

**RELIABILITY ANALYSIS OF SOME
INDUSTRIAL SYSTEMS
IN
CRISP AND FUZZY ENVIRONMENTS**

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**IN
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BY

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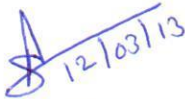
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March - 2013

CERTIFICATE

This is to certify that the thesis entitled, “**Reliability Analysis of Some Industrial Systems in Crisp and Fuzzy Environments**”, submitted by Ms. Sneh Lata in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidate’s own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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It is certified that the thesis is entirely my own and the ideas and references cited herein have been duly acknowledged.

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THE ALMIGHTY

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MY PARENTS

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Abstract

In this thesis, the drawback of an existing method, based on fuzzy Markov model, is pointed out and new methods are proposed to resolve this drawback. Also, the concept of fuzzy Markov model is extended by vague Markov model and new methods, based on proposed vague Markov model, are proposed to evaluate the vague reliability of industrial systems.

The chapter wise summary of the thesis is as follows:

Chapter 1

In this chapter, a brief review of the work done in the area of fuzzy reliability analysis and vague reliability analysis is presented.

Chapter 2

To evaluate the fuzzy reliability through fuzzy Markov modeling there is a need to solve fuzzy Kolmogorov's differential equations. Liu and Huang [106] used Laplace-Stieltjes transform for solving fuzzy Kolmogorov's differential equations. In this chapter, it is pointed out that the use of Laplace-Stieltjes transform is not genuine for solving fuzzy Kolmogorov's differential equations and the advantage of Buckley and Feuring method [24] over Liu and Huang method [106] is discussed. Also, fuzzy Kolmogorov's differential equations, developed by using fuzzy

Markov model of condensate system, are solved by applying Buckley and Feuring method [24] and with the help of obtained solution, fuzzy reliability of condensate system is evaluated.

Chapter 3

In the previous chapter, Buckley and Feuring method [24] is used for solving fuzzy Kolmogorov's differential equations obtained by fuzzy Markov model of condensate system. In this chapter, it is illustrated that if the same method is applied for solving fuzzy Kolmogorov's differential equations, obtained by fuzzy Markov model of piston manufacturing system, then the obtained solution does not define α -cut of a fuzzy number. Consequently, the solution of fuzzy Kolmogorov's differential equations, obtained by applying Buckley and Feuring method [24], cannot be used to evaluate the fuzzy reliability of piston manufacturing system. Also, by considering a system of fuzzy equations, it is confirmed that the drawback of Buckley and Feuring method [24] is occurring due to existing representation of α -cut. To resolve the drawback of Buckley and Feuring method [24], a new representation of α -cut is proposed. With the help of proposed α -cut, Buckley and Feuring method is modified. Also, fuzzy Kolmogorov's differential equations, obtained by fuzzy Markov model of piston manufacturing system, are solved by using the modified method and with the help of obtained solution, fuzzy reliability of piston manufacturing system is evaluated.

Chapter 4

In this chapter, a new method is proposed for solving n^{th} order fuzzy lin-

ear differential equations with the help of *JMD* trapezoidal fuzzy number [81] and it is shown that on solving the same fuzzy Kolmogorov's differential equations by applying the method, proposed in this chapter as well as by applying the modified method, proposed in previous chapter, same results are obtained. However, it is much easy to apply the method, proposed in this chapter, as compared to the modified method, proposed in previous chapter. To illustrate the proposed method, the fuzzy Kolmogorov's differential equations, developed in previous chapter, are solved and with the help of obtained solution, fuzzy reliability of piston manufacturing system is evaluated. The advantage of *JMD* trapezoidal fuzzy number over existing trapezoidal fuzzy number is also discussed.

Chapter 5

In real life, a person may assume that an object belongs to a set but it is possible that he (she) is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use vague set [55]. To evaluate the vague reliability of industrial systems through vague Markov modeling, there is a need to solve vague Kolmogorov's differential equations. Since, there is no method in the literature for solving vague differential equations. Therefore, several authors have used vague fault tree to evaluate the vague reliability of industrial systems but till now no one have utilized vague Markov model for the same. In this chapter, a new representation of trapezoidal vague set, named as *JMD* trapezoidal vague set, is proposed and with the help of

JMD trapezoidal vague set, a new method is proposed for solving n^{th} order vague linear differential equations. To illustrate the proposed method, vague Kolmogorov's differential equations, obtained by vague Markov model of piston manufacturing system, are solved and the obtained solution is used to evaluate the vague reliability of piston manufacturing system. The advantage of *JMD* trapezoidal vague set over existing trapezoidal vague set is also discussed.

Chapter 6

In real life situation, it is possible that at a particular time degree of acceptance of decision maker about a particular value is zero but the degree of rejection and degree of hesitation are not equal to zero. The trapezoidal vague set, presented in previous chapter, cannot be used to handle such real life situations. Therefore, Kumar [79] extended the concept of trapezoidal vague set by generalized trapezoidal vague set to handle such situations. On the same direction, in this chapter, the concept of *JMD* trapezoidal vague set is extended by generalized *JMD* trapezoidal vague set and by modifying the method, proposed in previous chapter, a new method is proposed for solving generalized vague differential equations. With the help of proposed method, generalized vague Kolmogorov's differential equations are solved and the obtained solution is used to evaluate the generalized vague reliability of piston manufacturing system. The advantage of generalized *JMD* trapezoidal vague set over existing generalized trapezoidal vague set is also discussed.

Chapter 7

Finally, in this chapter, based on the presented study, future work has been suggested.

List of Research Papers

1. **S. Lata**, A. Kumar, Mehar's method for analyzing the fuzzy reliability of piston manufacturing system, *Eksploatacja i Niezawodnosc - Maintenance and Reliability* 51 (2011) 26-39.
2. **S. Lata**, A. Kumar, A new method to solve time-dependent intuitionistic fuzzy differential equations and its application to analyze the intuitionistic fuzzy reliability of industrial systems, *Concurrent Engineering: Research and Applications* 20 (2012) 177-184.
3. **S. Lata**, A. Kumar, A new method for solving differential equations with vague parameters, *Applied Mathematical Modeling* 37 (2013) 3282-3295.
4. A. Kumar, **S. Lata**, A note on "Fuzzy initial value problem for N^{th} -order fuzzy linear differential equations", *Journal of Fuzzy Set Valued Analysis* 2012 (2012) 1-3.
5. A. Kumar, **S. Lata**, Reliability evaluation of condensate system using fuzzy Markov model, *Annals of Fuzzy Mathematics and Informatics* 4 (2012) 281-291.
6. A. Kumar, **S. Lata**, Reliability analysis of piston manufacturing system, *Journal of Reliability and Statistical Studies* 4 (2011) 43-55.

7. A. Kumar, **S. Lata**, Vague reliability analysis using vague differential equations, International Journal of Performability Engineering 8 (2012) 489-496.
8. **S. Lata**, A. Kumar, A new analytical method for solving fuzzy differential equations, Iranian Journal of Fuzzy Systems (Accepted).
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Chapter 1

INTRODUCTION

Due to growing complexity of modern engineering systems, reliability becomes very major issue of every system. The reliability of a system is defined as the probability that the system performs its assigned functions properly during a predefined period under a given environment.

Out of the various reliability tools, fault tree and Markov model are the common reliability tools that are used for analyzing the reliability of industrial systems. Although, fault trees are very useful but there exist some factors such as partial failures, coverage and repairable systems that are not covered well by fault tree approach. Markov modeling is a valuable tool for dealing with these factors [100].

In general, industrial systems are considered as repairable and their historical databases are in the form of failure and repair rates. It is well known that most databases, on which most of the reliability analysis depend, are either out of date or collected under different operating and environmental conditions.

To estimate failure and repair rates more accurately, large quantity of data will be required. Due to rare event of components, human error and economic restraints, it is difficult to obtain a large quantity of data. Thus, experts usually base their estimation on previous engineering experiences. From this point of view, fuzzy probabilities are better suited to model such judgements in a flexible and efficient

manner. The use of fuzzy sets to describe sub-system and component reliability allows input from a wide range of data sources collected under many different conditions.

In real life, a person may assume that an object belongs to a set but it is possible that he (she) is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use vague set [55]. Due to the same reason, in the literature, the concept of fuzzy reliability is extended by vague reliability.

1.1 Literature review

In this section, a brief review of the work done in the area of fuzzy reliability analysis and vague reliability analysis is presented.

To the best of our knowledge, Tanaka et al. [149] firstly proposed the concept of fault tree and reliability analysis using fuzzy set theory [169] in which the relative frequencies of the basic events are considered as fuzzy numbers.

Cai et al. [29] pointed out that due to the inaccuracy and uncertainty in data, the estimation of precise values of probability becomes very difficult in many systems and presented fuzzy-state assumption (the system failure can't be precisely defined in a reasonable way. At any time the system may be in one of the following two states: fuzzy success state or fuzzy failure state) and possibility assumption (the system behavior can be fully characterized in the context of possibility measures).

Further, Cai et al. [30] proposed reliability theories based on probability assumption with fuzzy-state assumption, possibility assumption with binary state

assumption and possibility assumption with fuzzy-state assumption. Cheng and Mon [46] proposed an approach, based on interval of confidence, for analyzing the fuzzy system reliability and discussed the advantage of proposed approach over the existing approach [149]. Chen [40] used arithmetic operations of fuzzy numbers to analyze the fuzzy reliability of series and parallel systems and shown that their method is much easy to apply as compared to the existing method [46].

Mon and Cheng [121] pointed out that in real life situations, it is not always possible to represent the failure probability of each basic event of fault tree by same type of fuzzy numbers and proposed a method for analyzing the fuzzy reliability of such systems in which the failure probabilities of all the basic components are represented by different types of fuzzy numbers.

Leuschen [100] proposed a method, based on fuzzy Markov model (Markov model in which the failure rate and repair rate of each sub-system are represented by fuzzy numbers), to analyze the fuzzy reliability.

Hong and Do [68] discussed the advantage of weakest t -norm based arithmetic operations over α -cut based arithmetic operations and proposed weakest t -norm based fuzzy fault tree for analyzing the fuzzy reliability. Chanda and Bhattacharjee [37] used trapezoidal fuzzy number based fault tree approach to evaluate the fuzzy reliability of transmission expansion planning. Chen [43] proposed the expressions for evaluating the vague reliability of series and parallel systems by assuming the reliability of each component as a triangular vague set.

Taniroven et al. [150] used fuzzy Markov model for analyzing the fuzzy reliability of a transmission system. Kumar et al. [86] developed a trapezoidal vague set based fault tree (fault tree in which probability of failure of each basic event is

represented by trapezoidal vague set) for evaluating the vague reliability of a marine power plant. Yuhua and Datao [168] used triangular fuzzy number based fuzzy fault tree to analyze the fuzzy reliability of oil and gas transmission pipelines.

Kumar et al. [87] discussed the advantage of interval valued trapezoidal vague set over trapezoidal vague set and developed an interval valued trapezoidal vague set based fault tree (fault tree in which probability of failure of each basic event is represented by an interval valued trapezoidal vague set) for analyzing the interval valued vague reliability of a computer system.

Shu et al. [142] developed a trapezoidal based vague fault tree for analyzing the vague reliability of a printed circuit board assembly. Chang et al. [38] developed a trapezoidal based vague fault tree to analyze the vague reliability of weapon systems.

Binh and Khoa [20] used fuzzy Markov model to evaluate the fuzzy reliability of power systems. Kumar et al. [92] proposed the weakest t -norm based arithmetic operations between interval valued vague sets and used it for analyzing the vague reliability of a basement flooding. Praba et al. [127] used fuzzy Markov model to evaluate the fuzzy reliability of communication network. Chang and Cheng [39] proposed a method for analyzing the fuzzy reliability of such industrial systems in which failure probabilities of all the components are represented by different type of vague numbers.

Liu and Huang [106] proposed a method, based on fuzzy Markov model, to analyze the fuzzy reliability of multi-state systems. Kumar and Yadav [84] proposed an approach for analyzing vague reliability using different types of vague sets. Kumar et al. [96] used vague fault tree approach to analyze the vague reliability

of computer security system. Kumar and Yadav [85] proposed the weakest t -norm based vague fault tree for analyzing the fuzzy system reliability.

After reviewing the literature, it is concluded that several authors have used fuzzy fault tree and vague fault tree to evaluate the fuzzy reliability and vague reliability of industrial systems. But, to the best of our knowledge, very few authors have used fuzzy Markov model to evaluate the fuzzy reliability of industrial systems and no one have used vague Markov model to evaluate the vague reliability of industrial systems.

In this thesis, the drawback of existing method [106], based on fuzzy Markov model, is pointed out and new methods are proposed to resolve this drawback. Also, the concept of fuzzy Markov model is extended by vague Markov model and new methods, based on proposed vague Markov model, are proposed to evaluate the vague reliability of industrial systems.

1.2 Organization of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

To evaluate the fuzzy reliability through fuzzy Markov modeling there is a need to solve fuzzy Kolmogorov's differential equations. Liu and Huang [106] used Laplace-Stieltjes transform for solving fuzzy Kolmogorov's differential equations. In this chapter, it is pointed out that the use of Laplace-Stieltjes transform is not genuine for solving fuzzy Kolmogorov's differential equations and the advantage of Buckley and Feuring method [24] over Liu and Huang method [106] is discussed. Also, fuzzy Kolmogorov's differential equations, developed by using fuzzy Markov

model of condensate system, are solved by applying Buckley and Feuring method [24] and with the help of obtained solution, fuzzy reliability of condensate system is evaluated.

Chapter 3

In the previous chapter, Buckley and Feuring method [24] is used for solving fuzzy Kolmogorov's differential equations obtained by fuzzy Markov model of condensate system. In this chapter, it is illustrated that if the same method is applied for solving fuzzy Kolmogorov's differential equations, obtained by fuzzy Markov model of piston manufacturing system, then the obtained solution does not define α -cut of a fuzzy number. Consequently, the solution of fuzzy Kolmogorov's differential equations, obtained by applying Buckley and Feuring method [24], cannot be used to evaluate the fuzzy reliability of piston manufacturing system. Also, by considering a system of fuzzy equations, it is confirmed that the drawback of Buckley and Feuring method [24] is occurring due to existing representation of α -cut. To resolve the drawback of Buckley and Feuring method [24], a new representation of α -cut is proposed. With the help of proposed α -cut, Buckley and Feuring method is modified. Also, fuzzy Kolmogorov's differential equations, obtained by fuzzy Markov model of piston manufacturing system, are solved by using the modified method and with the help of obtained solution, fuzzy reliability of piston manufacturing system is evaluated.

Chapter 4

In this chapter, a new method is proposed for solving n^{th} order fuzzy linear differential equations with the help of *JMD* trapezoidal fuzzy number [81] and

it is shown that on solving the same fuzzy Kolmogorov's differential equations by applying the method, proposed in this chapter as well as by applying the modified method, proposed in previous chapter, same results are obtained. However, it is much easy to apply the method, proposed in this chapter, as compared to the modified method, proposed in previous chapter. To illustrate the proposed method, the fuzzy Kolmogorov's differential equations, developed in previous chapter, are solved and with the help of obtained solution, fuzzy reliability of piston manufacturing system is evaluated. The advantage of *JMD* trapezoidal fuzzy number over existing trapezoidal fuzzy number is also discussed.

Chapter 5

In real life, a person may assume that an object belongs to a set but it is possible that he (she) is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use vague set [55]. To evaluate the vague reliability of industrial systems through vague Markov modeling, there is a need to solve vague Kolmogorov's differential equations. Since, there is no method in the literature for solving vague differential equations. Therefore, several authors have used vague fault tree to evaluate the vague reliability of industrial systems but till now no one have utilized vague Markov model for the same. In this chapter, a new representation of trapezoidal vague set, named as *JMD* trapezoidal vague set, is proposed and with the help of *JMD* trapezoidal vague set, a new method is proposed for solving n^{th} order vague linear differential equations. To illustrate the proposed method, vague Kolmogorov's

differential equations, obtained by vague Markov model of piston manufacturing system, are solved and the obtained solution is used to evaluate the vague reliability of piston manufacturing system. The advantage of *JMD* trapezoidal vague set over existing trapezoidal vague set is also discussed.

Chapter 6

In real life situation, it is possible that at a particular time degree of acceptance of decision maker about a particular value is zero but the degree of rejection and degree of hesitation are not equal to zero. The trapezoidal vague set, presented in previous chapter, cannot be used to handle such real life situations. Therefore, Kumar [79] extended the concept of trapezoidal vague set by generalized trapezoidal vague set to handle such situations. On the same direction, in this chapter, the concept of *JMD* trapezoidal vague set is extended by generalized *JMD* trapezoidal vague set and by modifying the method, proposed in previous chapter, a new method is proposed for solving generalized vague differential equations. With the help of proposed method, generalized vague Kolmogorov's differential equations are solved and the obtained solution is used to evaluate the generalized vague reliability of piston manufacturing system. The advantage of generalized *JMD* trapezoidal vague set over existing generalized trapezoidal vague set is also discussed.

Chapter 7

Finally, in this chapter, based on the presented study, future work has been suggested.

Chapter 2

FUZZY RELIABILITY ANALYSIS OF CONDENSATE SYSTEM

To evaluate the fuzzy reliability through fuzzy Markov modeling there is a need to solve fuzzy Kolmogorov's differential equations. Liu and Huang [106] used Laplace-Stieltjes transform for solving fuzzy Kolmogorov's differential equations. In this chapter, it is pointed out that the use of Laplace-Stieltjes transform is not genuine for solving fuzzy Kolmogorov's differential equations and the advantage of Buckley and Feuring method [24] over Liu and Huang method [106] is discussed. Also, fuzzy Kolmogorov's differential equations, developed by using fuzzy Markov model of condensate system, are solved by applying Buckley and Feuring method [24] and with the help of obtained solution, fuzzy reliability of condensate system is evaluated.

2.1 Preliminaries

In this section, some basic definitions, arithmetic operations between α -cut of trapezoidal fuzzy numbers and arithmetic operations between trapezoidal fuzzy numbers are presented [74].

The contents of this chapter are published in *Annals of Fuzzy Mathematics and Informatics* 4 (2012) 281-291.

2.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 2.1 Let X be a classical set of objects. Then, the set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ is called a fuzzy set in X . The evaluation function $\mu_{\tilde{A}}(x)$ is called the membership function.

Definition 2.2 Let \tilde{A} be a fuzzy set in X and $\alpha \in [0, 1]$ be a real number. Then, a classical set $A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$ is called an α -level set or α -cut of \tilde{A} .

Definition 2.3 A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a normalized fuzzy set if and only if $\text{Supremum}_{x \in X} \{\mu_{\tilde{A}}(x)\} = 1$.

Definition 2.4 A fuzzy set \tilde{A} is called a convex fuzzy set if and only if

$$\mu_{\tilde{A}}(\alpha x_1 + (1 - \alpha)x_2) \geq \text{Minimum}\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}, \quad \forall x_1, x_2 \in X, \alpha \in [0, 1].$$

Definition 2.5 A convex normalized fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a fuzzy number if and only if $\mu_{\tilde{A}}(x)$ is piecewise continuous in X .

Definition 2.6 A fuzzy number \tilde{A} is said to be non-negative fuzzy number if and only if $\mu_{\tilde{A}}(x) = 0 \quad \forall x < 0$.

Definition 2.7 A fuzzy number \tilde{A} defined on the universal set of real numbers \mathbb{R} , denoted as $\tilde{A} = (a, b, c, d)$, is said to be a trapezoidal fuzzy number, shown in Figure 2.1, if its membership function $\mu_{\tilde{A}}(x)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \leq x < b \\ 1, & b \leq x \leq c \\ \frac{(x-d)}{(c-d)}, & c < x \leq d \\ 0, & \text{otherwise} \end{cases}$$

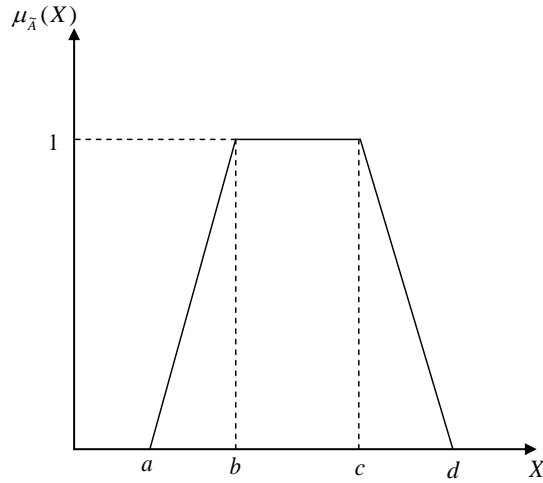


Figure 2.1 Trapezoidal fuzzy number \tilde{A}

Definition 2.8 Let $\tilde{A} = (a, b, c, d)$ be a trapezoidal fuzzy number. Then, its α -cut A_α is defined as follows:

$$A_\alpha = [a + (b - a)\alpha, d - (d - c)\alpha], \quad 0 \leq \alpha \leq 1$$

Definition 2.9 Let $A = [a_1, b_1]$ and $B = [a_2, b_2]$ be α -cuts of trapezoidal fuzzy numbers \tilde{A} and \tilde{B} respectively. Then, $\tilde{A} = \tilde{B}$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

Definition 2.10 A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is said to be non-negative trapezoidal fuzzy number if and only if $a \geq 0$.

Definition 2.11 A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is said to be zero trapezoidal fuzzy number if and only if $a = 0, b = 0, c = 0$ and $d = 0$.

Definition 2.12 Two trapezoidal fuzzy numbers $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ are said to be equal i.e., $\tilde{A}_1 = \tilde{A}_2$ if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_2$ and $d_1 = d_2$.

2.1.2 Arithmetic operations between α -cut of trapezoidal fuzzy numbers

In this section, some arithmetic operations between α -cut of trapezoidal fuzzy numbers are presented.

Let $A = [a_1, b_1]$ and $B = [a_2, b_2]$ be α -cuts of trapezoidal fuzzy numbers \tilde{A} and \tilde{B} respectively. Then,

$$(i) A + B = [a_1 + a_2, b_1 + b_2]$$

$$(ii) A - B = [a_1 - b_2, b_1 - a_2]$$

$$(iii) \lambda A = \begin{cases} [\lambda a_1, \lambda b_1], & \lambda \geq 0 \\ [\lambda b_1, \lambda a_1], & \lambda \leq 0 \end{cases}$$

$$(iv) AB = [a, b]$$

where,

$$a = \text{minimum } (a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2)$$

$$b = \text{maximum } (a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2)$$

2.1.3 Arithmetic operations between trapezoidal fuzzy numbers

In this section, some arithmetic operations between trapezoidal fuzzy numbers are presented.

Let $\tilde{A} = (a_1, b_1, c_1, d_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then,

$$(i) \tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

$$(ii) \tilde{A} \ominus \tilde{B} = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$$

$$(iii) \lambda \tilde{A} = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1), & \lambda \geq 0 \\ (\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1), & \lambda \leq 0 \end{cases}$$

$$(iv) \tilde{A} \otimes \tilde{B} = (a', b', c', d')$$

where,

$$a' = \text{minimum } (a_1 a_2, a_1 d_2, d_1 a_2, d_1 d_2)$$

$$b' = \text{minimum } (b_1b_2, b_1c_2, c_1b_2, c_1c_2)$$

$$c' = \text{maximum } (b_1b_2, b_1c_2, c_1b_2, c_1c_2)$$

$$d' = \text{maximum } (a_1a_2, a_1d_2, d_1a_2, d_1d_2)$$

Remark 2.1 If $b = c$ then a trapezoidal fuzzy number (a, b, c, d) is said to be triangular fuzzy number and is denoted as (a, b, b, d) or (a, c, c, d) or (a, b, d) or (a, c, d) .

2.2 Liu and Huang method for solving fuzzy Kolmogorov's differential equations

Liu and Huang [106] proposed the following method to evaluate the fuzzy probability at any time instant t .

Step 1: Use the fuzzy Markov model, depicted in Figure 2.2, to write the fuzzy Kolmogorov's differential equation (2.1)

$$\left\{ \begin{array}{l} \frac{d\tilde{p}_{(l,k_l)}(t)}{dt} = \ominus \tilde{p}_{(l,k_l)}(t) \otimes \sum_{j=1}^{k_l-1} \tilde{\lambda}_{(k_l,j)}^l \\ \frac{d\tilde{p}_{(l,i)}(t)}{dt} = \sum_{j=i+1}^{k_l} \tilde{\lambda}_{(j,i)}^l \otimes \tilde{p}_{(l,j)}(t) \ominus \tilde{p}_{(l,i)}(t) \otimes \sum_{j=1}^{i-1} \tilde{\lambda}_{(i,j)}^l, 1 \leq i \leq k_l, t \geq 0 \\ \frac{d\tilde{p}_{(l,1)}(t)}{dt} = \sum_{j=2}^{k_l} \tilde{\lambda}_{(j,1)}^l \otimes \tilde{p}_{(l,j)}(t) \end{array} \right. \quad (2.1)$$

with initial conditions:

$$\tilde{p}_{(l,k_l)}(0) = 1 \quad (2.2)$$

$$\tilde{p}_{(l,i)}(0) = 0 \text{ for } (i \neq k_l) \quad (2.3)$$

2.2.1 Shortcomings of Liu and Huang method

It is not genuine to apply Liu and Huang method [106], presented in Section 2.2, due to the following reasons:

1. In Step 3 of Liu and Huang method [106], it is assumed that

$$\tilde{A} \ominus \tilde{B} = \tilde{C} \ominus \tilde{D} \Rightarrow \tilde{A} \oplus \tilde{D} = \tilde{B} \oplus \tilde{C}$$

However, $\tilde{A} \ominus \tilde{B} = \tilde{C} \ominus \tilde{D} \not\Rightarrow \tilde{A} \oplus \tilde{D} = \tilde{B} \oplus \tilde{C}$.

e.g., If $\tilde{A} = (1, 3, 4, 6)$, $\tilde{B} = (2, 5, 10, 12)$, $\tilde{C} = (3, 5, 8, 10)$ and $\tilde{D} = (6, 9, 12, 14)$

then $\tilde{A} \ominus \tilde{B} = \tilde{C} \ominus \tilde{D} = (-11, -7, -1, 4)$

However, $\tilde{A} \oplus \tilde{D} = (7, 12, 16, 20) \neq \tilde{B} \oplus \tilde{C} = (5, 10, 18, 22)$.

2. In Step 4 of Liu and Huang method [106], it is assumed that

$$\tilde{A} \otimes \tilde{B} = \tilde{C} \Rightarrow \tilde{A} = \frac{\tilde{C}}{\tilde{B}}$$

However, $\tilde{A} \otimes \tilde{B} = \tilde{C} \not\Rightarrow \tilde{A} = \frac{\tilde{C}}{\tilde{B}}$.

e.g., If $\tilde{A} = (1, 2, 3, 4)$ and $\tilde{B} = (1, 3, 4, 6)$ then

$$\tilde{A} \otimes \tilde{B} = \tilde{C} = (1, 6, 12, 24)$$

However, $\frac{\tilde{C}}{\tilde{B}} = \frac{(1,6,12,24)}{(1,3,4,6)} = (1/6, 3/2, 4, 24) \neq \tilde{A}$

Remark 2.1: Let $\tilde{A} = (a_1, b_1, c_1, d_1)$, $\tilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers and $a_2 > 0$ or $d_2 < 0$. Then, $\frac{\tilde{A}}{\tilde{B}} = (a', b', c', d')$

where,

$$a' = \text{minimum } (a_1/a_2, a_1/d_2, d_1/a_2, d_1/d_2)$$

$$b' = \text{minimum } (b_1/b_2, b_1/c_2, c_1/b_2, c_1/c_2)$$

$$c' = \text{maximum } (b_1/b_2, b_1/c_2, c_1/b_2, c_1/c_2)$$

$$d' = \text{maximum } (a_1/a_2, a_1/d_2, d_1/a_2, d_1/d_2)$$

2.3 Buckley and Feuring method for solving n^{th} order fuzzy linear differential equations

Due to the shortcomings of Liu and Huang method [106] for solving fuzzy Kolmogorov's differential equations, discussed in Section 2.2.1, one may try to use any other existing method for solving fuzzy Kolmogorov's differential equations. To the best of our knowledge, there exist only two analytical methods [24] for solving n^{th} order fuzzy linear differential equations. The first method is to fuzzify the crisp solution and to check that the obtained solution satisfies the differential equation or not and the second method is to solve fuzzy differential equation and to check that it defines a fuzzy function or not. In this section, one of these existing methods [24] is presented.

The solution of n^{th} order fuzzy linear differential equation

$$\sum_{j=0}^n \tilde{a}_j \otimes \tilde{y}^{(j)} = \tilde{g} \quad (2.7)$$

with initial conditions:

$$\tilde{y}^{(j)}(x_0) = \tilde{\gamma}_j, \quad j = 0, 1, \dots, n-1 \quad (2.8)$$

where, $\tilde{y}^{(j)} = \frac{d^j \tilde{y}}{dx^j}$, \tilde{a}_j , \tilde{g} and $\tilde{\gamma}_j$ are trapezoidal fuzzy numbers, can be obtained by using the following steps.

Step 1: Find the α -cuts $[a_{j(1)}(x, \alpha), a_{j(2)}(x, \alpha)]$, $[y_1^{(j)}(x, \alpha), y_2^{(j)}(x, \alpha)]$, $[g_1(x, \alpha), g_2(x, \alpha)]$ and $[\gamma_{j(1)}(x_0, \alpha), \gamma_{j(2)}(x_0, \alpha)]$, corresponding to fuzzy parameters \tilde{a}_j , $\tilde{y}^{(j)}$, \tilde{g} and $\tilde{\gamma}_j$ respectively.

Step 2: Convert the n^{th} order fuzzy linear differential equation (2.7) with initial conditions (2.8) into the following n^{th} order linear differential equation:

$$\sum_{j=0}^n [a_{j(1)}(x, \alpha), a_{j(2)}(x, \alpha)] [y_1^{(j)}(x, \alpha), y_2^{(j)}(x, \alpha)] = [g_1(x, \alpha), g_2(x, \alpha)] \quad (2.9)$$

with initial conditions:

$$[y_1^{(j)}(x_0, \alpha), y_2^{(j)}(x_0, \alpha)] = [\gamma_{j(1)}(x_0, \alpha), \gamma_{j(2)}(x_0, \alpha)], j = 0, 1, \dots, n - 1 \quad (2.10)$$

Step 3: Using Definition 2.9 and Section 2.1.2, n^{th} order linear differential equation (2.9) with initial conditions (2.10) can be split into the following n^{th} order linear differential equations:

$$\sum_{j=0}^n \text{Minimum}(a_{j(1)}(x, \alpha)y_1^{(j)}(x, \alpha), a_{j(1)}(x, \alpha)y_2^{(j)}(x, \alpha), a_{j(2)}(x, \alpha)y_1^{(j)}(x, \alpha), a_{j(2)}(x, \alpha)y_2^{(j)}(x, \alpha)) = g_1(x, \alpha) \quad (2.11)$$

$$\sum_{j=0}^n \text{Maximum}(a_{j(1)}(x, \alpha)y_1^{(j)}(x, \alpha), a_{j(1)}(x, \alpha)y_2^{(j)}(x, \alpha), a_{j(2)}(x, \alpha)y_1^{(j)}(x, \alpha), a_{j(2)}(x, \alpha)y_2^{(j)}(x, \alpha)) = g_2(x, \alpha) \quad (2.12)$$

with initial conditions:

$$y_1^{(j)}(x_0, \alpha) = \gamma_{j(1)}(x_0, \alpha), j = 0, 1, \dots, n - 1 \quad (2.13)$$

$$y_2^{(j)}(x_0, \alpha) = \gamma_{j(2)}(x_0, \alpha), j = 0, 1, \dots, n - 1 \quad (2.14)$$

Step 4: Solve the n^{th} order linear differential equations (2.11) and (2.12) with initial conditions (2.13) and (2.14) to find the values of $y_1(x_1, \alpha)$ and $y_2(x_1, \alpha)$ corresponding to $x = x_1$, where x_1 is any real number.

Step 5: Check that $[y_1(x_1, \alpha), y_2(x_1, \alpha)]$ defines the α -cut of a fuzzy number or not i.e., for the values of $y_1(x_1, \alpha)$ and $y_2(x_1, \alpha)$, the following conditions are satisfied or not.

(i) $y_1(x_1, \alpha)$ is monotonically increasing function for $\alpha \in [0, 1]$.

(ii) $y_2(x_1, \alpha)$ is monotonically decreasing function for $\alpha \in [0, 1]$.

Case 1: If $[y_1(x_1, \alpha), y_2(x_1, \alpha)]$ defines α -cut of a fuzzy number then the solution $\tilde{y}(x_1)$ of fuzzy differential equation (2.7) with initial conditions (2.8) exists and $[y_1(x_1, \alpha), y_2(x_1, \alpha)]$ represents the α -cut corresponding to solution $\tilde{y}(x_1)$.

Case 2: If $[y_1(x_1, \alpha), y_2(x_1, \alpha)]$ does not define α -cut of a fuzzy number then the solution $\tilde{y}(x_1)$ of fuzzy differential equation (2.7) with initial conditions (2.8) does not exist.

2.3.1 Advantage of Buckley and Feuring method

As discussed in Section 2.2.1, for solving fuzzy differential equations by using Laplace-Stieltjes transform there is a need to apply the following properties:

$$\tilde{A} \ominus \tilde{B} = \tilde{C} \ominus \tilde{D} \Rightarrow \tilde{A} \oplus \tilde{D} = \tilde{B} \oplus \tilde{C}$$

$$\tilde{A} \otimes \tilde{B} = \tilde{C} \Rightarrow \tilde{A} = \frac{\tilde{C}}{\tilde{B}}$$

However, for solving fuzzy differential equations by using Buckley and Feuring method [24] there is no need to apply these properties. So, on solving fuzzy differential equations by using Buckley and Feuring method [24] all the shortcomings, occurring in the results, due to applying Liu and Huang method [106], are resolved. Hence, it is better to use Buckley and Feuring method [24] as compared to Liu and Huang method [106] for solving fuzzy Kolmogorov's differential equations.

2.4 Case Study

Gupta and Tewari [62] used Markov model with crisp parameters to evaluate the crisp reliability of condensate system. In this section, to explain the method for evaluating the fuzzy reliability of industrial systems, based on fuzzy Markov model, fuzzy Markov model of condensate system is obtained by replacing the crisp parameters with fuzzy parameters and with the help of obtained fuzzy Markov model, fuzzy Kolmogorov's differential equations are developed. Due to the shortcomings of Liu and Huang method [106], discussed in Section 2.2.1, the developed fuzzy Kolmogorov's differential equations are solved by using Buckley and Feuring

method [24] and the obtained solution is used to evaluate the fuzzy reliability of condensate system.

2.4.1 Fuzzy Markov modeling of condensate system

Condensate system helps the power plants to function efficiently and keeps them in continuous operation for optimal performance. Condensate system consists of six sub-systems namely condenser, gland steam condenser, drain cooler, low pressure heaters, deaerator, condensate extraction pumps connected in series. Fuzzy Markov model of condensate system is shown in Figure 2.3.

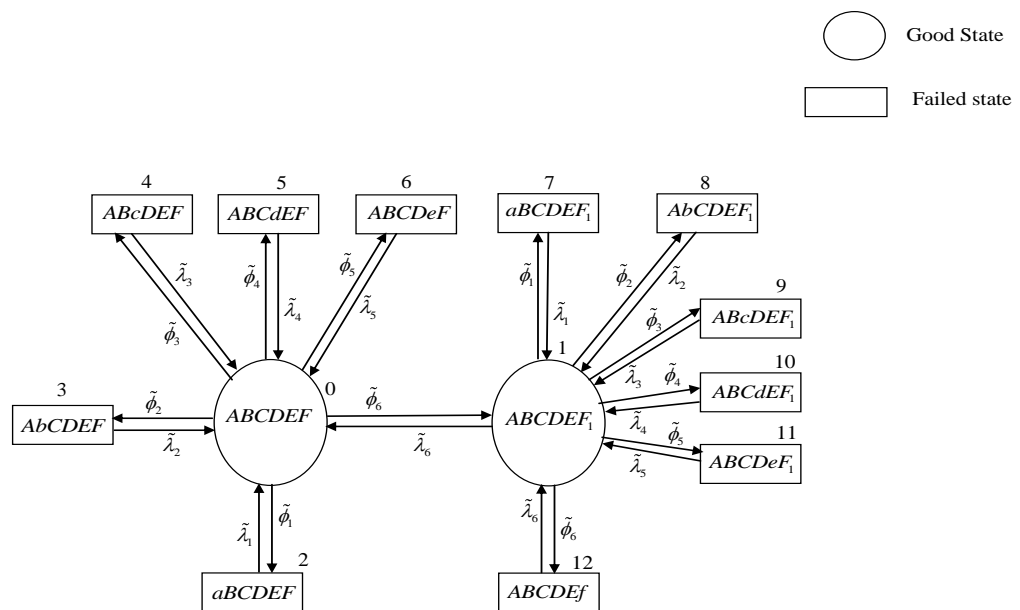


Figure 2.3 Fuzzy Markov model of condensate system

1. Sub-system *A* consists of condenser. It is single unit arranged in series. Failure of this unit causes the complete failure of the system.
2. Sub-system *B* consists of gland steam condenser arranged in series. Failure of this unit causes the complete failure of the system.
3. Sub-system *C* consists of one drain cooler arranged in series. Failure of this unit causes the complete failure of the system.

4. Sub-system D consists of three low pressure heaters arranged in series. Failure of any one unit causes the complete failure of the system.
5. Sub-system E consists of deaerator arranged in series. Failure of this unit causes the complete failure of the system.
6. Sub-system F consists of two condensate extraction pumps arranged in parallel; one operative and other in cold standby. Complete failure of the system will occur when both failed at a time.

2.4.2 Notation

In this section, notation that is used to evaluate the fuzzy reliability of condensate system are presented.

1. A, B, C, D, E, F represent full working states of sub-systems.
2. The symbols a, b, c, d, e, f represent failed states of sub-systems A, B, C, D, E, F respectively.
3. F_1 denote that the sub-system F is working on standby unit.
4. $\tilde{P}_0(t)$ represents fuzzy probability of the system working with full capacity at time t .
5. $\tilde{P}_1(t)$ represents fuzzy probability of the system in cold standby state.
6. $\tilde{P}_2(t)$ to $\tilde{P}_{12}(t)$ represent fuzzy probability of the system in failed state.
7. $\tilde{\phi}_i, i = 1$ to 6 represents fuzzy failure rates of sub-systems A, B, C, D, E and F respectively.
8. $\tilde{\lambda}_i, i = 1$ to 6 represents fuzzy repair rates of sub-systems A, B, C, D, E and F respectively.

9. d/dt represents the derivative with respect to time t .
10. $\tilde{R}(t)$ represents fuzzy reliability of condensate system.

2.4.3 Assumptions

In this section, the assumptions that are made to evaluate the fuzzy reliability of condensate system are presented.

1. Fuzzy failure rates and fuzzy repair rates are independent with each other and their unit is per day.
2. Failure rates and repair rates are represented by trapezoidal fuzzy numbers.
3. When one component fails, it is instantaneously replaced by one of the standby sub-systems if there is one.
4. A repaired system is as good as new, performance wise, for a specified duration and standby sub-systems if any are of the same nature and capacity as that of active systems.

2.4.4 Data

The assumed fuzzy failure rates and fuzzy repair rates for evaluating the fuzzy reliability of condensate system, represented by trapezoidal fuzzy numbers, are shown in Table 2.1.

Table 2.1 Fuzzy failure rates and fuzzy repair rates for different sub-systems of condensate system

Fuzzy failure rate	Fuzzy repair rate
$\tilde{\phi}_1 = (0.00615, 0.00684, 0.00836, 0.00919)$	$\tilde{\lambda}_1 = (0.243, 0.27, 0.33, 0.363)$
$\tilde{\phi}_2 = (0.00818, 0.00909, 0.01111, 0.01222)$	$\tilde{\lambda}_2 = (0.122, 0.135, 0.165, 0.182)$
$\tilde{\phi}_3 = (0.00332, 0.00369, 0.00451, 0.00496)$	$\tilde{\lambda}_3 = (0.284, 0.315, 0.385, 0.424)$
$\tilde{\phi}_4 = (0.00616, 0.00684, 0.00836, 0.00919)$	$\tilde{\lambda}_4 = (0.203, 0.225, 0.275, 0.303)$
$\tilde{\phi}_5 = (0.00267, 0.00297, 0.00363, 0.00399)$	$\tilde{\lambda}_5 = (0.151, 0.168, 0.206, 0.226)$
$\tilde{\phi}_6 = (0.0243, 0.027, 0.033, 0.0363)$	$\tilde{\lambda}_6 = (0.223, 0.248, 0.303, 0.333)$

2.4.5 Fuzzy Kolmogorov's differential equations

In this section, fuzzy Kolmogorov's differential equations are developed by using fuzzy Markov model of condensate system.

Fuzzy Kolmogorov's differential equations, developed by using fuzzy Markov model of condensate system, shown in Figure 2.3, are:

$$\frac{d\tilde{P}_0(t)}{dt} \oplus \tilde{\delta}_1 \otimes \tilde{P}_0(t) = \tilde{\lambda}_1 \otimes \tilde{P}_2(t) \oplus \tilde{\lambda}_2 \otimes \tilde{P}_3(t) \oplus \tilde{\lambda}_3 \otimes \tilde{P}_4(t) \oplus \tilde{\lambda}_4 \otimes \tilde{P}_5(t) \oplus \tilde{\lambda}_5 \otimes \tilde{P}_6(t) \oplus \tilde{\lambda}_6 \otimes \tilde{P}_1(t) \quad (2.15)$$

$$\frac{d\tilde{P}_1(t)}{dt} \oplus \tilde{\delta}_2 \otimes \tilde{P}_1(t) = \tilde{\lambda}_1 \otimes \tilde{P}_7(t) \oplus \tilde{\lambda}_2 \otimes \tilde{P}_8(t) \oplus \tilde{\lambda}_3 \otimes \tilde{P}_9(t) \oplus \tilde{\lambda}_4 \otimes \tilde{P}_{10}(t) \oplus \tilde{\lambda}_5 \otimes \tilde{P}_{11}(t) \oplus \tilde{\lambda}_6 \otimes \tilde{P}_{12}(t) \oplus \tilde{\phi}_6 \otimes \tilde{P}_0(t) \quad (2.16)$$

$$\frac{d\tilde{P}_2(t)}{dt} \oplus \tilde{\lambda}_1 \otimes \tilde{P}_2(t) = \tilde{\phi}_1 \otimes \tilde{P}_0(t) \quad (2.17)$$

$$\frac{d\tilde{P}_3(t)}{dt} \oplus \tilde{\lambda}_2 \otimes \tilde{P}_3(t) = \tilde{\phi}_2 \otimes \tilde{P}_0(t) \quad (2.18)$$

$$\frac{d\tilde{P}_4(t)}{dt} \oplus \tilde{\lambda}_3 \otimes \tilde{P}_4(t) = \tilde{\phi}_3 \otimes \tilde{P}_0(t) \quad (2.19)$$

$$\frac{d\tilde{P}_5(t)}{dt} \oplus \tilde{\lambda}_4 \otimes \tilde{P}_5(t) = \tilde{\phi}_4 \otimes \tilde{P}_0(t) \quad (2.20)$$

$$\frac{d\tilde{P}_6(t)}{dt} \oplus \tilde{\lambda}_5 \otimes \tilde{P}_6(t) = \tilde{\phi}_5 \otimes \tilde{P}_0(t) \quad (2.21)$$

$$\frac{d\tilde{P}_7(t)}{dt} \oplus \tilde{\lambda}_1 \otimes \tilde{P}_7(t) = \tilde{\phi}_1 \otimes \tilde{P}_1(t) \quad (2.22)$$

$$\frac{d\tilde{P}_8(t)}{dt} \oplus \tilde{\lambda}_2 \otimes \tilde{P}_8(t) = \tilde{\phi}_2 \otimes \tilde{P}_1(t) \quad (2.23)$$

$$\frac{d\tilde{P}_9(t)}{dt} \oplus \tilde{\lambda}_3 \otimes \tilde{P}_9(t) = \tilde{\phi}_3 \otimes \tilde{P}_1(t) \quad (2.24)$$

$$\frac{d\tilde{P}_{10}(t)}{dt} \oplus \tilde{\lambda}_4 \otimes \tilde{P}_{10}(t) = \tilde{\phi}_4 \otimes \tilde{P}_1(t) \quad (2.25)$$

$$\frac{d\tilde{P}_{11}(t)}{dt} \oplus \tilde{\lambda}_5 \otimes \tilde{P}_{11}(t) = \tilde{\phi}_5 \otimes \tilde{P}_1(t) \quad (2.26)$$

$$\frac{d\tilde{P}_{12}(t)}{dt} \oplus \tilde{\lambda}_6 \otimes \tilde{P}_{12}(t) = \tilde{\phi}_6 \otimes \tilde{P}_1(t) \quad (2.27)$$

where,

$$\tilde{\delta}_1 = \tilde{\phi}_1 \oplus \tilde{\phi}_2 \oplus \tilde{\phi}_3 \oplus \tilde{\phi}_4 \oplus \tilde{\phi}_5 \oplus \tilde{\phi}_6$$

$$\tilde{\delta}_2 = \tilde{\phi}_1 \oplus \tilde{\phi}_2 \oplus \tilde{\phi}_3 \oplus \tilde{\phi}_4 \oplus \tilde{\phi}_5 \oplus \tilde{\phi}_6 \oplus \tilde{\lambda}_6$$

with initial conditions:

$$\tilde{P}_0(0) = (0.95, 0.955, 0.965, 0.97) \quad (2.28)$$

$$\tilde{P}_1(0) = (0.004, 0.0045, 0.0055, 0.006) \quad (2.29)$$

$$\tilde{P}_j(0) = (0, 0, 0, 0), j=2 \text{ to } 12 \quad (2.30)$$

2.4.6 Solution of fuzzy Kolmogorov's differential equations

The solution of fuzzy Kolmogorov's differential equations (2.15) to (2.27) with initial conditions (2.28) to (2.30), obtained by using Buckley and Feuring method [24] for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$ at $t = 48$ days, is shown in Table 2.2.

