Group Theoretic Techniques for Solutions of Einstein Equations

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Certificate

This is to certify that the thesis entitled, **Group Theoretic Techniques for Solutions of Einstein Equations**, submitted by Ms. Lakhveer Kaur in the fulfillment of the requirements for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidates own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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Declaration

It is certified that the thesis is entirely my own and that the ideas and references cited herein have been duly acknowledged.

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Abstract

General relativity is a physical theory which plays a key role in astrophysics and is important for a number of ambitious experiments and space missions. Einstein field equations are basic equations of general relativity and are expressed in terms of coupled highly nonlinear partial differential equations describing the matter content of space-time. For this reason it is clear that the theory of partial differential equations is of immense importance in the study of Einstein field equations. The investigations carried out are confined to the applications of the group-theoretic methods, symmetry reduction method, Painlev$\acute{e}$ analysis and Generalized $\frac{G'}{G}$- expansion method to the system of nonlinear partial differential equations arising in general relativity and other important physical phenomenon from mathematical physics.

The thesis entitled **GROUP THEORETIC TECHNIQUES FOR SOLUTIONS OF EINSTEIN EQUATIONS** comprises eight chapters. This thesis is a condensed review of the exact solutions of Einstein field equations and ensuing phenomena.

In **Chapter 1**, some important features of Lie groups of transformations and symmetries are demonstrated. It presents primarily the methodologies utilized in the thesis and a brief account of the related studies made by various authors in the field.

Chapter 2 is devoted to the study of the system of partial differential equations cor-

responding to the Einstein-Maxwell equations for a static axially symmetric spacetime

$$\begin{aligned}&u_{\rho\rho}+\tfrac{u_\rho}{\rho}+u_{zz}=-\exp(-2u)(v_\rho^2+v_z^2)\\&v_{\rho\rho}+\tfrac{v_\rho}{\rho}+v_{zz}-2(u_\rho v_\rho+u_z v_z)=0\\&\tfrac{k_\rho}{\rho}=(u_\rho^2-u_z^2)+\exp(-2u)(v_\rho^2-v_z^2)\\&\tfrac{k_z}{\rho}=2u_\rho u_z+2\exp(-2u)(v_\rho v_z).\end{aligned}$$

By using Lie symmetry method, an optimal system of conjugacy inequivalent subalgebras is then identified with the adjoint action of symmetry group. For each basic vector field in optimal system, the above system is reduced to system of ODEs which is further examined with the aim of deriving certain exact solutions.

In **Chapter 3**, we have investigated Einstein field equations for perfect fluid distribution

$$(1-2u^2)(u_{tt}-u_{xx})+2u(u_t^2-u_x^2)=0,$$

and pure radiation fields

$$\begin{aligned}&u_{rr}+\tfrac{u_r}{r}-u_{tt}=0\\&v_r+v_t-r(u_r+u_t)^2=0\\&v_{rr}-v_{tt}+u_r^2-u_t^2=0.\end{aligned}$$

An optimal system of inequivalent subalgebras of the above system having basic vector fields is determined. Using the non-equivalent Lie ansätz for each essential vector field, the nonlinear ODEs and further exact solutions are constructed.

Chapter 4 is concerned with Einstein field equations corresponding to Weyl-Lewis-Papapetrou form for an axisymmetric rotating field

$$\begin{aligned}&u(u_{\rho\rho}+u_{zz}+\tfrac{u_\rho}{\rho})-u_\rho^2-u_z^2-\tfrac{(v_\rho^2+v_z^2)}{\rho^2}=0\\&u(v_{\rho\rho}+v_{zz}-\tfrac{v_\rho}{\rho})-2u_\rho v_\rho-2u_z v_z=0,\\&u^2w_\rho=uu_\rho+\tfrac{1}{2}\rho(u_\rho^2-u_z^2)-\tfrac{1}{2\rho}(v_\rho^2-v_z^2),\\&u^2w_z=uu_z+\rho u_\rho u_z-\tfrac{1}{\rho}v_\rho v_z.\end{aligned}$$

Using the invariance group properties of the governing system of partial differential equations (PDEs), admitting Lie group of point transformations with commuting infinitesimal generators, some appropriate canonical variables are characterized that transform the equations at hand to an equivalent system of ordinary differential equations and some physically important analytic solutions of field equations are constructed. Also, the class of axially symmetric solutions of Einstein field equations including the Papapetrou solution as particular case has been obtained.

In **Chapter 5**, the invariance under continuous groups of transformations of a system of nonlinear partial differential equations derived from a line element with axial symmetry for empty space containing an electrostatic field,

$$
\begin{aligned}
&\lambda_{11} + \lambda_{22} + \rho_1^2 + \tfrac{\lambda_2}{x_2} = 2\exp(-\rho)(\phi_1^2 - \phi_2^2)\\
&\lambda_{11} + \lambda_{22} + \rho_2^2 - \tfrac{\lambda_2}{x_2} - \tfrac{2\rho_2}{x_2} = -2\exp(-\rho)(\phi_1^2 - \phi_2^2)\\
&\rho_1\rho_2 - \tfrac{\rho_1}{x_2} - \tfrac{\lambda_1}{x_2} = 4\exp(-\rho)\phi_1\phi_2\\
&\rho_{11} + \rho_{22} + \tfrac{\rho_2}{x_2} = 2\exp(-\rho)(\phi_1^2 + \phi_2^2)\\
&\phi_{11} + \phi_{22} + \tfrac{\phi_2}{x_2} = (\rho_1\phi_1 + \rho_2\phi_2)
\end{aligned}
$$

where the suffix 1 and 2 after λ, ρ and ϕ means partial differentiation with respect to x_1 and x_2, has been examined. Corresponding to each basic vector field, the reductions of the above nonlinear systems to ODEs are obtained. These reduced systems of ODEs are further studied for exact solutions.

Chapter 6 deals with the study of Einstein - Maxwell equations

$$
\begin{aligned}
&V_{11} + \tfrac{V_1}{x^1} - V_{00} = \tfrac{1}{V}(V_1^2 - V_0^2 + C_0^2 - C_1^2),\\
&C_{11} + \tfrac{C_1}{x^1} - C_{00} = \tfrac{2}{V}(V_1C_1 - V_0C_0),\\
&\xi_0 = 2x^1V^{-2}(V_0V_1 - C_0C_1),\\
&\xi_1 = x^1V^{-2}(V_0^2 + V_1^2 + C_0^2 + C_1^2),\\
&\xi_{11} - \xi_{00} = V^{-2}(V_0^2 - V_1^2 + C_0^2 - C_1^2),
\end{aligned}
$$

where the lower suffix $1, 0$ denotes partial differentiation with respect to the corresponding variables x^1 and x^0.

Here, in this chapter, an investigation of similarity solutions of field equations of general relativity with an electromagnetic stress tensor as source and Maxwell's equations in curved space has been undertaken by using the generalized symmetry method based on Fr$\acute{e}$chet derivative of the differential operators. Metrics and electromagnetic fields as functions of two independent variables are considered and the field equations are presented in a simple form and certain exact solutions of these equations are derived by obtaining the infinitesimals of the group of transformations which leaves the system of field equations invariant.

Chapter 7, is devoted to the use of combination of Lie group method and generalized $(\frac{G'}{G})$-expansion method to variable coefficients Kawahara equation (VCKE)

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0,$$

and variable coefficients modified Kawahara equation (VCMKE)

$$u_t + \alpha(t)u^2u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0,$$

where $\alpha(t), \beta(t)$ and $\sigma(t)$ are arbitrary time-dependent coefficients. Firstly, the similarity reductions and exact solutions are derived by determining the complete sets of point symmetries of these equations. Then, with the use of generalized $(\frac{G'}{G})$- expansion method, more explicit traveling wave solutions involving arbitrary parameters are found out, which are expressed in terms of hyperbolic functions, the trigonometric functions and rational functions.

In **Chapter 8**, Painlev$\acute{e}$ analysis of variable coefficients Kuramoto-Sivashinsky (VCKS) equation,

$$u_t + \alpha(t)uu_x + \beta(t)u_{xx} + \sigma(t)u_{xxxx} = 0,$$

where $\alpha(t), \beta(t)$ and $\sigma(t)$ are arbitrary time-dependent coefficients, is performed to check the Painlev$\acute{e}$ property and further auto-B$\ddot{a}$cklund transformation is presented via the truncated Painlev$\acute{e}$ expansion. Then the exact solutions generated from group invariant reductions are presented. Moreover, the exact analytic solutions are also considered by the power series method.

List of Research Papers

1. Lakhveer Kaur, R. K. Gupta, ”On Symmetries and Exact Solutions of the Einstein-Maxwell Field Equations via the Symmetry Approach”, *Physica Scripta*, 87 (2013) 035003 (7pp). **(Impact Factor 1.204) (SCI)**

2. Lakhveer Kaur, R. K. Gupta, ”Symmetries and Exact Solutions of Einstein field equations for Perfect Fluid Distribution and Pure Radiation Fields”, *Maejo International Journal of Science and Technology*, 7 (2013) 133-144. **(Impact Factor 0.258) (SCI)**

3. Lakhveer Kaur, R. K. Gupta, ”On Certain New Exact Solutions of Einstein Equations for Axisymmetric Rotating Fields”, *Chinese Physics B*, 22 (2013) 100203 (6pp). **(Impact Factor 1.376) (SCI)**

4. Lakhveer Kaur, R. K. Gupta, ”Kawahara Equation and Modified Kawahara Equation with Time Dependent Coefficients: Symmetry Analysis and Generalized $\left(\frac{G'}{G}\right)$-Expansion Method”, *Mathematical Methods in the Applied Sciences*, 36 (2013) 584-600. **(Impact Factor 0.753) (SCI)**

5. Lakhveer Kaur, R. K. Gupta, ”Painlevé Analysis, Similarity Reductions and Exact Solutions of the Kuramoto-Sivashinsky Equation with Variable Coefficients”, *International Journal of Nonlinear Science*, 15 (2013) 139-149.

6. Lakhveer Kaur, R. K. Gupta, ”Some Invariant Solutions of Field Equations with Axial Symmetry for Empty Space Containing an Electrostatic Field”, *Applied Mathematics and Computation* (Under Minor Revision). **(Impact Factor 1.394) (SCI)**

7. R. K. Gupta, Lakhveer Kaur, ”Similarity Solutions of Field Equations in General Relativity”, (Communicated to *Communications in Theoretical Physics*). **(Impact Factor 0.954) (SCI)**

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Chapter 1

Introduction

1.1 Literature Survey and Motivation

In 1916, Albert Einstein published his famous theory of general relativity, which contains the rules of gravity and provides the basis for modern theories of astrophysics and cosmology. It describes phenomena on all scales in the universe, from compact objects such as black holes, neutron stars, and supernovae to large-scale structure formation such as that involved in creating the distribution of clusters of galaxies. For many years, physicists, astrophysicists and mathematicians have striven to develop techniques for unlocking the secrets contained in Einstein's theory of gravity. More recently, solutions of Einstein field equations have added their expertise to the endeavor. Those who study these objects face a daunting challenge that the equations are among the most complicated in mathematical physics. Together, they form a set of coupled, nonlinear, hyperbolic-elliptic partial differential equations that contain many thousands of terms.

Despite more than 95 years of intense analytical study, these equations have yielded only a handful of special solutions relevant for astrophysics and cosmology, giving only tantalizing snapshots of the dynamics that occur in our universe. Scientists have gradually realized that exact studies of Einstein's equations will play an essential role in uncovering the full picture. Progress here has been initially slow, due to the complexity of the equations and the lack of methods and the wide variety of algorithms and techniques.

After proposing special theory of relativity, Einstein came out with a more comprehensive general theory of relativity which also provided a very unusual description of the phenomenon of gravity as a manifestation of curved space-time around any presence of matter and energy. In 1915, Einstein presented his theory of general relativity. His theory was widely accepted once it was established that it explained the precession of Mercury's orbit and predicted the bending of starlight by the Sun. After the general relativity theory in 1915, Albert Einstein used it in an ambitious way to propose a model of the entire universe. This simple model assumed that the universe is homogeneous, isotropic and also static. The Einstein equation relates the space-time geometry and the physical contents of the universe.

Karl Schwarzschild determined the first solution to Einstein's equations in a vacuum and it was a spherically symmetric exact solution. Shortly after Einstein wrote down his gravitational field equations, in 1915, Karl Schwarzschild found a solution which describes a non-rotating spherical star or black hole. However, it is known that all stars rotate, and Schwarzchild's solution gives the best approximation. In the period 1915 to 1948, Gödel published research paper [62] in the field of general relativity. Thus, the theory of general relativity is very interesting from a number of reasons and it has become a gold mine for many known relativists. Gödel interests in cosmology continued and in 1950, he published the other paper [63].

From 1920's to the 1960's, there was considerable development of the theory of general relativity. After applying quantum physics and Relativity to Eddington's calculations, astrophysicist Chandrasekhar [37] gave an idea related to black holes showing that only stars of mass below a certain limit could become white dwarfs. This mass is known as the Chandrasekhar limit. On physical grounds it is expected that black holes should rotate and have a oblate shape which requires different solutions to the equations of General

Relativity other than that of Schwarzschild. The unique solution which has all the correct properties was obtained by Roy Kerr [92] and is known as Petrov Type D solution.

Bonnor [26] used canonical cylindrical coordinates of Weyl to derive certain exact solutions of the equations of general relativity with an electrostatic field and gave a physical interpretation. Miser [126] explained gravitational field equations of perfect fluid sphere. Cahill and Taub [32] studied spherically symmetric similarity solutions of the Einstein field equations for a perfect fluid. They showed that the metric coefficients of spherically symmetric space-times depend essentially on the single variable $z = \frac{r}{t}$, where r is a radial coordinate and t is the time and then the Einstein field equations reduces to ordinary differential equations. The solutions of Einstein field equations are analogous to the similarity solutions of the classical theory of hydrodynamics.

Kuchowicz [101] gave method of driving exact solution of spherical symmetry in the Einstein Cartan theory for a perfect fluid with a classical description of spin. Einstein Cartan theory has useful feature that, its equations in empty space are exactly same as that of Einstein theory and thus exact solutions of Einstein Cartan theory has been derived from known solutions of Einstein equations. Chakravarty [36] considered some exact solutions of Einstein field equations and the general relativistic exterior metric solution of an uncharged and rotating finite cylinder has been furnished. Kinnersley [95] described briefly all known vacuum solutions including electrovacuum inter-relationships.

Das and Banerji [45] constructed a method for generation of stationary solution of Einstein equations. Fischer [58] presented similarity solutions of the Einstein equations by the use of geometric technique of Harrison and Estabrook [74] by finding appropriate similarity variables to reduce partial differential equations to ordinary differential equations. Patel [143] studied Einstein field equations for axisymmetric gravitational collapse and obtained solutions for Einstein field equations. Das [46] proved that asymptotically flat stationary solutions of Einstein equations, including Kerr solution, are generated from solutions of Laplace equation. A new interior solution of Einstein's field equations for a spherically symmetric perfect fluid in shear-free motion are derived by Stephani [165] and also a new class of exact solutions is presented which contains two arbitrary functions of time and one additional parameter.

Matravers [121] solved Einstein's field equations with two parameter family of classical strings as the source for the gravitational field. Senovilla [154] has discovered an important solution representing a perfect fluid distribution with cylindrical symmetry. Das and Choudhuri [47] proposed that solutions of Einstein field equations are obtained from a non diagonal seed by 'inverse scattering method'. Castejon-Amenedo and Coley [35] studied exact solutions of Einstein's field equations admitting a Lie algebra of conformal Killing vectors and gave some examples of exact solutions and their particular conformal structures. Two exact solutions of the Einstein equations, representing the field of a static deformed mass, were obtained and these solutions are reduced to give well known Schwarzchild metric, hence used for the analysis of the physical properties of the gravitational field.

HernBndez Pastora and Martin [78] found a exact asymptotically flat solution of the Einstein equations and described the exterior gravitational field of a static mass possessing a quadrupole moment explicitly. Meinel and Neugebauer [123] presented a new class of exact solutions to the axisymmetric and stationary vacuum Einstein equations containing arbitrary complex parameters. The obtained solutions are related to Jacobi's inversion problem.

Marchildon [119] investigated Lie symmetries of Einstein vacuum equations in N dimensions, with a cosmological term. For this purpose, he first wrote down the second prolongation of the symmetry generating vector fields, and compute its action on Einsteins equations. Instead of setting to zero the coefficients of all independent partial derivatives, we set to zero the coefficients of derivatives that do not appear in Einsteins equations. This considerably constrained the coefficients of symmetry generating vector fields. Using the Lie algebra property of generators of symmetries and the fact that general coordinate transformations are symmetries of Einsteins equations, all the Lie symmetries are obtained.

Bhutani and Singh [15] used a new approach for the solution of partial differential equations corresponding to the metric function in a five dimensional flat space describing the perfect fluid distribution for generalized symmetries. In fact, the generalized symmetry approach [15] combined with variational symmetry has provided a breakthrough in obtaining the exact solutions of the coupled system of nonlinear partial differential equations corresponding to Einstein exterior equations. Also it has yielded new solutions of Ernst form of Einstein equations and enabled us to arrive at a generalized form of Weyl and Schwarzschild solutions.

Wang and U Mao-Wang [175] found a new generation theorem for Einstein-Maxwell field. Starting with Schwarzschild solution, a new solution of the Einstein-Maxwell equations is obtained by means of generation theorem. Therefore, by "generation techniques", appeared in 1970's, a new solution can be obtained from the known one through a certain transformation. The major research achievements in this field are mostly some important generation solution theorems developed by Ernst, Kinnersley, Chandrasekhar, Ehlers et al.[166], which resulted in a series of new exact solutions. The Einstein field equations for a static spherically symmetric distribution of perfect fluid have been investigated by many authors using different approaches.

Sharif and Iqbal [156] examined systematically Einstein field equations for non static spherically symmetric perfect fluid solutions. Bhutani et al. [16] found a certain class of exact solutions of Einstein field equations for rotating fields in conventional and non conventional form by using symmetry approach. They had also carried over the technique of invariant variational principle [116] and deduced generalized form of Weyl and Schwarzchild solutions for the case of no spin as particular cases. The symmetry approach to solve differential equations can be found from Olver [137], Bluman and Cole [21], Bluman and Kumei [23], Ovsiannikov [139].

Khugaev and Ahmedov [94] presented a class of axially symmetric solutions of vacuum Einstein field equations including the Papapetrou solution as particular case. Hansraj et al. [72] devised exact solutions to the Einstein field equations which arise when two spacetime geometries are conformally related and used the method of Lie analysis of differential equations to obtain new group invariant solutions. Wiltshire [186] studied Einstein's equations, in terms of the isotropy condition for fluid spheres. He considered the Lie symmetry approach to review symmetry solutions for comoving cases.

Negi [129] has derived some new exact solutions, which proves to be very helpful in constructing the appropriate core-envelope models of many stellar objects and may be used to test various equations of state for dense nuclear matter. Attallah et al. [8], studied well known exact solution of Einstein vacuum equations for stationary axially symmetric rotating fields with the generation technique of the isovector fields. They obtained all linearly independent isovector fields of Einstein vacuum equations for rotating fields and determined the isovector fields associated with Einstein vacuum equations corresponding to the most general axisymmetric metric by completely solving the equations for the isovector fields.

Janda [87] discussed certain aspects of Lie-point symmetries in spherically symmetric systems of gravitational physics. In case of perfect fluid, existence of symmetries appears to be helpful for solving these differential equations. Janda explained general concepts and a few examples of the equations with Lie symmetry method. Cabezas et.al. [31] established an approximate global stationary and axisymmetric solution of Einstein equations which can be considered as simple star model and described the gravitational field inside a ball of perfect fluid. Yingqin [188] presented a framework for getting a series of exact vacuum solutions of Einstein equations. This procedure of resolution is based on canonical form of metric.

Vilasi [171] described exact solutions of Einstein field equations invariant for a non-Abelian 2- dimensional Lie algebra of killing fields. A sub-class of these gravitational fields have a wave-like character. Davidson [48] derived two solutions of Einstein's equations, one representing rigid rotation of a perfect fluid and the other differential rotation. These are obtained as solutions of the two second order non-linear ordinary differential equations which describe a rotating family presented by Senovilla [155]. Ali [5] derived some new exact solution of Einstein vacuum equations for rotating axially symmetric fields with the use of isovector technique. This techniques is closely related to technique used by Stephani [165]. In order to obtain the new solutions, all linearly independent isovector fields of Einstein vacuum equations for rotating fields for axisymmetric metric in the general form of Weyl metric are determined.

Wang et al. [172] constructed a class of exact solutions of the non-commutative Einstein field equations in the vacuum. Chifu et al. [39] constructed the new analytical solutions to Einsteins geometrical field equations in prolate spheroidal regions and hence, found the solution which puts Einsteins geometrical theory of gravity on same footing with Newtons dynamical theory; with the dependence of the field on one and only one unknown function comparable to Newtons gravitational scalar potential. Einstein equations with cosmological constant has been integrated in a very general form by Vacaru [169] and then these equations has been reduced to a system of two nonlinear ordinary differential equations and thus presented the analytical and numerical solutions satisfying the dominant energy conditions. Goyal [64, 65] has considered Einstein - Rosen metric and derived some new exact solutions of the field equations for stationary axisymmetric Einstein - Maxwell fields by using Lie symmetry method.

These references provide a sample idea that these equations have been a subject of extensive and intensive study both by mathematicians and physicists. For the detail study of exact solutions of Einstein field equations, reader may refer to Stephani et al. [166]. Einstein equations, which play a central role in theory of general relativity, have symmetry consideration as one of the most important mathematical properties apart from their applications and implications for astrophysics and cosmology. That is why Einstein field equations, arising in a variety of applications, is a wonderful research area for many scientists and deserve clear scrutiny.

1.1.1 Einstein Field Equations

General Relativity is a unified theory of space, time and gravitation. The theory's roots extend over almost the entire previous history of physics and mathematics. General Relativity constitutes a triumph of the geometric approach to physical science. The connection between gravitation and Riemannian geometry arose in Einstein's mind in his effort to uncover the meaning of what in Newtonian theory is the fortuitous equality of the inertial and the gravitational mass. Identification, via the equivalence principle, of the gravitational tidal force with spacetime curvature at once gave a physical interpretation of curvature of the spacetime manifold and also revealed the geometrical meaning of gravitation.

The laws of General Relativity, Einstein's equations, constitute, when written in any system of local coordinates, a non-linear system of partial differential equations for the metric components. In practice, one of the great difficulties of relating the particular features of general relativity to real physical problems, arises from the high degree of non-linearity of Einstein field equations. Although the linearized theory has been used in some applications, its use is severely limited. Many of the most interesting properties of space-time, such as the occurrence of singularities, are consequences of the non-linearity of the equations. Therefore, it becomes very difficult to solve these equations unless certain symmetry restrictions are imposed on some space-time metric. These symmetry restrictions are expressed in terms of isometries possessed by space times. These isometries, which are also called killing vectors, give rise to conservational laws.

Symmetries in general relativity have been the subject of much study in recent years, partly because of the considerable simplification of Einstein's equations resulting from the assumption of one or more symmetries, partly because of interest in the geometric significance of the symmetries, which are described by vector fields of certain geometrical objects on the manifold, and partly because of the possible physical significance of the existence of these symmetries.

Einstein equations (without cosmological constant) linking the curvature of spacetime to its matter content are given by:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu}, \tag{1.1.1}$$

Here $T_{\mu\nu}$ is the energy-momentum tensor of matter, $G_{\mu\nu}$ the Einstein tensor, $R_{\mu\nu}$ the Ricci tensor, k is the Einstein gravitation constant and R the scalar curvature of the metric $g_{\mu\nu}$. From the original Bianchi identity

$$\nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0, \tag{1.1.2}$$

one obtains

$$\nabla^\nu G_{\mu\nu} = 0, \tag{1.1.3}$$

the twice contracted Bianchi identity. This identity (1.1.3) implies

$$\nabla^\nu T_{\mu\nu} = 0, \tag{1.1.4}$$

the equations of motion of matter. The Einstein vacuum equations

$$G_{\mu\nu} = 0, \tag{1.1.5}$$

correspond to the absence of matter: $T_{\mu\nu} = 0$. The equations are then equivalent to

$$R_{\mu\nu} = 0, \tag{1.1.6}$$

1.1.2 Differential Geometry

The general theory of relativity is a theory of gravitation in which gravitation emerges as the property of the space-time structure through the metric tensor g_{ij}. The metric tensor determines another object (of tensorial nature) known as Riemann curvature tensor. At any given event this tensorial object provides all information about the gravitational field in the neighbourhood of the event. It may, in real sense, be interpreted as describing the curvature of the space-time. The Riemann curvature tensor is the simplest non-trivial object one can build at a point; its vanishing is the criterion for the absence of genuine gravitational fields and its structure determines the relative motion of the neighbouring test particles via the equation of geodesic deviation.

The above discussion clearly illustrates the importance of the Riemann curvature tensor in general relativity and it is for these reasons that a study of this curvature tensor has been made here. Next, certain basic aspects of differential geometry are considered, which are necessary for later work. The aspects of differential geometry relevant to general relativity are briefly discussed in this section. The non vanishing components of the connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor are explicitly calculated for Einstein field equations. The coupling of the Einstein tensor and energy-momentum tensor is used to generate the Einstein field equations.

Riemann curvature tensor plays an important role for specifying the geometrical properties of spacetime. It is defined in terms of Christoffel symbols:

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\nu}_{\beta\delta}\Gamma^{\alpha}_{\nu\gamma} - \Gamma^{\nu}_{\beta\gamma}\Gamma^{\alpha}_{\nu\delta}, \tag{1.1.7}$$

where $\Gamma^{\alpha}_{\beta\delta,\gamma} = \frac{\partial(\Gamma^{\alpha}_{\beta\delta})}{\partial x^{\gamma}}$.

The spacetime is considered to be flat if the Riemann tensor vanishes everywhere. Riemann tensor can also be written directly in terms of spacetime metric

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(g_{\beta\gamma,\alpha\delta} + g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} - g_{\alpha\gamma,\beta\delta}) + g_{\mu\nu}\Delta^{\nu}_{\alpha\gamma}\delta^{\mu}_{\beta\delta} - g_{\mu\nu}\Delta^{\nu}_{\alpha\delta}\delta^{\mu}_{\beta\gamma}, \tag{1.1.8}$$

The Riemann tensor satisfy the following identities:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}, \\ R_{\alpha\beta\gamma\delta} &+ R_{\beta\delta\alpha\gamma} + R_{\alpha\delta\beta\gamma} = 0. \end{aligned} \tag{1.1.9}$$

Because of the symmetries above, the Riemann tensor in 4-dimensional spacetime has only 20 independent components. The general rule for computing the number of independent components is an N-dimensional spacetime is $\frac{N^2(N^2-1)}{12}$.

Ricci tensor is obtained from the Riemann tensor by simply contracting over two of the indices:

$$R_{\alpha\beta} \equiv R^{\gamma}_{\alpha\gamma\beta}. \tag{1.1.10}$$

It is symmetric, which means that it has at most 10 independent quantities. Ricci scalar is obtained by contracting the Ricci tensor over the remaining two indices:

$$\mathcal{R} \equiv g^{\alpha\beta} R_{\alpha\beta} \equiv R^{\alpha}_{\alpha}. \tag{1.1.11}$$

Bianchi identities are another important symmetry of the Riemann tensor

$$R_{\alpha\beta\gamma\delta;\nu} + R_{\beta\alpha\nu\gamma;\delta} + R_{\alpha\beta\delta\nu;\gamma} = 0, \tag{1.1.12}$$

which after contracting, leads to

$$R^{\alpha\beta}_{;\alpha} = \frac{1}{2} g^{\alpha\beta} \mathcal{R}_{;\alpha}. \tag{1.1.13}$$

Einstein Tensor is defined in terms of the Ricci tensor and Ricci scalar as

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R}. \tag{1.1.14}$$

From eq (1.1.13), a very important property of the Einstein tensor is derived

$$G_{\alpha\beta;\alpha} = 0. \tag{1.1.15}$$

The importance of G in gravity was first recognized by Einstein while developing the field equations for the theory of general relativity.

1.2 Methodology

In many situations, while translating a physical problems into mathematical terminology, one is often confronted with a single or a system of differential equations, that may be ordinary, partial, linear or nonlinear in nature. In the study of nonlinear partial differential equations the discovery of explicit solutions has great theocratical and practical importance. These explicit solutions for nonlinear systems are used as models for physical or numerical investigations and often reflect qualitatively on the behavior of more complicated solutions. As the scientific literature grew richer, the task of determining these special solutions posed over increasing challenge to the scientists.

The concept of symmetry fascinated through the centuries many artists and scientists, from the ancient Greeks to Kepler, to Newton, who embodied in the laws of mechanics as a symmetry principle the equivalence of motion in different inertial frames, to Einstein, who generalized the Galileos principle of relativity from mechanics to all the laws of physics. Presently, symmetries play an important role in mathematics, chemistry, engineering and in almost all branches of physics, including classical mechanics, quantum mechanics and general relativity etc. One reason for the overall prominence of the concept of symmetry is its nativeness and its simplicity. Intuitively speaking, a symmetry is a transformation of an object leaving that object invariant. This is clearly such a general property that it can be recovered almost everywhere in nature and, correspondingly, in numerous areas of science and art.

To be more specific, in the course of the thesis, our objects will be several differential equations associated with Einstein field equations and our transformations will be point transformations preserving these equations or relating them to each other. By definition, symmetries are attributes of their associated objects and thus in some sense provide an inverse way to characterize these objects. That is, by studying the transformations that leave an object invariant, we can already learn about the object itself. The most inspiring example of this finding stems from inverse group classification: Any differential equation can be represented as a function of the differential invariants of its admitted Lie symmetry group. In other words, the knowledge of the symmetries of a differential equation (i.e. the transformations) is a source to determine the differential equation (i.e. the object) itself. Indeed, this is a main motivation for investigating symmetries of differential equations. They help to understand these equations, which is of inestimable value especially for all those differential equations, for which it is difficult to determine their general solution(s) systematically.

Exact solutions for nonlinear equations are rare, and the methods, which can generate families of them, are not only increasingly popular, but increasingly sought. So far, a number of methods have been proposed to construct the exact solutions; the most effective methods include the classical Lie approach [6, 15, 20, 30, 64, 81, 128], the nonclassical approach [29, 59, 133, 145], Steinberg's symmetry reduction method [16, 164], the truncated Painlevé approach [3, 41, 42, 115, 182, 183], the transformation method [97, 111] the isovector method [5, 8, 167, 168] etc. But, the mathematical techniques which generate a wide range of solutions and applicable to all type of nonlinear differential equations are few. The study of the mathematical properties of the Einstein field equations and the techniques used to solve these equations are important in the context of general relativity. There are different techniques to solve these equations. The approach, which we have adopted in this thesis, is the symmetry analysis of differential equations, that was first formulated by Lie [110].

In the nineteenth century a great advance arose when the Norwegian mathematician Sophus Lie [109] began to investigate the continuous groups of transformations leaving differential equations invariant, creating what is now called the symmetry analysis of differential equations. The original Lie's aim was that of setting a general theory for the integration of ordinary differential equations similar to that developed by E. Galois and N. Abel for algebraic equations [187]. This theory enables to derive solutions of differential equations in a completely algorithmic way without appealing to special lucky guesses. Lie's approach to differential equations was not exploited for half a century and only the abstract theory of Lie groups grew. Further developments by E. Cartan [34] established Lie's theory as a cornerstone of mathematics and its physical applications. General references include [49, 152, 170].

It was in the forties of last century, with the work of G. Birkhoff [24] and I. Sedov [153] on dimensional analysis, that the theory gave relevant results in concrete applied problems. Further, L. V. Ovsiannikov [139] began to exploit systematically the methods of symmetry analysis of differential equations in the explicit construction of solutions of any sort of problems, even complicated, of mathematical physics. During the last few decades, there has been a revival of interest in Lies theory and significant progress has been made from either a theoretical or an applied point of view. Many monographies and textbooks are now available [22, 23, 33, 84, 137, 164] and an increasing number of research papers [79, 80, 88, 141, 151] are published.

Applications of Lie Group theory include integration of ordinary differential equations, determination of explicit group-invariant (similarity) solutions of partial differential equations, Noether's theorems [131] relating symmetries of variational problems and conservation laws [19, 90], bifurcation theory [61], asymptotics and blow-up [11], and the design of geometric numerical integration schemes [71]. Its more recent extensions to general Lie group and Lie pseudo-group actions [138], provides a general mechanism for construction and classification of differential invariants, with applications to differential geometry, the calculus of variations, soliton theory, computer vision, classical invariant theory and numerical methods.

Modern developments in applications of Lie group methods have proceeded in a variety of directions, general theories of infinite-dimensional Lie groups and algebras [89], and Lie pseudo-groups, arising in relativity, field theory, fluid mechanics, solitons, and geometry, remain elusive. Higher order or generalized symmetries, in which the infinitesimal generators also depend upon derivative coordinates, first proposed by Noether [131], have been used to classify integrable systems. Recursion operators are used to generate such higher order symmetries and via Noether's theorem, higher order conservation laws [137]. Most recursion operators are derived from a pair of compatible Hamiltonian structures and demonstrate the integrability of bi-Hamiltonian systems. The higher order symmetries also appear in series expansions of B$\ddot{a}$cklund transformations in the spectral parameter.

Lie's classical theory is a source for various generalizations. Among these generalizations there are the following techniques:

(i) Nonclassical method [21]

(ii) General method of differential constraints [144, 136]

(iii) Introduction of approximate symmetries [9, 85]

(iv) Generalized symmetries [137]

(v) Equivalence transformations [111]

(vi) Nonlocal symmetries [23, 107, 137]

The key idea of Lie's theory of symmetry analysis of differential equations relies on the invariance of the latter under a transformation of independent and dependent variables. This transformation forms a local group of point transformations establishing a diffeomorphism on the space of independent and dependent variables, mapping solutions of the equations to other solutions. Any transformation of the independent and dependent variables in turn induces a transformation of the derivatives. Some important recent contributions, in this direction, for obtaining exact solutions of various partial differential equations with the help of well known Lie classical method are [65, 103, 134, 158, 160].

There also exist alternative methods, which are not based on the applications of group theory, such as Direct method [40], B$\ddot{a}$cklund transformation [149], Painlev$\acute{e}$ analysis [182, 183], Inverse scattering transformation [2]. Recently, a variety of powerful methods, such as the tanh-sech method [118, 177], extended tanh method [51, 179], sine-cosine method [14, 176], Hirota method [180, 181], homogeneous balance method [55, 173], Jacobi elliptic function method [44, 56], F-expansion method [1, 191], homotopy perturbation method [60, 140], variational iteration method [75, 77], non-perturbative method [76], extended $\frac{G'}{G}$-expansion method [67], modified $\frac{G'}{G}$-expansion method [10, 124], generalized $\frac{G'}{G}$-expansion method [174, 192] are developed.

In the following sections, the relevant concepts of the Lie group of transformations are introduced and then we have provided the algorithmic descriptions of the techniques which are applied in the later chapters to derive the symmetry group of the systems under investigation. For more details on Lie groups and various theorems, their proofs and other concepts, we refer our reader to Bluman and Cole [22], Bluman and Kumei [23] and Olver [137].

1.2.1 Definitions

In this section, some basic definitions [33] and fundamentals of present work are given.

Definition 1: A kth-order ($k \geq 1$) system E of s differential equations is defined by

$$E^{\sigma}(x, u, u_{(1)}, ..., u_{(k)}) = 0, \;\; \sigma = 1, 2, ..., s, \tag{1.2.1}$$

where $u \equiv (u^1, u^2, ..., u^m)$ is the dependent vector, $x \equiv (x^1, x^2, ..., x^n)$ is the independent vector and $u_{(1)}, u_{(2)}, ..., u_{(k)}$ are respectively the collection of all first, second, up to kth-order partial derivatives. In expanded form

$$u_{(1)} = \{u^{\alpha}_i\}, u_{(2)} = \{u^{\alpha}_{ij}\}, u_{(k)} = \{u^{\alpha}_{i_1,...,i_k}\}, \tag{1.2.2}$$

where $\alpha = 1, 2, 3, ..., m$ and $i, j, i_1, ..., i_k = 1, 2, ..., n$.

Definition 2: Symmetry Transformations

A symmetry transformation of the system (1.2.1) is an invertible transformation of the variables x and u, namely

$$\bar{x}^i = f^i(x, u), \;\; \bar{u}^{\alpha} = \phi^{\alpha}(x, u), \;\; i = 1, 2, ..., n, \;\; \alpha = 1, 2, ..., m, \tag{1.2.3}$$

that leaves (1.2.1) form-invariant in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E^{\sigma}(\bar{x}, \bar{u}, \bar{u}_{(1)}, ..., \bar{u}_{(k)}) = 0, \;\; \sigma = 1, 2, ..., s, \tag{1.2.4}$$

whenever (1.2.1) is satisfied.

Definition 3: One-Parameter Lie Group of Transformations

A set G of transformations

$$T_a : \bar{x}^i = f^i(x, u, a), \;\; \bar{u}^{\alpha} = \phi^{\alpha}(x, u, a), \;\; i = 1, 2, ..., n, \;\; \alpha = 1, 2, ..., m, \tag{1.2.5}$$

where a is a real parameter which continuously takes values in a neighbourhood $\mathcal{D} \subseteq \mathbb{R}$ of $a = 0$ and f^i, ϕ^{α} are differentiable functions, is a continuous one-parameter (local) Lie group of transformations in $\mathbb{R}^{n+m}$ provided the group properties of closure, identity and inverses are satisfied, namely:

(i) Closure: If $T_a, T_b \in G$ and $a, b \in \mathcal{D}' \subset \mathcal{D}$, then

$$T_a T_b = T_c \in G, c = \phi(a, b) \in \mathcal{D}. \tag{1.2.6}$$

(ii) Identity: There exists $T_0 \in G$ such that

$$T_0 T_a = T_a T_0 = T_a, \tag{1.2.7}$$

for any $a \in$ *D' $\subset$ D*. T_0 is known as the identity of the group.

(iii) Inverses: There exists $T_a^{-1} = T_{a^{-1}}, a^{-1} \in \mathcal{D}$ such that

$$T_a^{-1} T_a = T_a T_a^{-1} = T_0. \tag{1.2.8}$$

for any $T_a \in G, a \in \mathcal{D}' \subset \mathcal{D}$.

Definition 4: Infinitesimal Generator and Lie's Equations

Lie's theory allows the construction of one-parameter group elements from their first order approximations

$$\bar{x}^i \approx x^i + a\xi^i(x, u), \;\; \bar{u}^\alpha \approx u^\alpha + a\eta^\alpha(x, u), \tag{1.2.9}$$

Equation (1.2.9) is the first-order Taylor expansion of T_a about $a = 0$ with the initial conditions

$$f^i|_{a=0} = x^i, \phi^\alpha|_{a=0} = u^\alpha. \tag{1.2.10}$$

Hence we have

$$\xi^i(x, u) = \frac{\partial f^i(x, u, a)}{\partial a}|_{a=0}, \eta^\alpha(x, u) = \frac{\partial \phi^\alpha(x, u, a)}{\partial a}|_{a=0}. \tag{1.2.11}$$

The vector (ξ^i, η^α) is the tangent vector at (x, u) to the curve $(\bar{x}, \bar{u})$ parametrized by a. The first-order approximations (1.2.9) can be written as

$$\bar{x}^i \approx (1 + aX)x^i, \bar{u}^i \approx (1 + aX)u^\alpha, \tag{1.2.12}$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{1.2.13}$$

The differential operator (1.2.13) is called the infinitesimal generator or vector field of the group G.

Lie's First Fundamental Theorem

For any given infinitesimal transformations (1.2.9), or an infinitesimal generator X, the corresponding one-parameter group G is obtained by solution of the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}), \tag{1.2.14}$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x^i, \bar{u}^\alpha|_{a=0} = u^\alpha. \tag{1.2.15}$$

The solution of Lie equations (1.2.14) involves exponentiating the generator X,

$$\bar{x}^i = \exp(aX)x^i, \bar{u}^\alpha = \exp(aX)u^\alpha, \tag{1.2.16}$$

where

$$\exp(aX) = 1 + aX + \frac{a^2}{2}X^2 + ... = \sum_{j=0}^{\infty} \frac{a^j}{j!}X^j, \tag{1.2.17}$$

is known as the Lie series operator.

Definition 5: A point $(x, u) \in \mathbb{R}^{n+m}$ is an invariant point of a group G with generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{1.2.18}$$

if and only if

$$\xi^i(x, u) = \eta^\alpha(x, u) = 0. \tag{1.2.19}$$

Definition 6: A function $F(x, u)$ is an invariant of a group G if and only if

$$F(\bar{x}, \bar{u}) = F(x, u), \forall x, u, a \in \mathcal{D}' \subset \mathcal{D}. \tag{1.2.20}$$

Definition 7: A function $F(x, u)$ is an invariant of a group G with the generator X if and only if

$$X(F) = 0. \tag{1.2.21}$$

The characteristic system for Eq. (1.2.21) is given by

$$\frac{dx^1}{\xi^1(x,u)} = ... = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = ... = \frac{du^m}{\eta^m(x,u)}. \qquad (1.2.22)$$

Thus an arbitrary invariant $F(x,u)$ of the group G is

$$F = \Lambda(I_1(x,u), ..., I_{m+n-1}(x,u)), \qquad (1.2.23)$$

where $I_1(x,u), ..., I_{m+n-1}(x,u)$ is called a basis of invariants of G. The basis is not unique. One can take, as basic invariants, the left hand side of $m+n-1$ first integrals

$$I_1(x,u), ..., I_{m+n-1}(x,u) = c_{m+n-1}. \qquad (1.2.24)$$

Prolongation Formulas

The transformations (1.2.5) form a symmetry group G of the system E if its invariant form (1.2.4) is satisfied whenever equation (1.2.1) holds. The transformed derivatives in (1.2.4) are obtained by employing the chain rule, $D_i = D_i(f^j)\bar{D}_j$, where

$$D_j = \frac{\partial}{\partial x^j} + u_j^\alpha \frac{\partial}{\partial u^\alpha} + u_{jk}^\alpha \frac{\partial}{\partial u_k^\alpha}; u_j^\alpha = D_j(u^\alpha), u_{jk}^\alpha = D_j(u_k^\alpha). \qquad (1.2.25)$$

is the total derivative operator with respect to x^i, as is $\bar{D}_i$ for the transformed variables. Applying $D_i = D_i(f^j)\bar{D}_j$ on $\bar{u}^\alpha$ and using the form of $\bar{u}^\alpha$ from (1.2.5) on the left hand side of the result, we arrive at

$$D_i(\phi^\alpha) = D_i(f^j)\bar{D}_j(\bar{u}^\alpha) = D_i(f^j)(\bar{u}_j^\alpha). \qquad (1.2.26)$$

When one expands, the last equation yields

$$\left(\frac{\partial f^j}{\partial x^i} + {u_i}^\beta \frac{\partial f^j}{\partial u^\beta}\right) \bar{u^\alpha}_j = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \qquad (1.2.27)$$

Now we solve equation (1.2.27) to get $\bar{u_i}^\alpha = {\psi_i}^\alpha(x, u, u_{(1)}, a)$. It can be verified that $\psi_i^\alpha|_{a=0} = u_i^\alpha$ and $\bar{u}_i^\alpha = {\psi_i}^\alpha(x, u, u_{(1)}, a)$ is locally solvable for $u_{(1)}$ given small a. The transformations (1.2.3) together with the transformations $\bar{u}_{(1)} = \psi(x, u, u_{(1)}, a)$ form a one-parameter group, $G^{[1]}$, which is the first prolonged group acting on the space $(x, u, u_{(1)})$. In similar fashion higher-order prolonged (extended) groups $G^{[2]}$ up to $G^{[k]}$

can be obtained.

