(q^1(t))^2\right)dt\ngeqq 0 \text{ for all}$$

$w\in W$ *and* $q^1(t)\in R$*.*

4.4 Dual formulation

We construct the following second-order symmetric NMFVP problem and its dual with cone constraints.

Primal Problem **(MFVP1)**

Minimize

$$\frac{\int\limits_c^d [h(t, w, \dot{w}, s, \dot{s}) + S(w|E) - s^T u - \frac{1}{2} p(t)^T \hat{R} p(t)]dt}{\int\limits_c^d [e(t, w, \dot{w}, s, \dot{s}) - S(w|A) + s^T v - \frac{1}{2} p(t)^T H p(t)]dt}$$

$$= \left(\frac{\int\limits_c^d [h^1(t,w,\dot{w},s,\dot{s}) + S(w|E_1) - s^T u^1 - \frac{1}{2}(p^1(t))^T R_1 p^1(t)]dt}{\int\limits_c^d [e^1(t,w,\dot{w},s,\dot{s}) - S(w|A_1) + s^T v^1 - \frac{1}{2}(p^1(t))^T H_1 p^1(t)]dt}, \ldots, \frac{\int\limits_c^d [h^l(t,w,\dot{w},s,\dot{s}) + S(w|E_l) - s^T u^l - \frac{1}{2}(p^l(t))^T R_l p^l(t)]dt}{\int\limits_c^d [e^l(t,w,\dot{w},s,\dot{s}) - S(w|A_l) + s^T v^l - \frac{1}{2}(p^l(t))^T H_l p^l(t)]dt} \right)$$

subject to

$$w(c) = 0 = w(d), \qquad s(c) = 0 = s(d),$$

$$\dot{w}(c) = 0 = \dot{w}(d), \qquad \dot{s}(c) = 0 = \dot{s}(d),$$

$$-\sum_{i=1}^{l} \lambda_i \left\{ \left[h^i_s - Dh^i_{\dot{s}} - u^i + R_i p^i(t)\right] - \left[e^i_s - De^i_{\dot{s}} + v^i + H_i p^i(t)\right] \frac{B^i(w, s)}{C^i(w, s)} \right\} \in C_2^*, \ t \in B,$$

$$s^T \sum_{i=1}^{l} \lambda_i \left\{ \left[h^i_s - Dh^i_{\dot{s}} - u^i + R_i p^i(t)\right] - \left[e^i_s - De^i_{\dot{s}} + v^i + H_i p^i(t)\right] \frac{B^i(w, s)}{C^i(w, s)} \right\} \geqq 0, \ t \in B,$$

$$\lambda > 0, \ w(t) \in C_1, \ t \in B,$$

$$u^i \in Y_i, \ v^i \in Z_i, \ i \in \hat{L}.$$

Dual Problem **(MFVD1)**

Maximize

$$\frac{\int\limits_c^d [h(t, z, \dot{z}, r, \dot{r}) - S(r|Y) + z^T x - \frac{1}{2} q(t)^T J q(t)]dt}{\int\limits_c^d [e(t, z, \dot{z}, r, \dot{r}) + S(r|Z) - z^T y - \frac{1}{2} q(t)^T K q(t)]dt}$$

$$= \left(\frac{\int\limits_c^d [h^1(t,z,\dot{z},r,\dot{r}) - S(r|Y_1) + z^T x^1 - \frac{1}{2}(q^1(t))^T J_1 q^1(t)]dt}{\int\limits_c^d [e^1(t,z,\dot{z},r,\dot{r}) + S(r|Z_1) - z^T y^1 - \frac{1}{2}(q^1(t))^T K_1 q^1(t)]dt}, \ldots, \frac{\int\limits_c^d [h^l(t,z,\dot{z},r,\dot{r}) - S(r|Y_l) + z^T x^l - \frac{1}{2}(q^l(t))^T J_l q^l(t)]dt}{\int\limits_c^d [e^l(t,z,\dot{z},r,\dot{r}) + S(r|Z_l) - z^T y^l - \frac{1}{2}(q^l(t))^T K_l q^l(t)]dt} \right)$$

subject to

$$z(c) = 0 = z(d), \qquad r(c) = 0 = r(d),$$

$$\dot{z}(c) = 0 = \dot{z}(d), \qquad \dot{r}(c) = 0 = \dot{r}(d),$$

$$\sum_{i=1}^{l} \lambda_i \left\{ \left[h^i_w - Dh^i_{\dot{w}} + x^i + J_i q^i(t)\right] - \left[e^i_w - De^i_{\dot{w}} - y^i + K_i q^i(t)\right] \frac{L^i(z, r)}{N^i(z, r)} \right\} \in C_1^*, \ t \in B$$

$$z^T \sum_{i=1}^{l} \lambda_i \left\{ \left[h^i_w - Dh^i_{\dot{w}} + x^i + J_i q^i(t)\right] - \left[e^i_w - De^i_{\dot{w}} - y^i + K_i q^i(t)\right] \frac{L^i(z, r)}{N^i(z, r)} \right\} \leqq 0, \ t \in B$$

$$\lambda > 0, \ r(t) \in C_2, \ t \in B$$

$$x^i \in E_i, \ y^i \in A_i, \ i \in \hat{L},$$

where

(*i*) $h^i : B \times W \times W \times \check{S} \times \check{S} \to R_+$ and $e^i : B \times W \times W \times \check{S} \times \check{S} \to R_+ \setminus \{0\}$, $i \in \hat{L}$, are C^2 functions,

(*ii*) $R_i(t, w, \dot{w}, \ddot{w}, s, \dot{s}, \ddot{s}) = h^i_{ss}(t, w, \dot{w}, s, \dot{s}) - Dh^i_{s\dot{s}}(t, w, \dot{w}, s, \dot{s}),\ t \in B,$

(*iii*) $H_i(t, w, \dot{w}, \ddot{w}, s, \dot{s}, \ddot{s}) = e^i_{ss}(t, w, \dot{w}, s, \dot{s}) - De^i_{s\dot{s}}(t, w, \dot{w}, s, \dot{s}),\ t \in B,$

(*iv*) $J_i(t, z, \dot{z}, \ddot{z}, r, \dot{r}, \ddot{r}) = h^i_{ww}(t, z, \dot{z}, r, \dot{r}) - Dh^i_{w\dot{w}}(t, z, \dot{z}, r, \dot{r}),\ t \in B,$

(*v*) $K_i(t, z, \dot{z}, \ddot{z}, r, \dot{r}, \ddot{r}) = e^i_{ww}(t, z, \dot{z}, r, \dot{r}) - De^i_{w\dot{w}}(t, z, \dot{z}, r, \dot{r}),\ t \in B,$

(*vi*) E_i, A_i, Y_i and Z_i are compact convex sets in R^n, R^n, R^m and R^m respectively,

(*vii*) $p^i : B \to R^m$ and $q^i : B \to R^n$ for $i \in \hat{L}$.

and

$$B^i(w, s) = \int_c^d [h^i(t, w, \dot{w}, s, \dot{s}) + S(w|E_i) - s^T u^i - \tfrac{1}{2}(p^i(t))^T R_i p^i(t)]dt,$$

$$C^i(w, s) = \int_c^d [e^i(t, w, \dot{w}, s, \dot{s}) - S(w|A_i) + s^T v^i - \tfrac{1}{2}(p^i(t))^T H_i p^i(t)]dt,$$

$$L^i(z, r) = \int_c^d [h^i(t, z, \dot{z}, r, \dot{r}) - S(r|Y_i) + z^T x^i - \tfrac{1}{2}(q^i(t))^T J_i q^i(t)]dt,$$

$$N^i(z, r) = \int_c^d [e^i(t, z, \dot{z}, r, \dot{r}) + S(r|Z_i) - z^T y^i - \tfrac{1}{2}(q^i(t))^T K_i q^i(t)]dt.$$

Assume that

$$C^i(w, s) > 0, \quad B^i(w, s) \geqq 0, \quad N^i(z, r) > 0, \quad L^i(z, r) \geqq 0, \quad \forall\ i \in \hat{L},$$

and let

$$k_i = \frac{B^i(w, s)}{C^i(w, s)} = \frac{\int_c^d [h^i(t, w, \dot{w}, s, \dot{s}) + S(w|E_i) - s^T u^i - \frac{1}{2}(p^i(t))^T R_i p^i(t)]dt}{\int_c^d [e^i(t, w, \dot{w}, s, \dot{s}) - S(w|A_i) + s^T v^i - \frac{1}{2}(p^i(t))^T H_i p^i(t)]dt}$$

and

$$n_i = \frac{L^i(z, r)}{N^i(z, r)} = \frac{\int_c^d [h^i(t, z, \dot{z}, r, \dot{r}) - S(r|Y_i) + z^T x^i - \frac{1}{2}(q^i(t))^T J_i q^i(t)]dt}{\int_c^d [e^i(t, z, \dot{z}, r, \dot{r}) + S(r|Z_i) - z^T y^i - \frac{1}{2}(q^i(t))^T K_i q^i(t)]dt}.$$

Express problems **(MFVP1)** and **(MFVD1)** equivalently as follows:

(EMFVP)

Minimize $k = (k_1, k_2, \ldots, k_l)$

subject to

$$w(c) = 0 = w(d), \qquad s(c) = 0 = s(d) \tag{4.4.1}$$

$$\dot{w}(c) = 0 = \dot{w}(d), \qquad \dot{s}(c) = 0 = \dot{s}(d) \tag{4.4.2}$$

$$\int_c^d [h^i(t,w,\dot{w},s,\dot{s}) + S(w|E_i) - s^T u^i - \tfrac{1}{2}(p^i(t))^T R_i p^i(t)]\mathrm{dt} - k_i \int_c^d [e^i(t,w,\dot{w},s,\dot{s}) - S(w|A_i) + s^T v^i - \tfrac{1}{2}(p^i(t))^T H_i p^i(t)]\mathrm{dt} = 0, \tag{4.4.3}$$

$$-\sum_{i=1}^{l} \lambda_i \{[h^i_s - Dh^i_{\dot{s}} - u^i + R_i p^i(t)] - k_i[e^i_s - De^i_{\dot{s}} + v^i + H_i p^i(t)]\} \in C_2^*,\ t \in B \tag{4.4.4}$$

$$s^T \sum_{i=1}^{l} \lambda_i \{[h^i_s - Dh^i_{\dot{s}} - u^i + R_i p^i(t)] - k_i[e^i_s - De^i_{\dot{s}} + v^i + H_i p^i(t)]\} \geqq 0,\ t \in B \tag{4.4.5}$$

$$\lambda > 0,\ w(t) \in C_1,\ t \in B \tag{4.4.6}$$

$$u^i \in Y_i,\ v^i \in Z_i,\ i \in \hat{L}. \tag{4.4.7}$$

(EMFVD)

Maximize $n = (n_1, n_2, \ldots, n_l)$

subject to

$$z(c) = 0 = z(d), \qquad r(c) = 0 = r(d) \tag{4.4.8}$$

$$\dot{z}(c) = 0 = \dot{z}(d), \qquad \dot{r}(c) = 0 = \dot{r}(d) \tag{4.4.9}$$

$$\int_c^d [h^i(t,z,\dot{z},r,\dot{r}) - S(r|Y_i) + z^T x^i - \tfrac{1}{2}(q^i(t))^T J_i q^i(t)]\mathrm{dt} - n_i \int_c^d [e^i(t,z,\dot{z},r,\dot{r}) + S(r|Z_i) - z^T y^i - \tfrac{1}{2}(q^i(t))^T K_i q^i(t)]\mathrm{dt} = 0, \tag{4.4.10}$$

$$\sum_{i=1}^{l} \lambda_i \{[h^i_w - Dh^i_{\dot{w}} + x^i + J_i q^i(t)] - n_i[e^i_w - De^i_{\dot{w}} - y^i + K_i q^i(t)]\} \in C_1^*,\ t \in B \tag{4.4.11}$$

$$z^T \sum_{i=1}^{l} \lambda_i \{[h^i_w - Dh^i_{\dot{w}} + x^i + J_i q^i(t)] - n_i[e^i_w - De^i_{\dot{w}} - y^i + K_i q^i(t)]\} \leqq 0,\ t \in B \tag{4.4.12}$$

$$\lambda > 0,\ r(t) \in C_2,\ t \in B \tag{4.4.13}$$

$$x^i \in E_i,\ y^i \in A_i,\ i \in \hat{L}. \tag{4.4.14}$$

In **(EMFVP)** and **(EMFVD)**, it can be seen that k and n are nonnegative. Let P and Q contain feasible solutions of **(EMFVP)** and **(EMFVD)**, respectively. Various duality results are established for **(EMFVP)** and **(EMFVD)** that are equally applicable to **(MFVP1)** and **(MFVD1)**.

Theorem 4.4.1 *(Weak Duality) Let* $(w,s,k,\lambda,u,v,p) \in P$ *and* $(z,r,n,\lambda,x,y,q) \in Q$. *Let the sublinear functionals* $F : B\times W\times W\times W\times W\times R^n \to R$ *and* $G : B\times \check{S}\times\check{S}\times\check{S}\times\check{S}\times R^m \to R$ *satisfy the following conditions:*

$$F(t,w,\dot{w},z,\dot{z};a) + \alpha_1^{-1} a^T z \geqq 0,\ \forall a \in C_1^*,\ t \in B \tag{4.4.15}$$

$$G(t,r,\dot{r},s,\dot{s};b) + \alpha_2^{-1} b^T s \geqq 0,\ \forall b \in C_2^*,\ t \in B \tag{4.4.16}$$

Suppose that

(i) $\sum_{i=1}^{l} \lambda_i \int_c^d \{(h^i(t,\cdot,\cdot,r,\dot{r}) + (\cdot)^T x^i) - n_i(e^i(t,\cdot,\cdot,r,\dot{r}) - (\cdot)^T y^i)\}dt$ *is second-order* $(F,\alpha_1,\rho_i^1,d_i^1)$*-pseudoconvex at* z,

(ii) $\sum\limits_{i=1}^{l} \lambda_i \int\limits_{c}^{d} \{(-h^i(t, w, \dot{w}, \cdot, \cdot) + (\cdot)^T u^i) + k_i(e^i(t, w, \dot{w}, \cdot, \cdot) + (\cdot)^T v^i)\}dt$ *is second-order* $(G, \alpha_2, \rho_i^2, d_i^2)$*-pseudoconvex at* s,

(iii) *either* $\sum\limits_{i=1}^{l} \lambda_i \rho_i^1 \int\limits_{c}^{d} [d_i^1(t, w, \dot{w}, z, \dot{z})]^2 dt + \sum\limits_{i=1}^{l} \lambda_i \rho_i^2 \int\limits_{c}^{d} [d_i^2(t, r, \dot{r}, s, \dot{s})]^2 dt \geqq 0$ *or* $\rho_i^1 \geqq 0$ & $\rho_i^2 \geqq 0, \ i \in \hat{L}$,

(iv) $\int\limits_{c}^{d} (e^i(t, w, \dot{w}, r, \dot{r}) - S(w|A_i) + r^T v^i)dt > 0, \ \forall \ i \in \hat{L}$

Then,

$$k \nleq n.$$

Proof Given that $(w, s, k, \lambda, u, v, p)$ and $(z, r, n, \lambda, x, y, q)$ are feasible solutions for **(EMFVP)** and **(EMFVD)** respectively.

Using $\alpha_1(w, z) > 0$ and (4.4.11), consider

$a = \alpha_1(w, z) \sum\limits_{i=1}^{l} \lambda_i \{h_w^i - Dh_{\dot{w}}^i + x^i + J_i q^i(t) - n_i[e_w^i - De_{\dot{w}}^i - y^i + K_i q^i(t)]\} \in C_1^*, \ t \in B,$

and so from (4.4.15), we get

$$\begin{aligned} &F(t, w, \dot{w}, z, \dot{z}; \alpha_1(w, z) \sum_{i=1}^{l} \lambda_i \{h_w^i - Dh_{\dot{w}}^i + x^i + J_i q^i(t) - n_i[e_w^i - De_{\dot{w}}^i - y^i + K_i q^i(t)]\}) \\ &\qquad \geqq -z^T \sum_{i=1}^{l} \lambda_i \{h_w^i - Dh_{\dot{w}}^i + x^i + J_i q^i(t) - n_i[e_w^i - De_{\dot{w}}^i - y^i + K_i q^i(t)]\} \\ &\qquad \geqq 0, \text{ (from (4.4.12))} \end{aligned}$$

which on using sublinearity of F and $\lambda_i > 0, \ i \in \hat{L}$ becomes

$\sum\limits_{i=1}^{l} \lambda_i F(t, w, \dot{w}, z, \dot{z}; \alpha_1(w, z)\{h_w^i - Dh_{\dot{w}}^i + x^i + J_i q^i(t) - n_i[e_w^i - De_{\dot{w}}^i - y^i + K_i q^i(t)]\}) \geqq 0,$

that is

$$\sum_{i=1}^{l} \lambda_i \int_{c}^{d} F(t, w, \dot{w}, z, \dot{z}; \alpha_1(w, z)\{h_w^i - Dh_{\dot{w}}^i + x^i + J_i q^i(t) - n_i[e_w^i - De_{\dot{w}}^i - y^i + K_i q^i(t)]\})dt \geqq 0. \tag{4.4.17}$$

Also second-order $(F, \alpha_1, \rho_i^1, d_i^1)$-pseudoconvexity of $\sum\limits_{i=1}^{l} \lambda_i \int\limits_{c}^{d} \{(h^i(t, \cdot, \cdot, r, \dot{r}) + (\cdot)^T x^i) - n_i(e^i(t, \cdot, \cdot, r, \dot{r}) - (\cdot)^T y^i)\}dt$ in w and $\dot{w}$ for fixed r and $\dot{r}$ gives

$$\sum_{i=1}^{l} \lambda_i \int_{c}^{d} \{(h^i(t, w, \dot{w}, r, \dot{r}) + w^T x^i) - n_i(e^i(t, w, \dot{w}, r, \dot{r}) - w^T y^i)\}dt - \sum_{i=1}^{l} \lambda_i \int_{c}^{d} \{(h^i(t, z, \dot{z}, r, \dot{r}) + z^T x^i) - n_i(e^i(t, z, \dot{z}, r, \dot{r}) - z^T y^i)\}dt + \sum_{i=1}^{l} \lambda_i \int_{c}^{d} (\tfrac{1}{2} q^i(t)^T J_i q^i(t) - n_i(\tfrac{1}{2} q^i(t)^T K_i q^i(t)))dt \geqq \sum_{i=1}^{l} \lambda_i \rho_i^1 \int_{c}^{d} [d_i^1(t, w, \dot{w}, z, \dot{z})]^2 dt \tag{4.4.18}$$

Substituting $w^T x^i \leqq S(w|E_i), \ x^i \in E_i$ and $w^T y^i \leqq S(w|A_i), \ y^i \in A_i, \ i \in \hat{L}$, (4.4.18) can be written as

$$\sum_{i=1}^{l}\lambda_i\int_c^d\{(h^i(t,w,\dot w,r,\dot r)+S(w|E_i))-n_i(e^i(t,w,\dot w,r,\dot r)-S(w|A_i))\}dt-\sum_{i=1}^{l}\lambda_i\int_c^d\{(h^i(t,z,\dot z,r,\dot r)+z^Tx^i)-n_i(e^i(t,z,\dot z,r,\dot r)-z^Ty^i)\}dt+\sum_{i=1}^{l}\lambda_i\int_c^d(\tfrac{1}{2}q^i(t)^TJ_iq^i(t)-n_i(\tfrac{1}{2}q^i(t)^TK_iq^i(t)))dt \geqq \sum_{i=1}^{l}\lambda_i\rho_i^1\int_c^d[d_i^1(t,w,\dot w,z,\dot z)]^2dt.$$

Using (4.4.10) along with $r^Tv^i \leqq S(r|Z_i),\ v^i\in Z_i,\ i\in\hat{L}$ in above inequality, we get

$$\sum_{i=1}^{l}\lambda_i\int_c^d\{(h^i(t,w,\dot w,r,\dot r)+S(w|E_i)-S(r|Y_i)))-n_i(e^i(t,w,\dot w,r,\dot r)-S(w|A_i)+r^Tv^i)\}dt \geqq \sum_{i=1}^{l}\lambda_i\rho_i^1\int_c^d[d_i^1(t,w,\dot w,z,\dot z)]^2dt. \tag{4.4.19}$$

Again using (4.4.4) and $\alpha_2(r,s)>0$, take $b=-\alpha_2(r,s)\sum_{i=1}^{l}\lambda_i\{[h_s^i-Dh_{\dot s}^i-u^i+R_ip_i(t)]-k_i[e_s^i-Dg_{\dot s}^i+v^i+H_ip^i(t)]\}\in C_2^*,\ t\in B$

From (4.4.16), we obtain

$$\begin{aligned}G(t,r,\dot r,s,\dot s;-\alpha_2(r,s)&\sum_{i=1}^{l}\lambda_i\{[h_s^i-Dh_{\dot s}^i-u^i+R_ip_i(t)]-k_i[e_s^i-Dg_{\dot s}^i+v^i+H_ip^i(t)]\})\\ &\geqq s^T\sum_{i=1}^{l}\lambda_i\{[h_s^i-Dh_{\dot s}^i-u^i+R_ip_i(t)]-k_i[e_s^i-Dg_{\dot s}^i+v^i+H_ip^i(t)]\}\\ &\geqq 0.\ \text{(from (4.4.5))}\end{aligned}$$

Now sublinearity of G and $\lambda_i>0,\ i\in\hat{L}$ gives

$$\sum_{i=1}^{l}\lambda_iG(t,r,\dot r,s,\dot s;-\alpha_2(r,s)\{[h_s^i-Dh_{\dot s}^i-u^i+R_ip_i(t)]-k_i[e_s^i-Dg_{\dot s}^i+v^i+H_ip^i(t)]\})\geqq 0$$

which implies

$$\sum_{i=1}^{l}\lambda_i\int_c^d G(t,r,\dot r,s,\dot s;-\alpha_2(r,s)\{[h_s^i-Dh_{\dot s}^i-u^i+R_ip_i(t)]-k_i[e_s^i-Dg_{\dot s}^i+v^i+H_ip^i(t)]\})dt\geqq 0. \tag{4.4.20}$$

Using hypothesis (ii), we have

$$\sum_{i=1}^{l}\lambda_i\int_c^d\{-h^i(t,w,\dot w,r,\dot r)+r^Tu^i+k_i(e^i(t,w,\dot w,r,\dot r)+r^Tv^i)\}dt-\sum_{i=1}^{l}\lambda_i\int_c^d\{-h^i(t,w,\dot w,s,\dot s)+s^Tu^i+k_i(e^i(t,w,\dot w,s,\dot s)+s^Tv^i)\}dt-\sum_{i=1}^{l}\lambda_i\int_c^d(\tfrac{1}{2}p^i(t)^TR_ip^i(t)-k_i(\tfrac{1}{2}p^i(t)^TH_ip^i(t)))dt \geqq \sum_{i=1}^{l}\lambda_i\rho_i^2\int_c^d[d_i^2(t,r,\dot r,s,\dot s)]^2dt. \tag{4.4.21}$$

Now (4.4.3), $r^Tu^i\leqq S(r|Y_i),\ u^i\in Y_i,\ i\in\hat{L}$, and (4.4.21) yields

$$\sum_{i=1}^{l}\lambda_i\int_c^d\{-h^i(t,w,\dot w,r,\dot r)-S(w|E_i)+S(r|Y_i)+k_i(e^i(t,w,\dot w,r,\dot r)-S(w|A_i)+r^Tv^i)\}dt \geqq \sum_{i=1}^{l}\lambda_i\rho_i^2\int_c^d[d_i^2(t,r,\dot r,s,\dot s)]^2dt. \tag{4.4.22}$$

Adding (4.4.19) and (4.4.22) and from (iii), we get

$$\sum_{i=1}^{l}\lambda_i(k_i-n_i)\int_c^d(e^i(t,w,\dot{w},r,\dot{r})-S(w|A_i)+r^Tv^i)dt \geqq 0. \qquad (4.4.23)$$

If $\forall\ i\in\hat{L},\ k_i\leqq n_i$ and $k_j<n_j$ for some $j\in\hat{L}$, then from $\lambda>0$ and assumption (iv) we get

$$\sum_{i=1}^{l}\lambda_i(k_i-n_i)\int_c^d e^i(t,w,\dot{w},r,\dot{r})-S(w|A_i)+r^Tv^i)dt<0,$$

which contradicts (4.4.23). Hence $k\nleq n$.

