

**A STUDY ON DERIVATIONS IN RINGS WITH ENGEL  
CONDITIONS**

Submitted in partial fulfillment of the requirements  
for the award of the degree of

**MASTER OF SCIENCE  
IN  
MATHEMATICS AND COMPUTING**

Submitted by

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**July -2016**

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# Certificate

I hereby declare that the work which is being presented here in the dissertation entitled "*A STUDY ON DERIVATIONS IN RINGS WITH ENGEL CONDITIONS*", which is being submitted by **Miss. Prashu** (Roll No. 301403011), in partial fulfillment of the requirements for the award of degree of **Master of Science** in "Mathematics and Computing", to the School of Mathematics, Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. M. S Khan**, Assistant Professor, SOM during the period from January 2016 to July 2016. The matter presented in this dissertation has not been submitted in any other University/Institute for the award of any degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

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## Acknowledgement

First and foremost, praises and thanks to the **God**, the Almighty, for His showers of blessings throughout my thesis work to complete the thesis successfully. He has given me the power to believe in myself and pursue my dreams.

I would like to express my deep sense of respect and sincere gratitude to my esteemed supervisor, **Dr. Mohammad Salahuddin Khan**, Assistant Professor School Of Mathematics, Thapar University, Patiala for his excellent guidance, sympathetic attitude, constant encouragement, logical suggestions, moral support all through the pursuance of this work. I learnt a lot from him, which I am sure will be useful in different stages of my life.

I wish to express my sincere thanks to **Dr. A. K. Lal**, Head, School Of Mathematics, Thapar University, Patiala and to the entire faculty and staff members for their direct and indirect help, cooperation, love and affection.

I am extremely grateful to my parents for their love, prayers, caring and sacrifices for educating and preparing me for my future. I express my thanks to my brother for his support and valuable prayers.

My Special thanks goes to my friends who have been always there for me, motivating me. I thank them all for their great company.

At last, I would like to pay my warmest tribute to my glorious and esteemed institute, **Thapar University, Patiala**.

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# Preface

Ring theory is a showpiece of mathematical unification, bringing together several branches of the subject and creating a powerful machine for the study of problems of considerable historical and mathematical importance. A ring will be defined as an abstract structure with a commutative addition, and a multiplication which may or may not be commutative. This distinction yields two quite different theories: the theory of respectively commutative or non-commutative rings. Rings with derivations are not the kind of subject that undergoes tremendous revolutions. However, this has been studied by many algebraists in the last 50 years, specially the relationships between derivations and the structure of rings. A classical problem of ring theory is to find combinations of properties that force a ring to be commutative.

The present thesis entitled “*A Study On Derivations In Rings With Engel Conditions*”. This exposition comprises two chapters and each chapter is subdivided into various sections.

Chapter 1 contains preliminary notions, basic definitions, examples and some important well-known results related to our study which may be needed for the development of the subject in the next chapter. This chapter is an attempt to make this thesis as self contained as possible. However, the basic knowledge of ring theory has been pre-assumed.

Chapter 2 deals with the study of Engel conditions with derivations in prime and semiprime rings. There is a technique for investigating commutativity of rings is the use of additive mappings like derivations and automorphisms of the ring  $R$ . The study of such mappings was initiated by Posner. In [41] Posner proved that if a prime ring  $R$  admits a nonzero derivation  $d$  such that  $[d(x), x] \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative. A number of authors have extended this theorem of Posner. Then in [34] Lee generalized Posner’s result. Lee states that if  $\text{char}(R) \neq 2$  and  $[d(x), x] \in Z$  for all  $x$  in a noncentral Lie ideal of  $R$ , then  $R$  is commutative.

In this chapter, I have studied the generalization of above results which is proved by Yu Wang [45] from the commutator type to the Engel condition. An additive mapping  $d : R \rightarrow R$  is called a derivation if  $d(xy) = d(x)y + xd(y)$  holds for all  $x, y \in R$ .  $R$  is always a prime ring with the center  $Z(R)$ .  $C$  is its extended centroid and  $Q$  is its Martindale quotient ring. For any  $x, y \in R$ , we set  $[x, y]_1 = [x, y] = xy - yx$  and  $[x, y]_n = [[x, y]_{n-1}, y]$ , where  $n > 1$  is an integer. Note that the engel condition is a polynomial  $[x, y]_n = \sum_{i=0}^n (-1)^i \binom{n}{i} y^i x y^{n-i}$ .

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# Chapter 1

## Some preliminaries

### 1.1 Introduction

The object of the present chapter is to introduce basic definitions, preliminary notions and some key results which we shall require for the development of the subject matter in the present thesis. The elementary knowledge of groups, rings, ideals, fields, homomorphism etc. have been pre-assumed. Throughout the thesis, unless otherwise mentioned,  $R$  will denote an associative ring (may be without identity) containing at least two elements. Most of the material included in this chapter occurs in standard literature like, Beidar et al. [2], Herstein [20], [21], Jacobson [22], [23].

### 1.2 Some definitions and examples

In the present section we give a brief exposition of some important terminologies in the theory of rings . Examples and counter examples are also included in this section to make the matter presented in the section self explanatory and to give a clear sketch of the various notions. We start our discussion with the following definitions:

**Definition 1.2.1** (Prime ideal). An ideal  $P$  of  $R$  is called a *prime ideal* of  $R$  if for any two ideals  $A$  and  $B$  of  $R$ ,  $AB \subseteq P$  implies  $A \subseteq P$  or  $B \subseteq P$ .

**Remark 1.2.1.** Equivalently, an ideal  $P$  in a ring  $R$  is prime if and only if any one of the following holds:

- (i) If  $a, b \in R$  such that  $aRb \subseteq P$ , then  $a \in P$  or  $b \in P$ .
- (ii) If  $(a)$  and  $(b)$  are principal ideals in  $R$  such that  $(a)(b) \subseteq P$ , then  $a \in P$  or  $b \in P$ .
- (iii) If  $U$  and  $V$  are left (right) ideals in  $R$  such that  $UV \subseteq P$ , then  $U \subseteq P$  or  $V \subseteq P$ .

**Definition 1.2.2** (Prime ring). A ring  $R$  is said to be a *prime ring* if and only if the zero ideal is a prime ideal in  $R$ .

**Remark 1.2.2.** Equivalently, a ring  $R$  is a prime ring if and only if any one of the following holds:

(i) If  $A$  and  $B$  are ideals in  $R$  such that  $AB = (0)$ , then  $A = (0)$  or  $B = (0)$ .

(ii) If  $a, b \in R$  such that  $aRb = (0)$ , then  $a = 0$  or  $b = 0$ .

**Definition 1.2.3** (Semiprime ideal). An ideal  $P$  in a ring  $R$  is said to be a *semiprime ideal* in  $R$  if for every ideal  $I$  of  $R$ ,  $I^2 \subseteq P$  implies  $I \subseteq P$ .

**Remark 1.2.3.** (i) A prime ideal is necessarily semiprime, but the converse need not be true in general.

(ii) Intersection of prime (semiprime) ideals is semiprime. Thus, in the ring  $\mathbb{Z}$  of integers, ideal  $(2) \cap (3) = (6)$  is semiprime which is not prime.

**Definition 1.2.4** (Nilpotent). An element  $a$  of a ring  $R$  is said to be a *nilpotent* if there exists a positive integer  $n$  such that  $a^n = 0$ .

**Remark 1.2.4.** It is trivial that the zero of a ring is nilpotent. Moreover, every nilpotent element is necessarily a divisor of zero. For if  $a \neq 0$  and  $n$  is the smallest positive integer such that  $a^n = 0$ , then  $n > 1$  and  $a(a^{n-1}) = 0$  with  $a^{n-1} \neq 0$ .

**Definition 1.2.5** (Nilpotent ideal). If  $A$  is an ideal such that  $A^n = (0)$  for some positive integer  $n$ ,  $A$  is said to be a *nilpotent ideal*.

**Definition 1.2.6** (Semiprime ring). A ring  $R$  which has no nonzero nilpotent ideal is said to be a *semiprime ring*.

**Remark 1.2.5.** (i) A ring  $R$  is semiprime if and only if for any  $a \in R$ ,  $aRa = (0)$  implies that  $a = 0$ .

(ii) The *radical* of  $R$ , denoted by  $\text{rad}(R)$ , is the intersection of all maximal ideals of  $R$ .

(iii) If  $\text{rad}(R) = (0)$ , then  $R$  is called semisimple.

