Existence of Fixed Points for Various Mappings in Abstract Spaces

A Thesis

submitted in the partial fulfillment of the requirements for the award of the degree of

Doctor of Philosophy

in

Mathematics

by

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June, 2022

Candidate's Declaration

I hereby declare that the work which is being presented in this thesis entitled **Existence of Fixed Points for Various Mappings in Abstract Spaces**, in partial fulfilment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work carried out during the period **July 27, 2016** to **May 31, 2022** under the supervision of **Dr. Jatinderdeep Kaur**, Associate Professor, School of Mathematics, and **Dr. S.S. Bhatia**, Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India.

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It is certified that the above statement made by the candidate is correct to the best of our knowledge.

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Dedicated
To
my
Family

Acknowledgement

At the end of this journey of Ph.D., I would like thank to all the people who helped me to complete my thesis and leave the unforgettable pictures of some beautiful moments on my mind.

First and above all, I praise **God**, the almighty for providing me with this opportunity and granting me the capability to proceed successfully.

I express my sincere regards and gratitude to my supervisors **Dr. Jatinderdeep Kaur**, Associate Professor, SOM, TIET, Patiala, and **Dr. S.S. Bhatia**, Professor, SOM, TIET, Patiala. Their expert guidance, cool temperament, valuable suggestions, support, advice, and continuous encouragement were key motivations throughout the period of my research work. I have learnt a lot from them, which will surely help me in different stages of my life.

I am thankful to **Dr. Mahesh Kumar Sharma**, Head, School of Mathematics, for providing necessary facilities to carry out this work, and to the Doctoral Committee members for their helpful and valuable advice.

I am also grateful to my friend **Ashish** and the other research scholars of SOM for their timely help. I am also thankful to **Dr. Vivek Sangwan** and the other staff members for their help and co-operation. I am also thankful to my colleagues **Dr. Sanjay Garg** and **Sh. Rajesh Deshwal** for the moral support they provided during my research work.

Words are inadequate in paying regard to my parents. I am forever grateful to my father, whose foresight and values paved the way for a privileged education and helped me to do better each day. I want to thank my mother, whose blessings supported me spiritually throughout my life. I express my deepest gratitude to my brother **Sahil Jain** and my sisters **Ashu Jain** and **Pooja Jain** for their cooperation, unconditional support, and encouragement. It is impossible for me to express in words the loving environment provided by my dear wife **Nisha Jain** and my son **Parv Jain**.

I would also like to thank all those persons who were involved directly or indirectly in getting this work done successfully.

I would like to pay high regard to **Saraswati**, the Goddess of knowledge, who gave me some intellect and wisdom to reach where I am today.

June, 2022 *(Kapil Jain)*

List of Symbols

$X = Y,\ X \neq Y$:	Equality and Inequality of sets
$X \cup Y,\ X \cap Y$:	Union and intersection of sets
X^n :	$\underbrace{X \times X \times \cdots \times X}$ (cartesian product of X n-times)
$f(x)$ or fx :	Image of x under f
sup :	Supremum (or least upper bound)
inf :	Infimum (or greatest lower bound)
max :	Maximum
min :	Minimum
ϕ :	empty set
$\mathbb{N}$:	The set of natural numbers
$\mathbb{R}$:	The set of real numbers
$\subset$ or $\subseteq$:	subset of
$\in$:	belong to or belonging to
$\notin$:	does not belong to
$\Rightarrow$:	implies
$\Leftrightarrow$:	logical equivalence
$<$:	is strictly less than
$\leq$:	is less than or equal to
$>$:	is strictly greater than
$\geq$:	is greater than or equal to
$\nleq$:	neither less than nor equal to
$\lvert x \rvert$:	absolute value of x
i.e. :	that is

Preface

The present thesis entitled **Existence of Fixed Points for Various Mappings in Abstract Spaces** comprises certain investigations carried out by me at the School of Mathematics (SOM), Thapar Institute of Engineering and Technology, Patiala (India), under the supervision of **Dr. Jatinderdeep Kaur**, Associate Professor, SOM, TIET, Patiala and **Dr. S.S. Bhatia**, Professor, SOM, TIET, Patiala.

Fixed point theory is an important branch of non-linear analysis. Many problems, occurring in different branches of mathematics, such as differential equations, optimization theory and variational analysis, can be converted into the equation $Tx = x$, where T is some non-linear operator defined on a certain space X. Solutions of this equation are called fixed point of T. Fixed point theory can be classified into three major areas: Metric fixed point theory, Topological fixed point theory and Discrete fixed point theory. The principal findings in these areas are Banach's fixed point theorem, Brouwer's fixed point theorem, and Tarski's fixed point theorem respectively.

Abstract space is a set of elements satisfying certain axioms. In 1906, the French mathematician Fréchet introduced the first abstract space, called metric space. In 1922, Polish mathematician Stefan Banach gave the first fixed point theorem for contraction mappings in metric spaces, and this theorem is famous as the Banach contraction principle. This principle states that *every contraction self-mapping defined on a complete metric space has a unique fixed point.* This result has become one of the most popular and effective tools in solving existence problem in many branches of mathematics.

Banach contraction principle has been generalized in several directions. There are two ways to extend or improve this principle. One way is to extend/improve the condition of contraction mappings, and the second approach is to replace complete metric space with a more general abstract space. In the first direction, there are numerous results in the literature proved by Kannan, Chatterjea, Reich, Hardy and Rogers, Ćirić, Wang *et al.*, Alber and Delabriae, Samet *et al.*, Shahi *et al.*, Wardowski and many more. In the second direction, we have several abstract spaces in the literature such as partial metric spaces, b-metric spaces, cone metric spaces, metric-like spaces, partial b-metric spaces, b-metric-like spaces, generalized metric spaces, F-metric spaces, 2-metric spaces, D-metric spaces, G-metric spaces, GP-metric spaces, generalized b-metric spaces etc.

In the present thesis, we will work in both directions. The present thesis consists of six chapters. **Chapter 1** is about the introduction related to our work. From the literature, a brief about various mappings related to fixed point theory and different abstract spaces are discussed in this chapter. At the end of the chapter, a brief plan of the results presented in the subsequent chapters is given.

In **Chapter 2**, inspired by the concept of b-metric space, G-metric space, and generalized b-metric space, a new abstract space (named generalized G_b-metric space) has been introduced. Some basic concepts and properties of new space have been studied. Various fixed point theorems in the framework of generalized G_b-metric space has been proved. Multiple examples have been presented for the authenticity of the main results. After that, an application of one of our main results has also been given.

In **Chapter 3**, another new class of abstract spaces (named G^*-metric space) has been introduced as a generalization of generalized G_b-metric spaces and GP-metric spaces. Some basic concepts in G^*-metric space have been studied. Some new types of Cauchy sequences have been noticed in this new abstract space. Various examples have also been presented for these new concepts.

Chapter 4 deals with fixed point results for contraction and quasi-contraction type mappings in G^*-metric space. As a consequence of these results, some fixed point theorems have been deduced in the framework of generalized G_b-metric space. Some examples have also been presented in support of the main results and consequences.

In **Chapter 5**, a new class of functions has been introduced. With the help of this new class of functions, some new contractive mappings in b-metric spaces have been introduced. We establish some fixed point results for these new contractive mappings in b-metric spaces. As a consequence of the main results, various fixed point theorems have also been presented. An example has also been illustrated in support of our results. At the end of the chapter, an application has been provided to prove the uniqueness of the solution to a system of simultaneous linear equations.

The aim of **Chapter 6** is to extend the main results of Chapter 5 in the framework of b-metric-like spaces. Also, some common fixed point theorems have presented for weakly compatible mappings. Suitable examples have been provided for the main results. Related to the main results, some corollaries have also been presented at the end of this chapter.

List of Published/Communicated Research Papers

(1) Kapil Jain and Jatinderdeep Kaur, *A Generalization of G-Metric Spaces and Related Fixed Point Theorems*, Mathematical Inequalities and Applications, 22(4), (2019), pp. 1145-1160, Article ID 79, DOI: 10.7153/mia-2019-22-79. (SCIE) (Published)

(2) Kapil Jain and Jatinderdeep Kaur, *Some Fixed Point Results in b-Metric Spaces and b-Metric-Like Spaces with New Contractive Mappings*, Axioms, 10(2), (2021), 15 pages, Article ID 55, DOI: 10.3390/axioms10020055. (SCIE) (Published)

(3) Kapil Jain, Jatinderdeep Kaur and S.S. Bhatia, *A Generalization of GP-Metric Space and Generalized G_b-Metric Space and Related Fixed Point Results*, Journal of Mathematical and Computational Science, 12, (2022), 27 pages, Article ID 132, DOI: 10.28919/jmcs/7294. (SCOPUS) (Published)

Conference Research Papers

(1) Presented a paper entitled "A New Approach to G-metric Space and Related Fixed Point Theorems", in the International conference on Mathematical Analysis and its Application held at Department of Mathematics, South Asian University, New Delhi during December 14-16, 2019.

(2) Presented a paper entitled "Fixed Point Theorems in a New Generalization of G-metric Space" in three days online International Conference on Recent Advances in Computational Mathematics and Engineering, held from 19.03.2021 to 21.03.2021 at B K Birla Institute of Engineering and Technology, Pilani, India.

Contents

CHAPTER 1

Introduction and Basics

1.1. Introduction

In modern mathematics, the theory of fixed points is one of the most powerful tools. Fixed point theory is a beautiful mixture of analysis, topology, and geometry. Fixed point theory has number of applications in many areas of mathematics. For instance, in finding the solution of system of linear equations, in proving the existence of solutions of ordinary differential equation, partial differential equation, integral equation etc. Fixed point theory has applications in economics, game theory, physics, engineering, computer science and many other disciplines also. Theorems concerning the existence and properties of fixed points are known as fixed point theorems.

Definition 1.1.1. Let X be a non-empty set and $T : X \to X$ be a mapping. Then a point $x \in X$ is called a *fixed point* of T if $T(x) = x$.

In other words, a fixed point of a mapping is a point that is mapped to itself, or a fixed point is such a point that remains invariant under the given mapping. In fact, fixed point is the point of intersection of the graph of curve $y = T(x)$ and the line $y = x$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping defined by $f(x) = x^3 + 2x^2 - 2$ for all $x \in \mathbb{R}$. Then $f(1) = 1$, $f(-1) = -1$ and $f(-2) = -2$, i.e., $1, -1$ and -2 are three fixed points of f.

In order to solve a system of equation(s), generally, in mathematics, equation(s) can be converted into the problem of finding the fixed point of a self-mapping on a certain set. For example, if we wish to find a solution to equation $F(x) = 0$, then its solution will be the fixed point of a mapping f, where $f(x) = F(x)+x$. The presence or absence of a fixed point is an intrinsic property of a map. The necessary and sufficient conditions for the existence and uniqueness of such points depend upon the algebraic or topological properties of the map or its domain. Over the last 120 years, the theory of fixed points has been revealed as a very classical and powerful tool in the study of non-linear phenomena.

In 1912, Brouwer [26] proved first fixed point theorem for a topological space which states that *"any continuous self-mapping defined on a closed ball in $\mathbb{R}^n$ has at least one fixed point"*. This result was applicable to finite-dimensional spaces, and forms a base for many fixed point results. In 1930, Schauder [126] proved another result in this field known as *"Schauder fixed point theorem"* stated as: *"If K is compact and convex subset of a Banach space V, then any continuous self-mapping defined on K has at least one fixed point"*. In fact, this result was an extension of Brouwer's theorem and was applicable to finite as well as infinite dimensional spaces. After these two famous theorems, the fixed point theory for topological space was enriched by some more famous results due to Tychonoff [139], Kakutani [74], Bohnenblust, and Karlin [22], Fan [47], Glicksberg [52], etc.

Meanwhile, in 1922, the first fixed point result for a metric space came into light which has been considered as the most celebrated result in the field of functional analysis. This result was introduced by Banach [18], and it is famous as *"Banach Contraction Principle"*. According to this principle, *A contraction self-mapping defined on a complete metric space has a unique fixed point.* Due to the simplicity of this fundamental theorem, it has become a very popular tool for proving existence and uniqueness theorems in various fields of Mathematical Analysis.

The present chapter is introductory in nature and is divided into four sections. In section 1.2, various abstract spaces has been presented. Further, some basic concepts and results concerning these abstract spaces have been presented in this section. Next section deals with various types of mappings related to fixed point theory. In section 1.4, objectives of the present thesis has been given. In the last section of this chapter, a brief chapterwise resume of the results contained in thesis has been included.

1.2. Various Abstract Spaces

An abstract space is a set of elements satisfying certain axioms. The core of abstract space is its set of axioms rather than the nature of its elements. We obtain different types of abstract spaces by choosing different sets of axioms. The idea of using abstract spaces was initiated by Fréchet, in 1906, by introducing Metric Space in his work *Sur quelques points du Calcul fonctionnel.* This initiative of Fréchet has been a great success. Since then, we had a number of abstract spaces in the literature. In the last two decades, these abstract spaces have increased in a rapid way. In this section, definitions of some abstract spaces and related basic concepts have been presented.

1.2.1 Metric Space

Definition 1.2.1. ([84], page 3) For a non-empty set X, a *metric* on X is a function d defined on $X \times X$ such that for all $x,\ y,\ z \in X$, we have:

$(M1)$ d is real-valued, finite and nonnegative;

$(M2)$ $d(x,y) = 0$ if and only if $x = y$;

$(M3)$ $d(x,y) = d(y,x)$ (Symmetry);

$(M4)$ $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle inequality).

The non-empty set X equipped with a metric d defined on it is called a *metric space* and is denoted by (X,d). Geometrically, $d(x,y)$ represents distance between two points x and y on the real line.

For example, the set $\mathbb{R}$ with d defined as usual distance $d(x,y) = |x-y|$, for all $x, y \in X$, is a metric space. In fact, it is so called as usual metric space.

Definition 1.2.2. ([84], page 18) Let (X,d) be a metric space. Then, for $x_0 \in X,\ r > 0$, the *open ball* in (X,d) with center x_0 and radius r is defined as:

$$B(x_0; r) = \{x \in X \mid d(x,x_0) < r\}.$$

A subset A of X is said to be an *open set* if, for each $a \in A$, there exists $r > 0$ such that $B(a; r) \subseteq A$.

Definition 1.2.3. ([84], page 25 and 28) Let (X,d) be a metric space. Then:

(i) A sequence $\{x_n\}$ in X is said to be a *convergent sequence* if there is an $x \in X$

such that $\lim_{n\to+\infty} d(x_n, x) = 0$. In this case, x is called the *limit* of $\{x_n\}$ and is denoted by $\lim_{n\to+\infty} x_n = x$; or, simply, $x_n \to x$.

(ii) A sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if, for each $\epsilon > 0$ there is *an* $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$, for every $m, n > n_0$.

(iii) (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Lemma 1.2.1 *([84], page 26) Let (X, d) be a metric space. Then:*

(i) A convergent sequence in X is bounded, and its limit is unique.

(ii) If $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.

Theorem 1.2.2 *([84], page 29) Every convergent sequence in a metric space is a Cauchy sequence.*

Definition 1.2.4. ([84], page 20) Let (X, d) and (Y, d') be two metric spaces. A mapping $T : X \to Y$ is said to be *continuous at a point* $x_0 \in X$ if, for each $\epsilon > 0$, there is a $\delta > 0$ such that $d'(Tx, Tx_0) < \epsilon$, for all x satisfying $d(x, x_0) < \delta$.

1.2.2 b-Metric Space

By weakening the triangle inequality in metric space, Bakhtin [17] has studied the concept of b-metric space as:

Definition 1.2.5. [17] Let X be a non-empty set. Then, a mapping $d : X \times X \to [0, +\infty)$ is called a *b-metric* if, for all $x, y, z \in X$, the following conditions hold:

$(bm1)$ $d(x, y) = 0$ if and only if $x = y$;

$(bm2)$ $d(x, y) = d(y, x)$;

$(bm3)$ $d(x, y) \leq 2(d(x, z) + d(z, y))$.

The pair (X, d) is called a *b-metric space*.

In 1998, Czerwik [40] presented the concept of b-metric space in the more general form as shown below:

Definition 1.2.6. [40] Let X be a non-empty set and $s \geq 1$ be a given real. Then a mapping $d : X \times X \to [0, +\infty)$ is called a *b-metric* if, for all $x, y, z \in X$, the following conditions hold:

$(b1)$ $d(x, y) = 0$ if and only if $x = y$;

$(b2)$ $d(x, y) = d(y, x)$;

$(b3)$ $d(x, y) \leq s(d(x, z) + d(z, y))$.

The pair (X, d) is called a *b-metric space*. In 2010, Khamsi and Hussain [81] used the notion of b-metric space under the name metric type space.

Example: Let $X = \mathbb{R}$ and $d : X \times X \to [0, +\infty)$ be a mapping defined by $d(x, y) = |x - y|^2$. Then d is a b-metric on X with $s = 2$, but it is not a metric on X as $d(1, 3) = 4 \nleq 2 = d(1, 2) + d(2, 3)$, i.e., $(M4)$ does not hold.

In 2009, Boriceanu [23] introduced the concepts of convergent sequence, Cauchy sequence, and completeness in the framework of b-metric space similar to that of metric space. Following remark is due to Boriceanu *et al.* [24].

Remark 1.2.7. [24] In a b-metric space (X, d), the following assertions hold:

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

The following example illustrates the fact that the mapping b-metric need not be continuous.

Example 1.2.8. [87] Let $X = \mathbb{N} \cup \{+\infty\}$ and let $d : X \times X \to [0, +\infty)$ be defined by

$$d(m,n) = \begin{cases} 0, & \text{if } m = n; \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m,n \text{ is even and the other is even or } +\infty; \\ 5, & \text{if one of } m,n \text{ is odd and the other is odd or } +\infty \text{ and } m \neq n; \\ 2, & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with $s = \frac{5}{2}$. Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$\lim_{n\to+\infty} d(2n, +\infty) = \lim_{n\to+\infty} \frac{1}{2n} = 0,$$

that is, $\lim\limits_{n\to+\infty} x_n = +\infty$, but $\lim\limits_{n\to+\infty} d(x_n, 1) = 2 \neq 5 = d(+\infty, 1)$.

Although b-metric is not a continuous mapping, but we have the following lemma due to Aghajani *et al.* [3].

Lemma 1.2.3 *[3] Let (X, d) be a b-metric space with $s \geq 1$ and suppose that sequences $\{x_n\}$ and $\{y_n\}$ converge to x and $y \in X$, respectively. Then*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to+\infty} d(x_n, y_n) \leq \limsup_{n\to+\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then $\lim\limits_{n\to+\infty} d(x_n, y_n) = 0$.

Moreover, for any $z \in X$,

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to+\infty} d(x_n, z) \leq \limsup_{n\to+\infty} d(x_n, z) \leq s d(x,z).$$

1.2.3 Partial Metric Space

In 1992, Matthews [89] presented the idea of partial metric space. In this space, the usual metric is replaced by a partial metric with a surprising property that the self-distance of any point of the space may not be zero.

Definition 1.2.9. [89] Let X be a non-empty set. Then a mapping $d : X \times X \to [0, +\infty)$ is called a *partial metric* if, for all $x, y, z \in X$,

$(p1)$ $x = y \Leftrightarrow d(x,x) = d(x,y) = d(y,y)$;

$(p2)$ $d(x,x) \leq d(x,y)$;

$(p3)$ $d(x,y) = d(y,x)$;

$(p4)$ $d(x,y) \leq d(x,z) + d(z,y) - d(z,z)$.

Such a pair (X, d) is called a *partial metric space.*

Remark 1.2.10. Every metric space is a partial metric space, but converse is not true. For example, if $X = [0, +\infty)$ and $d(x,y) = max\{x,y\}$, for all $x, y \in X$; then the pair (X, d) is a partial metric space, but not a metric space as $d(1,1) = 1 \neq 0$.

Matthews [89], also introduced the concepts of convergent sequence, Cauchy sequence, and completeness in the framework of partial metric space as follows:

Definition 1.2.11. [89] Let (X, d) be a partial metric space. Then:

(i) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n,m \to +\infty} d(x_n, x_m)$ exists and is finite.

(ii) A sequence $\{x_n\}$ in X *converges to* $x \in X$ if $\lim_{n \to +\infty} d(x_n, x) = d(x,x)$.

(iii) (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ *in* X converges to some $x \in X$ with $\lim_{n,m \to +\infty} d(x_n, x_m) = d(x,x)$.

Remark 1.2.12. In partial metric space, limit of a sequence need not be unique. Also, a convergent sequence need not be a Cauchy sequence.

Example: Let $X = [0, +\infty)$ and $d(x,y) = max\{x,y\}$, for all $x, y \in X$; then the pair (X, d) is a partial metric space. Define a sequence $\{x_n\}$ in X by $x_n = \frac{1}{n+1}$, then

$\lim_{n\to+\infty} d(x_n, 1) = d(1, 1)$ and $\lim_{n\to+\infty} d(x_n, 2) = d(2, 2)$. Thus, the limit of a sequence in partial metric space is not necessarily unique. Now, define another sequence $\{y_n\}$ in X by $y_n = 1$ and 2 according as n is odd and even, respectively. We see that $\lim_{n\to+\infty} d(y_n, 3) = d(3, 3)$, i.e., sequence $\{y_n\}$ is convergent, but $\{y_n\}$ is not a Cauchy sequence.

Definition 1.2.13. [2] Every partial metric p on X generates a T_0-topology τ_p on X with a base of the family of open p-balls $\{B_p(x, r) \mid x \in X,\ r > 0\}$, where $B_p(x, r) = \{y \in X \mid d(x, y) < d(x, x) + r\}$, for all $x \in X$ and $r > 0$.

Lemma 1.2.4 *[2] Let (X, d) be a partial metric space and let $\{x_n\}$ be a sequence in X such that $x_n \to x$ and $d(x, x) = 0$. Then, for every $y \in X$, $\lim_{n\to+\infty} d(x_n, y) = d(x, y)$.*

1.2.4 Metric-Like Space

In 2012, Harandi [60] generalized the concept of partial metric by establishing a new space named metric-like space. It is noticed that in metric-like space, self-distance of a point may be greater than the distance of that point to any other point. Harandi [60] defined metric-like space as:

Definition 1.2.14. [60] Let X be a non-empty set. Then a mapping $d : X \times X \to [0, +\infty)$ is called a *metric-like* if, for all $x, y, z \in X$,

(m_l1) $d(x, y) = 0 \ \Rightarrow\ x = y$;

(m_l2) $d(x, y) = d(y, x)$;

(m_l3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a *metric-like space*. A metric-like on X satisfies all of the conditions of a metric except that $d(x,x)$ may be positive for $x \in X$. Every partial metric space is a metric-like space, but the converse is not true.

Example 1.2.15. [60] Let $X = \{0,1\}$ and define $d : X \times X \to [0,+\infty)$ by

$$d(x,y) = \begin{cases} 2, & \text{if } x = y = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then, the pair (X,d) is a metric-like space, but not a partial metric space as $d(0,0) \nleq d(0,1)$.

Harandi [60], also defined the concepts of convergent sequence, Cauchy sequence, and completeness in the framework of metric-like space similar to that of partial metric space. Also, the concept of open set in the framework of metric-like space is defined as follows:

Definition 1.2.16. [60] Let (X,d) be a metric-like space. Then, for $x_0 \in X$, $r > 0$, the *open ball* in (X,d) with center x_0 and radius r is

$$B(x_0,r) = \{y \in X \mid |d(x_0,y) - d(x_0,x_0)| < r\}.$$

Also, a subset A of X is said to be an *open set* in X if, for each $a \in A$, there exists $r > 0$ such that $B(a,r) \subseteq A$. The family of all open subsets of X is denoted by τ_{ml} which is a T_0-topology on X.

Remark 1.2.17. In a metric-like space, limit of a convergent sequence need not be unique.

Example: Let $X = \{0,1\}$ and $d(x,y) = 1$ for all $x,y \in X$, then (X,d) is a metric-like space, Define a sequence $\{x_n\}$ in X by $x_n = 1$, then $\lim_{n\to+\infty} d(x_n,1) = d(1,1)$ and $\lim_{n\to+\infty} d(x_n,0) = d(0,0)$. Thus, limit of a sequence in metric-like space is not unique.

Also, in a metric-like space, a convergent sequence need not be a Cauchy sequence. Consider metric-like space as in Example 1.2.15. If, for each $n \in \mathbb{N}$,

$$x_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{otherwise,} \end{cases}$$

then $\lim_{n\to+\infty} d(x_n, 1) = d(1,1)$, but $\lim_{n,m\to+\infty} d(x_n, x_m)$ does not exist.

Proposition 1.2.5 *[140] Let $\{x_n\}$ be a sequence in a metric-like space (X, d) such that $x_n \to x$ and $d(x,x) = 0$. Then, for every $y \in X$, $\lim_{n\to+\infty} d(x_n, y) = d(x, y)$.*

1.2.5 Partial b-Metric Space

Shukla [133], in 2014, presented the notion of partial b-metric as an extension of partial metric and b-metric.

Definition 1.2.18. [133] Let X be a non-empty set. Then, a mapping $d : X \times X \to [0, +\infty)$ is called a *partial b-metric* if there exists a number $s \geq 1$ such that for all $x, y, z \in X$,

(p_b1) $x = y \Leftrightarrow d(x,x) = d(x,y) = d(y,y)$;

(p_b2) $d(x,x) \leq d(x,y)$;

(p_b3) $d(x,y) = d(y,x)$;

(p_b4) $d(x,y) \leq s(d(x,z) + d(z,y)) - d(z,z)$.

The pair (X, d) is called a *partial b-metric space.*

Every partial metric space and b-metric space are partial b-metric space, but converse is not true. See the following example:

Example: Let $X = [0, +\infty)$ and $P_b : X \times X \to [0, +\infty)$ is such that $d(x, y) =$

$[max\{x,y\}]^2 + |x-y|^2$, for all $x, y \in X$. Then (X, d) is a partial b-metric space on X with $s = 2 > 1$. But, d is not a b-metric as $d(1,1) = 1 \neq 0$. Also, d is not a partial metric on X as for $x = 1,\ y = 2$ and $z = 3$, $(p4)$ does not hold.

Shukla [133], also presented the concepts of convergent sequence, Cauchy sequence, and completeness in the framework of partial b-metric space analogous to that of partial metric space. Also, the concept of open set in the framework of partial b-metric space is defined as follows:

Definition 1.2.19. [133] Every partial b-metric d on X generates a topology τ_d on X with a base of the family of open d-balls $\{B_d(x,r) \mid\ x \in X,\ r > 0\}$, where $B_d(x,r) = \{y \in X \mid d(x,y) < d(x,x) + r\}$. Obviously, τ_d on partial b-metric space X is T_0, but need not be T_1.

1.2.6 b-Metric-Like Space

Alghamdi *et al.* [8] in 2013 introduced the concept of b-metric-like spaces that generalizes the notions of metric-like space and partial b-metric space.

Definition 1.2.20. [8] Let X be a non-empty set. Then, a mapping $d : X \times X \to [0, +\infty)$ is called a *b-metric-like* if there exists a number $s \geq 1$ such that for all $x, y, z \in X$,

$(b_{ml}1)$ $\quad d(x,y) = 0 \ \Rightarrow\ x = y$;

$(b_{ml}2)$ $\quad d(x,y) = d(y,x)$;

$(b_{ml}3)$ $\quad d(x,y) \leq s(d(x,z) + d(z,y))$.

In this case, the pair (X, d) is called a *b-metric-like space.*

Remark 1.2.21. Every metric-like space or partial b-metric space is a b-metric-like space, but the converse is not true.

Example: Let $X = [0, +\infty)$ and let the function $d : X \times X \to [0, +\infty)$ be defined by $d(x, y) = (x + y)^2$. Then (X, d) is a b-metric-like space with constant $s = 2$. For, $(b_{ml}3)$, let $x, y, z \in X$, then

$$\begin{aligned} d(x,y) &= (x+y)^2 \leq (x+z+z+y)^2 \\ &= (x+z)^2 + (z+y)^2 + 2(x+z)(z+y) \\ &\leq 2[(x+z)^2 + (z+y)^2] = 2[d(x,z) + d(z,y)]. \end{aligned}$$

Clearly, (X, d) is not a metric-like space as $d(1, 1) = 4 \nleq 2 = d(1, 0) + d(0, 1)$, i.e., (m_l3) does not hold. Also, (X, d) is not a partial b-metric space as $d(2, 2) = 16 \nleq 9 = d(2, 1)$, i.e., (p_b2) does not hold.

Alghamdi *et al.* [8], also introduced the concepts of convergent sequence, Cauchy sequence, and completeness in the framework of b-metric-like space as follows:

Definition 1.2.22. [8] Let (X, d) be a b-metric-like space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the *limit* of sequence $\{x_n\}$ if $\lim_{n\to+\infty} d(x, x_n) = d(x, x)$, and we say that the sequence $\{x_n\}$ is *convergent to* x, and denote it by $x_n \to x$ as $n \to +\infty$.

Definition 1.2.23. [8] Let (X, d) be a b-metric-like space.

(i) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n,m\to+\infty} d(x_n, x_m)$ exists and is finite.

(ii) (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ so that $\lim_{n,m\to+\infty} d(x_n, x_m) = d(x, x) = \lim_{n\to+\infty} d(x_n, x)$.

Alghamdi *et al.* [8], also proved the following proposition about the uniqueness of the limit of a sequence.

Proposition 1.2.6 *[8] Let (X, d) be a b-metric-like space with $s \geq 1$, and $\{x_n\}$ be a sequence in X such that for some $x \in X$, $\lim_{n\to+\infty} d(x_n, x) = 0$. Then*

(i) x is unique.

