

# Duality for Some Nonlinear Fractional Programming Problems Under Generalized Convexity

**A Thesis**

*submitted in partial fulfillment of the requirements for the award of the degree of*

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by

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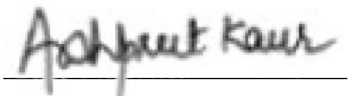
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# Certificate

I hereby certify that the work, which is being presented in the thesis, entitled **Duality for Some Nonlinear Fractional Programming Problems Under Generalized Convexity**, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work carried out under the supervision of **Dr. M. K. Sharma**, Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

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Date: May 11, 2020.

**Dr. M. K. Sharma**

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# Abstract

A common approach in direction of solving a mathematical programming problem is to transfer it from the primal domain to a dual domain. This method can prove to be very beneficial as the dual may have some simpler mathematical or geometrical structure. Besides, this transformation aids in getting the optimal solution with lesser algorithmic and computational efforts. As a dual provides a lower bound to the value of the primal problem, the second and higher-order duals give tighter bounds to the value of the primal problem. In this thesis higher-order dual models, for various forms of multiobjective fractional programming problems and a minimax fractional programming problem, are discussed.

Convexity plays crucial role in the study of optimality conditions and duality theory in mathematical programming problems. With the rising complexity of mathematical problems, convexity is turning into a hard criteria to be achieved. Moreover not all the properties of convex functions are required to be satisfied in achieving optimality and duality results for mathematical programs. So we are interested in finding further general classes of functions which broaden the scope of optimality conditions and duality theorems in mathematical programming problems. In this direction, we introduce higher-order  $(C, \alpha, \gamma, \rho, d)$  type-I functions, higher-order  $(C, \alpha, \gamma, \rho, d)$  convex functions over cones, higher-order  $(\Phi, \rho)$  convex functions over cones and higher-order  $(C, \alpha, \rho, d)$  convex functionals.

A multiobjective fractional programming problem containing support functions is considered. Then higher-order  $(C, \alpha, \gamma, \rho, d)$  type-I functions are introduced which are inclusive of the various previously existing classes of generalized convex functions. For the considered problem a higher-order Schaible type dual is constructed and the validity of this dual model is attested with the help of duality theorems using the above mentioned class of generalized convexity. Further, an extension of the discussed work is made to the case of fractional programs in which ordering is defined with respect to cones.

Another formulation of multiobjective fractional problems, in which the primal problem and the dual problem are symmetric to each other, are studied. Generally nonlinear programs are not symmetric. However assuming certain structural assumptions these programs can be modeled in the form of symmetric programs. On this line of action higher-order non-differentiable fractional symmetric programs over cones are studied. The relationship between the symmetric primal and dual problems is demonstrated with the help of generalized convex functions, namely higher-order  $K - (\Phi, \rho)$  convex functions. A further discussed problem, involve the decision variables of the form of functions or curves, called as variational problem. This problem is a multiobjective fractional variational symmetric problem, in which higher-order functional approximations are used. Two type of

generalized convex functions named as higher-order  $(F, \alpha, \rho, d)/(C, \alpha, \rho, d)$  convex functionals are introduced. Then the duality results are obtained for the symmetric programs with the help of generalized convex functionals.

In minimax programming problems, we consider a non-differentiable fractional minimax program containing support functions. For this problem higher-order parametric dual model is structured and by means of higher-order  $(C, \alpha, \gamma, \rho, d)$  convexity we attain duality relations between the minimax fractional problem and the parametric higher-order dual problems.

Then we study differentiable multiobjective programming problems in which the constraints are affected by data uncertainty. We assume that the uncertainty parameters lie in compact convex sets. These problems are solved using a deterministic technique for dealing with uncertainties, called robust optimization approach. So the problem with data uncertainty is transformed into a robust optimization problem which considers all the elements from uncertainty set including the worst possible case. Then for this robust problem we study sufficient optimality conditions for robust efficient solutions which are placid even under data uncertainty. Finally a robust dual model is constructed for the robust problem and the duality theorems are proved using pseudoconvexity and quasiconvexity assumptions. These results are further extended for multiobjective fractional programs under data uncertainty.

# List of Publications

1. Arshpreet Kaur, Navdeep Kailey and M. K. Sharma, Higher Order Duality of Multiobjective Constrained Ratio Optimization Problems, *FILOMAT*, 33(7), 1985-1998 (2019).
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3. Arshpreet Kaur and M.K. Sharma, Higher Order Symmetric Duality For Multiobjective Fractional Programming Problems Over Cones, *Journal of the Korean Mathematical Society* (under review).
4. Arshpreet Kaur, M.K. Sharma and Izhar Ahmad, Higher-order Symmetric Fractional Variational Programs under Generalized Convexity, *Journal of Mathematical Inequalities* (Communicated).
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# Chapter 1

## Introduction

The essence of optimization is choosing the best course of action amongst the given alternatives in a decision making problem. The first step in optimization is structuring the problem in the form of a mathematical model. A mathematical programming problem comprises a function to be optimized which is called objective function and restrictions on the possible choices called as constraints. The functions involved in constraints are called constraints functions. Then mathematical programming problems with linear objective and constraint functions are called as linear programming problem otherwise nonlinear programming problem. In the present study we consider only nonlinear programming problems.

The Karush-Kuhn-Tucker (KKT) optimality conditions have lead to various optimization techniques for constrained optimization problems. The KKT sufficient optimality conditions rigorously use the convexity assumptions. Moreover a point satisfying KKT conditions is a global optimal if the functions involved are convex. But the problems which arise in life, science and technology may not always involve convex functions. In the absence of convexity assumptions the KKT optimality conditions are only necessary. Further exploration revealed that the KKT sufficient optimality conditions hold for some general non-convex classes of functions too. So it became mandatory to study these generalized convex functions which give new directions in studying reliability of results for a broader spectrum of mathematical programming problems.

Problems containing single objective are called as scalar optimization problems and the problems involving two or more objectives are called as vector optimization problems or multiobjective optimization problems. The multiobjective problems pose several conflicting objectives into one frame. In such problems either the decision maker has a competing preference in the decision space or they may choose a vector mapping from decision space to some Euclidean space and the ordering in that Euclidean space defines the preferences in the decision space. Multiobjective optimization has its roots in a variety of other disciplines of mathematics and has gradually come out as an independent mathematical discipline.

In various optimization problems, ratio of two functions has to be maximized or minimized. These ratio optimization problems are termed as fractional programming problems. The study of fractional programs is rather useful as many of the real-life problems when for-

mulated into a mathematical model give rise to fractional programs. Such problems arise in management decision making, warehouse personnel allocation, information theory and in routing problems which tend to the maximization of profit to time ratio or the minimization of cost to time ratio. These further can be classified as single ratio or multi-ratio optimization problems. Fractional programming problems have been of better advantage in real life, as ratio optimization often describe an efficiency measure for a system. It is observed that a function as a ratio of convex to concave function is not convex in general. So different procedure has to be followed to deal with fractional programs. The study of fractional programs has benefitted a lot from the generalization of convexity. A bibliographical ensemble of various aspects of fractional programming is given in [193, 194].

In mathematical programming, optimality conditions provide a way to devise the efficient methods to find solutions. The dual program gives a bound to the value of primal problem so duality theory is used in the construction of efficient convergence algorithms. Generally speaking, duality is a relationship of a certain kind between two mathematical programming problems. The relationship defined by duality is between a Primal problem which is a constrained minimization (maximization) problem and other is called Dual, a constrained maximization (minimization) problem. The existence of the solution to one of the problems guarantees that the solution to the other problem also exists. In the words of Dennis [55] there is duality between current flow and voltage differences in electric networks. He emphasized that the concept of mathematical duality can be related to the electrical networks in the sense that two extremum problems are equivalent such that they describe the same physical situation; here they are said to bear a somewhat dual relationship to each other.

This Chapter is introductory in nature. In Section 1.1 we give preliminaries, notations as well as definitions of convex functions and their generalizations. Then in Sections 1.2 to 1.8 we give overview of some fundamental mathematical programs and their dual models. The literature survey of the major developments in various theoretical aspects of the programs is also given in the respective sections. Then in Section 1.9 we give a brief summary of the results obtained in the contributory part of this thesis.

In this thesis, the equations, definitions, theorems, examples, remarks and lemmas are numbered section-wise.

## 1.1 Preliminaries

In this section we give some maiden notations and definitions used in the thesis. For a natural number  $n$ ,  $\mathbb{R}^n$  denote an  $n$ - dimensional Euclidean space.  $\mathbb{R}_+^n$  denotes the non-

negative orthant of  $\mathbb{R}^n$ .

**Ordering in  $\mathbb{R}^n$ :** Suppose  $u, v \in \mathbb{R}^n$  are two vectors, then the partial ordering in  $\mathbb{R}^n$  is defined as follows

$$u \geq v \Leftrightarrow u_j \geq v_j, \text{ for every } j = 1, 2, \dots, n$$

$$u \geq v \Leftrightarrow u_j \geq v_j, \text{ for every } j = 1, 2, \dots, n \text{ and } u_i > v_i \text{ for some } i = 1, 2, \dots, n$$

$$u > v \Leftrightarrow u_j > v_j, \text{ for every } j = 1, 2, \dots, n$$

$$u = v \Leftrightarrow u_j = v_j, \text{ for every } j = 1, 2, \dots, n.$$

If  $n = 1$ , then ' $\geq$ ' and ' $>$ ' coincide.  $\mathbb{R}_+^n$  can be characterized as  $(x_1, x_2, \dots, x_n) = x \in \mathbb{R}_+^n$  implies  $x_i \geq 0$  for each  $i = 1, 2, \dots, n$ .

All the vectors are column vectors unless otherwise stated. The superscript  $T$  denotes transpose of a vector or matrix. Consider the following notations for  $a, b, c \in \mathbb{R}^k$

$$1) \text{ the symbol } \frac{ab}{c} \text{ denotes } \left( \frac{a_1 b_1}{c_1}, \frac{a_2 b_2}{c_2}, \dots, \frac{a_k b_k}{c_k} \right)^T$$

$$2) \frac{a+b}{c} \text{ denotes } \left( \frac{a_1+b_1}{c_1}, \frac{a_2+b_2}{c_2}, \dots, \frac{a_k+b_k}{c_k} \right)^T \text{ and } ab = (a_1 b_1, a_2 b_2, \dots, a_k b_k)^T.$$

**Definition 1.1.1** A subset  $\mathcal{A}$  of  $\mathbb{R}^m$  is called a **cone** if  $\alpha x \in \mathcal{A}$  for every  $x \in \mathcal{A}$  and  $\alpha \in \mathbb{R}_+$ . A particular example of cones are rays, which are defined as a non-zero vector  $x$  and all its multiplier of the form  $\lambda x (\lambda \geq 0)$  in  $\mathcal{A}$ .

Cone  $\mathcal{A}$  is called **convex cone** if and only if

$$a_1 + a_2 \in \mathcal{A}, \quad \forall \quad a_1, a_2 \in \mathcal{A}.$$

$\mathcal{A}$  is called **pointed cone** if  $\mathcal{A} \cap -\mathcal{A} = \{0\}$ .

**Ordering in cone  $\mathcal{A}$ :** Let us assume  $\mathcal{A}$  is a closed convex pointed cone in  $\mathbb{R}^m$  and  $\text{int}\mathcal{A} \neq \emptyset$ . Then we define the ordering with respect to cone  $\mathcal{A}$  as: for two points  $u, v \in \mathbb{R}^m$

$$u \geq_{\mathcal{A}} v \Leftrightarrow u - v \in \mathcal{A},$$

$$u \geq_{\mathcal{A}} v \Leftrightarrow u - v \in \mathcal{A} \setminus \{0\},$$

$$u >_{\mathcal{A}} v \Leftrightarrow u - v \in \text{int}\mathcal{A}.$$

Note that if  $\mathcal{A} = \mathbb{R}_+^m$  then this ordering turns into ordering in  $m$ -dimensional Euclidean space  $\mathbb{R}^m$ .

**Definition 1.1.2** [65] For a cone  $\mathcal{A}$ , the positive polar cone (or dual cone) of  $\mathcal{A}$ , denoted by  $\mathcal{A}^*$ , is defined as

$$\mathcal{A}^* = \{y \in \mathbb{R}^m : x^T y \geq 0 \forall x \in \mathcal{A}\}.$$

The dual cone is always convex even if the cone  $\mathcal{A}$  is not convex. Note that  $(\mathbb{R}_+^m)^* = \mathbb{R}_+^m$ .

Also note that if  $\mathcal{K}$  is a closed and convex cone then  $(\mathcal{K}^*)^* = \mathcal{K}$ , by bipolar theorem.

### 1.1.1 Convex and Generalized Convex Functions

We review here some preliminary convex and generalized convex functions. Consider  $X$  is an open convex subset of  $\mathbb{R}^n$  and function  $f$  given by  $f : X \rightarrow \mathbb{R}$ .

**Definition 1.1.3**  *$f$  is convex on  $X$ , if for all  $x, x_o \in X$*

$$f(\lambda x + (1 - \lambda)x_o) \leq \lambda f(x) + (1 - \lambda)f(x_o), \quad \text{for all } \lambda \in [0, 1].$$

Or

$$f(x) \geq f(x_o) + (x - x_o)^T \nabla f(x_o), \quad \text{when } f \text{ is differentiable at } x_o.$$

*$f$  is called Strictly convex if for all  $x, x_o \in X$ , but  $x \neq x_o$*

$$f(\lambda x + (1 - \lambda)x_o) < \lambda f(x) + (1 - \lambda)f(x_o), \quad \text{for all } \lambda \in (0, 1).$$

Convex functions have a property that their level sets are convex but the functions having convex level sets may not always be convex. Bruno de Fenetti was first to point out this fact and in [53] he introduced a class of non-convex functions whose level sets are convex. Fenchel [65] named such functions quasiconvex and studied the properties of their level sets.

**Definition 1.1.4**  *$f$  is quasiconvex at a point  $x_o$  if, for all  $x \in X$ ,*

$$f(x) \leq f(x_o) \Rightarrow f(\lambda x + (1 - \lambda)x_o) \leq f(x_o), \quad \text{for all } \lambda \in [0, 1],$$

or equivalently

$$f(x) \leq f(x_o) \Rightarrow (x - x_o)^T \nabla f(x_o) \leq 0, \quad \text{when } f \text{ is differentiable at } x_o.$$

*The function  $f$  is Strictly Quasiconvex if for all  $x \in X$  but  $(x \neq x_o)$ ,*

$$f(x) < f(x_o) \Rightarrow f(\lambda x + (1 - \lambda)x_o) < f(x_o), \quad \text{for all } \lambda \in (0, 1).$$

The concept of strictly quasiconvex functions is introduced as explicitly quasiconvex functions in literature by Martos [156] while Hanson [91] introduced them as functionally quasi-

convex functions. Karamardian [121] also generalized continuous convex functions to lower semi-continuous strictly quasiconvex functions. Mangasarian [154] introduced pseudoconvex functions in order to broaden the context of convexity in optimality. His motivation laid in the observation that for quasiconvex functions, a vanishing gradient does not lead to a point of optimality.

**Definition 1.1.5** [154]  *$f$  is pseudoconvex at  $x_o \in X$  if  $f$  is differentiable at  $x_o$  and for all  $x \in X$ ,*

$$(x - x_o)^T \nabla f(x_o) \geq 0 \Rightarrow f(x) \geq f(x_o),$$

*or equivalently,*

$$f(x) < f(x_o) \Rightarrow (x - x_o)^T \nabla f(x_o) < 0.$$

*The function  $f$  is **Strictly Pseudoconvex** at  $x_o$  if  $f$  is differentiable at  $x_o$  and for all  $x \in X$  but  $x \neq x_o$ ,*

$$f(x) \leq f(x_o) \Rightarrow (x - x_o)^T \nabla f(x_o) < 0$$

*or equivalently ,*

$$(x - x_o)^T \nabla f(x_o) \geq 0 \Rightarrow f(x) > f(x_o).$$

Ponstein [177] investigated that the various kinds of convex functions along with some of their stronger and weaker extensions imply that a local minimum of a constrained mathematical program is also a global minimum which is not always the case with quasiconvex functions. Hanson [92] speculated that there exist other classes of functions leading to the sufficient optimality conditions and are more general. He said that the definition of convexity can be extended by using a nonlinear functions (with prescribed nonlinear bounds) in a way that: instead of using 'x' some other nonlinear function can be used in Taylor series expansion of the function under consideration, thus giving invex functions (termed by Kaul and Kaur [123]) or  $\eta$ -convex functions (termed by Craven [48]).

**Definition 1.1.6** [92]  *$f$  is invex at  $x_o$  if there exist a function  $\eta : X \times X \rightarrow \mathbb{R}^n$  such that for all  $x \in X$*

$$f(x) - f(x_o) \geq \eta(x, x_o)^T \nabla f(x_o).$$

Hanson [92] also gave pseudo-invex and quasi-invex counterparts for invex functions. Liang et al. [145] used the notion of sublinear functionals to introduce  $(\mathcal{F}, \alpha, \rho, d)$  convex functions given below:

**Definition 1.1.7** [94]  *$\mathcal{F} : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R}$  (where  $X \subseteq \mathbb{R}^n$ ) is said to be sublinear functional if for  $(x, x_o) \in X$ , and  $b, c \in \mathbb{R}^n$ ,*

1.  $\mathcal{F}(x, x_o; b + c) \leq \mathcal{F}(x, x_o; a_1) + \mathcal{F}(x, x_o; c)$  and
2.  $\mathcal{F}(x, x_o; \alpha b) = \alpha \mathcal{F}(x, x_o; b)$  for any  $\alpha \in \mathbb{R}, \alpha \geq 0$ .

**Definition 1.1.8** [145]  $f$  is called  $(\mathcal{F}, \alpha, \rho, d)$ -convex at  $x_o$  if there exist functions  $\alpha : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}$  and  $d : X \times X \rightarrow \mathbb{R}$  and a scalar  $\rho \in \mathbb{R}$  such that for all  $x \in X$

$$f(x) - f(x_o) \geq \mathcal{F}(x, x_o; \alpha(x, x_o) \nabla f(x_o)) + \rho d^2(x, x_o).$$

We have the following instances under different assumptions.

1. When  $\alpha(x, x_o) = 1$ , then  $(\mathcal{F}, \alpha, \rho, d)$ -convexity reduce to  $(\mathcal{F}, \rho)$ -convexity introduced by Preda [178].
2. If  $\alpha(x, x_o) = 1$  and  $\rho = 0$  then  $f$  is called  $\mathcal{F}$ -convex as set by Hanson and Mond [94].
3. If  $\alpha(x, x_o) = 1$ ,  $\mathcal{F}(x, x_o; \alpha(x, x_o) \nabla f(x_o)) = \nabla f(x_o)^T(x - x_o)$ ,  $d = \|x - y\|$  then we have  $\rho$ -convex functions. These functions were defined by Vial [209].

Yuan et al. [221] further generalized these sublinear functionals to convex functionals and introduced  $(\mathcal{C}, \alpha, \rho, d)$ -convex functions. Moreover their definition requires  $f$  to be locally Lipschitzian only. The definition introduced by Yuan et al. [221] is as follows.