We have explored duality relations between variational fractional problems and second-order dual. An illustration is constructed below to validate the theoretical results obtained in this chapter.

Example 4.4.1 *Let* $B=[0,1]$. *Let* $W,\ \check{S}$ *be the space of piecewise smooth functions* $w:B\to[0,\infty)$ *and* $s:B\to[0,\infty)$ *respectively. Suppose that* $C_1=C_2=[0,\infty)$, *hence* $C_1^*=C_2^*=[0,\infty)$. *Let* $n=m=1$ *and* $l=2$, *therefore* $\hat{L}=\{1,2\}$.

Let the functions $h^i:B\times W\times W\times\check{S}\times\check{S}\to R_+$ *and* $e^i:B\times W\times W\times\check{S}\times\check{S}\to R_+\setminus\{0\},\ i\in\hat{L}$ *be defined as* $h^1=w+s+2,\ h^2=w^2+2,\ e^1=s+1,\ e^2=w+1.$
Hence $R_i=H_i=K_i=0$ *for* $i=1,2$ *and* $J_1=0,\ J_2=2.$
Let the functionals $F:B\times W\times W\times W\times W\times R\to R$ *and* $G:B\times\check{S}\times\check{S}\times\check{S}\times\check{S}\times R\to R$ *be considered as* $F(t,w,\dot{w},z,\dot{z};a)=a(w+z+1)$ *and* $G(t,r,\dot{r},s,\dot{s};b)=b(r+s+1)$.
Assume that $\alpha_1:W\times W\to R_+\setminus\{0\}$ *and* $\alpha_2:\check{S}\times\check{S}\to R_+\setminus\{0\}$ *are defined as* $\alpha_1(w,z)=\dfrac{1}{(w+z+1)}$ *and* $\alpha_2(r,s)=\dfrac{1}{(r+s+1)}$ *respectively. Take* $\rho_1^1=\rho_1^2=\frac{1}{3},\ \rho_2^1=\rho_2^2=1.$
Suppose $E_i=A_i=Y_i=Z_i=\{0\}$ *and hence* $S(w|E_i)=S(w|A_i)=S(r|Y_i)=S(r|Z_i)=0.$
Also $d^1:B\times W\times W\times W\times W\to R^2$ *is defined by* $d^1(t,w,\dot{w},z,\dot{z})=(d_1^1(t,w,\dot{w},z,\dot{z}),d_2^1(t,w,\dot{w},z,\dot{z}))$ *where* $d_1^1(t,w,\dot{w},z,\dot{z})=(w^2+z^2)^{\frac{1}{2}}$ *and* $d_2^1(t,w,\dot{w},z,\dot{z})=(w+z)^{\frac{1}{2}}$,
and
$d^2:B\times\check{S}\times\check{S}\times\check{S}\times\check{S}\to R^2$ *is defined by* $d^2(t,r,\dot{r},s,\dot{s})=(d_1^2(t,r,\dot{r},s,\dot{s}),d_2^2(t,r,\dot{r},s,\dot{s}))$ *where* $d_1^2(t,r,\dot{r},s,\dot{s})=(r+s)^{\frac{1}{2}}$ *and* $d_2^2(t,r,\dot{r},s,\dot{s})=(r+s)^{\frac{1}{2}}$. *Therefore, problems* ***(EMFVP)*** *and* ***(EMFVD)*** *becomes*

$$\text{Minimize } k=(k_1,k_2)$$
$$\text{subject to } w(0)=0=w(1),\quad s(0)=0=s(1)$$
$$\dot{w}(0)=0=\dot{w}(1),\quad \dot{s}(0)=0=\dot{s}(1)$$
$$\int_0^1(w+s+2)dt-k_1\int_0^1(s+1)dt=0$$
$$\int_0^1(w^2+2)dt-k_2\int_0^1(w+1)dt=0$$
$$-\lambda_1[1-k_1]\in C_2^*,\ t\in B$$

$$s\lambda_1(1-k_1) \geqq 0,\ t \in B$$

$$\lambda = (\lambda_1, \lambda_2) > 0,\ w(t) \in C_1,\ t \in B$$

and

$$\text{Maximize } n = (n_1, n_2)$$

$$\text{subject to } z(0) = 0 = z(1), \quad r(0) = 0 = r(1)$$

$$\dot{z}(0) = 0 = \dot{z}(1), \quad \dot{r}(0) = 0 = \dot{r}(1)$$

$$\int_0^1 (z + r + 2)dt - n_1 \int_0^1 (r + 1)dt = 0$$

$$\int_0^1 (z^2 + 2 - (q^2(t))^2)dt - n_2 \int_0^1 (z + 1)dt = 0$$

$$\lambda_1 + \lambda_2(2z + 2q^2(t) - n_2) \in C_1^*,\ t \in B$$

$$z[\lambda_1 + \lambda_2(2z + 2q^2(t) - n_2)] \leqq 0,\ t \in B$$

$$\lambda = (\lambda_1, \lambda_2) > 0,\ r(t) \in C_2,\ t \in B$$

where $B^1 = \int_0^1 (w + s + 2)dt \qquad B^2 = \int_0^1 (w^2 + 2)dt,$

$$C^1 = \int_0^1 (s + 1)dt \qquad C^2 = \int_0^1 (w + 1)dt,$$

$$L^1 = \int_0^1 (z + r + 2)dt \qquad L^2 = \int_0^1 (z^2 + 2 - (q^2(t))^2)dt,$$

$$N^1 = \int_0^1 (r + 1)dt \qquad N^2 = \int_0^1 (z + 1)dt$$

and

$$k_1 = \frac{B^1(w,s)}{C^1(w,s)} = \frac{\int_0^1 (w + s + 2)dt}{\int_0^1 (s + 1)dt},$$

$$k_2 = \frac{B^2(w,s)}{C^2(w,s)} = \frac{\int_0^1 (w^2 + 2)dt}{\int_0^1 (w + 1)dt},$$

$$n_1 = \frac{L^1(z,r)}{N^1(z,r)} = \frac{\int_0^1 (z + r + 2)dt}{\int_0^1 (r + 1)dt},$$

$$n_2 = \frac{L^2(z,r)}{N^2(z,r)} = \frac{\int_0^1 (z^2 + 2 - (q^2(t))^2)dt}{\int_0^1 (z + 1)dt}.$$

Now we see that $(w,s,k,\lambda,u,v,p)=(t^2(t-1)^2,0,(k_1,k_2),(3,1),0,0,(t,t^2))$ *and* $(z,r,n,\lambda,x,y,q)=(0,0,(n_1,n_2),(3,1),0,0,(t,t^2))$ *are feasible solutions of* ***(EMFVP)*** *and* ***(EMFVD)*** *respectively where* $k_1=\frac{61}{30},\ k_2=\frac{1261}{651},\ n_1=2$ *and* $n_2=\frac{9}{5}$.

Now we will check second-order $(F,\alpha_1,\rho_i^1,d_i^1)$*-pseudoconvexity of* $\sum\limits_{i=1}^{2}\lambda_i\int\limits_0^1\{(h^i(t,\cdot,\cdot,r,\dot r)+(\cdot)^Tx^i)-n_i(e^i(t,\cdot,\cdot,r,\dot r)-(\cdot)^Ty^i)\}dt$ *at* $z(t)=0$ *for fixed* $r(t)=0$. *We have*

$$\sum_{i=1}^{2}\lambda_i\int_0^1 F(t,w,\dot w,z,\dot z;\alpha_1(w,z)\{h_w^i-Dh_{\dot w}^i+x^i+J_iq^i(t)-n_i[e_w^i-De_{\dot w}^i-y^i+K_iq^i(t)]\})dt$$
$$=\int_0^1(3+(2t^2-\tfrac{9}{5}))dt\geqq 0$$

and $\sum\limits_{i=1}^{2}\lambda_i\int\limits_0^1\{(h^i(t,w,\dot w,r,\dot r)+w^Tx^i)-n_i(e^i(t,w,\dot w,r,\dot r)-w^Ty^i)\}dt-\sum\limits_{i=1}^{2}\lambda_i\int\limits_0^1\{(h^i(t,z,\dot z,r,\dot r)+z^Tx^i)-n_i(e^i(t,z,\dot z,r,\dot r)-z^Ty^i)\}dt+\sum\limits_{i=1}^{2}\lambda_i\int\limits_0^1(\frac{1}{2}q^i(t)^TJ_iq^i(t)-n_i(\frac{1}{2}q^i(t)^TK_iq^i(t)))dt-\sum\limits_{i=1}^{2}\lambda_i\rho_i^1\int\limits_0^1[d_i^1(t,w,\dot w,z,\dot z)]^2dt$

$$=\int_0^1(\tfrac{w}{5}+\tfrac{1}{5})dt\geqq 0 \text{ for } w\in W.$$

Therefore, $\sum\limits_{i=1}^{2}\lambda_i\int\limits_0^1\{(h^i(t,\cdot,\cdot,r,\dot r)+(\cdot)^Tx^i)-n_i(e^i(t,\cdot,\cdot,r,\dot r)-(\cdot)^Ty^i)\}dt$ *is second-order* $(F,\alpha_1,\rho_i^1,d_i^1)$*-pseudoconvex at* $z(t)=0$ *for fixed* $r(t)=0$.

Similarly, we can prove that $\sum\limits_{i=1}^{2}\lambda_i\int\limits_0^1\{(-h^i(t,w,\dot w,\cdot,\cdot)+(\cdot)^Tu^i)+k_i(e^i(t,w,\dot w,\cdot,\cdot)+(\cdot)^Tv^i)\}dt$ *is second-order* $(G,\alpha_2,\rho_i^2,d_i^2)$*-pseudoconvex at* $s(t)=0$ *for fixed* $w(t)=t^2(t-1)^2$.

Also we have $F(t,w,\dot w,z,\dot z;a)+\alpha_1^{-1}a^Tz=2t^2+\dfrac{6}{5}\geqq 0,\ \forall a\in C_1^*$,

Similarly $G(t,r,\dot r,s,\dot s;b)+\alpha_2^{-1}b^Ts\geqq 0,\ \forall b\in C_2^*$.

Further, hypotheses (iii) *and* (iv) *are also satisfied.*

Since all hypotheses of Theorem 4.4.1 are satisfied, we can see that by result of Theorem 4.4.1, $k\nleq n$ *that is* $(\frac{61}{30},\frac{1261}{651})\nleq(2,\frac{9}{5})$. *Hence verified.*

Any problem where λ is fixed as $\bar\lambda$, say **(EMFVD)**, will be considered as **(EMFVD)**$_{\bar\lambda}$.

Theorem 4.4.2 *(Strong Duality) Let* $(\bar w,\bar s,\bar k,\bar\lambda,\bar u,\bar v,\bar p)$ *be WES for* ***(EMFVP)****. Suppose*

(i) $R_i-\bar k_iH_i$ *are nonsingular and symmetric matrices* $\forall\ i\in\hat L$,

(ii) *the set of vectors* $\{h_s^i-Dh_{\dot s}^i-\bar u^i+R_i\bar p^i(t)-\bar k_i(e_s^i-De_{\dot s}^i+\bar v^i+H_i\bar p^i(t))\}, i\in\hat L$ *is LI.*

Then $\exists\ \bar x^i\in R^n,\ \bar y^i\in R^n,\ i\in\hat L$, *such that* $(\bar w,\bar s,\bar k,\bar x,\bar y,\bar q=0)$ *is feasible solution for* ***(EMFVD)***$_{\bar\lambda}$. *Also the objective values of* ***(EMFVP)*** *and* ***(EMFVD)***$_{\bar\lambda}$ *will be equal. Moreover if weak duality holds for all feasible solutions of* ***(EMFVP)***$_{\bar\lambda}$ *and* ***(EMFVD)***$_{\bar\lambda}$, *then* $(\bar w,\bar s,\bar k,\bar x,\bar y,\bar q=0)$ *is an ES of* ***(EMFVD)***$_{\bar\lambda}$.

Proof As $(\bar{w}, \bar{s}, \bar{k}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{p})$ is a WES of **(EMFVP)**, by Fritz John optimality conditions given in [122], $\exists\ \alpha \in R^l,\ \beta \in R^l$, piecewise smooth $\gamma(t) : B \to C_2,\ \xi(t) : B \to R_+$ and $\delta \in R^l_+$ such that

$$\hat{Q} = \sum_{i=1}^{l} \alpha_i \bar{k}_i + \sum_{i=1}^{l} \beta_i[(h^i + S(\bar{w}|E_i) - \bar{s}^T\bar{u}^i - \tfrac{1}{2}\bar{p}^i(t)^T R_i \bar{p}^i(t)) - \bar{k}_i(e^i - S(\bar{w}|A_i) + \bar{s}^T\bar{v}^i - \tfrac{1}{2}\bar{p}^i(t)^T H_i \bar{p}^i(t))] + (\gamma - \xi\bar{s})^T[\sum_{i=1}^{l} \bar{\lambda}_i\{h^i_s - Dh^i_{\dot{s}} - \bar{u}^i + R_i\bar{p}^i(t) - \bar{k}_i(e^i_s - De^i_{\dot{s}} + \bar{v}^i + H_i\bar{p}^i(t))\}] - \delta^T\bar{\lambda}$$

satisfies the following conditions at $(\bar{w}, \bar{s}, \bar{k}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{p})$:

$$[\hat{Q}_w - D\hat{Q}_{\dot{w}} + D^2\hat{Q}_{\ddot{w}}](w(t) - \bar{w}(t)) \geqq 0,\ \forall w(t) \in C_1,\ t \in B, \tag{4.4.24}$$

$$\hat{Q}_s - D\hat{Q}_{\dot{s}} + D^2\hat{Q}_{\ddot{s}} = 0,\ t \in B, \tag{4.4.25}$$

$$\hat{Q}_\lambda = 0,\ t \in B, \tag{4.4.26}$$

$$\hat{Q}_k = 0,\ t \in B, \tag{4.4.27}$$

$$\hat{Q}_p = 0,\ t \in B, \tag{4.4.28}$$

$$\int_c^d \beta_i[(h^i + S(\bar{w}|E_i) - \bar{s}^T\bar{u}^i - \tfrac{1}{2}\bar{p}^i(t)^T R_i\bar{p}^i(t)) - \bar{k}_i(e^i - S(\bar{w}|A_i) + \bar{s}^T\bar{v}^i - \tfrac{1}{2}\bar{p}^i(t)^T H_i\bar{p}^i(t))]dt = 0,\ i \in \hat{L},\ t \in B, \tag{4.4.29}$$

$$\gamma^T[\sum_{i=1}^{l} \bar{\lambda}_i\{h^i_s - Dh^i_{\dot{s}} - \bar{u}^i + R_i\bar{p}^i(t) - \bar{k}_i(e^i_s - De^i_{\dot{s}} + \bar{v}^i + H_i\bar{p}^i(t))\}] = 0,\ t \in B, \tag{4.4.30}$$

$$\xi\bar{s}^T[\sum_{i=1}^{l} \bar{\lambda}_i\{h^i_s - Dh^i_{\dot{s}} - \bar{u}^i + R_i\bar{p}^i(t) - \bar{k}_i(e^i_s - De^i_{\dot{s}} + \bar{v}^i + H_i\bar{p}^i(t))\}] = 0,\ t \in B, \tag{4.4.31}$$

$$\delta^T\bar{\lambda} = 0, \tag{4.4.32}$$

$$S(\bar{w}|E_i) = \bar{w}^T\eta_i,\ \eta_i \in E_i,\ i \in \hat{L}, \tag{4.4.33}$$

$$S(\bar{w}|A_i) = \bar{w}^T\theta_i,\ \theta_i \in A_i,\ i \in \hat{L}, \tag{4.4.34}$$

$$\beta_i\bar{s}^T + [\gamma - \xi\bar{s}]\bar{\lambda}_i \in N_{Y_i}(\bar{u}^i). \tag{4.4.35}$$

$$\bar{k}_i[\beta_i\bar{s}^T + (\gamma - \xi\bar{s})\bar{\lambda}_i] \in N_{Z_i}(\bar{v}^i). \tag{4.4.36}$$

$$(\alpha, \beta, \gamma, \xi, \delta) \neq 0,\ t \in B. \tag{4.4.37}$$

The above relation hold throughout the interval B, except at the corners of $(\bar{w}, \bar{s}, \bar{k}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{p})$ where (4.4.24) and (4.4.25) hold for unique right and left hand limits. γ and ξ are continuously differentiable except possibly of the corners of $(\bar{w}, \bar{s}, \bar{k}, \bar{\lambda}, \bar{u}, \bar{v}, \bar{p})$.

Now Eqs. (4.4.24)-(4.4.28) along with the observations on $Dh^i_{\dot{s}}$ and $De^i_{\dot{s}},\ i \in \hat{L}$, become

$$\begin{aligned}
\Big[\sum_{i=1}^{l} \beta_i\big[&h^i_w + \eta_i - (\tfrac{1}{2}\bar{p}^i(t)^T R_i\bar{p}^i(t))_w - \bar{k}_i(e^i_w - \theta_i - (\tfrac{1}{2}\bar{p}^i(t)^T H_i\bar{p}^i(t))_w) \\
&- D(h^i_{\dot{w}} - (\tfrac{1}{2}\bar{p}^i(t)^T R_i\bar{p}^i(t))_{\dot{w}} - \bar{k}_i(e^i_{\dot{w}} - (\tfrac{1}{2}\bar{p}^i(t)^T H_i\bar{p}^i(t))_{\dot{w}})) \\
&+ D^2(-(\tfrac{1}{2}\bar{p}^i(t)^T R_i\bar{p}^i(t))_{\ddot{w}} + \bar{k}_i(\tfrac{1}{2}\bar{p}^i(t)^T H_i\bar{p}^i(t))_{\ddot{w}})\big] \\
&+ (\gamma - \xi\bar{s})^T\big[\sum_{i=1}^{l} \bar{\lambda}_i\{h^i_{sw} - Dh^i_{\dot{s}w} + (R_i\bar{p}^i(t))_w - \bar{k}_i(e^i_{sw} - De^i_{\dot{s}w} + (H_i\bar{p}^i(t))_w)\}\big] \\
&- D\big[(\gamma - \xi\bar{s})^T[\sum_{i=1}^{l} \bar{\lambda}_i\{h^i_{s\dot{w}} - Dh^i_{\dot{s}\dot{w}} - h^i_{\dot{s}w} + (R_i\bar{p}^i(t))_{\dot{w}} - \bar{k}_i(e^i_{s\dot{w}} - De^i_{\dot{s}\dot{w}} - e^i_{\dot{s}w} + (H_i\bar{p}^i(t))_{\dot{w}})\}]\big]
\end{aligned}$$

$$+\ D^2\big[(\gamma-\xi\bar{s})^T[\sum_{i=1}^{l}\bar{\lambda}_i\{-h^i_{\dot{s}\dot{w}}+(R_i\bar{p}^i(t))_{\ddot{w}}-\bar{k}_i(-e^i_{\dot{s}\dot{w}}+(H_i\bar{p}^i(t))_{\ddot{w}})\}]\big]\Big](w(t)-\bar{w}(t)) \geqq 0,\ \forall\ w(t)\in C_1,\ t\in B, \tag{4.4.38}$$

$$\begin{aligned}
&\sum_{i=1}^{l}\beta_i\big[h^i_s-\bar{u}^i-(\tfrac{1}{2}\bar{p}^i(t)^T R_i\bar{p}^i(t))_s-\bar{k}_i(e^i_s+\bar{v}^i-(\tfrac{1}{2}\bar{p}^i(t)^T H_i\bar{p}^i(t))_s)\\
&\quad -D(h^i_{\dot{s}}-(\tfrac{1}{2}\bar{p}^i(t)^T R_i\bar{p}^i(t))_{\dot{s}}-\bar{k}_i(e^i_{\dot{s}}-(\tfrac{1}{2}\bar{p}^i(t)^T H_i\bar{p}^i(t))_{\dot{s}}))\\
&\quad +D^2(-(\tfrac{1}{2}\bar{p}^i(t)^T R_i\bar{p}^i(t))_{\ddot{s}}+\bar{k}_i(\tfrac{1}{2}\bar{p}^i(t)^T H_i\bar{p}^i(t))_{\ddot{s}})\big]\\
&\quad +(\gamma-\xi\bar{s})^T\big[\sum_{i=1}^{l}\bar{\lambda}_i\{h^i_{ss}-Dh^i_{\dot{s}s}+(R_i\bar{p}^i(t))_s-\bar{k}_i(e^i_{ss}-De^i_{\dot{s}s}+(H_i\bar{p}^i(t))_s)\}\big]\\
&\quad -\xi^T\big[\sum_{i=1}^{l}\bar{\lambda}_i\{h^i_s-Dh^i_{\dot{s}}-\bar{u}^i+R_i\bar{p}^i(t)-\bar{k}_i(e^i_s-De^i_{\dot{s}}+\bar{v}^i+H_i\bar{p}^i(t))\}\big]\\
&\quad -D\big[(\gamma-\xi\bar{s})^T[\sum_{i=1}^{l}\bar{\lambda}_i\{h^i_{s\dot{s}}-Dh^i_{\dot{s}\dot{s}}-h^i_{\dot{s}s}+(R_i\bar{p}^i(t))_{\dot{s}}-\bar{k}_i(e^i_{s\dot{s}}-De^i_{\dot{s}\dot{s}}-e^i_{\dot{s}s}+(H_i\bar{p}^i(t))_{\dot{s}})\}]\big]\\
&+D^2\big[(\gamma-\xi\bar{s})^T[\sum_{i=1}^{l}\bar{\lambda}_i\{-h^i_{\dot{s}\dot{s}}+(R_i\bar{p}^i(t))_{\ddot{s}}-\bar{k}_i(-e^i_{\dot{s}\dot{s}}+(H_i\bar{p}^i(t))_{\ddot{s}})\}]\big]=0,\ t\in B,
\end{aligned} \tag{4.4.39}$$

$$(\gamma-\xi\bar{s})^T[h^i_s-Dh^i_{\dot{s}}-\bar{u}^i+R_i\bar{p}^i(t)-\bar{k}_i(e^i_s-De^i_{\dot{s}}+\bar{v}^i+H_i\bar{p}^i(t))]-\delta_i=0,\ i\in\hat{L},\ t\in B, \tag{4.4.40}$$

$$\alpha_i-\beta_i(e^i-S(\bar{w}|A_i)+\bar{s}^T\bar{v}^i-\tfrac{1}{2}\bar{p}^i(t)^T H_i\bar{p}^i(t))-(\gamma-\xi\bar{s})^T[\bar{\lambda}_i(e^i_s-De^i_{\dot{s}}+\bar{v}^i+H_i\bar{p}^i(t))]=0,\ i\in\hat{L},\ t\in B, \tag{4.4.41}$$

$$-\beta_i[R_i\bar{p}^i(t)-\bar{k}_i(H_i\bar{p}^i(t))]+(\gamma-\xi\bar{s})^T[\bar{\lambda}_i(R_i-\bar{k}_iH_i)]=0,\ i\in\hat{L},\ t\in B, \tag{4.4.42}$$

As $\delta\geqq 0$ and $\bar{\lambda}>0$, (4.4.32) gives $\delta=0$. (4.4.43)

Consequently (4.4.40) becomes

$$(\gamma-\xi\bar{s})^T[h^i_s-Dh^i_{\dot{s}}-\bar{u}^i+R_i\bar{p}^i(t)-\bar{k}_i(e^i_s-De^i_{\dot{s}}+\bar{v}^i+H_i\bar{p}^i(t))]=0,\ i\in\hat{L},\ t\in B, \tag{4.4.44}$$

Since from hypothesis (i), $R_i-\bar{k}_iH_i$ are non singular and symmetric matrices, therefore from (4.4.42), we get

$$-\beta_i\bar{p}^i(t)+\bar{\lambda}_i(\gamma-\xi\bar{s})=0,\ i\in\hat{L}. \tag{4.4.45}$$

Now by hypothesis (ii) and (4.4.44), we have

$$\gamma-\xi\bar{s}=0. \tag{4.4.46}$$

From (4.4.45) and (4.4.46), we get

$$\beta_i\bar{p}^i(t)=0,\ i\in\hat{L} \tag{4.4.47}$$

Using (4.4.39), (4.4.46) and (4.4.47), we obtain

$$\sum_{i=1}^{l}(\beta_i-\xi\bar{\lambda}_i)(h^i_s-Dh^i_{\dot{s}}-\bar{u}^i+R_i\bar{p}^i(t)-\bar{k}_i(e^i_s-De^i_{\dot{s}}+\bar{v}^i+H_i\bar{p}^i(t)))=0,\ i\in\hat{L} \tag{4.4.48}$$

Now using hypothesis (ii) and (4.4.48), we have

$$\beta_i-\xi\bar{\lambda}_i=0. \tag{4.4.49}$$

or $\qquad \beta - \xi\bar{\lambda} = 0$ i.e.$\beta = \xi\bar{\lambda}$.