Table 2.2 Solution of fuzzy Kolmogorov's differential equations for condensate system at $t = 48$ days

j	$\tilde{p}_j(t)$ for $\alpha = 0$		$\tilde{p}_j(t)$ for $\alpha = 0.2$		$\tilde{p}_j(t)$ for $\alpha = 0.4$		$\tilde{p}_j(t)$ for $\alpha = 0.6$		$\tilde{p}_j(t)$ for $\alpha = 0.8$		$\tilde{p}_j(t)$ for $\alpha = 1$	
	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$
0	0.739958	0.756819	0.740764	0.755949	0.741571	0.755080	0.742377	0.754211	0.743184	0.753342	0.743991	0.752473
1	0.080578	0.082496	0.080656	0.082385	0.080734	0.082275	0.080813	0.082165	0.080891	0.082055	0.08097	0.081945
2	0.018731	0.01916	0.018754	0.019140	0.018778	0.019121	0.018801	0.019101	0.018825	0.019082	0.018849	0.019063
3	0.04955	0.050812	0.049652	0.050781	0.049755	0.050751	0.049858	0.05072	0.049961	0.05069	0.050064	0.05066
4	0.008651	0.008853	0.008663	0.008845	0.008676	0.008837	0.008689	0.008829	0.008702	0.008821	0.008715	0.008814
5	0.02246	0.022954	0.022492	0.022938	0.022524	0.022922	0.022556	0.022906	0.022588	0.02289	0.02262	0.022875
6	0.013085	0.013361	0.013098	0.01334	0.013112	0.01332	0.013126	0.013300	0.013140	0.013280	0.013154	0.01326
7	0.002038	0.002088	0.002040	0.002085	0.002042	0.002082	0.002045	0.002080	0.002047	0.002077	0.00205	0.002075
8	0.005365	0.005536	0.005351	0.005531	0.005338	0.005526	0.005325	0.005521	0.005312	0.005516	0.005299	0.005512
9	0.000941	0.000965	0.000942	0.000963	0.000943	0.000962	0.000945	0.000961	0.000946	0.000960	0.000948	0.000959
10	0.002442	0.0025	0.002445	0.002498	0.002449	0.002496	0.002452	0.002494	0.002456	0.002492	0.00246	0.00249
11	0.00142	0.001456	0.001421	0.001453	0.001423	0.001450	0.001425	0.001448	0.001427	0.001445	0.001429	0.001443
12	0.008774	0.008992	0.008781	0.008978	0.008789	0.008964	0.008796	0.008951	0.008804	0.008937	0.008812	0.008924

2.4.7 Fuzzy reliability evaluation of condensate system

In this section, the solution of fuzzy Kolmogorov's differential equations, shown in Table 2.2, is used to evaluate the fuzzy reliability of condensate system.

Using the obtained solution of fuzzy Kolmogorov's differential equations,

shown in Table 2.2, the α -cuts corresponding to fuzzy reliability $\tilde{R}(t) = \tilde{p}_0(t) \oplus \tilde{p}_1(t)$ i.e., $[R_1(t, \alpha), R_2(t, \alpha)] = [p_{0(1)}(t, \alpha) + p_{1(1)}(t, \alpha), p_{0(2)}(t, \alpha) + p_{1(2)}(t, \alpha)]$ of condensate system, computed for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$ at $t = 48$ days, are shown in Table 2.3.

Table 2.3 α -cuts corresponding to fuzzy reliability of condensate system at $t = 48$ days

$\alpha \downarrow$	$R_1(t, \alpha)$	$R_2(t, \alpha)$
0	0.820536	0.839315
0.2	0.82142	0.838334
0.4	0.822305	0.837355
0.6	0.82319	0.836376
0.8	0.824075	0.835397
1	0.824961	0.834418

Trapezoidal fuzzy number representing the fuzzy reliability of condensate system at $t = 48$ days corresponding to different presumption levels is shown in Figure 2.4.

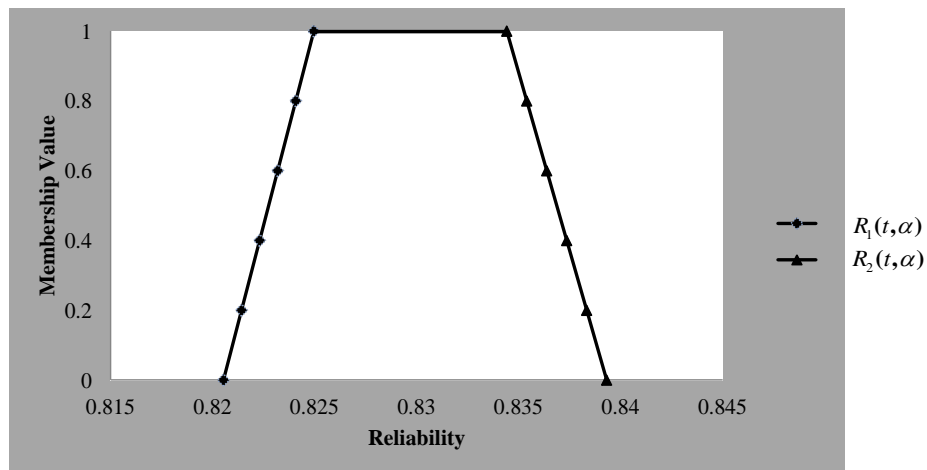


Figure 2.4 Trapezoidal fuzzy number representing the fuzzy reliability of condensate system at $t = 48$ days corresponding to different presumption levels

2.5 Physical interpretation of results

In this section, the physical interpretation of the obtained fuzzy reliability of condensate system at $t = 48$ days is interpreted.

It is obvious from Figure 2.4 that

1. The reliability of condensate system at $t = 48$ days lies between 0.820536 and 0.839315.
2. The degree of acceptance of the experts about the statement that the reliability of condensate system at $t = 48$ days lies between 0.824961 to 0.834418 is 1.
3. The degree of acceptance $\mu_{\tilde{R}}(r)$ for the remaining values of reliability r , $r \in [0.820536, 0.839315]$ can be obtained by putting r in:

$$\mu_{\tilde{R}}(r) = \begin{cases} \frac{(r-0.820536)}{0.004425}, & 0.820536 \leq r < 0.824961 \\ 1, & 0.824961 \leq r \leq 0.834418 \\ \frac{(0.839315-r)}{(0.004897)}, & 0.834418 < r \leq 0.839315 \end{cases}$$

Also, the variation in reliability of condensate system with time at $\alpha = 0$, $\alpha = 0.2$, $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$, $\alpha = 1$ is shown in Figure 2.5 to Figure 2.10 respectively.

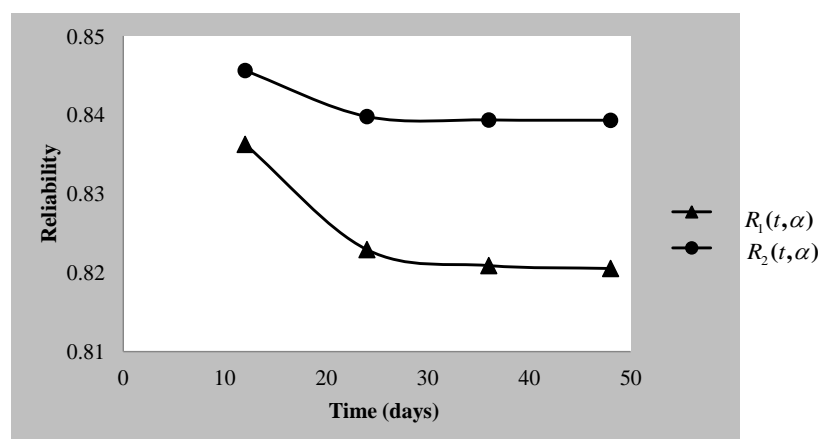


Figure 2.5 Variation in reliability with time at $\alpha = 0$

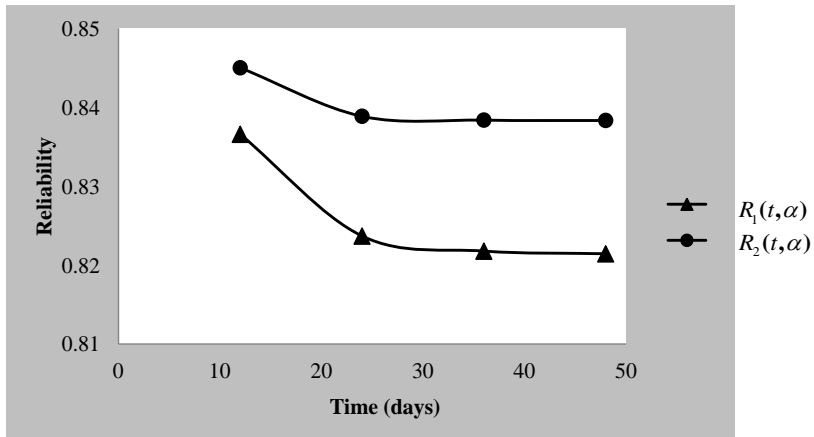


Figure 2.6 Variation in reliability with time at $\alpha = 0.2$

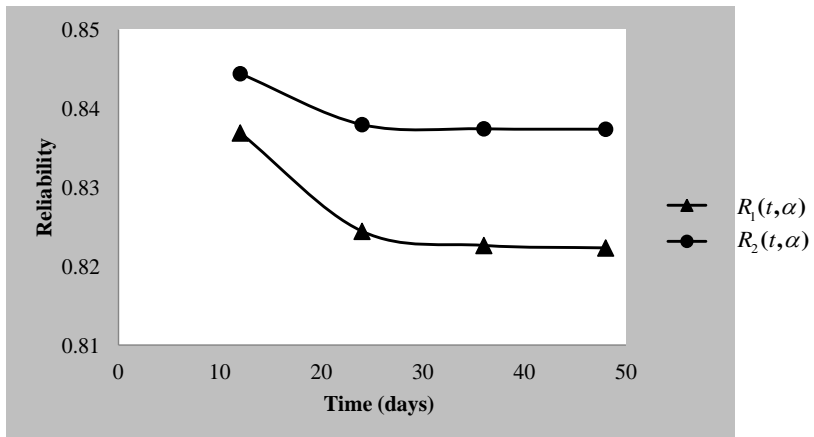


Figure 2.7 Variation in reliability with time at $\alpha = 0.4$

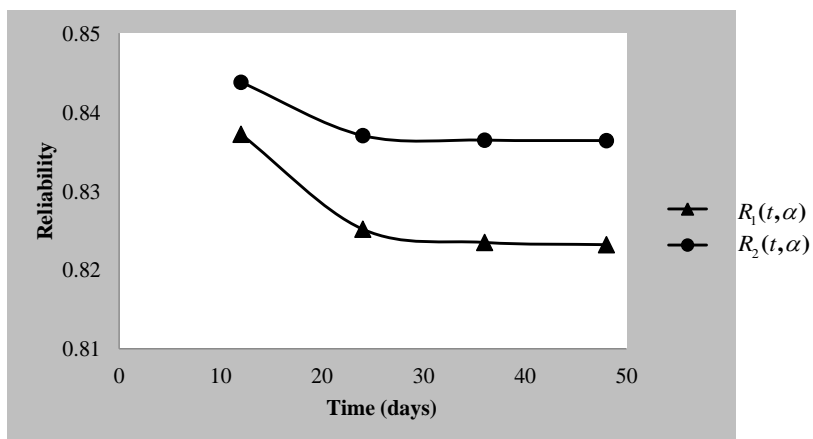


Figure 2.8 Variation in reliability with time at $\alpha = 0.6$

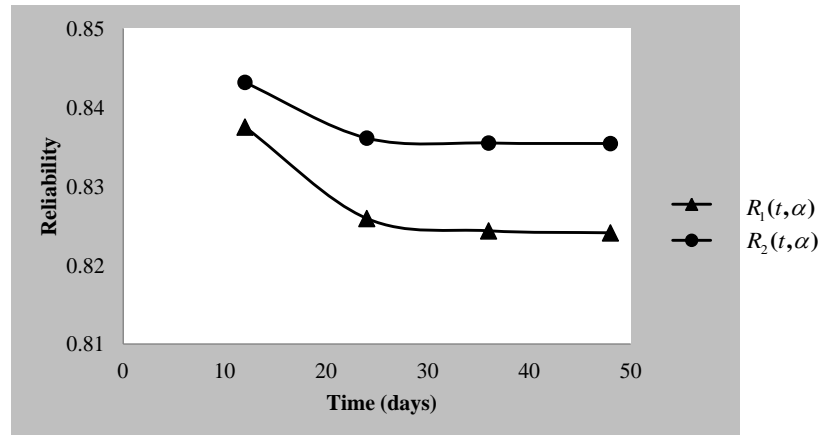


Figure 2.9 Variation in reliability with time at $\alpha = 0.8$

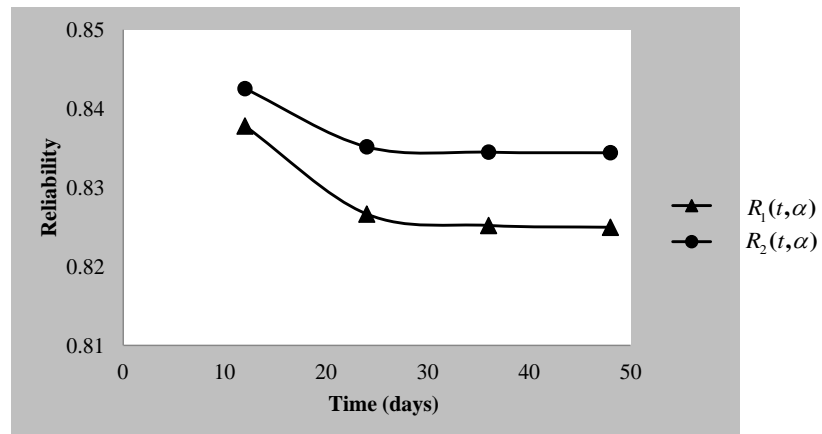


Figure 2.10 Variation in reliability with time at $\alpha = 1$

2.6 Conclusions

On the basis of presented study, it is concluded that it is better to use Buckley and Feuring method [24] as compared to Liu and Huang method [106] for solving fuzzy differential equations and hence for solving fuzzy Kolmogorov's differential equations to evaluate the fuzzy reliability of industrial systems.

Chapter 3

MODIFIED METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS AND ITS APPLICATION FOR ANALYZING THE FUZZY RELIABILITY OF PISTON MANUFACTURING SYSTEM

In the previous chapter, Buckley and Feuring method [24] is used for solving fuzzy Kolmogorov's differential equations obtained by fuzzy Markov model of condensate system. In this chapter, it is illustrated that if the same method is applied for solving fuzzy Kolmogorov's differential equations, obtained by fuzzy Markov model of piston manufacturing system, then the obtained solution does not define α -cut of a fuzzy number. Consequently, the solution of fuzzy Kolmogorov's differential equations, obtained by applying Buckley and Feuring method [24], cannot be used to evaluate the fuzzy reliability of piston manufacturing system. Also, by considering a system of fuzzy equations, it is confirmed that the drawback of Buckley and Feuring method [24] is occurring due to existing representation of α -cut. To resolve the drawback of Buckley and Feuring method [24], a new representation of α -cut is proposed. With the help of proposed α -cut, Buckley and Feuring method is modified. Also, fuzzy Kolmogorov's differential equations, obtained by fuzzy Markov

The contents of the paper, accepted for publication in *Applied Mathematical Modelling*, are the generalization of the contents of this chapter.

model of piston manufacturing system, are solved by using the modified method and with the help of obtained solution, fuzzy reliability of piston manufacturing system is evaluated.

3.1 Drawback of Buckley and Feuring method

In this section, it is shown that on solving the fuzzy Kolmogorov's differential equations, obtained by using fuzzy Markov model of piston manufacturing system, by using Buckley and Feuring method [24] the obtained solution is not appropriate. Due to which the obtained solution cannot be used to evaluate the fuzzy reliability of piston manufacturing system.

3.1.1 Fuzzy Markov modeling of piston manufacturing system

Piston manufacturing system consists of two sub-systems namely R_1 and R_2 , which are connected in series. Further, the sub-system R_1 consists of six sub-systems namely fixture seat machine, rough grooving and turning machine, rough pin hole boring machine, oil hole drilling machine, finishing grooving machine and finish profile turning machine. Similarly, six sub-systems namely finish pin hole boring machine, finish crown and cavity machine, valve milling machine, chamfering or radius machine, circlip grooving machine and piston cleaning machine constitute the sub-system R_2 . Fuzzy Markov models of sub-system R_1 and sub-system R_2 are shown in Figure 3.1 and Figure 3.2 respectively.

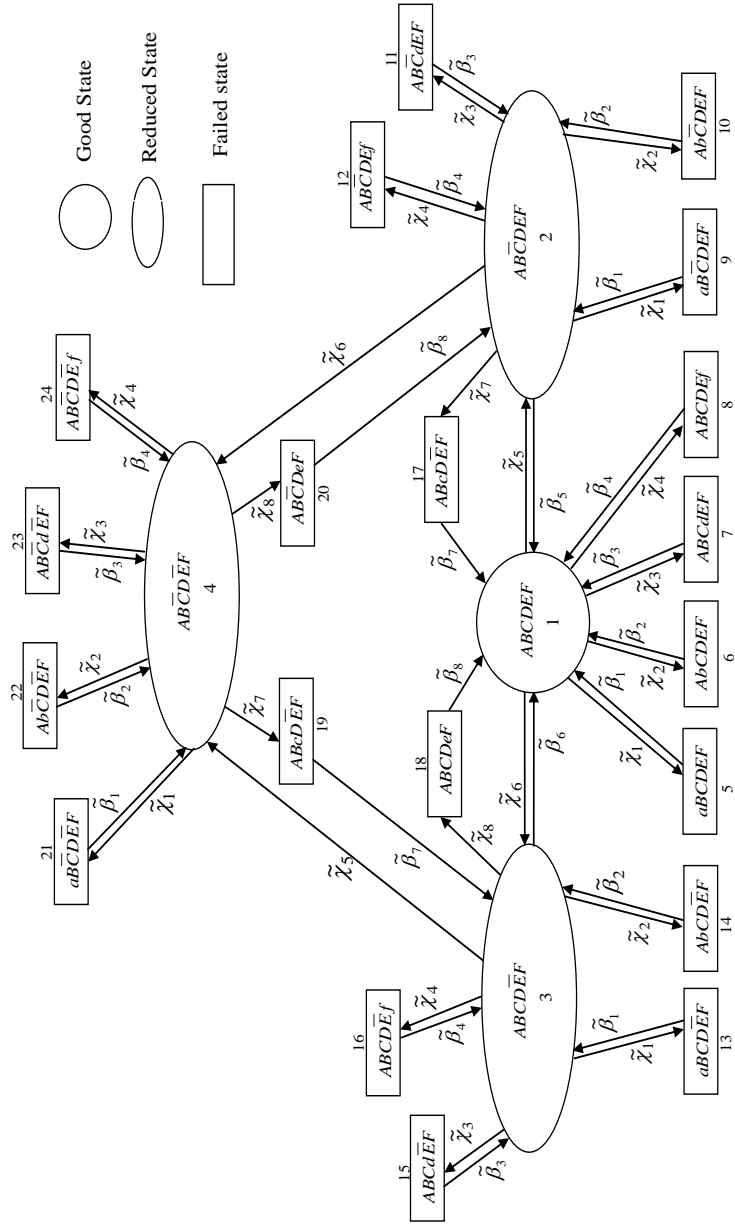


Figure 3.1 Fuzzy Markov model of sub-system R_1

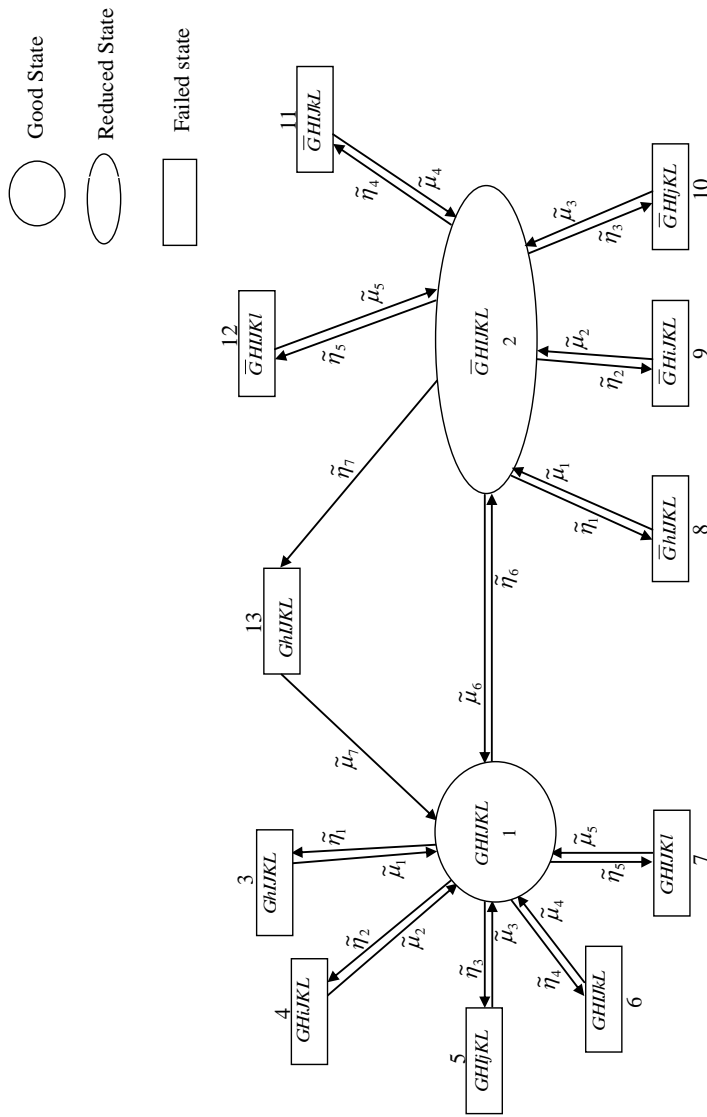


Figure 3.2 Fuzzy Markov model of sub-system R_2

The operations that are performed on these machines or sub-systems are as follows:

1. **Sub-system A (Fixture Seat Machine):** This machine is used to clamp the piston.
2. **Sub-system B (Rough Grooving and Turning Machine):** On this machine, rough grooves are made on piston. Turning operation is performed on this machine i.e., to bring the dia of piston to proper size.
3. **Sub-system C (Rough Pin Hole Boring Machine):** Pin hole boring operation is performed using this machine i.e., proper size is given to holes.
4. **Sub-system D (Oil Hole Drilling Machine):** On this machine, one hole is made on the piston to supply the oil. The oil is used to move piston in cylinder smoothly.
5. **Sub-system E (Finishing Grooving Machine):** On this machine, the finishing is given to rough grooves which are prepared using sub-system B.
6. **Sub-system F (Finish Profile Turning Machine):** Oval shape is given to piston using this machine.
7. **Sub-system G (Finish Pin Hole Boring Machine):** On this machine, finishing is given to the pin hole portion which is prepared using sub-system C.
8. **Sub-system H (Finish Crown and Cavity Machine):** On this machine, finishing operation is performed on the crown of piston.
9. **Sub-system I (Valve Milling Machine):** On this machine, valve recess is made on the piston.
10. **Sub-system J (Chamfering or Radius Machine):** This machine rounds off the corners of the piston so that it can run smoothly in the cylinder.

11. **Sub-system K (Circlip Grooving Machine):** On this machine, circlip grooves are made on the piston.
12. **Sub-system L (Piston Cleaning Machine):** This machine is used to clean the inside and outside portion of the piston.

3.1.2 Notation

In this section, notation that is used to evaluate the fuzzy reliability of piston manufacturing system are presented:

1. A, B, C, D, E, F and G, H, I, J, K, L represent full working states of sub-systems of R_1 and R_2 respectively.
2. The symbols $a, b, c, d, e, f, g, h, i, j, k$ and l represent failed state of sub-systems $A, B, C, D, E, F, G, H, I, J, K$ and L respectively.
3. \bar{C}, \bar{E} and \bar{G} indicate that the sub-systems C, E and G are working in reduced state.
4. $\tilde{\chi}_i (i = 1 \text{ to } 8)$ represents the fuzzy failure rates of the relevant sub-systems, when the transition is from A to a, B to b, D to d, F to f, C to \bar{C}, E to \bar{E}, \bar{C} to c and \bar{E} to e respectively.
5. $\tilde{\beta}_i (i = 1 \text{ to } 8)$ represents the fuzzy repair rates of the relevant sub-systems, when the transition is from a to A, b to B, d to D, f to F, \bar{C} to C, \bar{E} to E, c to C and e to E respectively.
6. $\tilde{\eta}_i (i = 1 \text{ to } 7)$ represents the fuzzy failure rates of the relevant sub-systems, when the transition is from H to h, I to i, J to j, K to k, L to l, G to \bar{G} and \bar{G} to g respectively.

7. $\tilde{\mu}_i (i = 1 \text{ to } 7)$ represents the fuzzy repair rates of the relevant sub-systems, when the transition is from h to H , i to I , j to J , k to K , l to L , \bar{G} to G and g to G respectively.
8. $\tilde{P}_j(t), j = 1, 2, \dots, n$ represents the fuzzy probability that the system is in state S_j at time t , where n is number of states. $\frac{d\tilde{P}_j(t)}{dt}, j = 1, 2, \dots, n$ represents derivative of $\tilde{P}_j(t)$ with respect to t .
9. $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ denote fuzzy reliability of sub-system R_1 and sub-system R_2 respectively.
10. $\tilde{R}(t)$ represents fuzzy reliability of whole system.

3.1.3 Assumptions

In this section, the assumptions that are made for evaluating the fuzzy reliability of piston manufacturing system are presented:

1. Fuzzy failure rates and fuzzy repair rates are independent with each other and their unit is per hour.
2. Failure rates and repair rates are represented by trapezoidal fuzzy numbers.
3. There are no simultaneous failures among the sub-systems.
4. Sub-systems C, E and G fails through reduced states only.

3.1.4 Data

The assumed fuzzy failure rates and fuzzy repair rates for different sub-systems of R_1 and R_2 are shown in Table 3.1 and Table 3.2 respectively.

Table 3.1 Fuzzy failure rates and fuzzy repair rates for different sub-systems of R_1

Fuzzy failure rate	Fuzzy repair rate
$\tilde{\chi}_1=(0.00105,0.00126,0.00154,0.00175)$	$\tilde{\beta}_1=(1.026,1.0584,1.1016,1.134)$
$\tilde{\chi}_2=(0.00045,0.00054,0.00066,0.00075)$	$\tilde{\beta}_2=(0.04085,0.04214,0.04386,0.04515)$
$\tilde{\chi}_3=(0.000675,0.00081,0.00099,0.001125)$	$\tilde{\beta}_3=(0.475,0.49,0.51,0.525)$
$\tilde{\chi}_4=(0.000675,0.00081,0.00099,0.001125)$	$\tilde{\beta}_4=(0.2717,0.28028,0.29172,0.3003)$
$\tilde{\chi}_5=(0.0156,0.01872,0.02288,0.026)$	$\tilde{\beta}_5=(0.1463,0.15092,0.15702,0.1617)$
$\tilde{\chi}_6=(0.0156,0.01872,0.02288,0.026)$	$\tilde{\beta}_6=(0.2375,0.245,0.255,0.2625)$
$\tilde{\chi}_7=(0.000675,0.00081,0.00099,0.001125)$	$\tilde{\beta}_7=(0.05605,0.05782,0.06018,0.06195)$
$\tilde{\chi}_8=(0.002925,0.00351,0.00429,0.004875)$	$\tilde{\beta}_8=(0.08265,0.08526,0.08874,0.09135)$

Table 3.2 Fuzzy failure rates and fuzzy repair rates for different sub-systems of R_2

Fuzzy failure rate	Fuzzy repair rate
$\tilde{\eta}_1=(0.00105,0.00126,0.00154,0.00175)$	$\tilde{\mu}_1=(0.3135,0.3234,0.3366,0.3465)$
$\tilde{\eta}_2=(0.00023,0.00027,0.00033,0.00038)$	$\tilde{\mu}_2=(0.475,0.49,0.51,0.525)$
$\tilde{\eta}_3=(0.00008,0.00009,0.00011,0.00013)$	$\tilde{\mu}_3=(0.6365,0.6566,0.6834,0.7035)$
$\tilde{\eta}_4=(0.00023,0.00027,0.00033,0.00038)$	$\tilde{\mu}_4=(0.03325,0.0343,0.0357,0.03675)$
$\tilde{\eta}_5=(0.00008,0.00009,0.00011,0.00013)$	$\tilde{\mu}_5=(2.8785,2.9694,3.0906,3.1815)$
$\tilde{\eta}_6=(0.0156,0.01872,0.02288,0.026)$	$\tilde{\mu}_6=(0.2109,0.21756,0.22644,0.2331)$
$\tilde{\eta}_7=(0.003,0.0036,0.0044,0.005)$	$\tilde{\mu}_7=(0.11875,0.1225,0.1275,0.13125)$

3.1.5 Fuzzy Kolmogorov's differential equations

In this section, fuzzy Kolmogorov's differential equations are developed by using fuzzy Markov model of sub-system R_1 and sub-system R_2 .

Fuzzy Kolmogorov's differential equations associated with fuzzy Markov model of sub-system R_1 , shown in Figure 3.1, are:

$$\begin{aligned} \frac{d\tilde{P}_1(t)}{dt} \oplus \tilde{\lambda}_1 \otimes \tilde{P}_1(t) &= \tilde{\beta}_1 \otimes \tilde{P}_5(t) \oplus \tilde{\beta}_2 \otimes \tilde{P}_6(t) \oplus \tilde{\beta}_3 \otimes \tilde{P}_7(t) \oplus \tilde{\beta}_4 \otimes \tilde{P}_8(t) \oplus \tilde{\beta}_5 \otimes \tilde{P}_2(t) \oplus \\ &\tilde{\beta}_6 \otimes \tilde{P}_3(t) \oplus \tilde{\beta}_7 \otimes \tilde{P}_{17}(t) \oplus \tilde{\beta}_8 \otimes \tilde{P}_{18}(t) \end{aligned} \quad (3.1)$$

$$\begin{aligned} \frac{d\tilde{P}_2(t)}{dt} \oplus \tilde{\lambda}_2 \otimes \tilde{P}_2(t) &= \tilde{\beta}_1 \otimes \tilde{P}_9(t) \oplus \tilde{\beta}_2 \otimes \tilde{P}_{10}(t) \oplus \tilde{\beta}_3 \otimes \tilde{P}_{11}(t) \oplus \tilde{\beta}_4 \otimes \tilde{P}_{12}(t) \oplus \tilde{\beta}_8 \otimes \\ &\tilde{P}_{20}(t) \oplus \tilde{\chi}_5 \otimes \tilde{P}_1(t) \end{aligned} \quad (3.2)$$

$$\frac{d\tilde{P}_3(t)}{dt} \oplus \tilde{\lambda}_3 \otimes \tilde{P}_3(t) = \tilde{\beta}_1 \otimes \tilde{P}_{13}(t) \oplus \tilde{\beta}_2 \otimes \tilde{P}_{14}(t) \oplus \tilde{\beta}_3 \otimes \tilde{P}_{15}(t) \oplus \tilde{\beta}_4 \otimes \tilde{P}_{16}(t) \oplus \tilde{\beta}_7 \otimes$$

$$\tilde{P}_{19}(t) \oplus \tilde{\chi}_6 \otimes \tilde{P}_1(t) \quad (3.3)$$

$$\begin{aligned} \frac{d\tilde{P}_4(t)}{dt} \oplus \tilde{\lambda}_4 \otimes \tilde{P}_4(t) &= \tilde{\beta}_1 \otimes \tilde{P}_{21}(t) \oplus \tilde{\beta}_2 \otimes \tilde{P}_{22}(t) \oplus \tilde{\beta}_3 \otimes \tilde{P}_{23}(t) \oplus \tilde{\beta}_4 \otimes \tilde{P}_{24}(t) \oplus \tilde{\chi}_5 \otimes \\ \tilde{P}_3(t) \oplus \tilde{\chi}_6 \otimes \tilde{P}_2(t) & \end{aligned} \quad (3.4)$$

$$\frac{d\tilde{P}_5(t)}{dt} \oplus \tilde{\beta}_1 \otimes \tilde{P}_5(t) = \tilde{\chi}_1 \otimes \tilde{P}_1(t) \quad (3.5)$$

$$\frac{d\tilde{P}_6(t)}{dt} \oplus \tilde{\beta}_2 \otimes \tilde{P}_6(t) = \tilde{\chi}_2 \otimes \tilde{P}_1(t) \quad (3.6)$$

$$\frac{d\tilde{P}_7(t)}{dt} \oplus \tilde{\beta}_3 \otimes \tilde{P}_7(t) = \tilde{\chi}_3 \otimes \tilde{P}_1(t) \quad (3.7)$$

$$\frac{d\tilde{P}_8(t)}{dt} \oplus \tilde{\beta}_4 \otimes \tilde{P}_8(t) = \tilde{\chi}_8 \otimes \tilde{P}_1(t) \quad (3.8)$$

$$\frac{d\tilde{P}_9(t)}{dt} \oplus \tilde{\beta}_1 \otimes \tilde{P}_9(t) = \tilde{\chi}_1 \otimes \tilde{P}_2(t) \quad (3.9)$$

$$\frac{d\tilde{P}_{10}(t)}{dt} \oplus \tilde{\beta}_2 \otimes \tilde{P}_{10}(t) = \tilde{\chi}_2 \otimes \tilde{P}_2(t) \quad (3.10)$$

$$\frac{d\tilde{P}_{11}(t)}{dt} \oplus \tilde{\beta}_3 \otimes \tilde{P}_{11}(t) = \tilde{\chi}_3 \otimes \tilde{P}_2(t) \quad (3.11)$$

$$\frac{d\tilde{P}_{12}(t)}{dt} \oplus \tilde{\beta}_4 \otimes \tilde{P}_{12}(t) = \tilde{\chi}_4 \otimes \tilde{P}_2(t) \quad (3.12)$$

$$\frac{d\tilde{P}_{13}(t)}{dt} \oplus \tilde{\beta}_1 \otimes \tilde{P}_{13}(t) = \tilde{\chi}_1 \otimes \tilde{P}_3(t) \quad (3.13)$$

$$\frac{d\tilde{P}_{14}(t)}{dt} \oplus \tilde{\beta}_2 \otimes \tilde{P}_{14}(t) = \tilde{\chi}_2 \otimes \tilde{P}_3(t) \quad (3.14)$$

$$\frac{d\tilde{P}_{15}(t)}{dt} \oplus \tilde{\beta}_3 \otimes \tilde{P}_{15}(t) = \tilde{\chi}_3 \otimes \tilde{P}_3(t) \quad (3.15)$$

$$\frac{d\tilde{P}_{16}(t)}{dt} \oplus \tilde{\beta}_4 \otimes \tilde{P}_{16}(t) = \tilde{\chi}_4 \otimes \tilde{P}_3(t) \quad (3.16)$$

$$\frac{d\tilde{P}_{17}(t)}{dt} \oplus \tilde{\beta}_7 \otimes \tilde{P}_{17}(t) = \tilde{\chi}_7 \otimes \tilde{P}_2(t) \quad (3.17)$$

$$\frac{d\tilde{P}_{18}(t)}{dt} \oplus \tilde{\beta}_8 \otimes \tilde{P}_{18}(t) = \tilde{\chi}_8 \otimes \tilde{P}_3(t) \quad (3.18)$$

$$\frac{d\tilde{P}_{19}(t)}{dt} \oplus \tilde{\beta}_7 \otimes \tilde{P}_{19}(t) = \tilde{\chi}_7 \otimes \tilde{P}_4(t) \quad (3.19)$$

$$\frac{d\tilde{P}_{20}(t)}{dt} \oplus \tilde{\beta}_8 \otimes \tilde{P}_{20}(t) = \tilde{\chi}_8 \otimes \tilde{P}_4(t) \quad (3.20)$$

$$\frac{d\tilde{P}_{21}(t)}{dt} \oplus \tilde{\beta}_1 \otimes \tilde{P}_{21}(t) = \tilde{\chi}_1 \otimes \tilde{P}_4(t) \quad (3.21)$$

$$\frac{d\tilde{P}_{22}(t)}{dt} \oplus \tilde{\beta}_2 \otimes \tilde{P}_{22}(t) = \tilde{\chi}_2 \otimes \tilde{P}_4(t) \quad (3.22)$$

$$\frac{d\tilde{P}_{23}(t)}{dt} \oplus \tilde{\beta}_3 \otimes \tilde{P}_{23}(t) = \tilde{\chi}_3 \otimes \tilde{P}_4(t) \quad (3.23)$$

$$\frac{d\tilde{P}_{24}(t)}{dt} \oplus \tilde{\beta}_4 \otimes \tilde{P}_{24}(t) = \tilde{\chi}_4 \otimes \tilde{P}_4(t) \quad (3.24)$$

where,

$$\tilde{\lambda}_1 = \tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3 \oplus \tilde{\chi}_4 \oplus \tilde{\chi}_5 \oplus \tilde{\chi}_6$$

$$\tilde{\lambda}_2 = \tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3 \oplus \tilde{\chi}_4 \oplus \tilde{\chi}_6 \oplus \tilde{\chi}_7 \oplus \tilde{\beta}_5$$

$$\tilde{\lambda}_3 = \tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3 \oplus \tilde{\chi}_4 \oplus \tilde{\chi}_5 \oplus \tilde{\chi}_8 \oplus \tilde{\beta}_6$$

$$\tilde{\lambda}_4 = \tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3 \oplus \tilde{\chi}_4 \oplus \tilde{\chi}_7 \oplus \tilde{\chi}_8$$

with initial conditions:

$$\tilde{P}_1(0) = (0.94, 0.945, 0.955, 0.96) \quad (3.25)$$

$$\tilde{P}_2(0) = (0.006, 0.0065, 0.0075, 0.008) \quad (3.26)$$

$$\tilde{P}_3(0) = (0.004, 0.0045, 0.0055, 0.006) \quad (3.27)$$

$$\tilde{P}_4(0) = (0.002, 0.0025, 0.0035, 0.004) \quad (3.28)$$

$$\tilde{P}_j(0) = (0, 0, 0, 0), \quad j=5 \text{ to } 24 \quad (3.29)$$

Fuzzy Kolmogorov's differential equations associated with fuzzy Markov model of sub-system R_2 , shown in Figure 3.2, are:

$$\begin{aligned} \frac{d\tilde{P}_1(t)}{dt} \oplus \tilde{\delta}_1 \otimes \tilde{P}_1(t) &= \tilde{\mu}_1 \otimes \tilde{P}_3(t) \oplus \tilde{\mu}_2 \otimes \tilde{P}_4(t) \oplus \tilde{\mu}_3 \otimes \tilde{P}_5(t) \oplus \tilde{\mu}_4 \otimes \tilde{P}_6(t) \oplus \tilde{\mu}_5 \otimes \tilde{P}_7(t) \oplus \tilde{\mu}_6 \otimes \\ \tilde{P}_2(t) \oplus \tilde{\mu}_7 \otimes \tilde{P}_{13}(t) \end{aligned} \quad (3.30)$$

$$\begin{aligned} \frac{d\tilde{P}_2(t)}{dt} \oplus \tilde{\delta}_2 \otimes \tilde{P}_2(t) &= \tilde{\mu}_1 \otimes \tilde{P}_8(t) \oplus \tilde{\mu}_2 \otimes \tilde{P}_9(t) \oplus \tilde{\mu}_3 \otimes \tilde{P}_{10}(t) \oplus \tilde{\mu}_4 \otimes \tilde{P}_{11}(t) \oplus \tilde{\mu}_5 \otimes \tilde{P}_{12}(t) \oplus \tilde{\eta}_6 \otimes \\ \tilde{P}_1(t) \end{aligned} \quad (3.31)$$

$$\frac{d\tilde{P}_3(t)}{dt} \oplus \tilde{\mu}_1 \otimes \tilde{P}_3(t) = \tilde{\eta}_1 \otimes \tilde{P}_1(t) \quad (3.32)$$

$$\frac{d\tilde{P}_4(t)}{dt} \oplus \tilde{\mu}_2 \otimes \tilde{P}_4(t) = \tilde{\eta}_2 \otimes \tilde{P}_1(t) \quad (3.33)$$

$$\frac{d\tilde{P}_5(t)}{dt} \oplus \tilde{\mu}_3 \otimes \tilde{P}_5(t) = \tilde{\eta}_3 \otimes \tilde{P}_1(t) \quad (3.34)$$

$$\frac{d\tilde{P}_6(t)}{dt} \oplus \tilde{\mu}_4 \otimes \tilde{P}_6(t) = \tilde{\eta}_4 \otimes \tilde{P}_1(t) \quad (3.35)$$

$$\frac{d\tilde{P}_7(t)}{dt} \oplus \tilde{\mu}_5 \otimes \tilde{P}_7(t) = \tilde{\eta}_5 \otimes \tilde{P}_1(t) \quad (3.36)$$

$$\frac{d\tilde{P}_8(t)}{dt} \oplus \tilde{\mu}_1 \otimes \tilde{P}_8(t) = \tilde{\eta}_1 \otimes \tilde{P}_2(t) \quad (3.37)$$

$$\frac{d\tilde{P}_9(t)}{dt} \oplus \tilde{\mu}_2 \otimes \tilde{P}_9(t) = \tilde{\eta}_2 \otimes \tilde{P}_2(t) \quad (3.38)$$

$$\frac{d\tilde{P}_{10}(t)}{dt} \oplus \tilde{\mu}_3 \otimes \tilde{P}_{10}(t) = \tilde{\eta}_3 \otimes \tilde{P}_2(t) \quad (3.39)$$

$$\frac{d\tilde{P}_{11}(t)}{dt} \oplus \tilde{\mu}_4 \otimes \tilde{P}_{11}(t) = \tilde{\eta}_4 \otimes \tilde{P}_2(t) \quad (3.40)$$

$$\frac{d\tilde{P}_{12}(t)}{dt} \oplus \tilde{\mu}_5 \otimes \tilde{P}_{12}(t) = \tilde{\eta}_5 \otimes \tilde{P}_2(t) \quad (3.41)$$

$$\frac{d\tilde{P}_{13}(t)}{dt} \oplus \tilde{\mu}_7 \otimes \tilde{P}_{13}(t) = \tilde{\eta}_7 \otimes \tilde{P}_2(t) \quad (3.42)$$

where,

$$\tilde{\delta}_1 = \tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \tilde{\eta}_3 \oplus \tilde{\eta}_4 \oplus \tilde{\eta}_5 \oplus \tilde{\eta}_6$$

$$\tilde{\delta}_2 = \tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \tilde{\eta}_3 \oplus \tilde{\eta}_4 \oplus \tilde{\eta}_5 \oplus \tilde{\eta}_7 \oplus \tilde{\mu}_6$$

with initial conditions:

$$\tilde{P}_1(0) = (0.95, 0.955, 0.965, 0.97) \quad (3.43)$$

$$\tilde{P}_2(0) = (0.004, 0.0045, 0.0055, 0.006) \quad (3.44)$$

$$\tilde{P}_j(0) = (0, 0, 0, 0), \quad j=3 \text{ to } 13 \quad (3.45)$$

3.1.6 Solution of fuzzy Kolmogorov's differential equations obtained by using Buckley and Feuring method

The solution of fuzzy Kolmogorov's differential equations (3.1) to (3.24) with initial conditions (3.25) to (3.29) and the solution of fuzzy Kolmogorov's differential equations (3.30) to (3.42) with initial conditions (3.43) to (3.45) for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$ at $t = 360$ hours, obtained by using Buckley and Feuring method [24] discussed in Chapter 2, is shown in Table 3.3 and Table 3.4 respectively.