(ii) $\frac{1}{s}d(x,y) \leq \lim_{n\to+\infty} d(x_n, y) \leq sd(x,y)$, for all $y \in X$.

Sen *et al.* [128] gave the following diagram to compare various abstract spaces discussed so far.

Metric space	$\Rightarrow$	Partial Metric space	$\Rightarrow$	Metric-like space
$\Downarrow$		$\Downarrow$		$\Downarrow$
b-Metric space	$\Rightarrow$	Partial b-Metric space	$\Rightarrow$	b-Metric-like space

1.2.7 Generalized Metric Space

After these extensions, in 2015, Jleli and Samet [67] introduced another interesting extension of metric space, as shown below:

Definition 1.2.24. [67] Let X be a non-empty set and $d : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, define the set

$$\mathcal{C}(d, X, x) = \left\{ \{x_n\} \subset X \mid \lim_{n\to+\infty} d(x_n, x) = 0 \right\}.$$

Then d is a *generalized metric* on X if there exists $C > 0$ such that the following conditions hold:

$(D1)$ $(x, y) \in X \times X$, $d(x, y) = 0$ implies $x = y$;

$(D2)$ $d(x, y) = d(y, x)$, for all $(x, y) \in X \times X$;

$(D3)$ if $\{x_n\} \in \mathcal{C}(d, X, x)$, then $d(x, y) \leq C \limsup_{n\to+\infty} d(x_n, y)$.

In this case, the pair (X, d) is a *generalized metric space.*

Example 1.2.25. [78] Let $X = \{0, 1\}$ and $d : X \times X \to [0, +\infty]$ be a mapping defined by $d(0, 0) = 0$ and $d(0, 1) = d(1, 0) = d(1, 1) = +\infty$. Then (X, d) is a generalized metric space.

Definition 1.2.26. [67] Let (X, d) be a generalized metric space. Then:

(i) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$.

(ii) A sequence $\{x_n\}$ in X *converges to* $x \in X$ if $\{x_n\} \in \mathcal{C}(d, X, x)$.

(iii) (X, d) is said to be *complete* if every Cauchy sequence in X is convergent to some element in X.

Proposition 1.2.7 *[67] Let (X, d) be a generalized metric space. Let $\{x_n\}$ be a sequence in X such that $\{x_n\}$ converges to some x in X and $\{x_n\}$ converges to some y in X, then $x = y$.*

1.2.8 Partial JS-Metric Space

In 2019, Asim and Imdad [11] introduced partial JS-metric space to generalize partial metric space as:

Definition 1.2.27. [11] Let X be a non-empty set and $d : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, define the set

$$\mathcal{K}(d, X, x) = \left\{ \{x_n\} \subset X \mid \lim_{n\to+\infty} d(x_n, x) = d(x, x) \right\}.$$

Then d is a *partial* JS-*metric* on X if, for all $x, y \in X$, it satisfies the following conditions:

(D_p1) if $d(x,x) = d(y,y) = d(x,y)$, then $x = y$;

(D_p2) $d(x,x) \leq d(x,y)$;

(D_p3) $d(x,y) = d(y,x)$;

(D_p4) there exists $C > 0$ such that if $(x,y) \in X \times X$, $\{x_n\} \in \mathcal{K}(d,X,x)$,

then $d(x,y) \leq C \limsup_{n\to+\infty} d(x_n,y) + (C-1)d(x,x)$.

The pair (X,d) is said to be *partial* JS-*metric space.*

Example 1.2.28. [11] Let $X = [0,1]$ and $d : X \times X \to [0,+\infty]$ be a mapping defined by

$$d(x,y) = \begin{cases} 20, & \text{if } (x,y) = (0,1) \text{ or } (x,y) = (1,0); \\ |x-y| + 3, & \text{otherwise.} \end{cases}$$

Then (X,d) is a partial JS-metric space.

Definition 1.2.29. [11] Let (X,d) be a partial JS-metric space. Then:

(i) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n,m\to+\infty} d(x_n,x_m)$ exists and is finite.

(ii) A sequence $\{x_n\}$ in X *converges to* $x \in X$ if $\{x_n\} \in \mathcal{K}(d,X,x)$.

(iii) (X,d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ so that $\lim_{n,m\to+\infty} d(x_n,x_m) = d(x,x) = \lim_{n\to+\infty} d(x_n,x)$.

1.2.9 G-Metric Space

In 1992, the notion of D-metric space was introduced by Dhage [42]. Most of the fundamental concepts introduced by Dhage in [42] were later proved inappropriate by Mustafa and Sims [105], Naidu *et al.* ([109], [110]). Alternatively, in 2006 Mustafa and Sims [106] introduced the notion called G-metric space.

Definition 1.2.30. [106] Let X be a non-empty set and $G : X \times X \times X \to [0, +\infty)$ be a function satisfying:

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$;

$(G2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

$(G3)$ $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables);

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the mapping G is called a *generalized metric* or a *G-metric* on X, and the pair (X, G) is a *G-metric space*. All above properties are satisfied when $G(x, y, z)$ is the perimeter of a triangle having vertices x, y, z in $\mathbb{R}^2$. A G-metric space (X, G) is called **symmetric** if $G(x, y, y) = G(x, x, y)$, for all $x, y \in X$.

Example 1.2.31. Let $X = \mathbb{R}$ be the set of real numbers.

Define $G : X \times X \times X \to [0, +\infty)$ as

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \quad \text{for all } x, y, z \in X.$$

Then G is a G-metric on X. Also, (X, G) is a symmetric G-metric space.

Example 1.2.32. [106] Let $X = \{a, b\}$ and define a map $G : X \times X \times X \to [0, +\infty)$ by $G(a, a, a) = G(b, b, b) = 0$, $G(a, a, b) = 1$, $G(a, b, b) = 2$ and $(G4)$ is assumed. Then (X, G) is a nonsymmetric G-metric space.

Following definitions and results are also presented by Mustafa and Sims [106].

Definition 1.2.33. [106] Let (X, G) be a G-metric space. Then, for $x_0 \in X$, $r > 0$, the *G-ball* with center x_0 and radius r is

$$B_G(x_0, r) = \{y \in X \mid G(x_0, y, y) < r\}.$$

Also, a subset A of X is said to be *G-open* if, for each $a \in A$, there exists $r > 0$ such that $B_G(a, r) \subseteq A$. The family of all G-open subsets of X is denoted by $\tau(G)$, called *G-metric topology* on X.

Definition 1.2.34. [106] Let (X, G) be a G-metric space. Then a sequence $\{x_n\}$ in X is said to be *G-convergent* to some $x \in X$ if, for each $\epsilon > 0$, there exists a natural number n_0 such that $G(x_n, x_n, x) < \epsilon$, for all $n \geq n_0$.

Proposition 1.2.8 *[106] Let (X, G) be a G-metric space. Then the following statements are equivalent.*

(i) Sequence $\{x_n\}$ is G-convergent to x in X.

(ii) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$.

(iii) $G(x_n, x, x) \to 0$ as $n \to +\infty$.

(iv) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

Definition 1.2.35. [106] Let (X, G) and (X', G') be two G-metric spaces. Then a mapping $f : X \to X'$ is said to be *G-continuous at* $x_0 \in X$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$G'(f(x_0), f(x), f(x)) < \epsilon \quad \text{whenever} \quad G(x_0, x, x) < \delta, \quad \text{for all} \quad x \in X..$$

Proposition 1.2.9 *[106] Let (X, G) and (X', G') be two G-metric spaces. Then a mapping $f : X \to X'$ is G-continuous at $x \in X$ if and only if whenever sequence $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Definition 1.2.36. [106] Let (X, G) be a G-metric space. Then a sequence $\{x_n\}$, $x_n \in X$, is said to be a *G-Cauchy sequence* if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq n_0$.

Proposition 1.2.10 *[106] Let (X, G) be a G-metric space. Then following statements are equivalent.*

(i) Sequence $\{x_n\}$ is G-Cauchy sequence in (X, G).

(ii) For each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq n_0$.

Proposition 1.2.11 *[106] Every G-convergent sequence in a G-metric space is a G-Cauchy sequence.*

Definition 1.2.37. [106] A G-metric space (X, G) is said to be G-*complete* if every G-Cauchy sequence is a G-convergent sequence.

1.2.10 GP-Metric Space

On the other hand, in 2011, Zand and Nezhad [144] introduced GP-metric space as a generalization of partial metric space and G-metric space.

Definition 1.2.38. [144] Let X be a non-empty set. Let $G : X \times X \times X \to [0, +\infty)$ be a function such that the following conditions hold:

(G_p1) $x = y = z$ if $G(x, y, z) = G(x, x, x) = G(y, y, y) = G(z, z, z)$;

(G_p2) $G(x, x, x) \leq G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$;

(G_p3) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables);

(G_p4) $G(x, y, z) \leq G(x, a, a) + G(a, y, z) - G(a, a, a)$, for all $x, y, z, a \in X$.

Then the function G is called a GP-*metric* on X, and the pair (X, G) is a GP-*metric space*.

Later on, in 2013, Parvaneh *et al.* [114] noticed that GP-metric spaces are symmetric due to (G_p2). Thus, GP-metric spaces are not a generalization of those G-metric spaces, which are nonsymmetric (see Example 1.2.32). In view of this, Parvaneh *et al.* [114] redefined GP-metric space by changing the inequality (G_p2) as:

$$(G_p2') \quad G(x,x,x) \leq G(x,x,y) \leq G(x,y,z), \text{ for all } x,y,z \in X \text{ with } y \neq z;$$

Example 1.2.39. [144] Let $X = [0,+\infty)$ and define a map $G : X \times X \times X \to [0,+\infty)$ as

$$G(x,y,z) = max\{x,y,z\}, \quad \text{for all } \; x,y,z \in X.$$

Then (X,G) is a GP-metric space, but is not a G-metric space as $G(1,1,1) = 1 \neq 0$, i.e., $(G1)$ does not hold.

Also, in [144], the concepts of open ball and convergent sequence are presented as follows:

Definition 1.2.40. [144] Let (X,G) be a GP-metric space. Then, for $x_0 \in X$, $r > 0$, the *GP-ball* with center x_0 and radius r is

$$B_{GP}(x_0,r) = \{y \in X \mid G(x_0,y,y) < r + G(x_0,x_0,x_0)\}.$$

Definition 1.2.41. [144] Let (X,G) be a GP-metric space. Then a sequence $\{x_n\}$, $x_n \in X$, is said to be *GP-convergent* to some $x \in X$ if $\lim\limits_{n,m\to+\infty} G(x_n,x_m,x) = G(x,x,x)$.

The concepts of Cauchy sequence and completeness in GP-metric space are presented by Aydi *et al.* [14] as follows:

Definition 1.2.42. [14] Let (X,G) be a GP-metric space. Then:

(i) A sequence $\{x_n\}$ in X is called a *GP-Cauchy sequence* if $\lim\limits_{n,m\to+\infty} G(x_n,x_m,x_m)$

exists and is finite.

(ii) (X, G) is said to be *GP-complete* if every GP-Cauchy sequence $\{x_n\}$ in X GP-converges to some $x \in X$ with $\lim_{n,m\to+\infty} G(x_n, x_m, x_m) = G(x, x, x)$.

Gajić *et al.* [51] presented the following results about convergent sequences and Cauchy sequences in GP-metric space.

Proposition 1.2.12 *[51] Let (X, G) be a symmetric GP-metric space. Then the following statements are equivalent.*

(i) *Sequence $\{x_n\}$ is GP-convergent to x in X.*

(ii) $G(x_n, x_n, x) \to G(x, x, x)$ *as* $n \to +\infty$.

(iii) $G(x_n, x, x) \to G(x, x, x)$ *as* $n \to +\infty$.

Remark 1.2.43. [51] In above proposition, if (X, G) is a nonsymmetric GP-metric space, then the result does not hold. This can be shown in the following example:

Example: Let $X = \{a, b\}$ and define a map $G : X \times X \times X \to [0, +\infty)$ by $G(a, a, a) = 0, G(b, b, b) = G(a, a, b) = 1,\ G(a, b, b) = 2$ and (G_p3) is assumed. Then (X, G) is a nonsymmetric GP-metric space. Define a sequence $\{x_n\},\ x_n \in X$ by $x_n = a$, for all $n \in \mathbb{N}$. Then

$$G(x_n, x_n, b) = G(a, a, b) \to 1 = G(b, b, b),$$

while

$$G(x_n, b, b) = G(a, b, b) \to 2 \neq G(b, b, b).$$

Proposition 1.2.13 *[51] Let (X, G) be a symmetric GP-metric space. Then the following statements are equivalent.*

(i) $\lim_{n,m\to+\infty} G(x_n, x_m, x_m) = r < +\infty$.

(ii) $\lim_{n,m,l\to+\infty} G(x_n, x_m, x_l) = r < +\infty$.

More about GP-metric spaces can be studied in ([10], [14], [19], [21], [37], [51], [118], [119], [143]).

1.2.11 G_b-Metric Space or Generalized b-Metric Space

In 2013, Aghajani *et al.* [4] initiated the concept of G_b-metric space by combining the concepts of G-metric space and b-metric space as:

Definition 1.2.44. [4] Let X be a non-empty set and $s \geq 1$ be a real number. Let $G : X \times X \times X \to [0, +\infty)$ be a function such that:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$;

(G_b2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G_b3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;

(G_b4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables);

(G_b5) $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)]$, for all $x, y, z, a \in X$.

Then the function G is called a *generalized b-metric* or a *G_b-metric* on X, and the pair (X, G) is a *generalized b-metric space* or *G_b-metric space*. Every G-metric space is a generalized b-metric space with $s = 1$, but the converse is not true in general.

Example 1.2.45. [4] Let $X = \mathbb{R}$ be the set of real numbers. Define $G : X \times X \times X \to [0, +\infty)$ as

$$G(x, y, z) = (|x - y| + |y - z| + |z - x|)^2, \quad \text{for all } x, y, z \in X.$$

Then (X, G) is a generalized b-metric space with $s = 2$. But (X, G) is not a G-metric space as for $x = 1,\ y = 2,\ z = 3$, and $a = 2$, $(G5)$ is not satisfied.

Following definitions and results are also presented by Aghajani *et al.* [4]:

Definition 1.2.46. [4] Let (X, G) be a generalized b-metric space. Then, for $x_0 \in X$ and $r > 0$, the G_b-*ball* with center x_0 and radius r is

$$B_G(x_0, r) = \{y \in X \mid G(x_0, y, y) < r\}.$$

Also, the family of all G_b-balls in X is a base of a topology $\tau(G)$ which is called G_b-*metric topology.*

Definition 1.2.47. [4] Let (X, G) be a generalized b-metric space. Then a sequence $\{x_n\}$, $x_n \in X$, is said to be G_b-*convergent* to some $x \in X$ if, for each $\epsilon > 0$, there exists a natural number n_0 such that $G(x_n, x_m, x) < \epsilon$, for all $n, m \geq n_0$.

Proposition 1.2.14 *[4] Let (X, G) be a generalized b-metric space. Then the following statements are equivalent.*

(i) Sequence $\{x_n\}$ is G_b-convergent to x in X.

(ii) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$.

(iii) $G(x_n, x, x) \to 0$ as $n \to +\infty$.

Definition 1.2.48. [4] Let (X, G) be a generalized b-metric space. Then a sequence $\{x_n\}$, $x_n \in X$, is said to be a G_b-*Cauchy sequence* if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq n_0$.

Proposition 1.2.15 *[4] Let (X, G) be a generalized b-metric space. Then the following statements are equivalent.*

(i) Sequence $\{x_n\}$ is G_b-Cauchy sequence in (X, G).

(ii) For each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq n_0$.

Proposition 1.2.16 *[4] Every G_b-convergent sequence in a generalized b-metric space is a G_b-Cauchy sequence.*

In general, a generalized b-metric need not be a continuous function. The following example illustrates this fact.

Example 1.2.49. [102] Consider b-metric space (X, d) as in Example 1.2.8. Define $G : X \times X \times X \to [0, +\infty)$ as

$$G(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\}, \quad \text{for all} \quad x, y, z \in X.$$

Then (X, G) is a generalized b-metric space with $s = \frac{5}{2}$. Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$\lim_{n\to+\infty} G(2n, +\infty, +\infty) = \lim_{n\to+\infty} \frac{1}{2n} = 0,$$

that is, $\lim_{n\to+\infty} x_n = +\infty$, but $\lim_{n\to+\infty} G(x_n, 1, 1) = 2 \neq 5 = G(+\infty, 1, 1)$.

Although generalized b-metric is not a continuous function (in general), we still have the following result due to Roshan *et al.* [124].

Lemma 1.2.17 *[124] Let (X, G) be a generalized b-metric space with $s \geq 1$.*

(I) Suppose that sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_b-convergent to x, y and $z \in X$, respectively. Then

$$\frac{1}{s^3} G(x, y, z) \leq \liminf_{n\to+\infty} G(x_n, y_n, z_n) \leq \limsup_{n\to+\infty} G(x_n, y_n, z_n) \leq s^3 G(x, y, z).$$

(II) If $z_n = c$ is constant, then

$$\frac{1}{s^2} G(x, y, c) \leq \liminf_{n\to+\infty} G(x_n, y_n, c) \leq \limsup_{n\to+\infty} G(x_n, y_n, c) \leq s^2 G(x, y, c).$$

(III) If $y_n = b$ and $z_n = c$ are constant, then

$$\frac{1}{s} G(x, b, c) \leq \liminf_{n\to+\infty} G(x_n, b, c) \leq \limsup_{n\to+\infty} G(x_n, b, c) \leq sG(x, b, c).$$

More details on generalized b-metric spaces can be studied in ([4],[15], [46], [57], [65], [82], [86], [100], [102], [124], [127], [132]).

1.3. Various Mappings Related to Fixed Point Theory

In this section, we present various types of mappings related to fixed point theory. In 1922, Banach [18] defined contraction for a metric space. This concept has a broad area of applications till today and has been generalized in various abstract spaces.

Definition 1.3.1. [18] Let (X, d) be a metric space, then a mapping $T : X \to X$ is said to be *Lipschitzian* if there exists a constant $\alpha \geq 0$ such that

$$d(T(x), T(y)) \leq \alpha d(x, y), \quad \text{for all} \quad x, y \in X.$$

It is noticed that a Lipschitzian map is evidently continuous. The smallest value of α, for which above condition holds true, is called the *Lipschitz constant* for T; and it is denoted by L. If $L < 1$, the mapping T is said to be a **contraction mapping**; whereas if $L = 1$, the mapping T is **nonexpansive**.

Example: Let $X = \mathbb{R}^2$ equipped with Euclidean metric d and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping such that $T(x) = \frac{1}{2}x$, for all $x \in \mathbb{R}^2$, where $x = (x_1, x_2)$. Then T is a contraction on the set X.

The main drawback of the results related to contraction mapping was the requirement of a continuous mapping. This problem was resolved by Kannan [75] in 1968. He proved fixed point results for discontinuous mappings, although these maps are continuous at their fixed points. His paper was widely used in numerous fixed point theorems for the next two decades. Kannan established the result using the following contractive condition:

$$d(Tx,Ty) \leq \lambda[d(x,Tx) + d(y,Ty)],$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$.

After that, Chatterjea [31] and Reich [121] presented the following results by using modified contractive conditions of Kannan.

Theorem 1.3.1 *[31] Let (X, d) be a complete metric space and T be a self-mapping on X satisfying*

$$d(Tx,Ty) \leq \lambda[d(x,Ty) + d(y,Tx)],$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$. Then T possesses a unique fixed point in X.

Theorem 1.3.2 *[121] Let (X, d) be a complete metric space and T be a self-mapping on X satisfying*

$$d(Tx,Ty) \leq \alpha d(x,Tx) + \beta d(y,Ty) + \gamma d(x,y),$$

for all $x, y \in X$, where α, β, γ are non-negative with $\alpha + \beta + \gamma < 1$. Then T possesses a unique fixed point in X.

Hardy and Rogers [61] in 1973, proved fixed point result with the following contractive condition:

$$d(Tx,Ty) \leq \alpha[d(x,Tx) + d(y,Ty)] + \beta[d(x,Ty) + d(y,Tx)] + \gamma d(x,y),$$

for all $x, y \in X$, where α, β, γ are non-negative with $2\alpha + 2\beta + \gamma < 1$.

Later on, in 1974, Ćirić [36] defined **quasi-contraction** in metric spaces as:

Definition 1.3.2. If (X, d) is a metric space, then a mapping $T : X \to X$ is said to be a *quasi-contraction mapping* if there exists a constant $q \in [0, 1)$ such that,

$$d(Tx, Ty) \leq q\ max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Rhoades [122] presented a comparison among various contractive mappings.

On the other hand, in 1976, Jungck [70] initiated the study of common fixed points. Two self-mappings f and g on the set X have a *common fixed point* x in X if $fx = gx = x$. Jungck [70] generalized the Banach contraction principle by proving a common fixed point theorem for commuting maps. Sessa [129], in 1982, further generalized the concept of commutativity by giving the notion of weakly commuting mappings.

Definition 1.3.3. [129] Two self-mappings T and S on a metric space (X, d) are said to be *weakly commuting* if, $d(TSx, STx) \leq d(Tx, Sx)$, for all x in X.

In 1986, Jungck [71] coined the term compatible mappings in order to generalize the notion of weakly commutativity.

Definition 1.3.4. [71] Two self-mappings T and S on a metric space (X, d) are said to be *compatible* if, $\lim_{n\to+\infty} d(TSx_n, STx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} Sx_n = t$, for some $t \in X$.

Further, Jungck and Rhoades [73], in 1998, weakened the concept of compatible mappings by giving the notion of weakly compatible mappings as:

Definition 1.3.5. [73] Two self-mappings T and S on a set X are said to be *weakly compatible* if, $TSx = STx$ whenever $Tx = Sx$, for some $x \in X$, i.e., if T and S

commute at their coincidence point. Here, x is a *coincidence point* of T and S, and the common value (i.e., $Tx = Sx$) is called a *point of coincidence* of T and S.

Thereafter, many fixed point results have been obtained for weakly compatible mappings, see ([1], [5], [35], [38], [69], [72], [73], [117]).

1.4. Objectives of the study

In specific terms, the objectives of this study are as follows:

Objective 1: To introduce new space(s) and study their properties.

Objective 2: To study the existence and uniqueness of fixed points/common fixed points for various mappings in newly defined space(s) and already known abstract spaces and their applications.

In view of these objectives, new abstract spaces, named generalized G_b-metric space and G^*-metric space, have been introduced in Chapter 2 and Chapter 3 respectively. Some properties of these spaces have also been studied in these chapters. Also, various fixed point results are proved in newly defined spaces in Chapter 2 and Chapter 4. Furthermore, we have introduced a new class of functions in Chapter 5. Using this new class of functions, some fixed point/common fixed point results are also studied in the context of b-metric spaces and b-metric-like spaces in Chapter 5 and Chapter 6 respectively.

1.5. Thesis Organization

Present thesis embodies six chapters. Each chapter is divided into various sections. The number for Theorem 2.4.1 indicates Theorem/Lemma/Proposition/Corollary 1 of section 4 in Chapter 2. The number for Example 4.3.5 indicates Remark/Definition/

Example 5 of Section 3 in Chapter 4. As usual, the numbers in square brackets refer to the references listed in the bibliography.

In **Chapter 2**, inspired by the concepts of b-metric space, G-metric space and generalized b-metric space, a new abstract space (named generalized G_b-metric space) has been introduced. Some basic concepts and properties of new space has been studied. Various fixed point theorems in the framework of generalized G_b-metric space have been proved in this chapter. Multiple examples are presented for the authenticity of the main results. After that, an application of one of our main results is also given.

In **Chapter 3**, another new class of abstract spaces (named G^*-metric space) has been introduced to generalize generalized G_b-metric spaces and GP-metric spaces. Some basic concepts in G^*-metric space have been studied. Some new types of Cauchy sequences have been noticed in this new abstract space. Various examples are also presented for these new concepts.

Chapter 4 deals with fixed point results for contraction and quasi-contraction type mappings in G^*-metric space. As a consequence of these results, some fixed point theorems are deduced in the framework of generalized G_b-metric space. Some examples are also presented in support of the main results and consequences.

In **Chapter 5**, a new class of functions has been introduced. With the help of this new class of functions, some new contractive mappings in b-metric spaces have been introduced. We establish fixed point results for these new contractive mappings in b-metric spaces. As a consequence of the main results, various fixed point theorems are also presented in this chapter. An example is also illustrated in support of our results. Towards the end of the chapter, an application is provided to prove the uniqueness of solution to a system of simultaneous linear equations.

The aim of **Chapter 6** is to extend the main results of Chapter 5 in the framework of b-metric-like spaces. Also, some common fixed point theorems are presented for weakly compatible mappings. Suitable examples are provided for the main results. Some corollaries related to the main results are also presented towards the end of this chapter.

The thesis is concluded by listing the Bibliography with various publications cited in this research work.

CHAPTER 2

Generalized G_b-Metric Space and Related Fixed Point Results

2.1. Introduction

The notion of metric space was introduced by Fréchet [48] in 1906 as an extension of usual distance. By weakening the triangle inequality in metric space, Bakhtin [17] studied the concept of b-metric space as:

Definition 2.1.1. [17] Let X be a non-empty set. Then a mapping $d : X \times X \to [0, +\infty)$ is called a *b-metric* if, for all $x, y, z \in X$, the following conditions hold:

$(bm1)$ $d(x, y) = 0$ if and only if $x = y$;

$(bm2)$ $d(x, y) = d(y, x)$;

$(bm3)$ $d(x, y) \leq 2(d(x, z) + d(z, y))$.

The pair (X, d) is called a *b-metric space.*

In 1998, Czerwik [40] presented the concept of b-metric space in more general form as shown below:

Part of the contents of this chapter has been published in **Mathematical Inequalities and Applications, 22(4), 2019, 1145-1160, Article ID 79, (SCIE).**

Definition 2.1.2. [40] Let X be a non-empty set and $s \geq 1$ be a given real. Then a mapping $d : X \times X \to [0, +\infty)$ is called a *b-metric* if, for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x, y) = 0$ if and only if $x = y$;

(b2) $d(x, y) = d(y, x)$;

(b3) $d(x, y) \leq s(d(x, z) + d(z, y))$.

The pair (X, d) is called a *b-metric space*. Clearly, the family of b-metric spaces is larger than the family of metric spaces. For $s = 1$, b-metric space is a metric space.

Example 2.1.3. Let $X = \mathbb{R}$ be the set of real numbers. Define $d : X \times X \to [0, +\infty)$ as

$$d(x, y) = |x - y|^2 \quad \text{for all} \quad x, y \in X.$$

Then d is a b-metric with coefficient $s = 2$.

In the last decade, the concept of metric space has been generalized in many ways (see, for detail, [8], [11], [60], [67], [68], [89], [133]). In 1992, the notion of D-metric space was introduced by Dhage [42]. Most of the fundamental concepts introduced by Dhage in [42] were proved inappropriate by Mustafa and Sims [105], Naidu *et al.* ([109], [110]). Alternatively, in 2006 Mustafa and Sims [106] introduced the notion called G-metric space.

Definition 2.1.4. [106] Let X be a non-empty set and $G : X \times X \times X \to [0, +\infty)$ be a function satisfying the following conditions:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

$(G3)$ $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables);

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the mapping G is called a *generalized metric* or a *G-metric* on X, and the pair (X, G) is called a *G-metric space*.

Example 2.1.5. Let $X = \mathbb{R}$ be the set of real numbers.

Define $G : X \times X \times X \to [0, +\infty)$ as

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \quad \text{for all} \quad x, y, z \in X.$$

Then G is a G-metric on X.

Various fixed point results in G-metric space can be found in ([66], [97], [98], [99], [101], [104], [107]). In 2013, Aghajani *et al.* [4] initiated the concept of G_b-metric space by combining the concepts of G-metric space and b-metric space as:

Definition 2.1.6. [4] Let X be a non-empty set and $s \geq 1$ be a real number. Let $G : X \times X \times X \to [0, +\infty)$ be a function such that:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$;

(G_b2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G_b3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;

(G_b4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables);

(G_b5) $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)]$, for all $x, y, z, a \in X$.

Then the function G is called a *generalized b-metric* or a *G_b-metric* on X, and the pair (X, G) is a *generalized b-metric space* or *G_b-metric space*. Every G-metric space is a generalized b-metric space with $s = 1$, but the converse is not true in general.

Example 2.1.7. [4] Let $X = \mathbb{R}$ be the set of real numbers. Define $G : X \times X \times X \to [0, +\infty)$ as

$$G(x, y, z) = \frac{1}{9}(|x - y| + |y - z| + |z - x|)^2, \quad \text{for all } x, y, z \in X.$$

Then G is a generalized b-metric on X.

The aim of this chapter is to generalize the concept of G_b-metric space of Aghajani *et al.* [4]. This chapter is organized as follows: In section 2.2, the concept of generalized G_b-metric space has been introduced with a suitable example. Also, some basic properties of this space has been studied in this section. Section 2.3 consists of some lemmas which are needed in the main results of this chapter. In section 2.4, some fixed point results have been established in context of newly defined space. Section 2.5 deals with an application of our main result.