**Definition 1.2.7** (R-Module). Let  $M$  be an additively written abelian group and  $R$  is a ring. Then  $M$  is said to be a (right) $R$ -module if a law of composition of  $M \times R$  into  $M$  is defined (that is, if  $x \in M, a \in R, xa$  is a uniquely determined element of  $M$ ) such that the following are true for  $x, y \in M$  and  $a, b \in R$ :

$$(i) \quad (x + y)a = xa + ya$$

$$(ii) \quad x(a + b) = xa + xb$$

$$(iii) \quad x(ab) = (xa)b.$$

**Definition 1.2.8** (Dense right (left) ideal). A right (resp. left) ideal  $I$  of  $R$  is said to be *dense right* (resp. *left*) *ideal* if for any  $0 \neq r_1 \in R$ ,  $r_2 \in R$  there exists  $r \in R$  such that  $r_1r \neq 0$  and  $r_2r \in I$  (resp.  $rr_1 \neq 0$  and  $rr_2 \in I$ ).

The collection of all dense right ideal of  $R$  will be denoted by  $D(R)$ .

**Remark 1.2.6.** Let  $R$  be a semiprime ring, and  $I, J, S \in D(R)$ . If  $f : I \rightarrow R$  is a homomorphism of right  $R$ -modules, then

$$(i) \quad R \in D(R).$$

$$(ii) \quad f^{-1}(J) = \{a \in I \mid f(a) \in J\} \in D(R).$$

$$(iii) \quad I \cap J \in D(R).$$

$$(iv) \quad IJ \in D(R).$$

**Definition 1.2.9** (Maximal right ring of quotients). Let  $R$  be a semiprime ring,  $\mathfrak{S}$  be the set of all pairs  $(U, f)$  where  $U \neq (0)$  is a dense right ideal of  $R$  and  $f : U \rightarrow R$  is a right  $R$ -module mapping of  $U$  into  $R$ . Define a relation ‘ $\sim$ ’ on  $\mathfrak{S}$  such that  $(U, f) \sim (V, g)$  if  $f = g$  on some dense right ideal  $W \neq (0)$  of  $R$  where  $W \subseteq U \cap V$ . It can be easily check that  $\sim$  is an equivalence relation on  $\mathfrak{S}$ . Let  $Q$  be the set of equivalence classes of  $\mathfrak{S}$ . Denote the equivalence class determined by  $(U, f)$  as  $\tilde{f}$ . For  $\tilde{f} = cl(U, f)$ ,  $\tilde{g} = cl(V, g) \in Q$ , define addition and multiplication on  $Q$  as  $\tilde{f} + \tilde{g} = cl(U \cap V, f + g)$  and  $\tilde{f} \cdot \tilde{g} = cl(g^{-1}(U), fg)$ . Thus  $Q$  forms an associative ring with identity relative to above defined operations known as *maximal right ring of quotients* or *right Utumi quotient ring of  $R$* .

**Definition 1.2.10** (Center of ring). The center of a ring  $R$  is the set of all those elements of  $R$  which commute with every element of  $R$  and is denoted as  $Z(R)$  i.e.,  $Z(R) = \{x \in R \mid xr = rx \text{ for all } r \in R\}$ .

Thus, a ring  $R$  is commutative if and only if  $Z(R) = R$ .

**Definition 1.2.11** (Centralizer). Let  $S$  be a nonempty subset of  $R$ . Then the *centralizer*  $C_R(S)$  of  $S$  in  $R$ , is defined by  $C_R(S) = \{x \in R \mid sx = xs \text{ for all } s \in S\}$ .

If  $a \in C_R(S)$ , then we say that  $a$  centralizes  $S$ . Evidently,  $C_R(R) = Z(R)$ .

**Remark 1.2.7.** (i) The center of a prime ring is free from zero divisors.

(ii) The center of a semiprime ring contains no nonzero nilpotent element.

**Definition 1.2.12** (Extended centroid). The center  $C$  of  $Q$  is known as extended centroid of  $R$ .

**Remark 1.2.8.** If  $R$  is a prime ring, then extended centroid of  $R$  is a field.

**Definition 1.2.13** (Central closure). Let  $R$  be a semiprime ring. Then the subring  $RC$  of  $Q$  is said to be the *central closure* of  $R$ . Further,  $R$  is called *centrally closed* if it coincides with its central closure i.e.,  $R = RC$ .

**Definition 1.2.14** (Characteristic of a ring). Let  $R$  be a ring. If there exists a positive integer  $n$  such that  $nx = 0$  for all  $x \in R$ , then the smallest positive integer with this property is called the characteristic of the ring  $R$  and is denoted by  $\text{char}(R)$ . If no such positive integer exists, then  $R$  is said to be of characteristic zero.

**Definition 1.2.15** (Maximal ideal). An ideal  $A$  in a ring  $R$  is called a *maximal ideal* if  $A \neq R$  and there exists no ideal  $B$  in  $R$  such that  $A \subset B \subset R$ .

**Definition 1.2.16** (Torsion free element). An element  $x \in R$  is called  $n$ -torsion free if  $nx = 0$  implies  $x = 0$ .

If  $nx = 0$  implies  $x = 0$  for all  $x \in R$ , we say that the ring  $R$  is  $n$ -torsion free.

**Definition 1.2.17** (Lie product). Let  $R$  be a ring. Then for all  $x, y \in R$ , the *Lie product* (*commutator*) is  $[x, y] = xy - yx$ .

**Remark 1.2.9.** For any  $x, y, z \in R$ , the following identities are obvious,

(i)  $[xy, z] = x[y, z] + [x, z]y$ ;

(ii)  $[x, yz] = [x, y]z + y[x, z]$ ;

(iii)  $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ ; (Jacobi identity).

**Remark 1.2.10.** An Engel condition is a polynomial  $[x, y]_n = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} y^i x y^{n-1-i}$  in noncommutative indeterminates  $x, y$ .

**Definition 1.2.18** (Derivation). An additive mapping  $d : R \rightarrow R$  is said to be a *derivation* on  $R$  if  $d(xy) = d(x)y + xd(y)$  holds for all  $x, y \in R$ .

**Example 1.2.1.** The most natural example of a non trivial derivation is the usual differentiation on the ring  $F[x]$  of polynomials defined over a field  $F$ .

**Definition 1.2.19** (Inner derivation). For a fixed  $a \in R$ , define  $d_a : R \rightarrow R$  such that  $d_a(x) = [a, x]$  for all  $x \in R$ . Then  $d$  is called an *inner derivation* of  $R$  associated with  $a$  and usually denoted by  $I_a$ .

It is obvious to see that every inner derivation on a ring  $R$  is a derivation. But the converse need not be true in general.

**Example 1.2.2.** Let  $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ . Define a mapping  $d : R \rightarrow R$  as follows:

$$d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} \text{ for all } a, b, c \in \mathbb{Z}.$$

It can be easily seen that  $d$  is a derivation on  $R$  which is not an inner derivation on  $R$ .

**Definition 1.2.20** (Tensor product). Let  $K$  be a commutative ring with identity, let  $V$  be a right  $K$ -module and let  $W$  be a left  $K$ -module. For  $P$  any additive abelian group we say that a map  $\phi : V \times W \rightarrow P$  is *balanced* if it is biadditive and satisfies  $(v\alpha, w)^\phi = (v, \alpha w)^\phi$ ,  $v \in V$ ,  $w \in W$ ,  $\alpha \in K$ . An abelian group  $T$  is called a *tensor product* of  $V$  and  $W$  over  $K$  if the following properties hold:

- (i) There is a balanced map  $\tau : V \times W \rightarrow T$  such that  $T$  is additively generated by the image of  $\tau$ .
- (ii) Given any abelian group  $P$  and any balanced map  $\rho : V \times W \rightarrow P$  there exists an additive map  $\psi : T \rightarrow P$  such that  $\rho = \psi\tau$ .

### 1.3 Some well-known results

The present section contains the results which will be used frequently in the next chapter. We begin with the following:

**Lemma 1.3.1** ([2, Corollary 6.13]). Let  $a_i, b_i \in Q$  for  $i = 1, \dots, m$  such that  $\sum_{i=1}^m a_i x b_i = 0$  for all  $x \in R$ . If  $a_1, \dots, a_m$  are linearly independent over  $C$ , then  $b_i = 0$  for all  $i = 1, \dots, m$ .