Let the infinitesimal transformations of the extended groups $G^{[1]}$ up to $G^{[k]}$ be given by (1.2.9) and

$$\begin{aligned}
\bar{u}_i^\alpha &\approx u_i^\alpha + a\zeta_i^\alpha(x, u, u_{(1)}),\\
\bar{u}_{ij}^\alpha &\approx u_{ij}^\alpha + a\zeta_{ij}^\alpha(x, u, u_{(1)}, u_{(2)}),\\
&\ \ \vdots\\
\bar{u}_{i_1,i_2,\ldots,i_k}^\alpha &\approx u_{i_1,i_2,\ldots,i_k}^\alpha + a\zeta_{i_1,i_2,\ldots,i_k}^\alpha(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}),
\end{aligned} \tag{1.2.28}$$

Then the functions ζ's are given recursively by the prolongation formulae,

$$\begin{aligned}
\zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ji}^\alpha,\\
\zeta_{ij}^\alpha &= D_i D_j(W^\alpha) + \xi^k u_{kj}^\alpha,\\
&\ \ \vdots\\
\zeta_{i_1,i_2,\ldots,i_k}^\alpha &= D_{i_1}\ldots D_{i_k}(W^\alpha) + \xi^j u_{ji_1,\ldots,i_k}^\alpha,
\end{aligned} \tag{1.2.29}$$

where

$$W^\alpha = \eta^\alpha - \xi^j {u_j}^\alpha, \tag{1.2.30}$$

is the so-called Lie characteristic function. Equivalently the formulae (1.2.29) can be written as

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j),\\
\zeta_{ij}^\alpha &= D_j(\zeta_i^\alpha) - u_{il}^\alpha D_j(\xi^l),\\
&\ \ \vdots\\
\zeta_{i_1,i_2,\ldots,i_k}^\alpha &= D_{i_k}(\zeta_{i_1\ldots i_{k-1}}^\alpha) - u_{i_1\ldots i_{k-1}l}^\alpha D_{i_k}(\xi^l).
\end{aligned} \tag{1.2.31}$$

We now illustrate how to obtain one of the prolongation formulae (1.2.29), say, ${\zeta_i}^\alpha$. Consider $D_i(\phi^\alpha) = D_i(f^j)(\bar{u^\alpha}_j)$. Using the infinitesimal transformations (1.2.9) and $\bar{u}_i^\alpha$ from (1.2.28), we obtain

$${u_i}^\alpha + a\zeta_i^\alpha + a u_j^\alpha D_i(\xi^j) = {u_i}^\alpha + aD_i(\eta^\alpha). \tag{1.2.32}$$

Therefore we have

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ji}^\alpha, \tag{1.2.33}$$

where the Einstein summation convention is adopted.

The corresponding prolonged generators of the prolonged groups $G^{[1]}$ up to $G^{[k]}$ are

$$\begin{aligned} X^{[1]} &= \xi^i(x,u)\tfrac{\partial}{\partial x^i} + \eta^\alpha(x,u)\tfrac{\partial}{\partial u^\alpha} + {\zeta_i}^\alpha(x,u,u_{(1)})\tfrac{\partial}{\partial {u_i}^\alpha} \\ &\cdot \\ &\cdot \\ &\cdot \\ X^{[k]} &= \xi^i(x,u)\tfrac{\partial}{\partial x^i} + \eta^\alpha(x,u)\tfrac{\partial}{\partial u^\alpha} + {\zeta_i}^\alpha(x,u,u_{(1)})\tfrac{\partial}{\partial {u_i}^\alpha} \\ &\quad + {\zeta_{i_i \dots i_k}}^\alpha(x,u,u_{(1)},\dots,u_{(k)})\tfrac{\partial}{\partial {u_{i_i \cdots i_k}}^\alpha}, \end{aligned} \tag{1.2.34}$$

where $X^{[1]}$ is the first prolongation of X. The converse is also true in that, given the prolonged operators (1.2.34) or their corresponding infinitesimal transformations (1.2.9), we can find the respective one-parameter prolonged groups. This is achieved by using the exponential map for the prolonged variables.

Definition 8: A differential function, $F(x, u, u_{(1)}, \dots, u_{(p)})$ for $p \geq 0$, is a pth-order differential invariant of a group G if

$$F(x,u,u_{(1)},\dots,u_{(p)}) = F(\bar{x},\bar{u},\bar{u}_{(1)},\dots,\bar{u}_{(p)}). \tag{1.2.35}$$

that means F is invariant under the prolonged group $G^{[p]}$, where for $p = 0$, $u_{(0)} = u$ and $G^{[0]} = G$.

Definition 9: A differential function, $F(x, u, u_{(1)}, \dots, u_{(p)})$ for $p \geq 0$, is a pth-order differential invariant of a group G if

$$X^{[p]}(F) = 0, \tag{1.2.36}$$

where $X_{[p]}$ is the pth prolongation of X and for $p = 0$, $X^{[0]} = X$. The differential invariants can be obtained by solving the characteristic equations for Eq. (1.2.36).

Determining Equations for Lie-Point Symmetries

In this section we introduce Lies algorithm for calculating point symmetries of PDEs:

Definition 10: An invertible transformation acting on the space (x, u) of E is a point symmetry of E provided every solution h of E is mapped onto another solution $\bar{h}$ of E.

Definition 11: Let G be a group of transformations (1.2.5) , admitted by the system E. Performing the first-order Taylor expansion of (1.2.4) around $a = 0$, we arrive at the fact that

$$X^{[k]}(E^{\sigma}(x, u, u_{(1)}, ..., u_{(k)})) = 0, \sigma = 1, ..., s, \tag{1.2.37}$$

whenever (1.2.1) is satisfied for every group operator X of G. Then G consists of symmetries of the system E. It can be shown that the converse is also true.

Equations (1.2.37) are the so-called determining equations. In general the determining equations comprise an over-determined system of linear homogeneous PDEs for the unknown coordinates ξ^i and η^{α} of the symmetry generator X. The solutions of the determining system form a vector space, that is, any finite linear combination of symmetries is again a symmetry. This stems from the fact that the determining equations are linear.

Lie Algebra

Definition 12: For an r-parameter Lie group of transformations with infinitesimal generators $X_{\alpha}, \;\; \alpha = 1, 2, ..., r$, the commutator (Lie Bracket) of X_{α} and X_{β} is a first-order operator defined by

$$[X_{\alpha}, X_{\beta}] = X_{\alpha}X_{\beta} - X_{\beta}X_{\alpha}. \tag{1.2.38}$$

It immediately follows that

$$[X_{\alpha}, X_{\beta}] = -[X_{\beta}, X_{\alpha}]. \tag{1.2.39}$$

Second Fundamental Theorem of Lie

The commutator of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator. In particular,

$$[X_{\alpha}, X_{\beta}] = \sum_{\gamma=1}^{r} C^{\gamma}_{\alpha\beta}X_{\gamma}. \tag{1.2.40}$$

where the coefficients $C^{\gamma}_{\alpha\beta}$ are constants called structure constants, $\alpha, \beta, \gamma = 1, 2, ..., r.$

Definition 13: Equations (1.2.40) are called the commutation relations of the r-parameter Lie group of transformations. For any three infinitesimal generators X_α, X_β and X_γ, by direct computation one can show that Jacobi'sidentity holds:

$$[X_\alpha, [X_\beta, X_\gamma] + [X_\beta, [X_\gamma, X_\alpha] + [X_\gamma, [X_\alpha, X_\beta] = 0. \tag{1.2.41}$$

Third Fundamental Theorem of Lie

The structure constants defined by the commutation relations (1.2.40) satisfy the relations

$$\begin{aligned} &C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha}, \\ &\sum_{\rho=1}^{r}[C^{\rho}_{\alpha\beta}C^{\delta}_{\rho\gamma} + C^{\rho}_{\beta\gamma}C^{\delta}_{\rho\alpha} + C^{\rho}_{\gamma\alpha}C^{\delta}_{\rho\beta}] = 0. \end{aligned} \tag{1.2.42}$$

In particular, these relations are equivalent to the commutator anti-symmetry property (1.2.39) and Jacobi's identity (1.2.41), respectively.

Definition 14: A Lie algebra L is a vector space over **R** or **C** with a bilinear bracket operation (the commutator) satisfying the properties (1.2.39), (1.2.41) and, most important, (1.2.40). In particular, the set of infinitesimal generators $\{X_\alpha\}$ $\alpha = 1, 2, ...r$, of an r-parameter Lie group of transformations forms an r-dimensional Lie algebra over **R**.

Let G be a Lie group with Lie algebra L. For each vector $v \in L$, the adjoint vector Adv at $w \in L$ is

$$adv|_w = [w, v] = -[v, w]. \tag{1.2.43}$$

The adjoint representation Ad G of the underlying Lie group can be reconstructed by summing the Lie series

$$\begin{aligned} Ad(\exp(\epsilon v))w_0 &= \sum_{n=0}^{\infty} \tfrac{\epsilon^n}{n!}(adv)^n(w_0), \\ &= w_0 - \epsilon[v, w_0] + \tfrac{\epsilon^2}{2!}[v, [v, w_0]] - ... \end{aligned} \tag{1.2.44}$$

Classification of Subalgebras and Group Invariant Solutions

Classification of subgroups of Lie symmetry groups of differential equations is an essential part in the study of these equations. This is since classification allows for an efficient computation of group-invariant solutions, without the possibility of an occurrence of equivalent solutions. Classifying subgroups may further lead to the construction of simple ansätze for the corresponding equivalence classes of reduced differential equations. Thereby, the classification also provides an important step for further investigations of properties of these reduced equations. The classification of subgroups of symmetry groups is usually done by the classification of the associated Lie subalgebras with respect to the adjoint representation [83, 111] and to compute the adjoint representation, we use the Lie series

$$Ad(\exp(\epsilon v))w_0 = w_0 - \epsilon[v, w_0] + \frac{\epsilon^2}{2!}[v, [v, w_0]] + ... \tag{1.2.45}$$

The classification of one-dimensional subalgebras of the whole symmetry algebra is done by an inductive approach [139]. Let $V_1, V_2, ..., V_r$ are basis of Lie algebra, then we start with the most general infinitesimal generator,

$$V = a_1V_1 + a_2V_2 + a_3V_3 + ... + a_rV_r, \tag{1.2.46}$$

and simplify it as much as possible by means of adjoint actions. Depending on the respective values of the coefficients $a_i, i = 1, ..., r$, we will find the list of inequivalent one-dimensional subalgebras. On using the inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra, group invariant reductions can be easily carried out which corresponds to group invariant solutions of studied equations.

We now provide the brief outlines of Lie classical method above. More emphasis has been laid on the implementation than on the mathematical intricacies of the techniques, thereby making the methods algorithmic in nature and thus easy to apply.

1.2.2 Classical Lie Method

Lies algorithm [22]

Below we give a layout of the steps involved in the execution of the procedure for calculating symmetries of E.

1. Let the one-parameter Lie group of point transformations (1.2.5) leaves invariant the system of PDEs (1.2.1).

2. Write the generator of symmetry

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.2.47}$$

   and prolong the symmetry generator X to the order which is the same as that of E,

$$\begin{aligned} X^{[k]} = \xi^i(x,u)\tfrac{\partial}{\partial x^i} + \eta^\alpha(x,u)\tfrac{\partial}{\partial u^\alpha} + {\zeta_i}^\alpha(x,u,u_{(1)})\tfrac{\partial}{\partial {u_i}^\alpha} + ... \\ + {\zeta_{i_i...i_k}}^\alpha(x,u,u_{(1)},...,u_{(k)})\tfrac{\partial}{\partial {u_{i_i...i_k}}^\alpha}, \end{aligned} \tag{1.2.48}$$

   where the variables ${\zeta_i}^\alpha$ are given by (1.2.31).

3. Apply the prolonged generator $X^{[k]}$ on E evaluated on the surface (1.2.1) yielding the symmetry conditions

$$X^{[k]}(E^\sigma(x,u,u_{(1)},...,u_{(k)})) = 0, \sigma = 1,...,s, \tag{1.2.49}$$

   when

$$E^\sigma(x,u,u_{(1)},...,u_{(k)}) = 0, \sigma = 1,...,s, \tag{1.2.50}$$

4. From the invariance condition, a system of linear PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generator X admitted by the given system of PDEs (1.2.1) is obtained.

5. The solutions of the determining equations will lead to the explicit forms of ξ and η.

6. Construct the corresponding characteristics equations (1.2.22) and obtain u in terms of $(n-1)$ new independent variables.

7. Rewrite the system (1.2.1) in these new coordinates to get the reduced form of the system.

1.2.3 Symmetry Reduction Method

A technique that has found an important place in literature on group theoretic methods for the determination of the solutions of a single or system of nonlinear partial differential equations is due to Steinberg [164] and is termed as Symmetry Reduction Method. Though the technique relies heavily on the theory of sophisticated use of nonlinear operators yet it has been cast in a form that it is easy to utilize by specialist and non-specialist alike. The algorithmic representation of the method makes the concepts clear and straight forward. Further, it bears a close relationship to the method of separation of variables in the case of linear equations. The technique has earlier been used to obtain the exact solutions of various nonlinear partial differential equations [15, 69, 157, 164].

The analytical execution of the technique consists of following three steps:[164]
i) Find the symmetries of the differential equations.
ii) Determine the canonical coordinates for symmetry or assume a separable form for the differential equation.
iii) Find the reduced problem in terms of the canonical coordinates.

For determining the symmetry operator of a system of differential equations, we need to proceed as follows: Let us consider a system of k nonlinear partial differential equations in k dependent variables $\bar{u} = (u_1, u_2, ..., u_k)$ and $(n + 1)$ independent variables $(t, x) = (t, x_1, x_2, ..., x_n)$. Let us assume that our system can be written in terms of nonlinear differential operator $\bar{N} \equiv (N_1, N_2, ..., N_k)$ as follows:

$$\bar{N}(\bar{u}) = \frac{\partial^p \bar{u}}{\partial t^p} - \bar{H}(\bar{u}), \tag{1.2.51}$$

where $\bar{u} = \bar{u}(t, \bar{x})$ and $\bar{H}$ may depend on $t, \bar{x}, \bar{u}$ and any derivative of $\bar{u}$ as long as the derivatives of $\bar{u}$ do not contain more than $(p - 1)$ derivatives of t. $\bar{H}$ can be nonlinear.
Next, we define symmetry operator for the system (1.2.51) called infinitesimal symmetries. These symmetries are quasi-linear partial differential operators of first order and

consequently must have the form

$$\bar{S}(\bar{u}) = A(t, \bar{x}, \bar{u})\frac{\partial \bar{u}}{\partial t} + \sum_{i=1}^{n} B_i(t, \bar{x}, \bar{u})\frac{\partial \bar{u}}{\partial x_i} + \bar{C}(t, \bar{x}, \bar{u}), \bar{C} = (C_1, C_2, ..., C_k). \quad (1.2.52)$$

The Fr$\acute{e}$chet derivative $\bar{F} = (F_1, F_2, ..., F_k)$ of $\bar{N}(\bar{u}) = (N_1, N_2, ..., N_k)$ in the direction of $\bar{v} = (v_1, v_2, ..., v_k)$ is given by

$$\bar{F}(\bar{N}, \bar{u}, \bar{v}) = \frac{d[\bar{N}(\bar{u} + \epsilon \bar{v})]}{d\epsilon}|\epsilon = 0. \quad (1.2.53)$$

The method mainly consists of determining the coefficients $A, B_i, i = 1, 2, ..., n$ and $C_j, j = 1, 2, ..., k$ in the symmetry operator $\bar{S}$. For this, we first find Fr$\acute{e}$chet derivative $\bar{F} = (F_1, F_2, ..., F_k)$ of $\bar{N}(\bar{u}) = (N_1, N_2, ..., N_k)$ by the equations (1.2.53), then $\bar{v} = (v_1, v_2, ..., v_k)$ is substituted by $\bar{S} = (S_1, S_2, ..., S_k)$ in order to evaluate them in the direction of the symmetry operator.

$$\bar{F}(\bar{N}, \bar{u}, \bar{S}) = \frac{d[\bar{N}(\bar{u} + \epsilon \bar{S})]}{d\epsilon}|\epsilon = 0. \quad (1.2.54)$$

For invariance of the system (1.2.51), we require that the Fr$\acute{e}$chet derivative (1.2.54) must vanish on the solution set of (1.2.51) in the direction of the symmetry operator . That is, we must have

$$\bar{F}(\bar{N}, \bar{\eta}, \bar{S})|_{\bar{N}=\bar{0}} = \bar{O}. \quad (1.2.55)$$

For this we substitute $\bar{H}(\bar{u})$ for $\frac{\partial^p \bar{u}}{\partial t^p}$ in (1.2.55) . The equations (1.2.55) when expanded, result in to polynomial expressions in various partial derivatives of $\bar{u}$. Equating the various coefficients of these derivative terms, we will get a set of linear partial differential equations called ”determining equations” for the group infinitesimals $A, B_i, i = 1, 2, ..., n$ and $C_j, j = 1, 2, ..., k$. Solve the resulting ”determining equations” for symmetries of the system (1.2.51).

Once this resulting set of partial differential equations is solved for coefficients of $\bar{S}$. The associated Lie algebra of infinitesimal symmetries of (1.2.51) is then the set of vector fields of the form

$$V = A(t, \bar{x}, \bar{u})\frac{\partial}{\partial t} + \sum_{i=1}^{n} B_i(t, \bar{x}, \bar{u})\frac{\partial}{\partial x_i} - \sum_{j=1}^{k} C_j(t, \bar{x}, \bar{u})\frac{\partial}{\partial u_j}, \quad (1.2.56)$$

Or, equivalently the one-parameter group of point transformations of (1.2.51) is as follows:

$$\begin{aligned} t^* &= t + \epsilon \bar{A}(t, \bar{x}, \bar{u}) + o(\epsilon^2), \\ \bar{x}^* &= x + \epsilon \bar{B}(t, \bar{x}, \bar{u}) + o(\epsilon^2), \\ \bar{u}^* &= u - \epsilon \bar{C}(t, \bar{x}, \bar{u}) + o(\epsilon^2), \end{aligned} \tag{1.2.57}$$

where $\bar{u}^* = (u_1^*, u_2^*, ..., u_k^*)$. Using the infinitesimal generators (1.2.57), one can obtain a reduction of system (1.2.51) to a system with number of independent variables one less than the original one. For this, first we solve the characteristic equations

$$\frac{dt}{A} = \frac{dx_1}{B_1} = \frac{dx_2}{B_2} = ... = \frac{dx_n}{B_n} = \frac{du_1}{-C_1} = \frac{du_2}{-C_2} = ... = \frac{du_k}{-C_k}. \tag{1.2.58}$$

From these equations, we obtain new canonical coordinates and then change the system (1.2.51) in these new coordinates to get the reduced form of the problem.

1.2.4 Generalized $\left(\frac{G'}{G}\right)$- Expansion Method

During the past few decades, search for exact solutions of nonlinear partial differential equations by using various different methods is the main goal for many researchers, and many powerful methods to construct exact solutions of nonlinear partial differential equations have been established and developed as discussed in earlier sections. More recently, a new method called Generalized $\left(\frac{G'}{G}\right)$- Expansion method [174, 192] has been proposed to seek exact solutions of nonlinear partial differential equations. Being concise and straightforward, this method can be applied to various nonlinear partial differential equations with variable coefficients. As a result, hyperbolic function solution, trigonometric function solution and rational solution with various parameters are obtained.

Description of The Generalized $\left(\frac{G'}{G}\right)$-Expansion Method [174]

Now, we have described the $\left(\frac{G'}{G}\right)$-expansion method for finding travelling wave solutions of nonlinear partial differential equations. Suppose that a nonlinear equation with independent variables $X = (x, y, z, ..., t)$ and dependent variable u is given by

$$F(u, u_t, u_x, u_y, u_z, ...u_{xt}, u_{yt}, u_{zt}, u_{tt}, ...) = 0, \tag{1.2.59}$$

where $u = u(x, y, z, ...t)$ is an unknown function, F is a polynomial in $u = u(x, y, z, ...t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of the generalized $\left(\frac{G'}{G}\right)$-expansion method.

Step 1: We suppose that the solution of Eq. (1.2.59) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u = \alpha_0(X) + \sum_{i=1}^{m} \alpha_i(X) \left(\frac{G'(\zeta)}{G(\zeta)} \right)^i, \;\; \alpha_m(X) \neq 0, \tag{1.2.60}$$

where $\alpha_0(X), \alpha_i(X), (i = 1, 2, ..., m)$ and $\zeta = \zeta(X)$ are all functions of X, to be determined later and $G = G(\zeta)$ satisfies following equation

$$G''(\zeta) + \lambda G'(\zeta) + \mu G(\zeta) = 0, \tag{1.2.61}$$

where $\zeta = p(t)x + q(t)$, $p(t)$ and $q(t)$ are functions to be determined.

Step 2: In order to determine u explicitly, we firstly find the value of integer m by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in Eq. (1.2.59).

Step 3: Substitute (1.2.60) along with Eq. (1.2.61) into Eq. (1.2.59) and collect all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left hand side of Eq. (1.2.59) is converted into a polynomial in $\left(\frac{G'}{G}\right)$. Then set each coefficient of this polynomial to zero to derive a set of over-determined partial differential equations for $\alpha_0(X), \alpha_i(X)$ and ζ.

Step 4: Solve the system of over-determined partial differential equations obtained in Step 3 for $\alpha_0(X), \alpha_i(X)$ and ζ.

Step 5: Use the results obtained in above steps to derive the solutions of Eq. (1.2.59) depending on $\left(\frac{G'}{G}\right)$, since the solutions of Eq. (1.2.61) have been well known to us depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, then the exact solutions of Eq. (1.2.59) are obtained.

1.2.5 Painlev$\acute{e}$ Analysis

Previous sections give us an idea of the considerable progress made in familiarizing one with nonlinear PDEs in terms of symmetries. Another important aspect in relation to the said dynamical system that deserves special attention is to trace out the progress made in developing an approach that helps in deciding whether it is integrable or not. In the case of ordinary differential equations; the singularity structure analysis (also called Painlev$\acute{e}$ test) of the solution in the complex plane has played an important role in deciding between integrable and non-integrable dynamical systems. More specifically, one could classify an ODE or a system of ODEs in the complex domain to be of Painlev$\acute{e}$ type if the only movable singularities of all its solutions are poles. Fundamental contribution connecting Painlev$\acute{e}$ property and integrability in the case of ODE has been made by Kovalevskaya [98], Yoshida [189], Erconlani and Siggia [54].

Weiss, Tabor and Carnevale [182, 183] have introduced Painlev$\acute{e}$ test for PDEs and have shown that there exists a close relationship between Painlev$\acute{e}$ property and integrability. This has been successfully carried over to KdV, KP and Boussinesq equation and hence leads to B$\ddot{a}$cklund transformation. In this section, the description of so called WTC technique has been given, with special reference to the main steps for its application to PDEs and different stages as leading order, resonance analysis and compatibility condition.

Painlev$\acute{e}$ Analysis for Partial Differential Equations [182, 183]

While extending the idea of connection between Painlev$\acute{e}$ property and its integrability in the case of ODE(s) or PDE(s), Weiss et al. [182, 183] have required that the solutions be single-valued around movable singularity manifolds. Further, they have pointed out that the singularity of PDEs are in general not isolated as the solutions are functions of several complex variable $(z_1, z_2, ..., z_n)$, but rather lie on manifolds determined by the condition

$$\phi(z_1, z_2, ..., z_n) = 0. \tag{1.2.62}$$

Consider the evolution equation

$$\frac{\partial u}{\partial t} = A(u), \tag{1.2.63}$$

where A is polynomial in u and its spatial derivatives.

Thus, if $u = u(z_1, z_2, ..., z_n)$ is a solution of the PDE (1.2.63) then we require that in the neighbourhood of the manifold, equation (1.2.62) can be expanded into

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j, \tag{1.2.64}$$

where $u_0 \neq 0$, $u_j = u_j(z_1, z_2, ..., z_n)$ and $\phi = \phi(z_1, z_2, ..., z_n)$ are analytic functions of $z_j$in a neighbourhood of the manifold (1.2.62) and that α is negative integer.

Implementation of this procedure is direct and follows algorithmically in a manner similar to that of the ODE(s). There are essentially four steps involved in the Painlev$\acute{e}$ analysis of PDE(s) [105].

1. Determine of the leading order behaviors.
2. Identification of the powers at which arbitrary functions can enter into the Laurent series called resonances.
3. Verifying that at the resonance values sufficient number of arbitrary functions exist without the introduction of movable critical manifolds.
4. Establishing connections with the solutions and other integrability properties.

The remarkable feature of the Painlev$\acute{e}$ analysis, particularly for soliton equations, is that a natural connection exists between Painlev$\acute{e}$ property and the linearization property, Lax pairs, B$\ddot{a}$cklund transformations, integrability, etc.

In the following, we outline briefly each of the stages.

1. Leading Order Analysis

   As pointed out earlier α occurring in the expansion (1.2.64) has to be so determined that α is negative integer so that no movable critical manifolds enter. Consequently, we start with the determination of all possible value(s) of α and u_0 in the expansion (1.2.64). For each value of α, the homogeneous terms with the highest degree may balance each other. These terms are called leading terms (or dominant terms). The values for u_0 can be determined by equating the coefficients of the dominant terms to zero and solving the resulting algebraic equation for u_0.

2. Resonance Analysis

   Next, one has to find the "resonance" values, j, that is the power(s) at which the coefficient u_j of the term $\phi^{j+\alpha}$ in the expansion (1.2.64) is arbitrary. To find these, we substitute (1.2.64) into the equation (1.2.63) and obtain appropriate recursion relation for u_j and extract the coefficient $\widetilde{Q}(j) = Q(j)u_j$ of the term $\phi^{j+\alpha-N}$, where N is the order of the PDE. Then $Q(j) = 0$ is called the resonance equation, for which -1 is always a root, which corresponds to the arbitrary nature of ϕ. In order to avoid any movable critical singular manifold, we require that these remaining roots are non-negative integers.

3. Arbitrary Functions

   Let j_s be the highest of the allowed resonance values. On substituting

$$u = \phi^{\alpha} \sum_{j=0}^{j_s} u_j \phi^{j+\alpha}, \tag{1.2.65}$$

   into equation (1.2.63) and collecting the coefficient of $\phi^{j+\alpha-N}$, we get

$$Q(j) + R_j = 0, \tag{1.2.66}$$

   where R_j is a polynomial in the partial derivatives of ϕ and u_k, $(k = 0, 1, ..., j-1)$. Since $Q(j) = 0$, for any resonance value j, R_j should identically vanish. In this case u_j is arbitrary. In case it is not so, we have to introduce logarithmic term of the form $a_j + b_j log(\phi)$ in the series. But due to this addition, logarithmic singularities will appear in the solution manifold. Thus, $R_j = 0$ is a condition to ensure that the solution is free from movable critical manifold at a particular resonance value j. In this way we can check that the general solution is free from movable critical manifolds.

4. B$\ddot{a}$cklund Transformation

   Assume that the evolution equations (1.2.63) posses the Painlev$\acute{e}$ test. Then B$\ddot{a}$cklund Transformation can be found as follows:

   (a) Find the Painlev$\acute{e}$ expansion (1.2.62),

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j. \tag{1.2.67}$$

   (b) Truncate the series at the constant level term by setting

$$u = u_0 \phi^{-n} + u_1 \phi^{-n+1} + ... + u_n. \tag{1.2.68}$$

   and from the recursion relations for u_j an over determined system of equations for $(\phi, u_j, j = 0, 1, 2, ..., n)$ will be obtained, where u_n will satisfy the original differential equation (1.2.63). This step provides us with an autob$\ddot{a}$cklund transformation. For further details of the cases that can occur on account of truncation of the expansion we refer to Newell et al. [130].

Chapter 2

Einstein-Maxwell Equations for a Static Axially Symmetric Spacetime [1]

2.1 Introduction

Weyl [184] had formulated and developed the static, axially symmetric problem in general relativity in a very most elegant manner. Weyl showed that the line element could be expressed in a diagonal form with two functions in vacuum or electrostatic vacuum with proper choice of coordinates. Using these coordinates, which are known as canonical coordinates, Weyl completely solved the problem for a pure gravitational field with axial symmetry and also obtained a particular class of solutions for an axially symmetric electrostatic field and that solutions involved a functional relation between the electrostatic potential A_0 and the component g_{00} of the metric tensor.

Reissner [148] and Nordstr$\ddot{o}$m[132] has obtained very important results for static solutions of the Einstein-Maxwell equations. It represents the external spherically symmetric gravitational field of a charged body and belongs to Weyl's family of static axisymmetric

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Einstein-Maxwell fields. The static axially symmetric Einstein-Maxwell equations has been studied by various authors [166] to find the solutions and its physical properties. The line element for the static axially symmetric Einstein - Maxwell field case may be written in the Weyl [184] form

$$ds^2 = \exp(2u)dt^2 - \exp(-2u)(\exp(2k)(d\rho^2 + dz^2) + \rho^2 d\phi^2), \tag{2.1.1}$$

where u and k are functions of ρ and z and ρ and z are weyl canonical coordinates.
The Einstein - Maxwell equations are

$$G_{ij} = -8\pi E_{ij}, \tag{2.1.2}$$

$$F^{ij}_{;j} = 0, \tag{2.1.3}$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, \tag{2.1.4}$$

$$E_{ij} = \frac{1}{4\pi}(g^{kl}F_{ik}F_{ji} - \frac{1}{4}g_{ij}F_{kl}F^{kl}). \tag{2.1.5}$$

The electromagnetic potential vector has only two nonvanishing components for this problem. There are various ways to define the electromagnetic fields in terms of the components of the four potential, the most common being $F_{ij} = A_{i,j} - A_{j,i}$. However, another definition given below has the advantage of reducing the nontrivial Maxwell equations as well as the components of the stress tensor (2.1.5) in a symmetrical form with respect to the components of the potential vector. Thus, we define

$$F^{31} = \frac{1}{\rho}\exp(2u - 2k)A_2, \tag{2.1.6}$$

$$F^{23} = \frac{1}{\rho}\exp(2u - 2k)A_1, \tag{2.1.7}$$

$$F_{01} = B_1, \tag{2.1.8}$$

$$F_{02} = B_2, \tag{2.1.9}$$

where $A(\rho, z)$ and $B(\rho, z)$ are the magnetic and electric potentials, respectively.
With the above substitutions, Einstein - Maxwell field equations becomes

$$u_{\rho\rho} + \frac{u_\rho}{\rho} + u_{zz} = -\exp(-2u)(A_\rho^2 + A_z^2 + B_\rho^2 + B_z^2), \tag{2.1.10}$$

$$A_{\rho\rho} + \frac{A_\rho}{\rho} + A_{zz} - 2(u_\rho A_\rho + u_z A_z) = 0, \tag{2.1.11}$$

$$B_{\rho\rho} + \frac{B_\rho}{\rho} + B_{zz} - 2(u_\rho B_\rho + u_z B_z) = 0, \tag{2.1.12}$$

$$\frac{k_\rho}{\rho} = (u_\rho^2 - u_z^2) + \exp(-2u)(A_\rho^2 - A_z^2 + B_\rho^2 - B_z^2), \tag{2.1.13}$$

$$\frac{k_z}{\rho} = 2u_\rho u_z + 2\exp(-2u)(A_\rho A_z + B_\rho B_z), \tag{2.1.14}$$

$$A_\rho B_z = A_z B_\rho. \tag{2.1.15}$$

The symmetrical occurrence of the potentials A and B in the equations (2.1.10)-(2.1.15) suggests a duality rotation $A = v\cos\beta$ and $B = v\sin\beta$, where v is a new potential and β is a constant.

Thus Einstein - Maxwell field equations (2.1.10)-(2.1.15) becomes

$$u_{\rho\rho} + \frac{u_\rho}{\rho} + u_{zz} = -\exp(-2u)(v_\rho^2 + v_z^2), \tag{2.1.16}$$

$$v_{\rho\rho} + \frac{v_\rho}{\rho} + v_{zz} - 2(u_\rho v_\rho + u_z v_z) = 0, \tag{2.1.17}$$

$$\frac{k_\rho}{\rho} = (u_\rho^2 - u_z^2) + \exp(-2u)(v_\rho^2 - v_z^2), \tag{2.1.18}$$

$$\frac{k_z}{\rho} = 2u_\rho u_z + 2\exp(-2u)(v_\rho v_z). \tag{2.1.19}$$

So, we have four equations (2.1.16)-(2.1.19) for the determination of three unknowns u, v and k and one can easily verify that equations constitute a completely determinate system of partial differential equations. Thus the basic equations are (2.1.16) and (2.1.17), since k can be obtained trivially from (2.1.18) and (2.1.19), once u and v are known. It may be pointed out that equations (2.1.16)-(2.1.17) are a set of coupled, second order, nonlinear partial differential equations in u and v, hence we will concentrate on these two equations. Due to nonlinearity of exponential order, it is difficult to solve equations (2.1.16)-(2.1.17) and hence study of symmetries and exact solutions of equations (2.1.16)-(2.1.17) is of great importance.

By using the transformation $t = \iota z$, the system (2.1.16)-(2.1.17) represents the Einstein equations for cylindrical gravitational waves [96] (in which case $Q = -u + log\rho$ and v

are metric coefficients) and the Einstein-Maxwell equations for colliding plane gravitational and plane electromagnetic waves [12] (in which case u is a metric coefficient and v is an electromagnetic potential), where ρ and t represents the cylindrical radial and time coordinates respectively. The importance of the equations (2.1.16)-(2.1.17) and the need to have some exact solutions are the main motive behind the present study. To have an insight, the explicit analytic solutions of the system (2.1.16)-(2.1.17) may enable one to better understand the phenomena which it describes. A detailed systematic analysis that leads to an exact analytic solution for (2.1.16)-(2.1.17) has not been performed and is therefore desirable.

Our intention is to systematically study of the system (2.1.16)-(2.1.17) and to obtain a deeper insight into the nature of solutions permitted using the Lie analysis of differential equations. In section (2.2), the Lie group analysis is used to generate the various symmetries of the system of partial differential equations (2.1.16)-(2.1.17), which are then used to identify the associated basic vector fields of the optimal system. Section (2.3) has been devoted to the systematically study of group invariant solutions admitted by the system (2.1.16)-(2.1.17). In section (2.4), some more exact solutions are also furnished.

2.2 Symmetry Analysis

Let us consider the Lie group of point transformations

$$\begin{aligned}
\rho^* &= \rho + \epsilon\xi(\overline{X}, \overline{\sigma}) + O(\epsilon^2), \\
z^* &= z + \epsilon\tau(\overline{X}, \overline{\sigma}) + O(\epsilon^2), \\
\overline{\sigma}^* &= \overline{\sigma} + \epsilon\overline{\phi}(\overline{X}, \overline{\sigma}) + O(\epsilon^2),
\end{aligned} \tag{2.2.1}$$

where $\overline{X} = (\rho, z)$, $\overline{\sigma} = (u, v)$, $\overline{\phi} = (\eta, \phi)$, which leaves the system (2.1.16)-(2.1.17) invariant.

The vector field associated with the above group of transformations can be written as follows:

$$V = \xi(\rho, z, u, v)\frac{\partial}{\partial \rho} + \tau(\rho, z, u, v)\frac{\partial}{\partial z} + \eta(\rho, z, u, v)\frac{\partial}{\partial u} + \phi(\rho, z, u, v)\frac{\partial}{\partial v}. \qquad (2.2.2)$$

The symmetry group of equations (2.1.16)-(2.1.17), will be generated by the vector field of the form (2.2.2).

Here, we have obtained the symmetry groups of system (2.1.16)-(2.1.17) by using the Lie classical method. We found that Lie symmetries (vector fields) under which the system (2.1.16)-(2.1.17) is invariant can be spanned by the following five linearly independent infinitesimal generators:

$$\begin{array}{l} V_1 = -2v\frac{\partial}{\partial u} + (-v^2 + \exp(2u))\frac{\partial}{\partial v}, \;\; V_2 = \rho\frac{\partial}{\partial \rho} + z\frac{\partial}{\partial z}, \\ V_3 = \frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, V_4 = \frac{\partial}{\partial z}, V_5 = \frac{\partial}{\partial v}, \end{array} \qquad (2.2.3)$$

with the nonzero Lie bracket relationships as follows:

$$[V_1, V_3] = -V_1, [V_1, V_5] = 2V_3, [V_2, V_4] = -V_4, [V_3, V_5] = -V_5, \qquad (2.2.4)$$

for the given fields. As a result, symmetries (2.2.4) form a five-dimensional Lie algebra.

In general, to each s parameter subgroup H of the full symmetry group G of a system of differential equations in $p > s$ independent variables, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective, systematic means of classifying these solutions, leading to an optimal system of group invariant solutions from which every other such solution can be derived. Since elements $g \in G$ not in the subgroup H will transform an H invariant solution to some other group-invariant solutions, only those solutions not so related need be listed in our optimal system.

Let G be a Lie group. An optimal system of s parameter subgroups is a list of conjugacy inequivalent s parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation [139] where the adjoint action is given by the Lie series:

$$Ad(\exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2}[V_i, [V_i, V_j]] - ..., \tag{2.2.5}$$

where $[V_i, V_j] = V_iV_j - V_jV_i$ is the commutator for the Lie algebra, and ϵ is a parameter. We use the sub algebraic structure of symmetries (2.2.4) of system (2.1.16)-(2.1.17) to construct an optimal system of one dimensional subgroups. Following [137] we deduce the following basic fields

$$\begin{aligned}
&(i)V_5,\\
&(ii)V_4 + \gamma V_5,\\
&(iii)V_3 + \mu V_4,\\
&(iv)V_2 + \alpha V_3,\\
&(v)V_1 + \beta V_2 + \lambda V_5,\\
&(vi)V_2 + \delta V_5,\\
&(vii)V_1 + \sigma V_4,\\
&(viii)V_2 + \nu V_1,
\end{aligned} \tag{2.2.6}$$

where γ, μ, α, β, λ, δ, σ and ν are arbitrary constants.

All solutions of the system (2.1.16)-(2.1.17) which are obtained via other combinations of point symmetries can be transformed by symmetry group transformations into the solutions obtained from the combinations above (2.2.6).

2.3 Group Invariant Solutions

In this section, the primary focus is on the reductions associated with the vector fields in the optimal system and attempt to furnish exact solutions.

(i) V_5

Corresponding to this vector field, no such invariant solutions of system (2.1.16)-(2.1.17) exist.

(ii) $V_4 + \gamma V_5$

For this vector field, the form of the similarity variable and similarity solution are as follows:

$\zeta = \rho$, $u(\rho, z) = F(\zeta)$, $v(\rho, z) = G(\zeta) + \gamma z$.

On using these in the system (2.1.16)-(2.1.17), the system of reduced ODEs:

$$F'' + \tfrac{F'}{\zeta} + \exp(-2F)(G'^2 + \gamma^2) = 0, G'' + \tfrac{G'}{\zeta} - 2F'G' = 0. \tag{2.3.1}$$

This can be further reduced as follows:

$$F(\zeta) = \tfrac{1}{2}\ln(G'\zeta k_1), G(\zeta) = \textstyle\int g(\zeta)d\zeta + k_2, \tag{2.3.2}$$

where $g(\zeta)$ is the solution of the following differential equation:

$$k_1\zeta g'' g = -(gk_1g' - g'^2 k_1\zeta + 2g^3 + 2g\gamma^2), \tag{2.3.3}$$

where k_1 and k_2 are arbitrary constants.

By integrating the eq. (2.3.3) for $\gamma = 0$, we get

$$g(\zeta) = \frac{-1}{4c_1\zeta}\left(-1 + \left(\tanh\left(\frac{1}{2}\frac{\sqrt{c_1k_1}\,(\ln(\zeta) + c_2)}{c_1k_1}\right)\right)^2\right). \tag{2.3.4}$$

Therefore, we have solution of system (2.1.16)-(2.1.17), as follows:

$$\begin{aligned} u(\rho, z) &= \tfrac{1}{2}\ln\left(\tfrac{k_1}{4c_1}\left(\cosh\left(\tfrac{1}{2}\tfrac{\sqrt{c_1k_1}(\ln(\rho)+c_2)}{c_1k_1}\right)\right)^{-2}\right), \\ v(\rho, z) &= \tfrac{1}{2}\tfrac{k_1}{\sqrt{c_1k_1}}\tanh\left(\tfrac{1}{2}\tfrac{\sqrt{c_1k_1}(\ln(\rho)+c_2)}{c_1k_1}\right) + k_2. \end{aligned} \tag{2.3.5}$$

Using (2.1.18)-(2.1.19), corresponding to these expressions of $u(\rho, z)$ and $v(\rho, z)$, we obtain

$$k(\rho, z) = \frac{1}{4}\frac{\ln(\rho)}{k_1c_1} + c_3, \tag{2.3.6}$$

where k_1, k_2, c_1, c_2 and c_3 are arbitrary constants.

(iii) $V_3 + \mu V_4$

Similarity variable and similarity solution are:

$\zeta = \rho$, $u(\rho, z) = F(\zeta) + \frac{z}{\mu}$, $v(\rho, z) = \exp(\frac{z}{\mu})G(\zeta)$.