2.2. Generalized G_b-Metric Space and its Basic Properties

Aghajani *et al.* [4] introduced the concept of G_b-metric space. **In [64], Jain and Kaur have also used the name G_b-metric space, but for some another abstract space. We, now rename it as generalized G_b-metric space which is defined as:**

Definition 2.2.1. Let X be a non-empty set and $s \geq 1$ be a real number. Let $G : X \times X \times X \to [0, +\infty)$ be a function satisfying the following conditions:

(gG_b1) $G(x, y, z) = 0$ if $x = y = z$;

(gG_b2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(gG_b3) $G(x, x, y) \leq s\, G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;

(gG_b4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetric in all three variables);

(gG_b5) $G(x,y,z) \leq s[G(x,a,a) + G(a,y,z)],$ for all $x,y,z,a \in X$.

Then the function G is called a *generalized G_b-metric* on X, and the pair (X,G) is a *generalized G_b-metric space.* Clearly, every generalized b-metric space is a generalized G_b-metric space, but the converse is not true. The following example depicts a generalized G_b-metric space that is not a generalized b-metric space.

Example 2.2.2. Define a mapping $G : \mathbb{R}\times \mathbb{R}\times \mathbb{R} \to [0,+\infty)$ as :

$$G(x,y,z) = |x-y|^2 + |y-z|^2 + |z-x|^2, \quad \text{for all} \quad x,y,z \in \mathbb{R}.$$

Then $(\mathbb{R},G)$ is a generalized G_b-metric space with $s = 2$, but $(\mathbb{R},G)$ is not a generalized b-metric space. For this, let $x = 1,\ y = 3,\ z = 2$, then $G(x,y,z) = |1-3|^2 + |3-2|^2 + |2-1|^2 = 6$ and $G(x,x,y) = 2|1-3|^2 = 8$.

Thus, $G(x,x,y) \nleq G(x,y,z)$, i.e., (G_b3) does not hold.

We now present some basic properties of generalized G_b-metric space. The proofs of these results are trivial and straightforward.

Proposition 2.2.1 *Let (X,G) be a generalized G_b-metric space. Then mapping $d_G : X\times X \to [0,+\infty)$ defined by*

$$d_G(x,y) = G(x,x,y) + G(x,y,y), \qquad \textit{for all} \quad x,y \in X,$$

is a b-metric on X.

Proposition 2.2.2 *Let (X,G) be a generalized G_b-metric space with constant $s \geq 1$. Then for all $x,y \in X$,*

(i) $G(x,y,z) = 0$ implies $x = y = z$;

(ii) $G(x,y,y) \leq 2sG(x,x,y)$;

(iii) $\frac{2s+1}{2s}G(x,y,y) \leq d_G(x,y) \leq (1+2s)G(x,y,y)$.

Definition 2.2.3. Let (X,G) be a generalized G_b-metric space. Then for $x_0 \in X$, $r > 0$, the *open ball* in (X,G) with center x_0 and radius r is defined as:

$$B_G(x_0,r) = \{y \in X \mid G(x_0,y,y) < r\}.$$

Proposition 2.2.3 *Let (X,G) be a generalized G_b-metric space with constant $s \geq 1$. Then for $x_0 \in X$ and $r > 0$,*

$$B_G(x_0,r) \subseteq B_{d_G}(x_0,(1+2s)r) \subseteq B_G(x_0,(1+2s)r).$$

Definition 2.2.4. Let (X,G) be a generalized G_b-metric space. Then a subset A of X is said to be a *generalized G_b-open set* if, for each $a \in A$, there exists $r > 0$ such that $B_G(a,r) \subseteq A$.

Definition 2.2.5. Let (X,G) be a generalized G_b-metric space. Then a sequence $\{x_n\}$, $x_n \in X$, is said to be a *generalized G_b-convergent sequence* in X if there exists some $x \in X$ such that for each $\epsilon > 0$, there exists a natural number n_0 such that $G(x_n,x_n,x) < \epsilon$, for all $n \geq n_0$.

Proposition 2.2.4 *Let (X,G) be a generalized G_b-metric space. Then the following statements are equivalent.*

(i) *Sequence $\{x_n\}$ is convergent to x in X.*

(ii) $G(x_n,x_n,x) \to 0$ *as* $n \to +\infty$.

(iii) $G(x_n,x,x) \to 0$ *as* $n \to +\infty$.

(iv) $G(x_n,x_m,x) \to 0$ *as* $n,m \to +\infty$.

(v) $d_G(x_n,x) \to 0$ *as* $n \to +\infty$.

Definition 2.2.6. Let (X, G) and (X', G') be two generalized G_b-metric spaces. Then a mapping $f : X \to X'$ is said to be *generalized G_b-continuous at $x_0 \in X$* if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$G'(f(x_0), f(x), f(x)) < \epsilon \quad \text{whenever} \quad G(x_0, x, x) < \delta, \quad \text{for all} \quad x \in X.$$

Proposition 2.2.5 *Let (X, G) and (X', G') be two generalized G_b-metric spaces. Then a mapping $f : X \to X'$ is generalized G_b-continuous at a point $x \in X$ if and only if generalized G_b-convergence of $\{x_n\}$ to x implies generalized G_b-convergence of $\{f(x_n)\}$ to $f(x)$.*

Definition 2.2.7. Let (X, G) be a generalized G_b-metric space. Then a sequence $\{x_n\}$, $x_n \in X$, is said to be a *generalized G_b-Cauchy sequence* if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq n_0$. In this case, we simply say that sequence $\{x_n\}$ is a *Cauchy sequence in* (X, G).

Proposition 2.2.6 *Let (X, G) be a generalized G_b-metric space. Then the following statements are equivalent.*

(i) Sequence $\{x_n\}$ is Cauchy sequence in (X, G).

(ii) For each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq n_0$.

(iii) Sequence $\{x_n\}$ is a Cauchy sequence in b-metric space (X, d_G).

Proposition 2.2.7 *Every generalized G_b-convergent sequence is a generalized G_b-Cauchy sequence.*

Definition 2.2.8. A generalized G_b-metric space (X, G) is said to be *generalized G_b-complete* if every generalized G_b-Cauchy sequence is a generalized G_b-convergent sequence.

Example 2.2.9. Let $X = \mathbb{R}$, and $G : X \times X \times X \to [0, +\infty)$ be defined as :

$$G(x, y, z) = |x - y|^2 + |y - z|^2 + |z - x|^2, \quad \text{for all} \quad x, y, z \in \mathbb{R}.$$

Then $(\mathbb{R}, G)$ is a generalized G_b-metric space with $s = 2$. Further, it is easy to prove that $(\mathbb{R}, G)$ is generalized G_b-complete with the help of Proposition 2.2.4 and Proposition 2.2.6.

2.3. Lemmas

The following lemmas are needed for the proof of the main results of this chapter:

Lemma 2.3.1 *Let (X, G) be a generalized G_b-metric space with constant $s \geq 1$ and let $\{x_n\}$ be any sequence in (X, G). Then*

$$G(x_0, x_k, x_k) \leq s^n \sum_{i=0}^{k-1} G(x_i, x_{i+1}, x_{i+1}),$$

for every $n \in \mathbb{N}$ and $k \in \{1, 2, 3, \cdots, 2^n\}$.

Proof. We shall prove the result by induction. Let $P(n)$ be the statement:

$$G(x_0, x_k, x_k) \leq s^n \sum_{i=0}^{k-1} G(x_i, x_{i+1}, x_{i+1}), \text{ for every } n \in \mathbb{N} \text{ and } k \in \{1, 2, 3, \cdots, 2^n\}.$$

It is easy to prove that $P(0)$ and $P(1)$ are true. Now, we show that $P(n)$ implies $P(n + 1)$.

Case 1: If $k \in \{1, 2, 3, \cdots, 2^n\}$, then using $P(n)$ and $s \geq 1$, we have

$$G(x_0, x_k, x_k) \leq s^n \sum_{i=0}^{k-1} G(x_i, x_{i+1}, x_{i+1}) \leq s^{n+1} \sum_{i=0}^{k-1} G(x_i, x_{i+1}, x_{i+1}).$$

Case 2: If $k \in \{2^n + 1, 2^n + 2, 2^n + 3, \cdots, 2^{n+1}\}$, then using $P(n)$, we get

$$\begin{aligned} G(x_0, x_k, x_k) &\leq s(G(x_0, x_{2^n}, x_{2^n}) + G(x_{2^n}, x_k, x_k)) \\ &\leq s\left(s^n \sum_{i=0}^{2^n-1} G(x_i, x_{i+1}, x_{i+1}) + s^n \sum_{i=2^n}^{k-1} G(x_i, x_{i+1}, x_{i+1})\right) \end{aligned}$$

$$= s^{n+1}\sum_{i=0}^{k-1} G(x_i, x_{i+1}, x_{i+1}),$$

which implies that $P(n+1)$ holds true. □

Lemma 2.3.2 *Let (X, G) be a generalized G_b-metric space with constant $s \geq 1$ and let $\{x_n\}$ be any sequence in (X, G) such that*

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n),$$

for every $n \in \mathbb{N}$ and for some $k \in [0, 1)$, then $\{x_n\}$ is a generalized G_b-Cauchy sequence.

Proof. If $k = 0$, then obviously, $\{x_n\}$ is a generalized G_b-Cauchy sequence. Now, let $k \in (0, 1)$; and on applying induction on the given condition, we get

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1). \tag{2.3.1}$$

Let $m, l \in \mathbb{N}$ with $l \geq 2$ and $p = [\log_2 l]$. Consider

$$\begin{aligned}
& G(x_{m+1}, x_{m+l}, x_{m+l}) \\
\leq\ & sG(x_{m+1}, x_{m+2}, x_{m+2}) + sG(x_{m+2}, x_{m+l}, x_{m+l}) \\
\leq\ & sG(x_{m+1}, x_{m+2}, x_{m+2}) + s^2G(x_{m+2}, x_{m+2^2}, x_{m+2^2}) + s^2G(x_{m+2^2}, x_{m+l}, x_{m+l}) \\
\leq\ & sG(x_{m+1}, x_{m+2}, x_{m+2}) + s^2G(x_{m+2}, x_{m+2^2}, x_{m+2^2}) + s^3G(x_{m+2^2}, x_{m+2^3}, x_{m+2^3}) + \\
& \quad s^3G(x_{m+2^3}, x_{m+l}, x_{m+l}) \\
\leq\ & \cdots \leq \sum_{n=1}^{p} s^n G(x_{m+2^{n-1}}, x_{m+2^n}, x_{m+2^n}) + s^{p+1}G(x_{m+2^p}, x_{m+l}, x_{m+l}) \\
\leq\ & \sum_{n=1}^{p} s^{2n}\left(\sum_{i=m}^{m+2^{n-1}-1} G(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}, x_{2^{n-1}+i+1})\right) + \\
& s^{2(p+1)}\left(\sum_{i=m}^{m+l-2^p-1} G(x_{2^p+i}, x_{2^p+i+1}, x_{2^p+i+1})\right) \\
\leq\ & \sum_{n=1}^{p+1} s^{2n}\left(\sum_{i=m}^{m+2^{n-1}-1} G(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}, x_{2^{n-1}+i+1})\right)
\end{aligned}$$

$$\begin{aligned}
&\leq G(x_0,x_1,x_1)\sum_{n=1}^{p+1} s^{2n}\left(\sum_{i=0}^{2^{n-1}-1} k^{m+2^{n-1}+i}\right)\\
&\leq \frac{G(x_0,x_1,x_1)k^m}{1-k}\sum_{n=1}^{p+1} s^{2n}k^{2^{n-1}}\\
&\leq \frac{G(x_0,x_1,x_1)k^m}{1-k}\sum_{n=1}^{\infty} k^{2n\log_k s+2^{n-1}}.
\end{aligned}$$

Since $\lim_{n\to+\infty}(2n\log_k s+2^{n-1}-n)=+\infty$, so for $\lambda>0$, there exists $n_0\in\mathbb{N}$ such that

$2n\log_k s+2^{n-1}-n>\lambda,\quad$ for all $\;n\geq n_0$;

which implies that

$$k^{2n\log_k s+2^{n-1}}\leq k^{\lambda+n},\quad \text{for all}\;\; n\geq n_0.$$

Hence, the series $\sum_{n=1}^{\infty} k^{2n\log_k s+2^{n-1}}$ is convergent and let its sum be μ, then

$$G(x_{m+1},x_{m+l},x_{m+l})\leq \frac{G(x_0,x_1,x_1)k^m\mu}{1-k}, \tag{2.3.2}$$

for all $m,l\in\mathbb{N}$. But $k\in(0,1)$, so by using Proposition 2.2.6, $\{x_n\}$ is a generalized G_b-Cauchy sequence. □

2.4. Fixed Point Theorems in Context of Generalized G_b-Metric Spaces

Main results of this section are the following theorems:

Theorem 2.4.1 *Let (X,G) be a generalized G_b-complete metric space with constant $s\geq 1$ and let $T:X\to X$ be a mapping such that*

$$G(Tx,Ty,Tz)\leq kG(x,y,z),\qquad for\ all\;\; x,y,z\in X,$$

where $k\in[0,\frac{1}{s})$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$,

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n).$$

On using induction, we get

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1).$$

For $n, m \in \mathbb{N}$ with $n < m$, consider

$$\begin{aligned}
& G(x_n, x_m, x_m) \\
\leq\ & s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)] \\
\leq\ & sG(x_n, x_{n+1}, x_{n+1}) + s^2[G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m)] \\
\leq\ & \cdots \leq sG(x_n, x_{n+1}, x_{n+1}) + s^2G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + s^{m-n}G(x_{m-1}, x_m, x_m) \\
\leq\ & (sk^n + s^2k^{n+1} + \cdots + s^{m-n}k^{m-1})G(x_0, x_1, x_1) \\
\leq\ & \frac{sk^n}{1 - sk}G(x_0, x_1, x_1).
\end{aligned}$$

This implies that $G(x_n, x_m, x_m) \to 0$ as $n, m \to +\infty$.

Hence, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized G_b-complete metric space, therefore, there exists $x' \in X$ such that sequence $\{x_n\}$ converges to x'. Therefore

$$T(x') = T(\lim_{n \to +\infty} x_n) = \lim_{n \to +\infty} T(x_n) = \lim_{n \to +\infty} x_{n+1} = x'.$$

This implies that x' is a fixed point of T.

Let y be another fixed point of T. Then

$$G(x', y, y) = G(Tx', Ty, Ty) \leq kG(x', y, y)$$

which implies that $G(x', y, y) = 0$ as $k \in [0, 1)$, therefore, $x' = y$, that is, x' is a unique fixed point of T. $\square$

Example 2.4.1. Let $X = \mathbb{R}$, and $G : X \times X \times X \to [0, +\infty)$ be defined as :

$$G(x, y, z) = |x - y|^2 + |y - z|^2 + |z - x|^2, \quad \text{for all} \quad x, y, z \in \mathbb{R}.$$

Then (X, G) is a generalized G_b-complete metric space with $s = 2$.

Define a mapping $T : X \to X$ by

$$T(x) = \frac{x}{2}, \quad \text{for all} \quad x \in X.$$

Then

$$\begin{aligned} G(Tx, Ty, Tz) &= G\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \\ &= \left|\frac{x}{2} - \frac{y}{2}\right|^2 + \left|\frac{y}{2} - \frac{z}{2}\right|^2 + \left|\frac{z}{2} - \frac{x}{2}\right|^2 \\ &\leq kG(x, y, z), \end{aligned}$$

where $k = \frac{1}{4} \in [0, \frac{1}{s})$. Also, T has a unique fixed point, namely 0.

Next theorem is an extension of Theorem 2.4.1 as the interval $[0, \frac{1}{s})$ is extended to $[0, 1)$.

Theorem 2.4.2 *Let (X, G) be a generalized G_b-complete metric space with constant $s \geq 1$ and let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \qquad \text{for all} \quad x, y, z \in X, \tag{2.4.2}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n),$$

so by the use of Lemma 2.3.2, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized G_b-complete metric space, therefore, there exists $x' \in X$ such that sequence $\{x_n\}$ converges to x'. Therefore

$$T(x') = T(\lim_{n\to+\infty} x_n) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_{n+1} = x'.$$

This implies that x' is a fixed point of T.

Let y be another fixed point of T. Then

$$G(x', y, y) = G(Tx', Ty, Ty) \leq kG(x', y, y)$$

which implies that $G(x', y, y) = 0$ as $k \in [0, 1)$, therefore, $x' = y$, that is, x' is a unique fixed point of T. □

In the following examples, it has been shown that the conditions of Theorem 2.4.2 are satisfied, but not of Theorem 2.4.1.

Example 2.4.3. Consider the generalized G_b-complete metric space (X, G) as described in Example 2.2.9. Let $T : X \to X$ be a mapping defined by

$$T(x) = \frac{3x}{4}, \quad \text{for all } x \in X.$$

Then

$$\begin{aligned} G(Tx, Ty, Tz) &= G\left(\frac{3x}{4}, \frac{3y}{4}, \frac{3z}{4}\right) \\ &= \left|\frac{3x}{4} - \frac{3y}{4}\right|^2 + \left|\frac{3y}{4} - \frac{3z}{4}\right|^2 + \left|\frac{3z}{4} - \frac{3x}{4}\right|^2 \\ &\leq kG(x, y, z), \end{aligned}$$

where $k = \frac{9}{16} \in [0, 1)$, but $k \notin [0, \frac{1}{s})$. Here, 0 is the only fixed point of T.

Example 2.4.4. Let $X = \{\alpha, \beta, \gamma\}$ and define a mapping $G : X\times X\times X \to [0, +\infty)$ as follows:

$G(x, x, x) = 0, \quad \text{for all} \quad x \in X,$

$G(\alpha, \beta, \beta) = G(\beta, \alpha, \beta) = G(\beta, \beta, \alpha) = G(\alpha, \alpha, \beta) = G(\alpha, \beta, \alpha) = G(\beta, \alpha, \alpha) = 1,$

$G(\alpha, \gamma, \gamma) = G(\gamma, \alpha, \gamma) = G(\gamma, \gamma, \alpha) = G(\alpha, \alpha, \gamma) = G(\alpha, \gamma, \alpha) = G(\gamma, \alpha, \alpha) = 1.2,$

$G(\beta, \gamma, \gamma) = G(\gamma, \beta, \gamma) = G(\gamma, \gamma, \beta) = G(\beta, \beta, \gamma) = G(\beta, \gamma, \beta) = G(\gamma, \beta, \beta) = 3.3,$

$G(x, y, z) = 3.2, \quad \text{for all} \quad x, y, z \in X \quad \text{with} \quad x \neq y \neq z \neq x.$

It is easy to prove that (X, G) is a generalized G_b-complete metric space with constant $s = 1.5$ (here, 1.5 is the smallest possible value of s).

However, it is noticed that, with $x = \beta,\ y = \alpha,\ z = \gamma$,

$$G(x, y, z) \nleq G(x, \alpha, \alpha) + G(\alpha, y, z);$$

thus, G is not a G-metric on X. Also, with $x = \beta,\ y = \gamma,\ z = \alpha$,

$$G(x, y, y) = 3.3 \nleq 3.2 = G(x, y, z);$$

thus, G is not a generalized b-metric on X.

Define a mapping $T : X \to X$ by $T\alpha = \alpha,\ T\beta = \alpha,\ T\gamma = \beta$.

Now, for $k = \frac{5}{6} \in [0, 1)$, it is not difficult to prove that

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \quad \text{for all} \quad x, y, z \in X,$$

and $\frac{5}{6}$ is the smallest value of such k. Here T has a unique fixed point, namely α, however, $k \notin [0, \frac{1}{s})$, but $k \in [0, 1)$.

In Theorem 2.4.2, condition (2.4.2) implies the continuity of mapping T. But, a discontinuous mapping may also have a fixed point as shown in the following theorem.

Theorem 2.4.3 *Let (X, G) be a generalized G_b-complete metric space with constant $s \geq 1$ and let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq k[G(x, y, z)+G(x, Tx, Tx)+G(y, Ty, Ty)+G(z, Tz, Tz)], \quad (2.4.5)$$

for all $x, y, z \in X$, where $k \in [0, \lambda)$ and $\lambda = min\{\frac{1}{4}, \frac{1}{2s}\}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq k[2G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})]$$

which implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{2k}{1 - 2k} G(x_{n-1}, x_n, x_n).$$

As $k < \frac{1}{4}$, therefore, $\frac{2k}{1-2k} < 1$ and hence by Lemma 2.3.2, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized G_b-complete metric space, therefore, there exists $x' \in X$ such that sequence $\{x_n\}$ converges to x'.

Now, Consider

$$\begin{aligned}
&G(x', Tx', Tx') \\
&\leq s[G(x', x_n, x_n) + G(x_n, Tx', Tx')] \\
&\leq sG(x', x_n, x_n) + sk[G(x_{n-1}, x', x') + G(x_{n-1}, x_n, x_n) + 2G(x', Tx', Tx')]
\end{aligned}$$

which gives that

$$(1 - 2ks)G(x', Tx', Tx') \leq sG(x', x_n, x_n) + sk[G(x_{n-1}, x', x') + G(x_{n-1}, x_n, x_n)].$$

Taking $n \to +\infty$, we get

$$G(x', Tx', Tx') = 0$$

which implies that $Tx' = x'$, that is, x' is a fixed point of T.

Let y be another fixed point of T. Then

$$G(x', y, y) = G(Tx', Ty, Ty) \leq kG(x', y, y)$$

which implies that $G(x', y, y) = 0$ as $k < \frac{1}{4}$, therefore, $x' = y$, that is, x' is a unique fixed point of T. □

Now, we provide an example that satisfy the hypothesis of Theorem 2.4.3 but not of Theorem 2.4.2; and also, the mapping involved is discontinuous.

Example 2.4.6. Let $X = [0, 1]$ and define a mapping $G : X \times\ X \times\ X \to [0, +\infty)$ as follows:

$$G(x, y, z) = |x - y|^2 + |y - z|^2 + |z - x|^2 \text{ for all } x, y, z\ X.$$

Clearly, (X, G) is a generalized G_b-complete metric space with constant $s = 2$.

Define a mapping $T : X \to X$ as:

$$T(x) = \begin{cases} \frac{x}{6}, & \text{if } x \in [0,1] - \{\frac{1}{2}\}; \\ 0, & \text{if } x = \frac{1}{2}. \end{cases}$$

Clearly, T is a discontinuous mapping. We now prove that (2.4.5) is true for $k = \frac{1}{5}$.

For this, let $x, y, z \in [0,1] - \{\frac{1}{2}\}$. Then

$$\begin{aligned} & G(Tx, Ty, Tz) = G(\frac{x}{6}, \frac{y}{6}, \frac{z}{6}) \\ & = \frac{1}{36}(|x-y|^2 + |y-z|^2 + |z-x|^2) \\ & \leq \frac{1}{5}[G(x,y,z) + G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)]. \end{aligned}$$

If $x = \frac{1}{2}$ and $y, z \in [0,1] - \{\frac{1}{2}\}$, then

$G(Tx, Ty, Tz) = G(0, \frac{y}{6}, \frac{z}{6}) = \frac{1}{36}(y^2 + |y-z|^2 + z^2) \leq \frac{3}{36}$; and

$G(x, Tx, Tx) = G(\frac{1}{2}, 0, 0) = 2|\frac{1}{2} - 1|^2 = \frac{1}{2}$; therefore,

$$\begin{aligned} & G(Tx, Ty, Tz) \leq \frac{3}{36} < \frac{1}{5} \times \frac{1}{2} \\ & \leq \frac{1}{5}[G(x,y,z) + G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)]. \end{aligned}$$

If $x = y = \frac{1}{2}$ and $z \in [0,1] - \{\frac{1}{2}\}$, then

$G(Tx, Ty, Tz) = G(0, 0, \frac{z}{6}) = \frac{2z^2}{36} \leq \frac{2}{36}$; and

$G(x, Tx, Tx) = G(\frac{1}{2}, 0, 0) = \frac{1}{2}$; therefore,

$$\begin{aligned} & G(Tx, Ty, Tz) \leq \frac{2}{36} < \frac{1}{5} \times \frac{1}{2} \\ & \leq \frac{1}{5}[G(x,y,z) + G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)]. \end{aligned}$$

Thus, conditions of Theorem 2.4.3 are satisfied. Moreover, for $x = y = \frac{1}{2}$ and $z = 0.51$, (2.4.2) does not hold.

The following theorem is a generalization of Theorem 2.4.2 and Theorem 2.4.3.

Theorem 2.4.4 *Let (X, G) be a generalized G_b-complete metric space with constant $s \geq 1$ and let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz), \tag{2.4.7}$$

for all $x, y, z \in X$, where $a+b+c+d < 1$, $s(c+d) < 1$ and $a+b \geq 0$. Then either T has a unique fixed point, or all elements of X are fixed points of T.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) \;&\leq\; aG(x_{n-1}, x_n, x_n) + bG(x_{n-1}, x_n, x_n) + cG(x_n, x_{n+1}, x_{n+1}) \\ &\quad +dG(x_n, x_{n+1}, x_{n+1}), \end{aligned}$$

which implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a+b}{1-(c+d)} G(x_{n-1}, x_n, x_n).$$

Since $a+b+c+d < 1$, $a+b \geq 0$, so by Lemma 2.3.2, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized G_b-complete metric space; therefore, there exists $x' \in X$ such that sequence $\{x_n\}$ converges to x'.

Now, Consider

$$\begin{aligned} &G(x', Tx', Tx') \\ &\leq\; s[G(x', x_n, x_n) + G(x_n, Tx', Tx')] \\ &\leq\; sG(x', x_n, x_n) + saG(x_{n-1}, x', x') + sbG(x_{n-1}, x_n, x_n) + s(c+d)G(x', Tx', Tx'), \end{aligned}$$

which gives that

$$(1-s(c+d))G(x',Tx',Tx') \leq sG(x',x_n,x_n)+s[aG(x_{n-1},x',x')+bG(x_{n-1},x_n,x_n)].$$

Taking limit as $n \to +\infty$, we get

$$G(x',Tx',Tx') = 0,$$

which gives that $Tx' = x'$, that is, x' is a fixed point of T. Now if all elements of X are not fixed points of T, then there exists some $x^* \in X$ such that $Tx^* \neq x^*$.

On putting $x = y = z = x^*$ in (2.4.7), we have

$$0 \leq (b+c+d)\ G(x^*,Tx^*,Tx^*)$$

which implies that

$$b+c+d \geq 0.$$

Thus $a < 1$ as $a+b+c+d < 1$. Let y be another fixed point of T. Then

$$\begin{aligned} G(x',y,y) &= G(Tx',Ty,Ty) \\ &\leq aG(x',y,y) + bG(x',Tx',Tx') + (c+d)G(y,Ty,Ty) \\ &= aG(x',y,y) \end{aligned}$$

which implies that $G(x',y,y) = 0$ as $a < 1$, therefore, $x' = y$, that is, x' is a unique fixed point of T. □

Now, we furnish some examples which show that the hypothesis of Theorem 2.4.4 is satisfied, but not of Theorem 2.4.2 and 2.4.3.

Example 2.4.8. Let $X = \{\alpha,\beta,\gamma,\delta\}$ and define a mapping $G : X \times\ X \times\ X \to [0,+\infty)$ as follows:

$G(x,x,x)=0,\quad \text{for all}\quad x\in X,$

$G(\alpha,\beta,\beta)=G(\beta,\alpha,\beta)=G(\beta,\beta,\alpha)=G(\alpha,\alpha,\beta)=G(\alpha,\beta,\alpha)=G(\beta,\alpha,\alpha)=2,$

$G(\alpha,\gamma,\gamma)=G(\gamma,\alpha,\gamma)=G(\gamma,\gamma,\alpha)=G(\alpha,\alpha,\gamma)=G(\alpha,\gamma,\alpha)=G(\gamma,\alpha,\alpha)=1,$

$G(\alpha,\delta,\delta)=G(\delta,\alpha,\delta)=G(\delta,\delta,\alpha)=G(\alpha,\alpha,\delta)=G(\alpha,\delta,\alpha)=G(\delta,\alpha,\alpha)=1,$

$G(\beta,\gamma,\gamma)=G(\gamma,\beta,\gamma)=G(\gamma,\gamma,\beta)=G(\beta,\beta,\gamma)=G(\beta,\gamma,\beta)=G(\gamma,\beta,\beta)=2.1,$

$G(\beta,\delta,\delta)=G(\delta,\beta,\delta)=G(\delta,\delta,\beta)=G(\beta,\beta,\delta)=G(\beta,\delta,\beta)=G(\delta,\beta,\beta)=1.3,$

$G(\gamma,\delta,\delta)=G(\delta,\gamma,\delta)=G(\delta,\delta,\gamma)=G(\gamma,\gamma,\delta)=G(\gamma,\delta,\gamma)=G(\delta,\gamma,\gamma)=4.3,$

$G(\alpha,\delta,\gamma)=G(\alpha,\gamma,\delta)=G(\delta,\alpha,\gamma)=G(\delta,\gamma,\alpha)=G(\gamma,\alpha,\delta)=G(\gamma,\delta,\alpha)=4.2,$

$G(x,y,z)=5,\quad \text{otherwise}.$

It is easy to prove that (X,G) is a generalized G_b-complete metric space with constant $s=2.5$. However, it is noticed that, with $x=\delta,\ y=\beta,\ z=\gamma$,

$$G(x,y,z)\nleq G(x,\beta,\beta)+G(\beta,y,z);$$

thus, G is not a G-metric on X. Also, with $x=\gamma,\ y=\delta,\ z=\alpha$,

$$G(x,y,y)=4.3\nleq 4.2=G(x,y,z);$$

which means G is not a generalized b-metric on X.

Define a mapping $T:X\to X$ by $T\alpha=\alpha,\ T\beta=\delta,\ T\gamma=\delta,\ T\delta=\alpha$.

