Solutions of Some Nonlinear Partial Differential Equations

Thesis Submitted

by

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Certificate

This is to certify that the thesis entitled, "**Solutions of Some Nonlinear Partial Differential Equations**" submitted by Anjali Verma in the partial fulfilment of the requirements for the award of the degree of Doctoral of Philosophy in the School of Mathematics, Thapar University, Patiala, is a record of the candidate's own work carried out by her under our supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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Declaration

It is certified that the thesis is entirely my own and that the idea and references cited here in have been duly acknowledged.

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Abstract

The prime objective and motivation in carrying out the proposed study is to demonstrate the importance and efficacy of group theoretic techniques and differential qudrature method to find analytical and numerical solutions of some nonlinear equations viz. family of nonlinear equations, two component shallow water wave system and three-coupled KdV equations, Klein-Gordon equation, Fisher's type equations and two dimensional hyperbolic equations with variable coefficients.

Our thesis comprises of six chapters. In the introductory part some important features of Lie group of transformations and differential quadrature method (DQM) are demonstrated and the mathematical fundamentals of continuous group theory and weighting coefficients of DQM are reviewed which are of great importance to the work dealt in Chapters 2-6.

Chapter 2 is concerned with family of nonlinear evolution equations. In mathematical physics the family of nonlinear evolution equations has been a subject of extensive study. It represents a class of nonlinear evolution equations. We investigate the symmetry of the equation by means of classical Lie symmetry method. The symmetry algebras and groups of family of nonlinear equation are obtained. Specially, the most general one-parameter group of symmetries is given and most general solutions are gained. We used $\left(G'/G\right)$-expansion method to find travelling wave solutions of one ODE. New explicit solutions of equation are derived.

Chapter 3 deals with two-component shallow water system and three-coupled KdV equations engendered by the Neumann system by using Lie classical method. The shallow water equations are a set of hyperbolic partial differential equations that describe the flow below a pressure surface in a fluid. KdV is a mathematical model of waves on shallow water surface. The KdV equation has infinitely many integrals of motion which do not change with time. we studied analytical solutions of two-component shallow water system and three-coupled KdV equations engendered by the Neumann system by Lie classical method. The symmetry group reduces the original equation to the simple one. By Lie classical method we

have investigated the symmetries. After having done standard change of dependent variables we used the basis of the infinitesimal operator written in the new variable in order to find some new classes of invariant solutions.

Chapter 4 is concerned with nonlinear Klein-Gordon equation. The nonlinear Klein-Gordon equation appears in many types of nonlinearities. The Klein-Gordon equation arises in relativistic quantum mechanics and field theory, which is of great importance for the high energy physicists, and is used to model many different phenomena, including the propagation of dislocations in crystals and the behaviour of elementary particles. A numerical scheme has been developed for the numerical simulation of nonlinear Klein-Gordon equation. The essential difficulty in the numerical solutions for the Klein-Gordon equation involves the unboundedness of the physical domain, but in the chapter we considered the nonlinear Klein-Gordon equation on a bounded domain. Gauss-Lobatto-Chebyshev grid points are used for the numerical simulation.

Chapter 5 deals with Fisher type's equations. In this chapter, analytic and numerical solutions of nonlinear diffusion equation of Fisher type's with the help of classical Lie symmetry method and differential qudrature method (DQM) are studied. The Fisher's equation occurs in chemical kinetics and neutron population in a nuclear reaction. Moreover, the same equation also occurs in logistic population growth models, flame propagation, neurophysiology, autocatalytic chemical reactions, and branching Brownian motion processes. Lie symmetry method is utilized to investigate the symmetries and invariant solutions of the equations. The infinitesimal generators in the optimal system are used for reductions and exact solutions. Finally, polynomial differential qudrature method is used to find the numerical solutions of the Fisher's type equations with the help of initial and boundary conditions taken from the analytic solutions obtained by classical Lie symmetry method. It is concluded that the numerical solutions are in good agreement with the analytical solutions.

Chapter 6 is devoted to derive numerical solutions of two dimensional hyperbolic partial differential equation with variable coefficients. Hyperbolic PDEs describe propagation of disturbances in space and time when the total energy if the disturbances remain conserved.

It’s the condition of energy conservation that makes the hyperbolic equation different from parabolic ones. In this chapter, cosine expansion based differential quadrature method extended for numerical simulation of the second order two dimensional hyperbolic equations. The main advantage of the present scheme is that the scheme gives very accurate and similar results to the exact solutions by choosing less number of grid points and the problem can be solved up to big time. The computed numerical results are compared with the results presented in some earlier works and it is found that the present numerical technique gives better results than the others.

List of Research Papers

1. Anjali Verma, Ram Jiwari and Satish Kumar, A numerical scheme based on differential quadrature method for numerical simulation of nonlinear Klein-Gordon equation, *International journal of Numerical methods for Heat and Fluid Flow*, 24 (2014) 1390-1404. **(Emerald, Impact factor 1.399) (SCI)**

2. Anjali Verma, Ram Jiwari and Mehmet Emir Koksal, Analytic and Numerical solutions of nonlinear diffusion equations via symmetry reduction, *Advances in Difference equations*,(2014) 1-13. **(Springer, Impact factor 0.64) (SCI)**

3. Anjali Verma and Ram Jiwari, Cosine expansion based Differential Qudrature algorithm for Numerical simulation of Two Dimensional Hyperbolic equations with variable coefficients, (Accepted in International Journal of Numerical methods for Heat and Fluid Flow). **(Emerald, Impact factor 1.399) (SCI)**

4. Anjali Verma and Ram Jiwari, Satish Kumar and Nisha Goyal, Lie symmetry method for finding exact solutions of family of nonlinear equations. (**Communicated**) (Acta Mathematica Scientia).

5. Anjali Verma and Ram Jiwari, Satish Kumar and Vikas Kumar, Symmetry Reduction and Analytical solutions of coupled equations (**Communicated)**

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Chapter 1

Introduction

1.1 Introduction

Problems of physical interest are often translated in terms of differential equations that may turn out to be linear or nonlinear, ordinary or partial. Partial Differential equations (PDE) play an important role in many branches of mathematical physics, where these equations provide a mathematical description of many natural phenomena. PDEs are the basic fields in the applied analysis area.

In applied mathematics, the studies of physics, fluid dynamics etc. involve many functions which donot depend upon single independent variable. To study the changes in these functions partial derivative with respect to more than one independent variable are required and this situation finally leads to partial differential equation in the considered field.

During the last half of nineteenth century, a large number of mathematicians became active in the investigation of numerous physical problems modelled by partial differential equations in order to see the physical behaviours of the problems. The solutions of partial differential equations can be found in analytical [40, 84, 98, 101, 115, 144, 141, 143, 160, 165, 196] and numerical [12, 13, 47, 60, 62, 184, 254, 255, 213, 214, 229, 230] form by using analytical and numerical methods. Analytic and numerical solutions of partial differential equations play an important role in the proper understanding of various features of many phenomena

It’s well known to us that most of the Ordinary Differential Equations (ODEs) are easier to solve as compared to Partial Differential Equations (PDEs). Thus in many applications it is desirable to reduce PDEs to ODE. The reduced system becomes easy to analyse in terms of both analytically and numerically.

In the thesis, we have used both analytical and numerical methods in order to find exact and numerical solutions. Differential Quadrature Method (DQM) is used for finding numerical solutions and Lie classical method for exact solutions. In current time, much attentation is given for obtaining exact and numerical solutions of nonlinear PDEs.

By using Symmetry group techniques, we can find obtain exact solutions of PDEs. In local language, symmetry is a transformation of an object leaving the object invariant. Lie [123-124] developed the theory of Lie symmetry groups of differential equations in the period of 1872-1899. Lie's classical theory of symmetries of differential equation is an inspiring source for linearzing, conservation laws, for obtaining explicit solutions.

The early expressions of Lie theory are found in books which are composed by Sophus Lie with his associate from 1888 to 1896. Geomety is the origins of Lie theory. Lie's seminal idea was to look at the action infinitesimally. The theory gave relevant results in concrete applied problems with the work of I. Sedov [97] and G. Birkhoff [71] on dimensional analysis, Further, it was the Russian school with L. V. Ovsiannikov [99, 102] that since 1960 began to exploit systematically the methods of symmetry analysis of differential equations in the explicit construction of solutions of any sort of problems, even complicated, of mathematical physics.

For finding exact solutions of nonlinear problems, Lie classical method is the most important applicable method. Lie's theory is a powerful tool to the development of systematic procedures and to the determination of invariant solutions of initial and boundary value problems. It is difficult to handle hundreds of equations to find a single solution. Today, we have access to powerful Computer Algebra Systems like Maple and Mathematica. Some recent contributions are from Nucci [104, 107, 110, 111], Biswas *et al.* [16, 32, 74, 76], Gupta *et al.* [223, 224], Hill [81, 85], Oliveri [65, 66] and Clarkson *et al.* [148, 150].

The basic object mediating between Lie groups and Lie algebras is the one-parameter group. There exist many other methods which are not dependent upon Lie group theory such as Inverse scattering transformation [114], direct method [55, 103], Backlund transformation [41] and Painleve analysis [89]. Presently, a variety of powerful methods such as Hirota method [20], Homogenous balance method [59], F-expansion method [87], Homotopy perturbation method [51], tanh-sech method [21], $\left(\frac{G'}{G}\right)$-expansion method [199] and modified $\left(\frac{G'}{G}\right)$-expansion method [221] etc are developed.

1.2 Preliminaries

A brief summary of a Lie group has been provided in this section.

1.2.1 Definition

A set of elements G is said to be group with composition f satisfying the following axioms:

(i) **Closure Property:** If p and q are elements of G, then $f(a,b)$ is an element of G.

(ii) **Associative Property:** If p,q and r are elements of G:

$$f(p, f(q,r)) = f(f(p,q),r). \tag{1.2.1}$$

(iii) **Identity element:** For any element p of G a unique identity element e of G exists:

$$f(p,e) = f(e,p) = p. \tag{1.2.2}$$

(iv) **Inverse element:** If p is a element of G, unique inverse element p^{-1} in G also exists such that

$$f(p,p^{-1}) = f(p^{-1},p) = e. \tag{1.2.3}$$

1.2.2 Definition: One parameter Lie Groups

Let the vectors $y = (y^1,...,y^n)$ lie in some continuous open set D on the n-dimensional Euclidean manifold R^n. Define the transformation

$$T^{\varepsilon} : \{y^* = Y(y,\varepsilon)\}. \tag{1.2.4}$$

The function Y is differential infinitely with respect to y and an analytical function of real continuous parameter ε, which lie on open interval S. The transformation T^{ε} is a **One-parameter Lie group** with respect to the binary operation of composition if and only if

(i) There is an identity element $\varepsilon \to \varepsilon_0$ such that

$$T^{\varepsilon_0} : \{Y(y;\varepsilon_0) = y\}. \tag{1.2.5}$$

(ii) For every value of ε, there is an inverse $\varepsilon \to \varepsilon_{inv}$ such that

$$T^{\varepsilon_{inv}} : \{Y(y^*;\varepsilon_{inv}) = y\} . \tag{1.2.6}$$

(iii) The binary operation of composition produces a transformation i.e. a member of the group $T^{\varepsilon_1}.T^{\varepsilon_2} = T^{\varepsilon_3}$, i.e., the group is closed. Consider two members of group

$$T^{\varepsilon_1} : \{y^{**} = Y(y^*;\varepsilon_1)\} \tag{1.2.7}$$

and

$$T^{\varepsilon_2} : \{y^* = Y(y;\varepsilon_2)\} \tag{1.2.8}$$

If we compose $T^{\varepsilon_1}, T^{\varepsilon_2}$ we get

$$T^{\varepsilon_3} : \{y^{**} = Y(y;\varepsilon_3)\}. \tag{1.2.9}$$

where $\varepsilon_3 = f(\varepsilon_1, \varepsilon_2) \in S$. The function f defining the law of composition of T^{ε} is an analytic function of $\varepsilon_1 \in S$ *and* $\varepsilon_2 \in S$ is commutative i.e., $f(\varepsilon_1, \varepsilon_2) = f(\varepsilon_2, \varepsilon_1)$; thus lie group is abelian.

(iv)The group is associative, i.e.

$$(T^{\varepsilon_1}.T^{\varepsilon_2}).T^{\varepsilon_3} = T^{\varepsilon_1}.(.T^{\varepsilon_2}.T^{\varepsilon_3}) . \tag{1.2.10}$$

1.2.3 Infinitesimal form of Lie Group

Consider a one-parameter Lie group of the form

$$T^{\varepsilon} : \{y^* = Y(y;\varepsilon)\} \tag{1.2.11}$$

If we expand (1.2.11) about $\varepsilon = 0$ in terms of Taylor's series we get

$$y^* = y + \varepsilon\left(\frac{\partial Y(y;\varepsilon)}{\partial \varepsilon}|\varepsilon = 0\right) + \frac{\varepsilon^2}{2}\left(\frac{\partial^2 Y(y;\varepsilon)}{\partial \varepsilon^2}|\varepsilon = 0\right) + \ldots \tag{1.2.12}$$

$$= y + \varepsilon\left(\frac{\partial Y(y;\varepsilon)}{\partial \varepsilon}|\varepsilon = 0\right) + o(\varepsilon^2).$$

The derivatives of $Y(y;\varepsilon)$ with respect to group parameter ε evaluated at $\varepsilon = 0$ are called the infinitesimals of the group and are denoted by $\xi(y)$, where $\xi(y) = \left(\frac{\partial Y(y;\varepsilon)}{\partial \varepsilon}|\varepsilon = 0\right)$.

1.2.4 Lie series, Group Operator and Infinitesimal Invariance Conditions for Function

Necessary and sufficient condidtion for a function $\phi(y)$ is to be invariant under Lie group of transformation (1.2.4) .If

$$\phi(y^*) = \phi(y) \tag{1.2.13}$$

or equivalently, $\phi(y)$ satisfies the condition

$$\sum_{i=1}^{n} \xi^i(y) \frac{\partial \phi(y)}{\partial y^i} = 0 \tag{1.2.14}$$

The operator

$$Y = Y(y) = \sum_{i=1}^{n} \xi^i(y) \frac{\partial}{\partial y^i} \quad . \tag{1.2.15}$$

is called the **group operator** or **infinitesimal generator** and $Y(\phi)$ is called Lie derivative of ϕ. Proceeding to higher order derivatives we can write $\phi(y^*)$ as a Taylor series around $\varepsilon = 0$:

$$\phi(y^*) = \phi(y) + \varepsilon Y\phi(y) + \frac{\varepsilon^2}{2} Y^2\phi(y) + ... + \frac{\varepsilon^n}{n!} Y^n\phi(y) + o(\varepsilon^{n+1}) \tag{1.2.16}$$

If the series converges, it is termed as **Lie series** and

$$\phi(y^*) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} Y^n \phi(y) \tag{1.2.17}$$

This can be written as

$$\phi(y^*) = \exp(\varepsilon y)\phi(y). \tag{1.2.18}$$

1.2.5 Lie Algebra

For the Lie group of transformation with infinitesimal generators V_1, V_2 the commutator (Lie bracket) of V_1, V_2 is the first order operator defined by

$$[V_1, V_2] = V_1V_2 - V_2V_1 \tag{1.2.19}$$

The commutator has following properties:

1. Bilinearity: $[pV_1 + qV_2, V_3] = p[V_1, V_3] + q[V_2, V_3]$ (1.2.20)

$$[V_1, pV_2 + qV_3] = p[V_1, V_2] + q[V_1, V_3]. \tag{1.2.21}$$

where $p, q \in R$.

2. Skew symmetry: $[V_1, V_2] = -[V_2, V_1]$. (1.2.22)

3. Jacobi Identity:

$$[V_1,[V_2,V_3]]+[V_2,[V_3,V_1]]+[V_3,[V_1,V_2]]=0\,. \tag{1.2.23}$$

Any vector space satisfying above three properties is called **Lie Algebra**.

1.2.6 Definition

Let G be a Lie group with Lie algebra L. If vector $Y \in L$, the adjoint vector $Ad\,Y\ at\ Z \in L$ is

$$Ad\,Y/Z=[Z,Y]=-[Y,Z] \tag{1.2.24}$$

The Lie group can be constructed by summing the Lie series

$$\begin{aligned} Ad(\exp \varepsilon Y)Z_0 &= \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}(adY)^n Z_0 \\ &= Z_0-\varepsilon[Y,Z_0]+\frac{\varepsilon^2}{2}[Y,[Y,Z_0]]-... \end{aligned} \tag{1.2.25}$$

1.2.7 Classification of Subgroup and Group Invariant Solutions

Let G be a Lie group. An optimal system of s-parameter subgroups is a list of equivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter sub algebras forms an optimal system if every s-parameter sub algebra of the Lie algebra L is equivalent to a unique member of the list under some element of the ad joint representation.

The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of sub algebras. For one dimensional sub algebras, this Classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional sub algebra is determined by a nonzero vector in L .

The classification done by associated Lie sub algebras with respect to the ad joint representation [68, 93] and to compute the ad joint representation, Lie series is used

$$Ad(\exp \varepsilon Y)Z_0 = Z_0 - \varepsilon[Y, Z_0] + \frac{\varepsilon^2}{2}[Y,[Y, Z_0]] - ... \tag{1.2.26}$$

Let $V_1, V_2 ....V_r$ be the basis of Lie algebra, then we start with general infinitesimals

$$V = P_1V_1 + P_2V_2 + ... + P_rV_r \ . \tag{1.2.27}$$

and simplify it as much as possible by means of adjoint actions. We will find the list of in equivalent one-dimensional sub algebras. On using the inequivalent one-dimensional sub algebras of the maximal Lie invariance algebra, group invariant reductions can be easily carried out.

1.2.8 Invariance for a System of PDEs

Let a system of NPDEs with m dependent variables $v = (v^1,...,v^m)$ and n independent variables $y = (y_1,..., y_n)$ given by

$$F^{\mu}(y, v, \partial v,..., \partial^l v) = 0, \ \mu = 1,..., N \ . \tag{1.2.28}$$

1.2.9 Definition

The one parameter Lie group of point transformations

$$y^* = Y(y, v; \varepsilon) \tag{1.2.29a}$$

$$v^* = V(y, v; \varepsilon) \ . \tag{1.2.29b}$$

If and only if its *lth* extension leaves invariant the N surface in $(y, v, \partial v,..., \partial^l v)$ space, then it is a point symmetry admitted by (1.2.28)

Theroem 1.1 (Infinitesimal criterion).

Let

$$Y=\xi_i(y,v)\frac{\partial}{\partial y_i}+\eta^u(y,v)\frac{\partial}{\partial v^u} \qquad (1.2.30)$$

the equation (1.2.30) be infinitesimal generator of the Lie group of point transformations

Consider

$$Y=\xi_i(y,v)\frac{\partial}{\partial y_i}+\eta^u(y,v)\frac{\partial}{\partial v^u}+\eta_i^{(1)^v}(y,v,\partial v)\frac{\partial}{\partial v_i^{\;u}}+....+\eta_{i_1,..,i_k}^{\;\;(l)^u}(y,v,\partial v..\partial^l v)\frac{\partial}{\partial v^u{}_{i_1,..,i_l}}$$

(1.2.31)

be infinitesimal generator of (1.2.30) *lth*-extended, where

$$\eta_i^{(1)u}=D_i\eta^\mu-(D_i\xi_j)v_j^{\;\mu} \qquad (1.2.32)$$

and $\eta_{i_1,..i_j}^{\;\;(j)\mu}=D_i^{\;j}\eta^{(j-1)\mu}{}_{i_1,..i_{j-1}}-(D_{i_j}\xi_j)v^\mu{}_{i_1..i_{j-1}}$, (1.2.33)

$v=1,2...m$ and $i_j=1,2..n$ for $j=1,2..k$ in terms of $\xi(y,v)=(\xi_1(y,v),\xi_2(y,v)....\xi(y,v))$, $\eta(y,v)=(\eta^1(y,v),\eta^2(y,v)...\eta^m(y,v))$. Then the one-parameter Lie group of point transformations is admitted by the system of PDEs iff

$$Y^{(l)}F^\sigma(y,v,\partial v,\partial^2 v,...,\partial^k v)=0 \qquad (1.2.34)$$

when v satisfies (1.2.28) for each $\sigma=1,2,...,N$.

1.2.10 Determining Equations

Let a system (1.2.28) of PDEs

$$v^{u_\mu}{}_{i_1,i_2,...,i_{l\mu}}=f^\mu(y,v,\partial v,\partial^2 v,...,\partial^l v)=0 \qquad (1.2.35)$$

In terms of some specific l_μth-order partial derivative of v^{u_μ} for some $u_\mu=1,2,...,m$ where $f^\mu(y,v,\partial v,\partial^2 v,..,\partial^l v)$ is independent explicitly $v^{u_\sigma}{}_{i_1,i_2,..i_{k\mu}}$, $\sigma=1,2,...,N,\ \mu=1,2,...,N$. From theorem 1.1, the point symmetry admits by system of PDEs (1.2.35)

$$Y=\xi_i(y,v)\frac{\partial}{\partial y_i}+\eta^u(y,v)\frac{\partial}{\partial v^u} \qquad (1.2.36)$$

with the *l*th extension of (1.2.36) given by (1.2.16), if and only if

$$\eta_{i_1..i_{k_\mu}}^{\;\;k_\mu(u_\mu)}=\xi_j\frac{\partial f^\mu}{\partial y_j}+\eta^v\frac{\partial f^\mu}{\partial y^u}+\eta_j^{(1)^u}\frac{\partial f^\mu}{\partial v_j^{\;u}}+...+\eta_{j_1,..j_l}^{\;\;(l)u}\frac{\partial f^\mu}{\partial v_{j_1...j_l}^{\;\;u}},\ \mu=1,...,N \qquad (1.2.37)$$

with

$$v^{u_\sigma}{}_{i_1 \ldots i_{k\sigma}} = f^{\sigma}(y, v, \partial v, \partial^2 v, \ldots, \partial^l v) = 0, \ \sigma = 1,2..N. \qquad (1.2.38)$$

we can easily see that $\eta^{(s)u}{}_{j_1,\ldots,j_s}$ is a polynomial in the components of $\partial v, \partial^2 v, \ldots, \partial^s v$. The coefficients are linear and homogeneous in the components of $\xi(y,v)$, $\eta(y,v)$. The order of derivative is s. First we eliminate the components $v^{u_\sigma}{}_{i_1 \ldots i_{l\sigma}}$ and their differential consequences from (1.2.37). Comopnent y, v and the remaining components of $\partial v, \partial^2 v, \ldots, \partial^l v$ that are in the symmetry determining equations (1.2.37) are independent variables. Since the resulting expression for (1.2.37) holds for any values of these independent variables, one obtains a system of linear homogeneous PDEs for ξ *and* η that constitutes a set of determining equations for the infinitesimal generators X admitted by the given system of PDEs (1.2.28). In particular, if each $f^{\mu}(y, v, \partial v, \partial^2 v, \ldots, \partial^l v), \mu = 1,2,\ldots,N$ is a Polynomial in the components of $\partial v, \partial^2 v, \ldots, \partial^l v$ then the system of symmetry determining equations (1.2.37) yields polynomial equations in the independent components of $\partial v, \partial^2 v, \ldots, \partial^l v$. The coefficients of these polynomial equations must vanish separately. This yields the set of linear determining equations for ξ *and* η. Typically, the numbers of determining equations are far greater than $n + m$.

1.2.11 Definition

An invariant solution resulting from admitted point symmetry with infinitesimal generator (1.2.27) is $v = \Theta(y)$, with components $v^u = \Theta(y), u = 1,2,\ldots,m$ if and only if:

(i) $v^u = \Theta^u(y)$ is an invariant surface of (1.2.30) for each $u = 1,2,\ldots,m$

(ii) $v = \Theta(y)$ solves (1.2.28).

$v = \Theta(y)$ is an invariant of the system of PDEs (1.2.28), if and only if $v = \Theta(y)$ satisfies:

(i) $Y(v^u - \Theta(y)) = 0$ when $v = \Theta(y), u = 1,2,\ldots,m$; i.e.,

$$\xi_i(y, \Theta^u(y)) \frac{\partial \Theta(y)}{\partial y_i} = n^u(y, \Theta(y)), u = 1,2,\ldots,m; \qquad (1.2.39)$$

(ii) $F^{\mu}(y, v, \partial v, \ldots, \partial^l v) = 0$ *when* $v = \Theta(y), \mu = 1,\ldots,N$ i.e.,

$$F^{\mu}(y, \Theta(y), \partial\Theta(y), \ldots, \partial^l \Theta(y)) = 0, \mu = 1,\ldots,N \qquad (1.2.40)$$

where $\partial^j \Theta(y)$ denotes $\dfrac{\partial^j \Theta^u(y)}{\partial y_{i_1},...,\partial y_i}, u=1,...,n \quad for \ j=1,2,...,l$.

Equation (1.2.39) is the invariant surface conditions for the invariant solutions of the system of PDEs (1.2.40). Invariant solutions can be determined by the following procedure:

Invariant form Method

Solve explicitly corresponding characteristics equations for $v=\Theta(y)$ given by

$$\frac{dy_1}{\xi_1(y,v)} = \frac{dy_2}{\xi_2(y,v)} = \frac{dy_n}{\xi_n(y,v)} = \frac{dv^1}{\eta^1(y,v)} = \frac{dv^2}{\eta^2(y,v)} = ...\frac{dv^m}{\eta^m(y,v)} \tag{1.2.41}$$

If $z_1(y,v), z_2(y,v),..., z_{n-1}, u^1(y,v), u^2(y,v),..., u^m(y,v)$ are $n+m-1$ functionally independent constants that arise from solving the system of $n+m-1$ first order ODEs (1.2.41) with the jacobian $\dfrac{\partial(u^1,u^2,...,u^m)}{\partial(v^1,v^2,...,v^m)} \neq 0$, then the general solution $v=\Theta(y)$ of the system of ODEs (1.2.32) is given implicitly

$$u^u(y,v) = \phi^u(z_1(y,v), z_2(y,v),..., z_{n-1}(y,v)) \tag{1.2.42}$$

where ϕ^u is an arbitrary differentiable function of

$z_1(x,u), z_2(x,u),..., z_{n-1}(x,u)$ for $u=1,2,...,m.$

Note that $y_1(x,u), y_2(x,u),..., y_{n-1}, v^1(x,u), v^2(x,u),..., v^m(x,u)$ are $n+m-1$ functionally independent group invariants of (1.2.30). Let $z_n(y,v)$ be $(n+m)th$ canonical coordinate satisfying

$$Yz_n = 1 \tag{1.2.43}$$

If we transform the system of PDEs (1.2.28) which contains independent variables $z_1,..., z_n$ and dependent variables $u^1,...,u^m$ then the transformed system of PDEs admits the one-parameter Lie group of transformations.

$$\begin{aligned} z^*{}_i &= z_i, \ i=1,2,...,n-1, \\ z^*{}_n &= z_{n+\varepsilon}, \\ u^{*^u} &= u^u, \ u=1,2,...,m. \end{aligned} \tag{1.2.44}$$

The variable z_n does not appear explicitly in the transformed system of ODEs and the solutions of transformed system of ODEs in the form (1.2.42). The solutions of PDEs (1.2.28) contain invariant solutions by the invariant form (1.2.30) given implicitly. The solutions are found by solving a reduced system of differential equations with $n-1$ independent variables $z_1,...,z_{n-1}$ and m dependent variables $u^1,...,u^m$. The variables $z_1,...,z_{n-1}$ are commonly called similarity variables. We can find the reduced system of differential equations by substituting the invariant form (1.2.42) into the given system of PDEs (1.2.28).

1.2.12 Methodology

In present work, we are dealing with the method of group invariant techniques, which are based on the theory of group transformation known as '**Lie groups'** which acts on the space of dependent and independent variables of the system. We also use Direct methods: $\left(\frac{G'}{G}\right)$-expansion method on some nonlinear partial differential equations in order to find exact solutions. For numerical solutions, we mainly use Differential Qudrature Method (DQM). We now provide the brief outlines of the methods mentioned above.

Because of simplicity, the group theoretic techniques are used to solve many equations. One arrives at an over determined linear system of differential equations for the group infinitesimals, when a system of PDEs is subject to invariance under one-parameter Lie group of transformation. To obtain the reduction of the system the infinitesimals of the transformation helps us.

1.2.12.1 Lie's Algorithm

The stepwise procedure of Lie's method is as follows:

Let a system of NPDEs with *m*, $v=(v^1,...,v^m)$ and *n*, $y=(y_1,..,y_n)$ dependent variables and independent variables respectively

$$F^{\mu}(y,v,\partial v,\partial^2 v,...,\partial^l v)=0, \mu=1,...,N. \tag{1.2.45}$$

1. Consider the one-parameter Lie group of point transformations (1.2.26a, b) is invariant

2. Apply the prolonged operator $Y^{(l)}$ given by (1.2.31) to each equation of the system (1.2.45) and require that

$$Y^{(l)}F^{\mu}\,|_{F^{u}=0}=0 \qquad \mu,u=1,2...N \tag{1.2.46}$$

The condition (1.2.46) means that $Y^{(l)}$ disperse on the solution set of the originally (1.2.45).

3. As defined in section (1.2.13) a system of linear PDEs that comprise the set of determining equations for the infinitesimal generator Y admitted by the given system of PDEs (1.2.45) is obtained.

4. Solving determining equations, the solutions will lead to the explicit form of ξ and η.

5. A new independent variable u in terms of $n-1$ is obtained by solving the characteristics equations (1.2.41).

6. To get the reduced form of the system rework the system (1.2.45).

1.2.12.2 The (G'/G) - Expansion Method

In 2008, Wang et al. [112] introduced a method for finding travelling wave solutions called (G'/G) - expansion method. According to this method, firstly we suppose that a nonlinear equation say into two independent variables x and t is given by

$$p(v,v_y,v_t,v_{yy},v_{tt},v_{yt},...)=0 \tag{1.2.47}$$

where p is polynomial in $v=v(y,t)$ and $v=v(y,t)$ are unknown function and its partial derivatives in which highest order derivatives and the nonlinear terms are involved. The main steps of this method are given below

Step 1 Let us suppose that $v=v(y,t)$. The travelling wave variable allows us reducing to an ODE for $v=v(\xi)$

$$p(v,v',v'',...)=0 \tag{1.2.48}$$

the prime denotes the derivative with respect to ξ.