On using these in system (2.1.16)-(2.1.17), the system of reduced ODEs:

$$\begin{aligned} &\zeta\mu^2 F'' + F'\mu^2 + \zeta \exp(-2F)(G'^2\mu^2 + G^2) = 0, \\ &\zeta\mu^2 G'' + G'\mu^2 - \zeta G - 2\zeta\mu^2 G' F' = 0. \end{aligned} \tag{2.3.7}$$

Let $G(\zeta) = \exp(F(\zeta))H(\zeta)$, $F'(\zeta) = N(\zeta)$.

Using these substitutions the system (2.3.7) reduces to

$$\begin{aligned} &\zeta\mu^2 N' + \mu^2 N + \zeta\mu^2 N^2 H^2 + 2\zeta\mu^2 N H H' + \zeta\mu^2 H'^2 + \zeta H^2 = 0, \\ &\zeta\mu^2 N' H - \zeta\mu^2 N^2 H + \zeta\mu^2 H'' + \mu^2 N H + \mu^2 H' - \zeta H = 0. \end{aligned} \tag{2.3.8}$$

We arrive at following cases:

Case (iii.1)

$H(\zeta) = 0$, that is not a physically interesting case.

Case (iii.2)

$H(\zeta) = \pm\iota$, where ι represents the complex number iota.

With this, our system (2.3.8) reduced to single equation:

$$\zeta\mu^2 N' - \zeta\mu^2 N^2 + \mu^2 N - \zeta = 0, \tag{2.3.9}$$

which can be further solved to give solution:

$$N(\zeta) = \frac{c_1 Y_1(\frac{\zeta}{\mu}) + J_1(\frac{\zeta}{\mu})}{\mu(c_1 Y_0(\frac{\zeta}{\mu}) + J_0(\frac{\zeta}{\mu}))}, \tag{2.3.10}$$

where c_1 is arbitrary constant and $J_v(x)$ and $Y_v(x)$ are the standard Bessel functions of the first and second kinds, respectively. They satisfy the Bessel equation:

$$x^2 Y'' + xY' + (x^2 - v^2)Y = 0. \tag{2.3.11}$$

Using these results we get final solution of system (2.1.16)-(2.1.17):

$$\begin{aligned} u(\rho,z) &= \tfrac{z}{\mu} - ln\left(c_1 Y_0\left(\tfrac{\rho}{\mu}\right) + J_0\left(\tfrac{\rho}{\mu}\right)\right) + c_2, \\ v(\rho,z) &= \pm\frac{\left(\iota\exp(\frac{z}{\mu})\exp(c_2)\right)}{c_1 Y_0\left(\frac{\rho}{\mu}\right)+J_0\left(\frac{\rho}{\mu}\right)}, \end{aligned} \tag{2.3.12}$$

Using above expressions for $u(\rho,z)$ and $v(\rho,z)$ in (2.1.18)-(2.1.19) and solving, we get

$$k(\rho,z) = c_3, \tag{2.3.13}$$

where c_1, c_2 and c_3 are arbitrary constants.

(**iv**) $V_2 + \alpha V_3$

Corresponding to this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = \frac{\rho}{z}$, $u(\rho,z) = F(\zeta) + \alpha\ln(z)$, $v(\rho,z) = z^{\alpha}G(\zeta)$.

On using these in system (2.1.16)-(2.1.17), the system of reduced ODEs:

$$\begin{aligned} &F'' + \tfrac{F'}{\zeta} + \zeta^2 F'' + 2\zeta F' - \alpha\zeta + \exp(-2F)(\zeta^2 G'^2 + G'^2 - 2\alpha\zeta G G' + \alpha^2 G^2) = 0, \\ &\zeta G'' + G' + \zeta^3 G'' + 2\zeta^2 G' - \alpha^2\zeta G - \alpha\zeta G - 2\zeta F'G' - 2\zeta^3 F'G' + 2\alpha\zeta^2 G F' = 0. \end{aligned} \tag{2.3.14}$$

Let $G(\xi) = \exp\left(F(\xi)\right)H(\xi)$, $F'(\xi) = N(\xi)$.

Then our system (2.3.14) becomes

$$\begin{aligned} &\zeta N' + N + \zeta^3 N' + 2\zeta^2 N - \alpha\zeta + \zeta^3 N^2 H^2 + 2\zeta^3 N H H' + \zeta^3 H'^2 + \zeta N^2 H^2 + 2\zeta N H H' \\ &+\zeta H'^2 - 2\alpha\zeta^2 N H^2 - 2\alpha\zeta^2 H H' + \alpha^2\zeta H^2 = 0, \\ &\zeta N' H - \zeta N^2 H + \zeta H'' + N H + H' + \zeta^3 N' H - \zeta^3 N^2 H + \zeta^3 H'' + 2\zeta^2 N H + 2\zeta^2 H' \\ &-\alpha^2\zeta H - \alpha\zeta H + 2\alpha\zeta^2 N H = 0. \end{aligned} \tag{2.3.15}$$

We arrive at following cases:

Case (iv.1)

$H(\zeta) = 0$, that is not a physically interesting case.

Case (iv.2)

$H(\zeta) = \pm\iota$, where ι represents the complex number iota.

With this, our system (2.3.15) reduced to single equation:

$$\zeta N + N + \zeta^3 N' + 2\zeta^2 N - \alpha\zeta - \zeta^3 N^2 - \zeta N^2 + 2\alpha\zeta^2 N - \alpha^2\zeta = 0. \tag{2.3.16}$$

For the function $w(\zeta) = \exp(-\int N(\zeta)d\zeta)$, the eq. (2.3.16) reduces to

$$(1+\zeta^2)w'' + \frac{(1+2(1+\alpha)\zeta^2)}{\zeta}w' + \alpha(1+\alpha)w = 0, \tag{2.3.17}$$

and hence, the solution of equation (2.3.17) is given by

$$\begin{aligned}
&N(\zeta) = -\{\tfrac{c_1\alpha(1+\alpha)\zeta}{2\alpha+1}F\left(\left[\tfrac{\alpha}{2}+1, \tfrac{3+\alpha}{2}\right], \left[\alpha+\tfrac{3}{2}\right], 1+\zeta^2\right) \\
&+(1-2\alpha)\zeta(1+\zeta^2)^{(-\alpha-\frac{1}{2})}F\left(\left[\tfrac{-\alpha}{2}+1, \tfrac{1-\alpha}{2}\right], \left[-\alpha+\tfrac{3}{2}\right], 1+\zeta^2\right) \\
&+\tfrac{(2-\alpha)(1-\alpha)\zeta(1+\zeta^2)^{(\frac{1}{2}-\alpha)}}{3-2\alpha}F\left(\left[\tfrac{-\alpha}{2}+2, \tfrac{3-\alpha}{2}\right], \left[-\alpha+\tfrac{5}{2}\right], 1+\zeta^2\right)\} \\
&\times\tfrac{1}{(c_1F([\frac{\alpha}{2},\frac{1+\alpha}{2}],[\alpha+\frac{1}{2}],1+\zeta^2)+c_2(1+\zeta^2)^{(\frac{1}{2}-\alpha)}F([\frac{-\alpha}{2}+1,\frac{1-\alpha}{2}],[-\alpha+\frac{3}{2}],1+\zeta^2))},
\end{aligned} \tag{2.3.18}$$

where$F$ is hypergeometric function and c_1 and c_2 are arbitrary constants.

Hence, the solution of the system (2.1.16)-(2.1.17) is

$$u(\rho, z) = \int N(\zeta)d\zeta + \alpha\ln(z), v(\rho, z) = \pm\iota z^\alpha \exp(\int N(\zeta)d\zeta). \tag{2.3.19}$$

(v) $V_1 + \beta V_2 + \lambda V_5$

Unfortunately, we are not able to reduce the ordinary differential equations corresponding to this case, this will be taken as future endeavor.

We consider the simplest form of transformation in view of above similarity variables and similarity functions as follow:

$v = \iota\exp(u)$ and $v = -\iota\exp(u)$.

Using above in the system (2.1.16)-(2.1.17), we get

$$u_{\rho\rho} - u_\rho^2 + \frac{u_\rho}{\rho} + u_{zz} - u_z^2 = 0, \tag{2.3.20}$$

and solution is

$$\begin{aligned}
&u(\rho,z)=-\tfrac{1}{2}\ln\left(\tfrac{(J_0(c_1\rho)c_2-c_3Y_0(c_1\rho))^2}{c_1^2\rho^2(-Y_0(c_1\rho)J_1(c_1\rho)+Y_1(c_1\rho)J_0(c_1\rho))^2}\right)+c_1z\\
&-\tfrac{1}{2}\ln\left(\tfrac{(c_2\exp(2c_1z)-c_3)^2}{4c_1^2}\right),\\
&v(\rho,z)=\pm 2\,\iota\,\mathrm{e}^{c_1z}\frac{1}{\sqrt{\frac{(J_0(c_1\rho)c_2-c_3Y_0(c_1\rho))^2}{c_1^2\rho^2(-Y_0(c_1\rho)J_1(c_1\rho)+Y_1(c_1\rho)J_0(c_1\rho))^2}}}\frac{1}{\sqrt{\frac{\left(c_2\mathrm{e}^{2\,c_1z}-c_3\right)^2}{c_1^2}}},
\end{aligned}\tag{2.3.21}$$

Using (2.1.18)-(2.1.19), corresponding to these expressions of $u(\rho,z)$ and $v(\rho,z)$, we obtain

$$k(\rho,z)=c_4,\tag{2.3.22}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Alternatively, we can also obtained the expressions for $u(\rho,z)$ as follows:

By letting $u(\rho,z)=\ln(w(\rho,z))$ and assuming the ansätz $w=R(\rho)Z(z)$ which leads to the equations

$$RR''-2R'^2+\tfrac{1}{\rho}RR'+\lambda R^2=0Z''-2Z'^2-\lambda Z^2=0.\tag{2.3.23}$$

Hence by solving the system (2.3.23), the solution of the system (2.1.16)-(2.1.17) is given by

$$\begin{aligned}
&u(x,t)=\ln\left(-\tfrac{\sqrt{\lambda}\left(-1+\sqrt{-\lambda}\rho J_1(\sqrt{-\lambda}\rho)Y_0(\sqrt{-\lambda}\rho)-\sqrt{-\lambda}\rho J_0(\sqrt{-\lambda}\rho)Y_1(\sqrt{-\lambda}\rho)\right)}{c_1\,\sin\left(\sqrt{\lambda}z\right)-c_2\,\cos\left(\sqrt{\lambda}z\right)}\right),\\
&v(x,t)=\pm\tfrac{\iota\sqrt{\lambda}\left(-1+\sqrt{-\lambda}\rho J_1(\sqrt{-\lambda}\rho)Y_0(\sqrt{-\lambda}\rho)-\sqrt{-\lambda}\rho J_0(\sqrt{-\lambda}\rho)Y_1(\sqrt{-\lambda}\rho)\right)}{c_1\,\sin\left(\sqrt{\lambda}z\right)-c_2\,\cos\left(\sqrt{\lambda}z\right)}.
\end{aligned}\tag{2.3.24}$$

Making substitutions of these expressions for $u(\rho,z)$ and $v(\rho,z)$ in (2.1.18)-(2.1.19) and solving for $k(\rho,z)$, we get

$$k(\rho,z)=\frac{-1}{2}\,\frac{-2\,c_3c_1\tan\left(\sqrt{\lambda}\rho\right)+2\,c_3c_2}{-c_2+c_1\tan\left(\sqrt{\lambda}\rho\right)},\tag{2.3.25}$$

where c_1, c_2 and c_3 are arbitrary constants.

(vi) $V_2+\delta V_5$

For this vector field, the similarity variable and the form of the similarity solution are as follows:

$\zeta = \frac{\rho}{z},\ u(\rho, z) = F(\zeta)\ \ v(\rho, z) = \delta \ln(\rho) + G(\zeta).$

On using these in the system (2.1.16)-(2.1.17), the system of reduced ODEs is given by

$$\begin{aligned}&\left(e^{-2F}\zeta^4 + e^{-2F}\zeta^2\right) G'^2 + 2\,e^{-2F}\delta\, G'\,\zeta + \left(2\,\zeta^3 + \zeta\right) F' + \left(\zeta^2 + \zeta^4\right) F'' + e^{-2F}\delta^2 = 0,\\ &\left(-\zeta - \zeta^3\right) G'' + \left(\left(2\,\zeta + 2\,\zeta^3\right) F' - 1 - 2\,\zeta^2\right) G' + 2\,F'\,\delta = 0,\end{aligned} \tag{2.3.26}$$

which is quite difficult to solve, so we consider the case with $\delta = 0$

$$\begin{aligned}&\zeta^3 F'' + \zeta^3 G'^2 e^{-2F} + 2\,F'\,\zeta^2 + \zeta\,F'' + \zeta\,G'^2 e^{-2F} + F' = 0,\\ &\left(-\zeta - \zeta^3\right) G'' + \left(\left(2\,\zeta + 2\,\zeta^3\right) F' - 1 - 2\,\zeta^2\right) G' = 0.\end{aligned} \tag{2.3.27}$$

The solution of reduced system of ODEs (2.3.27) are obtained and and reverting back to the original variables, solution of system (2.1.16)-(2.1.17) is deduced as:

$$\begin{aligned}u(\rho, z) &= -\ln\left(c_1\,\mathrm{arctanh}\left(\frac{1}{\sqrt{1+\frac{\rho^2}{z^2}}}\right) - c_2\right),\\ v(\rho, z) &= \pm\iota\left(c_1\,\mathrm{arctanh}\left(\frac{1}{\sqrt{1+\frac{\rho^2}{z^2}}}\right) - c_2\right)^{-1},\end{aligned} \tag{2.3.28}$$

where c_1 and c_2 are arbitrary constants.

(vii) $V_1 + \sigma V_4$

Corresponding to this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = \rho,\ u(\rho, z) = -\ln(F(\zeta) - \frac{z}{2\sigma\iota})$ and $v(\rho, z) = \pm\iota\exp(u(\rho, z))$.

Corresponding to these similarity variables, the reduced ODE is as follows:

$$\zeta F'' + F' = 0. \tag{2.3.29}$$

We can obtain solution of the eq. (2.3.29) and further by back substitution to original variables, the exact solution of equations (2.1.16)-(2.1.17) is given by:

$$\begin{aligned}u(\rho, z) &= -\ln\left(c_1 + c_2\ln(\rho) - \frac{z}{2\sigma\,\iota}\right),\\ v(\rho, z) &= \pm\iota\left(c_1 + c_2\ln(\rho) - \frac{z}{2\sigma\,\iota}\right)^{-1},\end{aligned} \tag{2.3.30}$$

where c_1 and c_2 are arbitrary constants.

(viii) $V_2 + \nu V_1$

The similarity variable and the form of similarity solution are as follows:
$\zeta = \frac{\rho}{z},\ u(\rho, z) = -\ln(F(\zeta) + 2\nu\iota\ln(\rho))$, and $v(\rho, z) = \pm\iota\exp(u(\rho, z))$.
Substituting these in equations (1.16)-(1.17), the reduced ODE is given by

$$(\zeta + \zeta^3)F'' + (1 + 2\zeta^2)F' = 0. \tag{2.3.31}$$

Solving equation (2.3.31) and reverting back to the original variables. Thus we get following exact solution of the equations (2.1.16)-(2.1.17):

$$\begin{aligned} u(\rho, z) &= -\ln\left(c_1 + \operatorname{arctanh}\left(\frac{1}{\sqrt{1+\frac{\rho^2}{z^2}}}\right)c_2 + 2\,\iota\,\nu\,\ln(\rho)\right), \\ v(\rho, z) &= \pm\iota\left(c_1 + \operatorname{arctanh}\left(\frac{1}{\sqrt{1+\frac{\rho^2}{z^2}}}\right)c_2 + 2\,\iota\,\nu\,\ln(\rho)\right)^{-1}, \end{aligned} \tag{2.3.32}$$

where c_1 and c_2 are arbitrary constants.

2.4 Some More Exact Solutions

In this section, we have found some more exact solutions of the nonlinear system (2.1.16)-(2.1.19) by assuming u or v depend on one variable ρ or z, these solutions are in two types of field : first, that in which the electric field is parallel to the axis of symmetry, called a longitudinal field; and secondly, that in which there is no component along this axis, called a radial field. To obtain the solutions corresponding to the longitudinal fields we have

$$v(\rho, z) = v(z).$$

So, system (2.1.16)-(2.1.17) becomes

$$u_{\rho\rho} + \frac{u_\rho}{\rho} + u_{zz} + \exp(-2u)v_z^2 = 0. \tag{2.4.1}$$

$$v_{zz} - 2u_z v_z = 0. \tag{2.4.2}$$

By solving equation (2.4.2) we obtain

$$u = \frac{1}{2}\ln(v_z) + F(\rho), \tag{2.4.3}$$

where $F(\rho)$ is an arbitrary function of ρ.

Substituting from (2.4.3) in (2.4.1), we get

$$2\rho F_{\rho\rho} v_z^2 + 2F_\rho v_z^2 + v_{zzz} v_z \rho - v_{zz}^2 \rho + 2v_z^3 \rho \exp(-2F) = 0. \tag{2.4.4}$$

since F is function of ρ only and v is function of z only, thus from (2.4.4) we evidently must have that F is constant or v_z is constant.

Let F is constant.

We take $F = \frac{-1}{2}\ln(a)$.

Then from equation (2.4.4), we have

$$\begin{aligned} u(\rho,z) &= \tfrac{1}{2}\ln\left(\frac{(1-(\tanh(\frac{z+c_2}{2c_1}))^2}{4ac_1^2}\right) - \tfrac{1}{2}\ln(a), \\ v(\rho,z) &= \tfrac{1}{2ac_1}\tanh\left(\tfrac{z+c_2}{2c_1}\right) + c_3, \end{aligned} \tag{2.4.5}$$

Making substitutions of these expressions for $u(\rho,z)$ and $v(\rho,z)$ in (2.1.18)-(2.1.19) and solving for $k(\rho,z)$, we get

$$k(\rho,z) = -\frac{\rho^2}{8{c_1}^2} + c_2, \tag{2.4.6}$$

where c_1, c_2 and c_3 are arbitrary constants.

Now we consider the other possibility that $v_z = c_4$, where c_4 is an arbitrary constant.

Equation (2.4.4) becomes

$$2\rho c_4^2 F_{\rho\rho} + 2c_4^2 F_\rho + 2c_4^3 \rho \exp(-2F) = 0, \tag{2.4.7}$$

Solving (2.4.7), we obtain

$$\begin{aligned} u(\rho,z) &= \tfrac{1}{2}\ln\left(\frac{c_4^2\rho^{2-c_1}(1-c_3^2\rho^{c_1})^2}{c_1^2c_3^2}\right), \\ v(\rho,z) &= c_4 z + c_2, \end{aligned} \tag{2.4.8}$$

Using above expressions for $u(\rho,z)$ and $v(\rho,z)$ in (2.1.18)-(2.1.19) and solving, we get

$$k(\rho,z) = \ln\left(c_6\rho^{\frac{(c_1-2)^2}{4}}(1-c_3^2\rho^{c_1})^2\right),$$

where c_i, $i = 1...6$ are arbitrary constants.

To obtain the solutions corresponding to the Radial fields we take

$$v(\rho, z) = v(\rho).$$

Thus, system (2.1.16)-(2.1.17) reduces as follows:

$$u_{\rho\rho} + \frac{u_\rho}{\rho} + u_{zz} + \exp(-2u)v_\rho^2 = 0. \qquad (2.4.9)$$

$$v_{\rho\rho} + \frac{v_\rho}{\rho} - 2u_\rho v_\rho = 0. \qquad (2.4.10)$$

By solving equation (2.4.10) we obtain

$$u = \frac{1}{2}\ln(\rho v_\rho) + F(z), \qquad (2.4.11)$$

where $F(z)$ is an arbitrary function of z.

Using (2.4.11) in (2.4.9), we get

$$2\rho F_{zz}v_\rho^2 - \rho v_{\rho\rho}^2 + v_\rho v_{\rho\rho} + \rho v_{\rho\rho\rho}v_\rho + 2v_\rho^3\exp(-2F) = 0, \qquad (2.4.12)$$

since F is function of z only and v is function of ρ only, thus from (2.4.12) we evidently must have that F is constant or $v_\rho = c\rho$ and $F_{zz}\exp(2F) = -c$.

Let F is constant.

We take $F = \frac{-1}{2}\ln(a)$.

Then from system (2.1.16)-(2.1.17), we have

$$\begin{aligned} u(\rho, z) &= \tfrac{1}{2}\ln\left(\frac{(1-(\tanh(\frac{(\ln(\rho)-c_2)}{2c_1})^2)}{4ac_1^2}\right) - \tfrac{1}{2}\ln(a), \\ v(\rho, z) &= \tfrac{1}{2ac_1}\tanh\left(\tfrac{\ln(\rho)-c_2}{2c_1}\right) + c_3, \end{aligned} \qquad (2.4.13)$$

By making substitutions of these expressions for $u(\rho, z)$ and $v(\rho, z)$ in (2.1.18)-(2.1.19) and solving for $k(\rho, z)$, we get

$$k = \frac{\ln(\rho)}{4{c_1}^2} + c_4, \qquad (2.4.14)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Now we consider the other possibility that $v_\rho = c\rho$ and $F_{zz}\exp(2F) = -c$.

we get the solution of system (2.1.16)-(2.1.17) as follows:

$$\begin{aligned} u(\rho, z) &= \ln\left(cc_1\rho\sin\left(\tfrac{z+c_2}{c_1}\right)\right), \\ v(\rho, z) &= \tfrac{1}{2}c\rho^2 + c_3, \end{aligned} \tag{2.4.15}$$

Using (2.1.18)-(2.1.19), corresponding to these expressions of $u(\rho, z)$ and $v(\rho, z)$, we obtain

$$k(\rho, z) = -\frac{1}{8}\,h^2\rho^2 + \ln(\rho) + 2\,\ln\left(\cosh\left(\frac{hz}{2} + k\right)\right) + D, \tag{2.4.16}$$

where c, c_1 c_2, c_3, h, k and D are arbitrary constants.

2.5 Concluding Remarks

From the beginning of theory of general relativity, there has been a sustained search for the new exact solutions of Einstein equations. We have investigated the symmetries and invariant solutions of the system of partial differential equations corresponding to Einstein - Maxwell equations for static axially symmetric spacetime with canonical coordinates. We have exploited the symmetries of Einstein - Maxwell equation to derive some ans$\ddot{a}$tz leading to the reduction of variables, where the analytic solutions are easier to obtain by considering the optimal system of conjugacy inequivalent subgroups. The exact solutions in which electromagnetic potential depends upon only one of the two coordinates are also obtained. The exact solutions, thus obtained, can be studied for its applications and implications in physics and astrophysics.

Chapter 3

Certain New Exact Solutions of Einstein Equations for Axisymmetric Rotating Fields[1]

3.1 Introduction

Here, we have considered Weyl-Lewis-Papapetrou [142] form for an axisymmetric rotating field as

$$ds^2 = fdt^2 - 2kd\phi dt - ld\phi^2 - \exp(\mu)(d\rho^2 + dz^2), \tag{3.1.1}$$

where f, k, l and μ are all functions of ρ and z and the coordinates ρ, z and ϕ corresponds to cylindrical polar coordinates. The external gravitational field is described by the symmetric Ricci tensor $R_{\mu\nu}$ which obeys the exterior Einstein vacuum equations

$$R_{\mu\nu} = 0. \tag{3.1.2}$$

[1]The contents of this chapter are accepted for publication in *Chinese Physics B*

By using (3.1.1), (3.1.2) can be written as

$$\begin{aligned}
2\exp(\mu)D^{-1}R_{tt} &= (D^{-1}f_\rho)_\rho + (D^{-1}f_z)_z + D^{-3}f(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0,\\
-2\exp(\mu)D^{-1}R_{t\phi} &= (D^{-1}k_\rho)_\rho + (D^{-1}k_z)_z + D^{-3}k(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0,\\
-2\exp(\mu)D^{-1}R_{\phi\phi} &= (D^{-1}l_\rho)_\rho + (D^{-1}l_z)_z + D^{-3}l(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0,
\end{aligned} \tag{3.1.3}$$

where $D^2 = fl + k^2$.

From (3.1.3) we get the following equation:

$$\exp(\mu)D^{-1}(lR_{tt} - 2kR_{t\phi} - fR_{\phi\phi}) = D_{\rho\rho} + D_{zz} = 0. \tag{3.1.4}$$

Therefore we found that the function D is satisfying the two-dimensional Laplace equation in the variables ρ and z. It follows that D can be considered as the real part of an analytic function $\Sigma(\rho + \iota z)$ of $(\rho + \iota z)$. Let E be the imaginary part of $\Sigma(\rho + \iota z)$, that is,

$$\Sigma(\rho + \iota z) = D(\rho, z) + \iota E(\rho, z). \tag{3.1.5}$$

Now we have considered the transformation from (ρ, z) to $(\bar{\rho}, \bar{z})$ given by

$$\bar{\rho} = D(\rho, z), \quad \bar{z} = E(\rho, z) \tag{3.1.6}$$

The Cauchy-Riemann equations imply that we have

$$(d\bar{\rho})^2 + (d\bar{z})^2 = (D_\rho^2 + D_z^2)(d\rho^2 + dz^2) \tag{3.1.7}$$

Equation (3.1.7) shows that the metric (3.1.1) is unaltered by the transformation (3.1.6), since we can define a new function $\bar{\mu}$ given by

$$\exp(\bar{\mu}) = \exp\mu({D_\rho}^2 + {E_\rho}^2)^{-1} \tag{3.1.8}$$

We can assume that all the functions $f, k, l, \bar{\mu}$ have been expressed in terms of the variables $(\bar{\rho}, \bar{z})$ obtained by substituting for (ρ, z) after solving for the latter from (3.1.6). Having expressed all functions in terms of $(\bar{\rho}, \bar{z})$, we can drop the bars so that because of (3.1.6) we are left with the following algebraic relation in f, k and l as

$$D^2 = fl + k^2 = \rho^2. \tag{3.1.9}$$

The above procedure was first used by Weyl (1917) for the axially symmetric static metric (with $k = 0$) and generalized to the present case by Lewis (1932). We drop the bar from $\bar{\mu}$ as well, so that the rest of the nontrivial Einstein equations (with the use of (3.1.9)) can be written as follows:

$$
\begin{aligned}
2R_{\rho\rho} &= -\mu_{\rho\rho} - \mu_{zz} + \rho^{-1}\mu_\rho + \rho^{-2}(f_\rho l_\rho + k_\rho^2) = 0, \\
2R_{\rho z} &= \rho^{-1}\mu_z + \tfrac{1}{2}\rho^{-2}(f_\rho l_z + f_z l_\rho + 2k_\rho k_z) = 0, \\
2R_{zz} &= -\mu_{\rho\rho} - \mu_{zz} - \rho^{-1}\mu_\rho + \rho^{-2}(f_z l_z + k_z^2) = 0.
\end{aligned} \tag{3.1.10}
$$

Because of (3.1.9) only two of (3.1.10) are independent. It is more convenient to use the function u and v instead of f and k respectively defined by $u = f^{-1}$ and $v = f^{-1}k$. By eliminating l and k from (3.1.3), Einstein field equations reduce to the following system of equations:

$$u(u_{\rho\rho} + u_{zz} + \frac{u_\rho}{\rho}) - u_\rho^2 - u_z^2 - \frac{(v_\rho^2 + v_z^2)}{\rho^2} = 0, \tag{3.1.11}$$

$$u(v_{\rho\rho} + v_{zz} - \frac{v_\rho}{\rho}) - 2u_\rho v_\rho - 2u_z v_z = 0, \tag{3.1.12}$$

$$u^2 w_\rho = u u_\rho + \frac{1}{2}\rho(u_\rho^2 - u_z^2) - \frac{1}{2\rho}(v_\rho^2 - v_z^2), \tag{3.1.13}$$

$$u^2 w_z = u u_z + \rho u_\rho u_z - \frac{1}{\rho} v_\rho v_z. \tag{3.1.14}$$

In order to determine the three unknowns $u(\rho, z)$, $v(\rho, z)$ and $w(\rho, z)$ from the nonlinear equations (3.1.11)-(3.1.14), we have firstly solved the equations (3.1.11) and (3.1.12) for obtaining values of $u(\rho, z)$, and $v(\rho, z)$. After that, by substituting these values of $u(\rho, z)$, and $v(\rho, z)$ into the equations (3.1.13) and (3.1.14), the values of $w(\rho, z)$ are furnished. The Lie classical method is utilized to the further investigation of nonlinear equations (3.1.11)-(3.1.14), here in this chapter. The method has yielded quite an exhaustive study and enabled us to recover some important results. Section (3.2) is devoted to generate various symmetries of nonlinear equations (3.1.11)-(3.1.12) and an optimal system comprising basic vector fields is identified. Section (3.3) contains the study of reduced ordinary differential equations (ODEs) and thus, new exact solutions of the Einstein vacuum field equations for axisymmetric rotating fields are furnished. In section (3.4), the extension of Papapetrou class of solutions is obtained. Some conclusions are drawn in Section (3.5).

3.2 Lie Symmetries

In this subsection, Lie group method is performed to the nonlinear equations (3.1.11)-(3.1.12). Firstly, a one parameter Lie group of infinitesimal transformations is considered as follows:

$$\begin{aligned}\rho^* &\longrightarrow \rho + \epsilon\xi_1(\overline{X},\overline{\sigma}) + O(\epsilon^2),\\ z^* &\longrightarrow z + \epsilon\xi_2(\overline{X},\overline{\sigma}) + O(\epsilon^2),\\ \overline{\sigma}^* &\longrightarrow \overline{\sigma} + \epsilon\overline{\phi}(\overline{X},\overline{\sigma}) + O(\epsilon^2),\end{aligned} \tag{3.2.1}$$

with small parameter $\epsilon \ll 1$ and where $\overline{X} = (\rho, z)$, $\overline{\sigma} = (u, v)$, $\overline{\phi} = (\eta_1, \eta_2)$, which leaves the equations (3.1.11)-(3.1.12) invariant. The method for determining the symmetry group of (3.1.11)-(3.1.12) consists of finding the infinitesimals $\xi_1,\ \xi_2,\ \eta_1$ and η_2, which are functions of $\rho,\ z,\ u$ and v.

Assuming that equations (3.1.11)-(3.1.12) are invariant under the transformations (3.2.1), the infinitesimals $\xi_1,\ \xi_2,\ \eta_1$ and η_2 must satisfy the symmetry conditions

$$\begin{aligned}&\eta_1(u_{\rho\rho} + u_{zz} + \tfrac{u_\rho}{\rho}) + u(\eta_1^{\rho\rho} + \eta_1^{zz} + \tfrac{\eta_1^\rho}{\rho} - \tfrac{\xi_1 u_\rho}{\rho^2}) - 2u_\rho\eta_1^\rho - 2u_z\eta_1^z - \tfrac{2v_\rho\eta_2^\rho + 2v_z\eta_2^z}{\rho^2} +\\ &\tfrac{2\xi_1(v_\rho^2 + v_z^2)}{\rho^3} = 0,\\ &\eta_1(v_{\rho\rho} + v_{zz} - \tfrac{v_\rho}{\rho}) + u(\eta_2^{\rho\rho} + \eta_2^{zz} - \tfrac{\eta_2^\rho}{\rho} + \tfrac{\xi_1 v_\rho}{\rho^2}) - 2u_\rho\eta_2^\rho - 2v_\rho\eta_1^\rho -\\ &-2u_z\eta_2^z - 2v_z\eta_1^z = 0,\end{aligned} \tag{3.2.2}$$

where $\eta_1^\rho,\ \eta_1^z,\ \eta_1^{\rho\rho},\ \eta_1^{zz},\ \eta_2^\rho,\ \eta_2^z,\ \eta_2^{\rho\rho}$ and η_2^{zz} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables (for more details the readers can refer to [23]. Substituting value of $\eta_1^\rho,\ \eta_1^z,\ \eta_1^{\rho\rho},\ \eta_1^{zz},\ \eta_2^\rho,\ \eta_2^z,\ \eta_2^{\rho\rho}$ and η_2^{zz}, into symmetry conditions (3.2.2), then equating the coefficients of the various monomials in the first, second and the other order partial derivatives of u and v and their powers, we can find the determining equations for the symmetry group of the Einstein field equations for axisymmetric rotating fields. Solving these equations, we get the following forms of infinitesimals:

$$\begin{aligned}&\xi_1 = a_1\rho,\ \ \xi_2 = a_1 z + a_2\\ &\eta_1 = -a_1 u + a_3 u,\\ &\eta_2 = a_3 v + a_4 + a_5(v^2 + \rho^2 u^2).\end{aligned} \tag{3.2.3}$$

where $a_1,\ a_2,\ a_3,\ a_4$ and a_5 are arbitrary constants.

Hence, the Lie algebra of infinitesimal symmetries of the equations (3.1.11)-(3.1.12) is spanned by the following five vector fields

$$\begin{aligned} &V_1 = 2uv\frac{\partial}{\partial u} + (v^2 + \rho^2 u^2)\frac{\partial}{\partial v},\ \ V_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ \ V_3 = \frac{\partial}{\partial v}, \\ &V_4 = \rho\frac{\partial}{\partial \rho} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u},\ \ V_5 = \frac{\partial}{\partial z}. \end{aligned} \tag{3.2.4}$$

In general, one may obtain the reduced system of ODEs from any linear combination of generators $V_j,\ j = 1, 2, ..., 5$. Since there exist infinite possibilities for such combinations, a systematic procedure to classify these reductions is based on the property that the transformations of the symmetry group will transform solutions of equations (3.1.11)-(3.1.12) into other solutions. Therefore, we classify the symmetry algebra of system into conjugacy inequivalent sub algebra under the adjoint action of the symmetry group. We will work out first an optimal system and then embark upon the various reductions associated with generators in the optimal system. We begin by considering a general element $V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5$ of symmetry algebra and subject it to various adjoint transformations to simplify it as much as possible [139]. The adjoint action is given by the Lie series

$$Ad(\exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2}[V_j, [V_i, V_j]] - ..., \tag{3.2.5}$$

where $[V_i, V_j] = V_iV_j - V_jV_i$ is the commutator for the Lie algebra, and ϵ is a parameter. The commutation relations of Lie algebra (3.2.4) are

$[V_1, V_2] = -[V_2, V_1] = -V_1,\ [V_1, V_3] = -[V_3, V_1] = -2V_2,\ [V_2, V_3] = -[V_3, V_2] = -V_3,\ [V_4, V_5] = -[V_5; V_4] = -V_5$ and $[V_i, V_j] = 0, \forall i,\ j$.

The optimal system of the equations (3.1.11)-(3.1.12) consists of the following basic vector fields:

$$\langle V_5,\ \ V_4,\ \ V_3 + \lambda V_4,\ \ V_2 + \mu V_4,\ \ V_1 + \alpha V_3 + \beta V_4\rangle, \tag{3.2.6}$$

where λ, μ, α and β are arbitrary constants.

Table 3.1: Adjoint Table

Index	V_1	V_2	V_3	V_4	V_5
V_1	V_1	$V_2+\epsilon V_1$	$V_3+2\epsilon V_2+V_1\epsilon^2$	V_4	V_5
V_2	$V_1\exp(-\epsilon)$	V_2	$V_3\exp(\epsilon)$	V_4	V_5
V_3	$V_1-2\epsilon V_2+V_3\epsilon^2$	$V_2-\epsilon V_3$	V_3	V_4	V_5
V_4	V_1	V_2	V_3	V_4	$V_5\exp(\epsilon)$
V_5	V_1	V_2	V_3	$V_4-\epsilon V_5$	V_5

3.3 Similarity Variables and Similarity Solutions

One of the main purposes for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. In the preceding section, we have obtained the vector fields and the optimal system of equations (3.1.11)-(3.1.12). In this section, the primary focus is on the reductions associated with the vector fields (3.2.6) and attempt to find some exact solutions.

(i) V_5

Corresponding to this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta=\rho, u(\rho,z)=F(\zeta), v(\rho,z)=G(\zeta).$

On using these in equations (3.1.11)-(3.1.12), the following system of reduced ODEs is obtained :

$$\begin{aligned}&\left(F''+\tfrac{F'}{\zeta}\right)F-F'^2-\tfrac{G'^2}{\zeta^2}=0\\&\left(G''-\tfrac{G'}{\zeta}\right)F-2F'G'=0.\end{aligned} \tag{3.3.1}$$

The solution of reduced system of ODEs (3.3.1) are found and then reverting back to the original variables, solution of system (3.1.11)-(3.1.12) is deduced as:

$$u(\rho,z)=\rho^{c_1}c_2,\quad v(\rho,z)=c_3, \tag{3.3.2}$$

and

$$\begin{aligned} u(\rho, z) &= \pm \frac{c_1^2}{\sqrt{-c_2 c_1^3 \rho \left(\cosh(\frac{\sqrt{c_1 c_2}(\ln \rho - c_4)}{2})\right)}} \\ v(\rho, z) &= -\left(\frac{c_2 c_3 + \tanh\left(\frac{c_4\sqrt{c_1 c_2}}{2} - \frac{\ln(\rho)\sqrt{c_1 c_2}}{2}\right)\sqrt{c_1 c_2}}{c_2}\right), \end{aligned} \tag{3.3.3}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

By making substitutions of these expressions (3.3.2) and (3.3.3) for $u(\rho, z)$ and $v(\rho, z)$ in (3.1.13)-(3.1.14) and solving for $w(\rho, z)$, we get

$$w(\rho, z) = \frac{1}{2} c_1^2 \ln(\rho) + c_1 \ln(\rho) + c_5, \tag{3.3.4}$$

and

$$w(\rho, z) = -\frac{1}{2} \ln(\rho) + \frac{c_1 c_2}{8} \ln(\rho) + c_6, \tag{3.3.5}$$

where c_5 and c_6 are arbitrary constants.

$(ii)\ V_4$

For this vector field, the form of the similarity variable and similarity solution are as follows:

$\zeta = \frac{\rho}{z}$, $u(\rho, z) = \frac{F(\zeta)}{z}$, $v(\rho, z) = G(\zeta)$.

Corresponding to these similarity variables, the reduced system of ODEs are as follows:

$$\begin{aligned} &\left(F''(1+\zeta^2) + F'\left(\frac{1}{\zeta} + 4\zeta\right) + 2F\right) F - F'^2 - \left(-F'\zeta - F\right)^2 - G'^2\left(\frac{1}{\zeta^2} + 1\right) = 0 \\ &\left(G''(1+\zeta^2) + G'\left(-\frac{1}{\zeta} + 2\zeta\right)\right) F - 2F'G' + 2\zeta G'\left(-F'\zeta - F\right) = 0. \end{aligned} \tag{3.3.6}$$

We can obtain solution of the system (3.3.6) and further by back substitution to original variables, the exact solution of equations (3.1.11)-(3.1.12) is given by:

$$\begin{aligned} u(\rho, z) &= \frac{\pm c_2 \iota}{\sqrt{\rho^2 + z^2}} \\ v(\rho, z) &= c_1 + \frac{c_2 z}{\sqrt{\rho^2 + z^2}}, \end{aligned} \tag{3.3.7}$$

where c_1 and c_2 are arbitrary constants.

Using above expressions for $u(\rho, z)$ and $v(\rho, z)$ in (3.1.13)-(3.1.14) and solving, we get

$$w(\rho, z) = -\frac{1}{2} \ln(\rho^2 + z^2) + c_3, \tag{3.3.8}$$

where c_3 is arbitrary constant.

$(iii)\ V_3 + \lambda V_4$

For this vector field, the similarity variable and the form of the similarity solution are as follows:

$\zeta = \frac{\rho}{z}$, $u(\rho, z) = \frac{F(\zeta)}{z}$, $v(\rho, z) = \frac{1}{\lambda}\ln(z) + G(\zeta)$.

On using these in equations (3.1.11)-(3.1.12), the system of reduced ODEs is given by

$$\begin{aligned} &\left(F''(1+\zeta^2) + F'(\tfrac{1}{\zeta} + 4\zeta) + 2F\right)F - F'^2 - (-F'\zeta - F)^2 - \tfrac{G'^2}{\zeta^2} - \left(-G' + \tfrac{1}{\lambda\zeta}\right)^2 = 0 \\ &\left(G''(1+\zeta^2) + G'(-\tfrac{1}{\zeta} + 2\zeta) - \tfrac{1}{\lambda}\right)F - 2F'G' - 2\left(-\zeta G' + \tfrac{1}{\lambda}\right)(-F'\zeta - F) = 0. \end{aligned} \tag{3.3.9}$$

After solving this system of ODEs (3.3.9), we obtain the solution of equations (3.1.11)-(3.1.12) as follows:

$$\begin{aligned} u(\rho, z) &= \frac{\pm\iota\left(\text{arctanh}\left(\frac{z}{\sqrt{\rho^2+z^2}}\right) - c_1\lambda\right)}{\lambda\sqrt{(\rho^2+z^2)}} \\ v(\rho, z) &= \frac{1}{\lambda}\ln(\rho) - \frac{c_1 z}{\sqrt{\rho^2+z^2}} + \frac{z}{\lambda\sqrt{\rho^2+z^2}}\text{arctanh}\left(\frac{z}{\sqrt{\rho^2+z^2}}\right) + c_2, \end{aligned} \tag{3.3.10}$$

where c_1 and c_2 are arbitrary constants.