Now for $a=0.7,\ b=0.07,\ c=0.08,\ d=0.09$, we have $a+b+c+d<1,\ s(c+d)<1$ and $a+b\geq 0$. Also it is not hard to prove that (2.4.7) holds true. Here, α is the unique fixed point of T.

However, for $x=y=\alpha,\ z=\gamma$, we notice that (2.4.2) does not hold true for any $k\in[0,1)$; and for $x=\alpha,\ y=z=\beta$, (2.4.5) does not hold true for any $k\in[0,\frac{1}{5})$.

Example 2.4.9. Let (X,G) be a generalized G_b-complete metric space as in Example 2.4.8.

Define a mapping $T : X \to X$ by $T\alpha = \alpha,\ T\beta = \beta,\ T\gamma = \gamma,\ T\delta = \delta$.

Then for $a = 1.4,\ b = -0.2,\ c = -3,\ d = -1$, we have $a+b+c+d < 1,\ s(c+d) < 1$ and $a + b \geq 0$. Also, it is easy to see that (2.4.7) holds true, however all elements of X are fixed points of T.

Now, we shall prove some fixed point results for quasi-contraction type mappings in context of generalized G_b-metric space.

Theorem 2.4.5 *Let (X, G) be a generalized G_b-complete metric space with constant $s \geq 1$ and let $T : X \to X$ be a mapping such that*

(i) $G(Tx, Ty, Tz) \leq k \max\{G(x, y, z),\ G(x, Tx, Tx),\ G(y, Ty, Ty),\ G(z, Tz, Tz)\}$,

$for\ all\ x, y, z \in X\ and\ for\ some\ k \in [0, 1)$;

(ii) T *is generalized G_b-continuous*; *or* $sk < 1$;

then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0),\ \text{for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq k \max\{G(x_{n-1}, x_n, x_n),\ G(x_{n-1}, x_n, x_n),\ G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad G(x_n, x_{n+1}, x_{n+1})\} \\ &= kG(x_{n-1}, x_n, x_n) \end{aligned}$$

As $k \in [0, 1)$, so by Lemma 2.3.2, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized G_b-complete metric space, therefore, there exists $x' \in X$ such that

$x_n \to x'$.

Case I: If T is generalized G_b-continuous, then

$T(x') = T(\lim_{n\to+\infty} x_n) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_{n+1} = x'$.

Case II: If $sk < 1$, then, consider

$$\begin{aligned} G(x', Tx', Tx') &\leq s[G(x', x_{n+1}, x_{n+1}) + G(x_{n+1}, Tx', Tx')] \\ &\leq sG(x', x_{n+1}, x_{n+1}) + sk\max\{G(x_n, x', x'),\ G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad G(x', Tx', Tx'),\ G(x', Tx', Tx')\}. \end{aligned}$$

Letting $n \to +\infty$, we have

$$G(x', Tx', Tx') \leq skG(x', Tx', Tx').$$

But $sk < 1$, therefore, $G(x', Tx', Tx') = 0$, which implies that $Tx' = x'$, that is, x' is a fixed point of T. Let y be another fixed point of T. Then, by Proposition 2.2.2,

$$\begin{aligned} G(x', y, y) &= G(Tx', Ty, Ty) \\ &\leq k\max\{G(x', y, y),\ G(x', Tx', Tx'),\ G(y, Ty, Ty),\ G(y, Ty, Ty)\} \\ &= k\max\{G(x', y, y),\ G(x', x', x'),\ G(y, y, y),\ G(y, y, y)\} \\ &= kG(x', y, y) \end{aligned}$$

which gives that $G(x', y, y) = 0$ as $k < 1$, therefore, $x' = y$. Thus T has a unique fixed point. □

Theorem 2.4.6 *Let (X, G) be a generalized G_b-complete metric space with constant $s \geq 1$ and let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq k\max\{G(x, y, z),\ G(x, Ty, Ty),\ G(y, Tx, Tx),\ G(z, Tz, Tz)\},$$

for all $x, y, z \in X$ and for some $k \in [0, \frac{1}{s})$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0) \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} & G(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+1}, x_n) \\ & \leq\; k \max\{G(x_n, x_n, x_{n-1}),\; G(x_n, x_{n+1}, x_{n+1}),\; G(x_n, x_{n+1}, x_{n+1}),\; G(x_{n-1}, x_n, x_n)\} \\ & =\; kG(x_{n-1}, x_n, x_n) \end{aligned}$$

As $k \in [0, 1)$, so by Lemma 2.3.2, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized G_b-complete metric space, therefore, there exists $x' \in X$ such that $x_n \to x'$.

Now,

$$\begin{aligned} G(x', Tx', Tx') &\leq s[G(x', x_{n+1}, x_{n+1}) + G(x_{n+1}, Tx', Tx')] \\ &= s[G(x', x_{n+1}, x_{n+1}) + G(Tx', Tx', x_{n+1})] \\ &\leq sG(x', x_{n+1}, x_{n+1}) + sk \max\{G(x', x', x_n),\; G(x', Tx', Tx'), \\ &\qquad G(x', Tx', Tx'),\; G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

Letting $n \to +\infty$, we have

$$G(x', Tx', Tx') \leq skG(x', Tx', Tx').$$

But $sk < 1$, therefore, $G(x', Tx', Tx') = 0$, which implies that $Tx' = x'$, that is, x'

is a fixed point of T. Let y be another fixed point of T. Then

$$\begin{aligned} G(y,y,x') &= G(Ty,Ty,Tx') \\ &\leq k\max\{G(y,y,x'),\ G(y,Ty,Ty),\ G(y,Ty,Ty),\ G(x',Tx',Tx')\} \\ &= k\max\{G(y,y,x'),\ G(y,y,y),\ G(y,y,y),\ G(x',x',x')\} \\ &= kG(y,y,x') \end{aligned}$$

which gives that $G(y,y,x') = 0$ as $k < 1$, therefore, $y = x'$. Thus T has a unique fixed point. □

2.5. Application

As an application of the fixed point theorem for contractions on a generalized G_b-complete metric space, the following result is provided about the existence and uniqueness of solution for a system of linear equations.

Theorem 2.5.1 *In a system of linear equations*

$$Ax = b, \tag{2.5.1}$$

where $A = [a_{ij}]_{n\times n}$ is a $n \times n$ matrix and $b = [b_i]_{n\times 1}$ is a column vector of constants and $x = [x_i]_{n\times 1}$ is a column matrix of n unknowns, if

$$|a_{ii}+1| + \sum_{j=1, j\neq i}^{n} |a_{ij}| < 1, \qquad \text{for all } \ i = 1,2,\cdots,n. \tag{2.5.2}$$

then the system has a unique solution.

Proof. Let $X = \{[x_i]_{n\times 1} \mid x_i \text{ is real for all } i = 1 \text{ to } n,\ n \text{ being fixed}\}$ and $G : X \times\ X \times\ X \to [0,+\infty)$ be defined as :

$$G(x,y,z) = \max_{i=1}^{n} |x_i - y_i|^2 + \max_{i=1}^{n} |y_i - z_i|^2 + \max_{i=1}^{n} |z_i - x_i|^2,$$

for all $x = [x_i]_{n\times 1},\ y = [y_i]_{n\times 1},\ z = [z_i]_{n\times 1} \in X$. Then clearly (X, G) is a generalized G_b-complete metric space with constant $s = 2$.

Now define a $n \times n$ matrix $C = [c_{ij}]_{n\times n}$ by

$$c_{ij} = \begin{cases} a_{ij} + 1, & \text{if } \ i = j; \\ a_{ij}, & \text{if } \ i \neq j. \end{cases}$$

Then given system (2.5.1) reduces to

$$x = Cx - b. \tag{2.5.3}$$

Also, given condition (2.5.2) becomes

$$\sum_{j=1}^{n} |c_{ij}| < 1, \qquad \text{for all } \ i = 1, 2, \cdots, n. \tag{2.5.4}$$

Now, define a mapping $T : X \to X$ by

$$T(x) = Cx - b, \qquad \text{where } \ x \in X.$$

Now, for $x = [x_i]_{n\times 1},\ \ y = [y_i]_{n\times 1},\ \ z = [z_i]_{n\times 1} \in X$, set $p = Tx,\ \ q = Ty,\ \ r = Tz$ and suppose that $p = [p_i]_{n\times 1},\ \ q = [q_i]_{n\times 1},\ \ r = [r_i]_{n\times 1}$, then

$$p_i = \sum_{j=1}^{n} a_{ij} x_i - b_i, \qquad \text{for all } \ i = 1, 2, \cdots, n, \ \ \text{etc.}$$

Consider

$$\begin{aligned}
& G_b(Tx, Ty, Tz) \\
= \ & \max_{i=1}^{n} |p_i - q_i|^2 + \max_{i=1}^{n} |q_i - r_i|^2 + \max_{i=1}^{n} |r_i - p_i|^2 \\
= \ & \max_{i=1}^{n} \left| \sum_{j=1}^{n} c_{ij}(x_i - y_i) \right|^2 + \max_{i=1}^{n} \left| \sum_{j=1}^{n} c_{ij}(y_i - z_i) \right|^2 + \max_{i=1}^{n} \left| \sum_{j=1}^{n} c_{ij}(z_i - x_i) \right|^2 \\
\leq \ & \max_{i=1}^{n} \left(\sum_{j=1}^{n} |c_{ij}||x_i - y_i| \right)^2 + \max_{i=1}^{n} \left(\sum_{j=1}^{n} |c_{ij}||y_i - z_i| \right)^2 + \max_{i=1}^{n} \left(\sum_{j=1}^{n} |c_{ij}||z_i - x_i| \right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\max_{k=1}^{n}|x_k - y_k|^2 + \max_{k=1}^{n}|y_k - z_k|^2 + \max_{k=1}^{n}|z_k - x_k|^2\right)\max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}|\right)^2 \\
&= G_b(x,y,z)\max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}|\right)^2 \\
&= \alpha G_b(x,y,z),
\end{aligned}
$$

where

$$\alpha = \max_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ij}|\right)^2.$$

By the condition (2.5.4), $\alpha \in [0,1)$, therefore, using Theorem 2.4.2, T has a unique fixed point, and hence, system (2.5.1) has a unique solution. □

CHAPTER 3

Generalization of GP-Metric Space and Generalized G_b-Metric Space

3.1. Introduction

In Chapter 1, various generalizations of metric spaces established in the literature have been presented. These generalizations are b-metric space, partial metric space, partial b-metric space, metric-like space, b-metric-like space etc. Sen *et al.* [128] have presented the process diagram of the various classes of these abstract spaces as under:

Metric space	$\Rightarrow$	Partial metric space	$\Rightarrow$	Metric-like space
$\Downarrow$		$\Downarrow$		$\Downarrow$
b-Metric space	$\Rightarrow$	Partial b-metric space	$\Rightarrow$	b-Metric-like space.

In 2015, Jleli and Samet [67] introduced another interesting extension of metric space as shown below:

Definition 3.1.1. [67] Let X be a non-empty set and $D : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, define the set

[1] Contents of this chapter have been published in **Journal of Mathematical and Computational Science, 12, 2022, 27 pages, Article ID 132, (SCOPUS).**

$$\mathcal{C}(D,X,x) = \left\{ \{x_n\} \subset X \mid \lim_{n\to+\infty} D(x_n,x) = 0 \right\}.$$

Then D is a *generalized metric* on X if there exists $C > 0$ such that the following conditions hold:

$(D1)$ $(x,y) \in X \times X,\ D(x,y) = 0$ implies $x = y$;

$(D2)$ $D(x,y) = D(y,x),$ for all $(x,y) \in X \times X$;

$(D3)$ if $(x,y) \in X \times X$ and $\{x_n\} \in \mathcal{C}(D,X,x),$ then $D(x,y) \leq C \limsup_{n\to+\infty} D(x_n,y).$

In this case, the pair (X,D) is a *generalized metric space.*

Example 3.1.2. [78] Let $X = \{0,1\}$ and $D : X \times X \to [0,+\infty]$ be a mapping defined by $D(0,0) = 0$ and $D(0,1) = D(1,0) = D(1,1) = +\infty$. Then (X,D) is a generalized metric space.

In 1992, Matthews [89] introduced partial metric space as:

Definition 3.1.3. [89] Let X be a non-empty set. Then a mapping $d : X \times X \to [0,+\infty)$ is called a *partial metric* if, for all $x,y,z \in X$,

$(p1)$ $x = y \Leftrightarrow d(x,x) = d(x,y) = d(y,y)$;

$(p2)$ $d(x,x) \leq d(x,y)$;

$(p3)$ $d(x,y) = d(y,x)$;

$(p4)$ $d(x,y) \leq d(x,z) + d(z,y) - d(z,z)$.

The pair (X,d) is called a *partial metric space.*

In 2019, Asim and Imdad [11] introduced partial JS-metric space as a generalization of partial metric space.

Definition 3.1.4. [11] Let X be a non-empty set and $D_p : X \times X \to [0,+\infty]$ be a given mapping. For every $x \in X$, define the set

$$\mathcal{K}(D_p, X, x) = \left\{ \{x_n\} \subset X \mid \lim_{n\to+\infty} D_p(x_n, x) = D_p(x, x) \right\}.$$

Then D_p is a *partial* JS-*metric* on X if, for all $x, y \in X$, it satisfies the following conditions:

(D_p1) if $D_p(x,x) = D_p(y,y) = D_p(x,y)$, then $x = y$;

(D_p2) $D_p(x,x) \leq D_p(x,y)$;

(D_p3) $D_p(x,y) = D_p(y,x)$;

(D_p4) there exists $C > 0$ such that if $(x,y) \in X \times X$, $\{x_n\} \in \mathcal{K}(D, X, x)$,

then $D_p(x,y) \leq C \limsup_{n\to+\infty} D_p(x_n, y) + (C-1)D_p(x,x)$.

The pair (X, D_p) is said to be *partial* JS-*metric space*.

Remark 3.1.5. We notice that partial JS-metric space is not a generalization of generalized metric space (i.e., JS-metric space), as shown in the following example:

Example 3.1.6. Let $X = [0,1]$ and $D : X \times X \to [0, +\infty]$ be a mapping defined by

$$D(x,y) = \begin{cases} 4, & \text{if } x, y \in (0,1]; \\ |x-y|, & \text{otherwise.} \end{cases}$$

Then $(D1)$ and $(D2)$ are obvious. For $(D3)$, let $(x,y) \in X \times X$ and $\{x_n\} \in \mathcal{C}(D, X, x)$, First we prove that $x = 0$. Suppose if $x > 0$, then $\{x_n\} \in \mathcal{C}(D, X, x)$ implies $\lim_{n\to+\infty} D(x_n, x) = 0$. Now as $x > 0$, therefore, there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq n_0$.

Hence, for every $n \geq n_0$, $D(x_n, x) = |x_n - x| = |x|$, and since $x > 0$, therefore, we have that $\lim_{n\to+\infty} D(x_n, x) \neq 0$, a contradiction. Thus, we have $x = 0$. Next, we prove

$(D3)$ by considering the following cases:

Case 1: If $y = 0$, then $D(x, y) = 0$ and, therefore, $(D3)$ holds in this case.

Case 2: If $y \neq 0$, then $D(x, y) = |0 - y| = |y|$ and

$$D(x_n, y) = \begin{cases} |y|, & \text{if } x_n = 0; \\ 4, & \text{if } x_n \neq 0. \end{cases}$$

Therefore, $D(x, y) \leq \limsup_{n \to +\infty} D(x_n, y)$, that is, $(D3)$ holds in this case for $C = 1$.

Hence, (X, D) is a generalized metric space. But, for $x = \frac{1}{2}$ and $y = 0$, $D(x, x) = 4 \nleq \frac{1}{2} = D(x, y)$. Thus (X, D) is not a partial JS-metric space.

Remark 3.1.7. We also notice that generalized metric space is itself an extension of b-metric-like space (and hence an extension of metric space, b-metric space, partial metric space, partial b-metric space, and metric-like space).

Remark 3.1.8. Although generalized metric space is an extension of b-metric-like space, the concepts of convergent sequence and Cauchy sequence (see Definition 1.2.26) in generalized metric space do not reduce to the concepts of convergent sequence and Cauchy sequence (see Definition 1.2.23) in b-metric-like space.

On the other hand, in 2011, Zand and Nezhad [144] introduced GP-metric space as a generalization of partial metric space and G-metric space.

Definition 3.1.9. [144] Let X be a non-empty set. Let $G : X \times X \times X \to [0, +\infty)$ be a function such that the following conditions hold:

(G_p1) $x = y = z$ if $G(x, y, z) = G(x, x, x) = G(y, y, y) = G(z, z, z)$;

(G_p2) $G(x, x, x) \leq G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$;

(G_p3) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables);

(G_p4) $G(x, y, z) \leq G(x, a, a) + G(a, y, z) - G(a, a, a)$, for all $x, y, z, a \in X$.

Then the function G is called a *GP-metric* on X, and the pair (X, G) is a *GP-metric space*. Also, (X, G) is called *symmetric GP-metric space* if $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$; otherwise, it is called *nonsymmetric* or *asymmetric GP-metric space*.

Later on, in 2013, Parvaneh *et al.* [114] noticed that GP-metric spaces are symmetric due to (G_p2). Thus, GP-metric spaces are not a generalization of those G-metric spaces, which are nonsymmetric (see Example 1.2.32). In view of this, Parvaneh *et al.* [114] redefined GP-metric space by changing the inequality (G_p2) as:

$$(G_p2') \quad G(x, x, x) \leq G(x, x, y) \leq G(x, y, z), \text{ for all } x, y, z \in X \text{ with } y \neq z;$$

More about GP-metric spaces can be studied in ([10], [14], [19], [21], [37], [51], [118], [119], [143]). Proposition 1.2.13 has been proved only for symmetric GP-metric space, but we prove that proposition hold true for both symmetric as well as nonsymmetric GP-metric space.

Proposition 3.1.1 *Let (X, G) be a GP-metric space and $\{x_n\}$ be any sequence in X. Then the following statements are equivalent.*

(I) $\displaystyle\lim_{n,m\to+\infty} G(x_n, x_m, x_m) = r < +\infty.$

(II) $\displaystyle\lim_{n,m,l\to+\infty} G(x_n, x_m, x_l) = r < +\infty.$

Proof. (II) implies (I) obviously. Now, we prove that (I) implies (II). Suppose that (I) holds. Then, for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$|G(x_n, x_m, x_m) - r| < \epsilon, \qquad \text{for all } n, m \geq n_0. \tag{3.1.10}$$

Define a set $A = \{(n,l) \mid n,l \in \mathbb{N} \text{ with } x_n \neq x_l\}$ and for each $k \in \mathbb{N}$, define

$$A_k = \{(n,l) \mid n,l \in \mathbb{N} \text{ with } n,l \geq k\}.$$

Now, if for every $k \in \mathbb{N}$, there exist infinitely many pairs $(n,l) \in A_k \cap A$, then by considering all pairs $(n,l) \in A_1 \cap A$ and $m \in \mathbb{N}$, and using (G_p4) and (G_p2'), we have

$$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x_l) - G(x_m, x_m, x_m)$$

which implies

$$0 \leq G(x_m, x_n, x_l) - G(x_m, x_m, x_l) \leq G(x_n, x_m, x_m) - G(x_m, x_m, x_m).$$

Taking limit $n,m,l \to +\infty$ with $(n,l) \in A_1 \cap A$, on both sides, we get

$$0 \leq \lim_{n,m,l \to +\infty, ((n,l) \in A_1 \cap A)} G(x_m, x_n, x_l) - r \leq r - r = 0.$$

Thus, there exists $n_0' \geq n_0$ such that

$$|G(x_m, x_n, x_l) - r| < \epsilon, \qquad \text{for all } \; m \geq n_0' \; \text{ and } \; (n,l) \in A_{n_0'} \cap A.$$

Also, for $(n,l) \in A_{n_0'} - A$ and $m \geq n_0'$, using (3.1.10),

$$|G(x_m, x_n, x_l) - r| < \epsilon.$$

Thus (II) holds. □

Inspired by the work of Jleli and Samet [67], we introduce a new abstract space named G^*-metric space in section 3.2 of this chapter. Further, some relevant examples and some properties of this newly defined abstract space have been studied in this section. Different types of Cauchy sequences have been observed in this space.

3.2. G^*-Metric Space

Definition 3.2.1. Let X be a non-empty set and let $G : X \times X \times X \to [0, +\infty]$ be a mapping. We say that G is a G^*-*metric* on X if there exists $\alpha > 0$ such that for all $x, y, z \in X$, the following conditions hold:

$(Gg1)$ $G(x, y, z) = 0$ implies $x = y = z$;

$(Gg2)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables);

$(Gg3)$ if $\{x_n\} \in C_X(G, x)$, then $G(x, y, z) \leq \alpha \left(\limsup_{n\to+\infty} G(x_n, y, z) + G(x, x, x) \right)$,

where $C_X(G, x) = \left\{ \{x_n\} \subset X \mid \lim_{n,m\to+\infty} G(x_n, x_m, x) = G(x, x, x) < +\infty \right\}$.

In this case, we call the pair (X, G) a G^*-*metric space* with *constant* α.

3.2.1 Some Basic Concepts

In this section, we present some basic concepts in the context of G^*-metric space.

Definition 3.2.2. Let (X, G) be a G^*-metric space. Let $\{x_n\}$ be a sequence in X. If there exists $x \in X$ such that $\{x_n\} \in C_X(G, x)$, that is, $\lim_{n,m\to+\infty} G(x_n, x_m, x) = G(x, x, x) < +\infty$, then we say that sequence $\{x_n\}$ is G^*-*convergent* and G^*-*converges to* x. Denote this by $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$. Also, in this case, we say x is a *limit* of sequence $\{x_n\}$.

We now present some new concepts in G^*-metric space as:

Definition 3.2.3. Let (X, G) be a G^*-metric space. Let $\{x_n\}$ be a sequence in X. Then we say that $\{x_n\}$ is:

(i) G^*-*Cauchy sequence* if there exists a real number r such that for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|G(x_n, x_m, x_l) - r| < \epsilon$, for all $n, m, l > n_0$.

Denote this by $\lim_{n,m,l\to+\infty} G(x_n, x_m, x_l) = r$.

(*ii*) *G^*-Cauchy-*1 *sequence* if there exists a real number r such that for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|G(x_n, x_{n+m}, x_{n+l}) - r| < \epsilon$, for all $n, m, l > n_0$.

Denote this by $\lim_{n,m,l\to+\infty} G(x_n, x_{n+m}, x_{n+l}) = r$.

(*iii*) *G^*-Cauchy-*2 *sequence* if there exists a real number r such that for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|G(x_n, x_m, x_{m+l}) - r| < \epsilon$, for all $n, m, l > n_0$ with $m \geq n$. Denote this by $\lim_{n,m(\geq n),l\to+\infty} G(x_n, x_m, x_{m+l}) = r$.

(*iv*) *G^*-Cauchy-*3 *sequence* if there exists a real number r such that for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|G(x_n, x_{n+m}, x_{n+m+l}) - r| < \epsilon$, for all $n, m, l > n_0$.

Denote this by $\lim_{n,m,l\to+\infty} G(x_n, x_{n+m}, x_{n+m+l}) = r$.

Definition 3.2.4. In Definition 3.2.3, if, in particular, we take $r = 0$, then the concepts in (i), (ii), (iii), and (iv) are called *0-G^*-Cauchy sequence*, *0-G^*-Cauchy-*1 *sequence*, *0-G^*-Cauchy-*2 *sequence*, and *0-G^*-Cauchy-*3 *sequence* respectively.

Remark 3.2.5. From elementary concepts in generalized G_b-metric space, one can prove that the concepts 0-G^*-Cauchy sequence, 0-G^*-Cauchy-1 sequence, 0-G^*-Cauchy-2 sequence, and 0-G^*-Cauchy-3 sequence are equivalent in the context of generalized G_b-metric space.

Definition 3.2.6. Let (X, G) be a G^*-metric space. Then we say that (X, G) is:

(*i*) *G^*-complete* if every G^*-Cauchy sequence $\{x_n\}$ in X is G^*-convergent to some $x \in X$ and $\lim_{n,m,l\to+\infty} G(x_n, x_m, x_l) = G(x, x, x) = \lim_{n,m\to+\infty} G(x_n, x_m, x)$.

(*ii*) *G^*-complete-*1 if every G^*-Cauchy-1 sequence $\{x_n\}$ in X is G^*-convergent to some $x \in X$ and $\lim_{n,m,l\to+\infty} G(x_n, x_{n+m}, x_{n+l}) = G(x, x, x) = \lim_{n,m\to+\infty} G(x_n, x_m, x)$.

(*iii*) *G^*-complete-*2 if every G^*-Cauchy-2 sequence $\{x_n\}$ in X is G^*-convergent to some $x \in X$ and $\lim_{n,m(\geq n),l \to +\infty} G(x_n, x_m, x_{m+l}) = G(x, x, x) = \lim_{n,m \to +\infty} G(x_n, x_m, x)$.

(iv) G^*-*complete*-3 if every G^*-Cauchy-3 sequence $\{x_n\}$ in X is G^*-convergent to some $x \in X$ and $\lim_{n,m,l \to +\infty} G(x_n, x_{n+m}, x_{n+m+l}) = G(x, x, x) = \lim_{n,m \to +\infty} G(x_n, x_m, x)$.

3.2.2 Some Remarks and Examples

The following remarks show that generalized G_b-metric spaces (see Definition 2.2.1) and GP-metric spaces are G^*-metric spaces.

Remark 3.2.7. Every generalized G_b-metric space is a G^*-metric space. Consider a generalized G_b-metric space (X, G). Then $(Gg1)$ and $(Gg2)$ are obvious. For $(Gg3)$, let $(x, y, z) \in X \times X \times X$ and $\{x_n\} \in C_X(G, x)$, then by (gG_b5),

$$\begin{aligned} G(x,y,z) &\leq s(G(x,x_n,x_n) + G(x_n,y,z)) \\ &\leq s\left(\limsup_{n\to+\infty} G(x,x_n,x_n) + \limsup_{n\to+\infty} G(x_n,y,z)\right) \\ &\leq s\left(G(x,x,x) + \limsup_{n\to+\infty} G(x_n,y,z)\right) \\ &= s\left(\limsup_{n\to+\infty} G(x_n,y,z) + G(x,x,x)\right), \end{aligned}$$

that is, $(Gg3)$ holds for $\alpha = s$. Thus, (X, G) is a G^*-metric space.

Remark 3.2.8. Every GP-metric space is a G^*-metric space. Consider a GP-metric space (X, G). Then $(Gg2)$ is obvious, and $(Gg3)$ is easy to check. For $(Gg1)$, let $(x, y, z) \in X \times X \times X$ such that $G(x, y, z) = 0$. Suppose that $y \neq z$, then by (G_p2'), we have $G(x, x, x) \leq G(x, x, y) \leq G(x, y, z) = 0$. Also, then by (G_p4), we have

$$G(y, y, z) \leq G(y, x, x) + G(x, y, z) - G(x, x, x) = 0 + 0 - 0 = 0.$$

So, again by (G_p2'), we have $G(y, y, y) \leq G(y, y, z) = 0$. Similarly, we can prove that

$G(z,z,z) = 0$. Thus, $G(x,y,z) = G(x,x,x) = G(y,y,y) = G(z,z,z) = 0$, therefore, by (G_p1), we have $x = y = z$. Thus, (X,G) is a G^*-metric space.

Remark 3.2.9. In light of this, these classes of abstract space can be represented by the following process diagram:

G-metric space $\Rightarrow$ GP-metric space $\Rightarrow$ G^*-metric space

$\Downarrow$

generalized b-metric space

$\Downarrow$

generalized G_b-metric space $\Rightarrow$ G^*-metric space

Now, we present some examples which assure that G^*-metric space generalizes the generalized G_b-metric space and GP-metric space. First we present an example of a symmetric G^*-metric space. After that, an example of nonsymmetric G^*-metric space is also presented.

Example 3.2.10. Let $X = [0,1]$ and $G : X \times X \times X \to [0,+\infty]$ be a mapping defined by

$$G(x,y,z) = \begin{cases} +\infty, & \text{if at least one of } x,y,z \text{ is } 1; \\ 2, & \text{if } x,y,z \in (\frac{1}{2},1); \\ |x-y| + |y-z| + |z-x|, & \text{otherwise.} \end{cases}$$

Then $(Gg1)$ and $(Gg2)$ are obvious. For $(Gg3)$, let $(x,y,z) \in X \times X \times X$ and $\{x_n\} \in C_X(G,x)$. Then

$$\lim_{n,m\to+\infty} G(x_n,x_m,x) = G(x,x,x) < +\infty. \tag{3.2.11}$$

Clearly, $x \neq 1$. Consider the following two cases:

Case I: If $y = 1$ or $z = 1$, then $(Gg3)$ holds obviously.

Case II: If $y \neq 1$ and $z \neq 1$, then consider two subcases:

Subcase 1: If $x \in [0, \frac{1}{2}]$, then by (3.2.11), $\lim_{n\to+\infty} G(x_n, x_n, x) = 0$, i.e., $\lim_{n\to+\infty} |x_n - x| = 0$, therefore, $\{x_n\}$ is any sequence in X such that $x_n \to x$ in the usual sense.

Also, $G(x, y, z) = |x - y| + |y - z| + |z - x|$ and

$$G(x_n, y, z) = \begin{cases} +\infty, & \text{if } x_n = 1; \\ 2, & \text{if } x_n, y, z \in (\frac{1}{2}, 1); \\ |x_n - y| + |y - z| + |z - x_n|, & \text{otherwise.} \end{cases}$$

Thus, $G(x, y, z) \leq \alpha \left(\limsup_{n\to+\infty} G(x_n, y, z) + G(x, x, x) \right)$ for $\alpha = 1$.