**Lemma 1.3.2** ([2, Theorem 6.4.1]). *Let  $R$  be a semiprime ring with extended centroid  $C$  and  $Q = Q_{mr}(R)$ . Suppose that  $0 \neq \phi \in Q_C < X >$  on  $R$  is a GPI on  $Q$ .*

**Lemma 1.3.3** ([6, Theorem 1]). *Let  $R$  be a prime ring and  $\rho$  a nonzero right ideal of  $R$ . If  $\rho$  possesses a central DI, then  $R$  is a PI-ring.*

**Lemma 1.3.4** ([10]).  *$R, Q$  and each nonzero ideal of  $R$  satisfy the same GPIs.*

**Lemma 1.3.5** ([10, Theorem 2]). *Assume that  $R$  is a prime ring and  $U$  is its Utumi quotient ring. For any rational submodule  $M$  of  $U$ , the GPIs satisfied by  $M$  are the same as the GPIs satisfied by  $U$ .*

**Lemma 1.3.6** ([19, Theorem 2.5]). *If  $A$  is a prime algebra over  $\Phi$ , then  $Q = A \otimes C/M$  is a closed prime algebra over  $C$ .*

**Lemma 1.3.7** ([19, Theorem 3.5]). *If  $A$  is a closed prime algebra over  $\Phi$  and  $F$  is an extension field of  $\Phi$ , then  $A \otimes_{\Phi} F$  is a closed prime algebra over  $F$ .*

**Lemma 1.3.8** ([21, Theorem]). *Suppose that  $a, b \in R$ ,  $a \notin Z(R)$  and  $b \notin Z(R)$ . If  $[b, [a, x]] = 0$  for all  $x \in R$ , then  $\text{char}R = 2$ ,  $a^2 \in Z(R)$  and  $b = \alpha a + \beta$  for some  $\alpha, \beta \in C$ .*

**Lemma 1.3.9** ([33]). *Let  $\lambda$  be a left ideal of  $R$  and  $a_i, b_i \in Q$  for  $i = 1, \dots, m$ . Suppose that  $\sum_{i=1}^m a_i x^n b_i = 0$  for all  $x \in \lambda$ , where  $n \geq 1$  is a fixed integer. Then  $\sum_{i=1}^m a_i x b_i = 0$  for all  $x \in \lambda$ .*

**Lemma 1.3.10** ([40]). *Let  $R$  be a prime ring and let  $S = RC$  be the central closure of  $R$ . Then  $S$  satisfies a generalized polynomial identity over  $C$  if and only if  $S$  contains a minimal right ideal  $eS$  (hence  $S$  is primitive) and  $eSe$  is a finite dimensional division algebra over  $C$ .*

It is well-known that any derivation of  $R$  can be uniquely extended to a derivation of  $Q$ . A derivation  $d$  of  $R$  is called  $Q$ -inner if its extension to  $Q$  is inner, that is, there exists  $b \in Q$  such that  $d(x) = [b, x]$  for all  $x \in R$ . Otherwise  $d$  is called  $Q$ -outer. Given two derivations  $d, \delta$  of  $R$ , we say that  $d$  and  $\delta$  are  $C$ -dependent module  $Q$ -inner derivations if there exists  $\alpha, \beta \in C$  not both zero, and  $b \in Q$  such that  $\alpha d(x) + \beta \delta(x) = [b, x]$  for all  $x \in R$ . To prove our results we need Kharchenko's Theorem [24, 25] on differential polynomial identities of prime rings. By a differential polynomial identity (abbreviated as DI) we mean a generalized polynomial identity involving indeterminates that are acted by derivations of  $R$ . We record several well-known lemmas that can be found in [24, 25].

**Lemma 1.3.11.** *Let  $d$  be a  $Q$ -outer derivations of  $R$ . If  $\phi(x_i, d(x_i))$  is a DI of  $R$ , then  $\phi(x_i, y_i)$  is a GPI of  $R$ .*

**Lemma 1.3.12.** *Let  $\delta, d$  be derivations of  $R$  such that  $\delta$  and  $d$  are not  $C$ -dependent module  $Q$ -inner derivation. If  $\phi(x_i, d(x_i), \delta(x_i), \delta d(x_i))$  is a DI of  $R$ , then  $\phi(x_i, y_i, z_i, w_i)$  is a GPI of  $R$ .*

**Lemma 1.3.13.** *Let  $d$  be a  $Q$ -outer derivation of  $R$ , If  $\text{char}R \neq 2$  and  $\phi(x_i, d(x_i), d^2(x_i))$  is a DI of  $R$ , then  $\phi(x_i, y_i, z_i)$  is a GPI of  $R$ .*

It easy to see that if  $\text{char}R = 2$  and  $d$  is a derivation of  $R$ , then  $d^2$  is also a derivation of  $R$ .

**Lemma 1.3.14.** *Suppose that  $\text{char}R = 2$  and  $d$  is a derivation of  $R$  such that  $d$  and  $d^2$  are not  $C$ -dependent module  $Q$ -inner derivations. If  $\phi(x_i, d(x_i), d^2(x_i))$  is a DI of  $R$ , then  $\phi(x_i, y_i, z_i)$  is a GPI of  $R$ .*

# Chapter 2

## On Engel conditions with derivations in rings

### 2.1 Introduction

A classical problem of ring theory is to find combinations of properties that force a ring to be commutative. There are now more than hundred papers in which conditions are given that determine commutativity for a ring or a special type of ring. There is a technique for investigating commutativity of rings is the use of additive mappings like derivations and automorphisms of the ring  $R$ . The study of such mappings was initiated by Posner. In [41] Posner proved that if a prime ring  $R$  admits a nonzero derivation  $d$  such that  $[d(x), x] \in Z(R)$  for all  $x \in R$ , then  $R$  is commutative. A number of authors have extended this theorem of Posner. Then in [34] Lee generalized Posner's result. Lee states that if  $\text{char}(R) \neq 2$  and  $[d(x), x] \in Z$  for all  $x$  in a noncentral Lie ideal of  $R$ , then  $R$  is commutative. In [27] Lanski proved that if  $[d(x), x]_n = 0$  for all  $x$  is a noncommutative Lie ideal of  $R$ , then  $\text{char}(R) = 2$  and  $R \subseteq M_2(F)$  for  $F$  a field. In [10] Lanski proved that if  $[d(x^k), x^k]_n = 0$  for all  $x$  in a nonzero left ideal of  $R$ , then  $R$  is commutative [28, Theorem 1]. In [32] Lee obtained the same results in [28].

On the other hand, in [5] Carini and De Filippis generalized Posner's result that when  $[d(x), x]^m \in Z$  for all  $x$  in a noncommutative Lie ideal of  $R$  with  $\text{char}(R) \neq 2$ . In [46] the author and You discussed the case of  $\text{char}(R) = 2$ . Recently, in [14] Du and author investigated the situation when  $[d(x^k), x^k]^m \in Z$  for all  $x$  in a nonzero ideal of  $R$  with  $\text{char}(R) \neq 2$ .

In this chapter, I have studied the generalization of the above results from the commutator type to the Engel condition which is proved by Yu Wang. Throughout this chapter,  $R$  is always a prime ring with the center  $Z(R)$ .  $C$  is its extended centroid and  $Q$  is its Martindale quotient ring. For any  $x, y \in R$ , we set  $[x, y]_1 = [x, y] =$

$xy - yx$  and  $[x, y]_n = [[x, y]_{n-1}, y]$ , where  $n > 1$  is an integer. Note that  $[x, y]_n = \sum_{i=0}^n (-1)^i \binom{n}{i} y^i x y^{n-i}$ .  $S_4$  denoted as the standard identity in 4 variables. By  $d$  we denote a nonzero derivation of  $R$ .

## 2.2 Proof of main results

**Lemma 2.2.1.** *Let  $R = M_2(F)$ , the ring of all  $2 \times 2$  matrices over a field  $F$  with  $\text{char}(F) \neq 2$ . if  $a$  is a nonzero element of  $R$  such that  $([a, x^k]_n)^m = 0$  for all  $x \in R$ , then  $a$  is central in  $R$ .*

*Proof.* Let  $a = \sum_{i,j=1}^2 a_{ij} e_{ij}$  with  $a_{ij} \in F$  be a non zero  $2 \times 2$  matrix. i.e

$$\begin{aligned} a &= a_{11}e_{11} + a_{12}e_{12} + a_{21}e_{21} + a_{22}e_{22} \\ &= a_{11} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \end{aligned}$$

We first claim that  $a$  is a diagonal matrix. As we know  $[x, y] = xy - yx$  and  $[x, y]_n = [[x, y]_{n-1}, y]$ .