**Definition 1.1.9** [221] A functional  $\mathcal{C} : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R}$  is convex with respect to third argument if and only if for any fixed  $x, x_o \in X \times X$ , the inequality

$$\mathcal{C}_{(x, x_o)}(\lambda \alpha_1 + (1 - \lambda) \alpha_2) \leq \lambda \mathcal{C}_{(x, x_o)}(\alpha_1) + (1 - \lambda) \mathcal{C}_{(x, x_o)}(\alpha_2) \quad \forall \lambda \in (0, 1)$$

holds for all  $\alpha_1, \alpha_2 \in \mathbb{R}^n$ . It is assumed that  $\mathcal{C}_{(x, x_o)}(0) = 0$  for all  $(x, x_o) \in X \times X$ . For  $\alpha$  and  $\rho$  as in previous definition and  $d : X \times X \rightarrow \mathbb{R}_+$  with the property that  $d(x, x_o) = 0 \Leftrightarrow x = x_o$ .

**Definition 1.1.10** [221] A locally Lipschitzian function  $f : X \rightarrow \mathbb{R}$  is called a  $(\mathcal{C}, \alpha, \rho, d)$ -convex function at  $x_o \in X$  if for all  $x \in X$

$$\frac{f(x) - f(x_o)}{\alpha(x, x_o)} \geq \mathcal{C}_{(x, x_o)}(\xi) + \rho \frac{d(x, x_o)}{\alpha(x, x_o)} \quad \forall \xi \in \partial f(x_o).$$

**Definition 1.1.11** [61] Let  $F : X \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $f : X \rightarrow \mathbb{R}$  be differentiable functions. Further  $d : X \times X \rightarrow \mathbb{R}_+$  with the property that  $d(x, x_o) = 0 \Leftrightarrow x = x_o$ ,  $\alpha, \gamma : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}$ ,  $d : X \times X \rightarrow \mathbb{R}$  and  $\rho \in \mathbb{R}$  and  $p, s \in \mathbb{R}^n$ .  $f$  is said to be higher order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$ -convex at  $x_o \in X$  if for all  $x \in X$

$$\begin{aligned} \frac{1}{\alpha(x, x_o)}[(f(x) - f(x_o))] &\geq \mathcal{C}_{(x, x_o)}(\nabla f(x_o) + \nabla_p F(u, p)) \\ &+ \frac{1}{\gamma(x, x_o)}[F(u, s) - s^T \nabla_s F(u, s)] + \frac{\rho d(x, x_o)}{\alpha(x, x_o)}. \end{aligned}$$

Some other classes of generalized convex functions can be studied from the work in [23, 127, 128, 129].

## 1.2 General Mathematical Programming Problem

Mathematical programming is a powerful technique used to model and solve optimization problems. A mathematical programming problem includes an objective function to be optimized (maximized/minimized) in the presence of a given number of constraints. Consider  $X \subseteq \mathbb{R}^n$  to be a nonempty and open set. A mathematical programming problem can be written in the following form:

$$\begin{array}{ll} \text{Problem (GMP)} & \text{Minimize/Maximize } f(x) \\ & \text{subject to } g_i(x) \leq 0, \ i = 1, 2, \dots, k, \ x \in X. \end{array}$$

Here the functions  $f, g_i : \mathbb{R}^n \rightarrow \mathbb{R}$  are differentiable functions on  $\mathbb{R}^n$ . The feasible set (or feasible region) is given as  $X_o = \{x \in X : g_i(x) \leq 0, \ i = 1, 2, \dots, k\}$  and each  $x \in X_o$  is called a feasible solution. The function  $f$  is objective function of the problem (GMP) and the functions  $g_i$  for  $i = 1, 2, \dots, k$  are called the constraint functions or simply constraints. If the objective function as well as the constraint functions are linear in variables  $x$  then we call (GMP) a linear program and otherwise (GMP) is called a nonlinear programming problem or a nonlinear program. Any  $u \in X_o$  which gives the minimum (or maximum, depending upon the problem) value of the objective function is said to solve the optimization problem and it is called an optimal solution of the problem.

Fritz John [117] pioneered the study into establishing the necessary and sufficient conditions for a critical point to be a minima of the problem. He extended the lagrangian multiplier method to the case where the subsidiary conditions were inequalities. But in Fritz John conditions the lagrangian multiplier corresponding to objective function is allowed to be zero which possibly leads to the loss of information pertaining to the objective function which is not desirable. Hence the constraint qualifications came into existence in order to give some kind of stability to the system; or simply, keeping the primary role of the objective function intact. This is done by putting restrictions on the feasible region or on the constraints responsible for the active feasible region. We consider the following constraint

qualifications to be used subsequently. Consider set  $I := \{i \in \{1, 2, \dots, k\} : g_i(x_o) = 0\}$ , for some  $x_o \in X_o$ .

1. **Kuhn Tucker Constraint Qualification:**[138] Suppose the vector function  $g = (g_1, g_2, \dots, g_k)$  is a differentiable at  $x_o \in X_o$ , then  $g$  is said to satisfy Kuhn-Tucker constraint qualification at  $x_o$  if for all  $d \in \mathbb{R}^n$  with  $\nabla g_I(x_o)^T d < 0$  implies there exist an  $n$ - dimensional vector function  $e : [0, 1] \rightarrow \mathbb{R}^n$  such that the following hold:
  - (i)  $e(0) = x_o$ ,
  - (ii)  $e(r) \in X_o$  for all  $r \in [0, 1]$  and
  - (iii)  $\frac{de}{dr}(0) = \alpha d$ , for some  $\alpha > 0$ .
2. **Linear Independence Constraint Qualification:** [122] If each  $g_i$  for  $i \notin I$  is continuous at  $x_o \in X$ , and  $\nabla g_i(x_o)$  for all  $i \in I$  are linearly independent then we say the problem satisfies linear independent constraint qualification.
3. **Slater's Constraint Qualification:**[189] If each  $g_i$  for  $i \in I$  is convex at  $X$ , for each  $g_i$  for  $i \notin I$  is continuous at  $x_o$ , and there is an  $x \in X$  such that  $g_i(x) < 0$  for all  $i \in I$ .

Kuhn and Tucker [138] obtained necessary and sufficient conditions for the solution of a maximization problem, via saddle point of a differentiable Lagrangian of non-negative arguments. They formulated constraint qualification to exclude singularities on the boundary of constraint set. Karush [122] obtained the similar optimality results in his master's thesis. These conditions are now called Karush-Kuhn-Tucker (KKT) optimality conditions.

**Theorem 1.2.1 (Karush-Kuhn-Tucker necessary optimality condition) [122, 138]**

*If  $x_o \in X_o$  solves problem (GMP) and Kuhn-Tucker constraint qualification is satisfied at  $x_o$ . Suppose that  $f$  and  $g_i$ , are differentiable at  $x_o$ , then there exist  $y \in \mathbb{R}^k$  such that the following hold*

$$\begin{aligned}
 \nabla f(x_o) + \sum_{j=1}^k y_j \nabla g_j(x_o) &= 0 \\
 y^T \nabla g(x_o) &= 0 \\
 y &\geq 0.
 \end{aligned}
 \tag{1.2.1}$$

However the KKT necessary conditions holds under any of the above constraint qualifications. A point satisfying the conditions in equations set (1.2.1) is called a KKT point.

The sufficient optimality conditions given by Karush [122] drew in the first order Taylor's

series expansion of the objective and constraint functions. However Kuhn-Tucker [138] used convexity assumptions on the Lagrangian. Arrow and Enthoven [14] gave sufficient optimality conditions for differentiable quasiconcave maximization problem. Mangasarian [154] obtained that KKT conditions are sufficient for optimality of a mathematical program with pseudoconvex objective function and quasiconvex constraints. Recently Dutta and Lalitha [62] studied optimality in case of non-convex constraints.

### 1.2.1 Duality in Mathematical Programming Problems

Duality has played a pivotal role in the development of mathematical programming problems. On one hand it gives a stopping level to find the solution of a primal problem and on the other hand it gives a complete new perspective to investigate various aspects of the primal problem. A feasible solution of the dual problem is also a KKT point for the primal problem. The first instance of duality in nonlinear programming is attributed to Dennis [55] who studied duality for a mathematical program with quadratic objective function and linear constraints. In another instance of introductory duality theory, Dorn [57] gave proofs of duality theorems for the same problem based on the duality results for linear programs. Dorn [58] further extended his conception of duality theory for convex programs under linear constraints.

Wolfe [216] had an independent approach for dealing with the duality for convex programs. He postulated that dual in nonlinear programs do not hold the same structural properties as the dual of linear programming problem. He obtained duality theorems for dual of the convex program and used the equivalence theorem constructed by Kuhn and Tucker [138] to prove strong duality theorem. Wolfe's work laid a firm ground for the further developments in duality theory. Afterwards the duality in nonlinear programming has undergone an independent development from the duality in linear programs.

The weak duality theorem gives the lower bound to the value of objective function while strong duality theorem gives that under a constraint qualification the primal and dual objective values are equal. Another theoretical aspect of duality theory is that there exist instances when the dual problem is much easier to deal with, as compared to the primal problem. So in such situations dual may help in reaching at the solution of the primal problem. Such occurrences in nonlinear programming is explored with the help of converse duality theorem and initially these theorems were studied by Hanson [90], Huard [99] and Mangasarian [153].

Huard [99] has given duality results for convex programs similar to the ones of Wolfe along with obtaining converse duality theorem by using KKT optimality conditions for the dual program. Hanson [90] shown that a certain class of non-convex functions also satisfy the

duality relations, thus extending the result of Wolfe, he named the functions anisotropically convex (concave) functions. Geoffrion [69] introduced stability criteria for convex differentiable mathematical program to ensure the existence of necessary optimality conditions which lead to duality theorems. He obtained that the stability of the convex program is equivalent to the constraint qualifications existing in literature.

### 1.2.1.1 Duality in Differentiable Mathematical Programming Problems

The differentiable mathematical programming problems have objective function and all the constraint functions differentiable. In this section, we study some of the basic dual models for minimization problems. For a differentiable mathematical program (GMP) the Lagrangian dual can be written as follows:

$$\begin{array}{ll} \text{Lagrangian Dual (LD)} & \text{Maximize} \quad \theta(y) \\ & \text{subject to} \quad y \geq 0 \end{array} .$$

where  $\theta(y) = \inf_{u \in X} L(u, y)$  is called Lagrangian dual function and  $L(u, y) = f(u) + y^T g(u)$  is called Lagrangian of (GMP). The Lagrangian dual provides a lower bound to the value of primal objective function. However under the convexity and differentiability assumptions along with a constraint qualification duality gap can be minimized to zero. Wolfe [216] formulated the following dual model for differentiable (GMP):

$$\begin{array}{ll} \text{Wolfe Dual (WD)} & \text{Maximize} \quad f(u) + \sum_{j=1}^k y_j g_j(u) \\ & \text{subject to} \quad \nabla f(u) + \sum_{j=1}^k y_j \nabla g_j(u) = 0, \\ & \quad u \in X, \quad y \geq 0. \end{array}$$

Wolfe [216] proved duality theorems for a convex minimization problem. Karamardian [121] generalized continuous convex functions to lower semi-continuous strictly quasiconvex functions and gave duality theorems. Hanson [92] obtained duality theorems for the above dual using invex functions.

Mangasarian [154] indicated that the duality results obtained for Wolfe Dual do not hold in case of relaxed convexity assumptions, for example in case of pseudoconvex functions. This consideration lead to the formation of Mond-Weir dual [173] which is as follows

$$\begin{array}{ll}
\text{Mond-Weir Dual (M-WD)} & \text{Maximize } f(u) \\
& \text{subject to } \nabla f(u) + \sum_{j=1}^k y_j \nabla g_j(u) = 0, \\
& y^T g(u) \geq 0, \ u \in X, \ y \geq 0.
\end{array}$$

### 1.2.1.2 Duality in Non-differentiable Mathematical Programming Problems

Non-differentiable problems arise in various situations of mathematical modelling, data analysis as well as in statistics dealing with random variables. The functions involving  $L_1$  or  $L_\infty$  norms, Euclidean distances, Chebychev distances or a maximum of convex functions are typical non-differentiable problems. The first problem of duality theory in nondifferentiable programs was introduced for a statistical problem involving random variables, by Sinha [188]. He modeled this stochastic problem into a nonlinear non-differentiable otherwise a convex program. In the absence of solutions techniques for such programs a differentiable dual is formulated in [188], then using the duality theorems the solution of the proposed problem is introduced. This work marked the beginning of duality in non-differentiable mathematical programming problems. Mond [164] gave converse duality theorem for this dual.

Mond [165] studied a problem with non-differentiable objective function with a square root term involving a symmetric matrix. Then Mond and Schechter [171] introduced another kind of nondifferentiable problems containing  $L_p$  norm, which generalized Mond's results [165]. Watson [210] solved problems with abstract norm, and using the subdifferential he obtained duality theorems. Schechter [183] came out with the idea that a positively homogeneous convex function which is lower semi-continuous too is nothing but a support function of a compact and convex set. Support functions are defined as follows.

**Definition 1.2.1 (Support Function)** *If  $C$  is a closed and convex set  $C \subseteq \mathbb{R}^n$ , then support function of  $C$  at a point  $x \in \mathbb{R}^n$  is defined as*

$$S(x|C) := \sup\{\langle x, y \rangle : y \in C\}.$$

**Lemma 1.2.2 ([183])** *If  $C$  is a compact and convex set then subdifferential of support function  $S(x|C)$  is given by the following set*

$$\partial S(x|C) = \{z \in C : \langle x, z \rangle = S(x|C)\}.$$

Schechter [183] considered the following non-differentiable program:

$$\begin{aligned} \text{Primal (NDP)} \quad & \text{Minimize} \quad f(x) + S(x|C) \\ & \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, 2, \dots, k, \quad x \in X. \end{aligned}$$

In this problem  $f, g_i$  are as in (GMP). The feasible set or region given by  $X_{ND} := \{x \in X : g_i(x) \leq 0, \text{ for all } i = 1, 2, \dots, k\}$ . Here the set  $C$  is a compact and convex subset of  $\mathbb{R}^n$  with  $S(x|C)$  being the support function of  $C$ .

The dual of (NDP) given by Schechter [183] is:

$$\begin{aligned} \text{Dual (NDD)} \quad & \text{Maximize} \quad f(u) + \langle w, u \rangle + \sum_{i=1}^k y_i g_i(u) \\ & \text{subject to} \quad \nabla f(u) + w + \sum_{i=1}^k y_i \nabla g_i(u) = 0, \\ & \quad \langle w, u \rangle = S(u|C), \quad u \in X, \quad y \geq 0. \end{aligned}$$

He obtained the weak duality and the strong duality theorems for above dual model for convex functions  $f$  and  $g_i, i = 1, 2, \dots, k$ . He also studied the duality results when only the constraints involve support functions. It is worth noting here that choosing  $C$  appropriately reduce the primal (NDP) and dual (NDD) problems to the preceding classes of nondifferentiable primal and dual problems. Later in [184], Schechter obtained the converse duality theorem and studied duality results for those problems which have each of the involved functions is non-differentiable.

In the upcoming sections we give an overview of mathematical programs having some special structures and their respective dual models existing in literature. These dual models provide a fundamental structure to the various duals considered in this thesis such that the chapters could be logically constructed over these basic models. The structural assumptions laid down in various sections have contributed significantly in constructing different aspects of this thesis.

### 1.3 Multiobjective Programming Problems

The earliest record of multiobjective (or multicriteria or vector) optimization dates back to 1881, in Francis Ysidro Edgeworth's (1845-1926) work on economic welfare theory [63]. In this book, Edgeworth studied the competing interest of the consumers from a utility

point of view and laid a groundwork for Pareto's generalizations. Pareto [176] developed the theory in economics particularly welfare economics with his concept of Pareto optimality and gave the concept of efficient solutions of multiobjective optimization. Pareto defined a term 'ophelimity' for any consumer objective (or requirement). He mentioned that at the position of maximum ophelimity it is impossible to increase the ophelimity of one consumer while keeping other consumer's ophelimity as well off as before. So it can be said the preliminary work in multiobjective optimization came out of economics and utility theory. Earlier records of multiobjective optimization are also reflected in the work of Cantor [30] and Hausdorff [96] as ordered sets, in the work of Borel [27] as the theory of psychological games and the notion of strategy. Along with these Koopmans [137] worked in developing efficient point set for linear programs in production theory.

A typical multiobjective programming problem can be expressed as

$$\begin{array}{ll} \text{Primal (MOP)} & \text{Minimize } f(x) = (f_1(x), f_2(x), \dots, f_m(x)) \\ & \text{subject to } g(x) \leq 0, x \in X. \end{array}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^k$  both are vector valued functions. Denote by  $X_{M0}$  the set of feasible solutions and it is given by  $X_{M0} := \{x \in X : g(x) \leq 0\}$ . So the problem (MOP) is about finding the efficient solutions which were defined as follows.

**Definition 1.3.1**  $x_o \in X_{M0}$  is an efficient solution of problem (MOP) if there does not exist any  $x \in X_{M0}$  such that

$$\begin{aligned} f_j(x) &\leq f_j(x_o), \quad \text{for every } j = 1, 2, \dots, m \\ f_i(x) &< f_i(x_o), \quad \text{for some } i = 1, 2, \dots, m. \end{aligned}$$

**Definition 1.3.2** A feasible solution  $x_o \in X_{M0}$  is said to be a weakly efficient solution of problem (MOP) if, there does not exist any  $x \in X_{M0}$  such that

$$f_j(x) < f_j(x_o), \quad \text{for every } j = 1, 2, \dots, m.$$

Kuhn and Tucker [138] initially worked to determine the solutions of multiobjective programs. They transformed the vector maximum problem into a scalar maximum problem by suitably choosing the positive constants. These constants can be interpreted as the prices (or values) to assigned to various objectives. They obtained Kuhn-Tucker constraint qualifications on the scalarized Lagrangian and studied necessary and sufficient optimality conditions for an efficient solution. Geoffrion [66, 67] developed computational algorithms

for solving a bi-objective programming problem.

Geoffrion [68] gave a restricted definition of efficient solutions to remove the anomalous behavior of some efficient solutions which give an infinite loss of one objective for a marginal gain in another objective. So he introduced properly efficient solutions in order to make certain that the marginal gain to loss ratio is bounded.

**Definition 1.3.3** [68] *A feasible solution  $x_o \in X_{M0}$  is called a properly efficient solution of problem (MOP) if it is efficient and there exist scalar  $M > 0$ , such that for each  $j$ ,  $f_j(x) < f_j(x_o)$  and  $x \in X_{M0}$  imply*

$$\frac{f_j(x_o) - f_j(x)}{f_i(x) - f_i(x_o)} \leq M$$

for some  $i = 1, 2, \dots, m$  such that  $f_i(x) > f_i(x_o)$ .

The properly efficient solutions can also be characterized as the solutions of a single objective programming problem as given by following lemma.

**Lemma 1.3.1** [68] *If for some fixed  $\mu(> 0)$  in  $\mathbb{R}^m$ ,  $\bar{x}$  is an optimal solution of the following scalar minimization problem*

$$\text{Problem } (P_\lambda) \quad \text{Min } \mu^T f(x) \quad \text{subject to } x \in X_{MO}.$$

*Then  $\bar{x}$  is proper efficient solution of the multiobjective program (MOP).*

Chandra et al. [34] introduced two kinds of regularity conditions for achieving optimality in multiobjective programs.