Subcase 2: If $x \in (\frac{1}{2}, 1)$, then by (3.2.11), $\lim_{n\to+\infty} G(x_n, x_n, x) = 2$, therefore, $\{x_n\}$ is any sequence in X such that $x_n \in (\frac{1}{2}, 1)$ for all $n \geq n_0$, for some natural n_0. Now

$$G(x, y, z) = \begin{cases} 2, & \text{if } y, z \in (\frac{1}{2}, 1); \\ |x - y| + |y - z| + |z - x|, & \text{otherwise.} \end{cases}$$

Thus, $G(x, y, z) \leq \alpha \left(\limsup_{n\to+\infty} G(x_n, y, z) + G(x, x, x) \right)$ for $\alpha = 1$.

Hence, (X, G) is a G^*-metric space. But (X, G) is not a generalized G_b-metric space as $G(0.9, 0.9, 0.9) = 2 \neq 0$. Also, (X, G) is not a GP-metric space as for $x = 0.6,\ y = 0.7,\ z = 0.8$, $G(x, y, z) = G(x, x, x) = G(y, y, y) = G(z, z, z) = 2$, but $x \neq y \neq z$, i.e., (G_p1) does not hold.

Example 3.2.12. Let $V = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $X = V \cup \{0\}$. Let $G : X \times X \times X \to [0, +\infty]$ be a mapping such that G satisfies $(Gg2)$ and

$$G(x, y, z) = \begin{cases} x + y + z, & \text{if at least one of } x, y, z \text{ is } 0; \text{ or} \\ & \text{if } x = \frac{1}{n}, y = \frac{1}{n+m}, z = \frac{1}{n+l}, \text{ where } n, m, l \geq 5; \\ 5, & \text{otherwise.} \end{cases}$$

Then $(Gg1)$ is obvious. For $(Gg3)$, let $(x, y, z) \in X \times X \times X$ and $\{x_n\} \in C_X(G, x)$.

Then

$$\lim_{n,m\to+\infty} G(x_n, x_m, x) = G(x, x, x) < +\infty. \tag{3.2.13}$$

Case I: If $x = 0$, then by (3.2.13), $\lim_{n\to+\infty} G(x_n, x_n, x) = 0$, i.e., $\lim_{n\to+\infty} x_n = 0$ (in usual sense). Thus,

$$\begin{aligned} G(x,y,z) &= y+z \\ &= \lim_{n\to+\infty}(x_n+y+z) \\ &\leq \limsup_{n\to+\infty} G(x_n,y,z) \\ &= \limsup_{n\to+\infty} G(x_n,y,z) + G(x,x,x). \end{aligned}$$

Case II: If $x = \frac{1}{k}$, for some $k \in \mathbb{N}$, then by (3.2.13), $\lim_{n\to+\infty} G(x_n, x_n, x) = G(x, x, x) = 5$.

Thus, $x_n \in \left\{\frac{1}{k+j} \mid j = -4,-3,-2,-1,0,1,2,3,4\right\}$ for all $n \in \mathbb{N}$ with $\frac{1}{x_m} - \frac{1}{x_l} \leq 4$ for all $m, l \in \mathbb{N}$. Also, clearly $G(x,y,z) \leq \alpha\left(\limsup_{n\to+\infty} G(x_n,y,z) + G(x,x,x)\right)$ for $\alpha = 1$.

Thus, (X, G) is a G^*-metric space. But (X, G) is not a generalized G_b-metric space as $G(0.5, 0.5, 0.5) = 5 \neq 0$. Also, (X, G) is not a GP-metric space as for $x = \frac{1}{10}$ and $y = \frac{1}{5}$, $G(x,x,x) = 5 \nleq \frac{2}{5} = G(x,x,y)$, i.e., (G_p2') does not hold. Further, we see that (X, G) is a nonsymmetric G^*-metric space, as $G(\frac{1}{5}, \frac{1}{10}, \frac{1}{10}) = \frac{2}{5}$ and $G(\frac{1}{5}, \frac{1}{5}, \frac{1}{10}) = 5$.

3.2.3 Some Results Related to Basic Concepts

We now present some results related to basic concepts in G^*-metric space as follows:

Proposition 3.2.1 *Let (X, G) be a G^*-metric space and $\{x_n\}$ be a sequence in X such that $\{x_n\} \in C_X(G, x)$ for some $x \in X$, then*

(I) $\lim_{n\to+\infty} G(x_n, x_n, x) = G(x, x, x)$;

(II) $\limsup_{n\to+\infty} G(x_n, x, x) \leq 2\alpha G(x, x, x)$.

Proof. (I) is obvious. We now prove (II). Let $n \in \mathbb{N}$, then using $(Gg3)$, we have

$$G(x, x_n, x) \leq \alpha \limsup_{m\to+\infty} G(x_m, x_n, x) + \alpha G(x, x, x).$$

Taking limit supremum $n \to +\infty$ on both sides, we have

$$\begin{aligned}
\limsup_{n\to+\infty} G(x, x_n, x) &\leq \alpha \limsup_{n\to+\infty} \left(\limsup_{m\to+\infty} G(x_m, x_n, x) \right) + \alpha G(x, x, x) \\
&= \alpha \lim_{n,m\to+\infty} G(x_m, x_n, x) + \alpha G(x, x, x) \\
&= \alpha G(x, x, x) + \alpha G(x, x, x) \\
&= 2\alpha G(x, x, x).
\end{aligned}$$

Hence (II) holds. □

Proposition 3.2.2 *Let (X, G) be a G^*-metric space and $\{x_n\}$ be a sequence in X such that $x_n \to x$ implies $G(x, x, x) = 0$. Then sequence $\{x_n\}$ has a unique limit in X.*

Proof. Suppose $x_n \to y$. Since $(y, x, x) \in X \times X \times X$, therefore, by $(Gg3)$, we get

$$\begin{aligned}
G(y, x, x) &\leq \alpha \limsup_{n\to+\infty} G(x_n, x, x) + \alpha G(y, y, y) \\
&\leq 2\alpha^2 G(x, x, x) + \alpha G(y, y, y) \\
&= 0 + 0 = 0.
\end{aligned}$$

Thus, $G(y, x, x) = 0$, i.e., $y = x$. □

Theorem 3.2.3 *In a G^*-metric space,*

(I) *Every G^*-Cauchy sequence is a G^*-Cauchy-1 sequence.*

(II) *Every G^*-Cauchy sequence is a G^*-Cauchy-2 sequence.*

(III) *Every G^*-Cauchy-1 sequence is a G^*-Cauchy-3 sequence.*

(IV) *Every G^*-Cauchy-2 sequence is a G^*-Cauchy-3 sequence.*

(V) *Every G^*-Cauchy sequence is a G^*-Cauchy-3 sequence.*

Proof. Proof of the theorem directly follows from definition 3.2.3. □

However, the reverse implication in all five statements of the above theorem doesn't hold good in general. Consider the following examples:

Example 3.2.14. Consider G^*-metric space as in Example in 3.2.12. Sequence $\{x_n\},\ x_n = \frac{1}{n},\ n \in \mathbb{N}$, is a G^*-Cauchy-1 sequence, but not a G^*-Cauchy sequence.

Example 3.2.15. Let $V = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $X = V \cup \{0\}$. Let $G : X \times X \times X \to [0, +\infty]$ be a mapping such that G satisfies $(Gg2)$ and

$$G(x, y, z) = \begin{cases} x + y + z, & \text{if at least one of } x, y, z \text{ is } 0; \text{ or} \\ & \text{if } x = \frac{1}{n}, y = \frac{1}{m}, z = \frac{1}{m+l}, \text{ where } n, m, l \geq 5 \text{ with } m \geq n; \\ 5, & \text{otherwise.} \end{cases}$$

Then (X, G) will be a G^*-metric space. Here, the sequence $\{x_n\},\ x_n = \frac{1}{n},\ n \in \mathbb{N}$, is a G^*-Cauchy-2 sequence, but not a G^*-Cauchy sequence.

Example 3.2.16. Let $V = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $X = V \cup \{0\}$. Let $G : X \times X \times X \to [0, +\infty]$ be a mapping such that G satisfies $(Gg2)$ and

$$G(x, y, z) = \begin{cases} x + y + z, & \text{if at least one of } x, y, z \text{ is } 0; \text{ or} \\ & \text{if } x = \frac{1}{n}, y = \frac{1}{n+m}, z = \frac{1}{n+m+l}, \text{ where } n, m, l \geq 5; \\ 5, & \text{otherwise.} \end{cases}$$

Then (X, G) will be a G^*-metric space. Here, the sequence $\{x_n\},\ x_n = \frac{1}{n},\ n \in \mathbb{N}$, is a G^*-Cauchy-3 sequence, but not a G^*-Cauchy-1 sequence and not a G^*-Cauchy-2 sequence.

Following remark shows yet another interesting fact in G^*-metric space:

Remark 3.2.17. In a G^*-metric space, a G^*-convergent sequence need not be a G^*-Cauchy sequence. For example, in Example 3.2.12, sequence $\{x_n\},\ x_n = \frac{1}{n},\ n \in \mathbb{N}$, is a G^*-convergent sequence and G^*-converges to 0 as $\lim\limits_{n,m\to+\infty} G\left(\frac{1}{n}, \frac{1}{m}, 0\right) = \lim\limits_{n,m\to+\infty}\left(\frac{1}{n} + \frac{1}{m} + 0\right) = 0 = G(0,0,0)$, but $\{x_n\}$ is a not G^*-Cauchy sequence as

$$G(x_n, x_m, x_l) = \begin{cases} \frac{1}{n} + \frac{1}{m} + \frac{1}{l}, & \text{if } n \geq 5, m \geq n+5, l \geq n+5; \\ 5, & \text{otherwise}; \end{cases}$$

implies that $\lim\limits_{n,m,l\to+\infty} G(x_n, x_m, x_l)$ does not exist.

Theorem 3.2.4 *Let (X, G) be a G^*-metric space.*

(I) If (X, G) is G^*-complete-1, then it is G^*-complete.

(II) If (X, G) is G^*-complete-2, then it is G^*-complete.

(III) If (X, G) is G^*-complete-3, then it is G^*-complete-1.

(IV) If (X, G) is G^*-complete-3, then it is G^*-complete-2.

(V) If (X, G) is G^*-complete-3, then it is G^*-complete.

Proof. (I) Let (X, G) be a G^*-complete-1 metric space. Let $\{x_n\}$ be a G^*-Cauchy sequence in X. Then there exists a real number r such that

$$\lim_{n,m,l\to+\infty} G(x_n, x_m, x_l) = r; \tag{3.2.18}$$

and by Theorem 3.2.3, sequence $\{x_n\}$ is a G^*-Cauchy-1 sequence. Now, as (X, G)

is G^*-complete-1, therefore there exists some $x \in X$ such that

$$r = \lim_{n,m,l \to +\infty} G(x_n, x_{n+m}, x_{n+l}) = G(x,x,x) = \lim_{n,m \to +\infty} G(x_n, x_m, x). \qquad (3.2.19)$$

Thus, by (3.2.18) and (3.2.19), we have

$$\lim_{n,m,l \to +\infty} G(x_n, x_m, x_l) = G(x,x,x) = \lim_{n,m \to +\infty} G(x_n, x_m, x).$$

Hence, (X, G) is G^*-complete.

Proofs of (II), (III), (IV), and (V) are similar. □

CHAPTER 4

Some Fixed Point Results in G^*-Metric Space

4.1. Introduction

The notion of generalized G_b-metric space has been introduced in Chapter 2. Further, in Chapter 3, we have introduced another abstract space named as G^*-metric space and defined as:

Definition 4.1.1. Let X be a non-empty set and let $G : X \times X \times X \to [0, +\infty]$ be a mapping. We say that G is a G^*-*metric* on X if there exists $\alpha > 0$ such that for all $x, y, z \in X$, the following conditions hold:

$(Gg1)$ $G(x, y, z) = 0$ implies $x = y = z$;

$(Gg2)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables);

$(Gg3)$ if $\{x_n\} \in C_X(G, x)$, then $G(x, y, z) \leq \alpha \left(\limsup_{n \to +\infty} G(x_n, y, z) + G(x, x, x) \right)$,

where $C_X(G, x) = \left\{ \{x_n\} \subset X \mid \lim_{n,m \to +\infty} G(x_n, x_m, x) = G(x, x, x) < +\infty \right\}$.

In this case, we call the pair (X, G) a G^*-*metric space* with *constant* α.

The concepts of convergent sequence and Cauchy sequence in the framework of

Contents of this chapter have been published in **Journal of Mathematical and Computational Science, 12, 2022, 27 pages, Article ID 132, (SCOPUS).**

G^*-metric space has also been defined in Chapter 3. In the present chapter, we prove some fixed point results in G^*-metric space. Some consequences of these results are also deduced in the framework of generalized G_b-metric space. Before we deduce these consequences, some lemmas are also proved in generalized G_b-metric space. We begin with the definition of k-contraction in the context of G^*-metric space as follows:

Definition 4.1.2. Let (X, G) be a G^*-metric space, $M : X \to X$ be a given mapping and $k \in (0, 1)$. Then M is a *k-contraction* in (X, G) if

$$G(Mx, My, Mz) \leq kG(x, y, z), \quad \text{for every } \ (x, y, z) \in X \times X \times X.$$

Also, for each $x \in X$, we define

$$\delta(G, M, x) = \sup\{G(M^p(x), M^q(x), M^r(x)) \mid p, q, r \in \mathbb{N}\}.$$

4.2. Fixed Point Results in G^*-Metric Space

The main results of this section are the following theorems:

Theorem 4.2.1 *Let (X, G) be a G^*-complete metric space and $M : X \to X$ a k-contraction mapping for some $k \in (0, 1)$. Also, let there exists $x_0 \in X$ such that $\delta(G, M, x_0) < +\infty$, then M has a fixed point (say θ). Further, if $M(\theta') = \theta'$ with $G(\theta, \theta, \theta') < +\infty$, then $\theta' = \theta$.*

Proof. Let $n \in \mathbb{N}$. Since M is a k-contraction, therefore, for all $p, q, r \in \mathbb{N}$, we have

$$G(M^{n+p}(x_0), M^{n+q}(x_0), M^{n+r}(x_0)) \ \leq kG(M^{n-1+p}(x_0), M^{n-1+q}(x_0), M^{n-1+r}(x_0)).$$

Taking supremum over all $p, q, r \in \mathbb{N}$ on both sides, we have

$$\delta(G, M, M^n(x_0)) \leq k\delta(G, M, M^{n-1}(x_0)).$$

and by the use of induction, we get

$$\delta(G, M, M^n(x_0)) \leq k^n \delta(G, M, x_0).$$

Now, for all $n, m, l \in \mathbb{N}$, with $n \leq m \leq l$, we have

$$G(M^{n+1}(x_0), M^{m+1}(x_0), M^{l+1}(x_0)) \leq \delta(G, M, M^n(x_0)) \leq k^n \delta(G, M, x_0).$$

Since $\delta(G, M, x_0) < +\infty$ and $k \in (0, 1)$, therefore,

$$\lim_{n,m,l \to +\infty} G(M^{n+1}(x_0), M^{m+1}(x_0), M^{l+1}(x_0)) = 0.$$

Hence, $\{M^n(x_0)\}$ is a G^*-Cauchy sequence in (X, G), but (X, G) is G^*-complete metric space therefore there exists $\theta \in X$ such that $M^n(x_0) \to \theta$ and

$$\lim_{n,m \to +\infty} G(M^n(x_0), M^m(x_0), \theta) = G(\theta, \theta, \theta) = \lim_{n,m,l \to +\infty} G(M^n(x_0), M^m(x_0), M^l(x_0)) = 0. \tag{4.2.1}$$

Now, as M is a k-contraction, therefore,

$$G(M(\theta), M(\theta), M(\theta)) \leq kG(\theta, \theta, \theta) = 0. \tag{4.2.2}$$

Again as M is a k-contraction, therefore, for all $n, m, l \in \mathbb{N}$, we have

$$G(M^{n+1}(x_0), M^{m+1}(x_0), M(\theta)) \leq kG(M^n(x_0), M^m(x_0), \theta).$$

Taking limit $n, m \to +\infty$ and using (4.2.1) and (4.2.2), we get

$$\lim_{n,m \to +\infty} G(M^{n+1}(x_0), M^{m+1}(x_0), M(\theta)) = 0 = G(M(\theta), M(\theta), M(\theta)),$$

which implies that $M^n(x_0) \to M(\theta)$. Thus in view of Proposition 3.2.2, $M(\theta) = \theta$, that is, θ is a fixed point of M.

Now, suppose that $\theta' \in X$ such that $M(\theta') = \theta'$ and $G(\theta, \theta, \theta') < +\infty$, then

$$G(\theta, \theta, \theta') = G(M\theta, M\theta, M\theta') \leq kG(\theta, \theta, \theta'),$$

and since $G(\theta,\theta,\theta') < +\infty$ and $k \in (0,1)$, therefore, $G(\theta,\theta,\theta') = 0$ which implies that $\theta = \theta'$. □

Now, we present the following fixed point result.

Theorem 4.2.2 *Let (X,G) be a G^*-complete metric space with constant α and let $M : X \to X$ be a mapping such that*

$$(T1) \quad G(Mx,My,Mz) \le \beta \max\{G(x,y,z),\ G(x,Mx,Mx),\ G(y,My,My),$$
$$G(z,Mz,Mz),\ G(x,My,My),\ G(y,Mz,Mz),\ G(z,Mx,Mx)\},$$
$$for\ all\ x,y,z \in X,\ and\ for\ some\ \beta \in [0,1)\ with\ \alpha\beta < 1;$$

$(T2)$ *there exists $x_0 \in X$ such that $\delta(G,M,x_0) < +\infty$;*

then sequence $\{M^n(x_0)\}$ G^*-converges to some $\theta \in X$. Also, If

$$\limsup_{n\to+\infty} G(M^n(x_0), M(\theta), M(\theta)) < +\infty, \tag{4.2.3}$$

then θ is a fixed point of M. Further, if θ' is another fixed point of M with $G(\theta,\theta,\theta') < +\infty$, $G(\theta,\theta',\theta') < +\infty$, and $G(\theta',\theta',\theta') < +\infty$, then $\theta = \theta'$.

Proof. Let $n \in \mathbb{N}$. By $(T1)$, for all $p,q,r \in \mathbb{N}$, we have

$$G(M^{n+p}(x_0), M^{n+q}(x_0), M^{n+r}(x_0))$$
$$\le \beta \max\{G(M^{n-1+p}(x_0), M^{n-1+q}(x_0), M^{n-1+r}(x_0)),\ G(M^{n-1+p}(x_0), M^{n+p}(x_0), M^{n+p}(x_0)),$$
$$G(M^{n-1+q}(x_0), M^{n+q}(x_0), M^{n+q}(x_0)),\ G(M^{n-1+r}(x_0), M^{n+r}(x_0), M^{n+r}(x_0))$$
$$G(M^{n-1+p}(x_0), M^{n+q}(x_0), M^{n+q}(x_0)),\ G(M^{n-1+q}(x_0), M^{n+r}(x_0), M^{n+r}(x_0)),$$
$$G(M^{n-1+r}(x_0), M^{n+p}(x_0), M^{n+p}(x_0))\},$$

which implies that

$$\delta(G, M, M^n(x_0)) \leq \beta\delta(G, M, M^{n-1}(x_0)).$$

By the use of induction, we get

$$\delta(G, M, M^n(x_0)) \leq \beta^n\delta(G, M, x_0).$$

Now, for all $n, m, l \in \mathbb{N}$, with $n \leq m \leq l$, we have

$$G(M^{n+1}(x_0), M^{m+1}(x_0), M^{l+1}(x_0)) \leq \delta(G, M, M^n(x_0)) \leq \beta^n\delta(G, M, x_0).$$

Since $\delta(G, M, x_0) < +\infty$ and $\beta \in [0, 1)$, therefore,

$$\lim_{n,m,l\to+\infty} G(M^{n+1}(x_0), M^{m+1}(x_0), M^{l+1}(x_0)) = 0.$$

Hence, $\{M^n(x_0)\}$ is a G^*-Cauchy sequence in (X, G), but (X, G) is a G^*-complete metric space therefore there exists $\theta \in X$ such that $M^n(x_0) \to \theta$ and

$$\lim_{n,m\to+\infty} G(M^n(x_0), M^m(x_0), \theta) = G(\theta, \theta, \theta) = \lim_{n,m,l\to+\infty} G(M^n(x_0), M^m(x_0), M^l(x_0)) = 0. \tag{4.2.4}$$

Also, by Proposition 3.2.1,

$$\limsup_{n\to+\infty} G(M^n(x_0), \theta, \theta) \leq 2\alpha G(\theta, \theta, \theta) = 0. \tag{4.2.5}$$

Now, using $(T1)$, we have

$$\begin{aligned} G(M^n(x_0), M(\theta), M(\theta)) \ &\leq \ \beta \max\{G(M^{n-1}(x_0), \theta, \theta),\ G(M^{n-1}(x_0), M^n(x_0), M^n(x_0)), \\ &\quad G(\theta, M(\theta), M(\theta)),\ G(M^{n-1}(x_0), M(\theta), M(\theta)),\ G(\theta, M^n(x_0), M^n(x_0))\}. \end{aligned}$$

Taking $\limsup n \to +\infty$ on both sides and using (4.2.5), (4.2.4), and (4.2.3), we have

$$\limsup_{n\to+\infty} G(M^n(x_0), M(\theta), M(\theta)) \ \leq \beta G(\theta, M(\theta), M(\theta)). \tag{4.2.6}$$

Now, by $(Gg3)$, (4.2.6), and (4.2.4), we have

$$\begin{aligned} G(\theta, M(\theta), M(\theta)) &\leq \alpha \limsup_{n\to+\infty} G(M^n(x_0), M(\theta), M(\theta)) + \alpha G(\theta,\theta,\theta) \quad (4.2.7) \\ &\leq \alpha\beta G(\theta, M(\theta), M(\theta)). \end{aligned}$$

Since $\alpha\beta < 1$ and $G(\theta, M(\theta), M(\theta)) < +\infty$ (in view of (4.2.7) and (4.2.3)), therefore, $G(\theta, M(\theta), M(\theta)) = 0$, which gives $M(\theta) = \theta$. Let θ' be another fixed point of M with $G(\theta,\theta,\theta') < +\infty$, $G(\theta,\theta',\theta') < +\infty$, and $G(\theta',\theta',\theta') < +\infty$.

Then

$$\begin{aligned} G(\theta',\theta',\theta') &= G(M(\theta'), M(\theta'), M(\theta')) \\ &\leq \beta \max\{G(\theta',\theta',\theta'),\ G(\theta', M(\theta'), M(\theta')), G(\theta', M(\theta'), M(\theta'))\} \\ &= \beta G(\theta',\theta',\theta'). \end{aligned}$$

Since $\beta \in [0,1)$ and $G(\theta',\theta',\theta') < \infty$, therefore, $G(\theta',\theta',\theta') = 0$.

Now

$$\begin{aligned} G(\theta,\theta,\theta') =&\ G(M(\theta), M(\theta), M(\theta')) \\ \leq&\ \beta \max\{G(\theta,\theta,\theta'),\ G(\theta, M(\theta), M(\theta)),\ G(\theta', M(\theta'), M(\theta')), \\ &\quad G(\theta, M(\theta'), M(\theta')),\ G(\theta', M(\theta), M(\theta))\} \\ =&\ \beta \max\{G(\theta,\theta,\theta'),\ G(\theta,\theta,\theta),\ G(\theta',\theta',\theta'),\ G(\theta,\theta',\theta'),\ G(\theta',\theta,\theta)\} \\ =&\ \beta \max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\}. \end{aligned}$$

Similarly, $G(\theta,\theta',\theta') \leq \beta \max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\}$, therefore,

$$\max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\} \leq \beta \max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\}.$$

Now, as $\beta \in [0,1)$, $G(\theta,\theta,\theta') < +\infty$, and $G(\theta,\theta',\theta') < +\infty$, therefore, $\max\{G(\theta,\theta,\theta'),\ G(\theta,\theta',\theta')\} = 0$, which gives that $\theta = \theta'$. □

Now, by omitting the conditions (4.2.3) and $G(\theta, \theta', \theta') < +\infty$, in the previous result, we have the following result:

Theorem 4.2.3 *Let (X, G) be a complete G^*-metric space with constant α and $M : X \to X$ be a mapping such that*

$$(T3) \quad G(Mx, My, Mz)$$

$$\leq \beta \max\{G(x,y,z),\ G(x,Mx,Mx),\ G(y,My,My),\ G(z,Mz,Mz)\},$$

$$for\ all\ x,y,z \in X,\ and\ for\ some\ \beta \in [0,1)\ with\ \alpha\beta < 1;$$

$(T4)$ *there exists $x_0 \in X$ such that $\delta(G, M, x_0) < +\infty$;*

then the sequence $\{M^n(x_0)\}$ G^*-converges to some $\theta \in X$. If $G(\theta, M(\theta), M(\theta)) < +\infty$, then θ is a fixed point of M. Further, if θ' is another fixed point of M with $G(\theta, \theta, \theta') < +\infty$ and $G(\theta', \theta', \theta') < +\infty$, then $\theta = \theta'$.

Proof. Proof of this result follows similar lines as that of the previous result. □

4.3. Consequences in Generalized G_b-Metric Space

4.3.1 Lemmas

Now, we prove the following lemmas in generalized G_b-metric space, in order to deduce the consequences of the results of previous section.

Lemma 4.3.1 *Let (X, G) be a generalized G_b-metric space with $s \geq 1$ and $M : X \to X$ be a k-contraction in (X, G), then for each $x \in X$, $\delta(G, M, x) < +\infty$.*

Proof. Let $x \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = M^n(x), \text{ for all } n \in \mathbb{N}.$$

Since M is a k-contraction, therefore,

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n).$$

Thus, using the inequality (2.3.2) of the proof of Lemma 2.3.2 (Chapter 2); and for every $q, r \in \mathbb{N}$, with $q \geq r$, we have

$$G(M^r(x), M^q(x), M^q(x)) \leq \frac{G(x, M(x), M(x))k^{r-1}\mu}{1-k} \leq \frac{G(x, M(x), M(x))\mu}{1-k}, \tag{4.3.1}$$

where $\mu = \sum_{n=1}^{\infty} s^{2n} k^{2^{n-1}} < +\infty$. Also, for every $p, q \in \mathbb{N}$, with $p \geq q$,

$$G(M^q(x), M^q(x), M^p(x)) \leq 2sG(M^q(x), M^p(x), M^p(x)) \leq \frac{2sG(x, M(x), M(x))\mu}{1-k}. \tag{4.3.2}$$

Now, using (4.3.1) and (4.3.2), for $p, q, r \in \mathbb{N}$ with $p \geq q \geq r$, we have

$$\begin{aligned} G(M^p(x), M^q(x), M^r(x)) &\leq sG(M^p(x), M^q(x), M^q(x)) + sG(M^q(x), M^q(x), M^r(x)) \\ &\leq \frac{2s^2G(x, M(x), M(x))\mu}{1-k} + \frac{sG(x, M(x), M(x))\mu}{1-k}, \end{aligned}$$

that is,

$$G(M^p(x), M^q(x), M^r(x)) \leq \frac{(2s^2 + s)G(x, M(x), M(x))\mu}{1-k}. \tag{4.3.3}$$

Thus, we get

$$\begin{aligned} \delta(G, M, x) &= \sup\{G(M^p(x), M^q(x), M^r(x)) \mid p, q, r \in \mathbb{N}\} \\ &= \sup\{G(M^p(x), M^q(x), M^r(x)) \mid p \geq q \geq r\} \\ &\leq \frac{(2s^2 + s)G(x, Mx, Mx)\mu}{1-k} < +\infty. \end{aligned}$$

Thus, $\delta(G, M, x) < +\infty$. □

Lemma 4.3.2 *Let (X, G) be a generalized G_b-metric space with $s \geq 1$ and $M : X \to X$ be a mapping such that there exists some $k \in [0, 1)$ and*

$$G(Mx, My, Mz) \leq k \max\{G(x, y, z),\ G(x, Mx, Mx),\ G(y, My, My),\ G(z, Mz, Mz)\}, \tag{4.3.4}$$

for all $x, y, z \in X$. Then for each $x \in X$, $\delta(G, M, x) < +\infty$.

Proof. Let $x \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = M^n(x), \text{ for all } n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$, using (4.3.4), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = kG(x_{n-1}, x_n, x_n).$$

Rest of the proof follows the same lines as that of the proof of Lemma 4.3.1. □

Lemma 4.3.3 *Let (X, G) be a generalized G_b-metric space with $s \geq 1$ and $M : X \to X$ be a mapping such that there exists some $k \in [0, \frac{1}{2s})$ and*

$$\begin{aligned} G(Mx, My, Mz) \ \leq \ & k \max\{G(x, y, z),\ G(x, Mx, Mx),\ G(y, My, My),\ G(z, Mz, Mz), \\ & G(x, My, My),\ G(y, Mz, Mz),\ G(z, Mx, Mx)\}, \end{aligned}$$

for all $x, y, z \in X$. Then for each $x \in X$, $\delta(G, M, x) < +\infty$.