Table 3.3 Solution of fuzzy Kolmogorov’s differential equations for sub-system R_1 obtained by using Buckley and Feuring method [24]

j	$\tilde{p}_j(t)$ for $\alpha = 0$		$\tilde{p}_j(t)$ for $\alpha = 0.2$		$\tilde{p}_j(t)$ for $\alpha = 0.4$		$\tilde{p}_j(t)$ for $\alpha = 0.6$		$\tilde{p}_j(t)$ for $\alpha = 0.8$		$\tilde{p}_j(t)$ for $\alpha = 1$	
	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$
1	0.525421	0.396102	0.515146	0.401844	0.504872	0.407586	0.494597	0.413328	0.484323	0.419071	0.474049	0.424813
2	0.055996	0.065514	0.05667	0.065057	0.057344	0.064601	0.058018	0.064144	0.058692	0.063687	0.059366	0.063231
3	0.032869	0.036752	0.033158	0.036576	0.033447	0.036401	0.033737	0.036226	0.034026	0.036051	0.034316	0.035876
4	0.306812	0.424067	0.315888	0.419049	0.324964	0.414032	0.33404	0.409015	0.343116	0.403998	0.352192	0.398981
5	0.000537	0.000611	0.000542	0.000607	0.000547	0.000604	0.000553	0.000601	0.000558	0.000597	0.000564	0.000594
6	0.005867	0.006627	0.005923	0.006592	0.00598	0.006557	0.006036	0.006522	0.006093	0.006487	0.00615	0.006453
7	0.000747	0.000849	0.000754	0.000844	0.000761	0.000839	0.000769	0.000834	0.000776	0.000829	0.000784	0.000825
8	0.001307	0.001485	0.00132	0.001476	0.001333	0.001468	0.001346	0.001459	0.001359	0.001451	0.001372	0.001443
9	0.000057	0.000101	0.000059	0.000098	0.000062	0.000095	0.000065	0.000093	0.000068	0.000091	0.000071	0.000088
10	0.000623	0.001093	0.000651	0.001065	0.000681	0.001038	0.000709	0.001011	0.000738	0.000984	0.000767	0.000957
11	0.000079	0.00014	0.000082	0.000136	0.000086	0.000133	0.00009	0.000129	0.000094	0.000126	0.000098	0.000122
12	0.000139	0.000245	0.000145	0.000238	0.000151	0.000232	0.000158	0.000226	0.000164	0.00022	0.000171	0.000214
13	0.000033	0.000056	0.000034	0.000054	0.000036	0.000053	0.000037	0.000052	0.000039	0.000051	0.000044	0.00005
14	0.000366	0.000614	0.000381	0.0006	0.000397	0.000586	0.000412	0.000572	0.000428	0.000558	0.000444	0.000544
15	0.000046	0.000078	0.000048	0.000076	0.00005	0.000074	0.000052	0.000072	0.000054	0.000071	0.000056	0.000069
16	0.000081	0.000137	0.000084	0.000134	0.000088	0.000131	0.000091	0.000127	0.000095	0.000124	0.000099	0.000121
17	0.000679	0.001193	0.00071	0.001163	0.000741	0.001133	0.000773	0.001103	0.000804	0.001073	0.000836	0.001044
18	0.00117	0.001967	0.00122	0.001921	0.00127	0.001876	0.00132	0.001831	0.00137	0.001786	0.00142	0.001741
19	0.003628	0.007661	0.003876	0.007431	0.004125	0.007201	0.004373	0.006972	0.004622	0.006742	0.004871	0.006513
20	0.010731	0.022557	0.01146	0.021884	0.01219	0.021211	0.012919	0.020538	0.013649	0.019865	0.014379	0.019193
21	0.000313	0.000654	0.000334	0.000634	0.000355	0.000615	0.000376	0.000595	0.000397	0.000576	0.000419	0.000557
22	0.003293	0.006991	0.00352	0.00678	0.003748	0.006569	0.003975	0.006358	0.004203	0.006147	0.004431	0.005936
23	0.000435	0.000908	0.000464	0.000881	0.000493	0.000853	0.000522	0.000826	0.000551	0.000799	0.000581	0.000772
24	0.000759	0.001587	0.00081	0.00154	0.000861	0.001493	0.000912	0.001446	0.000963	0.001399	0.001015	0.001352

Table 3.4 Solution of fuzzy Kolmogorov’s differential equations for sub-system R_2 obtained by using Buckley and Feuring method [24]

j	$\tilde{p}_j(t)$ for $\alpha = 0$		$\tilde{p}_j(t)$ for $\alpha = 0.2$		$\tilde{p}_j(t)$ for $\alpha = 0.4$		$\tilde{p}_j(t)$ for $\alpha = 0.6$		$\tilde{p}_j(t)$ for $\alpha = 0.8$		$\tilde{p}_j(t)$ for $\alpha = 1$	
	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$	$\tilde{p}_{j(1)}(t, \alpha)$	$\tilde{p}_{j(2)}(t, \alpha)$
1	0.87807	0.862585	0.876803	0.86353	0.875536	0.864525	0.87427	0.86552	0.873003	0.866515	0.871737	0.867511
2	0.064038	0.094192	0.065987	0.092551	0.067937	0.09091	0.069887	0.08927	0.071837	0.087629	0.073787	0.085989
3	0.002941	0.004356	0.003032	0.004278	0.003123	0.004201	0.003214	0.004123	0.003305	0.004046	0.003396	0.003969
4	0.000425	0.000624	0.000436	0.000611	0.000447	0.000598	0.000458	0.000586	0.000469	0.000573	0.00048	0.000561
5	0.00011	0.000159	0.000111	0.000155	0.000113	0.000151	0.000115	0.000147	0.000117	0.000143	0.000119	0.000139
6	0.006073	0.008919	0.00623	0.008739	0.006388	0.008559	0.006546	0.008379	0.006704	0.008199	0.006862	0.008019
7	0.000024	0.000035	0.000024	0.000034	0.000024	0.000033	0.000025	0.000032	0.000025	0.000031	0.000026	0.000031
8	0.000228	0.000475	0.000229	0.000459	0.000243	0.000442	0.000257	0.000425	0.000272	0.000409	0.000287	0.000393
9	0.000031	0.000068	0.000033	0.000065	0.000035	0.000062	0.000037	0.00006	0.000039	0.000057	0.000041	0.000055
10	0.000008	0.000017	0.000008	0.000016	0.000009	0.000015	0.000009	0.000014	0.00001	0.000013	0.00001	0.000013
11	0.000442	0.000973	0.000469	0.000937	0.000497	0.000901	0.000525	0.000865	0.000553	0.000829	0.000581	0.000794
12	0.000001	0.000004	0.000001	0.000003	0.000001	0.000003	0.000002	0.000003	0.000002	0.000003	0.000002	0.000003
13	0.001617	0.003588	0.001727	0.003463	0.001837	0.003339	0.001947	0.003215	0.002057	0.003091	0.002168	0.002967

3.1.7 Shortcoming of obtained results

It is obvious from the graphical representation of α -cut of a trapezoidal fuzzy number $\tilde{p}_j(t)$, shown in Figure 3.3, that if $p_{j(1)}(t, \alpha_1)$, $p_{j(1)}(t, \alpha_2)$ and $p_{j(2)}(t, \alpha_1)$,

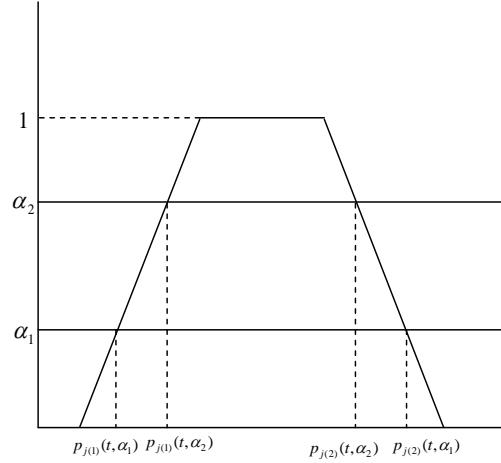


Figure 3.3 Lower and upper bounds of α -cut of a trapezoidal fuzzy number $\tilde{p}_j(t)$

$p_{j(2)}(t, \alpha_2)$ represent the left end points and right end points respectively of α -cuts of a trapezoidal fuzzy number $\tilde{p}_j(t)$ corresponding to different values of α , say α_1 and α_2 . Then, the following conditions should always be satisfied:

1. $p_{j(1)}(t, \alpha_1) \leq p_{j(2)}(t, \alpha_1) \forall \alpha_1 \in [0, 1]$
2. $p_{j(1)}(t, \alpha_2) \leq p_{j(2)}(t, \alpha_2) \forall \alpha_2 \in [0, 1]$
3. $p_{j(1)}(t, \alpha_1) \leq p_{j(1)}(t, \alpha_2)$ and $p_{j(2)}(t, \alpha_1) \geq p_{j(2)}(t, \alpha_2) \forall 0 \leq \alpha_1 \leq \alpha_2 \leq 1$
4. $p_{j(1)}(t, \alpha_1) \leq p_{j(1)}(t, \alpha = 1)$ and $p_{j(2)}(t, \alpha = 1) \leq p_{j(2)}(t, \alpha_1) \forall \alpha_1 \in [0, 1]$

However, it is obvious from the results, shown in Table 3.3 and Table 3.4, obtained by using Buckley and Feuring method [24] that for the values of $p_{1(1)}(t, \alpha)$ and $p_{1(2)}(t, \alpha)$, the conditions $p_{1(1)}(t, \alpha) \leq p_{1(2)}(t, \alpha)$, $\alpha \in [0, 1]$ and $p_{1(1)}(t, \alpha_1) \leq p_{1(1)}(t, \alpha_2)$, $p_{1(2)}(t, \alpha_1) \geq p_{1(2)}(t, \alpha_2)$, $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ are not satisfying. So, a trapezoidal fuzzy number representing $\tilde{p}_1(t)$ cannot be obtained by using the values of $p_{1(1)}(t, \alpha)$ and $p_{1(2)}(t, \alpha)$. Hence, it is not genuine to use the results, shown in Table 3.3 and Table 3.4, to evaluate the fuzzy reliability of piston manufacturing system.

Remark 3.1 In Section 3.1.7, it is shown that the solution obtained by using Buckley and Feuring method [24] cannot be used to evaluate the fuzzy reliability of piston

manufacturing system. Similarly, several real life problems can be found for which the solution of fuzzy Kolmogorov's differential equations, obtained by Buckley and Feuring method [24], is not valid.

3.2 Evidence for shortcoming

To show the shortcoming of Buckley and Feuring method [24], a system of fuzzy equations (3.46), chosen in Example 3.1, is solved by using Buckley and Feuring method [24] and it is shown that the same shortcoming, pointed out in Section 3.1.6, is also occurring in the obtained solution.

Example 3.1 Solve the following system of fuzzy equations:

$$\left. \begin{aligned} \tilde{a}_1 \oplus \tilde{a}_2 &= (10, 20, 30, 40) \\ \tilde{a}_3 \oplus \tilde{a}_4 &= (15, 20, 35, 40) \\ \tilde{a}_1 \oplus \tilde{a}_3 &= (12, 15, 25, 30) \end{aligned} \right\} \quad (3.46)$$

where, \tilde{a}_1 , \tilde{a}_2 , \tilde{a}_3 and \tilde{a}_4 are trapezoidal fuzzy numbers.

3.2.1 Solution of the chosen system of fuzzy equations

Using Buckley and Feuring method [24], the solution of system of fuzzy equations (3.46) can be obtained as follows:

Step 1: Replacing \tilde{a}_j by its α -cut $[a_{j(1)}(\alpha), a_{j(2)}(\alpha)]$, the system of fuzzy equations (3.46) can be written as:

$$\left. \begin{aligned} [a_{1(1)}(\alpha), a_{1(2)}(\alpha)] + [a_{2(1)}(\alpha), a_{2(2)}(\alpha)] &= [10 + 10\alpha, 40 - 10\alpha] \\ [a_{3(1)}(\alpha), a_{3(2)}(\alpha)] + [a_{4(1)}(\alpha), a_{4(2)}(\alpha)] &= [15 + 5\alpha, 40 - 5\alpha] \\ [a_{1(1)}(\alpha), a_{1(2)}(\alpha)] + [a_{3(1)}(\alpha), a_{3(2)}(\alpha)] &= [12 + 3\alpha, 30 - 5\alpha] \end{aligned} \right\} \quad (3.47)$$

Step 2: Using the arithmetic operations between α -cuts, presented in Section 2.1.2, the system of equations (3.47) can be written as:

$$\left. \begin{aligned} [a_{1(1)}(\alpha) + a_{2(1)}(\alpha), a_{1(2)}(\alpha) + a_{2(2)}(\alpha)] &= [10 + 10\alpha, 40 - 10\alpha] \\ [a_{3(1)}(\alpha) + a_{4(1)}(\alpha), a_{3(2)}(\alpha) + a_{4(2)}(\alpha)] &= [15 + 5\alpha, 40 - 5\alpha] \\ [a_{1(1)}(\alpha) + a_{3(1)}(\alpha), a_{1(2)}(\alpha) + a_{3(2)}(\alpha)] &= [12 + 3\alpha, 30 - 5\alpha] \end{aligned} \right\} \quad (3.48)$$

Step 3: Using Definition 2.9, the system of equations (3.48) can be written as:

$$\left. \begin{aligned} a_{1(1)}(\alpha) + a_{2(1)}(\alpha) &= 10 + 10\alpha \\ a_{1(2)}(\alpha) + a_{2(2)}(\alpha) &= 40 - 10\alpha \\ a_{3(1)}(\alpha) + a_{4(1)}(\alpha) &= 15 + 5\alpha \\ a_{3(2)}(\alpha) + a_{4(2)}(\alpha) &= 40 - 5\alpha \\ a_{1(1)}(\alpha) + a_{3(1)}(\alpha) &= 12 + 3\alpha \\ a_{1(2)}(\alpha) + a_{3(2)}(\alpha) &= 30 - 5\alpha \end{aligned} \right\} \quad (3.49)$$

Step 4: The system of equations (3.49) is solved by using simplex method [73] and the obtained solution is shown in Table 3.5.

Table 3.5 Solution of chosen system of fuzzy equations

$\alpha \downarrow$	$a_{1(1)}(\alpha)$	$a_{1(2)}(\alpha)$	$a_{2(1)}(\alpha)$	$a_{2(2)}(\alpha)$	$a_{3(1)}(\alpha)$	$a_{3(2)}(\alpha)$	$a_{4(1)}(\alpha)$	$a_{4(2)}(\alpha)$
0	0	30	10	10	12	0	3	40
0.1	1.5	29.5	9.5	9.5	10.8	0	4.7	39.5
0.2	3	29	9	9	9.6	0	6.4	39
0.3	4.5	28.5	8.5	8.5	8.4	0	8.1	38.5
0.4	6	28	8	8	7.2	0	9.8	38
0.5	7.5	27.5	7.5	7.5	6	0	11.5	37.5
0.6	9	27	7	7	4.8	0	13.2	37
0.7	10.5	26.5	6.5	6.5	3.6	0	14.9	36.5
0.8	12	26	6	6	2.4	0	16.6	36
0.9	13.5	25.5	5.5	5.5	1.2	0	18.3	35.5
1	15	25	5	5	0	0	20	35

3.2.2 Shortcoming of obtained result

Since, $a_{j(1)}(\alpha)$ and $a_{j(2)}(\alpha)$, $j = 1, 2, 3, 4$ are the left end points and right end points respectively of the trapezoidal fuzzy numbers \tilde{a}_j , $j = 1, 2, 3, 4$. So, as discussed in Section 3.1.7, the following conditions should be satisfied for the values of $a_{j(1)}(\alpha)$ and $a_{j(2)}(\alpha)$:

1. $a_{j(1)}(\alpha_1) \leq a_{j(2)}(\alpha_1) \forall \alpha_1 \in [0, 1]$
2. $a_{j(1)}(\alpha_2) \leq a_{j(2)}(\alpha_2) \forall \alpha_2 \in [0, 1]$
3. $a_{j(1)}(\alpha_1) \leq a_{j(1)}(\alpha_2)$ and $a_{j(2)}(\alpha_1) \geq a_{j(2)}(\alpha_2) \forall 0 \leq \alpha_1 \leq \alpha_2 \leq 1$
4. $a_{j(1)}(\alpha_1) \leq a_{j(1)}(\alpha = 1)$ and $a_{j(2)}(\alpha = 1) \leq a_{j(2)}(\alpha_1) \forall \alpha_1 \in [0, 1]$

However, it is obvious from Table 3.5 that for the values of $a_{2(1)}(\alpha)$, the condition $a_{2(1)}(\alpha_1) \leq a_{2(1)}(\alpha_2)$, $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and for the values of $a_{3(1)}(\alpha)$ and $a_{3(2)}(\alpha)$, the condition $a_{3(1)}(\alpha) \leq a_{3(2)}(\alpha)$, $\alpha \in [0, 1]$ are not satisfying. Hence, it is not genuine to use Buckley and Feuring method [24] for solving system of fuzzy equations.

3.3 *JMD* α -cut

In α -cut $[a, b]$, the condition $b \geq a$ should always be satisfied. However, in Buckley and Feuring method [24] the values of a and b are obtained independently. So, the condition $b \geq a$ may or may not be satisfied.

In this section, to resolve this shortcoming a new representation of α -cut, named as *JMD* α -cut, is proposed. Also, with the help of *JMD* α -cut, Buckley and Feuring method [24] is modified for solving n^{th} order fuzzy linear differential equations.

To show the advantage of modified method the system of fuzzy equations (3.46) as well as fuzzy Kolomogorov's differential equations, developed in Section 3.1.5, for which the solution obtained by using Buckley and Feuring method [24] is not appropriate, are solved by using modified method and it is shown that by applying the modified method all the shortcomings are resolved.

3.3.1 Basic definitions

In this section, definition of *JMD* α -cut and equality of two *JMD* α -cuts are introduced.

Definition 3.1 Let $[a, b]$ be an α -cut of a trapezoidal fuzzy number then its *JMD* representation is $[a, \alpha]_{JMD}$, where, $\alpha = b - a \geq 0$.

Definition 3.2 Two *JMD* α -cuts $A = [a_1, \alpha_1]_{JMD}$ and $B = [a_2, \alpha_2]_{JMD}$ are said to be equal i.e., $A = B$ if and only if $a_1 = a_2$ and $\alpha_1 = \alpha_2$.

3.3.2 Arithmetic operations

In this section, with the help of existing arithmetic operations, presented in Section 2.1.2, some arithmetic operations between *JMD* α -cuts are introduced.

Let $A_1 = [a_1, b_1], A_2 = [a_2, b_2]$ be α -cuts of trapezoidal fuzzy numbers and $A_1 = [a_1, \alpha_1]_{JMD}, A_2 = [a_2, \alpha_2]_{JMD}$ be their *JMD* representation, where $\alpha_1 = b_1 - a_1 \geq 0$ and $\alpha_2 = b_2 - a_2 \geq 0$. Then, the arithmetic operations, presented in the Section 2.1.2, are converted into the following arithmetic operations:

$$(i) A_1 + A_2 = [a_1 + a_2, \alpha_1 + \alpha_2]_{JMD}$$

$$(ii) A_1 - A_2 = [a_1 - a_2 - \alpha_2, \alpha_1 + \alpha_2]_{JMD}$$

$$(iii) \lambda A_1 = \begin{cases} [\lambda a_1, \lambda \alpha_1]_{JMD}, & \lambda \geq 0 \\ [\lambda a_1 + \lambda \alpha_1, -\lambda \alpha_1]_{JMD}, & \lambda \leq 0 \end{cases}$$

$$(iv) A_1 A_2 = [a, \alpha]_{JMD}$$

where,

$$a = \text{minimum}(a_1 a_2, (a_1 + \alpha_1) a_2, (a_2 + \alpha_2) a_1, (a_1 + \alpha_1)(a_2 + \alpha_2))$$

$$\alpha = \text{maximum}(a_1 a_2, (a_1 + \alpha_1) a_2, (a_2 + \alpha_2) a_1, (a_1 + \alpha_1)(a_2 + \alpha_2)) - a$$

3.4 Modified method

In this section, the steps of Buckley and Feuring method [24] are modified with the help of *JMD* α -cut.

The steps of modified method are as follows:

Step 1: Convert the α -cuts $[a_{j(1)}(x, \alpha), a_{j(2)}(x, \alpha)]$, $[y_1^{(j)}(x, \alpha), y_2^{(j)}(x, \alpha)]$, $[g_1(x, \alpha), g_2(x, \alpha)]$ and $[\gamma_{j(1)}(x_0, \alpha), \gamma_{j(2)}(x_0, \alpha)]$ corresponding to fuzzy parameters \tilde{a}_j , $\tilde{y}^{(j)}$, \tilde{g} and $\tilde{\gamma}_j$ respectively into *JMD* α -cuts. Assuming $[a_{j(1)}(x, \alpha), \beta_{j(1)}(x, \alpha)]_{JMD}$, $[y_1^{(j)}(x, \alpha), \alpha_1^{(j)}(x, \alpha)]_{JMD}$, $[g_1(x, \alpha), \xi_1(x, \alpha)]_{JMD}$ and $[\gamma_{j(1)}(x_0, \alpha), \zeta_{j(1)}(x_0, \alpha)]_{JMD}$ as *JMD* α -cuts corresponding to fuzzy parameters \tilde{a}_j , $\tilde{y}^{(j)}$, \tilde{g} and $\tilde{\gamma}_j$ respectively, the n^{th} order fuzzy linear differential equation (2.7) with initial conditions (2.8) can be written as:

$$\sum_{j=0}^n [a_{j(1)}(x, \alpha), \beta_{j(1)}(x, \alpha)]_{JMD} [y_1^{(j)}(x, \alpha), \alpha_1^{(j)}(x, \alpha)]_{JMD} = [g_1(x, \alpha), \xi_1(x, \alpha)]_{JMD} \quad (3.50)$$

with initial conditions:

$$[y_1^{(j)}(x_0, \alpha), \alpha_1^{(j)}(x_0, \alpha)]_{JMD} = [\gamma_{j(1)}(x_0, \alpha), \zeta_{j(1)}(x_0, \alpha)]_{JMD}, \quad j = 0, 1, \dots, n-1 \quad (3.51)$$

where,

$$\beta_{j(1)}(x, \alpha) = a_{j(2)}(x, \alpha) - a_{j(1)}(x, \alpha)$$

$$\alpha_1^{(j)}(x, \alpha) = y_2^{(j)}(x, \alpha) - y_1^{(j)}(x, \alpha)$$

$$\zeta_{j(1)}(x_0, \alpha) = \gamma_{j(2)}(x_0, \alpha) - \gamma_{j(1)}(x_0, \alpha)$$

$$\xi_1(x, \alpha) = g_2(x, \alpha) - g_1(x, \alpha)$$

Step 2: Using Definition 3.2 and Section 3.3.2, the n^{th} order linear differential equation (3.50) with initial conditions (3.51) can be split into the following two n^{th} order linear differential equations:

$$\sum_{j=0}^n \text{minimum } (b_{j(1)}(x, \alpha) z_1^{(j)}(x, \alpha)) = g_1(x, \alpha) \quad (3.52)$$

$$\sum_{j=0}^n \text{maximum } (b_{j(1)}(x, \alpha) z_1^{(j)}(x, \alpha)) = \xi_1(x, \alpha) \quad (3.53)$$

with initial conditions:

$$y_1^{(j)}(x_0, \alpha) = \gamma_{j(1)}(x_0, \alpha) \quad (3.54)$$

$$\alpha_1^{(j)}(x_0, \alpha) = \zeta_{j(1)}(x_0, \alpha) \quad (3.55)$$

where,

$$b_{j(1)}(x, \alpha) z_1^{(j)}(x, \alpha) = (a_{j(1)}(x, \alpha) y_1^{(j)}(x, \alpha), a_{j(1)}(x, \alpha) (y_1^{(j)}(x, \alpha) + \alpha_1^{(j)}(x, \alpha)), (a_{j(1)}(x, \alpha) + \beta_{j(1)}(x, \alpha)) y_1^{(j)}(x, \alpha), (a_{j(1)}(x, \alpha) + \beta_{j(1)}(x, \alpha)) (y_1^{(j)}(x, \alpha) + \alpha_1^{(j)}(x, \alpha)))$$

Step 3: Solve the n^{th} order linear differential equations (3.52) and (3.53) with initial conditions (3.54) and (3.55) to find the values of $y_1(x_1, \alpha)$ and $\alpha_1(x_1, \alpha)$ corresponding to $x = x_1$, where x_1 is any real number.

Step 4: Put the value of $y_1(x_1, \alpha)$ and $\alpha_1(x_1, \alpha)$ in $[y_1(x_1, \alpha), y_2(x_1, \alpha)] = [y_1(x_1, \alpha), y_1(x_1, \alpha) + \alpha_1(x_1, \alpha)]$ to find the solution of n^{th} order fuzzy linear differential equation (2.7) with initial conditions (2.8).

3.4.1 Advantage of modified method over Buckley and Feuring method

To show the advantage of modified method over Buckley and Feuring method [24], the system of fuzzy equations (3.46) is solved by using modified method and it is shown that all the shortcomings, occurring in the results due to applying Buckley and Feuring method [24], are resolved. Also, to show the advantage of modified method in real life problems, fuzzy Kolmogorov's differential equations (3.1) to (3.24) with initial conditions (3.25) to (3.29) and fuzzy Kolmogorov's differential equations (3.30) to (3.42) with initial conditions (3.43) to (3.45) are solved for $\alpha =$

0, 0.2, 0.4, 0.6, 0.8, 1 at $t = 360$ hours by using modified method and it is shown that all the shortcomings, occurring in the results due to applying Buckley and Feuring method [24], are resolved.

3.4.1.1 Solution of the chosen system of fuzzy equations

In this section, to show the advantage of modified method over Buckley and Feuring method [24], the system of fuzzy equations (3.46) for which the results obtained by using Buckley and Feuring method [24] are not appropriate, is solved by modified method and it is shown that all the shortcomings are resolved.

Using the modified method, the system of fuzzy equations (3.46) can be solved as follows:

Step 1: Replacing \tilde{a}_j by its *JMD* α -cut $[a_{j(1)}(\alpha), \beta_{j(1)}(\alpha)]_{JMD}$, the system of fuzzy equations (3.46) can be written as:

$$\left. \begin{aligned} [a_{1(1)}(\alpha), \beta_{1(1)}(\alpha)]_{JMD} + [a_{2(1)}(\alpha), \beta_{2(1)}(\alpha)]_{JMD} &= [10 + 10\alpha, 30 - 20\alpha]_{JMD} \\ [a_{3(1)}(\alpha), \beta_{3(1)}(\alpha)]_{JMD} + [a_{4(1)}(\alpha), \beta_{4(1)}(\alpha)]_{JMD} &= [15 + 5\alpha, 25 - 10\alpha]_{JMD} \\ [a_{1(1)}(\alpha), \beta_{1(1)}(\alpha)]_{JMD} + [a_{3(1)}(\alpha), \beta_{3(1)}(\alpha)]_{JMD} &= [12 + 3\alpha, 18 - 8\alpha]_{JMD} \end{aligned} \right\} (3.56)$$

Step 2: Using Section 3.3.2, the system of equations (3.56) can be written as:

$$\left. \begin{aligned} [a_{1(1)}(\alpha) + a_{2(1)}(\alpha), \beta_{1(1)}(\alpha) + \beta_{2(1)}(\alpha)]_{JMD} &= [10 + 10\alpha, 30 - 20\alpha]_{JMD} \\ [a_{3(1)}(\alpha) + a_{4(1)}(\alpha), \beta_{3(1)}(\alpha) + \beta_{4(1)}(\alpha)]_{JMD} &= [15 + 5\alpha, 25 - 10\alpha]_{JMD} \\ [a_{1(1)}(\alpha) + a_{3(1)}(\alpha), \beta_{1(1)}(\alpha) + \beta_{3(1)}(\alpha)]_{JMD} &= [12 + 3\alpha, 18 - 8\alpha]_{JMD} \end{aligned} \right\} (3.57)$$

Step 3: Using Definition 3.2, the system of equations (3.57) can be written as:

$$\left. \begin{aligned}
 a_{1(1)}(\alpha) + a_{2(1)}(\alpha) &= 10 + 10\alpha \\
 \beta_{1(1)}(\alpha) + \beta_{2(1)}(\alpha) &= 30 - 20\alpha \\
 a_{3(1)}(\alpha) + a_{4(1)}(\alpha) &= 15 + 5\alpha \\
 \beta_{3(1)}(\alpha) + \beta_{4(1)}(\alpha) &= 25 - 10\alpha \\
 a_{1(1)}(\alpha) + a_{3(1)}(\alpha) &= 12 + 3\alpha \\
 \beta_{1(1)}(\alpha) + \beta_{3(1)}(\alpha) &= 18 - 8\alpha
 \end{aligned} \right\} \quad (3.58)$$

Step 4: The system of equations (3.58) is solved by using simplex method [73] and the obtained solution is shown in Table 3.6.

Table 3.6 Solution of chosen system of fuzzy equations

$\alpha \downarrow$	$a_{1(1)}(\alpha)$	$a_{1(2)}(\alpha)$	$a_{2(1)}(\alpha)$	$a_{2(2)}(\alpha)$	$a_{3(1)}(\alpha)$	$a_{3(2)}(\alpha)$	$a_{4(1)}(\alpha)$	$a_{4(2)}(\alpha)$
0	0	15	10	25	12	15	3	25
0.1	0	14.5	11	24.5	12.3	15	3.2	24.5
0.2	0	14	12	24	12.6	15	3.4	24
0.3	0	13.5	13	23.5	12.9	15	3.6	23.5
0.4	0	13	14	23	13.2	15	3.8	23
0.5	0	12.5	15	22.5	13.5	15	4	22.5
0.6	0	12	16	22	13.8	15	4.2	22
0.7	0	11.5	17	21.5	14.1	15	4.4	21.5
0.8	0	11	18	21	14.4	15	4.6	21
0.9	0	10.5	19	20.5	14.7	15	4.8	20.5
1	0	10	20	20	15	15	5	20

It is obvious from Table 3.6 that for the values of $a_{j(1)}(\alpha)$ and $a_{j(2)}(\alpha)$, $j = 1, 2, 3, 4$, all the conditions, discussed in Section 3.2.2, are satisfying i.e., on solving the system of fuzzy equations by using modified method, all the shortcomings, occurring in the results, due to applying Buckley and Feuring method [24] are resolved.

3.4.1.2 Solution of fuzzy Kolmogorov's differential equations by using modified method

The solution of fuzzy Kolmogorov's differential equations (3.1) to (3.24) with initial conditions (3.25) to (3.29) and the solution of fuzzy Kolmogorov's dif-

erential equations (3.30) to (3.42) with initial conditions (3.43) to (3.45), obtained by using the modified method, is shown in Table 3.7 and Table 3.8 respectively.

Table 3.7 Solution of fuzzy Kolmogorov’s differential equations for sub-system R_1 obtained by using modified method

j	$\bar{p}_j(t)$ for $\alpha = 0$		$\bar{p}_j(t)$ for $\alpha = 0.2$		$\bar{p}_j(t)$ for $\alpha = 0.4$		$\bar{p}_j(t)$ for $\alpha = 0.6$		$\bar{p}_j(t)$ for $\alpha = 0.8$		$\bar{p}_j(t)$ for $\alpha = 1$	
	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$
1	0.525421	0.532877	0.525833	0.532468	0.526246	0.532059	0.526658	0.53165	0.527071	0.531241	0.527484	0.530833
2	0.055996	0.059671	0.056194	0.059473	0.056392	0.059276	0.056591	0.059079	0.056789	0.058882	0.056988	0.058685
3	0.032869	0.03525	0.032998	0.035118	0.033127	0.034986	0.033256	0.034854	0.033385	0.034722	0.033514	0.034591
4	0.306812	0.315675	0.307227	0.315258	0.307642	0.314841	0.308057	0.314424	0.308472	0.314007	0.308887	0.313591
5	0.000537	0.000586	0.000539	0.000583	0.000542	0.00058	0.000545	0.000578	0.000548	0.000575	0.000551	0.000573
6	0.005867	0.006177	0.005881	0.006162	0.005896	0.006147	0.005911	0.006132	0.005926	0.006117	0.005941	0.006103
7	0.000747	0.000821	0.000751	0.000817	0.000755	0.000813	0.000759	0.000809	0.000763	0.000805	0.000768	0.000801
8	0.001307	0.001652	0.001314	0.001644	0.001321	0.001637	0.001329	0.00163	0.001336	0.001623	0.001344	0.001616
9	0.000057	0.00008	0.000058	0.000078	0.000059	0.000077	0.00006	0.000076	0.000061	0.000075	0.000063	0.000074
10	0.000623	0.000737	0.000628	0.000731	0.000633	0.000726	0.000638	0.000721	0.000643	0.000716	0.000649	0.000711
11	0.000079	0.000109	0.000081	0.000108	0.000082	0.000107	0.000084	0.000106	0.000086	0.000105	0.000088	0.000104
12	0.000139	0.000272	0.000142	0.000269	0.000145	0.000266	0.000148	0.000263	0.000151	0.00026	0.000154	0.000257
13	0.000033	0.000048	0.000033	0.000047	0.000034	0.000046	0.000035	0.000045	0.000036	0.000044	0.000037	0.000044
14	0.000366	0.00045	0.000369	0.000446	0.000373	0.000442	0.000377	0.000438	0.000381	0.000434	0.000385	0.00043
15	0.000046	0.000068	0.000047	0.000066	0.000048	0.000065	0.000049	0.000064	0.000051	0.000063	0.000052	0.000062
16	0.000081	0.000179	0.000083	0.000177	0.000085	0.000175	0.000087	0.000172	0.000089	0.00017	0.000092	0.000168
17	0.000679	0.000836	0.000686	0.000828	0.000693	0.000821	0.0007	0.000814	0.000707	0.000807	0.000715	0.0008
18	0.00117	0.0016	0.00119	0.001579	0.001211	0.001558	0.001231	0.001537	0.001252	0.001516	0.001273	0.001495
19	0.003628	0.003872	0.003638	0.003862	0.003648	0.003852	0.003658	0.003842	0.003668	0.003832	0.003678	0.003822
20	0.010731	0.011673	0.01077	0.011633	0.010809	0.011594	0.010848	0.011555	0.010887	0.011516	0.010927	0.011477
21	0.000313	0.000366	0.000315	0.000363	0.000317	0.000361	0.00032	0.000358	0.000322	0.000356	0.000325	0.000354
22	0.003293	0.003467	0.0033	0.00346	0.003307	0.003453	0.003314	0.003446	0.003321	0.003439	0.003328	0.003432
23	0.000435	0.000506	0.000438	0.000502	0.000441	0.000499	0.000444	0.000496	0.000447	0.000493	0.000451	0.00049
24	0.000759	0.000977	0.000764	0.000972	0.000769	0.000967	0.000774	0.000962	0.000779	0.000957	0.000784	0.000952

Table 3.8 Solution of fuzzy Kolmogorov’s differential equations for sub-system R_2 obtained by using modified method

j	$\bar{p}_j(t)$ for $\alpha = 0$		$\bar{p}_j(t)$ for $\alpha = 0.2$		$\bar{p}_j(t)$ for $\alpha = 0.4$		$\bar{p}_j(t)$ for $\alpha = 0.6$		$\bar{p}_j(t)$ for $\alpha = 0.8$		$\bar{p}_j(t)$ for $\alpha = 1$	
	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$	$\bar{p}_{j(1)}(t, \alpha)$	$\bar{p}_{j(2)}(t, \alpha)$
1	0.87807	0.892416	0.878793	0.891695	0.879517	0.890974	0.880241	0.890253	0.880965	0.889532	0.881689	0.888812
2	0.064038	0.070146	0.064343	0.069841	0.064649	0.069537	0.064955	0.069232	0.065261	0.068928	0.065567	0.068624
3	0.003246	0.003239	0.002956	0.00323	0.002971	0.003215	0.002987	0.0032	0.003002	0.003185	0.003018	0.00317
4	0.000425	0.000467	0.000426	0.000464	0.000428	0.000462	0.00043	0.000459	0.000432	0.000457	0.000434	0.000455
5	0.00011	0.000119	0.00011	0.000118	0.00011	0.000117	0.00011	0.000117	0.00011	0.000116	0.000111	0.000116
6	0.006073	0.006309	0.006082	0.006297	0.006091	0.006285	0.006101	0.006273	0.00611	0.006261	0.00612	0.00625
7	0.000024	0.000026	0.000024	0.000025	0.000024	0.000025	0.000024	0.000025	0.000024	0.000025	0.000024	0.000025
8	0.000214	0.000337	0.00022	0.000331	0.000226	0.000325	0.000232	0.000319	0.000238	0.000313	0.000244	0.000307
9	0.000031	0.000047	0.000031	0.000046	0.000032	0.000045	0.000032	0.000044	0.000033	0.000043	0.000034	0.000043
10	0.000008	0.000032	0.0000082	0.000031	0.0000084	0.000031	0.0000086	0.00003	0.0000088	0.00003	0.000009	0.00003
11	0.000442	0.000522	0.000445	0.000518	0.000448	0.000514	0.000451	0.00051	0.000454	0.000506	0.000457	0.000503
12	0.000001	0.0000019	0.000001	0.0000018	0.000001	0.0000017	0.000001	0.0000017	0.000001	0.0000016	0.000001	0.0000016
13	0.001617	0.002328	0.001649	0.002295	0.001682	0.002262	0.001714	0.00223	0.001747	0.002197	0.00178	0.002165

It is obvious from Table 3.7 and Table 3.8 that all the necessary conditions, discussed in section 3.1.7, are satisfying for all values of $p_{j(1)}(t, \alpha)$ and $p_{j(2)}(t, \alpha)$ i.e.,

on solving fuzzy Kolmogorov's differential equations by using the modified method all the shortcomings, occurring in the results, due to applying Buckley and Feuring method [24], are resolved. Hence, the results shown in Table 3.7 and Table 3.8 can be used to evaluate the fuzzy reliability of piston manufacturing system.

3.4.2 Validity of the results obtained with the help of *JMD* α -cut

In Section 3.1.7 and Section 3.2.2, it is pointed out that on solving the system of fuzzy equations as well as fuzzy Kolmogorov's differential equations with existing α -cut the obtained solutions do not define α -cuts of fuzzy numbers. While, on solving the same equations with *JMD* α -cut, the obtained solutions always define α -cuts of fuzzy numbers. However, on the basis of numerical examples it cannot be concluded that by representing the parameters as *JMD* α -cuts of fuzzy numbers, the obtained solution will always represent α -cut of a fuzzy number. In this section, the theoretical reason for the validity of results, obtained with *JMD* α -cuts, is discussed.

To find the solution of n fuzzy equations with n variables represented by existing α -cut $\tilde{a}_j = [a_{j(1)}(\alpha), a_{j(2)}(\alpha)]$, $j = 1, 2, \dots, n$, these equations are converted into $2n$ crisp equations with $2n$ crisp variables $a_{j(1)}(\alpha), a_{j(2)}(\alpha)$, $j = 1, 2, \dots, n$ and the obtained crisp equations are solved by simplex method [73] to find the values of $a_{j(1)}(\alpha), a_{j(2)}(\alpha)$, $j = 1, 2, \dots, n$. Since, in the simplex method [73] there is only one restriction on the variables $a_{j(1)}(\alpha), a_{j(2)}(\alpha)$, $j = 1, 2, \dots, n$ that all these variables should be non-negative real numbers. So, the condition $a_{j(1)}(\alpha) \leq a_{j(2)}(\alpha)$, $j = 1, 2, \dots, n$ as well as other conditions, discussed in Section 3.2.2, may or may not be satisfied.

Similarly, for solving n^{th} order fuzzy linear differential equation with dependent variable represented by existing α -cut $\tilde{y} = [y_1(x, \alpha), y_2(x, \alpha)]$, it is converted into two n^{th} order crisp linear differential equations with dependent variables $y_1(x, \alpha)$ and $y_2(x, \alpha)$ and the obtained crisp linear differential equations are solved to find the values of $y_1(x, \alpha)$ and $y_2(x, \alpha)$. Since, it is not possible to apply any restriction on the dependent variables $y_1(x, \alpha)$ and $y_2(x, \alpha)$. So, the condition $y_1(x, \alpha) \leq y_2(x, \alpha)$ as well as other conditions, discussed in Section 3.1.7, may or may not be satisfied.

However, to find the solution of n fuzzy equations with n variables represented by *JMD* α -cut $\tilde{a}_j = [a_{j(1)}(\alpha), \beta_{j(1)}(\alpha)]_{JMD}$, $j = 1, 2, \dots, n$, these equations are converted into $2n$ crisp equations with $2n$ crisp variables $a_{j(1)}(\alpha), \beta_{j(1)}(\alpha)$, $j = 1, 2, \dots, n$ and the obtained crisp equations are solved by simplex method [73] to find the values of $a_{j(1)}(\alpha), \beta_{j(1)}(\alpha)$, $j = 1, 2, \dots, n$. Now, with the help of $a_{j(1)}(\alpha), \beta_{j(1)}(\alpha)$, the existing α -cut $[a_{j(1)}(\alpha), a_{j(1)}(\alpha) + \beta_{j(1)}(\alpha)]$ is obtained. Since, $\beta_{j(1)}(\alpha)$ is a non-negative real number. So, the condition $a_{j(1)}(\alpha) \leq a_{j(1)}(\alpha) + \beta_{j(1)}(\alpha)$ as well as other conditions, discussed in Section 3.2.2, will always be satisfied.

Similarly, for solving n^{th} order fuzzy linear differential equation with dependent variable represented by *JMD* α -cut $\tilde{y} = [y_1(x, \alpha), \alpha_1(x, \alpha)]_{JMD}$, it is converted into two n^{th} order crisp linear differential equations with dependent variables $y_1(x, \alpha)$ and $\alpha_1(x, \alpha)$ and the obtained crisp linear differential equations are solved to find the values of $y_1(x, \alpha)$ and $\alpha_1(x, \alpha)$. Since, $\alpha_1(x, \alpha)$ is a non-negative real number. So, the condition $y_1(x, \alpha) \leq y_1(x, \alpha) + \alpha_1(x, \alpha)$ as well as other conditions, discussed in Section 3.1.7, will always be satisfied.

3.5 Fuzzy reliability evaluation of piston manufacturing system

In Section 3.1.7, it is pointed out that the solution of fuzzy Kolmogorov's differential equations, obtained by using Buckley and Feuring method [24], does not define α -cut of a fuzzy number. Due to which, the obtained solution cannot be used to evaluate the fuzzy reliability of piston manufacturing system. However, in Section 3.4.1.2, it is shown that if the same fuzzy Kolmogorov's differential equations are solved by using the modified method then the obtained solution always define α -cut of a fuzzy number. In this section, the solution of fuzzy Kolmogorov's differential equations, shown in Table 3.7 and Table 3.8, obtained by using modified method, is used to evaluate the fuzzy reliability of piston manufacturing system.

Using the obtained solution of fuzzy Kolmogorov's differential equations, shown in Table 3.7 and Table 3.8, the α -cuts corresponding to fuzzy reliability $\tilde{R}_1(t) = \sum_{i=1}^4 \tilde{p}_i(t)$, $\tilde{R}_2(t) = \sum_{i=1}^2 \tilde{p}_i(t)$ and $\tilde{R}(t) = \tilde{R}_1(t) \otimes \tilde{R}_2(t)$ i.e., $[R_{1(1)}(t, \alpha), R_{1(2)}(t, \alpha)]$, $[R_{2(1)}(t, \alpha), R_{2(2)}(t, \alpha)]$ and $[R_1(t, \alpha), R_2(t, \alpha)] = [R_{1(1)}(t, \alpha), R_{1(2)}(t, \alpha)][R_{2(1)}(t, \alpha), R_{2(2)}(t, \alpha)]$ of sub-system R_1 , sub-system R_2 and whole system respectively computed for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$ at $t = 360$ hours are shown in Table 3.9.