Consider  $[a, e_{11}]_n$ , if we put  $n = 1$ , then

$$\begin{aligned} [a, e_{11}] &= ae_{11} - e_{11}a \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix} \\ &= (-1)^n a_{12}e_{12} + a_{21}e_{21}, n = 1. \end{aligned}$$

If we put  $n = 2$ , then

$$\begin{aligned}
[a, e_{11}]_2 &= [[a, e_{11}], e_{11}] \\
&= [a, e_{11}]e_{11} - e_{11}[a, e_{11}] \\
&= \begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \\
&= (-1)^n a_{12} e_{12} + a_{21} e_{21}, \quad n = 2 \\
&\vdots \\
[a, e_{11}]_n &= (-1)^n a_{12} e_{12} + a_{21} e_{21}.
\end{aligned}$$

Now consider  $([a, e_{11}]_n)^{2m}$ , if we put  $n = 1, m = 1$ , then

$$\begin{aligned}
([a, e_{11}]_1)^2 &= (ae_{11} - e_{11}a)^2 \\
&= \begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} -a_{12}a_{21} & 0 \\ 0 & -a_{12}a_{21} \end{pmatrix} \\
&= (-1)^{mn} (a_{12}a_{21})^m e_{11} + (-1)^{mn} (a_{12}a_{21})^m e_{22}, \quad m = 1, n = 1.
\end{aligned}$$

If we take  $n = 1, m = 2$ , then

$$\begin{aligned}
([a, e_{11}]_n)^{2m} &= ([a, e_{11}]_1)^4 \\
&= (ae_{11} - e_{11}a)^4 \\
&= \begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix}^4
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} (a_{12}a_{21})^2 & 0 \\ 0 & (a_{12}a_{21})^2 \end{pmatrix} \\
&= (-1)^{mn}(a_{12}a_{21})^m e_{11} + (-1)^{mn}(a_{12}a_{21})^m e_{22}, m = 2, n = 1 \\
&\vdots \\
([a, e_{11}]_n)^{2m} &= (-1)^{mn}(a_{12}a_{21})^m e_{11} + (-1)^{mn}(a_{12}a_{21})^m e_{22}.
\end{aligned}$$

By assumption we have

$$0 = ([a, e_{11}]_n)^{2m} = (-1)^{mn}(a_{12}a_{21})^m e_{11} + (-1)^{mn}(a_{12}a_{21})^m e_{22}.$$

Hence  $a_{12}a_{21} = 0$ , and so either  $a_{12} = 0$  or  $a_{21} = 0$ . We may assume without loss of generality that  $a_{21} = 0$ . Let  $\varphi$  be the inner automorphism of  $R$  given by  $\varphi(x) = (1 + e_{21})x(1 - e_{21})$ .

$$\begin{aligned}
\varphi(a) &= (1 + e_{21})a(1 - e_{21}) \\
&= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a_{11} - a_{12} & a_{12} \\ a_{11} - a_{12} - a_{22} & a_{12} + a_{22} \end{pmatrix} \\
&= (a_{11} - a_{12})e_{11} + a_{12}e_{12} + (a_{11} - a_{12} - a_{22})e_{21} + (a_{12} + a_{22})e_{22}.
\end{aligned}$$

Since  $([\varphi(a), x^k]_n)^m = 0$  for all  $x \in R$ , we can get as above that  $a_{12}(a_{11} - a_{12} - a_{22}) = 0$ .

So either  $a_{12} = 0$  or  $(a_{11} - a_{12} - a_{22}) = 0$ .

Suppose that  $(a_{11} - a_{12} - a_{22}) = 0$ . If  $n$  is odd, take  $n = 1$ , then

$$\begin{aligned}
[a, e_{11} + e_{21}]_n &= [a, e_{11} + e_{21}] \\
&= a(e_{11} + e_{21}) - (e_{11} + e_{21})a
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{12} & -a_{12} \\ a_{22} - a_{11} & -a_{12} \end{pmatrix} \\
&= \begin{pmatrix} a_{12} & -a_{12} \\ -a_{12} & -a_{12} \end{pmatrix} \\
&= a_{12}e_{11} - a_{12}e_{12} - a_{12}e_{21} - a_{12}e_{22}.
\end{aligned}$$

Take  $n = 3$ , then

$$\begin{aligned}
[a, e_{11} + e_{21}]_3 &= [[a, e_{11} + e_{21}]_2, e_{11} + e_{22}] \\
&= [[[a, e_{11} + e_{21}], e_{11} + e_{21}], e_{11} + e_{21}] \\
&= \left[ \left[ \begin{pmatrix} a_{12} & -a_{12} \\ -a_{12} & -a_{12} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right], \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right] \\
&= \left[ \begin{pmatrix} -a_{12} & a_{12} \\ -3a_{12} & a_{12} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} a_{12} & -a_{12} \\ -a_{12} & -a_{12} \end{pmatrix} \\
&= a_{12}e_{11} - a_{12}e_{12} - a_{12}e_{21} - a_{12}e_{22} \\
&\vdots \\
[a, e_{11} + e_{21}]_n &= a_{12}e_{11} - a_{12}e_{12} - a_{12}e_{21} - a_{12}e_{22}.
\end{aligned}$$

Now consider  $([a, e_{11} + e_{21}]_n)^2$ ,  $n$  is odd, put  $n = 1$ , then

$$([a, e_{11} + e_{21}])^2 = \begin{pmatrix} a_{12} & -a_{12} \\ -a_{12} & -a_{12} \end{pmatrix}^2$$

$$\begin{aligned}
&= 2a_{12}^2 e_{11} + 2a_{12}^2 e_{22} \\
&\vdots \\
([a, e_{11} + e_{21}]_n)^2 &= 2a_{12}^2 e_{11} + 2a_{12}^2 e_{22}.
\end{aligned}$$

Now consider  $([a, e_{11} + e_{21}]_n)^{2m}$ ,  $n$  is odd, take  $n = 1, m = 1$  then

$$([a, e_{11} + e_{21}])^2 = 2a_{12}^2 e_{11} + 2a_{12}^2 e_{22}$$

Take  $n = 1, m = 2$  then

$$\begin{aligned}
([a, e_{11} + e_{21}])^4 &= \begin{pmatrix} a_{12} & -a_{12} \\ -a_{12} & -a_{12} \end{pmatrix}^4 \\
&= \begin{pmatrix} 4(a_{12})^4 & 0 \\ 0 & 4(a_{12})^4 \end{pmatrix} \\
&= (2a_{12}^2)^2 e_{11} + (2a_{12}^2)^2 e_{22} \\
&\vdots \\
([a, e_{11} + e_{21}]_n)^{2m} &= (2a_{12}^2)^m e_{11} + (2a_{12}^2)^m e_{22}.
\end{aligned}$$

By assumption we have

$$0 = ([a, e_{11} + e_{21}]_n)^{2m} = (2a_{12}^2)^m e_{11} + (2a_{12}^2)^m e_{22}.$$

It implies that  $(2a_{12}^2)^m = 0$  and so  $a_{12} = 0$ .

If  $n$  is even, take  $n = 2$ , then  $[a, e_{11} + e_{21}]_n$

$$\begin{aligned}
[a, e_{11} + e_{21}]_2 &= [a, [e_{11} + e_{21}], [e_{11} + e_{21}]] \\
&= \left[ \begin{pmatrix} a_{12} & -a_{12} \\ -a_{12} & -a_{12} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} -a_{12} & a_{12} \\ -3a_{12} & a_{12} \end{pmatrix} \\
&= -a_{12} e_{11} + a_{12} e_{12} - 3a_{12} e_{21} + a_{12} e_{22} \\
&\vdots
\end{aligned}$$

$$[a, e_{11} + e_{21}]_n = -a_{12}e_{11} + a_{12}e_{12} - 3a_{12}e_{21} + a_{12}e_{22}.$$

Now consider  $([a, e_{11} + e_{21}]_n)^{2m}$ ,  $n$  is even, take  $n = 2, m = 1$  then

$$\begin{aligned} ([a, e_{11} + e_{21}]_2)^2 &= \begin{pmatrix} -a_{12} & a_{12} \\ -3a_{12} & a_{12} \end{pmatrix}^2 \\ &= (-2a_{12}^2)e_{11} + (-2a_{12}^2)e_{22}. \end{aligned}$$

Take  $n = 2, m = 2$  then

$$\begin{aligned} ([a, e_{11} + e_{21}]_2)^4 &= \begin{pmatrix} -a_{12} & a_{12} \\ -3a_{12} & a_{12} \end{pmatrix}^4 \\ &= (-2a_{12}^2)^2e_{11} + (-2a_{12}^2)^2e_{22} \\ &\vdots \\ ([a, e_{11} + e_{21}]_n)^{2m} &= (-2a_{12}^2)^me_{11} + (-2a_{12}^2)^me_{22}. \end{aligned}$$

By assumption we have

$$0 = ([a, e_{11} + e_{21}]_n)^{2m} = (-2a_{12}^2)^me_{11} + (-2a_{12}^2)^me_{22}.$$

Hence  $(-2a_{12}^2)^m = 0$  and so  $a_{12} = 0$ . Hence,  $a$  is a diagonal matrix.