### 1.3.1 Duality in Multiobjective Programming Problems

The preliminary work in duality for vector programming problems can be found in the work of several authors including [26, 43, 47, 74, 207]. In this section, we discuss the primary results for duality in multiobjective programs owing to Weir and Mond [212]. They extended the single objective results in duality theory given by Wolfe [216], Bector [19] and Mond and Weir [173] to vector optimization problems. For the multiobjective problem (MOP) the Mond-Weir type dual given by Weir and Mond [212] is as below:

$$\begin{aligned} \text{Dual (DMOP)} \quad & \text{Maximize } f(u) = (f_1(u), f_2(u), \dots, f_m(u)) \\ & \text{subject to } \nabla \mu^T f(u) + \nabla y^T g(u) = 0 \\ & y^T g(u) \geq 0, \quad y \geq 0 \end{aligned}$$

$$\mu > 0, \quad \mu^t e = 1, \quad (e = (1, 1, \dots, 1), \in R^m).$$

For this dual, Weir and Mond [212] obtained duality results for pseudoconvex objective and quasiconvex constraint functions. Mond et al. [170] obtained Mond-Weir and Wolfe type duals for multiobjective problems involving a square root of a predefined positive semi-definite matrix. Kannappan [119] suggested that the Fritz John and Kuhn Tucker optimality conditions for a multiobjective program can be obtained in the absence of differentiability assumptions by using the continuity and convexity only.

## 1.4 Fractional Programming Problems

Fractional programming problems are mathematical programs, with objective function taking the form of a ratio of two functions. The fractional programs in which all the functions involved are linear, are called as linear fractional programs and otherwise nonlinear fractional programs. Charnes and Cooper [37] used a variable transformation to convert a linear fractional programming problem into a linear programming problem while preserving the geometry of the problem. They proved that the parameterized program thus obtained, correctly characterizes the solutions of the corresponding linear fractional program. Isbell and Marlow [104] formulated an iterative scheme involving de-fractionalization. At each iteration a linear form is optimized which gives linear form to the successive stage. So the method generate a sequence of linear programs and the solutions of the programs lead to the solution of fractional program.

Jagannathan [105] obtained a parametrization of a nonlinear fractional program with linear denominator. Then Dinkelbach [56] proposed an parametric algorithm for solving nonlinear fractional programs, now known as Dinkelbach Algorithm. A general fractional programming problem may be written in the following form:

$$\begin{array}{ll} \text{Primal (FP)} & \text{Minimize } \frac{N(x)}{D(x)} \\ & \text{subject to } g(x) \leq 0, \quad x \in X. \end{array}$$

The feasible region is given by  $X_{fp} = \{x \in X : g_i(x) \leq 0, \quad i = 1, 2, \dots, k\}$  and  $N, D, g_i : X \rightarrow \mathbb{R}$  are continuous functions with  $D(\cdot) > 0$  in  $X$ . He obtained the following theorem to study optimality of fractional programming problems (FP) with compact and connected feasible region  $X_{fp}$ :

**Theorem 1.4.1**  $\lambda^0 = \text{Max} \left\{ \frac{N(x)}{D(x)} : x \in X_{fp} \right\}$  if and only if,  $\text{Max} \{N(x) - \lambda^0 D(x) : x \in X_{fp}\} = 0$ .

Dinkelbach has also pointed out that the ratio of concave to convex functions is a quasi-concave function. Bector [18] studied necessary and sufficient optimality conditions using quasiconvexity of a fractional programming problem.

### 1.4.1 Duality in Fractional Programming Problems

Bector [18] obtained duality results for the fractional programs which involve ratio of convex and strictly positive linear functions. However he has also stated that the results will hold if denominator is a concave function provided numerator is nonnegative. Whereas Jagannathan [106] gave duality results for the following dual (FD) of the fractional program (FP) when  $g_i(x)$  and  $N(x)$  are convex and differentiable on  $\mathbb{R}^n$  and  $D(x)$  is concave and differentiable on  $\mathbb{R}^n$ .

$$\begin{aligned}
\text{Dual (FD)} \quad & \text{Maximize } \lambda \\
& \text{subject to } \nabla N(u) - \lambda \nabla D(u) + \sum_{i=1}^k y_i \nabla g_i(u) = 0, \\
& N(u) - \lambda D(u) + \sum_{i=1}^k y_i g_i(u) \geq 0, \\
& y \in \mathbb{R}_+^k, \quad u \in X, \\
& \lambda \in \mathbb{R}, \quad \lambda \geq 0.
\end{aligned}$$

The above discussed results require the feasible region  $X_{fp}$  to be non-empty and convex. Schaible [182] removed this restriction on feasible region and gave parametric dual model for the fractional programs thus giving more general dual for fractional programs. Then Schaible [181] studied various duality results using his proposed parametric approach for differentiable as well as non-differentiable nonlinear fractional programs.

Further, Mond [167] studied a fractional program in which numerator and denominator both contain the square root terms and then formulated a parametric dual for this non-differentiable problem and studied duality results. Mond and Craven [168] generalized fractional program in which numerator and denominator both contain support functions of closed convex sets. Agarwal and Saxena [1] also studied a fractional program with square root term in numerator using convex functions. Singh [187] studied a non-differentiable primal problem with sum of square root terms in numerator and obtained duality relations using  $\mathcal{F}$ -convex functions.

### 1.4.2 Duality in Multiobjective Fractional Programming Problems

The multiobjective programming problems wherein every component of the objective function takes the form of a ratio of two functions are called as multiobjective fractional programming problems. Consider a multiobjective fractional program of the form:

$$\begin{array}{ll} \text{Primal (MOFP)} & \text{Minimize } \left( \frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_m(x)}{g_m(x)} \right) \\ & \text{subject to } h(x) \leq 0, x \in X. \end{array}$$

$f_j, g_j$  are continuously differentiable scalar functions defined on  $X$  for  $j = 1, 2, \dots, m$  and  $h : X \rightarrow \mathbb{R}^k$  is a vector valued differentiable function. Assume that  $f_j \geq 0$  and  $g_j > 0$  in  $X$ . Liang et al. [146] obtained duality results for (MOFP) using  $(\mathcal{F}, \alpha, \rho, d)$  convex functions. The dual of above (MOFP) given by Liang et al. [146] is as follows:

$$\begin{array}{ll} \text{Dual (DMOFP)} & \text{Maximize } v = (v_1, v_2, \dots, v_m) \\ & \text{subject to} \\ & \sum_{i=1}^m \tau_i \nabla(f_i(u) - v_i g_i(u)) + \sum_{j=1}^k y_j \nabla h_j(u) = 0, \\ & f_i(u) - v_i g_i(u) \geq 0, \quad i = 1, 2, \dots, m, \\ & y^T h(u) \geq 0, \quad y \in \mathbb{R}_+^k \\ & v \in \mathbb{R}_+^m, \tau \in \text{int } \mathbb{R}_+^m, \sum_{i=1}^m \tau_i = 1, u \in X. \end{array}$$

Egudo [64] studied multiobjective fractional programs with square root term to give optimality and duality results for several classes of generalized convexity assumptions (including  $\rho$  convexity,  $\rho$  pseudo-convexity,  $\rho$ -quasiconvexity etc.). Chinchuluun et al. [41] gave three types of duality models for multiobjective programs involving nondifferentiable functions via subdifferentials. Then Long et al. [150] used support functions of compact convex sets to add a nonsmooth factor to the multiobjective fractional programs and gave Mond-Weir type dual using  $(\mathcal{F}, \alpha, \rho, d)$  convex functions.

## 1.5 Minimax Programming Problems

Minimax methods are used in statistical decision theory for finding a solution among all the observations which minimizes the maximum risk or performs best in worst possible scenario. It is also used in game theory to minimize the maximum loss of a player or as a maximin strategy for maximizing the minimum payoff. The various other application areas of minimax decision making include non-probabilistic decision making, artificial intelligence in computer based board games and social justice philosophy.

Danskin [51] and Bram [28] are known to have contributed to the preliminary work in the theoretical development of max-min problems, obtaining the necessary optimality conditions in terms of directional derivatives. Danskin [51] studied a max-min problem in which a minimizing player plays after maximizing player with the full knowledge of the moves of the maximizing player, thus finding a solution to the long unsolved military allocation problem. Danskin [51] studied the problems with strict equality constraints and Bram [28] extended these results to minmax programs with several inequality constraints. A notable advancement in the study of minimax programming came with the work of Schmitendorf [185]. The problem studied in [185] is to choose that  $x \in \mathbb{R}^n$  which minimizes

$$\text{Primal (MXP)} \quad f(x) = \sup_{y \in Y} \phi(x, y), \quad \text{subject to } h(x) \leq 0,$$

here  $\phi : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}$  is a continuously differentiable function with respect to  $x$  but  $f$  is not differentiable in general.  $Y \subset \mathbb{R}^l$  is a compact set and the constraint function  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuously differentiable on  $\mathbb{R}^n$  and the feasible region is given by  $X_{mx} = \{x \in \mathbb{R}^n : h(x) \leq 0\}$ . Schmitendorf [185] studied a form of generalized Motkin's theorem for minimax problem (MXP) thus giving a refinement of the necessary optimality conditions for minimax programs.

### 1.5.1 Duality in Minimax Programming Problems

Using the necessary and sufficient optimality conditions for minimax programs derived by Schmitendorf [185], Tanimoto [206] introduced two types of dual models and discussed weak, strong and converse duality theorems involving convex functions. Consider the following sets, for each  $x \in \mathbb{R}^n$

$$I(x) = \{i \in M = \{1, 2, \dots, m\}; h_i(x) = 0\} \quad ,$$

$$Y(x) = \left\{ y \in Y; f(x, y) = \sup_{z \in Y} f(x, z) \right\}.$$

Denote by  $\mathcal{Z}$  the set of triplets  $(s, t, \bar{y})$ , where  $s$  ranges over the positive integers such that  $1 \leq s \leq n+1$ ,  $t = (t_1, \dots, t_s)$  an  $s$ -dimensional vector with  $t_i > 0$ ,  $\sum_{i=1}^s t_i = 1$  and  $y = (y_1, \dots, y_s)$  is an  $ls$ -dimensional vector such that  $\{y_1, y_2, \dots, y_s\} \subset Y(x)$ . For every  $(s, t, \bar{y}) \in \mathcal{Z}$ , define

$$\mathcal{K} = \left\{ (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \text{ and } \mu \text{ satisfying } \{y_1, \dots, y_s\} \subset Y(x) \text{ and } \sum_{i=1}^s t_i \nabla_x f(x, y_i) + \sum_{j=1}^m \mu_j \nabla h_j(x) = 0, \mu = (\mu_1, \dots, \mu_m) \geq 0 \right\}.$$

Tanimoto defined the dual for the problem (MXP) as follows:

Dual (MXD)

$$\underset{(s, t, \bar{y}) \in \mathcal{Z}}{\text{Maximize}} \quad \underset{(x, \mu) \in \mathcal{K}(s, t, \bar{y})}{\text{Sup}} \quad f(x) + \sum_{j=1}^m \mu_j g_j(x).$$

If, for an element  $(s, t, \bar{y}) \in \mathcal{Z}$  the set  $\mathcal{K}$  is empty, then the supremum will be taken as  $-\infty$ . Tanimoto proved the duality theorems under the convexity assumptions on  $f$  and  $h$ .

## 1.6 Mathematical Programming over Cones

The solutions of a mathematical programming problem must be in accordance with decision maker's inclinations. So their preference relation influence the domination structure among the alternatives. This motivated the study of finding solutions of vector problems in which ordering is specified with respect to some arbitrary cones. The preliminary work in optimality and duality for these problems was given by Craven [45]. Craven [45] defined weakly efficient solutions and obtained Fritz John type necessary optimality conditions for cone constrained problem. Craven [46] gave KKT type necessary and sufficient optimality conditions for a minimizing programming problem over cones. Tanino and Swaragi [207] gave optimality results for the set of non-dominated points as the extreme points of a set. Corley [43] gave three type of ordered relations over cones and different efficient solutions depending upon the ordered relationship. For a weakly efficient solution of (CP) as given below, Weir et al. [215] gave necessary and sufficient optimality conditions in terms of its vector valued Lagrangian:

Primal (CP) :  $\mathcal{A}$  – Minimize  $f(x)$  subject to  $-g(x) \in \mathcal{Q}$

Here  $\mathcal{A} \subseteq \mathbb{R}^m, \mathcal{Q} \subseteq \mathbb{R}^k$  are closed convex cones with non-empty interiors and  $f, g$  are differentiable functions given by  $f : X \rightarrow \mathbb{R}^m$  and  $g : X \rightarrow \mathbb{R}^k$ . Further, denote by  $X_{cp}$  the feasible region of (CP) where  $X_{cp} := \{x \in X : -g(x) \in \mathcal{Q}\}$ .

For (CP) the weakly efficient, efficient solutions and convex functions are defined as follows.

**Definition 1.6.1** [46] *A point  $u \in X_{cp}$  is a weakly efficient solution of (CP) if there does not exist any  $x \in X_{cp}$  satisfying*

$$f(u) - f(x) \in \text{int } \mathcal{A}.$$

**Definition 1.6.2** [46] *A point  $u \in X_{cp}$  is an efficient solution of (CP) if there does not exist any  $x \in X_{cp}$  satisfying*

$$f(u) - f(x) \in \mathcal{A} \setminus \{0\}.$$

**Definition 1.6.3** [45] *Given that  $\mathcal{A}$  is a convex cone. A function  $f : X \rightarrow \mathbb{R}^m$  is called an  $\mathcal{A}$ – convex function if whenever  $a, b \in X$  and  $0 < \lambda < 1$ ,*

$$\lambda f(a) + (1 - \lambda)f(b) - f(\lambda a + (1 - \lambda)b) \in \mathcal{A}.$$

**Definition 1.6.4** [214] *If  $\mathcal{A}$  is a closed convex cone then function  $f$  is an  $\mathcal{A}$ – convex if and only if for all  $x, u \in X$*

$$f(x) - f(u) - \nabla f(u)(x - u) \in \mathcal{A}$$

*and  $f$  is strictly  $\mathcal{A}$ – convex if and only if for all  $x, u \in X$ , but  $x \neq u$*

$$f(x) - f(u) - \nabla f(u)(x - u) \in \text{int } \mathcal{A}.$$

Cambini [29] used three different ordered relations for vectors with respect to cones by using  $\mathcal{A}$ ,  $\mathcal{A}^o$  for interior of cone  $\mathcal{A}$  and  $\mathcal{A}^\circ$  for  $\mathcal{A}$  excluding origin. Then he defined concavity with respect to each ordered relation. Doing so, he obtained a direct relation between the concavity over cones and componentwise concavity of vector valued objective function. He also introduced several forms of generalized convexity over cones along with establishing relationships amongst them.

To obtain the necessary optimality conditions for (CP) the following constraint qualifications are useful.

1. **Slater's Constraint Qualification:** [214] The constraint  $g$  is said to satisfy Slater's constraint qualification if  $-g(x) \in \text{int } \mathcal{Q}$  for some  $x = x^0 \in X_{cp}$ .
2. **Local Solvability:**[214] This constraint qualification is said to hold if the system  $-g(x) \in \mathcal{Q}$  is locally solvable at  $x = x^0 \in X_{cp}$ . This means if  $-g(x^0) \in \mathcal{Q}$ , then for any direction  $d$  which satisfies  $g(x^0) + \nabla g(x^0)d \in -\mathcal{Q}$  there exist a solution  $x = x^0 + \omega d + o(\omega)$  to  $-g(x) \in \mathcal{Q}$ , for sufficiently small  $\omega$ .

### 1.6.1 Duality for Mathematical Programs over Cones

Initially, Tanino and Swaragi [207] studied duality in vector programs by considering a dual map in terms of Lagrangian and proved duality theorems by investigating the properties of this dual map. A widely accepted approach for dealing with problems over cones is attributed to Weir et al. [215]. The dual for (CP) as formulated in [215] is given below:

$$\begin{aligned}
 \text{Dual (CPD)} \quad & \mathcal{A} - \text{Minimize } f(u) + y^T g(u)e \\
 & \text{subject to } \tau^T \nabla f(u) + y^T \nabla g(u) = 0, \\
 & \tau \in \mathcal{A}^*, y \in \mathcal{Q}^*, \tau^T e = 1, \\
 & e \text{ is a fixed element of } \text{int } \mathcal{Q}.
 \end{aligned}$$

The results were obtained for weakly minimum solutions of the problems.

## 1.7 Symmetric Duality

In the field of mathematical programming, there exist some programs in which the dual of the dual problem becomes primal itself. Such primal and dual programs are termed as symmetric programs. Linear programming problems fall into this category of programs. Whereas in the case of nonlinear programs such kind of occurrence is quite exceptional. However we can express the nonlinear programs in a specific way that the primal dual pair is symmetric.

Symmetric programs with nonlinear functions was originated by Dorn [57] for quadratic programs and an extension is proposed by Cottle [44]. Dantzig et al. [52] formulated the symmetric primal dual pair for nonlinear differentiable functions and established weak and

strong duality theorems. Gulati et al. [80] extended the theory of symmetric programs to the case of vector valued objective function. They proposed duality results for Wolfe and Mond-Weir type symmetric dual vector programs while some simpler formulations can be obtained from [174, 213]. The Mond-Weir type symmetric problems for a twice continuously differentiable function  $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^m$  in [80] is as follows:

$$\begin{array}{ll}
\text{Primal (SPP)} & \begin{array}{l}
\text{Minimize } f(u, v) = (f_1(u, v), f_2(u, v), \dots, f_m(u, v)) \\
\text{subject to } \nabla_v(\mu^T f(u, v)) \leq 0, \\
v^T \nabla_v(\mu^T f(u, v)) \geq 0, \\
\mu \in \mathbb{R}^m, \mu > 0.
\end{array}
\end{array}$$

$$\begin{array}{ll}
\text{Dual (SDP)} & \begin{array}{l}
\text{Maximize } f(x, y) = (f_1(x, y), f_2(x, y), \dots, f_m(x, y)) \\
\text{subject to } \nabla_u(\mu^T f(x, y)) \geq 0, \\
x^T \nabla_u(\mu^T f(x, y)) \leq 0, \\
\mu \in \mathbb{R}^m, \mu > 0.
\end{array}
\end{array}$$

$\nabla_v f(u, v)$  is  $n_2 \times m$  a matrix of first order partial derivatives of the function  $f$  with respect to its second variable  $v$  and  $\nabla_u f(u, v)$  is  $n_1 \times m$  a matrix of first order partial derivatives of  $f$  with respect to  $u$ . Further  $\nabla_v(\mu^T f)$  and  $\nabla_u(\mu^T f)$  are gradient vectors of  $\mu^T f$  (with dimensions  $n_2$  and  $n_1$  respectively).

## 1.8 Second-Order and Higher-Order Duality

As it is known that the dual program for a constrained optimization problem gives a lower bound to the objective function value of the primal problem which means; the feasible solution of the dual can be used as a stopping point in algorithms meant for finding the primal optimal solutions. The introduction of second-order duality and higher order duality [155] in constrained optimization problems has significantly improved the solution finding algorithms by providing them with tighter lower bounds.