Consequently, $w(\rho, z)$ is given by:

$$\begin{aligned} w(\rho, z) = &\int \frac{\left(c_1\lambda\, z - z\text{arctanh}\left(\frac{z}{\sqrt{\rho^2+z^2}}\right) + \sqrt{\rho^2+z^2}\right)dz}{(\rho^2+z^2)\left(\text{arctanh}\left(\frac{z}{\sqrt{\rho^2+z^2}}\right) - c_1\lambda\right)} + \int \Big(-(\rho^2+z^2)\big(-\text{arctanh}\,\big(\tfrac{z}{\sqrt{\rho^2+z^2}}\big) + c_1\lambda\big) \\ &\int \big(2\sqrt{\rho^2+z^2}\rho^2 z\big(\text{arctanh}\,\big(\tfrac{z}{\sqrt{\rho^2+z^2}}\big)\big)^2 - \rho^2\big(4\,z\sqrt{\rho^2+z^2}c_1\lambda + z^2 + \rho^2\big)\text{arctanh}\,\big(\tfrac{z}{\sqrt{\rho^2+z^2}}\big) \\ &+ z\big(\big(2\,{c_1}^2\lambda^2 + 1\big)\rho^2 + z^2\big)\sqrt{\rho^2+z^2} + \rho^2 c_1\lambda\big(\rho^2+z^2\big)\big)\big(-\text{arctanh}\,\big(\tfrac{z}{\sqrt{\rho^2+z^2}}\big) + c_1\lambda\big)^{-2} \\ &\big(\rho^2+z^2\big)^{\frac{-5}{2}}dz + \text{arctanh}\,\big(\tfrac{z}{\sqrt{\rho^2+z^2}}\big)\rho^2 + \sqrt{\rho^2+z^2}t - \rho^2 c_1\lambda\Big)\big(\rho^2+z^2\big)^{-1} \\ &\big(-\text{arctanh}\,\big(\tfrac{z}{\sqrt{\rho^2+z^2}}\big) + c_1\lambda\big)^{-2}d\rho + c_3, \end{aligned}$$

where c_3 is an arbitrary constant.

$(iv)\ V_2 + \mu V_4$

The similarity variable and the form of similarity solution are as follows:

$\zeta = \frac{\rho}{z}$, $u(\rho, z) = F(\zeta)z^{(\frac{1-\mu}{\mu})}$, $v(\rho, z) = z^{\frac{1}{\mu}}G(\zeta)$.

The reduced ODEs in this case is as follows:

$$\left(F''(1+\zeta^2)+F'\left(\frac{1}{\zeta}+2\zeta-\frac{2\zeta(1-\mu)}{\mu}\right)+F\frac{(1-\mu)^2}{\mu^2}-F\frac{(1-\mu)}{\mu}\right)F-F'^2-\left(-F'\zeta+F\frac{(1-\mu)}{\mu}\right)^2$$
$$-\frac{G'^2}{\zeta^2}-\left(-G'+\frac{G}{\mu\zeta}\right)^2=0$$
$$\left(G''(1+\zeta^2)+G'\left(-\frac{1}{\zeta}+2\zeta-\frac{2\zeta}{\mu}\right)+G\left(\frac{1}{\mu^2}-\frac{1}{\mu}\right)\right)F-2\left(-\zeta G'+\frac{G}{\mu}\right)\left(-F'\zeta+F\frac{(1-\mu)}{\mu}\right)$$
$$-2F'G'=0. \tag{3.3.11}$$

For $\mu=1$

we get final solution of equations (3.1.11)-(3.1.12):

$$\begin{aligned} u(\rho,z) &= c_1\arctan\left(\frac{z}{\sqrt{-\rho^2-z^2}}\right)+c_2 \\ v(\rho,z) &= c_1\sqrt{\rho^2+z^2}, \end{aligned} \tag{3.3.12}$$

where c_1 and c_2 are arbitrary constants.

Making substitutions of these expressions for $u(\rho,z)$ and $v(\rho,z)$ in (3.1.13)-(3.1.14) and solving for $w(\rho,z)$, we get

$$w(\rho,z)=\ln\left(\arctan\left(\frac{z}{\sqrt{(-\rho^2-z^2)}}\right)c_1+c_2\right)+c_3, \tag{3.3.13}$$

where c_3 is an arbitrary constant.

(v) $V_1+\alpha V_3+\beta V_4$

Corresponding to this case, the following form of the similarity variable and similarity solution is found:

$$u(\rho,z)=\pm\frac{v(\rho,z)}{\rho},\, v(\rho,z)=\frac{1}{\frac{-2}{\beta}\ln(\rho)+\ln(G(\zeta))}.$$

Substituting these in equations (3.1.11)-(3.1.12), the reduced ODE is given by

$$-G''G\zeta+\zeta G'^2-G'G-GG''\zeta^3-2\zeta^2G'G+G'^2\zeta^3=0. \tag{3.3.14}$$

Solving equation (3.3.14) and reverting back to the original variables. Thus we get following exact solution of the equations (3.1.11)-(3.1.14):

$$\begin{aligned} u(\rho,z) &= \frac{\pm 1}{\rho\left(\frac{-2\ln(\rho)}{\beta}+c_2-c_1\operatorname{arctanh}\left(\frac{z}{\sqrt{\rho^2+z^2}}\right)\right)} \\ v(\rho,z) &= \frac{1}{\frac{-2\ln(\rho)}{\beta}+c_2-c_1\operatorname{arctanh}\left(\frac{z}{\sqrt{\rho^2+z^2}}\right)}, \end{aligned} \tag{3.3.15}$$

and

$$w(\rho, z) = \frac{-1}{2}\ln(\rho) + c_3, \tag{3.3.16}$$

where c_1, c_2 and c_3 are arbitrary constants.

Also, by using the transformation $v(\rho, z) = \pm\rho u(\rho, z)$, in equations (3.1.11)-(3.1.12), we get

$$u(u_{\rho\rho} - \tfrac{u_\rho}{\rho} + u_{zz}) - 2u_\rho^2 - 2u_z^2 - u^2 = 0. \tag{3.3.17}$$

Thus, the solution to equations (3.1.11)-(3.1.12) is finally given by

$$\begin{aligned} u(\rho, z) &= \tfrac{\iota k_5(-J_1(\sqrt{-k_5}\rho)Y_0(\sqrt{-k_5}\rho)+J_0(\sqrt{-k_5}\rho)Y_1(\sqrt{-k_5}\rho))}{(k_1 J_0(\sqrt{-k_5}\rho)-k_2 Y_0(\sqrt{-k_5}\rho))(k_3\sin(\sqrt{k_5}z)-k_4\cos(\sqrt{k_5}z))} \\ v(\rho, z) &= \pm\tfrac{-\iota k_5\rho(-J_1(\sqrt{-k_5}\rho)Y_0(\sqrt{-k_5}\rho)+J_0(\sqrt{-k_5}\rho)Y_1(\sqrt{-k_5}\rho))}{(k_1 J_0(\sqrt{-k_5}\rho)-k_2 Y_0(\sqrt{-k_5}\rho))(k_3\sin(\sqrt{k_5}z)-k_4\cos(\sqrt{k_5}z))}, \end{aligned} \tag{3.3.18}$$

where k_1, k_2, k_3, k_4 and k_5 are arbitrary constants.

Using (3.1.13)-(3.1.14), corresponding to these expressions of $u(\rho, z)$ and $v(\rho, z)$, we obtain

$$w(\rho, z) = -\frac{1}{2}\ln(\rho) + k_6, \tag{3.3.19}$$

where k_6 is an arbitrary constant.

3.4 Extension of Papapetrou Class of Solutions

Following to the approach given by [86], one can select the function $v(\rho, z)$ satisfying the equation

$$v_{\rho\rho} + v_{zz} - \frac{v_\rho}{\rho} = 0 \tag{3.4.1}$$

as

$$v(\rho, z) = k\rho\sigma_\rho, \tag{3.4.2}$$

where k is a constant.

By using (3.4.2), equation (3.4.1) can be written as:

$$\sigma_{\rho\rho\rho} + \sigma_{\rho zz} + (\rho^{-1}\sigma_\rho)_\rho = 0. \tag{3.4.3}$$

Integrating (3.4.3), we get

$$\sigma_{\rho\rho} + \sigma_{zz} + \frac{\sigma_\rho}{\rho} = \theta(z). \tag{3.4.4}$$

Therefore, we introduced an arbitrary function $\theta = \theta(z)$. In case of $\theta(z) = 0$ the equation (3.4.4) becomes Laplace equation, which is similar to expression given in [86]. However, for $\theta(z) \neq 0$, we have Poisson like equation (3.4.4) for the function θ. By using (3.4.2) for the $v(\rho, z)$ and the equation (3.4.1), equation (3.1.12) becomes

$$-u_\rho(\sigma_{zz} - \theta) + u_z\sigma_{\rho z} = 0. \tag{3.4.5}$$

From (3.4.4) and (3.4.5), the general solution for the metric function u takes the following form

$$u = u(\bar{\sigma}), \tag{3.4.6}$$

where $\bar{\sigma} = \sigma_z - \int \theta(z)dz$ and $u' = \frac{\partial u}{\partial \bar{\sigma}}$.

Therefore, we can rewrite the derivatives of the metric functions u in the following ways

$$\begin{aligned} &u_\rho = u'\sigma_{\rho z}, \quad u_{\rho\rho} = u''\sigma_{\rho z}^2 + u'\sigma_{\rho\rho z}, \quad u_z = u'(\sigma_{zz} - \theta(z)), \\ &u_{zz} = u''(\sigma_{zz} - \theta(z))^2 + u'(\sigma_{zzz} - \theta(z)_z). \end{aligned} \tag{3.4.7}$$

Using the equations (3.4.7) and (3.4.4), we get

$$\begin{aligned} &u_{\rho\rho} + u_{zz} + \tfrac{u_\rho}{\rho} = u''(\sigma_{\rho z}^2 + (\sigma_{zz} - \theta)^2) \\ &u_\rho^2 + u_z^2 = u'^2(\sigma_{\rho z}^2 + (\sigma_{zz} - \theta)^2) \\ &v_\rho^2 + v_z^2 = k^2\rho^2(\sigma_{\rho z}^2 + (\sigma_{zz} - \theta)^2). \end{aligned} \tag{3.4.8}$$

Finally, after substituting the derived expressions into (3.1.11), we obtain

$$uu'' - u'^2 - k^2 = 0, \tag{3.4.9}$$

which gives the extension of Papapetrou class of solutions

$$u(\rho, z) = \alpha\cosh(\sigma_z - \textstyle\int \theta(z)dz) + \beta\sinh(\sigma_z - \textstyle\int \theta(z)dz), \tag{3.4.10}$$

where the functions θ and σ satisfies the (3.4.4) and $k^2 = \alpha^2 - \beta^2$. Here the parameters α and β have the same meaning as in the Papapetrou solutions. As a particular case, in the limit $\theta(z) = 0$, the solutions (3.4.10) reduces to Papapetrou solutions.

In the case of the Papapetrou solutions in the Newtonian limit that is at large arguments $r = \sqrt{\rho^2 + z^2} \longrightarrow \infty$, the asymptotics looks like

$$u(\rho, z) = \alpha^{-1}(1 + \tfrac{\beta z}{\alpha r^3} + O(r^{-2})), \tag{3.4.11}$$

and does not contain a term being proportional to r^{-1}. We expect that in general case when $\theta \neq 0$, it would be possible to construct a solution, which gives flat asymptotics at infinity for massive sources. As an example, one of the possible solutions could be obtained in the following way. Assume that the function $\sigma(\rho, z) = f_1(\rho) f_2(z)$ could be taken as a product of two functions f_1 and f_2. Then we got from equation (3.4.4)

$$f_2 f_{1\rho\rho} + f_1 f_{2zz} + \tfrac{1}{\rho} f_{1\rho} f_2 = \theta(z). \tag{3.4.12}$$

We have considered the function f_2 in the form $f_2 = f_{2zz} = \theta(z)$, with $\theta(z) = C_0(\exp(z) - \exp(-z))$, For the function f_1, we have the following equation

$$g_{1\rho\rho} + g_1 + \tfrac{1}{\rho} g_{1\rho} = 0, \tag{3.4.13}$$

where $f_1 = 1 + g_1$ and C_0 is constant.

The last equation (3.4.13) is the Bessel one and $f_1 = 1 + J_0(\rho)$, where $J_0(\rho)$ is zero rank Bessel function. Finally

$$\sigma(\rho, z) = C_0(1 + J_0(\rho))(\exp(z) - \exp(-z)), \tag{3.4.14}$$

and hence

$$\bar{\sigma} = C_0 J_0(\rho)(\exp(z) + \exp(-z)), \tag{3.4.15}$$

which takes the simple form $\bar{\sigma} = 2C_0 J_0(\rho)$ at the plane $z = 0$. For the large arguments $\rho \longrightarrow \infty$, zero rank Bessel function has the following asymptotic behavior

$$\lim_{\rho \longrightarrow \infty} J_0(\rho) = \sqrt{\frac{2}{\pi\rho}} \cos(\rho - \frac{\pi}{4}), \tag{3.4.16}$$

and consequently

$$\bar{\sigma}_{|z=0} = 2C_0 \sqrt{\frac{2}{\pi\rho}} \cos(\rho - \frac{\pi}{4}) = \sqrt{\frac{1}{\rho}} \cos(\rho - \frac{\pi}{4}) \equiv x, \tag{3.4.17}$$

where we put, for the simplification $C_0 = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. Using asymptotic properties of hyperbolic functions one could write

$$\begin{aligned} \cosh(x) &= 1 + \tfrac{1}{2}\tfrac{1}{\rho}(\cos(\rho - \tfrac{\pi}{4}))^2 + \tfrac{1}{24}\tfrac{1}{\rho^2}(\cos(\rho - \tfrac{\pi}{4}))^4 + O(\rho^{-2}) \\ \sinh(x) &= \rho^{\frac{-1}{2}}(\cos(\rho - \tfrac{\pi}{4})) + \tfrac{1}{6}\rho^{\frac{-3}{2}}(\cos(\rho - \tfrac{\pi}{4}))^3 + O(\rho^{-2}). \end{aligned} \tag{3.4.18}$$

Substituting obtained asymptotic expressions (3.4.18) in solution (3.4.10), we can write the asymptotics of function $u(\rho, z)$, at the plane $z = 0$ when $\rho \longrightarrow \infty$, in the following form

$$\begin{aligned} \lim_{\rho\longrightarrow\infty} u_{|z=0} &= \alpha(1 + \tfrac{1}{4}\rho^{-1} + \tfrac{1}{4}\rho^{-1}\sin(2\rho) + \tfrac{1}{24}\rho^{-2}(\cos(\rho - \tfrac{\pi}{4}))^4 + \tfrac{\beta}{\alpha}\rho^{\frac{-1}{2}}(\cos(\rho - \tfrac{\pi}{4})) \\ &\quad + \tfrac{\beta}{6\alpha}\rho^{\frac{-3}{2}}(\cos(\rho - \tfrac{\pi}{4}))^3 + O(\rho^{-2})). \end{aligned} \tag{3.4.19}$$

We can see from the asymptotics (3.4.19) that in contrast to the Papapetrou one (3.4.11) there is a term being proportional to ρ^{-1} and at the same time we have asymptotically flat solution at $z = 0$ plane.

3.5 Discussion and Concluding Remarks

We have analysed the underlying system of nonlinear partial differential equations which arises in the study of Einstein field equations corresponding to Weyl-Lewis-Papapetrou form for an axisymmetric rotating field. Our interest in this system lies in discussing new solutions that can be found by means of Lie point symmetries. Firstly, the Lie classical method is utilized for the purpose of obtaining the group infinitesimals. The basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations. Several new families of exact solutions are found explicitly. We also obtained solution (3.4.10), which is an extension of the well known Papapetrou class of solutions, which can be applied to the physical systems, presenting rotating bounded nonzero masses. We have constructed the solution for the particular function $\theta(z)$, which in some sense, demonstrates the existence of ρ^{-1} term in asymptotics.

Chapter 4

Symmetries and Exact Solutions of Einstein Field Equations for Perfect Fluid Distribution and Pure Radiation Fields[1]

4.1 Introduction

The study of exact solutions of Einstein's field equations is an important part of the theory of general relativity. This importance derives from the growing applications of general relativity for the explanation of various phenomena. Einstein field equations, which play a central role in Einstein theory of general relativity, have symmetry consideration as one of the most important mathematical properties apart from their applications and implications for astrophysics. The heart of the classification schemes for the solutions of these equations are the symmetry methods based on the Lie groups. Einstein field equations are

[1]The contents of this chapter has been published in *Maejo International Journal of Science and Technology* 7 (2013) 133-144

studied by various authors to establish exact solutions by using Lie group analysis, some recent contributions are in [5, 68, 69, 102, 119, 120].

4.2 Einstein Field Equation for Perfect Fluid Distribution

Einstein [52] firstly pointed the possibility of existence of gravitational waves, propagated with the speed of light, in the case of weak gravitational field. The usual procedure, in cartesian co-ordinates, is to start with a field

$$g_{ik} = \eta_{ik} + h_{ik}, \quad i, k = 1, 2, 3, 4, \tag{4.2.1}$$

where η_{ik} is the Galilean metric and h_{ik} describe the modifications due to a weak gravitational field. In view of the linearized field equations $R_{ik} = 0$, coupled with a set of co ordinates conditions, the h_{ik} satisfy the wave equation. In particular, when h_{ik} depends on t and x only, there exists a coordinate system [13] in which one can take all the components h_{ik} to vanish, except

$$h_{22} = -h_{33} \neq 0, h_{23} = h_{32} \neq 0, \tag{4.2.2}$$

where the non vanishing components are arbitrary functions of the argument $(t-x)$. Since general relativity is essentially a non linear theory, its intrinsic consequences cannot be based on a weak field approximation and there must be certain reservations about the conclusions drawn from the linearized field. Bondi, Pirani and Robinson [25] demonstrated the existence of plane gravitational waves described by exact solution of Einstein field equations for empty spacetime. Now, we have considered the exact gravitational field equations

$$R_{ik} = -8\Pi(T_{ik} - \frac{1}{2}g_{ik}T), \tag{4.2.3}$$

on the Riemannian fourfold

$$ds^2 = dt^2 - dx^2 - (1-u)dy^2 - (1+u)dz^2 + 2vdydz, \tag{4.2.4}$$

where u and v are functions of t and x only.

In the case of line element (4.2.4), the non zero components of the curvature tensor and the Ricci tensor are given as follows:

$$\begin{aligned}
R_{yzyz} &= \tfrac{u_x^2-u_t^2+v_x^2-v_t^2}{4}\\
R_{z\mu z\nu} &= \tfrac{2Pu_{\mu\nu}-(1-u)u_\mu u_\nu-(1+u)v_\mu v_\nu+v(u_\mu v_\nu+u_\nu v_\mu)}{4P}\\
R_{y\mu y\nu} &= \tfrac{-(2Pu_{\mu\nu}+(1+u)u_\mu u_\nu+(1-u)v_\mu v_\nu+v(u_\mu v_\nu+u_\nu v_\mu))}{4P}\\
R_{y\mu z\nu} &= \tfrac{-(2Pv_{\mu\nu}-(1-u)u_\mu v_\nu+(1+u)u_\nu v_\mu-v(u_\mu u_\nu-v_\nu v_\mu))}{4P}\\
R_{yztx} &= \tfrac{u_t v_x-u_t v_x}{2P}.\\
R_{\mu\nu} &= -\tfrac{(2P(uu_{\mu\nu}+vv_{\mu\nu})+(1+u^2-v^2)u_\mu u_\nu+(1-u^2+v^2)v_\mu v_\nu+2uv(u_\mu v_\nu+u_\nu v_\mu))}{2P^2}\\
R_{yy}+R_{zz} &= \tfrac{u_x^2-u_t^2+v_x^2-v_t^2}{P}\\
R_{yy}-R_{zz} &= \tfrac{P(u_{xx}-u_{tt})-u(v_x^2-v_t^2)+v(u_x v_x-u_t v_t)}{P}\\
R_{yz}=R_{zy} &= \tfrac{P(v_{xx}-v_{tt})-v(u_x^2-u_t^2)+u(u_x v_x-u_t v_t)}{2P},
\end{aligned} \tag{4.2.5}$$

where μ and ν take the values t and x only and $u_i \equiv \frac{\partial u}{\partial x^i}$, $u_{ik} \equiv \frac{\partial u}{\partial x^i \partial x^k}, \cdots$ etc. and $(x^1, x^2, x^3, x^4) = (t, y, z, x)$ and $P = (1 - u^2 - v^2)$.

4.2.1 The Perfect Fluid Distribution

The compatibility of perfect fluid distribution of matter is defined by the field equations

$$R_{ik} = -8\pi[(p+\rho)v_i v_k - \frac{1}{2}g_{ik}(\rho - p)], g^{ik}v_i v_k = 1, \tag{4.2.6}$$

where p and ρ are the proper pressure and proper density respectively and v_i is the flow vector. In view of (4.2.5) and (4.2.6), we have the following four relations:

$$\begin{aligned}
&(1-u)^{-1}R_{yy} = (1+u)^{-1}R_{zz} = -v^{-1}R_{yz},\\
&((1-u)R_{tt} - R_{yy})((1-u)R_{xx} + R_{yy}) = (1-u)^2 R_{tx}^2.
\end{aligned} \tag{4.2.7}$$

Two of the relations, contained in the first set of the above equations, give

$$\begin{aligned}
&P(u_{xx} - u_{tt}) + u(u_x^2 - u_t^2) + v(u_x v_x - u_t v_t) = 0,\\
&P(v_{xx} - v_{tt}) + v(v_x^2 - v_t^2) + u(u_x v_x - u_t v_t) = 0.
\end{aligned} \tag{4.2.8}$$

Since, a perfect fluid distribution of matter is possible if u=v.

Thus these relations are compatible for perfect fluid distribution of matter if $u = v$ and the resulting single equation is as follows:

$$(1-2u^2)(u_{tt}-u_{xx})+2u(u_t^2-u_x^2)=0, \tag{4.2.9}$$

4.2.2 Lie Symmetry Analysis

In this section, Lie's method [110] of infinitesimal transformation groups is applied on system of Einstein equations for perfect fluids. On considering a one point group transformations of point-like transformations acting on the space of independent variables (x,t) and dependent variables u, the associated infinitesimal generator is given by

$$X=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u}. \tag{4.2.10}$$

Then, it is required that this transformation leave the set of solutions of PDE (4.2.9) invariant. This yields an over determined linear system of equations for the infinitesimals $\xi(x,t,u),\ \ \tau(x,t,u)$ and $\eta(x,t,u)$. After the infinitesimals are determined, the symmetry variables are found by solving the invariant-surface conditions

$$\Phi=\xi(x,t,u)\frac{\partial u}{\partial x}+\tau(x,t,u)\frac{\partial u}{\partial t}-\eta(x,t,u). \tag{4.2.11}$$

The structure of the determining equations prompts the selection of the following forms of infinitesimals:

$$\begin{aligned}\xi &= F_1(t+x)-F_2(t-x)\\ \tau &= F_1(t+x)+F_2(t-x)\\ \eta &= 0,\end{aligned} \tag{4.2.12}$$

where $F_1(t+x)$ and $F_2(t-x)$ are arbitrary functions. Thus the equation (4.2.9) admits a set of infinite-dimensional Lie algebra.

For the symmetries described in (4.2.12), the similarity variable $\zeta=\zeta(x,t)$ and the corresponding form of u as the function of the new independent variable ζ are as follows:

$$\begin{aligned}&\zeta=\int\tfrac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx+\int\tfrac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt,\\ &u(x,t)=F(\zeta).\end{aligned} \tag{4.2.13}$$

In the above set of equations (4.2.13), the function F is a function of ζ and is determined by substitution of (4.2.13) into (4.2.9) and solving the resulting nonlinear ordinary differential equation which is as follows:

$$2F'(\zeta)^2F(\zeta) + F''(\zeta) - 2F(\zeta)^2F''(\zeta) = 0, \tag{4.2.14}$$

where prime (') denotes the differentiation with respect to the variable ζ. Solving this equation (4.2.14) and reverting back to the original variables, we obtain the following group-invariant solution of equation (4.2.9):

(a) Solutions in terms of cos() function

$$\begin{aligned}
&(i)u(x,t) = \pm\tfrac{\sqrt{2}}{2}\cos(c_1 + c_2(\textstyle\int \frac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx + \int \frac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt))\\
&(ii)u(x,t) = \pm\tfrac{\sqrt{2}}{2} \mp \sqrt{2}\cos(c_1 + c_2(\textstyle\int \frac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx + \int \frac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt))^2\\
&(iii)u(x,t) = \pm\tfrac{3\sqrt{2}}{2}\cos(c_1 + c_2(\textstyle\int \frac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx + \int \frac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt))\\
&\qquad \mp 2\sqrt{2}\cos(c_1 + c_2((\textstyle\int \frac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx + \int \frac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt))^3,
\end{aligned} \tag{4.2.15}$$

where c_1 and c_2 are arbitrary constants.

(b) Solutions in terms of sin() function

$$\begin{aligned}
&(i)u(x,t) = \pm\tfrac{\sqrt{2}}{2}\sin(c_1 + c_2(\textstyle\int \frac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx + \int \frac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt))\\
&(ii)u(x,t) = \pm\tfrac{\sqrt{2}}{2} \mp \sqrt{2}\sin(c_1 + c_2(\textstyle\int \frac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx + \int \frac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt))^2\\
&(iii)u(x,t) = \pm\tfrac{3\sqrt{2}}{2}\sin(c_1 + c_2(\textstyle\int \frac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx + \int \frac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt))\\
&\qquad \mp 2\sqrt{2}\sin(c_1 + c_2((\textstyle\int \frac{(F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dx + \int \frac{(-F_1(t+x)+F_2(t-x))}{(F_1(t+x)F_2(t-x))}dt))^3,
\end{aligned} \tag{4.2.16}$$

where c_1 and c_2 are arbitrary constants.

4.3 Einstein Field Equations for Pure Radiation Fields

The Riemann curvature tensor plays the most fundamental role in Einstein theory of gravitation. The algebraic and differential properties of this tensor have characterized wave fields in general relativity in great detail. The problem of pure radiation fields has been discussed by several authors [4, 66, 108, 99, 100, 147, 185].

The field equations corresponding to the pure radiation fields are

$$R_i^j = \kappa\omega_i\omega^j, \tag{4.3.1}$$

where κ is a scalar. When $\kappa = 0$ one gets pure gravitational radiation. The more general waves given by (4.3.1) ($\kappa \neq 0$) are distinct from pure gravitational waves. We will derive some of the exact solutions of the Einstein Rosen [53] cylindrically symmetric space-times corresponding to pure radiation fields.

4.3.1 The Metric Form and The Field Equations

Consider Einstein Rosen metric [53] in cylindrical polar coordinates r, ϕ, z and time t as

$$ds^2 = \exp(2v - 2u)(dt^2 - dr^2) - r^2 \exp(-2u)d\phi^2 - \exp(2u)dz^2, \tag{4.3.2}$$

where u and v are functions of r and t only. The non zero components of curvature tensor obtained from (4.3.2) are

$$\begin{aligned}
R_r^r &= \exp(2u - 2v)(-v_{rr} + v_{tt} + u_{rr} - u_{tt} - 2u_r^2 + \tfrac{v_r + u_r}{r}) \\
R_\phi^\phi &= -R_z^z = \exp(2u - 2v)(u_{rr} - u_{tt} + \tfrac{u_r}{r}) \\
R_t^t &= \exp(2u - 2v)(-v_{rr} + v_{tt} + u_{rr} - u_{tt} + 2u_t^2 + \tfrac{u_r - v_r}{r}) \\
R_r^t &= -R_t^r = \exp(2u - 2v)(2u_r u_t - \tfrac{v_t}{r}).
\end{aligned} \tag{4.3.3}$$

Pure radiation fields with null vector ω^i such that $\omega^r = 1$, $\omega^\phi = 0$, $\omega^z = 0$, $\omega^t = 1$ for the metric (4.3.2), by using (4.3.1), obey the field equations

$$\begin{aligned}
R_r^r + R_t^t &= 0 \\
R_r^r + R_t^r &= 0 \\
R_\phi^\phi = R_z^z &= 0.
\end{aligned} \tag{4.3.4}$$

Making use of expressions for R_i^i given in (4.3.3), the relations (4.3.4) give the system of partial differential equations:

$$\begin{aligned}
u_{rr} + \tfrac{u_r}{r} - u_{tt} &= 0 \\
v_r + v_t - r(u_r + u_t)^2 &= 0 \\
v_{rr} - v_{tt} + u_r^2 - u_t^2 &= 0.
\end{aligned} \tag{4.3.5}$$

So we have three equations for the determination of two unknowns u and v and one can easily verify that these three equations are all consistent. Therefore we drop third equation in system (4.3.5) and solve the remaining two equations for u and v. Hence we get a system of partial differential equations

$$\begin{aligned} &u_{rr} + \tfrac{u_r}{r} - u_{tt} = 0 \\ &v_r + v_t - r(u_r + u_t)^2 = 0. \end{aligned} \tag{4.3.6}$$

Now, we will derive various symmetries of system (4.3.6) by using Lie group method and an optimal system comprising basic vector fields is identified. Further, the reduced systems of ordinary differential equations (ODEs) and some of the exact solutions of equation (4.3.6) are presented.

First of all, by using Lie symmetry analysis method, we obtain the vector field of the system (4.3.6) as follows:

$$\begin{aligned} &X_1 = u\tfrac{\partial}{\partial u} + 2v\tfrac{\partial}{\partial v}, \quad X_2 = \ln(r)\tfrac{\partial}{\partial u} + 2u\tfrac{\partial}{\partial v}, \quad X_3 = \tfrac{\partial}{\partial v}, \quad X_4 = \tfrac{\partial}{\partial u}, \\ &X_5 = r\tfrac{\partial}{\partial r} + t\tfrac{\partial}{\partial t}, \quad X_6 = \tfrac{\partial}{\partial t}. \end{aligned} \tag{4.3.7}$$

By calculation, we have showed that the symmetries of system (4.3.6) form a six-dimensional Lie algebra generated by vector fields (4.3.7). To find non equivalent branches of solutions, the one-dimensional optimal system of subalgebras is constructed. The corresponding generators of the optimal system of subalgebras are

$$(i)X_1 + \alpha X_5, \quad (ii)X_2 + \beta X_5, \quad (iii)X_3 + \delta X_5, \quad (iv)X_4 + \mu X_5, \quad (v)X_5, \quad (vi)X_6, \tag{4.3.8}$$

where $\alpha,\ \beta,\ \delta$ and μ are arbitrary constants.

Next, similarity variables and similarity solutions for all six essential vector fields with optimal system are derived by solving characterstic equations

$$\tfrac{dv}{\psi} = \tfrac{du}{\eta} = \tfrac{dx}{\xi} = \tfrac{dt}{\tau}. \tag{4.3.9}$$

The general solution of these equations involves three constants; one becomes the new independent variable ζ and the others, say F and G, plays the role of new dependent variables. On substituting these solutions of (4.3.9) in system (4.3.6), one gets the reduced system of ordinary differential equations.

4.3.2 Symmetry Reductions and Exact Solutions

Now, reduction of system (4.3.6) into ODEs are obtained corresponds to each vector field in the optimal system. Some exact solutions of these ODEs and hence the system (4.3.6) are also obtained.

$(i)\ X_1 + \alpha X_5$

Corresponding to this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = \frac{r}{t}$, $u(r,t) = t^{\frac{1}{\alpha}} F(\zeta)$, $v(r,t) = t^{\frac{2}{\alpha}} G(\zeta)$.

On using these in system , the reduced system of ODEs is as follows:

$$\begin{aligned} &F''(\zeta)(1-\zeta^2) + \tfrac{2\zeta F'(\zeta)}{\alpha} + \tfrac{F'(\zeta)}{\zeta} - 2F'(\zeta)\zeta - \tfrac{F(\zeta)}{\alpha^2} + \tfrac{F(\zeta)}{\alpha} = 0 \\ &G'(\zeta) + \tfrac{2G(\zeta)}{\alpha} - G'(\zeta)\zeta - \zeta\left(\tfrac{F(\zeta)}{\alpha} - \zeta F'(\zeta) + F'(\zeta)\right)^2 = 0. \end{aligned} \tag{4.3.10}$$

The solution of the reduced system of ODEs (4.3.10) is as follows:

$$\begin{aligned} F(\zeta) &= c_1 \text{hypergeom}\left(\left[\tfrac{-1}{2\alpha}, \tfrac{\alpha-1}{2\alpha}\right], \left[\tfrac{-2+\alpha}{2\alpha}\right], 1-\zeta^2\right) \\ &\quad + c_2(-1+\zeta^2)^{\left(\frac{\alpha+2}{2\alpha}\right)} \text{hypergeom}\left(\left[\tfrac{1+2\alpha}{2\alpha}, \tfrac{1+\alpha}{2\alpha}\right], \left[\tfrac{3\alpha+2}{2\alpha}\right], 1-\zeta^2\right) \\ G(\zeta) &= \left(\int \tfrac{-\zeta(F'(\zeta)\zeta\alpha - F(\zeta) - F'(\zeta)\alpha)^2(-1+\zeta)^{\frac{(-\alpha-2)}{2}}}{\alpha} d\zeta + c_1\right)(-1+\zeta)^{\frac{2}{\alpha}}. \end{aligned} \tag{4.3.11}$$

Hence, the solution of system (4.3.6) is as follows:

$$\begin{aligned} u(r,t) &= \left(c_1 \text{hypergeom}\left(\left[\tfrac{-1}{2\alpha}, \tfrac{\alpha-1}{2\alpha}\right], \left[\tfrac{-2+\alpha}{2\alpha}\right], 1-\tfrac{r^2}{t^2}\right)\right) t^{\frac{1}{\alpha}} \\ &\quad + \left(c_2(-1+\tfrac{r^2}{t^2})^{\left(\frac{\alpha+2}{2\alpha}\right)} \text{hypergeom}\left(\left[\tfrac{1+2\alpha}{2\alpha}, \tfrac{1+\alpha}{2\alpha}\right], \left[\tfrac{3\alpha+2}{2\alpha}\right], 1-\tfrac{r^2}{t^2}\right)\right) t^{\frac{1}{\alpha}} \\ v(r,t) &= \left(\left(\int \tfrac{-\zeta(F'(\zeta)\zeta\alpha - F(\zeta) - F'(\zeta)\alpha)^2(-1+\zeta)^{\frac{(-\alpha-2)}{2}}}{\alpha} d\zeta + c_1\right)(-1+\zeta)^{\frac{2}{\alpha}}\right) t^{\frac{2}{\alpha}}. \end{aligned} \tag{4.3.12}$$

where c_1 and c_2 are arbitrary constants and hypergeom stands for hypergeometric function.

$(ii)\ X_2 + \beta X_5$

For this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = \frac{r}{t}, u(r,t) = \frac{(\ln(r))^2}{2\beta} + F(\zeta), v(r,t) = \frac{(\ln(r))^3}{3\beta^2} + \frac{2F(\zeta)}{\beta}\ln(r) + G(\zeta).$

On using these in system (4.3.6), the system of reduced ODEs:

$$\begin{aligned} &F''(\zeta)\beta\zeta^2(1-\zeta^2) - 2\beta\zeta^3 F'(\zeta) + \beta\zeta F'(\zeta) + 1 = 0 \\ &\beta\zeta^2 G'(\zeta) - G'(\zeta)\beta\zeta + (F(\zeta))^2\zeta^4\beta - 2\zeta^3\beta(F'(\zeta))^2 + \zeta^2\beta(F'(\zeta))^2 - 2F(\zeta) = 0. \end{aligned} \tag{4.3.13}$$

The solution of the system (4.3.13) is given by:

$$\begin{aligned} F(\zeta) &= \int \frac{-\arctan\left(\frac{1}{\sqrt{-1+\zeta^2}}\right)\sqrt{\zeta-1}\sqrt{\zeta+1} + c_2\beta\sqrt{-1+\zeta^2}}{\sqrt{-1+\zeta^2}\beta\sqrt{\zeta+1}\sqrt{\zeta-1}\zeta} d\zeta + c_3 \\ G(\zeta) &= \int \frac{-\zeta^2\beta F'^2 + 2F - F^2\zeta^4\beta + 2\zeta^3\beta F'^2}{\beta\zeta(\zeta-1)} d\zeta + c_1, \end{aligned} \tag{4.3.14}$$

where c_1, c_2 and c_3 are arbitrary constants.

Thus, we can get the following solution of system (4.3.6) by using (4.3.14), followed by reverting back to original variables

$$\begin{aligned} u(r,t) &= \frac{(\ln(r))^2}{2\beta} + F(\zeta) \\ v(r,t) &= \frac{(\ln(r))^3}{3\beta^2} + \frac{2F(\zeta)}{\beta}\ln(r) + G(\zeta). \end{aligned} \tag{4.3.15}$$

$(iii)\;\; X_3 + \delta X_5$

For this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = \frac{r}{t}, u(r,t) = F(\zeta), v(r,t) = G(\zeta) + \frac{\ln(t)}{\delta}.$

Using these substitutions, system (4.3.6) reduces to

$$\begin{aligned} &F''(\zeta)(1-\zeta^2) - 2\zeta(F'(\zeta)) + \frac{F'(\zeta)}{\zeta} = 0, \\ &G'(\zeta)(1-\zeta) - \zeta(-\zeta F'(\zeta) + F'(\zeta))^2 + \frac{1}{\delta} = 0. \end{aligned} \tag{4.3.16}$$

The solution of reduced system of ODEs (4.3.16) are obtained and the solution of system (4.3.6) is as follows:

$$\begin{aligned} u(r,t) &= \arctan\left(\frac{t}{\sqrt{(r^2-t^2)}}\right) c_2 + c_1, \\ v(r,t) &= -c_2^2\ln(r) + c_2^2\ln(r+t) + \frac{1}{\delta}\ln(r-t) + c_3, \end{aligned} \tag{4.3.17}$$

where c_1, c_2 and c_3 are arbitrary constants.

$(iv)\;\; X_4 + \delta X_5$

In this case, the form of the similarity variable and similarity solution is as follows:

$\zeta = \frac{r}{t}$, $u(r,t) = F(\zeta) + \frac{\ln(r)}{\delta}$, $v(r,t) = G(\zeta)$.

Using these substitutions in system (4.3.6), we get following reduced system of ODEs:

$$\begin{aligned} &F''(\zeta)(1-\zeta^2) - 2\zeta(F'(\zeta)) + \frac{F'(\zeta)}{\zeta} + \frac{1}{\delta} = 0 \\ &G'(\zeta)(1-\zeta) - \zeta\left(-\zeta F'(\zeta) + F'(\zeta) + \frac{1}{\delta}\right)^2 = 0. \end{aligned} \tag{4.3.18}$$

The solution of reduced ODEs (4.3.18) are obtained and hence the solution of system (4.3.6) is as follows:

$$\begin{aligned} u(r,t) &= \frac{\ln(r)}{\delta} - c_1 \arctan\left(\frac{t}{\sqrt{(r^2-t^2)}}\right) + c_2 \\ v(r,t) &= -\frac{\ln(r-t)}{\delta^2} + \frac{\ln(r)}{\delta^2} - c_1^2\ln(r) + c_1^2\ln(r+t) - \frac{2c_1 \arctan\left(\frac{t}{\sqrt{(r^2-t^2)}}\right)}{\delta} + c_3, \end{aligned} \tag{4.3.19}$$

where c_1, c_2 and c_3 are arbitrary constants.

(v) X_5

For this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = \frac{r}{t}$, $u(r,t) = F(\zeta)$, $v(r,t) = G(\zeta)$.

Using these substitutions, system (4.3.6) reduce to

$$\begin{aligned} &F''(\zeta)(1-\zeta^2) - 2\zeta(F'(\zeta)) + \frac{F'(\zeta)}{\zeta} = 0 \\ &G'(\zeta)(1-\zeta) - \zeta(-\zeta F'(\zeta) + F'(\zeta))^2 = 0. \end{aligned} \tag{4.3.20}$$

The solution of reduced ODEs (4.3.20) are furnished and the solution of system (4.3.6) is as follows:

$$\begin{aligned} u(r,t) &= \arctan\left(\frac{t}{\sqrt{(r^2-t^2)}}\right) c_1 + c_2 \\ v(r,t) &= -c_1^2\ln(r) + c_1^2\ln(r+t) + c_3. \end{aligned} \tag{4.3.21}$$

where c_1, c_2 and c_3 are arbitrary constants.

(vi) X_6

Corresponding to this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = r$, $u(r,t) = F(\zeta)$, $v(r,t) = G(\zeta)$.

On using these in system (4.3.6), the system of reduced ODEs:

$$\begin{aligned} &\zeta F''(\zeta) + F'(\zeta) = 0, \\ &G'(\zeta) - \zeta(F'(\zeta))^2 = 0. \end{aligned} \tag{4.3.22}$$

The solution of reduced ODEs (4.3.22) are obtained and the solution of system (4.3.6) is deduced as:

$$\begin{aligned} u(r,t) &= c_1 + c_3 \ln(r) \\ v(r,t) &= c_2 + c_3^2 \ln(r), \end{aligned} \tag{4.3.23}$$

where c_1, c_2 and c_3 are arbitrary constants.

Since, after reduction to ODEs, the further attempt to apply Lie group analysis to ODEs has been made, but no further physically important nontrivial symmetries comes out, hence the solutions of ODEs are obtained directly. After attaining the reductions and exact solutions corresponding to essential vector fields of the optimal system, we observed that in each of physically relevant case, the similarity variable is of the form $\frac{r}{t}$. Since reductions can be obtained from any linear combination of basic vector fields (4.3.7), hence we can consider other linear combinations for physically significant reductions and exact solutions.

For example, we consider linear combination $X_1 + \mu X_2 + \lambda X_6$ of vector fields, where μ and λ are arbitrary constants. For this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = r$, $u(r,t) = \exp\left(\frac{t}{\lambda}\right)F(\zeta) - \mu \ln(\zeta)$

$v(r,t) = -2\mu \exp\left(\frac{t}{\lambda}\right)F(\zeta) + \mu^2 \ln(\zeta) + G(\zeta)\exp(\frac{2t}{\lambda})$.

On using these in system (4.3.6), the reduced system of ODEs is as follows:

$$\begin{aligned} &-F''(\zeta)\zeta\lambda^2 + F(\zeta)\zeta - \lambda^2 F'(\zeta) = 0, \\ &-G'(\zeta)\lambda^2 - 2G(\zeta)\lambda + F(\zeta)^2\zeta + 2F(\zeta)F'(\zeta)\zeta\lambda + F'(\zeta)^2\lambda^2\zeta = 0. \end{aligned} \tag{4.3.24}$$

The solution of reduced ODEs are obtained and the solution of system (4.3.6) is deduced as:

$$\begin{aligned} u(r,t) &= \left(c_2 J_0\left(\tfrac{r}{\lambda}\right) + c_3 Y_0\left(\tfrac{r}{\lambda}\right)\right)\exp\left(\tfrac{t}{\lambda}\right) - \mu\ln r \\ v(r,t) &= \left(\int r\exp(\tfrac{2r}{\lambda})\left(c_2 J_1(\tfrac{r}{\lambda}) - c_3 Y_1(\tfrac{r}{\lambda}) + c_2 J_0(\tfrac{r}{\lambda}) - c_3 J_0(\tfrac{r}{\lambda})\right)^2 dr + c_1\lambda^2\right) \\ &\qquad \frac{\exp(\frac{-2r}{\lambda})\exp(\frac{2t}{\lambda})}{\lambda^2} + \mu^2\ln(r) - 2\mu\exp(\tfrac{t}{\lambda})(c_2 J_0(\tfrac{r}{\lambda}) + c_3 Y_0(\tfrac{r}{\lambda})), \end{aligned} \tag{4.3.25}$$

where $J_v(x)$ and $Y_v(x)$ are the modified Bessel functions of the first and second kinds, respectively. They satisfy the modified Bessel equation:

$$x^2Y'' + xY' - (x^2 + v^2)Y = 0,$$

and c_1, c_2 and c_3 are arbitrary constants.