Also as we assume that  $(a_{11} - a_{12} - a_{22}) = 0$  and  $a_{12} = 0$ , It implies that  $a_{11} = a_{12}$ , so  $a$  is central in  $R$ .  $\square$

**Lemma 2.2.2.** *Let  $R = M_s(F)$ , the ring of all  $s \times s$  matrices over a field  $F$  with  $\text{char}(F) \neq 2$  and  $s > 2$ . If  $a \in R$  such that  $([a, x^k]_n)^m \in F \cdot I_s$  for all  $x \in R$ , where  $I_s$  is the identity matrix of  $R$ , then  $a \in F \cdot I_s$ .*

*Proof.* Let  $a = \sum_{i,j=1}^s a_{ij}e_{ij}$  with  $a_{ij} \in F$  be a non zero  $s \times s$  matrix. i.e

$$\begin{aligned} a &= (a_{11}e_{11} + a_{12}e_{12} + \dots + a_{1s}e_{1s}) + (a_{21}e_{21} + a_{22}e_{22} + \dots + a_{2s}e_{2s}) \\ &\quad + (a_{s1}e_{s1} + a_{s2}e_{s2} + \dots + a_{ss}e_{ss}) \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1s} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2s} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s1} & a_{s2} & a_{s3} & \cdot & \cdot & \cdot & a_{ss} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & A \\ B & C \end{pmatrix}
\end{aligned}$$

where  $A = (a_{12}, \dots, a_{1s})$ ,  $B = (a_{21}, \dots, a_{s1})^T$ , and  $C = (a_{ij})$  with  $2 \leq i, j \leq s$ . Consider  $[a, e_{11}]_n$ , take  $n = 1$ , then

$$[a, e_{11}] = ae_{11} - e_{11}a$$

$$\begin{aligned}
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1s} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2s} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s1} & a_{s2} & a_{s3} & \cdot & \cdot & \cdot & a_{ss} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \\
&- \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1s} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2s} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s1} & a_{s2} & a_{s3} & \cdot & \cdot & \cdot & a_{ss} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} 0 & -a_{12} & -a_{13} & \dots & -a_{1s} \\ a_{21} & 0 & 0 & \dots & 0 \\ a_{31} & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_{s1} & 0 & 0 & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (-1)^n A \\ B & 0 \end{pmatrix}, n = 1.
\end{aligned}$$

Take  $n = 2$ , then  $[a, e_{11}]_n$  is

$$\begin{aligned}
[a, e_{11}]_2 &= [[a, e_{11}], e_{11}] \\
&= [a, e_{11}]e_{11} - e_{11}[a, e_{11}] \\
&= \begin{pmatrix} 0 & -a_{12} & -a_{13} & \dots & -a_{1s} \\ a_{21} & 0 & 0 & \dots & 0 \\ a_{31} & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_{s1} & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & -a_{12} & -a_{13} & \dots & -a_{1s} \\ a_{21} & 0 & 0 & \dots & 0 \\ a_{31} & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_{s1} & 0 & 0 & \dots & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1s} \\ a_{21} & 0 & 0 & \dots & 0 \\ a_{31} & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_{s1} & 0 & 0 & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (-1)^n A \\ B & 0 \end{pmatrix}, n = 2 \\
&\vdots \\
[a, e_{11}]_n &= \begin{pmatrix} 0 & (-1)^n A \\ B & 0 \end{pmatrix}.
\end{aligned}$$

Now consider  $([a, e_{11}]_n)^{2m}$ , if we put  $n = 1, m = 1$ , then

$$\begin{aligned}
([a, e_{11}]_1)^2 &= (ae_{11} - e_{11}a)^2 \\
&= \begin{pmatrix} 0 & -a_{12} & -a_{13} & \dots & -a_{1s} \\ a_{21} & 0 & 0 & \dots & 0 \\ a_{31} & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_{s1} & 0 & 0 & \dots & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} -(a_{12}a_{21} + a_{13}a_{31} + \dots + a_{1s}a_{s1}) & 0 & 0 & \dots & 0 \\ 0 & -a_{21}a_{12} & -a_{21}a_{13} & \dots & -a_{21}a_{1s} \\ 0 & -a_{31}a_{12} & -a_{31}a_{13} & \dots & -a_{31}a_{1s} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & -a_{s1}a_{12} & -a_{s1}a_{13} & \dots & -a_{s1}a_{1s} \end{pmatrix} \\
&= \begin{pmatrix} (-1)^{mn}(AB)^m & 0 \\ 0 & (-1)^{mn}(BA)^m \end{pmatrix}, m = 1, n = 1.
\end{aligned}$$

Take  $n = 2, m = 1$ , then  $([a, e_{11}]_n)^{2m}$  is

$$\begin{aligned}
([a, e_{11}]_2)^2 &= \begin{pmatrix} 0 & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1s} \\ a_{21} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a_{31} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s1} & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} (a_{12}a_{21} + a_{13}a_{31} + \dots + a_{1s}a_{s1}) & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & a_{21}a_{12} & a_{21}a_{13} & \cdot & \cdot & \cdot & a_{21}a_{1s} \\ 0 & a_{31}a_{12} & a_{31}a_{13} & \cdot & \cdot & \cdot & a_{31}a_{1s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{s1}a_{12} & a_{s1}a_{13} & \cdot & \cdot & \cdot & a_{s1}a_{1s} \end{pmatrix} \\
&= \begin{pmatrix} (-1)^{mn}(AB)^m & 0 \\ 0 & (-1)^{mn}(BA)^m \end{pmatrix}, \quad m = 1, n = 2 \\
&\vdots \\
([a, e_{11}]_2)^{2m} &= \begin{pmatrix} (-1)^{mn}(AB)^m & 0 \\ 0 & (-1)^{mn}(BA)^m \end{pmatrix}.
\end{aligned}$$

By assumption, we have

$$([a, e_{11}]_n)^{2m} = \begin{pmatrix} (-1)^{mn}(AB)^m & 0 \\ 0 & (-1)^{mn}(BA)^m \end{pmatrix} \in F \cdot I_s.$$

Since  $\text{rank}((-1)^{mn}(AB)^m) \leq \text{rank}(A) \leq 1$  and  $\text{rank}((-1)^{mn}(BA)^m) \leq \text{rank}(A) \leq 1$ , we see that  $\text{rank}([(a, e_{11}]_n)^{2m}] \leq 2$ , which forcing  $([a, e_{11}]_n)^{2m} = 0$  as  $s > 2$ . In particular,  $(-1)^{mn}(AB)^m = 0$  and so  $AB = 0$ .

Let  $\varphi_{ij}$  be an inner automorphism of  $R$  given by  $\varphi_{ij}(x) = (1 + e_{ij})x(1 - e_{ij})$  for all  $x \in R$ . Write  $1 + e_{21} = \begin{pmatrix} 1 & 0 \\ E_2 & I_{s-1} \end{pmatrix}$ , where  $E_2 = (1, 0, \dots, 0)^T$  and  $I_{s-1}$  is the  $(s - 1)$ -identity matrix.

$$\begin{aligned}
\varphi_{21}(a) &= (1 + e_{21})a(1 - e_{21}) \\
&= \begin{pmatrix} 1 & 0 \\ E_2 & I_{s-1} \end{pmatrix} \begin{pmatrix} a_{11} & A \\ B & C \end{pmatrix} \begin{pmatrix} 1 & O \\ -E_2 & I_{s-1} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & A \\ a_{11}E_2 + I_{s-1}B & E_2A + I_{s-1}C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -E_2 & I_{s-1} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} - AE_2 & AI_{s-1} \\ a_{11}E_2 + I_{s-1}B - E_2AE_2 - I_{s-1}CE_2 & E_2AI_{s-1} + I_{s-1}CI_{s-1} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} - a_{12} & A \\ a_{11}E_2 - a_{12}E_2 + B - CE_2 & E_2A + C \end{pmatrix}.
\end{aligned}$$

As what has been shown, we have

$$\begin{aligned}
A(a_{11}E_2 - a_{12}E_2 + B - CE_2) &= 0 \\
a_{11}AE_2 - a_{12}AE_2 + AB - ACE_2 &= 0 \\
a_{11}a_{12} - a_{12}^2 + 0 - ACE_2 &= 0 \\
a_{11}a_{12} - a_{12}^2 - ACE_2 &= 0.
\end{aligned}$$