Proof. Let $x \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = M^n(x), \text{ for all } n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} & G(x_n, x_{n+1}, x_{n+1}) \\ \leq \ & k \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1})\}. \\ = \ & k \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1})\}. \end{aligned}$$

$$\leq k \max\{G(x_{n-1}, x_n, x_n), s[G(x_{n-1}, x_n, x_n) + G(x_n, x_n, x_{n+1})]\}.$$
$$= ks[G(x_{n-1}, x_n, x_n) + G(x_n, x_n, x_{n+1})],$$

which implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{ks}{1-ks} G(x_{n-1}, x_n, x_n).$$

Now, as $k \in [0, \frac{1}{2s})$, the rest of the proof is similar to that of Lemma 4.3.1. □

4.3.2 Consequences

Due to Lemma 4.3.1, the following result (Theorem 2.4.2 in Chapter 2) is a consequence of Theorem 4.2.1.

Corollary 4.3.4 *(Theorem 2.4.2, Chapter 2) Let (X, G) be a complete generalized G_b-metric space with and $M : X \to X$ a k-contraction mapping for some $k \in (0, 1)$. Then M has a unique fixed point.*

Next, we provide an example for which the hypothesis of Theorem 4.2.1 holds, but not of Corollary 4.3.4.

Example 4.3.5. Consider a G^*-metric space as in Example 3.2.10, which is a G^*-complete metric space. Now, define a mapping $M : X \to X$ by

$$M(x) = \begin{cases} 1, & \text{if } x = 1; \\ \frac{x}{2}, & \text{if } x \in [0, 1). \end{cases}$$

Then M is a k-contraction for any $k \in (\frac{1}{2}, 1)$. Also for $x_0 \in [0, 1)$, $\delta(G, M, x_0) < +\infty$, therefore, the hypothesis of Theorem 4.2.1 is satisfied. Further, we see that sequence $M^n(x_0) = \frac{x_0}{2^n} \to 0$, and 0 is a fixed point of M. Also, M has another fixed point namely 1, but then $G(0, 0, 1) = +\infty$.

Also, in view of Lemma 4.3.3, the following result is a consequence of Theorem 4.2.2.

Corollary 4.3.5 *Let (X, G) be a complete generalized G_b-metric space with $s \geq 1$ and $M : X \to X$ be a mapping such that there exists some $\beta \in [0, \frac{1}{2s})$ and*

$$\begin{aligned} G(Mx, My, Mz) \;\leq\; & \beta \max\{G(x, y, z),\ G(x, Mx, Mx),\ G(y, My, My),\ G(z, Mz, Mz), \\ & G(x, My, My),\ G(y, Mz, Mz),\ G(z, Mx, Mx)\}, \end{aligned}$$

for all $x, y, z \in X$. Then M has a unique fixed point.

The following example shows that the hypothesis of Theorem 4.2.2 is satisfied, but not of Theorem 4.2.1 and Corollary 4.3.5.

Example 4.3.6. Consider a G^*-metric space as in Example 3.2.10, which is a G^*-complete metric space with constant $\alpha = 1$. Now, define a mapping $M : X \to X$ by

$$M(x) = \begin{cases} 1, & \text{if } x = 1 \text{ or } \frac{1}{2}; \\ \frac{x}{3}, & \text{otherwise.} \end{cases}$$

Then M is not a k-contraction as for $x = y = z = \frac{1}{2}$, $G(Mx, My, Mz) = +\infty \not\leq 0 = kG(x, y, z)$ for any $k \in [0, 1)$. Thus, hypothesis of Theorem 4.2.1 are not satisfied. But we can check that $(T1)$ is satisfied for any $\beta \in (\frac{1}{3}, \frac{1}{\alpha})$. Also, for $x_0 \in [0, 1) - \{\frac{1}{2}\}$, $\delta(G, M, x_0) < +\infty$, therefore, hypothesis of Theorem 4.2.2 are satisfied. Further, we see that sequence $M^n(x_0) = \frac{x_0}{3^n} \to 0$, and 0 is a fixed point of M. Also, M has another fixed point namely 1, but then $G(0, 0, 1) = +\infty$ and $G(1, 1, 1) = +\infty$.

Also, the following result (in generalized G_b-metric space) is a consequence of Theorem 4.2.3 due to Lemma 4.3.2.

Corollary 4.3.6 *(Theorem 2.4.5, Chapter 2) Let (X, G) be a complete generalized G_b-metric space with $s \geq 1$ and $M : X \to X$ be a mapping such that there exists some $\beta \in [0, \frac{1}{s})$ and*

$$G(Mx, My, Mz) \leq \beta \max\{G(x, y, z),\ G(x, Mx, Mx),\ G(y, My, My),\ G(z, Mz, Mz)\},$$

for all $x, y, z \in X$. Then M has a unique fixed point.

CHAPTER 5

Some Fixed Point Results in b-Metric Spaces with New Contractive Mappings

5.1. Introduction

In the theory of metric space, Banach contraction principle [18] is one of the most important theorems and a powerful tool. A mapping $T : X \to X$, where (X, d) is a metric space, is called a contraction mapping if there exists $\alpha < 1$ such that for all $x, y \in X$, $d(Tx, Ty) \leq \alpha d(x, y)$. If the metric space (X, d) is complete, then T has a unique fixed point. Condition of contraction mapping implies the continuity of contraction. But, later in 1968, Kannan [75] proved the following result which gives the fixed point for discontinuous mapping. Let $T : X \to X$ be a mapping on a complete metric space (X, d) with

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty)),$$

where $\alpha \in [0, \frac{1}{2})$ and $x, y \in X$. Then T has a unique fixed point.

Contraction mappings have been extended or generalized in several directions by various authors (see, for example, [36], [91], [108], [122], [123], [125], [142]). Not

Contents of this chapter have been published in **Axioms, 10(2), 2021, 15 pages, Article ID 55, (SCIE).**

only contraction mappings, but the concept of metric space has also been extended in many ways in the literature (see, for example, [8], [17], [39], [40], [60], [67], [68], [89], [133]).

By weakening the triangle inequality in metric space, Bakhtin [17] introduced the idea of b-metric space with $s = 2$. In 1998, Czerwik [40] presented the notion of b-metric space in the following form:

Definition 5.1.1. [40] Let X be a non-empty set and $s \geq 1$ be a given real. Then a mapping $d : X \times X \to [0, +\infty)$ is called a *b-metric* if, for all $x, y, z \in X$,

$(b1)$ $d(x, y) = 0$ if and only if $x = y$;

$(b2)$ $d(x, y) = d(y, x)$;

$(b3)$ $d(x, y) \leq s(d(x, z) + d(z, y))$.

The pair (X, d) is called a *b-metric space.* Clearly, every metric space is a b-metric space with $s = 1$, but the converse is not true in general. In fact, the class of b-metric spaces is larger than the class of metric spaces.

In [40], Banach contraction principle is proved in the framework of b-metric spaces. In 2013, Kir and Kiziltunc [83] established the results in b-metric spaces, which generalized the Kannan and Chatterjea type mappings. Aleksić *et al.* [7] introduced the following result, which improves Theorem 1 in [45].

Theorem 5.1.1 *[7] Let (X, d) be a complete b-metric space with a constant $s \geq 1$. If $T : X \to X$ satisfies the inequality:*

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 (d(x, Ty) + d(Tx, y)),$$

where $\lambda_i \geq 0$ for all $i = 1, 2, 3, 4$ and $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$ for $s \in [1, 2]$ and $\frac{2}{s} < \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$ for $s \in (2, +\infty)$, then T has a unique fixed point.

Ćirić [36] introduced quasi-contraction mapping in metric space (X, d) as:
A mapping $T : X \to X$ is said to be a *quasi-contraction mapping* if there exists $0 \leq q < 1$ such that for any $x, y, \in X$,

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\}.$$

Many authors proved fixed point theorems for quasi-contraction mappings in b-metric spaces with some more restriction on values of q (see, for example, [7], [13], [69], [95], [113], [134], [135]). More on b-metric spaces can be found in ([53], [63], [76], [77], [81], [83], [87], [92], [93] [94], [103], [112], [120], [136], [137]).

In the present chapter, we define a new class of functions. After that, we define some new contractive mappings which combine the terms $d(x, y)$, $d(x, Tx)$, $d(y, Ty)$, $d(x, Ty)$, and $d(Tx, y)$ by means of the member of a newly defined class. The contents of this chapter are divided into different sections. Section 5.2 presents some preliminaries related to this chapter. In section 5.3, some fixed point results have been proved for newly defined contractive mappings. Section 5.4 deals with some consequences of the results presented in the previous section. Towards the end, an application to solve a system of linear equations is provided in section 5.5.

5.2. Preliminaries

This section presents some elementary definitions and results from literature and some new concepts that are to be used in the later sections of this chapter.

Definition 5.2.1. [23] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called :

(*i*) *Cauchy sequence* if, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$, for all $n, m \geq n_0$.

(*ii*) *Convergent sequence* if there exists $l \in X$ such that for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, l) < \epsilon,$ for all $n \geq n_0$. In this case, the sequence $\{x_n\}$ is said to *converge* to l.

Definition 5.2.2. [23] A b-metric space (X, d) is said to be a *complete b-metric space* if every Cauchy sequence is convergent in it.

Following lemma proved by Aghajani *et al.* [3] in 2014 plays a crucial role in the study of b-metric space.

Lemma 5.2.1 *[3] Let (X, d) be a b-metric space with $s \geq 1$ and suppose that sequences $\{x_n\}$ and $\{y_n\}$ converge to x and $y \in X$, respectively. Then*

$$\frac{1}{s^2}d(x, y) \leq \liminf_{n\to+\infty} d(x_n, y_n) \leq \limsup_{n\to+\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then $\lim_{n\to+\infty} d(x_n, y_n) = 0$.

Moreover, for any $z \in X$,

$$\frac{1}{s}d(x, z) \leq \liminf_{n\to+\infty} d(x_n, z) \leq \limsup_{n\to+\infty} d(x_n, z) \leq sd(x, z).$$

In 2017, Miculescu and Mihail [92] and Suzuki [137] presented the following lemma about the Cauchy sequence in b-metric space.

Lemma 5.2.2 *[92, 137] Every sequence $\{x_n\}$ of elements from a b-metric space (X, d), having the property that there exists $\lambda \in [0, 1)$ such that $d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$ for every $n \in \mathbb{N}$, is a Cauchy sequence.*

We, now introduce a new class of functions and some new contractive mappings in b-metric space:

Definition 5.2.3. For any $m \in \mathbb{N}$, define Ξ_m to be the set of all functions $\xi : [0, +\infty)^m \to [0, +\infty)$ such that

(ξ_1) $\xi(t_1, t_2, \cdots, t_m) < \max\{t_1, t_2, \cdots, t_m\}$ if $(t_1, t_2, \cdots, t_m) \neq (0, 0, \cdots, 0)$;

(ξ_2) if $\{t_i^{(n)}\}_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are m sequences in $[0, +\infty)$ such that

$\limsup_{n\to+\infty} t_i^{(n)} = t_i < +\infty$ for all $i = 1$ to m, then

$\liminf_{n\to+\infty} \xi\left(t_1^{(n)}, t_2^{(n)}, \cdots, t_m^{(n)}\right) \leq \xi(t_1, t_2, \cdots, t_m)$.

Definition 5.2.4. Let (X, d) be a b-metric space with $s \geq 1$. The mapping $T : X \to X$ is said to be an *ξ-contractive mapping of type-I* if there exists $\xi \in \Xi_4$ and

$$d(Tx, Ty) \leq \frac{1}{s}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s}\right), \tag{5.2.5}$$

for all $x, y \in X$.

Definition 5.2.6. Let (X, d) be a b-metric space with $s \geq 1$. The mapping $T : X \to X$ is said to be an *ξ-contractive mapping of type-II* if there exists $\xi \in \Xi_5$ and

$$d(Tx, Ty) \leq \frac{1}{s}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)}{2s}, d(Tx, y)\right), \tag{5.2.7}$$

for all $x, y \in X$.

5.3. Fixed Point Results in b-Metric Spaces

The first main result of this chapter is as follows:

Theorem 5.3.1 *Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be an ξ-contractive mapping of type-I. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as

$$x_n = T(x_{n-1}), \text{ for all } n \in \mathbb{N}.$$

If $x_m = x_{m+1}$ for some natural number m, then x_m is a fixed point of T.

Now, assume that any two consecutive terms of the sequence $\{x_n\}$ are distinct. First, we prove that $\{x_n\}$ is a Cauchy sequence. For this, let $n \in \mathbb{N}$. Then, by using (5.2.5), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{s}\xi\left(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\right) \quad (5.3.1)\\ &< \frac{1}{s}\max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\right\}\\ &= \frac{1}{s}\max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_{n+1})}{2s}\right\}\\ &\leq \frac{1}{s}\max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\}, \end{aligned}$$

which implies that

$$d(x_n, x_{n+1}) < \frac{1}{s}d(x_{n-1}, x_n), \qquad \text{for all } n \geq 1. \tag{5.3.2}$$

Case 1: If $s > 1$, then by Lemma 5.2.2, $\{x_n\}$ is a Cauchy sequence.

Case 2: If $s = 1$, then by (5.3.2), the sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing and bounded below. Therefore, sequence $\{d(x_n, x_{n+1})\}$ converges to a real k, where $k \geq 0$. Suppose that $k > 0$. Now taking $\liminf\ n \to +\infty$ in (5.3.1), we have $k \leq \xi(k, k, k, k')$,

where

$$k' = \limsup_{n\to+\infty} \frac{d(x_{n-1}, x_{n+1})}{2} \leq \limsup_{n\to+\infty} \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} = k.$$

Now, $k \leq \xi(k, k, k, k') < \max\{k, k, k, k'\} = k$, a contradiction, therefore,

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{5.3.3}$$

Suppose that $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ such that for any $r \in \mathbb{N}$, there exists $m_r > n_r \geq r$ such that

$$d(x_{m_r}, x_{n_r}) \geq \varepsilon. \tag{5.3.4}$$

Also, assume that m_r is the smallest natural number greater than n_r such that (5.3.4) holds. Then,

$$\begin{aligned} \varepsilon &\leq d(x_{m_r}, x_{n_r}) \\ &\leq d(x_{m_r}, x_{m_r-1}) + d(x_{m_r-1}, x_{n_r}) \\ &< d(x_{m_r}, x_{m_r-1}) + \varepsilon \\ &< d(x_r, x_{r-1}) + \varepsilon, \end{aligned}$$

thus, by using (5.3.3) and taking lim $r \to +\infty$, we get

$$\lim_{r \to +\infty} d(x_{m_r}, x_{n_r}) = \varepsilon. \tag{5.3.5}$$

Now, by using (5.2.5)

$$\begin{aligned} &d(x_{m_r+1}, x_{n_r+1}) \\ &\leq \xi\left(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{m_r+1}), d(x_{n_r}, x_{n_r+1}), \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r})}{2}\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &d(x_{m_r}, x_{n_r}) \\ &\leq d(x_{m_r}, x_{m_r+1}) + d(x_{m_r+1}, x_{n_r+1}) + d(x_{n_r+1}, x_{n_r}) \\ &\leq d(x_{m_r}, x_{m_r+1}) + d(x_{n_r+1}, x_{n_r}) + \\ &\quad \xi\left(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{m_r+1}), d(x_{n_r}, x_{n_r+1}), \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r})}{2}\right). \end{aligned}$$

Thus, by taking lim $r \to +\infty$ on both sides and also using (5.3.3) and (5.3.5), we get

$$\varepsilon \leq 0 + 0 + \xi(\varepsilon, 0, 0, \varepsilon'),$$

where

$$\begin{aligned}
\varepsilon' &= \limsup_{r\to+\infty} \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r})}{2} \\
&\leq \limsup_{r\to+\infty} \frac{d(x_{m_r}, x_{n_r}) + d(x_{n_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{m_r}) + d(x_{m_r}, x_{n_r})}{2} \\
&= \frac{\varepsilon + 0 + 0 + \varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, $\varepsilon \leq \xi(\varepsilon, 0, 0, \varepsilon') < \max\{\varepsilon, 0, 0, \varepsilon'\} = \varepsilon$, a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete b-metric space, therefore, there exists $x \in X$ such that sequence $\{x_n\}$ converges to x.

Now, by using (5.2.5), we have

$$d(Tx_n, Tx) \leq \frac{1}{s}\xi\left(d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n)}{2s}\right),$$

which implies that

$$d(x_{n+1}, Tx) \leq \frac{1}{s}\xi\left(d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{d(x_n, Tx) + d(x, x_{n+1})}{2s}\right).$$

Taking lim inf $n \to +\infty$ on both sides and using Lemma 5.2.1, we get

$$\frac{1}{s}d(x, Tx) \leq \frac{1}{s}\xi\left(0, 0, d(x, Tx), l\right),$$

that is,

$$d(x, Tx) \leq \xi\left(0, 0, d(x, Tx), l\right),$$

where

$$l = \limsup_{n\to+\infty} \frac{d(x_n, Tx) + d(x, x_{n+1})}{2s} \leq \limsup_{n\to+\infty} \frac{sd(x, Tx) + 0}{2s} = \frac{d(x, Tx)}{2}.$$

Hence,

$$d(x, Tx) \leq \xi\,(0, 0, d(x, Tx), l) < \max\{0, 0, d(x, Tx), l\} = d(x, Tx),$$

which is a contradiction. Therefore, $Tx = x$.

Let $Ty = y$ for some $y \in X$, and suppose that $x \neq y$. Then by using (5.2.5), we get

$$\begin{aligned} d(x, y) = d(Tx, Ty) &\leq \frac{1}{s}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right) \\ &\leq \frac{1}{s}\xi\left(d(x, y), 0, 0, \frac{d(x, y)}{s}\right) \\ &< \frac{1}{s}\max\left\{d(x, y), 0, 0, \frac{d(x, y)}{s}\right\} \\ &= \frac{d(x, y)}{s}, \end{aligned}$$

a contradiction. Therefore, $x = y$. □

The following remark improves Theorem 5.3.1.

Remark 5.3.6. Theorem 5.3.1 is also valid if the term $\frac{d(x,Ty)+d(Tx,y)}{2s}$ in (5.2.5) is replaced by $\frac{d(x,Ty)+d(Tx,y)}{\delta s}$, where δ is a real number defined by

$$\delta = \begin{cases} 2, & \text{if} \quad s = 1, \\ \delta', & \text{if} \quad 1 < s \leq 2, \\ 1, & \text{if} \quad s > 2, \end{cases}$$

where δ' is any number in $\left(\frac{2}{s}, 1 + \frac{1}{s}\right)$.

The proof of next result goes in a similar manner as the proof of Theorem 5.3.1.

Theorem 5.3.2 *Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be an ξ-contractive mapping of type-II. Then T has a unique fixed point.*

Following remark improves Theorem 5.3.2.

Remark 5.3.7. Theorem 5.3.2 is also valid if the term $\frac{d(x,Ty)}{2s}$ in (5.2.7) is replaced by $\frac{d(x,Ty)}{\delta s}$, where δ is the same as in Remark 5.3.6.

5.4. Consequences

This section presents some results which are consequences of the results presented in the previous section. Following results are consequences of Theorem 5.3.1.

Corollary 5.4.1 *Let* (X, d) *be a complete b-metric space with* $s \geq 1$ *and* $T : X \to X$ *be a mapping such that there exists* $q \in [0, \frac{1}{s})$ *and*

$$d(Tx, Ty) \leq q \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s}\right\}, \qquad (5.4.1)$$

for all $x, y \in X$*. Then* T *has a unique fixed point.*

Proof. Let $\xi \in \Xi_4$ be defined by $\xi(t_1, t_2, t_3, t_4) = qs \max\{t_1, t_2, t_3, t_4\}$. Then by Theorem 5.3.1, T has a unique fixed point. □

In the following example, it is seen that conditions of Theorem 5.3.1 are satisfied, but Corollary 5.4.1 is not applicable.

Example 5.4.2. Let $X = \left\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}\right\} \bigcup \{0\}$. Define $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x - y|^2$, for all $x, y \in X$. Then d is a b-metric on X with $s = 2$.
Define $T : X \to X$ by $T\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{2(n+1)}}$, for all $n \in \mathbb{N}$ and T(0)=0. Define

$$\xi(t_1, t_2, t_3, t_4) = \begin{cases} \frac{\max\{t_1, t_2, t_3, t_4\}}{1+t_1}, & \text{if } t_1 > 0; \\ \frac{1}{2}\max\{t_2, t_3, t_4\}, & \text{otherwise.} \end{cases}$$

Then, for all $x, y \in X$, (5.2.5) is satisfied and hence the conditions of Theorem 5.3.1 are satisfied.

But, we see that if (5.4.1) is satisfied for all $x, y \in X$, then we have

$$d(Tx, Ty) \leq qN(x, y),$$

for all $x, y \in X$, where $N(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(Tx,y)}{2s}\right\}$.

So, in particular, we have

$$d\left(\frac{1}{\sqrt{2(n+1)}}, \frac{1}{\sqrt{2(m+1)}}\right) \leq qN\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{m}}\right), \quad \text{for all } \ m, n \in \mathbb{N},\ m \neq n;$$

that is,

$$\frac{\left|\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{m+1}}\right|^2}{N\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{m}}\right)} \leq 2q, \quad \text{for all } \ m, n \in \mathbb{N},\ m \neq n.$$

Now take lim $n, m \to +\infty$, we get $2q \geq 1$, a contradiction. Hence, Corollary 5.4.1 is not applicable for this example.

Remark 5.4.3. In view of Remark 5.3.6, Corollary 5.4.1 is also valid, if the term $\frac{d(x,Ty)+d(Tx,y)}{2s}$ is replaced by $\frac{d(x,Ty)+d(Tx,y)}{\delta s}$, where δ is the same as defined in Remark 5.3.6.

Corollary 5.4.2 *Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4(d(x, Ty) + d(Tx, y)), \quad (5.4.4)$$

for all $x, y \in X$, where $\lambda_1 + \lambda_2 + \lambda_3 + \delta s\lambda_4 < \frac{1}{s}$ and $\lambda_i \geq 0$ for all $i = 1$ to 4. Then T has a unique fixed point.

Proof. Let $\xi \in \Xi_4$ be defined by $\xi(t_1, t_2, t_3, t_4) = s(\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3 + \delta s\lambda_4 t_4)$. Then by Theorem 5.3.1 and Remark 5.4.3, T has a unique fixed point. □

Following results are consequences of Theorem 5.3.2.

Corollary 5.4.3 *Let* (X, d) *be a complete b-metric space with* $s \geq 1$ *and* $T : X \to X$ *be a mapping such that there exists* $q \in [0, \frac{1}{s})$ *and*

$$d(Tx, Ty) \leq q \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)}{\delta s}, d(Tx, y)\right\}, \quad (5.4.5)$$

for all $x, y \in X$*. Then* T *has a unique fixed point.*

Proof. Let $\xi \in \Xi_5$ be defined by $\xi(t_1, t_2, t_3, t_4, t_5) = qs \max\{t_1, t_2, t_3, t_4, t_5\}$. Then, by Theorem 5.3.2, T has a unique fixed point. □

Corollary 5.4.4 *Let* (X, d) *be a complete b-metric space with* $s \geq 1$ *and* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 d(x, Ty) + \lambda_5 d(Tx, y), \quad (5.4.6)$$

for all $x, y \in X$*, where* $\lambda_1 + \lambda_2 + \lambda_3 + \delta s \lambda_4 + \lambda_5 < \frac{1}{s}$ *and* $\lambda_i \geq 0$ *for all* $i = 1$ *to* 5*. Then* T *has a unique fixed point.*

Proof. Let $\xi \in \Xi_5$ be defined by $\xi(t_1, t_2, t_3, t_4, t_5) = s(\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3 + \delta s \lambda_4 t_4 + \lambda_5 t_5)$. Then, by Theorem 5.3.2, T has a unique fixed point. □

5.5. Application

As an application of Theorem 5.3.1, the following result is presented in this section.

Theorem 5.5.1 *Consider a system of linear equations*

$$Ax = b, \quad (5.5.1)$$

where $A = [a_{ij}]_{n\times n}$ is an $n\times n$ matrix and $b = [b_i]_{n\times 1}$ is a column vector of constants and $x = [x_i]_{n\times 1}$ is a column matrix of n unknowns. If for each $x = [x_i]_{n\times 1}$, $y = [y_i]_{n\times 1}$ and $i = 1$ to n,

$$|(a_{ii}+1)(x_i-y_i) + \sum_{j=1, j\neq i}^{n} a_{ij}(x_j-y_j)|(1+\max_{k=1}^{n}|x_k-y_k|) \leq |x_i-y_i|, \tag{5.5.2}$$

then system has a unique solution.

Proof. Let $X = \{[x_i]_{n\times 1} \mid \ x_i$ is real for all $i = 1$ to n, n being fixed$\}$ and $d : X\times \ X \to [0,+\infty)$ be defined as :

$$d(x,y) = \max_{i=1}^{n}|x_i-y_i|,$$

for all $x = [x_i]_{n\times 1}$, $y = [y_i]_{n\times 1} \in X$. Then clearly (X,d) is a complete b-metric space with constant $s = 1$ (i.e., (X,d) is a complete metric space).

Now define a $n\times n$ matrix $C = [c_{ij}]_{n\times n}$ by

$$c_{ij} = \begin{cases} a_{ij}+1, & \text{if } \ i = j, \\ a_{ij}, & \text{if } \ i \neq j. \end{cases}$$

Then the given system (5.5.1) reduces to

$$x = Cx - b. \tag{5.5.3}$$

Condition (5.5.2) becomes

$$|\sum_{j=1}^{n} c_{ij}(x_j-y_j)|(1+\max_{k=1}^{n}|x_k-y_k|) \leq |x_i-y_i|, \qquad \text{for all } \ i = 1,2,\cdots,n. \tag{5.5.4}$$

Now, define a mapping $T : X \to \ X$ by

$$T(x) = Cx - b, \qquad \text{where } \ x \in X.$$

For $x = [x_i]_{n\times 1}$ and $y = [y_i]_{n\times 1}$, suppose that $Tx = u = [u_i]_{n\times 1}$ and $Ty = v = [v_i]_{n\times 1}$, then

$$u_i = \sum_{j=1}^{n} c_{ij}x_j - b_i, \quad \text{for all} \quad i = 1, 2, \cdots, n$$

and

$$v_i = \sum_{j=1}^{n} c_{ij}y_j - b_i, \quad \text{for all} \quad i = 1, 2, \cdots, n.$$

Define

$$\xi(t_1, t_2, t_3, t_4) = \begin{cases} \frac{\max\{t_1,t_2,t_3,t_4\}}{1+t_1}, & \text{if} \quad t_1 > 0; \\ \frac{1}{2}\max\{t_2, t_3, t_4\}, & \text{otherwise.} \end{cases}$$

Now, using condition (5.5.4), we get

$$\begin{aligned} d(Tx, Ty) &= \max_{i=1}^{n} |u_i - v_i| \\ &= \max_{i=1}^{n} \Big| \sum_{j=1}^{n} c_{ij}(x_j - y_j) \Big| \\ &\leq \max_{i=1}^{n} \left(\frac{|x_i - y_i|}{1 + \max_{k=1}^{n} |x_k - y_k|} \right) \\ &\leq \xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s} \right). \end{aligned}$$

Thus, it is straightforward to see that the hypothesis of Theorem 5.3.1 is satisfied. Therefore, T has a unique fixed point, and hence system (5.5.1) has a unique solution.

□

CHAPTER 6

Some Fixed Point Theorems in b-Metric-Like Spaces

6.1. Introduction

Partial metric spaces were introduced by Matthews [89] in 1992 as a generalization of metric spaces. In 2012, Harandi [60] generalized the concept of partial metric by establishing a new space named metric-like-space. It is noticed that in metric-like space, self-distance of a point may be greater than the distance of that point to any other point (see Example 1.2.15). Later on, Shukla [133] in 2014 gave the idea of partial b-metric as a generalization of partial metric and b-metric. Meanwhile, in 2013, Alghamdi *et al.* [8] introduced the concept of b-metric-like spaces that generalized the notions of partial b-metric space and metric-like space. Obviously, b-metric-like space generalizes all abstract spaces that are mentioned in this paragraph. For the sake of clarity, we recall the definitions of these abstract spaces.

Definition 6.1.1. [89] Let X be a non-empty set. Then a mapping $d : X \times X \to [0, +\infty)$ is called a *partial metric* if, for all $x, y, z \in X$,

Part of the contents of this chapter has been published in **Axioms, 10(2), 2021, 15 pages, Article ID 55, (SCIE).**

$(p1)$ $x = y \Leftrightarrow d(x,x) = d(x,y) = d(y,y)$;

$(p2)$ $d(x,x) \leq d(x,y)$;

$(p3)$ $d(x,y) = d(y,x)$;

$(p4)$ $d(x,y) \leq d(x,z) + d(z,y) - d(z,z)$.

The pair (X,d) is called a *partial metric space*.

Definition 6.1.2. [60] Let X be a non-empty set. Then a mapping $d : X \times X \to [0,+\infty)$ is called a *metric-like* if, for all $x,y,z \in X$,

(m_l1) $d(x,y) = 0 \Rightarrow x = y$;

(m_l2) $d(x,y) = d(y,x)$;

(m_l3) $d(x,y) \leq d(x,z) + d(z,y)$.

The pair (X,d) is called a *metric-like space*.

Definition 6.1.3. [133] Let X be a non-empty set. Then a mapping $d : X \times X \to [0,+\infty)$ is called a *partial b-metric* if there exists a number $s \geq 1$ such that for all $x,y,z \in X$,

(p_b1) $x = y$ if and only if $d(x,x) = d(x,y) = d(y,y)$;

(p_b2) $d(x,x) \leq d(x,y)$;

(p_b3) $d(x,y) = d(y,x)$;

(p_b4) $d(x,y) \leq s(d(x,z) + d(z,y)) - d(z,z)$.