4.4 Summary

In this work, we have studied Einstein field equations for perfect fluid distribution and the system of partial differential equations corresponding to Einstein Rosen cylindrically symmetric space time for pure radiation fields by using Lie symmetry analysis method. Especially, all similarity reductions and exact solutions based on the Lie group method are obtained by generating the group infinitesimals. The partial differential equations are reduced to ordinary differential equations, which are further studied with the aim of deriving certain exact solutions. It is worth to mention here that the authenticity of all the solutions has been checked with the aid of software Maple. Thus we found new exact solutions that might prove to be interesting for further applications.

Chapter 5

Einstein Field Equations with Axial Symmetry for Empty Space Containing an Electrostatic Field [1]

5.1 Introduction

In the theory of general relativity, the field equations for regions containing electromagnetic fields but no matter are [50] as follows:

$$\begin{aligned} G_{\mu\nu} &= -8\pi E_{\mu\nu} \\ E_{\mu}^{\nu} &= -F^{\nu\alpha}F_{\mu\alpha} + \tfrac{1}{4}g_{\mu}^{\nu}F^{\alpha\beta}F_{\alpha\beta}, \end{aligned} \tag{5.1.1}$$

where g_{μ}^{ν} is metric tensor, $G_{\mu\nu}$ is the contracted Riemann-Christoffel tensor and $F_{\mu\nu}$ is the electromagnetic field tensor. This last tensor satisfies Maxwell's equations if we write

$$\begin{aligned} F_{\mu\nu} &= \kappa_{\mu,\nu} - \kappa_{\nu,\mu} \\ \Im^{\mu\nu}_{,\nu} &= \Im^{\mu}, \end{aligned} \tag{5.1.2}$$

where κ_{μ} is the four-potential, and $\Im^{\mu}$ is the charge and current density which is equal to zero for the region free of matter.

[1] The contents of this chapter are communicated for publication to *Applied Mathematics and Computation*

Weyl [184] has found a class of solutions of the above equations corresponding to certain axially symmetric electrostatic fields. In such fields the potential has only one non-vanishing component κ_4. Weyl's solution is for the axially symmetric case where there is a functional relation between g_{44} and ϕ of the form

$$g_{44} = A + B\phi + \phi^2, \tag{5.1.3}$$

where A and B are arbitrary constants.

The solutions of Equations (5.1.1) and (5.1.2) have been considered by Majumdar [117] and Papapetrou [142], when no spatial symmetry is assumed, and the general solution has been given when there is a relationship between g_{44} and ϕ, of the form

$$g_{44} = (C + \phi)^2, \tag{5.1.4}$$

where C is a constant. Majumdar has also proved that (5.1.3) is the only possible functional relationship between g_{44} and ϕ, whether or not there is spatial symmetry.

With the above-mentioned exception, the only exact electrostatic solutions reported appear to be special cases of Weyl's solution. Among the latter are the following: the well-known solution for a charged mass-point [50], the axially symmetric solution of Curzon [43] for several charged mass-points when the relation (5.1.4) exists between g_{44} and ϕ, the case of an electric field of uniform direction studied by McVittie [122], the solution of Mukherji [127] for a charged line-mass, and a solution corresponding to a particular uniform electric field given by Papapetrou [142].

In present study, we have derived certain axially symmetric electrostatic solutions of the field equations for empty space by using the canonical cylindrical coordinates introduced by Weyl and obtained complete sets of solutions in these coordinates.

In canonical coordinates, the line element for a field with axial symmetry is

$$ds^2 = -\exp(\lambda)(dx_1^2 + dx_2^2) - \exp(-\rho)x_2^2 dx_3^2 + \exp(\rho)dx_4^2, \tag{5.1.5}$$

where the origin of coordinates is on the axis of symmetry x_1, x_2 is a radial coordinate, x_3 is an angular coordinate and x_4 is time-like. λ and ρ are functions of x_1 and x_2 only. The equations (5.1.1) and (5.1.2), with $\Im^{\mu} = 0$, yield the following set which has previously been given by [43]:

$$\lambda_{11} + \lambda_{22} + \rho_1^2 + \frac{\lambda_2}{x_2} = 2\exp(-\rho)(\phi_1^2 - \phi_2^2). \tag{5.1.6}$$

$$\lambda_{11} + \lambda_{22} + \rho_2^2 - \frac{\lambda_2}{x_2} - \frac{2\rho_2}{x_2} = -2\exp(-\rho)(\phi_1^2 - \phi_2^2). \tag{5.1.7}$$

$$\rho_1\rho_2 - \frac{\rho_1}{x_2} - \frac{\lambda_1}{x_2} = 4\exp(-\rho)\phi_1\phi_2. \tag{5.1.8}$$

$$\rho_{11} + \rho_{22} + \frac{\rho_2}{x_2} = 2\exp(-\rho)(\phi_1^2 + \phi_2^2). \tag{5.1.9}$$

$$\phi_{11} + \phi_{22} + \frac{\phi_2}{x_2} = (\rho_1\phi_1 + \rho_2\phi_2). \tag{5.1.10}$$

where the suffix 1 and 2 after λ, ρ and ϕ means partial differentiation with respect to x_1 and x_2.

The field equations which determine $\lambda,\ \rho$ and ϕ are (5.1.6)-(5.1.10), one can easily verify that these all are consistent. We will first find the expression for ρ and ϕ by solving the equations (5.1.9) and (5.1.10) and then by substitution of these expressions, λ can later be obtained from (5.1.6)-(5.1.8). Rewriting the Equations (5.1.9) and (5.1.10)

$$\begin{aligned} &\rho_{11} + \rho_{22} + \tfrac{\rho_2}{x_2} = 2\exp(-\rho)(\phi_1^2 + \phi_2^2) \\ &\phi_{11} + \phi_{22} + \tfrac{\phi_2}{x_2} = (\rho_1\phi_1 + \rho_2\phi_2). \end{aligned} \tag{5.1.11}$$

In view of the nonlinear character of expression involved, it is difficult to obtain exact solutions of the system (5.1.11) and the exact solutions of the system (5.1.11) may enable one to better understand the phenomena which it describes. A detailed systematic analysis that leads to an exact analytic solution for (5.1.11) has not been performed.

The detail plan of the chapter is as follows: in Section (5.2), Lie classical method is used to derive the Lie symmetries of the system (5.1.11). The optimal system of non-conjugate sub-algebras of the full symmetry algebra is identified under the adjoint action of the symmetry group. Section (5.3) is devoted in finding the reduced system of ODEs using various Lie ansätze associated with each basic field in the optimal system of sub-algebras. Further the systems of reduced ODEs are examined for certain exact solutions. Finally, in the last section we made some concluding remarks.

5.2 Lie Point Symmetries and Classification of Subalgebras

In present section, Lie classical method [23] to the system (5.1.11) is applied by considering the one-parameter Lie group of infinitesimal transformations in $x_1,\ x_2,\ \rho,\ \phi,\ \xi^1(x_1,\ x_2)$ $\xi^2(x_1,\ x_2),\ \eta^1(x_1,\ x_2)$ and $\eta^2(x_1,\ x_2)$. This transformation leaves invariant the following set

$$S_\Delta \equiv \{\rho(x_1,\ x_2),\ \phi(x_1,\ x_2) : \Delta_1(\rho,\ \phi) = 0,\ \Delta_2(\rho,\ \phi) = 0\}, \tag{5.2.1}$$

of solutions of the system (5.1.11), where

$$\begin{aligned} \Delta_1 &= \rho_{11} + \rho_{22} + \tfrac{\rho_2}{x_2} - 2\exp(-\rho)(\phi_1^2 + \phi_2^2), \\ \Delta_2 &= \phi_{11} + \phi_{22} + \tfrac{\phi_2}{x_2} - ((\rho_1\phi_1 + \rho_2\phi_2)). \end{aligned}$$

The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\Gamma \equiv \xi^1 \frac{\partial}{\partial x_1} + \xi^2 \frac{\partial}{\partial x_2} + \eta^1 \frac{\partial}{\partial \rho} + \eta^2 \frac{\partial}{\partial \phi}. \tag{5.2.2}$$

The set S_Δ is invariant under the one-parameter transformations provided that $Pr^{(2)}(\Gamma)|_{\Delta=0} = 0$, where $Pr^{(2)}(\Gamma)$ is the second prolongation of the vector field Γ, which is explicitly given in terms of $\xi^1,\ \xi^2,\ \eta^1$ and η^2. This procedure yields an over determined system of linear PDEs. Solving this system of PDEs, Lie symmetries of the system are

$$\begin{aligned} \xi^1 &= a_1 x_1 + a_2, \\ \xi^2 &= a_1 x_2, \\ \eta^1 &= 4a_3\phi + 2a_4, \\ \eta^2 &= (\phi^2 + \exp(\rho))a_3 + a3\phi + a_5, \end{aligned} \tag{5.2.3}$$

where $a_1,\ a_2,\ a_3,\ a_4$ and a_5 are arbitrary constants.

After determining the infinitesimals of the system (5.1.11), the similarity variables are derived by solving invariant surface conditions

$$\begin{aligned} \Phi_1 &\equiv \xi^1\rho_{x_1} + \xi^2\rho_{x_2} - \eta^1 = 0, \\ \Phi_2 &\equiv \xi^1\phi_{x_1} + \xi^2\phi_{x_2} - \eta^2 = 0. \end{aligned} \tag{5.2.4}$$

The symmetries under which the system (5.1.11) is invariant can be spanned by the following five linearly independent infinitesimal generators:

$$\begin{aligned}&\Gamma_1 = 4\phi\frac{\partial}{\partial\rho} + (\phi^2 + \exp(\rho))\frac{\partial}{\partial\phi}, \quad \Gamma_2 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, \quad \Gamma_3 = 2\frac{\partial}{\partial\rho} + \phi\frac{\partial}{\partial\phi},\\ &\Gamma_4 = \frac{\partial}{\partial x_1}, \quad \Gamma_5 = \frac{\partial}{\partial\phi},\end{aligned} \tag{5.2.5}$$

with the nonzero Lie bracket relationships

$$[\Gamma_1, \Gamma_3] = -\Gamma_1, [\Gamma_1, \Gamma_5] = -2\Gamma_3, [\Gamma_2, \Gamma_4] = -\Gamma_4, [\Gamma_3, \Gamma_5] = -\Gamma_5. \tag{5.2.6}$$

As a result, symmetries (5.2.5) form a five-dimensional Lie algebra. We use the subalgebraic structure of symmetries (5.2.5) to construct an optimal system [139] of one dimensional subgroups. Such an optimal system of subgroups is determined by classifying the orbits of the infinitesimal adjoint representation of a Lie group on its Lie algebra obtained by using its infinitesimal generators.

The optimal system yields only the following symmetry combinations:

$$(i)\Gamma_5, \quad (ii)\Gamma_4 + \mu\Gamma_5, \quad (iii)\Gamma_3 + \delta\Gamma_4, \quad (iv)\Gamma_2 + \gamma\Gamma_3, \quad (v)\Gamma_1 + \alpha\Gamma_2 + \beta\Gamma_5, \tag{5.2.7}$$

where μ, δ, γ, α and β are arbitrary constants.

5.3 Solutions of Einstein Field Equations by Symmetry Reduction

Our main goal is to derive exact solutions of system (5.1.11) as exact solutions are helpful for mathematical as well as physical description. In this section, we will look for some exact solutions of system (5.1.11) by the reducing it into a system of ODEs using symmetry variables.

(i) V_5

Corresponding to this vector field, we get only constant solution of the system (5.1.11).

(ii) $V_4 + \mu V_5$

For this vector field, the form of the similarity variable and similarity solution are as follows:

$\zeta = x_2$, $u(x_1, x_2) = F(\zeta)$, $v(x_1, x_2) = G(\zeta) + \gamma x_1$.

On using these in the system (5.1.11), the system of reduced ODEs are as follows:

$$\begin{aligned} &F'' + \tfrac{F'}{\zeta} - \exp(-F)(G'^2 + \mu^2) = 0 \\ &G'' + \tfrac{G'}{\zeta} - F'G' = 0. \end{aligned} \tag{5.3.1}$$

This can be further reduced as follows:

$$\begin{aligned} &F(\zeta) = \ln(G'\zeta c_1) \\ &G(\zeta) = \textstyle\int g(\zeta) d\zeta + c_2, \end{aligned} \tag{5.3.2}$$

where $g(\zeta)$ is the solution of the following differential equation:

$$c_1 \zeta g'' g = -g c_1 g' + g'^2 c_1 \zeta + 2g^3 + 2g\mu^2, \tag{5.3.3}$$

where c_1 and c_2 are arbitrary constants.

(iii) $V_3 + \delta V_4$

For this vector field, we obtained the following similarity variable and similarity solution:

$\zeta = x_2$, $u(x_1, x_2) = F(\zeta) + \frac{2x_1}{\mu}$, $v(x_1, x_2) = \exp(\frac{x_1}{\mu})G(\zeta)$.

On using these expressions in system (5.1.11), the following system of reduced ODEs can be easily obtained:

$$\begin{aligned} &\zeta\delta^2 F'' + F'\delta^2 - 2\zeta \exp(-F)(G'^2\delta^2 + G^2) = 0 \\ &\zeta\delta^2 G'' + G'\delta^2 - \zeta G - \zeta\delta^2 G'F' = 0. \end{aligned} \tag{5.3.4}$$

Let $G(\zeta) = \exp(\frac{1}{2}F(\zeta))H(\zeta)$, $F'(\zeta) = N(\zeta)$.

Using these substitutions the system (5.3.4) reduces to

$$\begin{aligned} &\zeta\delta^2 N' + \delta^2 N - \tfrac{1}{2}\zeta\delta^2 N^2 H^2 - 2\zeta\delta^2 NHH' - 2\zeta\delta^2 H'^2 - 2\zeta H^2 = 0 \\ &-2\zeta\delta^2 N'H + \zeta\delta^2 N^2 H - 4\zeta\delta^2 H'' - 2\delta^2 NH - 4\delta^2 H' + 4\zeta H = 0. \end{aligned} \tag{5.3.5}$$

We arrive at following cases:

Case (iii.1)

$H(\zeta) = 0$, that is not a physically interesting case.

Case (iii.2)

$H(\zeta) = \pm 1.$

With this, our system (5.3.5) reduced to single equation:

$$\zeta\delta^2 N' - \frac{1}{2}\zeta\delta^2 N^2 + \delta^2 N - 2\zeta = 0, \tag{5.3.6}$$

which can be further solve to give solution:

$$N(\zeta) = \frac{4c_1 Y_1(\frac{-\iota\zeta}{\delta}) + 2J_1(\frac{\zeta}{\delta})}{\delta(2\iota c_1 Y_0(\frac{-\iota\zeta}{\delta}) + J_0(\frac{\zeta}{\delta}))}, \tag{5.3.7}$$

where c_1 is an arbitrary constant and $J_v(x)$ and $Y_v(x)$ are the modified Bessel functions of the first and second kinds, respectively and satisfy the modified Bessel equation:

$$x^2 y'' + xy' - (x^2 + v^2)y = 0.$$

Using these results we get final solution of system (5.1.11):

$$\begin{aligned} \rho(x_1, x_2) &= \tfrac{2x_1}{\mu} - 2ln\left(2\iota c_1 Y_0\left(\tfrac{-\iota x_2}{\delta}\right) + J_0\left(\tfrac{x_2}{\delta}\right)\right) \\ \phi(x_1, x_2) &= \tfrac{\pm\exp(\frac{x_1}{\delta})}{\left(2\iota c_1 Y_0\left(\frac{-\iota x_2}{\delta}\right) + J_0\left(\frac{x_2}{\delta}\right)\right)}, \end{aligned} \tag{5.3.8}$$

Using above expressions for $u(x_1, x_2)$ and $v(x_1, x_2)$ in (5.1.6)-(5.1.8) and solving, we get

$$\lambda(x_1, x_2) = -\frac{2x_1}{\mu} + 2\ ln\left(2\iota c_1 Y_0\left(\frac{-\iota x_2}{\delta}\right) + J_0\left(\frac{x_2}{\delta}\right)\right) + c_2$$

where c_2 is an arbitrary constant.

(iv) $V_2 + \gamma V_3$

Corresponding to this vector field, the form of the similarity variable and similarity solution are as follows:

$\zeta = \frac{x_1}{x_2}$, $u(x_1, x_2) = F(\zeta) + 2\gamma \ln(x_2)$, $v(x_1, x_2) = x_2^{\gamma} G(\zeta)$.

On using these similarity variable and similarity solution in system (5.1.11), the system of reduced ODEs:

$$\begin{aligned} &F'' + \zeta F' + \zeta^2 F'' - \exp(-F)(2G'^2 - 4\gamma G G' \zeta + 2\zeta^2 G'^2 + 2\gamma^2 G^2) = 0 \\ &G'' + \zeta G' + \zeta^2 G'' - \gamma^2 G - F'G' - \zeta^2 F'G' + \gamma \zeta F' G = 0. \end{aligned} \tag{5.3.9}$$

Let $G(\xi) = \exp\left(\frac{1}{2}F(\xi)\right)H(\xi)$, $F'(\xi) = N(\xi)$.

Then our system (5.3.9) reduces to following equations:

$$\begin{aligned} &-N' - \zeta N - \zeta^2 N' + \tfrac{1}{2}N^2 H^2 + 2NHH' + 2H'^2 + \tfrac{1}{2}\zeta^2 N^2 H^2 + 2\zeta^2 NHH' + 2\zeta^2 H'^2 \\ &-2\gamma\zeta N H^2 - 4\gamma\zeta H H' + 2\gamma^2 H^2 = 0 \\ &-2N'H + N^2 H - 4H'' - 2\zeta N H - 4\zeta H' - 2\zeta^2 N' H + \zeta^2 N^2 H - 4\zeta^2 H'' + 4\gamma^2 H \\ &-4\gamma\zeta N H = 0. \end{aligned} \tag{5.3.10}$$

Case $(iv.1)$

$H(\zeta) = 0$, that is not a physically interesting case.

Case $(iv.2)$

$H(\zeta) = \pm 1$.

With this, our system (5.3.10) reduced to single equation:

$$N' + \zeta N + \zeta^2 N' - \frac{1}{2}N^2(1 + \zeta^2) + 2\zeta\gamma N - 2\gamma^2 = 0. \tag{5.3.11}$$

which can be further solve to give solution:

$$\begin{aligned} N(\zeta) = &\frac{2(ic_1 - ic_1\gamma)\mathrm{LegendreQ}\left(\frac{1}{2}, -\frac{1}{2}+\gamma, i\zeta\right)}{\left(c_1 \mathrm{LegendreQ}\left(-\frac{1}{2}, -\frac{1}{2}+\gamma, i\zeta\right) + \mathrm{LegendreP}\left(-\frac{1}{2}, -\frac{1}{2}+\gamma, i\zeta\right)\right)(1+\zeta^2)} \\ &+ \frac{2(c_1\zeta\,\mathrm{LegendreQ}\left(-\frac{1}{2}, -\frac{1}{2}+\gamma, i\zeta\right)\gamma + (i - i\gamma)\mathrm{LegendreP}\left(\frac{1}{2}, -\frac{1}{2}+\gamma, i\zeta\right) + \zeta\,\mathrm{LegendreP}\left(-\frac{1}{2}, -\frac{1}{2}+\gamma, i\zeta\right)\gamma)}{\left(c_1 \mathrm{LegendreQ}\left(-\frac{1}{2}, -\frac{1}{2}+\gamma, i\zeta\right) + \mathrm{LegendreP}\left(-\frac{1}{2}, -\frac{1}{2}+\gamma, i\zeta\right)\right)(1+\zeta^2)}, \end{aligned} \tag{5.3.12}$$

where c_1 ia an arbitrary constant and $\mathrm{LegendreP}\,(\mu, \lambda, x)$ and $\mathrm{LegendreQ}\,(\mu, \lambda, x)$ are the Legendre functions of the first and second kinds, respectively and satisfy the Legendre equation:

$$(1 - x^2)y'' - 2xy' + \left(\lambda(\lambda + 1) - \frac{\mu^2}{(1 - x^2)}\right) y = 0,$$

where complex numbers λ and μ are degree and order of associated Legendre functions respectively.

Hence, the solution of the system (5.1.11) is

$$\begin{aligned}\rho(x_1,x_2) &= 2\gamma\ln(x_2)+\ln\left(1+\frac{{x_1}^2}{{x_2}^2}\right)\gamma-\frac{1}{2}\ \ln\left(1+\frac{{x_1}^2}{{x_2}^2}\right)\\ &\quad -2\ \ln\left(c_1\text{LegendreQ}\ \left(-\frac{1}{2},\gamma-\frac{1}{2},\frac{ix_1}{x_2}\right)+\text{LegendreP}\ \left(-\frac{1}{2},\gamma-\frac{1}{2},\frac{ix_1}{x_2}\right)\right)\\ \phi(x_1,x_2) &= \frac{x_2^{\gamma}\left(1+\frac{x_1^2}{{x_2}^2}\right)^{\frac{1}{2}\gamma-\frac{1}{4}}}{\left(c_1\text{LegendreQ}\left(-\frac{1}{2},\gamma-\frac{1}{2},\frac{ix_1}{x_2}\right)+\text{LegendreP}\left(-\frac{1}{2},\gamma-\frac{1}{2},\frac{ix_1}{x_2}\right)\right)}.\end{aligned}\tag{5.3.13}$$

Consequently, $\lambda(x_1,x_2)$ is given by:

$$\begin{aligned}\lambda(x_1,x_2) &= -\ln\left(1+\frac{{x_1}^2}{{x_2}^2}\right)\gamma+\frac{1}{2}\ \ln\left(1+\frac{{x_1}^2}{{x_2}^2}\right)-2\,\gamma\ \ln\left(x_2\right)\\ &\quad +2\ \ln\left(c_1\text{LegendreQ}\ \left(-\frac{1}{2},\gamma-\frac{1}{2},\frac{ix_1}{x_2}\right)+\text{LegendreP}\ \left(-\frac{1}{2},\gamma-\frac{1}{2},\frac{ix_1}{x_2}\right)\right)+c_2,\end{aligned}$$

where c_2 is an arbitrary constant.

(v) $V_1+\alpha V_4+\beta V_5$

In this case, we are able to obtain the solutions only when $\beta=0$. Similarity variable and similarity solution, for $\beta=0$, are as follows:

$\zeta=\frac{x_1}{x_2}$, $u(x_1,x_2)=2\ln(v(x_1,x_2))$, $v(x_1,x_2)=\frac{-1}{\frac{\ln(x_1)}{\alpha}+\ln(G(\zeta))}$.

By using above substitutions, the system (5.1.11) is reduced to a single ODE:

$$-\alpha\zeta^2G''G+\alpha\zeta^2G'^2-\alpha\zeta^4G''G-\alpha\zeta^3G'G+\alpha\zeta^4G'^2+G^2=0.\tag{5.3.14}$$

Solving the equation (5.3.14) and then reverting back to the original variables, we obtain the following solution of the system (5.1.11):

$$\begin{aligned}\rho(x_1,x_2) &= 2\ln\left(\frac{-1}{\ln(c_1)+\frac{\ln(x_2)}{\alpha}-c_2\text{arcsinh}\left(\frac{x_1}{x_2}\right)}\right)\\ \phi(x_1,x_2) &= \frac{-1}{\ln(c_1)+\frac{\ln(x_2)}{\alpha}-c_2\text{arcsinh}\left(\frac{x_1}{x_2}\right)},\end{aligned}\tag{5.3.15}$$

where c_1 and c_2 are arbitrary constants.

Making substitutions of these expressions for $\rho(x_1,x_2)$ and $\phi(x_1,x_2)$ in (5.1.6)-(5.1.8) and solving for $\lambda(x_1,x_2)$, we get

$$\lambda(x_1,x_2)=-2\ln\left(\frac{-1}{\ln(c_1)+\frac{\ln(x_2)}{\alpha}-c_2\text{arcsinh}\left(\frac{x_1}{x_2}\right)}\right)+c_3,$$

where c_3 is an arbitrary constant.

5.4 Conclusion and Outlook

We have investigated the exact solutions of the coupled system of highly nonlinear partial differential equations of second order which arises in general relativity corresponding to empty space containing an electrostatic field. The Lie symmetry method is utilized for the purpose of obtaining the symmetries and invariant solutions of the system (5.1.11). We completely solved the determining equations for the infinitesimal generators of Lie groups and obtained all linearly independent vector fields of the system (5.1.11). Using the adjoint action of the symmetry group an optimal system is identified. The basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations. For each element in the optimal system, some similarity solutions are attempted for the system (5.1.11). Thus we found some new exact solutions that might prove to be interesting for further applications.

Chapter 6

Similarity Solutions of Field Equations in General Relativity [1]

6.1 Introduction

In this chapter, we have worked with the field equations of general relativity with electromagnetism

$$R_{ij} - \frac{1}{2} g_{ij} R = kT_{ij}, \tag{6.1.1}$$

with

$$k = 8\pi G c^{-4}, \tag{6.1.2}$$

and

$$T_{ij} = (4\pi)^{-1}(F_{il}F^l_j - \frac{1}{4}F_{lm}F^{lm}g_{ij}), \tag{6.1.3}$$

where F_{ij} is the electromagnetic tensor, g_{ij} is the metric tensor and conventions as to the metric signature [106].

[1] The contents of this chapter are communicated in *Communications in Theoretical Physics*

Einstein-Maxwell equations [125] in curved space are

$$\begin{aligned}&[ijkl]\tfrac{\partial F_{jk}}{\partial x^l} = 0,\\ &\tfrac{\partial}{\partial x^k}[(-g)^{\frac{1}{2}} g^{ij} g^{kl} F_{jl}] = 0\end{aligned} \tag{6.1.4}$$

where $[ijkl] = (+1, -1)$ for (even, odd) permutation of i, j, k, l and $[ijkl] = 0$ if any two of i, j, k, l are equal.

We assumed that the metric to be diagonal

$$g_{ij} = \delta_{ij} e_i \exp(2f_i), \tag{6.1.5}$$

with

$$e_0 = -1, \;\; e_1 = e_2 = e_3 = 1. \tag{6.1.6}$$

Although there are various ways to defining potentials. We considered the following form of potentials

$$G_{ij} = A_{i,j} - A_{j,i}, \quad (i, j = 0, 1, 2), \tag{6.1.7}$$

and

$$G_{i3} = e_i \exp(f_i - f_j - f_k + f_3)(B_{j,k} - B_{k,j}), \tag{6.1.8}$$

$(i, j, k = 0, 1, 2)$ in cyclic order. Here, we introduce a new potential C as $A = C \cos\alpha$ and $B = C \sin\alpha$. The metric [73]

$$-ds^2 = V^{-2}(\exp(2\xi))((-dx^0)^2 + (dx^1)^2) + (x^1)^2 V^{-2}(dx^2)^2 + V^2(dx^3)^2. \tag{6.1.9}$$

With the above substitutions [73], the Einstein-Maxwell field equations become

$$V_{11} + \frac{V_1}{x^1} - V_{00} = \frac{1}{V}(V_1^2 - V_0^2 + C_0^2 - C_1^2), \tag{6.1.10}$$

$$C_{11} + \frac{C_1}{x^1} - C_{00} = \frac{2}{V}(V_1 C_1 - V_0 C_0), \tag{6.1.11}$$

$$\xi_0 = 2x^1 V^{-2}(V_0 V_1 - C_0 C_1), \tag{6.1.12}$$

$$\xi_1 = x^1 V^{-2}(V_0^2 + V_1^2 + C_0^2 + C_1^2), \tag{6.1.13}$$

$$\xi_{11} - \xi_{00} = V^{-2}(V_0^2 - V_1^2 + C_0^2 - C_1^2). \tag{6.1.14}$$

where the lower suffix $1, 0$ denotes partial differentiation with respect to the corresponding variable x^1, x^0. Equation (6.1.14) is derivable from Eqs. (6.1.10)-(6.1.13); ξ_{01} as calculated from Eq. (6.1.12) is identical with that calculated from Eq. (6.1.13), with Eqs. (6.1.10) and (6.1.11) assumed. Thus we can find ξ from V and C, and Eqs. (6.1.10) and (6.1.11) are the main equations. They are seen to be quasilinear wave equations for V and C. Since, the nonlinear system of partial differential equations (PDEs) (6.1.10)-(6.1.14) represents mathematically and physically important phenomena for electromagnetic fields and gravitational fields in the theory of general relativity.

This chapter is structured as follows: In Section 6.2, the generalized symmetry method is used to derive the Lie symmetries of nonlinear Eqs (6.1.10) and (6.1.11). The optimal system of non-conjugate sub-algebras of the full symmetry algebra is identified under the adjoint action of the symmetry group in Section 6.3. Section 6.4 is devoted in finding the reduced system of ODEs using various Lie ans$\ddot{a}$tze associated with each basic field in the optimal system of sub-algebras. The systems of reduced ODEs are further solved to find exact solutions. Finally, in the last section we made some concluding remarks.

6.2 Symmetry Group and Optimal System

In order to determine the Lie group of transformations of Eqs (6.1.10) and (6.1.11), we have exploited the generalized symmetry method given by Steinberg [164], which is based on the Fr$\acute{e}$chet derivatives of the nonlinear operators.

Let the Eqs (6.1.10) and (6.1.11) be considered as a manifold $\bar{M} = M_1,\ M_2$,

$$\begin{aligned} M_1(V,C) &= V_{11} + \tfrac{V_1}{x^1} - V_{00} - \tfrac{1}{V}(V_1^2 - V_0^2 + C_0^2 - C_1^2) = 0, \\ M_2(V,C) &= C_{11} + \tfrac{C_1}{x^1} - C_{00} - \tfrac{2}{V}(V_1 C_1 - V_0 C_0) = 0, \end{aligned} \tag{6.2.1}$$

in the space of variables $\bar{X} = (x^1, x^0),\ \bar{\eta} = (V, C)$.

The one-parameter group of local point transformations that leaves Eq. (6.1.10) and

(6.1.11) invariant corresponds to the vector fields of the form

$$W = A(\bar{X},\bar{\eta})\frac{\partial}{\partial x^1} + B(\bar{X},\bar{\eta})\frac{\partial}{\partial x^0} - D(\bar{X},\bar{\eta})\frac{\partial}{\partial V} - E(\bar{X},\bar{\eta})\frac{\partial}{\partial C}. \tag{6.2.2}$$

The group infinitesimals A, B, D and E are to be found under the following conditions:

$$F_i(\bar{M}_i, \bar{\eta}, \bar{S})|_{\bar{N}=0} = \bar{0}, \tag{6.2.3}$$

for $i = 1,\ 2$.

In Eq. (6.2.3), $F_i(\bar{M}_i, \bar{\eta}, \bar{S})$ denotes the Fréchet derivative of M_i at $\bar{\eta} = (V, C)$ in the direction of the quasi-linear symmetry operator $\bar{S} = (S_1,\ S_2)$ and is defined by

$$\bar{F}(\bar{N}, \bar{\eta}, \bar{S}) = \frac{d[\bar{N}(\bar{\eta} + \epsilon\bar{\eta}_1)]}{d\epsilon}|\epsilon = 0. \tag{6.2.4}$$

The symmetry operator $\bar{S} = (S_1,\ S_2)$ has the following form:

$$S_1(V) = A(\bar{X},\bar{\eta})\frac{\partial V}{\partial x^0} + B(\bar{X},\bar{\eta})\frac{\partial V}{\partial x^1} + D(\bar{X},\bar{\eta}), \tag{6.2.5}$$

$$S_2(V) = A(\bar{X},\bar{\eta})\frac{\partial C}{\partial x^0} + B(\bar{X},\bar{\eta})\frac{\partial C}{\partial x^1} + E(\bar{X},\bar{\eta}), \tag{6.2.6}$$

Eq. (6.2.4) is used to find the Fréchet derivative of each of the three nonlinear operators defined through Eq. (6.2.1), and we arrive at the following expressions:

$$F_1(\bar{M}_1, \bar{\eta}, \bar{S}) = S_1(V_{11} + \tfrac{V_1}{x^1} - V_{00}) + V([S_1]_{11} + \tfrac{[S_1]_1}{x_1} - [S_1]_{00}) - 2(V_1[S_1]_1 - V_0[S_1]_0 + C_1[S_2]_1 - C_0[S_2]_0), \tag{6.2.7}$$

$$F_2(\bar{M}_2, \bar{\eta}, \bar{S}) = S_1(C_{11} + \tfrac{C_1}{x^1} - C_{00}) + V([S_2]_{11} + \tfrac{[S_2]_1}{x_1} - [S_2]_{00}) - 2(V_1[S_2]_1 + [S_1]_1 C_1 - V_0[S_2]_0 + [S_1]_0 C_0). \tag{6.2.8}$$

In view of conditions (6.2.3), Eqs. (6.2.7) are used to get the determining equations for the group infinitesimals A, B, D and E. In other words, Eqs (6.2.7) is expanded and the temporal derivatives of $V(x^1, x^0)$ and $C(x^1, x^0)$ are substituted with the help of Eq. (6.1.10) and (6.1.11). This leads to the polynomial expressions in various partial derivatives of $V(x^1, x^0)$ and $C(x^1, x^0)$ with respect to the spatial variable. On equating the coefficients of various derivative terms to zero in these expressions, set of determining equations for the group infinitesimals A, B, D and E are obtained. Without going into the details of algebraic calculations, we list here the simplified version of the determining equations. The set of equations obtained from Eq. (6.2.7) is as follows:

$$\begin{aligned}
&A_V = 0,\ \ A_C = 0,\ \ B_V = 0,\ \ B_C = 0,\\
&A_1 - B_0 = 0,\ \ A_0 - B_1 = 0,\\
&VD_{0C} + E_C = 0,\ \ VD_{VC} + E_V = 0,\\
&D + V^2 D_{VV} - VD_V = 0,\ \ 2D_{1C} + E_1 = 0,\\
&-D - VD_V + V^2 D_{CC} + 2VE_C = 0,\ \ x^1 D_{11} - x^1 D_{00} + D_1 = 0,\\
&Vx^1 A_{11} + 2x^1 D_0 - Vx^1 A_{00} - 2Vx^1 D_{0V} + VA_1 = 0,\\
&-2(x^1)^2 D_1 + 2V(x^1)^2 D_{1V} - Vx^1 B_1 + VB + V(x^1)^2 B_{11} - VB_{00} = 0.
\end{aligned} \tag{6.2.9}$$

Similarly, the equation (6.2.8) brings-in the following additional equations. It is being mentioned here that these equations have been obtained keeping in view the consequences on the infinitesimals as affected by the set of equation (6.2.9).

$$\begin{aligned}
&VE_{1V} - E_1 = 0,\ \ \ -VD_{0V} + E_0 = 0,\\
&VE_{VV} - E_V = 0,\ \ x^1 E_{11} + E_1 - x^1 E_{00} = 0,\\
&-VD_V + V^2 E_{VC} + D = 0,\ \ VE_{CC} - E_V - 2D_C = 0,\\
&2V(x^1)^2 E_{1C} - V(x^1)^2 B_{00} - 2(x^1)^2 D_1 - Vx^1 B_1 + VB + V(x^1)^2 B_{11} = 0,\\
&-D - 2VB_1 - V^2 E_{VC} + VD_V + 2VA_0 - V^2 A_{0V} = 0,\\
&-2Vx^1 E_{0C} - Vx^1 A_{00} + Vx^1 A_{11} + 2x^1 D_0 + VA_1 = 0.
\end{aligned} \tag{6.2.10}$$

Now, the two sets of equations (6.2.9) and (6.2.10) are combined, and simplified to the extent possible for the determination of the infinitesimals A, B, D and E. Without presenting any calculations, we provide the following form of the generalized symmetries:

$$\begin{aligned}
A &= a_1 x^0 + a_2,\\
B &= a_1 x^1,\\
D &= -2(a_3 C - \tfrac{a_4}{2})V,\\
E &= (-C^2 + V^2)a_3 + a_4 C + a_5,
\end{aligned} \tag{6.2.11}$$

where $a_j, j = 1, 2, 3....5$ are arbitrary constants. The symmetries under which the equation (1.11) is invariant can be spanned by the following five linearly independent infinitesimal generators:

$$W_1 = 2VC\frac{\partial}{\partial V} + (C^2 - V^2)\frac{\partial}{\partial C},\ \ W_2 = -V\frac{\partial}{\partial V} - C\frac{\partial}{\partial C},\ \ W_3 = x^1\frac{\partial}{\partial x^1} + x^0\frac{\partial}{\partial x^0},$$
$$W_4 = -\frac{\partial}{\partial C},\ \ W_5 = \frac{\partial}{\partial x^0}. \tag{6.2.12}$$

The adjoint action is given by the Lie series

$$Ad(\exp(\epsilon W_i))W_j = W_j - \epsilon[W_i, W_j] + \frac{\epsilon^2}{2}[W_i, [W_i, W_j]] - ..., \tag{6.2.13}$$

where $[W_i, W_j] = W_iW_j - W_jW_i$ is the commutator for the Lie algebra, and ϵ is a parameter. With the help of Lie series (6.2.13), the Commutator table and adjoint table for Lie algebra (6.2.12) can be easily constructed as shown in Table (6.1) and Table (6.2).

Table 6.1: Commutator Table

Index	W_1	W_2	W_3	W_4	W_5
W_1	0	$-W_1$	0	$-W_2$	0
W_2	W_1	0	0	W_4	0
W_3	0	0	0	0	$-W_5$
W_4	W_2	W_4	0	0	0
W_5	0	0	W_5	0	0

Table 6.2: Adjoint Table

Index	W_1	W_2	W_3	W_4	W_5
W_1	W_1	$W_2 + \epsilon W_1$	$W_3 + 2\epsilon W_1$	$W_4 + \epsilon W_2 + \frac{\epsilon^2}{2} W_1$	W_5
W_2	$W_1 \exp(-\epsilon)$	W_2	W_3	$W_4 \exp \epsilon$	W_5
W_3	W_1	W_2	W_3	W_4	$W_5 \exp(\epsilon)$
W_4	$W_1 - \epsilon W_2 - \frac{\epsilon^2}{2} W_4$	$W_2 - \epsilon W_4$	W_3	W_4	W_5
W_5	W_1	W_2	$W_3 - \epsilon W_5$	W_4	W_5

We thus deduce the following basic fields which form the following optimal system [139] for Eqs (6.1.10) and (6.1.11):

$$
\begin{aligned}
&(i) W_5\\
&(ii) W_4 + \alpha W_5\\
&(iii) W_3 + \beta W_4\\
&(iv) W_2 + \gamma W_3\\
&(v) W_1 + \delta W_3 + \lambda W_4,
\end{aligned}
\tag{6.2.14}
$$

where $\alpha, \beta, \gamma, \delta$ and λ are arbitrary constants.

6.3 Reductions and Exact Solutions

In the following we consider, corresponding to each generator in the optimal system of sub algebras, the reductions of PDEs (6.1.10) and (6.1.11) into ODEs in terms of similarity variable ζ and the new dependent variables F and G. Some exact solutions of each reduced system are then attempted.

$(i)\ W_5$

The vector field, W_5, in the optimal system defines the similarity variable and similarity solution as follows:

$\zeta = x^1,\ V(x^1, x^0) = F(\zeta),\ C(x^1, x^0) = G(\zeta).$

Using the similarity variable and the forms of the similarity solution, PDEs (6.1.10) and (6.1.11) reduces to the following system of ODEs:

$$
\begin{aligned}
&\zeta F F'' + F F' - \zeta F'^2 + \zeta G'^2 = 0,\\
&-\zeta F G'' - F G' + 2\zeta F' G' = 0.
\end{aligned}
\tag{6.3.1}
$$

After solving this system of ODEs (6.3.1), we obtain the following solution of equations (6.1.10) and (6.1.11):

$$
\begin{aligned}
C(x^1, x^0) &= \frac{-c_2 c_3 + \tanh(\frac{-1}{2}\ln(x^1)\sqrt{c_1 c_2} + \frac{c_4\sqrt{c_1 c_2}}{2})\sqrt{c_1 c_2}}{c_2},\\
V(x^1, x^0) &= \pm\frac{\sqrt{-\zeta(2\zeta G' G''' - 2\zeta G''^2 + 2G' G'')G'^2}}{\zeta G' G''' - \zeta G''^2 + G' G''},
\end{aligned}
\tag{6.3.2}
$$

where $c_1,\ c_2,\ c_3$ and c_4 are arbitrary constants.

$(ii)\ W_4 + \alpha W_5$

For this generator, the associated similarity variable and similarity solution are as follows: $\zeta = x^1, V(x^1, x^0) = F(\zeta), C(x^1, x^0) = \frac{x^0}{\alpha} + G(\zeta)$.