Now consider  $[a, e_{11} + e_{21}]_n$ , assume first that  $n$  is odd, take  $n = 1$ , we get

$$\begin{aligned}
[a, e_{11} + e_{21}] &= a(e_{11} + e_{21}) - (e_{11} + e_{21})a \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1s} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2s} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s1} & a_{s2} & a_{s3} & \cdot & \cdot & \cdot & a_{ss} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& - \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1s} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2s} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3s} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{s1} & a_{s2} & a_{s3} & \dots & a_{ss} \end{pmatrix} \\
& = \begin{pmatrix} a_{11} + a_{12} & 0 & 0 & \dots & 0 \\ a_{21} + a_{22} & 0 & 0 & \dots & 0 \\ a_{31} + a_{32} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{s1} + a_{s2} & 0 & 0 & \dots & 0 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1s} \\ a_{11} & a_{12} & a_{13} & \dots & a_{1s} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
& = \begin{pmatrix} a_{12} & -a_{12} & -a_{13} & \dots & -a_{1s} \\ a_{21} + a_{22} - a_{11} & -a_{12} & -a_{13} & \dots & -a_{1s} \\ a_{31} + a_{32} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{s1} + a_{s2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
& = \begin{pmatrix} a_{12} & -A \\ D_1 & -E_2A \end{pmatrix}
\end{aligned}$$

where  $D_1 = B + CE_2 - E_2a_{11}$ . So

$$\begin{aligned}
([a, e_{11} + e_{21}]_n)^2 &= \begin{pmatrix} a_{12} & -A \\ D_1 & -E_2A \end{pmatrix}^2 \\
&= \begin{pmatrix} a_{12}^2 - AD_1 & 0 \\ * & -D_1A + a_{12}E_2A \end{pmatrix}.
\end{aligned}$$

Now

$$AD_1 = A(B + CE_2 - E_2a_{11})$$

$$\begin{aligned}
&= AB + ACE_2 - AE_2a_{11} \\
&= 0 + ACE_2 - a_{11}a_{12}.
\end{aligned}$$

Also  $a_{11}a_{12} - a_{12}^2 - ACE_2 = 0$ . It implies  $AD_1 = -a_{12}^2$ . Thus

$$([a, e_{11} + e_{21}]_n)^2 = \begin{pmatrix} 2a_{12}^2 & 0 \\ * & -D_1A + a_{12}E_2A \end{pmatrix}.$$

$$\text{Hence } ([a, e_{11} + e_{21}]_n)^{2m} = \begin{pmatrix} (2a_{12}^2)^m & 0 \\ * & (-D_1A + a_{12}E_2A)^m \end{pmatrix} \in F \cdot I_s.$$

Since  $\text{rank}((-DA + a_{12}E_2A)^m) \leq \text{rank}(A) \leq 1$ , we see that  $\text{rank}((([a, e_{11} + e_{21}]_n)^{2m}) \leq 2$ , which implies that  $([a, e_{11} + e_{21}]_n)^{2m} = 0$  as  $s > 2$ . In particular,  $(2a_{12}^2)^m = 0$  and so  $a_{12} = 0$ .

Suppose next that  $n$  is even, put  $n = 2$  then  $[a, e_{11} + e_{12}]_n$  is

$$\begin{aligned}
[a, e_{11} + e_{12}]_2 &= [[a, e_{11} + e_{12}], e_{11} + e_{12}] \\
&= \begin{pmatrix} a_{12} & -a_{12} & -a_{13} & \cdot & \cdot & \cdot & -a_{1s} \\ a_{21} + a_{22} - a_{11} & -a_{12} & -a_{13} & \cdot & \cdot & \cdot & -a_{1s} \\ a_{31} + a_{32} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s1} + a_{s2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} a_{12} & -a_{12} & -a_{13} & \cdot & \cdot & \cdot & -a_{1s} \\ a_{21} + a_{22} - a_{11} & -a_{12} & -a_{13} & \cdot & \cdot & \cdot & -a_{1s} \\ a_{31} + a_{32} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s1} + a_{s2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} -a_{12} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1s} \\ a_{21} + a_{22} - a_{11} - 2a_{12} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1s} \\ a_{31} + a_{32} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s1} + a_{s2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -a_{12} & A \\ D_2 & E_2A \end{pmatrix}
\end{aligned}$$

where  $D_2 = B + CE_2 - (a_{11} + 2a_{12})E_2$ . Thus

$$\begin{aligned}
([a, e_{11} + e_{21}]_n)^2 &= \begin{pmatrix} -a_{12} & A \\ D_2 & E_2A \end{pmatrix}^2 \\
&= \begin{pmatrix} a_{12}^2 + AD_2 & 0 \\ * & D_2A + a_{12}E_2A \end{pmatrix}.
\end{aligned}$$

Now

$$\begin{aligned}
AD_2 &= A(B + CE_2 - (a_{11} + 2a_{12})E_2) \\
&= AB + ACE_2 - (a_{11} + 2a_{12})AE_2 \\
&= ACE_2 - a - 11a_{12} - 2a_{12}^2
\end{aligned}$$

Also  $a_{11}a_{12} - a_{12}^2 - ACE_2 = 0$ . It implies  $AD_2 = -3a_{12}^2$ . Hence

$$([a, e_{11} + e_{21}]_n)^2 = \begin{pmatrix} -2a_{12}^2 & 0 \\ * & D_2A + a_{12}E_2A \end{pmatrix}$$

and so

$$([a, e_{11} + e_{21}]_n)^{2m} = \begin{pmatrix} (-2a_{12}^2)^m & 0 \\ * & (D_2A + a_{12}E_2A)^m \end{pmatrix} \in F \cdot I_s.$$

Since  $\text{rank}((D_2A + a_{12}E_2A)^m) \leq \text{rank}(A) \leq 1$ , we see that the rank  $(([a, e_{11} + e_{21}]_n)^{2m}) \leq 2$ , which implies the  $([a, e_{11} + e_{21}]_n)^{2m} = 0$  as  $s > 2$ . In particular,  $(-2a_{12}^2)^m = 0$ , and so  $a_{12} = 0$ .

Now we claim that  $a$  is a diagonal matrix. Since  $([\varphi_{j2}(a), x^k]_n)^m \in F \cdot I_s$  for all  $x \in R$ , where  $j > 2$ , as what has been shown, we get that  $-a_{1j} = \varphi_{j1}(a)_{12} = 0$ . So

$a_{1j} = 0$  for  $j > 1$ . For  $1 < j < t \leq s$ , we get from  $([\varphi_{1j}(a), x^k]_n)^m \in F \cdot I_s$  for all  $x \in R$ , that  $a_{jt} = \varphi_{1j}(a)_{1t} = 0$ . This shows that  $a$  must be lower triangular. Since the transpose of  $a$  satisfies the same condition,  $a$  is indeed diagonal.

We have shown that  $a = \sum_{i=1}^m a_{ii} e_{ii}$  with  $a_{ii} \in F$ . For  $1 \leq i \neq j \leq s$ , as above we get that  $\varphi_{ij}(a)$  is a diagonal matrix. On the other hand,  $\varphi(a) = a + (a_{jj} - a_{ii})e_{ij}$ , we infer that  $a_{jj} = a_{ii}$  and so  $a \in F \cdot I_s$ . This proves the result.  $\square$

**Lemma 2.2.3.** *Let  $R$  be a prime ring of  $\text{char} R \neq 2$  and  $a \notin C$  such that  $([a, x^k]_n)^m = 0$  for all  $x \in R$ . Then  $R$  satisfies a nonzero generalized polynomial identity (GPI).*

*Proof.* By both lemma 1.3.2 and 1.3.5, we have that  $([a, x^k]_n)^m = 0$  for all  $x \in Q$ . That is, the element  $([a, X^k]_n)^m$  in the free product  $T = Q *_C X$  is a GPI on  $R$ . Since  $a \notin C$ , we easily see that the term  $(aX^{kn})^m$  appears in the expansion of  $([a, X^k]_n)^m$  nontrivially. So  $([a, X^k]_n)^m$  is a nonzero element in  $T = Q *_C CX$ . Therefore,  $R$  satisfies a nonzero GPI.  $\square$

**Lemma 2.2.4.** *Let  $x, e \in R$  and  $e = e^2$ . Then  $[x, e]_{2n+1} = [x, e]$  for all integers  $n \geq 0$ . And,  $[x, e]_n = [x, e]$  for all integers  $n \geq 1$  if  $\text{char} R = 2$ .*

*Proof.* Recall that  $[x, e]_3 = \sum_{i=0}^3 (-1)^i \binom{3}{i} e^i x e^{3-i} = xe^3 - 3exe^2 + 3e^2xe - e^3x = xe - ex = [x, e]$ . This implies that  $[x, e]_{2n+1} = [x, e]$ . If  $\text{char} R = 2$ , then  $[x, e]_2 = \sum_{i=0}^2 (-1)^i \binom{2}{i} e^i x e^{2-i} = xe^2 - 2exe + e^2x = xe + ex = xe - ex = [x, e]$ . Consequently  $[x, e]_n = [x, e]$ .  $\square$