If $\beta_i = 0$, for some $i \in \hat{L}$, from (4.4.49), $\xi = 0$ as $\bar{\lambda}_i > 0$. Therefore, $\beta_i = 0$ for all i and using (4.4.41) and (4.4.46), we obtain, $\alpha_i = 0$ and from (4.4.40), $\delta_i = 0,\ i \in \hat{L}$.
Now $\xi = 0$ and (4.4.46) implies $\gamma = 0$. So $(\alpha, \beta, \gamma, \xi, \delta) = 0$, contradicts (4.4.37).
Thus $\beta_i \neq 0\ \forall\ i \in \hat{L}$ i.e. $\beta \neq 0$. (4.4.50)
Therefore from (4.4.47) and (4.4.50), we get $\bar{p}^i(t) = 0$. (4.4.51)
Since $\beta \neq 0$ and $\bar{\lambda} > 0$, therefore (4.4.49) implies $\xi \neq 0$ that is $\xi > 0$.
From (4.4.46), we have

$$\bar{s} = \tfrac{\gamma}{\xi} \in C_2,\ t \in B. \tag{4.4.52}$$

Now (4.4.38), (4.4.46), (4.4.49) and (4.4.51) along with $\xi > 0$ gives

$$\sum_{i=1}^{l} \bar{\lambda}_i[h^i_w + \eta_i - \bar{k}_i(e^i_w - \theta_i) - D(h^i_{\dot{w}} - \bar{k}_i e^i_{\dot{w}})](w(t) - \bar{w}(t)) \geqq 0,\ t \in B, \tag{4.4.53}$$

Let $w(t) \in C_1$. Then $w(t) + \bar{w}(t) \in C_1,\ t \in B$ and so (4.4.53) becomes

$$\sum_{i=1}^{l} \bar{\lambda}_i[h^i_w + \eta_i - \bar{k}_i(e^i_w - \theta_i) - D(h^i_{\dot{w}} - \bar{k}_i e^i_{\dot{w}})]w(t) \geqq 0,\ t \in B,$$

that is

$$\sum_{i=1}^{l} \bar{\lambda}_i[h^i_w + \eta_i - \bar{k}_i(e^i_w - \theta_i) - D(h^i_{\dot{w}} - \bar{k}_i e^i_{\dot{w}})] \in C_1^*,\ t \in B. \tag{4.4.54}$$

Now taking $w(t) = 0$ and $w(t) = 2\bar{w}(t)$ in (4.4.53) yields

$$\bar{w}(t)^T \sum_{i=1}^{l} \bar{\lambda}_i[h^i_w + \eta_i - \bar{k}_i(e^i_w - \theta_i) - D(h^i_{\dot{w}} - \bar{k}_i e^i_{\dot{w}})] = 0,\ t \in B. \tag{4.4.55}$$

Further (4.4.35), (4.4.46) and (4.4.49) with $\xi > 0,\ t \in B$ gives

$\bar{\lambda}_i \bar{s} \in N_{Y_i}(\bar{u}^i)$, for $i \in \hat{L}$

or $\quad \bar{s} \in N_{Y_i}(\bar{u}^i)$ as $\bar{\lambda}_i > 0$.

Now Y_i is compact convex set in R^m, $\bar{s}^T\bar{u}^i = S(\bar{s}|Y_i),\ i \in \hat{L}$.
Also, from (4.4.36), (4.4.46) and (4.4.49) with $\xi > 0,\ t \in B$, we have for $i \in \hat{L}$,

$\bar{k}_i\bar{\lambda}_i\bar{s} \in N_{Z_i}(\bar{v}^i)$, for $i \in \hat{L}$

or $\quad \bar{s} \in N_{Z_i}(\bar{v}^i)$ using $\bar{\lambda}_i > 0$.

Also Z_i is compact convex set in R^m, $\bar{s}^T\bar{v}^i = S(\bar{s}|Z_i),\ i \in \hat{L}$.
Therefore, from (4.4.52), (4.4.54) and (4.4.55), it shows that $(\bar{w}, \bar{s}, \bar{k}, \bar{x} = \eta, \bar{y} = \theta, \bar{q} = 0)$ is feasible solution for **(EMFVD)**$_{\bar{\lambda}}$. Thus **(EMFVP)** and **(EMFVD)**$_{\bar{\lambda}}$ have equal objective values (i.e.$\bar{k} = \bar{n}$).
If $(\bar{w}, \bar{s}, \bar{k}, \bar{x}, \bar{y}, \bar{q} = 0)$ is not an ES of **EMFVD**$_{\bar{\lambda}}$, then $\exists\ (\bar{z}, \bar{r}, \bar{n}, \bar{x}, \bar{y}, \bar{q} = 0)$ feasible for **EMFVD**$_{\bar{\lambda}}$ such that

$$\bar{k} \leq \bar{n},$$

which contradicts Theorem 4.4.1. Thus $(\bar{w}, \bar{s}, \bar{k}, \bar{x}, \bar{y}, \bar{q} = 0)$ is an ES of **(EMFVD)**$_{\bar{\lambda}}$.

Hence proved.

Theorem 4.4.3 *(Converse Duality) Let* $(\bar{z}, \bar{r}, \bar{n}, \bar{\lambda}, \bar{x}, \bar{y}, \bar{q})$ *be a WES for* ***(EMFVD)****. Assume that*

(i) *$J_i - \bar{n}_i K_i$ are nonsingular and symmetric matrices,*

(ii) *the set of vectors* $\{h^i_w - Dh^i_{\dot{w}} + \bar{x}^i + J_i\bar{q}^i(t) - \bar{n}_i(e^i_w - De^i_{\dot{w}} - \bar{y}^i + K_i\bar{q}^i(t))\}, i \in \hat{L}$ *is LI.*

Then $\exists\ \bar{u}^i \in R^m,\ \bar{v}^i \in R^m,\ i \in \hat{L}$, *such that* $(\bar{z}, \bar{r}, \bar{n}, \bar{u}, \bar{v}, \bar{p} = 0)$ *is feasible solution for* ***(EMFVP)***$_{\bar{\lambda}}$ *and the objective values of* ***(EMFVP)*** *and* ***(EMFVD)***$_{\bar{\lambda}}$ *are equal. Also if weak duality holds for all FSs of* ***(EMFVP)***$_{\bar{\lambda}}$ *and* ***(EMFVD)***$_{\bar{\lambda}}$*, then* $(\bar{z}, \bar{r}, \bar{n}, \bar{u}, \bar{v}, \bar{p} = 0)$ *is an ES of* ***(EMFVP)***$_{\bar{\lambda}}$.

Proof Same on the lines of Theorem 4.4.2.

4.5 Special cases

(i) If we take $p^i(t) = 0$ and $q^i(t) = 0,\ i \in \hat{L}$ in **(MFVP1)** and **(MFVD1)** respectively then, our dual model will be reduced to the dual model given by [73].

(ii) If $l = 1$, $E = A = Y = Z = \{0\}$ and $p^i(t) = q^i(t) = 0,\ i \in \hat{L}$ then our problem will be reduced to the problem studied in [13].

(iii) If $l = 1$ and $E = A = Y = Z = \{0\}$ then our problem will become the problem studied in [67].

4.6 Conclusion

This chapter presents a symmetric second-order dual for NMFVP problem and usual results are acquired with second-order (F, α, ρ, d)-pseudoconvexity suppositions.

Chapter 5

Higher-order symmetric duality in nondifferentiable multiobjective optimization over cones[1]

5.1 Introduction

The approach of symmetric duality in quadratic programming was initiated by Dorn [35]. His results were extended to convex NLP problems in Dantzig et al. [32] and then in Bazaraa and Goode [18] over arbitrary cones by assuming the kernel function $f(a, b)$ to be convex in a and concave in b. Mond and Weir [107] proposed symmetric dual programs which admit the relaxation of the convexity/concavity supposition to pseudoconvexity/pseudoconcavity.

Higher-order duality in nonlinear programs has been studied by several researchers [11, 44, 52, 53, 108]. Mond and Zhang [108] developed relations for higher-order dual with invexity suppositions. Wolfe type higher-order symmetric dual was constructed by Gulati and Gupta [52]. Ahmad et al. [11] established Mond-Weir dual for MP and acquired results. Optimality conditions for nonconvex quadratic-exponential minimization problems were discussed by Gao and Ruan [45]. Mishra et al. [97] obtained optimality conditions and relations between primal and dual models for nonsmooth MP with generalized type I functions. Usual duality relations are acquired by Saini and Gulati [114] for second-order Wolfe dual.

Thakur and Priya [126] discussed duality results for MP with (ϕ, ρ)-univexity. Symmetric second-order dual programs for MP over arbitrary cones were introduced by Gupta and Kailey [56] and duality relations were derived with K-η-bonvexity suppositions. Recently, Gao [44] constructed symmetric higher-order Mond-Weir dual. Motivated by [44, 56, 126], we construct a nondifferentiable higher-order symmetric dual for MP. We acquire duality relations with higher-order K-η-convexity suppositions. An illustration is provided for higher-order K-η-convex functions. Self duality relations are acquired with suitable suppositions. Some special cases are also obtained.

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5.2 Notations and preliminaries

Let K be a closed convex pointed cone in R^k with int $K \neq \phi$. Let C_1 and C_2 be closed convex cones with nonempty interiors in R^n and R^m, respectively and $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ be open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

Definition 5.2.1 *A differentiable function $\phi : S_1 \times S_2 \to R^k$ is called higher-order K-η_1-convex in the first variable at $u \in S_1$ for fixed $w \in S_2$ w.r.t. $g : S_1 \times S_2 \times R^n \to R^k$, if $\exists$ $\eta_1 : S_1 \times S_1 \to R^n$ such that $\forall$ $a \in S_1,\ q_i \in R^n,\ i = 1, 2, \ldots, k,$*

$$\Big(\phi_1(a,w)-\phi_1(u,w)-g_1(u,w,q_1)+q_1^T\nabla_{q_1}g_1(u,w,q_1)-\eta_1^T(a,u)[\nabla_a\phi_1(u,w)+\nabla_{q_1}g_1(u,w,q_1)], \ldots,$$
$$\phi_k(a,w)-\phi_k(u,w)-g_k(u,w,q_k)+q_k^T\nabla_{q_k}g_k(u,w,q_k)-\eta_1^T(a,u)[\nabla_a\phi_k(u,w)+\nabla_{q_k}g_k(u,w,q_k)]\Big) \in K,$$

and $\phi(a,b)$ is called higher-order K-η_2-convex in the second variable at $w \in S_2$ for fixed $u \in S_1$ w.r.t. $h : S_1 \times S_2 \times R^m \to R^k$, if $\exists$ $\eta_2 : S_2 \times S_2 \to R^m$ such that $\forall$ $b \in S_2,\ p_i \in R^m,\ i = 1, 2, \ldots, k,$

$$\Big(\phi_1(u,b)-\phi_1(u,w)-h_1(u,w,p_1)+p_1^T\nabla_{p_1}h_1(u,w,p_1)-\eta_2^T(b,w)[\nabla_b\phi_1(u,w)+\nabla_{p_1}h_1(u,w,p_1)], \ldots,$$
$$\phi_k(u,b)-\phi_k(u,w)-h_k(u,w,p_k)+p_k^T\nabla_{p_k}h_k(u,w,p_k)-\eta_2^T(b,w)[\nabla_b\phi_k(u,w)+\nabla_{p_k}h_k(u,w,p_k)]\Big) \in K.$$

Remark 5.2.1 *(i) If we take $g_i(u,w,q_i) = \frac{1}{2}q_i^T\nabla_{aa}\phi_i(u,w)q_i$ and $h_i(u,w,p_i) = \frac{1}{2}p_i^T\nabla_{bb}\phi_i(u,w)p_i$, then higher-order K-η_1-convexity and K-η_2-convexity reduces to K-η_1-bonvexity and K-η_2-bonvexity [56] respectively.*
(ii) The above definition can be reduced to η-convexity/invexity [103], η-bonvexity [47, 123] and K-convexity [122] as given in Remark 1 of [56].

Example 5.2.1 *Let $X = (1.95, 2.4) \subset R,\ n = m = 1,\ k = 2$ and $K = \{(a,b) : a \geq 0,\ b \geq 0\}$. Take $\phi : X \to R^2$ as $\phi(a) = (\phi_1, \phi_2)$, where*

$\phi_1(a) = 8\cos^2 a, \qquad \phi_2(a) = \cos 3a,$

and $\eta : X \times X \to R$ as $\eta(a,u) = -1 - u$. Assume $g : X \times R \to R^2$ as $g(u,q) = (g_1(u,q_1), g_2(u,q_2))$, where

$g_1(u,q_1) = q_1(u^2+1), \qquad g_2(u,q_2) = q_2(u^2-1).$

We justify that ϕ is higher-order K-η-convex.

That is,

$$\Big(\phi_1(a)-\phi_1(u)-g_1(u,q_1)+q_1^T\nabla_{q_1}g_1(u,q_1)-\eta^T(a,u)[\nabla_a\phi_1(u)+\nabla_{q_1}g_1(u,q_1)],$$
$$\phi_2(a)-\phi_2(u)-g_2(u,q_2)+q_2^T\nabla_{q_2}g_2(u,q_2)-\eta^T(a,u)[\nabla_a\phi_2(u)+\nabla_{q_2}g_2(u,q_2)]\Big)\in K,$$
or

$$(8\cos^2 a-8\cos^2 u+(1+u)(-8\sin 2u+u^2+1),\ \cos 3a-\cos 3u+(1+u)(-3\sin 3u+u^2-1))\in K$$

Let $L=(8\cos^2 a-8\cos^2 u+(1+u)(-8\sin 2u+u^2+1),\ \cos 3a-\cos 3u+(1+u)(-3\sin 3u+u^2-1))$
$\quad = (L_1, L_2),$

where
$L_1 = 8\cos^2 a-8\cos^2 u+(1+u)(-8\sin 2u+u^2+1)$
$\quad \geq 0\ \forall a,\ u\in X$ *(see Figure 5.1)*

and
$L_2=\cos 3a-\cos 3u+(1+u)(-3\sin 3u+u^2-1)$
$\quad \geq 0\ \forall a,\ u\in X$ *(see Figure 5.2)*

Next, we need to prove that ϕ *is not* K*-*η*-bonvex. That is*

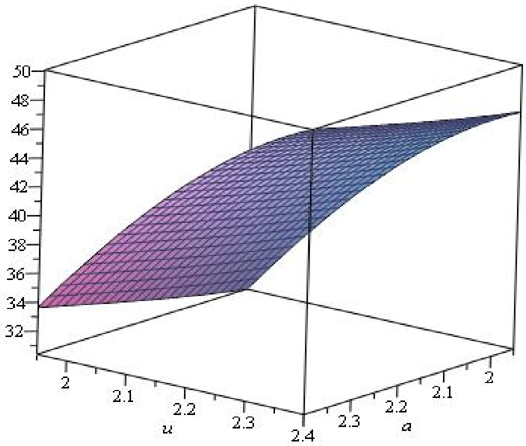

Figure 5.1: graph of L_1

$$M=\big(\phi_1(a)-\phi_1(u)+\frac{1}{2}q_1^T(\nabla_{aa}\phi_1(u)q_1)-\eta^T(a,u)[\nabla_a\phi_1(u)+\nabla_{aa}\phi_1(u)q_1],$$
$$\phi_2(a)-\phi_2(u)+\frac{1}{2}q_2^T(\nabla_{aa}\phi_2(u)q_2)-\eta^T(a,u)[\nabla_a\phi_2(u)+\nabla_{aa}\phi_2(u)q_2]\big)\notin K,$$
i.e., either

$$\phi_1(a)-\phi_1(u)+\frac{1}{2}q_1^T(\nabla_{aa}\phi_1(u)q_1)-\eta^T(a,u)[\nabla_a\phi_1(u)+\nabla_{aa}\phi_1(u)q_1]\ngeq 0$$
or

$$\phi_2(a)-\phi_2(u)+\frac{1}{2}q_2^T(\nabla_{aa}\phi_2(u)q_2)-\eta^T(a,u)[\nabla_a\phi_2(u)+\nabla_{aa}\phi_2(u)q_2]\ngeq 0.$$
Since

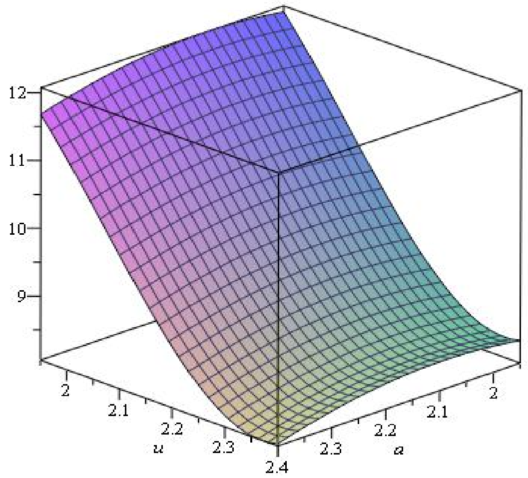

Figure 5.2: graph of L_2

$$
\begin{aligned}
&\phi_2(a)-\phi_2(u)+\frac{1}{2}q_2^T(\nabla_{aa}\phi_2(u)q_2)-\eta^T(a,u)[\nabla_a\phi_2(u)+\nabla_{aa}\phi_2(u)q_2]\\
&\qquad=\cos 3a-\cos 3u-\frac{9}{2}q_2^2\cos 3u+(1+u)(-3\sin 3u-9q_2\cos 3u)\\
&\qquad\leq 0 \ (\textit{ for } a=2,\ u=2.1 \textit{ and } q_2\in(-10^{18},10^{18}))
\end{aligned}
$$

Therefore $M \notin K$. Hence ϕ is not K-η-bonvex function.

5.3 Problem formulation

Consider the following pair.
(SPP) K-minimize

$$
\begin{aligned}
S(a,b,\delta,p)= \quad & \big(\phi(a,b)+S(a|E)e_k-b^T\sum_{i=1}^{k}\delta_i(\nabla_b\phi_i(a,b)+\nabla_{p_i}h_i(a,b,p_i))e_k\\
&+\sum_{i=1}^{k}\delta_i h_i(a,b,p_i)e_k-\sum_{i=1}^{k}\delta_i(p_i^T\nabla_{p_i}h_i(a,b,p_i))e_k\big)\\
\text{subject to}\quad & -\sum_{i=1}^{k}\delta_i(\nabla_b\phi_i(a,b)-z+\nabla_{p_i}h_i(a,b,p_i))\in C_2^*, && (5.3.1)\\
& z\in D && (5.3.2)\\
& \delta^T e_k=1 && (5.3.3)\\
& \delta\in \text{int } K^*,\ a\in C_1 && (5.3.4)
\end{aligned}
$$

(SDP) K-maximize

$$
\begin{aligned}
T(u,w,\delta,q) = \quad & \phi(u,w) - S(w|D)e_k - u^T \sum_{i=1}^{k} \delta_i(\nabla_b \phi_i(u,w) + \nabla_{q_i} g_i(u,w,q_i))e_k \\
& + \sum_{i=1}^{k} \delta_i g_i(u,w,q_i)e_k - \sum_{i=1}^{k} \delta_i(q_i^T \nabla_{q_i} g_i(u,w,q_i))e_k \\
\text{subject to} \quad & \sum_{i=1}^{k} \delta_i(\nabla_a \phi_i(u,w) + v + \nabla_{q_i} g_i(u,w,q_i)) \in C_1^*, && (5.3.5) \\
& v \in E && (5.3.6) \\
& \delta^T e_k = 1 && (5.3.7) \\
& \delta \in \text{int } K^*, \ w \in C_2 && (5.3.8)
\end{aligned}
$$

where

(i) $\phi_i : S_1 \times S_2 \to R,\ h_i : S_1 \times S_2 \times R^m \to R$ and $g_i : S_1 \times S_2 \times R^n \to R,\ i = 1,2,\ldots,k$ are differentiable functions, where $h(a,b,p)$ denotes $(h_1(a,b,p_1), h_2(a,b,p_2), \ldots, h_k(a,b,p_k))$ and $g(u,w,q)$ denotes $(g_1(u,w,q_1), g(u,w,q_2), \ldots, g(u,w,q_k))$, $e_k = (1,\ldots,1)^T \in R^k$, $\delta = (\delta_1, \delta_2, \ldots, \delta_k)$,

(ii) C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively,

(iii) q_i and p_i are vectors in R^n and R^m, respectively for $i = 1,2,\ldots,k$.