The corresponding reduced system of ODEs is given by

$$\begin{aligned}&\alpha^2\zeta FF'' + \alpha^2 FF' - \alpha^2\zeta F'^2 + \alpha^2\zeta G'2 - \zeta = 0,\\ &\zeta FG'' + FG' - 2\zeta F'G' = 0.\end{aligned} \tag{6.3.3}$$

In this case, we are only able to produce the following solution:

$$F(\zeta) = c_1\sqrt{\zeta},\ \ G(\zeta) = \pm\frac{\zeta}{\alpha} + c_2, \tag{6.3.4}$$

and

$$F(\zeta) = \frac{1}{2}c_1c_3(\zeta^{(\frac{1}{\alpha c_1})}c_3^{-2} + \zeta^{(\frac{-1}{\alpha c_1})}),\ \ G(\zeta) = c_4. \tag{6.3.5}$$

Thus, solution of PDEs (6.1.10)-(6.1.14) can be expressed as follows:

$$\begin{aligned}&V(x^1, x^0) = c_1\sqrt{x^1},\ \ C(x^1, x^0) = \pm\tfrac{x^1}{\alpha} + \tfrac{x^0}{\alpha} + c_2,\\ &\xi(x^1, x^0) = \tfrac{\mp 2x^0}{c_1^2\alpha^2} + \tfrac{2x^1}{c_1^2\alpha^2} + \tfrac{1}{4}\ln(x^1) + c_5,\end{aligned} \tag{6.3.6}$$

and

$$\begin{aligned}&V(x^1, x^0) = \tfrac{1}{2}c_1c_3((x^1)^{(\frac{1}{\alpha c_1})}c_3^{-2} + (x^1)^{(\frac{-1}{\alpha c_1})}),\ \ C(x^1, x^0) = \tfrac{x^0}{\alpha} + c_4,\\ &\xi(x^1, x^0) = \frac{\frac{-c_3^2(-1+2\alpha c_1-\alpha^2c_1^2)}{\alpha c_1} - \frac{(-1+2\alpha c_1-\alpha^2c_1^2)\ln(x^1)(x^1)^{\frac{2}{\alpha c_1}}}{\alpha c_1}}{c_1\alpha((x^1)^{\frac{2}{\alpha c_1}} + c_3^2)} + \ln((x^1)^{\frac{2}{\alpha c_1}} + c_3^2) + c_4,\end{aligned} \tag{6.3.7}$$

where $c_1,\ c_2,\ c_3$ and c_4 are arbitrary constants.

$(iii)\ W_3 + \beta W_4$

For this vector field, the form of the similarity variable and similarity solution is as follows:

$\zeta = \frac{x^1}{x^0},\ V(x^1, x^0) = F(\zeta),\ C(x^1, x^0) = \beta\ln(x^0) + G(\zeta)$.

On substituting these in PDEs (6.1.10) and (6.1.11), the reduced system of ODEs is given by

$$\begin{aligned}&\zeta FF'' - \zeta^3 FF'' - 2\zeta^2 FF' + FF' - \zeta F^2 + \zeta^3 F'^2 + 2\beta\zeta^2 G' - \zeta^3 G'^2 + \zeta G'^2 - \beta^2\zeta = 0,\\ &-\zeta FG'' - \beta\zeta F + \zeta^3 FG'' + 2\zeta^2 FG' - FG' + 2\zeta F'G' + 2\beta\zeta^2 F' - 2\zeta^3 F'G' = 0,\end{aligned} \tag{6.3.8}$$

which is quite difficult to solve. Therefore by taking $\beta = 0$, ODEs (6.3.8) becomes

$$\begin{aligned}&\zeta FF'' - \zeta^3 FF'' - 2\zeta^2 FF' + FF' - \zeta F^2 + \zeta^3 F'^2 - \zeta^3 G'^2 + \zeta G'^2 = 0,\\ &-\zeta FG'' + \zeta^3 FG'' + 2\zeta^2 FG' - FG' + 2\zeta F'G' - 2\zeta^3 F'G' = 0.\end{aligned} \tag{6.3.9}$$

which can be further solve to give following solution

$$\begin{aligned}F(\zeta) &= c_1\sqrt{\zeta}(\zeta^2-1)^{\frac{1}{4}}\sqrt{\left(\frac{1}{2\zeta\sqrt{\zeta^2-1}c_2^2\left(\cosh\left(\frac{arctan(\frac{1}{\sqrt{\zeta^2-1}})+c_3}{c_1c_2}\right)+1\right)}\right)},\\ G(\zeta) &= \int\left(\frac{1}{2\zeta\sqrt{\zeta^2-1}c_2^2\left(\cosh\left(\frac{\arctan(\frac{1}{\sqrt{\zeta^2-1}})+c_3}{c_1c_2}\right)+1\right)}\right)d\zeta + c_4,\end{aligned} \tag{6.3.10}$$

where $c_1,\ c_2,\ c_3$ and c_4 are arbitrary constants.

Using these results we get final solution of Eqs. (6.1.10) and (6.1.11):

$$\begin{aligned}V(x^1,x^0) &= c_1\sqrt{(\tfrac{x^1}{x^0})}((\tfrac{x^1}{x^0})^2-1)^{\frac{1}{4}}\sqrt{\left(\frac{1}{2(\frac{x^1}{x^0})\sqrt{(\frac{x^1}{x^0})^2-1}c_2^2\left(\cosh\left(\frac{arctan(\frac{1}{\sqrt{(\frac{x^1}{x^0})^2-1}})+c_3}{c_1c_2}\right)+1\right)}\right)},\\ C(x^1,x^0) &= \int\left(\frac{1}{2\zeta\sqrt{\zeta^2-1}c_2^2\left(\cosh(\frac{\arctan(\frac{1}{\sqrt{\zeta^2-1}})+c_3}{c_1c_2})+1\right)}\right)d\zeta + c_4.\end{aligned} \tag{6.3.11}$$

$(iv)\ W_2 + \gamma W_3$

In this case of vector field in the optimal system, we obtain

$\zeta = \frac{x^1}{x^0},\ V(x^1,x^0) = (x^1)^{(\frac{-1}{\gamma})}F(\zeta),\ C(x^1,x^0) = (x^1)^{(\frac{-1}{\gamma})}G(\zeta).$

For the case under consideration the reduced ODEs are

$$\begin{aligned}&\gamma^2FF'' - \zeta^2\gamma^2FF'' - 2\zeta\gamma^2FF' + \tfrac{\gamma^2FF'}{\zeta} - \gamma^2F'^2 + \zeta^2\gamma^2F'^2 - \zeta^2\gamma^2G'^2 + \tfrac{-2\gamma GG'}{\zeta} + \\ &\gamma^2G'^2 + \tfrac{G^2}{\zeta^2} = 0, \\ &-\tfrac{FG}{\zeta^2} + \gamma^2G''F - \zeta^2\gamma^2FG'' - 2\zeta\gamma^2FG' + \tfrac{\gamma^2FG'}{\zeta} + \tfrac{2\gamma GF'}{\zeta} - 2\gamma^2G'F' + 2\zeta^2\gamma^2F'G' = 0.\end{aligned} \tag{6.3.12}$$

Solution of this system of ODEs (6.3.12) is as follow:

Let $G(\zeta) = \pm\iota F(\zeta)$. Using these substitutions our system reduce to a following single equation

$$\zeta^2\gamma^2FF'' - \zeta^4\gamma^2FF'' - 2\zeta^3\gamma^2FF' + \zeta\gamma^2FF' - 2\zeta^2\gamma^2F'^2 + 2\zeta^4\gamma^2F'^2 + 2\zeta\gamma FF' - F^2 = 0, \tag{6.3.13}$$

hence we arrive at following solution:

$$F(\zeta) = \frac{-c_1\exp\left(\int \frac{(-(-1+\alpha)\zeta^2 \text{hypergeom}(p,(-\zeta^2+1)) + \text{hypergeom}(q,-\zeta^2+1)(\alpha-2))}{\text{hypergeom}(r,\zeta^2-1)(\alpha-2)\alpha\zeta}d\zeta\right)}{\int\left(\left(\frac{(\zeta^2-1)^{\frac{(2-\alpha)}{2\alpha}}}{\zeta}\right)\exp\left(\frac{-2}{\alpha(\alpha-2)}\left((\alpha-1)\int\left(\frac{\zeta^3}{\zeta^2-1}\frac{\text{hypergeom}(p,-\zeta^2+1)}{\text{hypergeom}(q,-\zeta^2+1)}d\zeta\right) + (1-\alpha)\int\left(\frac{\zeta}{\zeta^2-1}\frac{\text{hypergeom}(p,-\zeta^2+1)}{\text{hypergeom}(q,-\zeta^2+1)}d\zeta\right)\right)\right)\right)d\zeta} \tag{6.3.14}$$

where $p = [\frac{-1+2\alpha}{2\alpha}, \frac{-1+3\alpha}{2\alpha}], [\frac{3\alpha-2}{2\alpha}]$, $q = [\frac{-1}{2\alpha}, \frac{-1+\alpha}{2\alpha}], [\frac{\alpha-2}{2\alpha}]$ and c_1 and c_2 are arbitrary constants. Making substitutions of these expressions for $F(\zeta)$, the solution of Eqs.(6.1.10) and (6.1.11) are furnished as follows:

$$\begin{aligned}V(x^1,x^0) &= (x^1)^{(\frac{-1}{\gamma})}F(\zeta), \\ C(x^1,x^0) &= \pm\iota(x^1)^{(\frac{-1}{\gamma})}F(\zeta).\end{aligned} \tag{6.3.15}$$

(iii) $W_1 + \delta W_3 + \lambda W_4$

Unfortunately, we are not able to reduce the ODEs corresponding to this case; this will be taken up in a future endeavor. We consider the simplest form of transformation in view of the above similarity variables and similarity functions as follows:

$C = \pm\exp(V)$.

Using this in the Eqs. (6.1.10) and (6.1.11), we obtain

$$x^1VV_{11} - x^1VV_{00} + VV_1 - 2x^1V_1^2 + 2x^1V_0^2 = 0. \tag{6.3.16}$$

and the solution is

$$\begin{aligned} V(x^1,x^0) &= \tfrac{x^1c_1^3(J_1(c_1x^1)Y_0(c_1x^1)-J_0(c_1x^1)Y_1(c_1x^1))}{(-c_3J_0(c_1x^1)+c_4Y_0(c_1x^1))(c_3\sin(c_1x^0)-c_4\cos(c_1x^0))}, \\ C(x^1,x^0) &= \pm\iota\tfrac{x^1c_1^3(J_1(c_1x^1)Y_0(c_1x^1)-J_0(c_1x^1)Y_1(c_1x^1))}{(-c_3J_0(c_1x^1)+c_4Y_0(c_1x^1))(c_3\sin(c_1x^0)-c_4\cos(c_1x^0))} \end{aligned} \tag{6.3.17}$$

where c_1, c_3 and c_4 are arbitrary constants and $J_v(x)$ and $Y_v(x)$ are the modified Bessel functions of the first and second kinds, respectively. They satisfy the modified Bessel equation:

$$x^2Y'' + xY' - (x^2 + v^2)Y = 0.$$

Using above expressions for $V(x^1, x^0)$ and $C(x^1, x^0)$ in Eqs. (6.1.12)-(6.1.14) and solving, we get

$$\xi(x^1, x^0) = c_5, \tag{6.3.18}$$

where c_3 is arbitrary constant.

6.4 Concluding Remarks

In summary, we have utilized the symmetry method based on the Fréchet derivative of the differential operators to obtain the Lie symmetries admitted by field equations of general relativity with an electromagnetic stress tensor as source and Maxwell's equations in curved space, in which metric coefficients and electromagnetic fields are restricted to be the functions of two independent variables only. We completely solved the determining equations for the infinitesimal generators of Lie groups. Further, the group classification from the point of view of the optimal system of non-conjugate sub-algebras of the symmetry algebra of the nonlinear system has been performed under the adjoint action of the symmetry group. The various fields in the optimal system have been then used to get the reductions of PDEs into ODEs. Some exact solutions are attempted for the reduced systems that might prove to be interesting for further applications. The software package MAPLE has been used to check the correctness of the various solutions being reported through this work.

Chapter 7

Kawahara Equation and Modified Kawahara Equation with Variable Coefficients[1]

7.1 Introduction

The Kawahara equation [91]

$$u_t + auu_x + bu_{xxx} + cu_{xxxxx} = 0, \qquad (7.1.1)$$

and modified Kawahara equation [83]

$$u_t + au^2u_x + bu_{xxx} + cu_{xxxxx} = 0, \qquad (7.1.2)$$

where a, b and c are arbitrary constants, occurs in the theory of magneto-acoustic waves in plasma and in theory of shallow water waves with surface tension. Eq. (7.1.1) was first proposed by Kawahara in 1972, as a model equation describing solitary-wave propagation

[1]The contents of this chapter has been published in *Mathematical Methods in the Applied Sciences*, 36 (2013) 584-600

in media [91]. In the literature this equation is also referred as fifth-order KdV equation or singularly perturbed KdV equation [27]. The dispersive equation (7.1.2) was proposed by Kawahara as an important dispersive equation that arises in the context of shallow water waves. These equations have been the subject of research work in recent publications [27, 38, 161, 178] for its various important applications in the theory of magneto-acoustic waves in a plasma and in the theory of shallow water waves with surface tension. The paper [38] is mainly concerned with the local well-posedness of the initial-value problems for the Kawahara and the modified Kawahara equations in Sobolev spaces.

However, the physical situations in which nonlinear equations arise tend to be highly idealized due to assumption of constant coefficients. Due to this, much attention has been paid on study of nonlinear equations with variable coefficients [17, 70, 93, 159, 160]. The exact solutions of these equations are very useful to discuss and examine the sensitivity of physical phenomena with several important parameters described by variable coefficients. The exact solutions are also helpful in designing and testing of numerical algorithm.

In the present chapter, we have studied the variable coefficients version of the Kawahara equation (VCKE)

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \tag{7.1.3}$$

and the modified Kawahara equation (VCMKE)

$$u_t + \alpha(t)u^2u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \tag{7.1.4}$$

where $\alpha(t), \beta(t)$ and $\sigma(t)$ are arbitrary time-dependent coefficients, for exact solutions with the help of Lie symmetry analysis.

The chapter has been organized as follows: in Section 7.2, Lie classical method is applied to generate various symmetries of VCKE and VCMKE and all the similarity reductions and exact solutions to VCKE and VCMKE are derived. In Section 7.3, the power series solutions to the equations (7.1.3) and (7.1.4) are investigated by means of the power series method. In section 7.4, we have applied generalized $\left(\frac{G'}{G}\right)$-expansion method to VCKE and VCMKE. Finally, the conclusions and remarks are given in Section 7.5.

7.2 Lie Symmetry Analysis

7.2.1 Symmetry Reductions and Exact Solutions of VCKE

In this section, we first determined the Lie point symmetries of (7.1.3) and then use them to furnish the reduced ordinary differential equations.

A Lie point symmetry of a partial differential equation (PDE) is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. In general, determining the symmetries of a partial differential equation is a formidable task. However, Sophus Lie observed that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries. Lie symmetry analysis is applied to VCKE (7.1.3) as follows:

1. Let us first consider the Lie group of point transformations

$$\begin{aligned} u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2) \\ x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2) \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \end{aligned} \tag{7.2.1}$$

   which leaves the system (7.1.3) invariant. In other words, the transformations are such that if u is a solution of system (7.1.3), then u^* is also a solution. The method for determining the symmetry group of (7.1.3) mainly consists of finding the infinitesimals η, ξ and τ which are functions of x, t, u.

2. Assuming that the system (7.1.3) is invariant under the transformations (7.2.1), we

get the following relations from the coefficients of the first order of ϵ:

$$\begin{aligned}&\eta^t + \alpha(t)(\eta u_x + u\eta^x) + \tau\alpha'(t)uu_x + \beta\eta^{xxx} + \tau\beta'(t)u_{xxx} + \sigma\eta^{xxxxx} \\ &+\tau\sigma'(t)u_{xxxxx} = 0,\end{aligned} \tag{7.2.2}$$

where $\eta^t, \eta^x, \eta^{xxx}$ and η^{xxxxx} are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables u_t, u_x, u_{xxx} and u_{xxxxx}. The infinitesimals are determined from invariance condition (7.2.2), by setting the coefficients of different differentials equal to zero. The general solution of this system of linear PDEs provides the following forms for the infinitesimal elements η, ξ and τ and admissible forms of various coefficients in Eq. (7.1.3):

$$\begin{aligned}&\eta = k_1 u + k_2, \\ &\xi = k_3 x + \int \alpha(t) k_2 dt + k_4, \\ &\tau = \frac{(5k_3 \int \sigma(t)dt + k_5)}{\sigma(t)},\end{aligned} \tag{7.2.3}$$

and the functions $\alpha(t), \beta(t)$ and $\sigma(t)$ are given by:

$$\begin{aligned}&\alpha(t) = k_6\sigma(t)(5k_3 \int \sigma(t)dt + k_5)^{-\frac{(k_1+4k_3)}{5k_3}}, \\ &\beta(t) = k_7\sigma(t)(5k_3 \int \sigma(t)dt + k_5)^{-\frac{2}{5}},\end{aligned} \tag{7.2.4}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ and k_7 are arbitrary constants and $\sigma(t)$ is an arbitrary function of t.

Here, we are not able to consider optimal system, as the expression for ξ is in some power of $(5k_3 \int \sigma(t)dt + k_5)$. So we have derived the functional form of the similarity solution of Eq. (7.1.3), by solving the following set of characteristic equations:

$$\frac{du}{\eta} = \frac{dx}{\xi} = \frac{dt}{\tau}. \tag{7.2.5}$$

For the choice of the symmetries given in (7.2.3), Eq. (7.2.5) yield the similarity variable $\zeta = \zeta(x, t)$ and the corresponding form of u as function of new independent variable ζ as follows:

$$\begin{aligned}&\zeta(x,t) = -\frac{\left(-xk_3{k_1}^2 + x{k_3}^2 k_1 - k_4{k_1}^2 + k_4k_1k_3 + k_6k_2k_3\left(5\,k_3\int\sigma(t)dt + k_5\right)^{-\frac{k_1-k_3}{5k_3}}\right)}{\sqrt[5]{5\,k_3\int\sigma(t)dt + k_5}\,(k_1-k_3)k_3k_1} \\ &u(x,t) = \frac{\left(-k_2 + \left(5\,k_3\int\sigma(t)dt + k_5\right)^{\frac{k_1}{5k_3}} F(\zeta)k_1\right)}{k_1}.\end{aligned} \tag{7.2.6}$$

Using the similarity variable, the forms of the similarity solution and the coefficient functions in the Eq. (7.1.3), we arrive at the following ODE:

$$F''''' + k_7 F''' + k_6 F F' - k_3 \zeta F' + k_1 F = 0, \tag{7.2.7}$$

where prime ($'$) denotes the differentiation with respect to the variable ζ. Note that the reduced equation (7.2.7) is higher-order nonlinear ODE, we will deal with this equations in the section (7.3).
Moreover, for $k_1 = -k_3$, Eq. (7.2.7) becomes:

$$F''''' + k_7 F''' + k_6 F F' - k_3 \zeta F' - k_3 F = 0. \tag{7.2.8}$$

Thus, by solving (7.2.8) and reverting back to the original variables, we obtain the following group-invariant solution of the Kawahara equation (7.1.3) as follows:

$$u(x,t) = \frac{2(x k_3 + k_4)}{\left(5\, k_3 \int \sigma\,(t)\, dt + k_5\right)^{2/5} k_6}. \tag{7.2.9}$$

7.2.2 Lie Point Symmetries and Group Invariant Solutions of VCMKE

Assuming that the equation (7.1.4) is invariant under the transformations (7.2.1), we get the following relations from the coefficients of the first order of ϵ:

$$\begin{aligned}&\eta^t + \alpha(t)(2u\eta u_x + u^2\eta^x) + \tau\alpha'(t)u^2u_x + \beta(t)\eta^{xxx} + \tau\beta'(t)u_{xxx} + \sigma(t)\eta^{xxxxx}\\ &+\tau\sigma'(t)u_{xxxxx} = 0.\end{aligned} \tag{7.2.10}$$

After some straightforward, albeit tedious and lengthy calculations, the above system gives the following infinitesimals:

$$\begin{aligned}\eta &= k_1 u\\ \xi &= k_3 x + k_2\\ \tau &= \tfrac{(5k_3 \int \sigma(t)dt + k_4)}{\sigma(t)},\end{aligned} \tag{7.2.11}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants and the functions $\alpha(t), \beta(t)$ and $\sigma(t)$ are governed by following relations:

$$
\begin{aligned}
&2\beta(t)\sigma(t)k_3 - \tau\sigma'(t)\beta(t) + \tau\sigma(t)\beta'(t) = 0\\
&4\alpha(t)\sigma(t)k_3 + 2\alpha(t)\sigma(t)k_1 + \tau\alpha'(t)\sigma(t) - \tau\alpha(t)\sigma'(t) = 0.
\end{aligned} \tag{7.2.12}
$$

Thus the Lie algebra of infinitesimal symmetries of equations (7.1.4) is spanned by the following four linearly independent infinitesimal generators:

$$
\begin{aligned}
X_1 &= \tfrac{5\int\sigma(t)dt}{\sigma(t)}\tfrac{\partial}{\partial t} + x\tfrac{\partial}{\partial x}\\
X_2 &= u\tfrac{\partial}{\partial u}\\
X_3 &= \tfrac{1}{\sigma(t)}\tfrac{\partial}{\partial t}\\
X_4 &= \tfrac{\partial}{\partial x}.
\end{aligned} \tag{7.2.13}
$$

It is easy to verify that X_j, $j = 1, 2, 3, 4$ is closed under the Lie bracket.

However, if the two lie algebras are similar, i.e. they are connected to each other by a transformation from the symmetry group, then their corresponding invariant solutions are connected to each other by the same transformation. Therefore, it is sufficient to put all similar subalgebras into one class and select a representative from each class. The set of all these representatives is called an optimal system [137].

For brevity we omit the details, and just state the result that an optimal system of generators is

$$
\begin{aligned}
&(i)X_1 + \mu X_2\\
&(ii)X_2 + \lambda X_3 + V_4\\
&(iii)X_2 + \lambda X_3 - V_4\\
&(iv)X_2 + \gamma X_3\\
&(v)X_3 + X_4\\
&(vi)X_3 - X_4\\
&(vii)X_3\\
&(viii)X_4,
\end{aligned} \tag{7.2.14}
$$

where μ, λ and γ are arbitrary constants. Because the discrete symmetry $(x, t, u) \longmapsto (-x, t, u)$ will map (ii),(v) to (iii), (vi) respectively, and therefore, in the optimal system, we confine ourselves to remaining six essential vector fields of the optimal system, while neglecting the other two. It seems reasonable now to construct Lie ans$\ddot{a}$tze and to seek exact solutions of the nonlinear Eq. (7.1.4). With this in mind, consider its Lie symmetry generated by the basic operators in the optimal system. In Table (7.1), we have listed the Lie ansätze for all the essential fields comprising the optimal system and also the coefficient functions of the Eq. (7.1.4). Using the ansätze mentioned in Table (7.1), we can reduce the nonlinear equation (7.1.4) to ordinary differential equation (ODE). Having exact solutions of ODE and using the relevant ansätze one obtains the solutions of the nonlinear equation (7.1.4).

Now, we deal with the symmetry reductions and exact solutions to Eq. (7.1.4). We will consider the similarity reductions and group-invariant solutions. From an optimal system of group-invariant solutions to an Eq. (7.1.4), every other such solution to the equation can be derived.

Table 7.1: Similarity Variables, Similarity Functions and the Coefficient Functions of Modified Kawahara Equation

Essential Vector Fields	Similarity Variables	Similarity Forms	Coefficient Functions
$X_1 + \mu X_2$	$x(\int \sigma(t)dt)^{-\frac{1}{5}}$	$(\int \sigma(t)dt)^{\frac{\mu}{5}} F(\zeta)$	$\alpha(t) = \sigma(t)k_5(\int \sigma(t)dt)^{\frac{-(4+2\mu)}{5}}$, $\beta(t) = \sigma(t)k_6(\int \sigma(t)dt)^{\frac{-2}{5}}$
$X_2 + \lambda X_3 + X_4$	$-x + \frac{1}{\lambda}\int \sigma(t)dt$	$\exp(\frac{1}{\lambda}\int \sigma(t)dt)F(\zeta)$	$\alpha(t) = \sigma(t)k_5 \exp(\frac{-2}{\lambda}\int(\sigma(t)dt))$ $\beta(t) = k_6\sigma(t)$
$X_2 + \gamma X_3$	x	$\exp(\frac{1}{\gamma}\int \sigma(t)dt)F(\zeta)$	$\alpha(t) = \sigma(t)k_5 \exp(\frac{-2}{\gamma}\int \sigma(t)dt)$ $\beta(t) = k_6\sigma(t)$
$X_3 + X_4$	$-x + \int \sigma(t)dt$	$F(\zeta)$	$\alpha(t) = k_5\sigma(t), \beta(t) = k_6\sigma(t)$
X_3	x	$F(\zeta)$	$\alpha(t) = k_5\sigma(t), \beta(t) = k_6\sigma(t)$
X_4	t	$F(\zeta)$	$\alpha(t) = k_5\sigma(t), \beta(t) = k_6\sigma(t)$

where k_5 and k_6 are arbitrary constants and $\sigma(t)$ is an arbitrary function of t.

Vector Field $X_1 + \mu X_2$

Subsituting the forms of the similarity solution and the coefficient functions, corresponding to this vector field, Eq. (7.1.4) yields the following ODE:

$$5F''''' + 5k_6 F''' + 5k_5 F^2 F' - \zeta F' + \mu F = 0, \tag{7.2.15}$$

where prime ($'$) denotes the differentiation with respect to the variable ζ.

Vector Field $X_2 + \lambda X_3 + X_4$

For the case under consideration, the reduced ODE is as follows:

$$\lambda F''''' + k_6 \lambda F''' + k_5 \lambda F^2 F' - F' - F = 0. \tag{7.2.16}$$

Vector Field $X_2 + \gamma X_3$

Using the similarity variable, the forms of the similarity solution and the coefficient functions, Eq. (7.1.4) is reduced to the following ODE:

$$\gamma F''''' + k_6 \gamma F''' + k_5 \gamma F^2 F' + F = 0. \tag{7.2.17}$$

The reduced equations (7.2.15), (7.2.16) and (7.2.17) are higher-order nonlinear ODEs and the solutions of these equations are not found in terms of elementary or known functions of mathematical physics and so on. Therefore, we apply the power series method to furnish the solutions of equations (7.2.15), (7.2.16) and (7.2.17) in the next section.

Vector Field $X_3 + X_4$

For this vector field, Eq. (7.1.4) is transformed into the following fifth order nonlinear ODE:

$$F''''' + k_6 F''' + k_5 F^2 F' - F' = 0. \tag{7.2.18}$$

Solving this equation (7.2.18) and reverting back to the original variables, we obtain the following group-invariant solutions of the modified Kawahara equation (7.1.4):

Solutions in terms of tanh() function

$$\begin{aligned}(i)u(x,t) &= \pm\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\tanh\left(c_1+\left(\frac{1}{60}-\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{2\sqrt{k_5}}\\ &\mp\frac{\sqrt{6}\left(-\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}+2\,{k_6}^2+20\right)}{6\sqrt{k_5}\sqrt{{k_6}^2+10}}\\ (ii)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\tanh\left(c_1+\left(\frac{1}{60}+\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}}\\ &\mp\frac{1}{6}\,\frac{\sqrt{6}\left(\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}+2\,{k_6}^2+20\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}},\end{aligned} \tag{7.2.19}$$

where c_1 is an arbitrary constant.

Solutions in terms of tan() function

$$\begin{aligned}(i)u(x,t) &= \pm\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\tan\left(c_1+\left(\frac{1}{60}-\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{2\sqrt{k_5}}\\ &\mp\frac{\sqrt{6}\left(-\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}-2\,{k_6}^2-20\right)}{6\sqrt{k_5}\sqrt{{k_6}^2+10}}\\ (ii)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\tan\left(c_1+\left(\frac{1}{60}+\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}}\\ &\mp\frac{1}{6}\,\frac{\sqrt{6}\left(\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}-2\,{k_6}^2-20\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}},\end{aligned} \tag{7.2.20}$$

where c_1 is an arbitrary constant.

Solutions in terms of coth() function

$$\begin{aligned}(i)u(x,t) &= \pm\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\coth\left(c_1+\left(\frac{1}{60}-\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{2\sqrt{k_5}}\\ &\mp\frac{\sqrt{6}\left(-\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}+2\,{k_6}^2+20\right)}{6\sqrt{k_5}\sqrt{{k_6}^2+10}}\\ (ii)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\coth\left(c_1+\left(\frac{1}{60}+\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}}\\ &\mp\frac{1}{6}\,\frac{\sqrt{6}\left(\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}+2\,{k_6}^2+20\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}},\end{aligned} \tag{7.2.21}$$

where c_1 is an arbitrary constant.

Solutions in terms of cot()function

$$\begin{aligned}(i)u(x,t) &= \pm\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\cot\left(c_1+\left(\frac{1}{60}-\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{2\sqrt{k_5}}\\ &\mp\frac{\sqrt{6}\left(-\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}-2\,{k_6}^2-20\right)}{6\sqrt{k_5}\sqrt{{k_6}^2+10}}\\ (ii)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\cot\left(c_1+\left(\frac{1}{60}+\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}}\\ &\mp\frac{1}{6}\,\frac{\sqrt{6}\left(\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}-2\,{k_6}^2-20\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}},\end{aligned} \tag{7.2.22}$$

where c_1 is an arbitrary constant.

Solutions in terms of cosech() function

$$\begin{aligned}
(i)u(x,t) &= \pm \frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\operatorname{cosech}\left(c_1+\left(\frac{1}{60}-\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{2\sqrt{k_5}} \\
&\mp \frac{\sqrt{6}\left(-\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}-{k_6}^2-10\right)}{6\sqrt{k_5}\sqrt{{k_6}^2+10}} \\
(ii)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\operatorname{cosech}\left(c_1+\left(\frac{1}{60}+\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}} \\
&\mp\frac{1}{6}\,\frac{\sqrt{6}\left(\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}-{k_6}^2-10\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}},
\end{aligned} \tag{7.2.23}$$

where c_1 is an arbitrary constant.

Solutions in terms of cosec() function

$$\begin{aligned}
(i)u(x,t) &= \pm \frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\operatorname{cosec}\left(c_1+\left(\frac{1}{60}-\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{2\sqrt{k_5}} \\
&\mp \frac{\sqrt{6}\left(-\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}+{k_6}^2+10\right)}{6\sqrt{k_5}\sqrt{{k_6}^2+10}} \\
(ii)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\operatorname{cosec}\left(c_1+\left(\frac{1}{60}+\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}} \\
&\mp\frac{1}{6}\,\frac{\sqrt{6}\left(\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}+{k_6}^2+10\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}},
\end{aligned} \tag{7.2.24}$$

where c_1 is an arbitrary constant.

Solutions in terms of sech() function

$$\begin{aligned}
(i)u(x,t) &= \pm \frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\operatorname{sech}\left(c_1+\left(\frac{1}{60}-\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{2\sqrt{k_5}} \\
&\pm \frac{\sqrt{6}\left(-\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}-{k_6}^2-10\right)}{6\sqrt{k_5}\sqrt{{k_6}^2+10}} \\
(ii)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\operatorname{sech}\left(c_1+\left(\frac{1}{60}+\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}} \\
&\mp\frac{1}{6}\,\frac{\sqrt{6}\left(\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}-{k_6}^2-10\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}},
\end{aligned} \tag{7.2.25}$$

where c_1 is an arbitrary constant.

Solutions in terms of sec() function

$$\begin{aligned}
(i)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\sec\left(c_1+\left(\frac{1}{60}-\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}} \\
&\mp\frac{1}{6}\,\frac{\sqrt{6}\left(-\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}+{k_6}^2+10\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}} \\
(ii)u(x,t) &= \pm\frac{1}{2}\,\frac{\sqrt{6}\sqrt{{k_6}^2+10}\left(\sec\left(c_1+\left(\frac{1}{60}+\frac{1}{60}\iota\right)\sqrt{30}(15\,{k_6}^2+150)^{\frac{1}{4}}(-x+\int\sigma(t)dt)\right)\right)^2}{\sqrt{k_5}} \\
&\mp\frac{1}{6}\,\frac{\sqrt{6}\left(\frac{1}{5}\iota k_6\sqrt{15\,{k_6}^2+150}+{k_6}^2+10\right)}{\sqrt{k_5}\sqrt{{k_6}^2+10}},
\end{aligned} \tag{7.2.26}$$

where c_1 is an arbitrary constant.

Solutions in terms of Weierstrass P function () function

$$(i) u(x,t) = \pm \frac{k_6\sqrt{-10\,k_5}}{10k_5} \mp \frac{60\,{c_4}^2}{\sqrt{-10\,k_5}} \wp \left(c_3 + c_4 \left(-x + \int \sigma\left(t\right) dt\right), -\frac{1}{180}\,\frac{{k_6}^2+10}{{c_4}^4}, c_2 \right), \tag{7.2.27}$$

where c_2, c_3 and c_4 are arbitrary constants and $\wp$ denotes Weierstrass P function.

Vector Field X_3

The reduced ODE, for this case, is as follows:

$$F''''' + k_6 F''' + k_5 F^2 F' = 0. \tag{7.2.28}$$

We can easily obtain the solutions of equation (7.2.28) in terms of trigonometric functions, hyperbolic functions and elliptic functions with the help of any of mathematical software. But in this case, the similarity variable is x. Hence we get the solutions in terms of variable x only, which may not really correspond to a physically interesting case. Thus we omit the details here.

Vector Field X_4

Corresponding to this vector field, we get the trivial solution of Eq. (7.1.4) as
$u(x,t) = c$, where c is an arbitrary constant.

7.3 Power Series Solutions

In previous section, we have obtained the reduced ordinary differential equations by using Lie symmetry reductions. Now, we have examined the nonlinear ODEs (7.2.7), (7.2.15), (7.2.16) and (7.2.17). In general, we can not get the exact explicit solutions for these nonlinear ODEs by using the elementary functions and integrals. But the power series can be used to solve ODEs, including many complicated differential equations with nonconstant coefficients [7, 112, 113, 114]. Now we consider the power series solutions for these reduced equations.

Power Series Solutions to Eq. (7.2.7)

Firstly, we will seek a solution of Eq. (7.2.7) in a power series of the form

$$F(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n. \tag{7.3.1}$$

Substituting (7.3.1) into (7.2.7), we have

$$\begin{aligned}&120c_5 + \sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}\zeta^n + 6c_3k_7\\ &+k_7\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)c_{n+3}\zeta^n + k_6\sum_{n=1}^{\infty}(\sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k})\zeta^n\\ &-\zeta k_3\sum_{n=0}^{\infty}(n+1)c_{n+1}\zeta^n + k_1\sum_{n=0}^{\infty}c_n\zeta^n = 0.\end{aligned} \tag{7.3.2}$$

From Eq (7.3.2), on comparing coefficients of ζ^n, for $n = 0$, we obtain,

$$c_5 = \frac{1}{120}(-6c_3k_7 - c_0k_1). \tag{7.3.3}$$

In general, for $n \geq 1$, we have

$$\begin{aligned}c_{n+5} = \tfrac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}\Big(&-k_7(n+1)(n+2)(n+3)c_{n+3} + k_3nc_n\\ &-k_1c_n - k_6(\sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k})\Big).\end{aligned} \tag{7.3.4}$$

From (7.3.3) and (7.3.4), we can get all the coefficients c_n, $n \geq 5$ of the power series (7.3.1), as following,

$$\begin{aligned}c_6 &= \tfrac{1}{720}(-24k_7c_4 + c_1k_3 - c_1k_1 - k_6(2c_0c_2 + c_1^2))\\ c_7 &= \tfrac{1}{2520}(-60k_7c_5 + 2c_2k_3 - c_2k_1 - 3k_6(c_0c_3 + c_1c_2)),\end{aligned} \tag{7.3.5}$$

and so on.

Thus, for arbitrary chosen constant numbers c_0, c_1, c_2, c_3 and c_4, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (7.3.3) and (7.3.4) in a unique manner. This implies that for Eq. (7.2.7), there exists a power series solution (7.3.1) with the coefficients given by (7.3.3) and (7.3.4). Furthermore, it is easy to prove the convergence of the power series (7.3.1) with the coefficients given by (7.3.3) and (7.3.4) [112, 113, 114]. Therefore, this power series solution (7.3.1) to Eq. (7.2.7) is an power seriessolution.

Now we will show the convergence of the power series solution (7.3.1) of Eq. (7.2.7). From (7.3.4), we have

$$|c_{n+5}| \leq M(|c_{n+3}| + |c_n| + \sum_{k=0}^{n} |c_k||c_{n+1-k}|), n = 1, 2, 3... \tag{7.3.6}$$

where $M = \max\{|k_7|, |k_3 + k_1|, |k_6|\}$.

If we define a power series

$$\mu = P(\zeta) = \sum_{n=0}^{\infty} p_n \zeta^n, \tag{7.3.7}$$

by $p_0 = |c_0|,\ p_1 = |c_1|,\ p_2 = |c_2|,\ p_3 = |c_3|,\ p_4 = |c_4|,\ p_5 = |c_5|$ and

$p_{n+5} = M(p_{n+3} + p_n + \sum_{k=0}^{n} p_k p_{n+1-k}),\ n = 1, 2, 3...$

then it is easily seen that

$|c_n| \leq p_n, n = 0, 1, 2, 3...$

In other words, the series $\mu = P(\zeta) = \sum_{n=0}^{\infty} p_n \zeta^n$ is a majorant series of (7.3.1). Next, we show that this series $\mu = P(\zeta)$ has a positive radius of convergence. Indeed, note that by formal calculation, we have

$$\begin{aligned}
P(\zeta) &= p_0 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + \sum_{n=1}^{\infty} p_{n+5}\zeta^{n+5} \\
&= p_0 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + M(\sum_{n=1}^{\infty} p_{n+3}\zeta^{n+5} + \sum_{n=1}^{\infty} p_n\zeta^{n+5} \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{n} p_k p_{n+1-k}\zeta^{n+5}) \\
&= p_0 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + M\Big(- P^2(\zeta) + p_0^2 + 2p_0p_1\zeta + (p_1^2 + 2p_0p_2)\zeta^2 \\
&\quad +(2p_0p_3 + 2p_1p_2)\zeta^3 + (2p_0p_4 + p_2^2 + 2p_1p_3)\zeta^4 + (2p_0p_5 + 2p_1p_4 + 2p_3p_2)\zeta^5 \\
&\quad +(p_4 + p_1 + 2p_0p_6 + 2p_1p_5 + 2p_2p_4 + p_1^2)\zeta^6\Big).
\end{aligned} \tag{7.3.8}$$

Consider now the implicit functional equation

$$\begin{aligned}
F(\zeta, \mu) &= \mu - p_0 - p_1\zeta - p_2\zeta^2 - p_3\zeta^3 - p_4\zeta^4 - p_5\zeta^5 - M\Big(- \mu^2 + p_0^2 + 2p_0p_1\zeta \\
&\quad +(p_1^2 + 2p_0p_2)\zeta^2 + (2p_0p_3 + 2p_1p_2)\zeta^3 + (2p_0p_4 + p_2^2 + 2p_1p_3)\zeta^4 \\
&\quad +(2p_0p_5 + 2p_1p_4 + 2p_3p_2)\zeta^5 + (p_4 + p_1 + 2p_0p_6 + 2p_1p_5 + 2p_2p_4 + p_1^2)\zeta^6\Big) = 0.
\end{aligned} \tag{7.3.9}$$

Since F is analytic in the (ζ, μ)-plane and $F(0, p_0) = 0$, $F'_\mu(0, p_0) = 1+2p_0M \neq 0$, by the implicit function theorem [57, 150], we see that $\mu = P(\zeta)$ is analytic in a neighborhood of the point $(0, p_0)$ of the plane and with a positive radius. This implies that the power series (7.3.1) converges in a neighborhood of the point $(0, p_0)$ of the plane. This completes the proof.

Hence, the power series solution of Eq. (7.1.3) can be written as following:

$$u(x,t) = \frac{(c_0+c_1\zeta+c_2\zeta^2+c_3\zeta^3+c_4\zeta^4+\sum_{n=1}^{\infty} c_{n+4}\zeta^{n+4})(5k_3\int\sigma(t)dt+k_5)^{\frac{k_1}{5k_3}}k_1-k_2}{k_1} \quad (7.3.10)$$

$$\begin{aligned} u(x,t) = & \frac{(c_0+c_1\zeta+c_2\zeta^2+c_3\zeta^3+c_4\zeta^4+(\frac{1}{120}(-6c_3k_7-c_0k_1-k_6c_0c_1))\zeta^5)(5k_3\int\sigma(t)dt+k_5)^{\frac{k_1}{5k_3}}k_1-k_2}{k_1} \\ & +\frac{k_1(5k_3\int\sigma(t)dt+k_5)^{\frac{k_1}{5k_3}}}{k_1}\Big(\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}(-k_7(n+1)(n+2) \\ & (n+3)c_{n+3}+k_3nc_n-k_1c_n-k_6(\sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k}))\zeta^{n+5}\Big), \end{aligned} \quad (7.3.11)$$

where ζ is given by

$$\zeta = \frac{-xk_3k_1^2+xk_3^2k_1-k_4k_1^2+k_3k_1k_4+k_2k_3(5k_3\int\sigma(t)dt+k_5)^{-\frac{(k_1-k_3)}{5k_3}}}{(k_1-k_3)k_1k_3(5k_3\int\sigma(t)dt+k_5)^{\frac{1}{5}}}, \quad (7.3.12)$$

and c_0, c_1, c_2, c_3 and c_4 are arbitrary constants, the other coefficients c_n, $n \geq 5$ can be determined successively from (7.3.3) and (7.3.4).

Power Series Solutions to Eq. (7.2.16)

Now, we seek a solution of Eq. (7.2.16) in a power series of the form (7.3.1). Substituting (7.3.1) into (7.2.16), and comparing coefficients, we obtain

$$\begin{aligned} c_{n+5} = & \frac{1}{5(n+1)(n+2)(n+3)(n+4)(n+5)}\Big(-5k_6(n+1)(n+2)(n+3)c_{n+3}+(n-\mu)c_n \\ & -5k_5(\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)c_jc_{k-j}c_{n+1-k})\Big), \\ & n = 0,1,2,3,... \end{aligned} \quad (7.3.13)$$

In view of (7.3.13), we can get all the coefficients $c_n, n \geq 1$ of the power series (7.3.1),

$$\begin{aligned} c_5 &= \tfrac{1}{600}(-\mu c_0 - 5k_5c_1c_0^2 - 30k_6c_3) \\ c_6 &= \tfrac{1}{3600}(-120k_6c_4 - \mu c_1 + c_1 - 10k_5(c_0c_1^2 + c_2c_0^2)) \\ c_7 &= \tfrac{1}{12600}(-300k_6c_4 + (-\mu+2)c_2 - 5k_5(3c_0^2c_3 + 6c_0c_1c_2 + c_1^3)), \end{aligned} \quad (7.3.14)$$

and so on. Thus, for arbitrary chosen constant numbers c_0, c_1, c_2, c_3 and c_4, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (7.3.13) in a unique manner. This implies that for Eq. (7.2.15), there exists a power series solution (7.3.1) with the coefficients given by (7.3.13).