Such a pair (X,d) is called a *partial b-metric space*.

Definition 6.1.4. [8] Let X be a non-empty set. Then a mapping $d : X \times X \to [0,+\infty)$ is called a *b-metric-like* if there exists a number $s \geq 1$ such that for all $x,y,z \in X$,

$(b_{ml}1)$ $d(x,y) = 0 \Rightarrow x = y$;

$(b_{ml}2)$ $d(x, y) = d(y, x)$;

$(b_{ml}3)$ $d(x, y) \leq s(d(x, z) + d(z, y))$.

Such a pair (X, d) is called a *b-metric-like space*.

Sen *et al.* [128] have presented the process diagram of the various classes of these abstract spaces as follows:

$$\begin{array}{ccccc} \text{Metric space} & \Rightarrow & \text{Partial Metric space} & \Rightarrow & \text{Metric-like space} \\ \Downarrow & & \Downarrow & & \Downarrow \\ b\text{-Metric space} & \Rightarrow & \text{Partial } b\text{-Metric space} & \Rightarrow & b\text{-Metric-like space} \end{array}$$

The aim of this chapter is to extend the main results of Chapter 5 in the framework of b-metric-like spaces. Also, some common fixed point theorems are presented for weakly compatible mappings. This chapter is organized as follows: In section 6.2, some basic concepts related to common fixed points have been presented. After that, some elementary definitions and results in context of b-metric-like spaces have been presented in this section. Section 6.3 consists of the main results of the chapter. Towards the end, some consequences of main results are deduced in section 6.4.

6.2. Preliminaries

Two self-mappings f and g on the set X have a *common fixed point* x in X if $fx = gx = x$. In 1976, Jungck [70] initiated the study of common fixed points. Jungck [70] generalized the Banach contraction principle by proving a common fixed point theorem for commuting maps. Sessa [129], in 1982, further generalized the concept of commutativity by giving the notion of weakly commuting mappings.

Definition 6.2.1. [129] Two self-mappings T and S on a metric space (X, d) are said to be *weakly commuting* if, $d(TSx, STx) \leq d(Tx, Sx)$, for all x in X.

In 1986, Jungck [71] coined the term compatible mappings in order to generalize the notion of weakly commutativity.

Definition 6.2.2. [71] Two self-mappings T and S on a metric space (X, d) are said to be *compatible* if, $\lim_{n\to+\infty} d(TSx_n, STx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} Sx_n = t$, for some $t \in X$.

Further, Jungck and Rhoades [73] in 1998 weakened the concept of compatible mappings by giving the notion of weakly compatible mappings as:

Definition 6.2.3. [73] Two self-mappings T and S on a set X are said to be *weakly compatible* if, $TSx = STx$ whenever $Tx = Sx$, for some $x \in X$, i.e., if T and S commute at their coincidence point. Here, x is a *coincidence point* of T and S, and the common value (i.e., $Tx = Sx$) is called a *point of coincidence* of T and S.

Thereafter, many fixed point results have been obtained for weakly compatible mappings (see [1], [5], [35], [38], [69], [72], [73], [117]).

Proposition 6.2.1 *[1] If two self-mappings f, $g : X \to X$ are weakly compatible and have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

On the other hand, in 2013, Alghamdi *et al.* [8] introduced the concepts of convergent sequence, Cauchy sequence, and completeness in the framework of b-metric-like space as:

Definition 6.2.4. [8] Let (X, d) be a b-metric-like space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the *limit* of sequence $\{x_n\}$ if $\lim_{n\to+\infty} d(x, x_n) = d(x, x)$, and we say that the sequence $\{x_n\}$ is *convergent to* x and denote it by $x_n \to x$ as $n \to +\infty$.

Definition 6.2.5. [8] Let (X, d) be a b-metric-like space.

(i) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n,m\to+\infty} d(x_n, x_m)$ exists and is finite.

(ii) (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ so that $\lim_{n,m\to+\infty} d(x_n, x_m) = d(x, x) = \lim_{n\to+\infty} d(x_n, x)$.

Alghamdi *et al.* [8] also proved the following proposition about the uniqueness of the limit of a sequence in a b-metric-like space.

Proposition 6.2.2 *[8] Let (X, d) be a b-metric-like space with $s \geq 1$, and $\{x_n\}$ be a sequence in X such that for some $x \in X$, $\lim_{n\to+\infty} d(x_n, x) = 0$. Then*

(i) *x is unique.*

(ii) $\frac{1}{s}d(x, y) \leq \lim_{n\to+\infty} d(x_n, y) \leq sd(x, y), \quad \textit{for all } y \in X.$

For Cauchy sequence in a b-metric-like space, the following lemma is presented by Sen *et al.* [128].

Lemma 6.2.3 *[128] Let (X, d) be a b-metric-like space, and $\{x_n\}$ be a sequence in X such that*

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n),$$

for some $\lambda \in [0, 1)$ and for each $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence with $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$.

6.3. Main Results

In this section, we present the main theorems of this chapter. The following theorem is an extended version of Theorem 5.3.1 in the framework of a b-metric-like space. Towards the end of this theorem, an example is provided to illustrate the result.

Theorem 6.3.1 *Let (X,d) be a complete b-metric-like space with $s \geq 1$. Let $T : X \to X$ be a mapping such that there exists $\xi \in \Xi_4$ and*

$$d(Tx,Ty) \leq \frac{1}{s}\xi\left(d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)-d(y,y)}{2s}\right), \tag{6.3.1}$$

for all $x, y \in X$ with $d(x,Ty)+d(Tx,y) \geq d(y,y)$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as

$$x_n = T(x_{n-1}), \text{ for all } n \in \mathbb{N}.$$

Assume that any two consecutive terms of the sequence $\{x_n\}$ are distinct, otherwise T has a fixed point. First, we prove that $\{x_n\}$ is a Cauchy sequence. For this, let $n \in \mathbb{N}$.

Now,

$$d(x_{n-1},Tx_n) + d(Tx_{n-1},x_n) = d(x_{n-1},x_{n+1}) + d(x_n,x_n) \geq d(x_n,x_n),$$

therefore, by using (6.3.1), we have

$$d(x_n,x_{n+1}) \leq \frac{1}{s}\xi\left(d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)-d(x_n,x_n)}{2s}\right) \tag{6.3.2}$$

$$
\begin{aligned}
&< \frac{1}{s}\max\left\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})}{2s}\right\}\\
&= \frac{1}{s}\max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_{n+1})}{2s}\right\}\\
&\leq \frac{1}{s}\max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\},
\end{aligned}
$$

which implies that

$$
d(x_n,x_{n+1}) < \frac{1}{s}d(x_{n-1},x_n), \qquad \text{for all } n \geq 1. \tag{6.3.3}
$$

Case 1: If $s > 1$, then by Lemma 6.2.3 and in view of (6.3.3), $\{x_n\}$ is a Cauchy sequence in (X,d) and $\lim_{n,m\to+\infty} d(x_n,x_m) = 0$.

Case 2: If $s = 1$, then by (6.3.3), the sequence $\{d(x_n,x_{n+1})\}$ is monotonically decreasing and bounded below. Therefore, sequence $\{d(x_n,x_{n+1})\}$ converges to a real k, where $k \geq 0$. Suppose that $k > 0$. Now taking $\liminf\ n \to +\infty$ in (6.3.2), we have $k \leq \xi(k,k,k,k')$,

where

$$
k' = \limsup_{n\to+\infty} \frac{d(x_{n-1},x_{n+1})}{2} \leq \limsup_{n\to+\infty} \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2} = k.
$$

Now, $k \leq \xi(k,k,k,k') < \max\{k,k,k,k'\} = k$, a contradiction, therefore,

$$
\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0. \tag{6.3.4}
$$

Also,

$$
d(x_n,x_n) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_n),
$$

taking $\lim\ n \to +\infty$, and using (6.3.4) we get

$$
\lim_{n\to+\infty} d(x_n,x_n) = 0, \tag{6.3.5}
$$

Suppose that $\lim\limits_{n,m\to+\infty} d(x_n, x_m) \neq 0$, then there exists $\varepsilon > 0$ such that for any $r \in \mathbb{N}$, there exists $m_r > n_r \geq r$ such that

$$d(x_{m_r}, x_{n_r}) \geq \varepsilon. \tag{6.3.6}$$

Also, assume that m_r is the smallest natural number greater than n_r such that (6.3.6) holds. Then,

$$\begin{aligned} \varepsilon &\leq d(x_{m_r}, x_{n_r}) \\ &\leq d(x_{m_r}, x_{m_r-1}) + d(x_{m_r-1}, x_{n_r}) \\ &< d(x_{m_r}, x_{m_r-1}) + \varepsilon \\ &< d(x_r, x_{r-1}) + \varepsilon. \end{aligned}$$

Thus, by using (6.3.4) and taking lim $r \to +\infty$, we get

$$\lim_{r\to+\infty} d(x_{m_r}, x_{n_r}) = \varepsilon. \tag{6.3.7}$$

Now, suppose that there exists infinitely many r such that

$$d(x_{m_r}, Tx_{n_r}) + d(Tx_{m_r}, x_{n_r}) < d(x_{n_r}, x_{n_r}).$$

Taking lim $r \to +\infty$, and using (6.3.5), we get

$$\lim_{r\to+\infty} (d(x_{m_r}, Tx_{n_r}) + d(Tx_{m_r}, x_{n_r})) = 0,$$

this implies that

$$\lim_{r\to+\infty} d(x_{m_r}, x_{n_r+1}) = \lim_{r\to+\infty} d(x_{m_r+1}, x_{n_r}) = 0.$$

Now,

$$\varepsilon = \lim_{r\to+\infty} d(x_{m_r}, x_{n_r}) \leq \limsup_{r\to+\infty}((d(x_{m_r}, x_{n_r+1}) + d(x_{n_r+1}, x_{n_r})) = 0,$$

a contradiction. Therefore, there exists $r_0 \in \mathbb{N}$ such that for all $r \geq r_0$,

$$d(x_{m_r}, Tx_{n_r}) + d(Tx_{m_r}, x_{n_r}) \geq d(x_{n_r}, x_{n_r}).$$

Thus, for all $r \geq r_0$, using (6.3.1), we get

$$\begin{aligned}
&d(x_{m_r+1}, x_{n_r+1}) \\
&\leq \xi\left(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{m_r+1}), d(x_{n_r}, x_{n_r+1}), \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r}) - d(x_{n_r}, x_{n_r})}{2}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
&d(x_{m_r}, x_{n_r}) \\
&\leq\ d(x_{m_r}, x_{m_r+1}) + d(x_{m_r+1}, x_{n_r+1}) + d(x_{n_r+1}, x_{n_r}) \\
&\leq\ d(x_{m_r}, x_{m_r+1}) + d(x_{n_r+1}, x_{n_r}) + \\
&\quad \xi\left(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{m_r+1}), d(x_{n_r}, x_{n_r+1}), \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r}) - d(x_{n_r}, x_{n_r})}{2}\right).
\end{aligned}$$

Thus, by taking lim inf $r \to +\infty$ on both sides, and also using (6.3.4) and (6.3.7), we get

$$\varepsilon \leq 0 + 0 + \xi(\varepsilon, 0, 0, \varepsilon'),$$

where

$$\begin{aligned}
\varepsilon' &= \limsup_{r\to+\infty} \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r}) - d(x_{n_r}, x_{n_r})}{2} \\
&\leq \limsup_{r\to+\infty} \frac{d(x_{m_r}, x_{n_r}) + d(x_{n_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{m_r}) + d(x_{m_r}, x_{n_r}) - 0}{2} \\
&= \frac{\varepsilon + 0 + 0 + \varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, $\varepsilon \leq \xi(\varepsilon, 0, 0, \varepsilon') < \max\{\varepsilon, 0, 0, \varepsilon'\} = \varepsilon$, a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in (X, d) with $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$. Since (X, d) is a complete

b-metric-like space; therefore, there exists $x \in X$ such that sequence $\{x_n\}$ converges to x and

$$d(x,x) = \lim_{n\to+\infty} d(x_n, x) = \lim_{n,m\to+\infty} d(x_n, x_m) = 0.$$

Suppose that $Tx \neq x$. Then, by using (6.3.1), we have

$$d(Tx_n, Tx) \leq \frac{1}{s}\xi\left(d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n) - d(x,x)}{2s}\right),$$

that is,

$$d(x_{n+1}, Tx) \leq \frac{1}{s}\xi\left(d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{d(x_n, Tx) + d(x, x_{n+1})}{2s}\right).$$

Taking $\liminf\ n \to +\infty$ on both sides and using Proposition 6.2.2, we get

$$\frac{1}{s}d(x, Tx) \leq \frac{1}{s}\xi\left(0, 0, d(x, Tx), l\right),$$

that is,

$$d(x, Tx) \leq \xi\left(0, 0, d(x, Tx), l\right),$$

where

$$l = \limsup_{n\to+\infty} \frac{d(x_n, Tx) + d(x, x_{n+1})}{2s} \leq \limsup_{n\to+\infty} \frac{sd(x, Tx) + 0}{2s} = \frac{d(x, Tx)}{2}.$$

Hence,

$$d(x, Tx) \leq \xi\left(0, 0, d(x, Tx), l\right) < \max\{0, 0, d(x, Tx), l\} = d(x, Tx),$$

which is a contradiction. Therefore, $Tx = x$.

Let $Ty = y$ for some $y \in X$, then by (6.3.1), $d(y, y) = 0$. Now, suppose that $x \neq y$. Then by (6.3.1), we have

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\leq \frac{1}{s}\xi\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx) - d(x,x)}{2s}\right), \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{s}\xi\left(d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right),\\
&\leq \frac{1}{s}\xi\left(d(x,y), 0, 0, \frac{d(x,y)}{s}\right)\\
&< \frac{1}{s}\max\left\{d(x,y), 0, 0, \frac{d(x,y)}{s}\right\}\\
&= \frac{d(x,y)}{s},
\end{aligned}
$$

a contradiction. Therefore, $x = y$. □

Example 6.3.8. Let $X = [0, +\infty)$. Define $d : X \times X \to [0, +\infty)$ by

$$d(x,y) = (x+y)^2, \qquad \text{for all } \; x, y \in X.$$

Then d is a b-metric-like on X with $s = 2$, but d is not a b-metric on X.

Define $T : X \to X$ by $T(x) = \frac{x}{2}$. Also, define $\xi(t_1, t_2, t_3, t_4) = \frac{1}{2}\max\{t_1, t_2, t_3, t_4\}$. Then, for all $x, y \in X$ with $d(x, Ty) + d(Tx, y) \geq d(y, y)$, (6.3.1) in Theorem 6.3.1 is satisfied; and T has a unique fixed point 0.

Next, we prove some common fixed point results for weakly compatible mappings in the context of b-metric-like space.

Theorem 6.3.2 *Let (X, d) be a complete b-metric-like space with $s \geq 1$ and let $T,\ S : X \to X$ be two mappings such that $T(X) \subseteq S(X)$, one of these subsets is complete, and there exists $\xi \in \Xi_4$ and*

$$d(Tx, Ty) \leq \frac{1}{s}\xi\left(d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Tx, Sy) - d(Sy, Sy)}{2s}\right), \tag{6.3.9}$$

for all $x, y \in X$ with $d(Sx, Ty) + d(Tx, Sy) \geq d(Sy, Sy)$. If $d(Tz, Sz) = 0$ for some z in X, then z is the unique point of coincidence of T and S. If, moreover, the pair (T, S) is weakly compatible, then T and S have a unique common fixed point.

Proof. Since $d(Tz, Sz) = 0$, therefore, $Sz = Tz$, thus T and S have a point of coincidence $Tz = Sz = w$ (say). Let $Tz_1 = Sz_1 = w_1$ be another point of coincidence of T and S. Suppose that $d(Sz_1, Sz_1) \neq 0$.

Since $d(Sz_1, Tz_1) + d(Tz_1, Sz_1) = 2d(Sz_1, Sz_1) \geq d(Sz_1, Sz_1)$, so by using (6.3.9), we have

$$\begin{aligned}
& d(Sz_1, Sz_1) \\
= \ & d(Tz_1, Tz_1) \\
\leq \ & \frac{1}{s}\xi\left(d(Sz_1, Sz_1), d(Sz_1, Tz_1), d(Sz_1, Tz_1), \frac{d(Sz_1, Tz_1) + d(Sz_1, Tz_1) - d(Sz_1, Sz_1)}{2s}\right), \\
= \ & \frac{1}{s}\xi\left(d(Sz_1, Sz_1), d(Sz_1, Sz_1), d(Sz_1, Sz_1), \frac{d(Sz_1, Sz_1) + d(Sz_1, Sz_1) - d(Sz_1, Sz_1)}{2s}\right), \\
< \ & \frac{1}{s}\max\left\{d(Sz_1, Sz_1), d(Sz_1, Sz_1), d(Sz_1, Sz_1), \frac{d(Sz_1, Sz_1)}{2s}\right\} \\
= \ & \frac{d(Sz_1, Sz_1)}{s},
\end{aligned}$$

which is a contradiction, therefore, $d(Sz_1, Sz_1) = 0$. Next, suppose that $d(w, w_1) \neq 0$, and since $d(Sz, Tz_1) + d(Tz, Sz_1) \geq 0 = d(Sz_1, Sz_1)$, so by (6.3.9), we have

$$\begin{aligned}
& d(w, w_1) \\
= \ & d(Tz, Tz_1) \\
\leq \ & \frac{1}{s}\xi\left(d(Sz, Sz_1), d(Sz, Tz), d(Sz_1, Tz_1), \frac{d(Sz, Tz_1) + d(Sz_1, Tz) - d(Sz_1, Sz_1)}{2s}\right), \\
= \ & \frac{1}{s}\xi\left(d(w, w_1), 0, 0, \frac{d(w, w_1) + d(w_1, w)}{2s}\right), \\
\leq \ & \frac{1}{s}\xi\left(d(w, w_1), 0, 0, \frac{d(w, w_1)}{s}\right) \\
< \ & \frac{1}{s}\max\left\{d(w, w_1), 0, 0, \frac{d(w, w_1)}{s}\right\} \\
= \ & \frac{d(w, w_1)}{s},
\end{aligned}$$

a contradiction. Therefore, $d(w, w_1) = 0$, i.e., $w = w_1$. Thus, by Proposition 6.2.1, if T and S are weakly compatible, then T and S have a unique common fixed point. □

Theorem 6.3.2 is still valid even if the condition " $d(Tz, Sz) = 0$ *for some* z *in* X" is removed as shown in the following theorem.

Theorem 6.3.3 *Let* (X, d) *be a complete b-metric-like space with* $s \geq 1$. *Let* $T,\ S : X \to X$ *be two mappings such that* $T(X) \subseteq S(X)$, *one of these subsets is complete, and there exists* $\xi \in \Xi_4$ *such that (6.3.9) holds for all* $x, y \in X$ *with* $d(Sx, Ty) + d(Tx, Sy) \geq d(Sy, Sy)$. *Then* T *and* S *have a unique point of coincidence. If, moreover, the pair* (T, S) *is weakly compatible, then* T *and* S *have a unique common fixed point.*

Proof. Consider $S(X)$ is complete. Let $x_0 \in X$. As $T(X) \subseteq S(X)$, so define a sequence $\{y_n\}$ in $S(X)$ by

$$y_n = Tx_n = Sx_{n+1}, \text{ for all } n \geq 0.$$

If $d(y_m, y_{m+1}) = 0$ for some m, then $d(Tz, Sz) = 0$, where $z = x_{m+1}$. Thus, by Theorem 6.3.2, proof is complete. We now prove the result when $d(y_m, y_{m+1}) \neq 0$ for each $m \in \mathbb{N}$. For this, we first prove that $\{y_n\}$ is a Cauchy sequence. Let $n \in \mathbb{N}$, then we have

$$d(Sx_n, Tx_{n+1}) + d(Tx_n, Sx_{n+1}) = d(y_{n-1}, y_{n+1}) + d(y_n, y_n) \geq d(y_n, y_n) = d(Sx_{n+1}, Sx_{n+1}).$$

Therefore, by using (6.3.9), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) \leq & \frac{1}{s}\xi(d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), \\ & \frac{d(Sx_n, Tx_{n+1}) + d(Tx_{n+1}, Sx_{n+1}) - d(Sx_{n+1}, Sx_{n+1})}{2s}), \end{aligned}$$

that is,

$$d(y_n, y_{n+1}) \le \frac{1}{s}\xi\left(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{d(y_{n-1}, y_{n+1}) + d(y_n, y_n) - d(y_n, y_n)}{2s}\right) \tag{6.3.10}$$

$$\begin{aligned} &< \frac{1}{s}\max\left\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{d(y_{n-1}, y_{n+1})}{2s}\right\} \\ &= \frac{1}{s}\max\left\{d(y_{n-1}, y_n), \frac{d(y_{n-1}, y_{n+1})}{2s}\right\} \\ &\le \frac{1}{s}\max\left\{d(y_{n-1}, y_n), \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{2}\right\}, \end{aligned}$$

which implies that

$$d(y_n, y_{n+1}) < \frac{1}{s} d(y_{n-1}, y_n), \qquad \text{for all } n \ge 1. \tag{6.3.11}$$

Case 1: If $s > 1$, then by Lemma 6.2.3 and in view of (6.3.11), $\{y_n\}$ is a Cauchy sequence in $(S(X), d)$ and $\lim_{n,m\to+\infty} d(y_n, y_m) = 0$.

Case 2: If $s = 1$, then by (6.3.11), the sequence $\{d(y_n, y_{n+1})\}$ is monotonically decreasing and bounded below. Therefore, sequence $\{d(y_n, y_{n+1})\}$ converges to a real k, where $k \ge 0$. Suppose that $k > 0$. Now taking $\liminf\ n \to +\infty$ in (6.3.10), we have $k \le \xi(k, k, k, k')$,

where

$$k' = \limsup_{n\to+\infty} \frac{d(y_{n-1}, y_{n+1})}{2} \le \limsup_{n\to+\infty} \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{2} = k.$$

Now, $k \le \xi(k, k, k, k') < \max\{k, k, k, k'\} = k$, a contradiction, therefore,

$$\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0. \tag{6.3.12}$$

Also

$$d(y_n, y_n) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_n).$$

Taking lim $n \to +\infty$, and using (6.3.12), we get

$$\lim_{n\to+\infty} d(y_n, y_n) = 0. \tag{6.3.13}$$

Suppose that $\lim_{n,m\to+\infty} d(y_n, y_m) \neq 0$, then there exists $\varepsilon > 0$ such that for any $r \in \mathbb{N}$, there exists $m_r > n_r \geq r$ such that

$$d(y_{m_r}, y_{n_r}) \geq \varepsilon. \tag{6.3.14}$$

Also, assume that m_r is the smallest natural number greater than n_r such that (6.3.14) holds. Then,

$$\begin{aligned} \varepsilon &\leq d(y_{m_r}, y_{n_r}) \\ &\leq d(y_{m_r}, y_{m_r-1}) + d(y_{m_r-1}, y_{n_r}) \\ &< d(y_{m_r}, y_{m_r-1}) + \varepsilon \\ &< d(y_r, y_{r-1}) + \varepsilon. \end{aligned}$$

Thus, by using (6.3.12) and taking lim $r \to +\infty$, we get

$$\lim_{r\to+\infty} d(y_{m_r}, y_{n_r}) = \varepsilon. \tag{6.3.15}$$

Now, suppose that there exists infinitely many r such that

$$d(Sx_{m_r+1}, Tx_{n_r+1}) + d(Tx_{m_r+1}, Sx_{n_r+1}) < d(Sx_{n_r+1}, Sx_{n_r+1}) = d(y_{n_r}, y_{n_r}).$$

Taking lim $r \to +\infty$, and using (6.3.13), we get

$$\lim_{r\to+\infty} (d(y_{m_r}, y_{n_r+1}) + d(y_{m_r+1}, y_{n_r})) = 0,$$

which gives that

$$\lim_{r\to+\infty} d(y_{m_r}, y_{n_r+1}) = \lim_{r\to+\infty} d(y_{m_r+1}, y_{n_r}) = 0.$$

Now,

$$\varepsilon = \lim_{r\to+\infty} d(y_{m_r}, y_{n_r}) \le \limsup_{r\to+\infty}((d(y_{m_r}, y_{m_r+1}) + d(y_{m_r+1}, y_{n_r})) = 0,$$

a contradiction. Therefore, there exists $r_0 \in \mathbb{N}$ such that for all $r \ge r_0$,

$$d(Sx_{m_r+1}, Tx_{n_r+1}) + d(Tx_{m_r+1}, Sx_{n_r+1}) \ge d(Sx_{n_r+1}, Sx_{n_r+1}).$$

Thus, for all $r \ge r_0$, using (6.3.9), we have

$$\begin{aligned}
&d(Tx_{m_r+1}, Tx_{n_r+1}) \\
\le\ &\xi(d(Sx_{m_r+1}, Sx_{n_r+1}), d(Sx_{m_r+1}, Tx_{m_r+1}), d(Sx_{n_r+1}, Tx_{n_r+1}), \\
&\quad \frac{d(Sx_{m_r+1}, Tx_{n_r+1}) + d(Tx_{m_r+1}, Sx_{n_r+1}) - d(Sx_{n_r+1}, Sx_{n_r+1})}{2}),
\end{aligned}$$

that is,

$$\begin{aligned}
&d(y_{m_r+1}, y_{n_r+1}) \\
\le\ &\xi\left(d(y_{m_r}, y_{n_r}), d(y_{m_r}, y_{m_r+1}), d(y_{n_r}, y_{n_r+1}), \frac{d(y_{m_r}, y_{n_r+1}) + d(y_{m_r+1}, y_{n_r}) - d(y_{n_r}, y_{n_r})}{2}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
&d(y_{m_r}, y_{n_r}) \\
\le\ &d(y_{m_r}, y_{m_r+1}) + d(y_{m_r+1}, y_{n_r+1}) + d(y_{n_r+1}, y_{n_r}) \\
\le\ &d(y_{m_r}, y_{m_r+1}) + d(y_{n_r+1}, y_{n_r}) + \\
&\xi\left(d(y_{m_r}, y_{n_r}), d(y_{m_r}, y_{m_r+1}), d(y_{n_r}, y_{n_r+1}), \frac{d(y_{m_r}, y_{n_r+1}) + d(y_{m_r+1}, y_{n_r}) - d(y_{n_r}, y_{n_r})}{2}\right).
\end{aligned}$$

Thus, by taking $\liminf\ r \to +\infty$ on both sides, and also using (6.3.12) and (6.3.15), we get $\varepsilon \leq 0 + 0 + \xi(\varepsilon, 0, 0, \varepsilon')$, where

$$\begin{aligned} \varepsilon' &= \limsup_{r\to+\infty} \frac{d(y_{m_r}, y_{n_r+1}) + d(y_{m_r+1}, y_{n_r}) - d(y_{n_r}, y_{n_r})}{2} \\ &\leq \limsup_{r\to+\infty} \frac{d(y_{m_r}, y_{n_r}) + d(y_{n_r}, y_{n_r+1}) + d(y_{m_r+1}, y_{m_r}) + d(y_{m_r}, y_{n_r}) - 0}{2} \\ &= \frac{\varepsilon + 0 + 0 + \varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $\varepsilon \leq \xi(\varepsilon, 0, 0, \varepsilon') < \max\{\varepsilon, 0, 0, \varepsilon'\} = \varepsilon$, a contradiction. Hence, $\{y_n\}$ is a Cauchy sequence in $(S(X), d)$ with $\lim\limits_{n,m\to+\infty} d(y_n, y_m) = 0$. Since $(S(X), d)$ is a complete b-metric-like space, therefore, there exists $z \in X$ such that sequence $\{y_n = Tx_n = Sx_{n+1}\}$ converges to Sz, and

$$d(Sz, Sz) = \lim_{n\to+\infty} d(y_n, Sz) = \lim_{n,m\to+\infty} d(y_n, y_m) = 0.$$

Suppose that $d(Sz, Tz) \neq 0$. Now, by using (6.3.9), we have

$$d(Tx_n, Tz) \leq \frac{1}{s}\xi\left(d(Sx_n, Sz), d(Sx_n, Tx_n), d(Sz, Tz), \frac{d(Sx_n, Tz) + d(Sz, Tx_n) - d(Sz, Sz)}{2s}\right),$$

that is,

$$d(y_n, Tz) \leq \frac{1}{s}\xi\left(d(y_{n-1}, Sz), d(y_{n-1}, y_n), d(Sz, Tz), \frac{d(y_{n-1}, Tz) + d(Sz, y_n)}{2s}\right).$$

Taking $\liminf\ n \to +\infty$ on both sides and using Proposition 6.2.2, we get

$$\frac{1}{s}d(Sz, Tz) \leq \frac{1}{s}\xi\left(0, 0, d(Sz, Tz), l\right),$$

that is,

$$d(Sz, Tz) \leq \xi\left(0, 0, d(Sz, Tz), l\right),$$

where

$$l = \limsup_{n\to+\infty} \frac{d(y_{n-1}, Tz) + d(Sz, y_n)}{2s} \leq \limsup_{n\to+\infty} \frac{sd(Sz, Tz) + 0}{2s} = \frac{d(Sz, Tz)}{2}.$$

Hence,

$$d(Sz,Tz) \leq \xi\left(0,0,d(Sz,Tz),l\right) < \max\{0,0,d(Sz,Tz),l\} = d(Sz,Tz),$$

which is a contradiction, therefore, $d(Sz,Tz) = 0$. Hence, by Theorem 6.3.2, proof is complete. □

Theorem 6.3.4 *Let (X,d) be a complete b-metric-like space with $s \geq 1$. Let $T,\ S : X \to X$ be two mappings such that $T(X) \subseteq S(X)$, one of these subsets is complete, and there exists $\xi \in \Xi_5$ and*

$$d(Tx,Ty) \leq \frac{1}{s}\xi\left(d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),\frac{d(Sx,Ty)}{2s},d(Tx,Sy)-d(Sy,Sy)\right),$$

for all $x,y \in X$ with $d(Tx,Sy) \geq d(Sy,Sy)$. Then T and S have a unique point of coincidence. If, moreover, the pair (T,S) is weakly compatible, then T and S have a unique common fixed point.