Step 2 According to this method we suppose that solution of equation is in polynomial in (G'/G) as follows:

$$v(\xi)=\sum_{i=0}^{N} d_i\left(\frac{G'(\xi)}{G(\xi)}\right) \quad . \tag{1.2.50}$$

where d_i are real constants with $d_i \neq 0$ to be determined N is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \alpha G'(\xi) + \beta G(\xi) = 0. \qquad (1.2.51)$$

where α and β are real constants to be determined.

Step 3 In this step, substitue (1.2.50) into (1.2.49) and using second order linear ordinary differential equation (1.2.51). All the terms with same order of (G'/G) together are separated, the equation (1.2.49) is converted into another polynomial in (G'/G). Now, equate each coefficient of polynomial to zero. Then we get algebraic equations for $d_i, ..., \alpha$ and β.

Step 4 The general solution of equation (1.2.51) has been well known for us, then substituting d_i, d and general solution of equation (1.2.51) into (1.2.50). We have travelling wave solutions of nonlinear partial differential equation (1.2.48).

1.3 Numerical Solutions of Partial Differential equations

There are many PDEs in the literature which can't be solved analytically. So these solutions can be approximated numerically. In the literature there are many numerical techniques to find the numerical solutions of partial differential equations among them are finite difference method (FD) [40, 92], finite volume method (FV) [10, 13], finite element method (FE) [40] etc.

In the nineteen century, almost all the engineering problems were solved by low order methods such as finite difference method, finite volume method and finite element method using a large number of grid points. But presently, Differential Qudrature method (DQM) is a higher order numerical technique used for solving partial differential equations. In DQ method the derivative of a function is approximated by a weighting sum of function values at all discrete points. Richards Bellman and his associates [166] introduced the method of Differential Qudrature method in the early 1970s. Important contributions to this method and its applications have been made by many Scholars. To simplify computing efforts of the evaluation of DQ weighting coefficients for higher order derivatives, Mingle [90] proposed linear transformations in the DQ method. Civan *et al* [68] extended this method to multi-dimensional problems and integro-differential equations. The explicit formulas to compute

accurately and efficiently the DQ weighting coefficients for the first and second order derivative was given by Qian *et al.* [97]. Chang et al. [43] employed the proper functions as basic function instead of polynomial function in the DQ method for dealing with problems involving steep gradients successfully. Recently, the DQ method is mostly used to analyze deflections, vibration and buckling of linear and nonlinear structural components by Bert et al. [44], jang et al. [200], striz et al. [17], Kukretic et al. [24] and Malik [117]. DQM has been efficiently employed in variety of problems such as engineering and physical sciences.
Some basic definitions and fundamentals for weighting coefficients of Differential Qudrature method are given in this section.

1.3.1 Weighting Coefficients for the First Order Derivative

Consider a one dimensional problem over a closed interval $\left[p,q\right]$ and there are N grid points with coordinates as $p = y_1,...,y_N = q$. It is assumed that a function $f(y)$ is sufficiently smooth over the interval $\left[p,q\right]$ so that its first order derivative $f^{(1)}(y)$ at any grid point can be approximated by the following formulation.

$$f_y^{(1)}(y_i) = \sum_{j=1}^{N} p_{ij} f(y_j), i = 1,2,..., N \tag{1.3.1}$$

where $f(y_j)$ represents the functional value at a grid point y_j, $f_y^{(1)}(y_i)$, indicates the first order derivative of $f(y)$ at y_i and p_{ij} is the weighting coefficients of the first order derivative. The key procedure is determination of weighting coefficients.

1.3.2 Bellman's Approach

Bellman et al. [166] proposed two approaches to compute the weighting coefficient p_{ij} in equation (1.3.1). Both approaches based on two different test functions.

Bellman's first approach: In this approach following test function is chosen:

$$g_l(y) = y^l, \quad l = 0,1,..., N-1 \ . \tag{1.3.2}$$

equation (1.3.2) gives N test functions. In equation (1.3.1) i and j are taken 1 to N. The total number of weighting coefficients is $N \times N$. To obtain these weighting coefficients, the N test functions should be applied at N grid points $y_1, y_2, ..., y_N$. For p_{ij} the following $N \times N$ algebraic equations are obtained:

$$\begin{cases} \sum_{j=1}^{N} p_{ij} = 0 \\ \sum_{j=1}^{N} p_{ij} . y_j = 1 \qquad\qquad \text{for } i = 1,2,...,N \ . \\ \sum_{j=1}^{N} p_{ij} . y_j{}^{k} = l . y_i{}^{l-1}, \ l = 2,3...N-1 \end{cases} \tag{1.3.3}$$

System of equations (1.3.3) has a unique solution because its matrix is of vandermonde form. Unfortunately, when N is large, the matrix is ill-conditioned and its inversion is difficult. Usually, N is chosen less than 13.

Bellman's second approach: Bellman's second approach is similar to first approach, but uses different test functions

$$g_l(y) = \frac{L_N(y)}{(y - y_1) . L_N{}^{(1)}(y_l)}, \ l = 1,2,..., N \ . \tag{1.3.4}$$

where $L_N(y)$ is the Legendre Polynomial of degree N and $L_N{}^{(1)}(y)$ is the first order derivative of $L_N(y)$. By choosing y_l to be the root of the shifted Legendre Polynomial and applying equation (1.3.4) at N grid points $y_1, ..., y_N$. Bellman et al. [99] obtained a simple algebraic formulation to compute p_{ij}.

$$p_{ij} = \frac{L_N{}^{(1)}(y)}{(y_i - y_j) . L_N{}^{(1)}(y_j)} \qquad \textit{for} \ \ j \neq i \tag{1.3.5a}$$

$$p_{ii} = \frac{1 - 2y_i}{2y_i(y_i - 1)} \qquad \textit{for} \ \ j = i \ . \tag{1.3.5b}$$

Using equation (1.3.5) the computation of weighting coefficient is simple. However, this approach is not as flexible as the first approach because the coordinates of the grid points in this approach cannot be chosen arbitrary. They should be chosen as the root of the Legendre

Polynomial of degree N . So equation (1.3.5) is a special case. In practical applications, first approach is usually adopted.

1.3.3 Quan and Chang's Approach

In computing the weighting coefficients, many attempts have been made to improve Bellman's approach. One of the most useful approach was introduced by Quan and Chang [94, 97]. Quan and Chang used following Lagrange's interpolation Polynomial as the test function.

$$g_l(y) = \frac{M(y)}{(y - y_l).M^{(1)}(y_l)}, \quad l = 1,2,..., N \tag{1.3.6}$$

where $M(y) = (y - y_1)(y - y_2)...(y - y_N)$ (1.3.7)

$$M^{(1)}(y_i) = \prod_{l=1, l\neq i}^{N} (y_i - y_l) \tag{1.3.8}$$

After applying equations (1.3.6) to N grid points, they obtained the following algebraic formulations to compute the weighting coefficient p_{ij}

$$p_{ij} = \frac{1}{(y_j - y_i)} \prod_{l=1, l\neq i}^{N} \frac{l_i - l_l}{l_j - l_l}, \quad for \;\; j \neq i \tag{1.3.9a}$$

$$p_{ii} = \sum_{l=1, l\neq i}^{N} \frac{1}{y_i - y_l} \tag{1.3.9b}$$

There is no restriction on the choice of grid points in the equation (1.3.9)

1.3.4 Shu's general approach

Shu's general approach was inspired from Bellman's approach. It covers all the weighting coefficients by all approaches are same due to polynomial approximation and linear vector space. Shu uses the following sets of the base polynomials:

$$g_l(y) = \frac{M(y)}{(y - y_l)M^{(1)}(y_l)}, \quad l = 1,..., N \tag{1.3.10a}$$

$$g_l(y) = y^{l-1}, \qquad l = 1,2...N \tag{1.3.10b}$$

where $M(y)$ and $M^{(1)}(y_i)$ are the same as in equation (1.3.7) and (1.3.8), but the computation of weighting coefficients is different. For simplicity, we set

$$M(y) = N(y, y_l).(y - y_l), \quad l = 1,2,..., N \tag{1.3.11}$$

with $N(y_i, y_j) = M^{(1)}(y_i).\Delta_{ij}$ where Δ_{ij} is the kronecker operator. Using equation (1.3.11), equation (1.3.10) can be simplified to

$$g_l(y) = \frac{N(y, y_l)}{M^{(1)}(y_l)}, \quad l = 1,..., N \tag{1.3.12}$$

Substituting equation (1.3.11) into (1.3.1) gives

$$p_{ij} = \frac{N^{(1)}(y_i, y_j)}{M^{(1)}(y_j)} \tag{1.3.13}$$

In equation (1.3.13), $M^{(1)}(y_j)$ can be easily computed from the equation (1.3.8). To evaluate $N^{(1)}(y_i, y_j)$, we successively differentiate equation (1.3.11) with respect to y and obtain the following recurrence formulation

$$M^{(m)}(y) = N^{(m)}(y, y_l).(y - y_l) + m.N^{(m-1)}(y, y_l) \tag{1.3.14}$$

for $m = 1,2,..., N-1;\ l = 1,2,..., N$

where $M^{(m)}(y)$ and $N^{(m)}(y, y_l)$ indicate the *mth* order derivative of $M(y)$ and $N(y, y_l)$. From equation (1.3.14), we can obtain the expression for $N^{(1)}(y_i, y_j)$ as

$$N^{(1)}(y_i, y_j) = \frac{M^{(1)}(y_i)}{y_i - y_j} \qquad \textit{for} \quad j \neq i \tag{1.3.15a}$$

$$N^{(1)}(y_i, y_i) = \frac{M^{(2)}(y_i)}{2} \tag{1.3.15b}$$

Substituting equation (1.3.15) into equation (1.3.13) we obtain

$$p_{ij} = \frac{M^{(1)}(y_i)}{(y_i - y_j)M^{(1)}(y_j)} \qquad \textit{for} \quad j \neq i \tag{1.3.16a}$$

$$a_{ii} = \frac{M^{(2)}(y_i)}{2.M^{(1)}(y_i)} \tag{1.3.16b}$$

If x_i is given, it is easy to compute $M^{(1)}(y_i)$ from equation (1.3.16), and hence p_{ij} for $i \neq j$. The calculation of p_{ii} is not easy to calculate. This difficulty can be eliminated by using the second set of base polynomial (1.3.8). According to linear vector space property, if one set of base polynomials satisfies a linear operator, say equation (1.3.1) so does another set of base polynomials. The system of equations for determination of p_{ij} derived from the Lagrange interpolation polynomial equation (1.3.10a) should be equivalent to that derived from another set of base polynomials $y^{l-1}, l = 1,2,..., N$ equations (1.3.11b). Thus p_{ij} satisfies the following equations which are obtained by base polynomial y^{l-1} where $l = 1$

$$\sum_{j=1}^{N} p_{ij} = 0 \quad \text{or} \quad p_{ii} = -\sum_{j=1, j\neq i}^{N} p_{ij} \tag{1.3.17}$$

equation (1.3.16a) and (1.3.17) are two formulations to compute the weighting coefficients a_{ij}.

1.3.5 Weighting Coefficients for the second order derivative

For second order derivative, we introduced a similar approximation form given by

$$f_y^{(2)}(y_i) = \sum_{j=1}^{N} q_{ij} f(y_j), \qquad i = 1,2,..., N. \tag{1.3.18}$$

where $f_y^{(2)}(y_i)$ indicates the second order derivative of $f(y)$ at y_i, and q_{ij} is the weighting coefficient of the second order derivative. Obviously equation (1.3.18) is a linear operator. The equation (1.3.18) is same as equation (1.3.1). The difference between these equations is the use of different weighting coefficients. Two approaches can be used to determine q_{ij} that are following:

1.3.6 Quan and Chang's Approach

In this approach Quan and Chnag [94, 97] used the Lagrange Interpolation polynomials as the test functions and then derived

$$q_{ij} = \frac{2}{(y_j - y_i)}\left(\prod_{l=1, l\neq i,j}^{N} \frac{y_i - y_l}{y_j - y_l}\right)\left(\sum_{k=1,k\neq i}^{N} \frac{1}{x_i - x_k}\right), \; for \; i \neq j \tag{1.3.19a}$$

$$q_{ii} = 2\sum_{l=1,l\neq i}^{N}\left[\frac{1}{y_i - y_k}\left(\sum_{k=l+1,k\neq i}^{N}\frac{1}{y_i - y_k}\right)\right]. \tag{1.3.19b}$$

1.3.7 Shu's General Approach

Shu's [46] general approach is also based on polynomial approximation and linear vector space analyses. Two sets of base polynomials (1.3.10) are used substituting equations (1.3.12) into equations (1.3.18) gives

$$q_{ij} = \frac{N^{(2)}(y_i, y_j)}{M^{(1)}(y_j)} \tag{1.3.20}$$

from equation (1.3.14) we obtain

$$N^{(2)}(y_i, y_j) = \frac{M^{(2)}(y_i) - 2N^{(1)}(y_i, y_j)}{y_i - y_j} \quad for \quad i \neq j \tag{1.3.21a}$$

$$N^{(2)}(y_i, y_i) = \frac{M^{(3)}(y_i)}{3} \tag{1.3.22b}$$

Substituting equations (1.3.21) into equation (1.3.20) yields

$$q_{ij} = \frac{M^{(2)}(y_i) - 2N^{(1)}(y_i, y_j)}{(y_i - y_j).M^{(1)}(y_j)} \quad for \;\; i \neq j \tag{1.3.23a}$$

$$q_{ii} = \frac{M^{(3)}(y_i)}{3M^{(1)}(y_i)} \tag{1.2.23b}$$

finally, by substituting equation (1.3.16) into equation (1.3.22a) we obtain

$$q_{ij} = 2p_{ij}\left(p_{ii} - \frac{1}{y_i - y_j}\right) \quad for \; i \neq j \tag{1.3.24}$$

when $i \neq j, q_{ij}$ can be easily computed from equation (1.3.24). However, from equation (1.3.22a), it can be seen that computation of q_{ii} involves the third order derivative $M^{(3)}(y_i)$ which cannot be computed easily. Similarly to the analysis for the case of first order derivative, the equation system for q_{ij} derived from the Lagrange Interpolation polynomials (1.3.22a) should be equivalent to that derived from another set of base polynomials y^{l-1}, $l = 1,2...,.N$ (1.3.24b). Thus q_{ij} satisfies the following equation which is obtained by base polynomial y^{l-1} when $l = 1$

$$\sum_{j=1}^{N} q_{ij} = 0 \qquad \text{or} \qquad q_{ii} = -\sum_{j=1,\, j\neq i}^{N} q_{ij} \tag{1.3.25}$$

for the application of Shu's general approach, q_{ij} is firstly computed from equation (1.3.24) when $i \neq j$ and q_{ii} is computed from equation (1.2.25)

1.3.8 Shu's Recurrence Formulation for higher order derivatives

For the discretization of higher order derivatives, the following two linear operators are applied.

$$f_y^{(m-1)}(y_i) = \sum_{j=1}^{N} w_{ij}^{(m-1)} f(y_j) \tag{1.3.26}$$

$$f_y^{(m)} = \sum_{j=1}^{N} w_{ij}^{(m)} f(y_j) \tag{1.3.27}$$

For $i = 1,2,...,N;\ m = 2,3,...,N-1$

where $f_y^{(m-1)}(y_i), f_y^{(m)}(y_i)$ indicates the $(m-1)^{th}$ and $(m)^{th}$ order derivatives of $f(y)$ with respect to y at y_i. $w_{ij}^{(m-1)}, w_{ij}^{(m)}$ are the weighting coefficient related to $f_y^{(m-1)}(y_i)$ and $f_y^{(m)}(y_i)$. Two sets of base polynomials will also be used to derive explicit formulation for $w_{ij}^{(m)}$. The first set of base polynomials is given by equations (1.3.12). Substituting equations (1.3.12) into equations (1.3.26) and (1.3.27) gives

$$w_{ij}^{(m-1)} = \frac{N^{(m-1)}(y_i, y_j)}{M^{(1)}(y_j)} \tag{1.3.28}$$

$$w_{ij}^{(m)} = \frac{N^{(m)}(y_i, y_j)}{M^{(1)}(y_j)} \tag{1.3.29}$$

By rewritten equation (1.3.28), we obtain

$$N^{(m-1)}(y_i, y_j) = w_{ij}{}^{(m-1)} .M^{(1)}(y_j) \tag{1.3.30}$$

equation (1.3.30) is valid for any i and j on the other hand, from the recurrence formulation (1.3.14) we have

$$N^{(m-1)}(y_i, y_i) = \frac{M^{(m)}(y_i)}{m} \tag{1.3.31}$$

$$N^{(m)}(y_i, y_j) = \frac{M^{(m)}(y_i) - mN^{(m-1)}(y_i, y_j)}{y_i - y_j} \quad for\ i \neq j \tag{1.3.32}$$

$$N^{(m)}(y_i, y_i) = \frac{M^{(m+1)}(y_i)}{m+1} \tag{1.3.33}$$

Substituting equation (1.3.31) into equation (1.3.32) leads to

$$N^{(m)}(y_i, y_j) = \frac{m(N^{(m-1)}(y_i, y_i) - N^{(m-1)}(y_i, y_j)}{y_i - y_j} \quad for\ \ i \neq j \tag{1.3.34}$$

equation (1.3.34) can be further simplified using equation (1.3.30)

$$N^{(m)}(y_i, y_j) = \frac{m(w_{ii}{}^{(m-1)} M^{(1)}(y_i) - w_{ij}{}^{(m-1)} M^{(1)}(y_j))}{y_i - y_j} \quad for\ i \neq j \tag{1.3.35}$$

Substituting equation (1.3.35) into equation (1.3.29) using equation (1.3.16a), a recurrence formulation is obtained as follows:

$$w_{ij} = m\left(p_{ij} w_{ii}{}^{(m-1)} - \frac{w_{ij}{}^{(m-1)}}{y_i - y_j} \right) \ for\ i, j = 1,..., N;\ m = 2,..., N-1 \tag{1.3.36}$$

where p_{ij} is the weighting coefficient of the first order derivative described above. The formulation for $w_{ii}{}^{(m)}$ can be obtained by substituting equation (1.3.32) into equation (1.3.28), which gives

$$w_{ii}^{(m)} = \frac{M^{(m+1)}(y_i)}{(m+1).M^{(1)}(y_i)} \qquad for \;\; i, j = 1,..., N; m = 2,..., N-1. \qquad (1.3.37)$$

Its way to compute the weighting coefficients $w_{ij}^{(m)}$ for $i \neq j$ from (1.3.36), but it is very difficult to apply equation (1.3.37) to compute the weighting coefficients $w_{ii}^{(m)}$. This difficulty can be overcome by the properties of linear vector space. In terms of the analysis of the $N-$dimensional linear vector space, the equation system for $w_{ij}^{(m)}$ derived from the Lagrange's interpolation polynomials should be equivalent to that derived from the base polynomials y^{l-1}, $l = 1,2,..., N$. Thus $w_{ij}^{(m)}$ should satisfy the following equation obtained from base polynomial y^{l-1} when $l = 1$

$$\sum_{j=1}^{N} w_{ij}^{(m)} = 0 \quad \text{or} \quad w_{ii}^{(m)} = -\sum_{j=1, j=i}^{N} w_{ij}^{(m)} \qquad (1.3.38)$$

From this formulation, $w_{ii}^{(m)} (i \neq j)$

1.3.9 Weighting Coefficients for multi-dimensional case

In general, most of the problems are related to two or three dimensional cases. In this section, we will explain Cosine expansion–based differential Quadrature method for two dimensional cases.

1.3.9.1 Cosine Expansion –based Differential Quadrature Method

If a function $v(y,z,t)$ is sufficiently smooth over the whole domain, the first and second order partial derivatives of the function $v(y,z,t)$ with respect to y and z at a grid point (y_i, z_j) are approximated by a linear sum of all functional values in the whole domain that is

$$v_y^{(1)}(y_i, z_j, t) \cong \sum_{l=0}^{N} w_{il}^{(1)} v(y_l, z_j, t), \qquad (1.3.39a)$$

$$v_{yy}^{(2)}(y_i, z_j, t) \cong \sum_{l=0}^{N} w_{il}^{(2)} v(y_l, z_j, t), \qquad (1.3.39b)$$

$$v_z^{(1)}(y_i, z_j, t) \cong \sum_{l=0}^{M} w_{jl}^{-(1)} v(y_i, z_l, t), \tag{1.3.40a}$$

$$v_{zz}^{(2)}(y_i, z_j, t) \cong \sum_{l=0}^{M} w_{jl}^{-(2)} v(y_i, z_l, t) \ \ for\, i = 0,1,..., N, \ \ j = 0,1,..., M \,. \tag{1.3.40b}$$

where $v_y^{(1)}(y_i, z_j, t), v_y^{(1)}(y_i, z_j, t)$ and $v_{yy}^{(2)}(y_i, z_j, t), v_{zz}^{(2)}(y_i, z_j, t)$ are the first and second order partial derivatives of $v(y, z, t)$ with respect to y and z at a grid point (y_i, z_j) and $w_{il}^{(1)}, w_{jl}^{-(1)}, w_{il}^{(2)}, w_{jl}^{-(2)}$ are corresponding weighting coefficients.

For finding the weighting coefficients Shu [47] proposed the most general approach by using Lagrange's interpolation as base functions. Recently, the most frequently used differential quadrature procedures to solve one and two dimensional differential equations are Lagrange interpolation polynomials based differential quadrature method (PDQM), Cosine expansion based differential quadrature method (CDQ) and other methods [11, 14, 15, 19, 30 35, 80, 118, 161-164, 217].

The weighting coefficients in equation (1.3.39)-(1.3.40) can be determined by using a set base functions. For CDQM, the following base functions are used to obtain weighting coefficients [47]

$$d_l(y) = \cos(ly), \; l = 0,1,..., N \,, \tag{1.3.41}$$

$$d_l(y) = \frac{d(y)}{\Phi(y_l)(\cos(y) - \cos(y_l))}, \quad l = 0,1,..., N \,, \tag{1.3.42}$$

$$e_l(z) = \frac{d(y)}{\phi(z_l)(\cos(z) - \cos(z_l)}, \ \ l = 0,1,..., N \,. \tag{1.3.43}$$

where

$$d(y) = \prod_{l=0}^{N} (\cos(y) - \cos(y_l)), \qquad c(y) = \prod_{k=0}^{N} (\cos(y) - \cos(y_k) \tag{1.3.44}$$

$$\phi(x_k) = \prod_{j=0,\, j \neq l}^{N} (\cos(y_l) - \cos(y_j)), \; \phi(z_l) = \prod_{j=0,\, j \neq l}^{N} (\cos(z_l) - \cos(z_j)) \tag{1.3.45}$$

In fact, the test function given in equation (1.3.42) is Lagrange interpolated cosine function and obtained using the transformations $w = \cos(y)$ and $w_l = \cos(y_l)$ to the Lagrange interpolation polynomials.

$$r_l(w) = \frac{L(w)}{(w - w_l)L^{(1)}(w_l)}, \quad k = 0,1,....,N \tag{1.3.46}$$

where

$$L(w) = (w - w_0)(w - w_1)...(w - w_N) \tag{1.3.47}$$

$$L^{(1)}(w_k) = \prod_{j=0,\, j\neq l}^{N} (w_l - w_j) \tag{1.3.48}$$

when the problem domain is chosen as $[0, \pi]$ for independent variable ω.

Using the set of base functions given in equation (1.3.41) and (1.3.42), the weighting coefficients of the first order derivative are found as

$$w_{ij}^{(1)} = \frac{-\phi(y_i)\sin(y_i)}{(\cos(y_i) - \cos(y_j)\phi(y_j))}, i, j = 0,1,....,N \quad for\ i \neq j \tag{1.3.49}$$

$$w_{ii}^{(1)} = \sum_{j=0,\, j\neq i}^{N} w_{ij}^{(1)}, \quad i = 0,1,...,N \tag{1.3.50}$$

$$w_{ij}^{-(1)} = \frac{-\phi(z_i)\sin(z_i)}{(\cos(z_i) - \cos(z_j))\phi(z_j)}, \quad i, j = 0,1,...,M \quad for \quad i \neq j \tag{1.3.51}$$

$$w_{ii}^{-(1)} = -\sum_{j=0,\, j\neq i}^{N} w_{ij}^{-(1)}, \quad i = 0,1,...,M \tag{1.3.52}$$

Similarly the weighting coefficients for the second order partial derivatives are given by the formula [47]

$$w_{ij}^{(2)} = w_{ij}^{(1)}(2w_{ii}^{(1)} + \frac{2\sin(y_i)}{\cos(y_i) - \cos(y_j)} + \cot(y_i)), \ i, j = 0,...,N \quad for\ i \neq j \tag{1.3.53}$$

$$w_{ii}^{(2)} = -\sum_{j=0,\, j\neq i}^{N} w_{ij}^{(2)}, \qquad i = 0,...,N \tag{1.3.54}$$

$$w_{ij}^{-(2)} = w_{ij}^{-(1)}(2w_{ii}^{-(1)} + \frac{2\sin(z_i)}{\cos(z_i)-\cos(z_j)} + \cot(z_i))\ , i, j = 0,1,...,M, \quad for\ i \neq j \quad (1.3.55)$$

$$w_{ii}^{-(2)} = -\sum_{j=0, j\neq i}^{N} w_{ij}^{-(2)}, \quad i = 0,1,...,M\ . \quad (1.3.56)$$

For the weighting coefficients of a function defined on the interval $[a,b]\times[c,d]$ a transformation can be prescribed as [47]

$$\xi = \frac{y-p}{q-p}\pi, \quad \psi = \frac{z-d}{e-d}\pi \quad (1.3.57)$$

To map the interval $[p,q]\times[d,e]$ in y and z domain to the interval $[0,\pi]\times[0,\pi]$ in the ξ and ψ domain. Basically, there are two ways to solve the problems defined on the interval $[p,q]\times[d,e]$. One way is to transform the all the derivatives with respect to ξ and ψ. Then the discretization is made in the transformed domain. For this case, the above weighting coefficients (1.3.49) to (1.3.56) can be applied directly. Another way is to make discretization directly in the in the y and z domains, but for this case the weighting coefficients (1.3.49) to (1.3.57) should be modified accordingly using the transformation (1.3.58), we have

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} = \frac{\pi}{q-p}\frac{\partial v}{\partial \xi} \quad (1.3.58)$$

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial \psi}\frac{\partial \psi}{\partial z} = \frac{\pi}{e-d}\frac{\partial v}{\partial \psi} \quad (1.3.59)$$

$$\frac{\partial^2 v}{\partial y^2} = \left(\frac{\pi}{q-p}\right)^2 \frac{\partial^2 v}{\partial \xi^2}, \qquad \frac{\partial^2 v}{\partial z^2} = \left(\frac{\pi}{e-d}\right)^2 \frac{\partial^2 v}{\partial \psi^2} \quad (1.3.60)$$

Then using equations (1.3.59) and (1.3.60), the explicit formulations given for the weighting coefficients in the y and z domain can be modified as

$$w_{ij}^{(1)} = \frac{-\pi\phi(\xi_i)\sin(\xi_i)}{(q-p)(\cos(\xi_i)-\cos(\xi_j))\phi(\xi_j)}, \quad i, j = 0,1,...,N, \quad for\ i \neq j \quad (1.3.61)$$

$$w_{ij}^{-(1)} = \frac{-\pi\phi(\psi_i)\sin(\psi_i)}{(q-p)(\cos(\psi_i)-\cos(\psi_j))\phi(\psi_j)}, \; i, j = 0,\ldots,M \quad for \;\; i \neq j \tag{1.3.62}$$

$$w_{ij}^{(2)} = w_{ij}^{(1)}(2w_{ii}^{(1)} + \frac{2\pi\sin(\xi_i)}{(q-p)(\cos(\xi_i)-\cos(\xi_j))} + \frac{\pi}{(q-p)}\cot(\xi_i)), \; i, j = 0,1,\ldots,N \quad for \; i \neq j \tag{1.3.63}$$

$$w_{ij}^{-(2)} = w_{ij}^{-(1)}(2w_{ii}^{-(1)} + \frac{2\pi\sin(\psi_i)}{(q-p)(\cos(\psi_i)-\cos(\psi_j))} + \frac{\pi}{(e-d)}\cot(\psi_i)), \; i, j = 0,1,\ldots,M \quad for \; i \neq j \tag{1.3.64}$$

In the calculation of the weighting coefficients $w_{ij}^{(2)}$ and $w_{ij}^{-(2)}$ using equation (1.3.63)–(1.3.64) the values of the function $\cot(\xi)$ is undefined at the problem boundaries 0 and π

Chapter 2

Exact Solutions of a Family of Nonlinear Evolution Equations

2.1 Introduction

In mathematical physics the family of equations

$$\alpha(u_t + 3uu_x) + \beta(u_{txx} + 2u_x u_{xx} + uu_{xxx}) - \gamma u_{xxx} = 0 \qquad (2.1.1)$$

has been a subject of extensive study. It represents a class of nonlinear evolution equations. The equation (2.1.1) represents version of Korteweg-de Vries (KdV) equation $\alpha = 1, \beta = 0$ [7], Camssa-Holm equation $\alpha = 1, \beta = 1$ [6], the Hunter-Saxton equation $\alpha = 0, \beta = 0$ [83] and Inviscid Burgers' equation $u_t + 3uu_x = 0$ [34].