Accordingly, the exact traveling wave solution to Eq. (7.1.4) is given by

$$\begin{aligned} u(x,t) = (\textstyle\int \sigma(t)dt)^{\frac{\mu}{5}} \Big(& c_0 + c_1(x \textstyle\int(\sigma(t))dt)^{\frac{-1}{5}} + c_2(x \textstyle\int(\sigma(t))dt)^{\frac{-2}{5}} + c_3(x \textstyle\int(\sigma(t))dt)^{\frac{-3}{5}} \\ & + c_4(x \textstyle\int(\sigma(t))dt)^{\frac{-4}{5}} + \tfrac{1}{600}(-\mu c_0 - 5k_5 c_1 c_0^2 - 30k_6 c_3)(x \textstyle\int(\sigma(t))dt)^{-1} \\ & + \sum_{n=1}^{\infty} \tfrac{1}{5(n+1)(n+2)(n+3)(n+4)(n+5)} (-5k_6(n+1)(n+2)(n+3)c_{n+3} + (n-\mu)c_n \\ & - 5k_5 (\sum_{k=0}^{n} \sum_{j=0}^{k} (n+1-k)c_j c_{k-j} c_{n+1-k}))(x \textstyle\int(\sigma(t))dt)^{\frac{-(n+5)}{5}} \Big), \end{aligned} \tag{7.3.15}$$

where c_0, c_1, c_2, c_3 and c_4 are arbitrary constants, the other coefficients $c_{n+5}(n = 0, 1, 2, ...)$ are given by (7.3.13) successively.

Power Series Solutions to Eq. (7.2.16)

Similarly, we seek a solution of Eq. (7.2.16) in a power series of the form (7.3.1). Substituting it into (7.2.16), and comparing coefficients, we obtain

$$\begin{aligned} c_{n+5} = \tfrac{1}{\lambda(n+1)(n+2)(n+3)(n+4)(n+5)} \Big(& -k_6(n+1)(n+2)(n+3)c_{n+3}\lambda + c_n + (n+1)c_{n+1} \\ & - k_5\lambda (\sum_{k=0}^{n} \sum_{j=0}^{k} (n+1-k)c_j c_{k-j} c_{n+1-k}) \Big), \end{aligned} \tag{7.3.16}$$

$n = 0, 1, 2, ...$

In view of (7.3.16), the power series solution (7.3.1) of Eq. (7.2.16) can be written as follows:

$$\begin{aligned} F(\zeta) = & c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4 + \Big(\tfrac{-6c_3k_6\lambda - k_5\lambda c_0^2 c_1 + c_0 + c_1}{120\lambda}\Big)\zeta^5 \\ & + \sum_{n=1}^{\infty} \tfrac{1}{\lambda(n+1)(n+2)(n+3)(n+4)(n+5)} \Big(-k_6\lambda(n+1)(n+2)(n+3)c_{n+3} + c_n + (n+1)c_{n+1} \\ & - k_5\lambda (\sum_{k=0}^{n} \sum_{j=0}^{k} (n+1-k)c_j c_{k-j} c_{n+1-k}) \Big)\zeta^{(n+5)}. \end{aligned} \tag{7.3.17}$$

Therefore, the power series solution of Eq. (7.1.4) is given by

$$
\begin{aligned}
u(x,t) = \exp(\textstyle\int \frac{\sigma(t)}{\lambda}dt)\Big(&c_0 + c_1(-x + \tfrac{1}{\lambda}\textstyle\int(\sigma(t))dt) + c_2(-x + \tfrac{1}{\lambda}\textstyle\int(\sigma(t))dt)^2 \\
&+c_3(-x + \tfrac{1}{\lambda}\textstyle\int(\sigma(t))dt)^3 + c_4(-x + \tfrac{1}{\lambda}\textstyle\int(\sigma(t))dt)^4 + \left(\tfrac{-6c_3k_6\lambda - k_5\lambda c_0^2 c_1 + c_0 + c_1}{120\lambda}\right) \\
&(-x + \tfrac{1}{\lambda}\textstyle\int(\sigma(t))dt)^5 + \sum\limits_{n=1}^{\infty} \tfrac{1}{\lambda(n+1)(n+2)(n+3)(n+4)(n+5)}(-k_6\lambda(n+1)(n+2)(n+3)c_{n+3} \\
&+c_n + (n+1)c_{n+1} - k_5\lambda(\sum\limits_{k=0}^{n}\sum\limits_{j=0}^{k}(n+1-k)c_jc_{k-j}c_{n+1-k}))(-x + \tfrac{1}{\lambda}\textstyle\int(\sigma(t))dt)^{n+5}\Big).
\end{aligned}
\tag{7.3.18}
$$

where c_0, c_1, c_2, c_3 and c_4 are arbitrary constants.

Power Series Solutions to Eq. (7.2.17)

Similarly, we have derived the power series solution of Eq. (7.2.17) and hence, analytic solution of Eq. (7.1.4) is as follows:

$$
\begin{aligned}
u(x,t) = \exp(\textstyle\int \frac{\sigma(t)}{\gamma}dt)\Big(&c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \left(\tfrac{-6c_3k_6\gamma - k_5\gamma c_0^2c_1 - c_0}{120\gamma}\right)x^5 \\
&+ \sum\limits_{n=1}^{\infty} \tfrac{1}{\gamma(n+1)(n+2)(n+3)(n+4)(n+5)}(-k_6\gamma(n+1)(n+2)(n+3)c_{n+3} - c_n \\
&-k_5\gamma(\sum\limits_{k=0}^{n}\sum\limits_{j=0}^{k}(n+1-k)c_jc_{k-j}c_{n+1-k}))x^{n+5}\Big),
\end{aligned}
\tag{7.3.19}
$$

where c_0, c_1, c_2, c_3 and c_4 are arbitrary constants.

7.4 Application of Generalized $\left(\frac{G'}{G}\right)$ Method to VCKE and VCMKE

In this section, we have applied the generalized $\left(\frac{G'}{G}\right)$-expansion method to VCKE (7.1.3) and VCMKE (7.1.4). As a result, hyperbolic function solutions, trigonometric function solutions and rational solutions with parameters are obtained.
The solution of Eq. (7.1.3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$
u = \alpha_0(t) + \sum_{i=1}^{m}\alpha_i(t)\left(\frac{G'}{G}\right)^i, \tag{7.4.1}
$$

where $G = G(\theta)$ satisfies following equation

$$G''(\theta) + \lambda G'(\theta) + \mu G(\theta) = 0, \tag{7.4.2}$$

where $\theta = p(t)x + q(t)$, $p(t)$ and $q(t)$ are functions to be determined.

By balancing the highest order nonlinear term and the highest order partial derivative term of u in (7.1.3), we get $m = 4$. In order to search for explicit solutions, we suppose that (7.1.3) has the following formal solution:

$$u = \alpha_0(t) + \alpha_1(t)\left(\frac{G'}{G}\right) + \alpha_2(t)\left(\frac{G'}{G}\right)^2 + \alpha_3(t)\left(\frac{G'}{G}\right)^3 + \alpha_4(t)\left(\frac{G'}{G}\right)^4, \tag{7.4.3}$$

Substituting (7.4.3) into (7.1.3) and using (7.4.2), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq. (7.1.3) is converted into polynomial in $x^j\left(\frac{G'}{G}\right)$, $(j = 0, 1)$. Setting each coefficient of this polynomial to zero, we derive a set of overdetermined differential equations for $\alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), p(t)$ and $q(t)$. Solving this set of equations, we have obtained the following results:

Case 1:

$$\begin{aligned}
&p(t) = c_1,\ \ q(t) = c_3 + c_4 \int \sigma(t)\,dt,\ \ \beta(t) = -13\,{c_1}^2\lambda^2\sigma(t) + 52\,{c_1}^2\sigma(t)\,\mu,\\
&\alpha(t) = c_2\sigma(t),\ \ \alpha_4(t) = \tfrac{-1680{c_1}^4}{c_2},\ \ \alpha_3(t) = \tfrac{-3360{c_1}^4\lambda}{c_2},\ \ \alpha_1(t) = \tfrac{-3360{c_1}^4\lambda\mu}{c_2},\\
&\alpha_2(t) = -\tfrac{280}{13}\,\tfrac{{c_1}^2\left(78\,{c_1}^2\lambda^2 + 156\,{c_1}^2\mu\right)}{c_2},\ \ \alpha_0(t) = \left(-\tfrac{c_4 - 36\,{c_1}^5\lambda^4 + 288\,{c_1}^5\lambda^2\mu + 1104\,{c_1}^5\mu^2}{c_2 c_1}\right),\\
&\sigma(t) = \sigma(t),
\end{aligned} \tag{7.4.4}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Substituting the general solutions of (7.4.2) into (7.4.3) and using (7.4.4), we have three types of exact solutions of (7.1.3) as follows:

When $\lambda^2 - 4\mu > 0$, we have obtained hyperbolic function solution in the form

$$u(x,t) = -\frac{c_4 - 36\,c_1{}^5\lambda^4 + 288\,c_1{}^5\lambda^2\mu + 1104\,c_1{}^5\mu^2}{c_2 c_1} - \frac{1680\,c_1{}^4\lambda\,\mu}{c_2}\left(\frac{\sqrt{\lambda^2-4\,\mu}(a_1\sinh(\Lambda_1)+a_2\cosh(\Lambda_1))}{a_2\sinh(\Lambda_1)+a_1\cosh(\Lambda_1)} - \lambda\right)$$
$$-\frac{420\left(\lambda^2+2\,\mu\right)c_1{}^4}{c_2}\left(\frac{\sqrt{\lambda^2-4\,\mu}(a_1\sinh(\Lambda_1)+a_2\cosh(\Lambda_1))}{a_2\sinh(\Lambda_1)+a_1\cosh(\Lambda_1)} - \lambda\right)^2$$
$$-\frac{420\,c_1{}^4\lambda}{c_2}\left(\frac{\sqrt{\lambda^2-4\,\mu}(a_1\sinh(\Lambda_1)+a_2\cosh(\Lambda_1))}{a_2\sinh(\Lambda_1)+a_1\cosh(\Lambda_1)} - \lambda\right)^3$$
$$-\frac{105\,c_1{}^4}{c_2}\left(\frac{\sqrt{\lambda^2-4\,\mu}(a_1\sinh(\Lambda_1)+a_2\cosh(\Lambda_1))}{a_2\sinh(\Lambda_1)+a_1\cosh(\Lambda_1)} - \lambda\right)^4, \tag{7.4.5}$$

where $\Lambda_1 = \frac{\sqrt{\lambda^2-4\,\mu}}{2}\left(c_1x + c_3 + c_4\int\sigma\left(t\right)dt\right).$

When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$u(x,t) = -\frac{c_4 - 36\,c_1{}^5\lambda^4 + 288\,c_1{}^5\lambda^2\mu + 1104\,c_1{}^5\mu^2}{c_1 c_2} - \frac{1680\,c_1{}^4\lambda\,\mu}{c_2}\left(\frac{\sqrt{-\lambda^2+4\,\mu}(-a_2\sin(\Lambda_2)+a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2)+a_2\cos(\Lambda_2)} - \lambda\right)$$
$$-\frac{420\left(\lambda^2+2\,\mu\right)c_1{}^4}{c_2}\left(\frac{\sqrt{-\lambda^2+4\,\mu}(-a_2\sin(\Lambda_2)+a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2)+a_2\cos(\Lambda_2)} - \lambda\right)^2$$
$$-\frac{420\,c_1{}^4\lambda}{c_2}\left(\frac{\sqrt{-\lambda^2+4\,\mu}(-a_2\sin(\Lambda_2)+a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2)+a_2\cos(\Lambda_2)} - \lambda\right)^3$$
$$-\frac{105\,c_1{}^4}{c_2}\left(\frac{\sqrt{-\lambda^2+4\,\mu}(-a_2\sin(\Lambda_2)+a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2)+a_2\cos(\Lambda_2)} - \lambda\right)^4, \tag{7.4.6}$$

where $\Lambda_2 = \frac{\sqrt{4\,\mu-\lambda^2}}{2}\left(c_1x + c_3 + c_4\int\sigma\left(t\right)dt\right).$

When $\lambda^2 - 4\mu = 0$, we get rational solution as follows:

$$u(x,t) = \frac{1}{507}\frac{\left(-507c_4 - 53235c_1{}^5\lambda^4\right)}{c_1c_2} - \frac{840\,c_1{}^4}{c_2}\lambda^3\left(\frac{a_4}{a_3+a_4\left(c_1x+c_3+c_4\int\sigma(t)dt\right)} - \frac{\lambda}{2}\right)$$
$$-\frac{2520\,c_1{}^4}{c_2}\lambda^2\left(\frac{a_4}{a_3+a_4\left(c_1x+c_3+c_4\int\sigma(t)dt\right)} - \frac{\lambda}{2}\right)^2 - \frac{3360\,c_1{}^4\lambda}{c_2}\left(\frac{a_4}{a_3+a_4\left(c_1x+c_3+c_4\int\sigma(t)dt\right)} - \frac{\lambda}{2}\right)^3$$
$$-\frac{1680\,c_1{}^4}{c_2}\left(\frac{a_4}{a_3+a_4\left(c_1x+c_3+c_4\int\sigma(t)dt\right)} - \frac{\lambda}{2}\right)^4, \tag{7.4.7}$$

where a_1, a_2, a_3 and a_4 are arbitrary constants.

Case 2:

$$p\,(t) = c_1,\;\; q\,(t) = c_3 + c_4 \int \sigma\,(t)\,dt,\;\; \alpha\,(t) = c_2 \sigma\,(t)\,,\;\; \sigma(t) = \sigma(t),$$

$$\alpha_4\,(t) = \frac{-1680 {c_1}^4}{c_2},\;\; \alpha_3\,(t) = \frac{-3360 {c_1}^4 \lambda}{c_2},\;\; \beta\,(t) = 13\,\left(-\tfrac{1}{2} + \tfrac{3}{62}\, i\sqrt{31}\right) {c_1}^2 \left(4\,\mu - \lambda^2\right) \sigma\,(t)\,,$$

$$\alpha_2\,(t) = -\frac{420}{31}\,\frac{{c_1}^4 \left(124\,\mu + 155\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2}$$

$$\alpha_1\,(t) = -\frac{280}{13}\,\frac{{c_1}^2 \lambda \left(104\,\sigma(t){c_1}^2\mu + 13\,\sigma(t){c_1}^2\lambda^2 + 13\left(\frac{-1}{2} + \frac{3}{62}\,i\sqrt{31}\right){c_1}^2\left(4\,\mu - \lambda^2\right)\sigma(t)\right)}{c_2 \sigma(t)},$$

$$\alpha_0(t) = -\frac{1}{62}\,\frac{62\,c_4 + 465\,{c_1}^5\lambda^4 + 22320\,{c_1}^5\lambda^2\mu + 7440\,{c_1}^5\mu^2 + 2736\,i{c_1}^5\mu^2\sqrt{31} - 528\,i{c_1}^5\mu\,\sqrt{31}\lambda^2 - 39\,i{c_1}^5\sqrt{31}\lambda^4}{c_1 c_2}, \tag{7.4.8}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Consequently, we have the following three types of exact solutions of (7.1.3):

If $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution in the form

$$\begin{aligned} u(x,t) = &-\frac{1}{62}\,\frac{62\,c_3 + 465\,{c_1}^5\lambda^4 + 22320\,{c_1}^5\lambda^2\mu + 7440\,{c_1}^5\mu^2 + 2736\,i{c_1}^5\mu^2\sqrt{31} - 528\,i{c_1}^5\mu\,\sqrt{31}\lambda^2 - 39\,i{c_1}^5\sqrt{31}\lambda^4}{c_1 c_2} \\ &- \frac{210}{31}\,\frac{{c_1}^4\lambda\left(124\,\mu + 31\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2}\left(\frac{\sqrt{\lambda^2 - 4\,\mu}\,(a_1\sinh(\Lambda_1) + a_2\cosh(\Lambda_1))}{a_2\sinh(\Lambda_1) + a_1\cosh(\Lambda_1)} - \lambda\right) \\ &- \frac{105}{31}\,\frac{{c_1}^4\left(124\,\mu + 155\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2}\left(\frac{\sqrt{\lambda^2 - 4\,\mu}\,(a_1\sinh(\Lambda_1) + a_2\cosh(\Lambda_1))}{a_2\sinh(\Lambda_1) + a_1\cosh(\Lambda_1)} - \lambda\right)^2 \\ &- \frac{420 {c_1}^4\lambda}{c_2}\left(\frac{\sqrt{\lambda^2 - 4\,\mu}\,(a_1\sinh(\Lambda_1) + a_2\cosh(\Lambda_1))}{a_2\sinh(\Lambda_1) + a_1\cosh(\Lambda_1)} - \lambda\right)^3 \\ &- \frac{105 {c_1}^4}{c_2}\left(\frac{\sqrt{\lambda^2 - 4\,\mu}\,(a_1\sinh(\Lambda_1) + a_2\cosh(\Lambda_1))}{a_2\sinh(\Lambda_1) + a_1\cosh(\Lambda_1)} - \lambda\right)^4, \end{aligned} \tag{7.4.9}$$

where $\Lambda_1 = \frac{\sqrt{\lambda^2 - 4\,\mu}}{2}\,\left(c_1 x + c_3 + c_4 \int \sigma\,(t)\,dt\right).$

If $\lambda^2 - 4\mu < 0$, we have trigonometric function solution in the following form:

$$\begin{aligned} u(x,t) = &-\frac{1}{62}\,\frac{62\,c_4 + 465\,{c_1}^5\lambda^4 + 22320\,{c_1}^5\lambda^2\mu + 7440\,{c_1}^5\mu^2 + 2736\,i{c_1}^5\mu^2\sqrt{31} - 528\,i{c_1}^5\mu\,\sqrt{31}\lambda^2 - 39\,i{c_1}^5\sqrt{31}\lambda^4}{c_1 c_2} \\ &- \frac{210}{31}\,\frac{{c_1}^4\lambda\left(124\,\mu + 31\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2}\left(\frac{\sqrt{4\,\mu - \lambda^2}\,(-a_2\sin(\Lambda_2) + a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2) + a_2\cos(\Lambda_2)} - \lambda\right) \\ &- \frac{105}{31}\,\frac{{c_1}^4\left(124\,\mu + 155\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2}\left(\frac{\sqrt{4\,\mu - \lambda^2}\,(-a_2\sin(\Lambda_2) + a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2) + a_2\cos(\Lambda_2)} - \lambda\right)^2 \\ &- \frac{420\,{c_1}^4\lambda}{c_2}\left(\frac{\sqrt{4\,\mu - \lambda^2}\,(-a_2\sin(\Lambda_2) + a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2) + a_2\cos(\Lambda_2)} - \lambda\right)^3 \\ &- \frac{105\,{c_1}^4}{c_2}\left(\frac{\sqrt{4\,\mu - \lambda^2}\,(-a_2\sin(\Lambda_2) + a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2) + a_2\cos(\Lambda_2)} - \lambda\right)^4, \end{aligned} \tag{7.4.10}$$

where $\Lambda_2 = \frac{\sqrt{4\,\mu - \lambda^2}}{2}\ \left(c_1 x + c_3 + c_4 \int \sigma\left(t\right) dt\right).$

If $\lambda^2 - 4\mu = 0$, we get rational solution as follows:

$$u(x,t) = -\frac{c_4 + 105\,{c_1}^5 \lambda^4}{c_1 c_2} - \frac{840\,{c_1}^4 \lambda^3}{c_2} \left(\frac{a_4}{a_3 + a_4\left(c_1 x + c_3 + c_4 \int \sigma(t) dt\right)} - \frac{\lambda}{2} \right)$$
$$- \frac{2520\,{c_1}^4 \lambda^2}{c_2} \left(\frac{a_4}{a_3 + a_4\left(c_1 x + c_3 + c_4 \int \sigma(t) dt\right)} - \frac{\lambda}{2} \right)^2 - \frac{3360\,{c_1}^4 \lambda}{c_2} \left(\frac{a_4}{a_3 + a_4\left(c_1 x + c_3 + c_4 \int \sigma(t) dt\right)} - \frac{\lambda}{2} \right)^3$$
$$- \frac{1680\,{c_1}^4}{c_2} \left(\frac{a_4}{a_3 + a_4\left(c_1 x + c_3 + c_4 \int \sigma(t) dt\right)} - \frac{\lambda}{2} \right)^4, \tag{7.4.11}$$

where a_1, a_2, a_3 and a_4 are arbitrary constants.

Case 3:

$$p\left(t\right) = c_1,\ \ q\left(t\right) = c_3 + c_4 \int \sigma\left(t\right) dt,\ \ \beta\left(t\right) = 13\ \left(-\tfrac{1}{2} - \tfrac{3}{62}\, i\sqrt{31}\right) {c_1}^2 \left(4\,\mu - \lambda^2\right) \sigma\left(t\right)$$

$$\alpha\left(t\right) = c_2 \sigma\left(t\right),\ \ \alpha_4\left(t\right) = -\frac{1680 {c_1}^4}{c_2},\ \ \alpha_3\left(t\right) = -\frac{3360 {c_1}^4 \lambda}{c_2},\ \ \sigma(t) = \sigma(t),$$

$$\alpha_2\left(t\right) = \frac{420}{31}\ \frac{{c_1}^4\left(-124\,\mu - 155\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2},\ \ \alpha_1\left(t\right) = \frac{420}{31}\ \frac{{c_1}^4 \lambda\left(-124\,\mu - 31\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2},$$

$$\alpha_0\left(t\right) = \frac{1}{62}\ \frac{-62 c_4 - 465\,{c_1}^5 \lambda^4 - 22320\,{c_1}^5 \lambda^2 \mu - 7440\,{c_1}^5 \mu^2 + 2736\,i{c_1}^5 \mu^2 \sqrt{31} - 528\,i{c_1}^5 \mu\,\sqrt{31}\lambda^2 - 39\,i{c_1}^5 \sqrt{31}\lambda^4}{c_1 c_2}, \tag{7.4.12}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

When $\lambda^2 - 4\mu > 0$, we derived hyperbolic function solution of Eq. (7.1.3) in the following form

$$u(x,t) = \frac{1}{62}\ \frac{-62 c_4 - 465\,{c_1}^5 \lambda^4 - 22320\,{c_1}^5 \lambda^2 \mu - 7440\,{c_1}^5 \mu^2 + 2736\,i{c_1}^5 \mu^2 \sqrt{31} - 528\,i{c_1}^5 \mu\,\sqrt{31}\lambda^2 - 39\,i{c_1}^5 \sqrt{31}\lambda^4}{c_1 c_2}$$
$$+ \frac{210}{31}\ \frac{{c_1}^4 \lambda\left(-124\,\mu - 31\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2} \left(\frac{\sqrt{\lambda^2 - 4\,\mu}\left(a_1 \sinh(\Lambda_1) + a_2 \cosh(\Lambda_1)\right)}{a_2 \sinh(\Lambda_1) + a_1 \cosh(\Lambda_1)} - \lambda \right)$$
$$+ \frac{105}{31}\ \frac{{c_1}^4\left(-124\,\mu - 155\,\lambda^2 + 4\,i\sqrt{31}\mu - i\sqrt{31}\lambda^2\right)}{c_2} \left(\frac{\sqrt{\lambda^2 - 4\,\mu}\left(a_1 \sinh(\Lambda_1) + a_2 \cosh(\Lambda_1)\right)}{a_2 \sinh(\Lambda_1) + a_1 \cosh(\Lambda_1)} - \lambda \right)^2$$
$$- \frac{420 {c_1}^4 \lambda}{c_2} \left(\frac{\sqrt{\lambda^2 - 4\,\mu}\left(a_1 \sinh(\Lambda_1) + a_2 \cosh(\Lambda_1)\right)}{a_2 \sinh(\Lambda_1) + a_1 \cosh(\Lambda_1)} - \lambda \right)^3$$
$$- \frac{105 {c_1}^4}{c_2} \left(\frac{\sqrt{\lambda^2 - 4\,\mu}\left(a_1 \sinh(\Lambda_1) + a_2 \cosh(\Lambda_1)\right)}{a_2 \sinh(\Lambda_1) + a_1 \cosh(\Lambda_1)} - \lambda \right)^4, \tag{7.4.13}$$

where $\Lambda_1 = \frac{\sqrt{\lambda^2-4\,\mu}}{2}\ \left(c_1 x + c_3 + c_4 \int \sigma\left(t\right) dt\right).$

When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution of Eq. (7.1.3) as follows:

$$\begin{aligned}
u(x,t) = {} & \frac{1}{62}\ \frac{-62c_4-465\,{c_1}^5\lambda^4-22320\,{c_1}^5\lambda^2\mu-7440\,{c_1}^5\mu^2+2736\,i{c_1}^5\mu^2\sqrt{31}-528\,i{c_1}^5\mu\,\sqrt{31}\lambda^2-39\,i{c_1}^5\sqrt{31}\lambda^4}{c_1c_2} \\
& +\frac{210}{31}\ \frac{{c_1}^4\lambda\left(-124\,\mu-31\,\lambda^2+4\,i\sqrt{31}\mu-i\sqrt{31}\lambda^2\right)}{c_2}\ \left(\frac{\sqrt{4\,\mu-\lambda^2}(-a_2\sin(\Lambda_2)+a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2)+a_2\cos(\Lambda_2)}-\ \lambda\right) \\
& +\frac{105}{31}\ \frac{{c_1}^4\left(-124\,\mu-155\,\lambda^2+4\,i\sqrt{31}\mu-i\sqrt{31}\lambda^2\right)}{c_2}\ \left(\frac{\sqrt{4\,\mu-\lambda^2}(-a_2\sin(\Lambda_2)+a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2)+a_2\cos(\Lambda_2)}-\ \lambda\right)^2 \\
& -\frac{420{c_1}^4\lambda}{c_2}\ \left(\frac{\sqrt{4\,\mu-\lambda^2}(-a_2\sin(\Lambda_2)+a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2)+a_2\cos(\Lambda_2)}-\ \lambda\right)^3 \\
& -\frac{105{c_1}^4}{c_2}\ \left(\frac{\sqrt{4\,\mu-\lambda^2}(-a_2\sin(\Lambda_2)+a_1\cos(\Lambda_2))}{a_1\sin(\Lambda_2)+a_2\cos(\Lambda_2)}-\ \lambda\right)^4,
\end{aligned} \tag{7.4.14}$$

where $\Lambda_2 = \frac{\sqrt{4\,\mu-\lambda^2}}{2}\ \left(c_1 x + c_3 + c_4 \int \sigma\left(t\right) dt\right).$

When $\lambda^2 - 4\mu = 0$, we get rational solution of Eq. (7.1.3) as follows:

$$\begin{aligned}
u(x,t) = {} & -\frac{c_4+105\,{c_1}^5\lambda^4}{c_1c_2}-\frac{840\,{c_1}^4\lambda^3}{c_2}\ \left(\frac{a_4}{a_3+a_4\left(c_1x+c_3+c_4\int\sigma(t)dt\right)}-\frac{\lambda}{2}\right) \\
& -\frac{2520\,{c_1}^4\lambda^2}{c_2}\ \left(\frac{a_4}{a_3+a_4\left(c_1x+c_3+c_4\int\sigma(t)dt\right)}-\frac{\lambda}{2}\right)^2-\frac{3360\,{c_1}^4\lambda}{c_2}\ \left(\frac{a_4}{a_3+a_4\left(c_1x+c_3+c_4\int\sigma(t)dt\right)}-\frac{\lambda}{2}\right)^3 \\
& -\frac{1680\,{c_1}^4}{c_2}\ \left(\frac{a_4}{a_3+a_4\left(c_1x+c_3+c_4\int\sigma(t)dt\right)}-\frac{\lambda}{2}\right)^4,
\end{aligned} \tag{7.4.15}$$

where a_1, a_2, a_3 and a_4 are arbitrary constants.

Now, we will obtain the new exact solutions of Eq. (7.1.4) involving parameters, expressed by three types of functions which are hyperbolic, trigonometric and rational function solutions, by using generalized $\left(\frac{G'}{G}\right)$ method. On balancing the highest-order derivatives with the nonlinear terms appearing in (7.1.4), we get m = 2. Substituting (7.4.1) into (7.1.4) and using (7.4.2), collecting all terms with the various powers of $\left(\frac{G'}{G}\right)$ and $\left(x\left(\frac{G'}{G}\right)\right)$ together, and equating each coefficient of them to zero, yield a set of equations for $\alpha_0(t), \alpha_1(t), \alpha_2(t), p(t)$ and $q(t)$. Solving that set of equations, we get

$$p\left(t\right) = k_4,\;\; \alpha\left(t\right) = k_3\sigma\left(t\right),\;\; \beta\left(t\right) = k_1\sigma\left(t\right),\;\; \sigma\left(t\right) = \sigma\left(t\right),$$

$$q\left(t\right) = \tfrac{k_4}{10}\left(-120\,{k_4}^4\mu\,\lambda^2 + 15\,{k_4}^4\lambda^4 + 240\,{k_4}^4\mu^2 + {k_1}^2\right)\int\sigma\left(t\right)dt + k_2,$$

$$\alpha_0\left(t\right) = \frac{\sqrt{10}\sqrt{-k_3}\left(40\,{k_4}^2\mu + 5\,{k_4}^2\lambda^2 + k_1\right)}{10k_3},\;\; \alpha_1\left(t\right) = \frac{6\sqrt{-10\,k_3}{k_4}^2\lambda}{k_3},\;\; \alpha_2\left(t\right) = \frac{6\sqrt{-10\,k_3}{k_4}^2}{k_3}, \tag{7.4.16}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants.

Substituting (7.4.16) into (7.4.3) and using the general solutions of (7.4.2), we deduce the following three types of traveling wave solutions of VCMKE (7.1.4):

If $\lambda^2 - 4\mu > 0$, we obtained hyperbolic function solution

$$u(x,t) = \frac{\sqrt{10}\sqrt{-k_3}\left(40\,{k_4}^2\mu + 5\,{k_4}^2\lambda^2 + k_1\right)}{10k_3} + \frac{3\sqrt{-10\,k_3}{k_4}^2\lambda}{k_3}\left(\frac{\sqrt{\lambda^2-4\,\mu}(a_1\sinh(\Lambda_3)+a_2\cosh(\Lambda_3))}{a_2\sinh(\Lambda_3)+a_1\cosh(\Lambda_3)} - \lambda\right) + \frac{3\sqrt{-10\,k_3}{k_4}^2}{2k_3}\left(\frac{\sqrt{\lambda^2-4\,\mu}(a_1\sinh(\Lambda_3)+a_2\cosh(\Lambda_3))}{a_2\sinh(\Lambda_3)+a_1\cosh(\Lambda_3)} - \lambda\right)^2, \tag{7.4.17}$$

where $\Lambda_3 = \frac{\sqrt{\lambda^2-4\,\mu}}{2}\;\left(k_4x + \frac{k_4}{10}\left(-120\,{k_4}^4\mu\,\lambda^2 + 15\,{k_4}^4\lambda^4 + 240\,{k_4}^4\mu^2 + {k_1}^2\right)\int\sigma\left(t\right)dt + k_2\right).$

If $\lambda^2 - 4\mu < 0$, we derived trigonometric function solution

$$u(x,t) = \frac{\sqrt{10}\sqrt{-k_3}\left(40\,{k_4}^2\mu + 5\,{k_4}^2\lambda^2 + k_1\right)}{10k_3} + \frac{3\sqrt{-10\,k_3}{k_4}^2\lambda}{k_3}\left(\frac{\sqrt{-\lambda^2+4\,\mu}(-a_2\sin(\Lambda_4)+a_1\cos(\Lambda_4))}{a_1\sin(\Lambda_4)+a_2\cos(\Lambda_4)} - \lambda\right) + \frac{3\sqrt{-10\,k_3}{k_4}^2}{2k_3}\left(\frac{\sqrt{-\lambda^2+4\,\mu}(-a_2\sin(\Lambda_4)+a_1\cos(\Lambda_4))}{a_1\sin(\Lambda_4)+a_2\cos(\Lambda_4)} - \lambda\right)^2, \tag{7.4.18}$$

where $\Lambda_4 = \frac{\sqrt{-\lambda^2+4\,\mu}}{2}\;\left(k_4x + \frac{k_4}{10}\left(-120\,{k_4}^4\mu\,\lambda^2 + 15\,{k_4}^4\lambda^4 + 240\,{k_4}^4\mu^2 + {k_1}^2\right)\int\sigma\left(t\right)dt + k_2\right).$

If $\lambda^2 - 4\mu = 0$, we have rational function solution in the form

$$u(x,t) = \frac{\sqrt{-10\,k_3}\left(15\,{k_4}^2\lambda^2 + k_1\right)}{10k_3} + \frac{6\sqrt{-10\,k_3}{k_4}^2\lambda}{k_3}\left(\frac{a_4}{a_3 + a_4\left(k_4x + \int\frac{1}{10}\,k_4\sigma(t){k_1}^2dt + k_2\right)} - \frac{\lambda}{2}\right) + \frac{6\,\sqrt{-10\,k_3}{k_4}^2}{k_3}\left(\frac{a_4}{a_3 + a_4\left(k_4x + \int\frac{1}{10}\,k_4\sigma(t){k_1}^2dt + k_2\right)} - \frac{\lambda}{2}\right)^2, \tag{7.4.19}$$

where a_1, a_2, a_3 and a_4 are arbitrary constants.

7.5 Conclusions

In this work, we have considered VCKE and VCMKE by using Lie symmetry analysis. Especially, all similarity reductions and exact solutions based on the Lie group method are obtained by generating the group infinitesimals. Then the some power series solutions are investigated by using the power series method. Further, we successfully obtained exact and explicit analytic solutions with arbitrary parameters to VCKE (7.1.3) and VCMKE (7.1.4) via generalized $\left(\frac{G'}{G}\right)$-expansion method. These solutions are expressed in terms of the hyperbolic functions, trigonometric functions and rational functions. In almost all the cases, the solutions obtained are such that one can choose the arbitrary function $\sigma(t)$, along with various other arbitrary parameters, in a suitable manner, to simulate physical situations governed by the equation (7.1.3) and (7.1.4) and to attain solutions with some desired features. Thus we found some new exact solutions that might prove to be potentially useful for applications in mathematical physics and applied mathematics.

Chapter 8

Variable Coefficients Kuramoto-Sivashinsky Equation[1]

8.1 Introduction

The Kuramoto-Sivashinsky (KS) equation [42]

$$u_t + auu_x + bu_{xx} + cu_{xxxx} = 0, \tag{8.1.1}$$

where a, b and c are arbitrary constants. Kuramoto-Sivashinsky (KS) equation which is a canonical nonlinear evolution equation arising in a variety of physical contexts, e.g. long waves on thin films, long waves on the interface between two viscous fluids [82], unstable drift waves in plasmas, reaction diffusion systems [104] and flame front instability [162]. It describes the fluctuations of the position of a flame front, the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium [42]. This equation was examined as a prototypical example of spatiotemporal chaos in one space dimension [146].

[1]The contents of this chapter are published *International Journal of Nonlinear Sciences*, 15 (2013) 139-149

Here, we have studied Kuramoto-Sivashinsky (VCKS) equation with time dependent variable coefficients

$$u_t + \alpha(t)uu_x + \beta(t)u_{xx} + \sigma(t)u_{xxxx} = 0, \tag{8.1.2}$$

for exact solutions with the help of Lie symmetry analysis.

In this chapter, power series solutions of variable-coefficients Kuramoto-Sivashinsky (VCKS) equation are derived. All the physical parameters in the solutions are evaluated as functions of varying coefficients. Moreover, by direct integration, it is difficult to solve the equations with higher degree nonlinear terms with variable coefficients. Exact solutions of nonlinear partial differential equations with variable coefficients [70, 103, 115, 157, 158] are very helpful for understanding the physical behavior of nonlinear phenomena and dynamical process modeled by these nonlinear models. Although, nonlinear models may posses constant or variable dependent coefficients which provide a rich variety of shape preserving waves and their interesting properties.

In the following sections: Painlevé analysis of Variable Coefficients Kuramoto-Sivashinsky (VCKS) equation is performed to check the Painlevé property and further auto-Bäcklund transformation is presented via the truncated Painlevé expansion in Section (8.2). Section (8.3) is devoted to outline of Lie classical method to generate various symmetries of VCKS equation. Section (8.4) contains the corresponding ordinary differential equations (ODEs) and their exact solutions. In Section (8.5), we deal with the power series solutions by using the power series method and some concluding remarks are given in last section (8.6).

8.2 Painlevé Analysis for VCKS Equation

Weiss et al. [182, 183] have defined the Painlevé property for PDEs and developed a method for testing a common particular type of movable singularity, without studying any similarity reductions. A PDE is said to possess the Painlevé property if the only singularities of the general integral which can live on arbitrary non-characteristic (movable) hypersurfaces are poles. This Painlevé test has proved to be a useful criterion for the identification of completely integrable PDE. The leading order of the solution of equation (8.1.2) is assumed as

$$u(x,t) \approx u_0 g^p \tag{8.2.1}$$

where $u = u(x,t)$ and $g = g(x,t)$ are analytic functions of (x,t). On substituting equation (8.2.1) into (8.1.2) and equating the most dominant terms, the following results are obtained:

$$p = -3, u_0 = \tfrac{120\sigma(t)g_x^3}{\alpha(t)} \tag{8.2.2}$$

where g_x denotes the partial differentiation of $g(x,t)$ with respect to x. For finding the resonances, the full Laurent series:

$$u(x,t) = u_0 g^{-1} + \sum u_r g^{r-3} \tag{8.2.3}$$

where $u_r = u_r(x,t)$, is substituted into (8.1.2) and by equating the coefficients of like terms, the polynomial equation is derived as

$$\begin{aligned}&360\sigma(t)g_x(x,t)^4 g(x,t)^{r-1}u_r\alpha(t)^2 + 222r\sigma(t)g_x(x,t)^4 g(x,t)^{r-1}u_r\alpha(t)^2\\ &-\sigma(t)g_x(x,t)^4 g(x,t)^{r-1}r^4u_r\alpha(t)^2 + 18r^3\sigma(t)g_x(x,t)^4 g(x,t)^{r-1}u_r\alpha(t)^2\\ &-119\sigma(t)g_x(x,t)^4 g(x,t)^{r-1}r^2u_r\alpha(t)^2.\end{aligned} \tag{8.2.4}$$

Using equation above, the resonances are found to be

$$r = -1, 6, \frac{13}{2} \pm \iota\frac{\sqrt{71}}{2} \tag{8.2.5}$$

As usual, the resonance at $r = -1$ corresponds to the arbitrariness of the singular manifold $g(x,t) = 0$. Since, the three resonances are not all located at positive integers, therefore we may obtain a solution of (8.1.2) depending on two arbitrary functions. In order to check the nature of resonance located at $r = 6$, the full Laurent expansion (8.2.3) is substituted in (8.1.2). After the detailed calculation, we observe that (8.1.2) admits the arbitrary functions g and u_6, thus, it is concluded that (8.1.2) passes the Painlevé -test not in strict sense, but in a weak sense.

B$\ddot{a}$cklund transformation is a powerful tool in the study on the solutions to the nonlinear evolution equations. The Painlev$\acute{e}$ truncation provides us a straightforward way to obtain auto-B$\ddot{a}$cklund transformation [163, 182, 183]. To achieve auto-B$\ddot{a}$cklund transformation, we must work with the general form $g(x,t) = 0$ of the noncharacteristic singularity manifold. With leading-order analysis, we obtain the truncated Painlev$\acute{e}$ expansion at the constant level term as

$$u = u_0 g^{-3} + u_1 g^{-2} + u_2 g^{-1} + u_3. \tag{8.2.6}$$

Substituting Eq. (8.2.6) into Eq. (8.1.2) and making the coefficients of like powers of g vanish with symbolic computation yield,

$$\begin{aligned}
&360\sigma(t)u_0 g_x^4 - 3\alpha(t)u_0^2 g_x = 0.\\
&86400\sigma(t)^2 g_x^5 g_{xx} + 480\sigma(t)u_1\alpha(t)g_x^4 = 0.\\
&19\alpha(t)u_2 - 60\beta(t)g_x - 19\sigma(t)g_{xxx} = 0.\\
&30\beta(t)g_x^2 g_{xx} + 285 g_x^2 g_{xxxx} + 285\sigma(t)g_{xx}^3 - 570\sigma(t)g_x g_{xx} g_{xxx} + 19 g_x^2 g_t + 19 u_3 g_x^3 \alpha(t) = 0.\\
&-361\sigma(t)g_x\alpha'(t) + 361\sigma(t)g_x^3\alpha(t)^2 u_{3x} + 361\alpha(t)\sigma'(t)g_x^3 + 5415\sigma(t)^2\alpha(t)g_{xxx}g_{xx}^2\\
&-18050\sigma(t)^2\alpha(t)g_x g_{xxx}^2 + 7581\sigma(t)^2\alpha(t)g_x^2 g_{xxxxx} + 2166 u_3\sigma(t)\alpha(t)^2 g_x^2 g_{xx}\\
&+5415\sigma(t)^2\alpha(t)g_x g_{xx} g_{xxxx} + 1083\sigma(t)\alpha(t)g_x g_{xx} g_t + 2280\sigma(t)\beta(t)\alpha(t)g_x g_{xx}^2\\
&+1083\sigma(t)\alpha(t)g_x^2 g_{xt} + 950\sigma(t)\beta(t)\alpha(t)g_x^2 g_{xxx} - 11\beta(t)^2\alpha(t)g_x^3 = 0.\\
&1444\alpha(t)^2\sigma(t)u_3 g_x g_{xxx} + 1083\alpha(t)\sigma(t)g_{xt}g_{xx} - 1083\alpha'(t)\sigma(t)g_x g_{xx} + 1083\alpha(t)^2\sigma(t)g_x g_{xx} u_{3x}\\
&+1083\alpha(t)\sigma'(t)g_x g_{xx} + 1083\alpha(t)\sigma(t)g_x g_{xxt} - 9025\alpha(t)\sigma(t)^2 g_{xxx}g_{xxxx} - 3\beta^2\alpha(t)g_x g_{xx}\\
&+2527\alpha(t)\sigma(t)^2 g_x g_{xxxxxx} + 7581\alpha(t)\sigma(t)^2 g_{xx}g_{xxxxx} + 1083\alpha(t)^2\sigma(t)u_3 g_{xx}^2\\
&+19\alpha(t)\beta(t)g_x g_t + 361\alpha(t)\sigma(t)g_t g_{xxx} + 19 u_3\alpha(t)^2\beta(t)g_x^2 + 2660\alpha(t)\beta(t)\sigma(t)g_{xx}g_{xxx}\\
&+760\alpha(t)\beta(t)\sigma(t)g_x g_{xxxx} = 0.\\
&20\alpha(t)\beta(t)\sigma(t)g_{xxxxx} + 19\alpha(t)\sigma(t)^2 g_{xxxxxxx} + \alpha(t)\beta'(t)g_x + \alpha(t)\beta(t)g_{xt} + 19\alpha(t)\sigma'(t)g_{xxx}\\
&+19\alpha(t)\sigma(t)g_{xxxt} - \alpha'(t)\beta(t)g_x - 19\alpha'(t)\sigma(t)g_{xxx} + \alpha(t)\beta(t)^2 g_{xxx} + \alpha(t)^2\beta(t)g_x u_{3x}\\
&+19\alpha(t)^2\sigma(t)g_{xxx}u_{3x} + u_3\alpha(t)^2\beta(t)g_{xx} + 19 u_3\alpha(t)^2\sigma(t)g_{xxxx} = 0.\\
&u_{3t} + \alpha(t)u_3 u_{3x} + \beta(t)u_{3xx} + \sigma(t)u_{3xxxx} = 0.
\end{aligned} \tag{8.2.7}$$

where u_3 is a solution of Eq. (8.1.2), and g satisfies the overdetermined Eqs. (8.2.7). Therefore, we obtain an auto-B$\ddot{a}$cklund transformation of Eq. (8.1.2) as follows:

$$u = \frac{60\sigma(t)}{\alpha(t)}(\ln(g))_{xxx} + \frac{60\beta(t)}{19\alpha(t)}(\ln(g))_x + u_3. \tag{8.2.8}$$

8.3 Invariance and Infinitesimal Characterization

A Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. Determining all the symmetries of a differential equation is a formidable task. However, the Norwegian mathematician Sophus Lie (1842-1899) realized that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end detailed study of Lie group theory the interested reader is referred to the well-known books [22, 23, 85, 137]. The technique has earlier been used to obtain the exact solutions of various nonlinear partial differential equations.