Table 3.9 α -cuts corresponding to fuzzy reliability of sub-system R_1 , sub-system R_2 and whole system at $t = 360$ hours

$\alpha \downarrow$	$R_{1(1)}(t, \alpha)$	$R_{1(2)}(t, \alpha)$	$R_{2(1)}(t, \alpha)$	$R_{2(2)}(t, \alpha)$	$R_1(t, \alpha)$	$R_2(t, \alpha)$
0	0.921098	0.943473	0.942108	0.962562	0.867774	0.908151
0.2	0.922252	0.942317	0.943136	0.961536	0.869809	0.906072
0.4	0.923407	0.941162	0.944166	0.960511	0.87185	0.903996
0.6	0.924562	0.940007	0.945196	0.959485	0.873892	0.901923
0.8	0.925717	0.938852	0.946226	0.95846	0.875938	0.899852
1	0.926873	0.9377	0.947256	0.957436	0.877986	0.897788

Trapezoidal fuzzy number representing the fuzzy reliability of piston manufacturing system at $t = 360$ hours corresponding to different presumption levels is shown in Figure 3.4.

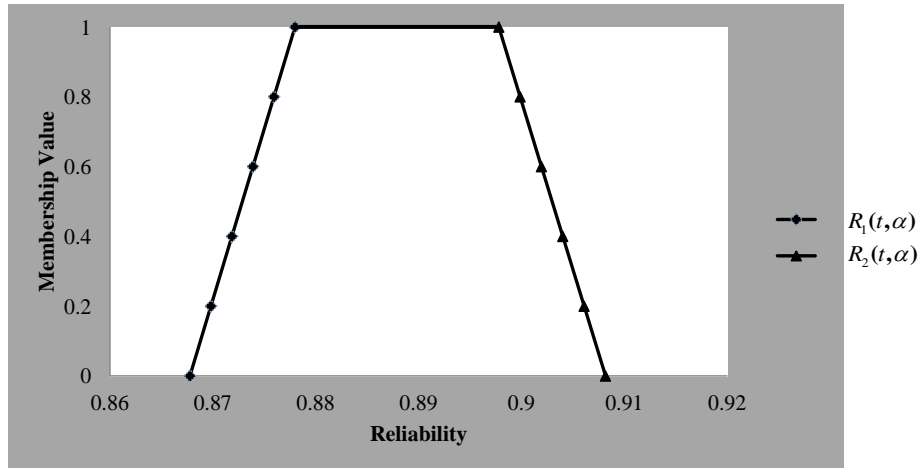


Figure 3.4 Trapezoidal fuzzy number representing the fuzzy reliability of piston manufacturing system at $t = 360$ hours corresponding to different presumption levels

3.6 Physical interpretation of results

In this section, the physical interpretation of the obtained fuzzy reliability of piston manufacturing system at $t = 360$ hours is interpreted.

It is obvious from Figure 3.4 that

1. The reliability of piston manufacturing system at $t = 360$ hours lies between 0.867774 and 0.908151.
2. The degree of acceptance of the experts about the statement that the reliability of piston manufacturing system at $t = 360$ hours lies between 0.877786 to 0.897788 is 1.
3. The degree of acceptance $\mu_{\tilde{R}}(r)$ for the remaining values of reliability r , $r \in [0.867774, 0.908151]$ can be obtained by putting r in:

$$\mu_{\tilde{R}}(r) = \begin{cases} \frac{(r-0.867774)}{(0.010212)}, & 0.867774 \leq r < 0.877986 \\ 1, & 0.877986 \leq r \leq 0.897788 \\ \frac{(0.908151-r)}{(0.010363)}, & 0.897788 < r \leq 0.908151 \end{cases}$$

Also, the variation in reliability of piston manufacturing system with time at $\alpha = 0, \alpha = 0.2, \alpha = 0.4, \alpha = 0.6, \alpha = 0.8, \alpha = 1$ is shown in Figure 3.5 to Figure 3.10 respectively.

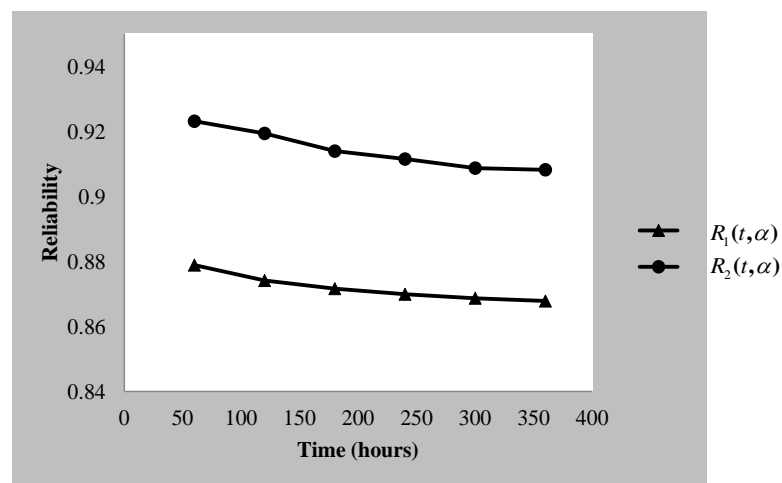


Figure 3.5 Variation in reliability with time at $\alpha = 0$

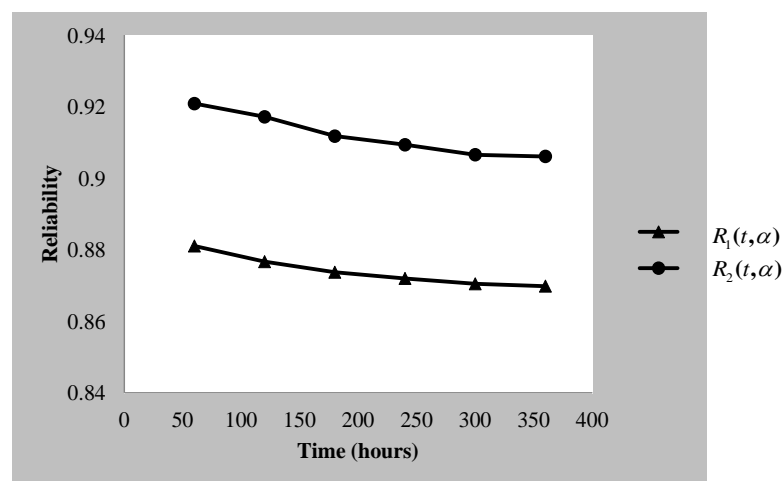


Figure 3.6 Variation in reliability with time at $\alpha = 0.2$

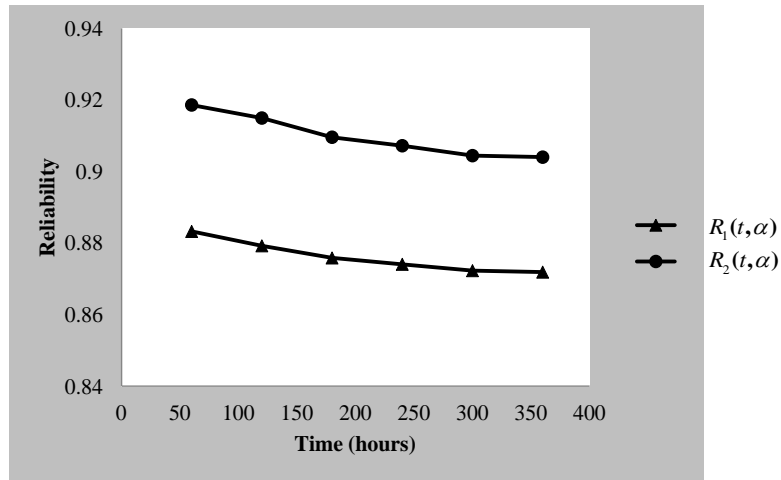


Figure 3.7 Variation in reliability with time at $\alpha = 0.4$

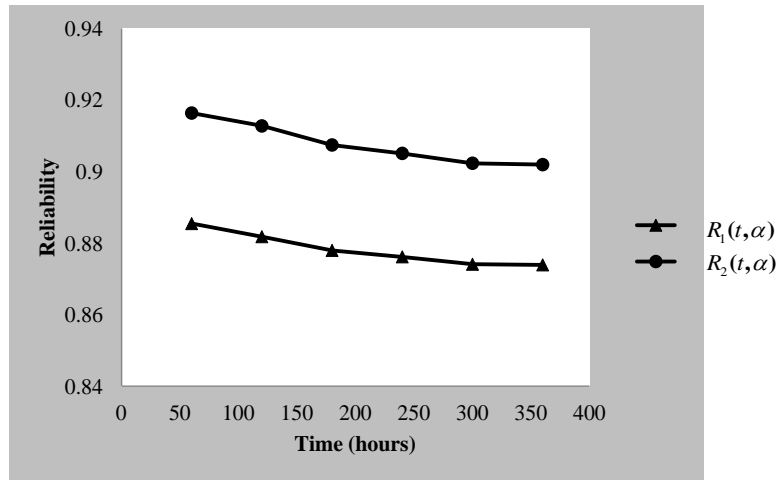


Figure 3.8 Variation in reliability with time at $\alpha = 0.6$

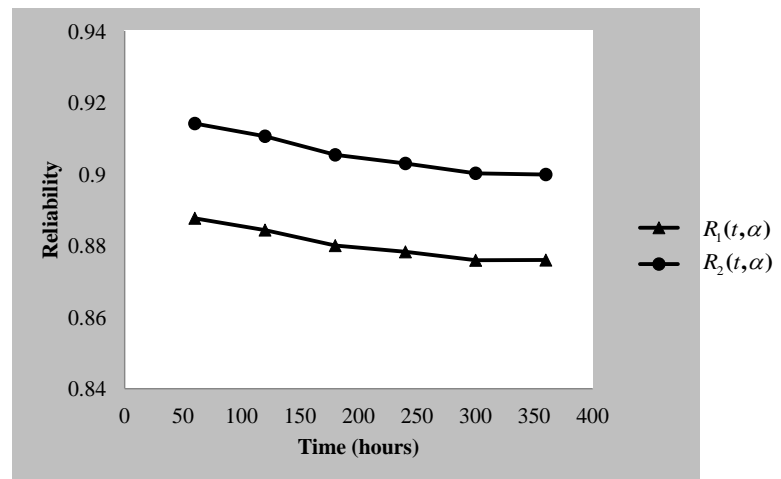


Figure 3.9 Variation in reliability with time at $\alpha = 0.8$

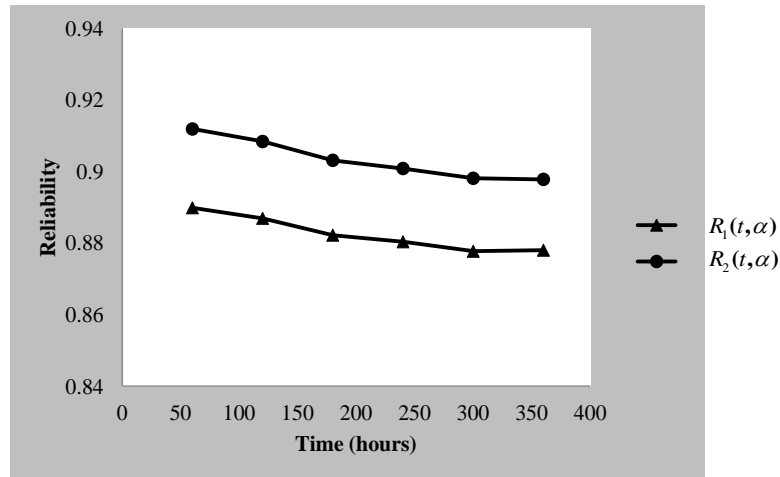


Figure 3.10 Variation in reliability with time at $\alpha = 1$

3.7 Conclusions

On the basis of the presented study, it is concluded that it is better to use the modified method as compared to Buckley and Feuring method [24] for solving n^{th} order fuzzy linear differential equations and hence for solving fuzzy Kolmogorov's differential equations to evaluate the fuzzy reliability of industrial systems.

Chapter 4

AN EFFICIENT METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS AND ITS APPLICATION FOR ANALYZING THE FUZZY RELIABILITY OF PISTON MANUFACTURING SYSTEM

In this chapter, a new method is proposed for solving n^{th} order fuzzy linear differential equations with the help of *JMD* trapezoidal fuzzy number [81] and it is shown that on solving the same fuzzy Kolmogorov's differential equations by applying the method, proposed in this chapter as well as by applying the modified method, proposed in previous chapter, same results are obtained. However, it is much easy to apply the method, proposed in this chapter, as compared to the modified method, proposed in previous chapter. To illustrate the proposed method, the fuzzy Kolmogorov's differential equations, developed in previous chapter, are solved and with the help of obtained solution, fuzzy reliability of piston manufacturing system is evaluated. The advantage of *JMD* trapezoidal fuzzy number over existing trapezoidal fuzzy number is also discussed.

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4.1 *JMD* trapezoidal fuzzy number

Kumar and Kaur [81] proposed *JMD* trapezoidal fuzzy number and shown that it is better to use the proposed representation of trapezoidal fuzzy number instead of existing representation of trapezoidal fuzzy number to find the fuzzy optimal solution of fuzzy transportation problems.

In this section, some basic definitions and arithmetic operations between *JMD* trapezoidal fuzzy numbers are presented [81].

4.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 4.1 Let (a, b, c, d) be a trapezoidal fuzzy number then its *JMD* representation is $(a, \alpha, \beta, \gamma)_{JMD}$.

where, $\alpha = b - a \geq 0, \beta = c - b \geq 0, \gamma = d - c \geq 0$.

Definition 4.2 A *JMD* trapezoidal fuzzy number $\tilde{A} = (a, \alpha, \beta, \gamma)_{JMD}$ is said to be zero *JMD* trapezoidal fuzzy number if and only if $a = 0, \alpha = 0, \beta = 0$ and $\gamma = 0$.

Definition 4.3 A *JMD* trapezoidal fuzzy number $\tilde{A} = (a, \alpha, \beta, \gamma)_{JMD}$ is said to be non-negative *JMD* trapezoidal fuzzy number if and only if $a \geq 0$.

Definition 4.4 Two *JMD* trapezoidal fuzzy numbers $\tilde{A} = (a_1, \alpha_1, \beta_1, \gamma_1)_{JMD}$ and $\tilde{B} = (a_2, \alpha_2, \beta_2, \gamma_2)_{JMD}$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ if and only if $a_1 = a_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$.

4.1.2 Arithmetic operations between *JMD* trapezoidal fuzzy numbers

In this section, arithmetic operations between *JMD* trapezoidal fuzzy numbers are presented.

Let $\tilde{A}_1 = (a_1, \alpha_1, \beta_1, \gamma_1)_{JMD}$ and $\tilde{A}_2 = (a_2, \alpha_2, \beta_2, \gamma_2)_{JMD}$ be two *JMD* trapezoidal fuzzy numbers. Then,

$$(i) \tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2)_{JMD}$$

$$(ii) \tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - a_2 - \alpha_2 - \beta_2 - \gamma_2, \alpha_1 + \gamma_2, \beta_1 + \beta_2, \alpha_2 + \gamma_1)_{JMD}$$

$$(iii) \lambda A_1 = \begin{cases} (\lambda a_1, \lambda \alpha_1, \lambda \beta_1, \lambda \gamma_1)_{JMD}, & \lambda \geq 0 \\ (\lambda a_1 + \lambda \alpha_1 + \lambda \beta_1 + \lambda \gamma_1, -\lambda \gamma_1, -\lambda \beta_1, -\lambda \alpha_1)_{JMD}, & \lambda \leq 0 \end{cases}$$

$$(iv) \tilde{A}_1 \otimes \tilde{A}_2 = (\text{minimum}(a), \text{minimum}(b) - \text{minimum}(a), \text{maximum}(b) - \text{minimum}(b), \text{maximum}(a) - \text{maximum}(b))$$

where,

$$a = (a_1 a_2, a_1(a_2 + \alpha_2 + \beta_2 + \gamma_2), (a_1 + \alpha_1 + \beta_1 + \gamma_1)a_2, (a_1 + \alpha_1 + \beta_1 + \gamma_1)(a_2 + \alpha_2 + \beta_2 + \gamma_2))$$

$$b = ((a_1 + \alpha_1)(a_2 + \alpha_2), (a_1 + \alpha_1)(a_2 + \alpha_2 + \beta_2), (a_1 + \alpha_1 + \beta_1)(a_2 + \alpha_2), (a_1 + \alpha_1 + \beta_1)(a_2 + \alpha_2 + \beta_2))$$

4.2 Proposed method for solving n^{th} order fuzzy linear differential equations

To find the solution of fuzzy differential equations by using the modified method, proposed in previous chapter, there is a need to repeat the whole process for different values of α . In this section, to reduce the complexity of the method, proposed in Chapter 3, a new method is proposed to find the solution of n^{th} order fuzzy linear differential equations with the help of *JMD* trapezoidal fuzzy numbers.

The solution of n^{th} order fuzzy linear differential equation (2.7) with initial conditions (2.8) can be obtained by using the following steps of proposed method.

Step 1: Convert all the parameters of fuzzy differential equation, represented by trapezoidal fuzzy number (a, b, c, d) , into *JMD* trapezoidal fuzzy number. Assuming $\tilde{a}_j = (a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)})_{JMD}$, $\tilde{y}^{(j)} = (y_1^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)})_{JMD}$, $\tilde{\gamma}_j = (\gamma_{j(1)}, \zeta_{j(1)}, \zeta_{j(2)})$,

$\zeta_{j(3)}_{JMD}$ and $\tilde{g} = (g_1, \xi_1, \xi_2, \xi_3)_{JMD}$, the n^{th} order fuzzy linear differential equation (2.7) with initial conditions (2.8) can be written as:

$$\sum_{j=0}^n [(a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)})_{JMD} \otimes (y_1^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)})_{JMD}] = (g_1, \xi_1, \xi_2, \xi_3)_{JMD} \quad (4.1)$$

with initial conditions:

$$(y_1^{(j)}(x_0), \alpha_1^{(j)}(x_0), \alpha_2^{(j)}(x_0), \alpha_3^{(j)}(x_0))_{JMD} = (\gamma_{j(1)}, \zeta_{j(1)}, \zeta_{j(2)}, \zeta_{j(3)})_{JMD},$$

$$j = 0, 1, \dots, n-1 \quad (4.2)$$

Step 2: Using Definition 4.4 and Section 4.1.2, the n^{th} order fuzzy linear differential equation (4.1) with initial conditions (4.2) can be split into the following four n^{th} order linear differential equations:

$$\sum_{j=0}^n \text{minimum}(b_j z^{(j)}) = g \quad (4.3)$$

$$\sum_{j=0}^n [\text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_1 \quad (4.4)$$

$$\sum_{j=0}^n [\text{maximum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)}) - \text{minimum}(c_j z^{(j)})] = \xi_2 \quad (4.5)$$

$$\sum_{j=0}^n [\text{maximum}(b_j z^{(j)}) - \text{minimum}(b_j z^{(j)}) - \text{minimum}(c_j z^{(j)}) - \text{maximum}(c_j z^{(j)})] = \xi_3 \quad (4.6)$$

with initial conditions:

$$y_1^{(j)}(x_0) = \gamma_{j(1)}, j = 0, 1, \dots, n-1 \quad (4.7)$$

$$\alpha_1^{(j)}(x_0) = \zeta_{j(1)}, j = 0, 1, \dots, n-1 \quad (4.8)$$

$$\alpha_2^{(j)}(x_0) = \zeta_{j(2)}, j = 0, 1, \dots, n-1 \quad (4.9)$$

$$\alpha_3^{(j)}(x_0) = \zeta_{j(3)}, j = 0, 1, \dots, n-1 \quad (4.10)$$

where,

$$b_j z^{(j)} = (a_{j(1)} y_1^{(j)}, a_{j(1)}(y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)}) y_1^{(j)}, (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)})(y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)}))$$

$$c_j z^{(j)} = ((a_{j(1)} + \beta_{j(1)})(y_1^{(j)} + \alpha_1^{(j)}), (a_{j(1)} + \beta_{j(1)})(y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)}), (a_{j(1)} + \beta_{j(1)} +$$

$$\beta_{j(2)}(y_1^{(j)} + \alpha_1^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)})(y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)})$$

Step 3: Solve the n^{th} order linear differential equations (4.3) to (4.6) with initial conditions (4.7) to (4.10) to find the values of y_1 , α_1 , α_2 and α_3 .

Step 4: Put the values of y_1 , α_1 , α_2 and α_3 in $\tilde{y} = (y_1, \alpha_1, \alpha_2, \alpha_3)_{JMD}$ to find the solution of n^{th} order fuzzy linear differential equation (2.7) with initial conditions (2.8).

Step 5: Convert $\tilde{y} = (y_1, \alpha_1, \alpha_2, \alpha_3)_{JMD}$ into $\tilde{y} = (a, b, c, d)$, where, $a = y_1$, $b = y_1 + \alpha_1$, $c = y_1 + \alpha_1 + \alpha_2$ and $d = y_1 + \alpha_1 + \alpha_2 + \alpha_3$.

4.3 Fuzzy reliability evaluation of piston manufacturing system

In this section, to illustrate the proposed method the fuzzy Kolmogorov's differential equations of piston manufacturing system, developed in previous chapter, are solved by using the proposed method and with the help of obtained solution, fuzzy reliability of piston manufacturing system is evaluated.

4.3.1 Solution of fuzzy Kolmogorov's differential equations

The solution of fuzzy Kolmogorov's differential equations (3.1) to (3.24) with initial conditions (3.25) to (3.29) and the solution of fuzzy Kolmogorov's differential equations (3.30) to (3.42) with initial conditions (3.43) to (3.45), obtained by using the proposed method, is shown in Table 4.1 and Table 4.2 respectively.

Table 4.1 Solution of fuzzy Kolmogorov's differential equations of sub-system R_1 obtained by using proposed method

j	$\bar{p}_j(t)$ for $t=60$		$\bar{p}_j(t)$ for $t=120$		$\bar{p}_j(t)$ for $t=180$		$\bar{p}_j(t)$ for $t=240$		$\bar{p}_j(t)$ for $t=300$		$\bar{p}_j(t)$ for $t=360$	
	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$
1	(0.7220279, 0.724095, 0.731181, 0.734988)	(0.652165, 0.655343, 0.66104, 0.664203)	(0.603616, 0.606389, 0.611232, 0.613988)	(0.568691, 0.571173, 0.5754, 0.577863)	(0.543538, 0.54602, 0.550247, 0.55271)	(0.525421, 0.527484, 0.530833, 0.532877)	(0.509596, 0.512169, 0.515742, 0.519315)	(0.495498, 0.499171, 0.502844, 0.506517)	(0.482849, 0.486522, 0.490195, 0.493868)	(0.471641, 0.475314, 0.478987, 0.482660)	(0.461882, 0.465555, 0.469228, 0.472901)	(0.453585, 0.457258, 0.460931, 0.464604)
2	(0.071153, 0.072092, 0.074107, 0.075045)	(0.065846, 0.066938, 0.069151, 0.070288)	(0.062073, 0.063193, 0.065337, 0.066453)	(0.059359, 0.060453, 0.06345, 0.063539)	(0.057404, 0.058498, 0.060495, 0.061584)	(0.055936, 0.056988, 0.058685, 0.059671)	(0.055196, 0.056248, 0.057841, 0.059434)	(0.054996, 0.056048, 0.057641, 0.059234)	(0.054898, 0.055950, 0.057543, 0.059136)	(0.054800, 0.055852, 0.057445, 0.059038)	(0.054702, 0.055754, 0.057347, 0.058940)	(0.054604, 0.055656, 0.057249, 0.058842)
3	(0.044381, 0.045198, 0.046916, 0.047741)	(0.042925, 0.043805, 0.044963, 0.046121)	(0.040352, 0.041218, 0.042925, 0.044632)	(0.037551, 0.038333, 0.040146, 0.041959)	(0.035423, 0.036189, 0.037996, 0.039809)	(0.032869, 0.033514, 0.034666, 0.035813)	(0.032869, 0.033514, 0.034666, 0.035813)	(0.032869, 0.033514, 0.034666, 0.035813)	(0.032869, 0.033514, 0.034666, 0.035813)	(0.032869, 0.033514, 0.034666, 0.035813)	(0.032869, 0.033514, 0.034666, 0.035813)	(0.032869, 0.033514, 0.034666, 0.035813)
4	(0.096162, 0.096893, 0.098549, 0.099281)	(0.169323, 0.172501, 0.174963, 0.175998)	(0.221951, 0.223285, 0.22648, 0.227818)	(0.259851, 0.261463, 0.265271, 0.266889)	(0.287149, 0.28731, 0.291118, 0.292736)	(0.306812, 0.308887, 0.313391, 0.319675)	(0.306812, 0.308887, 0.313391, 0.319675)	(0.306812, 0.308887, 0.313391, 0.319675)	(0.306812, 0.308887, 0.313391, 0.319675)	(0.306812, 0.308887, 0.313391, 0.319675)	(0.306812, 0.308887, 0.313391, 0.319675)	(0.306812, 0.308887, 0.313391, 0.319675)
5	(0.000738, 0.000761, 0.000807, 0.00083)	(0.000688, 0.00069, 0.000729, 0.00075)	(0.000618, 0.000637, 0.00067, 0.000689)	(0.000582, 0.000599, 0.000627, 0.000643)	(0.000556, 0.000573, 0.000601, 0.000617)	(0.000537, 0.000551, 0.000573, 0.000586)	(0.000537, 0.000551, 0.000573, 0.000586)	(0.000537, 0.000551, 0.000573, 0.000586)	(0.000537, 0.000551, 0.000573, 0.000586)	(0.000537, 0.000551, 0.000573, 0.000586)	(0.000537, 0.000551, 0.000573, 0.000586)	(0.000537, 0.000551, 0.000573, 0.000586)
6	(0.000756, 0.000785, 0.007641, 0.007663)	(0.007414, 0.007452, 0.007546, 0.007584)	(0.006856, 0.006907, 0.007028, 0.007079)	(0.006417, 0.006478, 0.00658, 0.006679)	(0.00617, 0.006224, 0.00633, 0.006417)	(0.006009, 0.006063, 0.00618, 0.006251)	(0.006009, 0.006063, 0.00618, 0.006251)	(0.006009, 0.006063, 0.00618, 0.006251)	(0.006009, 0.006063, 0.00618, 0.006251)	(0.006009, 0.006063, 0.00618, 0.006251)	(0.006009, 0.006063, 0.00618, 0.006251)	(0.006009, 0.006063, 0.00618, 0.006251)
7	(0.000719, 0.000722, 0.000733, 0.000736)	(0.00074, 0.000749, 0.000773, 0.000781)	(0.000699, 0.000713, 0.00075, 0.000764)	(0.000659, 0.000685, 0.00073, 0.000744)	(0.000636, 0.000661, 0.000709, 0.000724)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)
8	(0.001802, 0.001829, 0.001915, 0.001942)	(0.001629, 0.001667, 0.001813, 0.00185)	(0.001505, 0.001546, 0.001737, 0.001778)	(0.001417, 0.001458, 0.001684, 0.001725)	(0.001353, 0.001394, 0.00162, 0.001661)	(0.001307, 0.001344, 0.001616, 0.001652)	(0.001307, 0.001344, 0.001616, 0.001652)	(0.001307, 0.001344, 0.001616, 0.001652)	(0.001307, 0.001344, 0.001616, 0.001652)	(0.001307, 0.001344, 0.001616, 0.001652)	(0.001307, 0.001344, 0.001616, 0.001652)	(0.001307, 0.001344, 0.001616, 0.001652)
9	(0.000072, 0.000076, 0.000087, 0.000091)	(0.000067, 0.000073, 0.000086, 0.000092)	(0.000063, 0.00007, 0.000084, 0.000091)	(0.000061, 0.000068, 0.000081, 0.000088)	(0.000058, 0.000065, 0.000078, 0.000085)	(0.000057, 0.000063, 0.000074, 0.000081)	(0.000057, 0.000063, 0.000074, 0.000081)	(0.000057, 0.000063, 0.000074, 0.000081)	(0.000057, 0.000063, 0.000074, 0.000081)	(0.000057, 0.000063, 0.000074, 0.000081)	(0.000057, 0.000063, 0.000074, 0.000081)	(0.000057, 0.000063, 0.000074, 0.000081)
10	(0.000719, 0.000722, 0.000733, 0.000736)	(0.00074, 0.000749, 0.000773, 0.000781)	(0.000699, 0.000713, 0.00075, 0.000764)	(0.000659, 0.000685, 0.00073, 0.000744)	(0.000636, 0.000661, 0.000709, 0.000724)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)	(0.000623, 0.000649, 0.000681, 0.000697)
11	(0.000101, 0.000105, 0.000115, 0.000119)	(0.000093, 0.0001, 0.000116, 0.000123)	(0.000088, 0.000096, 0.000114, 0.000122)	(0.000084, 0.000093, 0.000111, 0.00012)	(0.000081, 0.00009, 0.000108, 0.000117)	(0.000079, 0.000088, 0.000104, 0.000109)	(0.000079, 0.000088, 0.000104, 0.000109)	(0.000079, 0.000088, 0.000104, 0.000109)	(0.000079, 0.000088, 0.000104, 0.000109)	(0.000079, 0.000088, 0.000104, 0.000109)	(0.000079, 0.000088, 0.000104, 0.000109)	(0.000079, 0.000088, 0.000104, 0.000109)
12	(0.000177, 0.000181, 0.000197, 0.000201)	(0.000164, 0.000173, 0.000211, 0.00022)	(0.000154, 0.000166, 0.000224, 0.000236)	(0.000147, 0.000161, 0.000237, 0.000251)	(0.000142, 0.000156, 0.000246, 0.00026)	(0.000139, 0.000154, 0.000257, 0.000272)	(0.000139, 0.000154, 0.000257, 0.000272)	(0.000139, 0.000154, 0.000257, 0.000272)	(0.000139, 0.000154, 0.000257, 0.000272)	(0.000139, 0.000154, 0.000257, 0.000272)	(0.000139, 0.000154, 0.000257, 0.000272)	(0.000139, 0.000154, 0.000257, 0.000272)
13	(0.000045, 0.000049, 0.000058, 0.000062)	(0.000041, 0.000046, 0.000057, 0.000062)	(0.000038, 0.000043, 0.000053, 0.000058)	(0.000036, 0.000041, 0.000055, 0.00006)	(0.000034, 0.000039, 0.000048, 0.000053)	(0.000033, 0.000037, 0.000044, 0.000048)	(0.000033, 0.000037, 0.000044, 0.000048)	(0.000033, 0.000037, 0.000044, 0.000048)	(0.000033, 0.000037, 0.000044, 0.000048)	(0.000033, 0.000037, 0.000044, 0.000048)	(0.000033, 0.000037, 0.000044, 0.000048)	(0.000033, 0.000037, 0.000044, 0.000048)
14	(0.000457, 0.00046, 0.00047, 0.000473)	(0.000457, 0.000464, 0.000485, 0.000492)	(0.000453, 0.000463, 0.000477, 0.000487)	(0.00045, 0.000465, 0.00048, 0.000495)	(0.000448, 0.000464, 0.000481, 0.000496)	(0.000446, 0.000463, 0.00048, 0.000495)	(0.000446, 0.000463, 0.00048, 0.000495)	(0.000446, 0.000463, 0.00048, 0.000495)	(0.000446, 0.000463, 0.00048, 0.000495)	(0.000446, 0.000463, 0.00048, 0.000495)	(0.000446, 0.000463, 0.00048, 0.000495)	(0.000446, 0.000463, 0.00048, 0.000495)
15	(0.000063, 0.000066, 0.000075, 0.000078)	(0.000057, 0.000063, 0.000076, 0.000082)	(0.000053, 0.00006, 0.000074, 0.000081)	(0.00005, 0.000057, 0.00007, 0.000077)	(0.000048, 0.000055, 0.00007, 0.000075)	(0.000046, 0.000053, 0.000068, 0.000075)	(0.000046, 0.000053, 0.000068, 0.000075)	(0.000046, 0.000053, 0.000068, 0.000075)	(0.000046, 0.000053, 0.000068, 0.000075)	(0.000046, 0.000053, 0.000068, 0.000075)	(0.000046, 0.000053, 0.000068, 0.000075)	(0.000046, 0.000053, 0.000068, 0.000075)
16	(0.00011, 0.000115, 0.00013, 0.000134)	(0.000101, 0.000109, 0.000141, 0.000149)	(0.000093, 0.000103, 0.00015, 0.00016)	(0.000088, 0.000099, 0.000158, 0.000169)	(0.000084, 0.000095, 0.000168, 0.000179)	(0.000081, 0.000092, 0.000171, 0.000182)	(0.000081, 0.000092, 0.000171, 0.000182)	(0.000081, 0.000092, 0.000171, 0.000182)	(0.000081, 0.000092, 0.000171, 0.000182)	(0.000081, 0.000092, 0.000171, 0.000182)	(0.000081, 0.000092, 0.000171, 0.000182)	(0.000081, 0.000092, 0.000171, 0.000182)
17	(0.000835, 0.00084, 0.000856, 0.000861)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)	(0.000809, 0.000822, 0.000857, 0.00087)
18	(0.001591, 0.001613, 0.001675, 0.001697)	(0.001453, 0.0015, 0.001624, 0.001671)	(0.001345, 0.001412, 0.001581, 0.001649)	(0.001266, 0.001335, 0.001548, 0.001633)	(0.001211, 0.001295, 0.001493, 0.001578)	(0.00117, 0.001273, 0.001455, 0.00156)	(0.00117, 0.001273, 0.001455, 0.00156)	(0.00117, 0.001273, 0.001455, 0.00156)	(0.00117, 0.001273, 0.001455, 0.00156)	(0.00117, 0.001273, 0.001455, 0.00156)	(0.00117, 0.001273, 0.001455, 0.00156)	(0.00117, 0.001273, 0.001455, 0.00156)
19	(0.000835, 0.000839, 0.000852, 0.000856)	(0.000794, 0.000804, 0.000836, 0.000846)	(0.000752, 0.000759, 0.000819, 0.000829)	(0.000715, 0.000722, 0.000789, 0.000796)	(0.000685, 0.000692, 0.000761, 0.000768)	(0.000655, 0.000662, 0.000734, 0.000741)	(0.000655, 0.000662, 0.000734, 0.000741)	(0.000655, 0.000662, 0.000734, 0.000741)	(0.000655, 0.000662, 0.000734, 0.000741)	(0.000655, 0.000662, 0.000734, 0.000741)	(0.000655, 0.000662, 0.000734, 0.000741)	(0.000655, 0.000662, 0.000734, 0.000741)
20	(0.002753, 0.002772, 0.002827, 0.002846)	(0.002552, 0.002597, 0.002729, 0.002774)	(0.002333, 0.002383, 0.002526, 0.002579)	(0.002142, 0.002192, 0.002347, 0.002401)	(0.001971, 0.002021, 0.002187, 0.002241)	(0.001819, 0.001874, 0.002051, 0.002105)	(0.001819, 0.001874, 0.002051, 0.002105)	(0.001819, 0.001874, 0.002051, 0.002105)	(0.001819, 0.001874, 0.002051, 0.002105)	(0.001819, 0.001874, 0.002051, 0.002105)	(0.001819, 0.001874, 0.002051, 0.002105)	(0.001819, 0.001874, 0.002051, 0.002105)
21	(0.000096, 0.000099, 0.000107, 0.00011)	(0.000172, 0.000177, 0.00019, 0.000195)	(0.000226, 0.000233, 0.000251, 0.000258)	(0.000297, 0.000306, 0.000325, 0.000334)	(0.00035, 0.00036, 0.00038, 0.00039)	(0.000404, 0.000414, 0.000434, 0.000444)	(0.000404, 0.000414, 0.000434, 0.000444)	(0.000404, 0.000414, 0.000434, 0.000444)	(0.000404, 0.000414, 0.000434, 0.000444)	(0.000404, 0.000414, 0.000434, 0.000444)	(0.000404, 0.000414, 0.000434, 0.000444)	(0.000404, 0.000414, 0.000434, 0.000444)
22	(0.000672, 0.000675, 0.000683, 0.000686)	(0.001549, 0.001556, 0.001578, 0.001585)	(0.002266, 0.002279, 0.002337, 0.002343)	(0.002696, 0.002745, 0.00								

4.3.2 Fuzzy reliability evaluation

Using the obtained solution of fuzzy Kolmogorov's differential equations, shown in Table 4.1 and Table 4.2, the obtained fuzzy reliability $\tilde{R}_1(t) = \sum_{i=1}^4 \tilde{p}_i(t)$, $\tilde{R}_2(t) = \sum_{i=1}^2 \tilde{p}_i(t)$ and $\tilde{R}(t) = \tilde{R}_1(t) \otimes \tilde{R}_2(t)$ i.e., $(R_{1(1)}(t), R_{(2)}(t), R_{1(3)}(t), R_{1(4)}(t))$, $(R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t))$ and $(R_1(t), R_2(t), R_3(t), R_4(t)) = (R_{1(1)}(t), R_{1(2)}(t), R_{1(3)}(t), R_{1(4)}(t)) \otimes (R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t))$ of sub-system R_1 , sub-system R_2 and whole system respectively are shown in Table 4.3.

Table 4.3 Fuzzy reliability of sub-system R_1 , sub-system R_2 and whole system obtained by using proposed method

Fuzzy Reliability →	$\tilde{R}_1(t)$	$\tilde{R}_2(t)$	$\tilde{R}(t)$
Time (t) ↓	$(R_{1(1)}(t), R_{1(2)}(t), R_{1(3)}(t), R_{1(4)}(t))$	$(R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t))$	$(R_1(t), R_2(t), R_3(t), R_4(t))$
60	(0.931975, 0.938278, 0.950753, 0.957055)	(0.942978, 0.948376, 0.959108, 0.964498)	(0.878831, 0.88984, 0.911874, 0.923077)
120	(0.927686, 0.936, 0.948079, 0.954246)	(0.942224, 0.947547, 0.958096, 0.963408)	(0.874088, 0.886903, 0.908351, 0.919328)
180	(0.925126, 0.931181, 0.942902, 0.948956)	(0.942123, 0.947386, 0.957802, 0.963051)	(0.871582, 0.882187, 0.903113, 0.913893)
240	(0.923384, 0.929278, 0.940672, 0.946624)	(0.94211, 0.947326, 0.957642, 0.962841)	(0.869929, 0.880329, 0.900827, 0.911448)
300	(0.922029, 0.926532, 0.937926, 0.943878)	(0.942109, 0.947288, 0.957527, 0.962686)	(0.868651, 0.877692, 0.898089, 0.908658)
360	(0.921098, 0.926873, 0.9377, 0.943473)	(0.942108, 0.947256, 0.957436, 0.962562)	(0.867774, 0.877986, 0.897788, 0.908151)

The variation in reliability of piston manufacturing system at $t = 60$ hours, $t = 120$ hours, $t = 180$ hours, $t = 240$ hours, $t = 300$ hours, $t = 360$ hours is shown in Figure 4.1 to Figure 4.6 respectively.

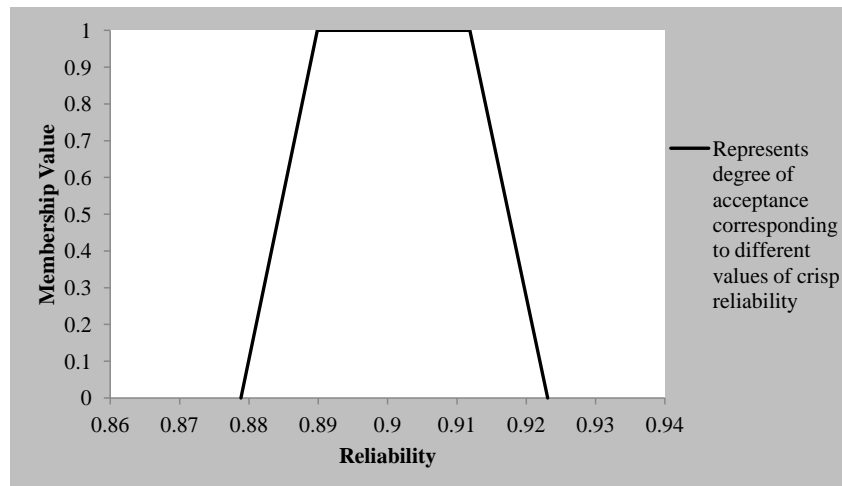


Figure 4.1 Trapezoidal fuzzy number representing the fuzzy reliability of piston manufacturing system at $t = 60$ hours

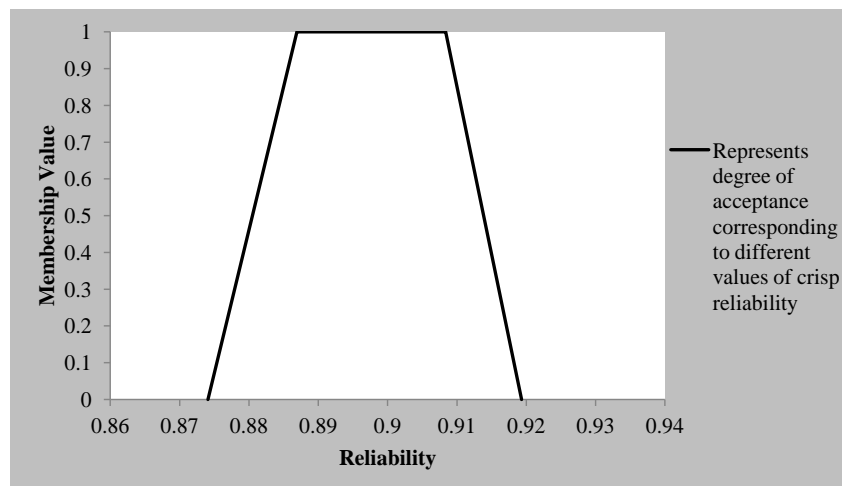


Figure 4.2 Trapezoidal fuzzy number representing the fuzzy reliability of piston manufacturing system at $t = 120$ hours

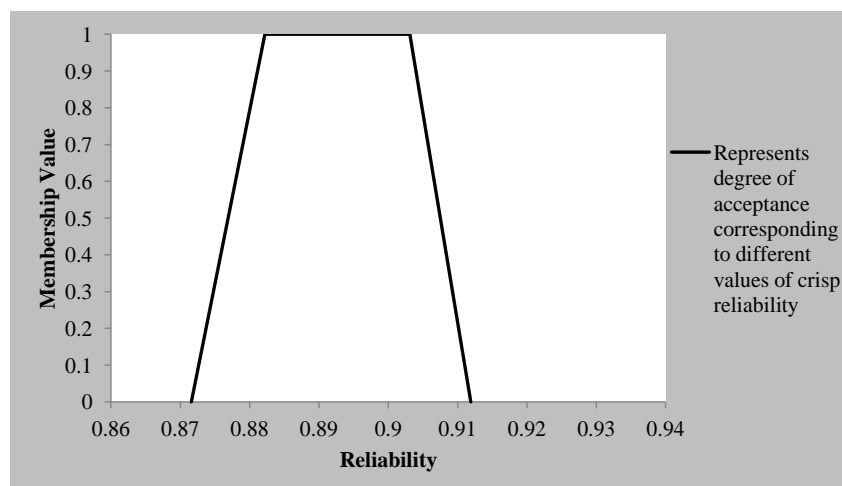


Figure 4.3 Trapezoidal fuzzy number representing the fuzzy reliability of piston manufacturing system at $t = 180$ hours

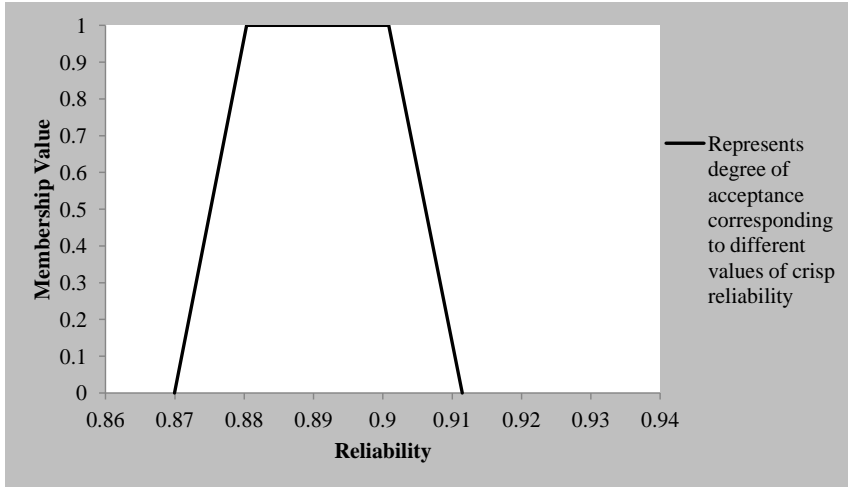


Figure 4.4 Trapezoidal fuzzy number representing the fuzzy reliability of piston manufacturing system at $t = 240$ hours

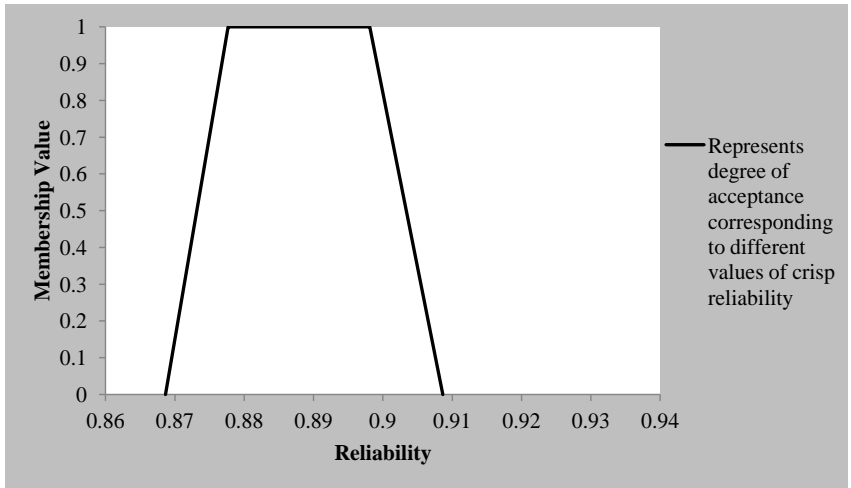


Figure 4.5 Trapezoidal fuzzy number representing the fuzzy reliability of piston manufacturing system at $t = 300$ hours

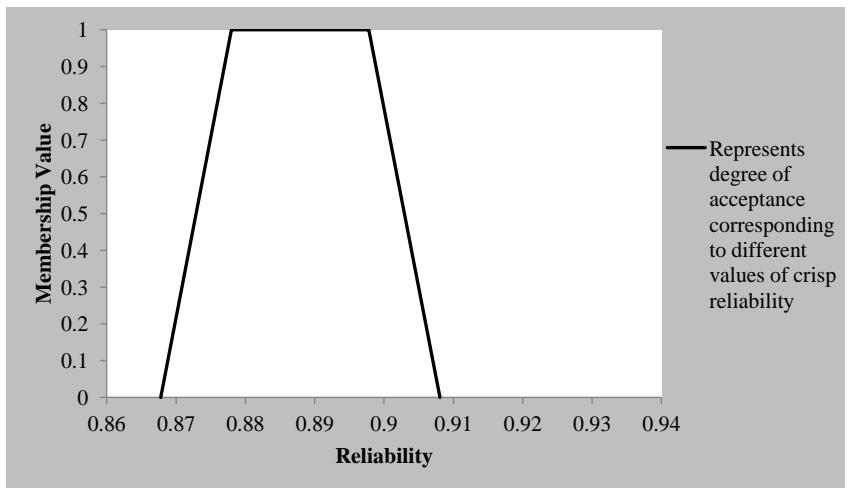


Figure 4.6 Trapezoidal fuzzy number representing the fuzzy reliability of piston manufacturing system at $t = 360$ hours

Also, the variation in reliability of piston manufacturing system with time at presumption level 0 is shown in Figure 4.7.