In mathematics, the KdV is a mathematical model of waves on shallow water wave's surface. More than half a century elapsed, since Scott Ruessell's discovery before two Dutch scientists Korteweg and de Vries [7] derived in 1895, the equation that governs the propagation of solitary wave in shallow channel. Shi *et al.* [227] proved that the modified KdV equation admits a cantor family of small amplitude, quasio-periodic solution. Dutykh *et al.* [50] examine the energy transfer mechanism during the nonlinear stage of the modulation instability in the modified KdV equation. The particularity of this study consists in considering the problem essentially in the Fourier space. Triki *et al.* [77] study two families of fifth order KdV equation with time-dependent coefficients and linear damping term. These models apply to the characterization of envelope wave dynamics in inhomogeneous systems modelled by KdV-type equation.

In order to study shallow water waves, Camassa and Holm [6] derived a partial differential equation which is now called the Camassa- Holm equation and is considered one of the most fascinating PDEs in mathematical physics. In fluid dynamic, the Camassa-Holm equation is integrable, dimensionless and nonlinear partial differential equation. Rehman *et* *al.* [208] apply two recent analytical approaches to investigate the possible classes of travelling wave solutions of some members of a recently attain integrable family of generalized Camassa-Holm equation. Li *et al.* [127] consider the numerical approximation of the Camassa-Holm equation which supports peaken solution. They develop and test a high order central discontinuous Galerkin-finite element method for solving the equation. Jia *et al.* [91] used asymptotic densities to study an asymptotic property of the dispersion Camassa-Holm equation.

The Hunter-Saxton equation describes the propagation of waves in a massive director field of nematic liquid crystal with orientation of molecules described by the field of unit vectors. Hunter and Saxton investigated the case when viscous damping is ignored and a kinetic energy term is included in the model. As is well known, the Burgers equation can be obtained as a limiting case of Navier-Stokes equation and serves transport problems. For the inviscid case, the Burgers equation reduces to $u_t + uu_x = 0$ where $u(x,t)$ is particle velocity.

Symmetry group techniques provide one method for obtaining exact solutions of PDEs. The theory of Lie symmetry groups of differential equations was developed by Lie. Several applications of the Lie groups in the theory of differential equations were discussed in literature; most important ones are the original equation can be reduced to an equation with fewer independent variables, reduction of order of ordinary differential equations, construction of invariant solutions by solving the characteristic equation, mapping solutions to other solutions and detection of linear zing transformations (for many other applications of Lie Symmetries, see [157, 73, 94]).

In this chapter, by using the Lie point symmetry method, we have investigated equation (2.1.1). We found the preliminary classification of group-invariant solutions by looking at the representation of the obtained symmetry group on its Lie algebra and then we reduced the equation (2.1) to ordinary differential equation (ODEs). Then we apply $\left(G'/G\right)$-expansion method on one ODE and get travelling wave solutions.

The work is organized as follows. In Section 2.2, we derived the general form of an Infinitesimal generator admitted by (2.1.1) and an optimal system of one-dimensional symmetry group of (2.1.1) is constructed. Section 2.3 is devoted to the corresponding one-parameter reductions and group-invariant solutions and (G'/G) -expansion method has been applied on one reduced ODE. Finally, some conclusions and discussions are given in Section 2.4.

2.2 Lie Symmetry Analysis

In real-life applications, models are expressed in terms of PDEs as opposed to ODEs which are relatively easier to solve. Fortunately, symmetries can be used to construct invariant solutions (i.e., solutions that are unchanged under the action of a subgroup of a symmetry group) which in turn allows the reduction of PDEs to ODEs.

An equivalence transformation of a system is a change of both dependent and independent variables into new dependent and independent variables taking the original system into a system of the same form.

In this section, we will apply Lie's method of infinitesimal transformation groups on family of equations. We let $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ are infinitesimals corresponding to x,t and u respectively and impose the condition of invariance on (2.1.1). The invariance under infinitesimal transformations means that if u is solution of equation (2.1.1), and then u^* is also a solution of it.

The following relation from the coefficients of the first order is derived, on invoking the invariance criterion:

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2), \qquad (2.2.1) \\ t^* &= x + \varepsilon\tau(x,t,u) + O(\varepsilon^2). \end{aligned}$$

where are $\eta_t, \eta_x, \eta_{xx}, \eta_{xxx}$ and η_{txx} extended (prolonged) infinitesimals corresponding to $u_t, u_x, u_{xx}, u_{xxx}$ and u_{txx} respectively. For the infinitesimals ξ, τ and η the determining equation's set which is obtained from invariance condition, after equating the coefficients of various derivative terms to zero, is as follows

$$\xi_t = \frac{-3\tau_t\gamma}{2\beta},\ \xi_u = 0,\ \xi_x = 0,\ \tau_u = 0,\ \tau_{tt} = 0$$
$$\tau_x = 0,\ \eta = \frac{-\tau_t(2\beta u + \gamma)}{2\beta} \tag{2.2.2}$$

The set of equations (2.2.2) helps us to correspondes the infinitesimals ξ, τ and η as follows:

$$\xi = a_1 - a_3\left(\frac{3\gamma t}{2\beta}\right),\ \tau = a_2 + a_3 t,\ \eta = \frac{-\tau_t a_3(2\beta u - \gamma)}{2\beta}\ . \tag{2.2.3}$$

where a_1, a_2 and a_3 are arbitrary constants. The Lie algebra associated with equation (2.1.1) consists of the following three vector fields

$$V_1 = \frac{\partial}{\partial x},$$
$$V_2 = \frac{\partial}{\partial t}, \tag{2.2.4}$$
$$V_3 = \frac{-3\gamma\tau_t}{2\beta}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{-3\gamma\tau_t}{2\beta}\frac{\partial}{\partial u}.$$

The similarity variable and form can be obtained by solving the characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta} \tag{2.2.5}$$

The general solution of these equations involves two constants; one becomes the new independent variable and the other, says F, plays the role of new dependent variable. On substituting these solutions of (2.2.5) in equation (2.1.1), one gets the reduced ordinary differential equation.

As mentioned in Olver [65], the commutator Table 2.1 and the ad joint Table 2.2 for above Lie algebra can be easily constructed as follows

2.1 COMMUTATOR TABLE

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	V_1+V_2
V_3	0	$-(V_1+V_2)$	0

2.2 ADJOINT TABLE

	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	V_3 - $\varepsilon(V_1+V_2)$
V_3	V_1	$-(V_1+V_2)e^{\varepsilon}$	V_3

In general, to each s-parameter subgroup of the full symmetry group, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is usually not feasible to list all possible group-invariant solutions to the system. That needs an effective, systematic means of classifying these solutions, leading to an optimal system of group-invariant solutions from which every other solution can be derived. About the optimal system, a lot of excellent work has been done by experts [157, 99 and 73] and some examples of optimal system can also be found in [142]. Up to now, several methods have been developed to construct the optimal system. The adjoint representation of the Lie group on its Lie algebras was also known to Lie. Its use in classifying group-invariant solutions appeared in [143] and [94] which are written by Ovsiannikov [99] and Oliver [157] respectively. The latter reference contains more details on how to perform the classification of subgroup under the adjoint action. Here, we have used Oliver's method, which only depends upon fragments of the theory of Lie algebras to construct the optimal system of equation (2.1.1).

An optimal system comprises of three vector fields viz. $(i)\, V_1$ $(ii)\, V_2+c_1V_1$ $(iii)\, V_3$. Now our primary focus on the reductions associated with these vectors fields and attempts to find some exact solutions.

2.3 Reductions and Exact Solutions of Family of Nonlinear Equations

In this section, we will derive some exact solutions of family of equations. One way to obtain exact solutions of (2.1.1) is by reducing it to ordinary differential equations. This can be achieved with the use of Lie point symmetries admitted by (2.1.1). It is well known that the reduction of a partial differential equation with respect to r-dimensional (solvable) sub algebra of its Lie symmetry algebra leads to reducing the number of independent variables by r.

2.3.1 Subalgebra V_3

The reduced ODEs, in this case are

$$F' = 0, \tag{2.3.6}$$

The obvious solution of (2.2.1) is

$$u(x, t) = \text{ constant}, \tag{2.3.7}$$

2.3.2 Subalgebra $V_2 + a_1 V_1$

On using the characteristic equations, the corresponding variable and the form of the corresponding solution are as follows

$$\zeta = x - a_1 t, \tag{2.3.8}$$

$$u(x,t) = F(\zeta) \tag{2.3.9}$$

On using these in equation (2.1.1), the reduced ODE is given by

$$-\alpha cF' + 3\alpha FF' - \beta cF''' + 2\beta F'F'' + \beta FF''' - a_1 F''' = 0 \tag{2.3.10}$$

In this section, we seek solution of equation (2.4.1) by (G'/G)-expansion method as described in section (1.2.12). The reduced ODE is

$$-\alpha cF' + 3\alpha FF' - \beta cF''' + 2\beta F'F'' + \beta FF''' - a_1 F''' = 0 \tag{2.3.11}$$

The solutions of equation (2.3.10) can be expressed by a polynomial in (G'/G) as follows:

$$F(\zeta) = \sum_{i=0}^{N} b_i \left(\frac{G'(\zeta)}{G(\zeta)} \right) \tag{2.3.12}$$

where N is a positive integer to be determined. The function $G(\zeta)$ is the solution of auxiliary linear ordinary differential equation

$$G''(\zeta) + \lambda G'(\zeta) + \mu G(\zeta) = 0 \tag{2.3.13}$$

where λ and μ are real constants to be determined.

Now balancing F''' with FF' gives $N = 2$. Therefore we can write solutions of equation (2.3.10) in the following form:

$$F(\zeta) = b_0 + b_1 \left(\frac{G'}{G} \right) + b_2 \left(\frac{G'}{G} \right)^2 \tag{2.3.14}$$

Substituting equation (2.3.14) into equation (2.3.11) and using (2.3.13) collecting all terms with the same order of together, the left hand side of equation (2.3.10) is converted into polynomial in $(G'/G)^j$, $(j = 0,1,...,7)$ setting each coefficient of this polynomial to zero, we derive a set over determined differential equation for b_0, b_1 and b_2.

$$(G'/G)^7 : -48\beta b_2{}^2 \tag{2.3.15}$$

$$(G'/G)^6 : -118\beta b_2{}^2\lambda - 50\beta b_2 b_1$$

(2.3.16)

$$(G'/G)^5 : -6\alpha b_2{}^2 - 96\beta b_2{}^2\mu + 24a_1 b_2 - 24\beta c b_2 - 94\beta b_2{}^2\mu - 10\beta b_1{}^2 - 118 b_2 b_1 \lambda - 24\beta b_0 b_2 \tag{2.3.17}$$

$$\begin{aligned}(G'/G)^4 :& -52\beta b_2{}^2\lambda\mu - 54\beta b_0 b_1 \lambda - 4\beta b_2 a b_1 \lambda^2 - 92\beta b_2 b_1 \mu + 54 a_1 b_2 \lambda - 24\beta b_2{}^2\mu\lambda \\ & - 6\alpha b_2{}^2\lambda - 85\beta b_1 b_2 \lambda^2 - 9\alpha b_2 b_1 - 24\beta b_2{}^2\lambda^3 + 6a_1 b_1 - 72\beta b_2{}^2\lambda\mu - 6\beta b_0 b_1 + 54\beta c b_2 \lambda \\ & - 22\beta b_1{}^2\lambda + 6\beta c b_1\end{aligned} \tag{2.3.18}$$

$$\begin{aligned}(G'/G)^3 :& -6\alpha b_2 b_0 - 9\alpha b_2 \lambda b_1 + 40\beta c b_2 \mu + 38\beta c b_2 \lambda^2 + 12\beta c b_1 \lambda - 64\beta b_2 b_1 \lambda\mu - 2\beta b_1{}^2 \\ & a\lambda^2 - 12\beta b_1 b_2 \mu\lambda - 4\beta b_2 \lambda^3 a b_1 - 24\beta b_2{}^2\lambda^2\mu - 30\beta b_2{}^2\mu\lambda^2 - 12\beta b_0 b_1 \lambda - 40\beta b_0 b_2 \mu - \\ & 38\beta b_0 b_2 \lambda^2 + 40 a_1 b_2 \mu - 6\alpha b_2{}^2\mu - 48\beta b_2{}^2\mu^2 + 12 a_1 b_1 \lambda + 2\alpha c b_2 - 16\beta b_1{}^2\mu + 38 a_1 b_2 \lambda^2 \\ & - 3\alpha b_1{}^2 - 56\beta b_2 \lambda b_1 \mu - 21\beta b_2 b_1 \lambda^3\end{aligned} \tag{2.3.19}$$

$$(G'/G)^2 :-3\alpha b_1 b_0 + 2\alpha c b_2\lambda - 6\alpha b_2\lambda b_0 + 52\beta c b_2\lambda\mu + 8\beta c b_2\lambda^3 + 7\beta c b_1\lambda^2 + 8\beta c b_1\mu$$
$$-10\beta b_1^{\,2}\lambda\mu - 42\beta b_2\mu^2 b_1 - 24\beta b_2^{\,2}\mu^2\lambda - 52\beta b_0 b_2\lambda\mu - 8\beta b_0 b_2\lambda^3 - 7\beta b_0 b_1\lambda^2 - 8\beta b_0 b_1$$
$$\mu - 10\beta b_1^{\,2}\lambda\mu + 52 a_1 b_2\lambda\mu + 8a_1 b_1\mu - 3\beta b_1^{\,2}\lambda^3 + 7a_1 b_1\lambda^2 - 3\alpha b_1^{\,2}\lambda + 8a_1 b_2\lambda^3 - 6\beta b_2^{\,2}$$
$$\lambda\mu^2 + \alpha c b_1 - 4\beta b_2\lambda^2 b_1\mu - 12\beta b_1 b_2\lambda^2\mu - 27\beta b_2\mu b_1\lambda^2 - 9\alpha b_2\mu b_1 \tag{2.3.20}$$

$$(G'/G)^1 : \beta c b_1^{\,2}\lambda^3 - 4\beta b_2\mu b_1\lambda\mu^2 + 14\beta c b_2\lambda^2\mu - \beta b_0 b_1\lambda^3 - 14\beta b_0 b_2\lambda^2\mu - \beta b_0 b_1\lambda^3 -$$
$$14\beta b_0 b_2\lambda^2\mu - 3\beta b_1^{\,2}\mu\lambda^2 + a_1 b_1\lambda^3 - 12\beta b_1\mu^2 b_2\lambda - 6\alpha b_2\mu b_0 + 2\alpha c b_2\mu - 16\beta b_0 b_2\mu^2 +$$
$$16\beta c b_2\mu^2 - 3\alpha b_1^{\,2}\mu - 3\alpha b_1\lambda b_0 - 6\beta b_1 b_2\lambda\mu^2 + 8\beta c b_1\lambda\mu - 6\beta b_1^{\,2}\mu^2 + \alpha c b_1\lambda + 14 a_1 b_2$$
$$\lambda^2\mu + 16 a_1 b_2\mu^2 - 2\beta b_1^{\,2}\lambda^2\mu + 8a_1 b_1\lambda\mu - 8\beta b_0 b_1\lambda\mu \tag{2.3.21}$$

$$(G'/G)^0 : -2\beta b_1^{\,2}\mu^2\lambda + 2\beta c b_1\mu^2 + \beta c b_1\lambda^2\mu - 2\beta b_0 b_1\mu^2 + a_1 b_1\lambda^2\mu - 3\alpha b_1\mu b_0 +$$
$$2a_1 b_1\mu^2 + 6\beta c b_2\lambda\mu^2 + 6a_1 b_2\lambda\mu^2 - 6\beta b_0 b_2\lambda^2 + \alpha c b_1\mu - \beta b_0 b_1\lambda^2\mu \tag{2.3.22}$$

Solving these systems of algebraic equations, we have following results

Case 1

$$c = \frac{-(b_1^{\,2} + 8b_2^{\,4}\mu - 12 b_0 b_2)}{4b_2},\ \mu = \mu,\ \alpha = \alpha,\ \beta = 0, \lambda = \frac{b_1}{b_2}, a_1 = \frac{1}{4}\alpha b_2$$
$$b_0 = b_0,\ b_1 = b_1,\ b_2 = b_2 \tag{2.3.23}$$

Substituting the general solutions of equation (2.3.13) into equation (2.3.14), we have three types of exact solutions of equation (2.3.10). Then we get the solutions of PDE by changing the function ζ

when $\lambda^2 - 4\mu > 0$, then we get hyperbolic function solutions

$$(i)\, u_{11}(x,t) = b_0 + b_1\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-a_1t)\right) + c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-a_1t)\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-a_1t)\right) + c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-a_1t)\right)}\right) - \frac{\lambda}{2}\right)$$
$$+ b_2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-a_1t)\right) + c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-a_1t)\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-a_1t)\right) + c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-a_1t)\right)}\right) - \frac{\lambda}{2}\right)^2 \tag{2.3.24}$$

when $\lambda^2 - 4\mu < 0$, then we get trigonometric function solutions

$$(ii)\ u_{12}(x,t) = b_0 + b_1\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-a_1t)\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-a_1t)\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-a_1t)\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-a_1t)\right)}\right)-\frac{\lambda}{2}\right)$$

$$+ b_2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-a_1t)\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-a_1t)\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-a_1t)\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-a_1t)\right)}\right)-\frac{\lambda}{2}\right)^2 \qquad (2.3.25)$$

when $\lambda^2 - 4\mu = 0$ we obtain rational function solutions

$$(iii)\ u_{13}(x,t) = \frac{c_2}{c_1 + c_2(x-a_1t)} - \frac{\lambda}{2} \qquad (2.3.26)$$

Graphically representation of solutions has given in Figure 2.1-2.6.

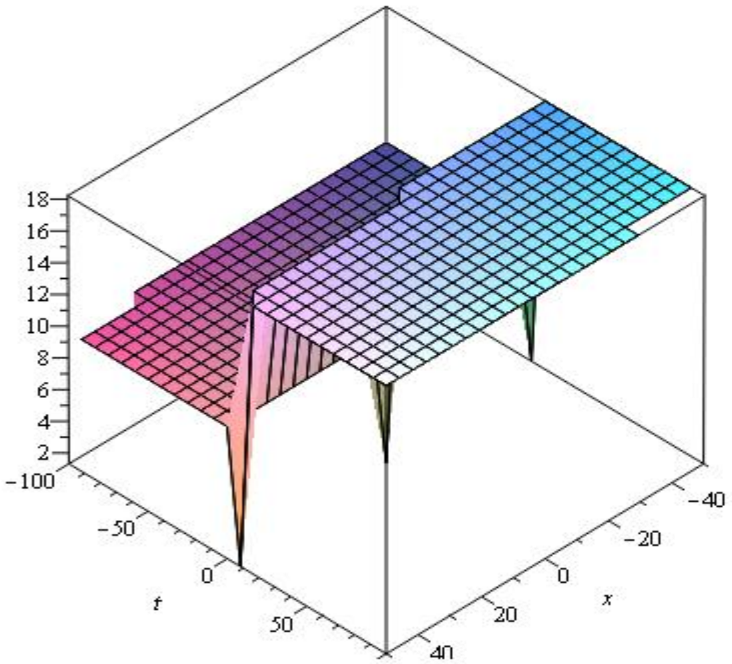


Figure 2.1: The distribution of some singularities of the solution (i) for

$b_0 = 1, b_1 = 9, b_2 = 3, a_1 = 6, \lambda = 3\ \mu = 1, c_1 = 4, c_2 = 1.$

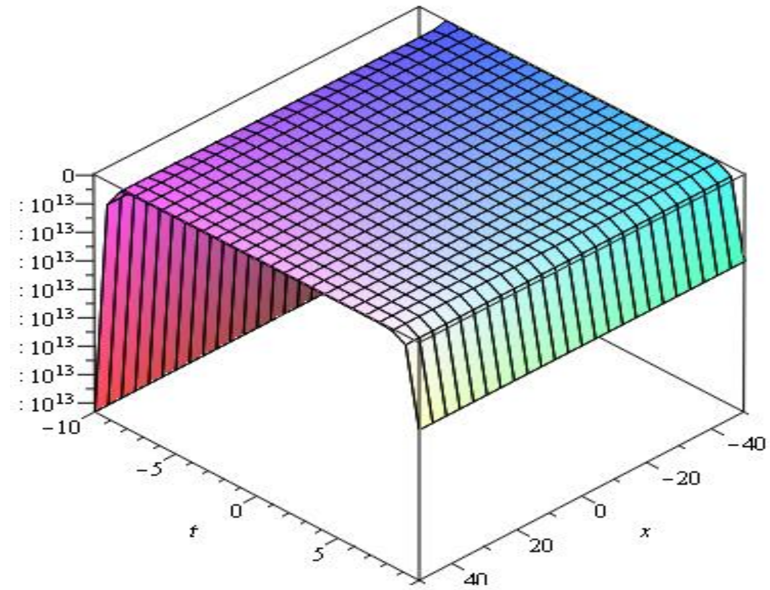


Figure 2.2: The kink wave solution (i) for $b_0 = 6, b_1 = 32, b_2 = 8, a_1 = 8,$

$\lambda = 4, \mu = 2, c_1 = -10, c_2 = 20$

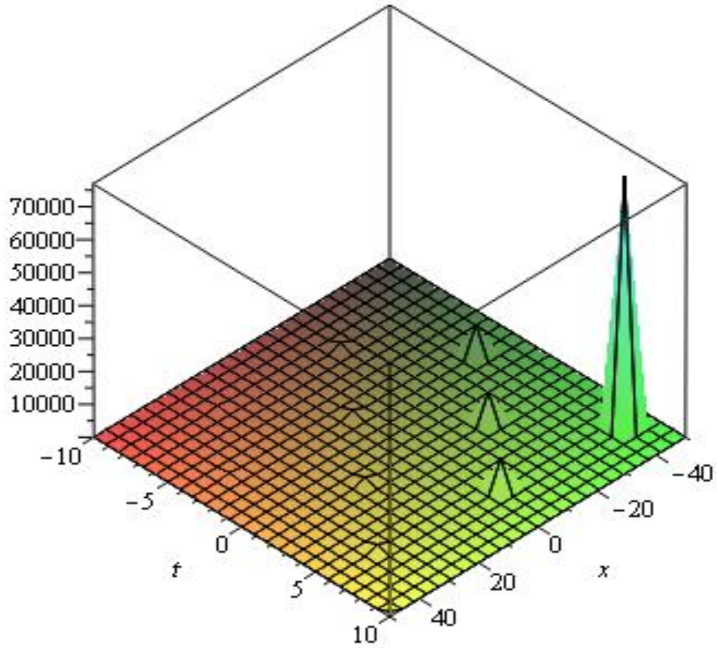


Figure 2.3: The distribution of some singularities of the solution (ii) for $b_0 = 1, b_1 = 4, b_2 = 2, a_1 = 4, \ \lambda = 2, \mu = 2, c_1 = -2, c_2 = 3$.

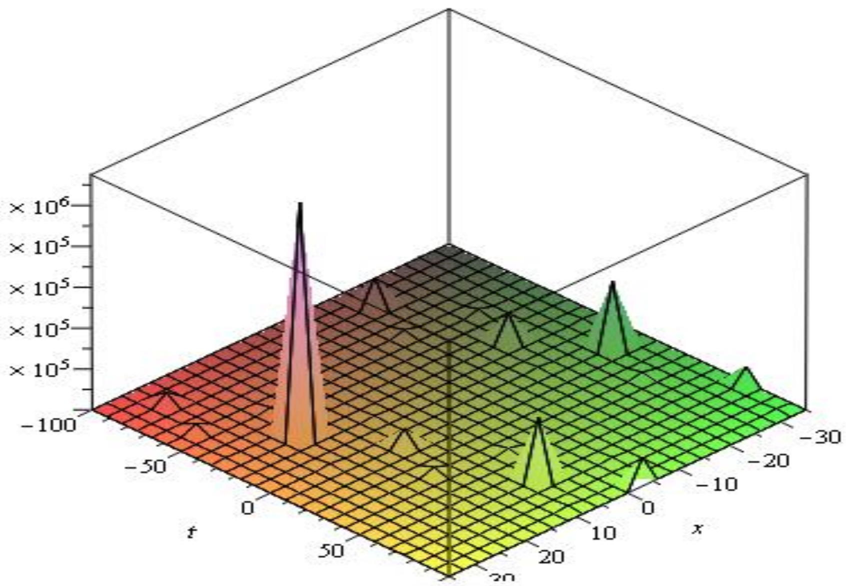


Figure 2.4: The distribution of some singularities of the solution (ii) for $b_0 = -10, b_1 = 1, b_2 = 1, a_1 = 4, \ \lambda = 1, \mu = 5, c_1 = 25, c_2 = -7$.

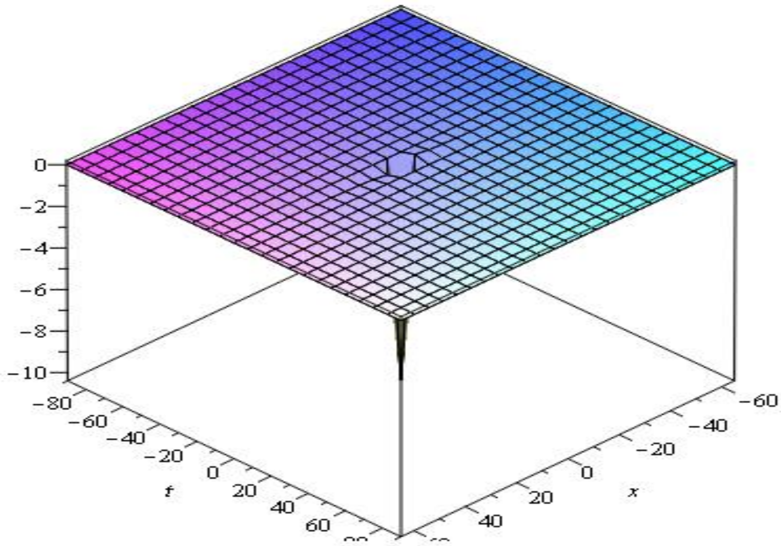


Figure 2.5: The Singularity at $t = 0$ of solution (iii) for $a_1 = 48, c_1 = -5, c_2 = 52$.

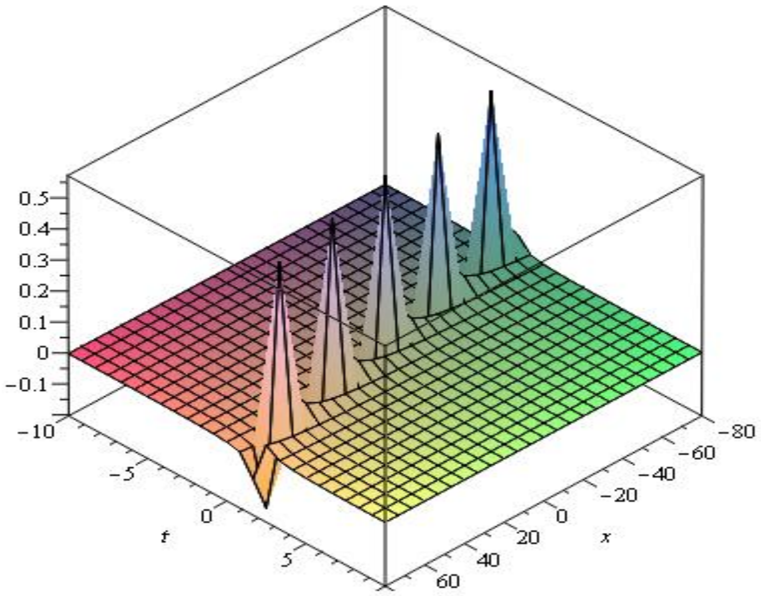


Figure 2.6: The periodic solution (iii) for $a_1 = 40, c_1 = -70, c_2 = 40$.