In this section, we considered the symmetry groups using the Lie's classical method. A vector field

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} \tag{8.3.1}$$

is a generator of point symmetry of equation (8.1.2) if

$$V^{[5]}(u_t + \alpha(t)uu_x + \beta(t)u_{xx} + \sigma(t)u_{xxxx}) = 0, \tag{8.3.2}$$

whenever

$$u_t + \alpha(t)uu_x + \beta(t)u_{xx} + \sigma(t)u_{xxxx} = 0. \tag{8.3.3}$$

where the operator $V^{[5]}$ is the third prolongation of the operator V defined by

$$V^{[5]} = V + U^x\frac{\partial}{\partial u_x} + U^t\frac{\partial}{\partial u_t} + U^{xx}\frac{\partial}{\partial u_{xx}} + U^{tt}\frac{\partial}{\partial u_{tt}} + ... + +U^{xxxxx}\frac{\partial}{\partial u_{xxxxx}}, \tag{8.3.4}$$

where $U^x,\ U^t,\ U^{xx},\ U^{tt}, ..., U^{xxxxx}$ are extended (prolonged) infinitesimals acting on an enlarged space corresponding to $u_x,\ u_t,\ u_{xx},\ u_{tt}, ..., u_{xxxxx}$ respectively. The method for determining the symmetry group mainly consists of finding the infinitesimals X, Y, U which are functions of x, t, u. The set of determining equations for the group infinitesimals $X,\ T$ and U which we get from (8.3.2) after equating the coefficients of various derivative terms to zero, is as follows:

$$
\begin{aligned}
&X_u = 0,\ T_x = 0\ T_u = 0\ U_{uu} = 0\\
&-3X_{xx} + 4U_{xu} = 0,\\
&\alpha(t)uU_x + U_t + U_{uuuu} + \beta(t)U_{uu} = 0,\\
&-T\sigma'(t) - T_t\sigma(t) + 4\sigma X_x = 0,\\
&2\beta X_x\sigma(t) + 6(\sigma(t))^2U_{xxu} - 4(\sigma(t))^2X_{xxx} - T\sigma'(t)\beta + T\beta'(t)\sigma(t) = 0,\\
&-T\sigma'(t)\alpha(t)u + T\alpha'(t)u\sigma(t) - (\sigma(t))^2X_{xxxx} + 3\alpha(t)uX_x\sigma(t) + \alpha(t)\sigma(t)U - \sigma(t)X_t\\
&+2\beta(t)\sigma(t)U_{xu} - \beta\sigma X_{xx} + 4(\sigma(t))^2U_{xxxu} = 0.
\end{aligned}
\tag{8.3.5}
$$

The general solution of equations of this large system provides following forms for the infinitesimal elements $X,\ T$ and U and admissible forms of various coefficients in Eq. (8.1.2):

$$
\begin{aligned}
&X = k_1x + k_2\\
&T = \frac{\left(4k_1\int\sigma(t)dt + k_3\right)}{\sigma(t)}\\
&U = k_4u,
\end{aligned}
\tag{8.3.6}
$$

where k_1, k_2, k_3 and k_4 are arbitrary constants.

The functions $\alpha(t), \beta(t)$ and $\sigma(t)$ are governed by the following conditions:

$$
\begin{aligned}
&2\beta k_1 - T\sigma'(t)\beta(t) + T\sigma(t)\beta'(t) = 0\\
&3\alpha(t)\sigma(t)k_1 + \alpha(t)\sigma(t)k_4 + T\alpha'(t)\sigma(t) - T\alpha(t)\sigma'(t) = 0.
\end{aligned}
\tag{8.3.7}
$$

The vector fields associated with equation (8.1.2) are following :

$$
\begin{aligned}
&\Lambda_1 = x\frac{\partial}{\partial x} + \left(\frac{4}{\sigma(t)}\int\sigma(t)dt\right)\frac{\partial}{\partial t}\\
&\Lambda_2 = u\frac{\partial}{\partial u}\\
&\Lambda_3 = \frac{1}{\sigma(t)}\frac{\partial}{\partial t}\\
&\Lambda_4 = \frac{\partial}{\partial x}.
\end{aligned}
\tag{8.3.8}
$$

It is easy to check that the vector fields $\Lambda_1, \Lambda_2, \Lambda_3$ and Λ_4 are closed under the Lie bracket. For Eq. (8.1.2), we have $[\Lambda_1, \Lambda_1] = ... = [\Lambda_4, \Lambda_4] = 0$, $[\Lambda_1, \Lambda_2] = -[\Lambda_2, \Lambda_1] = 0$, $[V_2, \Lambda_3] = -[\Lambda_3, \Lambda_2] = 0$, $[\Lambda_2, \Lambda_4] = -[\Lambda_4, \Lambda_2] = 0$, $[\Lambda_3, \Lambda_4] = -[\Lambda_4, \Lambda_3] = 0$,

$[\Lambda_1, \Lambda_3] = -[\Lambda_3, \Lambda_1] = -4\Lambda_3$, $[\Lambda_1, \Lambda_4] = -[\Lambda_4, \Lambda_1] = -\Lambda_4$.

As a result, symmetries (8.3.8) form a four-dimensional Lie algebra.

As we know, classification of subgroups of Lie symmetry groups of differential equations is an essential part in study of equations. This is since classification allows for an efficient computation of group-invariant solutions, without the possibility of an occurence of equivalent solutions. The classification of subgroups of symmetry groups is usually done by the classification of the associated Lie subalgebras w.r.t. the adjoint representation. Following [137] we deduce the following basic fields which form an optimal system for Eq. (8.1.2)

(i) $\Lambda_1 + \mu\Lambda_2$, (ii) $\Lambda_2 + \lambda\Lambda_3 + \Lambda_4$, (iii) $\Lambda_2 + \lambda\Lambda_3 - \Lambda_4$, (iv) $\Lambda_2 + \rho\Lambda_3$, (v) $\Lambda_3 + \Lambda_4$,

(vi) $\Lambda_3 - \Lambda_4$, (vii) Λ_3, (viii)Λ_4,

where μ, λ and ρ are arbitrary constants. The discrete symmetry $(t, x, u) \longrightarrow (t, -x, u)$ will map (ii) and (v) to (iii) and (vi) respectively, so we will deal only with the (i), (ii), (iv), (v), (vii) and (viii) cases.

8.4 Reduced ODEs and Exact Solutions

In this section, we have utilized the symmetries calculated in the previous section to deduce exact solutions of (8.1.2). One way to obtain exact solutions of (8.1.2) is by reducing it to ordinary differential equations. This can be achieved with the use of Lie point symmetries admitted by (8.1.2). It is well known that the reduction of a partial differential equation with respect to r-dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r.

In the table (8.1), we now list the the similarity variables and the similarity solutions and also the coefficient functions of the Eq. (8.1.2), for the six essential vector fields of the optimal system.

Table 8.1: Similarity Variables, Similarity Functions and the Coefficient Functions of Kuramoto-Sivashinsky (VCKS) equation

Essential Vector Fields	Similarity Variables	Similarity Forms	Coefficient Functions
$\Lambda_1 + \mu\Lambda_2$	$x(\int \sigma(t)dt)^{-\frac{1}{4}}$	$(\int \sigma(t)dt)^{\frac{\mu}{4}} F(\zeta)$	$\alpha(t) = \sigma(t)k_5(\int \sigma(t)dt)^{\frac{-(3+\mu)}{4}}$, $\beta(t) = \sigma(t)k_6(\int \sigma(t)dt)^{\frac{-1}{2}}$
$\Lambda_2 + \lambda\Lambda_3 + \Lambda_4$	$-x + \frac{1}{\lambda}\int \sigma(t)dt$	$\exp(\frac{1}{\lambda}\int \sigma(t)dt)F(\zeta)$	$\alpha(t) = \sigma(t)k_5 \exp(\frac{-1}{\lambda}\int \sigma(t)dt)$, $\beta(t) = k_6\sigma(t)$
$\Lambda_2 + \rho\Lambda_3$	x	$\exp(\frac{1}{\rho}\int \sigma(t)dt)F(\zeta)$	$\alpha(t) = \sigma(t)k_5 \exp(\frac{-1}{\rho}\int \sigma(t)dt)$, $\beta(t) = k_6\sigma(t)$
$\Lambda_3 + \Lambda_4$	$-x + \int \sigma(t)dt$	$F(\zeta)$	$\alpha(t) = k_5\sigma(t), \beta(t) = k_6\sigma(t)$
Λ_3	x	$F(\zeta)$	$\alpha(t) = k_5\sigma(t), \beta(t) = k_6\sigma(t)$
Λ_4	t	$F(\zeta)$	$\alpha(t) = k_5\sigma(t), \beta(t) = k_6\sigma(t)$

where k_5 and k_6 are arbitrary constants.

Vector field $\Lambda_1 + \mu\Lambda_2$

Using the similarity variable, the forms of the similarity solution and the coefficient functions, the (8.1.2) is reduced to the following ODE:

$$4F'''' + 4k_6F'' + 4k_5FF' - \zeta F' + \mu F = 0, \tag{8.4.1}$$

where prime ($'$) denotes the differentiation with respect to the variable ζ.

Now solving equation (8.4.1) and thus, solution of Eq. (8.1.2) can be given as:

$$u(x,t) = \frac{(1-\mu)x}{4k_5}\left(\int \sigma(t)dt\right)^{\frac{\mu}{4}-\frac{1}{4}}. \tag{8.4.2}$$

Note that the reduced equation (8.4.1) is higher-order nonlinear or nonautonomous ODE, we will deal with this equations in the next section.

Vector field $\Lambda_2 + \lambda\Lambda_3 + \Lambda_4$

Corresponding to vector field, the reduced ODE is as follows:

$$\lambda F'''' + k_6\lambda F'' - \lambda k_5 FF' + F' + F = 0. \tag{8.4.3}$$

The solution $F(\zeta)$ leads by back substitution to the exact solution of the equation (8.1.2) of the form, for $k_6 = -1, k_5 = 0$ and $\lambda = 1$, is as follows:

$$\begin{aligned} u(x,t) = \exp\left(\textstyle\int \sigma(t)dt\right)\left(\exp\left(-\left(-x + \textstyle\int \sigma\left(t\right)dt\right)\right)c_1 + \exp\left(a_1\left(-x + \textstyle\int \sigma\left(t\right)dt\right)\right)c_2\right) \\ + \exp\left(\textstyle\int \sigma(t)dt\right)\left(\exp\left(a_2\left(-x + \textstyle\int \sigma\left(t\right)dt\right)\right)c_3 + \exp\left(a_3\left(-x + \textstyle\int \sigma\left(t\right)dt\right)\right)c_4\right), \end{aligned} \tag{8.4.4}$$

where

$$a_1 = -\frac{\left(\left(100+12\sqrt{3}\sqrt{23}\right)^{2/3}+4-2\sqrt[3]{100+12\sqrt{3}\sqrt{23}}\right)}{6\sqrt[3]{100+12\sqrt{3}\sqrt{23}}}$$

$$a_2 = -\frac{\left(\sqrt[3]{100+12\sqrt{3}\sqrt{23}}+2\right)\left(i\sqrt[3]{100+12\sqrt{3}\sqrt{23}}\sqrt{3}-\sqrt[3]{100+12\sqrt{3}\sqrt{23}}-2-2i\sqrt{3}\right)}{12\sqrt[3]{100+12\sqrt{3}\sqrt{23}}}$$

$$a_3 = \frac{\left(\sqrt[3]{100+12\sqrt{3}\sqrt{23}}+2\right)\left(i\sqrt[3]{100+12\sqrt{3}\sqrt{23}}\sqrt{3}+\sqrt[3]{100+12\sqrt{3}\sqrt{23}}-2i\sqrt{3}+2\right)}{12\sqrt[3]{100+12\sqrt{3}\sqrt{23}}}$$

and c_1, c_2, c_3 and c_4 are arbitrary constants.

Corresponding to $k_6 = -1, k_5 = 0$ and $\lambda = -1$, the solution of (8.1.2) is given by:

$$u(x,t) = \mathrm{e}^{-\int\sigma(t)dt}\left(c_1\mathrm{e}^{-x-\int\sigma(t)dt} + c_2\mathrm{e}^{-\frac{\left((44+12\sqrt{69})^{2/3}-20+2\sqrt[3]{44+12\sqrt{69}}\right)(-x-\int\sigma(t)dt)}{6\sqrt[3]{44+12\sqrt{69}}}}\right)$$
$$+\mathrm{e}^{-\int\sigma(t)dt}\left(c_3\mathrm{e}^{\frac{\left((44+12\sqrt{69})^{2/3}-20-4\sqrt[3]{44+12\sqrt{69}}\right)(-x-\int\sigma(t)dt)}{12\sqrt[3]{44+12\sqrt{69}}}}\sin\left(\frac{\left(\sqrt{3}\left(44+12\sqrt{69}\right)^{2/3}+20\sqrt{3}\right)\left(-x-\int\sigma(t)dt\right)}{12\sqrt[3]{44+12\sqrt{69}}}\right)\right)$$
$$+\mathrm{e}^{-\int\sigma(t)dt}\left(c_4\mathrm{e}^{\frac{\left((44+12\sqrt{69})^{2/3}-20-4\sqrt[3]{44+12\sqrt{69}}\right)(-x-\int\sigma(t)dt)}{12\sqrt[3]{44+12\sqrt{69}}}}\cos\left(\frac{\left(\sqrt{3}\left(44+12\sqrt{69}\right)^{2/3}+20\sqrt{3}\right)\left(-x-\int\sigma(t)dt\right)}{12\sqrt[3]{44+12\sqrt{69}}}\right)\right), \tag{8.4.5}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Note that we will also deal with this higher order nonlinear or nonautonomous ODE in the next section.

Vector field $\Lambda_2 + \rho\Lambda_3$

Subsituting the forms of the similarity solution and the coefficient functions, corresponding to this vector field, the (8.1.2) yields the following ODE:

$$\rho F'''' + k_6\rho F'' + \rho k_5 FF' + F = 0. \tag{8.4.6}$$

Eq. (8.4.6) has the solution

$$F(\zeta) = c_3 - \frac{c_1 + c_2\zeta}{\rho k_5 c_2}, \tag{8.4.7}$$

where c_1, c_2 and c_3 are arbitrary constants.

Thus, we obtain the solution of Eq. (8.1.2) as follows:

$$u(x,t) = \exp\left(\int\left(\frac{\sigma(t)}{\rho}\right)dt\right)\left(c_3 - \frac{c_1 + c_2 x}{\rho k_5 c_2}\right). \tag{8.4.8}$$

Vector field $\Lambda_3 + \Lambda_4$

For this vector field, using the similarity variable, the forms of the similarity solution and the coefficient functions, the (8.1.2) is transformed into the following fourth order nonlinear ODE:

$$F'''' + k_6 F'' - k_5 FF' + F' = 0. \tag{8.4.9}$$

Solving this equation (8.4.9) and reverting back to the original variables, we obtain the following group-invariant solution of the Kuramoto-Sivashinsky (KS) equation (8.1.2):

Solutions in terms of tanh() function

$$\begin{aligned}(i)u(x,t) &= \tfrac{1}{k_5} - \tfrac{45k_6\sqrt{-19k_6}}{361k_5}\tanh\left(c_1 + \tfrac{\sqrt{-19k_6}(-x+\int\sigma(t)dt)}{38}\right)\\ &\quad + \tfrac{15k_6\sqrt{-19k_6}}{361k_5}\left(\tanh\left(c_1 + \tfrac{\sqrt{-19k_6}(-x+\int\sigma(t)dt)}{38}\right)\right)^3\\ (ii)u(x,t) &= \tfrac{1}{k_5} + \tfrac{135k_6^{\frac{3}{2}}\sqrt{209}}{361k_5}\tanh\left(c_1 + \tfrac{\sqrt{209k_6}(-x+\int\sigma(t)dt)}{38}\right)\\ &\quad - \tfrac{165k_6^{\frac{3}{2}}\sqrt{209}}{361k_5}\left(\tanh\left(c_1 + \tfrac{\sqrt{209k_6}(-x+\int\sigma(t)dt)}{38}\right)\right)^3,\end{aligned} \tag{8.4.10}$$

By the analysis of solutions obtained here, we conclude that solutions depends upon certain arbitrary constants and function $\sigma(t)$. Depending upon these constants and function of time, we obtain certain different solutions. For $k_5 = k_6 = 1, c_1 = 1$ and $\sigma(t) = 1$, solution ((8.4.10)(ii)) behaves as periodic waves as shown in Figure (7.1).

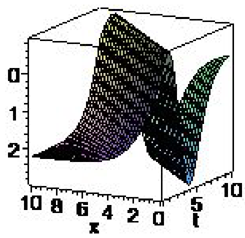

Figure 8.1: Periodic Wave Solution of ((8.4.10)(ii)) for $k_5 = k_6 = 1, c_1 = 1$ and $\sigma(t) = 1$

Solutions in terms of tan() function

$$\begin{aligned}(i)u(x,t) &= \tfrac{1}{k_5} + \tfrac{45k_6^{\frac{3}{2}}\sqrt{19}}{361k_5}\tan\left(c_1 + \tfrac{\sqrt{19k_6}(-x+\int\sigma(t)dt)}{38}\right)\\ &\quad + \tfrac{15k_6^{\frac{3}{2}}\sqrt{19}}{361k_5}\left(\tan\left(c_1 + \tfrac{\sqrt{19k_6}(-x+\int\sigma(t)dt)}{38}\right)\right)^3\\ (ii)u(x,t) &= \tfrac{1}{k_5} - \tfrac{135k_6\sqrt{-209k_6}}{361k_5}\tan\left(c_1 + \tfrac{\sqrt{-209k_6}(-x+\int\sigma(t)dt)}{38}\right)\\ &\quad - \tfrac{165k_6\sqrt{-209k_6}}{361k_5}\left(\tan\left(c_1 + \tfrac{\sqrt{-209k_6}(-x+\int\sigma(t)dt)}{38}\right)\right)^3,\end{aligned} \tag{8.4.11}$$

By the analysis of solutions obtained corresponding to $\Lambda_3+\Lambda_4$, we conclude that solutions behaves as periodic solutions for different values of constants $k_5,\ k_6,\ c_1$ and arbitrary function $\sigma(t)$ involved.

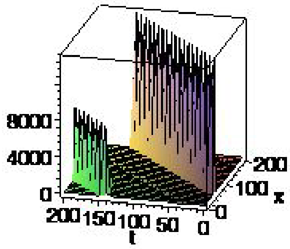

Figure 8.2: Periodic Wave Solution of ((8.4.11)(ii)) for $k_5 = k_6 = 1, c_1 = 1$ and $\sigma(t) = 1$

Solutions in terms of coth() function

$$\begin{aligned}
(i)u(x,t) &= \tfrac{1}{k_5} - \tfrac{45k_6\sqrt{-19k_6}}{361k_5}\coth\left(c_1 + \tfrac{\sqrt{-19k_6}(-x+\int\sigma(t)dt)}{38}\right) \\
&\quad + \tfrac{15k_6\sqrt{-19k_6}}{361k_5}\left(\coth\left(c_1 + \tfrac{\sqrt{-19k_6}(-x+\int\sigma(t)dt)}{38}\right)\right)^3 \\
(ii)u(x,t) &= \tfrac{1}{k_5} + \tfrac{135k_6^{\frac{3}{2}}\sqrt{209}}{361k_5}\coth\left(c_1 + \tfrac{\sqrt{209k_6}(-x+\int\sigma(t)dt)}{38}\right) \\
&\quad - \tfrac{165k_6^{\frac{3}{2}}\sqrt{209}}{361k_5}\left(\coth\left(c_1 + \tfrac{\sqrt{209k_6}(-x+\int\sigma(t)dt)}{38}\right)\right)^3,
\end{aligned} \tag{8.4.12}$$

The solutions, thus derived, takes the form of periodic solutions as shown in Fig. (7.3)

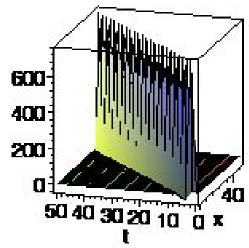

Figure 8.3: Periodic Wave Solution of ((8.4.12)(ii)) for $k_5 = k_6 = 1, c_1 = 1$ and $\sigma(t) = 1$

Solutions in terms of cot() function

$$\begin{aligned}
(i)u(x,t) &= \tfrac{1}{k_5} - \tfrac{45k_6^{\frac{3}{2}}\sqrt{19}}{361k_5}\cot\left(c_1 + \tfrac{\sqrt{19k_6}(-x+\int\sigma(t)dt)}{38}\right) \\
&\quad - \tfrac{15k_6^{\frac{3}{2}}\sqrt{19}}{361k_5}\left(\cot\left(c_1 + \tfrac{\sqrt{19k_6}(-x+\int\sigma(t)dt)}{38}\right)\right)^3 \\
(ii)u(x,t) &= \tfrac{1}{k_5} + \tfrac{135k_6\sqrt{-209k_6}}{361k_5}\cot\left(c_1 + \tfrac{\sqrt{-209k_6}(-x+\int\sigma(t)dt)}{38}\right) \\
&\quad + \tfrac{165k_6\sqrt{-209k_6}}{361k_5}\left(\cot\left(c_1 + \tfrac{\sqrt{-209k_6}(-x+\int\sigma(t)dt)}{38}\right)\right)^3,
\end{aligned} \tag{8.4.13}$$

where c_1 is an arbitrary constant.

The solutions obtained in this section behaves as periodic waves for different values of arbitrary constants and function $\sigma(t)$.

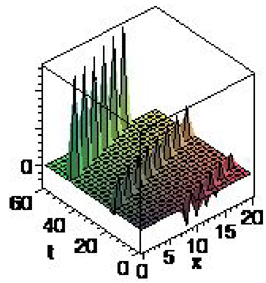

Figure 8.4: Periodic Wave Solution of ((8.4.13)(ii)) for $k_5 = k_6 = 1, c_1 = 1$ and $\sigma(t) = 1$

Vector field Λ_3

In this case, the reduced ODE is as follows:

$$F'''' + k_6 F'' + k_5 F F' = 0, \tag{8.4.14}$$

which is a fourth-order nonlinear ODE. Now solving this equation (8.4.14) and reverting back to the original variables, solutions of Eq. (8.1.2) can be given as:

Solutions in terms of tanh() function

$$(i) u(x,t) = \frac{45k_6\sqrt{-19k_6}}{361k_5}\tanh\left(c_1 + \frac{\sqrt{-19k_6}x}{38}\right) - \frac{15k_6\sqrt{-19k_6}}{361k_5}\left(\tanh\left(c_1 + \frac{\sqrt{-19k_6}x}{38}\right)\right)^3$$
$$(ii) u(x,t) = -\frac{135k_6^{\frac{3}{2}}\sqrt{209}}{361k_5}\tanh\left(c_1 + \frac{\sqrt{209k_6}x}{38}\right) + \frac{165k_6^{\frac{3}{2}}\sqrt{209}}{361k_5}\left(\tanh\left(c_1 + \frac{\sqrt{209k_6}x}{38}\right)\right)^3, \tag{8.4.15}$$

Solutions in terms of tan() function

$$(i) u(x,t) = -\frac{45k_6^{\frac{3}{2}}\sqrt{19}}{361k_5}\tan\left(c_1 + \frac{\sqrt{19k_6}x}{38}\right) - \frac{15k_6^{\frac{3}{2}}\sqrt{19}}{361k_5}\left(\tan\left(c_1 + \frac{\sqrt{19k_6}x}{38}\right)\right)^3$$
$$(ii) u(x,t) = \frac{135k_6\sqrt{-209k_6}}{361k_5}\tan\left(c_1 + \frac{\sqrt{-209k_6}x}{38}\right) + \frac{165k_6\sqrt{-209k_6}}{361k_5}\left(\tan\left(c_1 + \frac{\sqrt{-209k_6}x}{38}\right)\right)^3, \tag{8.4.16}$$

Solutions in terms of coth() function

$$(i)u(x,t) = \tfrac{45k_6\sqrt{-19k_6}}{361k_5}\coth\left(c_1 + \tfrac{\sqrt{-19k_6}x}{38}\right) - \tfrac{15k_6\sqrt{-19k_6}}{361k_5}\left(\coth\left(c_1 + \tfrac{\sqrt{-19k_6}x}{38}\right)\right)^3$$
$$(ii)u(x,t) = -\tfrac{135k_6^{\frac{3}{2}}\sqrt{209}}{361k_5}\coth\left(c_1 + \tfrac{\sqrt{209k_6}x}{38}\right) + \tfrac{165k_6^{\frac{3}{2}}\sqrt{209}}{361k_5}\left(\coth\left(c_1 + \tfrac{\sqrt{209k_6}x}{38}\right)\right)^3, \tag{8.4.17}$$

Solutions in terms of cot() function

$$(i)u(x,t) = \tfrac{45k_6^{\frac{3}{2}}\sqrt{19}}{361k_5}\cot\left(c_1 + \tfrac{\sqrt{19k_6}x}{38}\right) + \tfrac{15k_6^{\frac{3}{2}}\sqrt{19}}{361k_5}\left(\cot\left(c_1 + \tfrac{\sqrt{19k_6}x}{38}\right)\right)^3$$
$$(ii)u(x,t) = -\tfrac{135k_6\sqrt{-209k_6}}{361k_5}\cot\left(c_1 + \tfrac{\sqrt{-209k_6}x}{38}\right) - \tfrac{165k_6\sqrt{-209k_6}}{361k_5}\left(\cot\left(c_1 + \tfrac{\sqrt{-209k_6}x}{38}\right)\right)^3, \tag{8.4.18}$$

where c_1 is an arbitrary constant.

Vector field Λ_4

Corresponding to this vector field, we get the trivial solution of Eq. (8.1.2) is $u(x,t) = c$, where c is an arbitrary constant.

8.5 Power Series Solutions

In Section (8.3), we obtained the reduced equations by using Lie symmetry reductions and further attempted for some exact solutions. In this section, we will find the solutions of the nonlinear ODEs (8.4.1), (8.4.3) and (8.4.6). In general, we can not get the exact explicit solutions for the nonlinear ODEs by using the elementary functions and integrals. But the power series can be used to solve ODEs, including many complicated differential equations with nonconstant coefficients [7, 112, 113, 114]. Now we consider the power series solutions for the reduced equations.

Analytic solutions to Eq. (8.4.1)

Firstly, we will find a solution of Eq. (8.4.1) in a power series of the form

$$F(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n. \tag{8.5.1}$$

Substituting (8.5.1) into (8.4.1), we have

$$\begin{aligned} &96c_4 + 4\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)c_{n+4}\zeta^n + 8c_2k_6 + 4k_6\sum_{n=1}^{\infty}(n+1)(n+2)c_{n+2}\zeta^n \\ &+4k_5\sum_{n=1}^{\infty}(\sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k})\zeta^n - \zeta\sum_{n=0}^{\infty}(n+1)c_{n+1}\zeta^n + \mu\sum_{n=0}^{\infty}c_n\zeta^n = 0. \end{aligned} \tag{8.5.2}$$

From Eq. (8.5.2), comparing coefficients, for $n = 0$, we obtain

$$c_4 = \frac{1}{96}(-8c_2k_6 - \mu c_0 - 4k_5c_0c_1). \tag{8.5.3}$$

Generally, for $n \geq 1$, we have

$$\begin{aligned} c_{n+4} = &\tfrac{1}{4(n+1)(n+2)(n+3)(n+4)}(-4k_6(n+1)(n+2)c_{n+2} - 4k_5(\sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k}) \\ &+(n-\mu)c_n). \end{aligned} \tag{8.5.4}$$

From (8.5.3) and (8.5.4), we can get all the coefficients c_n, $n \geq 1$ of the power series (8.5.1), as following,

$$\begin{aligned} c_5 &= \tfrac{1}{480}(-24c_3k_6 - \mu c_1 + c_1 - 4k_5(2c_0c_2 + c_1^2)) \\ c_6 &= \tfrac{1}{1440}(-48c_4k_6 - \mu c_2 + 2c_2 - 4k_5(3c_0c_3 + 3c_1c_2)), \end{aligned} \tag{8.5.5}$$

and so on.

Thus, for arbitrary chosen constant numbers c_0, c_1, c_2 and c_3, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (8.5.3) and (8.5.4) in a unique manner. This implies that for Eq. (8.4.1), there exists a power series solution (8.5.1) with the coefficients given by (8.5.3) and (8.5.4). Therefore, this power series solution (8.5.1) to Eq. (8.4.1) is an power series solution.

Hence, the power series solution of Eq. (8.1.2) can be written as following:

$$
\begin{aligned}
u(x,t) = {} & (\int \sigma(t)dt)^{\frac{\mu}{4}}(c_0 + c_1(x\int \sigma(t)dt)^{\frac{-1}{4}} + c_2(x\int \sigma(t)dt)^{\frac{-2}{4}} + c_3(x\int \sigma(t)dt)^{\frac{-3}{4}} \\
& + \tfrac{1}{96}(-8c_2k_6 - \mu c_0 - 4k_5c_0c_1)(x\int \sigma(t)dt)^{-1} \\
& + \sum_{n=1}^{\infty} \tfrac{1}{4(n+1)(n+2)(n+3)(n+4)}(-4k_6(n+1)(n+2)c_{n+2} - 4k_5(\sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k}) \\
& + (n-\mu)c_n)(x\int \sigma(t)dt)^{\frac{-(n+4)}{4}}),
\end{aligned}
\tag{8.5.6}
$$

where $c_i (i = 0, 1, 2, 3)$ and μ are arbitrary constants, the other coefficients c_n, $n \geq 4$ can be determined successively from (8.5.3) and (8.5.4).

In physical applications, it will be convenient to write the solution of Eq. (8.1.2) in the approximate form, in terms of the above computation as follows:

$$
\begin{aligned}
u(x,t) = {} & (\int \sigma(t)dt)^{\frac{\mu}{4}}(c_0 + c_1(x\int \sigma(t)dt)^{\frac{-1}{4}} + c_2(x\int \sigma(t)dt)^{\frac{-2}{4}} + c_3(x\int \sigma(t)dt)^{\frac{-3}{4}} \\
& + \tfrac{1}{96}(-8c_2k_6 - \mu c_0 - 4k_5c_0c_1)(x\int \sigma(t)dt)^{-1} \\
& + \tfrac{1}{480}(-24c_3k_6 - \mu c_1 + c_1 - 4k_5(2c_0c_2 + c_1^2))(x\int \sigma(t)dt)^{\frac{-5}{4}} \\
& + \tfrac{1}{1440}(-48c_4k_6 - \mu c_2 + c_2 - 4k_5(3c_0c_3 + 3c_1c_2))(x\int \sigma(t)dt)^{\frac{-6}{4}}) + ...
\end{aligned}
\tag{8.5.7}
$$

power series solutions to Eq. (8.4.3)

Similarly, we seek a solution of Eq. (8.4.3) in a power series of the form (8.5.1). Substituting it into (8.4.3), and comparing coefficients, we obtain

$$
\begin{aligned}
c_{n+4} = {} & \tfrac{1}{\lambda(n+1)(n+2)(n+3)(n+4)}(-k_6\lambda(n+1)(n+2)c_{n+2} + k_5\lambda\sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k} \\
& -(n+1)c_{n+1} - c_n),
\end{aligned}
\tag{8.5.8}
$$

$n = 0, 1, 2...$

In view of Eq. (8.5.8), we can get all the coefficients c_n, $n \geq 4$ of the power series (**??**) such as

$$
\begin{aligned}
c_4 &= \tfrac{1}{24\lambda}(-2c_2k_6\lambda + \lambda k_5c_0c_1 - c_0 - c_1) \\
c_5 &= \tfrac{1}{120\lambda}(-6c_3k_6\lambda + k_5\lambda(2c_0c_2 + c_1^2) - 2c_2 - c_1) \\
c_6 &= \tfrac{1}{360\lambda}(-12c_4k_6\lambda + k_5\lambda(3c_0c_3 + 3c_1c_2) - 3c_3 - c_2),
\end{aligned}
\tag{8.5.9}
$$

and so on.

Thus, for arbitrary chosen constant numbers c_1, c_2 and c_3, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (8.5.8) in a unique manner. This implies that for Eq. (8.4.3), there exists a power series solution (8.5.1) with the coefficients given by (8.5.8). The exact solution of Eq. (8.1.2) is given by

$$\begin{aligned}u(x,t) &= (\exp(\int (\tfrac{\sigma(t)}{\lambda})dt)(c_0 + c_1(-x + \tfrac{1}{\lambda}\int \sigma(t)dt) + c_2(-x + \tfrac{1}{\lambda}\int \sigma(t)dt)^2 \\ &+c_3(-x + \tfrac{1}{\lambda}\int \sigma(t)dt)^3 + \tfrac{1}{24\lambda}(-2c_2k_6\lambda + \lambda k_5c_0c_1 - c_0 - c_1)(-x + \tfrac{1}{\lambda}\int \sigma(t)dt)^4 \\ &+ \sum_{n=1}^{\infty} \tfrac{1}{\lambda(n+1)(n+2)(n+3)(n+4)}(-k_6\lambda(n+1)(n+2)c_{n+2} + k_5\lambda \sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k} \\ &-(n+1)c_{n+1} - c_n)(-x + \tfrac{1}{\lambda}\int \sigma(t)dt)^{n+4}),\end{aligned} \tag{8.5.10}$$

and the solution in the approximate form can be written in terms of the above computation. The details are omitted here.

power series solutions to Eq. (8.4.6)

Similarly, we can derive the power series solution of Eq. (8.4.6) and hence, analytic solution of Eq. (8.1.2) is:

$$\begin{aligned}u(x,t) &= \exp(\int \tfrac{\sigma(t)}{\rho}dt)(c_0 + c_1x + c_2x^2 + c_3x^3 + (\tfrac{1}{24\rho}(-2c_2k_6\rho - \rho k_5c_0c_1 - c_0))x^4 \\ &+ \left(\tfrac{-6c_3k_6\rho - k_5\rho(2c_0c_2 + c_1^2) - c_1}{120\rho}\right)x^5 + \sum_{n=1}^{\infty} \tfrac{1}{\rho(n+1)(n+2)(n+3)(n+4)}(-k_6\rho(n+1)(n+2)c_{n+2} \\ &-c_n - k_5\rho(\sum_{k=0}^{n}(n+1-k)c_kc_{n+1-k}))),\end{aligned} \tag{8.5.11}$$

where $c_i, i = 0, 1, ...3$ are arbitrary constants.

8.6 Discussions

We have studied Painlev$\acute{e}$ property of VCKS equation, which describes many interesting fluid motions. The equation (8.1.2) passes the Painlev$\acute{e}$ test in a weak sense and the auto- B$\ddot{a}$cklund transformation via the truncated Painlev$\acute{e}$ expansion is given out. We have investigated the symmetries and similarity reductions of VCKS equation by using Lie symmetry analysis. The infinitesimals of the group of transformations which leaves VCKS equation invariant and the admissible forms of the coefficients are furnished. An optimal system of conjugacy inequivalent subgroups is then identified with the adjoint action of the symmetry group and all the group-invariant solutions to the equation are considered. Then the power series solutions are investigated by using the power series method. It is also worth mentioning here that the arbitrary physical parameter $\sigma(t)$ in various solutions plays a crucial role as the remaining coefficients of VCKS equation have all been expressed in terms of it. In fact, the various other arbitrary constants occurring in the solutions, along with $\sigma(t)$, provide further freedom to simulate the desired physical situations. In almost all the cases the solutions obtained may also help to recover certain solutions available in literature for the particular models with constant coefficients.

Summary

Partial differential equations (PDEs) arise frequently in the formulation of fundamental laws of nature and in the mathematical analysis of a wide variety of problems in physics as well as in other natural and applied sciences. In the recent decades, there has been tremendous emphasis on understanding and modeling of nonlinear processes: such processes are often governed by non linear partial differential equations. However, exact solutions to nonlinear PDE(s) play an important role for understanding of qualitative as well as quantitative features of many phenomena and processes and provide answer to various scientific problems. Thus the search for exact solutions of such equations is one of the most important stages for mathematical as well as physical description of nature.

The basic partial differential equations, of general relativity, are Einstein field equations. In general, these equations are expressed in terms of coupled partial differential equations describing the matter content of space-time. It is necessary to employ many different techniques in order to obtain exact solutions of these equations. The theoretical and practical importance of Einstein equations due to their common occurrence in the study of various physical phenomena has been the prime reasons for the studies carried out in this thesis. The exact solutions for Einstein equations play a central role for physical or numerical investigations and reflect qualitatively on the behaviour of more complicated

solutions.

Many of mathematical problems involving partial differential equations that arise in Einstein equations are highly nonlinear and so as to obtain manageable problems it is often necessary to make assumptions. A frequent way of doing this is to make symmetry assumptions and hence, find the symmetry class of solutions. Thus by using symmetry considerations many physical phenomenon are explained. However, it is easy to use different formulation of Einstein equations to get a qualitative and sometimes even quantitative understanding of some consequences of general relativity.

In order to avoid the cumbersome process of solving nonlinear equations, different solution generation techniques are used to find solution of field equations of interacting field. This problem is being studied by various authors to establish exact solutions of Einstein field equations in various spacetimes.

The thesis entitled **GROUP THEORETIC TECHNIQUES FOR SOLUTIONS OF EINSTEIN EQUATIONS** is an attempt to obtain symmetries and the exact solutions of Einstein Maxwell equations and Einstein equations for various fields. Some other physically relevant systems of nonlinear partial differential equations representing interesting physical phenomena are also studied for investigating symmetries and to construct exact solutions. To determine the admissible symmetries, the methods - Lie classical approach and other based on the Fréchet derivative of the nonlinear operators have been utilized. Both these techniques consist of steps which can be applied in a very systematic manner. After obtaining the point symmetries of the system under investigation, a formal approach of identifying an optimal system of Lie sub algebras has been adopted with help of the adjoint action of the Lie algebra. The basic generators contained in the optimal system have been exploited to achieve the desired reduction of PDEs to ODEs. The resulting ODEs have been examined subsequently for various types of exact solutions via some techniques which are essentially based on special functions such as Bessel functions, Hyperbolic functions etc.

In chapters 2, 3, 4 and 5, Lie symmetry method has been applied to investigate the symmetries and invariant solutions of Einstein Maxwell equations and Einstein equations for various fields. The vector fields of the optimal system lead to reductions of the nonlinear system of PDEs to ODEs and some new exact solutions are derived. Chapter 6 is devoted to utilize the symmetry method based on the Fr$\acute{e}$chet derivative of the differential operators to obtain the Lie symmetries admitted by field equations of general relativity with an electromagnetic stress tensor as source and Maxwell's equations in curved space, in which metric coefficients and electromagnetic fields are restricted to be the functions of two independent variables only. The symmetries of these equations are exploited to derive some ans$\ddot{a}$tz leading to the reduction of variables, where the analytic solutions are easier to obtain by considering the optimal system of conjugacy inequivalent subgroups.

Kawahara equation and Modified Kawahara equation with variable coefficients and variable coefficients Kuramoto-Sivashinsky equation are studied in chapter 7 and 8. The admissible forms of coefficients have been derived for which these nonlinear equations possess the Lie symmetries. Further, we have successfully established exact and explicit analytic solutions with arbitrary parameters to variable coefficients Kawahara equation and variable coefficients Modified Kawahara equation via generalized $\left(\frac{G'}{G}\right)$-expansion method. These solutions are expressed in terms of the hyperbolic functions, trigonometric functions and rational functions. Most of the solutions obtained involve an arbitrary coefficient function and it may enable one to control and discuss the behaviour of solution as governed by the choice of this arbitrary function.

Another important aspect that deserves special attention is to study Painlev$\acute{e}$ property that helps in deciding whether system under investigation is integrable or not. A differential equation is said to have the Painlev$\acute{e}$ property if all the movable singularities of all its solutions are poles. In chapter 8, we have performed Painlev$\acute{e}$ analysis of variable coefficients Kuramoto-Sivashinsky equation. We found that variable coefficients Kuramoto-Sivashinsky equation passes the Painlev$\acute{e}$ test in a weak sense and the auto-B$\ddot{a}$cklund transformation via the truncated Painlev$\acute{e}$ expansion is given out.

Finally, it is worth mentioning that in spite of the focus on the exact solutions, the author found it really difficult at times to handle the resulting systems of ODEs for extracting the solutions. Since, after reduction to ODEs, the further attempt to apply Lie group analysis to ODEs has been made, but no further physically important nontrivial symmetries comes out, hence the solutions of ODEs are obtained directly.

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