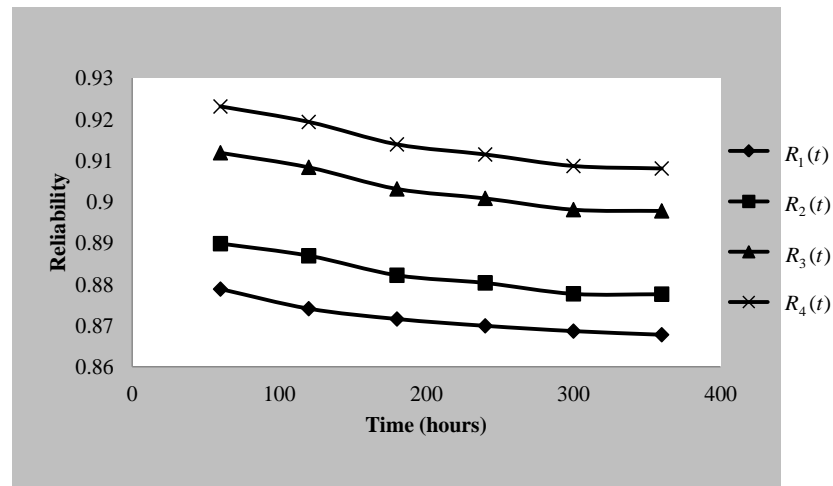


Figure 4.7 Variation in reliability with time at presumption level 0

4.3.3 Advantage of proposed method

The main advantage of proposed method over the method proposed in previous chapter is that on solving the same fuzzy Kolmogorov's differential equations by applying the method, proposed in this chapter as well as by applying the modified method, proposed in previous chapter, same results are obtained. However, on solving fuzzy differential equations by using the proposed method directly trapezoidal fuzzy numbers are obtained which represent the solution of fuzzy Kolmogorov's differential equations. Whereas, by using the method, proposed in previous chapter, to obtain the same trapezoidal fuzzy number, the whole process is repeated for different values of α . Therefore, the method, proposed in this chapter is much easy to apply as compared to the method, proposed in previous chapter, for solving fuzzy Kolmogorov's differential equations.

4.4 Advantage of *JMD* trapezoidal fuzzy number over existing trapezoidal fuzzy number

In the previous chapter, by considering a system of fuzzy equations as well as fuzzy Kolmogorov's differential equations, it is shown that it is better to use *JMD* α -cut as compared to existing α -cut. On the same direction, in this section the advantage of *JMD* trapezoidal fuzzy number over existing trapezoidal fuzzy number is discussed.

4.4.1 Solution of the chosen system of fuzzy equations

In this section, the system of fuzzy equations (3.46) is solved by representing all the parameters as existing trapezoidal fuzzy numbers as well as *JMD* trapezoidal fuzzy numbers and it is shown that on solving the system of fuzzy equations with existing trapezoidal fuzzy numbers, the obtained solution does not define a fuzzy number. While, on solving the same system of fuzzy equations with *JMD* trapezoidal fuzzy numbers, the obtained solution defines a fuzzy number.

4.4.1.1 Solution of system of fuzzy equations with existing trapezoidal fuzzy number

The solution of system of fuzzy equations (3.46) can be obtained by using the following steps.

Step 1: Replacing \tilde{a}_j by existing trapezoidal fuzzy number $(a_{j(1)}, a_{j(2)}, a_{j(3)}, a_{j(4)})$,

the system of fuzzy equations (3.46) can be written as:

$$\left. \begin{aligned} (a_{1(1)}, a_{1(2)}, a_{1(3)}, a_{1(4)}) \oplus (a_{2(1)}, a_{2(2)}, a_{2(3)}, a_{2(4)}) &= (10, 20, 30, 40) \\ (a_{3(1)}, a_{3(2)}, a_{3(3)}, a_{3(4)}) \oplus (a_{4(1)}, a_{4(2)}, a_{4(3)}, a_{4(4)}) &= (15, 20, 35, 40) \\ (a_{1(1)}, a_{1(2)}, a_{1(3)}, a_{1(4)}) \oplus (a_{3(1)}, a_{3(2)}, a_{3(3)}, a_{3(4)}) &= (12, 15, 25, 30) \end{aligned} \right\} \quad (4.11)$$

Step 2: Using Section 2.1.3, the system of equations (4.11) can be written as:

$$\left. \begin{aligned} (a_{1(1)} + a_{2(1)}, a_{1(2)} + a_{2(2)}, a_{1(3)} + a_{2(3)}, a_{1(4)} + a_{2(4)}) &= (10, 20, 30, 40) \\ (a_{3(1)} + a_{4(1)}, a_{3(2)} + a_{4(2)}, a_{3(3)} + a_{4(3)}, a_{3(4)} + a_{4(4)}) &= (15, 20, 35, 40) \\ (a_{1(1)} + a_{3(1)}, a_{1(2)} + a_{3(2)}, a_{1(3)} + a_{3(3)}, a_{1(4)} + a_{3(4)}) &= (12, 15, 25, 30) \end{aligned} \right\} \quad (4.12)$$

Step 3: Using Definition 2.12, the system of equations (4.12) can be written as:

$$\left. \begin{aligned} a_{1(1)} + a_{2(1)} &= 10, a_{1(2)} + a_{2(2)} = 20, a_{1(3)} + a_{2(3)} = 30, a_{1(4)} + a_{2(4)} = 40 \\ a_{3(1)} + a_{4(1)} &= 15, a_{3(2)} + a_{4(2)} = 20, a_{3(3)} + a_{4(3)} = 35, a_{3(4)} + a_{4(4)} = 40 \\ a_{1(1)} + a_{3(1)} &= 12, a_{1(2)} + a_{3(2)} = 15, a_{1(3)} + a_{3(3)} = 25, a_{1(4)} + a_{3(4)} = 30 \end{aligned} \right\} \quad (4.13)$$

Step 4: On solving the system of equations (4.13) by using simplex method [73],

the following solution is obtained:

$$a_{1(1)} = 0, a_{1(2)} = 15, a_{1(3)} = 25, a_{1(4)} = 30$$

$$a_{2(1)} = 10, a_{2(2)} = 5, a_{2(3)} = 5, a_{2(4)} = 10$$

$$a_{3(1)} = 12, a_{3(2)} = 0, a_{3(3)} = 0, a_{3(4)} = 0$$

$$a_{4(1)} = 3, a_{4(2)} = 20, a_{4(3)} = 35, a_{4(4)} = 40$$

4.4.1.2 Solution of system of fuzzy equations with *JMD* trapezoidal fuzzy number

The solution of system of fuzzy equations (3.46) can be obtained by using the following steps.

Step 1: Replacing \tilde{a}_j by *JMD* trapezoidal fuzzy number $(a_{j(1)}, \alpha_{j(1)}, \alpha_{j(2)}, \alpha_{j(3)})_{JMD}$,

the system of fuzzy equations (3.46) can be written as:

$$\left. \begin{aligned} (a_{1(1)}, \alpha_{1(1)}, \alpha_{1(2)}, \alpha_{1(3)})_{JMD} + (a_{2(1)}, \alpha_{2(1)}, \alpha_{2(2)}, \alpha_{2(3)})_{JMD} \\ = (10, 10, 10, 10)_{JMD} \\ (a_{3(1)}, \alpha_{3(1)}, \alpha_{3(2)}, \alpha_{3(3)})_{JMD} + (a_{4(1)}, \alpha_{4(1)}, \alpha_{4(2)}, \alpha_{4(3)})_{JMD} \\ = (15, 5, 15, 5)_{JMD} \\ (a_{1(1)}, \alpha_{1(1)}, \alpha_{1(2)}, \alpha_{1(3)})_{JMD} + (a_{3(1)}, \alpha_{3(1)}, \alpha_{3(2)}, \alpha_{3(3)})_{JMD} \\ = (12, 3, 10, 5)_{JMD} \end{aligned} \right\} \quad (4.14)$$

Step 2: Using the arithmetic operations between *JMD* trapezoidal fuzzy numbers, presented in Section 4.1.2, the system of equations (4.14) can be written as:

$$\left. \begin{aligned}
& (a_{1(1)} + a_{2(1)}, \alpha_{1(1)} + \alpha_{2(1)}, \alpha_{1(2)} + \alpha_{2(2)}, \alpha_{1(3)} + \alpha_{2(3)})_{JMD} \\
& = (10, 10, 10, 10)_{JMD} \\
& (a_{3(1)} + a_{4(1)}, \alpha_{3(1)} + \alpha_{4(1)}, \alpha_{3(2)} + \alpha_{4(2)}, \alpha_{3(3)} + \alpha_{4(3)})_{JMD} \\
& = (15, 5, 15, 5)_{JMD} \\
& (a_{1(1)} + a_{3(1)}, \alpha_{1(1)} + \alpha_{3(1)}, \alpha_{1(2)} + \alpha_{3(2)}, \alpha_{1(3)} + \alpha_{3(3)})_{JMD} \\
& = (12, 3, 10, 5)_{JMD}
\end{aligned} \right\} \quad (4.15)$$

Step 3: Using Definition 4.4, the system of equations (4.15) can be written as:

$$\left. \begin{aligned}
& a_{1(1)} + a_{2(1)} = 10, \alpha_{1(1)} + \alpha_{2(1)} = 10, \alpha_{1(2)} + \alpha_{2(2)} = 10, \alpha_{1(3)} + \alpha_{2(3)} = 10 \\
& a_{3(1)} + a_{4(1)} = 15, \alpha_{3(1)} + \alpha_{4(1)} = 5, \alpha_{3(2)} + \alpha_{4(2)} = 15, \alpha_{3(3)} + \alpha_{4(3)} = 5 \\
& a_{1(1)} + a_{3(1)} = 12, \alpha_{1(1)} + \alpha_{3(1)} = 3, \alpha_{1(2)} + \alpha_{3(2)} = 10, \alpha_{1(3)} + \alpha_{3(3)} = 5
\end{aligned} \right\} \quad (4.16)$$

Step 4: On solving the system of equations (4.16) by using simplex method [73],

the following solution is obtained:

$$a_{1(1)} = 0, \alpha_{1(1)} = 0, \alpha_{1(2)} = 10, \alpha_{1(3)} = 5$$

$$a_{2(1)} = 10, \alpha_{2(1)} = 10, \alpha_{2(2)} = 0, \alpha_{2(3)} = 5$$

$$a_{3(1)} = 12, \alpha_{3(1)} = 3, \alpha_{3(2)} = 0, \alpha_{3(3)} = 0$$

$$a_{4(1)} = 3, \alpha_{4(1)} = 2, \alpha_{4(2)} = 15, \alpha_{4(3)} = 5$$

Step 5: Using the values, obtained in Step 4, the solution of system of fuzzy equations (3.16) is as follows:

$$\tilde{a}_1 = (a_{1(1)}, \alpha_{1(1)}, \alpha_{1(2)}, \alpha_{1(3)})_{JMD} = (0, 0, 10, 5)_{JMD}$$

$$\tilde{a}_2 = (a_{2(1)}, \alpha_{2(1)}, \alpha_{2(2)}, \alpha_{2(3)})_{JMD} = (10, 10, 0, 5)_{JMD}$$

$$\tilde{a}_3 = (a_{3(1)}, \alpha_{3(1)}, \alpha_{3(2)}, \alpha_{3(3)})_{JMD} = (12, 3, 0, 0)_{JMD}$$

$$\tilde{a}_4 = (a_{4(1)}, \alpha_{4(1)}, \alpha_{4(2)}, \alpha_{4(3)})_{JMD} = (3, 2, 15, 5)_{JMD}$$

Step 6: Using *JMD* trapezoidal fuzzy numbers, obtained in Step 5, the existing trapezoidal fuzzy numbers are:

$$\tilde{a}_1 = (a_{1(1)}, a_{1(1)} + \alpha_{1(1)}, a_{1(1)} + \alpha_{1(1)} + \alpha_{1(2)}, a_{1(1)} + \alpha_{1(1)} + \alpha_{1(2)} + \alpha_{1(3)}) = (0, 0, 10, 15)$$

$$\tilde{a}_2 = (a_{2(1)}, a_{2(1)} + \alpha_{2(1)}, a_{2(1)} + \alpha_{2(1)} + \alpha_{2(2)}, a_{2(1)} + \alpha_{2(1)} + \alpha_{2(2)} + \alpha_{2(3)}) = (10, 20, 20, 25)$$

$$\tilde{a}_3 = (a_{3(1)}, a_{3(1)} + \alpha_{3(1)}, a_{3(1)} + \alpha_{3(1)} + \alpha_{3(2)}, a_{3(1)} + \alpha_{3(1)} + \alpha_{3(2)} + \alpha_{3(3)}) = (12, 15, 15, 15)$$

$$\tilde{a}_4 = (a_{4(1)}, a_{4(1)} + \alpha_{4(1)}, a_{4(1)} + \alpha_{4(1)} + \alpha_{4(2)}, a_{4(1)} + \alpha_{4(1)} + \alpha_{4(2)} + \alpha_{4(3)}) = (3, 5, 20, 25)$$

4.4.2 Solution of fuzzy Kolmogorov's differential equations with existing trapezoidal fuzzy number

The solution of fuzzy Kolmogorov's differential equations (3.1) to (3.24) with initial conditions (3.25) to (3.29) and the solution of fuzzy Kolmogorov's differential equations (3.30) to (3.42) with initial conditions (3.43) to (3.45), obtained by using proposed method with existing trapezoidal fuzzy number, is shown in Table 4.4 and Table 4.5 respectively.

Table 4.4 Solution of fuzzy Kolmogorov's differential equations for sub-system R_1 obtained by using proposed method with existing trapezoidal fuzzy numbers

j	$\tilde{p}_j(t)$ for $t=60$		$\tilde{p}_j(t)$ for $t=120$		$\tilde{p}_j(t)$ for $t=180$		$\tilde{p}_j(t)$ for $t=240$		$\tilde{p}_j(t)$ for $t=300$		$\tilde{p}_j(t)$ for $t=360$	
	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$	$(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t))$
1	(0.720279, 0.681444, 0.635114, 0.600719)	(0.652165, 0.600728, 0.541703, 0.501324)	(0.603616, 0.547756, 0.487123, 0.448286)	(0.568691, 0.512688, 0.454965, 0.41975)	(0.539359, 0.484983, 0.433938, 0.385552)	(0.512688, 0.454965, 0.403364, 0.352192)	(0.484983, 0.433938, 0.385552, 0.339359)	(0.454965, 0.403364, 0.352192, 0.309835)	(0.424813, 0.396102, 0.366437, 0.339359)	(0.396102, 0.366437, 0.339359, 0.312688)	(0.366437, 0.339359, 0.312688, 0.287149)	(0.339359, 0.312688, 0.287149, 0.262802)
2	(0.071153, 0.077829, 0.084789, 0.088344)	(0.065846, 0.070633, 0.075201, 0.077243)	(0.062073, 0.065921, 0.069611, 0.071331)	(0.059359, 0.062802, 0.066318, 0.06815)	(0.057404, 0.060736, 0.064376, 0.066437)	(0.055996, 0.059366, 0.063231, 0.065514)	(0.054381, 0.057740, 0.061417, 0.064486)	(0.052802, 0.056181, 0.059835, 0.063509)	(0.051372, 0.054751, 0.058424, 0.062102)	(0.049957, 0.053336, 0.057021, 0.060677)	(0.048541, 0.051920, 0.055599, 0.059273)	(0.047124, 0.050503, 0.054181, 0.057857)
3	(0.044381, 0.048441, 0.052512, 0.054488)	(0.040352, 0.042938, 0.045117, 0.045866)	(0.037486, 0.039332, 0.040802, 0.041272)	(0.035423, 0.036946, 0.038259, 0.038801)	(0.033938, 0.035423, 0.036946, 0.038259)	(0.032802, 0.034286, 0.035770, 0.036254)	(0.031666, 0.033150, 0.034634, 0.035118)	(0.030530, 0.032014, 0.033498, 0.034982)	(0.029394, 0.030878, 0.032362, 0.033846)	(0.028258, 0.029742, 0.031226, 0.032710)	(0.027122, 0.028606, 0.030090, 0.031574)	(0.025986, 0.027470, 0.028954, 0.030438)
4	(0.096162, 0.125721, 0.166747, 0.196529)	(0.169323, 0.213389, 0.26945, 0.306637)	(0.221951, 0.271397, 0.329903, 0.365802)	(0.259851, 0.309835, 0.365552, 0.397663)	(0.297845, 0.347781, 0.407667, 0.44824)	(0.335739, 0.385675, 0.445561, 0.496447)	(0.373633, 0.423569, 0.483455, 0.534341)	(0.411527, 0.461463, 0.521349, 0.572235)	(0.449421, 0.499357, 0.559243, 0.609129)	(0.487315, 0.537251, 0.597137, 0.647023)	(0.525209, 0.575145, 0.635031, 0.684917)	(0.563103, 0.613039, 0.672925, 0.722811)
5	(0.000738, 0.000813, 0.00089, 0.00093)	(0.00688, 0.000716, 0.00758, 0.000775)	(0.00618, 0.000652, 0.00681, 0.000692)	(0.00552, 0.00052, 0.00584, 0.000564)	(0.00486, 0.000486, 0.00518, 0.000518)	(0.00420, 0.000420, 0.00452, 0.000452)	(0.00354, 0.000354, 0.00386, 0.000386)	(0.00288, 0.000288, 0.00320, 0.000320)	(0.00222, 0.000222, 0.00254, 0.000254)	(0.00156, 0.000156, 0.00188, 0.000188)	(0.00090, 0.000090, 0.00122, 0.000122)	(0.00024, 0.000024, 0.00056, 0.000056)
6	(0.007563, 0.008481, 0.009508, 0.010101)	(0.007414, 0.008024, 0.008591, 0.008833)	(0.007273, 0.007621, 0.007974)	(0.007136, 0.007484, 0.007835)	(0.007000, 0.007348, 0.007699)	(0.006863, 0.007211, 0.007562)	(0.006726, 0.007074, 0.007425)	(0.006589, 0.006937, 0.007288)	(0.006452, 0.006800, 0.007151)	(0.006315, 0.006663, 0.007014)	(0.006178, 0.006526, 0.006875)	(0.006041, 0.006389, 0.006736)
7	(0.01027, 0.001132, 0.01241, 0.001296)	(0.00929, 0.000996, 0.01056, 0.001079)	(0.00831, 0.000867, 0.00958, 0.000946)	(0.00733, 0.000747, 0.00860, 0.000824)	(0.00635, 0.000659, 0.00762, 0.000736)	(0.00537, 0.000571, 0.00664, 0.000646)	(0.00439, 0.000481, 0.00566, 0.000554)	(0.00341, 0.000393, 0.00468, 0.000462)	(0.00243, 0.000295, 0.00370, 0.000364)	(0.00145, 0.000197, 0.00272, 0.000272)	(0.00047, 0.000069, 0.00174, 0.000174)	(0.00009, 0.000011, 0.00036, 0.000036)
8	(0.001802, 0.001987, 0.002179, 0.002279)	(0.01629, 0.001747, 0.01852, 0.001893)	(0.00063, 0.000078, 0.00097, 0.000119)	(0.00058, 0.000071, 0.00086, 0.000112)	(0.00053, 0.000064, 0.00081, 0.000105)	(0.00048, 0.000057, 0.00076, 0.000098)	(0.00043, 0.000050, 0.00068, 0.000091)	(0.00038, 0.000043, 0.00063, 0.000084)	(0.00033, 0.000038, 0.00058, 0.000077)	(0.00028, 0.000033, 0.00053, 0.000070)	(0.00023, 0.000028, 0.00048, 0.000063)	(0.00018, 0.000023, 0.00043, 0.000056)
9	(0.00072, 0.000092, 0.000118, 0.000136)	(0.000067, 0.000084, 0.000105, 0.000119)	(0.000063, 0.000078, 0.000097, 0.000111)	(0.000058, 0.000071, 0.000086, 0.000105)	(0.000053, 0.000064, 0.000076, 0.000098)	(0.000048, 0.000057, 0.000068, 0.000091)	(0.000043, 0.000050, 0.000061, 0.000084)	(0.000038, 0.000043, 0.000054, 0.000077)	(0.000033, 0.000038, 0.000049, 0.000072)	(0.000028, 0.000033, 0.000044, 0.000065)	(0.000023, 0.000028, 0.000039, 0.000056)	(0.000018, 0.000023, 0.000034, 0.000047)
10	(0.000719, 0.000932, 0.001221, 0.001426)	(0.0074, 0.000931, 0.00874, 0.001335)	(0.0063, 0.000867, 0.0080, 0.001281)	(0.0053, 0.000819, 0.0073, 0.001218)	(0.0043, 0.000771, 0.0066, 0.001163)	(0.0033, 0.000723, 0.0060, 0.001115)	(0.0023, 0.000675, 0.0050, 0.001067)	(0.0013, 0.000627, 0.0040, 0.001020)	(0.0003, 0.000579, 0.0030, 0.000973)	(0.0000, 0.000531, 0.0020, 0.000926)	(0.0000, 0.000483, 0.0010, 0.000879)	(0.0000, 0.000435, 0.0009, 0.000832)
11	(0.000101, 0.000129, 0.0001651, 0.00019)	(0.00093, 0.000117, 0.00146, 0.000166)	(0.00088, 0.000109, 0.00135, 0.000153)	(0.00083, 0.000103, 0.00128, 0.000146)	(0.00078, 0.000097, 0.00122, 0.000140)	(0.00073, 0.000091, 0.00116, 0.000133)	(0.00068, 0.000084, 0.00110, 0.000126)	(0.00063, 0.000077, 0.00104, 0.000119)	(0.00058, 0.000070, 0.00098, 0.000112)	(0.00053, 0.000063, 0.00092, 0.000105)	(0.00048, 0.000056, 0.00086, 0.000098)	(0.00043, 0.000049, 0.00080, 0.000091)
12	(0.000177, 0.000226, 0.00029, 0.000334)	(0.000256, 0.000291, 0.00033, 0.000374)	(0.000237, 0.000268, 0.00027, 0.000308)	(0.000225, 0.000255, 0.00026, 0.000294)	(0.000213, 0.000243, 0.00025, 0.000280)	(0.000201, 0.000231, 0.00024, 0.000266)	(0.000189, 0.000219, 0.00022, 0.000254)	(0.000177, 0.000207, 0.00021, 0.000242)	(0.000165, 0.000195, 0.00020, 0.000234)	(0.000153, 0.000183, 0.00019, 0.000226)	(0.000141, 0.000171, 0.00017, 0.000218)	(0.000129, 0.000159, 0.00016, 0.000210)
13	(0.000045, 0.000057, 0.000073, 0.000084)	(0.000041, 0.000051, 0.000057, 0.000067)	(0.000038, 0.000046, 0.000043, 0.000053)	(0.000035, 0.000043, 0.000040, 0.000050)	(0.000032, 0.000040, 0.000037, 0.000047)	(0.000029, 0.000037, 0.000034, 0.000044)	(0.000026, 0.000034, 0.000031, 0.000041)	(0.000023, 0.000031, 0.000028, 0.000038)	(0.000020, 0.000028, 0.000025, 0.000035)	(0.000017, 0.000025, 0.000022, 0.000032)	(0.000014, 0.000022, 0.000019, 0.000029)	(0.000011, 0.000019, 0.000016, 0.000026)
14	(0.000457, 0.000592, 0.000772, 0.000901)	(0.00457, 0.000571, 0.00636, 0.000804)	(0.0035, 0.000521, 0.00534, 0.000711)	(0.0025, 0.000471, 0.00430, 0.000633)	(0.0015, 0.000421, 0.00326, 0.000552)	(0.0005, 0.000371, 0.00202, 0.000471)	(0.0000, 0.000321, 0.00108, 0.000390)	(0.0000, 0.000271, 0.00074, 0.000340)	(0.0000, 0.000221, 0.00051, 0.000299)	(0.0000, 0.000171, 0.00040, 0.000258)	(0.0000, 0.000121, 0.00033, 0.000217)	(0.0000, 0.000071, 0.00026, 0.000176)
15	(0.000063, 0.00008, 0.000102, 0.000117)	(0.000087, 0.000098, 0.000101, 0.000124)	(0.000083, 0.000114, 0.000139, 0.000155)	(0.000079, 0.000108, 0.000135, 0.000153)	(0.000075, 0.000104, 0.000131, 0.000149)	(0.000071, 0.000101, 0.000128, 0.000146)	(0.000067, 0.000096, 0.000123, 0.000141)	(0.000063, 0.000092, 0.000119, 0.000137)	(0.000059, 0.000088, 0.000115, 0.000133)	(0.000055, 0.000084, 0.000111, 0.000129)	(0.000051, 0.000080, 0.000107, 0.000125)	(0.000047, 0.000076, 0.000103, 0.000121)
16	(0.00011, 0.000141, 0.00018, 0.000206)	(0.000101, 0.000124, 0.000154, 0.000173)	(0.000139, 0.000155, 0.000191, 0.000219)	(0.000177, 0.000191, 0.000243, 0.000268)	(0.000215, 0.000229, 0.000281, 0.000306)	(0.000253, 0.000267, 0.000319, 0.000344)	(0.000291, 0.000305, 0.000357, 0.000382)	(0.000329, 0.000343, 0.000395, 0.000420)	(0.000367, 0.000381, 0.000433, 0.000458)	(0.000405, 0.000419, 0.000471, 0.000496)	(0.000443, 0.000457, 0.000509, 0.000534)	(0.000481, 0.000495, 0.000547, 0.000572)
17	(0.000835, 0.001076, 0.001398, 0.001626)	(0.000809, 0.001014, 0.001274, 0.001447)	(0.00076, 0.000941, 0.001167, 0.00132)	(0.00071, 0.000891, 0.001117, 0.001251)	(0.00066, 0.000841, 0.001066, 0.001200)	(0.00061, 0.000811, 0.001026, 0.001160)	(0.00056, 0.000781, 0.001001, 0.001135)	(0.00051, 0.000751, 0.000976, 0.001110)	(0.00046, 0.000721, 0.000946, 0.001085)	(0.00041, 0.000691, 0.000916, 0.001054)	(0.00036, 0.000661, 0.000891, 0.001023)	(0.00031, 0.000631, 0.000866, 0.000992)
18	(0.001591, 0.002036, 0.002617, 0.003017)	(0.001453, 0.001806, 0.002237, 0.002515)	(0.001345, 0.001644, 0.002005, 0.002238)	(0.001248, 0.001547, 0.001919, 0.002141)	(0.001151, 0.001450, 0.001821, 0.002023)	(0.001054, 0.001353, 0.001724, 0.001906)	(0.000957, 0.001256, 0.001627, 0.001810)	(0.000860, 0.001159, 0.001530, 0.001703)	(0.000763, 0.001058, 0.001429, 0.001606)	(0.000666, 0.000957, 0.001328, 0.001499)	(0.000569, 0.000858, 0.001227, 0.001400)	(0.000472, 0.000757, 0.001106, 0.001273)
19	(0.000835, 0.001296, 0.002071, 0.002741)	(0.001794, 0.002665, 0.004017, 0.005103)	(0.002496, 0.003587, 0.005182, 0.006392)	(0.003198, 0.004289, 0.006076, 0.007483)	(0.003899, 0.005090, 0.007076, 0.008583)	(0.004599, 0.005790, 0.007876, 0.009478)	(0.005299, 0.006490, 0.008574, 0.010269)	(0.005999, 0.007190, 0.009274, 0.011064)	(0.006699, 0.007890, 0.009968, 0.011859)	(0.007399, 0.008590, 0.010672, 0.012654)	(0.008099, 0.009290, 0.011356, 0.013449)	(0.008799, 0.009990, 0.012041, 0.013844)
20	(0.002753, 0.004246, 0.006735, 0.008871)	(0.00552, 0.008165, 0.012241, 0.015492)	(0.007515, 0.010762, 0.015486, 0.019052)	(0.009500, 0.012807, 0.018541, 0.022322)	(0.011485, 0.015030, 0.020764, 0.024495)	(0.013469, 0.017014, 0.022758, 0.026489)	(0.015453, 0.019000, 0.024744, 0.028484)	(0.017437, 0.020984, 0.026728, 0.030479)	(0.019421, 0.022968, 0.028712, 0.032474)	(0.021405, 0.024892, 0.030636, 0.034131)	(0.023389, 0.026876, 0.032620, 0.036116)	(0.025373, 0.028860, 0.034604, 0.038111)
21	(0.000996, 0.000147, 0.00023, 0.000299)	(0.000172, 0.000252, 0.000375, 0.000471)	(0.000226, 0.000322, 0.000445, 0.000563)	(0.000280, 0.000376, 0.000499, 0.000613)	(0.000334, 0.000430, 0.000553, 0.000667)	(0.000388, 0.000484, 0.000607, 0.000721)	(0.000442, 0.000538, 0.000661, 0.000775)	(0.000496, 0.000592, 0.000715, 0.000829)	(0.000550, 0.000646, 0.000769, 0.000883)	(0.000604, 0.000700, 0.000823, 0.000937)	(0.000658, 0.000754, 0.000877, 0.000991)	(0.000712, 0.000808, 0.000931, 0.001045)
22	(0.000672, 0.001047, 0.001684, 0.002201)	(0.001549, 0.002312, 0.003506, 0.004472)	(0.002244, 0.003194, 0.004636, 0.005737)	(0.002938, 0.003888, 0.005330, 0.006483)	(0.003632, 0.004582, 0.006024, 0.007179)	(0.004326, 0.005276, 0.006718, 0.007813)	(0.005020, 0.005970, 0.007412, 0.008507)	(0.005714, 0.006664, 0.008106, 0.009199)	(0			

4.4.3 Discussion

For a trapezoidal fuzzy number (a, b, c, d) , the condition $a \leq b \leq c \leq d$ should always be satisfied. However, it is obvious that for the solution of system of fuzzy equations, obtained by using the proposed method with existing trapezoidal fuzzy numbers, this condition is not satisfying. Also, it obvious that for the results of fuzzy Kolmogorov's differential equations, obtained by using the proposed method with existing trapezoidal fuzzy numbers, shown in Table 4.4 and Table 4.5, the required condition $p_{j(1)}(t) \leq p_{j(2)}(t) \leq p_{j(3)}(t) \leq p_{j(4)}(t)$ is not satisfying for $j=1$ i.e., the obtained solution does not define a fuzzy number.

While, it is obvious that for the solution of system of fuzzy equations, obtained by using proposed method with *JMD* trapezoidal fuzzy numbers, this condition is satisfying. Also, for the solution of fuzzy Kolmogorov's differential equations, obtained by using proposed method with *JMD* trapezoidal fuzzy numbers, shown in Table 4.1 and Table 4.2, the required condition $p_{j(1)}(t) \leq p_{j(2)}(t) \leq p_{j(3)}(t) \leq p_{j(4)}(t)$ is satisfying for all values of j .

4.5 Validity of the results obtained with the help of *JMD* trapezoidal fuzzy number

Although, in Section 4.4.3, it is pointed out that on solving the system of fuzzy equations as well as fuzzy Kolmogorov's differential equations with existing trapezoidal fuzzy number the obtained solutions do not define trapezoidal fuzzy numbers. While, on solving the same equations with *JMD* trapezoidal fuzzy numbers, the obtained solutions always define trapezoidal fuzzy numbers. However, on the basis of numerical examples it cannot be concluded that by representing the pa-

rameters as trapezoidal fuzzy numbers, the obtained solution will always represent a trapezoidal fuzzy number. In this section, the theoretical reason for the validity of results, obtained with *JMD* trapezoidal fuzzy numbers, is discussed.

To find the solution of n fuzzy equations with n variables represented by existing trapezoidal fuzzy number $\tilde{a}_j = (a_{j(1)}, a_{j(2)}, a_{j(3)}, a_{j(4)})$, $j = 1, 2, \dots, n$, these equations are converted into $4n$ crisp equations with $4n$ crisp variables $a_{j(1)}, a_{j(2)}, a_{j(3)}, a_{j(4)}$, $j = 1, 2, \dots, n$ and the obtained crisp equations are solved by simplex method [73] to find the values of $a_{j(1)}, a_{j(2)}, a_{j(3)}, a_{j(4)}$, $j = 1, 2, \dots, n$. Since, in the simplex method [73], there is only one restriction on the variables $a_{j(1)}, a_{j(2)}, a_{j(3)}, a_{j(4)}$, $j = 1, 2, \dots, n$ that all these variables should be non negative real numbers. So, the condition $a_{j(1)} \leq a_{j(2)} \leq a_{j(3)} \leq a_{j(4)}$, $j = 1, 2, \dots, n$ may or may not be satisfied.

Similarly, for solving n^{th} order fuzzy linear differential equation with dependent variable represented by existing trapezoidal fuzzy number $\tilde{y} = (y_1, y_2, y_3, y_4)$, it is converted into four n^{th} order crisp linear differential equations with dependent variables y_1, y_2, y_3 and y_4 and the obtained crisp linear differential equations are solved to find the values of y_1, y_2, y_3 and y_4 . Since, it is not possible to apply any restriction on the dependent variables y_1, y_2, y_3 and y_4 . So, the condition $y_1 \leq y_2 \leq y_3 \leq y_4$ may or may not be satisfied.

However, to find the solution of n fuzzy equations with n variables represented by *JMD* trapezoidal fuzzy number $\tilde{a}_j = (a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)})_{JMD}$, $j = 1, 2, \dots, n$, it is converted into $4n$ crisp equations with $4n$ crisp variables $a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)}$, $j = 1, 2, \dots, n$ and the obtained crisp equations are solved by simplex method [73] to find the values of $a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)}$, $j = 1, 2, \dots, n$. Now, with the help of $a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)}$, the existing trapezoidal fuzzy number $(a_{j(1)}, a_{j(1)} +$

$\beta_{j(1)}, a_{j(1)} + \beta_{j(1)}, a_{j(1)} + \beta_{j(1)} + \beta_{j(2)}, a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)}$ is obtained. Since, $\beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)}$ are non negative real numbers. So, the condition $a_{j(1)} \leq a_{j(1)} + \beta_{j(1)} \leq a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} \leq a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)}$ will always be satisfied.

Similarly, for solving n^{th} order fuzzy linear differential equation with dependent variable represented by *JMD* trapezoidal fuzzy number $\tilde{y} = (y_1, \alpha_1, \alpha_2, \alpha_3)_{JMD}$, it is converted into four n^{th} order crisp linear differential equations with dependent variables y_1, α_1, α_2 and α_3 and the obtained crisp linear differential equations are solved to find the values of y_1, α_1, α_2 and α_3 . Since, α_1, α_2 and α_3 are non negative real numbers. So, the condition $y_1 \leq y_1 + \alpha_1 \leq y_1 + \alpha_1 + \alpha_2 \leq y_1 + \alpha_1 + \alpha_2 + \alpha_3$ will always be satisfied.

4.6 Conclusions

On the basis of presented study, it is concluded that it is better to use the method, proposed in this chapter, as compared to the method, proposed in previous chapter for solving fuzzy differential equations and hence for solving fuzzy Kolmogorov's differential equations to evaluate the fuzzy reliability of industrial systems.

Chapter 5

A NEW METHOD FOR SOLVING VAGUE DIFFERENTIAL EQUATIONS AND ITS APPLICATION FOR ANALYZING THE VAGUE RELIABILITY OF PISTON MANUFACTURING SYSTEM

In real life, a person may assume that an object belongs to a set but it is possible that he (she) is not sure about it. In other words, there may be hesitation or confusion that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or confusion. A possible solution is to use vague set [55]. To evaluate the vague reliability of industrial systems through vague Markov modeling, there is a need to solve vague Kolmogorov's differential equations. Since, there is no method in the literature for solving vague differential equations. Therefore, several authors have used vague fault tree to evaluate the vague reliability of industrial systems but till now no one have utilized vague Markov model for the same. In this chapter, a new representation of trapezoidal vague set, named as *JMD* trapezoidal vague set, is proposed and with the help of *JMD* trapezoidal vague set, a new method is proposed for solving n^{th} order vague linear differential equations. To illustrate the proposed method, vague Kolmogorov's

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differential equations, obtained by vague Markov model of piston manufacturing system, are solved and the obtained solution is used to evaluate the vague reliability of piston manufacturing system. The advantage of *JMD* trapezoidal vague set over existing trapezoidal vague set is also discussed.

5.1 Preliminaries

In this section, some basic definitions and arithmetic operations between trapezoidal vague sets are presented [38].

5.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 5.1 A vague set $\tilde{A} = \langle [x; \mu_{\tilde{A}}(x), (1 - \nu_{\tilde{A}}(x)) | x \in X] \rangle$, defined on the universal set X , shown in Figure 5.1, is characterized by a truth membership function $\mu_{\tilde{A}}, \mu_{\tilde{A}} : X \rightarrow [0, 1]$ and complement of a false membership function $(1 - \nu_{\tilde{A}}), (1 - \nu_{\tilde{A}}) : X \rightarrow [0, 1]$. The values $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ represents the degree of acceptance and degree of rejection for $x \in X$ and always satisfies the condition $\mu_{\tilde{A}}(x) \leq (1 - \nu_{\tilde{A}}(x)) \forall x \in X$. The value $(1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x))$ represents the degree of hesitation for $x \in X$.

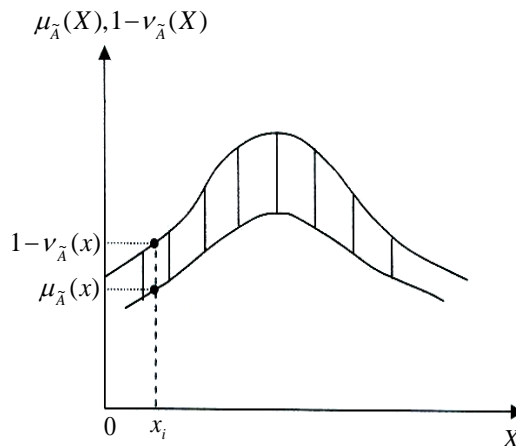


Figure 5.1 Vague set

Definition 5.2 A vague set \tilde{A} , defined on the universal set of real numbers \mathbb{R} , denoted as $\tilde{A} = \langle [(a, b, c, d); \delta, \rho] \rangle$, where $a \leq b \leq c \leq d$ and $\delta \leq \rho$, is said to be a trapezoidal vague set, shown in Figure 5.2, if degree of membership $\mu_{\tilde{A}}(x)$ and degree of non-membership $\nu_{\tilde{A}}(x)$ are given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{\delta(x-a)}{(b-a)}, & a \leq x < b \\ \delta, & b \leq x \leq c \\ \frac{\delta(x-d)}{(c-d)}, & c < x \leq d \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{\rho(x-a)}{(b-a)}, & a \leq x < b \\ 1 - \rho, & b \leq x \leq c \\ 1 - \frac{\rho(x-d)}{(c-d)}, & c < x \leq d \\ 1, & \text{otherwise} \end{cases}$$

where,

$$\delta = \text{Supremum}\{\mu_{\tilde{A}}(x) : x \in \mathbb{R}\}$$

$$\rho = \text{Supremum}\{(1 - \nu_{\tilde{A}}(x)) : x \in \mathbb{R}\}$$

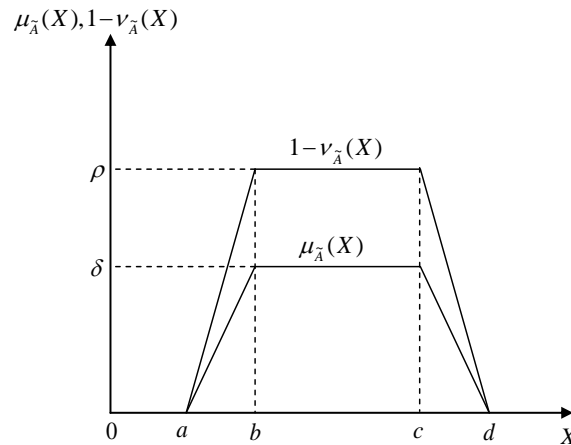


Figure 5.2 Trapezoidal vague set \tilde{A}

Definition 5.3 Two trapezoidal vague sets $\tilde{A} = \langle [(a_1, b_1, c_1, d_1); \delta, \rho] \rangle$ and $\tilde{B} = \langle [(a_2, b_2, c_2, d_2); \delta, \rho] \rangle$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$.

5.1.2 Arithmetic operations between trapezoidal vague sets

In this section, some arithmetic operations between trapezoidal vague sets, defined on universal set of real numbers \mathbb{R} , are presented.

Let $\tilde{A} = \langle [(a_1, b_1, c_1, d_1); \delta, \rho] \rangle$ and $\tilde{B} = \langle [(a_2, b_2, c_2, d_2); \delta, \rho] \rangle$ be two trapezoidal vague sets. Then,

$$(i) \tilde{A} \oplus \tilde{B} = \langle [(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); \delta, \rho] \rangle$$

$$(ii) \tilde{A} \ominus \tilde{B} = \langle [(a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2); \delta, \rho] \rangle$$

$$(iii) \lambda \tilde{A} = \begin{cases} \langle [(\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1); \delta, \rho] \rangle, & \lambda \geq 0 \\ \langle [(\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1); \delta, \rho] \rangle, & \lambda \leq 0 \end{cases}$$

$$(iv) \tilde{A} \otimes \tilde{B} = \langle [(\text{minimum}(a), \text{minimum}(b), \text{maximum}(b), \text{maximum}(a)); \delta, \rho] \rangle$$

where,

$$a = (a_1 a_2, a_1 d_2, d_1 a_2, d_1 d_2)$$

$$b = (b_1 b_2, b_1 c_2, c_1 b_2, c_1 c_2)$$

5.2 *JMD* trapezoidal vague set

In the previous chapter, it is proved that it is better to use *JMD* trapezoidal fuzzy number instead of existing trapezoidal fuzzy number to find the solution of fuzzy differential equations.