2.3.3 Subalgebra V_3

Using characteristic equations (2.2.5) the similarity variable and the form of the similarity solution are as follows:

$$\zeta = 2\beta x + 3\gamma t\,,$$
$$u(x,t) = \frac{1}{2\beta}\left(\frac{F(2\beta x + 3\gamma t)}{t}\right). \tag{2.3.27}$$

On using these in equation (2.1.1), the reduced ODE is given by

$$-\alpha F' + 3\alpha FF' - 4\beta^3 F'' + 8\beta^3 F'F'' + 4\beta^3 FF''' = 0 \tag{2.3.28}$$

Solutions, corresponding to this ODE are:

$$(i)\,\alpha = -4\beta^3 {c_3}^2, F(\zeta) = c_5 \sin(c_2 + c_3\zeta) \tag{2.3.29}$$

$$(ii)\,\alpha = 16\beta^3 {c_3}^2, F(\zeta) = \frac{-1}{2}c_6 + c_6(\sin(c_2 + c_3\zeta))^2 \tag{2.3.30}$$

$$(iii)\,\alpha = 36\beta^3 {c_3}^2, F(\zeta) = \frac{-3}{4}c_7 \sin(c_2 + c_3\zeta) + c_7(\sin(c_2 + c_3\zeta))^3 \tag{2.3.31}$$

$$(iv)\,\alpha = 4\beta^3 {c_2}^2, F(\zeta) = c_4 \cos(c_1 + c_2\zeta) \tag{2.3.32}$$

$$(v)\,\alpha = 16\beta^3 {c_2}^2, F(\zeta) = \frac{-1}{2}c_5 + c_5(\cos(c_1 + c_2\zeta))^2 \tag{2.3.33}$$

$$(vi)\,\alpha = 36\beta^3 {c_2}^2, F(\xi) = \frac{-3}{4}c_6 \cos(c_1 + c_2\zeta) + c_6(\cos(c_1 + c_2\zeta))^3 \tag{2.3.34}$$

$$(vii)\,\alpha = -4\beta^3 {c_2}^2, F(\zeta) = c_4 \cosh(c_1 + c_2\zeta) \tag{2.3.35}$$

$$(viii)\,\alpha = -16\beta^3 {c_2}^2, F(\zeta) = \frac{-1}{2}c_5 + c_5(\cosh(c_1 + c_2\zeta))^2 \tag{2.3.36}$$

$$(ix)\,\alpha = -36\beta^3 {c_2}^2, F(\xi) = \frac{-3}{4}c_6 \cosh(c_1 + c_2\zeta) + c_6(\cosh(c_1 + c_2\zeta))^3 \tag{2.3.37}$$

$$(x)\,\alpha = -4\beta^3 {c_2}^2, F(\zeta) = c_4 e^{c_1 + c_2\zeta} \tag{2.3.38}$$

$$(xi)\,\alpha = -36\beta^3 {c_2}^2,\ F(\zeta) = c_6 e^{c_1 + c_2\zeta^3} \tag{2.3.39}$$

$$(xii)\,\alpha = -16\beta^3 {c_2}^2, F(\zeta) = c_5 e^{c_1 + c_2\zeta^2} \tag{2.3.40}$$

$$(xiii)\,\alpha = 4\beta^3 {c_2}^2,\ F(\zeta) = c_4 \sin(c_1 + c_2\zeta) \tag{2.3.41}$$

$$(xiv)\,\alpha = 16\beta^3 {c_2}^2,\; F(\zeta) = \frac{-1}{2} c_5 + c_5 (\sin(c_1 + c_2\zeta))^2 \tag{2.3.42}$$

$$(xv)\,\alpha = 36\beta^3 {c_2}^2, F(\zeta) = \frac{-3}{4} c_6 \sin(c_1 + c_2\zeta) + c_6 (\sin(c_1 + c_2\zeta))^3 \tag{2.3.43}$$

$$(xvi)\,\alpha = -4\beta^3 {c_2}^2, F(\zeta) = c_4 \sinh(c_1 + c_2\zeta) \tag{2.3.44}$$

$$(xvii)\,\alpha = -16\beta^3 {c_2}^2, F(\zeta) = \frac{1}{2} c_5 + c_5 (\sinh(c_1 + c_2\zeta))^2 \tag{2.3.45}$$

$$(xviii)\,\alpha = -36\beta^3 {c_2}^2, F(\zeta) = \frac{3}{4} c_6 \sinh(c_1 + c_2\zeta) + c_6 (\sinh(c_1 + c_2\zeta))^3 \tag{2.3.46}$$

After substituting $\zeta = 2\beta x + 3\gamma t$ and using $u(x,t) = \frac{1}{2}\beta\left(\frac{F(\zeta)}{t} - \gamma\right)$, we can get solutions of equation (2.1.1) as following

Solution 1:

$$(i)\,\alpha = -4\beta^3 {c_3}^2, u(x,t) = c_5 \sin(c_2 + c_3(2\beta x + 3\gamma t)) \tag{2.3.47}$$

Solution 2:

$$(ii)\,\alpha = 16\beta^3 {c_3}^2, u(x,t) = \frac{-1}{2} c_6 + c_6 (\sin(c_2 + c_3(2\beta x + 3\gamma t)))^2 \tag{2.3.48}$$

Solution 3:

$$(iii)\,\alpha = 36\beta^3 {c_3}^2, u(x,t) = \frac{-3}{4} c_7 \sin(c_2 + c_3\zeta) + c_7 (\sin(c_2 + c_3(2\beta x + 3\gamma t)))^3 \tag{2.3.49}$$

Solution 4:

$$(iv)\,\alpha = 4\beta^3 {c_2}^2, u(x,t) = c_4 \cos(c_1 + c_2(2\beta x + 3\gamma t)) \tag{2.3.50}$$

Solution 5:

$$(v)\,\alpha = 16\beta^3 {c_2}^2, u(x,t) = \frac{-1}{2} c_5 + c_5 (\cos(c_1 + c_2(2\beta x + 3\gamma t)))^2 \tag{2.3.51}$$

Solution 6:

$$(vi)\,\alpha = 36\beta^3 {c_2}^2, u(x,t) = \frac{-3}{4} c_6 \cos(c_1 + c_2(2\beta x + 3\gamma t)) + c_6 (\cos(c_1 + c_2(2\beta x + 3\gamma t)))^3 \tag{2.3.52}$$

Solution 7:

$$(vii)\,\alpha = -4\beta^3 {c_2}^2, u(x,t) = c_4 \cosh(c_1 + c_2(2\beta x + 3\gamma t)) \quad (2.3.53)$$

Solution 8:

$$(viii)\,\alpha = -16\beta^3 {c_2}^2, u(x,t) = \frac{-1}{2} c_5 + c_5 (\cosh(c_1 + c_2(2\beta x + 3\gamma t)))^2 \quad (2.3.54)$$

Solution 9:

$$(ix)\,\alpha = -36\beta^3 {c_2}^2, u(x,t) = \frac{-3}{4} c_6 \cosh(c_1 + c_2(2\beta x + 3\gamma t)) + c_6 (\cosh(c_1 + c_2(2\beta x + 3\gamma t)))^3$$

(2.3.55)

Solution 10:

$$(x)\,\alpha = -4\beta^3 {c_2}^2, u(x,t) = c_4 e^{c_1 + c_2(2\beta x + 3\gamma t)} \quad (2.3.56)$$

Solution 12:

$$(xi)\,\alpha = -36\beta^3 {c_2}^2,\ u(x,t) = c_6 e^{c_1 + c_2(2\beta x + 3\gamma t)^3} \quad (2.3.57)$$

Solution 13:

$$(xii)\,\alpha = -16\beta^3 {c_2}^2, u(x,t) = c_5 e^{c_1 + c_2(2\beta x + 3\gamma t)^2} \quad (2.3.58)$$

Solution 14:

$$(xiii)\,\alpha = 4\beta^3 {c_2}^2,\ u(x,t) = c_4 \sin(c_1 + c_2(2\beta x + 3\gamma t)) \quad (2.3.59)$$

Solution 15:

$$(xiv)\,\alpha = 16\beta^3 {c_2}^2,\ u(x,t) = \frac{-1}{2} c_5 + c_5 (\sin(c_1 + c_2(2\beta x + 3\gamma t)))^2 \quad (2.3.60)$$

Solution 16:

$$(xv)\,\alpha = 36\beta^3 {c_2}^2, u(x,t) = \frac{-3}{4} c_6 \sin(c_1 + c_2(2\beta x + 3\gamma t)) + c_6 (\sin(c_1 + c_2(2\beta x + 3\gamma t)))^3 \quad (2.3.61)$$

Solution 17:

$$(xvi)\,\alpha=-4\beta^{3}c_{2}{}^{2},u(x,t)=c_{4}\sinh(c_{1}+c_{2}(2\beta x+3\gamma t)) \tag{2.3.62}$$

Solution 18:

$$(xvii)\,\alpha=-16\beta^{3}c_{2}{}^{2},u(x,t)=\frac{1}{2}c_{5}+c_{5}(\sinh(c_{1}+c_{2}(2\beta x+3\gamma t)))^{2} \tag{2.3.63}$$

Solution 19:

$$(xviii)\,\alpha=-36\beta^{3}c_{2}{}^{2},u(x,t)=\frac{3}{4}c_{6}\sinh(c_{1}+c_{2}(2\beta x+3\gamma t))+c_{6}(\sinh(c_{1}+c_{2}(2\beta x+3\gamma t)))^{3}\ . \tag{2.3.64}$$

On choosing arbitrary constants equal to one, graphical representation of some solutions of equation (2.4.2) are shown in Figure 2.7-2.10.

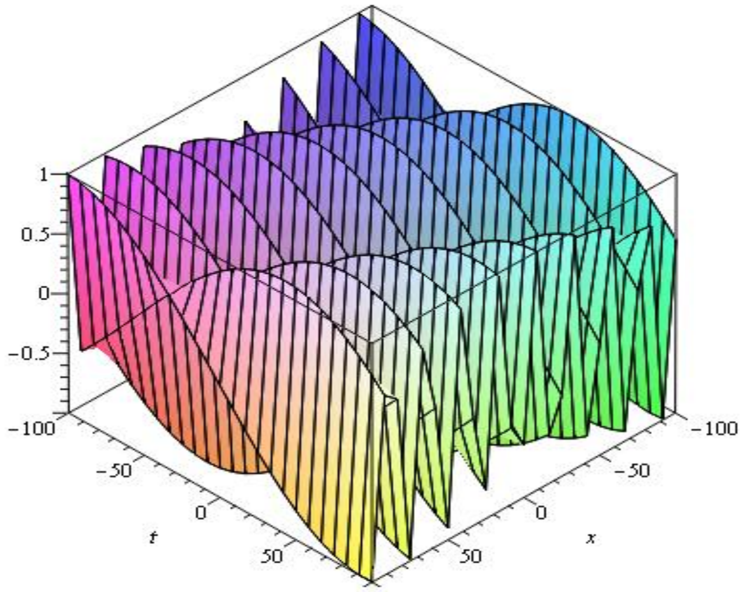


Figure 2.7: The periodic solution (i) for $c_2 = c_3 = c_5 = \beta = \gamma = 1$.

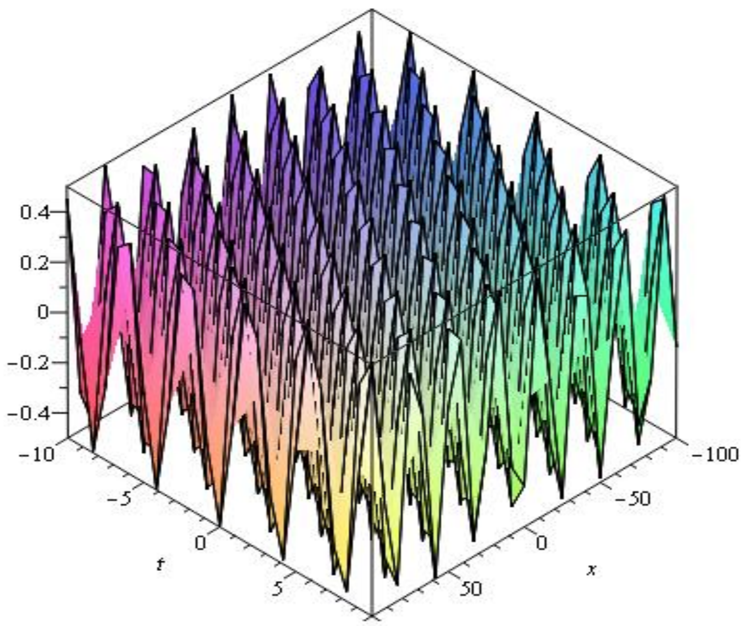


Figure 2.8: The periodic solution (ii) for $c_2 = c_3 = c_6 = \beta = \gamma = 1$.

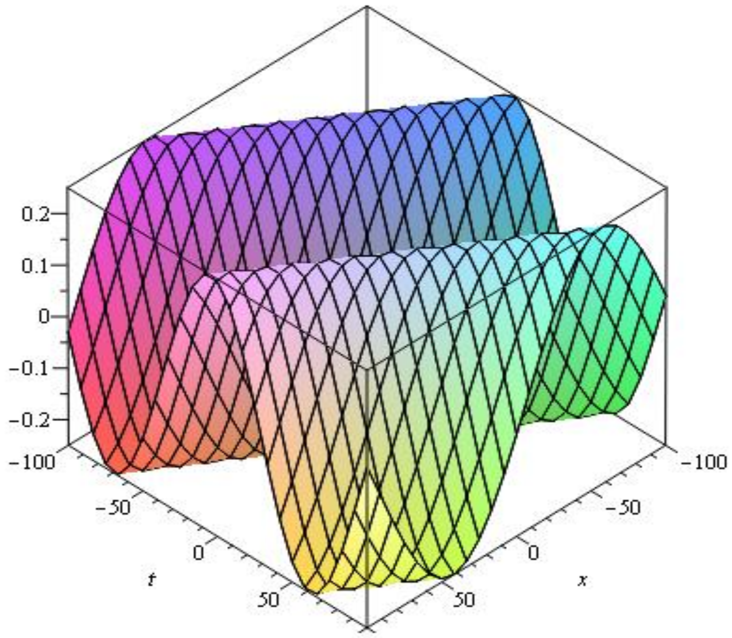


Figure 2.9: The kink wave solution (vi) for $c_1 = c_2 = c_4 = \beta = \gamma = 1$.

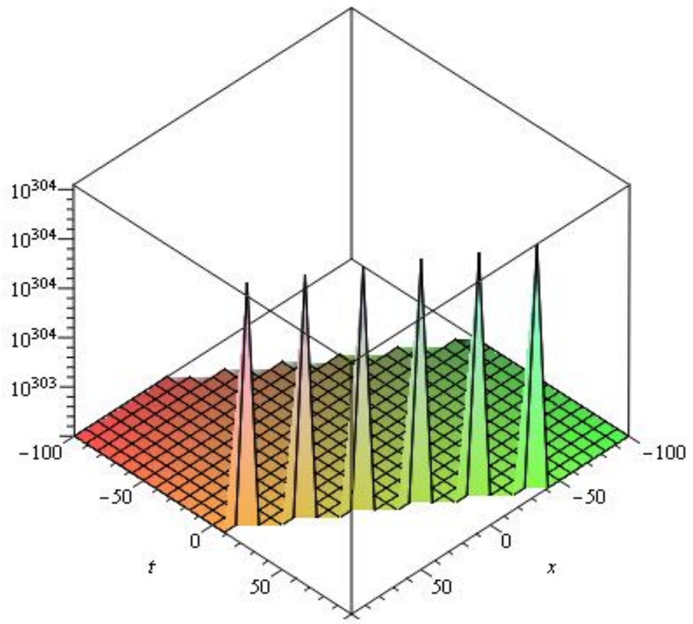


Figure 2.10: The periodic solution (ix) for $c_1 = c_2 = c_6 = \beta = \gamma = 1$.

2.4 Conclusion

In summary, we investigate the symmetry of the equation (2.1.1) by means of classical Lie symmetry method. The symmetry algebras and groups of (2.1.1) are obtained. Specially, the most general one-parameter group of symmetries is given and most general solutions are gained. Next, we have classified the one-dimensional sub algebras of the Lie algebra of (2.1.1). Then, the reductions and some solutions of equation (2.1.1) by using the associated vector fields of the obtained symmetry are given out. By one-dimensional sub algebras, (2.1.1) is reduced to ODEs. We used (G'/G)-expansion method to find travelling wave solutions of one ODE. New explicit solutions of equation (2.1.1) are derived. We have also shown graphical representation of solutions in figures.

Chapter 3

Symmetry Reduction and Solutions of Shallow Water and KdV-System of Equations

3.1 Introduction

In this chapter, we have studied two-component shallow water system and three-coupled Korteweg-de Vries (KdV) equations engendered by the Neumann system by using Lie classical method. The two-component shallow water system is in the form

$$
\begin{aligned}
&u_t - u_{xxt} + (a+b)uu_x - au_x u_{xx} - buu_{xxx} + \sigma w w_x = 0, \\
&w_t + k_2 u w_x + (k_1 + k_2) w u_x = 0
\end{aligned}
\tag{3.1.1}
$$

The shallow water equations are a set of hyperbolic partial differential equations that describe the flow below a pressure surface in a fluid. The equations are derived from depth integrating the Navier-Stokes equations, in the case where the horizontal length scale is much greater than the vertical length scale. Liu *et al.* [228] consider the Cauchy problem for 2D viscous shallow water system in Besov space. Castrodiaz *et al.* [113] describes the development of high-order methods for the shallow water system that preserve every stationary solution of the system not only the water-at-rest ones. Liu *et al.* [153] studied (2+1)-dimensional displacement shallow water wave system for perfect fluid. Luna *et al.* [209] compare some first order well-balanced numerical schemes for shallow water system with special interest in applications where there are abrupt variations of the topography. Bova *et al.* [202] introducing the total energy of the water column to motivate a change of variables which symmetrizes the shallow water conservation system and streamline upwind Petrov-Galerkin scheme is developed based on the proposed symmetric form. Imani *et al.* [1] used reconstruction of variational iteration method for computing the coupled shallow water wave equation. Hagen *et al.* [186] used Commdity-type graphics card to simulate nonlinear water waves described by a system a balance laws called the shallow water system and to solve this hyperbolic system exploit high resolution central-up wind schemes is used. Asuncion *et al.* [206] tackle the acceleration of the shallow water simulation in triangular meshes by exploiting the combined power of several CUDA-enabled GPUs in a GPU cluster. Lastra *et al.* [134] obtained numerical solutions of the shallow water wave system in 2D domains by using modern graphics processing units. Levin *et al.* [90] present the derivation of the discrete Euler-Lagrange equations for an inverse spectral element ocean model based on the shallow water equations.

The three-coupled KdV equations engendered by the Neumann system is

$$
\begin{aligned}
&u_t + \frac{1}{4} u_{xxx} + \frac{3}{4} w u_x - \frac{3}{2} v v_x = 0, \\
&v_t + \frac{1}{4} v_{xxx} + \frac{3}{4} w v_x + \frac{3}{4} v w_x = 0, \\
&w_t - \frac{1}{8} w_{xxx} - \frac{3}{4} w w_x - \frac{3}{4} u_x = 0.
\end{aligned}
\tag{3.1.2}
$$

KdV is a mathematical model of waves on shallow water surface. The KdV equation has infinitely many integrals of motion which do not change with time. Li *et al.* [93] used method of dynamical system to study the coupled KdV system and some explicit parametric representation of the solitary wave and periodic wave solutions are obtained in given parameter region. Yu *et al.* [67] present the generalized Kupershmidt deformation of the fifth order coupled KdV equation hierarchy with self-consistent sources and its lax representation. Zho *et al.* [53] investigated three-coupled KdV equations corresponding to the Neumann system of the fourth order eigen value problem. Adem *et al.* [9] used Lie symmetry analysis on the coupled KdV system and conservation laws of the coupled KdV system are derived using the multiplier approach and a conservation theorem. Hu *et al.* [79] constructed new position, negaton and complexion solutions of the coupled KdV-mKdV system from zero seed solution by means of Darboux transformation. Wazwaz [8] study two completely integrable coupled KdV and KP system and Hirota's bilinear method is employed to formally derive multiple soliton solutions and multiple singular solutions for each system. Fan *et al.* [61] obtained several kinds of exact solutions for a system of coupled KdV equation by using an improved Homogenous balance method. Zheng *et al.* [45] used extended mapping method and a linear variable separation method and new types of variable separation solutions with two arbitrary functions for (2+1)-dimensional Kortweg-de Vries system is derived. Lee [49] presents the differential operators for the fifth order KdV equation.

The work put in this chapter has been structured as follows: In section (3.2), Lie symmetries are used on two-component generalization of shallow water. In section (3.3), the exact solutions of two-component generalization of shallow water are found. In section (3.4) and (3.5) Lie classical method is applied on three-coupled KdV equations engendered by the Neumann system and exact solutions are obtained. In section (3.6), concluding remarks are given.

3.2 Lie Symmetry Analysis for Two-component Generalization of Shallow Water

The application of the classical Lie group method requires considering a one-parameter Lie group of infinitesimal transformations in the variables (x,t,u,w) as follows

$$\begin{aligned}
x^* &= x + \varepsilon X(x,t,u,w) + O(\varepsilon^2),\\
t^* &= t + \varepsilon T(x,t,u,w) + O(\varepsilon^2),\\
u^* &= u + \varepsilon U(x,t,u,w) + O(\varepsilon^2),\\
w^* &= w + \varepsilon W(x,t,u,w) + O(\varepsilon^2),
\end{aligned} \tag{3.2.1}$$

which leaves the system (3.1.1) invariant. The method for determining the symmetry group of (3.1.1) consists of finding the infinitesimals X, T, U and W, which are functions of x, t, u and w.

Assuming that system (3.1.1) is invariant under the transformations (3.2.1), the infinitesimals X, T, U and W must satisfy the symmetry conditions

$$\begin{aligned}
&U_t - U_{xxt} + (a+b)(Uu_x + uU_x) - a(u_{xx}U_x + u_xU_{xx}) - b(u_{xxx}U - uU_{xxx}) + \sigma(w_x\beta + w\beta_x) = 0,\\
&W_t + k_2(w_xU + W_xu) + (k_1+k_2)(u_xW + U_xw) = 0
\end{aligned} \tag{3.2.2}$$

where $U_t, U_x, W_x, W_t, W_{xx}$ and U_{xxx} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables (for more details, see [99, 72, 149]). Substituting values of $U_t, U_x, W_x, W_t, W_{xx}$ and U_{xxx}, into symmetry conditions (3.2.2), then equating the coefficients of the various monomials in the first, second and the other order partial derivatives of u and w and their powers, we can find the determining equations for the symmetry group of the equations (3.1.1). Solving these equations, we get the following forms of infinitesimals:

$$\begin{aligned} X &= c_3, \\ T &= c_1 t + c_2, \\ U &= -c_1 u, \\ W &= -c_1 w. \end{aligned} \tag{3.2.3}$$

The Lie algebra associated with system (3.1.1) consists of following three vector fields

$$\begin{aligned} V_1 &= t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - w\frac{\partial}{\partial w}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= \frac{\partial}{\partial x}. \end{aligned} \tag{3.2.4}$$

It is easy to verify that $V_j,\ j = 1, 2, 3, 4, 5$ is closed under the Lie bracket.

The classification of one-dimensional subalgebras of the whole symmetry algebra (3.2.4) is done by an inductive approach [72]. We will work out first an optimal system and then embark upon the various reductions associated with generators in the optimal system. We begin by considering a general element $V = a_1 V_1 + a_2 V_2 + a_3 V_3$ of symmetry algebra and subject it to various adjoint transformations to simplify it as much as possible. The adjoint action is given by the Lie series

$$\mathrm{Ad}\left(\exp(\varepsilon V_i)\right)V_j = V_j - \varepsilon\left[V_i,\ V_j\right] + \frac{\varepsilon^2}{2}\left[V_i, \left[V_i,\ V_j\right]\right] - \cdots, \tag{3.2.5}$$

where $\left[V_i, V_j\right] = V_i V_j - V_j V_i$ is the commutator for the Lie algebra, and ε is a parameter.

The types of one-dimensional subalgebras of system (3.1.1) are as follows:

(i) $V_1 + \lambda_1 V_3$,

(iii) $V_2 + \lambda_2 V_3$,

(viii) V_3. (3.2.6)

where λ_1 and λ_2 are arbitrary constants.

Once the subalgebra of the equation (3.1.1) have been determined, the symmetry variables for the associated reduction can be found by solving the characteristic equation

$$\frac{dx}{X(x,t,u,w)} = \frac{dt}{T(x,t,u,w)} = \frac{du}{U(x,t,u,w)} = \frac{dw}{W(x,t,u,w)}. \tag{3.2.7}$$

On solving the equation (3.2.7) for the various operators in the optimal system, we obtain a set of non-equivalent Lie ansätze (similarity variable, similarity solutions and forms of the coefficient functions) for the function $u(x, t)$ and $w(x, t)$. Using these ansätze, we can reduce the nonlinear system (3.1.1) to the system of ODEs.

3.3 Exact Solutions of Shallow Water Wave Equation

In this section, the primary focus is on the reductions associated with the subalgebras in the optimal system. Corresponding, to every essential subalgebra in the optimal system, the reduced ODEs are obtained. Each reduced system of ODEs can be further studied for exact and numerical solutions

3.3.1 Subalgebra $V_1 + \lambda_1 V_2$

In this subalgebra we obtain the following corresponding variables and solutions for the reduction as:

$$\xi = x - \lambda_1 \log(t),$$

$$u(x,t) = t^{-1} F(\xi),$$

$$w(x,t) = t^{-1} G(\xi) . \tag{3.3.1}$$

By using the similarity variable and solution (3.3.1), the system of PDEs (3.1.1) reduces to the following system of ODEs:

$$-F - \lambda_1 F' - F'' - \lambda_1 F''' + (a+b) F F' - a F' F'' - b F F''' + \sigma G G' = 0, \tag{3.3.2}$$

$$-G - \lambda_1 G' + k_2 F G' + (k_1 + k_2) G F' = 0$$

Herein, we seek a special solution for (3.3.2) in the following form:

$$F = \alpha \exp(m\xi),$$

$$G = \beta \exp(m\xi) \tag{3.3.3}$$

By substituting these expressions for F and G in (3.3.3), we get the following relations among the various parameters:

$$\lambda_1 = -\frac{1}{m}, \; k_1 = -2k_2, \; \sigma = \frac{\alpha^2 \left(\alpha m^2 + \beta m^2 - \alpha - \beta\right)}{\beta^2} . \tag{3.3.4}$$

where α, β, m and k_2 are arbitrary constants.

Thus, on using the back transformation, we get the solution of the system (3.1.1) as follows

$$u(x,t) = \alpha t^{-1} \exp\left(m(x - \lambda_1 \log(t))\right) ,$$

$$w(x,t) = \beta t^{-1} \exp\left(m(x - \lambda_1 \log(t))\right). \tag{3.3.5}$$

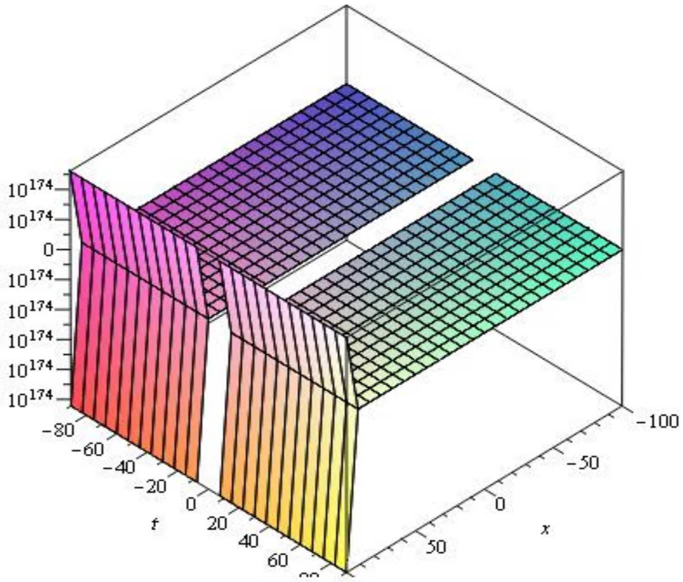


Figure 3.1: Graphical representation of solution (3.3.5) for $\lambda_1 = -\frac{1}{4}, \alpha = 5, \beta = -10, m = 4$.

3.3.2 Subalgebra $V_2 + \lambda_3 V_3$

Under this subalgebra, following is the optimal system describes the corresponding variable and corresponding solutions as follows:

$$\xi = x - \lambda_2 t,$$

$$u(x,t) = F(\xi),$$

$$w(x,t) = G(\xi). \tag{3.3.6}$$

Again by using the corresponding variable and solution, the system of PDEs (3.1.1) reduces to the following system of ODEs:

$$-\lambda_2 F' - \lambda_2 F''' + (a+b)\, F\ F' - a F'F'' - bFF''' + \sigma G\ G' = 0,$$

$$-\lambda_2 G' + k_2 FG' + (k_1 + k_2) GF' = 0 \qquad \text{(3.3.7a, b)}$$

when $\lambda_2 = k_1 = 0$, Eq. (3.3.7b) integrates once easily to yield

$$G = K F^{-1}$$

where K is an arbitrary constant. On substituting G in Eq. (3.3.7a), we get

$$(a+b)\ F\ F' - a F'F'' - bFF''' + \sigma K^2 F^{-3} F' = 0, \tag{3.3.8}$$

Thus, by solving equation (3.3.8) and reverting back to the original variables, we obtain the group-invariant solutions of the equation (3.1.1) as follows

Solution 1:

$$a = 0, b = b, K = K, \sigma = 0, u(x,t) = \frac{1}{c_4\left(\tanh\left(\frac{1}{2}(x-\lambda_2 t)+c_1\right)-1\right)} \tag{3.3.9}$$

Solution 2:

$$a = 0, b = b, K = 0, \sigma = \sigma, u(x,t) = \frac{1}{c_4\left(\tanh\left(\frac{1}{2}(x-\lambda_2 t)+c_1\right)+1\right)} \tag{3.3.10}$$

Solution 3:

$$a = -3b, b = b, K = K, \sigma = 0, u(x,t) = \frac{1}{c_4\left(\tanh\left(\frac{1}{2}I\sqrt{2}(x-\lambda_2 t)+c_1\right)\right)} \tag{3.3.11}$$

Solution 4:

$$a = -3b, b = b, K = 0, \sigma = \sigma, u(x,t) = \frac{1}{c_4\left(\tanh\left(\frac{1}{2}I\sqrt{2}(x-\lambda_2 t)+c_1\right)\right)} \tag{3.3.12}$$

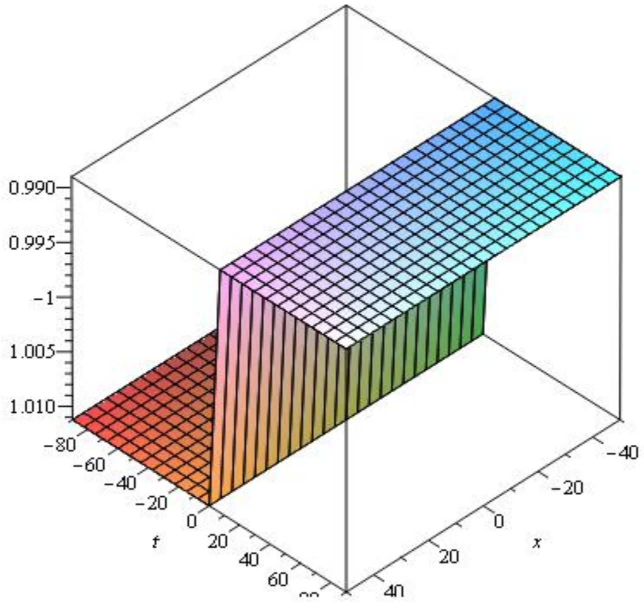


Figure 3.2: Graphical representation of solution (3.3.9) for $\lambda_2 = 70, c_1 = 58, c_4 = \frac{1}{90}$.