**Lemma 2.2.5.** *Suppose that  $a \in R$  and  $\alpha \in Z(R)$  such that  $[a, [x, y]] = \alpha[x, y]$  for all  $x, y \in R$ . Then  $a \in Z(R)$ .*

*Proof.* Clearly, we may assume that  $R$  is noncommutative. Let  $x, y, r \in R$ . Then  $[\alpha[x, y], r] = [[a, [x, y]], r] = [a, [[x, y], r]] + [[a, r], [x, y]] = \alpha[[x, y], r] + [[a, r], [x, y]]$ , so  $[[a, r], [x, y]] = 0$ . Since  $R$  is noncommutative, the centralizer of  $[R, R]$  is  $Z(R)$ , so  $[a, r] \in Z(R)$  for all  $r \in R$ , implying  $a \in Z(R)$ .  $\square$

**Theorem 2.2.1.** *Let  $R$  be a prime ring of characteristic different from 2 with  $d$  a nonzero derivation of  $R$  such that  $([d(x^k), x^k]_n)^m = 0$  for all  $x \in R$ , where  $k, m, n$  are fixed positive integers. Then  $R$  is commutative.*

*Proof.* Suppose on the contrary that  $R$  is not commutative. Assume first that  $d$  is  $Q$ -inner, that is,  $d(x) = [a, x]$  for all  $x \in R$ , where  $a$  is a noncentral element in  $Q$ . By assumption, we get  $([a, x^k]_{n+1})^m = 0$  for  $x \in R$ . It follows from both Lemma 1.3.2 and 1.3.5 that  $([a, x^k]_{n+1})^m = 0$  for all  $x \in Q$ . In case  $C$  is infinite, we have  $([a, x^k]_{n+1})^m = 0$  for all  $x \in Q \otimes_C \bar{C}$ , where  $\bar{C}$  is the algebraic closure of  $C$ . Since both  $Q$  and  $Q \otimes_C \bar{C}$  are centrally closed by both Lemma 1.3.6 and 1.3.7, we may replace  $R$  by  $Q$  or  $Q \otimes_C \bar{C}$  according as  $C$  is finite or infinite. Thus we may assume that  $R$  is centrally closed over  $C$  which is either finite or algebraically closed and  $([a, x^k]_{n+1})^m = 0$  for all  $x \in R$ . By Lemma 2.2.3, we have that  $R$  is a nontrivial  $GPI$  prime ring.

By Lemma 1.3.10  $R$  is a primitive ring and so isomorphic to a dense subring of linear transformations on a vector space  $V$  over  $C$ . Suppose that  $V$  is infinite dimensional over  $C$ . For any given  $v \in V$ , we claim that  $v$  and  $va$  are  $C$ -dependent. Suppose on the contrary that  $v$  and  $va$  are  $C$ -independent. We choose  $v_1, \dots, v_{k(n+1)-1}$  such that  $v, va, v_1, \dots, v_{k(n+1)-1}$  are  $C$ -independent. By the density of  $R$  on  $V$ , there exists  $r \in R$  such that  $vr = 0, var = v_1, v_i r = v_{i+1}, v_{k(n+1)-1} r = v$ , where  $i = 1, \dots, k(n+1)-2$ . Thus

$$v[a, r^k]_{n+1} = var^{k(n+1)} = v_1 r^{k(n+1)-1} = v_2 r^{k(n+1)-2} = \dots = v_{k(n+1)-1} r = v$$

and so  $0 = v([a, r^k]_{n+1})^m = v$ , a contradiction. Therefore,  $v$  and  $va$  are  $C$ -dependent for any  $v \in V$ . A standard argument shows that  $a \in C$ , a contradiction. So  $V$  must be a finite dimension. That is  $R \cong M_s(C)$  for some  $s > 1$ . In view of both Lemma 2.2.1 and Lemma 2.2.2, we see that  $a \in C$ , a contradiction.

Suppose next  $d$  is not  $Q$ -inner. By assumption, we have

$$\left( \left[ \sum_{i=0}^{k-1} x^i d(x) x^{k-i-1}, x^k \right]_n \right)^m = 0$$

for all  $x \in R$ . Applying Kharchenko's theorem [7], we get

$$\left( \left[ \sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k \right]_n \right)^m = 0$$

for all  $x, y \in R$ . Substituting  $y$  with  $[a, x]$ , where  $a$  is a noncentral element of  $R$ , we obtain that  $([a, x^k]_{n+1})^m = 0$  for all  $x \in R$ . Then the result follows from the above discussion.

□

**Theorem 2.2.2.** *Let  $R$  be a prime ring of characteristic different from 2 with  $d$  a nonzero derivation of  $R$  such that  $([d(x^k), x^k]_n)^m \in Z$  for all  $x \in R$ , where  $k, m, n$  are fixed positive integers. Then  $R$  satisfies  $S_4$ , the standard identity in 4 variables.*

*Proof.* suppose on the contrary that  $R$  does not satisfy  $S_4$ . By assumption, we have

$$([d(x^k), x^k]_n)^m \in Z \quad \text{for all } x \in R.$$

If  $([d(x^k), x^k]_n)^m = 0$  for all  $x \in R$ , then the result follows from Theorem 2.2.1. Otherwise, there exists  $r \in R$  such that  $([d(r^k), r^k]_n)^m \neq 0$ . That is,  $R$  satisfies the central differential identity  $([d(X^k), X^k]_n)^m$ . In view Lemma 1.3.3, we have that  $R$  is a  $PI$ -ring and so is  $Q$ .

Assume first that  $d$  is a  $Q$ -inner derivation induced by a noncentral element  $a \in Q$ . Thus  $([a, x^k]_{n+1})^m \in Z$  for all  $x \in R$ . In view of both Lemma 1.3.2 and 1.3.5, we have that  $([a, x^k]_{n+1})^m \in Z$  for all  $x \in Q$ . Since there exists  $r \in I$  such that  $([a, r^k]_{n+1})^m \neq 0$ , we see that  $([a, X^k]_{n+1})^m, Y$  is a non zero GPI on  $Q$ . By Lemma 1.3.10,  $Q$  is primitive ring. it follows from the famous Kaplansky's theorem [23, Theorem 6.1.10] that  $Q$  is a finite dimensional central simple algebra over  $C$ . In view of [27, Lemma 2], there exists a suitable field  $F$  of  $\text{char}(F) \neq 2$  such that  $Q \subseteq M_s(F)$ , where  $s > 2$ , and moreover,  $Q$  and  $M_s(F)$  satisfy the same GPI. Then Lemma 2.2.2 tells us that  $a \in C$ , a contradiction.

Assume next that  $d$  is not  $Q$ -inner. By assumption, we have

$$\left( \left[ \sum_{i=0}^{k-1} x^i d(x) x^{k-i-1}, x^k \right]_n \right)^m \in Z$$

for all  $x \in R$ . Applying Kharchenko's theorem, we get

$$\left( \left[ \sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k \right]_n \right)^m \in Z$$

for all  $x \in R$ . Substituting  $y$  with  $[a, x]$ , where  $a$  is a noncentral element of  $R$ , we obtain that  $([a, x^k]_{n+1})^m \in Z$  for all  $x \in R$ . Then the result follows the above discussion.  $\square$

**Theorem 2.2.3.** *Let  $R$  be a noncommutative prime ring and  $a, b \in Q$ . Suppose that  $[b, [a, x]_n] = 0$  for all  $x \in R$ , where  $n \geq 2$  is a fixed integer. If  $a \notin C$  and  $b \notin C$ , then  $\text{char}R = 2, a^2 \in C$  and  $b = \alpha a + \beta$  for some  $\alpha, \beta \in C$ .*

*Proof.* Set  $\phi(x) = [b, [a, x]_n]$ . Then by assumption,  $\phi(x) = 0$  for all  $x \in R$ . Thus  $\phi(x)$  is *GPI* of  $R$ . Note that

$$\begin{aligned}
\phi(x) &= [b, [a, x]_n] \\
&= b[a, x]_n - [a, x]_n b \\
&= b \left( \sum_{i=0}^n (-1)^i \binom{n}{i} x^i a x^{n-i} \right) - \left( \sum_{i=0}^n (-1)^i \binom{n}{i} x^i a x^{n-i} \right) b \\
&= \binom{n}{0} b a x^n - \binom{n}{1} b x a x^{n-1} + \cdots + (-1)^{n-1} \binom{n}{n-1} b x^{n-1} a x \\
&\quad + (-1)^n \binom{n}{n} b x^n a - \binom{n}{0} a x^n b + \binom{n}{1} x a x^{n-1} b + \cdots + (-1)^n \binom{n}{n-1} \\
&\quad \quad \quad x^{n-1} a x b + (-1)^{n+1} \binom{n}{n} x^n a b.
\end{aligned}$$