(iv) E and D are compact convex sets in R^n and R^m, respectively, and

(v) $S(a|E)$ and $S(w|D)$ are the support functions of E and D, respectively.

Theorem 5.3.1 *(Weak Duality) Suppose (a,b,δ,z,p) is feasible for **(SPP)** and (u,w,δ,v,q) is feasible for **(SDP)**. Let*

(i) $\phi(\cdot,w) + (\cdot)^T v e_k$ be higher-order K-η_1-convex at u w.r.t. $g(u,w,q)$ for fixed w,

(ii) $-\phi(a,\cdot) + (\cdot)^T z e_k$ be higher-order K-η_2-convex at b w.r.t. $-h(a,b,p)$ for fixed a,

(iii) $R_+^k \subseteq K$,

(iv) $\eta_1(a,u) + u \in C_1$, for all $a \in C_1$,

(v) $\eta_2(w,b) + b \in C_2$, for all $w \in C_2$.

Then

$$S(a,b,\delta,p) - T(u,w,\delta,q) \notin -K \setminus \{0\}.$$

Proof Suppose

$$S(a,b,\delta,p) - T(u,w,\delta,q) \in -K \setminus \{0\}.$$

that is

$$\begin{aligned}
&\Bigg\{\Bigg[\phi(a,b) + S(a|E)e_k - b^T \sum_{i=1}^{k} \delta_i(\nabla_b \phi_i(a,b) + \nabla_{p_i} h_i(a,b,p_i))e_k \\
&+ \sum_{i=1}^{k} \delta_i h_i(a,b,p_i)e_k - \sum_{i=1}^{k} \delta_i(p_i^T \nabla_{p_i} h_i(a,b,p_i))e_k\Bigg] \\
&-\Bigg[\phi(u,w) - S(w|D)e_k - u^T \sum_{i=1}^{k} \delta_i(\nabla_a \phi_i(u,w) + \nabla_{q_i} g_i(u,w,q_i))e_k \\
&+ \sum_{i=1}^{k} \delta_i g_i(u,w,q_i)e_k - \sum_{i=1}^{k} \delta_i(q_i^T \nabla_{q_i} g_i(u,w,q_i))e_k\Bigg]\Bigg\} \in -K \setminus \{0\}
\end{aligned}$$

As $\delta \in$ int $K^* \subseteq$ int R_+^k (by hypothesis (iii)), hence $\delta > 0$. Therefore, we get

$$\begin{aligned}
&\sum_{i=1}^{k} \delta_i \phi_i(a,b) + S(a|E) - b^T \sum_{i=1}^{k} \delta_i(\nabla_b \phi_i(a,b) + \nabla_{p_i} h_i(a,b,p_i)) \\
&+ \sum_{i=1}^{k} \delta_i h_i(a,b,p_i) - \sum_{i=1}^{k} \delta_i(p_i^T \nabla_{p_i} h_i(a,b,p_i)) \\
&-(\sum_{i=1}^{k} \delta_i \phi_i(u,w) - S(w|D) - u^T \sum_{i=1}^{k} \delta_i(\nabla_a \phi_i(u,w) + \nabla_{q_i} g_i(u,w,q_i)) \\
&+ \sum_{i=1}^{k} \delta_i g_i(u,w,q_i) - \sum_{i=1}^{k} \delta_i(q_i^T \nabla_{q_i} g_i(u,w,q_i))) < 0. \qquad (5.3.9)
\end{aligned}$$

By higher-order K-η_1-convexity of $\phi(\cdot,w) + (\cdot)^T v e_k$, we acquire

$$\begin{aligned}
&\Big(\phi_1(a,w) + a^T v - \phi_1(u,w) - u^T v - g_1(u,w,q_1) + q_1^T \nabla_{q_1} g_1(u,w,q_1) \\
&-\eta_1^T(a,u)[\nabla_a \phi_1(u,w) + v + \nabla_{q_1} g_1(u,w,q_1)], \ldots, \\
&\phi_k(a,w) + a^T v - \phi_k(u,w) - u^T v - g_k(u,w,q_k) + q_k^T \nabla_{q_k} g_k(u,w,q_k) \\
&-\eta_1^T(a,u)[\nabla_a \phi_k(u,w) + v + \nabla_{q_k} g_k(u,w,q_k)]\Big) \in K.
\end{aligned}$$

Using $\delta \in$ int K^*, we get

$$\sum_{i=1}^{k} \delta_i \Big\{ \phi_i(a,w) + a^T v - \phi_i(u,w) - u^T v - g_i(u,w,q_i) + q_i^T \nabla_{q_i} g_i(u,w,q_i)$$

$$\left.-\eta_1^T(a,u)[\nabla_a\phi_i(u,w)+v+\nabla_{q_i}g_i(u,w,q_i)]\right\}\geqq 0. \tag{5.3.10}$$

As (u,w,δ,v,q) is feasible for **(SDP)**, using (5.3.5) and (iv), we conclude

$$[\eta_1(a,u)+u]^T\sum_{i=1}^{k}\delta_i(\nabla_a\phi_i(u,w)+v+\nabla_{q_i}g_i(u,w,q_i))\geqq 0.$$

which implies

$$\begin{aligned}&\eta_1^T(a,u)\sum_{i=1}^{k}\delta_i(\nabla_a\phi_i(u,w)+v+\nabla_{q_i}g_i(u,w,q_i))\\&\geqq -u^T\sum_{i=1}^{k}\delta_i(\nabla_a\phi_i(u,w)+v+\nabla_{q_i}g_i(u,w,q_i)).\end{aligned}\tag{5.3.11}$$

Using (5.3.10), (5.3.11) and $\delta^Te_k=1$, we obtain

$$\begin{aligned}&\sum_{i=1}^{k}\delta_i(\phi_i(a,w)-\phi_i(u,w)-g_i(u,w,q_i)+q_i^T\nabla_{q_i}g_i(u,w,q_i))+a^Tv-u^Tv\\&\geqq -u^T\sum_{i=1}^{k}\delta_i(\nabla_a\phi_i(u,w)+\nabla_{q_i}g_i(u,w,q_i))-u^Tv.\end{aligned}\tag{5.3.12}$$

Similarly, by higher-order K-η_2-convexity of $-\phi(a,\cdot)+(\cdot)^Tze_k$, from (5.3.1) and hypothesis (v), we get

$$\begin{aligned}&\sum_{i=1}^{k}\delta_i(\phi_i(a,b)-\phi_i(a,w)+h_i(a,b,p_i)-p_i^T\nabla_{p_i}h_i(a,b,p_i))-b^Tz+w^Tz\\&\geqq b^T\sum_{i=1}^{k}\delta_i(\nabla_b\phi_i(a,b)+\nabla_{p_i}h_i(a,b,p_i))-b^Tz.\end{aligned}\tag{5.3.13}$$

Adding inequalities (5.3.12) and (5.3.13), we have

$$\begin{aligned}&\Bigg(\sum_{i=1}^{k}\delta_i\phi_i(a,b)+a^Tv-b^T\sum_{i=1}^{k}\delta_i(\nabla_b\phi_i(a,b)+\nabla_{p_i}h_i(a,b,p_i))\\&+\sum_{i=1}^{k}\delta_ih_i(a,b,p_i)-\sum_{i=1}^{k}\delta_i(p_i^T\nabla_{p_i}h_i(a,b,p_i))\Bigg)\\&\geqq\Bigg(\sum_{i=1}^{k}\delta_i\phi_i(u,w)-w^Tz-u^T\sum_{i=1}^{k}\delta_i(\nabla_a\phi_i(u,w)+\nabla_{q_i}g_i(u,w,q_i))\end{aligned}$$

$$+\sum_{i=1}^{k}\delta_i g_i(u,w,q_i) - \sum_{i=1}^{k}\delta_i(q_i^T\nabla_{q_i}g_i(u,w,q_i))\Big),$$

By using $a^Tv \leqq S(a|E)$ and $w^Tz \leqq S(w|D)$ in above inequality, we obtain

$$\Big(\sum_{i=1}^{k}\delta_i\phi_i(a,b) + S(a|E) - b^T\sum_{i=1}^{k}\delta_i(\nabla_b\phi_i(a,b) + \nabla_{p_i}h_i(a,b,p_i))$$
$$+\sum_{i=1}^{k}\delta_i h_i(a,b,p_i) - \sum_{i=1}^{k}\delta_i(p_i^T\nabla_{p_i}h_i(a,b,p_i))\Big)$$
$$-\Big(\sum_{i=1}^{k}\delta_i\phi_i(u,w) - S(w|D) - u^T\sum_{i=1}^{k}\delta_i(\nabla_a\phi_i(u,w) + \nabla_{q_i}g_i(u,w,q_i))$$
$$+\sum_{i=1}^{k}\delta_i g_i(u,w,q_i) - \sum_{i=1}^{k}\delta_i(q_i^T\nabla_{q_i}g_i(u,w,q_i))\Big) \geqq 0,$$

which contradicts (5.3.9).

If δ in **(SPP)** and **(SDP)** is fixed as $\bar{\delta}$, we express these problems as **(SPP)**$_{\bar{\delta}}$ and **(SDP)**$_{\bar{\delta}}$, respectively.

Theorem 5.3.2 *(Strong Duality) Suppose $(\bar{a},\bar{b},\bar{\delta},\bar{z},\bar{p})$ is WES of* ***(SPP)****. Suppose*

(i) $\nabla_{p_ip_i}h_i,\ \forall\ i=1,2,\dots,k,$ *is positive or negative definite.*

(ii) the set of vectors $\nabla_b\phi_1(\bar{a},\bar{b}),\dots,\nabla_b\phi_k(\bar{a},\bar{b})$ *is LI,*

(iii) $\sum_{i=1}^{k}\bar{\delta}_i\nabla_{bb}\phi_i\bar{p}_i \notin span\{\nabla_b\phi_i + \nabla_{p_i}h_i - \nabla_b h_i, \nabla_b\phi_i,\ i=1,2,\dots,k\}\setminus\{0\}$,

(iv) $\bar{p}_i \neq 0$, *for some* $i\in\{1,2,\dots,k\}$ *imply* $\sum_{i=1}^{k}\bar{\delta}_i\nabla_{bb}\phi_i\bar{p}_i \neq 0$,

(v) $\sum_{i=1}^{k}\bar{\delta}_i h_i(\bar{a},\bar{b},0) = \sum_{i=1}^{k}\bar{\delta}_i g_i(\bar{a},\bar{b},0),\ \sum_{i=1}^{k}\bar{\delta}_i\nabla_b h_i(\bar{a},\bar{b},0)=0,\ \sum_{i=1}^{k}\bar{\delta}_i\nabla_{p_i}h_i(\bar{a},\bar{b},0)=0,$
$\sum_{i=1}^{k}\bar{\delta}_i\nabla_a h_i(\bar{a},\bar{b},0) = \sum_{i=1}^{k}\bar{\delta}_i\nabla_{q_i}g_i(\bar{a},\bar{b},0)$ *and*

(vi) K *is a closed convex pointed cone with* $R_+^k \subseteq K$.

Then,

(I) $\exists\ \bar{v}\in E$ *such that* $(\bar{a},\bar{b},\bar{\delta},\bar{v},\bar{q}=0)$ *is feasible for* ***(SDP)***$_{\bar{\delta}}$, *and*

(II) $S(\bar{a},\bar{b},\bar{\delta},\bar{p}) = T(\bar{a},\bar{b},\bar{\delta},\bar{q})$.

Furthermore, if conditions of Theorem 5.3.1 hold for all feasible solutions of ***(SPP)*** *and*

***(SDP)$_{\bar{\delta}}$**, then $(\bar{a},\bar{b},\bar{\delta},\bar{v},\bar{q}=0)$ is an ES for **(SDP)$_{\bar{\delta}}$**.*

Proof As $(\bar{a},\bar{b},\bar{\delta},\bar{z},\bar{p})$ is WES of **(SPP)**, $\exists\ \bar{\alpha}\in K^*,\ \bar{\beta}\in C_2,\ \bar{\eta}\in R$, such that by Fritz-John conditions ([122], Lemma 1), the following hold at $(\bar{a},\bar{b},\bar{\delta},\bar{z},\bar{p})$ (for simplicity, we write $\nabla_a\phi_i,\ \nabla_{ab}\phi_i$ instead of $\nabla_a\phi_i(\bar{a},\bar{b}),\ \nabla_{ab}\phi_i(\bar{a},\bar{b})$ etc.):

$$(a-\bar{a})^T\Bigg\{\sum_{i=1}^{k}\bar{\alpha}_i(\nabla_a\phi_i+\bar{\gamma})+\sum_{i=1}^{k}\bar{\delta}_i(\nabla_{ab}\phi_i)^T[\bar{\beta}-(\bar{\alpha}^Te_k)\bar{b}]+\sum_{i=1}^{k}\bar{\delta}_i\nabla_a h_i(\bar{\alpha}^Te_k)$$
$$+\sum_{i=1}^{k}\bar{\delta}_i(\nabla_{ap_i}h_i)^T\left[\bar{\beta}-\bar{\alpha}^Te_k(\bar{b}+\bar{p}_i)\right]\Bigg\}\geqq 0,\ \text{for all } a\in C_1, \tag{5.3.14}$$

$$\sum_{i=1}^{k}\bar{\alpha}_i\nabla_b\phi_i+\sum_{i=1}^{k}\bar{\delta}_i(\nabla_{bb}\phi_i)^T[\bar{\beta}-(\bar{\alpha}^Te_k)\bar{b}]+\sum_{i=1}^{k}\bar{\delta}_i\nabla_b h_i(\bar{\alpha}^Te_k)$$
$$+\sum_{i=1}^{k}\bar{\delta}_i(\nabla_{bp_i}h_i)^T\left[\bar{\beta}-\bar{\alpha}^Te_k(\bar{b}+\bar{p}_i)\right]-\sum_{i=1}^{k}\bar{\delta}_i[\nabla_b\phi_i+\nabla_{p_i}h_i](\bar{\alpha}^Te_k)=0, \tag{5.3.15}$$

$$(\nabla_b\phi_i)^T[\bar{\beta}-\bar{\alpha}^Te_k\bar{b}]+h_i(\bar{a},\bar{b},\bar{p}_i)(\bar{\alpha}^Te_k)+(\nabla_{p_i}h_i(\bar{a},\bar{b},\bar{p}_i))^T\Big[\bar{\beta}-(\bar{\alpha}^Te_k)(\bar{b}+\bar{p}_i)\Big]$$
$$+\bar{\eta}=0,\ i=1,2,\ldots,k, \tag{5.3.16}$$

$$[(\bar{\beta}-(\bar{\alpha}^Te_k)(\bar{b}+\bar{p}_i))\bar{\delta}_i]^T\nabla_{p_ip_i}h_i=0,\ i=1,2,\ldots,k, \tag{5.3.17}$$

$$\bar{\beta}^T\sum_{i=1}^{k}\bar{\delta}_i(\nabla_b\phi_i-\bar{z}+\nabla_{p_i}h_i)=0, \tag{5.3.18}$$

$$\bar{\eta}^T[\bar{\delta}^Te_k-1]=0, \tag{5.3.19}$$

$$\bar{\beta}\in N_D(\bar{z}), \tag{5.3.20}$$

$$\bar{\gamma}\in E,\ \bar{\gamma}^T\bar{a}=S(\bar{a}|E) \tag{5.3.21}$$

$$(\bar{\alpha},\bar{\beta},\bar{\eta})\neq 0. \tag{5.3.22}$$

Since $R_+^k\subseteq K\Rightarrow K^*\subseteq R_+^k$, it gives int$(K^*)\subseteq$int$(R_+^k)$.
As $\bar{\delta}\in\text{int}(K^*)$, $\bar{\delta}>0$.
Now hypothesis (i), $\bar{\delta}_i>0$ for $i=1,2,\ldots,k$, and (5.3.17) imply that

$$\bar{\beta}=(\bar{\alpha}^Te_k)(\bar{b}+\bar{p}_i),i=1,2,\ldots,k, \tag{5.3.23}$$

If $\bar{\alpha}=0$ then (5.3.23) yields $\bar{\beta}=0$. Further, (5.3.16) provides $\bar{\eta}=0$. Hence $(\bar{\alpha},\bar{\beta},\bar{\eta})=0$ contradict to (5.3.22). So, $\bar{\alpha}\neq 0$. Also, $\bar{\alpha}\in K^*\subseteq R_+^k$ gives

$$\bar{\alpha}^Te_k>0. \tag{5.3.24}$$

Using (5.3.23) and (5.3.24) in (5.3.15), we acquire

$$\sum_{i=1}^{k} \bar{\delta}_i \nabla_{bb}\phi_i \bar{p}_i = \sum_{i=1}^{k} \bar{\delta}_i (\nabla_b \phi_i + \nabla_{p_i} h_i - \nabla_b h_i) - \frac{1}{(\bar{\alpha}^T e_k)} \sum_{i=1}^{k} \bar{\alpha}_i \nabla_b \phi_i, \tag{5.3.25}$$

which yields

$$\sum_{i=1}^{k} \bar{\delta}_i \nabla_{bb}\phi_i \bar{p}_i \in \operatorname{span}\{\nabla_b \phi_i + \nabla_{p_i} h_i - \nabla_b h_i, \nabla_b \phi_i,\ i = 1, 2, \ldots, k\}. \tag{5.3.26}$$

We justify $\bar{p}_i = 0\ \forall\ i$. On the contrary, let $\bar{p}_i \neq 0$ for some i, then using hypothesis (iv), we achieve

$$\sum_{i=1}^{k} \bar{\delta}_i \nabla_{bb}\phi_i \bar{p}_i \neq 0. \tag{5.3.27}$$

This contradicts hypothesis (iii) (by (5.3.26) and (5.3.27)). Hence

$$\bar{p}_i = 0\ \forall\ i. \tag{5.3.28}$$

From (5.3.23), we conclude

$$\bar{\beta} = (\bar{\alpha}^T e_k)\bar{b}. \tag{5.3.29}$$

Using hypothesis (v) and (5.3.28) in (5.3.25) yields

$$\sum_{i=1}^{k} \nabla_b \phi_i [\bar{\alpha}_i - (\bar{\alpha}^T e_k)\bar{\delta}_i] = 0,$$

and from hypothesis (ii), we acquire

$$\bar{\alpha}_i = (\bar{\alpha}^T e_k)\bar{\delta}_i,\ i = 1, 2, \ldots, k, \tag{5.3.30}$$

Using (5.3.24), (5.3.28), (5.3.29) and (5.3.30) in (5.3.14), we have

$$(a - \bar{a})^T \left\{ \sum_{i=1}^{k} \bar{\delta}_i (\nabla_a \phi_i(\bar{a}, \bar{b}) + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\delta}_i \nabla_a h_i(\bar{a}, \bar{b}, \bar{p}_i) \right\} \geqq 0,\ \forall\ a \in C_1.$$

For $\bar{q}_i = 0$, it follows from the hypothesis (v) and (5.3.28) that

$$(a - \bar{a})^T \left\{ \sum_{i=1}^{k} \bar{\delta}_i (\nabla_a \phi_i(\bar{a}, \bar{b}) + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\delta}_i \nabla_{q_i} g_i(\bar{a}, \bar{b}, \bar{q}_i) \right\} \geqq 0. \tag{5.3.31}$$

Let $a \in C_1$. Then $a + \bar{a} \in C_1$ and so (5.3.31) implies

$$a^T \left\{ \sum_{i=1}^{k} \bar{\delta}_i (\nabla_a \phi_i + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\delta}_i \nabla_{q_i} g_i \right\} \geqq 0, \text{for all } a \in C_1.$$

Therefore,

$$\sum_{i=1}^{k} \bar{\delta}_i (\nabla_a \phi_i + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\delta}_i \nabla_{q_i} g_i \in C_1^*. \tag{5.3.32}$$

Also from (5.3.24) and (5.3.29), we have

$$\bar{b} = \frac{\bar{\beta}}{\bar{\alpha}^T e_k} \in C_2.$$

Thus $(\bar{a}, \bar{b}, \bar{\delta}, \bar{v} = \bar{\gamma}, \bar{q} = 0)$ is feasible solution for $\mathbf{(SDP)}_{\bar{\delta}}$.
By taking $a = 0$ and $a = 2\bar{a}$ in (5.3.31) gives

$$\bar{a}^T \left\{ \sum_{i=1}^{k} \bar{\delta}_i (\nabla_a \phi_i + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\delta}_i \nabla_{q_i} g_i \right\} = 0$$

or

$$\bar{a}^T \left\{ \sum_{i=1}^{k} \bar{\delta}_i \nabla_a \phi_i + \nabla_{q_i} g_i \right\} = -\bar{a}^T \bar{\gamma} = -S(\bar{a}|E). \tag{5.3.33}$$

From (5.3.20) and (5.3.29), $(\bar{\alpha}^T e_k)\bar{b} \in N_D(\bar{z})$. Since $\bar{\alpha}^T e_k > 0$, $\bar{b} \in N_D(\bar{z})$. Since D is a compact convex set in R^m, $\bar{b}^T \bar{z} = S(\bar{b}|D)$.
Further from (5.3.18), (5.3.24) and (5.3.29) and the above relation, we obtain

$$\bar{b}^T \sum_{i=1}^{k} \bar{\delta}_i (\nabla_b \phi_i + \nabla_{p_i} h_i) = \bar{b}^T \bar{z} = S(\bar{b}|D). \tag{5.3.34}$$