For a constrained nonlinear problem, Mangasarian [155] associated second and higher order duals. He did this by taking the non-linear approximations of objective function and the constraints around some arbitrary but fixed point. For (GMP) the approximation of  $f$  and  $g$  are given by  $f(\tilde{x}) + F(\tilde{x}, p)$  and  $g(\tilde{x}) + G(\tilde{x}, p)$  respectively, where  $F$  and  $G$  are given by  $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $G : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k$ .

The nonlinear problem (GMP) is replaced by

$$\underset{p \in \mathbb{R}^n}{Min} \left\{ f(\tilde{x}) + F(\tilde{x}, p) \mid g_i(\tilde{x}) + G_i(\tilde{x}, p) \leq 0, i = 1, 2, \dots, k \right\}.$$

Then its dual is given as below

$$\underset{y \in \mathbb{R}^k, p \in \mathbb{R}^n}{Max} \left\{ f(\tilde{x}) + F(\tilde{x}, p) + \sum_{i=1}^k y_i g_i(\tilde{x}) + \sum_{i=1}^k y_i G_i(\tilde{x}, p) \mid \right. \\ \left. \nabla_p F(\tilde{x}, p) + \sum_{i=1}^k y_i \nabla_p G_i(\tilde{x}, p) = 0, y \geq 0 \right\}$$

Removing the restriction from  $x$  the higher order dual of problem (GMP) is obtained as follows

$$\begin{aligned} \text{Dual (HD)} \quad & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}^k, p \in \mathbb{R}^n}{\text{Maximize}} \quad f(x) + F(x, p) + \sum_{i=1}^k y_i g_i(x) + \sum_{i=1}^k y_i G_i(x, p) \\ & \text{subject to} \quad \nabla_p F(x, p) + \sum_{i=1}^k y_i \nabla_p G_i(x, p) = 0, y \geq 0. \end{aligned}$$

In the above dual if the higher order approximations  $F(x, p)$  and  $G_i(x, p)$  are replaced with  $\nabla f(x)p + \frac{1}{2}p\nabla^2 f(x)p$  and  $\nabla g_i(x)p + \frac{1}{2}p\nabla^2 g_i(x)p$  respectively, then it turns into second order dual. Mond [166] gave their rendering of second order duality theory for mathematical programs. Zhang and Mond [222] studied nondifferentiable program and obtained second order generalized Mond-Weir type dual using second order  $\eta$ -convex and  $\eta$ -pseudo/quasi convex functions.

Motivated by the above discussed notions of duality we investigate and extend duality theory further, especially higher order duality for fractional programming problems under various settings. The major contribution of the present thesis is the study of duality results for different classes of fractional programs, including non-differentiable multiobjective fractional programs, multiobjective fractional programs involving integrals, minimax fractional programs and multiobjective fractional programs under data uncertainty. In the next Section we give summary of the contributory part of this thesis.

## 1.9 Summary of the Thesis

**In Chapter 2**, we study optimality and higher order duality for a constrained non-differentiable multiobjective fractional programming problem. The problem is non-differentiable due to the presence of support functions. We have extended the previously existing classes of generalized convex functions by taking motivation from the generalized convexity assumptions of Yuan et al. [221] and Dubey et al. [61] to introduce higher order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions. To demonstrate the existence of such functions, we identify a function lying exclusively in the class of higher order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions but not in the class of  $(\mathcal{C}, \alpha, \rho, d)$  type-I functions. Using the properties of defined functions we obtained necessary and sufficient efficiency conditions for this non-differentiable programme. Further, by parameterizing the objective functions we attained a Schaible type dual model which also involves higher order approximations of the objective functions and constraints. Then we have established weak, strong and strict converse duality theorems for this dual problem under the notion of aforesaid function.

**In Chapter 3** we study a similar fractional program, but here the preference relation among the alternatives is taken with respect to closed convex cones. We have studied duality relations for this program with a parametric vector program, which comprises of the higher order approximations of the functions appearing in fractional program. In this direction we introduce generalized cone convex functions namely higher order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions. These functions generalize higher order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions and its preceding classes of generalized convex functions. We illustrate with the help of an example that there exist numerical functions which lie solely in our class of functions not in the existing classes of generalized convex functions. Then we consider Schaible dual for the above said program and its existence is justified using the weak and strong duality theorems.

**In Chapter 4**, we considered higher order nondifferentiable fractional vector programs over cones. These programs differ from the one discussed in Chapters 2 and 3 in the sense that, here the primal and the corresponding dual are symmetric programs. An equivalent symmetric parametric vector program over cones is devised, using Schaible's parametrization approach for fractional programs. We have defined a new class of generalized convex functions calling them higher order  $\mathcal{K} - (\Phi, \rho)$  convex functions which extend various other classes of generalized convex functions. The given fractional symmetric programs are transformed into equivalent parametric symmetric programs. Then utilizing the prop-

erties of the newly defined functions we obtain weak, strong and converse duality theorems.

**In Chapter 5**, we have studied higher order multiobjective fractional variational symmetric problems under generalized convexity. We first introduced two types of higher-order generalized convex functionals (higher order  $(\mathcal{F}, \alpha, \rho, d)/(\mathcal{C}, \alpha, \rho, d)$  convex functionals) and shown the existence of such functions using non-trivial numerical examples. Then we formulated higher order multiobjective fractional variational symmetric primal-dual pair and gave an equivalent parametric formulation of these programs. After that we studied the duality relations between the symmetric parametric problems with the help of generalized convex functionals. We have also constructed an example of weak duality theorem.

**In Chapter 6**, we considered non-differentiable fractional minimax programming problems containing support functions. We have obtained sufficient optimality conditions using higher order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions. Then we have formulated higher order parametric dual model for this non-differentiable minimax program. For this dual model, duality theorems were proved under the assumptions of generalized convexity. This work generalizes various dual models and results from the literature.

**In Chapter 7**, we considered a semi-infinite multiobjective programming problem in the face of data uncertainty in the constraints. Robust sufficient optimality conditions are established for weakly robust efficient, robust efficient and properly robust efficient solutions of the problem. The Mond-Weir type dual problem is structured and then the weak, strong and converse duality results are obtained. We have also discussed two nontrivial examples to validate the existence of sufficient optimality conditions and weak duality theorem. Then we extended the results for a robust fractional semi-infinite multiobjective problem and proved sufficient optimality conditions and duality results under convexity/generalized convexity assumptions.



# Chapter 2

## Higher-Order Duality for Multiobjective Fractional Programming Problems

### 2.1 Introduction

For a mathematical programming problem with inequality constraints KKT conditions are both necessary and sufficient (N&S) when the functions are convex and a suitable constraint qualification is satisfied. Later it is determined that if the objective function is pseudoconvex and constraints are quasiconvex then also the sufficient KKT conditions are achievable. Hanson and Mond [95] made a further attempt to find an exhaustive class of functions which is sufficient for the existence of primal optimal solution by defining two closely related classes of functions named as type-I functions and type-II functions. They also proved that, the existence of type-I functions is sufficient for primal optimality whereas the existence of the type-II functions lead to dual optimality. Since the primal optimality conditions help in structuring the dual problem so type-I function were extended for further analysis of mathematical programs.

Rueda and Hanson [180] further widened the type-I class to pseudo type-I and quasi type-I functions and studied how various classes of convexity are related to each other. Kaul et al. [124] introduced different types of generalized type-I functions and with the help of these functions they studied optimality and duality for multiobjective programs. Aghezaff and Hachimi [2] further defined different classes of weak and strong pseudoquasi type-I and obtained duality results for mixed type dual model for multiobjective programs. Then Hachimi and Aghezaff [88] introduced  $(\mathcal{F}, \alpha, \rho, d)$  type-I functions which extend  $(\mathcal{F}, \alpha, \rho, d)$  convex functions owing to Liang et al. [145]. In case of non-differentiable programs, the optimality and duality results were studied by Suneja and Srivastava [204] using d-type-I functions in terms of directional derivatives, and by Gulati and Agarwal [76] using generalized  $(\mathcal{F}, \alpha, \rho, d)$  type-I functions in terms of Clarke's generalized directional derivatives. Hanson [93] introduced second order type-I functions and gave duality theorems for Wolfe type dual model. Then Srivastava and Govil [191] proposed second order  $(F, \rho, \sigma)$  type-I functions. They formulated second order Mond-Weir type dual models for a multiobjective program and studied their duality relations. Later Hachimi and Aghezaff [89] proposed

definition of second order  $(\mathcal{F}, \alpha, \rho, p, d)$  type-I functions along with their generalized counterparts to study mixed type duality for multiobjective programs. They studied different forms of duality theorems using these functions. Jayswal et al. [110] studied second order duality relations for non-differentiable programs involving square root of bilinear forms using second order  $(\mathcal{F}, \alpha, \rho, d)$  V type-I functions.

Ahmad et al. [7] extended the second order functions given in [89] to higher-order  $(\mathcal{F}, \alpha, \rho, d)$  type -I functions. Then they constructed a higher-order generalized Mond-Weir dual for a multiobjective non-differentiable program. Then Gulati and Agarwal [77] also studied non-differentiable multiobjective programs, to give their higher-order Wolfe and generalized Mond-Weir dual. Ahmad et al. [4] considered a non-differentiable fractional multiobjective program and formulated its higher-order Schaible type dual model. They utilized higher-order  $(\mathcal{F}, \alpha, \rho, d)$  type-I functions to obtain duality results.

For  $(\mathcal{F}, \alpha, \rho, d)$  convex functions,  $\mathcal{F}$  is assumed to be a sublinear functional in its third argument. The sublinear functional  $\mathcal{F}$  is extended to convex functional  $\mathcal{C}$  by Yuan et al. [221] to give  $(\mathcal{C}, \alpha, \rho, d)$  convex functions. The corresponding type-I functions are introduced in [220]. Gupta et al. [85] put forth second order  $(\mathcal{C}, \alpha, \rho, d)$  type-I functions and studied second order dual models for non-differentiable fractional programs. This led Dubey et al. [61] to give concept of higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions. Using these functions Dubey et al. [61] studied optimality conditions for non-differentiable multiobjective fractional problems and considered its duality relations with a Mond-Weir type dual. The extension of second order to higher-order functions adds a new parameter  $\gamma$  which follows the same assumptions as the parameter  $\alpha$ .

On this line of action we have moved a step further to propose higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions. We have illustrated with the help of an example that there exist some problems in mathematical programs with functions lying in the class of higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions but not in the previous classes. We derive necessary and sufficient optimality conditions for multiobjective nonlinear fractional programming problem which contain support functions in both objectives and constraints for the defined class of functions. Further we construct a Schaible type dual model and study relevant duality theorems.

This Chapter is structured as follows. In Section 2.2, we give some preliminaries and introduce higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions along with its numerical example. Then in Section 2.3, we consider a non-differentiable multiobjective fractional programming problem. Section 2.4 contains necessary and sufficient (N&S) optimality conditions. In Section 2.5, we give the dual formulation along with weak, strong and converse duality theorems.

## 2.2 Preliminaries

The type-I functions introduced by Hanson and Mond [95] are defined as follows. Consider a mathematical program: Minimize  $\phi(x)$  subject to  $x \in X_o = \{x \in X : h(x) \leq 0\}$  where  $\phi : X \rightarrow \mathbb{R}$  and  $h : X \rightarrow \mathbb{R}^m$  are differentiable functions and  $X$  is an open subset of  $\mathbb{R}^n$ .

**Definition 2.2.1**  $\phi$  and  $h$  are called type-I objective and constraints with respect to a function  $\eta$  at  $x_o \in X_o$  if, for all  $x \in X_o$

$$\begin{aligned}\phi(x) - \phi(x_o) &\geq [\nabla_x \phi(x_o)]^T \eta(x) \\ -h(x_o) &\geq [\nabla_x h(x_o)]^T \eta(x).\end{aligned}$$

Now we define higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions.

Consider  $\mathcal{C} : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a functional, convex in its third argument as given in Definition 1.1.9 in Chapter 1. Throughout this thesis, it is assumed that  $\mathcal{C}_{(x,u)}(0) = 0$ , for every  $(x, u) \in X \times X$ .

Now we introduce higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions. Let  $\phi : X \rightarrow \mathbb{R}^k$  and  $h : X \rightarrow \mathbb{R}^m$  be vector valued differentiable functions on  $X$ . Now consider higher-order approximations of  $\phi$  and  $h$  to be  $F = (F_1, F_2, \dots, F_k)$  and  $H = (H_1, H_2, \dots, H_m)$  respectively such that  $F_i, H_j : X \times \mathbb{R}^n \rightarrow \mathbb{R}$  are differentiable functions on  $X \times \mathbb{R}^n$ .  $\alpha = (\alpha^1, \alpha^2) = (\alpha_1^1, \dots, \alpha_k^1, \alpha_1^2, \dots, \alpha_m^2)$ ,  $d = (d_1^1, \dots, d_k^1, d_1^2, \dots, d_m^2)$ ,  $\rho = (\rho_1^1, \dots, \rho_k^1, \rho_1^2, \dots, \rho_m^2)$ ,  $\alpha_i^1, \gamma_i^1, \alpha_j^2, \gamma_j^2 : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}$  and  $d_i^1, d_j^2 : X \times X \rightarrow \mathbb{R}_+$  satisfying  $d(x, x_0) = 0 \Leftrightarrow x = x_0$  and  $p, q, r, s \in \mathbb{R}^n, i = 1, 2, \dots, k, j = 1, 2, \dots, m$ .

**Definition 2.2.2** The function  $(\phi, h)$  is called higher-order (strictly)  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $F, p, r$  and  $H, q, s$  respectively, if for each  $x \in X$ ,

$$\begin{aligned}\frac{1}{\alpha^1(x, u)}[\phi(x) - \phi(u)] &\geq (>) \mathcal{C}_{(x,u)}(\nabla \phi(u) + \nabla_p F(u, p)) \\ &+ \frac{1}{\gamma^1(x, u)}[F(u, r) - r^T \nabla_r F(u, r)] + \frac{\rho^1 d^1(x, u)}{\alpha^1(x, u)},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\alpha^2(x, u)}(-h(u)) &\geq (>) \mathcal{C}_{(x,u)}(\nabla h(u) + \nabla_q H(u, q)) \\ &+ \frac{1}{\gamma^2(x, u)}[H(u, s) - s^T \nabla_s H(u, s)] + \frac{\rho^2 d^2(x, u)}{\alpha^2(x, u)}.\end{aligned}$$

For convenience we use the following notations throughout the Chapter.

$\mathcal{C}_{(x,u)}(\nabla\phi(u) + \nabla_p F(u, p))$  means vector

$(\mathcal{C}_{(x,u)}(\nabla\phi_1(u) + \nabla_p F_1(u, p)), \mathcal{C}_{(x,u)}(\nabla\phi_2(u) + \nabla_p F_2(u, p)), \dots, \mathcal{C}_{(x,u)}(\nabla\phi_k(u) + \nabla_p F_k(u, p)))$ .

### Special Cases:

1. If  $F_i(u, \cdot) = 0$  and  $H_j(u, \cdot) = 0$  then the definition (2.2.2) becomes that of  $(\mathcal{C}, \alpha, \rho, d)$  type-I as defined by Yuan et al. [220].
2. For  $F_i(u, \cdot) = \frac{1}{2}(\cdot)^t \nabla^2 f_i(u)(\cdot)$ ,  $\alpha(x, u) = \gamma(x, u)$ ,  $H_j(u, \cdot) = \frac{1}{2}(\cdot)^t \nabla^2 h_j(u)(\cdot)$ ,  $p = r$  and  $q = s$ .
  - (a) If  $k = 1, m = 1$  then definition (2.2.2) becomes second order  $(\mathcal{C}, \alpha, \rho, d)$  type-I function given by Gupta et al. [85].
  - (b) If  $\mathcal{C}$  is sublinear with respect to third variable then the functions in definition (2.2.2) are second order  $(\mathcal{F}, \alpha, \rho, p, d)$  type-I given by Hachimi and Aghezaff [89]. In addition to this, if  $p = 0$  and  $q = 0$  the function defined above becomes  $(\mathcal{F}, \alpha, \rho, d)$  type-I function given by Hachimi and Aghezaff [88].
3. Further if we take  $k = 1, F_i(u, \cdot) = 0$  and  $H_j(u, \cdot) = 0$ ,  $\alpha^1 = 1, \alpha^2 = (1, 1, \dots, 1) \in \mathbb{R}^k, \rho = \mathbf{0}, \mathcal{C}_{(x,u)}(a) = a^T(\eta(x))$ , for any function  $\eta$  of  $x$ , then our definition reduces to the one defined by Hanson and Mond [95].

**Example 2.2.1** Let  $k = 1, m = 1, X = \mathbb{R}_+ \subset \mathbb{R}$ ,  $\phi : X \rightarrow \mathbb{R}^1$ ,  $h : X \rightarrow \mathbb{R}^1$ ,  $\mathcal{C} : X \times X \times \mathbb{R}^1 \rightarrow \mathbb{R}$ ,  $F, H : X \times \mathbb{R} \rightarrow \mathbb{R}^1$ ,  $d^1, d^2 : X \times X \rightarrow \mathbb{R}_+$  be defined as follows

$$\begin{aligned} \phi(x) &= \frac{(x+1)^4 + x^2 - 3\sin^2(x)}{x+1}, & h(x) &= \exp(x) - x^2, & d^1, d^2 &= (x-u)^2, \\ F(u, \cdot) &= \frac{(\cdot)^2}{u+1}, & H(u, \cdot) &= 2(\cdot)^2(u+1), & \mathcal{C}_{(x,u)}(a) &= \frac{a^2}{10}. \end{aligned}$$

For  $\alpha = (\alpha^1, \alpha^2), \rho = (\rho^1, \rho^2), \gamma = (\gamma^1, \gamma^2)$ . Let  $\alpha = (3/4, 1), \gamma = (3/4, 1), \rho = (1, -1), p = r = q = s = -1$ . At a point  $u = 0$ , for all  $x \in X$ ,

$$\begin{aligned} \Psi &= \frac{1}{\alpha^1(x, u)}[\phi(x) - \phi(u)] - \mathcal{C}_{(x,u)}(\nabla\phi(u) + \nabla_p F(u, p)) - \frac{1}{\gamma^1(x, u)}[F(u, r) - r^T \nabla_r F(u, r)] \\ &\quad - \frac{\rho^1 d^1(x, u)}{\alpha^1(x, u)} \\ &= \frac{4}{3} \left[ \frac{(x+1)^4 + x^2 - 3\sin^2(x)}{x+1} - \frac{(0+1)^4 + 0^2 - 3\sin^2(0)}{0+1} \right] - \mathcal{C}_{(x,0)} \left( 3 + \frac{2p}{0+1} \right) \\ &\quad - \frac{4}{3} \left[ \frac{r^2}{0+1} - r \cdot \frac{2r}{0+1} \right] - \frac{4x^2}{3} \end{aligned}$$

$$\begin{aligned}
&= \frac{4}{3} \left[ \frac{(x+1)^4 + x^2 - 3\sin^2(x) - x - 1}{(x+1)} \right] - \mathcal{C}_{(x,0)}(3-2) - \left(\frac{4}{3}\right) \left(\frac{-r^2}{1}\right) - \frac{4x^2}{3}. \\
&= \frac{4}{3} \left[ \frac{(x+1)^4 + x^2 - 3\sin^2(x) - x - 1}{(x+1)} \right] - \frac{1}{10} + \frac{4}{3} - \frac{4x^2}{3}. \\
&= \frac{40x^4 + 120x^3 + 240x^2 + 157x - 120\sin^2(x) + 37}{30(x+1)} \geq 0, \quad \forall x \in X,
\end{aligned}$$

This can also be seen from Figure [2.1]. Now for function  $h$ ,

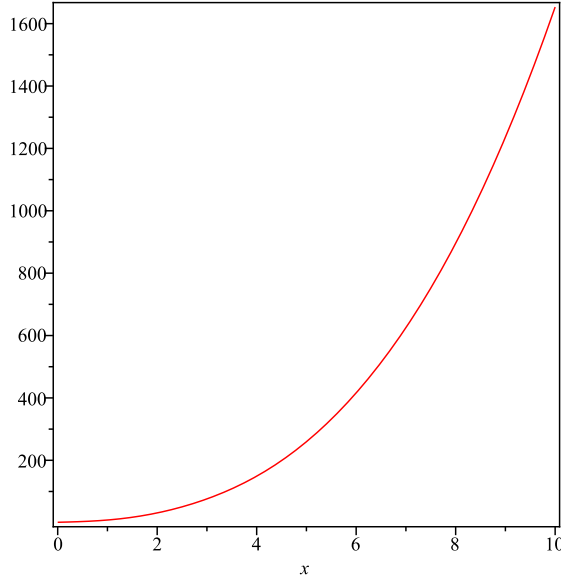


Figure 2.1: Graph of  $\Psi(x)$

$$\begin{aligned}
&\frac{1}{\alpha^2(x,u)}(-h(u)) - \mathcal{C}_{(x,u)}(\nabla h(u) + \nabla_q H(u,q)) - \frac{1}{\gamma^2(x,u)}[H(u,s) - s^T \nabla_s H(u,s)] - \frac{\rho^2 d^2(x,u)}{\alpha^2(x,u)} \\
&= -(exp(0) - (0)^2) - C_{(x,0)}(exp(0) - 2(0) + 4q(0+1)) - 1(2s^2(0+1) - s(4s(0+1))) + x^2 \\
&= -(1) - \mathcal{C}_{(x,0)}(1 - 4(1)) - 1(2s^2(1) - 4s^2(1)) + x^2 \\
&= -1 - \frac{9}{10} + 2(s^2)(1) + x^2 = -1.9 + 2(1) + x^2 = 0.1 + x^2 \geq 0, \quad \text{for all } x \in X = \mathbb{R}_+.
\end{aligned}$$

So,  $(\phi, h)$  is higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u = 0$ .