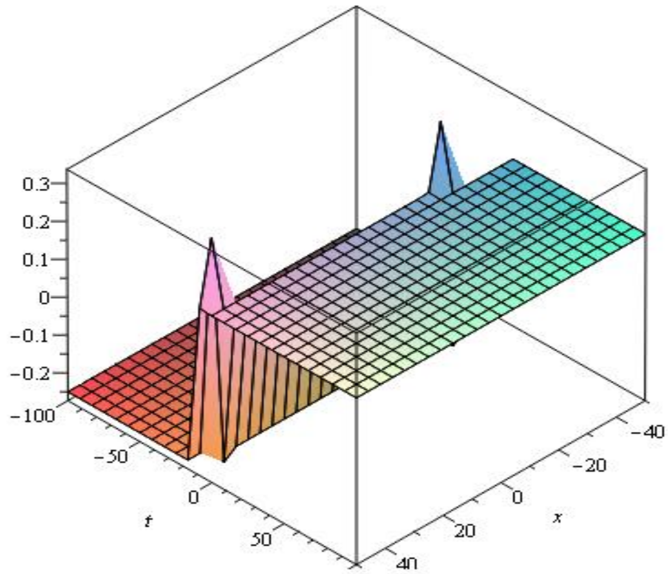


Figure 3.3: Graphical representation of solution (3.3.10) for $\lambda_2 = -9, c_1 = 15, c_4 = 5$.

3.3.3 Subalgebra V_3

The reduced ODEs, in this case are

$$F' = 0, \qquad G' = 0.$$

equation (3.3.25) the obvious solution of (3.1.1) is

$$u(x,\, t) = \text{ constant}, w(x,\, t) = \text{ constant}. \tag{3.3.26}$$

3.4 Lie Symmetry Analysis Three-coupled KdV Equations Engendered by the Neumann System

The application of the classical Lie group method requires considering a one-parameter Lie group of infinitesimal transformations in the variables (x,t,u,w) as follows

$$\begin{aligned}
x^* &= x + \varepsilon\, X(x,t,u,v,w) + O(\varepsilon^2),\\
t^* &= t + \varepsilon\, T(x,t,u,v,w) + O(\varepsilon^2),\\
u^* &= u + \varepsilon\, U(x,t,u,v,w) + O(\varepsilon^2),\\
v^* &= v + \varepsilon\, V(x,t,u,v,w) + O(\varepsilon^2),\\
w^* &= w + \varepsilon\, W(x,t,u,v,w) + O(\varepsilon^2).
\end{aligned} \tag{3.4.1}$$

which leaves the system (3.1.2) invariant. The method for determining the symmetry group of (3.1.2) consists of finding the infinitesimals X, T, U, V and W, which are functions of x, t, u, v and w.

Assuming that system (3.1.2) is invariant under the transformations (3.4.1), the infinitesimals X, T, U, V and W must satisfy the symmetry conditions

$$U_t + \frac{1}{4}U_{xxx} + \frac{3}{4}(wU_x + u_x W) - \frac{3}{2}(Vv_x + vV_x) = 0\,,$$

$$V_t + \frac{1}{4}V_{xxx} + \frac{3}{4}(wV_x + v_x W) + \frac{3}{4}(vW_x + w_x V) = 0\,, \tag{3.4.2}$$

$$W_t - \frac{1}{8}W_{xxx} - \frac{3}{4}(wW_x + w_x W) - \frac{3}{4}U_x = 0\,.$$

where $U_t, U_x, W_x, W_t, W_{xx}$ and U_{xxx} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables (for more details, see [99, 72, 149]). Substituting values of $U_t, U_x, W_x, W_t, W_{xx}$ and U_{xxx}, into symmetry conditions (3.4.2), then equating the coefficients of the various monomials in the first, second and the other order partial derivatives of u and w and their powers, we can find the determining equations for the symmetry group of the equations (3.2.1). Solving these equations, we get the following forms of infinitesimals:

$$X = \frac{1}{3}c_1 x + c_3\,,$$

$$T = c_1 t + c_2\,, \tag{3.4.3}$$

$$U = -\frac{4}{3}c_1 u + c_4\,,$$

$$V = -c_1 v\,,$$

$$W = -\frac{2}{3}c_1 w\,.$$

where c_1, c_2, c_3 and c_1 are arbitrary constants. The Lie algebra associated with system (3.1.2) consists of following four vector fields

$$V_1 = \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{4}{3}u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - \frac{2}{3}w\frac{\partial}{\partial w}\,,$$

$$V_2 = \frac{\partial}{\partial t}\,,$$

$$V_3 = \frac{\partial}{\partial x},$$

$$V_4 = \frac{\partial}{\partial u}, \tag{3.4.4}$$

It is easy to verify that $V_j,\ j=1, 2, 3, 4, 5$ is closed under the Lie bracket.

The classification of one-dimensional subalgebras of the whole symmetry algebra (3.4.4) is done by an inductive approach [72]. We will work out first an optimal system and then embark upon the various reductions associated with generators in the optimal system. We begin by considering a general element $V = a_1 V_1 + a_2 V_2 + a_3 V_3 + a_4 V_4$ of symmetry algebra and subject it to various adjoint transformations to simplify it as much as possible. The adjoint action is given by the Lie series

$$\mathrm{Ad}\left(\exp\left(\varepsilon V_i\right)\right)V_j = V_j - \varepsilon\left[V_i,\ V_j\right] + \frac{\varepsilon^2}{2}\left[V_i, \left[V_i,\ V_j\right]\right] - \cdot\cdot\cdot, \tag{3.4.5}$$

where $\left[V_i, V_j\right] = V_i V_j - V_j V_i$ is the commutator for the Lie algebra, and ε is a parameter.

The types of one-dimensional subalgebras of system (3.1.2) are as follows:

(i) V_1,

(ii) $V_2 + \lambda_1 V_3 + \lambda_2 V_4$,

(iii) $V_3 + \lambda_3 V_4$,

(iv) V_4. (3.4.6)

where λ_1, λ_2 and λ_3 are arbitrary constants.

Once the subalgebra of the equation (3.1.2) have been determined, the symmetry variables for the associated reduction can be found by solving the characteristic equation

$$\frac{dx}{X\left(x,t,u,v,w\right)} = \frac{dt}{T\left(x,t,u,v,w\right)} = \frac{du}{U\left(x,t,u,v,w\right)} = \frac{dv}{V\left(x,t,u,v,w\right)} = \frac{dw}{W\left(x,t,u,w\right)} \tag{3.4.7}$$

On solving the equation (2.8) for the various operators in the optimal system, we obtain a set of non-equivalent Lie ansätze (similarity variable, similarity solutions and forms of the coefficient functions) for the function $u(x, t)$ and $w(x, t)$. Using these ansätze, we can reduce the nonlinear system (3.1.2) to the system of ODEs.

3.5 Exact Solutions of Three-coupled KdV Equations

In this section, we will drive some exact solutions of Three-coupled KdV Equations. One way to obtain exact solutions of (3.1.2) is by reducing it to ordinary differential equations. This can be achieved with the use of Lie point symmetries admitted by (3.1.2).

3.5.1 Subalgebra V_1

In this subalgebra we obtain the similarity variables and similarity solutions for the reduction as follows:

$$\begin{aligned}
\xi &= x^3 t^{-1}, \\
u(x,t) &= t^{-\frac{4}{3}} F(\xi), \\
v(x,t) &= t^{-1} G(\xi), \\
w(x,t) &= t^{-\frac{2}{3}} H(\xi).
\end{aligned} \tag{3.5.1}$$

Using the similarity variable and solution (3.5.1), the system of PDEs (3.1.2) reduces to the following system of ODEs:

$$\begin{aligned}
&-\frac{4}{3}F+\left(\frac{6}{4}-\xi\right)F'+\left(\frac{18}{4}+9\right)\xi F''+\frac{27}{4}\xi^2 F'''+\frac{9}{4}(\xi)^{\frac{2}{3}}HF'-\frac{9}{2}(\xi)^{\frac{2}{3}}GG'=0, \\
&-G+\left(\frac{6}{4}-\xi\right)G'+\left(\frac{18}{4}+9\right)\xi G''+\frac{27}{4}\xi^2 G'''+\frac{9}{4}(\xi)^{\frac{2}{3}}HG'+\frac{9}{4}(\xi)^{\frac{2}{3}}GH'=0, \\
&-\frac{2}{3}H-\left(\frac{6}{8}+\xi\right)H'-\left(\frac{18}{8}+9\right)\xi H''-\frac{27}{8}\xi^2 H'''-\frac{9}{4}(\xi)^{\frac{2}{3}}HH'-\frac{9}{4}(\xi)^{\frac{2}{3}}F'=0.
\end{aligned} \tag{3.5.2}$$

we are unable to find the solutions of ODE (3.5.2)

3.5.2 Subalgebra $V_2 + \lambda_1 V_3 + \lambda_2 V_4$

Under this subalgebra, the following is the optimal system decribe the corresponding variable and solutions as follows:

$$\xi = x - \lambda_1 t,$$

$$u(x,t) = \lambda_2 t + F(\xi),$$

$$v(x,t) = G(\xi),$$

$$w(x,t) = H(\xi). \tag{3.5.3}$$

Using the corresponding variable and solution, the system of PDEs (3.1.2) reduces to the following system of ODEs:

$$-\lambda_2 - \lambda_1 F' + \frac{1}{4}F''' + \frac{3}{4}HF' - \frac{3}{2}GG' = 0,$$

$$-\lambda_1 G' + \frac{1}{4}G''' + \frac{3}{4}HG' + \frac{3}{4}GH' = 0, \tag{3.5.4}$$

$$-\lambda_1 H' - \frac{1}{8}H''' - \frac{3}{4}HH' - \frac{3}{4}F' = 0.$$

Thus, by solving equation (3.5.4) and reverting back to the original variables, we obtain the group-invariant solutions of the equation (3.1.2) as follows

Solution 1:

$$u(x,t) = \frac{\tanh(c_2(x-\lambda_1 t)+c_1)^2 c_5{}^2}{c_2{}^2} + c_3, w(x,t) = -2\tanh(c_2(x-\lambda_1 t)+c_1)^2 c_2{}^2 + c_6$$
$$v(x,t) = \tanh(c_2(x-\lambda_1 t)+c_1)^2 c_5 - \frac{1}{4}\frac{c_5(8c_2{}^6 - 4c_2{}^4 c_6 + c_5{}^2)}{c_2{}^6} \tag{3.5.5}$$

where $\lambda_2 = 0, \lambda_1 = \dfrac{8c_2{}^6 - 6c_2{}^4 c_6 + 3c_5{}^2}{c_2{}^4}$

Solution 2:

$$u(x,t) = -4\tanh(c_2(x-\lambda_1 t)+c_1)^4 c_2{}^4 + \tanh(c_2(x-\lambda_1 t)+c_1)^2\left(\frac{16}{3}c_2{}^4 + \frac{32}{3}\lambda_1 c_2{}^2\right) + c_3,$$

$$v(x,t) = -4I\tanh(c_2(x-\lambda_1 t)+c_1)^3 c_2{}^3 + 4I\tanh(c_2(x-\lambda_1 t+c_1)c_2{}^3, \tag{3.5.6}$$

$$w(x,t) = -4\tanh(c_2(x-\lambda_1 t)+c_1)^3 c_2{}^2 + \frac{8}{3}c_2{}^2 + \frac{4}{3}\lambda I$$

where $\lambda_1 = \lambda_1, \lambda_2 = 0$.

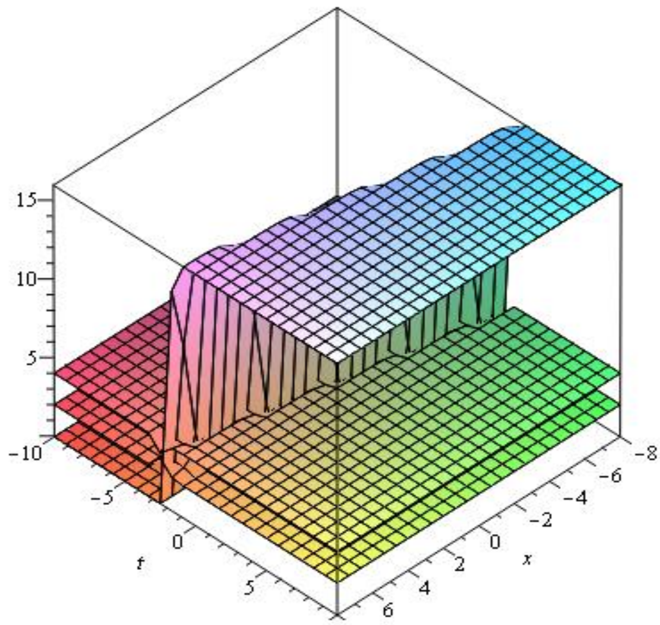


Figure 3.4: Graphical representation of solution (3.5.5) for $\lambda_1 = -4, c_1 = 1, c_2 = 1, c_3 = -\frac{1}{19}$,

$c_5 = 2, c_6 = 4$.

3.5.3 Subalgebra $V_3 + \lambda_3 V_4$

For this subalgebra, we obtain the following corresponding variables and solutions for the reduction

$$\xi = t,$$

$$u(x,t) = \lambda_3 x + F(\xi),$$

$$v(x,t) = G(\xi),$$

$$w(x,t) = H(\xi). \tag{3.5.10}$$

Using the corresponding variable and solutions, the system of PDEs (3.1.2) reduces to the following system of ODEs:

$$F' + \frac{3}{4}\lambda_3 H = 0,$$

$$G' = 0,$$

$$H' - \frac{3}{4}\lambda_3 = 0. \tag{3.5.11}$$

Thus, by solving equation (3.5.11) and reverting back to the original variables, we obtain the group-invariant solutions of the equation (3.1.2) as follows

Solution 1:

$$u(x,t) = -\frac{9}{32}t^2\lambda_3^2 - \frac{3}{4}\lambda_3 c_2 t + c_1,$$

$$v(x,t) = c_3,$$

$$w(x,t) = \frac{3}{4}\lambda_3 t + c_2, \tag{3.5.12}$$

3.5.4. Subalgebra V_4

The reduced ODEs, in this case are

$$F' = 0, \quad G' = 0, \quad H' = 0. \tag{3.5.13}$$

The obvious solution of (3.1.2) is

$$u(x,t) = \text{constant}, \; v(x,t) = \text{constant}. \; w(x,t) = \text{constant} \;. \tag{3.5.14}$$

3.6 Conclusion

In this chapter, we have studied analytical solutions of two-component shallow water system and three-coupled KdV equations engendered by the Neumann system by Lie classical method. Our main goal was to find exact solutions using the admitted Lie symmetries. The symmetry group reduces the original equation to the simple one. By Lie classical method we have investigated the symmetries. After having done standard change of dependent variables we used the basis of the infinitesimal operator written in the new variable in order to find some new classes of invariant solutions Graphical representation of the solutions are also given.

Chapter 4

Numerical Solution of Klein-Gordon Equation

4.1 Introduction

Nonlinear phenomena that appear in many areas of scientific fields can be modelled by partial differential equations. In this chapter, we have studied the numerical approximation of the nonlinear Klein-Gordon equation with quadratic (i.e. $g(u)=u^2$) and cubic (i.e. $g(u)=u^3$) nonlinearity. The equation was named after the physicists Oskar Klein and Walter Gordon, who in 1926 proposed that it describes relativistic electrons. The Klein-Gordon equation was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie waves. The nonlinear Klein-Gordon equation appears in many types of nonlinearities. The Klein-Gordon equation arises in relativistic quantum mechanics and field theory [22], which is of great importance for the high energy physicists [218], and is used to model many different phenomena, including the propagation of dislocations in crystals and the behaviour of elementary particles.

The Klein-Gordon equation plays a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory [26] and mathematical physics [23, 34, 192]. The equation has attracted much attention in studying solitons and condensed matter physics [152], in investigating the interaction of solitons in collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [168]. Several methods are developed to solve the Klein-Gordon-type equations, such as the Weierstrass elliptic function method, the elliptic equation rational expansion method and the extended F-function method [191]. Parkes *et al.* [63] and Fu *et al.* [231] used the Jacobi elliptic function expansion method to find double periodic solutions. Sirendaoreji [191] used the auxiliary equation method to construct more types of new exact travelling wave solutions of the nonlinear Klein-Gordon equation with quadratic and cubic nonlinearity. Guo *et al.* [151] presented a numerical analysis of the one-dimensional Klein-Gordon equation with quadratic and cubic nonlinearity, using the element free reproducing kernel particle Ritz method. Davender *et al.* [54] presented a reliable algorithm based on the homotopy analysis transform method to solve linear and nonlinear Klein-Gordon equation. This method provides the solution in the form of a rapidly. Fabio *et al.* [70] proved results of periodic waves for the Klein- Gordon equation with quintic nonlinearity. Wenting *et al.* [215] studied the numerical solutions of one-dimensional Klein-Gordon and Sine-Gordon equation using the chebyshev Tau meshless method based on the integration differentiation and introduce the domain decomposition method in space and the block-marching technique in time for problems defined in large interval and long time computing. Yue *et al.* [225] studied instability problems of standing waves for the nonlinear Klein-Gordon equation. They also showed that those special wave solutions may be strongly unstable by blow up in finite time depending on the range of wave's frequency. Camille [42] studied the internal stabilization and control of the critical nonlinear Klein- Gordon equation on 3-D compact manifolds. Under a geometric assumption slightly stronger than the classical geometric control condition, he proves exponential decay for some solutions bounded in the energy space but small in lower norm.

The numerical solutions to the nonlinear Klein-Gordon equation have received considerable attention in the literature. Lynch [119] proposed a variety of second order finite difference schemes, the alternative approaches using spectral and pseudo-spectral methods presented in [33, 219], Jiminez and Vazquez [190] discussed some finite difference schemes and observed undesirable characteristics in some of the numerical schemes, Dehghan *et al.* [120] proposed a fourth-order compact finite difference method to solve the one-dimensional nonlinear Klein-Gordon equation, Shakeri and Dehghan [69] developed a semi-numerical approach based on the variational iteration method for the numerical solutions of nonlinear Klein-Gordon equation. For more works on this equation see [2, 23, 25, 27, 121, 122, 124, 158, 159]. Recently, Dehghan *et al.* [124] developed techniques based on the dual reciprocity boundary element method and collocation and finite difference-collocation methods to obtain the approximate solutions of the nonlinear Klein-Gordon equation. Rashidinia *et al.* [96, 98] proposed tension spline approach and a three-level spline-difference scheme to solve the one dimensional Klein-Gordon equation. Pekmen and Tezer-Sezgin [34] used differential quadrature method for the solution of nonlinear Klein-Gordon and sine-Gordon equations. Dehghan *et al.* [118, 122, 136, 181] proposed Chebyshev pseudospectral multidomain method, Legendre pseudospectral viscosity technique, local radial basis functions based differential quadrature collocation method for some well nonlinear physical problems.

In this chapter, a scheme based on polynomial differential quadrature method is developed to find numerical solutions of the nonlinear Klein-Gordon equation. By this method, we approximate the spatial derivatives of unknown function at any grid points using weighted sum of all the functional values at certain points in the whole computational domain. Differential quadrature method (DQM), first introduced by Belman *et al.* [166] in 1972, is one of the most efficient numerical methods to solve differential equations. Since then, many authors have developed various types of differential quadrature methods using some test functions such as Legendre polynomials, Lagrange interpolation polynomials, spline functions, Lagrange interpolated cosine functions, etc. [48, 97, 169]. The most frequently used differential quadrature methods are based on Lagrange interpolation polynomials and sine-cosine expansion. Korkmaz and Dag [18, 19] proposed sine differential quadrature method and cosine expansion based differential quadrature method for many nonlinear partial differential equations. Mittal *et al.* [167, 161, 163, 164, 167] have used polynomial based differential quadrature method for numerical solutions of some nonlinear partial differential equations.

The aim of this chapter is to develop a numerical scheme based on polynomial differential quadrature method to find the numerical solutions of Klein-Gordon equation with Dirichlet and Neumann boundary conditions. The work put in this chapter has been structured as follows. In section 4.2, Klein-Gordon is given with Dirichlet and Neumann boundary conditions. Methodology of polynomial differential quadrature method is given in section 4.3. In section 4.4, we first semi-discretize the original non-linear equation in the temporal direction by forward difference formula with the constant time step which produces a set of nonlinear second order ordinary differential equations. Then, the quasilinearization process [175, 180] is used to linearize the nonlinear second order ordinary differential equations. For totally discrete scheme, we discretize the set of linear problems by using polynomial differential quadrature method. In section 4.5, five test examples of nonlinear Klein-Gordon equations are considered to test the accuracy and efficiency of the developed scheme.

4.2 Klein-Gordon Equation

The nonlinear one-dimensional Klein-Gordon equation is in the form

$$\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial^2 u}{\partial x^2}+\beta u+\gamma\, g(u)= f(x,t), \quad x\in[L_0, L_1],\ t>t_0 \tag{4.2.1}$$

Subject to the initial conditions

$$u(x,t_0) = \phi(x), \qquad u_t(x,t_0) = \psi(x), \qquad x \in [L_0, L_1], \tag{4.2.2}$$

and with appropriate Dirichlet and Neumann boundary conditions

$$u(L_0,t) = g_1(t), \qquad u(L_1,t) = g_2(t), \qquad t \ge t_0 \quad \text{(Dirichlet conditions)} \tag{4.2.3a}$$

$$u_x(L_0,t) = g_3(t), \qquad u_x(L_1,t) = g_4(t), \qquad t \ge t_0 \quad \text{(Neumann conditions)} \tag{4.2.3b}$$

where $u(x,t)$ represents the wave displacement at position x and time t, $\alpha < 0$, β, γ are constants, $g(u)$ is the nonlinear force and the functions $\phi(x)$ and $\psi(x)$ are wave modes or kinks and velocity, respectively. The above nonlinear Klein-Gordon equation is Hamiltonian partial differential equation, and for a wide class of force $g(u)$, it has the conserved Hamiltonian quantity (or energy)

$$H = \int \left(\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + G(u) \right) \text{ where } G^{'}(u) = g(u).$$

4.3 Polynomial Differential Quadrature Method

In this chapter, polynomial differential quadrature method (PDQM) has used to approximate the spatial derivatives of the problem. By polynomial differential quadrature the derivatives of the problem (4.2.1) are discretized at a point x_i in the following manner

$$u_x(x_i,t) = \sum_{j=1}^{N} a_{ij}\, u(x_j,t), \qquad u_{xx}(x_i,t) = \sum_{j=1}^{N} b_{ij}\, u(x_j,t), \tag{4.3.1}$$

where a_{ij} and b_{ij} represent the weighting coefficients [30], $i = 1,2,...,N$. The first order and second order weighting coefficients are as follows (for more detail see Chapter 1)

$$a_{ij} = \frac{L^{(1)}(x_i)}{(x_i - x_j)L^{(1)}(x_j)}, \quad k = 1,2,...,N, \quad i \ne j, \tag{4.3.2}$$

$$a_{ii} = -\sum_{j=1,\, j\neq i}^{N} a_{ij}\,, \quad i = 1,2,...,N\,, \tag{4.3.3}$$

$$b_{ij} = 2a_{ij}\left(a_{ii} - \frac{1}{x_i - x_j}\right), \;\; i, j = 1,2,...,N,\; i \neq j\,, \tag{4.3.4}$$

$$b_{ii} = -\sum_{j=1,\, j\neq i}^{N} b_{ij}\,, \quad i = 1,2,...,N\,, \tag{4.3.5}$$

4.4 Numerical Scheme for Klein-Gordon Equation

The development of the numerical scheme for Klein-Gordon equation is divided into the following subsections

4.4.1 Time discretization and Quasilinearization Process

The first step in the construction of numerical scheme consists of discretizing the time variable with forward difference formula for time derivative of the unknown u , after applying the forward difference formula the equation reduces to following system of second order nonlinear ordinary differential equations

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta t^2} + \alpha\left(u_{j+1}\right)_{xx} + \beta u_{j+1} + \gamma\, g\left(u_{j+1}\right) = f\left(x, t_{j+1}\right), \quad j = 0,1,2,...,M-1 \tag{4.4.1}$$

with initial conditions

$$u(x, t_0) = \phi(x), \qquad u_t(x, t_0) = \psi(x), \tag{4.4.2}$$

and both type of boundary conditions

$$u\left(L_0, t_{j+1}\right) = g_1\left(t_{j+1}\right), \quad u\left(L_1, t_{j+1}\right) = g_2\left(t_{j+1}\right), \tag{4.4.3a}$$

$$u_x\left(L_0, t_{j+1}\right) = g_3\left(t_{j+1}\right), \quad u_x\left(L_1, t_{j+1}\right) = g_4\left(t_{j+1}\right), \quad j = 0,1,...,M-1 \tag{4.4.3b}$$

where $u_{j+1} = u\left(x, t_j + \Delta t\right)$, $t_{j+1} = t_j + \Delta t$ and Δt step length in time direction.

Now, linearize the non-linear problem (4.4.1) by using quasilinearization process, which yields

$$\Delta t^2 \alpha \left(u_{j+1}^{(n+1)}\right)_{xx} + \left(1+\Delta t^2 \beta\right) u_{j+1}^{(n+1)} + \Delta t^2 \gamma \left(g\left(u_{j+1}^{(n)}\right) + \left(u_{j+1}^{(n+1)} - u_{j+1}^{(n)}\right) g_{u_{j+1}}\left(u_{j+1}^{(n)}\right)\right)$$
$$= \Delta t^2 f\left(x, t_{j+1}\right) + 2u_j^{(n+1)} - u_{j-1}^{(n+1)}, \quad j = 0,1,2,\ldots, M-1 \tag{4.4.4}$$

with initial conditions

$$u^{(n+1)}\left(x, t_0\right) = \phi(x), \qquad u_t^{(n+1)}\left(x, t_0\right) = \psi(x), \tag{4.4.5}$$

and both type of boundary conditions

$$u_{j+1}^{(n+1)}\left(L_0\right) = g_1\left(t_{j+1}\right), \quad u_{j+1}^{(n+1)}\left(L_1\right) = g_2\left(t_{j+1}\right), \tag{4.4.6a}$$

$$u_x\left(L_0, t_{j+1}\right) = g_3\left(t_{j+1}\right), \quad u_x\left(L_1, t_{j+1}\right) = g_4\left(t_{j+1}\right), \quad j = 0,1,\ldots, M-1 \tag{4.4.6b}$$

where $u^{(0)}$ the initial guess and *n* is is the iteration index.

4.4.2 Spatial Discretization

The equation (4.4.4) is a system of second order linear ordinary differential equations. The space derivatives in the equation are approximated by the polynomial differential qudrature method, and then we have the following algebraic system

$$\Delta t^2 \alpha \sum_{k=1}^{N} b_{ik}\, u_{k,j+1}^{(n+1)} + \left(1+\Delta t^2 \beta\right) u_{i,j+1}^{(n+1)} + \Delta t^2 \gamma \left(g\left(u_{i,j+1}^{(n)}\right) + \left(u_{i,j+1}^{(n+1)} - u_{i,j+1}^{(n)}\right) g_{u_{i,j+1}}\left(u_{i,j+1}^{(n)}\right)\right)$$
$$= \Delta t^2 f\left(x, t_{j+1}\right) + 2u_{i,j}^{(n+1)} - u_{i,j-1}^{(n+1)}, \; i = 1,2,\ldots, N, \; j = 0,1,2,\ldots, M-1 \tag{4.4.7}$$

4.4.3 Implementation of Boundary Conditions

The Dirichlet boundary conditions (4.4.6*a*) are directly applied in the PDQM and gives numerical solutions on boundary in the following way

$$u_{1,j+1}^{(n+1)} = g_1\left(t_{j+1}\right), \quad u_{N,j+1}^{(n+1)} = g_2\left(t_{j+1}\right), \tag{4.4.8}$$

The Neumann boundary conditions (4.4.6b) at boundary points can be approximated as

$$\sum_{k=1}^{N} a_{1k} u_{k,j+1}^{(n+1)} = g_3\left(t_{j+1}\right),$$
$$\sum_{k=1}^{N} a_{Nk} u_{k,j+1}^{(n+1)} = g_4\left(t_{j+1}\right), \tag{4.4.9}$$

The equation (4.4.9) becomes

$$a_{11}u_{1,j+1}^{(n+1)}+a_{1N}u_{N,j+1}^{(n+1)}=g_3(t_{j+1})-\sum_{k=2}^{N-1}a_{1k}\,u_{k,j+1}^{(n+1)}, \tag{4.4.10a}$$

$$a_{N1}u_{1,j+1}^{(n+1)}+a_{NN}\,u_{N,j+1}^{(n+1)}=g_4(t_{j+1})-\sum_{k=2}^{N-1}a_{Nk}\,u_{k,j+1}^{(n+1)}, \tag{4.4.10b}$$

Multiplying equation (4.4.10b) by a_{11} and (4.4.10a) by a_{N1} and subtract the both equations, we have

$$(a_{1N}a_{N1}-a_{NN}a_{11})u_{N,j+1}^{(n+1)}=a_{N1}\left(g_3(t_{j+1})-S_1\right)-a_{11}\left(g_4(t_{j+1})-S_2\right), \tag{4.4.11}$$

Again multiplying equation (4.4.10a) by a_{NN} and (4.4.10b) by a_{1N} and subtracting the both equations, we have

$$(a_{NN}a_{11}-a_{1N}a_{N1})u_{1,j+1}^{(n+1)}=a_{NN}\left(g_3(t_{j+1})-S_1\right)-a_{1N}\left(g_4(t_{j+1})-S_2\right) \tag{4.4.12}$$

Then after simplification, we have numerical solutions on boundary in the following way

$$\begin{aligned}
u_{1,\,j+1}^{(n+1)}&=\frac{a_{NN}\left(g_3\left(t_{j+1}\right)-S_1,\right)-a_{1N}\left(g_4\left(t_{j+1}\right)-S_2\right)}{a_{11}\,a_{NN}-a_{N1}\,a_{1N}}\\
u_{N,\,j+1}^{(n+1)}&=\frac{a_{N1}\left(g_3\left(t_{j+1}\right)-S_1,\right)-a_{11}\left(g_4\left(t_{j+1}\right)-S_2\right)}{a_{1N}\,a_{N1}-a_{11}\,a_{NN}}
\end{aligned} \tag{4.4.13}$$

where $S_1=\sum_{k=2}^{N-1}a_{1k}\,u_{k,j+1}^{(n+1)}$ and $S_2=\sum_{k=2}^{N-1}a_{Nk}\,u_{k,j+1}^{(n+1)}$.