Proof. Proof of the theorem follows a similar line as in Theorem 6.3.3. □

6.4. Consequences

Considering S be an identity mapping in Theorem 6.3.3 and Theorem 6.3.4, we obtained, respectively Theorem 6.3.1 and the following result

Corollary 6.4.1 *Let (X,d) be a complete b-metric-like space with $s \geq 1$. Let $T : X \to X$ be a mapping such that there exists $\xi \in \Xi_5$ and*

$$d(Tx,Ty) \leq \frac{1}{s}\xi\left(d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)}{2s},d(Tx,y)-d(y,y)\right),$$

for all $x,y \in X$ with $d(Tx,y) \geq d(y,y)$. Then T has a unique fixed point.

Remark 6.4.1. As $d(y, y) = 0$ in b-metric space, so Theorem 6.3.1 and Corollary 6.4.1 reduces to Theorem 5.3.1 and Theorem 5.3.2 respectively.

Bibliography

[1] Abbas, M.; and Jungck, G.; *Common fixed point results for noncommuting mappings without continiuty in cone metric spaces*, Journal of Mathematical Analysis and Applications, **341(1)**, (2008), 416-420.

[2] Abdeljawad, T.; Karapinar, E.; and Taş, K.; *Existence and uniqueness of a common fixed point on partial metric spaces*, Applied Mathematics Letters, **24(11)**, (2011), 1900-1904, doi: 10.1016/j.aml.2011.05.014

[3] Aghajani, A.; Abbas, M.; and Roshan, J.R.; *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Mathematica Slovaca, **64(4)**, (2014), 941-960, doi: 10.2478/s12175-014-0250-6.

[4] Aghajani, A.; Abbas, M.; and Roshan, J.R.; *Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces*, Filomat, **28:6**, (2014), 1087-1101, doi: 10.2298/Fil1406087A.

[5] Ahmed, M.A.; *Common fixed point theorems for weakly compatible mappings*, The Rocky Mountain Journal of Mathematics, **33(4)**, (2003), 1189-1203.

[6] Alber, Ya.; Guerre-Delabriere, S.; *Principle of weakly contractive maps in Hilbert spaces*, Operator Theory: Advances and Applications, **98**, (1997), 7-22.

[7] Aleksić, S.; Mitrović, Z.D.; and Radenović, S.; *On some recent fixed point results for single and multivalued mappings in b-metric spaces*, Fasciculi Mathematici, (2018), doi: 10.1515/fascmath-2018-0013.

[8] Alghamdi, M.A.; Hussain, N.; and Salimi, P.; *Fixed point and coupled fixed point theorems on b-metric-like spaces*, Journal of Inequalities and Applications, (2013), **402**, 1-25.

[9] Ali, J.; Imdad, M.; and Bahuguna, D.; *Common fixed point theorems in Menger spaces with common property (E.A)*, Computers and Mathematics with Applications, (2010), **60**, 3152-3159.

[10] Ansari, A.H.; Vetro, P.; and Radenović, S.; *Existence of fixed point for $GP_{(\Lambda,\Theta)}$-contractive mappings in G_p-metric spaces*, Filomat, **31(8)**, (2017), 2211-2218, doi: 10.2298/FIL1708211A.

[11] Asim, M.; and Imdad, M.; *Partial JS-metric spaces and fixed point results*, Indian Journal of Mathematics, **61(2)**, (2019), 175-186.

[12] Asl, J.H.; Rezapour, S.; and Shahzad, N.; *On fixed points of α-ψ-contractive multifunctions*, Fixed Point Theory and Applications, (2012), **2012:212**.

[13] Aydi, H.; Bota, M.F.; Karapinar, E.; and Mitrović, S.; *A fixed point theorem for set-valued quasi-contractions in b-metric spaces*, Fixed Point Theory and Applications, (2012), **2012:88**.

[14] Aydi, H.; Karapinar, E.; and Salimi, P.; *Some fixed point results in G_p-metric spaces*, Journal of Applied Mathematics, **2012**, 16 pages, article ID 891713.

[15] Aydi, H.; Rakić, D.; Aghajani, A.; Dosenović, T.; Noorani, M.S.M.; and Qawaqneh, H.; *On fixed point results in G_b-metric spaces*, Mathematics, **7(7)**:617, (2019).

[16] Bahuguna, D.; and Sharma, A.; *On the convergence of a new iterative algorithm of three infinite families of generalized nonexpansive multi-valued mappings*, Proceedings of Indian Academy of Sciences (Mathematical Sciences), (2018), **128:52**.

[17] Bakhtin, I.A.; *The contraction mapping principle in almost metric spaces*, Functional Analysis, **30**, (1989), 26-37.

[18] Banach, S.; *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematicae, **3**, (1922), 133-181.

[19] Barakat, M.A.; and Zidan, A.M.; *A common fixed point theorem for weak contractive maps in G_p-metric spaces*, Journal of Egptian Mathematical Society, **23**, (2015), 309-314.

[20] Belluce, L.P.; and Kirk, W.A.; *Nonexpansive mappings and fixed points in Banach spaces*, Illinois Journal of Mathematics, **11**, (1967), 474-479.

[21] Bilgili, N.; Karapinar, E.; and Salimi, P.; *Fixed point theorems for generalized contractions on G_p-metric spaces*, Journal of Inequalities and Applications, (2013), **2013:39**.

[22] Bohnenblust, H.F.; and Karlin, S.; *On a theorem of Ville-Contributions to the theory of games*, Univ. Press Princeton, **I**, (1950), 155-160.

[23] Boriceanu, M.; *Fixed point theory for multivalued generalized contraction on a set with two b-metrics*, Studia Universitatis Babeş-Bolyai Mathematica, **LIV(3)**, (2009), 1-14.

[24] Boriceanu, M.; Bota, M.; and Petruşel A.; *Multivalued fractals in b-metric spaces*, Central European Journal of Mathematics, **8(2)**, (2010), 367-377, doi: 10.2478/s11533-010-0009-4.

[25] Bourbaki, N.; *Topologie Generale*, Herman: Paris, France, (1974).

[26] Brouwer, L.E.J.; *Uber Abbildungen Von Mannigfaltigkeiten*, Mathematische Analen, **77** (1912), 97-115.

[27] Browder, F.E.; *Nonexpansive nonlinear operators in a Banach space*, Proceedings of the National Academy of Sciences USA, **54**, (1965), 1041-1044.

[28] Bruck, R.E.; *A common fixed point theorem for a commuting family of nonexpansive mappings* , Pacific Journal of Mathematics, **53**, (1974), 59-71.

[29] Çanak, I.; Dik, M., and Dik, F.; *Conditions for convergence and subsequential convergence*, Applied Mathematics Letters, **19**, (2006), 1042-1045.

[30] Chandok, S.; and Narang, T.D.; *Common fixed points for generalized asymptotically nonexpansive mappings*, Tamkang Journal of Mathematics, **44(1)**, (2013), 23-29.

[31] Chatterjea, S.K.; *Fixed point theorems*, Comptes rendus de l′Academie bulgare des Sciences, **25**, (1972), 727-730.

[32] Chen, C.; Dong, J.; and Zhu, C.; *Some fixed point theorems in b-metric-like spaces*, Fixed Point Theory and Applications, (2015), doi: 10.1186/s13663-015-0369-3.

[33] Choudhury, B.S.; *A common unique fixed point result in metric spaces involving generalised altering distances*, Mathematical Communications, **10**, (2005), 105-110.

[34] Choudhury, B.S.; and Metiya, N.; *Multivalued and singlevalued fixed point results in partially ordered metric spaces*, Arab Journal of Mathematical Sciences, **17**, (2011), 135-151.

[35] Chugh, R; and Kumar, S.; *Common fixed points for weakly compatible maps*, Proceedings of the Indian Academy of Sciences (Mathematical Sciences), **111(2)**, (2001), 241-247.

[36] Ćirić, Lj.B.; *A generalization of Banach's contraction principle*, Proceeding of the American Mathematical Society, **45**, (1974), 267-273.

[37] Ćirić, Lj.; Alsulami, S.M.; Parvaneh, V.; and Roshan, J.R.; *Some fixed point results in ordered G_p-metric spaces*, Fixed Point Theory and Applications, (2013), **2013:317**.

[38] Ćirić, Lj.B.; and Ume, J.S.; *Some Common fixed point theorems for weakly compatible mappings*, Journal of Mathematical Analysis and applications, **314(2)**, (2006), 488-499.

[39] Czerwik, S.; *Contraction mappings in b-metric spaces*, Acta Mathematica et Informatica Universitatis Ostraviensis, **30**, (1993), 5-11.

[40] Czerwik, S.; *Nonlinear set-valued contraction mapping in b-metric spaces*, Atti del Seminaro Matematico e Fisico dell'Universita di Modena e Reeggio Emilia, **46**, (1998), 263-276.

[41] Dhage, B.C.; *A study of some fixed point theorem*, Ph.D. Thesis, Marathwada University of Aurangabad, India, (1984).

[42] Dhage, B.C.; *Generalized metric spaces and mappings with fixed point*, Bulletin of the Calcutta Mathematical Society, **84**, (1992), 329-336.

[43] Dhage, B.C.; *On generalized metric spaces and topological structure-II*, Pure and Applied Mathematika Sciences, **40**, (1994), 37-41.

[44] Dhage, B.C.; *Generalized Metric Spaces and Topological Structure I*, Analele ştinţifice Ale Universităţii "Alexandru Ioan Cuza" Iaşi Tomul Xlvi s.I a Matematică, **46**, (2000), 3-24.

[45] Dubey, A.K.; Shukla, R.; and Dubey, R.; *Some fixed point results in b-metric spaces*, Asian Journal of Mathematics and Applications, (2014), 1-6, article ID ama0147.

[46] Ege, O.; Park, C.; and Ansari, A.H.; *A different approach to complex valued* G_b*-metric spaces*, Advances in Difference Equations, **2020**:152, (2020).

[47] Fan, Ky.; *Fixed point and minimax theorems in locally convex topological linear spaces*, Proceedings of the National Academy of Sciences of the United States of America, **38**, (1952), 121-126.

[48] Fréchet, M.; *Sur quelques points du Calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, **22**, (1906), 1-74.

[49] Gähler, S.; *2 Metrische Räme und Ihr Topological Struktur*, Mathematische Nachrichten, **26**, (1963), 115-148.

[50] Gähler, S.; *Zur Geometric 2-Metriche Raume*, Revue Roumaine de Mathematiques Pures et Appliquees, **40**, (1966), 664-669.

[51] Gajić, L.; Kadelburg, Z.; and Radenović, S.; *G_p-metric spaces−symmetric and asymmetric*, University of Novi Paza Series A Applied Mathematics Informatics and Mechanics, **9(1)**, (2017), 37-46.

[52] Glicksberg, I.; *A further generalization of Kakutani fixed point theorem, with application to Nash equilibrium points*, Proceedings of the American Mathematical Society, **3**, (1952), 170-174.

[53] Goswami, N.; Hokip, N.; and Mishra, V.N.; *F-contrative type mappings in b-metric spaces and some related fixed point results*, Fixed Point Theory and Applications, (2019), **2019:13**, doi: 10.1186/s13663-019-0663-6.

[54] Guang, H.L.; and Xian, Z.; *Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications, **332**, (2007), 1468-1476.

[55] Gupta, V.; and Acu, A.M.; *On difference of operators with different basis functions*, Filomat, **33(10)**, (2019), 3023-3034.

[56] Gupta, V.; and Agrawal, P.N.; *Approximation by modified Păltănea operators*, Publications De L'Institut Mathématique, **107(121)**, (2020), 157-164.

[57] Gupta, V.; Ege, O.; Saini, R.; and Sen, M.D.L.; *Various fixed point results in complete G_b-metric spaces*, Dynamic Systems and Applications, **30(2)**, (2021), 277-293.

[58] Ha, K.S.; Cho, Y.J.; and White, A.; *Strictly convex and strictly 2-Convex 2-normed spaces*, Mathematica Japonica, **33**, (1988), 375-384.

[59] Han, Y.; and Xu, S.; *Some new theorems of expanding mappings without continuity in cone metric spaces*, Fixed Point Theory and Applications, (2013), **2013:3**.

[60] Haranadi, A.A.; *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory and Applications, (2012), **2012:204**.

[61] Hardy, G.E.; and Rogers, T.D.; *A generalization of a fixed point theorems of Reich*, Canadian Mathematical Bulletin, **16(2)**, (1973), 201-206.

[62] Heckmann, R.; *Approximation of metric spaces by partial metric spaces*, Applied Categorical Structures, **7**, (1999), 71-83.

[63] Huang, H.; Deng, G.; and Radenović, S.; *Fixed point theorems in b-metric spaces with applications to differential equations*, Journal of Fixed Point Theory and Applications, (2018), doi: 10.1007/s11784-018-0491-z.

[64] Jain, K.; and Kaur, J.; *A generalization of G-metric spaces and related fixed point theorems*, Mathematical Inequalities and Applications, **22**, (2019), 1145-1160.

[65] Jaradat, M.M.M.; Mustafa, Z.; Khan, S.U.; Arshad, M.; and Ahmad, J.; *Some fixed point results on G-metric and G_b-metric spaces*, Demonstratio Mathematica, **50**, (2017), 190-207.

[66] Jleli, M.; and Samet, B.; *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory and Applications, (2012), **2012:210**.

[67] Jleli, M.; and Samet, B.; *A generalized metric space and related fixed point theorems*, Fixed Point Theory and Applications, (2015), doi: 10.1186/s13663-015-0312-7, 14 pages.

[68] Jleli, M.; and Samet, B.; *On a new generalization of metric spaces*, Journal of Fixed Point Theory and Application, **20**, (2018), 128.

[69] Jovanović, M.; Kadelburg, Z.; and Radenović, S.; *Common fixed point results in metric-type spaces*, Fixed Point Theory and Applications, (2010), doi: 10.1155/2010/978121.

[70] Jungck, G.; *Commuting mappings and fixed points*, The American Mathematical Monthly, **83**, (1976), 261-263.

[71] Jungck, G.; *Compatible mappings and common fixed points*, International Journal of Mathematics and Mathematical Sciences, **9(4)**, (1986), 771-779.

[72] Jungck, G.; Radenović, S.; Radojević, S.; and Rakočević, V.; *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory and Applications, (2009), Article ID 643840, 13 pages.

[73] Jungck, G.; and Rhoades, B.E.; *Fixed point for set-valued functions without continuity*, Indian Journal of Pure and Applied Mathematics, **29**, (1998), 227-238.

[74] Kakutani, S.; *A generalization of Brouwer's fixed point theorem*, Duke Mathematical Journal, **8**, (1941), 457-459.

[75] Kannan, R.; *Some results on fixed points*, Bulletin of the Calcutta Mathematical Society, **60**, (1968), 71-76.

[76] Karapinar, E.; Czerwik, S.; and Aydi, H.; (α, ψ)*-Meir-Keeler contraction mappings in generalized b-metric spaces*, Journal of Function Spaces, (2018), doi: 10.1155/2018/3264620.

[77] Karapinar, E.; Mitrović, Z.D.; Öztürk, A.; and Radenović, S.; *On a theorem of Ćirić in b-metric spaces*, Rendiconti del Circolo Matematico di Palermo Aeries 2, March 2020, doi: 10.1007/s12215-020-00491-9.

[78] Karapinar, E.; O'Regan, D.; Hierro, A.; and Shahzad, N.; *Fixed point theorems in new generalized metric spaces*, Journal of Fixed Point Theory and Applications, **18**, (2016), 645-671.

[79] Karapinar, E.; and Samet, B.; *Generalized α-ψ-contractive type mappings and related fixed point theorems with applications*, Abstract and Applied Analysis, (2012), Article ID 793486, 17 pages.

[80] Kaushik, P.; Kumar, S.; and Tas, K.; *A new class of contraction in b-metric spaces and applications*, Abstract and Applied Analysis, (2017), Article ID 9718535, 10 pages.

[81] Khamsi, M.A.; and Hussain, N.; *KKM mappings in metric type spaces*, Nonlinear Analysis, **73**, (2010), 3123-3129.

[82] Khomdram, B.; Rohen, Y.; and Singh, T.C.; *Coupled fixed point theorems in G_b-metric space satisfying some rational contractive conditions*, SpringerPlus, **5**, (2016), 1261.

[83] Kir, M.; and Kiziltunc, H.; *On some well known fixed point theorems in b-metric spaces*, Turkish Journal of Analysis and Number Theory, **1**, (2013), 13-16.

[84] Kreyszig, E.; *Introductory Functional Analysis With Application.* John Wiley and Sons, New York Santa Barbara London Sydney Toronto, (1978).

[85] Kumar, S.; and Rani, A.; *Some common fixed point theorems in menger space*, Applied Mathematics, **3**, (2012), 235-245.

[86] Kumar, J.; and Vashistha, S.; *Coupled fixed point theorems in complex valued G_b-metric spaces*, Advances in Fixed Point Theory, **6**, (2016), 341-351.

[87] Latif, A.; Parvaneh, V.; Salimi, P.; and Al-Mazrooei, A.E.; *Various Suzuki type theorems in b-metric spaces*, Journal of Nonlinear Science and Applications, **8(4)**, (2015), 363-377.

[88] Lim, T.C.; *A fixed point theorem for families of nonexpansive mappings*, Pacific Journal of Mathematics, **53(2)**, (1974), 487-493.

[89] Matthews, G.S.; *Partial metric topology*, Research Report 212, Deptt. of Computer Science, University of Warwick, 1992.

[90] Matthews, G.S.; *Partial metric topology*, Annals of the New York Academy of Sciences, **728**, (1994), 183-197.

[91] Meir, A.; and Keeler, E.; *A theorem on contraction mappings*, Journal of Mathematical Analysis and Applications, **28**, (1969), 326-329.

[92] Miculescu, R.; and Mihail, A.; *New fixed point theorems for set-valued contractions in b-metric spaces*, Journal of Fixed Point Theory and Applications, **19(3)**, (2017), 2153-2163.

[93] Mishra, P.K.; Sachdeva, S.; and Banerjee, S.K.; *Some fixed point theorems in b-metric space*, Turkish Journal of Analysis and Number Theory, **2(1)**, (2014), 19-22.

[94] Mitrović, Z.D.; Parvaneh, V.; Mlaiki, N.; Hussain N.; and Radenović, S.; *On some new generalizations of Nadler contraction in b-metric spaces*, Cogent Mathematics and Statistics, (2020), **7:1**, 1760189.

[95] Mlaiki, N.; Dedovic, N.; Aydi, H.; Filipović, M.G.; Mohsin, B.B.; and Radenović, S.; *Some new observations on Geraghty and Ćirić type results in b-metric spaces*, Mathematics, **7**, (2019), doi: 10.3390/math7070643.

[96] Mursaleen, M.; Mohiuddine, S.A.; and Agarwal, R.P.; *Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces*, Fixed Point Theory and Application, (2012), **2012:228**.

[97] Mustafa, Z.; *A new structure for generalized metric spaces with applications to fixed point theory*, Ph.D. Thesis, The University of Newcastle, Callaghan, Australia, (2005).

[98] Mustafa, Z.; Arshad, M.; Khan, S.U.; Ahmad, J.; and Jaradat, M.M.M.; *Common fixed points for multivalued mappings in G-metric spaces with applications*, Journal of Nonlinear Sciences and Applications, (2017), 2550-2564, doi: 10.22436/jnsa.010.05.23.

[99] Mustafa, Z.; Awawdeh, F.; and Shatanawi, W.; *Fixed point theorem for expansive mappings in G-metric spaces*, International Journal of Contemporary Mathematical Sciences, **5**, (2010), 2463-2472.

[100] Mustafa, Z.; Jaradat, M.M.M.; Aydi, H.; and Alrhayyel, A.; *Some common fixed points of six mappings on G_b-metric spaces using (E.A) property*, European Journal of Pure and Applied Mathematics, **11**, (2018), 90-109.

[101] Mustafa, Z.; Obiedat, H.; and Awawdeh, F.; *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory and Applications, (2008), Article ID 189870, doi: 10.1155/2008/189870.

[102] Mustafa, Z.; Roshan, J.R.; and Parvaneh, V.; *Coupled coincidence point results for (ψ, ϕ)-weakly contractive mappings in partially ordered G_b-metric spaces*, Fixed Point Theory and Applications, (2013), **2013**, 206.

[103] Mustafa, Z.; Roshan, J.R.; Parvaneh, V.; and Kadelburg, Z.; *Some common fixed point results in ordered partial b-metric spaces*, Journal of Inequalities and Applications, **2013(1)**, (2013), 1-26.

[104] Mustafa, Z.; Shatanawi, W.; and Bataineh, M.; *Existence of fixed point results in G-metric spaces*, International Journal of Mathematics and Mathematical Sciences, (2009), Article ID 283028, 10 pages, doi: 10.1155/2009/283028.

[105] Mustafa, Z.; and Sims, B.; *Some remarks concerning D-metric spaces*, Proceedings of the International Conference on Fixed Point Theory and Applications, Valencia(Spain), July(2003), 189-198.

[106] Mustafa, Z.; and Sims, B.; *A new approach to generalized metric spaces*, Journal of Nonlinear and Convex Analysis, **7**, (2006), 289-297.

[107] Mustafa, Z.; and Sims, B.; *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory and Applications, (2009), Article ID 917175, doi: 10.1155/2009/917175.

[108] Nadler, Jr.S.B.; *Multivalued contraction mappings*, Pacific Journal of Mathematics, **30**, (1969), 475-488.

[109] Naidu, S.V.R.; Rao, K.P.R. and Rao, N.S.; *On the concepts of Balls in a D-metric space*, International Journal of Mathematics and Mathematical Sciences, **2005(1)**, (2005), 133-141, doi: 10.1155/IJMMS.2005.133.

[110] Naidu, S.V.R.; Rao, K.P.R. and Rao, N.S.; *On convergent sequences and fixed point theorems in D-metric spaces*, International Journal of Mathematics and Mathematical Sciences, (2005), 1969-1988, doi: 10.1155/IJMMS.2005.1969.

[111] Nashine, H.K.; Agarwal, R.P.; and Kadelburg, Z.; *H^+-Multivalued contractions and their application to homotopy theory*, Journal of Fixed Point Theory and Application, (2017), doi: 10.1007/s11784-017-0425-1, Springer International Publishing 2017.

[112] Ozturk, V.; and Radenović, S.; *Some remarks on b-(E.A)-property in b-metric spaces*, Springer Plus, (2016), **5:544**, doi: 10.1186/s40064-016-2163-z.

[113] Pant, R.; and Panicker, R.; *Geraghty and Ćirić type fixed point theorems in b-metric spaces*, Journal of Nonlinear Science and Applications, **9**, (2016), 5741-5755.

[114] Parvaneh, V.; Roshan J.R.; and Kadelburg, Z.; *On generalized weakly G_p-contractive mappings in ordered G_p-metric spaces*, Gulf Journal of Mathematics, **1**, (2013), 78-97.

[115] Pathak, H.K.; and Shahzad, N.; *A generalization of Nadler's fixed point theorem and its application to nonconvex integral inclusions*, Topological Methods in Nonlinear Analysis, **41(1)**, (2013), 207-227.

[116] Poincare, H; *Surless courbes define barles equations differentiate less*, Journal of de Math, **2**, (1886), 54-65.

[117] Popa, V.; *A general fixed point theorem for four weakly compatible mappings satisfying on implicit relation*, Univerzitet u Nisu. Prirodno-Matematicki Fakultet, Filomat, **19**, (2005), 45-51.

[118] Popa, V.; *A general fixed point theorem for two pairs of absorbing mappings in G_p-metric spaces*, Annales Mathematicae Silesianae, (2020), doi: 10.2478/amsil-2020-0004.

[119] Popa, V.; and Patriciu, A.M.; *Two general fixed point theorems for a sequence of mappings satisfying implicit relation in G_p-metric spaces*, Applied General Topology, **16(2)**, (2015), 225-231.

[120] Qawaqneh, H.; Noorani, M.S.M.; Shatanawi, W.; Aydi, H.; and Alsamir, H.; *Fixed point results for multi-valued contractions in b-metric spaces and an application*, Mathematics, **7(132)**, (2019), doi: 10.3390/math7020132.

[121] Reich, S.; *Some remarks concerning contraction mappings*, Canadian Mathematical Bulletin, **14**, (1971), 121-124.

[122] Rhoades, B.E.; *A comparison of various definitions of contractive mappings*, Transactions of the American Mathematical Society, **226**, (1977), 257-290.

[123] Rhoades, B.E.; *Some theorems on weakly contractive maps*, Non-linear Analysis TMA, **47(4)**, (2001), 2683-2694.

[124] Roshan, J.R.; Shobkolaei, N.; Sedghi, S.; Parvaneh V. and Radenović, S.; *Common fixed point theorems for three maps in discontinuous G_b-metric spaces*, Acta Mathematica Scientia, **34B(5)**, (2014), 1643-1654.

[125] Samet, B.; Vetro, C.; and Vetro, P.; *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Analysis: Theory, Methods and Applications, **75(4)**, (2012), 2154–2165.

[126] Schauder, J.; *Der Fixpunktsatz in Funktionalraumen*, Studia Mathematica, **2**, (1930), 171-180.

[127] Sedghi, S.; Shobkolaei, N.; Roshan J.R.; and Shatanawi, W.; *Coupled fixed point theorems in G_b-metric spaces*, Matematicki Vesnik, (2014), **66(2)**:190-201.

[128] Sen, M.D.; Nikolić, N.; Došenović, T.; Pavlović, M.; and Radenović, S.; *Some results on (s-q)-graphic contraction mappings in b-metric-like spaces*, Mathematics, **7**, (2019), doi: 10.3390/math7121190.

[129] Sessa, S.; *On a weak commutativity conditions of mappings in fixed point consideration*, Publications De L'institut Mathematique, **32**, (1982), 149-153.

[130] Shahi, P.; Kaur, J.; and Bhatia, S.S.; *Fixed point theorems for (ξ, α)-expansive mappings in complete metric spaces*, Fixed Point Theory and Applications, (2012), **2012:157**.

[131] Shahi, P.; Kaur, J.; and Bhatia, S.S.; *Common fixed points of expansive mappings in generalized metric spaces*, Journal of Applied Research and Technology, (2014), **12**, 607-614.

[132] Shahkoohi, R.J.; Kazemipour, S.A.; and Eyvali, A.R.; *Tripled coincidence point under ϕ-contractions in ordered G_b-metric spaces*, Journal of Linear and Topological Algebra, **03(03)**, (2014), 131-147.

[133] Shukla, S.; *Partial b-metric spaces and fixed point theorems*, Mediterranean Journal of Mathematics, **11**, (2014), 703-711.

[134] Singh, S.L.; Bhatnagar, C.; and Mishra, S.N.; *Stability of iterative procedures for multivalued maps in metric spaces*, Demonstratio Mathematica, **38(4)**, (2005), 905-916.

[135] Singh, S.L.; Czerwik, S.; Krol, K.; and Singh, A.; *Coincidences and fixed points of hybrid contractions*, Tamsui Oxford Journal of Mathematical Sciences, **24(4)**, (2008), 401-416.

[136] Sintunavarat, W.; Plubtieng, S.; and Katchang, P.; *Fixed point result and applications on a b-metric space endowed with an arbitrary binary relation*, Fixed Point Theory and Applications, (2013), **2013:296**.

[137] Suzuki, T.; *Basic inequality on a b-metric space and its applications*, Journal of Inequalities and Applications, (2017), **2017:256**, doi: 10.1186/s13660-017-1528-3.

[138] Totur, Ü.; *On the limit inferior and limit superior for double sequences*, Journal of Classical Analysis, **8(1)**, (2016), 53-58.

[139] Tychonoff, A.; *Ein fixpunktsatz*, Mathematische Annalen, **111**, (1935), 767-776.

[140] Vujaković, J.; Mitrović, S.; Mitrović, Z.D.; and Radenović, S.; *On* **F**-*contractions for weak* α*-admissible mappings in metric-like spaces*, Mathematics, **8**, (2020), doi: 10.3390/math8091629.

[141] Wang, S.Z.; Li, B.Y.; Gao, Z.M.; and Iseki, K.; *Some fixed point theorems on expansion mappings*, Mathematica Japonica, **29**, (1984), 631-636.

[142] Wardowski, D.; *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Applications, (2012), **2012:94**, doi: 10.1186/1687-1812-2012-94.

[143] Yazdi, H.G.; Zand, M.R.A.; and Radenović, S.; *Coupled fixed point on* G_p*-metric spaces−symmetric and asymmetric*, Advances and Applications in Mathematical Sciences, **17(10)**, (2018), 681-692.

[144] Zand, M.R.A.; and Nezhad, A.D.; *A generalization of partial metric spaces*, Journal of Contemporary Applied Mathematics, **24**, (2011), 86-93.

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