Now it can be verified that this function is not  $(\mathcal{C}, \alpha, \rho, d)$  type-I at  $u = 0$  as defined in [220]. For  $\phi, h$  to be  $(\mathcal{C}, \alpha, \rho, d)$  type-I the following should hold.

$$\begin{aligned}
&\frac{1}{\alpha^1(x,u)}[\phi(x) - \phi(u)] - \mathcal{C}_{(x,u)}(\nabla \phi(u)) - \frac{\rho^1 d^1(x,u)}{\alpha^1(x,u)} \geq 0 \\
&\text{and} \\
&\frac{1}{\alpha^2(x,u)}(-h(u)) - \mathcal{C}_{(x,u)}(\nabla h(u)) - \frac{\rho^2 d^2(x,u)}{\alpha^2(x,u)} \geq 0, \quad \forall x \in X.
\end{aligned}$$

But at  $x = 0.1$

$$\begin{aligned} & \frac{1}{\alpha^1(x, u)} [\phi(x) - \phi(u)] - \mathcal{C}_{(x, u)}(\nabla \phi(u)) - \frac{\rho^1 d^1(x, u)}{\alpha^1(x, u)} \\ &= \frac{4(x+1)^4 + x^2 - 3\sin^2(x) - x - 1}{3(x+1)} - \frac{9}{10} - \frac{4x^2}{3} \\ &= -0.496 \end{aligned}$$

and

$$\frac{1}{\alpha^2(x, u)} (-h(u)) - \mathcal{C}_{(x, u)}(\nabla h(u)) - \frac{\rho^2 d^2(x, u)}{\alpha^2(x, u)} = -1.09.$$

So we have shown that  $(\phi, h)$  is not  $(\mathcal{C}, \alpha, \rho, d)$  type-I at  $u = 0$  but it is higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u = 0$ .

## 2.3 Problem Formulation

Consider the following non-differentiable multiobjective fractional programming problem:

$$\begin{aligned} \text{Primal (NMFP)} \quad & \text{Minimize } F(x) = \left( \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \frac{f_2(x) + S(x|C_2)}{g_2(x) - S(x|D_2)}, \dots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right) \\ & \text{subject to} \\ & h_j(x) + S(x|E_j) \leq 0, \quad j = 1, 2, \dots, m, \quad x \in X. \end{aligned}$$

The set of feasible solutions is given by  $X_{nf} := \{x \in X \subseteq \mathbb{R}^n \mid h_j(x) + S(x|E_j) \leq 0, \quad j = 1, 2, \dots, m\}$ .

For  $i = 1, 2, \dots, k, j = 1, 2, \dots, m$  the sets  $C_i, D_i$  and  $E_j$  are compact convex subsets of  $\mathbb{R}^n$  and  $S(x|C_i), S(x|D_i), S(x|E_j)$  denote support functions of  $C_i, D_i$  and  $E_j$  respectively, for  $i = 1, 2, \dots, k, j = 1, 2, \dots, m$ . The functions  $f_i, g_i, h_j : X \rightarrow \mathbb{R}$  are continuously differentiable.  $f_i(\cdot) + S(\cdot|C_i) \geq 0$  and  $g_i(\cdot) - S(\cdot|D_i) > 0$  on  $X$ . For this problem the efficient and weak efficient solutions are defined as follows.

**Definition 2.3.1** [61] A point  $u \in X_{nf}$  is weakly efficient solution of (NMFP), if there exists no  $x \in X_{nf}$  such that for every  $i = 1, 2, \dots, k$ ,

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} < \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}.$$

**Definition 2.3.2** [61] A point  $u \in X_{nf}$  is an efficient solution of (NMFP), if there exists no  $x \in X_{nf}$  such that for every  $i = 1, 2, \dots, k$ ,

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}$$

and for some  $r = 1, 2, \dots, k$ ,

$$\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(u) + S(u|C_r)}{g_r(u) - S(u|D_r)}.$$

**Lemma 2.3.1** [205] If  $u$  is efficient solution of problem (NMFP), then  $u$  solves the problem  $(FP_\beta)$  for each  $r = 1, 2, \dots, k$  where  $(FP_\beta)$  is given as

$$\begin{aligned} (FP_\beta) \quad & \text{Minimize} \quad \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} \\ & \text{subject to} \\ & \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \beta_i, \quad i = 1, 2, \dots, k, i \neq r, \\ & h_j(x) + S(x|E_j) \leq 0, \quad j = 1, 2, \dots, m, \end{aligned}$$

for  $\beta_i = \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}$ ,  $i = 1, 2, \dots, k$ .

Now since  $g_i(\cdot) - S(\cdot|D_i) > 0$ , for all  $i = 1, 2, \dots, k$  therefore  $(FP_\beta)$  can be re-formulated as

$$\begin{aligned} (FP_\beta^1) \quad & \text{Minimize} \quad \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} \\ & \text{subject to} \\ & f_i(x) + S(x|C_i) - \beta_i(g_i(x) - S(x|D_i)) \leq 0, \quad i = 1, 2, \dots, k, i \neq r, \\ & h_j(x) + S(x|E_j) \leq 0, \quad j = 1, 2, \dots, m. \end{aligned}$$

**Lemma 2.3.2** [205]  $u$  is efficient solution of (NMFP) if and only if  $u$  solves  $(FP_\beta^1)$  for each  $r = 1, 2, \dots, k$ , where  $\beta_i$  are defined as above.

We use the following notations for convenience

$$(1.) \quad f(\cdot) + S(\cdot|C) = (f_1(\cdot) + S(\cdot|C_1), f_2(\cdot) + S(\cdot|C_2), \dots, f_k(\cdot) + S(\cdot|C_k))$$

- (2.)  $f(\cdot) + (\cdot)^T z = (f_1(\cdot) + (\cdot)^T z_1, f_2(\cdot) + (\cdot)^T z_2, \dots, f_k(\cdot) + (\cdot)^T z_k)$  each  $z_i \in \mathbb{R}^n$
- (3.)  $g(\cdot) - S(\cdot|D) = (g_1(\cdot) - S(\cdot|D_1), g_2(\cdot) - S(\cdot|D_2), \dots, g_k(\cdot) - S(\cdot|D_k))$
- (4.)  $g(\cdot) - (\cdot)^T v = (g_1(\cdot) - (\cdot)^T v_1, g_2(\cdot) - (\cdot)^T v_2, \dots, g_k(\cdot) - (\cdot)^T v_k)$ , each  $v_i \in \mathbb{R}^n$
- (5.)  $h(\cdot) + (\cdot)^T w = (h_1(\cdot) + (\cdot)^T w_1, h_2(\cdot) + (\cdot)^T w_2, \dots, h_m(\cdot) + (\cdot)^T w_m)$ , each  $w_i \in \mathbb{R}^n$

## 2.4 Optimality Conditions

In this section, we obtain N&S optimality conditions for the above considered problem. The necessary optimality conditions for (NMFP) is given by Dubey et al. [61] and we will use them to give a modified form of the necessary optimality conditions. Then we derive sufficient optimality conditions using higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I assumptions.

**Theorem 2.4.1 (Necessary Optimality Condition)** [61] *Assume that  $u$  is an efficient solution of (NMFP) and Slater's constraint qualification is satisfied on  $X$ . Then for  $i = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, m$ , there exist some  $\bar{\lambda} \in \mathbb{R}^k$ ,  $\mu \in \mathbb{R}^m$ ,  $z_i \in \mathbb{R}^n$ ,  $v_i \in \mathbb{R}^n$ ,  $w_j \in \mathbb{R}^n$ , such that*

$$\begin{aligned} \sum_{i=1}^k \bar{\lambda}_i \nabla \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{j=1}^m \mu_j \nabla (h_j(u) + u^T w_j) &= 0, \\ \sum_{j=1}^m \mu_j (h_j(u) + u^T w_j) &= 0, \\ u^T z_i &= S(u|C_i), \quad u^T v_i = S(u|D_i), \quad u^T w_j = S(u|E_j), \\ z_i &\in C_i, \quad v_i \in D_i, \quad w_j \in E_j, \quad \bar{\lambda}_i > 0, \quad \mu_j \geq 0. \end{aligned}$$

**Theorem 2.4.2 (Equivalent Necessary Optimality Condition)** *Assume that  $u$  is an efficient solution of (NMFP) and Slater's constraint qualification is satisfied on  $X$ . Then for  $i = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, m$ , there exist  $\lambda \in \mathbb{R}^k$ ,  $\mu \in \mathbb{R}^m$ ,  $z_i \in \mathbb{R}^n$ ,  $v_i \in \mathbb{R}^n$ ,  $w_j \in \mathbb{R}^n$ , such that*

$$\sum_{i=1}^k \lambda_i (\nabla (f_i(u) + u^T z_i) - \beta_i \nabla (g_i(u) - u^T v_i)) + \sum_{j=1}^m \mu_j \nabla (h_j(u) + u^T w_j) = 0, \quad (2.4.1)$$

$$f_i(u) + u^T z_i - \beta_i (g_i(u) - u^T v_i) = 0, \quad (2.4.2)$$

$$\sum_{j=1}^m \mu_j (h_j(u) + u^T w_j) = 0, \quad (2.4.3)$$

$$u^T z_i = S(u|C_i), \quad u^T v_i = S(u|D_i), \quad u^T w_j = S(u|E_j), \quad (2.4.4)$$

$$z_i \in C_i, v_i \in D_i, w_j \in E_j, \lambda_i > 0, \mu_j \geq 0. \quad (2.4.5)$$

**Theorem 2.4.3 (Sufficient Optimality Condition)** *Consider  $u$  to be a feasible solution of (NMFP). Assume that for  $i = 1, 2, \dots, k$ , and  $j = 1, 2, \dots, m$ , there exist  $\lambda_i > 0$  and  $\mu_j \geq 0$ , for such that equivalent necessary optimality conditions (2.4.1)-(2.4.5) hold. Let for any , the following hold*

- (i)  $(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$  is higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $K, p, r$  and  $H, q, s$  respectively,
- (ii)  $(- (g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$  is higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $-G, p, r$  and  $H, q, s$  respectively,
- (iii)  $\sum_{i=1}^k \frac{\lambda_i \rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} (1 + \beta_i) + \sum_{j=1}^m \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \geq 0$ ,
- (iv)  $\gamma_i^1(x, u) = \xi(x, u), \quad i = 1, 2, \dots, k$  &  $\alpha_j^2(x, u) = \zeta(x, u)$  and  $\gamma_j^2(x, u) = \sigma(x, u), \quad j = 1, 2, \dots, m$ ,
- (v)  $\sum_{j=1}^m \mu_j (H_j(u, s) - s^T \nabla_s H_j(u, s)) \geq 0, \quad \sum_{i=1}^k \lambda_i (K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i (G_i(u, r) - r^T \nabla_r G_i(u, r))) \geq 0$  and  $\sum_{i=1}^k \lambda_i (\nabla_p K_i(u, p) - \beta_i \nabla_p G_i(u, p)) + \sum_{j=1}^m \mu_j \nabla_q H_j(u, q) = 0$ .

Then  $u$  is an efficient solution of (NMFP).

**Proof.** If  $u$  is not an efficient solution of (NMFP), then there exists some  $x \in X_{nf}$  satisfying the following;

$$\begin{aligned} \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} &\leq \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}, & \text{for all } i = 1, 2, \dots, k, \\ \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} &< \frac{f_r(u) + S(u|C_r)}{g_r(u) - S(u|D_r)} & \text{for some } r = 1, 2, \dots, k. \end{aligned} \quad (2.4.6)$$

For some  $z_i \in C_i$  and  $v_i \in D_i$ , it is given that  $u^T z_i = S(u|C_i)$ ,  $u^T v_i = S(u|D_i)$ ,  $i = 1, 2, \dots, k$ , gives

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)} = \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} = \beta_i,$$

for all  $i=1,2,\dots,k$ , and for some  $r = 1, 2, \dots, k$

$$\frac{f_r(x) + x^T z_r}{g_r(x) - x^T v_r} \leq \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(u) + S(u|C_r)}{g_r(u) - S(u|D_r)} = \frac{f_r(u) + u^T z_r}{g_r(u) - u^T v_r} = \beta_r$$

so that

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \beta_i, \quad \text{for all } i = 1, 2, \dots, k$$

and

$$\frac{f_r(x) + x^T z_r}{g_r(x) - x^T v_r} < \beta_r, \quad \text{for some } r = 1, 2, \dots, k.$$

As  $g_i(x) - S(x|D_i) > 0$ , so we get

$$f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) \leq 0, \quad \text{for all } i = 1, 2, \dots, k$$

and

$$f_r(x) + x^T z_r - \beta_r(g_r(x) - x^T v_r) < 0, \quad \text{for some } r = 1, 2, \dots, k.$$

For  $\lambda_i > 0$  and  $\alpha_i^1(x, u) \in \mathbb{R}_+ \setminus \{0\}$ ,

$$\sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) < 0 \quad (2.4.7)$$

Using hypotheses (i) and (ii) we get

$$\begin{aligned} \frac{f_i(x) + x^T z_i - (f_i(u) + u^T z_i)}{\alpha_i^1(x, u)} &\geq \mathcal{C}_{(x,u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \\ &\quad \frac{1}{\gamma_i^1(x, u)} (K_i(u, r) - r^T \nabla_r K_i(u, r)) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} \end{aligned} \quad (2.4.8)$$

$$\begin{aligned} \frac{-(g_i(x) - x^T v_i - (g_i(u) - u^T v_i))}{\alpha_i^1(x, u)} &\geq \mathcal{C}_{(x,u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)) + \\ &\quad \frac{1}{\gamma_i^1(x, u)} (-(G_i(u, r) - r^T \nabla_r G_i(u, r))) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} \end{aligned} \quad (2.4.9)$$

$$\begin{aligned} \frac{-(h_j(u) + u^T w_j)}{\alpha_j^2(x, u)} &\geq \mathcal{C}_{(x,u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, \underline{q})) + \\ &\quad \frac{1}{\gamma_j^2(x, u)} (H_j(u, s) - s^T \nabla_s H_j(u, s)) + \frac{\rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \end{aligned} \quad (2.4.10)$$

Adding the two inequalities (2.4.8) and (2.4.9) after multiplying (2.4.9) by  $\beta_i, (\beta_i \geq 0) i =$

$1, 2, \dots, k$ , we get

$$\begin{aligned}
& \frac{1}{\alpha_i^1(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i))) \\
& \geq \mathcal{C}_{(x, u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \frac{1}{\gamma_i^1(x, u)} (K_i(u, r) - r^T \nabla_r K_i(u, r)) \\
& \quad + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} + \beta_i \mathcal{C}_{(x, u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)) \\
& \quad - \beta_i \frac{1}{\gamma_i^1(x, u)} (G_i(u, r) - r^T \nabla_r G_i(u, r)) + \beta_i \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}. \tag{2.4.11}
\end{aligned}$$

Multiply inequality (2.4.10) by  $\mu_j (\mu_j \geq 0)$ , and (2.4.11) by  $\lambda_i (\lambda_i > 0)$ ,  $i = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, m$ , then adding over their ranges, we have

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i))) \\
& \quad + \sum_{j=1}^m \frac{\mu_j}{\alpha_j^2(x, u)} (-(h_j(u) + u^T w_j)) \\
& \geq \sum_{i=1}^k \lambda_i [\mathcal{C}_{(x, u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \beta_i \mathcal{C}_{(x, u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p))] \\
& \quad + \sum_{i=1}^k \frac{\lambda_i}{\gamma_i^1(x, u)} (K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i (G_i(u, r) - r^T \nabla_r G_i(u, r))) \\
& \quad + \sum_{i=1}^k \frac{\lambda_i \rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} (1 + \beta_i) + \sum_{j=1}^m \mu_j \mathcal{C}_{(x, u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)) \\
& \quad + \sum_{j=1}^m \frac{\mu_j}{\gamma_j^2(x, u)} (H_j(u, s) - s^T \nabla_s H_j(u, s)) + \sum_{j=1}^m \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)}. \tag{2.4.12}
\end{aligned}$$

Taking  $\tau = \sum_{i=1}^k \lambda_i (1 + \beta_i) + \sum_{j=1}^m \mu_j$ , so  $\tau > 0$ , divide the equation by  $\tau$  and using convexity of  $\mathcal{C}$  we obtain

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i^1(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i))) \\
& \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \alpha_j^2(x, u)} (-(h_j(u) + u^T w_j))
\end{aligned}$$

$$\begin{aligned}
&\geq \mathfrak{C}_{(x,u)} \left[ \frac{1}{\tau} \sum_{i=1}^k \lambda_i (\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p) - \beta_i \nabla(g_i(u) - u^T v_i) - \beta_i \nabla_p G_i(u, p)) \right. \\
&\quad \left. + \frac{1}{\tau} \sum_{j=1}^m \mu_j (\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)) \right] + \sum_{j=1}^m \frac{\mu_j}{\tau \gamma_j^2(x, u)} (H_j(u, s) - s^T \nabla_s H_j(u, s)) \\
&\quad + \sum_{i=1}^k \frac{\lambda_i}{\tau \gamma_i^1(x, u)} (K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i (G_i(u, r) - r^T \nabla_r G_i(u, r))) \\
&\quad + \sum_{i=1}^k \frac{\lambda_i \rho_i^1 d_i^1(x, u)}{\tau \alpha_i^1(x, u)} (1 + \beta_i) + \sum_{j=1}^m \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\tau \alpha_j^2(x, u)}. \tag{2.4.13}
\end{aligned}$$