In this chapter, we have solved Klein-Gordon equation with quadratic (i.e. $g(u)=u^2$) and cubic (i.e. $g(u)=u^3$) nonlinearity. Then the equation (4.4.7) will be different for both two cases.

Case 1: When $g(u)=u^2$ and after applying the boundary conditions (4.2.3), the equation (4.4.7) becomes

$$\begin{aligned}
\Delta t^2\alpha\sum_{k=2}^{N-1}b_{ik}\,u_{k,\,j+1}^{(n+1)}+\left(1+\Delta t^2\left(\beta+2\gamma\left(u_{i,\,j+1}^{(n)}\right)^2\right)\right)u_{i,\,j+1}^{(n+1)}&=\Delta t^2 f\left(x_i,t_{j+1}\right)+2u_{i,\,j}^{(n+1)}-u_{i,\,j-1}^{(n+1)}\\
&+\Delta t^2\gamma\left(u_{i,\,j+1}^{(n)}\right)^2-\left(1+\Delta t^2\left(\beta+2\gamma\left(u_{i,\,j}^{(n+1)}\right)^2\right)\right)\left(u_{1,\,j+1}^{(n+1)}+u_{N,\,j+1}^{(n+1)}\right)\\
&-\Delta t^2\alpha\left(b_{i1}\,u_{1,\,j+1}^{(n+1)}+b_{iN}u_{N,\,j+1}^{(n+1)}\right)\\
&i=2,3,\ldots,N-1,\ j=0,1,2,\ldots,M-1
\end{aligned} \tag{4.4.14}$$

Case 2: When $g(u)=u^3$ and after applying the boundary conditions (4.2.3), the equation (4.4.7) becomes

$$\Delta t^2\alpha\sum_{k=2}^{N-1} b_{ik}\, u_{k,j+1}^{(n+1)} + \left(1+\Delta t^2\left(\beta+3\gamma\left(u_{i,j+1}^{(n)}\right)^3\right)\right)u_{i,j+1}^{(n+1)} = \Delta t^2 f\left(x_i, t_{j+1}\right) + 2u_{i,j}^{(n+1)} - u_{i,j-1}^{(n+1)}$$
$$+ 2\Delta t^2\gamma\left(u_{i,j+1}^{(n)}\right)^3 - \left(1+\Delta t^2\left(\beta+3\gamma\left(u_{i,j}^{(n+1)}\right)^3\right)\right)\left(u_{1,j+1}^{(n+1)} + u_{N,j+1}^{(n+1)}\right) \qquad (4.4.15)$$
$$-\Delta t^2\alpha\left(b_{i1}\, u_{1,j+1}^{(n+1)} + b_{iN}\, u_{N,j+1}^{(n+1)}\right)$$
$$i = 2,3,\ldots, N-1,\ j = 0,1,2,\ldots, M-1$$

The values of $u_{1,j+1}^{(n+1)}$ and $u_{N,j+1}^{(n+1)}$ in the right hand sides of (4.4.14) and (4.4.15) are taken from the equations (4.4.8) and (4.4.13) according to the boundary conditions. After that the equations (4.4.14) and (4.4.15) are solved by the Gauss-elimination method successively by starting the process with the following terms

$$u^{(n+1)}(x,t_0)=\phi(x), \qquad u_t^{(n+1)}(x,t_0)=\psi(x), \qquad (4.4.16)$$

4.5 Numerical Experiments

In this section, five test examples of nonlinear Klein-Gordon equations are considered to demonstrate the efficiency and accuracy of the proposed numerical scheme and the whole computation work is done with the help of Gauss-Lobatto-Chebyshev grid points. The Gauss-Lobatto-Chebyshev grid points are defined as follows over the interval $[a,b]$

$$x_i = a + \frac{1}{2}\left(1-\cos\frac{((i-1)\pi)}{N-1}\right)(b-a), \quad i = 1,2,\ldots,N \qquad (4.5.1)$$

In numerical experiments L_2 , L_∞ and root mean square (RMS) errors are calculated by the formulas

$$L_2 \text{ Error} = \left(\sum_{i=1}^{n} e_i^2\right)^{1/2} \qquad (4.5.2)$$

$$L_\infty \text{ Error} = \max_{1\le i\le n}\left|e_i\right| \qquad (4.5.3)$$

$$\text{(RMS): } RMS \text{ Error} = \left(\sum_{i=1}^{n}\frac{e_i^2}{n}\right)^{1/2} \qquad (4.5.4)$$

where $e_i = (u_i - U_i)$, u_i are approximated solutions and U_i are exact solutions.

Example 1

In this Example, we have considered the nonlinear Klein-Gordon equation (4.2.1) with quadratic nonlinearity. The constants are $\alpha = -1, \beta = 0$ and $\gamma = 1$ in the domain $-1 \le x \le 1$. The initial conditions are given by

$$\begin{cases} u(x,0) = x, & -1 \le x \le 1 \\ u_t(x,0) = 0, & -1 \le x \le 1 \end{cases} \tag{4.5.5}$$

and the right-hand side function is

$$f(x,t) = -x\cos(t) + x^2\cos^2(t) \tag{4.5.6}$$

The analytical solution is given in [15] as

$$u(x,t) = x\cos(t) \tag{4.5.7}$$

The boundary conditions are taken from the analytical solution. Numerical results are presented in Table (4.1) and Figure (4.1). Table (4.1) shows the L_2, RMS and L_∞ errors and made a comparison with [122]. Figure (4.1) compares the numerical and exact solutions at different times in form of 3D graph.

Example 2

Consider the nonlinear Klein-Gordon equation (4.2.1) with quadratic nonlinearity in the interval $0 \le x \le 1$ with $\alpha = -1, \beta = 0$ and $\gamma = 1$. The initial conditions and the right-hand side function are given by

$$\begin{cases} u(x,0) = 0, & 0 \le x \le 1 \\ u_t(x,0) = 0, & 0 \le x \le 1 \end{cases} \tag{4.5.8}$$

$$f(x,t) = 6xt(x^2 - t^2) + x^6 t^6 \tag{4.5.9}$$

The analytical solution is given in [26] as

$$u(x,t) = x^3 t^3 \tag{4.5.10}$$

Table (4.2) presents a comparison of present scheme and the scheme developed in [81] by comparing the different types of errors at different time. Figure (4.2) delineates the numerical and exact solutions at different times which are very similar to each other.

Example 3

In this example, we have considered the nonlinear Klein-Gordon equation (4.2.1) with cubic nonlinearity. The constants are $\alpha = -1,\ \beta = 1$ and $\gamma = 1$ in the domain $-1 \le x \le 1$. The initial conditions are given by

$$\begin{cases} u(x,0) = x^2 \cosh(x), & -1 \le x \le 1 \\ u_t(x,0) = x^2 \cosh(x), & -1 \le x \le 1 \end{cases} \tag{4.5.11}$$

and the right-hand side function is

$$f(x,t) = (x^2 - 2)\cosh(x+t) - 4x\sinh(x+t) + x^6 \cosh^3(x+t) \tag{4.5.12}$$

The analytical solution is given by

$$u(x,t) = x^2 \cosh(x+t) \tag{4.5.13}$$

Table (4.3) and Figure (4.3) present the numerical results of the Example in form of errors, numerical and exact solutions. The L_2, RMS and L_∞ errors are compared with [122] in Table (4.3) while Figure (4.3) compares the numerical and exact solutions at different times in form of 3D graph.

Example 4

Consider the nonlinear Klein-Gordon equation (4.2.1) with cubic nonlinearity in the interval $0 \le x \le 1$ with $\alpha = -2.5,\ \beta = 1$ and $\gamma = 1.5$. The initial and boundary conditions are given by

$$\begin{cases} u(x,0) = \sqrt{\frac{2}{3}} \tan\left(\sqrt{\frac{-1}{2c^2-5}}\, x\right), & -1 \le x \le 1 \\ u_t(x,0) = c\sqrt{\frac{-2}{3(2c^2-5)}} \sec^2\left(\sqrt{\frac{-1}{2c^2-5}}\, x\right),, & -1 \le x \le 1 \end{cases} \tag{4.5.14}$$

$$\begin{cases} u_x(0,t) = \sqrt{\frac{-2}{3(2c^2-5)}} \sec^2\left(\sqrt{\frac{-1}{2c^2-5}}\,(c\,t)\right) \\ u_x(1,t) = \sqrt{\frac{-2}{3(2c^2-5)}} \sec^2\left(\sqrt{\frac{-1}{2c^2-5}}\,(1+c\,t)\right), \end{cases} \tag{4.5.15}$$

The analytical solution is given by [23]

$$u(x,t) = \sqrt{\frac{2}{3}} \tan\left(\sqrt{\frac{-1}{2c^2-5}}\,(x+c\,t)\right) \tag{4.5.16}$$

In this Example, we have taken the right-hand side function $f(x,t)=0$. Table (4.4) compares the L_2, RMS and L_∞ errors and CPU time with [122]. Figure (4.4) depicts a comparison of numerical and exact solutions at different times in form of 3D graph.

Example 5 (Kink Wave)

We consider the nonlinear Klein-Gordon equation with initial and Neumann boundary conditions $u_{tt} - \alpha^2 u_{xx} + \alpha u - \beta u^3 = 0, \quad x \in [-10, 10]$,

$$\begin{cases} u(x,0) = \sqrt{\frac{\alpha}{\beta}} \tanh(\kappa x) \\ u_t(x,0) = -c\,\kappa \sqrt{\frac{\alpha}{\beta}} \sec h^2(\kappa x) \end{cases} \tag{4.5.17}$$

and

$$\begin{cases} u_x(-10,t) = \kappa\sqrt{\dfrac{\alpha}{\beta}}\,\sec h^2\left(\kappa(-10-ct)\right), \\ u_x(10,t) = \kappa\sqrt{\dfrac{\alpha}{\beta}}\,\sec h^2\left(\kappa(10-ct)\right), \end{cases} \tag{4.5.18}$$

where $\kappa = \sqrt{\dfrac{\alpha}{2(c^2-\alpha^2)}}$ and $\alpha,\ \beta,\ c^2-\alpha^2 > 0$.The exact solution is $u(x,t) = \sqrt{\dfrac{\alpha}{\beta}}\tanh(\kappa(x-ct))$.The results of the Example are presented in Table (4.5) and Figure (4.5). Table 4.5 shows the accuracy of the computed solutions with the variations of grid points in space and $\alpha = 0.2,\ \beta = 1,\ c = 0.3$ at time $t = 1.0$. The accuracy is improved when the number of grid points in space direction is increased. Figure 4.5 delineates the space-time graph of numerical and exact solutions which are very similar and behave like a kink wave

Table 4.1 Comparison of errors of Example 1 at different time.

t	L_∞ Error	RMS Error	L_2 Error	CPU Time (s)
Present Scheme				
1	1.09044×10^{-5}	6.81437×10^{-6}	3.12274×10^{-5}	1
3	7.40829×10^{-6}	4.64986×10^{-6}	2.13083×10^{-5}	1
5	1.62080×10^{-5}	1.11963×10^{-5}	5.13081×10^{-5}	2
7	2.02889×10^{-5}	1.24906×10^{-5}	5.72392×10^{-5}	2
10	8.17517×10^{-6}	6.25475×10^{-6}	2.86629×10^{-5}	3
Dehghan and Shokri [122]				
1	1.2540×10^{-5}	6.5097×10^{-6}	6.5422×10^{-5}	5
3	1.5554×10^{-5}	1.1659×10^{-5}	1.1717×10^{-4}	22
5	3.3792×10^{-5}	2.1902×10^{-5}	2.2011×10^{-4}	49
7	3.7753×10^{-5}	2.5763×10^{-5}	2.5892×10^{-4}	83
10	1.3086×10^{-5}	7.9458×10^{-6}	7.9854×10^{-5}	150

Table 4.2 Comparison of errors of Example 2 at different time.

t	L_∞ Error	RMS Error	L_2 Error	CPU Time (s)
Present Scheme				
1	1.43637×10^{-5}	1.00730×10^{-5}	4.89982×10^{-5}	1
2	1.32882×10^{-4}	1.03085×10^{-4}	4.72397×10^{-4}	1
3	5.16169×10^{-4}	2.94028×10^{-4}	1.34741×10^{-3}	1
4	1.41447×10^{-3}	7.22379×10^{-4}	3.31035×10^{-3}	2
5	2.70151×10^{-3}	1.33844×10^{-3}	6.13349×10^{-3}	2
Dehghan and Shokri [122]				
1	1.1012×10^{-5}	5.4725×10^{-6}	5.4998×10^{-5}	6
2	1.6496×10^{-4}	1.1465×10^{-4}	1.1522×10^{-3}	14
3	5.9728×10^{-4}	3.2426×10^{-4}	3.2588×10^{-3}	25
4	1.8264×10^{-3}	9.7704×10^{-4}	9.8191×10^{-3}	37
5	3.6915×10^{-3}	1.9044×10^{-3}	1.9139×10^{-2}	52

Table 4.3 Comparison of errors of Example 3 at different time.

t	L_∞ Error	RMS Error	L_2 Error	CPU Time (s)
Present Scheme				
1	6.29309×10^{-5}	3.49452×10^{-5}	1.60139×10^{-4}	1
2	3.76638×10^{-4}	1.54973×10^{-4}	7.10177×10^{-4}	2
3	1.38838×10^{-3}	5.49316×10^{-4}	2.51728×10^{-3}	2
4	3.50357×10^{-3}	1.68884×10^{-3}	7.73923×10^{-3}	3
5	1.11616×10^{-2}	4.80751×10^{-3}	2.20308×10^{-2}	4
Dehghan and Shokri [122]				
1	5.0705×10^{-5}	2.0789×10^{-5}	2.9474×10^{-4}	23
2	5.0260×10^{-4}	1.9102×10^{-4}	2.7082×10^{-3}	51
3	2.0612×10^{-3}	6.8592×10^{-4}	9.7246×10^{-3}	84
4	6.5720×10^{-3}	1.9666×10^{-3}	2.7881×10^{-2}	120
5	1.9067×10^{-2}	5.4549×10^{-3}	7.7337×10^{-2}	160

Table 4.4 Comparison of errors of Example 4 at different time with $c = 0.5$

t	L_∞ Error	RMS Error	L_2 Error	CPU Time (s)
Present Scheme				
1	2.7279×10^{-5}	1.8583×10^{-5}	3.9683×10^{-5}	0
2	1.7644×10^{-5}	1.4710×10^{-5}	1.3697×10^{-5}	0
3	5.1961×10^{-5}	3.7627×10^{-5}	2.8671×10^{-5}	1
4	1.4827×10^{-4}	7.5186×10^{-5}	2.6051×10^{-5}	1
Dehghan and Shokri [122]				
1	5.9964×10^{-6}	4.0559×10^{-6}	4.0761×10^{-5}	0
2	2.1973×10^{-5}	1.5691×10^{-5}	1.5769×10^{-4}	1
3	9.0893×10^{-5}	6.4470×10^{-5}	6.492×10^{-4}	1
4	8.2945×10^{-4}	5.3306×10^{-4}	5.3572×10^{-3}	2

Table 4.5 Comparison of errors of Example 5 at time $t = 1.0$ with $\alpha = 0.2,\ \beta = 1,\ c = 0.3$

	Pekmen and Tezer-Sezgin [34]		Present Scheme	
N	L_∞ *error*	*RMS error*	L_∞ *error*	*RMS error*
16	1.13×10^{-2}	4.42×10^{-3}	1.64×10^{-3}	9.53×10^{-4}
32	3.12×10^{-3}	9.40×10^{-4}	8.13×10^{-4}	6.15×10^{-4}
64	1.55×10^{-4}	4.10×10^{-5}	1.37×10^{-4}	3.13×10^{-5}
128	1.71×10^{-7}	6.72×10^{-8}	1.61×10^{-7}	4.36×10^{-8}

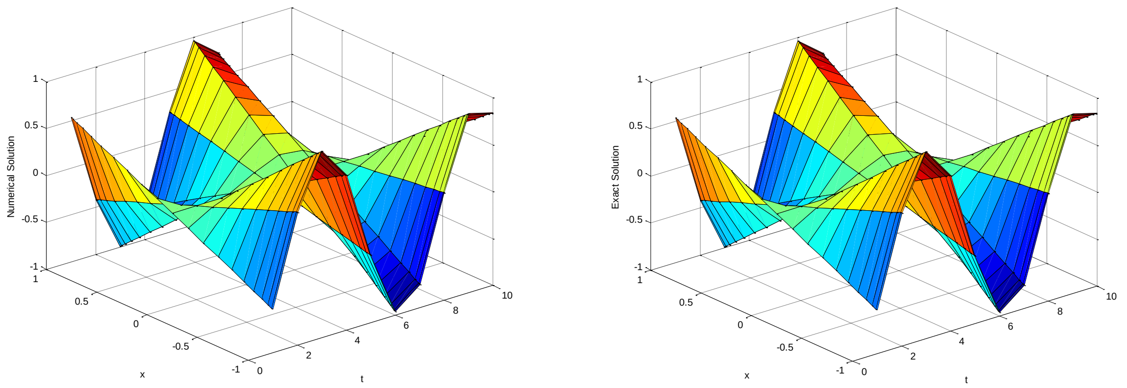


Figure 4.1: Numerical (Left) and exact (Right) solutions of Example 1 up to $t = 10s$, in space-time graph form.

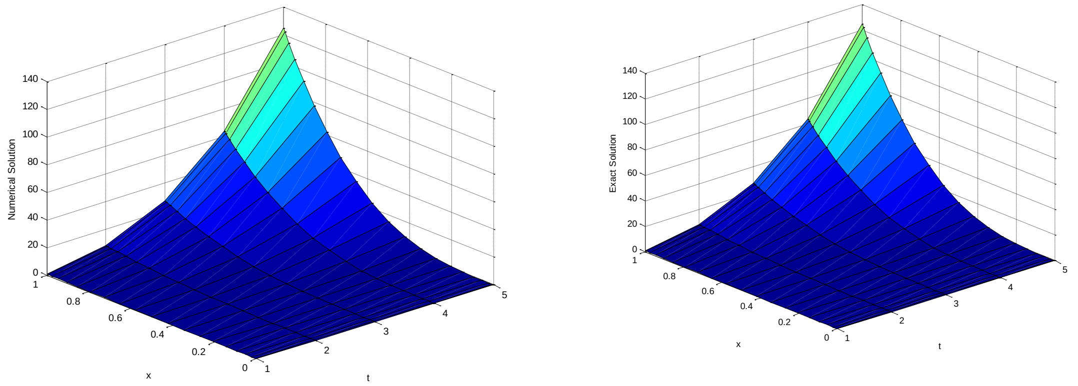


Figure 4.2: Numerical (Left) and exact (Right) solutions of Example 2 up to $t = 5s$, in space-time graph form.

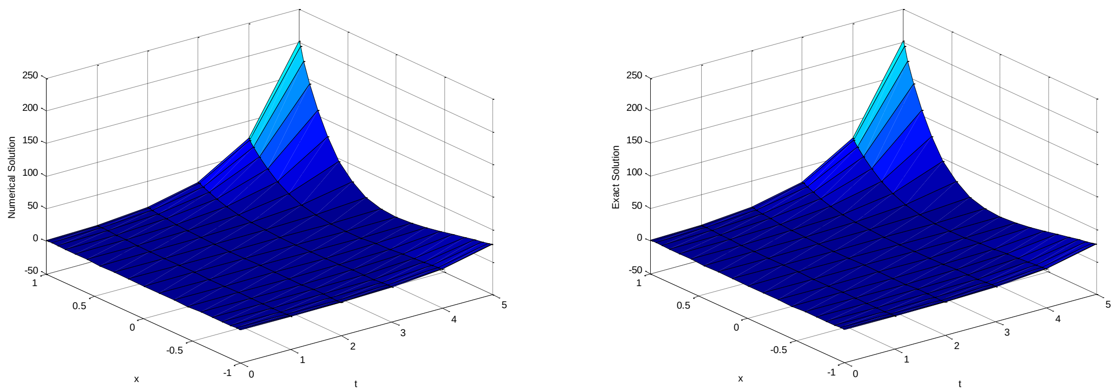


Figure 4.3: Numerical (Left) and exact (Right) solutions of Example 3 up to $t = 5s$, in space-time graph form.

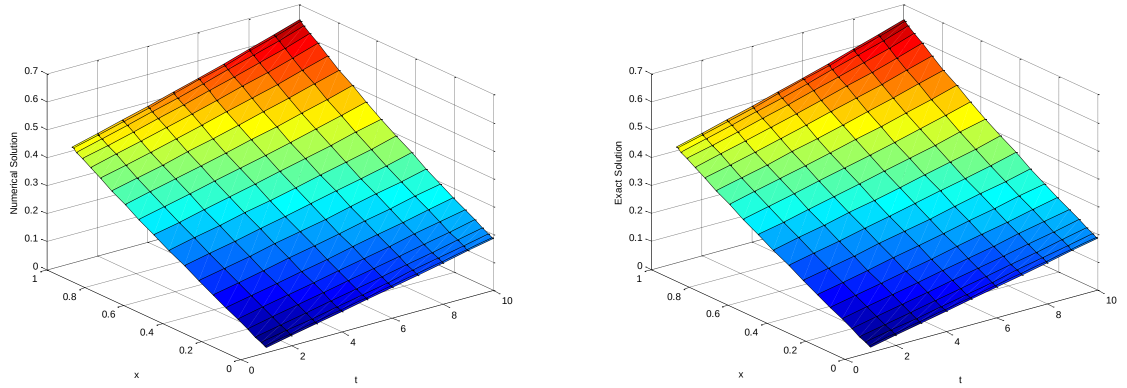


Figure 4.4: Numerical (Left) and exact (Right) solutions of Example 4 up to $t = 10\,s$, in space-time graph form.

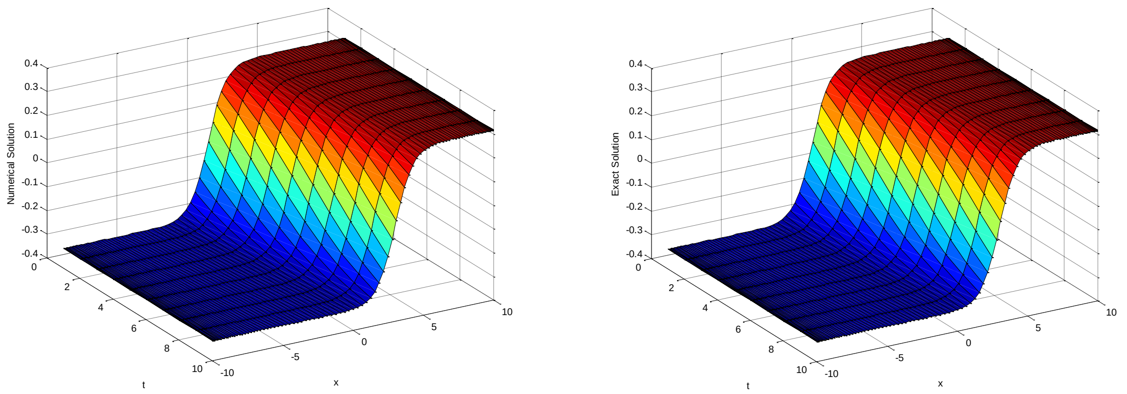


Figure 4.5: Numerical (Left) and exact (Right) solutions of Example 5 up to $t = 10\,s$, in space-time graph form.

4.6 Conclusion

In this chapter, a numerical scheme has been developed for the numerical simulation of nonlinear Klein-Gordon equation. The essential difficulty in the numerical solutions for the Klein-Gordon equation involves the unboundedness of the physical domain, but in the chapter we considered the nonlinear Klein-Gordon equation on a bounded domain. Gauss-Lobatto-Chebyshev grid points are used for the numerical simulation. In all the examples, L_2, L_∞ RMS errors and CPU times are presented in Tables 4.1-4.5 with time step length $\Delta t = 0.001$ and $N = 21$. The main advantage of the present scheme is that the scheme gives very accurate and similar results to the exact solutions by choosing less number of grid points. Secondly, the scheme gives better accuracy than [34, 122] by choosing less number of grid points and big time step length Δt. Also, the scheme can be extended for multidimensional problems.

Chapter 5

Comparative Solution Study of Fisher's Type Equations

5.1 Introduction

In this chapter, analytic and numerical solutions of nonlinear diffusion equation of Fisher type's with the help of classical Lie symmetry method and differential qudrature method (DQM) are studied. We have consider nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + g(u) \tag{5.1.1}$$

when $g(u) = \alpha u(1-u)$, (5.1.1) decribe the Fisher equation, which was proposed by Fisher [182] as a model for the spatial and temporal propagation of a viral gene in an infinite medium. This equation also represents a one-dimensional reaction diffusion model for the evolution of the infected population. The equation is defined by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \quad \text{where } \alpha > 0 \text{ is a parameter.} \tag{5.1.2}$$

In equation (5.1.2), energy let out by non-linear term balances energy absorbed by diffusion, resulting in travelling waves or fronts [75]. In chemistry, biology and medicine [68] travelling wave fronts have important applications. Such wave fronts were first investigated by Fisher in 1930s by studying the equation (5.1.2).

The Fisher's equation (5.1.2) arises in chemical kinetics [201] and neutron population in a nuclear reaction. Furthermore, the same equation also arises in logistic population growth models [156], flame propagation, neurophysiology, autocatalytic chemical reactions [183] and branching Brownian motion processes. The reaction diffusion equation (5.1.2) also represents a model equation for the evolution of a neutron population in a nuclear reactor [99] and also arises in the study of chemical wave propagation [76]. This equation includes the effects of linear diffusion via u_{xx} and nonlinear local multiplication or reaction via $u(1-u)$.

The mathematical properties and a lot of discussion of the Fisher's equation are present in the literature. Brazhnik and Tyson [156], Kawahara and Tanaka [204] and Larson [56] provide excellent brief statments of Fisher's equation. Wang [220] has introduced the exact and explicit solitary wave solutions for the generalized Fisher's equation. Hammond and Bortz [92] provide analytical solutions to generalized Fisher's equation with time variable coefficients by using tanh method. Ablowitz and Zepetelle [139] have also introduced explicit solutions of Fisher's equation for a special wave speed. Wazwaz and Gorguis [29] considered the analytic study of Fisher's equation by applying Adomain decomposition method. But the numerical solutions of the Fisher's equation were not proposed in the literature till 1974. First numerical solutions of Fisher's equation were introduced by Gazdag and Canosa [91] with a pseudo-spectral approach. After that, a number of investigater has solved Fisher's equation numerically. Parekh and Puri [145] and Twizell *et al.* [64] have introduced the implicit and explicit finite differences algorithms to analize the numerical study of Fisher's equation. Hagstrom and Keller [210] have evolved asymptotic boundary conditions by using centered finite difference algorithm. Tang and Weber [201] presented a Galerkin finite element method and Rizwan [211] compared the nodal integral method and non standard finite-difference schemes. Garey and Shen [75] presented a least-squares finite element method and Khaled [93] proposed sinc collocation method. Mickens [189] presented a best finite-difference scheme for Fisher's equation. Qiu and Sloan [226] used a moving mesh method for numerical solution of Fisher's equation. Daniel and Shizgal [147] presented pseudo spectral method for the numerical solution of Fisher's equation. Evan and Sahimi [57] used an alternating group explicit iterative method to solve the equation and have obtained satisfactory results, of a qualitatively similar nature. El-Azab [105] solved Fisher's equation by fully different numerical scheme. The equation is discretized in time by Rothe's method and in space by Wavelet-Galerkin method. Aghamohamadi *et al.* [106] presented tension spline method for the nonlinear Fisher's equation with initial and boundary conditions and a three time-level implicit method based on the non-polynomial cubic tension spline developed for the solution of the nonlinear reaction-diffusion equation. The numerical scheme mentioned in [64] is quite complicated and cause unexpected high-frequency oscillations, which must be filtered out at each time step. Mittal *et al.* [176] have investigated Fisher's equation by applying wavelet Galerkin method while Jiwari *et al.* [163, 164, 167, 177-179] studied numerical solutions of some nonlinear evolution equations by using differential quadrature method.

The purpose of this paper is to present the analytic and numerical solutions via symmetry reductions of nonlinear diffusion equations of Fisher's-type defined as:

F1: Fisher's equation

$$u_t - u_{xx} - \alpha u(1-u) = 0 \tag{5.1.3}$$

F2: Fisher's type equation

$$u_{t.} - u_{xx} - u^2(1-u) = 0, \tag{5.1.4}$$

F3: Fisher's type equation

$$u_t - u_{xx} - u(1-u)(u-a) = 0, \tag{5.1.5}$$

where a is parameter.

The work put in this chapter has been structured as follows. Lie symmetries are used to reduce the equations into ordinary differential equations. In section (5.2) Lie group classification with respect to time dependent coefficient and optimal system of one-dimensional sub algebras is obtained. Then, sub algebras are used to construct symmetry reduction and analytic solutions. Finally, in section (5.3) numerical solutions of nonlinear diffusion equations are obtained by using polynomial differential quadrature method. Absolute, root mean square (RMS) and L_∞ errors are calculated.

5.2 Lie Classical Analysis for Nonlinear Diffusion Fisher's Type Equations

In this section, we have studied the infinitesimal transformations and reductions by one-dimensional sub algebras of the equations (5.1.3)-(5.1.5) by applying the classical Lie symmetry method one by one. The general case of nonlinear system of PDEs is describe by

$$\Delta_v(x,t,u^{(k)}) = 0, \quad v = 1,2,...m. \tag{5.2.1}$$

where Δ_v represents one of the m coupled equations in n independent variables $x = (x^1,...,x^n)$, q independent variables $t = (t^1,...,t^q)$ and m dependent variables $u = (u^1,...,u^m)$, $u^{(k)}$ denotes all derivatives of the dependent variable u with respect to the independent variables x up to order k. We assume that the Δ_v are smooth functions.