Therefore, using (5.3.28), (5.3.33), (5.3.34) and the hypothesis (v), for $\bar{q}_i = 0$, we get

$$\begin{aligned}
&\phi(\bar{a}, \bar{b}) + S(\bar{a}|E)e_k - \bar{b}^T \sum_{i=1}^{k} \bar{\delta}_i (\nabla_b \phi_i(\bar{a}, \bar{b}) + \nabla_{p_i} h_i(\bar{a}, \bar{b}, \bar{p}_i))e_k \\
&+ \sum_{i=1}^{k} \bar{\delta}_i h_i(\bar{a}, \bar{b}, \bar{p}_i)e_k - \sum_{i=1}^{k} \bar{\delta}_i (\bar{p}_i^T \nabla_{p_i} h_i(\bar{a}, \bar{b}, \bar{p}_i))e_k \\
&= \phi(\bar{a}, \bar{b}) - S(\bar{b}|D)e_k - \bar{a}^T \sum_{i=1}^{k} \bar{\delta}_i (\nabla_a \phi_i(\bar{a}, \bar{b}) + \nabla_{q_i} g_i(\bar{a}, \bar{b}, \bar{q}_i))e_k \\
&+ \sum_{i=1}^{k} \bar{\delta}_i g_i(\bar{a}, \bar{b}, \bar{q}_i)e_k - \sum_{i=1}^{k} \bar{\delta}_i (\bar{q}_i^T \nabla_{q_i} g_i(\bar{a}, \bar{b}, \bar{q}_i))e_k
\end{aligned}$$

i.e., objectives attain equal values.
Consider $(\bar{a}, \bar{b}, \bar{\delta}, \bar{v}, \bar{q} = 0)$ is not an ES of $\mathbf{(SDP)}_{\bar{\delta}}$, then $\exists$ $(\bar{u}, \bar{w}, \bar{\delta}, \bar{v}, \bar{q} = 0)$ feasible for

(SDP)$_{\bar{\delta}}$ such that

$$\begin{aligned}&\{\phi(\bar{a},\bar{b}) - S(\bar{b}|D)e_k - \bar{a}^T \sum_{i=1}^{k} \bar{\delta}_i(\nabla_a \phi_i(\bar{a},\bar{b}) + \nabla_{q_i} g_i(\bar{a},\bar{b},\bar{q}_i))e_k \\ &+ \sum_{i=1}^{k} \bar{\delta}_i g_i(\bar{a},\bar{b},\bar{q}_i)e_k - \sum_{i=1}^{k} \bar{\delta}_i(\bar{q}_i^T \nabla_{q_i} g_i(\bar{a},\bar{b},\bar{q}_i))e_k\} \\ &-\{\phi(\bar{u},\bar{w}) - S(\bar{w}|D)e_k - \bar{u}^T \sum_{i=1}^{k} \bar{\delta}_i(\nabla_a \phi_i(\bar{u},\bar{w}) + \nabla_{q_i} g_i(\bar{u},\bar{w},\bar{q}_i))e_k \\ &+ \sum_{i=1}^{k} \bar{\delta}_i g_i(\bar{u},\bar{w},\bar{q}_i)e_k - \sum_{i=1}^{k} \bar{\delta}_i(\bar{q}_i^T \nabla_{q_i} g_i(\bar{u},\bar{w},\bar{q}_i))e_k\} \in -K \setminus \{0\},\end{aligned}$$

Using (5.3.28), (5.3.33), (5.3.34) and the hypothesis (v), for $\bar{q}_i = 0$, we obtain

$$\begin{aligned}&\{\phi(\bar{a},\bar{b}) + S(\bar{a}|E)e_k - \bar{b}^T \sum_{i=1}^{k} \bar{\delta}_i(\nabla_b \phi_i(\bar{a},\bar{b}) + \nabla_{p_i} h_i(\bar{a},\bar{b},\bar{p}_i))e_k \\ &+ \sum_{i=1}^{k} \bar{\delta}_i h_i(\bar{a},\bar{b},\bar{p}_i)e_k - \sum_{i=1}^{k} \bar{\delta}_i(\bar{p}_i^T \nabla_{p_i} h_i(\bar{a},\bar{b},\bar{p}_i))e_k\} \\ &-\{\phi(\bar{u},\bar{w}) - S(\bar{w}|D)e_k - \bar{u}^T \sum_{i=1}^{k} \bar{\delta}_i(\nabla_a \phi_i(\bar{u},\bar{w}) + \nabla_{q_i} g_i(\bar{u},\bar{w},\bar{q}_i))e_k \\ &+ \sum_{i=1}^{k} \bar{\delta}_i g_i(\bar{u},\bar{w},\bar{q}_i)e_k - \sum_{i=1}^{k} \bar{\delta}_i(\bar{q}_i^T \nabla_{q_i} g_i(\bar{u},\bar{w},\bar{q}_i))e_k\} \in -K \setminus \{0\},\end{aligned}$$

that is

$$S(\bar{a},\bar{b},\bar{\delta},\bar{p}) - T(\bar{u},\bar{w},\bar{\delta},\bar{q}) \in -K \setminus \{0\}.$$

which contradicts Theorem 5.3.1. Hence $(\bar{a},\bar{b},\bar{\delta},\bar{v},\bar{q}=0)$ is an ES of **(SDP)**$_{\bar{\delta}}$.

Theorem 5.3.3 *(Converse Duality) Suppose* $(\bar{u},\bar{w},\bar{\delta},\bar{v},\bar{q})$ *is a WES of* ***(SDP)****. Suppose*

(i) the Hessian matrix $\nabla_{q_i q_i} g_i$, $\forall\ i = 1,2,\ldots,k$ *is positive or negative definite.*

(ii) the set of vectors $\nabla_a \phi_1(\bar{u},\bar{w}),\ldots,\nabla_a \phi_k(\bar{u},\bar{w})$ *is LI,*

(iii) $\sum_{i=1}^{k} \bar{\delta}_i \nabla_{aa} \phi_i \bar{q}_i \notin$ *span*$\{\nabla_a \phi_i + \nabla_{q_i} g_i - \nabla_a g_i, \nabla_a \phi_i,\ i = 1,2,\ldots,k\} \setminus \{0\}$,

(iv) $\bar{q}_i \neq 0$, *for some* $i \in \{1,2,\ldots,k\}$ *imply that* $\sum_{i=1}^{k} \bar{\delta}_i \nabla_{aa} \phi_i \bar{q}_i \neq 0$,

(v) $\sum_{i=1}^{k} \bar{\delta}_i g_i(\bar{u},\bar{w},0) = \sum_{i=1}^{k} \bar{\delta}_i h_i(\bar{u},\bar{w},0)$, $\sum_{i=1}^{k} \bar{\delta}_i \nabla_a g_i(\bar{u},\bar{w},0) = 0$, $\sum_{i=1}^{k} \bar{\delta}_i \nabla_{q_i} g_i(\bar{u},\bar{w},0) = 0$,

$\sum_{i=1}^{k} \bar{\delta}_i \nabla_b g_i(\bar{u}, \bar{w}, 0) = \sum_{i=1}^{k} \bar{\delta}_i \nabla_{p_i} h_i(\bar{u}, \bar{w}, 0)$ *and*

(vi) K *is a closed convex pointed cone with* $R_+^k \subseteq K$.

Then,

(I) $\exists\ \bar{z} \in D$ *such that* $(\bar{u}, \bar{w}, \bar{\delta}, \bar{z}, \bar{p} = 0)$ *is feasible for* ***(SPP)***$_{\bar{\delta}}$ *and*

(II) $S(\bar{u}, \bar{w}, \bar{\delta}, \bar{p}) = T(\bar{u}, \bar{w}, \bar{\delta}, \bar{q})$.

Furthermore, if conditions of Theorem 5.3.1 hold for all feasible solutions of ***(SPP)***$_{\bar{\delta}}$ *and* ***(SDP)****, then* $(\bar{u}, \bar{w}, \bar{\delta}, \bar{z}, \bar{p} = 0)$ *is an ES for* ***(SPP)***$_{\bar{\delta}}$.

5.4 Self-duality

Here **(SPP)** and **(SDP)** are not self dual if we do not add conditions on ϕ, g and h. For **(SPP)** and **(SDP)**, self duality exists with these suppositions:
(i) $m = n$, (ii) $C_1 = C_2$, (iii) $D = E$, (iv) the vector functions $\phi : R^n \times R^m \to R^k$ and $g : R^n \times R^m \times R^n \to R^k$ to be skew symmetric, i.e.,$\phi_i(a, b) = -\phi_i(b, a)$ and $g_i(u, w, q_i) = -g_i(w, u, q_i),\ i \in \{1, 2, \ldots, k\}$.
Now recasting **(SDP)** as a minimization problem:
(SDP1) K-minimize

$$\Big(- \phi(u, w) + S(w|D)e_k + u^T \sum_{i=1}^{k} \delta_i (\nabla_a \phi_i(u, w) + \nabla_{q_i} g_i(u, w, q_i))e_k \\ - \sum_{i=1}^{k} \delta_i g_i(u, w, q_i)e_k + \sum_{i=1}^{k} \delta_i (q_i^T \nabla_{q_i} g_i(u, w, q_i))e_k \Big)$$

$$\begin{aligned} \text{subject to} \quad & \sum_{i=1}^{k} \delta_i (\nabla_a \phi_i(u, w) + v + \nabla_{q_i} g_i(u, w, q_i)) \in C_1^*, \\ & v \in E \\ & \delta^T e_k = 1 \\ & \delta \in \text{int } K^*,\ w \in C_2 \end{aligned}$$

Now ϕ and g are skew symmetric,
i.e., $\nabla_a \phi_i(u, w) = -\nabla_b \phi_i(w, u)$ and $\nabla_{q_i} g_i(u, w, q_i) = -\nabla_{q_i} g_i(w, u, q_i)$ for $i = 1, \ldots, k$. Therefore, **(SDP1)** becomes

K-minimize

$$\Bigg(\phi(w,u)+S(w|E)e_k-u^T\sum_{i=1}^{k}\delta_i(\nabla_b\phi_i(w,u)+\nabla_{q_i}g_i(w,u,q_i))e_k$$
$$+\sum_{i=1}^{k}\delta_i g_i(w,u,q_i)e_k-\sum_{i=1}^{k}\delta_i(q_i^T\nabla_{q_i}g_i(w,u,q_i))e_k\Bigg)$$

subject to
$$-\sum_{i=1}^{k}\delta_i(\nabla_b\phi_i(w,u)-v+\nabla_{q_i}g_i(w,u,q_i))\in C_2^*,$$
$$v\in D$$
$$\delta^T e_k=1$$
$$\delta\in \text{int } K^*,\ w\in C_1$$

Here **(SDP1)** is formally identical to **(SPP)**. Therefore, **(SPP)** is self dual. The feasibility of (a,b,δ,z,p) for **(SPP)** assures the feasibility of (b,a,δ,z,p) for **(SDP)** and conversely.

5.5 Special cases

In all these cases, if $\sum_{i=1}^{k}\delta_i h_i(a,b,p_i)=\sum_{i=1}^{k}\delta_i\frac{1}{2}p_i^T\nabla_{bb}\phi_i(a,b)p_i$ and $\sum_{i=1}^{k}\delta_i g_i(u,w,q_i)=\sum_{i=1}^{k}\delta_i\frac{1}{2}q_i^T\nabla_{aa}\phi_i(u,w)q_i$.

(i) If $E=\{0\}$ and $D=\{0\}$, then **(SPP)** and **(SDP)** become the problems studied in Gupta and Kailey [56].

(ii) For $K=R_+^k,\ C_1=R_+^n,\ C_2=R_+^m,\ k=1,\ q_i=q,\ p_i=p$, $E=\{Bb: b^TBb\leqq 1\}$, $D=\{Ca: a^TCa\leqq 1\}$, where B and C are positive semidefinite matrices, $(a^TBa)^{\frac{1}{2}}=S(a|E)$ and $(b^TCb)^{\frac{1}{2}}=S(b|D)$, **(SPP)** and **(SDP)** reduce to the problems presented in Ahmad and Hussain [6].

(iii) If $K=R_+^k,\ C_1=R_+^n,\ C_2=R_+^m,\ k=1,\ q_i=q,\ p_i=p$, then **(SPP)** and **(SDP)** reduce to the programs presented in Yang et al. [135].

(iv) The cases given in Gupta and Kailey [56] can also be extracted from our problems.

5.6 Conclusion

A new pair of higher-order symmetric dual for MP has been proposed. We have given an illustration of higher-order K-η-convex functions. Duality results with higher-order K-η-convexity suppositions have been provided. Some results acquired in Ahmad and Hussain [6], Gupta and Kailey [56] and Yang et al. [135] are special cases of our work.

Chapter 6

Higher-order mixed symmetric duality in multiobjective programming involving support functions

6.1 Introduction

Mixed duality is a fruitful result that unifies two different dual models. Xu [131] constructed two mixed duals for MP and fractional MP problems. Bector et al. [21] acquired duality results for mixed symmetric MP problems. Yang et al. [133] derived duality relations for mixed nondifferentiable programming problems. Second-order mixed dual for MP problems was constructed by Aghezzaf [1]. Ahmad and Husain [9] proposed mixed symmetric dual for MP problem over cones. They acquired results with K-preinvexity/K-pseudoinvexity suppositions. Gao [44] formulated Mond-Weir type higher-order symmetrical dual pair for MP and proved duality results under generalized invexity. Recently Verma et al. [128] obtained mixed type higher-order symmetric dual model for scalar-objective programming problem.

In this chapter, we focus on higher-order duality for nondifferentiable mixed symmetric MP problem with higher-order K-(F, α, ρ, d)-convexity suppositions. This chapter is organized as follows. Section 6.2 reviews the definitions. Higher-order mixed symmetric duality for nondifferentiable MP problem and its duality results are discussed with aforesaid assumptions in Section 6.3. The chapter closes with conclusions in Section 6.5.

6.2 Definitions and preliminaries

Let K be a closed convex pointed cone in R^k with int $K \neq \phi$ and Q be a closed convex cone with nonempty interior in R^m. Let $F : S \times S \times R^n \to R$(where $S \subseteq R^n$) be sublinear functional in its third variable. Let $\zeta_j : S \times R^n \to R$ be differentiable function.

Definition 6.2.1 *A differentiable function* $\psi : S \to R^k$ *is called higher-order* K-(F, α, ρ, d)-

convex at $b \in S$ *on* S *w.r.t.* ζ *if* $\forall\ a \in S$ *and* $q_j \in R^n$, $j = 1, 2, \ldots, k$, $\exists$ *vector* $\rho \in R^k$, *a real valued function* $\alpha : S \times S \to R_+ \setminus \{0\}$ *and* $d : S \times S \to R^k$ *such that* $(\psi_1(a)-\psi_1(b)-\zeta_1(b,q_1)+q_1^T\nabla_{q_1}\zeta_1(b,q_1)-F(a,b;\alpha(a,b)(\nabla_a\psi_1(b)+\nabla_{q_1}\zeta_1(b,q_1)))-\rho_1 d_1^2(a,b),\ldots,$ $\psi_k(a)-\psi_k(b)-\zeta_k(b,q_k)+q_k^T\nabla_{q_k}\zeta_k(b,q_k)-F(a,b;\alpha(a,b)(\nabla_a\psi_k(b)+\nabla_{q_k}\zeta_k(b,q_k)))-\rho_k d_k^2(a,b)) \in K$.

6.3 Mixed higher-order symmetric dual program

For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, let $N_1 \subseteq N,\ M_1 \subseteq M,\ N_2 = N \setminus N_1$ and $M_2 = M \setminus M_1$ where $|N_1|$ denotes the number of elements in the set N_1. $|N_2|$, $|M_1|$, $|M_2|$ are defined similarly. Let $a^1 \in R^{|N_1|}, a^2 \in R^{|N_2|}$. Then any $a \in R^n$ can be written as (a^1, a^2). Also for $b^1 \in R^{|M_1|}$, $b^2 \in R^{|M_2|}$, $b \in R^m$ can be written as (b^1, b^2). Now if $N_1 = \phi$, then $|N_1| = 0,\ N_2 = N$ and hence $|N_2| = n$. In this situation, $R^{|N_1|}$, $R^{|N_2|}$ and $R^{|N_1|} \times R^{|N_2|}$ will be zero-dimensional, n-dimensional and $|N_2|$-dimensional, respectively. Also $N_2 = \phi,\ M_1 = \phi$ or $M_2 = \phi$ can be interpret similarly.

Now we consider the following pair of multiobjective mixed higher-order symmetric dual programs:

Primal Problem **(NMSP)**

K-minimize

$P(a^1, b^1, a^2, b^2, \lambda, r, s) = (P_1(a^1, b^1, a^2, b^2, \lambda, r_1, s_1), P_2(a^1, b^1, a^2, b^2, \lambda, r_2, s_2), \ldots,$
$P_k(a^1, b^1, a^2, b^2, \lambda, r_k, s_k))$

subject to

$$-\sum_{j=1}^{k} \lambda_j(\nabla_{b^1}\varphi_j(a^1, b^1) - q^1 + \nabla_{r_j} h_j^1(a^1, b^1, r_j)) \in C_3^*, \tag{6.3.1}$$

$$-\sum_{j=1}^{k} \lambda_j(\nabla_{b^2}\phi_j(a^2, b^2) - q_j^2 + \nabla_{s_j} g_j^1(a^2, b^2, s_j)) \in C_4^*, \tag{6.3.2}$$

$$(b^2)^T \sum_{j=1}^{k} \lambda_j(\nabla_{b^2}\phi_j(a^2, b^2) - q_j^2 + \nabla_{s_j} g_j^1(a^2, b^2, s_j)) \geqq 0, \tag{6.3.3}$$

$$\lambda^T e_k = 1, \tag{6.3.4}$$

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \text{int } K^*, \quad a^1 \in C_1, \quad a^2 \in C_2, \tag{6.3.5}$$

$$q^1 \in Q^1,\ q_j^2 \in Q_j^2,\ j = 1, 2, \ldots, k.$$

Dual Problem **(NMSD)**

K-maximize

$T(u^1,w^1,u^2,w^2,\lambda,c,d)=(T_1(u^1,w^1,u^2,w^2,\lambda,c_1,d_1),T_2(u^1,w^1,u^2,w^2,\lambda,c_2,d_2),\ldots,$
$T_k(u^1,w^1,u^2,w^2,\lambda,c_k,d_k))$
subject to

$$\sum_{j=1}^{k}\lambda_j(\nabla_{a^1}\varphi_j(u^1,w^1)+v^1+\nabla_{c_j}h_j^2(u^1,w^1,c_j))\in C_1^*, \tag{6.3.6}$$

$$\sum_{j=1}^{k}\lambda_j(\nabla_{a^2}\phi_j(u^2,w^2)+v_j^2+\nabla_{d_j}g_j^2(u^2,w^2,d_j))\in C_2^*, \tag{6.3.7}$$

$$(u^2)^T\sum_{j=1}^{k}\lambda_j(\nabla_{a^2}\phi_j(u^2,w^2)+v_j^2+\nabla_{d_j}g_j^2(u^2,w^2,d_j))\leqq 0, \tag{6.3.8}$$

$$\lambda^Te_k=1, \tag{6.3.9}$$

$$\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)\in \text{int } K^*,\quad w^1\in C_3,\quad w^2\in C_4, \tag{6.3.10}$$

$$v^1\in V^1,\ v_j^2\in V_j^2,\ j=1,2,\ldots,k,$$

where
$P_j(a^1,b^1,a^2,b^2,\lambda,r_j,s_j)=\varphi_j(a^1,b^1)+S(a^1|V^1)+\phi_j(a^2,b^2)+S(a^2|V_j^2)-(b^2)^Tq_j^2$
$-(b^1)^T\sum_{j=1}^{k}\lambda_j(\nabla_{b^1}\varphi_j(a^1,b^1)+\nabla_{r_j}h_j^1(a^1,b^1,r_j))+h_j^1(a^1,b^1,r_j)-r_j^T\nabla_{r_j}h_j^1(a^1,b^1,r_j)$
$+g_j^1(a^2,b^2,s_j)-s_j^T\nabla_{s_j}g_j^1(a^2,b^2,s_j)$.
$T_j(u^1,w^1,u^2,w^2,\lambda,c_j,d_j)=\varphi_j(u^1,w^1)-S(w^1|Q^1)+\phi_j(u^2,w^2)-S(w^2|Q_j^2)+(u^2)^Tv_j^2$
$-(u^1)^T\sum_{j=1}^{k}\lambda_j(\nabla_{a^1}\varphi_j(u^1,w^1)+\nabla_{c_j}h_j^2(u^1,w^1,c_j))+h_j^2(u^1,w^1,c_j)-c_j^T\nabla_{c_j}h_j^2(u^1,w^1,c_j)+$
$g_j^2(u^2,w^2,d_j)-d_j^T\nabla_{d_j}g_j^2(u^2,w^2,d_j)$.
and

(i) $\varphi_j: R^{|N_1|}\times R^{|M_1|}\to R,\ \phi_j: R^{|N_2|}\times R^{|M_2|}\to R,\ h_j^1: R^{|N_1|}\times R^{|M_1|}\times R^{|M_1|}\to R,\ g_j^1: R^{|N_2|}\times R^{|M_2|}\times R^{|M_2|}\to R,\ h_j^2: R^{|N_1|}\times R^{|M_1|}\times R^{|N_1|}\to R$ and $g_j^2: R^{|N_2|}\times R^{|M_2|}\times R^{|N_2|}\to R$ are differentiable functions, respectively.

(ii) $V^1,\ V_j^2,\ Q^1$ and Q_j^2 are compact convex sets in $R^{|N_1|},\ R^{|N_2|},\ R^{|M_1|}$ and $R^{|M_2|}$, respectively.

(iii) $C_1,\ C_2,\ C_3$ and C_4 are closed convex cones in $R^{|N_1|},\ R^{|N_2|},\ R^{|M_1|}$ and $R^{|M_2|}$, respectively and $C_1^*,\ C_2^*,\ C_3^*$ and C_4^* are their respective positive polar cones.

(iv) Also $h^1(a^1,b^1,r)$ denotes $(h_1^1(a^1,b^1,r_1),...,h_k^1(a^1,b^1,r_k))$, $h^2(u^1,w^1,c)$ denotes $(h_1^2(u^1,w^1,c_1),...,h_k^2(u^1,w^1,c_k))$, $g^1(a^2,b^2,s)$ denotes $(g_1^1(a^2,b^2,s_1),...,g_k^1(a^2,b^2,s_k))$ and $g^2(u^2,w^2,d)$ denotes $(g_1^2(u^2,w^2,d_1),...,g_k^2(u^2,w^2,d_k))$.