It follows from hypotheses (iii) and (iv) that

$$\begin{aligned}
&\sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i^1(x, u)} (f_i(x) + x^T z_i - \beta_i (g_i(x) - x^T v_i)) \\
&\geq \mathfrak{C}_{(x,u)} \left[ \frac{1}{\tau} \sum_{i=1}^k \lambda_i (\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p) - \beta_i \nabla(g_i(u) - u^T v_i) - \beta_i \nabla_p G_i(u, p)) \right. \\
&\quad \left. + \frac{1}{\tau} \sum_{j=1}^m \mu_j (\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)) \right] \\
&\quad + \sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i^1(x, u)} (f_i(u) + u^T z_i - \beta_i (g_i(u) - u^T v_i)) \\
&\quad + \sum_{j=1}^m \frac{\mu_j}{\tau \zeta(x, u)} (h_j(u) + u^T w_j) + \sum_{j=1}^m \frac{\mu_j}{\tau \sigma(x, u)} (H_j(u, s) - s^T \nabla_s H_j(u, s)) \\
&\quad + \sum_{i=1}^k \frac{\lambda_i}{\tau \xi(x, u)} (K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i (G_i(u, r) - r^T \nabla_r G_i(u, r))). \tag{2.4.14}
\end{aligned}$$

Further, using hypothesis (v) we have

$$\begin{aligned}
&\sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i^1(x, u)} (f_i(x) + x^T z_i - \beta_i (g_i(x) - x^T v_i)) \\
&\geq \mathfrak{C}_{(x,u)} \left[ \frac{1}{\tau} \left( \sum_{i=1}^k \lambda_i (\nabla(f_i(u) + u^T z_i) - \beta_i \nabla(g_i(u) - u^T v_i)) + \sum_{j=1}^m \mu_j \nabla(h_j(u) + u^T w_j) \right) \right]
\end{aligned}$$

$$+ \sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i^1(x, u)} (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)) + \frac{1}{\tau \zeta(x, u)} \sum_{j=1}^m \mu_j (h_j(u) + u^T w_j). \quad (2.4.15)$$

Finally, using  $f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i) = 0$ ,  $\sum_{j=1}^m \mu_j (h_j(u) + u^T w_j) = 0$  and  $\mathcal{C}_{(x,u)}(0) = 0$ , it follows that

$$\sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i^1(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \geq 0, \quad (2.4.16)$$

which contradicts (2.4.7). Hence  $u$  is an efficient solution of (NMFP).

## 2.5 Schaible Dual

In this section, we consider higher-order Schaible type dual model of (NMFP):

Dual (MFD)

Maximize  $\beta = (\beta_1, \beta_2, \dots, \beta_k)$

subject to

$$\begin{aligned} \sum_{i=1}^k \lambda_i [\nabla(f_i(u) + u^T z_i) - \beta_i \nabla(g_i(u) - u^T v_i)] + \sum_{j=1}^m \mu_j \nabla(h_j(u) + u^T w_j) \\ + \sum_{i=1}^k \lambda_i (\nabla_p K_i(u, p) - \beta_i \nabla_p G_i(u, p)) + \sum_{j=1}^m \mu_j \nabla_q H_j(u, q) = 0, \end{aligned} \quad (2.5.1)$$

$$\begin{aligned} \sum_{i=1}^k \lambda_i (f_i(u) + u^T z_i - \beta_i (g_i(u) - u^T v_i) + (K_i(u, r) - r^T \nabla_r K_i(u, r)) \\ - \beta_i (G_i(u, r) - r^T \nabla_r G_i(u, r))) \geq 0, \end{aligned} \quad (2.5.2)$$

$$\sum_{j=1}^m \mu_j (h_j(u) + u^T w_j + H_j(u, s) - s^T \nabla_s H_j(u, s)) \geq 0, \quad (2.5.3)$$

$$\begin{aligned} \lambda_i > 0, \quad \mu_j \geq 0, \quad \text{and} \quad \beta_i \geq 0, \quad \text{for} \quad i = 1, 2, \dots, k \quad \text{and} \quad j = 1, 2, \dots, m, \\ z_i \in C_i, \quad v_i \in D_i, \quad w_j \in E_j. \end{aligned} \quad (2.5.4)$$

**Theorem 2.5.1 (Weak Duality Theorem)** *Let  $x$  be a feasible solution of primal (NMFP) and  $(u, \beta, \lambda, \mu, z, v, w, p, q, r, s)$  be feasible solution of dual (NMFD). Suppose that*

- (i)  $(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$  is higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $K, p, r$  and  $H, q, s$  respectively,
- (ii)  $(-(g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$  is higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $-G, p, r$  and  $H, q, s$  respectively,
- (iii)  $\sum_{i=1}^k \frac{\lambda_i \rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} (1 + \beta_i) + \sum_{j=1}^m \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \geq 0$ ,
- (iv)  $\alpha_i^1(x, u) = \gamma_i^1(x, u) = a(x, u)$  and  $\alpha_j^2(x, u) = \gamma_j^2(x, u) = b(x, u)$ .

Then the following can not hold:

$$\begin{aligned} \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} &\leq \beta_i, & \text{for all, } i = 1, 2, \dots, k, \\ \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} &< \beta_r, & \text{for some } r = 1, 2, \dots, k. \end{aligned} \quad (2.5.5)$$

**Proof.** Let if possible (2.5.5) hold. Then for  $\lambda_i > 0$  and  $a(x, u) \in \mathbb{R}_+ \setminus \{0\}$ , for  $z_i \in C_i$  and  $v_i \in D_i$  for  $i = 1, 2, \dots, k$  and using the definition of support function  $x^T z_i \leq S(x|C_i)$ ,  $x^T v_i \leq S(x|D_i)$ , we get the following

$$\sum_{i=1}^k \frac{\lambda_i}{a(x, u)} (f_i(x) + x^T z_i - \beta_i (g_i(x) - x^T v_i)) < 0. \quad (2.5.6)$$

Using higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I convexity of  $(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$  with respect to  $K, p, r$  and  $H, q, s$  respectively and that of  $(-(g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$  with respect to  $-G, p, r$  and  $H, q, s$  respectively, we have the following

$$\begin{aligned} \frac{f_i(x) + x^T z_i - (f_i(u) + u^T z_i)}{\alpha_i^1(x, u)} &\geq \mathcal{C}_{(x, u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \\ &\quad \frac{1}{\gamma_i^1(x, u)} (K_i(u, r) - r^T \nabla_r K_i(u, r)) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}, \end{aligned} \quad (2.5.7)$$

$$\begin{aligned} \frac{-(g_i(x) - x^T v_i - (g_i(u) - u^T v_i))}{\alpha_i^1(x, u)} &\geq \mathcal{C}_{(x, u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)) + \\ &\quad \frac{1}{\gamma_i^1(x, u)} (-(G_i(u, r) - r^T \nabla_r G_i(u, r))) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}, \end{aligned} \quad (2.5.8)$$

$$\frac{-(h_j(u) + u^T w_j)}{\alpha_j^2(x, u)} \geq \mathcal{C}_{(x, u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)) +$$

$$\frac{1}{\gamma_j^2(x, u)}(H_j(u, s) - s^T \nabla_s H_j(u, s)) + \frac{\rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)}. \quad (2.5.9)$$

Multiply equation (2.5.8) by  $\beta_i (\beta_i \geq 0)$ ,  $i = 1, 2, \dots, k$ , then adding equations (2.5.7) and (2.5.8), we get

$$\begin{aligned} & \frac{1}{\alpha_i^1(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i))) \\ & \geq \mathcal{C}_{(x, u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \frac{1}{\gamma_i^1(x, u)}(K_i(u, r) - r^T \nabla_r K_i(u, r)) \\ & \quad + \beta_i \mathcal{C}_{(x, u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)) - \beta_i \frac{1}{\gamma_i^1(x, u)}(G_i(u, r) - r^T \nabla_r G_i(u, r)) \\ & \quad + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} + \beta_i \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}. \end{aligned} \quad (2.5.10)$$

Adding the inequalities (2.5.9) and (2.5.10) over their ranges after multiplying (2.5.9) by  $\mu_j (\mu_j \geq 0)$  and (2.5.10) by  $\lambda_i (\lambda_i > 0)$ ,  $i = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, m$ , we have

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\alpha_i^1(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i))) \\ & \quad + \sum_{j=1}^m \frac{\mu_j}{\alpha_j^2(x, u)}(-(h_j(u) + u^T w_j)) \\ & \geq \sum_{i=1}^k \lambda_i [\mathcal{C}_{(x, u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \beta_i \mathcal{C}_{(x, u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p))] \\ & \quad + \sum_{i=1}^k \frac{\lambda_i}{\gamma_i^1(x, u)}(K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r))) \\ & \quad + \sum_{i=1}^k \frac{\lambda_i \rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}(1 + \beta_i) + \sum_{j=1}^m \mu_j \mathcal{C}_{(x, u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)) \\ & \quad + \sum_{j=1}^m \frac{\mu_j}{\gamma_j^2(x, u)}(H_j(u, s) - s^T \nabla_s H_j(u, s)) + \sum_{j=1}^m \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)}. \end{aligned} \quad (2.5.11)$$

Using (iii) and taking  $\tau = \sum_{i=1}^k \lambda_i(1 + \beta_i) + \sum_{j=1}^m \mu_j$ , so  $\tau > 0$ , divide the equation by  $\tau$

and using convexity of  $\mathcal{C}$  we obtain

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i^1(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i))) \\
& \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \alpha_j^2(x, u)} (-(h_j(u) + u^T w_j)) \\
& \geq \mathcal{C}_{(x, u)} \left[ \frac{1}{\tau} \sum_{i=1}^k \lambda_i (\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p) - \beta_i \nabla(g_i(u) - u^T v_i) - \beta_i \nabla_p G_i(u, p)) \right. \\
& \quad \left. + \frac{1}{\tau} \sum_{j=1}^m \mu_j (\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)) \right] \\
& \quad + \sum_{i=1}^k \frac{\lambda_i}{\tau \gamma_i^1(x, u)} (K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r))) \\
& \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \gamma_j^2(x, u)} (H_j(u, s) - s^T \nabla_s H_j(u, s)). \tag{2.5.12}
\end{aligned}$$

Now using (2.5.1), hypothesis (iv) and the assumption that  $\mathcal{C}_{(x, u)}(0) = 0$ , the previous inequality yields

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\tau a(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \\
& \geq \frac{1}{\tau b(x, u)} \sum_{j=1}^m \mu_j (H_j(u, s) - s^T \nabla_s H_j(u, s)) + \frac{1}{\tau b(x, u)} \sum_{j=1}^m \mu_j ((h_j(u) + u^T w_j)) \\
& \quad + \frac{1}{\tau a(x, u)} \sum_{i=1}^k \lambda_i (K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r))) \\
& \quad + \frac{1}{\tau a(x, u)} \sum_{i=1}^k \lambda_i (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)). \tag{2.5.13}
\end{aligned}$$

Finally using the dual feasibility conditions (2.5.2) and (2.5.3), the equation (2.5.13) becomes

$$\sum_{i=1}^k \frac{\lambda_i}{a(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \geq 0,$$

which contradicts (2.5.6). Hence the result.

**Theorem 2.5.2 (Weak Duality Theorem)** *Let  $x$  be a feasible solution of primal (NMFP) and  $(u, \beta, \lambda, \mu, z, v, w, p, q, r, s)$  be feasible solution of dual (NMFD). Suppose that*

(i)  $(f(\cdot) + (\cdot)^T z - \beta(g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$  is higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $(K - \beta G), p, r$  and  $H, q, s$  respectively

$$(ii) \sum_{i=1}^k \frac{\lambda_i \rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} + \sum_{j=1}^m \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \geq 0$$

$$(iii) \alpha_i^1(x, u) = \gamma_i^1(x, u) = a(x, u) \text{ and } \alpha_j^2(x, u) = \gamma_j^2(x, u) = b(x, u).$$

Then the following can not hold

$$\begin{aligned} \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} &\leq \beta_i, & \text{for all } i = 1, 2, \dots, k, \\ \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} &< \beta_r, & \text{for some } r = 1, 2, \dots, k. \end{aligned} \quad (2.5.14)$$

**Proof.** It follows on the lines of Theorem 2.5.1.

**Theorem 2.5.3 (Strong Duality Theorem)** Let  $u$  be an efficient solution of primal problem (NMFP). Assume that Slater's constraint qualification holds on  $X$ . If

$$K_i(u, 0) = \nabla_p K_i(u, 0) = \nabla_r K_i(u, 0) = 0,$$

$$G_i(u, 0) = \nabla_p G_i(u, 0) = \nabla_r G_i(u, 0) = 0,$$

$$H_j(u, 0) = \nabla_s H_j(u, 0) = \nabla_q H_j(u, 0) = 0.$$

Then  $\exists \lambda > 0, \lambda \in \mathbb{R}^k, \mu \in \mathbb{R}^m, \mu \geq 0, \beta \geq 0, \beta \in \mathbb{R}^k, z \in \mathbb{R}^n, v \in \mathbb{R}^n, w \in \mathbb{R}^n$ , such that  $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$  satisfies dual feasibility conditions, i.e., this is feasible solution of (NMFD). Also the objective function values of (NMFP) and (NMFD) are equal. In addition to this, if the conditions of weak duality theorem hold for each feasible solutions (NMFP) and each feasible solution  $(u', \beta', \lambda', \mu' z', v', w', p', q', r', s')$  of (NMFD) then  $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$  is an efficient solution of (NMFD).

**Proof.** Given that  $u$  is an efficient solution of (NMFP) and Slater's constraint qualification holds then the Theorem 2.4.2 implies that  $\exists \lambda \in \mathbb{R}^k, \mu \in \mathbb{R}^m, z_i \in \mathbb{R}^n, v_i \in \mathbb{R}^n, w_j \in \mathbb{R}^n$ , such that

$$\sum_{i=1}^k \lambda_i (\nabla(f_i(u) + u^T z_i) - \beta_i \nabla(g_i(u) - u^T v_i)) + \sum_{j=1}^m \mu_j \nabla(h_j(u) + u^T w_j) = 0, \quad (2.5.15)$$

$$f_i(u) + u^T z_i - \beta_i (g_i(u) - u^T v_i) = 0, \quad (2.5.16)$$

$$\sum_{j=1}^m \mu_j (h_j(u) + u^T w_j) = 0, \quad (2.5.17)$$

$$\lambda_i > 0, \mu_j \geq 0, \text{ for all } i = 1, 2, \dots, k, \quad j = 1, 2, \dots, m, \quad (2.5.18)$$

$$\begin{aligned}
u^T z_i &= S(u|C_i), \quad u^T v_i = S(u|D_i), \quad u^T w_j = S(u|E_j), \\
z_i &\in C_i, \quad v_i \in D_i, \quad w_j \in E_j.
\end{aligned} \tag{2.5.19}$$

So we have  $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$  as a feasible solution of (NMFD). Due to (2.5.16) and (2.5.19) the objective values of both programs are equal.

Now we claim that  $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$  is an efficient solution of (NMFD).

If not, then there exist a feasible solution of dual say,  $(u', \beta', \lambda', \mu', z', v', w', p', q', r', s')$  such that,

$$\beta \leq \beta'.$$

But this contradicts weak duality theorem. Hence  $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$  is an efficient solution of (NMFD).

**Theorem 2.5.4 (Strict Converse Duality Theorem)** *Let  $x$  be an efficient solution of (NMFP) and  $(u, \beta, \lambda, \mu, z, v, w, p, q, r, s)$  be an efficient solution of (NMFD). Further assume that:*

- (i)  $(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$  is higher-order strictly  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $K, p, r$  and  $H, q, s$  respectively,
- (ii)  $(-g(\cdot) - (\cdot)^T v, h(\cdot) + (\cdot)^T w)$  is higher-order strictly  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $-G, p, r$  and  $H, q, s$  respectively,
- (iii)  $\sum_{i=1}^k \frac{\lambda_i \rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} (1 + \beta_i) + \sum_{j=1}^m \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \geq 0$ ,
- (iv)  $\alpha_i^1(x, u) = \gamma_i^1(x, u) = a(x, u)$  for  $i = 1, 2, \dots, k$  and  $\alpha_j^2(x, u) = \gamma_j^2(x, u) = b(x, u)$ , for  $j = 1, 2, \dots, m$
- (v)  $\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \beta_i$ .

Then  $x = u$ .

**Proof.** Suppose  $x \neq u$ . Then due to hypotheses (i) and (ii) we have

$$\begin{aligned}
\frac{f_i(x) + x^T z_i - (f_i(u) + u^T z_i)}{\alpha_i^1(x, u)} &> \mathcal{C}_{(x, u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \\
&\quad \frac{1}{\gamma_i^1(x, u)} (K_i(u, r) - r^T \nabla_r K_i(u, r)) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} \tag{2.5.20}
\end{aligned}$$

$$\frac{-(g_i(x) - x^T v_i - (g_i(u) - u^T v_i))}{\alpha_i^1(x, u)} > \mathcal{C}_{(x, u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)) +$$

$$\frac{1}{\gamma_i^1(x, u)}(-(G_i(u, r) - r^T \nabla_r G_i(u, r))) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} \quad (2.5.21)$$

$$\begin{aligned} \frac{-(h_j(u) + u^T w_j)}{\alpha_j^2(x, u)} &> \mathfrak{C}_{(x, u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)) + \\ &\quad \frac{1}{\gamma_j^2(x, u)}(H_j(u, s) - s^T \nabla_s H_j(u, s)) + \frac{\rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \end{aligned} \quad (2.5.22)$$

Now we proceed with the similar manner as in Theorem 2.5.1 and we are left with

$$\sum_{i=1}^k \frac{\lambda_i}{a(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) > 0. \quad (2.5.23)$$

It follows from hypothesis (v) that

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \beta_i,$$

which further imply that

$$\sum_{i=1}^k \lambda_i(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \leq 0, \quad \text{for } \lambda_i > 0 \quad (2.5.24)$$

contradicting (2.5.23). Hence  $x = u$ .



# Chapter 3

## Higher-Order Duality for Fractional Programming Problems over Cones

### 3.1 Introduction

This Chapter deals with a dual model formulation for non-differentiable vector programming problems over arbitrary cones. As discussed in Section 1.6 the introductory theory for convex mathematical programs over cones is developed in [45, 46, 212]. Giorgi and Guerraggio [70] defined a class of nonsmooth nonconvex functions and used them to obtain optimality and duality results for mathematical programs over cones. Reiland [179] defined  $\mathcal{K}$ -invex functions for a closed convex cone  $\mathcal{K}$  and obtained duality relations for a nondifferentiable mathematical program with cone constraints. Khurana [126] defined  $\mathcal{K}$ -pseudo-invex and  $\mathcal{K}$ -quasi-invex functions and obtained duality results for symmetric type mathematical programming problem in which ordering is defined with respect to a convex cone  $\mathcal{K}$ .