Consider the one-parameter group of infinitesimal transformations in (x,t,u) given by

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),$$

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \tag{5.2.2}$$

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),$$

where ε is the group parameter. The functions ξ, τ, η are the infinitesimal of the transformations for the variables *x*, *t* and *u*, respectively. We shall denote the infinitesimal for u_t, u_{xx} by η^x, η^{xx}. The infinitesimal are as follows

$$\eta^x \equiv (\eta - \xi u_x - \tau u_t)_x + \xi u_{xx} + \tau u_{tx},$$

$$\eta^{xx} \equiv (\eta - \xi u_x - \tau u_t)_{xx} + \xi u_{xxx} + \tau u_{txx}. \tag{5.2.3}$$

Using these various extensions, the infinitesimal criteria for the invariance of (5.1.3)-(5.1.5) under the group (5.3.1) is given by

$$VH\big|_{H=0} \equiv 0. \tag{5.2.4}$$

where $H = (x^*, t^*, v^*, v_{x^*}, v_{t^*}, v_{x^*x^*})$ and $v = \theta(x^*, t^*)$ are also solutions of (5.1.3)-(5.1.5). In equation (5.2.4), the prolongation of the tangent vector field *V* is given by

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \dots \tag{5.2.5}$$

Now, substitute the equations (5.2.3) and (5.2.5) into the equation (5.2.4). Then, we collect together the coefficients of $u, u_x, u_t, u_{xx}, u_{tt}$ and set all of them to zero. At last, we get a system of linear partial differential equations from which we can find ξ, τ and η in practice.

Once we know the infinitesimals, we also know the infinitesimals generator X. The explicit transformation acting on the space of independent and dependent variables can be calculated with the infinitesimal generator, called Lie series

$$(x^*, u^*, t^*) = \exp[eX](x,u,t) \tag{5.2.6}$$

Knowing explicit transformations in (5.2.6), new solutions can be obtained from old ones.

There is also another method to construct a class of function which solve (5.2.1) and are invariant under a subgroup of the fully symmetry group. The group invariants can be calculated by solving the characteristic equations

$$\frac{dx^1}{\xi^1(x,u,t)} = \dots = \frac{dx^n}{\xi^n(x,u,t)} = \frac{dt^1}{\tau^1(x,u,t)} = \dots \frac{dt^q}{\tau^q(x,u)} = \frac{du^1}{\eta^1(x,u)} = \dots = \frac{du^m}{\eta^m(x,u)} \tag{5.2.7}$$

Then the original system (5.2.1) can be rewritten in terms of group invariants and thus the number of independent variables is reduced.

Next we will use the above method to find the Lie symmetry group of the Fisher's type equations (5.1.3)-(5.1.5). The prolongation of the tangent vector field of (5.1.3) is given by

$$V = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{tt} \frac{\partial}{\partial u_{tt}} \tag{5.2.8}$$

After substituting the equations (5.2.3) and (5.2.8) into the equation (5.2.4), we have

$$\eta^t - \eta^{xx} - \eta + 2u\eta = 0 \tag{5.2.9}$$

Now, substituting (5.2.3) in (5.2.9), we get a system of linear partial differential equations. We set all the coefficients of u, u_x, u_t, u_{xx} to zero. Then, we obtain

$$\xi = a, \tau = b, \eta = 0. \tag{5.2.10}$$

where *a* and *b* are arbitrary constants. The associated vector fields for the one-parameter Lie group of infinitesimal transformations are V_1 and V_2 given by

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}. \tag{5.2.11}$$

Firstly, we construct an optimal system to classify the group-invariant solutions of (5.1.3) and the problem of finding an optimal of subgroups is equivalent to that of finding an optimal system of sub algebras. Here, by using the method presented in [155], we will construct an optimal system of one-dimensional sub algebras of (5.3.1). The general one-parameter group of symmetries can be obtained by considering linear combination $V = a_1V_1 + a_2V_2$ of given vector fields. But the explicit formulae for the above transformations are very complicated. Factually, it can be represented uniquely in the form

$$g = \exp(\varepsilon_1 V_1) + \exp(\varepsilon_2 V_2). \tag{5.2.12}$$

Our task is to simplify as many of the coefficients $\varepsilon_i,\ i = 1,2$ as possible though judicious applications of adjoint maps to *V*. By taking $a_2 \neq 0, a_2 = 1$, we have $V_2 + a_1 V_1$.

5.2.1 Reductions by one-dimensional Sub algebras

Case 1: Thus making use of this group transformation, the similarity variable and similarity solution are given by

$$\xi = x - ct,\ u(x,t) = F(\xi) \tag{5.2.13}$$

After substituting these similarity variables in equation (5.1.3), the equation reduced into ordinary differential equation (ODE) given by

$$-cF'(\xi) - F''(\xi) - \alpha F(\xi)(1 - F(\xi)) = 0, \tag{5.2.14}$$

The exact solution corresponding to this ODE is given

$$c = -\frac{5\sqrt{\alpha}}{\sqrt{6}}, F(\xi) = \frac{1}{4} - \frac{1}{2}\tanh\left(-c_1 + \frac{1}{12}\sqrt{6\alpha}\,\xi\right) + \frac{1}{4}\tanh\left(-c_1 + \frac{1}{12}\sqrt{6\alpha}\,\xi\right)^2, \tag{5.2.15}$$

-where c_1 is constant.

Thus, the exact solution of Fisher equation (5.1.3) is given by

$$u(x,t) = \frac{1}{4} - \frac{1}{2}\tanh(-c_1 + \frac{1}{12}\sqrt{6\alpha}(x - ct)) + \frac{1}{4}\tanh(c_1 + \frac{1}{12}\sqrt{6\alpha}(x - ct))^2 \tag{5.2.16}$$

In the similar way, we can apply the Lie classical method to the equations (5.1.4) and (5.1.5) and can study the exact solutions of these equations.

Case 2: Now, by applying the Lie classical method to equation (5.1.4), the following vector fields are obtained:

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}. \tag{5.2.17}$$

The optimal system of these vector fields is:

(i) $V_2 + a_2 V_1$ (5.2.18)

(ii) V_1,

where a_2 is arbitrary constant.

Similarity variable and similarity solution corresponding to the basic vector field $V_2 + a_2 V_1$ is given by

$$\xi = x - lt,\ u(x,t) = F(\xi), \quad \text{where} \quad l = \frac{1}{a_2}, a_2 \neq 0.$$

On using this similarity variable and similarity solution into equation (5.1.4), the equation (5.1.4) reduces to

$$F'(\xi) - F''(\xi) - F^2(\xi)(1 - F(\xi)) = 0. \tag{5.2.19}$$

The solution of the above ODE is given by

$$l = \frac{1}{\sqrt{2}}, F(\xi) = \frac{1}{2} - \frac{1}{2}\tanh\left(c_2 + \frac{1}{4}\sqrt{2}\xi\right). \tag{5.2.20}$$

Thus, the exact solution of Fisher equation (5.1.4) is given by

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(c_2 + \frac{1}{4}\sqrt{2}\left(x - \frac{1}{\sqrt{2}}t\right)\right). \tag{5.2.21}$$

where c_2 is constant.

Corresponding to the basic vector field V_1 in optimal system, we can obtain only constant solution.

Case 3: After utilizing the Lie classical method to equation (5.1.5), the following vector fields are obtained

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t} . \tag{5.2.22}$$

The optimal systems of these vector fields contain the following sub algebras:

(i) $V_2 + a_3 V_1,$ (5.2.23)

(ii) $V_1.$

where a_3 is arbitrary constant.

For the sub algebra, $V_2 + a_3 V_1$, the similarity variable and similarity solution are

$$\xi = x - mt,\ \ u(x,t) = F(\xi), \quad \text{where } m = \frac{1}{a_3}, a_3 \neq 0.$$

Using these similarity variables in nonlinear diffusion equation (5.1.5), the reduced ODE is:

$$-m\, F'(\xi) - F''(\xi) - F(\xi)(1 - F(\xi))(F(\xi) - a) = 0. \tag{5.2.24}$$

The exact solution of the above ODE is given

$$m = \frac{1}{\sqrt{2}}(1+a), F(\zeta) = \frac{1}{2} + \frac{1}{2}a + \left(\frac{1}{2} - \frac{1}{2}a\right)\tanh\left(c_3 + \frac{1}{4}\sqrt{2}(-1+a)\zeta\right). \tag{5.2.25}$$

where c_3 is arbitrary constant.

Now, the exact solution of the nonlinear diffusion equation (5.1.5) is given by

$$u(x,t) = \frac{1}{2} + \frac{1}{2}a + \left(\frac{1}{2} - \frac{1}{2}a\right)\tanh\left(c_3 + \frac{1}{4}\sqrt{2}(-1+a)(x - mt)\right) \tag{5.2.26}$$

For the sub algebra V_1, only constant can be obtained.

5.3 Comparative Study of Exact and Numerical Solutions of Fisher's Type Equations

In this section, a comparative study of exact and numerical solutions is made by finding numerical solution of the equations with the help of polynomial differential quadrature method. For this purpose, the exact solutions obtained in section 5.2 are used for initial and boundary conditions to find the numerical solution by polynomial differential quadrature method (PDQM). Discretized the spatial derivatives of equations (5.1.3)-(5.1.5) by using PDQM at the point x_i , we have

$$\frac{du_i}{dt} = \sum_{j=1}^{N} b_{ij}\, u_j + \alpha u_i (1-u_i), \quad i = 1,2,...,N \tag{5.3.1}$$

$$\frac{du_{i.}}{dt} = \sum_{j=1}^{N} b_{ij}\, u_j + u_i^2 (1-u_i), \; i = 1,2,...,N\,, \tag{5.3.2}$$

$$\frac{du_{i.}}{dt} = \sum_{j=1}^{N} b_{ij}\, u_j + u_i (1-u_i)(u_i - a), \; i = 1,2,...,N\,. \tag{5.3.3}$$

where α and a are parameters, b_{ij} are weighting coefficients of the second order partial derivative and $u_i = u(x_i, t)$. The equations (5.3.1)-(5.3.3) are systems of first order nonlinear differential equations. The initial and boundary conditions are taken from the analytic solutions obtained by Lie symmetry method in section 5.2. Finally, the systems of initial and boundary value problems are solved by Pike and Roe's fourth-stage RK4 [95].

5.3.1 Numerical Experiments and Discussion

In this subsection, three particular numerical examples are considered with fix values of the arbitrary constants occur in solutions of the equations (5.1.3)-(5.1.5) and numerical solutions are obtained by using polynomial differential quadrature method. The whole computation work is done by software MATLAB and DEV C++. Absolute errors L_∞, root mean square error (RMS) and L_2 are computed according the following formulas:

$$L_\infty - Error = \max_{1 \le i \le n} |e_i| \qquad RMS - Error = \left(\sum_{i=1}^{n} \frac{e_i^2}{n} \right)^{1/2},$$

$$L_2 - Error = \left(\sum_{i=1}^{n} e_i^2 \right)^{1/2}$$

where $e_i = (u_i - U_i)$, u_i are approximated solutions and U_i are exact solutions.

Example 1: Consider the equation (5.1.3) over the domain $[0,1]$ with the following initial conditions

$$u(x,0) = \frac{1}{4} - \frac{1}{2}\tanh(-c_1 + \frac{1}{12}\sqrt{6\alpha}(x)) + \frac{1}{4}\tanh(c_1 + \frac{1}{12}\sqrt{6\alpha}(x))^2$$

and boundary conditions

$$u(0,t)=\frac{1}{4}-\frac{1}{2}\tanh(-c_1+\frac{1}{12}\sqrt{6\alpha}(-ct))+\frac{1}{4}\tanh(c_1+\frac{1}{12}\sqrt{6\alpha}(-ct))^2,$$

$$u(x,t)=\frac{1}{4}-\frac{1}{2}\tanh(-c_1+\frac{1}{12}\sqrt{6\alpha}(1-ct))+\frac{1}{4}\tanh(c_1+\frac{1}{12}\sqrt{6\alpha}(1-ct))^2.$$

where $c=\frac{-5\sqrt{\alpha}}{\sqrt{6}}$ and c_1 is arbitrary constant.

The analytical solution of the equation is taken from the equation (5.2.16). Numerical solutions in form of errors of the example are given in Table 5.1. The Table shows that the errors are small and negligible. Figure 5.1 compares the numerical and exact solutions in 3D form and it is concluded that the solutions are very similar.

Example 2: In this example, we have considered the Fisher's equation (5.1.4) over the domain $[0,1]$ with the following initial conditions

$$u(x,t)=\frac{1}{2}-\frac{1}{2}\tanh\left(c_2+\frac{1}{4}\sqrt{2}(x)\right)$$

and boundary conditions

$$u(x,t)=\frac{1}{2}-\frac{1}{2}\tanh\left(c_2+\frac{1}{4}\sqrt{2}\left(-\frac{1}{\sqrt{2}}t\right)\right)$$

$$u(x,t)=\frac{1}{2}-\frac{1}{2}\tanh\left(c_2+\frac{1}{4}\sqrt{2}\left(1-\frac{1}{\sqrt{2}}t\right)\right)$$

The analytical solution of the equation is taken from the equation (5.2.21). Numerical solutions of the example are specified in Table 5.2 and Figure 5.2. The Table represents the errors at different time and constants which are small and negligible. Figure 5.2 depicts a comparison of numerical and exact solutions in 3D form and it is concluded that the solutions are very similar.

Example 3: Consider the Fisher's equation (5.1.5) over the domain $[0,1]$ with the following initial and boundary conditions

$$u(x,0)=\frac{1}{2}+\frac{1}{2}a+\left(\frac{1}{2}-\frac{1}{2}a\right)\tanh\left(c_3+\frac{1}{4}\sqrt{2}(-1+a)(x)\right),$$

$$u(0,t)=\frac{1}{2}+\frac{1}{2}a+\left(\frac{1}{2}-\frac{1}{2}a\right)\tanh\left(c_3+\frac{1}{4}\sqrt{2}(-1+a)(-mt)\right),$$

$$u(1,t)=\frac{1}{2}+\frac{1}{2}a+\left(\frac{1}{2}-\frac{1}{2}a\right)\tanh\left(c_3+\frac{1}{4}\sqrt{2}(-1+a)(1-mt)\right).$$

In this example, analytical solution is taken from the equation (5.2.26). Table 3.3 and Figure 5.3 present the numerical results of the example. Table 5.3 presents the errors at different time and constants which are small and negligible. Figure 5.3 depicts a comparison of numerical and exact solutions in 3D form and it is concluded that the solutions are very similar.

Table 5.1: L_∞, RMS and L_2 errors of Example 1 at different time *t* and constants.

	$c_3=1$			$c_3=3$		
t	L_∞	RMS	L_2	L_∞	RMS	L_2
0.2	5.659E-06	6.500E-06	1.578E-05	1.362E-07	3.685E-08	1.801E-05
0.5	6.306E-06	7.278E-06	1.547E-05	1.581E-07	4.278E-08	1.800E-05
1.0	7.457E-06	8.691E-06	1.489E-05	2.026E-07	5.485E-07	1.798E-05
3.0	1.175E-05	1.471E-05	1.165E-05	5.432E-07	1.472E-07	1.786E-05
5.0	1.175E-05	1.705E-05	7.666E-06	1.422E-06	3.870E-07	1.753E-05

Table 5.2: L_∞, RMS and L_2 errors of Example 2 at different time *t* and constants.

	$\alpha=1, c_1=1$			$\alpha=2,\ c_1=1$		
t	L_∞	RMS	L_2	L_∞	RMS	L_2
0.2	2.105E-05	3.633E-03	4.475E-03	4.217E-05	1.903E-03	2.300E-03
0.5	1.736E-05	3.821E-03	4.444E-03	2.699E-05	1.401E-03	1.548E-03
1.0	1.103E-05	2.464E-03	2.690E-03	9.083E-06	4.226E-03	4.383E-03
3.0	3.788E-05	2.245E-05	2.277E-05	8.711E-05	9.755E-05	9.768E-05
5.0	2.777E-07	1.818E-07	1.823E-07	5.871E-06	3.486E-06	3.486E-06

Table 5.3: L_∞, RMS and L_2 errors of Example 1 at different time *t* and constants.

	$a=0.5, c_3=1$			$a=1.5, c_3=1$		
t	L_∞	RMS	L_2	L_∞	RMS	L_2
0.2	2.412E-06	3.045E-06	7.903E-07	2.105E-05	5.726E-03	7.054E-03
0.5	2.227E-06	2.802E-06	7.225E-07	1.736E-05	5.340E-03	6.210E-03
1.0	1.936E-06	2.425E-06	6.193E-07	1.103E-05	3.055E-03	3.335E-03
3.0	1.039E-06	1.285E-06	3.196E-07	6.243E-05	3.717E-05	3.767E-05
5.0	5.228E-07	6.425E-07	1.577E-07	6.174E-07	3.750E-07	3.760E-07

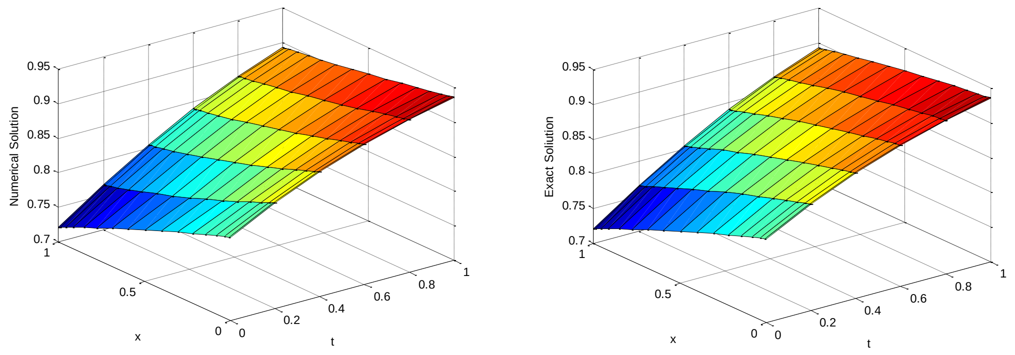


Figure 5.1: Numerical (left) and exact (right) solution of Example 2 for $c_1 = \alpha = 1$ up to time $t \le 1$.

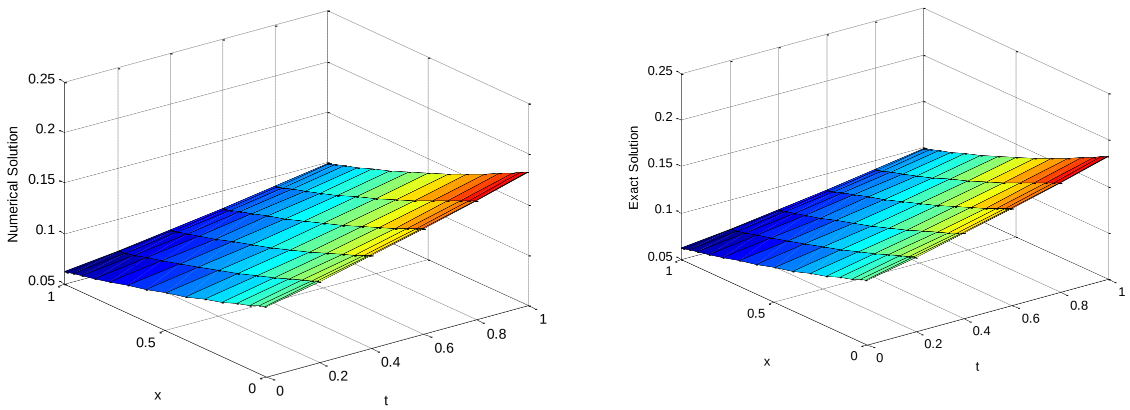


Figure 5.2: Numerical (left) and exact (right) solution of Example 1 for $c_2 = 1$ up to time $t \le 1$.

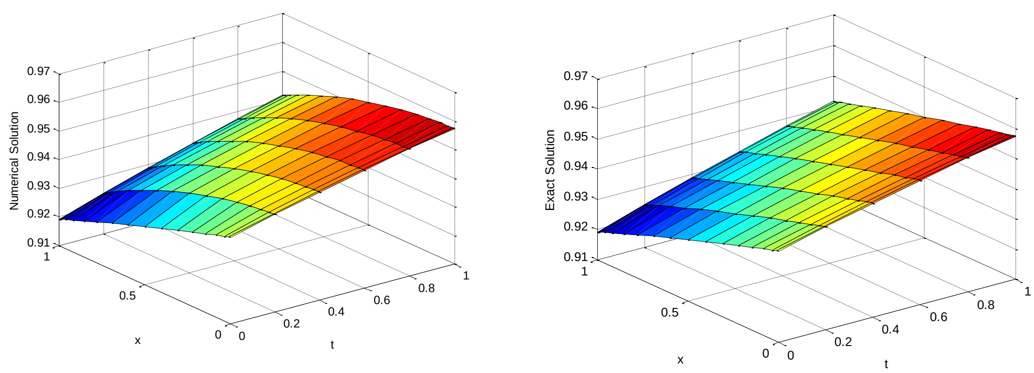


Figure 5.3: Numerical (left) and exact (right) solution of Example 3 for $c_3 = 1, a = 0.5$ up to time $t \le 1$.

5.4 Concluding Remarks

In this chapter, we studied analytic and numerical solutions of the Fisher's type equations with the help of classical Lie symmetry method and polynomial differential quadrature method. Lie symmetry method is utilized to investigate the symmetries and invariant solutions of the equations. By determining the transformation group under which a given system is invariant, we obtained information about the invariants and symmetries of that equation. This information, in turn, was used to determine similarity variables that reduce the number of independent variables. The vector fields of the optimal system lead to reduction of the nonlinear system of partial differential equations to ordinary differential equations. The infinitesimal generators in the optimal system are used for reductions and exact solutions. Finally, polynomial differential qudrature method is used to find the numerical solutions of the Fisher's type equations with the help of initial and boundary conditions taken from the analytic solutions obtained by classical Lie symmetry method. It is concluded that the numerical solutions are in good agreement with the analytical solutions. L_∞, RMS and L_2 errors are calculated for each equation with particular values of arbitrary constants which are small and negligible.

Chapter 6

Two Dimensional Hyperbolic Equations with Variable Coefficients

6.1 Introduction

Consider the two dimensional hyperbolic partial differential equation with variable coefficients

$$\frac{\partial^2 u}{\partial t^2} = A(x,y,t)\frac{\partial^2 u}{\partial x^2} + B(x,y,t)\frac{\partial^2 u}{\partial x \partial y} + C(x,y,t)\frac{\partial^2 u}{\partial y^2} + f(x,y,t,u,u_x,u_y,u_t) \qquad (6.1.1)$$
$$(x,y,t) \in [a,b]\times[c,d]\times[t>0]$$

with initial conditions

$$\begin{aligned} u(x,y,0) &= g_1(x,y), \qquad (x,y)\in[a,b]\times[c,d] \\ u(x,y,0) &= g_2(x,y), \qquad (x,y)\in[a,b]\times[c,d] \end{aligned} \qquad (6.1.2)$$

Subject to Dirichlet boundary conditions

$$u(0,y,t) = h_1(x,t),\ (y,t)\in[c,d]\times[t>0], \quad u(1,y,t) = h_2(y,t),\ (y,t)\in[c,d]\times[t>0]$$

$$u(x,0,t) = h_3(x,t)\ ,(x,t)\in[a,b]\times[t>0], \quad u(x,1,t) = h_4(x,t),\ (x,t)\in[a,b]\times[t>0]$$

$$(6.1.3)$$

and Neumann boundary conditions

$$u_x(0,y,t) = f_1(y,t),\ (y,t)\in[c,d]\times[t>0], \quad u_x(1,y,t) = f_2(y,t),\ (y,t)\in[c,d]\times[t>0]$$

$$u_y(x,0,t) = f_3(x,t),\ (x,t)\in[a,b]\times[t>0], \quad u_y(x,1,t) = f_4(x,t),\ (x,t)\in[a,b]\times[t>0]$$

$$(6.1.4)$$

The equation (6.1.1) is semi linear and assumed to be hyperbolic if $A(x,y,t) > 0$ and $B(x,y,t) > 0$ in the solutions region solution $\Omega \equiv \{(x,y,t):(x,y,t)\in[a,b]\times[c,d]\times[t>0]\}$.

Further, it is assumed that $u(x,y,t) \in C^2$, $A(x,y,t)$, $B(x,y,t) \in C^2, g_1(x,y), g_2(x,y)$ and $f(x,y,t,u,u_x,u_y,u_t)$ are sufficiently differentiable functions are required. Hyperbolic PDEs describe propagation of disturbances in space and time when the total energy if the disturbances remain conserved. It's the condition of energy conservation that makes the hyperbolic equation different from parabolic ones.

The mathematical model of second-order hyperbolic partial differential equation (6.1.1) with appropriate initial and boundary conditions have too many applications in the various areas such as in the theory of superconducting electrodynamics where it describes the propagation of electromagnetic waves [185, 186] in superconducting media, in the transmission of electrical impulses in the axons of nerve and muscle cells, in propagation of pressure waves which occur in pulsatile blood flow in arteries and in many branches physics, fluid dynamics and aerodynamics, theory of elasticity, optics, etc

The wave equation with nonlinear first derivative terms has been studied by [116] and Greenspan [58]. Ciment and Leventhat [107, 108] have developed fourth order difference schemes for linear two-dimensional wave equation by using operator compact implicit method. Jain, Iyengar and Mittal [109, 193] proposed unconditionally stable ADI schemes of $O(h^2+k^2)$ and $O(h^4+k^2)$ for the wave equation with constant coefficients and variable coefficients. After that, Mohanty *et al.* [172] developed a high order conditionally stable numerical method for the solution of two-space dimensional second-order nonlinear hyperbolic equation. Later, Mohanty *et al.* [173] have extended their technique to solve second order quasilinear hyperbolic equations by using seven evaluations of the function and 19-grid points. Mohanty [171] used an operator splitting method for an unconditionally stable difference scheme for a linear hyperbolic equation with variable coefficients in two space dimensions. Mohanty and Jain [174] developed an unconditionally stable ADI scheme for the two space dimensional linear hyperbolic equation. Schwartzkopff *et al.* [205] has used and analyzed a higher order ADER finite volume schemes for linear hyperbolic equations using Cartesian meshes while Dumbser and Kaser [125] used arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems. Dehghan and Mehebbi [126] proposed a high order implicit collocation method for solving the two dimensional hyperbolic equation. Dehghan and Shokri [140] solved the one and two dimensional hyperbolic equations using collocation points and used the thin plate-spline radial basis functions to approximate the solution. In [137-138], Dehghan and Mohebbi have discussed some numerical techniques for two dimensional wave equation and Schrodinger equation. Recently, Mohanty and Singh [170] developed a new compact three-level implicit numerical method of accuracy two in time and four in space for the solution of two space dimensional quasilinear hyperbolic equation and shown to be unconditionally stable. Merad *et al.* [3] used Laplace transform and prove the existence, uniqueness and continuous dependence upon the data of solution to integro-differential hyperbolic equation with purely nonlocal conditions. Jin *et al.* [203] developed a set of stochastic numerical schemes for hyperbolic and transport equations with diffusive scaling and subject to random inputs. Wang *et al.* [100] presented second order accurate numerical scheme for a nonlinear hyperbolic equation with an exponential nonlinear term. The solution to such an equation is proven to have a conservative nonlinear energy. Vabishchevich *et al.* [154] consider explicit schemes of the alternating triangle method for the numerical solution of boundary value problems for hyperbolic equations of the second order. The study is based on the general theory of stability for operator-difference schemes. Mohanty *et al.* [187] described a new compact three level implicit method of order four in time and space based on half-step discretization for one space dimensional quasilinear hyperbolic equation defined in the semi-infinite solution region. Mohanty and Gopal [188] proposed a new three-level implicit nine point compact non-polynomial spline in compression finite difference method of order two in time and four in space directions, based on non-polynomial spline approximation in *x*-direction and central difference approximation in t-direction for the numerical solution of one-space dimensional second order quasi-linear hyperbolic partial differential equations with first order space derivative term. Different numerical schemes have been developed for such types of equations in [4, 5, 36, 38, 123, 129-133, 207, 216].