In this section, a new representation of trapezoidal vague set, named as *JMD* trapezoidal vague set, is proposed.

5.2.1 Basic definitions

In this section, definition of *JMD* trapezoidal vague set and equality of two *JMD* trapezoidal vague sets are introduced.

Definition 5.4 Let $\tilde{A} = \langle [(a, b, c, d); \delta, \rho] \rangle$ be a trapezoidal vague set. Then, its *JMD* representation is $\tilde{A} = \langle [(a, \alpha, \beta, \gamma); \delta, \rho]_{JMD} \rangle$.

where, $\alpha = b - a \geq 0$, $\beta = c - b \geq 0$ and $\gamma = d - c \geq 0$.

Definition 5.5 Two *JMD* trapezoidal vague sets $\tilde{A} = \langle [(a_1, \alpha_1, \beta_1, \gamma_1); \delta, \rho]_{JMD} \rangle$ and $\tilde{B} = \langle [(a_2, \alpha_2, \beta_2, \gamma_2); \delta, \rho] \rangle$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ if and only if $a_1 = a_2$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$.

5.2.2 Arithmetic operations between *JMD* trapezoidal vague sets

In this section, with the help of existing arithmetic operations, presented in Section 5.1.2, some arithmetic operations between *JMD* trapezoidal vague sets are introduced.

Let $\tilde{A} = \langle [(a_1, \alpha_1, \beta_1, \gamma_1); \delta, \rho]_{JMD} \rangle$ and $\tilde{B} = \langle [(a_2, \alpha_2, \beta_2, \gamma_2); \delta, \rho]_{JMD} \rangle$

be two *JMD* trapezoidal vague sets. Then,

- (i) $\tilde{A} \oplus \tilde{B} = \langle [(a_1 + a_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2); \delta, \rho]_{JMD} \rangle$
- (ii) $\tilde{A} \ominus \tilde{B} = \langle [(a_1 - a_2 - \alpha_2 - \beta_2 - \gamma_2, \alpha_1 + \gamma_2, \beta_1 + \beta_2, \alpha_2 + \gamma_1); \delta, \rho]_{JMD} \rangle$
- (iii) $\lambda \tilde{A} = \begin{cases} \langle [(\lambda a_1, \lambda \alpha_1, \lambda \beta_1, \lambda \gamma_1); \delta, \rho]_{JMD} \rangle, & \lambda \geq 0 \\ \langle [(\lambda a_1 + \lambda \alpha_1 + \lambda \beta_1 + \lambda \gamma_1, -\lambda \gamma_1, -\lambda \beta_1, -\lambda \alpha_1); \delta, \rho]_{JMD} \rangle, & \lambda \leq 0 \end{cases}$
- (iv) $\tilde{A} \otimes \tilde{B} = \langle [(\text{minimum}(a), \text{minimum}(b) - \text{minimum}(a), \text{maximum}(b) - \text{minimum}(b), \text{maximum}(a) - \text{maximum}(b)); \delta, \rho]_{JMD} \rangle$

where,

$$a = (a_1 a_2, a_1(a_2 + \alpha_2 + \beta_2 + \gamma_2), (a_1 + \alpha_1 + \beta_1 + \gamma_1)a_2, (a_1 + \alpha_1 + \beta_1 + \gamma_1)(a_2 + \alpha_2 + \beta_2 + \gamma_2))$$

$$b = ((a_1 + \alpha_1)(a_2 + \alpha_2), (a_1 + \alpha_1)(a_2 + \alpha_2 + \beta_2), (a_1 + \alpha_1 + \beta_1)(a_2 + \alpha_2), (a_1 + \alpha_1 + \beta_1)(a_2 + \alpha_2 + \beta_2))$$

5.3 Proposed method for solving n^{th} order vague linear differential equations

In this section, a new method is proposed for solving n^{th} order vague linear differential equations.

Any n^{th} order vague linear differential equation can be written as:

$$\sum_{j=0}^n \tilde{a}_j \otimes \tilde{y}^{(j)} = \tilde{g} \quad (5.1)$$

with initial conditions:

$$\tilde{y}^{(j)}(x_0) = \tilde{\gamma}_j, \quad j = 0, 1, \dots, n-1 \quad (5.2)$$

where, $\tilde{y}^{(j)} = \frac{d^j \tilde{y}}{dx^j}$, \tilde{a}_j , \tilde{g} and $\tilde{\gamma}_j$ are trapezoidal vague sets.

The solution of equation (5.1) with initial conditions (5.2) can be obtained as follows:

Step 1: Convert all the parameters \tilde{a}_j , $\tilde{y}^{(j)}$, \tilde{g} and $\tilde{\gamma}_j$ of n^{th} order vague linear differential equation (5.1) with initial conditions (5.2) into *JMD* trapezoidal vague sets.

Step 2: Assuming $\tilde{a}_j = \langle [(a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)}); \delta, \rho]_{JMD} \rangle$, $\tilde{y}^{(j)} = \langle [(y_1^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}); \delta, \rho]_{JMD} \rangle$, $\tilde{g} = \langle [(g, \xi_1, \xi_2, \xi_3); \delta, \rho]_{JMD} \rangle$ and $\tilde{\gamma}_j = \langle [(\gamma_{j(1)}, \zeta_{j(1)}, \zeta_{j(2)}, \zeta_{j(3)}); \delta, \rho]_{JMD} \rangle$, the n^{th} order vague linear differential equation (5.1) with initial conditions (5.2), can be written as:

$$\sum_{j=0}^n \langle [(a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)}); \delta, \rho]_{JMD} \rangle \otimes \langle [(y_1^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}); \delta, \rho]_{JMD} \rangle = \langle [(g, \xi_1, \xi_2, \xi_3); \delta, \rho]_{JMD} \rangle \quad (5.3)$$

with initial conditions:

$$\begin{aligned} & \langle [(y_1^{(j)}(x_0), \alpha_1^{(j)}(x_0), \alpha_2^{(j)}(x_0), \alpha_3^{(j)}(x_0)); \delta, \rho]_{JMD} \rangle = \\ & \langle [(\gamma_{j(1)}, \zeta_{j(1)}, \zeta_{j(2)}, \zeta_{j(3)}); \delta, \rho]_{JMD} \rangle, \quad j = 0, 1, \dots, n-1 \end{aligned} \quad (5.4)$$

Step 3: Using Definition 5.4 and Section 5.2.2, n^{th} order vague linear differential

equation (5.3) with initial conditions (5.4) can be transformed into the following n^{th} order linear differential equations:

$$\sum_{j=0}^n \text{minimum}(b_j z^{(j)}) = g \quad (5.5)$$

$$\sum_{j=0}^n [\text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_1 \quad (5.6)$$

$$\sum_{j=0}^n [\text{maximum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)}) - \text{minimum}(c_j z^{(j)})] = \xi_2 \quad (5.7)$$

$$\sum_{j=0}^n [\text{maximum}(b_j z^{(j)}) - \text{minimum}(b_j z^{(j)}) - \text{minimum}(c_j z^{(j)}) - \text{maximum}(c_j z^{(j)})] = \xi_3 \quad (5.8)$$

with initial conditions:

$$y_1^{(j)}(0) = \gamma_{j(1)}, \quad j = 0, 1, \dots, n-1 \quad (5.9)$$

$$\alpha_1^{(j)}(0) = \zeta_{j(1)}, \quad j = 0, 1, \dots, n-1 \quad (5.10)$$

$$\alpha_2^{(j)}(0) = \zeta_{j(2)}, \quad j = 0, 1, \dots, n-1 \quad (5.11)$$

$$\alpha_3^{(j)}(0) = \zeta_{j(3)}, \quad j = 0, 1, \dots, n-1 \quad (5.12)$$

where,

$$b_j z^{(j)} = (a_{j(1)} y_1^{(j)}, a_{j(1)}(y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)}) y_1^{(j)}, (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)})(y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)}))$$

$$c_j z^{(j)} = ((a_{j(1)} + \beta_{j(1)})(y_1^{(j)} + \alpha_1^{(j)}), (a_{j(1)} + \beta_{j(1)})(y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)})(y_1^{(j)} + \alpha_1^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)})(y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)}))$$

Step 4: Solve the n^{th} order linear differential equations (5.5) to (5.8) with initial conditions (5.9) to (5.12) to find the values of y_1 , α_1 , α_2 and α_3 .

Step 5: Put the values of y_1 , α_1 , α_2 and α_3 in $\tilde{y} = \langle [(y_1, \alpha_1, \alpha_2, \alpha_3); \delta, \rho]_{JMD} \rangle$ to find the solution of n^{th} order vague linear differential equation (5.1) with initial conditions (5.2).

Step 6: Convert $\tilde{y} = \langle [(y_1, \alpha_1, \alpha_2, \alpha_3); \delta, \rho]_{JMD} \rangle$ into $\langle [(a, b, c, d); \delta, \rho] \rangle$, where, $a = y_1$, $b = y_1 + \alpha_1$, $c = y_1 + \alpha_1 + \alpha_2$ and $d = y_1 + \alpha_1 + \alpha_2 + \alpha_3$.

5.4 Solution of vague Kolmogorov's differential equations

In this section, to illustrate the proposed method, the vague Kolmogorov's differential equations of sub-system R_1 , obtained by replacing the fuzzy parameters

$$\tilde{p}_j(t) = (p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)),$$

$$\tilde{P}_1(0) = (0.94, 0.945, 0.955, 0.96),$$

$$\tilde{P}_2(0) = (0.006, 0.0065, 0.0075, 0.008),$$

$$\tilde{P}_3(0) = (0.004, 0.0045, 0.0055, 0.006),$$

$$\tilde{P}_4(0) = ((0.002, 0.0025, 0.0035, 0.004),$$

$$\tilde{P}_j(0) = ((0, 0, 0, 0), j=5 \text{ to } 24$$

and fuzzy failure rates, fuzzy repair rates, presented in Table 3.1, by the vague parameters $\tilde{p}_j(t) = \langle [(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)); 0.7, 0.8] \rangle$,

$$\tilde{P}_1(0) = \langle [(0.94, 0.945, 0.955, 0.96); 0.7, 0.8] \rangle,$$

$$\tilde{P}_2(0) = \langle [((0.006, 0.0065, 0.0075, 0.008); 0.7, 0.8] \rangle,$$

$$\tilde{P}_3(0) = \langle [((0.004, 0.0045, 0.0055, 0.006); 0.7, 0.8] \rangle,$$

$$\tilde{P}_4(0) = \langle [((0.002, 0.0025, 0.0035, 0.004); 0.7, 0.8] \rangle,$$

$$\tilde{P}_j(0) = \langle [((0, 0, 0, 0); 0.7, 0.8] \rangle, j=5 \text{ to } 24$$

and vague failure rates, vague repair rates, presented in Table 5.1 as well as the vague Kolmogorov's differential equations of sub-system R_2 , obtained by replacing the fuzzy parameters

$$\tilde{p}_j(t) = (p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)),$$

$$\tilde{P}_1(0) = (0.95, 0.955, 0.965, 0.97),$$

$$\tilde{P}_2(0) = (0.004, 0.0045, 0.0055, 0.006),$$

$$\tilde{P}_j(0) = (0, 0, 0, 0), j=3 \text{ to } 13$$

and fuzzy failure rates, fuzzy repair rates, presented in Table 3.2, by the vague parameters $\tilde{p}_j(t) = \langle [(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)); 0.7, 0.8] \rangle$,

$$\tilde{P}_1(0) = \langle [(0.95, 0.955, 0.965, 0.97); 0.7, 0.8] \rangle,$$

$$\tilde{P}_2(0) = \langle [(0.004, 0.0045, 0.0055, 0.006); 0.7, 0.8] \rangle,$$

$$\tilde{P}_j(0) = \langle [(0, 0, 0, 0); 0.7, 0.8] \rangle, \quad j=3 \text{ to } 13$$

and vague failure rates, vague repair rates, presented in Table 5.2, are solved by using the proposed method at $t = 60$ hours, $t = 120$ hours, $t = 180$ hours, $t = 240$ hours, $t = 300$ hours, $t = 360$ hours and the obtained results are shown in Table 5.3 and Table 5.4 respectively.

Table 5.1 Vague failure rates and vague repair rates for different sub-systems of R_1

Vague failure rate	Vague repair rate
$\tilde{\chi}_1 = \langle [(0.00105, 0.00126, 0.00154, 0.00175); 0.7, 0.8] \rangle$	$\tilde{\beta}_1 = \langle [(1.026, 1.0584, 1.1016, 1.134); 0.7, 0.8] \rangle$
$\tilde{\chi}_2 = \langle [(0.00045, 0.00054, 0.00066, 0.00075); 0.7, 0.8] \rangle$	$\tilde{\beta}_2 = \langle [(0.04085, 0.04214, 0.04386, 0.04515); 0.7, 0.8] \rangle$
$\tilde{\chi}_3 = \langle [(0.000675, 0.00081, 0.00099, 0.001125); 0.7, 0.8] \rangle$	$\tilde{\beta}_3 = \langle [(0.475, 0.49, 0.51, 0.525); 0.7, 0.8] \rangle$
$\tilde{\chi}_4 = \langle [(0.000675, 0.00081, 0.00099, 0.001125); 0.7, 0.8] \rangle$	$\tilde{\beta}_4 = \langle [(0.2717, 0.28028, 0.29172, 0.3003); 0.7, 0.8] \rangle$
$\tilde{\chi}_5 = \langle [(0.0156, 0.01872, 0.02288, 0.026); 0.7, 0.8] \rangle$	$\tilde{\beta}_5 = \langle [(0.1463, 0.15092, 0.15702, 0.1617); 0.7, 0.8] \rangle$
$\tilde{\chi}_6 = \langle [(0.0156, 0.01872, 0.02288, 0.026); 0.7, 0.8] \rangle$	$\tilde{\beta}_6 = \langle [(0.2375, 0.245, 0.255, 0.2625); 0.7, 0.8] \rangle$
$\tilde{\chi}_7 = \langle [(0.000675, 0.00081, 0.00099, 0.001125); 0.7, 0.8] \rangle$	$\tilde{\beta}_7 = \langle [(0.05605, 0.05782, 0.06018, 0.06195); 0.7, 0.8] \rangle$
$\tilde{\chi}_8 = \langle [(0.002925, 0.00351, 0.00429, 0.004875); 0.7, 0.8] \rangle$	$\tilde{\beta}_8 = \langle [(0.08265, 0.08526, 0.08874, 0.09135); 0.7, 0.8] \rangle$

Table 5.2 Vague failure rates and vague repair rates for the different sub-systems of R_2

Vague failure rate	Vague repair rate
$\tilde{\eta}_1 = \langle [(0.00105, 0.00126, 0.00154, 0.00175); 0.7, 0.8] \rangle$	$\tilde{\mu}_1 = \langle [(0.3135, 0.3234, 0.3366, 0.3465); 0.7, 0.8] \rangle$
$\tilde{\eta}_2 = \langle [(0.00023, 0.00027, 0.00033, 0.00038); 0.7, 0.8] \rangle$	$\tilde{\mu}_2 = \langle [(0.475, 0.49, 0.51, 0.525); 0.7, 0.8] \rangle$
$\tilde{\eta}_3 = \langle [(0.00008, 0.00009, 0.00011, 0.00013); 0.7, 0.8] \rangle$	$\tilde{\mu}_3 = \langle [(0.6365, 0.6566, 0.6834, 0.7035); 0.7, 0.8] \rangle$
$\tilde{\eta}_4 = \langle [(0.00023, 0.00027, 0.00033, 0.00038); 0.7, 0.8] \rangle$	$\tilde{\mu}_4 = \langle [(0.03325, 0.0343, 0.0357, 0.03675); 0.7, 0.8] \rangle$
$\tilde{\eta}_5 = \langle [(0.00008, 0.00009, 0.00011, 0.00013); 0.7, 0.8] \rangle$	$\tilde{\mu}_5 = \langle [(2.8785, 2.9694, 3.0906, 3.1815); 0.7, 0.8] \rangle$
$\tilde{\eta}_6 = \langle [(0.0156, 0.01872, 0.02288, 0.026); 0.7, 0.8] \rangle$	$\tilde{\mu}_6 = \langle [(0.2109, 0.21756, 0.22644, 0.2331); 0.7, 0.8] \rangle$
$\tilde{\eta}_7 = \langle [(0.003, 0.0036, 0.0044, 0.005); 0.7, 0.8] \rangle$	$\tilde{\mu}_7 = \langle [(0.11875, 0.1225, 0.1275, 0.13125); 0.7, 0.8] \rangle$

Table 5.4 Solution of vague Kolmogorov's differential equations for sub-system R_2 obtained by using proposed method

j	$\bar{p}_j(t)$ for $t=60$	$\bar{p}_j(t)$ for $t=120$	$\bar{p}_j(t)$ for $t=180$	$\bar{p}_j(t)$ for $t=240$	$\bar{p}_j(t)$ for $t=300$	$\bar{p}_j(t)$ for $t=360$
	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$
1	$\langle [0.87882, 0.853263, 0.891744, 0.896118]; 0.7, 0.8 \rangle$	$\langle [0.878179, 0.882218, 0.890005, 0.894036]; 0.7, 0.8 \rangle$	$\langle [0.878084, 0.889381, 0.889381, 0.892316]; 0.7, 0.8 \rangle$	$\langle [0.878072, 0.881804, 0.889083, 0.892302]; 0.7, 0.8 \rangle$	$\langle [0.878071, 0.881734, 0.888916, 0.892365]; 0.7, 0.8 \rangle$	$\langle [0.87807, 0.881689, 0.888812, 0.892416]; 0.7, 0.8 \rangle$
2	$\langle [0.064096, 0.065113, 0.067364, 0.06838]; 0.7, 0.8 \rangle$	$\langle [0.064046, 0.065329, 0.068091, 0.069372]; 0.7, 0.8 \rangle$	$\langle [0.064039, 0.065457, 0.068421, 0.069835]; 0.7, 0.8 \rangle$	$\langle [0.064038, 0.065522, 0.068559, 0.070038]; 0.7, 0.8 \rangle$	$\langle [0.064038, 0.065554, 0.068611, 0.070121]; 0.7, 0.8 \rangle$	$\langle [0.064038, 0.065567, 0.068624, 0.070146]; 0.7, 0.8 \rangle$
3	$\langle [0.002943, 0.002986, 0.00309, 0.003133]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003004, 0.003145, 0.003208]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003013, 0.003164, 0.003236]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003016, 0.003169, 0.003244]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003018, 0.003171, 0.003247]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003018, 0.00317, 0.003246]; 0.7, 0.8 \rangle$
4	$\langle [0.000425, 0.000432, 0.00045, 0.000459]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.000434, 0.000456, 0.000467]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.000435, 0.000457, 0.000469]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.000435, 0.000457, 0.000469]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.000434, 0.000455, 0.000467]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.000434, 0.000455, 0.000467]; 0.7, 0.8 \rangle$
5	$\langle [0.00011, 0.00011, 0.000116, 0.000119]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000111, 0.000116, 0.000119]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000111, 0.000116, 0.000119]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000111, 0.000116, 0.000119]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000111, 0.000116, 0.000119]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000111, 0.000116, 0.000119]; 0.7, 0.8 \rangle$
6	$\langle [0.005269, 0.00528, 0.005311, 0.005324]; 0.7, 0.8 \rangle$	$\langle [0.005966, 0.005985, 0.006041, 0.006065]; 0.7, 0.8 \rangle$	$\langle [0.006059, 0.006086, 0.006164, 0.006198]; 0.7, 0.8 \rangle$	$\langle [0.006071, 0.006105, 0.006202, 0.006206]; 0.7, 0.8 \rangle$	$\langle [0.006073, 0.006114, 0.006228, 0.006279]; 0.7, 0.8 \rangle$	$\langle [0.006073, 0.00612, 0.00625, 0.006309]; 0.7, 0.8 \rangle$
7	$\langle [0.00024, 0.00025, 0.00026, 0.00027]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.000244, 0.000254, 0.00026]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.000244, 0.000245, 0.000253]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.000244, 0.000245, 0.000262]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.000244, 0.000254, 0.00026]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.000243, 0.00025, 0.00026]; 0.7, 0.8 \rangle$
8	$\langle [0.00214, 0.00221, 0.00242, 0.00249]; 0.7, 0.8 \rangle$	$\langle [0.00214, 0.00229, 0.00268, 0.00283]; 0.7, 0.8 \rangle$	$\langle [0.00214, 0.00235, 0.00286, 0.00307]; 0.7, 0.8 \rangle$	$\langle [0.00214, 0.00239, 0.00297, 0.00322]; 0.7, 0.8 \rangle$	$\langle [0.00214, 0.00242, 0.00303, 0.00331]; 0.7, 0.8 \rangle$	$\langle [0.00214, 0.00244, 0.00307, 0.00337]; 0.7, 0.8 \rangle$
9	$\langle [0.00031, 0.00032, 0.00035, 0.00036]; 0.7, 0.8 \rangle$	$\langle [0.00031, 0.00033, 0.00039, 0.00042]; 0.7, 0.8 \rangle$	$\langle [0.00031, 0.00034, 0.00042, 0.00045]; 0.7, 0.8 \rangle$	$\langle [0.00031, 0.00034, 0.00042, 0.00046]; 0.7, 0.8 \rangle$	$\langle [0.00031, 0.00034, 0.00043, 0.00047]; 0.7, 0.8 \rangle$	$\langle [0.00031, 0.00034, 0.00043, 0.00047]; 0.7, 0.8 \rangle$
10	$\langle [0.00008, 0.000082, 0.000093, 0.000098]; 0.7, 0.8 \rangle$	$\langle [0.00008, 0.000085, 0.0001, 0.00011]; 0.7, 0.8 \rangle$	$\langle [0.00008, 0.000086, 0.000106, 0.000116]; 0.7, 0.8 \rangle$	$\langle [0.00008, 0.000086, 0.000106, 0.000116]; 0.7, 0.8 \rangle$	$\langle [0.00008, 0.000086, 0.00011, 0.00012]; 0.7, 0.8 \rangle$	$\langle [0.00008, 0.000086, 0.00011, 0.00012]; 0.7, 0.8 \rangle$
11	$\langle [0.000373, 0.000374, 0.000379, 0.000381]; 0.7, 0.8 \rangle$	$\langle [0.000433, 0.000437, 0.000451, 0.000456]; 0.7, 0.8 \rangle$	$\langle [0.000441, 0.000448, 0.000471, 0.00048]; 0.7, 0.8 \rangle$	$\langle [0.000442, 0.000452, 0.000483, 0.000495]; 0.7, 0.8 \rangle$	$\langle [0.000442, 0.000455, 0.000494, 0.00051]; 0.7, 0.8 \rangle$	$\langle [0.000442, 0.000457, 0.000503, 0.000522]; 0.7, 0.8 \rangle$
12	$\langle [0.00001, 0.000021, 0.000024, 0.000026]; 0.7, 0.8 \rangle$	$\langle [0.00001, 0.000011, 0.000015, 0.000017]; 0.7, 0.8 \rangle$	$\langle [0.00001, 0.000011, 0.000015, 0.000018]; 0.7, 0.8 \rangle$	$\langle [0.00001, 0.000011, 0.000016, 0.000019]; 0.7, 0.8 \rangle$	$\langle [0.00001, 0.000011, 0.000016, 0.000019]; 0.7, 0.8 \rangle$	$\langle [0.00001, 0.000011, 0.000016, 0.000019]; 0.7, 0.8 \rangle$
13	$\langle [0.001617, 0.001642, 0.001714, 0.001739]; 0.7, 0.8 \rangle$	$\langle [0.001618, 0.001676, 0.001835, 0.001951]; 0.7, 0.8 \rangle$	$\langle [0.001617, 0.001707, 0.001944, 0.002034]; 0.7, 0.8 \rangle$	$\langle [0.001617, 0.001736, 0.002037, 0.002155]; 0.7, 0.8 \rangle$	$\langle [0.001617, 0.00176, 0.002109, 0.002252]; 0.7, 0.8 \rangle$	$\langle [0.001617, 0.00178, 0.002165, 0.002328]; 0.7, 0.8 \rangle$

5.4.1 Vague reliability evaluation of piston manufacturing system

In this section, the solution of vague Kolmogorov’s differential equations, shown in Table 5.3 and Table 5.4, obtained by using proposed method, is used to evaluate the vague reliability of piston manufacturing system.

Using the obtained solution of vague Kolmogorov’s differential equations, shown in Table 5.3 and Table 5.4, the obtained values of vague reliability $\tilde{R}_1(t) = \sum_{i=1}^4 \tilde{p}_i(t)$, $\tilde{R}_2(t) = \sum_{i=1}^2 \tilde{p}_i(t)$ and $\tilde{R}(t) = \tilde{R}_1(t) \otimes \tilde{R}_2(t)$ i.e., $\langle [(R_{1(1)}(t), R_{1(2)}(t), R_{1(3)}(t), R_{1(4)}(t)); 0.7, 0.8] \rangle$, $\langle [(R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t)); 0.7, 0.8] \rangle$ and $\langle [(R_1(t), R_2(t), R_3(t), R_4(t)); 0.7, 0.8] \rangle = \langle [(R_{1(1)}(t), R_{1(2)}(t), R_{1(3)}(t), R_{1(4)}(t)); 0.7, 0.8] \rangle \otimes \langle [(R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t)); 0.7, 0.8] \rangle$ of sub-system R_1 , sub-system R_2 and whole system respectively are shown in Table 5.5.

Table 5.5 Vague reliability of sub-system R_1 , sub-system R_2 and whole system obtained by using proposed method

Vague Reliability →	$\tilde{R}_1(t)$	$\tilde{R}_2(t)$	$\tilde{R}(t)$
Time (t) ↓ (hours)	$\langle [(R_{1(1)}(t), R_{1(2)}(t), R_{1(3)}(t), R_{1(4)}(t)); 0.7, 0.8] \rangle$	$\langle [(R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t)); 0.7, 0.8] \rangle$	$\langle [(R_1(t), R_2(t), R_3(t), R_4(t)); 0.7, 0.8] \rangle$
60	$\langle [(0.931975, 0.938278, 0.950753, 0.957055); 0.7, 0.8] \rangle$	$\langle [(0.942978, 0.948376, 0.959108, 0.964498); 0.7, 0.8] \rangle$	$\langle [(0.878831, 0.88984, 0.911874, 0.923077); 0.7, 0.8] \rangle$
120	$\langle [(0.927686, 0.936, 0.948079, 0.954246); 0.7, 0.8] \rangle$	$\langle [(0.942224, 0.947547, 0.958096, 0.963408); 0.7, 0.8] \rangle$	$\langle [(0.874088, 0.886903, 0.908351, 0.919328); 0.7, 0.8] \rangle$
180	$\langle [(0.925126, 0.931181, 0.942902, 0.948956); 0.7, 0.8] \rangle$	$\langle [(0.942123, 0.947386, 0.957802, 0.960951); 0.7, 0.8] \rangle$	$\langle [(0.871582, 0.882187, 0.903113, 0.9119); 0.7, 0.8] \rangle$
240	$\langle [(0.923384, 0.929278, 0.940672, 0.946624); 0.7, 0.8] \rangle$	$\langle [(0.94211, 0.947326, 0.957642, 0.962841); 0.7, 0.8] \rangle$	$\langle [(0.869929, 0.880329, 0.900827, 0.911448); 0.7, 0.8] \rangle$
300	$\langle [(0.922029, 0.926532, 0.937926, 0.943878); 0.7, 0.8] \rangle$	$\langle [(0.942109, 0.947288, 0.957527, 0.962686); 0.7, 0.8] \rangle$	$\langle [(0.868651, 0.877692, 0.898089, 0.908658); 0.7, 0.8] \rangle$
360	$\langle [(0.921098, 0.926873, 0.9377, 0.943473); 0.7, 0.8] \rangle$	$\langle [(0.942108, 0.947256, 0.957436, 0.962562); 0.7, 0.8] \rangle$	$\langle [(0.867774, 0.877986, 0.897788, 0.908151); 0.7, 0.8] \rangle$

The variation in reliability of piston manufacturing system at $t = 60$ hours, $t = 120$ hours, $t = 180$ hours, $t = 240$ hours, $t = 300$ hours, $t = 360$ hours is shown in Figure 5.3 to Figure 5.8 respectively.

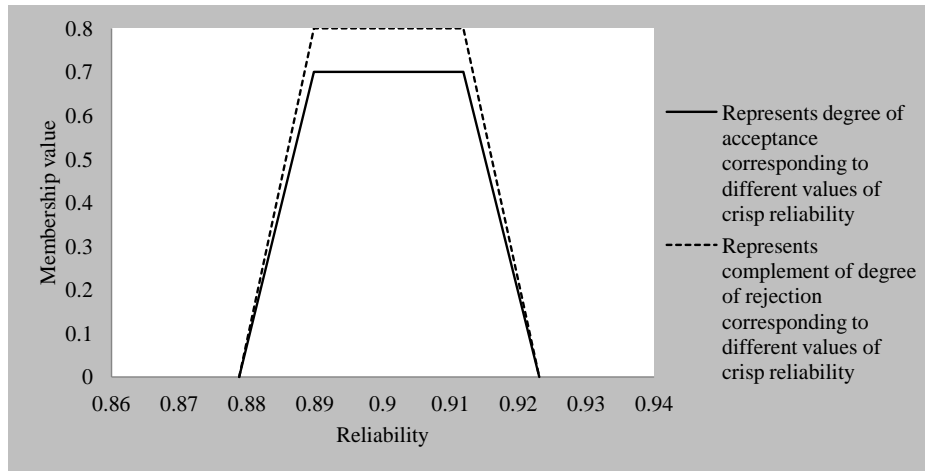


Figure 5.3 Trapezoidal vague set representing the vague reliability of piston manufacturing system at $t = 60$ hours

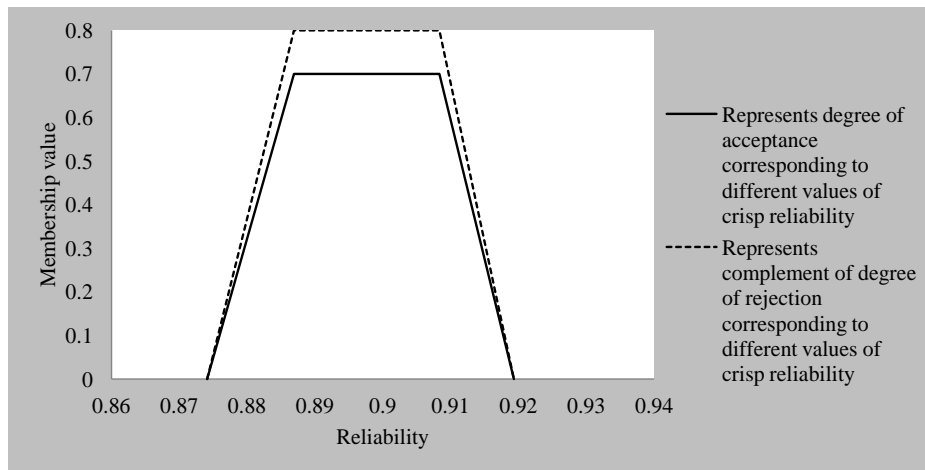


Figure 5.4 Trapezoidal vague set representing the vague reliability of piston manufacturing system at $t = 120$ hours

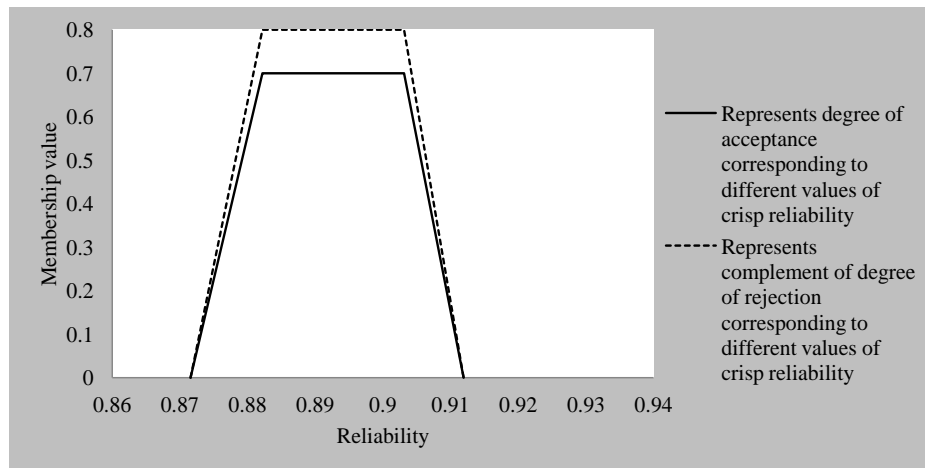


Figure 5.5 Trapezoidal vague set representing the vague reliability of piston manufacturing system at $t = 180$ hours

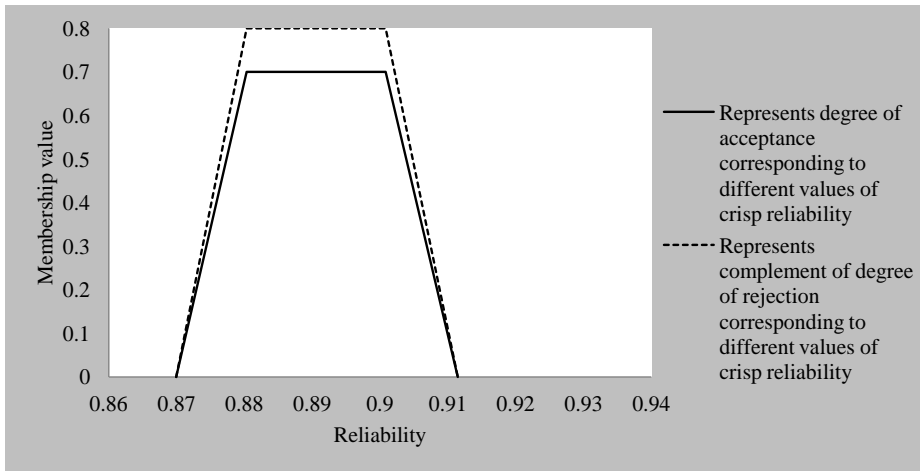


Figure 5.6 Trapezoidal vague set representing the vague reliability of piston manufacturing system at $t = 240$ hours

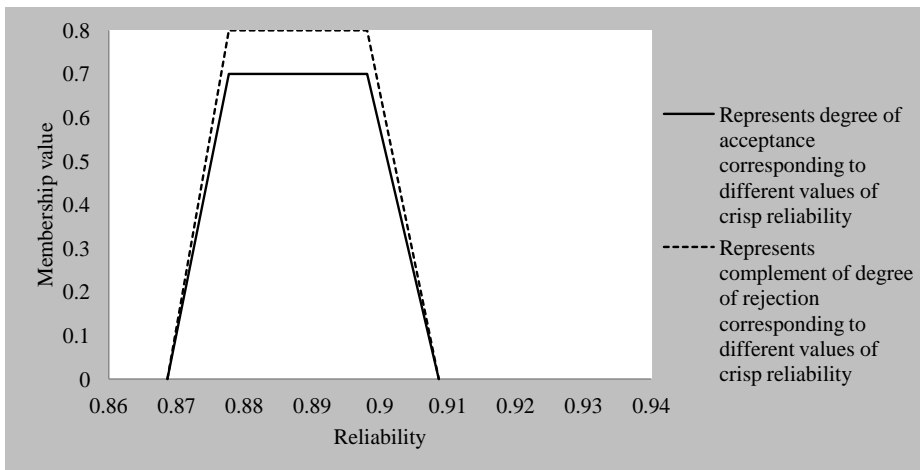


Figure 5.7 Trapezoidal vague set representing the vague reliability of piston manufacturing system at $t = 300$ hours

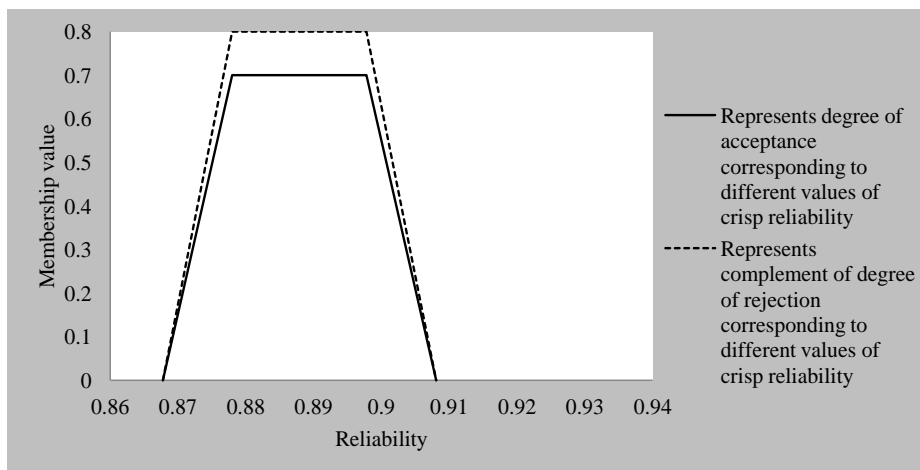


Figure 5.8 Trapezoidal vague set representing the vague reliability of piston manufacturing system at $t = 360$ hours

5.5 Physical interpretation of results

In this section, the obtained vague reliability of piston manufacturing system at $t = 360$ hours is physically interpreted.

It is obvious from Figure 5.8 that

1. The reliability of piston manufacturing system at $t = 360$ hours lies between 0.867774 and 0.908151.
2. The degree of acceptance of the experts about the statement that reliability of piston manufacturing system at $t = 360$ hours lies between 0.877986 to 0.897788 is 0.7.
3. The degree of rejection of the experts about the statement that reliability of piston manufacturing system at $t = 360$ hours lies between 0.877986 to 0.897788 is 0.2.
4. The degree of hesitation of the experts about the statement that reliability of piston manufacturing system at $t = 360$ hours lies between 0.877986 to 0.897788 is 0.1.
5. The degree of acceptance $\mu_{\tilde{R}}(r)$, degree of rejection $\nu_{\tilde{R}}(r)$ and degree of hesitation $1 - \mu_{\tilde{R}}(r) - \nu_{\tilde{R}}(r)$ for the remaining values of reliability r , $r \in [0.867774, 0.908151]$ can be obtained by putting r in:

$$\mu_{\tilde{R}}(r) = \begin{cases} \frac{0.7(r-0.867774)}{(0.010212)}, & 0.867774 \leq r < 0.877986 \\ 0.7, & 0.877986 \leq r \leq 0.897788 \\ \frac{0.7(0.908151-r)}{(0.010363)}, & 0.897788 < r \leq 0.908151 \end{cases}$$

$$\nu_{\tilde{R}}(r) = \begin{cases} 1 - \frac{0.8(r-0.867774)}{(0.010212)}, & 0.867774 \leq r < 0.877986 \\ 0.2, & 0.877986 \leq r \leq 0.897788 \\ 1 - \frac{0.8(0.908151-r)}{(0.010363)}, & 0.897788 < r \leq 0.908151 \end{cases}$$

and

$$1 - \mu_{\tilde{R}}(r) - \nu_{\tilde{R}}(r) = \begin{cases} \frac{0.8(r-0.867774)}{(0.010212)} - \frac{0.7(r-0.867774)}{(0.010212)}, & 0.867774 \leq r < 0.877986 \\ 0.1, & 0.877986 \leq r \leq 0.897788 \\ \frac{0.8(0.908151-r)}{(0.010363)} - \frac{0.7(0.908151-r)}{(0.010363)}, & 0.897788 < r \leq 0.908151 \end{cases}$$

5.6 Advantage of *JMD* trapezoidal vague set over existing trapezoidal vague set

To show the advantage of *JMD* trapezoidal vague set over existing trapezoidal vague set, the vague Kolmogorov's differential equations, for sub-system R_1 and sub-system R_2 , are solved by using proposed method with existing trapezoidal vague set and the obtained results are shown in Table 5.6 and Table 5.7 respectively.