Suneja et al. [199] defined generalized nonsmooth invex functions over cones. Using the properties of these functions, they gave various optimality results for optimization problems involving nonsmooth functions and studied duality results for Mond-Weir type dual model. Then Suneja et al. [198] further introduced  $\mathcal{K}$ -generalized invex functions,  $\mathcal{K} \times \mathcal{Q}$ -generalized type-I functions and their extensions to explore the duality and optimality for the same problem.

Suneja et al. [205] introduced second order cone convex functions and their generalizations. They gave Mond-Weir dual for vector optimization problems over cones. Bhatia [25] defined higher-order cone convex, pseudo-convex and quasi-convex functions and associated a higher-order dual program to vector optimization problem over cones. Suneja et al. [202] defined higher-order strongly  $\mathcal{K}$ -pseudoconvex, higher-order naturally  $\mathcal{K}$ -pseudoconvex and several other generalizations to study relationship among different classes of cone convex functions. They also considered a generalized Mond Weir type dual for a vector program over cones and obtained duality relations using various cone convex functions.

Louhan and Suneja [151] studied a fractional vector program over cones having same denominator in each component and gave Mond-Weir and Schaible type dual models for frac-

tional programming problem over cones and studied duality results. Grover and Kapoor [75] defined higher-order  $(\mathcal{F}, \rho, \sigma)$  type-I functions over cones. They considered vector optimization problems in which objective functions and constraints involve support function of convex compact sets and obtained duality theorems for a higher-order dual making use of aforesaid functions. Kapoor et al. [197] studied optimality and duality results for fractional programs using  $\mathcal{K}$ -convex functions. Debnath and Gupta [54] considered a more general fractional problem to study its parametric and non-parametric dual models. Their duality results were attained using higher-order  $(\mathcal{K} \times \mathcal{Q}) - \mathcal{F}$  type-I functions.

In this Chapter an attempt is made to further extend the scope of duality for non-differentiable fractional problems over cones. We introduce a generalization of higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions, named as higher-order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions. An example of such functions, which lie in this class only, has also been identified. Higher order Schaible type dual for a non-differentiable fractional vector program over cones is formulated. In this program each component of objective function and constraints contain support functions of compact convex sets. Further, we formulated Higher order Schaible type dual model and establish weak and strong duality theorems using higher-order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions.

This Chapter is organized as follows. In Section 3.2, we give preliminaries and the definitions. Section 3.3 constitutes problem formulation and Section 3.4 contains the dual model and duality theorems.

## 3.2 Preliminaries and Definitions

Consider  $X$  as a non-empty open subset of  $\mathbb{R}^n$ . Also consider  $\mathcal{K}$  and  $\mathcal{Q}$  to be closed convex pointed cones in  $\mathbb{R}^k$  and  $\mathbb{R}^m$  respectively such that  $\text{int } \mathcal{K} \neq \emptyset$ ,  $\text{int } \mathcal{Q} \neq \emptyset$  with non-empty interiors. Note that for cone  $\mathcal{K}$ ,  $a \in \mathcal{K} \Leftrightarrow b^T a > 0$ ,  $\forall b \in \mathcal{K}^* \setminus \{0\}$ , where  $\mathcal{K}^*$  denote the dual cone of  $\mathcal{K}$  (as defined in Chapter 1). Now we will define  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions.

**Definition 3.2.1** *We assume that the function  $\mathcal{C}$  defined previously satisfies  $\mathcal{C}_{(x,u)}(0) = 0$ ,  $\forall (x, u) \in X \times X$ . Let us assume function  $\phi : X \rightarrow \mathbb{R}^k$  to be differentiable on  $X$  and the function  $H(u, p) = (H_1(u, p), H_2(u, p), \dots, H_k(u, p))$  defined by  $H : X \times \mathbb{R}^n \rightarrow \mathbb{R}^k$ , is a differentiable function on  $X \times \mathbb{R}^n$ . Further assume functions  $\alpha(x, u) = (\alpha_1(x, u), \alpha_2(x, u), \dots, \alpha_k(x, u))$  and  $\gamma(x, u) = (\gamma_1(x, u), \gamma_2(x, u), \dots, \gamma_k(x, u))$  such that each  $\alpha_i, \gamma_i : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}$ . Further take  $d(x, u) = (d_1(x, u), d_2(x, u), \dots, d_k(x, u))$  such that  $d_i : X \times X \rightarrow \mathbb{R}_+$  satisfying the property that  $d_i(x, u) = 0$  if and only if  $x = u, \forall (x, u) \in X \times X$  for  $i = 1, 2, \dots, k$  and  $p, s \in \mathbb{R}^n$ . A function  $\phi$  is said to be higher-order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$*

convex at  $u$  with respect to  $H, p$  and  $s$  if for each  $x \in X$ ,

$$\begin{pmatrix} \frac{1}{\alpha_1(x, u)}[\phi_1(x) - \phi_1(u)] - \mathcal{C}_{(x, u)}(\nabla \phi_1(u) + \nabla_p H_1(u, p)) - \frac{1}{\gamma_1(x, u)}[H_1(u, s) \\ - s^T \nabla_s H_1(u, s)] - \frac{\rho_1 d_1(x, u)}{\alpha_1(x, u)} \\ \frac{1}{\alpha_2(x, u)}[\phi_2(x) - \phi_2(u)] - \mathcal{C}_{(x, u)}(\nabla \phi_2(u) + \nabla_p H_2(u, p)) - \frac{1}{\gamma_2(x, u)}[H_2(u, s) \\ - s^T \nabla_s H_2(u, s)] - \frac{\rho_2 d_2(x, u)}{\alpha_2(x, u)} \\ \text{---} \\ \text{---} \\ \frac{1}{\alpha_k(x, u)}[\phi_k(x) - \phi_k(u)] - \mathcal{C}_{(x, u)}(\nabla \phi_k(u) + \nabla_p H_k(u, p)) - \frac{1}{\gamma_k(x, u)}[H_k(u, s) \\ - s^T \nabla_s H_k(u, s)] - \frac{\rho_k d_k(x, u)}{\alpha_k(x, u)} \end{pmatrix} \in \mathcal{K}$$

### Special Cases:

1. If each  $\alpha_i(x, u) = 1 = \gamma_i(x, u)$ ,  $\mathcal{C}_{x,u}(\cdot) = \mathcal{F}_{x,u}(\cdot)$  where  $\mathcal{F}$  is a sublinear functional and  $\rho_i = 0$  then we have the first type of functions defined in [54] (see, first inequality of Definition 2.4, [54]).
2. If  $k = m$ ,  $H(u, p) = \frac{1}{2}p^T \nabla^2 f(u)p$  in our definition along with  $\alpha_i(x, u) = 1 = \gamma_i(x, u)$ ,  $\rho_i = 0$  and  $\mathcal{C}_{x,u}(\cdot) = (x - u)^T(\cdot)$  then we have definition of second order  $\mathcal{K}$ -convex functions given in [205].
3. In addition to case (2.) if  $p = 0$  we get  $\mathcal{K}$ -convex functions defined in [212].
4. If  $k = 1$  and  $\mathcal{K} = \mathbb{R}^+$  then we get higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions as defined in [61].
5. Along with  $p = 0$  the case (4.) reduce to  $(\mathcal{C}, \alpha, \rho, d)$  convexity as defined in [221].
6. In addition to (5.) if  $\mathcal{C}$  is sublinear we get  $(\mathcal{F}, \alpha, \rho, d)$  convex functions defined in [145].

**Example 3.2.1** We consider a convex pointed cone  $\mathcal{K} = \{(x, y) : x \leq 0, y \geq x\}$ . Let us suppose a subset  $X \subseteq \mathbb{R}^1$  such that  $X = [0, 1.8)$  and a function  $\phi : X \rightarrow \mathbb{R}^2$  defined as

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} x^2 - e^{-2x} \\ 2 + \sin(x) - x^2 \end{pmatrix}$$

Assume that  $\mathcal{C} : X \times X \times \mathbb{R}^1 \rightarrow \mathbb{R}$  and  $H : X \times \mathbb{R}^1 \rightarrow \mathbb{R}^2$  given by

$$\mathcal{C}_{(x,u)}(a) = a^4(x-u)^2, \quad H_1(u, p) = u^2(p-1), \quad H_2(u, p) = p \sin(u)$$

. Further  $\alpha_i, \gamma_i : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}$ ,  $i = 1, 2$   $\rho = (\rho_1, \rho_2)$ ,  $p, s \in \mathbb{R}$  where

$$\alpha = \left( \frac{1}{x^2 + u^2 + 0.01}, 10 \right), \quad \gamma = (1, 1), \quad p = 1, s = 1, \quad \rho = \left( \frac{1}{2}, \frac{1}{2} \right).$$

Now  $d(x, u) = ((x-u)^2, (x-u)^2)$ . Now at  $u = 0$

$$\begin{aligned} \Psi 1 &= \left( \frac{1}{\alpha_1(x, u)}(\phi_1(x) - \phi_1(u)) - \mathcal{C}_{(x,u)}(\nabla \phi_1(u) + \nabla_p H_1(u, p)) - \frac{1}{\gamma_1(x, u)}(H_1(u, s) \right. \\ &\quad \left. - s^T \nabla_s H_1(u, s)) - \frac{\rho_1 d_1(x, u)}{\alpha_1(x, u)}, \frac{1}{\alpha_2(x, u)}(\phi_2(x) - \phi_2(u)) - \mathcal{C}_{(x,u)}(\nabla \phi_2(u) + \nabla_p H_2(u, p)) \right. \\ &\quad \left. - \frac{1}{\gamma_2(x, u)}(H_2(u, s) - s^T \nabla_s H_2(u, s)) - \frac{\rho_2 d_2(x, u)}{\alpha_2(x, u)} \right) \\ &= \left( \left( \frac{x^4}{2} - x^2 e^{-2x} - 14.995x^2 - 0.01e^{-2x} + 0.01 \right), \frac{1}{10}(\sin(x) - 11.5x^2) \right) \in \mathcal{K} \end{aligned}$$

So we get an example of higher-order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions.

But  $\Psi 1 \not\geq 0$ , for every  $x \in X$ . say at  $u = 1$ ,  $\Psi 1 = (-14.6217, -1.0659)$ . So this function  $\phi$  is not in the class of higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  convex functions (which are defined in [61]). It can also be verified that the function does not lie in the class of higher-order  $\mathcal{K}$ -convex functions defined in [25] and  $\mathcal{K}$ -convex functions defined in [212] respectively as shown below.

If  $\phi$  is higher-order higher-order  $\mathcal{K}$ -convex functions then the following should hold:

$$\begin{aligned} \Psi_B &= \left( \phi_1(x) - \phi_1(u) - (x-u)^T(\nabla \phi_1(u) + \nabla_p H_1(u, p)) - H_1(u, p) + p^T \nabla_p H_1(u, p), \right. \\ &\quad \left. \phi_2(x) - \phi_2(u) - (x-u)^T(\nabla \phi_2(u) + \nabla_p H_2(u, p)) - H_2(u, p) + p^T \nabla_p H_2(u, p) \right) \in \mathcal{K}. \end{aligned}$$

For this example  $\Psi_B = (x^2 - 2x - e^{-2x} + 1, \sin x - x^2 - x)$ . Putting  $x = 1$  we get  $\Psi_B = (-0.13, -1.15)$  which again does not lie in  $\mathcal{K}$ . Similarly for function  $\phi$  to lie in the class of  $\mathcal{K}$  convex functions following has to be satisfied.

$$\Psi_W = (\phi_1(x) - \phi_1(u) - (x-u)^T \nabla \phi_1(u), \phi_2(x) - \phi_2(u) - (x-u)^T \nabla \phi_2(u)) \in \mathcal{K}.$$

But  $\Psi_W = (x^2 - 2x - e^{-2x} + 1, \sin x - x^2 - x) = (-0.13, -1.15)$  at  $x = 1$  which does not lie in  $\mathcal{K}$ .

So we got that there exist functions lying exclusively in the class of  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$ -convex functions and not in the previous classes.

Now we define another class of functions namely higher-order  $(\mathcal{K} \times \mathcal{Q}) - (\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions. This class of functions is an immediate extension of the higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions defined in Chapter 2. Consider differentiable functions  $\phi$  and  $h$  as  $\phi : X \rightarrow \mathbb{R}^k$  and  $h : X \rightarrow \mathbb{R}^m$ . Now consider higher-order approximations of  $\phi$  and  $h$  to be  $F(u, p) = (F_1(u, p), F_2(u, p), \dots, F_k(u, p))$  and  $H(u, p) = (H_1(u, p), H_2(u, p), \dots, H_m(u, p))$  respectively such that  $F_i, H_j : X \times \mathbb{R}^n \rightarrow \mathbb{R}$  are differentiable functions on  $X \times \mathbb{R}^n$ .  $\alpha = (\alpha^1, \alpha^2) = (\alpha_1^1, \dots, \alpha_k^1, \alpha_1^2, \dots, \alpha_m^2)$ ,  $\gamma = (\gamma^1, \gamma^2) = (\gamma_1^1, \dots, \gamma_k^1, \gamma_1^2, \dots, \gamma_m^2)$  such that  $\alpha_i^1, \gamma_i^1, \alpha_j^2, \gamma_j^2 : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}$ ,  $d = (d^1, d^2) = (d_1^1, \dots, d_k^1, d_1^2, \dots, d_m^2)$  such that  $d_i^1, d_j^2 : X \times X \rightarrow \mathbb{R}_+$  satisfying  $d(x, x_0) = 0 \Leftrightarrow x = x_0$ ,  $\rho = (\rho^1, \rho^2) = (\rho_1^1, \dots, \rho_k^1, \rho_1^2, \dots, \rho_m^2)$  and  $p, q, r, s \in \mathbb{R}^n, i = 1, 2, \dots, k, j = 1, 2, \dots, m$ .

**Definition 3.2.2** The function  $(\phi, h)$  is said to be higher-order  $(\mathcal{K} \times \mathcal{Q}) - (\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $F, p, r$  and  $H, q, s$  respectively, if for each  $x \in X$ ,

$$\begin{pmatrix} \frac{1}{\alpha_1^1(x, u)} [\phi_1(x) - \phi_1(u)] - \mathcal{C}_{(x, u)}(\nabla \phi_1(u) + \nabla_p F_1(u, p)) - \frac{1}{\gamma_1^1(x, u)} [F_1(u, s) - s^T \nabla_s F_1(u, s)] - \frac{\rho_1^1 d_1^1(x, u)}{\alpha_1^1(x, u)} \\ \frac{1}{\alpha_2^1(x, u)} [\phi_2(x) - \phi_2(u)] - \mathcal{C}_{(x, u)}(\nabla \phi_2(u) + \nabla_p F_2(u, p)) - \frac{1}{\gamma_2^1(x, u)} [F_2(u, s) - s^T \nabla_s F_2(u, s)] - \frac{\rho_2^1 d_2^1(x, u)}{\alpha_2^1(x, u)} \\ \dots \\ \frac{1}{\alpha_k^1(x, u)} [\phi_k(x) - \phi_k(u)] - \mathcal{C}_{(x, u)}(\nabla \phi_k(u) + \nabla_p F_k(u, p)) - \frac{1}{\gamma_k^1(x, u)} [F_k(u, s) - s^T \nabla_s F_k(u, s)] - \frac{\rho_k^1 d_k^1(x, u)}{\alpha_k^1(x, u)} \end{pmatrix} \in \mathcal{K}$$

and

$$\begin{pmatrix} \frac{1}{\alpha_1^2(x, u)} [-h_1(u)] - \mathcal{C}_{(x, u)}(\nabla(h_1(u)) + \nabla_q H_1(u, q)) - \frac{1}{\gamma_1^2(x, u)} [H_1(u, s) - s^T \nabla_s H_1(u, s)] - \frac{\rho_1^2 d_1^2(x, u)}{\alpha_1^2(x, u)} \\ \frac{1}{\alpha_2^2(x, u)} [-h_2(u)] - \mathcal{C}_{(x, u)}(\nabla(h_2(u)) + \nabla_q H_2(u, q)) - \frac{1}{\gamma_2^2(x, u)} [H_2(u, s) - s^T \nabla_s H_2(u, s)] - \frac{\rho_2^2 d_2^2(x, u)}{\alpha_2^2(x, u)} \\ \dots \\ \frac{1}{\alpha_m^2(x, u)} [-h_m(u)] - \mathcal{C}_{(x, u)}(\nabla(h_m(u)) + \nabla_q H_m(u, q)) - \frac{1}{\gamma_m^2(x, u)} [H_m(u, s) - s^T \nabla_s H_m(u, s)] - \frac{\rho_m^2 d_m^2(x, u)}{\alpha_m^2(x, u)} \end{pmatrix} \in \mathcal{Q}$$

**Remark 3.2.1** In this definition if  $\mathcal{K} = \mathbb{R}_+^k$  and  $\mathcal{Q} = \mathbb{R}_+^m$  then these functions will reduce to the higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I functions introduced in Chapter 2.

### 3.3 Problem Formulation

The problem of interest in this Chapter is a nondifferentiable vector fractional problem (NVFP) and is given as:

$$\begin{aligned} \text{Primal (NVFP)} \quad & \mathcal{K} - \text{Min } \overline{f(x)} = \left( \frac{f_1(x) + S(x | B_1)}{g_1(x) - S(x | D_1)}, \dots, \frac{f_k(x) + S(x | B_k)}{g_k(x) - S(x | D_k)} \right) \\ & \text{subject to } -(h(x) + E(x)) \in \mathcal{Q}, x \in X \end{aligned}$$

where  $h(x) + E(x) = (h_1(x) + S(x | E_1), h_2(x) + S(x | E_2), \dots, h_m(x) + S(x | E_m))$ .

The functions  $f, g : X \rightarrow \mathbb{R}^k$  and  $h : X \rightarrow \mathbb{R}^m$  are continuously differentiable functions on  $X$ .

The non-differentiability of this problem is due to support functions  $S(x|B_i)$ ,  $S(x|D_i)$ ,  $S(x|E_j)$  which are support functions of the sets  $B_i, D_i$  and  $E_j$  respectively, for  $i = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, m$ . These  $B_i, D_i$  and  $E_j$  are some compact convex sets in  $\mathbb{R}^n$ . It is assumed that  $f_i(\cdot) + S(\cdot|B_i) \geq 0$  and  $g_i(\cdot) - S(\cdot|D_i) > 0$  in  $X$  for each  $i = 1, 2, \dots, k$ . So the feasible region is;  $X_{VP} := \{x \in X | -(h(x) + E(x)) \in \mathcal{Q}\}$ .