6.2 Cosine Expansion-Based Differential Quadrature Method

If a function $u(x,y,t)$ is sufficiently smooth over the whole domain, the first and second order partial derivatives of the function $u(x,y,t)$ with respect to x and y at a grid point (x_i, y_j) are approximated as follows (for more detail see chapter 1)

$$\frac{\partial u(x_i, y_j, t)}{\partial x} \cong \sum_{k=0}^{N} w_{ik}^{(1)} u\left(x_k,\ y_j, t\right),\ i = 0,1,...,N; \quad j = 0,1,...,M\ , \tag{6.2.1}$$

$$\frac{\partial u(x_i, y_j)}{\partial y} \cong \sum_{k=0}^{M} \overline{w}_{jk}^{(1)} u(x_i, y_k, t); \; i = 0,1,..., N; \;\; j = 0,1,..., M \,. \tag{6.2.2}$$

where the weighting coefficients $w_{ij}^{(1)}$ and $\overline{w}_{ij}^{(1)}$ are as follows

$$w_{ij}^{(1)} = \frac{-\phi(x_i)\sin(x_i)}{(\cos(x_i) - \cos(x_j)\phi(x_j))}, i, j = 0,1,..., N \quad \mathit{for} \;\; i \neq j \tag{6.2.3}$$

$$w_{ii}^{(1)} = \sum_{j=0, j\neq i}^{N} w_{ij}^{(1)}, \quad i = 0,1,..., N \tag{6.2.4}$$

$$\overline{w}_{ij}^{(1)} = \frac{-\phi(y_i)\sin(y_i)}{(\cos(y_i) - \cos(y_j))\phi(y_j)}, \quad i, j = 0,1,..., M \quad \mathit{for} \quad i \neq j \tag{6.2.5}$$

$$\overline{w}_{ii}^{(1)} = -\sum_{j=0, j\neq i}^{N} \overline{w}_{ij}^{(1)}, \quad i = 0,1,..., M \tag{6.2.6}$$

Similarly the weighting coefficients for the second order partial derivatives are given by the formula [46]

$$w_{ij}^{(2)} = w_{ij}^{(1)}(2w_{ii}^{(1)} + \frac{2\sin(x_i)}{\cos(x_i) - \cos(x_j)} + \cot(x_i)), \; i, j = 0,..., N \quad \mathit{for} \;\; i \neq j \tag{6.2.7}$$

$$w_{ii}^{(2)} = -\sum_{j=0, j\neq i}^{N} w_{ij}^{(2)}, \qquad i = 0,..., N \tag{6.2.8}$$

$$\overline{w}_{ij}^{(2)} = \overline{w}_{ij}^{(1)}(2\overline{w}_{ii}^{(1)} + \frac{2\sin(y_i)}{\cos(y_i) - \cos(y_j)} + \cot(y_i)) \;, i, j = 0,1,..., M, \quad \mathit{for} \;\; i \neq j \tag{6.2.9}$$

$$\overline{w}_{ii}^{(2)} = -\sum_{j=0, j\neq i}^{N} \overline{w}_{ij}^{(2)}, \quad i = 0,1,..., M \tag{6.2.10}$$

6.3 Numerical Scheme for Two Dimensional Hyperbolic Equations with Variable Coefficients

The space derivatives in the two-dimensional hyperbolic equation (6.1.1) are approximated by the cosine expansion based differential qudrature method. The equation (6.1.1) changes into the following form.

$$\frac{d^2u_{i,j}}{dt^2} = A(x_i, y_j, t)\sum_{k=0}^{N} w_{i,k}^{(2)}u_{k,j} + B(x_i, y_j, t)\sum_{l=0}^{N}\sum_{k=0}^{M} w_{i,l}^{(1)}\overline{w}_{j,k}^{(1)}u_{i,k} + c(x_i, y_j, t)\sum_{k=0}^{M} \overline{w}_{j,k}^{(2)}u_{i,k} \quad (6.3.1)$$

with initial conditions

$$u(x_i, y_j, 0) = g_1(x_i, y_j) \ , \quad (6.3.2)$$

$$\frac{du}{dt}(x_i, y_j, 0) = g_2(x_i, y_j). \quad (6.3.3)$$

where $u_{i,j} = u(x_i, y_j, t)$

The system (6.3.1) is a system of second order differential equations.

Now let

$$\frac{du}{dt}(x_i, y_j, t) = v(x_i, y_j, t) \qquad \text{then} \qquad \frac{d^2u}{dt^2}(x_i, y_j, t) = \frac{dv}{dt}(x_i, y_j, t) \quad (6.3.4)$$

Using the assumption of equation (6.3.4) into the equation (6.3.1), the equation (6.3.1) changed as follows

$$\frac{du}{dt}(x_i, y_j, t) = v(x_i, y_j, t) \quad (6.3.5)$$

$$\frac{dv_{i,j}}{dt} = A(x_i, y_j, t)\sum_{k=0}^{N} w_{i,k}^{(2)}u_{k,j} + B(x_i, y_j, t)\sum_{l=0}^{N}\sum_{k=0}^{M} w_{i,l}^{(1)}\overline{w}_{j,k}^{(1)}u_{lk} + c(x_i, y_j, t)\sum_{k=0}^{M} \overline{w}_{j,k}^{(2)}u_{i,k} +$$
$$f(x_i, y_j, t, u_{ij}, \sum_{k=0}^{M} \overline{w}_{j,k}^{(1)}u_{i,k}, v_{i,j}), \quad (x_i, y_j, t) \in [a,b]\times[c,d]\times[t>0] \ \ i = 0,1,\ldots,N, \ j = 0,1,\ldots,M$$

(6.3.6)

with initial conditions

$$u(x_i, y_j, 0) = g_1(x_i, y_j), \quad (6.3.7)$$

$v(x_i, y_j, 0) = g_2(x_i, y_j)$. (6.3.8)

6.3.1 Implementation of Dirichlet Boundary Conditions

The dirichlet boundary conditions are directly applied in the CDQM and we can get numerical solutions at the rectangular or square boundary in the following way

$$\begin{aligned} u_{0,j} &= h_1(y_j, t), \qquad (y_j, t) \in [c,d]\times[t>0] \\ u_{N,j} &= h_2(y_j, t), \qquad (y_j, t) \in [c,d]\times[t>0], \qquad j = 0,1...,M \end{aligned} \tag{6.3.9}$$

$$\begin{aligned} u_{i,0} &= h_3(x_i, t), \qquad (x_i, t) \in [a,b]\times[t>0] \\ u_{i,M} &= h_4(x_i, t) \qquad (x_i, t) \in [a,b]\times[t>0] \qquad i = 0,1,...,N \end{aligned} \tag{6.3.10}$$

6.3.2 Implementation of Neumann Boundary Conditions

The Neumann boundary conditions (6.1.4) at $x = a$ and $x = b$ can be approximated as

$$\sum_{k=1}^{N} w_{1,k}^{(1)} u_{k,j} = f_1(a, y, t),$$

(6.3.11a)

$$\sum_{k=1}^{N} w_{N,k}^{(1)} u_{k,j} = f_2(b, y, t), \qquad j = 0,1...,M \tag{6.3.12b}$$

The equation (6.3.11a) and (6.3.12b) becomes

$$w_{1,1}^{(1)} u_{1,j} + w_{1,N}^{(1)} u_{N,j} = f_1(a, y, t) - \sum_{k=2}^{N-1} w_{1,k}^{(1)} u_{k,j}, \tag{6.3.13}$$

$$w_{N,1}^{(1)} u_{1,j} + w_{N,N}^{(1)} u_{N,j} = f_2(b, y, t) - \sum_{k=2}^{N-1} w_{N,k}^{(1)} u_{k,j}. \qquad , \tag{6.3.14}$$

Multiplying equation (6.3.13) by $w_{N,1}$ and (6.3.14) by $w_{1,1}$ and subtract the both equations, we have

$$(w_{1,N}^{(1)} w_{N,1}^{(1)} - w_{1,1}^{(1)} w_{N,N}^{(1)}) u_{N,j} = w_{1,1}^{(1)} (f_2 - s_2) - w_{N,1}^{(1)} (f_1 - s_1) \tag{6.3.15}$$

Again multiplying equation (6.3.13) by $w_{N,N}$ and (6.3.14) by $w_{1,N}$ and subtracting the both equations, we have

$$(w_{N,1}^{(1)} w_{1,N}^{(1)} - w_{1,1}^{(1)} w_{N,N}^{(1)}) u_{1,j} = w_{1,N}^{(1)} (f_2 - s_2) - w_{N,N}^{(1)} (f_1 - s_1) \qquad (6.3.16)$$

Then after simplification, we have numerical solutions on boundary in the following way

$$u_{1,j} = \frac{w_{1,N}^{(1)} (f_2 - s_2) - w_{N,N}^{(1)} (f_1 - s_1)}{(w_{1,N}^{(1)} w_{N,1}^{(1)} - w_{1,1}^{(1)} w_{N,N}^{(1)})}, \qquad j = 0,1....M \qquad (6.3.17)$$

$$u_{N,j} = \frac{w_{N,1}^{(1)} (f_1 - s_1) - w_{1,1}^{(1)} (f_2 - s_2)}{(w_{1,N}^{(1)} w_{N,1}^{(1)} - w_{1,1}^{(1)} w_{N,N}^{(1)})}, \qquad j = 0,1....M \qquad (6.3.18)$$

where $S_1 = \sum_{k=2}^{N-1} w_{i,k}^{(1)} u_{k,j}, \quad S_2 = \sum_{k=2}^{N-1} w_{N,k}^{(1)} u_{k,j}, \; f_1 = f_1(a,y,t) \; and \; f_2 = f_2(b,y,t)$ (6.3.19)

In a similar way, the Neumann boundary conditions at (6.1.4) $y = c$ and $y = d$ can be approximated as above and we get solutions at boundaries

$$u_{i,1} = \frac{\overline{w}_{1,M}^{(1)} (f_4 - s_4) - \overline{w}_{M,M}^{(1)} (f_3 - s_3)}{(w_{1,M}^{-(1)} w_{M,1}^{-(1)} - w_{1,1}^{-(1)} w_{M,M}^{-(1)})}, \qquad i = 0,1...., N \qquad (6.3.20)$$

$$u_{i,M} = \frac{\overline{w}_{M,1}^{(1)} (f_3 - s_3) - \overline{w}_{1,1}^{(1)} (f_4 - s_4)}{(\overline{w}_{1,M}^{(1)} \overline{w}_{M,1}^{(1)} - \overline{w}_{1,1}^{(1)} \overline{w}_{M,M}^{(1)})}, \qquad i = 0,1...., N \qquad (6.3.21)$$

where $S_3 = \sum_{k=2}^{M-1} \overline{w}_{1,k}^{(1)} u_{i,k}, \; S_4 = \sum_{k=2}^{M-1} \overline{w}_{M,k}^{(1)} u_{i,k}, \; f_3 = f_3(x,c,t)$ and $f_4 = f_4(x,d,t)$

The system of ordinary differential equations (6.3.5)-(6.3.6) with above mentioned initial conditions and boundary conditions discussed in subsection (6.3.1) and (6.3.2) cannot be solved directly by standard RK4 method since the method is used to solve initial value problems (IVP). So, after applying the boundary conditions on the system of equations (6.3.5)-(6.3.6), the system (6.3.5)-(6.3.6), is changed into a system of IVPs. Then, the obtained system is solved by RK4 method.

6.4 Numerical Experiments and Discussion

In this section, the proposed scheme based on cosine expansion differential quadrature method applied on several test problems of hyperbolic equations to check the efficiency and accuracy of the proposed numerical scheme. In each problem relative errors, root mean square (RMS) errors and absolute errors are computed by using the following formulas

Relative Error= $\sqrt{\left[\dfrac{\sum_{i=0}^{N}\sum_{j=0}^{M}(u_{ij}-\overline{u_{ij}})^2}{\sum_{i=0}^{N}\sum_{j=0}^{M}u_{ij}^{\ 2}}\right]}$

RMS Error= $\sqrt{\dfrac{\sum_{i=0}^{N}\sum_{j=0}^{M}(u_{ij}-\overline{u_{ij}})^2}{N\times M}}$

Absolute Error= $\left|u_{ij}-\overline{u_{ij}}\right|$

where u_{ij} and $\overline{u_{ij}}$ denote the exact and numerical solution of the problem.

Problem 1

In this problem, a hyperbolic wave equation with variable coefficients is considered over the square domain $[0,1]\times[0,1],\ t>0$

$$u_{tt}+2e^{x+y}+\sin^2(x+y)u=(1+x^2)u_{xx}+(1+y^2)u_{yy}+f(x,y,t) \tag{6.4.1}$$

With initial and Dirichlet boundary conditions

$$\begin{cases} u(x,y,0)=\sinh(\pi x)\sinh(\pi y) \\ u_t(x,y,0)=-\sinh(\pi x)\sinh(\pi y) \end{cases}$$

$$\begin{cases} u(0,y,t)=u(1,y,t)=0 \\ u(x,0,t)=u(x,1,t)=\ 0 \end{cases}$$

The exact solution of the problem is given by

$u(x,y,t)=e^{-t}\sinh(\pi x)\sinh(\pi y),\ \ t>0$

Table 6.1 presents maximum (max) absolute errors, RMS errors, relative errors and CPU time in seconds of Example 1. The RMS errors are compared with [174] and it can be seen from the Table 6.2 that the errors of the present scheme are better than the scheme developed in [174]. Figure 6.1 shows the numerical and exact solutions at time $t = 1.0$ and it is concluded that the numerical results are in good agreements with the exact solutions. Figure 6.2 shows physical behaviour of the problem in contour form at $t = 1.0, 3.0$.

Problem 2

In this problem, we considered a hyperbolic wave equation with variable coefficients over the region $[0,1]\times[0,1], t > 0$

$$\begin{aligned} u_{tt} &= (1+x^2)u_{xx} + (1+y^2)u_{yy} + \gamma u(u_x + u_y + u_t) + (\gamma e^{-t}(\cosh(x)\cosh(y) - \sin(x+y)) \\ &\quad -1 - x^2 - y^2)e^{-t}\cosh(x)\cosh(y) \end{aligned} \tag{6.4.2}$$

The initial and Dirichlet boundary conditions are given by

$$\begin{cases} u(x,y,0) = \cosh(x)\cosh(y) \\ u_t(x,y,0) = -\cosh(x)\cosh(y) \end{cases}$$

$$\begin{cases} u(0,y,t) = e^{-t}\cosh(y) & 0 \le y \le 1, x = 0 \\ u(1,y,t) = e^{-t}\cosh(1)\cosh(y) & 0 \le y \le 1, x = 1 \\ u(x,0,t) = e^{-t}\cosh(x) & 0 \le x \le 1, y = 0 \\ u(x,1,t) = e^{-t}\cosh(x)\cosh(1) & 0 \le x \le 1, y = 1 \end{cases}$$

The exact solution of the hyperbolic wave equation (6.4.2) is given $u(x,y,t) = e^{-t}\cosh(x)\cosh(y)$

The numerical results of the Example 2 are presented in Table 6.3 in form of maximum absolute errors, RMS errors, relative errors and CPU time in seconds. Table 6.4 compares the max absolute errors with [172] at different values of γ and found that the present scheme gives better results than the scheme developed in [172]. The Figure 6.3 shows the numerical and exact solutions at time $t = 1.0$ and shows that the numerical results are in good agreements with the exact solution. The Figure 6.4 shows the contour plot of the considered equation (6.4.2) at $t = 1.0, 3.0$

Problem 3

Consider a hyperbolic wave equation with variable coefficients over the region $[0,1]\times[0,1],\ t>0$

$$u_{tt}=(1+e^{t}\cos(x)\sin(y))u_{xx}+(1+e^{t}\cos(y)\sin(x))u_{yy}+\in(u_x+u_y+u_t)u+f(x,y,t) \quad (6.4.3)$$

With initial and Dirichlet boundary conditions

$$\begin{cases} u(x,y,0)=\sin(x)\sin(y) \\ u_t(x,y,0)=0 \end{cases}$$

$$\begin{cases} \dfrac{\partial u(0,y,t)}{\partial x}=\sin(y)\cos(t) & 0\le y\le 1 \\ \dfrac{\partial u(1,y,t)}{\partial x}=\cos(1)\sin(y)\cos(t) & 0\le y\le 1 \\ \dfrac{\partial u(x,0,t)}{\partial y}=\sin(x)\cos(t) & 0\le x\le 1 \\ \dfrac{\partial u(x,1,t)}{\partial y}=\cos(1)\sin(x)\cos(t) & 0\le x\le 1 \end{cases}$$

The exact solution is given by
$u(x,y,t)=\sin(x)\sin(y)\cos(t), t\ge 0.$

The max absolute errors, RMS errors, relative errors and CPU time in seconds of Example 3 are shown in Table 6.5. The Table shows that the errors are small and negligible. Table 6.6 shows a comparison of maximum absolute errors of present scheme with the scheme developed in [172]. The numerical and exact solutions at time $t=1.0$ are shown in Figure 6.5. The Figure show that the numerical results are in good agreements with the exact solutions. The Figure 6.6 shows physical behaviour of the problem in contour form at $t=1.0, t=3.0$

Table 6.1 Max Absolute, RMS, Relative error and CPU time of example 1 at different times for $N = M = 21$.

T	CDQM Max Absolute Error	CDQM RMS Error	CDQM Relative Error	CPU Time(s)
0.25	1.313×10^{-4}	3.996×10^{-5}	1.060×10^{-4}	05
0.50	1.279×10^{-4}	3.985×10^{-5}	1.362×10^{-4}	12
1.0	1.225×10^{-4}	3.920×10^{-5}	2.201×10^{-4}	23
1.5	1.191×10^{-4}	3.819×10^{-5}	3.539×10^{-4}	35
2.0	1.117×10^{-4}	3.777×10^{-5}	5.777×10^{-4}	48
3.0	1.114×10^{-4}	3.712×10^{-5}	4.673×10^{-4}	71
5.0	1.012×10^{-4}	3.016×10^{-5}	3.128×10^{-5}	131

Table 6.2 Comparison of RMS errors of example 1 at different number of grid points.

$N = M$	[174]		CDQM	
	$t = 1.0$	$t = 2.0$	$t = 1.0$	$t = 2.0$
16	1.208×10^{-3}	3.576×10^{-4}	1.863×10^{-5}	3.196×10^{-5}
32	2.802×10^{-4}	8.912×10^{-5}	3.769×10^{-5}	3.627×10^{-5}
64	6.791×10^{-5}	2.206×10^{-5}	3.927×10^{-5}	1.773×10^{-5}

Table 6.3 Max Absolute, RMS, Relative errors and CPU time of example 2 at different times for $N = M = 21, \ \gamma = 1$

T	CDQM Max Absolute Error	CDQM RMS Error	CDQM Relative Error	CPU Time(s)
0.25	2.106×10^{-6}	1.983×10^{-6}	6.147×10^{-6}	7
0.50	6.643×10^{-6}	1.917×10^{-6}	2.126×10^{-6}	13
1.0	2.643×10^{-6}	1.056×10^{-6}	1.932×10^{-5}	26
1.5	4.123×10^{-5}	1.452×10^{-5}	1.537×10^{-5}	40
2.0	7.615×10^{-5}	2.465×10^{-5}	1.225×10^{-5}	54
3.0	5.430×10^{-5}	2.236×10^{-5}	3.020×10^{-4}	83
5.0	4.942×10^{-5}	1.430×10^{-5}	1.427×10^{-4}	138

Table 6.4 Comparison of Max Absolute errors of example 2 at different number of grid points at $T = 1.0$

$N = M$	[172]			CDQM		
	$\gamma = 1.0$	$\gamma = 2.0$	$\gamma = 5.0$	$\gamma = 1.0$	$\gamma = 2.0$	$\gamma = 5.0$
04	7.776×10^{-4}	9.220×10^{-4}	2.318×10^{-2}	4.843×10^{-5}	7.429×10^{-4}	5.770×10^{-3}
08	4.418×10^{-5}	6.072×10^{-5}	1.272×10^{-3}	3.055×10^{-5}	4.257×10^{-5}	5.563×10^{-4}
16	2.218×10^{-6}	3.664×10^{-6}	8.814×10^{-5}	1.205×10^{-6}	2.142×10^{-6}	1.854×10^{-5}

Table 6.5 Max Absolute, RMS, Relative errors and CPU time of example 3 at different times for $N = M = 21, \in = 10$

T	CDQM Max Absolute Error	CDQM RMS Error	CDQM Relative Error	CPU Time(s)
0.25	1.877×10^{-6}	3.194×10^{-6}	7.229×10^{-6}	06
0.50	1.128×10^{-5}	1.255×10^{-5}	3.135×10^{-5}	13
1.0	1.258×10^{-5}	3.249×10^{-5}	1.318×10^{-4}	27
1.5	1.385×10^{-5}	2.020×10^{-5}	6.270×10^{-4}	40
2.0	1.305×10^{-5}	1.071×10^{-5}	5.640×10^{-5}	55
3.0	9.695×10^{-6}	3.883×10^{-6}	8.599×10^{-6}	81
5.0	1.831×10^{-6}	8.605×10^{-6}	6.650×10^{-6}	135

Table 6.6 Comparison of Max Absolute error of example 3 at different number of grid points $T = 0.5$

$N = M$	[172]			CDQM		
	$\in = 10.0$	$\in = 0.1$	$\in = 0.01$	$\in = 10.0$	$\in = 0.1$	$\in = 0.01$
04	3.069×10^{-3}	5.282×10^{-4}	5.400×10^{-4}	1.280×10^{-4}	3.905×10^{-4}	3.837×10^{-4}
08	7.136×10^{-4}	1.545×10^{-4}	1.574×10^{-4}	2.924×10^{-5}	2.228×10^{-5}	4.875×10^{-5}
16	1.829×10^{-4}	4.015×10^{-5}	4.091×10^{-5}	3.261×10^{-5}	1.419×10^{-6}	1.739×10^{-5}

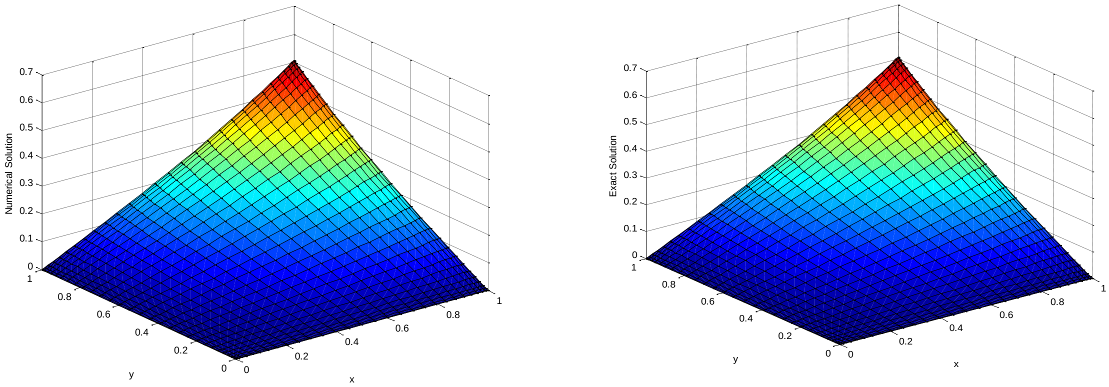


Figure 6.1: Numerical (Left) and Exact (Right) solutions of Example 1 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 1.0$.

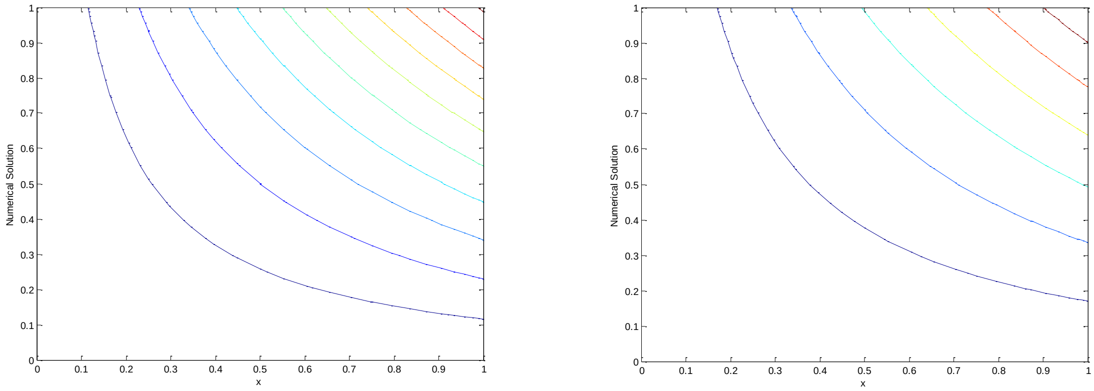


Figure 6.2: Contours plots of Example 1 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 1.0,\ 3.0$ respectively.

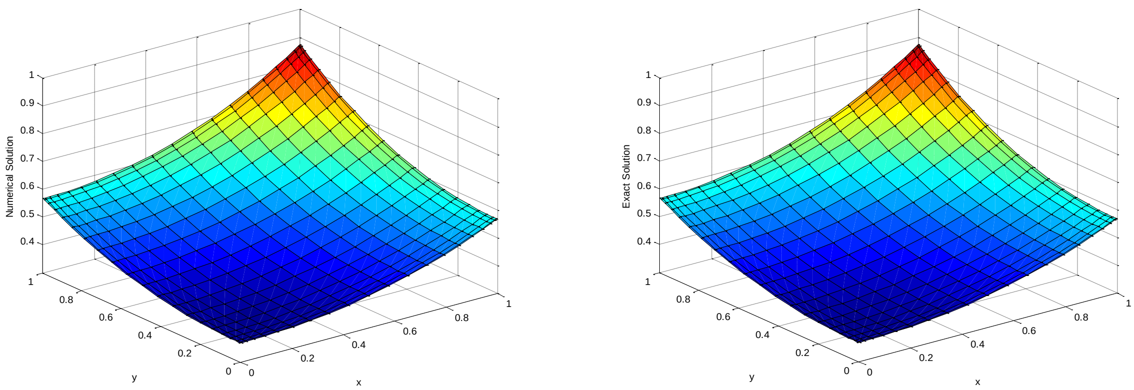


Figure 6.3: Numerical (Left) and Exact (Right) solutions of Example 2 with time step length $\Delta t = 0.001,\ \gamma = 1$ at $t = 1.0$.

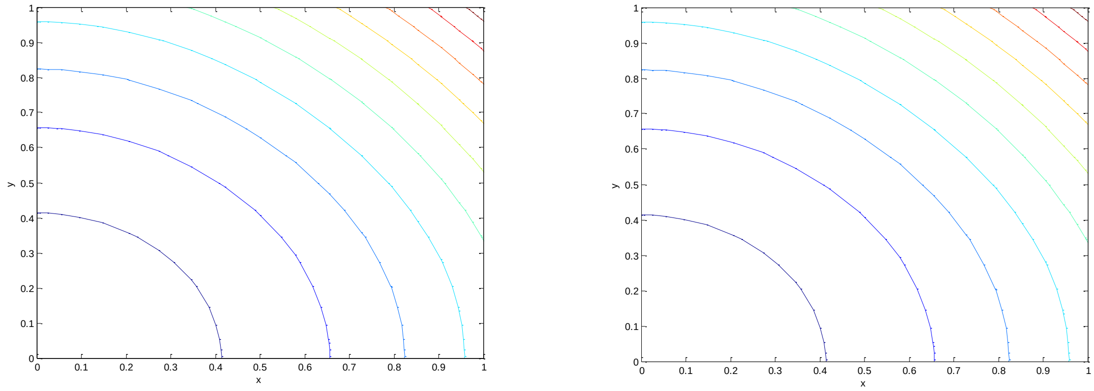


Figure 6.4: Contours plots of Example 2 with time step length $\Delta t = 0.001, \gamma = 1$ at $t = 1.0,\ 3.0$ respectively.

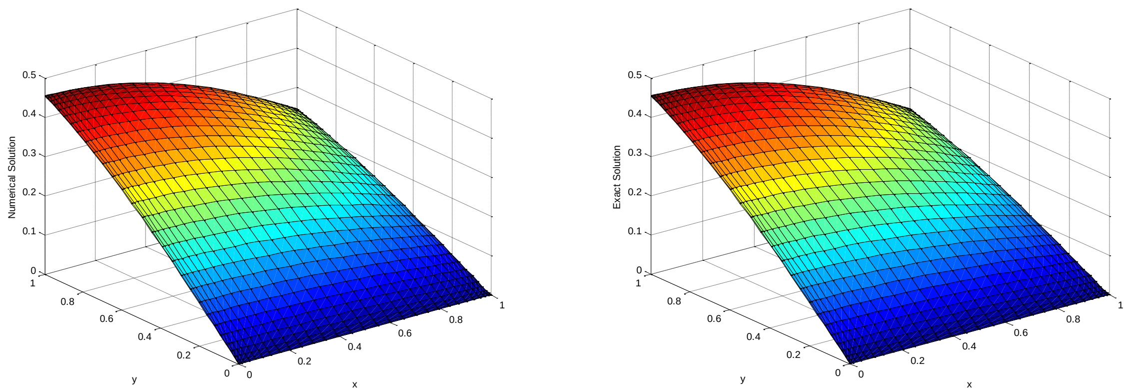
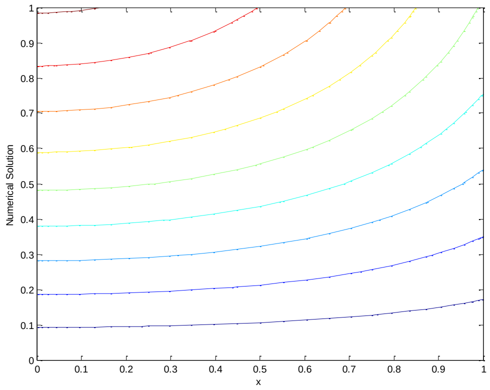


Figure 6.5: Numerical (Left) and Exact (Right) solutions of Example 3 with time step length $\Delta t = 0.001,\ \varepsilon = 10.0$ at $t = 1.0$.



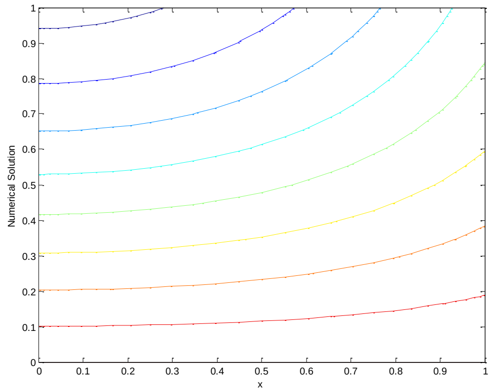


Figure 6.6: Contours plots of Example 3 with time step length $\Delta t = 0.001, \varepsilon = 10.0$ at $t = 1.0,\ 3.0$ respectively.

6.5 Conclusion

In this chapter, cosine expansion based differential quadrature method has been extended for numerical simulation of the second order two dimensional hyperbolic equations. In all problems, max absolute errors, RMS errors, relative errors and CPU times are presented in Tables 6.1-6.6 with time step length $\Delta t = 0.001$ and the Tables show that the errors are negligible. The main advantage of the present scheme is that the scheme gives very accurate and similar results to the exact solutions by choosing less number of grid points and the problem can be solved up to big time. The computed numerical results are compared with the results presented in some earlier works and it is found that the present numerical technique gives better results than the others. The multidimensional problems can be easily handled by the present numerical technique. Due to easiness, the present scheme is very useful for simulation of engineering and physical problems.

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