Table 5.6 Solution of vague Kolmogorov's differential equations for sub-system R_1 obtained by using proposed method with existing trapezoidal vague set

j	$\tilde{P}_j(t)$ for $t=60$	$\tilde{P}_j(t)$ for $t=120$	$\tilde{P}_j(t)$ for $t=180$	$\tilde{P}_j(t)$ for $t=240$	$\tilde{P}_j(t)$ for $t=300$	$\tilde{P}_j(t)$ for $t=360$
	$\langle [P_{(1)}(t), P_{(2)}(t)] \rangle$	$\langle [P_{(1)}(t), P_{(2)}(t)] \rangle$	$\langle [P_{(1)}(t), P_{(2)}(t)] \rangle$	$\langle [P_{(1)}(t), P_{(2)}(t)] \rangle$	$\langle [P_{(1)}(t), P_{(2)}(t)] \rangle$	$\langle [P_{(1)}(t), P_{(2)}(t)] \rangle$
1	$\langle [0.720279, 0.681444] \rangle$	$\langle [0.652165, 0.600728] \rangle$	$\langle [0.603616, 0.547756] \rangle$	$\langle [0.568691, 0.512688] \rangle$	$\langle [0.543538, 0.489445] \rangle$	$\langle [0.525421, 0.474049] \rangle$
2	$\langle [0.071153, 0.077829] \rangle$	$\langle [0.065846, 0.070633] \rangle$	$\langle [0.062073, 0.065921] \rangle$	$\langle [0.059359, 0.062802] \rangle$	$\langle [0.057404, 0.060736] \rangle$	$\langle [0.055996, 0.059366] \rangle$
3	$\langle [0.044381, 0.048441] \rangle$	$\langle [0.040352, 0.042938] \rangle$	$\langle [0.037431, 0.039832] \rangle$	$\langle [0.035423, 0.036946] \rangle$	$\langle [0.033938, 0.035364] \rangle$	$\langle [0.032869, 0.034316] \rangle$
4	$\langle [0.096162, 0.125721] \rangle$	$\langle [0.169323, 0.213389] \rangle$	$\langle [0.221951, 0.271397] \rangle$	$\langle [0.265552, 0.309835] \rangle$	$\langle [0.287149, 0.335309] \rangle$	$\langle [0.306812, 0.352192] \rangle$
5	$\langle [0.007738, 0.008113] \rangle$	$\langle [0.006888, 0.007016] \rangle$	$\langle [0.006188, 0.006652] \rangle$	$\langle [0.005582, 0.006061] \rangle$	$\langle [0.005056, 0.005583] \rangle$	$\langle [0.004537, 0.005056] \rangle$
6	$\langle [0.005608, 0.010101] \rangle$	$\langle [0.007414, 0.008024] \rangle$	$\langle [0.006856, 0.007273] \rangle$	$\langle [0.006417, 0.006741] \rangle$	$\langle [0.006097, 0.006385] \rangle$	$\langle [0.005867, 0.00615] \rangle$
7	$\langle [0.001027, 0.001132] \rangle$	$\langle [0.000929, 0.000996] \rangle$	$\langle [0.000859, 0.000907] \rangle$	$\langle [0.000809, 0.000849] \rangle$	$\langle [0.000773, 0.00081] \rangle$	$\langle [0.000747, 0.000784] \rangle$
8	$\langle [0.001802, 0.001987] \rangle$	$\langle [0.001629, 0.001747] \rangle$	$\langle [0.001505, 0.001591] \rangle$	$\langle [0.001417, 0.001486] \rangle$	$\langle [0.001353, 0.001417] \rangle$	$\langle [0.001307, 0.001372] \rangle$
9	$\langle [0.002179, 0.002279] \rangle$	$\langle [0.001852, 0.001893] \rangle$	$\langle [0.001661, 0.001687] \rangle$	$\langle [0.001548, 0.001576] \rangle$	$\langle [0.001482, 0.001517] \rangle$	$\langle [0.001443, 0.001485] \rangle$
10	$\langle [0.001118, 0.001132] \rangle$	$\langle [0.001056, 0.001070] \rangle$	$\langle [0.000997, 0.001011] \rangle$	$\langle [0.000922, 0.000935] \rangle$	$\langle [0.000899, 0.000912] \rangle$	$\langle [0.000888, 0.000901] \rangle$
11	$\langle [0.000101, 0.000129] \rangle$	$\langle [0.000093, 0.000117] \rangle$	$\langle [0.000088, 0.000109] \rangle$	$\langle [0.000084, 0.000103] \rangle$	$\langle [0.000081, 0.000101] \rangle$	$\langle [0.000079, 0.000098] \rangle$
12	$\langle [0.000177, 0.000226] \rangle$	$\langle [0.000164, 0.000205] \rangle$	$\langle [0.000153, 0.000191] \rangle$	$\langle [0.000146, 0.000181] \rangle$	$\langle [0.000142, 0.000175] \rangle$	$\langle [0.000139, 0.000171] \rangle$
13	$\langle [0.000045, 0.000057] \rangle$	$\langle [0.000041, 0.000051] \rangle$	$\langle [0.000038, 0.000046] \rangle$	$\langle [0.000036, 0.000044] \rangle$	$\langle [0.000034, 0.000042] \rangle$	$\langle [0.000033, 0.000041] \rangle$
14	$\langle [0.000457, 0.000592] \rangle$	$\langle [0.000457, 0.000571] \rangle$	$\langle [0.000425, 0.000521] \rangle$	$\langle [0.000399, 0.000485] \rangle$	$\langle [0.000380, 0.000461] \rangle$	$\langle [0.000366, 0.000444] \rangle$
15	$\langle [0.000633, 0.00008] \rangle$	$\langle [0.000057, 0.000071] \rangle$	$\langle [0.000053, 0.000065] \rangle$	$\langle [0.000050, 0.000061] \rangle$	$\langle [0.000048, 0.000058] \rangle$	$\langle [0.000046, 0.000056] \rangle$
16	$\langle [0.000102, 0.000117] \rangle$	$\langle [0.000087, 0.000098] \rangle$	$\langle [0.000083, 0.000093] \rangle$	$\langle [0.000079, 0.000088] \rangle$	$\langle [0.000077, 0.000086] \rangle$	$\langle [0.000076, 0.000085] \rangle$
17	$\langle [0.000835, 0.001076] \rangle$	$\langle [0.000712, 0.000804] \rangle$	$\langle [0.000636, 0.000711] \rangle$	$\langle [0.000589, 0.000658] \rangle$	$\langle [0.000561, 0.000639] \rangle$	$\langle [0.000544, 0.000614] \rangle$
18	$\langle [0.001591, 0.002036] \rangle$	$\langle [0.001453, 0.001806] \rangle$	$\langle [0.001345, 0.001644] \rangle$	$\langle [0.001266, 0.001537] \rangle$	$\langle [0.001211, 0.001467] \rangle$	$\langle [0.001170, 0.00142] \rangle$
19	$\langle [0.002071, 0.002741] \rangle$	$\langle [0.001794, 0.002665] \rangle$	$\langle [0.001639, 0.002515] \rangle$	$\langle [0.001589, 0.002398] \rangle$	$\langle [0.001569, 0.002302] \rangle$	$\langle [0.001553, 0.002257] \rangle$
20	$\langle [0.002753, 0.004246] \rangle$	$\langle [0.002532, 0.003615] \rangle$	$\langle [0.002337, 0.003165] \rangle$	$\langle [0.002180, 0.002802] \rangle$	$\langle [0.002098, 0.002632] \rangle$	$\langle [0.002031, 0.002510] \rangle$
21	$\langle [0.000096, 0.000147] \rangle$	$\langle [0.000072, 0.000252] \rangle$	$\langle [0.000066, 0.000322] \rangle$	$\langle [0.000061, 0.000368] \rangle$	$\langle [0.000057, 0.000398] \rangle$	$\langle [0.000054, 0.000419] \rangle$
22	$\langle [0.000672, 0.001047] \rangle$	$\langle [0.000549, 0.002312] \rangle$	$\langle [0.000471, 0.004472] \rangle$	$\langle [0.000406, 0.006563] \rangle$	$\langle [0.000359, 0.009639] \rangle$	$\langle [0.000329, 0.014431] \rangle$
23	$\langle [0.001684, 0.002241] \rangle$	$\langle [0.001506, 0.004472] \rangle$	$\langle [0.001332, 0.007537] \rangle$	$\langle [0.001179, 0.012483] \rangle$	$\langle [0.001049, 0.018362] \rangle$	$\langle [0.000943, 0.025577] \rangle$
24	$\langle [0.000225, 0.000344] \rangle$	$\langle [0.000111, 0.000604] \rangle$	$\langle [0.000054, 0.000776] \rangle$	$\langle [0.000028, 0.000889] \rangle$	$\langle [0.000015, 0.000965] \rangle$	$\langle [0.000008, 0.001015] \rangle$

Table 5.7 Solution of vague Kolmogorov's differential equations for sub-system R_2 obtained by using proposed method with existing trapezoidal vague set

j	$\tilde{p}_j(t)$ for $t=60$	$\tilde{p}_j(t)$ for $t=120$	$\tilde{p}_j(t)$ for $t=180$	$\tilde{p}_j(t)$ for $t=240$	$\tilde{p}_j(t)$ for $t=300$	$\tilde{p}_j(t)$ for $t=360$
	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)]; 0.7, 0.8 \rangle$
1	$\langle [0.878882, 0.872595, 0.868478, 0.863537]; 0.7, 0.8 \rangle$	$\langle [0.878178, 0.871845, 0.867666, 0.862687]; 0.7, 0.8 \rangle$	$\langle [0.878084, 0.871751, 0.867573, 0.862596]; 0.7, 0.8 \rangle$	$\langle [0.878072, 0.871739, 0.867561, 0.862585]; 0.7, 0.8 \rangle$	$\langle [0.878071, 0.871737, 0.867561, 0.862585]; 0.7, 0.8 \rangle$	$\langle [0.87807, 0.871737, 0.867561, 0.862585]; 0.7, 0.8 \rangle$
2	$\langle [0.064096, 0.073858, 0.086077, 0.094293]; 0.7, 0.8 \rangle$	$\langle [0.064046, 0.073796, 0.085999, 0.094203]; 0.7, 0.8 \rangle$	$\langle [0.064039, 0.073788, 0.085991, 0.094193]; 0.7, 0.8 \rangle$	$\langle [0.064038, 0.073788, 0.085989, 0.094192]; 0.7, 0.8 \rangle$	$\langle [0.064038, 0.073787, 0.085989, 0.094192]; 0.7, 0.8 \rangle$	$\langle [0.064038, 0.073787, 0.085989, 0.094192]; 0.7, 0.8 \rangle$
3	$\langle [0.002943, 0.0034, 0.003973, 0.004361]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003396, 0.003969, 0.004357]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003396, 0.003969, 0.004356]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003396, 0.003969, 0.004356]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003396, 0.003969, 0.004356]; 0.7, 0.8 \rangle$	$\langle [0.002941, 0.003396, 0.003969, 0.004356]; 0.7, 0.8 \rangle$
4	$\langle [0.000425, 0.000481, 0.000562, 0.000625]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.00048, 0.000561, 0.000624]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.00048, 0.000561, 0.000624]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.00048, 0.000561, 0.000624]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.00048, 0.000561, 0.000624]; 0.7, 0.8 \rangle$	$\langle [0.000425, 0.00048, 0.000561, 0.000624]; 0.7, 0.8 \rangle$
5	$\langle [0.00011, 0.000119, 0.000139, 0.000159]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000119, 0.000139, 0.000159]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000119, 0.000139, 0.000159]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000119, 0.000139, 0.000159]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000119, 0.000139, 0.000159]; 0.7, 0.8 \rangle$	$\langle [0.00011, 0.000119, 0.000139, 0.000159]; 0.7, 0.8 \rangle$
6	$\langle [0.005369, 0.006013, 0.007112, 0.007977]; 0.7, 0.8 \rangle$	$\langle [0.005966, 0.006755, 0.007915, 0.008817]; 0.7, 0.8 \rangle$	$\langle [0.005969, 0.006848, 0.008007, 0.008908]; 0.7, 0.8 \rangle$	$\langle [0.006071, 0.00686, 0.008018, 0.008918]; 0.7, 0.8 \rangle$	$\langle [0.006073, 0.006861, 0.008019, 0.008919]; 0.7, 0.8 \rangle$	$\langle [0.006073, 0.006862, 0.008019, 0.008919]; 0.7, 0.8 \rangle$
7	$\langle [0.00024, 0.00026, 0.00031, 0.00035]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.00026, 0.00031, 0.00035]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.00026, 0.00031, 0.00035]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.00026, 0.00031, 0.00035]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.00026, 0.00031, 0.00035]; 0.7, 0.8 \rangle$	$\langle [0.00024, 0.00026, 0.00031, 0.00035]; 0.7, 0.8 \rangle$
8	$\langle [0.000214, 0.000287, 0.000393, 0.000476]; 0.7, 0.8 \rangle$	$\langle [0.000214, 0.000287, 0.000393, 0.000475]; 0.7, 0.8 \rangle$	$\langle [0.000214, 0.000287, 0.000393, 0.000475]; 0.7, 0.8 \rangle$	$\langle [0.000214, 0.000287, 0.000393, 0.000475]; 0.7, 0.8 \rangle$	$\langle [0.000214, 0.000287, 0.000393, 0.000475]; 0.7, 0.8 \rangle$	$\langle [0.000214, 0.000287, 0.000393, 0.000475]; 0.7, 0.8 \rangle$
9	$\langle [0.000031, 0.000041, 0.000055, 0.000068]; 0.7, 0.8 \rangle$	$\langle [0.000031, 0.000041, 0.000055, 0.000068]; 0.7, 0.8 \rangle$	$\langle [0.000031, 0.000041, 0.000055, 0.000068]; 0.7, 0.8 \rangle$	$\langle [0.000031, 0.000041, 0.000055, 0.000068]; 0.7, 0.8 \rangle$	$\langle [0.000031, 0.000041, 0.000055, 0.000068]; 0.7, 0.8 \rangle$	$\langle [0.000031, 0.000041, 0.000055, 0.000068]; 0.7, 0.8 \rangle$
10	$\langle [0.000008, 0.00001, 0.000013, 0.000017]; 0.7, 0.8 \rangle$	$\langle [0.000008, 0.00001, 0.000013, 0.000017]; 0.7, 0.8 \rangle$	$\langle [0.000008, 0.00001, 0.000013, 0.000017]; 0.7, 0.8 \rangle$	$\langle [0.000008, 0.00001, 0.000013, 0.000017]; 0.7, 0.8 \rangle$	$\langle [0.000008, 0.00001, 0.000013, 0.000017]; 0.7, 0.8 \rangle$	$\langle [0.000008, 0.00001, 0.000013, 0.000017]; 0.7, 0.8 \rangle$
11	$\langle [0.000373, 0.000496, 0.000689, 0.000852]; 0.7, 0.8 \rangle$	$\langle [0.000433, 0.00057, 0.000782, 0.000961]; 0.7, 0.8 \rangle$	$\langle [0.000441, 0.000579, 0.000793, 0.000972]; 0.7, 0.8 \rangle$	$\langle [0.000442, 0.000581, 0.000794, 0.000973]; 0.7, 0.8 \rangle$	$\langle [0.000442, 0.000581, 0.000794, 0.000973]; 0.7, 0.8 \rangle$	$\langle [0.000442, 0.000581, 0.000794, 0.000973]; 0.7, 0.8 \rangle$
12	$\langle [0.000001, 0.000002, 0.000003, 0.000004]; 0.7, 0.8 \rangle$	$\langle [0.000001, 0.000002, 0.000003, 0.000004]; 0.7, 0.8 \rangle$	$\langle [0.000001, 0.000002, 0.000003, 0.000004]; 0.7, 0.8 \rangle$	$\langle [0.000001, 0.000002, 0.000003, 0.000004]; 0.7, 0.8 \rangle$	$\langle [0.000001, 0.000002, 0.000003, 0.000004]; 0.7, 0.8 \rangle$	$\langle [0.000001, 0.000002, 0.000003, 0.000004]; 0.7, 0.8 \rangle$
13	$\langle [0.001617, 0.002168, 0.002969, 0.003591]; 0.7, 0.8 \rangle$	$\langle [0.001618, 0.002168, 0.002967, 0.003588]; 0.7, 0.8 \rangle$	$\langle [0.001617, 0.002168, 0.002967, 0.003588]; 0.7, 0.8 \rangle$	$\langle [0.001617, 0.002168, 0.002967, 0.003588]; 0.7, 0.8 \rangle$	$\langle [0.001617, 0.002168, 0.002967, 0.003588]; 0.7, 0.8 \rangle$	$\langle [0.001617, 0.002168, 0.002967, 0.003588]; 0.7, 0.8 \rangle$

It is obvious from Table 5.6 and Table 5.7 that on solving the vague Kolmogorov's differential equations by using the proposed method with existing representation of trapezoidal vague set, in the obtained values of $\tilde{p}_j(t)$ i.e., $\langle [(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)); 0.7, 0.8] \rangle$, the restriction $p_{1(1)}(t) \leq p_{1(2)}(t) \leq p_{1(3)}(t) \leq p_{1(4)}(t)$ is not satisfying. So, $\tilde{p}_j(t)$ does not define a trapezoidal vague set for $j = 1$.

However, it is obvious from Table 5.3 and Table 5.4 that on solving the same vague Kolmogorov's differential equations by using the proposed method with *JMD* trapezoidal vague set, in the obtained values of $\langle [(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)); 0.7, 0.8] \rangle$, the restriction $p_{j(1)}(t) \leq p_{j(2)}(t) \leq p_{j(3)}(t) \leq p_{j(4)}(t)$ is satisfying for all values of j . Hence, it is better to use *JMD* trapezoidal vague set instead of existing trapezoidal vague set.

5.7 Conclusions

On the basis of presented study, it is concluded that there is no method in the literature for solving vague differential equations and no one have used Markov model to evaluate the vague reliability of industrial systems. In this chapter, a new method is proposed for solving vague differential equations which can be used for solving vague Kolmogorov's differential equations to evaluate the vague reliability of industrial systems.

Chapter 6

A NEW METHOD FOR SOLVING GENERALIZED VAGUE DIFFERENTIAL EQUATIONS AND ITS APPLICATION FOR ANALYZING THE GENERALIZED VAGUE RELIABILITY OF PISTON MANUFACTURING SYSTEM

In real life situation, it is possible that at a particular time degree of acceptance of decision maker about a particular value is zero but the degree of rejection and degree of hesitation are not equal to zero. The trapezoidal vague set, presented in previous chapter, cannot be used to handle such real life situations. Therefore, Kumar [79] extended the concept of trapezoidal vague set by generalized trapezoidal vague set to handle such situations. On the same direction, in this chapter, the concept of *JMD* trapezoidal vague set is extended by generalized *JMD* trapezoidal vague set and by modifying the method, proposed in previous chapter, a new method is proposed for solving generalized vague differential equations. With the help of proposed method, generalized vague Kolmogorov's differential equations are solved and the obtained solution is used to evaluate the generalized vague reliability of piston manufacturing system. The advantage of generalized *JMD* trapezoidal

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vague set over existing generalized trapezoidal vague set is also discussed.

6.1 Preliminaries

In this section, some basic definitions and arithmetic operations between generalized trapezoidal vague sets are presented.

6.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 6.1 A vague set \tilde{A} , defined on the universal set of real numbers \mathbb{R} , denoted as $\tilde{A} = \langle [(b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4); \delta, \rho] \rangle$, is said to be generalized trapezoidal vague set, shown in Figure 6.1, if degree of membership $\mu_{\tilde{A}}(x)$ and degree of non-membership $\nu_{\tilde{A}}(x)$ are given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \delta \left(\frac{x-a_1}{a_2-a_1} \right), & a_1 \leq x < a_2 \\ \delta, & a_2 \leq x < a_3 \\ \delta \left(\frac{a_4-x}{a_4-a_3} \right), & a_3 \leq x < a_4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{\tilde{A}}(x) = \begin{cases} 1 - \rho \left(\frac{x-b_1}{b_2-b_1} \right), & b_1 \leq x < b_2 \\ 1 - \rho, & b_2 \leq x < b_3 \\ 1 - \rho \left(\frac{b_4-x}{b_4-b_3} \right), & b_3 \leq x < b_4 \\ 1, & \text{otherwise} \end{cases}$$

where,

$$b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4 \text{ and } \delta \leq \rho.$$

$$\delta = \text{Supremum}\{\mu_{\tilde{A}}(x) : x \in \mathbb{R}\}$$

$$\rho = \text{Supremum}\{(1 - \nu_{\tilde{A}}(x)) : x \in \mathbb{R}\}$$

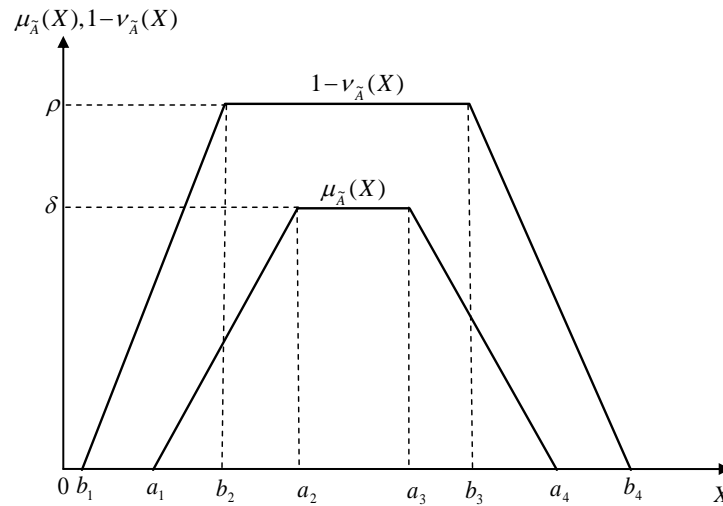


Figure 6.1 Generalized Trapezoidal vague set \tilde{A}

Definition 6.2 Two generalized trapezoidal vague sets $\tilde{A} = \langle [(b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4); \delta, \rho] \rangle$ and $\tilde{B} = \langle [(c_1, d_1, c_2, d_2, d_3, c_3, d_4, c_4); \delta, \rho] \rangle$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ if and only if $b_1 = c_1, a_1 = d_1, b_2 = c_2, a_2 = d_2, a_3 = d_3, b_3 = c_3, a_4 = d_4$ and $b_4 = c_4$.

6.1.2 Arithmetic operations between generalized trapezoidal vague sets

In this section, some arithmetic operations between generalized trapezoidal vague sets, defined on universal set of real numbers \mathbb{R} , are presented.

Let $\tilde{A} = \langle [(b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4); \delta, \rho] \rangle$ and $\tilde{B} = \langle [(c_1, d_1, c_2, d_2, d_3, c_3, d_4, c_4); \delta, \rho] \rangle$ be two generalized trapezoidal vague sets. Then,

$$(i) \tilde{A} \oplus \tilde{B} = \langle [(b_1 + c_1, a_1 + d_1, b_2 + c_2, a_2 + d_2, a_3 + d_3, b_3 + c_3, a_4 + d_4, b_4 + c_4); \delta, \rho] \rangle$$

$$(ii) \tilde{A} \ominus \tilde{B} = \langle [(b_1 - c_4, a_1 - d_4, b_2 - c_3, a_2 - d_3, a_3 - d_2, b_3 - c_2, a_4 - d_1, b_4 - c_1); \delta, \rho] \rangle$$

$$(iii) \lambda \tilde{A} = \begin{cases} \langle [(\lambda b_1, \lambda a_1, \lambda b_2, \lambda a_2, \lambda a_3, \lambda b_3, \lambda a_4, \lambda b_4); \delta, \rho] \rangle, & \lambda \geq 0 \\ \langle [(\lambda b_4, \lambda a_4, \lambda b_3, \lambda a_3, \lambda a_2, \lambda b_2, \lambda a_1, \lambda b_1); \delta, \rho] \rangle, & \lambda \leq 0 \end{cases}$$

$$(iv) \tilde{A} \otimes \tilde{B} = \langle [(\text{minimum}(a), \text{minimum}(b), \text{minimum}(c), \text{minimum}(d), \text{maximum}(d), \text{maximum}(c), \text{maximum}(b), \text{maximum}(a)); \delta, \rho] \rangle$$

where,

$$a = (b_1c_1, b_1c_4, b_4c_1, b_4c_4)$$

$$b = (a_1d_1, a_1d_4, a_4d_1, a_4d_4)$$

$$c = (b_2c_2, b_2c_3, b_3c_2, b_3c_3)$$

$$d = (a_2d_2, a_2d_3, a_3d_2, a_3d_3)$$

6.2 *JMD* generalized trapezoidal vague set

In this section, a new representation of generalized trapezoidal vague set, named as *JMD* generalized trapezoidal vague set, is proposed.

6.2.1 Basic definitions

In this section, definition of *JMD* generalized trapezoidal vague set and equality of two *JMD* generalized trapezoidal vague sets are introduced.

Definition 6.3 Let $\tilde{A} = \langle [(b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4); \delta, \rho] \rangle$ be a generalized trapezoidal vague set then its *JMD* representation is $\tilde{A} = \langle [(x_1, \alpha_1, \beta_1, \gamma_1, \delta_1, \omega_1, \phi_1, \psi_1); \delta, \rho]_{JMD} \rangle$, where, $x_1 = b_1$, $\alpha_1 = a_1 - b_1 \geq 0$, $\beta_1 = b_2 - a_1 \geq 0$, $\gamma_1 = a_2 - b_2 \geq 0$, $\delta_1 = a_3 - a_2 \geq 0$, $\omega_1 = b_3 - a_3 \geq 0$, $\phi_1 = a_4 - b_3 \geq 0$ and $\psi_1 = b_4 - a_4 \geq 0$.

Definition 6.4 Two *JMD* generalized trapezoidal vague sets $\tilde{A} = \langle [(x_1, \alpha_1, \beta_1, \gamma_1, \delta_1, \omega_1, \phi_1, \psi_1); \delta, \rho]_{JMD} \rangle$ and $\tilde{B} = \langle [(x_2, \alpha_2, \beta_2, \gamma_2, \delta_2, \omega_2, \phi_2, \psi_2); \delta, \rho]_{JMD} \rangle$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ if and only if $x_1 = x_2$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$, $\omega_1 = \omega_2$, $\phi_1 = \phi_2$, $\psi_1 = \psi_2$.

6.2.2 Arithmetic operations between *JMD* generalized trapezoidal vague sets

In this section, with the help of existing arithmetic operations, presented

in Section 6.1.2, some arithmetic operations between JMD generalized trapezoidal vague sets are introduced.

Let $\tilde{A} = \langle [(x_1, \alpha_1, \beta_1, \gamma_1, \delta_1, \omega_1, \phi_1, \psi_1); \delta, \rho]_{JMD} \rangle$ and $\tilde{B} = \langle [(x_2, \alpha_2, \beta_2, \gamma_2, \delta_2, \omega_2, \phi_2, \psi_2); \delta, \rho]_{JMD} \rangle$ be two JMD generalized trapezoidal vague sets. Then,

$$(i) \tilde{A} \oplus \tilde{B} = \langle [(x_1 + x_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2, \delta_1 + \delta_2, \omega_1 + \omega_2, \phi_1 + \phi_2, \psi_1 + \psi_2); \delta, \rho]_{JMD} \rangle$$

$$(ii) \tilde{A} \ominus \tilde{B} = \langle [(x_1 - x_2 - \alpha_2 - \beta_2 - \gamma_2 - \delta_2 - \omega_2 - \phi_2 - \psi_2, \alpha_1 + \psi_2, \beta_1 + \phi_2, \gamma_1 + \omega_2, \delta_1 + \delta_2, \omega_1 + \delta_2, \phi_1 + \beta_2, \psi_1 + \alpha_2); \delta, \rho]_{JMD} \rangle$$

$$(iii) \lambda \tilde{A} = \begin{cases} \langle [(\lambda x_1, \lambda \alpha_1, \lambda \beta_1, \lambda \gamma_1, \lambda \delta_1, \lambda \omega_1, \lambda \phi_1, \lambda \psi_1); \delta, \rho]_{JMD} \rangle, & \lambda \geq 0 \\ \langle [(\lambda x_1 + \lambda \alpha_1 + \lambda \beta_1 + \lambda \gamma_1 + \lambda \delta_1 + \lambda \omega_1 + \lambda \phi_1 + \lambda \psi_1, -\lambda \psi_1, -\lambda \phi_1, -\lambda \omega_1, -\lambda \delta_1, -\lambda \gamma_1, -\lambda \beta_1, -\lambda \alpha_1); \delta, \rho]_{JMD} \rangle, & \lambda \leq 0 \end{cases}$$

$$(iv) \tilde{A} \otimes \tilde{B} = \langle [(\text{minimum}(a), \text{minimum}(b) - \text{minimum}(a), \text{minimum}(c) - \text{minimum}(b), \text{minimum}(d) - \text{minimum}(c), \text{maximum}(d) - \text{minimum}(d), \text{maximum}(c) - \text{maximum}(d), \text{maximum}(b) - \text{maximum}(c), \text{maximum}(a) - \text{maximum}(b)); \delta, \rho]_{JMD} \rangle$$

where,

$$a = \left(x_1 x_2, x_1(x_2 + \alpha_2 + \beta_2 + \gamma_2 + \delta_2 + \omega_2 + \phi_2 + \psi_2), (x_1 + \alpha_1 + \beta_1 + \gamma_1 + \delta_1 + \omega_1 + \phi_1 + \psi_1)x_2, (x_1 + \alpha_1 + \beta_1 + \gamma_1 + \delta_1 + \omega_1 + \phi_1 + \psi_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2 + \delta_2 + \omega_2 + \phi_2 + \psi_2) \right)$$

$$b = \left((x_1 + \alpha_1)(x_2 + \alpha_2), (x_1 + \alpha_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2 + \delta_2 + \omega_2 + \phi_2), (x_1 + \alpha_1 + \beta_1 + \gamma_1 + \delta_1 + \omega_1 + \phi_1)(x_2 + \alpha_2), (x_1 + \alpha_1 + \beta_1 + \gamma_1 + \delta_1 + \omega_1 + \phi_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2 + \delta_2 + \omega_2 + \phi_2) \right)$$

$$c = \left((x_1 + \alpha_1 + \beta_1)(x_2 + \alpha_2 + \beta_2), (x_1 + \alpha_1 + \beta_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2 + \delta_2 + \omega_2), (x_1 + \alpha_1 + \beta_1 + \gamma_1 + \delta_1 + \omega_1)(x_2 + \alpha_2 + \beta_2), (x_1 + \alpha_1 + \beta_1 + \gamma_1 + \delta_1 + \omega_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2 + \delta_2 + \omega_2) \right)$$

$$d = \left(x_1 + \alpha_1 + \beta_1 + \gamma_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2), (x_1 + \alpha_1 + \beta_1 + \gamma_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2 + \delta_2), (x_1 + \alpha_1 + \beta_1 + \gamma_1 + \delta_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2), (x_1 + \alpha_1 + \beta_1 + \gamma_1 + \delta_1)(x_2 + \alpha_2 + \beta_2 + \gamma_2 + \delta_2) \right)$$

6.3 Proposed method for solving n^{th} order generalized vague linear differential equations

In this section, a new method is proposed for solving n^{th} order generalized vague linear differential equations.

Any n^{th} order generalized vague linear differential equation can be written

as:

$$\sum_{j=0}^n \tilde{a}_j \otimes \tilde{y}^{(j)} = \tilde{g} \quad (6.1)$$

with initial conditions:

$$\tilde{y}^{(j)}(x_0) = \tilde{\gamma}_j, \quad j = 0, 1, \dots, n-1 \quad (6.2)$$

where, $\tilde{y}^{(j)} = \frac{d^j \tilde{y}}{dx^j}$, \tilde{a}_j , \tilde{g} and $\tilde{\gamma}_j$ are generalized trapezoidal vague sets.

The solution of equation (6.1) can be obtained as follows:

Step 1: Assuming $\tilde{a}_j = \langle [(a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)}, \beta_{j(4)}, \beta_{j(5)}, \beta_{j(6)}, \beta_{j(7)}); \delta, \rho]_{JMD} \rangle$,

$$\tilde{y}^{(j)} = \langle [(y_1^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}, \alpha_4^{(j)}, \alpha_5^{(j)}, \alpha_6^{(j)}, \alpha_7^{(j)}); \delta, \rho]_{JMD} \rangle,$$

$$\tilde{g} = \langle [(g, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7); \delta, \rho]_{JMD} \rangle$$

$$\text{and } \tilde{\gamma}_j = \langle [(\gamma_{j(1)}, \zeta_{j(1)}, \zeta_{j(2)}, \zeta_{j(3)}, \zeta_{j(4)}, \zeta_{j(5)}, \zeta_{j(6)}, \zeta_{j(7)}); \delta, \rho]_{JMD} \rangle,$$

n^{th} order generalized vague linear differential equation (6.1) with initial conditions

(6.2) can be written as

$$\sum_{j=0}^n \langle [(a_{j(1)}, \beta_{j(1)}, \beta_{j(2)}, \beta_{j(3)}, \beta_{j(4)}, \beta_{j(5)}, \beta_{j(6)}, \beta_{j(7)}); \delta, \rho]_{JMD} \rangle \otimes \langle [(y_1^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}, \alpha_4^{(j)}, \alpha_5^{(j)}, \alpha_6^{(j)}, \alpha_7^{(j)}); \delta, \rho]_{JMD} \rangle = \langle [(g, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7); \delta, \rho]_{JMD} \rangle \quad (6.3)$$

with initial conditions:

$$\begin{aligned}
&< [(y_1^{(j)}(x_0), \alpha_1^{(j)}(x_0), \alpha_2^{(j)}(x_0), \alpha_3^{(j)}(x_0), \alpha_4^{(j)}(x_0), \alpha_5^{(j)}(x_0), \alpha_6^{(j)}(x_0), \alpha_7^{(j)}(x_0)); \delta, \rho]_{JMD} > \\
&= < [(\gamma_{j(1)}, \zeta_{j(1)}, \zeta_{j(2)}, \zeta_{j(3)}, \zeta_{j(4)}, \zeta_{j(5)}, \zeta_{j(6)}, \zeta_{j(7)}); \delta, \rho]_{JMD} >, \quad j = 0, 1, \dots, n-1 \quad (6.4)
\end{aligned}$$

Step 2: Using Definition 6.4 and Section 6.2.2, n^{th} order generalized vague linear differential equation (6.3) with initial conditions (6.4) can be transformed into the following n^{th} order linear differential equations:

$$\sum_{j=0}^n \text{minimum}(b_j z^{(j)}) = g \quad (6.5)$$

$$\sum_{j=0}^n [\text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_1 \quad (6.6)$$

$$\sum_{j=0}^n [\text{minimum}(d_j z^{(j)}) - \text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_2 \quad (6.7)$$

$$\sum_{j=0}^n [\text{minimum}(e_j z^{(j)}) - \text{minimum}(d_j z^{(j)}) - \text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_3 \quad (6.8)$$

$$\sum_{j=0}^n [\text{maximum}(e_j z^{(j)}) - \text{minimum}(e_j z^{(j)}) - \text{minimum}(d_j z^{(j)}) - \text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_4 \quad (6.9)$$

$$\sum_{j=0}^n [\text{maximum}(d_j z^{(j)}) - \text{maximum}(e_j z^{(j)}) - \text{minimum}(e_j z^{(j)}) - \text{minimum}(d_j z^{(j)}) - \text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_5 \quad (6.10)$$

$$\sum_{j=0}^n [\text{maximum}(c_j z^{(j)}) - \text{maximum}(d_j z^{(j)}) - \text{maximum}(e_j z^{(j)}) - \text{minimum}(e_j z^{(j)}) - \text{minimum}(d_j z^{(j)}) - \text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_6 \quad (6.11)$$

$$\sum_{j=0}^n [\text{maximum}(b_j z^{(j)}) - \text{maximum}(c_j z^{(j)}) - \text{maximum}(d_j z^{(j)}) - \text{maximum}(e_j z^{(j)}) - \text{minimum}(e_j z^{(j)}) - \text{minimum}(d_j z^{(j)}) - \text{minimum}(c_j z^{(j)}) - \text{minimum}(b_j z^{(j)})] = \xi_7 \quad (6.12)$$

with initial conditions:

$$y_1^{(j)}(0) = \gamma_{j(1)}, \quad j = 0, 1, \dots, n-1 \quad (6.13)$$

$$\alpha_1^{(j)}(0) = \zeta_{j(1)}, \quad j = 0, 1, \dots, n-1 \quad (6.14)$$

$$\alpha_2^{(j)}(0) = \zeta_{j(2)}, \quad j = 0, 1, \dots, n-1 \quad (6.15)$$

$$\alpha_3^{(j)}(0) = \zeta_{j(3)}, \quad j = 0, 1, \dots, n-1 \quad (6.16)$$

$$\alpha_4^{(j)}(0) = \zeta_{j(4)}, \quad j = 0, 1, \dots, n-1 \quad (6.17)$$

$$\alpha_5^{(j)}(0) = \zeta_{j(5)}, \quad j = 0, 1, \dots, n-1 \quad (6.18)$$

$$\alpha_6^{(j)}(0) = \zeta_{j(6)}, \quad j = 0, 1, \dots, n-1 \quad (6.19)$$

$$\alpha_7^{(j)}(0) = \zeta_{j(7)}, \quad j = 0, 1, \dots, n-1 \quad (6.20)$$

where,

$$b_j z^{(j)} = \left(a_{j(1)} y_1^{(j)}, a_{j(1)} (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \alpha_4^{(j)} + \alpha_5^{(j)} + \alpha_6^{(j)} + \alpha_7^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)} + \beta_{j(4)} + \beta_{j(5)} + \beta_{j(6)} + \beta_{j(7)}) y_1^{(j)}, (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)} + \beta_{j(4)} + \beta_{j(5)} + \beta_{j(6)} + \beta_{j(7)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \alpha_4^{(j)} + \alpha_5^{(j)} + \alpha_6^{(j)} + \alpha_7^{(j)}) \right)$$

$$c_j z^{(j)} = \left((a_{j(1)} + \beta_{j(1)}) (y_1^{(j)} + \alpha_1^{(j)}), (a_{j(1)} + \beta_{j(1)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \alpha_4^{(j)} + \alpha_5^{(j)} + \alpha_6^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)} + \beta_{j(4)} + \beta_{j(5)} + \beta_{j(6)}) (y_1^{(j)} + \alpha_1^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)} + \beta_{j(4)} + \beta_{j(5)} + \beta_{j(6)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \alpha_4^{(j)} + \alpha_5^{(j)} + \alpha_6^{(j)}) \right)$$

$$d_j z^{(j)} = \left((a_{j(1)} + \beta_{j(1)} + \beta_{j(2)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \alpha_4^{(j)} + \alpha_5^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)} + \beta_{j(4)} + \beta_{j(5)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)} + \beta_{j(4)} + \beta_{j(5)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \alpha_4^{(j)} + \alpha_5^{(j)}) \right)$$

$$e_j z^{(j)} = \left((a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \alpha_4^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)} + \beta_{j(4)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)}), (a_{j(1)} + \beta_{j(1)} + \beta_{j(2)} + \beta_{j(3)} + \beta_{j(4)}) (y_1^{(j)} + \alpha_1^{(j)} + \alpha_2^{(j)} + \alpha_3^{(j)} + \alpha_4^{(j)}) \right)$$

Step 3: Solve the n^{th} order linear differential equations (6.5) to (6.12) with initial

conditions (6.13) to (6.20) to find the values of $y_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ and α_7 .

Step 4: Put the values of $y_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ and α_7 in $\tilde{y} = \langle [(y_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4,$

$\alpha_5, \alpha_6, \alpha_7); \delta, \rho]_{JMD} \rangle$ to find the solution of n^{th} order generalized vague linear dif-

ferential equation (6.1) with initial conditions (6.2).

Step 5: Convert $\tilde{y} = \langle [(y_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7); \delta, \rho]_{JMD} \rangle$ into $\langle [(b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4); \delta, \rho] \rangle$, where, $b_1 = y_1, a_1 = y_1 + \alpha_1, b_2 = y_1 + \alpha_1 + \alpha_2, a_2 = y_1 + \alpha_1 + \alpha_2 + \alpha_3, a_3 = y_1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, b_3 = y_1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, a_4 = y_1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ and $b_4 = y_1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$.

6.4 Solution of generalized vague Kolmogorov's differential equations

In this section, to illustrate the proposed method, the generalized vague Kolmogorov's differential equations of sub-system R_1 , obtained by replacing the fuzzy parameters

$$\tilde{p}_j(t) = (p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)),$$

$$\tilde{P}_1(0) = (0.94, 0.945, 0.955, 0.96),$$

$$\tilde{P}_2(0) = (0.006, 0.0065, 0.0075, 0.008),$$

$$\tilde{P}_3(0) = (0.004, 0.0045, 0.0055, 0.006),$$

$$\tilde{P}_4(0) = ((0.002, 0.0025, 0.0035, 0.004),$$

$$\tilde{P}_j(0) = ((0, 0, 0, 0), j=5 \text{ to } 24$$

and fuzzy failure rates, fuzzy repair rates, presented in Table 3.1 by the generalized vague parameters

$$\tilde{p}_j(t) = \langle [(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t), p_{j(5)}(t), p_{j(6)}(t), p_{j(7)}(t), p_{j(8)}(t)); 0.7, 0.8] \rangle,$$

$$\tilde{P}_1(0) = \langle [(0.93, 0.935, 0.94, 0.945, 0.955, 0.96, 0.965, 0.97); 0.7, 0.8] \rangle,$$

$$\tilde{P}_2(0) = \langle [(0.005, 0.0055, 0.006, 0.0065, 0.0075, 0.008, 0.0085, 0.009); 0.7, 0.8] \rangle,$$

$$\tilde{P}_3(0) = \langle [(0.003, 0.0035, 0.004, 0.0045, 0.0055, 0.006, 0.0065, 0.007); 0.7, 0.8] \rangle,$$

$$\tilde{P}_4(0) = \langle [(0.001, 0.0015, 0.002, 0.0025, 0.0035, 0.004, 0.0045, 0.005); 0.7, 0.8] \rangle,$$

$$\tilde{P}_j(0) = \langle [(0, 0, 0, 0, 0, 0, 0, 0); 0.7, 0.8] \rangle, j=5 \text{ to } 24$$

and generalized vague failure rates, generalized vague repair rates, presented in Table 6.1, as well as the generalized vague Kolmogorov's differential equations of sub-system R_2 , obtained by replacing the fuzzy parameters

$$\tilde{p}_j(t) = (p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t)),$$

$$\tilde{P}_1(0) = (0.95, 0.955, 0.965, 0.97),$$

$$\tilde{P}_2(0) = (0.004, 0.0045, 0.0055, 0.006),$$

$$\tilde{P}_j(0) = (0, 0, 0, 0), \quad j=3 \text{ to } 13$$

and fuzzy failure rates, fuzzy repair rates, presented in Table 3.2, by the generalized vague parameters

$$\tilde{p}_j(t) = \langle [(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t), p_{j(5)}(t), p_{j(6)}(t), p_{j(7)}(t), p_{j(8)}(t)); 0.7, 0.8] \rangle,$$

$$\tilde{P}_1(0) = \langle [(0.93, 0.935, 0.94, 0.945, 0.955, 0.96, 0.965, 0.97); 0.7, 0.8] \rangle,$$

$$\tilde{P}_2(0) = \langle [(0.003, 0.0035, 0.004, 0.0045, 0.0055, 0.006, 0.0065, 0.007); 0.7, 0.8] \rangle,$$

$$\tilde{P}_j(0) = \langle [(0, 0, 0, 0, 0, 0, 0, 0); 0.7, 0.8] \rangle, \quad j=3 \text{ to } 13$$

and generalized vague failure rates, generalized vague repair rates, presented in Table 6.2, are solved by using the proposed method at $t = 60$ hours, $t = 120$ hours, $t = 180$ hours, $t = 240$ hours, $t = 300$ hours, $t = 360$ hours and the obtained results are shown in Table 6.3 and Table 6.4 respectively.