Theorem 6.3.1 *(Weak Duality) Let $(a^1, b^1, a^2, b^2, q^1, q^2, \lambda, r, s)$ be feasible solution of* ***(NMSP)*** *and $(u^1, w^1, u^2, w^2, v^1, v^2, \lambda, c, d)$ be feasible solution of* ***(NMSD)****. Let the sublinear functionals $F_1 : R^{|N_1|} \times R^{|N_1|} \times R^{|N_1|} \to R$, $F_2 : R^{|M_1|} \times R^{|M_1|} \times R^{|M_1|} \to R$, $G_1 : R^{|N_2|} \times R^{|N_2|} \times R^{|N_2|} \to R$ and $G_2 : R^{|M_2|} \times R^{|M_2|} \times R^{|M_2|} \to R$ satisfy the following conditions:*

$$F_1(a^1, u^1; \nu^1) + (\alpha_1^1)^{-1}(\nu^1)^T u^1 \geqq 0, \quad \forall \nu^1 \in C_1^*, \tag{6.3.11}$$

$$F_2(w^1, b^1; \nu^2) + (\alpha_1^2)^{-1}(\nu^2)^T b^1 \geqq 0, \quad \forall \nu^2 \in C_3^*, \tag{6.3.12}$$

$$G_1(a^2, u^2; \nu^3) + (\alpha_2^1)^{-1}(\nu^3)^T u^2 \geqq 0, \quad \forall \nu^3 \in C_2^*, \tag{6.3.13}$$

$$G_2(w^2, b^2; \nu^4) + (\alpha_2^2)^{-1}(\nu^4)^T b^2 \geqq 0, \quad \forall \nu^4 \in C_4^*. \tag{6.3.14}$$

Suppose that

- *(i)* $\varphi(\cdot, w^1) + (\cdot)^T v^1 e_k$ *is higher-order* K*-*$(F_1, \alpha_1^1, \rho^{11}, d^{11})$*-convex at* u^1 *w.r.t.* $h^2(u^1, w^1, c)$ *and* $-(\varphi(a^1, \cdot) - (\cdot)^T q^1 e_k)$ *is higher-order* K*-*$(F_2, \alpha_1^2, \rho^{12}, d^{12})$*-convex at* b^1 *w.r.t.* $-h^1(a^1, b^1, r)$,
- *(ii)* $\phi(\cdot, w^2) + (\cdot)^T v^2$ *is higher-order* K*-*$(G_1, \alpha_2^1, \rho^{21}, d^{21})$*-convex at* u^2 *w.r.t.* $g^2(u^2, w^2, d)$ *and* $-(\phi(a^2, \cdot) - (\cdot)^T q^2)$ *is higher-order* K*-*$(G_2, \alpha_2^2, \rho^{22}, d^{22})$*-convex at* b^2 *w.r.t.* $-g^1(a^2, b^2, s)$,
- *(iii)* $\sum\limits_{j=1}^{k} \lambda_j[\rho_j^{11}(d_j^{11}(a^1, u^1))^2 + \rho_j^{12}(d_j^{12}(w^1, b^1))^2] \geqq 0$ *and*
  $\sum\limits_{j=1}^{k} \lambda_j[\rho_j^{21}(d_j^{21}(a^2, u^2))^2 + \rho_j^{22}(d_j^{22}(w^2, b^2))^2] \geqq 0$,
- *(iv)* $R_+^k \subseteq K$.

Then, $P(a^1, b^1, a^2, b^2, \lambda, r, s) - T(u^1, w^1, u^2, w^2, \lambda, c, d) \notin -K \setminus \{0\}$.

Proof Suppose that $P(a^1, b^1, a^2, b^2, \lambda, r, s) - T(u^1, w^1, u^2, w^2, \lambda, c, d) \in -K \setminus \{0\}$,
That is

$$\Bigg[\varphi_1(a^1, b^1) + S(a^1|V^1) + \phi_1(a^2, b^2) + S(a^2|V_1^2) - (b^2)^T q_1^2 - (b^1)^T \sum_{j=1}^{k} \lambda_j(\nabla_{b^1}\varphi_j(a^1, b^1) +$$
$$\nabla_{r_j} h_j^1(a^1, b^1, r_j)) + h_1^1(a^1, b^1, r_1) - r_1^T \nabla_{r_1} h_1^1(a^1, b^1, r_1) + g_1^1(a^2, b^2, s_1) - s_1^T \nabla_{s_1} g_1^1(a^2, b^2, s_1), \ldots,$$
$$\varphi_k(a^1, b^1) + S(a^1|V^1) + \phi_k(a^2, b^2) + S(a^2|V_k^2) - (b^2)^T q_k^2 - (b^1)^T \sum_{j=1}^{k} \lambda_j(\nabla_{b^1}\varphi_j(a^1, b^1)$$
$$+ \nabla_{r_j} h_j^1(a^1, b^1, r_j)) + h_k^1(a^1, b^1, r_k) - r_k^T \nabla_{r_k} h_k^1(a^1, b^1, r_k) + g_k^1(a^2, b^2, s_k) - s_k^T \nabla_{s_k} g_k^1(a^2, b^2, s_k)\Bigg]$$
$$- \Bigg[\varphi_1(u^1, w^1) - S(w^1|Q^1) + \phi_1(u^2, w^2) - S(w^2|Q_1^2) + (u^2)^T v_1^2 - (u^1)^T \sum_{j=1}^{k} \lambda_j(\nabla_{a^1}\varphi_j(u^1, w^1) +$$
$$\nabla_{c_j} h_j^2(u^1, w^1, c_j)) + h_1^2(u^1, w^1, c_1) - c_1^T \nabla_{c_1} h_1^2(u^1, w^1, c_1) + g_1^2(u^2, w^2, d_1) - d_1^T \nabla_{d_1} g_1^2(u^2, w^2, d_1), \ldots,$$
$$\varphi_k(u^1, w^1) - S(w^1|Q^1) + \phi_k(u^2, w^2) - S(w^2|Q_k^2) + (u^2)^T v_k^2 - (u^1)^T \sum_{j=1}^{k} \lambda_j(\nabla_{a^1}\varphi_j(u^1, w^1) +$$
$$\nabla_{c_j} h_j^2(u^1, w^1, c_j)) + h_k^2(u^1, w^1, c_k) - c_k^T \nabla_{c_k} h_k^2(u^1, w^1, c_k) + g_k^2(u^2, w^2, d_k) - d_k^T \nabla_{d_k} g_k^2(u^2, w^2, d_k)\Bigg] \in$$

$-K \setminus \{0\}$.

Since $\lambda \in \text{int}K^* \subseteq \text{int}R_+^k$ (by hypothesis (iv)), hence $\lambda > 0$. Therefore, we get

$$\begin{aligned}
&\sum_{j=1}^k \lambda_j \Bigg\{ \varphi_j(a^1, b^1) + S(a^1|V^1) + \phi_j(a^2, b^2) + S(a^2|V_j^2) - (b^2)^T q_j^2 - (b^1)^T \sum_{j=1}^k \lambda_j (\nabla_{b^1} \varphi_j(a^1, b^1) \\
&+ \nabla_{r_j} h_j^1(a^1, b^1, r_j)) + h_j^1(a^1, b^1, r_j) - r_j^T \nabla_{r_j} h_j^1(a^1, b^1, r_j) + g_j^1(a^2, b^2, s_j) - s_j^T \nabla_{s_j} g_j^1(a^2, b^2, s_j) \Bigg\} \\
&- \sum_{j=1}^k \lambda_j \Bigg\{ \varphi_j(u^1, w^1) - S(w^1|Q^1) + \phi_j(u^2, w^2) - S(w^2|Q_j^2) + (u^2)^T v_j^2 - \\
&(u^1)^T \sum_{j=1}^k \lambda_j (\nabla_{a^1} \varphi_j(u^1, w^1) + \nabla_{c_j} h_j^2(u^1, w^1, c_j)) + h_j^2(u^1, w^1, c_j) - c_j^T \nabla_{c_j} h_j^2(u^1, w^1, c_j) \\
&+ g_j^2(u^2, w^2, d_j) - d_j^T \nabla_{d_j} g_j^2(u^2, w^2, d_j) \Bigg\} < 0 \qquad (6.3.15)
\end{aligned}$$

As $\varphi(\cdot, w^1) + (\cdot)^T v^1 e_k$ is higher-order K-$(F_1, \alpha_1^1, \rho^{11}, d^{11})$-convex at u^1 and $-(\varphi(a^1, \cdot) - (\cdot)^T q^1 e_k)$ is higher-order K-$(F_2, \alpha_1^2, \rho^{12}, d^{12})$-convex at b^1, we acquire

$$\begin{aligned}
&(\varphi_1(a^1, w^1) + {a^1}^T v^1 - \varphi_1(u^1, w^1) - (u^1)^T v^1 - h_1^2(u^1, w^1, c_1) + c_1^T \nabla_{c_1} h_1^2(u^1, w^1, c_1) - \\
&F_1(a^1, u^1; \alpha_1^1(a^1, u^1)(\nabla_{a^1} \varphi_1(u^1, w^1) + v^1 + \nabla_{c_1} h_1^2(u^1, w^1, c_1))) - \rho_1^{11}(d_1^{11}(a^1, u^1))^2, ..., \\
&\varphi_k(a^1, w^1) + {a^1}^T v^1 - \varphi_k(u^1, w^1) - (u^1)^T v^1 - h_k^2(u^1, w^1, c_k) + c_k^T \nabla_{c_k} h_k^2(u^1, w^1, c_k) - \\
&F_1(a^1, u^1; \alpha_1^1(a^1, u^1)(\nabla_{a^1} \varphi_k(u^1, w^1) + v^1 + \nabla_{c_k} h_k^2(u^1, w^1, c_k))) - \rho_k^{11}(d_k^{11}(a^1, u^1))^2) \in K,
\end{aligned}$$

and

$$\begin{aligned}
&(-\varphi_1(a^1, w^1) + {w^1}^T q^1 + \varphi_1(a^1, b^1) - (b^1)^T q^1 + h_1^1(a^1, b^1, r_1) - {r_1}^T \nabla_{r_1} h_1^1(a^1, b^1, r_1) - \\
&F_1(w^1, b^1; -\alpha_1^2(w^1, b^1)(\nabla_{b^1} \varphi_1(a^1, b^1) - q^1 + \nabla_{r_1} h_1^1(a^1, b^1, r_1))) - \rho_1^{12}(d_1^{12}(w^1, b^1))^2, ..., \\
&-\varphi_k(a^1, w^1) + {w^1}^T q^1 + \varphi_k(a^1, b^1) - (b^1)^T q^1 + h_k^1(a^1, b^1, r_k) - {r_k}^T \nabla_{r_k} h_k^1(a^1, b^1, r_k) - \\
&F_1(w^1, b^1; -\alpha_1^2(w^1, b^1)(\nabla_{b^1} \varphi_k(a^1, b^1) - q^1 + \nabla_{r_k} h_k^1(a^1, b^1, r_k))) - \rho_k^{12}(d_k^{12}(w^1, b^1))^2) \in K.
\end{aligned}$$

Using $\lambda \in \text{int}K^*$, we get

$$\begin{aligned}
&\sum_{j=1}^k \lambda_j \{\varphi_j(a^1, w^1) + {a^1}^T v^1 - \varphi_j(u^1, w^1) - (u^1)^T v^1 - h_j^2(u^1, w^1, c_j) + c_j^T \nabla_{c_j} h_j^2(u^1, w^1, c_j)\} \geqq \\
&\sum_{j=1}^k \lambda_j F_1(a^1, u^1; \alpha_1^1(a^1, u^1)(\nabla_{a^1} \varphi_j(u^1, w^1) + v^1 + \nabla_{c_j} h_j^2(u^1, w^1, c_j))) + \sum_{j=1}^k \lambda_j \rho_j^{11}(d_j^{11}(a^1, u^1))^2,
\end{aligned}$$

and

$$\sum_{j=1}^{k}\lambda_j\{-\varphi_j(a^1,w^1)+{w^1}^Tq^1+\varphi_j(a^1,b^1)-(b^1)^Tq^1+h_j^1(a^1,b^1,r_j)-{r_j}^T\nabla_{r_j}h_j^1(a^1,b^1,r_j)\}\geqq$$
$$\sum_{j=1}^{k}\lambda_jF_1(w^1,b^1;-\alpha_1^2(w^1,b^1)(\nabla_{b^1}\varphi_j(a^1,b^1)-q^1+\nabla_{r_j}h_j^1(a^1,b^1,r_j)))+\sum_{j=1}^{k}\lambda_j\rho_j^{12}(d_j^{12}(w^1,b^1))^2.$$

Using (6.3.5) and sublinearity of F in above expressions, we get

$$\begin{aligned}&\sum_{j=1}^{k}\lambda_j\{\varphi_j(a^1,w^1)+{a^1}^Tv^1-\varphi_j(u^1,w^1)-(u^1)^Tv^1-h_j^2(u^1,w^1,c_j)+c_j^T\nabla_{c_j}h_j^2(u^1,w^1,c_j)\}\\&\qquad\geqq F_1(a^1,u^1;\alpha_1^1(a^1,u^1)\sum_{j=1}^{k}\lambda_j(\nabla_{a^1}\varphi_j(u^1,w^1)+v^1+\nabla_{c_j}h_j^2(u^1,w^1,c_j)))\\&\qquad\quad+\sum_{j=1}^{k}\lambda_j\rho_j^{11}(d_j^{11}(a^1,u^1))^2\end{aligned}\tag{6.3.16}$$

and

$$\begin{aligned}&\sum_{j=1}^{k}\lambda_j\{-\varphi_j(a^1,w^1)+{w^1}^Tq^1+\varphi_j(a^1,b^1)-(b^1)^Tq^1+h_j^1(a^1,b^1,r_j)-{r_j}^T\nabla_{r_j}h_j^1(a^1,b^1,r_j)\}\\&\qquad\geqq F_1(w^1,b^1;-\alpha_1^2(w^1,b^1)\sum_{j=1}^{k}\lambda_j(\nabla_{b^1}\varphi_j(a^1,b^1)-q^1+\nabla_{r_j}h_j^1(a^1,b^1,r_j)))\\&\qquad\quad+\sum_{j=1}^{k}\lambda_j\rho_j^{12}(d_j^{12}(w^1,b^1))^2.\end{aligned}\tag{6.3.17}$$

Using $\alpha_1^1(a^1,u^1)>0$ and (6.3.6), we obtain
$\nu^1=\alpha_1^1(a^1,u^1)\sum_{j=1}^{k}\lambda_j(\nabla_{a^1}\varphi_j(u^1,w^1)+v^1+\nabla_{c_j}h_j^2(u^1,w^1,c_j))\in C_1^*$,
and so from (6.3.11),

$$F_1(a^1,u^1;\nu^1)+(\alpha_1^1)^{-1}(\nu^1)^Tu^1\geqq 0.\tag{6.3.18}$$

Similarly

$$F_2(w^1,b^1;\nu^2)+(\alpha_1^2)^{-1}(\nu^2)^Tb^1\geqq 0,\tag{6.3.19}$$

as $\nu^2 = -\alpha_1^2(w^1, b^1) \sum_{j=1}^{k} \lambda_j(\nabla_{b^1}\varphi_j(a^1, b^1) - q^1 + \nabla_{r_j} h_j^1(a^1, b^1, r_j)) \in C_3^*$.
From (6.3.16) and (6.3.18), we get

$$\sum_{j=1}^{k} \lambda_j\{\varphi_j(a^1, w^1) + {a^1}^T v^1 - \varphi_j(u^1, w^1) - (u^1)^T v^1 - h_j^2(u^1, w^1, c_j) + {c_j}^T \nabla_{c_j} h_j^2(u^1, w^1, c_j)\}$$
$$- \sum_{j=1}^{k} \lambda_j \rho_j^{11}(d_j^{11}(a^1, u^1))^2 \geqq -(\alpha_1^1)^{-1}(\nu^1)^T u^1. \tag{6.3.20}$$

Using (6.3.17) and (6.3.19), we obtain

$$\sum_{j=1}^{k} \lambda_j\{-\varphi_j(a^1, w^1) + {w^1}^T q^1 + \varphi_j(a^1, b^1) - (b^1)^T q^1 + h_j^1(a^1, b^1, r_j) - {r_j}^T \nabla_{r_j} h_j^1(a^1, b^1, r_j)\}$$
$$- \sum_{j=1}^{k} \lambda_j \rho_j^{12}(d_j^{12}(w^1, b^1))^2 \geqq -(\alpha_1^2)^{-1}(\nu^2)^T b^1. \tag{6.3.21}$$

Now by adding (6.3.20) and (6.3.21), we have
$\sum_{j=1}^{k} \lambda_j\{\varphi_j(a^1, b^1) + {w^1}^T q^1 - (b^1)^T q^1 + h_j^1(a^1, b^1, r_j) - {r_j}^T \nabla_{r_j} h_j^1(a^1, b^1, r_j)\} + (\alpha_1^2)^{-1}(\nu^2)^T b^1 -$
$\sum_{j=1}^{k} \lambda_j\{\varphi_j(u^1, w^1) - {a^1}^T v^1 + (u^1)^T v^1 + h_j^2(u^1, w^1, c_j) - {c_j}^T \nabla_{c_j} h_j^2(u^1, w^1, c_j)\} + (\alpha_1^1)^{-1}(\nu^1)^T u^1 \geqq$
$\sum_{j=1}^{k} \lambda_j \rho_j^{11}(d_j^{11}(a^1, u^1))^2 + \sum_{j=1}^{k} \lambda_j \rho_j^{12}(d_j^{12}(w^1, b^1))^2$,
which along with (iii) gives

$$\sum_{j=1}^{k} \lambda_j\{\varphi_j(a^1, b^1) + (w^1)^T q^1 - (b^1)^T q^1 + h_j^1(a^1, b^1, r_j) - {r_j}^T \nabla_{r_j} h_j^1(a^1, b^1, r_j)\} + (\alpha_1^2)^{-1}(\nu^2)^T b^1$$
$$- \sum_{j=1}^{k} \lambda_j\{\varphi_j(u^1, w^1) - (a^1)^T v^1 + (u^1)^T v^1 + h_j^2(u^1, w^1, c_j) - {c_j}^T \nabla_{c_j} h_j^2(u^1, w^1, c_j)\}$$
$$+ (\alpha_1^1)^{-1}(\nu^1)^T u^1 \geqq 0. \tag{6.3.22}$$

As we know $(w^1)^T q^1 \leqq S(w^1|Q^1)$, $(a^1)^T v^1 \leqq S(a^1|V^1)$ and substituting the values of ν^1 and ν^2 in above, we get

$$\sum_{j=1}^{k} \lambda_j\{\varphi_j(a^1, b^1) + S(w^1|Q^1) + h_j^1(a^1, b^1, r_j) - {r_j}^T \nabla_{r_j} h_j^1(a^1, b^1, r_j)\} - (b^1)^T \sum_{j=1}^{k} \lambda_j(\nabla_{b^1}\varphi_j(a^1, b^1)$$
$$+ \nabla_{r_j} h_j^1(a^1, b^1, r_j)) - \sum_{j=1}^{k} \lambda_j\{\varphi_j(u^1, w^1) - S(a^1|V^1) + h_j^2(u^1, w^1, c_j) - {c_j}^T \nabla_{c_j} h_j^2(u^1, w^1, c_j)\}$$

$$+(u^1)^T \sum_{j=1}^{k} \lambda_j(\nabla_{a^1}\varphi_j(u^1, w^1) + \nabla_{c_j} h_j^2(u^1, w^1, c_j)) \geqq 0. \tag{6.3.23}$$

Using $\alpha_2^1(a^2, u^2) > 0$ and (6.3.7), we acquire
$\nu^3 = \alpha_2^1(a^2, u^2) \sum\limits_{j=1}^{k} \lambda_j(\nabla_{a^2}\phi_j(u^2, w^2) + v_j^2 + \nabla_{d_j} g_j^2(u^2, w^2, d_j)) \in C_2^*$.
Now from (6.3.13) and above, we have

$$\begin{aligned}
&G_1(a^2, u^2; \alpha_2^1(a^2, u^2) \sum_{j=1}^{k} \lambda_j(\nabla_{a^2}\phi_j(u^2, w^2) + v_j^2 + \nabla_{d_j} g_j^2(u^2, w^2, d_j)))\\
&\geqq -(u^2)^T \sum_{j=1}^{k} \lambda_j(\nabla_{a^2}\phi_j(u^2, w^2) + v_j^2 + \nabla_{d_j} g_j^2(u^2, w^2, d_j))\\
&\geqq 0. (\text{Using } (6.3.8))
\end{aligned} \tag{6.3.24}$$

Since $\phi(\cdot, w^2) + (\cdot)^T v^2$ is higher-order K-$(G_1, \alpha_2^1, \rho^{21}, d^{21})$-convex at u^2, therefore we have
$(\phi_1(a^2, w^2) + (a^2)^T v_1^2 - \phi_1(u^2, w^2) - (u^2)^T v_1^2 - g_1^2(u^2, w^2, d_1) + {d_1}^T \nabla_{d_1} g_1^2(u^2, w^2, d_1)$
$- G_1(a^2, u^2; \alpha_2^1(a^2, u^2)(\nabla_{a^2}\phi_1(u^2, w^2) + v_1^2 + \nabla_{d_1} g_1^2(u^2, w^2, d_1))) - \rho_1^{21}(d_1^{21}(a^2, u^2))^2, ...,$
$\phi_k(a^2, w^2) + (a^2)^T v_k^2 - \phi_k(u^2, w^2) - (u^2)^T v_k^2 - g_k^2(u^2, w^2, d_k) + {d_k}^T \nabla_{d_k} g_k^2(u^2, w^2, d_k)$
$- G_1(a^2, u^2; \alpha_2^1(a^2, u^2)(\nabla_{a^2}\phi_k(u^2, w^2) + v_k^2 + \nabla_{d_k} g_k^2(u^2, w^2, d_k))) - \rho_k^{21}(d_k^{21}(a^2, u^2))^2) \in K,$

Using $\lambda \in \text{int } K^*$ and sublinearity of F, we obtain
$\sum\limits_{j=1}^{k} \lambda_j\{\phi_j(a^2, w^2) + (a^2)^T v_j^2 - \phi_j(u^2, w^2) - (u^2)^T v_j^2 - g_j^2(u^2, w^2, d_j) + {d_j}^T \nabla_{d_j} g_j^2(u^2, w^2, d_j)\}$
$\geqq G_1(a^2, u^2; \alpha_2^1(a^2, u^2) \sum\limits_{j=1}^{k} \lambda_j(\nabla_{a^2}\phi_j(u^2, w^2) + v_j^2 + \nabla_{d_j} g_j^2(u^2, w^2, d_j))) + \sum\limits_{j=1}^{k} \lambda_j \rho_j^{21}(d_j^{21}(a^2, u^2))^2,$

which on using (6.3.24), we get

$$\begin{aligned}
&\sum_{j=1}^{k} \lambda_j\{\phi_j(a^2, w^2) + (a^2)^T v_j^2 - \phi_j(u^2, w^2) - (u^2)^T v_j^2 - g_j^2(u^2, w^2, d_j) + {d_j}^T \nabla_{d_j} g_j^2(u^2, w^2, d_j)\}\\
&-\sum_{j=1}^{k} \lambda_j \rho_j^{21}(d_j^{21}(a^2, u^2))^2 \geqq 0.
\end{aligned} \tag{6.3.25}$$

Similarly (6.3.14), $\lambda \in \text{int } K^*$ and higher-order K-$(G_2, \alpha_2^2, \rho^{22}, d^{22})$-convexity of $-(\phi(a^2, \cdot) - (\cdot)^T q^2)$ at b^2 gives

$$\begin{aligned}
&\sum_{j=1}^{k} \lambda_j\{-\phi_j(a^2, w^2) + (w^2)^T q_j^2 + \phi_j(a^2, b^2) - (b^2)^T q_j^2 + g_j^1(a^2, b^2, s_j) - {s_j}^T \nabla_{s_j} g_j^1(a^2, b^2, s_j)\}\\
&-\sum_{j=1}^{k} \lambda_j \rho_j^{22}(d_j^{22}(w^2, b^2))^2 \geqq 0.
\end{aligned} \tag{6.3.26}$$

Adding (6.3.25) and (6.3.26), we obtain

$$\sum_{j=1}^{k} \lambda_j \{\phi_j(a^2,b^2) + (w^2)^T q_j^2 - \phi_j(u^2,w^2) + (a^2)^T v_j^2 - (b^2)^T q_j^2 - (u^2)^T v_j^2 + g_j^1(a^2,b^2,s_j)$$
$$- {s_j}^T \nabla_{s_j} g_j^1(a^2,b^2,s_j) - g_j^2(u^2,w^2,d_j) + {d_j}^T \nabla_{d_j} g_j^2(u^2,w^2,d_j)\} - \sum_{j=1}^{k} \lambda_j \rho_j^{21}(d_j^{21}(a^2,u^2))^2$$
$$- \sum_{j=1}^{k} \lambda_j \rho_j^{22}(d_j^{22}(w^2,b^2))^2 \geqq 0.$$

Now using $(w^2)^T q_j^2 \leqq S(w^2|Q_j^2)$, $(a^2)^T v_j^2 \leqq S(a^2|V_j^2)$ and (iii), we get

$$\sum_{j=1}^{k} \lambda_j \{\phi_j(a^2,b^2) + S(w^2|Q_j^2) - \phi_j(u^2,w^2) + S(a^2|V_j^2) - (b^2)^T q_j^2 - (u^2)^T v_j^2 + g_j^1(a^2,b^2,s_j)$$
$$-{s_j}^T \nabla_{s_j} g_j^1(a^2,b^2,s_j) - g_j^2(u^2,w^2,d_j) + {d_j}^T \nabla_{d_j} g_j^2(u^2,w^2,d_j)\} \geqq 0. \qquad (6.3.27)$$

Summing up (6.3.23) and (6.3.27) gives

$$\sum_{j=1}^{k} \lambda_j \Bigg\{ \varphi_j(a^1,b^1) + S(a^1|V^1) + \phi_j(a^2,b^2) + S(a^2|V_j^2) - (b^2)^T q_j^2 - (b^1)^T \sum_{j=1}^{k} \lambda_j (\nabla_{b^1}\varphi_j(a^1,b^1) +$$
$$\nabla_{r_j} h_j^1(a^1,b^1,r_j)) + h_j^1(a^1,b^1,r_j) - r_j^T \nabla_{r_j} h_j^1(a^1,b^1,r_j) + g_j^1(a^2,b^2,s_j) - s_j^T \nabla_{s_j} g_j^1(a^2,b^2,s_j) \Bigg\}$$
$$- \sum_{j=1}^{k} \lambda_j \Bigg\{ \varphi_j(u^1,w^1) - S(w^1|Q^1) + \phi_j(u^2,w^2) - S(w^2|Q_j^2) + (u^2)^T v_j^2 - (u^1)^T \sum_{j=1}^{k} \lambda_j (\nabla_{a^1}\varphi_j(u^1,w^1) +$$
$$\nabla_{c_j} h_j^2(u^1,w^1,c_j)) + h_j^2(u^1,w^1,c_j) - c_j^T \nabla_{c_j} h_j^2(u^1,w^1,c_j) + g_j^2(u^2,w^2,d_j) - d_j^T \nabla_{d_j} g_j^2(u^2,w^2,d_j) \Bigg\} \geqq$$
$$0,$$

which contradicts (6.3.15). Hence proved.