Assume first that  $\phi x$  is a trivial *GPI* of  $R$ . Since  $a \notin C$  and  $b \notin C$ , above equation implies that  $\binom{n}{i} b x^i a x^{n-i}$  is a trivial monomial for all  $1 \leq i \leq n-1$ . Thus  $\binom{n}{i} = 0$  for all  $1 \leq i \leq n-1$ . In particular, from  $0 = \binom{n}{1} = n$  it follows that  $\text{char} R = p$ , a prime number and  $p$  divides  $n$ . Moreover, using  $\binom{n}{i} = 0$  for all  $1 \leq i \leq n-1$ , we see that  $n = p^k$  for some  $k \geq 1$ . Now above equation reduces to

$$\begin{aligned}
\phi(x) &= b a x^n + (-1)^n b x^n a - a x^n b + (-1)^{n+1} x^n a b \\
&= b a x^{p^k} + (-1)^{p^k} b x^{p^k} a - a x^{p^k} b + (-1)^{p^k+1} x^{p^k} a b \\
&= b a x^{p^k} - b x^{p^k} a - a x^{p^k} b + x^{p^k} a b \\
&= [b, [a, x^{p^k}]].
\end{aligned}$$

Then  $\phi(x) = [b, [a, x^{p^k}]] = 0$  for all  $x \in R$ . By Lemma 1.3.9,  $[b, [a, x]] = 0$  for all  $x \in R$  and hence for all  $x \in Q$  by Lemma 1.3.4. Applying Lemma 1.3.8 to this identity, we obtain  $\text{char} R = 2$ ,  $a^2 \in C$  and  $b = \alpha a + \beta$  for some  $\alpha, \beta \in C$ , proving the theorem. Hence from now on we may assume that  $\phi(x)$  is a nontrivial *GPI* of  $R$ . By Lemma 1.3.4,  $\phi(x)$  is also a *GPI* of  $Q$ . Let  $F$  be the algebraic closure of  $C$  if  $C$  is infinite

and set  $F = C$  for  $C$  finite. Let  $\tilde{Q} = Q \otimes_C F$ . Clearly  $Q \cong Q \otimes_C C \subseteq \tilde{Q}$ . So we may regard  $Q$  as a subring of  $\tilde{Q}$ . By [31, proposition],  $\phi(x)$  is also a *GPI* of  $\tilde{Q}$ . Moreover, in view of Lemma 1.3.7,  $\tilde{Q}$  is a prime ring with  $F$  as its extended centroid. Thus  $\tilde{Q}$  is a prime ring that satisfies a nontrivial *GPI*  $\phi(x)$  and its extended centroid  $F$  is either an algebraically closed field or a finite field. By Lemma 1.3.10,  $\tilde{Q}$  is a primitive ring having nonzero socle with the field  $F$  as its associated division ring. By [22, p. 75]  $\tilde{Q}$  is isomorphic to a dense subring of the ring of linear transformation on a vector space  $V$  over  $F$ , containing nonzero linear transformation of finite rank. By assumption,  $R$  is noncommutative so  $\tilde{Q}$  is noncommutative. Hence  $\dim V_F \geq 2$ . Let  $H$  be the set of all finite rank elements in  $\tilde{Q}$ . Then  $H$  is a nonzero ideal of  $\tilde{Q}$ . Further,  $H$  is a simple ring with nontrivial idempotents.

Assume first that  $n$  is odd or  $\text{char} R = 2$ . Let  $e = e^2$  be an idempotent in  $H$ . By Lemma 2.2.4,  $[a, e]_n = [a, e]$  and so  $0 = [b, [a, e]]$ . Suppose  $\dim V_F = k < \infty$ . Then  $\tilde{Q} = H \cong M_k(F)$ , the  $k \times k$  matrix algebra over  $F$ . In this case  $\tilde{Q}$  is  $F$ -linear spanned by the idempotents in  $H$ . So we have  $[b, [a, x]] = 0$  for all  $x \in \tilde{Q}$ . Thus  $[b, [a, Q]] = 0$ . By Lemma 1.3.8, we have done. Suppose now that  $\dim V_F = \infty$ . Let  $E$  be the additive subgroup of  $H$  generated by the idempotents in  $H$ . Then  $[b, [a, E]] = 0$ . By [20, p. 18, Corollary],  $[H, H] \subseteq E$ , where  $[H, H]$  stands for the additive subgroup of  $\tilde{Q}$  generated by  $[x, y]$  for all  $x, y \in H$ . Thus  $[b, [a, [x, y]]] = 0$  for all  $x, y \in H$ . Since  $H$  is a nonzero ideal of  $\tilde{Q}$ , by Lemma 1.3.4,  $[b, [a, [x, y]]] = 0$  for all  $x, y \in \tilde{Q}$ . Then  $[b, [a, [\tilde{Q}, \tilde{Q}]]] = 0$ , where  $[\tilde{Q}, \tilde{Q}]$  stands for the additive subgroup of  $\tilde{Q}$  generated by  $[x, y]$  for all  $x, y \in \tilde{Q}$ . By [26, Theorem 1],  $[b, [a, Q]] = 0$ . By Lemma 1.3.8, we are done.

Assume now that  $n$  is even and  $\text{char} R \neq 2$ . Suppose first that  $ea(1 - e) \neq 0$  for some idempotents  $e = e^2 \in H$ . Let  $x \in \tilde{Q}$  and  $f = e + (1 - e)xe \in H$ . Then  $f = f^2$ ,  $f(1 - e) = 0$ , and  $ef = e$ . By Fact 9, we have

$$0 = [b, [a, f]_n] = [b, [a, f]_2] = b(af - 2faf + fa) - (af - 2faf + fa)b.$$

Multiplying above identity by  $1 - e$  from the right hand side and by  $e$  from the left hand side and using the fact that  $f(1 - e) = 0$  and  $ef = e$ , we obtain  $ebfa(1 - e) - eafb(1 - e) + 2eafb(1 - e) - eab(1 - e) = 0$ . Hence

$$eb(e + (1 - e)xe)a(1 - e) + ea(e + (1 - e)xe)b(1 - e) - eab(1 - e) = 0$$

for all  $x \in \tilde{Q}$ . Setting  $x = 0$  in above equation, we get

$$ebeb(1-e) + eae(1-e) - eab(1-e) = 0.$$

The difference of above two equations yields

$$eb(1-e)xea(1-e) + ea(1-e)xeb(1-e) = 0$$

for all  $x \in \tilde{Q}$ . Since  $ea(1-e) \neq 0$ , applying Fact 1 to above equation, we conclude that  $eb(1-e)$  and  $ea(1-e)$  are  $F$ -dependent. So  $eb(1-e) = \alpha ea(1-e)$  for some  $\alpha \in F$ . Then above equation reduces to  $2\alpha ea(1-e)xea(1-e) = 0$  for all  $x \in \tilde{Q}$ . Recall that  $\text{char}R \neq 2$ . Then 2 is invertible in  $C$  and so  $\alpha ea(1-e)xea(1-e) = 0$  for all  $x \in \tilde{Q}$ . By primeness of  $\tilde{Q}$  and  $ea(1-e) \neq 0$ , we have  $\alpha = 0$ . Thus  $eb(1-e) = 0$ . Replacing  $x$  with  $xe$ , we have

$$b \left( \sum_{i=0}^n (-1)^i \binom{n}{i} (xe)^i a (xe)^{n-i} \right) - \left( \sum_{i=0}^n (-1)^i \binom{n}{i} (xe)^i a (xe)^{n-i} \right) b = 0$$

for all  $x \in \tilde{Q}$ . Multiplying by  $(1-e)$  from the right hand side and using  $eb(1-e) = 0$ , we obtain  $(-1)^n b (xe)^n a (1-e) - (-1)^n (xe)^n a b (1-e) = 0$  for all  $x \in \tilde{Q}$ . Apply Lemma 1.3.9 to obtain

$$(-1)^n b x e a (1-e) - (-1)^n x e a b (1-e) = 0$$

for all  $x \in \tilde{Q}$ . Since  $b \notin C$ , 1 and  $b$  are  $F$ -independent. By Fact 1,  $ea(1-e)=0$ , a contradiction.

Hence we may assume  $ea(1-e) = 0$  for all  $e = e^2 \in H$ . Then  $e^2 a = ea = eae$  for all  $e = e^2 \in H$ . In particular,  $0 = e(ae - ea) = e[a, e]$  for all  $e = e^2 \in H$ . By [36, Lemma 1.1],  $a \in F$ , implying  $a \in C$ , a contradiction. The proof is thereby complete.  $\square$

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