Table 6.1 Generalized vague failure rates and generalized vague repair rates for different sub-systems of R_1

	Generalized vague failure rate	Generalized vague repair rate
$\tilde{\chi}_1 = <$	$[(0.00042, 0.00077, 0.00105, 0.00126, 0.00154, 0.00175, 0.00203, 0.00238); 0.7, 0.8] >$	$\tilde{\beta}_1 = < [(0.864, 0.972, 1.026, 1.0584, 1.1016, 1.134, 1.188, 1.296); 0.7, 0.8] >$
$\tilde{\chi}_2 = <$	$[(0.00018, 0.00033, 0.00045, 0.00054, 0.00066, 0.00075, 0.00087, 0.00102); 0.7, 0.8] >$	$\tilde{\beta}_2 = < [(0.0344, 0.0387, 0.04085, 0.04214, 0.04386, 0.04515, 0.0473, 0.0516); 0.7, 0.8] >$
$\tilde{\chi}_3 = <$	$[(0.00027, 0.000495, 0.000675, 0.00081, 0.00099, 0.001125, 0.001305, 0.00153); 0.7, 0.8] >$	$\tilde{\beta}_3 = < [(0.4, 0.45, 0.475, 0.49, 0.51, 0.525, 0.55, 0.6); 0.7, 0.8] >$
$\tilde{\chi}_4 = <$	$[(0.00027, 0.000495, 0.000675, 0.00081, 0.00099, 0.001125, 0.001305, 0.00153); 0.7, 0.8] >$	$\tilde{\beta}_4 = < [(0.2288, 0.2574, 0.2717, 0.28028, 0.29172, 0.30063, 0.3146); 0.7, 0.8] >$
$\tilde{\chi}_5 = <$	$[(0.0062, 0.0114, 0.0156, 0.01872, 0.02288, 0.026, 0.0302, 0.0354); 0.7, 0.8] >$	$\tilde{\beta}_5 = < [(0.1232, 0.1386, 0.1463, 0.15092, 0.15702, 0.1617, 0.1694, 0.1848); 0.7, 0.8] >$
$\tilde{\chi}_6 = <$	$[(0.0062, 0.0114, 0.0156, 0.01872, 0.02288, 0.026, 0.0302, 0.0354); 0.7, 0.8] >$	$\tilde{\beta}_6 = < [(0.2, 0.225, 0.2375, 0.245, 0.255, 0.2625, 0.275, 0.3); 0.7, 0.8] >$
$\tilde{\chi}_7 = <$	$[(0.00027, 0.000495, 0.000675, 0.00081, 0.00099, 0.001125, 0.001305, 0.00153); 0.7, 0.8] >$	$\tilde{\beta}_7 = < [(0.0472, 0.0531, 0.05605, 0.05782, 0.06018, 0.06195, 0.0649, 0.0708); 0.7, 0.8] >$
$\tilde{\chi}_8 = <$	$[(0.00117, 0.002145, 0.002925, 0.00351, 0.00429, 0.004875, 0.005655, 0.00663); 0.7, 0.8] >$	$\tilde{\beta}_8 = < [(0.0696, 0.0783, 0.08265, 0.08526, 0.08874, 0.091350, 0.0957, 0.1044); 0.7, 0.8] >$

Table 6.2 Generalized vague failure rates and generalized vague repair rates for the different sub-systems of R_2

	Generalized vague failure rate	Generalized vague repair rate
$\tilde{\eta}_1 = <$	$[(0.00042, 0.00077, 0.00105, 0.00126, 0.00154, 0.00175, 0.00203, 0.00238); 0.7, 0.8] >$	$\tilde{\mu}_1 = < [(0.264, 0.297, 0.3135, 0.3294, 0.3366, 0.3465, 0.363, 0.396); 0.7, 0.8] >$
$\tilde{\eta}_2 = <$	$[(0.00017, 0.00023, 0.00027, 0.00033, 0.00038, 0.00044, 0.00052); 0.7, 0.8] >$	$\tilde{\mu}_2 = < [(0.4, 0.45, 0.475, 0.49, 0.51, 0.525, 0.55, 0.6); 0.7, 0.8] >$
$\tilde{\eta}_3 = <$	$[(0.00004, 0.00006, 0.00008, 0.00009, 0.00011, 0.00013, 0.00015, 0.00017); 0.7, 0.8] >$	$\tilde{\mu}_3 = < [(0.536, 0.603, 0.6365, 0.6566, 0.6834, 0.7035, 0.737, 0.804); 0.7, 0.8] >$
$\tilde{\eta}_4 = <$	$[(0.0001, 0.00017, 0.00023, 0.00027, 0.00033, 0.00038, 0.00044, 0.00052); 0.7, 0.8] >$	$\tilde{\mu}_4 = < [(0.028, 0.0315, 0.03325, 0.0343, 0.0357, 0.03675, 0.0385, 0.042); 0.7, 0.8] >$
$\tilde{\eta}_5 = <$	$[(0.00004, 0.00006, 0.00008, 0.00009, 0.00011, 0.00013, 0.00015, 0.00017); 0.7, 0.8] >$	$\tilde{\mu}_5 = < [(2.424, 2.727, 2.8785, 2.9694, 3.0906, 3.1815, 3.333, 3.636); 0.7, 0.8] >$
$\tilde{\eta}_6 = <$	$[(0.0062, 0.0114, 0.0156, 0.01872, 0.02288, 0.026, 0.0302, 0.0354); 0.7, 0.8] >$	$\tilde{\mu}_6 = < [(0.1776, 0.1998, 0.2109, 0.21756, 0.22644, 0.2331, 0.2442, 0.2664); 0.7, 0.8] >$
$\tilde{\eta}_7 = <$	$[(0.0012, 0.0022, 0.003, 0.0036, 0.0044, 0.005, 0.0058, 0.0068); 0.7, 0.8] >$	$\tilde{\mu}_7 = < [(0.1, 0.1125, 0.11875, 0.1225, 0.1275, 0.13125, 0.1375, 0.15); 0.7, 0.8] >$

Table 6.3 Solution of generalized vague Kolmogorov's differential equations for sub-system R_1 obtained by using proposed method

j	$\bar{p}_j(t)$ for $t=60$	$\bar{p}_j(t)$ for $t=180$	$\bar{p}_j(t)$ for $t=240$	$\bar{p}_j(t)$ for $t=360$
	$\langle (P_{(1)}(t), P_{(2)}(t), P_{(3)}(t), P_{(4)}(t), P_{(5)}(t), P_{(6)}(t), P_{(7)}(t), P_{(8)}(t)) \rangle$	$\langle (P_{(1)}(t), P_{(2)}(t), P_{(3)}(t), P_{(4)}(t), P_{(5)}(t), P_{(6)}(t), P_{(7)}(t), P_{(8)}(t)) \rangle$	$\langle (P_{(1)}(t), P_{(2)}(t), P_{(3)}(t), P_{(4)}(t), P_{(5)}(t), P_{(6)}(t), P_{(7)}(t), P_{(8)}(t)) \rangle$	$\langle (P_{(1)}(t), P_{(2)}(t), P_{(3)}(t), P_{(4)}(t), P_{(5)}(t), P_{(6)}(t), P_{(7)}(t), P_{(8)}(t)) \rangle$
1	$\langle (0.843224, 0.847045, 0.869075, 0.872896, 0.7, 0.8) \rangle$	$\langle (0.804154, 0.80717, 0.828885, 0.825901, 0.7, 0.8) \rangle$	$\langle (0.787853, 0.790616, 0.804566, 0.807329, 0.7, 0.8) \rangle$	$\langle (0.760161, 0.762544, 0.776327, 0.7, 0.8) \rangle$
2	$\langle (0.040567, 0.041445, 0.04242, 0.043359, 0.044378, 0.046317, 0.047292, 0.04817) \rangle$	$\langle (0.039061, 0.039896, 0.04093, 0.042047, 0.043286, 0.044377, 0.045352, 0.046136) \rangle$	$\langle (0.038430, 0.039215, 0.04019, 0.04128, 0.042368, 0.043437, 0.044511, 0.04559) \rangle$	$\langle (0.037359, 0.038064, 0.038918, 0.039905, 0.041611, 0.042598, 0.043452, 0.044157) \rangle$
3	$\langle (0.025281, 0.025933, 0.02674, 0.027565, 0.028283, 0.030108, 0.030915, 0.031567) \rangle$	$\langle (0.024155, 0.024689, 0.025488, 0.026251, 0.027789, 0.028632, 0.029351, 0.029885) \rangle$	$\langle (0.023694, 0.024237, 0.024833, 0.025603, 0.026964, 0.027746, 0.028393, 0.028882) \rangle$	$\langle (0.022892, 0.023316, 0.02385, 0.024509, 0.025584, 0.026243, 0.026777, 0.027201) \rangle$
4	$\langle (0.022481, 0.023338, 0.024161, 0.024893, 0.025655, 0.027282, 0.028105, 0.028692) \rangle$	$\langle (0.021658, 0.022529, 0.023329, 0.024141, 0.024999, 0.025829, 0.026633, 0.027411) \rangle$	$\langle (0.021072, 0.021929, 0.022709, 0.023511, 0.024366, 0.025171, 0.025987, 0.026764) \rangle$	$\langle (0.020533, 0.021379, 0.022136, 0.022904, 0.023684, 0.024474, 0.025274, 0.026074) \rangle$
5	$\langle (0.00941, 0.009475, 0.009494, 0.009517, 0.009563, 0.009586, 0.009605, 0.009627) \rangle$	$\langle (0.009391, 0.009433, 0.009471, 0.009509, 0.009547, 0.009585, 0.009623, 0.009661) \rangle$	$\langle (0.009372, 0.009414, 0.009452, 0.00949, 0.009528, 0.009566, 0.009604, 0.009642) \rangle$	$\langle (0.009353, 0.009395, 0.009433, 0.009471, 0.009509, 0.009547, 0.009585, 0.009623) \rangle$
6	$\langle (0.003901, 0.003935, 0.003963, 0.003985, 0.00401, 0.004034, 0.004058, 0.004083) \rangle$	$\langle (0.003881, 0.003915, 0.003943, 0.003971, 0.003999, 0.004027, 0.004055, 0.004083) \rangle$	$\langle (0.003861, 0.003895, 0.003923, 0.003951, 0.003979, 0.004007, 0.004035, 0.004063) \rangle$	$\langle (0.003841, 0.003875, 0.003903, 0.003931, 0.003959, 0.003987, 0.004015, 0.004043) \rangle$
7	$\langle (0.000569, 0.000586, 0.000608, 0.00063, 0.000658, 0.000682, 0.000706, 0.00073) \rangle$	$\langle (0.000559, 0.000576, 0.000594, 0.000612, 0.00063, 0.000648, 0.000666, 0.000684) \rangle$	$\langle (0.000549, 0.000566, 0.000584, 0.000602, 0.00062, 0.000638, 0.000656, 0.000674) \rangle$	$\langle (0.000539, 0.000556, 0.000574, 0.000592, 0.00061, 0.000628, 0.000646, 0.000664) \rangle$
8	$\langle (0.000997, 0.001024, 0.001053, 0.00108, 0.001166, 0.001193, 0.001222, 0.001249) \rangle$	$\langle (0.000987, 0.000997, 0.001007, 0.001017, 0.001027, 0.001037, 0.001047, 0.001057) \rangle$	$\langle (0.000977, 0.000987, 0.000997, 0.001007, 0.001017, 0.001027, 0.001037, 0.001047) \rangle$	$\langle (0.000967, 0.000977, 0.000987, 0.000997, 0.001007, 0.001017, 0.001027, 0.001037) \rangle$
9	$\langle (0.000019, 0.00002, 0.000021, 0.000022, 0.000023, 0.000024, 0.000025, 0.000026) \rangle$	$\langle (0.000019, 0.00002, 0.000021, 0.000022, 0.000023, 0.000024, 0.000025, 0.000026) \rangle$	$\langle (0.000019, 0.00002, 0.000021, 0.000022, 0.000023, 0.000024, 0.000025, 0.000026) \rangle$	$\langle (0.000019, 0.00002, 0.000021, 0.000022, 0.000023, 0.000024, 0.000025, 0.000026) \rangle$
10	$\langle (0.000178, 0.000184, 0.000189, 0.000193, 0.000197, 0.000201, 0.000205, 0.000209) \rangle$	$\langle (0.000178, 0.000184, 0.000189, 0.000193, 0.000197, 0.000201, 0.000205, 0.000209) \rangle$	$\langle (0.000178, 0.000184, 0.000189, 0.000193, 0.000197, 0.000201, 0.000205, 0.000209) \rangle$	$\langle (0.000178, 0.000184, 0.000189, 0.000193, 0.000197, 0.000201, 0.000205, 0.000209) \rangle$
11	$\langle (0.000027, 0.000031, 0.000035, 0.000039, 0.000043, 0.000047, 0.000051, 0.000055) \rangle$	$\langle (0.000027, 0.000031, 0.000035, 0.000039, 0.000043, 0.000047, 0.000051, 0.000055) \rangle$	$\langle (0.000027, 0.000031, 0.000035, 0.000039, 0.000043, 0.000047, 0.000051, 0.000055) \rangle$	$\langle (0.000027, 0.000031, 0.000035, 0.000039, 0.000043, 0.000047, 0.000051, 0.000055) \rangle$
12	$\langle (0.000017, 0.000018, 0.000019, 0.00002, 0.000021, 0.000022, 0.000023, 0.000024) \rangle$	$\langle (0.000017, 0.000018, 0.000019, 0.00002, 0.000021, 0.000022, 0.000023, 0.000024) \rangle$	$\langle (0.000017, 0.000018, 0.000019, 0.00002, 0.000021, 0.000022, 0.000023, 0.000024) \rangle$	$\langle (0.000017, 0.000018, 0.000019, 0.00002, 0.000021, 0.000022, 0.000023, 0.000024) \rangle$
13	$\langle (0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012) \rangle$	$\langle (0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012) \rangle$	$\langle (0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012) \rangle$	$\langle (0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012, 0.000012) \rangle$
14	$\langle (0.000137, 0.000141, 0.000146, 0.000151, 0.000155, 0.000159, 0.000163, 0.000167) \rangle$	$\langle (0.000137, 0.000141, 0.000146, 0.000151, 0.000155, 0.000159, 0.000163, 0.000167) \rangle$	$\langle (0.000137, 0.000141, 0.000146, 0.000151, 0.000155, 0.000159, 0.000163, 0.000167) \rangle$	$\langle (0.000137, 0.000141, 0.000146, 0.000151, 0.000155, 0.000159, 0.000163, 0.000167) \rangle$
15	$\langle (0.000037, 0.000041, 0.000045, 0.000048, 0.000052, 0.000056, 0.00006, 0.000064) \rangle$	$\langle (0.000037, 0.000041, 0.000045, 0.000048, 0.000052, 0.000056, 0.00006, 0.000064) \rangle$	$\langle (0.000037, 0.000041, 0.000045, 0.000048, 0.000052, 0.000056, 0.00006, 0.000064) \rangle$	$\langle (0.000037, 0.000041, 0.000045, 0.000048, 0.000052, 0.000056, 0.00006, 0.000064) \rangle$
16	$\langle (0.000033, 0.000034, 0.000035, 0.000036, 0.000037, 0.000038, 0.000039, 0.00004) \rangle$	$\langle (0.000033, 0.000034, 0.000035, 0.000036, 0.000037, 0.000038, 0.000039, 0.00004) \rangle$	$\langle (0.000033, 0.000034, 0.000035, 0.000036, 0.000037, 0.000038, 0.000039, 0.00004) \rangle$	$\langle (0.000033, 0.000034, 0.000035, 0.000036, 0.000037, 0.000038, 0.000039, 0.00004) \rangle$
17	$\langle (0.000213, 0.000222, 0.00023, 0.000236, 0.000245, 0.000254, 0.000263, 0.000272) \rangle$	$\langle (0.000213, 0.000222, 0.00023, 0.000236, 0.000245, 0.000254, 0.000263, 0.000272) \rangle$	$\langle (0.000213, 0.000222, 0.00023, 0.000236, 0.000245, 0.000254, 0.000263, 0.000272) \rangle$	$\langle (0.000213, 0.000222, 0.00023, 0.000236, 0.000245, 0.000254, 0.000263, 0.000272) \rangle$
18	$\langle (0.000418, 0.000447, 0.000476, 0.000499, 0.000528, 0.000557, 0.000586, 0.000615) \rangle$	$\langle (0.000418, 0.000447, 0.000476, 0.000499, 0.000528, 0.000557, 0.000586, 0.000615) \rangle$	$\langle (0.000418, 0.000447, 0.000476, 0.000499, 0.000528, 0.000557, 0.000586, 0.000615) \rangle$	$\langle (0.000418, 0.000447, 0.000476, 0.000499, 0.000528, 0.000557, 0.000586, 0.000615) \rangle$
19	$\langle (0.000085, 0.000093, 0.000099, 0.000104, 0.00011, 0.000117, 0.000122, 0.000128) \rangle$	$\langle (0.000085, 0.000093, 0.000099, 0.000104, 0.00011, 0.000117, 0.000122, 0.000128) \rangle$	$\langle (0.000085, 0.000093, 0.000099, 0.000104, 0.00011, 0.000117, 0.000122, 0.000128) \rangle$	$\langle (0.000085, 0.000093, 0.000099, 0.000104, 0.00011, 0.000117, 0.000122, 0.000128) \rangle$
20	$\langle (0.000287, 0.000318, 0.000345, 0.000364, 0.000383, 0.000402, 0.000421, 0.00044) \rangle$	$\langle (0.000287, 0.000318, 0.000345, 0.000364, 0.000383, 0.000402, 0.000421, 0.00044) \rangle$	$\langle (0.000287, 0.000318, 0.000345, 0.000364, 0.000383, 0.000402, 0.000421, 0.00044) \rangle$	$\langle (0.000287, 0.000318, 0.000345, 0.000364, 0.000383, 0.000402, 0.000421, 0.00044) \rangle$
21	$\langle (0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011) \rangle$	$\langle (0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011) \rangle$	$\langle (0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011) \rangle$	$\langle (0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011, 0.000011) \rangle$
22	$\langle (0.000067, 0.000072, 0.000076, 0.000079, 0.000083, 0.000087, 0.000091, 0.000095) \rangle$	$\langle (0.000067, 0.000072, 0.000076, 0.000079, 0.000083, 0.000087, 0.000091, 0.000095) \rangle$	$\langle (0.000067, 0.000072, 0.000076, 0.000079, 0.000083, 0.000087, 0.000091, 0.000095) \rangle$	$\langle (0.000067, 0.000072, 0.000076, 0.000079, 0.000083, 0.000087, 0.000091, 0.000095) \rangle$
23	$\langle (0.00015, 0.00018, 0.00022, 0.00025, 0.00028, 0.00031, 0.00034, 0.00037) \rangle$	$\langle (0.00015, 0.00018, 0.00022, 0.00025, 0.00028, 0.00031, 0.00034, 0.00037) \rangle$	$\langle (0.00015, 0.00018, 0.00022, 0.00025, 0.00028, 0.00031, 0.00034, 0.00037) \rangle$	$\langle (0.00015, 0.00018, 0.00022, 0.00025, 0.00028, 0.00031, 0.00034, 0.00037) \rangle$
24	$\langle (0.000025, 0.00003, 0.000035, 0.000039, 0.000043, 0.000047, 0.000051, 0.000055) \rangle$	$\langle (0.000025, 0.00003, 0.000035, 0.000039, 0.000043, 0.000047, 0.000051, 0.000055) \rangle$	$\langle (0.000025, 0.00003, 0.000035, 0.000039, 0.000043, 0.000047, 0.000051, 0.000055) \rangle$	$\langle (0.000025, 0.00003, 0.000035, 0.000039, 0.000043, 0.000047, 0.000051, 0.000055) \rangle$

6.4.1 Generalized vague reliability evaluation of piston manufacturing system

In this section, the solution of generalized vague Kolmogorov's differential equations, shown in Table 6.3 and Table 6.4, obtained by using proposed method, is used to evaluate the generalized vague reliability of piston manufacturing system.

Using the obtained solution of generalized vague Kolmogorov's differential equations, shown in Table 6.3 and Table 6.4, the obtained values of generalized

vague reliability $\tilde{R}_1(t) = \sum_{i=1}^4 \tilde{p}_i(t)$, $\tilde{R}_2(t) = \sum_{i=1}^2 \tilde{p}_i(t)$ and $\tilde{R}(t) = \tilde{R}_1(t) \otimes \tilde{R}_2(t)$ i.e.,
 $\langle [(R_{1(1)}(t), R_{1(2)}(t), R_{1(3)}(t), R_{1(4)}(t), R_{1(5)}(t), R_{1(6)}(t), R_{1(7)}(t), R_{1(8)}(t)); 0.7, 0.8] \rangle$,
 $\langle [(R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t), R_{2(5)}(t), R_{2(6)}(t), R_{2(7)}(t), R_{2(8)}(t)); 0.7, 0.8] \rangle$
and $\langle [(R_1(t), R_2(t), R_3(t), R_4(t), R_5(t), R_6(t), R_7(t), R_8(t)); 0.7, 0.8] \rangle =$
 $\langle [(R_{1(1)}(t), R_{1(2)}(t), R_{1(3)}(t), R_{1(4)}(t), R_{1(5)}(t), R_{1(6)}(t), R_{1(7)}(t), R_{1(8)}(t)); 0.7, 0.8] \rangle$
 $\otimes \langle [(R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t), R_{2(5)}(t), R_{2(6)}(t), R_{2(7)}(t), R_{2(8)}(t)); 0.7, 0.8] \rangle$
of sub-system R_1 , sub-system R_2 and the whole system respectively are shown in Table 6.5.

Table 6.5 Generalized vague reliability of sub-system R_1 , sub-system R_2 and whole system obtained by using proposed method

Generalized vague Reliability →	$\tilde{R}_1(t)$	$\tilde{R}_2(t)$	$\tilde{R}(t)$
Time (t) ↓ (hours)	$\langle [(R_{1(1)}(t), R_{1(2)}(t), R_{1(3)}(t), R_{1(4)}(t), R_{1(5)}(t), R_{1(6)}(t), R_{1(7)}(t), R_{1(8)}(t)); 0.7, 0.8] \rangle$	$\langle [(R_{2(1)}(t), R_{2(2)}(t), R_{2(3)}(t), R_{2(4)}(t), R_{2(5)}(t), R_{2(6)}(t), R_{2(7)}(t), R_{2(8)}(t)); 0.7, 0.8] \rangle$	$\langle [(R_1(t), R_2(t), R_3(t), R_4(t), R_5(t), R_6(t), R_7(t), R_8(t)); 0.7, 0.8] \rangle$
60	$\langle [(0.931553, 0.937761, 0.944033, 0.950336, 0.962812, 0.969115, 0.975387, 0.981595); 0.7, 0.8] \rangle$	$\langle [(0.938083, 0.943471, 0.948847, 0.954246, 0.959622, 0.965013, 0.970389, 0.975773); 0.7, 0.8] \rangle$	$\langle [(0.873874, 0.88475, 0.895743, 0.906854, 0.923956, 0.935209, 0.946505, 0.957814); 0.7, 0.8] \rangle$
180	$\langle [(0.93003, 0.935966, 0.941976, 0.948029, 0.959753, 0.965806, 0.971816, 0.977752); 0.7, 0.8] \rangle$	$\langle [(0.937484, 0.9428, 0.948048, 0.953313, 0.958562, 0.963812, 0.969061, 0.97437); 0.7, 0.8] \rangle$	$\langle [(0.871888, 0.882429, 0.893038, 0.903768, 0.919983, 0.930855, 0.941749, 0.952692); 0.7, 0.8] \rangle$
240	$\langle [(0.929642, 0.935497, 0.941403, 0.947355, 0.958751, 0.964704, 0.97061, 0.976464); 0.7, 0.8] \rangle$	$\langle [(0.937467, 0.942768, 0.94798, 0.953199, 0.958411, 0.963613, 0.968825, 0.974117); 0.7, 0.8] \rangle$	$\langle [(0.871509, 0.881957, 0.892431, 0.903018, 0.918878, 0.929601, 0.940351, 0.951119); 0.7, 0.8] \rangle$
360	$\langle [(0.928996, 0.93474, 0.940472, 0.946245, 0.957077, 0.96285, 0.968582, 0.974326); 0.7, 0.8] \rangle$	$\langle [(0.937464, 0.942746, 0.94791, 0.953061, 0.958225, 0.963355, 0.968519, 0.973789); 0.7, 0.8] \rangle$	$\langle [(0.8709, 0.881222, 0.891483, 0.901829, 0.917095, 0.927566, 0.93809, 0.948788); 0.7, 0.8] \rangle$

The variation in reliability of piston manufacturing system at $t = 60$ hours, $t = 180$ hours, $t = 240$ hours and $t = 360$ hours is shown in Figure 6.2 to Figure 6.5 respectively.

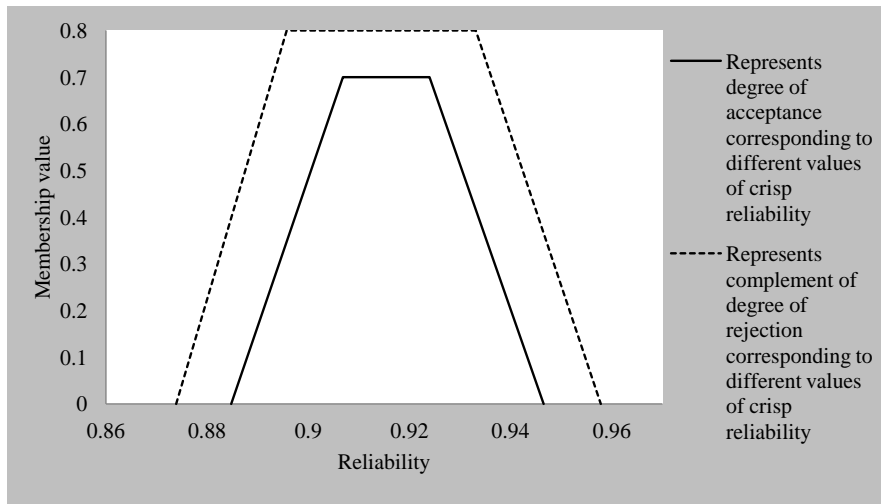


Figure 6.2 Generalized trapezoidal vague set representing the generalized vague reliability of piston manufacturing system at $t = 60$ hours

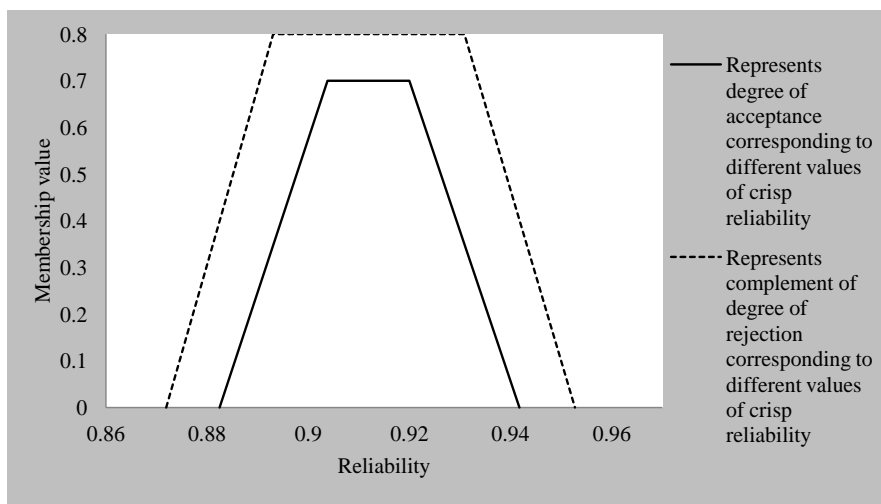


Figure 6.3 Generalized trapezoidal vague set representing the generalized vague reliability of piston manufacturing system at $t = 180$ hours

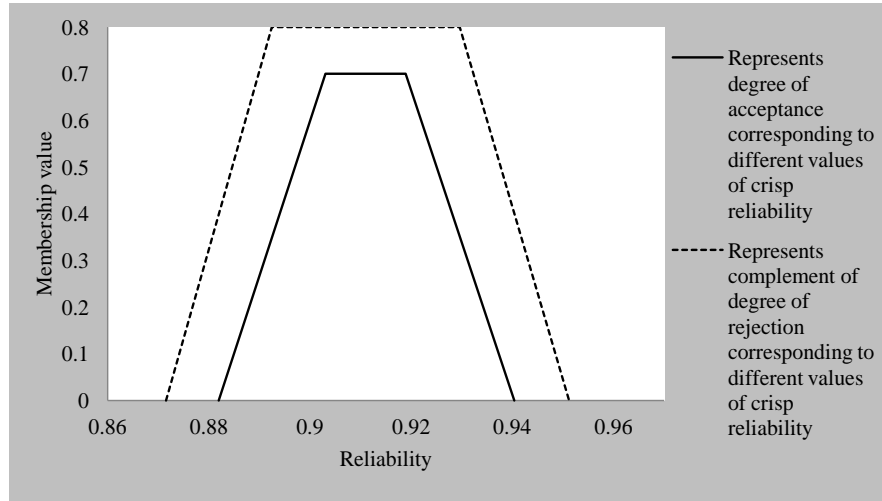


Figure 6.4 Generalized trapezoidal vague set representing the generalized vague reliability of piston manufacturing system at $t = 240$ hours

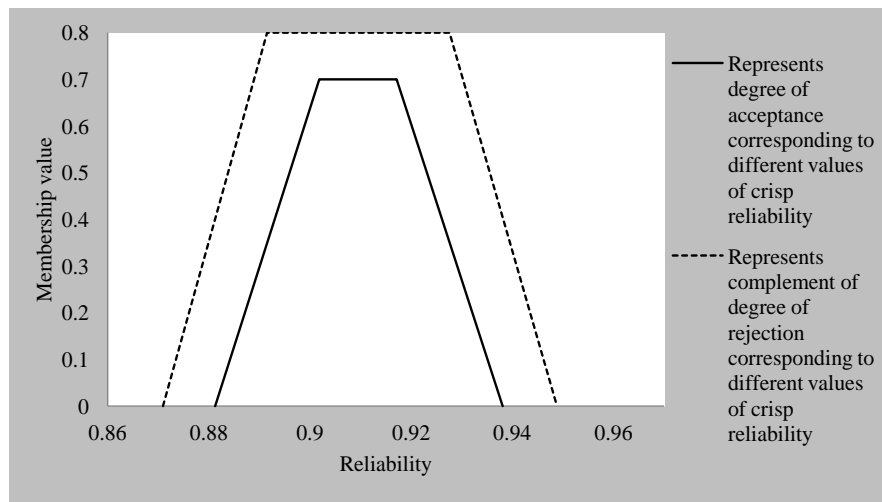


Figure 6.5 Generalized trapezoidal vague set representing the generalized vague reliability of piston manufacturing system at $t = 360$ hours

6.5 Physical interpretation of results

In this section, the obtained generalized vague reliability of piston manufacturing system at $t = 360$ hours is physically interpreted.

It is obvious from Figure 6.5 that

1. The reliability of piston manufacturing system at $t = 360$ hours lies between 0.8709 and 0.948788.
2. The degree of acceptance of the experts about the statement that reliability of piston manufacturing system at $t = 360$ hours lies between 0.901829 to 0.917095 is 0.7.
3. The degree of rejection of the experts about the statement that reliability of piston manufacturing system at $t = 360$ hours lies between 0.901829 to 0.917095 is 0.2.
4. The degree of hesitation of the experts about the statement that reliability of piston manufacturing system at $t = 360$ hours lies between 0.901829 to 0.917095 is 0.1.
5. The degree of acceptance of the experts about the statement that the reliability of piston manufacturing system at $t = 360$ hours lies between 0.8709 to 0.881222 is 0. The degree of rejection $\nu_{\tilde{R}}(r)$ and the degree of hesitation $1 - \mu_{\tilde{R}}(r) - \nu_{\tilde{R}}(r)$ for the values of reliability r , $r \in [0.870900, 0.881222]$ can be obtained as follows:

$$\nu_{\tilde{R}}(r) = 1 - \frac{0.8(r-0.870900)}{(0.010322)}, 0.870900 \leq r \leq 0.881222$$

$$1 - \mu_{\tilde{R}}(r) - \nu_{\tilde{R}}(r) = \frac{0.8(r-0.870900)}{(0.010322)}, 0.870900 \leq r \leq 0.881222$$

The degree of acceptance of the experts about the statement that the reliability of piston manufacturing system at $t = 360$ hours lies between 0.93809 to 0.948788 is 0. The degree of rejection $\nu_{\tilde{R}}(r)$ and the degree of hesitation $1 - \mu_{\tilde{R}}(r) - \nu_{\tilde{R}}(r)$ for the values of reliability r , $r \in [0.938090, 0.948788]$ can be obtained as follows:

$$\nu_{\tilde{R}}(r) = 1 - \frac{0.8(0.948788-r)}{(0.010698)}, 0.938090 \leq r \leq 0.948788$$

$$1 - \mu_{\tilde{R}}(r) - \nu_{\tilde{R}}(r) = \frac{0.8(0.948788-r)}{(0.010698)}, 0.938090 \leq r \leq 0.948788$$

The degree of acceptance $\mu_{\tilde{R}}(r)$, degree of rejection $\nu_{\tilde{R}}(r)$ and degree of hesitation $1 - \mu_{\tilde{R}}(r) - \nu_{\tilde{R}}(r)$ for the remaining values of reliability r , $r \in [0.881222, 0.938090]$ can be obtained as follows:

$$\mu_{\tilde{R}}(r) = \begin{cases} \frac{0.7(r-0.881222)}{(0.020607)}, & 0.881222 \leq r < 0.901829 \\ 0.7, & 0.901829 \leq r \leq 0.917095 \\ \frac{0.7(0.938090-r)}{(0.020995)}, & 0.917095 < r \leq 0.938090 \end{cases}$$

$$\nu_{\tilde{R}}(r) = \begin{cases} 1 - \frac{0.8(r-0.881222)}{(0.020607)}, & 0.881222 \leq r < 0.901829 \\ 0.2, & 0.901829 \leq r \leq 0.917095 \\ 1 - \frac{0.8(0.938090-r)}{(0.020995)}, & 0.917095 < r \leq 0.938090 \end{cases}$$

and

$$1 - \mu_{\tilde{R}}(r) - \nu_{\tilde{R}}(r) = \begin{cases} \frac{0.8(r-0.881222)}{(0.020607)} - \frac{0.7(r-0.881222)}{(0.020607)}, & 0.881222 \leq r < 0.901829 \\ 0.1, & 0.901829 \leq r \leq 0.917095 \\ \frac{0.8(0.938090-r)}{(0.020995)} - \frac{0.7(0.938090-r)}{(0.020995)}, & 0.917095 < r \leq 0.938090 \end{cases}$$

Also, the variation in reliability of piston manufacturing system with time at presumption level 0 is shown in Figure 6.6.

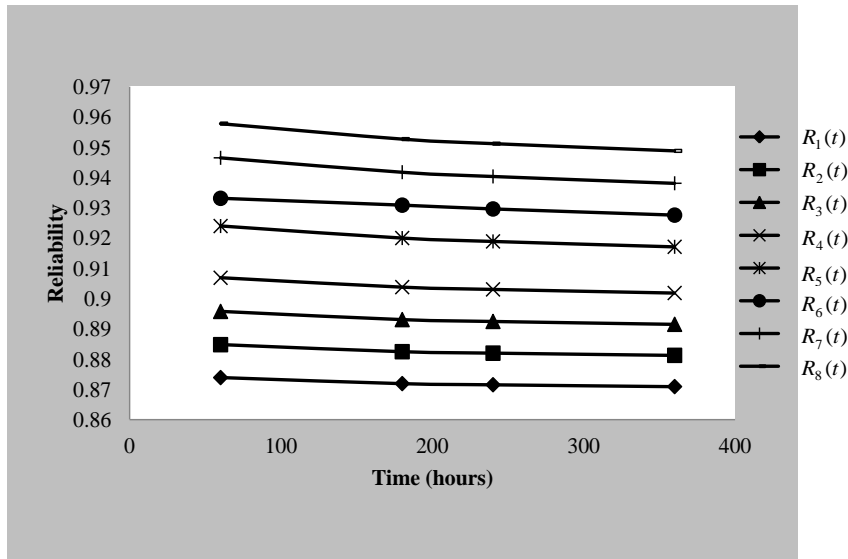


Figure 6.6 Variation in reliability with time at presumption level 0

6.6 Advantage of *JMD* generalized trapezoidal vague set over existing generalized trapezoidal vague set

To show the advantage of *JMD* generalized trapezoidal vague set over existing generalized trapezoidal vague set, the generalized vague Kolmogorov's differential equations, for sub-system R_1 and sub-system R_2 , are solved by using proposed method with existing generalized trapezoidal vague set and the obtained results are shown in Table 6.6 and Table 6.7 respectively.

Table 6.7 Solution of generalized vague Kolmogorov's differential equations for sub-system R_2 obtained by using proposed method with existing generalized trapezoidal vague set

j	$\tilde{p}_j(t)$ for $t=60$	$\tilde{p}_j(t)$ for $t=180$	$\tilde{p}_j(t)$ for $t=240$	$\tilde{p}_j(t)$ for $t=360$
	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t), p_{j(5)}(t), p_{j(6)}(t), p_{j(7)}(t), p_{j(8)}(t)]; 0.7, 0.8 \rangle$ $\langle [0.906646; 0.890069; 0.878883; 0.872596; 0.868478; 0.863538; 0.857282; 0.853516]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t), p_{j(5)}(t), p_{j(6)}(t), p_{j(7)}(t), p_{j(8)}(t)]; 0.7, 0.8 \rangle$ $\langle [0.906066; 0.889368; 0.878085; 0.871754; 0.867574; 0.862596; 0.856347; 0.852709]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t), p_{j(5)}(t), p_{j(6)}(t), p_{j(7)}(t), p_{j(8)}(t)]; 0.7, 0.8 \rangle$ $\langle [0.906049; 0.889354; 0.878072; 0.871739; 0.867563; 0.862587; 0.856347; 0.852704]; 0.7, 0.8 \rangle$	$\langle [p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t), p_{j(5)}(t), p_{j(6)}(t), p_{j(7)}(t), p_{j(8)}(t)]; 0.7, 0.8 \rangle$ $\langle [0.906046; 0.889352; 0.87807, 0.871738, 0.867562; 0.862585; 0.856347; 0.852704]; 0.7, 0.8 \rangle$
1	$\langle [0.031437; 0.05023; 0.064096; 0.073859; 0.086078; 0.094294; 0.103557; 0.110592]; 0.7, 0.8 \rangle$	$\langle [0.031418; 0.050192; 0.064039; 0.073789; 0.085591; 0.094194; 0.103448; 0.110489]; 0.7, 0.8 \rangle$	$\langle [0.031418; 0.050191; 0.064039; 0.073788; 0.085589; 0.094193; 0.103447; 0.110489]; 0.7, 0.8 \rangle$	$\langle [0.031418; 0.050191; 0.064039; 0.073788; 0.085589; 0.094192; 0.103447; 0.110489]; 0.7, 0.8 \rangle$
2	$\langle [0.001442; 0.002308; 0.002944; 0.003396; 0.003974; 0.004362; 0.004795; 0.00513]; 0.7, 0.8 \rangle$	$\langle [0.001441; 0.002306; 0.002941; 0.003396; 0.003969; 0.004357; 0.004789; 0.005125]; 0.7, 0.8 \rangle$	$\langle [0.001441; 0.002306; 0.002941; 0.003396; 0.003969; 0.004356; 0.004789; 0.005125]; 0.7, 0.8 \rangle$	$\langle [0.001441; 0.002306; 0.002941; 0.003396; 0.003969; 0.004356; 0.004789; 0.005125]; 0.7, 0.8 \rangle$
3	$\langle [0.000227; 0.000336; 0.000426; 0.000481; 0.000562; 0.000625; 0.000686; 0.000739]; 0.7, 0.8 \rangle$	$\langle [0.000227; 0.000336; 0.000425; 0.00048; 0.000561; 0.000624; 0.000685; 0.000739]; 0.7, 0.8 \rangle$	$\langle [0.000227; 0.000336; 0.000425; 0.00048; 0.000561; 0.000624; 0.000685; 0.000739]; 0.7, 0.8 \rangle$	$\langle [0.000227; 0.000336; 0.000425; 0.00048; 0.000561; 0.000624; 0.000685; 0.000739]; 0.7, 0.8 \rangle$
4	$\langle [0.000068; 0.000089; 0.00011; 0.000119; 0.000139; 0.000159; 0.000174; 0.00018]; 0.7, 0.8 \rangle$	$\langle [0.000068; 0.000088; 0.00011; 0.000119; 0.000139; 0.000159; 0.000174; 0.00018]; 0.7, 0.8 \rangle$	$\langle [0.000068; 0.000088; 0.00011; 0.000119; 0.000139; 0.000159; 0.000174; 0.00018]; 0.7, 0.8 \rangle$	$\langle [0.000068; 0.000089; 0.00011; 0.000119; 0.000139; 0.000159; 0.000174; 0.00018]; 0.7, 0.8 \rangle$
5	$\langle [0.00264; 0.004089; 0.005269; 0.006013; 0.007113; 0.007977; 0.008861; 0.009753]; 0.7, 0.8 \rangle$	$\langle [0.002615; 0.004078; 0.005259; 0.006008; 0.007007; 0.007875; 0.008758; 0.009652]; 0.7, 0.8 \rangle$	$\langle [0.002615; 0.004078; 0.005259; 0.006008; 0.007007; 0.007875; 0.008758; 0.009652]; 0.7, 0.8 \rangle$	$\langle [0.002615; 0.004078; 0.005259; 0.006008; 0.007007; 0.007875; 0.008758; 0.009652]; 0.7, 0.8 \rangle$
6	$\langle [0.000015; 0.000019; 0.000024; 0.000026; 0.000031; 0.000035; 0.000039; 0.000043]; 0.7, 0.8 \rangle$	$\langle [0.000015; 0.000019; 0.000024; 0.000026; 0.000031; 0.000035; 0.000039; 0.000043]; 0.7, 0.8 \rangle$	$\langle [0.000015; 0.000019; 0.000024; 0.000026; 0.000031; 0.000035; 0.000039; 0.000043]; 0.7, 0.8 \rangle$	$\langle [0.000015; 0.000019; 0.000024; 0.000026; 0.000031; 0.000035; 0.000039; 0.000043]; 0.7, 0.8 \rangle$
7	$\langle [0.000005; 0.000013; 0.0000215; 0.000288; 0.000394; 0.000476; 0.000579; 0.000665]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000013; 0.000021; 0.000287; 0.000393; 0.000476; 0.000579; 0.000664]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000013; 0.000021; 0.000287; 0.000393; 0.000476; 0.000579; 0.000664]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000013; 0.000021; 0.000287; 0.000393; 0.000476; 0.000579; 0.000664]; 0.7, 0.8 \rangle$
8	$\langle [0.000008; 0.000019; 0.000031; 0.000041; 0.000056; 0.000068; 0.000083; 0.000096]; 0.7, 0.8 \rangle$	$\langle [0.000008; 0.000019; 0.000031; 0.000041; 0.000056; 0.000068; 0.000083; 0.000096]; 0.7, 0.8 \rangle$	$\langle [0.000008; 0.000019; 0.000031; 0.000041; 0.000056; 0.000068; 0.000083; 0.000096]; 0.7, 0.8 \rangle$	$\langle [0.000008; 0.000019; 0.000031; 0.000041; 0.000056; 0.000068; 0.000083; 0.000096]; 0.7, 0.8 \rangle$
9	$\langle [0.000002; 0.000005; 0.000008; 0.00001; 0.000014; 0.000017; 0.000021; 0.000023]; 0.7, 0.8 \rangle$	$\langle [0.000002; 0.000005; 0.000008; 0.00001; 0.000014; 0.000017; 0.000021; 0.000023]; 0.7, 0.8 \rangle$	$\langle [0.000002; 0.000005; 0.000008; 0.00001; 0.000014; 0.000017; 0.000021; 0.000023]; 0.7, 0.8 \rangle$	$\langle [0.000002; 0.000005; 0.000008; 0.00001; 0.000014; 0.000017; 0.000021; 0.000023]; 0.7, 0.8 \rangle$
10	$\langle [0.000888; 0.000224; 0.000374; 0.000496; 0.000689; 0.000853; 0.00105; 0.001245]; 0.7, 0.8 \rangle$	$\langle [0.000888; 0.000224; 0.000374; 0.000496; 0.000689; 0.000853; 0.00105; 0.001245]; 0.7, 0.8 \rangle$	$\langle [0.000888; 0.000224; 0.000374; 0.000496; 0.000689; 0.000853; 0.00105; 0.001245]; 0.7, 0.8 \rangle$	$\langle [0.000888; 0.000224; 0.000374; 0.000496; 0.000689; 0.000853; 0.00105; 0.001245]; 0.7, 0.8 \rangle$
11	$\langle [0.000005; 0.000011; 0.000018; 0.000022; 0.000031; 0.000039; 0.000047; 0.000052]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000011; 0.000018; 0.000022; 0.000031; 0.000038; 0.000047; 0.000052]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000011; 0.000018; 0.000022; 0.000031; 0.000038; 0.000047; 0.000052]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000011; 0.000018; 0.000022; 0.000031; 0.000038; 0.000047; 0.000052]; 0.7, 0.8 \rangle$
12	$\langle [0.000375; 0.00098; 0.001617; 0.002169; 0.002969; 0.003591; 0.004368; 0.005014]; 0.7, 0.8 \rangle$	$\langle [0.000377; 0.000982; 0.001618; 0.002169; 0.002968; 0.003588; 0.004364; 0.005009]; 0.7, 0.8 \rangle$	$\langle [0.000377; 0.000982; 0.001618; 0.002169; 0.002967; 0.003588; 0.004364; 0.005009]; 0.7, 0.8 \rangle$	$\langle [0.000377; 0.000982; 0.001618; 0.002169; 0.002967; 0.003588; 0.004364; 0.005009]; 0.7, 0.8 \rangle$
13	$\langle [0.000005; 0.000011; 0.000018; 0.000022; 0.000031; 0.000039; 0.000047; 0.000052]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000011; 0.000018; 0.000022; 0.000031; 0.000038; 0.000047; 0.000052]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000011; 0.000018; 0.000022; 0.000031; 0.000038; 0.000047; 0.000052]; 0.7, 0.8 \rangle$	$\langle [0.000005; 0.000011; 0.000018; 0.000022; 0.000031; 0.000038; 0.000047; 0.000052]; 0.7, 0.8 \rangle$

It is obvious from Table 6.6 and Table 6.7 that on solving the generalized vague Kolmogorov's differential equations by using the proposed method with existing generalized trapezoidal vague set, in the obtained values of $\tilde{p}_j(t)$ i.e., $\langle [(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t), p_{j(5)}(t), p_{j(6)}(t), p_{j(7)}(t), p_{j(8)}(t)); 0.7, 0.8] \rangle$, the restriction $p_{1(1)}(t) \leq p_{1(2)}(t) \leq p_{1(3)}(t) \leq p_{1(4)}(t) \leq p_{1(5)}(t) \leq p_{1(6)}(t) \leq p_{1(7)}(t) \leq p_{1(8)}(t)$ is not satisfying. So, $\tilde{p}_j(t)$ does not define generalized trapezoidal vague set for $j = 1$ and it is obvious from Table 6.3 and Table 6.4 that on solving the same generalized vague Kolmogorov's differential equations by using the proposed method with *JMD* generalized trapezoidal vague set, in the obtained values of $\langle [(p_{j(1)}(t), p_{j(2)}(t), p_{j(3)}(t), p_{j(4)}(t), p_{j(5)}(t), p_{j(6)}(t), p_{j(7)}(t), p_{j(8)}(t)); 0.7, 0.8] \rangle$, the restriction $p_{j(1)}(t) \leq p_{j(2)}(t) \leq p_{j(3)}(t) \leq p_{j(4)}(t) \leq p_{j(5)}(t) \leq p_{j(6)}(t) \leq p_{j(7)}(t) \leq p_{j(8)}(t)$ is satisfying for all values of j . Hence, it is better to use *JMD* generalized trapezoidal vague set instead of existing generalized trapezoidal vague set.

6.7 Conclusions

On the basis of presented study, it is concluded that there is no method in the literature for solving generalized vague differential equations and no one have used Markov model to evaluate the generalized vague reliability of industrial systems. In this chapter, a new method is proposed for solving generalized vague differential equations which can be used for solving generalized vague Kolmogorov's differential equations to evaluate the generalized vague reliability of industrial systems.

Chapter 7

FUTURE SCOPE

The following may be treated as future directions:

- (1) To illustrate the proposed methods, the crisp failure rates and repair rates are converted into fuzzy failure rates and repair rates, vague failure rates and repair rates, generalized vague failure rates and repair rates. In future, it may be tried to construct fuzzy failure rates and repair rates, vague failure rates and repair rates, generalized vague failure rates and repair rates on the basis of expert opinions then the fuzzy reliability, vague reliability and generalized vague reliability of the chosen industrial system can be obtained by using the proposed methods.
- (2) There are several other important factors like availability, maintainability, risk analysis etc. which are not evaluated in vague environment. In future, it may be tried to develop the methods for the same.

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