If λ in **(NMSP)** and **(NMSD)** is fixed as $\bar{\lambda}$, we express these problems as **(NMSP)**$_{\bar{\lambda}}$ and **(NMSD)**$_{\bar{\lambda}}$, respectively.

Theorem 6.3.2 *(Strong Duality)Let* $(\bar{a}^1, \bar{b}^1, \bar{a}^2, \bar{b}^2, \bar{q}^1, \bar{q}^2, \bar{\lambda}, \bar{r}, \bar{s})$ *be a WES of* ***(NMSP)****. Let*

(i) $\nabla_{r_j r_j} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j)$ *is positive or negative definite* $\forall\ j = 1, 2, \ldots, k$*;*

(ii) $\nabla_{s_j s_j} g_j^1(\bar{a}^2, \bar{b}^2, \bar{s}_j)$ *is positive or negative definite* $\forall\ j = 1, 2, \ldots, k$*;*

(iii) $\bar{r}_j \neq 0$ *for some* $j \in \{1, 2, \ldots, k\}$ *imply that* $\sum_{j=1}^{k} \alpha_j (\nabla_{b^1 b^1} \varphi_j(\bar{a}^1, \bar{b}^1))^T \bar{r}_j \neq 0$ *for all* $\alpha \in K^*$*;*

(iv) $\sum_{j=1}^{k} \alpha_j (\nabla_{b^1 b^1} \varphi_j(\bar{a}^1, \bar{b}^1))^T \bar{r}_j \notin$ *span*$\{\nabla_{b^1} \varphi_j(\bar{a}^1, \bar{b}^1) + \nabla_{b^1} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j),\ \nabla_{b^1} \varphi_j(\bar{a}^1, \bar{b}^1) + \nabla_{r_j} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j),\ j = 1, 2, \ldots, k\} \setminus \{0\}$, *for all* $\alpha \in K^*$*;*

(v) the set of vectors $\{\nabla_{b^2} \phi_j(\bar{a}^2, \bar{b}^2) - \bar{q}_j^2 + \nabla_{b^2} g_j^1(\bar{a}^2, \bar{b}^2, \bar{s}_j), \nabla_{b^2} \phi_j(\bar{a}^2, \bar{b}^2) - \bar{q}_j^2 + \nabla_{s_j} g_j^1(\bar{a}^2, \bar{b}^2, \bar{s}_j)\}$,

$j = 1, 2, \ldots, k$ is LI;

(vi) the set of vectors $\{\nabla_{b^2}\phi_j(\bar{a}^2, \bar{b}^2) - \bar{q}_j^2 + \nabla_{s_j} g_j^1(\bar{a}^2, \bar{b}^2, \bar{s}_j)\}$, $j = 1, 2, \ldots, k$ is LI;

(vii) $g_j^1(\bar{a}^2, \bar{b}^2, 0) = 0 = g_j^2(\bar{a}^2, \bar{b}^2, 0)$, $\nabla_{a^2} g_j^1(\bar{a}^2, \bar{b}^2, 0) = \nabla_{d_j} g_j^2(\bar{a}^2, \bar{b}^2, 0)$, $\nabla_{b^2} g_j^1(\bar{a}^2, \bar{b}^2, 0) = 0 = \nabla_{s_j} g_j^1(\bar{a}^2, \bar{b}^2, 0)$, $h_j^1(\bar{a}^1, \bar{b}^1, 0) = h_j^2(\bar{a}^1, \bar{b}^1, 0)$ and $\nabla_{a^1} h_j^1(\bar{a}^1, \bar{b}^1, 0) = \nabla_{c_j} h_j^2(\bar{a}^1, \bar{b}^1, 0)$, $j = 1, 2, \ldots, k$;

(viii) K is closed convex pointed cone with $R_+^k \subseteq K$.

Then,

*(I) $\exists\ \bar{v}^1 \in V^1$ and $\bar{v}_j^2 \in V_j^2$ such that $(\bar{a}^1, \bar{b}^1, \bar{a}^2, \bar{b}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{c} = 0, \bar{d} = 0)$ be feasible solution of **(NMSD)**$_{\bar{\lambda}}$, and*

(II) $P(\bar{a}^1, \bar{b}^1, \bar{a}^2, \bar{b}^2, \bar{\lambda}, \bar{r}, \bar{s}) = T(\bar{a}^1, \bar{b}^1, \bar{a}^2, \bar{b}^2, \bar{\lambda}, \bar{c}, \bar{d})$.

*Further, if the suppositions of Theorem 6.3.1 are satisfied for all feasible solutions of **(NMSP)** and **(NMSD)**$_{\bar{\lambda}}$, then $(\bar{a}^1, \bar{b}^1, \bar{a}^2, \bar{b}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{c} = 0, \bar{d} = 0)$ is an ES of **(NMSD)**$_{\bar{\lambda}}$.*

Proof Since $(\bar{a}^1, \bar{b}^1, \bar{a}^2, \bar{b}^2, \bar{q}^1, \bar{q}^2, \bar{\lambda}, \bar{r}, \bar{s})$ is a WES of **(NMSP)**, by Fritz John necessary optimality conditions ([122], Lemma 1), $\exists\ \alpha \in K^*$, $\beta \in C_3$, $\gamma \in C_4$, $\delta \in R_+$ and $\eta \in R$ such that

$$\Big[\sum_{j=1}^{k} \alpha_j(\nabla_{a^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nu + \nabla_{a^1} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j)) + \sum_{j=1}^{k} \bar{\lambda}_j(\nabla_{b^1 a^1}\varphi_j(\bar{a}^1, \bar{b}^1))^T(\beta - (\alpha^T e_k)\bar{b}^1)$$
$$+ \sum_{j=1}^{k}(\nabla_{r_j a^1} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j))^T(\bar{\lambda}_j\beta - \bar{\lambda}_j(\alpha^T e_k)\bar{b}^1 - \alpha_j \bar{r}_j)\Big]^T (a^1 - \bar{a}^1) \geqq 0,\ \forall\ a^1 \in C_1, \tag{6.3.28}$$

$$\Big[\sum_{j=1}^{k} \alpha_j(\nabla_{a^2}\phi_j(\bar{a}^2, \bar{b}^2) + \xi_j + \nabla_{a^2} g_j^1(\bar{a}^2, \bar{b}^2, \bar{s}_j)) + \sum_{j=1}^{k} \bar{\lambda}_j(\nabla_{b^2 a^2}\phi_j(\bar{a}^2, \bar{b}^2))^T(\gamma - (\delta)\bar{b}^2)$$
$$+ \sum_{j=1}^{k}(\nabla_{s_j a^2} g_j^1(\bar{a}^2, \bar{b}^2, \bar{s}_j))^T(\bar{\lambda}_j\gamma - \alpha_j \bar{s}_j - \bar{\lambda}_j(\delta)\bar{b}^2)\Big]^T (a^2 - \bar{a}^2) \geqq 0,\ \forall\ a^2 \in C_2, \tag{6.3.29}$$

$$\sum_{j=1}^{k} \alpha_j(\nabla_{b^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nabla_{b^1} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j)) - \sum_{j=1}^{k}(\alpha^T e_k)\bar{\lambda}_j\{\nabla_{b^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nabla_{r_j} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j)\}$$
$$+ \sum_{j=1}^{k} \bar{\lambda}_j(\nabla_{b^1 b^1}\varphi_j(\bar{a}^1, \bar{b}^1))^T(\beta - (\alpha^T e_k)\bar{b}^1) + \sum_{j=1}^{k}(\nabla_{r_j b^1} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j))^T(\bar{\lambda}_j\beta - \bar{\lambda}_j(\alpha^T e_k)\bar{b}^1$$
$$- \alpha_j \bar{r}_j) = 0, \tag{6.3.30}$$

$$\sum_{j=1}^{k}\alpha_j(\nabla_{b^2}\phi_j(\bar{a}^2,\bar{b}^2)-\bar{q}_j^2+\nabla_{b^2}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j))+\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{b^2b^2}\phi_j(\bar{a}^2,\bar{b}^2))^T(\gamma-\delta\bar{b}^2)+$$
$$\sum_{j=1}^{k}(\nabla_{s_jb^2}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j))^T(\bar{\lambda}_j\gamma-\alpha_j\bar{s}_j-\bar{\lambda}_j\delta\bar{b}^2)-\delta(\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{b^2}\phi_j(\bar{a}^2,\bar{b}^2)-\bar{q}_j^2$$
$$+\nabla_{s_j}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j)))=0, \tag{6.3.31}$$
$$(\beta-(\alpha^Te_k)\bar{b}^1)^T(\nabla_{b^1}\varphi_j(\bar{a}^1,\bar{b}^1)+\nabla_{r_j}h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j))+(\gamma-\delta\bar{b}^2)^T[\nabla_{b^2}\phi_j(\bar{a}^2,\bar{b}^2)$$
$$-\bar{q}_j^2+\nabla_{s_j}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j)]+\eta=0,\ j=1,2,\ldots,k, \tag{6.3.32}$$
$$(\nabla_{r_jr_j}h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j))^T(\bar{\lambda}_j\beta-\bar{\lambda}_j(\alpha^Te_k)\bar{b}^1-\alpha_j\bar{r}_j)=0, \tag{6.3.33}$$
$$(\nabla_{s_js_j}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j))^T(\bar{\lambda}_j\gamma-\alpha_j\bar{s}_j-\bar{\lambda}_j\delta\bar{b}^2)=0, \tag{6.3.34}$$
$$\beta^T\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{b^1}\varphi_j(\bar{a}^1,\bar{b}^1)-\bar{q}^1+\nabla_{r_j}h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j))=0, \tag{6.3.35}$$
$$\gamma^T\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{b^2}\phi_j(\bar{a}^2,\bar{b}^2)-\bar{q}_j^2+\nabla_{s_j}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j))=0, \tag{6.3.36}$$
$$\delta(\bar{b}^2)^T\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{b^2}\phi_j(\bar{a}^2,\bar{b}^2)-\bar{q}_j^2+\nabla_{s_j}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j))=0, \tag{6.3.37}$$
$$\eta(\bar{\lambda}^Te_k-1)=0, \tag{6.3.38}$$
$$\beta\in N_{Q^1}(\bar{q}^1), \tag{6.3.39}$$
$$\alpha_j\bar{b}^2+\bar{\lambda}_j\gamma-\bar{\lambda}_j\delta\bar{b}^2\in N_{Q_j^2}(\bar{q}_j^2),\ j=1,2,\ldots,k, \tag{6.3.40}$$
$$\nu^T\bar{a}^1=S(\bar{a}^1|V^1),\ \nu\in V^1, \tag{6.3.41}$$
$$\xi_j^T\bar{a}^2=S(\bar{a}^2|V_j^2),\ \xi_j\in V_j^2, \tag{6.3.42}$$
$$(\alpha,\beta,\gamma,\delta,\eta)\neq 0. \tag{6.3.43}$$

From hypothesis (i) and (6.3.33), we get

$$\bar{\lambda}_j\beta-\bar{\lambda}_j(\alpha^Te_k)\bar{b}^1-\alpha_j\bar{r}_j=0. \tag{6.3.44}$$

Using (6.3.44) in (6.3.30), we have

$$\sum_{j=1}^{k}\alpha_j(\nabla_{b^1b^1}\varphi_j(\bar{a}^1,\bar{b}^1))^T\bar{r}_j=(\alpha^Te_k)\sum_{j=1}^{k}\bar{\lambda}_j\{\nabla_{b^1}\varphi_j(\bar{a}^1,\bar{b}^1)+\nabla_{r_j}h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j)\}$$
$$-\sum_{j=1}^{k}\alpha_j(\nabla_{b^1}\varphi_j(\bar{a}^1,\bar{b}^1)+\nabla_{b^1}h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j)) \tag{6.3.45}$$

that is

$$\sum_{j=1}^{k}\alpha_j(\nabla_{b^1b^1}\varphi_j(\bar a^1,\bar b^1))^T\bar r_j \in span\{\nabla_{b^1}\varphi_j(\bar a^1,\bar b^1)+\nabla_{b^1}h_j^1(\bar a^1,\bar b^1,\bar r_j),$$
$$\nabla_{b^1}\varphi_j(\bar a^1,\bar b^1)+\nabla_{r_j}h_j^1(\bar a^1,\bar b^1,\bar r_j),\ j=1,2,\dots,k\}. \tag{6.3.46}$$

Now we prove $\bar r_j = 0\ \forall\ j=1,2,\dots,k$.

On contrary suppose that for some $j\in\{1,2,\dots,k\}$, $\bar r_j\neq 0$, then using (iii), we have

$$\sum_{j=1}^{k}\alpha_j(\nabla_{b^1b^1}\varphi_j(\bar a^1,\bar b^1))^T\bar r_j\neq 0, \tag{6.3.47}$$

This contradicts (iv) (by (6.3.46) & (6.3.47)). Hence

$$\bar r_j = 0\ \forall\ j=1,2,\dots,k. \tag{6.3.48}$$

Using (6.3.48) in (6.3.44) implies

$$\bar\lambda_j\beta=\bar\lambda_j(\alpha^Te_k)\bar b^1,\ j=1,2,\dots,k.$$

Since $R_+^k\subseteq K$ implies $K^*\subseteq R_+^k$. As $\bar\lambda\in \mathrm{int}K^*$, therefore $\bar\lambda>0$. Hence from above we get

$$\beta=(\alpha^Te_k)\bar b^1. \tag{6.3.49}$$

Now from (ii) and (6.3.34), we obtain

$$\bar\lambda_j\gamma-\alpha_j\bar s_j-\bar\lambda_j\delta\bar b^2=0. \tag{6.3.50}$$

Now we prove that $\alpha_j\neq 0\ \forall\ j$. Let $\alpha_j=0$ for some j. As $\bar\lambda>0$, (6.3.50) gives

$$\gamma=\delta\bar b^2. \tag{6.3.51}$$

Using (6.3.50) and (6.3.51) in (6.3.31), we obtain

$$\sum_{j=1}^{k}\alpha_j(\nabla_{b^2}\phi_j(\bar a^2,\bar b^2)-\bar q_j^2+\nabla_{b^2}g_j^1(\bar a^2,\bar b^2,\bar s_j))-(\sum_{j=1}^{k}\delta\bar\lambda_j(\nabla_{b^2}\phi_j(\bar a^2,\bar b^2)$$
$$-\bar q_j^2+\nabla_{s_j}g_j^1(\bar a^2,\bar b^2,\bar s_j)))=0,$$

which along with hypothesis (v) gives

$$\begin{aligned} \alpha_j &= 0,\ j = 1, 2, \ldots, k, \\ \delta\bar{\lambda}_j &= 0,\ j = 1, 2, \ldots, k, \end{aligned} \tag{6.3.52}$$

which implies $\delta = 0$ as $\bar{\lambda}_j > 0$. Also $\delta = 0$ and (6.3.51) gives

$$\gamma = 0. \tag{6.3.53}$$

Using (6.3.49) and (6.3.51) in (6.3.32), we get

$$\eta = 0. \tag{6.3.54}$$

From (6.3.49) and (6.3.52), we obtain

$$\beta = 0.$$

Consequently $(\alpha, \beta, \gamma, \delta, \eta) = 0$ which contradicts (6.3.43).
Therefore, $\alpha_j \neq 0\ \forall\ j$. Since $\alpha \in K^*$ and $K^* \subseteq R^k_+$, therefore $\alpha_j > 0\ \forall\ j$ and hence $\alpha^T e_k > 0$. Subtract (6.3.37) from (6.3.36), we obtain

$$(\gamma - \delta(\bar{b}^2))^T \sum_{j=1}^{k} \bar{\lambda}_j (\nabla_{b^2}\phi_j(\bar{a}^2, \bar{b}^2) - \bar{q}_j^2 + \nabla_{s_j} g_j^1(\bar{a}^2, \bar{b}^2, \bar{s}_j)) = 0, \tag{6.3.55}$$

Now (6.3.50), (6.3.55) and $\bar{\lambda} > 0$ implies

$$\sum_{j=1}^{k} \alpha_j \bar{s}_j^T (\nabla_{b^2}\phi_j(\bar{a}^2, \bar{b}^2) - \bar{q}_j^2 + \nabla_{s_j} g_j^1(\bar{a}^2, \bar{b}^2, \bar{s}_j)) = 0,$$

which using $\alpha_j > 0$ and hypothesis (vi) gives

$$\bar{s}_j = 0. \tag{6.3.56}$$

From (6.3.50), (6.3.56) and $\bar{\lambda} > 0$, we get

$$\gamma = \delta\bar{b}^2. \tag{6.3.57}$$

Using (6.3.50), (6.3.56), (6.3.57) and hypothesis (vii), (6.3.29) and (6.3.31) reduces to

$$\left[\sum_{j=1}^{k}\alpha_j(\nabla_{a^2}\phi_j(\bar{a}^2,\bar{b}^2)+\xi_j+\nabla_{a^2}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j))\right]^T(a^2-\bar{a}^2)\geqq 0, \tag{6.3.58}$$

and

$$\sum_{j=1}^{k}(\alpha_j-\delta\bar{\lambda}_j)(\nabla_{b^2}\phi_j(\bar{a}^2,\bar{b}^2)-\bar{q}_j^2+\nabla_{s_j}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j))=0, \tag{6.3.59}$$

From hypothesis (vi) and (6.3.59), we obtain

$$\alpha_j=\delta\bar{\lambda}_j,\ j=1,2,\ldots,k. \tag{6.3.60}$$

Since $\bar{\lambda}_j>0$ and $\alpha_j>0,\ j=1,2,\ldots,k$, then (6.3.60) gives $\delta>0$. Hence from (6.3.58) and (6.3.60), we acquire

$$\left[\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^2}\phi_j(\bar{a}^2,\bar{b}^2)+\xi_j+\nabla_{a^2}g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j))\right]^T(a^2-\bar{a}^2)\geqq 0,\ \forall\ a^2\in C_2.$$

For $\bar{d}_j=0$, by using condition (vii) and (6.3.56) in above inequality, we get

$$\left[\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^2}\phi_j(\bar{a}^2,\bar{b}^2)+\xi_j+\nabla_{d_j}g_j^2(\bar{a}^2,\bar{b}^2,\bar{d}_j))\right]^T(a^2-\bar{a}^2)\geqq 0. \tag{6.3.61}$$

Let $a^2\in C_2$. Then $a^2+\bar{a}^2\in C_2$, as C_2 is a closed convex cone and so (6.3.61) shows that for every $a^2\in C_2$

$$a^{2T}\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^2}\phi_j(\bar{a}^2,\bar{b}^2)+\xi_j+\nabla_{d_j}g_j^2(\bar{a}^2,\bar{b}^2,\bar{d}_j))\geqq 0,$$

which gives

$$\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^2}\phi_j(\bar{a}^2,\bar{b}^2)+\xi_j+\nabla_{d_j}g_j^2(\bar{a}^2,\bar{b}^2,\bar{d}_j))\in C_2^*, \tag{6.3.62}$$

Now if $a^2=0$, (6.3.61) becomes

$$(\bar{a}^2)^T\sum_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^2}\phi_j(\bar{a}^2,\bar{b}^2)+\xi_j+\nabla_{d_j}g_j^2(\bar{a}^2,\bar{b}^2,\bar{d}_j))\leqq 0, \tag{6.3.63}$$

From (6.3.57), we get

$$\bar{b}^2 = \frac{\gamma}{\delta} \in C_4. \tag{6.3.64}$$

Also from (6.3.49), we obtain

$$\bar{b}^1 = \frac{\beta}{\alpha^T e_k} \in C_3.$$

Using (6.3.44), (6.3.49), (6.3.60) and $\delta > 0$ in (6.3.28), we get

$$\left[\sum_{j=1}^{k} \bar{\lambda}_j (\nabla_{a^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nu + \nabla_{a^1} h_j^1(\bar{a}^1, \bar{b}^1, \bar{r}_j))\right]^T (a^1 - \bar{a}^1) \geqq 0, \ \forall \ a^1 \in C_1.$$

Now for $\bar{c}_j = 0$, using (6.3.48) and hypothesis (vii) with above inequality, we obtain

$$\left[\sum_{j=1}^{k} \bar{\lambda}_j (\nabla_{a^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nu + \nabla_{c_j} h_j^2(\bar{a}^1, \bar{b}^1, \bar{c}_j))\right]^T (a^1 - \bar{a}^1) \geqq 0, \tag{6.3.65}$$

Let $a^1 \in C_1$. Then $\bar{a}^1 + a^1 \in C_1$ and from (6.3.65), we have

$$a^{1T} \sum_{j=1}^{k} \bar{\lambda}_j (\nabla_{a^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nu + \nabla_{c_j} h_j^2(\bar{a}^1, \bar{b}^1, \bar{c}_j)) \geqq 0,$$

which gives

$$\sum_{j=1}^{k} \bar{\lambda}_j (\nabla_{a^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nu + \nabla_{c_j} h_j^2(\bar{a}^1, \bar{b}^1, \bar{c}_j)) \in C_1^*,$$

By taking $\nu = \bar{v}^1$ and $\xi_j = \bar{v}_j^2$, we can see that $(\bar{a}^1, \bar{b}^1, \bar{a}^2, \bar{b}^2, \nu = \bar{v}^1, \xi = \bar{v}^2, \bar{\lambda}, \bar{c} = 0, \bar{d} = 0)$ satisfies the constraints of **(NMSD)**$_{\bar{\lambda}}$. Therefore it is a feasible solution for the dual problem.

Now by taking $a^1 = 0$ and $a^1 = 2\bar{a}^1$, simultaneously in (6.3.65), we achieve

$$(\bar{a}^1)^T \sum_{j=1}^{k} \bar{\lambda}_j (\nabla_{a^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nu + \nabla_{c_j} h_j^2(\bar{a}^1, \bar{b}^1, \bar{c}_j)) = 0,$$

which implies

$$(\bar{a}^1)^T \sum_{j=1}^{k} \bar{\lambda}_j (\nabla_{a^1}\varphi_j(\bar{a}^1, \bar{b}^1) + \nabla_{c_j} h_j^2(\bar{a}^1, \bar{b}^1, \bar{c}_j)) = -(\bar{a}^1)^T \nu = -S(\bar{a}^1 | V^1), \tag{6.3.66}$$

Now (6.3.39) and (6.3.49) gives $(\alpha^T e_k)\bar{b}^1 \in N_{Q^1}(\bar{q}^1)$. Since $\alpha^T e_k > 0$, we have $\bar{b}^1 \in N_{Q^1}(\bar{q}^1)$. Now as Q^1 is a compact convex set in $R^{|M_1|}$, $(\bar{b}^1)^T \bar{q}^1 = S(\bar{b}^1 | Q^1)$.

From (6.3.35), (6.3.49) and $\alpha^T e_k > 0$, we get

$$(\bar{b}^1)^T \sum_{j=1}^{k} \lambda_j(\nabla_{b^1}\varphi_j(\bar{a}^1,\bar{b}^1) - \bar{q}^1 + \nabla_{r_j} h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j)) = 0,$$

which implies

$$(\bar{b}^1)^T \sum_{j=1}^{k} \lambda_j(\nabla_{b^1}\varphi_j(\bar{a}^1,\bar{b}^1) + \nabla_{r_j} h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j)) = (\bar{b}^1)^T \bar{q}^1 = S(\bar{b}^1|Q^1). \tag{6.3.67}$$

Using (6.3.40), $\alpha_j > 0$ and (6.3.57), we get

$$\bar{b}^2 \in N_{Q_j^2(\bar{q}_j^2)}.$$

Since Q_j^2 is compact convex set in $R^{|M_2|}$,

$$S(\bar{b}^2|Q_j^2) = (\bar{b}^2)^T \bar{q}_j^2,\ j = 1,2,\ldots,k. \tag{6.3.68}$$

By hypothesis (vii) for $\bar{c}_j = 0,\ \bar{d}_j = 0$, (6.3.48), (6.3.56), (6.3.66)-(6.3.68) gives

$$\begin{aligned}
&\varphi_j(\bar{a}^1,\bar{b}^1) + S(\bar{a}^1|V^1) + \phi_j(\bar{a}^2,\bar{b}^2) + S(\bar{a}^2|V_j^2) - (\bar{b}^2)^T\bar{q}_j^2 - (\bar{b}^1)^T \sum_{j=1}^{k} \bar{\lambda}_j(\nabla_{b^1}\varphi_j(\bar{a}^1,\bar{b}^1) \\
&+\nabla_{r_j} h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j)) + h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j) - \bar{r}_j^T \nabla_{r_j} h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j) + g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j) - \bar{s}_j^T \nabla_{s_j} g_j^1(\bar{a}^2,\bar{b}^2,\bar{s}_j) \\
&= \varphi_j(\bar{a}^1,\bar{b}^1) - S(\bar{b}^1|Q^1) + \phi_j(\bar{a}^2,\bar{b}^2) - S(\bar{b}^2|Q_j^2) + (\bar{a}^2)^T\bar{v}_j^2 - (\bar{a}^1)^T \sum_{j=1}^{k} \bar{\lambda}_j(\nabla_{a^1}\varphi_j(\bar{a}^1,\bar{b}^1) \\
&+\nabla_{c_j} h_j^2(\bar{a}^1,\bar{b}^1,\bar{c}_j)) + h_j^2(\bar{a}^1,\bar{b}^1,\bar{c}_j) - \bar{c}_j^T \nabla_{c_j} h_j^2(\bar{a}^1,\bar{b}^1,\bar{c}_j) + g_j^2(\bar{a}^2,\bar{b}^2,\bar{d}_j) - \bar{d}_j^T \nabla_{d_j} g_j^2(\bar{a}^2,\bar{b}^2,\bar{d}_j).
\end{aligned}$$

Now let $(\bar{a}^1,\bar{b}^1,\bar{a}^2,\bar{b}^2,\nu = \bar{v}^1,\xi = \bar{v}^2,\bar{\lambda},\bar{c} = 0,\bar{d} = 0)$ be not an ES of **(NMSD)**$_{\bar{\lambda}}$, then $\exists$ another feasible solution $(\bar{u}^1,\bar{w}^1,\bar{u}^2,\bar{w}^2,\bar{v}^1,\bar{v}^2,\bar{\lambda},\bar{c},\bar{d})$ of **(NMSD)**$_{\bar{\lambda}}$ such that

$$\begin{aligned}
&\Big(\varphi_1(\bar{a}^1,\bar{b}^1) - S(\bar{b}^1|Q^1) + \phi_1(\bar{a}^2,\bar{b}^2) - S(\bar{b}^2|Q_1^2) + (\bar{a}^2)^T\bar{v}_1^2 - (\bar{a}^1)^T \sum_{j=1}^{k} \bar{\lambda}_j(\nabla_{a^1}\varphi_j(\bar{a}^1,\bar{b}^1) \\
&+\nabla_{c_j} h_j^2(\bar{a}^1,\bar{b}^1,\bar{c}_j)) + h_1^2(\bar{a}^1,\bar{b}^1,\bar{c}_1) - \bar{c}_1^T \nabla_{c_1} h_1^2(\bar{a}^1,\bar{b}^1,\bar{c}_1) + g_1^2(\bar{a}^2,\bar{b}^2,\bar{d}_1) - \bar{d}_1^T \nabla_{d_1} g_1^2(\bar{a}^2,\bar{b}^2,\bar{d}_1),\ldots, \\
&\varphi_k(\bar{a}^1,\bar{b}^1) - S(\bar{b}^1|Q^1) + \phi_k(\bar{a}^2,\bar{b}^2) - S(\bar{b}^2|Q_k^2) + (\bar{a}^2)^T\bar{v}_k^2 - (\bar{a}^1)^T \sum_{j=1}^{k} \bar{\lambda}_j(\nabla_{a^1}\varphi_j(\bar{a}^1,\bar{b}^1) \\
&+\nabla_{c_j} h_j^2(\bar{a}^1,\bar{b}^1,\bar{c}_j)) + h_k^2(\bar{a}^1,\bar{b}^1,\bar{c}_k) - \bar{c}_k^T \nabla_{c_k} h_k^2(\bar{a}^1,\bar{b}^1,\bar{c}_k) + g_k^2(\bar{a}^2,\bar{b}^2,\bar{d}_k) - \bar{d}_k^T \nabla_{d_k} g_k^2(\bar{a}^2,\bar{b}^2,\bar{d}_k)\Big)
\end{aligned}$$

$-\Bigg(\varphi_1(\bar{u}^1,\bar{w}^1)-S(\bar{w}^1|Q^1)+\phi_1(\bar{u}^2,\bar{w}^2)-S(\bar{w}^2|Q_1^2)+(\bar{u}^2)^T\bar{v}_1^2-(\bar{u}^1)^T\sum\limits_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^1}\varphi_j(\bar{u}^1,\bar{w}^1)+$

$\nabla_{c_j}h_j^2(\bar{u}^1,\bar{w}^1,\bar{c}_j))+h_1^2(\bar{u}^1,\bar{w}^1,\bar{c}_1)-\bar{c}_1^T\nabla_{c_1}h_1^2(\bar{u}^1,\bar{w}^1,\bar{c}_1)+g_1^2(\bar{u}^2,\bar{w}^2,\bar{d}_1)-\bar{d}_1^T\nabla_{d_1}g_1^2(\bar{u}^2,\bar{w}^2,\bar{d}_1),\ldots,$

$\varphi_k(\bar{u}^1,\bar{w}^1)-S(\bar{w}^1|Q^1)+\phi_k(\bar{u}^2,\bar{w}^2)-S(\bar{w}^2|Q_k^2)+(\bar{u}^2)^T\bar{v}_k^2-(\bar{u}^1)^T\sum\limits_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^1}\varphi_j(\bar{u}^1,\bar{w}^1)+$

$\nabla_{c_j}h_j^2(\bar{u}^1,\bar{w}^1,\bar{c}_j))+h_k^2(\bar{u}^1,\bar{w}^1,\bar{c}_k)-\bar{c}_k^T\nabla_{c_k}h_k^2(\bar{u}^1,\bar{w}^1,\bar{c}_k)+g_k^2(\bar{u}^2,\bar{w}^2,\bar{d}_k)-\bar{d}_k^T\nabla_{d_k}g_k^2(\bar{u}^2,\bar{w}^2,\bar{d}_k)\Bigg)\in$

$-K\setminus\{0\},$

Using (6.3.48), (6.3.56), (6.3.66)-(6.3.68) and hypothesis (vii) for $\bar{c}=0,\ \bar{d}=0$, we get

$\Bigg(\varphi_1(\bar{a}^1,\bar{b}^1)+S(\bar{a}^1|V^1)+\phi_1(\bar{a}^2,\bar{b}^2)+S(\bar{a}^2|V_1^2)-(\bar{b}^2)^T\bar{q}_1^2-(\bar{b}^1)^T\sum\limits_{j=1}^{k}\bar{\lambda}_j(\nabla_{\bar{b}^1}\varphi_j(\bar{a}^1,\bar{b}^1)$

$+\nabla_{r_j}h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j))+h_1^1(\bar{a}^1,\bar{b}^1,\bar{r}_1)-\bar{r}_1^T\nabla_{r_1}h_1^1(\bar{a}^1,\bar{b}^1,\bar{r}_1)+g_1^1(\bar{a}^2,\bar{b}^2,\bar{s}_1)-\bar{s}_1^T\nabla_{s_1}g_1^1(\bar{a}^2,\bar{b}^2,\bar{s}_1),\ldots,$

$\varphi_k(\bar{a}^1,\bar{b}^1)+S(\bar{a}^1|V^1)+\phi_k(\bar{a}^2,\bar{b}^2)+S(\bar{a}^2|V_k^2)-(\bar{b}^2)^T\bar{q}_k^2-(\bar{b}^1)^T\sum\limits_{j=1}^{k}\bar{\lambda}_j(\nabla_{\bar{b}^1}\varphi_j(\bar{a}^1,\bar{b}^1)$

$+\nabla_{r_j}h_j^1(\bar{a}^1,\bar{b}^1,\bar{r}_j))+h_k^1(\bar{a}^1,\bar{b}^1,\bar{r}_k)-\bar{r}_k^T\nabla_{r_k}h_k^1(\bar{a}^1,\bar{b}^1,\bar{r}_k)+g_k^1(\bar{a}^2,\bar{b}^2,\bar{s}_k)-\bar{s}_k^T\nabla_{s_k}g_k^1(\bar{a}^2,\bar{b}^2,\bar{s}_k)\Bigg)$

$-\Bigg(\varphi_1(\bar{u}^1,\bar{w}^1)-S(\bar{w}^1|Q^1)+\phi_j(\bar{u}^2,\bar{w}^2)-S(\bar{w}^2|Q_1^2)+(\bar{u}^2)^T\bar{v}_1^2-(\bar{u}^1)^T\sum\limits_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^1}\varphi_j(\bar{u}^1,\bar{w}^1)+$

$\nabla_{c_j}h_j^2(\bar{u}^1,\bar{w}^1,\bar{c}_j))+h_1^2(\bar{u}^1,\bar{w}^1,\bar{c}_1)-\bar{c}_1^T\nabla_{c_1}h_1^2(\bar{u}^1,\bar{w}^1,\bar{c}_1)+g_1^2(\bar{u}^2,\bar{w}^2,\bar{d}_1)-\bar{d}_1^T\nabla_{d_1}g_1^2(\bar{u}^2,\bar{w}^2,\bar{d}_1),\ldots,$

$\varphi_k(\bar{u}^1,\bar{w}^1)-S(\bar{w}^1|Q^1)+\phi_k(\bar{u}^2,\bar{w}^2)-S(\bar{w}^2|Q_k^2)+(\bar{u}^2)^T\bar{v}_k^2-(\bar{u}^1)^T\sum\limits_{j=1}^{k}\bar{\lambda}_j(\nabla_{a^1}\varphi_j(\bar{u}^1,\bar{w}^1)+$

$\nabla_{c_j}h_j^2(\bar{u}^1,\bar{w}^1,\bar{c}_j))+h_k^2(\bar{u}^1,\bar{w}^1,\bar{c}_k)-\bar{c}_k^T\nabla_{c_k}h_k^2(\bar{u}^1,\bar{w}^1,\bar{c}_k)+g_k^2(\bar{u}^2,\bar{w}^2,\bar{d}_k)-\bar{d}_k^T\nabla_{d_k}g_k^2(\bar{u}^2,\bar{w}^2,\bar{d}_k)\Bigg)\in$

$-K\setminus\{0\}$

which contradicts Theorem 6.3.1. Hence $(\bar{a}^1,\bar{b}^1,\bar{a}^2,\bar{b}^2,\bar{v}^1,\bar{v}^2,\bar{\lambda},\bar{c}=0,\bar{d}=0)$ is an ES of **(NMSD)**$_{\bar{\lambda}}$.

Theorem 6.3.3 *(Converse duality)Let* $(\bar{u}^1,\bar{w}^1,\bar{u}^2,\bar{w}^2,\bar{v}^1,\bar{v}^2,\bar{\lambda},\bar{c},\bar{d})$ *be a WES of* ***(NMSD)****. Let*

(i) $\nabla_{c_jc_j}h_j^2(\bar{u}^1,\bar{w}^1,\bar{c}_j)$ *is positive or negative definite* $\forall\ j=1,2,\ldots,k$*;*

(ii) $\nabla_{d_jd_j}g_j^2(\bar{u}^2,\bar{w}^2,\bar{d}_j)$ *is positive or negative definite* $\forall\ j=1,2,\ldots,k$*;*

(iii) $\bar{c}_j\neq 0$ *for some* $j\in\{1,2,\ldots,k\}$ *imply that* $\sum\limits_{j=1}^{k}\alpha_j(\nabla_{a^1a^1}\varphi_j(\bar{u}^1,\bar{w}^1))^T\bar{c}_j\neq 0$ *for all* $\alpha\in K^*$*;*

(iv) $\sum\limits_{j=1}^{k}\alpha_j(\nabla_{a^1a^1}\varphi_j(\bar{u}^1,\bar{w}^1))^T\bar{c}_j\notin$ *span*$\{\nabla_{a^1}\varphi_j(\bar{u}^1,\bar{w}^1)+\nabla_{a^1}h_j^2(\bar{u}^1,\bar{w}^1,\bar{c}_j),\ \nabla_{a^1}\varphi_j(\bar{u}^1,\bar{w}^1)+$

$\nabla_{c_j} h_j^2(\bar{u}^1, \bar{w}^1, \bar{c}_j),\ j = 1, 2, \ldots, k\} \setminus \{0\}$, for all $\alpha \in K^*$;

(v) the set of vectors $\{\nabla_{a^2}\phi_j(\bar{u}^2, \bar{w}^2) - \bar{v}_j^2 + \nabla_{a^2} g_j^2(\bar{u}^2, \bar{w}^2, \bar{d}_j), \nabla_{a^2}\phi_j(\bar{u}^2, \bar{w}^2) - \bar{v}_j^2 + \nabla_{d_j} g_j^2(\bar{u}^2, \bar{w}^2, \bar{d}_j)\},\ j = 1, 2, \ldots, k$ is LI;

(vi) the set of vectors $\{\nabla_{a^2}\phi_j(\bar{u}^2, \bar{w}^2) - \bar{v}_j^2 + \nabla_{d_j} g_j^2(\bar{u}^2, \bar{w}^2, \bar{d}_j)\},\ j = 1, 2, \ldots, k$ is LI;

(vii) $g_j^2(\bar{u}^2, \bar{w}^2, 0) = g_j^1(\bar{u}^2, \bar{w}^2, 0),\ \nabla_{b^2} g_j^2(\bar{u}^2, \bar{w}^2, 0) = \nabla_{s_j} g_j^1(\bar{u}^2, \bar{w}^2, 0),\ \nabla_{a^2} g_j^2(\bar{u}^2, \bar{w}^2, 0) = \nabla_{d_j} g_j^2(\bar{u}^2, \bar{w}^2, 0),\ h_j^2(\bar{u}^1, \bar{w}^1, 0) = h_j^1(\bar{u}^1, \bar{w}^1, 0)$ and $\nabla_{b^1} h_j^2(\bar{u}^1, \bar{w}^1, 0) = \nabla_{r_j} h_j^1(\bar{u}^1, \bar{w}^1, 0)$, $j = 1, 2, \ldots, k$;

(viii) K is closed convex pointed cone with $R_+^k \subseteq K$.

Then,

*(I) $\exists\ \bar{q}^1 \in Q^1$ and $\bar{q}_j^2 \in Q_j^2$ such that $(\bar{u}^1, \bar{w}^1, \bar{u}^2, \bar{w}^2, \bar{q}^1, \bar{q}^2, \bar{\lambda}, \bar{r} = 0, \bar{s} = 0)$ be feasible solution of **(NMSP)**$_{\bar{\lambda}}$, and*

(II) $P(\bar{u}^1, \bar{w}^1, \bar{u}^2, \bar{w}^2, \bar{\lambda}, \bar{c}, \bar{d}) = T(\bar{u}^1, \bar{w}^1, \bar{u}^2, \bar{w}^2, \bar{\lambda}, \bar{r}, \bar{s})$.

*Further, if the suppositions of Theorem 6.3.1 are satisfied for all feasible solutions of **(NMSD)** and **(NMSP)**$_{\bar{\lambda}}$, then $(\bar{u}^1, \bar{w}^1, \bar{u}^2, \bar{w}^2, \bar{q}^1, \bar{q}^2, \bar{\lambda}, \bar{r} = 0, \bar{s} = 0)$ is an ES of **(NMSP)**$_{\bar{\lambda}}$.*

Proof It follows on the lines of Theorem 6.3.2.

6.4 Special Cases

(i) If we take $N_1 = \phi$ and $M_1 = \phi$, then our primal and dual model reduce to the problem studied by Agarwal et al. [3].

(ii) If we consider $h_j^1(a^1, b^1, r_j) = \frac{1}{2} r_j^T \nabla_{b^1 b^1} \varphi_j(a^1, b^1) r_j,\ g_j^1(a^2, b^2, s_j) = \frac{1}{2} s_j^T \nabla_{b^2 b^2} \phi_j(a^2, b^2) s_j$, $h_j^2(u^1, w^1, c_j) = \frac{1}{2} c_j^T \nabla_{a^1 a^1} \varphi_j(u^1, w^1) c_j,\ g_j^2(u^2, w^2, d_j) = \frac{1}{2} d_j^T \nabla_{a^2 a^2} \phi_j(u^2, w^2) d_j$ and $K = R_+^k$,

   (a) For $C_1 = R_+^{|N_1|},\ C_2 = R_+^{|N_2|},\ C_3 = R_+^{|M_1|},\ C_4 = R_+^{|M_2|},\ k = 1,\ r_j = r,\ s_j = s,\ c_j = c,\ d_j = d,\ q_j^2 = q^2,\ v_j^2 = v^2$ our problem reduces to the problem studied in Agarwal et al. [2].

   (b) For $r_j = r,\ c_j = c,\ k = l,\ Q^1 = \{0\},\ Q_j^2 = \{0\},\ V^1 = \{0\}$ and $V_j^2 = \{0\}$, **(NMSP)** and **(NMSD)** becomes the problem considered in Kailey et al. [74].

   (c) The cases given in Agarwal et al. [2] can also be extracted from our problem.

6.5 Conclusion

This chapter presents a multiobjective higher-order mixed symmetric dual including support functions over arbitrary cones. Duality results are acquired with higher-order K-(F, α, ρ, d)-convexity suppositions.

Conclusion and Future Scope

In this thesis, we have studied duality results for NMFP problems and acquired duality results using generalized convexity. We have formulated a parametric dual model for NMFP problems and obtained usual duality relations under (p, r)-ρ-(η, θ)-invex suppositions. Two types of second-order dual models have been constructed for NMFP problem and achieved results using second-order B-(p, r)-invex functions. After that we established higher-order duality results for NMFP problems. We have also solved a NMFP problem using various optimality conditions. Then we focused on multiobjective VP problems. A parametric second-order dual model for NMFVP problem has been formulated. Duality results have been acquired under second-order (F, α, ρ, d)-pseudoconvexity assumptions. It was quite interesting to study symmetric duality results for MP problems. We have constructed a new pair of multiobjective higher-order symmetric dual programs involving support functions over arbitrary cones. Weak, strong and converse duality theorems under higher-order K-η-convexity assumptions have also been established. A mixed-type higher-order symmetric dual model has been formulated for nondifferentiable MP problems over arbitrary cones. Our work can be used to solve various types of NMFP and MP problems. One can extend the second-order duality results achieved for multiobjective VP problems to higher-order case. Mixed type higher-order symmetric duality results for MP problems can be further extended to fractional MP problems using generalized convexity suppositions. There is wide scope for new researchers to develop algorithms using optimality conditions for solving fractional and multiobjective programming problems.

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