**Definition 3.3.1**  $u \in X_{VP}$  is called an efficient solution of (NVFP) if there is no  $x \in X_{VP}$  such that

$$\overline{f(u)} - \overline{f(x)} \in \mathcal{K} \setminus \{0\}.$$

**Definition 3.3.2**  $u \in X_{VP}$  is called a weakly efficient solution of (NVFP) if there is no  $x \in X_{VP}$  such that

$$\overline{f(u)} - \overline{f(x)} \in \text{int}\mathcal{K}.$$

We use the following notations in this Chapter

$$(1) \quad -\beta(g(\cdot) - S(\cdot|D)) = (-\beta_1(g_1(\cdot) - S(\cdot|D_1)), -\beta_2(g_2(\cdot) - S(\cdot|D_2)), \dots, -\beta_k(g_k(\cdot) - S(\cdot|D_k))).$$

$$(2) \quad -\beta(g(\cdot) - (\cdot)^T v) = (-\beta_1(g_1(\cdot) - (\cdot)^T v_1), -\beta_2(g_2(\cdot) - (\cdot)^T v_2), \dots, -\beta_k(g_k(\cdot) - (\cdot)^T v_k)),$$

each  $v_i \in \mathbb{R}^n$

### 3.4 Schaible Dual

The higher-order Schaible dual for (NVFP) is

Dual (NVFD)

$$\mathcal{K}\text{-Maximize } \beta = (\beta_1, \beta_2, \dots, \beta_k)$$

subject to

$$\begin{aligned} \sum_{i=1}^k \lambda_i [\nabla(f_i(u) + u^T z_i) - \beta_i \nabla(g_i(u) - u^T v_i)] + \sum_{j=1}^m \mu_j \nabla(h_j(u) + u^T w_j) \\ + \sum_{i=1}^k \lambda_i \nabla_p(F_i(u, p) - \beta_i G_i(u, p)) + \sum_{j=1}^m \mu_j \nabla_q H_j(u, q) = 0. \end{aligned} \quad (3.4.1)$$

$$\begin{aligned} \sum_{i=1}^k \lambda_i (f_i(u) + u^T z_i - \beta_i \nabla(g_i(u) - u^T v_i) + F_i(u, r) - \beta_i G_i(u, r) \\ - r^T (\nabla_r(F_i(u, r) - \beta_i G_i(u, r)))) \geq 0. \end{aligned} \quad (3.4.2)$$

$$\sum_{j=1}^m \mu_j (h_j(u) + u^T w_j + H_j(u, s) - s^T \nabla_s H_j(u, s)) \geq 0. \quad (3.4.3)$$

$$\lambda \in \text{int}\mathcal{K}^*, \mu \in \mathcal{Q}^*, \text{ and } \beta \in \mathbb{R}_+^k,$$

$$z_i \in B_i, \quad v_i \in D_i, \quad w_j \in E_j, \quad i = 1, 2, \dots, k, \text{ and } j = 1, 2, \dots, m. \quad (3.4.4)$$

To validate the existence this dual weak and strong duality theorems are proved below. Weak duality theorem gives us a bound to the value of primal problem whereas strong duality theorem reduces this duality gap to zero so that the objective values of primal and dual are same.

**Theorem 3.4.1 (Weak Duality Theorem)** *Let  $x$  and  $(u, \beta, z, v, \mu, \lambda, w, p, q, r, s)$  be feasible solutions of primal (NVFP) and its proposed dual (NVFD), respectively. Suppose that*

(i)  $(f(\cdot) + (\cdot)^T z)$  is higher-order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex at  $u$  with respect to  $F, p$  and  $r$  and  $-\beta(g(\cdot) - (\cdot)^T v)$  is higher-order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex at  $u$  with respect to  $- \beta G, p$  and  $r$ ,

(ii)  $(h(\cdot) + (\cdot)^T w)$  is higher-order  $\mathcal{Q} - (\mathcal{C}, \xi, \delta, \eta, c)$ -convex at  $u$  with respect to  $H, q$  and  $s$ ,

$$(iii) \sum_{i=1}^k \frac{2\lambda_i \rho_i d_i(x, u)}{\alpha_i(x, u)} + \sum_{j=1}^m \frac{\mu_j \eta_j c_j(x, u)}{\xi_j(x, u)} \geq 0,$$

(iv)  $a(x, u) = \alpha_i(x, u) = \gamma_i(x, u)$  for  $i = 1, 2, \dots, k$   
and  $b(x, u) = \delta_j(x, u) = \xi_j(x, u)$  for  $j = 1, 2, \dots, m$ ,

(v)  $\mathbb{R}_+^k \subseteq \mathcal{K}$  &  $\mathbb{R}_+^m \subseteq \mathcal{Q}$ . And for any element  $\bar{r} \in \text{int}\mathbb{R}_+^k$  and  $\bar{k} \in \mathcal{K} \setminus \{0\}$ ,  $\bar{r}\bar{k} \in \mathcal{K} \setminus \{0\}$ .

Then the following can not hold

$$(\beta - \overline{f(x)}) \in \mathcal{K} \setminus \{0\}. \quad (3.4.5)$$

**Proof.** Suppose (3.4.5) holds.

$$\text{Then } (\beta - \overline{f(x)}) \in \mathcal{K} \setminus \{0\} \Leftrightarrow (\beta - \frac{f(x) + S(x|B)}{g(x) - S(x|D)}) \in \mathcal{K} \setminus \{0\}$$

$$\Leftrightarrow (\frac{\beta(g(x) - S(x|D)) - f(x) - S(x|B)}{g(x) - S(x|D)}) \in \mathcal{K} \setminus \{0\},$$

we get the following, due to (v),

$$g(x) - S(x|D) (\frac{\beta(g(x) - S(x|D)) - f(x) - S(x|B)}{g(x) - S(x|D)}) \in \mathcal{K} \setminus \{0\}.$$

Now the definition of dual cone implies there exist some  $\lambda \in \text{int}\mathcal{K}^*$  such that

$$\begin{aligned} & \lambda^T (\beta(g(x) - S(x|D)) - f(x) - S(x|B)) > 0. \\ & \Leftrightarrow \sum_{i=1}^k \lambda_i \beta_i (g_i(x) - S(x|D_i)) > \sum_{i=1}^k \lambda_i (f_i(x) + S(x|B_i)). \end{aligned}$$

Now as  $\lambda \in \text{int}\mathcal{K}^*$  and  $\mathbb{R}_+^k \subseteq \mathcal{K}$  we get  $\lambda > 0, \beta \in \mathbb{R}_+^k$ , as well as the fact that  $x^T z_i \leq S(x|B_i)$  and  $x^T v_i \leq S(x|D_i)$  we get

$$\sum_{i=1}^k \lambda_i \beta_i (g_i(x) - x^T v_i) > \sum_{i=1}^k \lambda_i (f_i(x) + x^T z_i).$$

which is equivalent to

$$\sum_{i=1}^k \frac{\lambda_i}{a(x, u)} (f_i(x) + x^T z_i - \beta_i (g_i(x) - x^T v_i)) < 0, \text{ for some } a(x, u) \in \mathbb{R}_+. \quad (3.4.6)$$

$f(\cdot) + (\cdot)^T z$  is higher-order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex at  $u$  with respect to  $F, p, r$  and  $-\beta(g(\cdot) - (\cdot)^T v)$  is higher-order  $\mathcal{K} - (\mathcal{C}, \alpha, \gamma, \rho, d)$  convex at  $u$  with respect to  $-\beta G, p, r$ .

So

$$\left( \begin{array}{l} \frac{1}{\alpha_1(x, u)}[f_1(x) + x^T z_1 - f_1(u) - u^T z_1] - \mathcal{C}_{(x, u)}(\nabla(f_1(u) + u^T z_1) + \nabla_p F_1(u, p)) \\ \quad - \frac{1}{\gamma_1(x, u)}[F_1(u, r) - r^T \nabla_r F_1(u, r)] - \frac{\rho_1 d_1(x, u)}{\alpha_1(x, u)} \\ \frac{1}{\alpha_2(x, u)}[f_2(x) + x^T z_2 - f_2(u) - u^T z_2] - \mathcal{C}_{(x, u)}(\nabla(f_2(u) + u^T z_2) + \nabla_p F_2(u, p)) \\ \quad - \frac{1}{\gamma_2(x, u)}[F_2(u, r) - r^T \nabla_r F_2(u, r)] - \frac{\rho_2 d_2(x, u)}{\alpha_2(x, u)} \\ \quad \quad \quad - - - \\ \frac{1}{\alpha_k(x, u)}[f_k(x) + x^T z_k - f_k(u) - u^T z_k] - \mathcal{C}_{(x, u)}(\nabla(f_k(u) + u^T z_k) + \nabla_p F_k(u, p)) \\ \quad - \frac{1}{\gamma_k(x, u)}[F_k(u, r) - r^T \nabla_r F_k(u, r)] - \frac{\rho_k d_k(x, u)}{\alpha_k(x, u)} \end{array} \right) \in \mathcal{K}$$

$$\left( \begin{array}{l} \frac{1}{\alpha_1(x, u)}[-\beta_1(g_1(x) - x^T v_1 - g_1(u) + u^T v_1)] - \mathcal{C}_{(x, u)}(\nabla(-\beta_1(g_1(u) - u^T v_1)) \\ \quad - \beta_1 \nabla_p G_1(u, p)) + \frac{\beta_1}{\gamma_1(x, u)}[G_1(u, r) - r^T \nabla_r G_1(u, r)] - \frac{\rho_1 d_1(x, u)}{\alpha_1(x, u)} \\ \frac{1}{\alpha_2(x, u)}[-\beta_2(g_2(x) - x^T v_2 - g_2(u) + u^T v_2)] - \mathcal{C}_{(x, u)}(\nabla(-\beta_2(g_2(u) - u^T v_2)) \\ \quad - \beta_2 \nabla_p G_2(u, p)) + \frac{\beta_2}{\gamma_2(x, u)}[G_2(u, r) - r^T \nabla_r G_2(u, r)] - \frac{\rho_2 d_2(x, u)}{\alpha_2(x, u)} \\ \quad \quad \quad - - - \\ \frac{1}{\alpha_k(x, u)}[-\beta_k(g_k(x) - x^T v_k - g_k(u) + u^T v_k)] - \mathcal{C}_{(x, u)}(\nabla(-\beta_k(g_k(u) - u^T v_k)) \\ \quad - \beta_k \nabla_p G_k(u, p)) + \frac{\beta_k}{\gamma_k(x, u)}[G_k(u, r) - r^T \nabla_r G_k(u, r)] - \frac{\rho_k d_k(x, u)}{\alpha_k(x, u)} \end{array} \right) \in \mathcal{K}$$

Also  $h(\cdot) + (\cdot)^T w$  is  $\mathcal{Q} - (\mathcal{C}, \xi, \delta, \eta, c)$  convex at  $u$  with respect to  $H, q$ , and  $s$ . Then

$$\left( \begin{array}{l} \frac{1}{\xi_1(x, u)}[h_1(x) + x^T w_1 - h_1(u) - u^T w_1] - \mathcal{C}_{(x, u)}(\nabla(h_1(u) + u^T w_1) + \nabla_q H_1(u, q)) \\ \quad - \frac{1}{\delta_1(x, u)}[H_1(u, s) - s^T \nabla_s H_1(u, s)] - \frac{\eta_1 c_1(x, u)}{\xi_1(x, u)} \\ \frac{1}{\xi_2(x, u)}[h_2(x) + x^T w_2 - h_2(u) - u^T w_2] - \mathcal{C}_{(x, u)}(\nabla(h_2(u) + u^T w_2) + \nabla_q H_2(u, q)) \\ \quad - \frac{1}{\delta_2(x, u)}[H_2(u, s) - s^T \nabla_s H_2(u, s)] - \frac{\eta_2 c_2(x, u)}{\xi_2(x, u)} \\ \quad \quad \quad - - - \\ \frac{1}{\xi_m(x, u)}[h_m(x) + x^T w_m - h_m(u) - u^T w_m] - \mathcal{C}_{(x, u)}(\nabla(h_m(u) + u^T w_m) + \nabla_q H_m(u, q)) \\ \quad - \frac{1}{\delta_m(x, u)}[H_m(u, s) - s^T \nabla_s H_m(u, s)] - \frac{\eta_m c_m(x, u)}{\xi_m(x, u)} \end{array} \right) \in \mathcal{Q}$$

It is given that  $\lambda \in \text{int}\mathcal{K}^*$  and  $\mu \in \mathcal{Q}^*$ , thus one can obtain the following:

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\alpha_i(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i))) \\
& + \sum_{j=1}^m \frac{\mu_j}{\xi_j(x, u)} (h_j(x) + x^T w_j - (h_j(u) + u^T w_j)) - \sum_{i=1}^k \lambda_i [\mathcal{C}_{(x, u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p F_i(u, p))] \\
& - \sum_{i=1}^k \lambda_i [\mathcal{C}_{(x, u)}(-\beta_i(g_i(u) - u^T v_i) - \beta_i \nabla_p G_i(u, p))] - \sum_{i=1}^k \frac{2\lambda_i \rho_i d_i(x, u)}{\alpha_i(x, u)} \\
& - \sum_{i=1}^k \frac{\lambda_i}{\gamma_i(x, u)} (F_i(u, r) - r^T \nabla_r F_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r))) \\
& - \sum_{j=1}^m \mu_j \mathcal{C}_{(x, u)}(\nabla(h_j(u) - u^T w_j) + \nabla_q H_j(u, q)) \\
& - \sum_{j=1}^m \frac{\mu_j}{\delta_j(x, u)} (H_j(u, s) - s^T \nabla_s H_j(u, s)) - \sum_{j=1}^m \frac{\mu_j \eta_j c_j(x, u)}{\xi_j(x, u)} \geq 0
\end{aligned} \tag{3.4.7}$$

Taking  $\tau = \sum_{i=1}^k 2\lambda_i + \sum_{j=1}^m \mu_j$ . So that  $\sum_{i=1}^k \frac{2\lambda_i}{\tau} + \sum_{j=1}^m \frac{\mu_j}{\tau} = 1$  then using convexity of  $\mathcal{C}$  in third argument

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i))) \\
& + \sum_{j=1}^m \frac{\mu_j}{\tau \xi_j(x, u)} (h_j(x) + x^T w_j - (h_j(u) + u^T w_j)) \\
& \geq \mathcal{C}_{(x, u)} \left( \frac{1}{\tau} \sum_{i=1}^k \lambda_i \left( \nabla(f_i(u) + u^T z_i) + \nabla_p F_i(u, p) - \beta_i \nabla(g_i(u) - u^T v_i) - \beta_i \nabla_p G_i(u, p) \right) \right. \\
& \quad \left. + \frac{1}{\tau} \sum_{j=1}^m \mu_j \left( \nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q) \right) \right) \\
& + \sum_{i=1}^k \frac{\lambda_i}{\tau \gamma_i(x, u)} (F_i(u, r) - r^T \nabla_r F_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r))) \\
& + \sum_{j=1}^m \frac{\mu_j}{\tau \delta_j(x, u)} (H_j(u, s) - s^T \nabla_s H_j(u, s)) \\
& + \sum_{i=1}^k \frac{2\lambda_i \rho_i d_i(x, u)}{\tau \alpha_i(x, u)} + \sum_{j=1}^m \frac{\mu_j \eta_j c_j(x, u)}{\tau \xi_j(x, u)}.
\end{aligned} \tag{3.4.8}$$

Using assumption (iii), dual feasibility condition (3.4.1) and property of  $\mathcal{C}$  i.e.,  $\mathcal{C}_{x, u}(0) =$

0,  $\forall x, u \in X$ , we get

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) + \sum_{j=1}^m \frac{\mu_j}{\tau \xi_j(x, u)} (h_j(x) + x^T w_j) \\
& \geq \sum_{i=1}^k \frac{\lambda_i}{\tau \alpha_i(x, u)} (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)) \\
& \quad + \sum_{i=1}^k \frac{\lambda_i}{\tau \gamma_i(x, u)} (F_i(u, r) - r^T \nabla_r F_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r))) \\
& \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \xi_j(x, u)} (h_j(u) + u^T w_j) + \sum_{j=1}^m \frac{\mu_j}{\tau \delta_j(x, u)} (H_j(u, s) - s^T \nabla_s H_j(u, s)). \quad (3.4.9)
\end{aligned}$$

Further, by using the assumption (iv) then dual feasibility conditions (3.4.2) and (3.4.3) and the fact that  $\tau > 0$  we get the following

$$\sum_{i=1}^k \frac{\lambda_i}{a(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) + \sum_{j=1}^m \frac{\mu_j}{b(x, u)} (h_j(x) + x^T w_j) \geq 0. \quad (3.4.10)$$

Using the feasibility of the primal problem, assumption (v) and the definitions of support function the previous equation reduces to

$$\sum_{i=1}^k \frac{\lambda_i}{a(x, u)} (f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \geq 0. \quad (3.4.11)$$

This contradicts (3.4.6).

Hence we have weak duality theorem.

Now we give another weak duality theorem and we use the second type of functions defined in this Chapter to achieve the desired results.

**Theorem 3.4.2 (Weak Duality Theorem)** *Let  $x$  and  $(u, \beta, z, v, \mu, \lambda, w, p, q, r, s)$  be feasible solutions of (NVFP) and (NVFD) respectively. Suppose that*

- (i)  $(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$  is higher-order  $(\mathcal{K} \times \mathcal{Q}) - (\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect to  $F, p, r$  and  $H, q, s$ , respectively
- (ii)  $(-\beta(g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$  is higher-order  $(\mathcal{K} \times \mathcal{Q}) - (\mathcal{C}, \alpha, \gamma, \rho, d)$  type-I at  $u$  with respect  $-\beta G, p, r$  and  $H, q, s$ , respectively,
- (iii)  $\sum_{i=1}^k \frac{2\lambda_i \rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} + \sum_{j=1}^m \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \geq 0,$
- (iv)  $\alpha_i^1(x, u) = \gamma_i^1(x, u) = a(x, u)$  and  $\alpha_j^2(x, u) = \gamma_j^2(x, u) = b(x, u),$

(v)  $\mathbb{R}_+^k \subseteq \mathcal{K}$  &  $\mathbb{R}_+^m \subseteq \mathcal{Q}$ . And for any element  $\bar{r} \in \text{int}\mathbb{R}_+^k$  and  $\bar{k} \in \mathcal{K} \setminus \{0\}$ ,  $\bar{r}\bar{k} \in \mathcal{K} \setminus \{0\}$ .

Then the following can not hold

$$(\beta - \overline{f(x)}) \in \mathcal{K} \setminus \{0\}. \quad (3.4.12)$$

**Proof.** The proof follows on the lines of proofs of Theorems 3.4.1 and 2.5.1.

We will consider the following lemma to prove strong duality theorem

**Lemma 3.4.3** [49] *Consider a problem (P)  $\mathcal{K}$ -Min  $\phi(x)$  subject to  $-\psi(x) \in \mathcal{Q}$ , where  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$  and  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  are locally Lipschitzian functions. If  $u$  is a weakly minimum of (P) then there exist  $(\lambda, \mu) \in (\mathcal{K}^*, \mathcal{Q}^*)$  and  $(\lambda, \mu)$ ,  $\lambda \neq 0$  such that*

$$0 \in \partial^c(\lambda^T \phi + \mu^T \psi)(u) \quad (3.4.13)$$

$$\mu^T \psi(u) = 0 \quad (3.4.14)$$

We will now give the strong duality theorem.

**Theorem 3.4.4 (Strong Duality Theorem)** *If  $\bar{u}$  is weak efficient solution of (NVFP) and let Slater's constraint qualification is satisfied at  $\bar{u}$ . Also if*

$$H(\bar{u}, 0) = 0, \quad \nabla_s H(\bar{u}, 0) = 0, \quad \nabla_q H(\bar{u}, 0) = 0,$$

$$F(\bar{u}, 0) = 0, \quad \nabla_p F(\bar{u}, 0) = 0, \quad \nabla_r F(\bar{u}, 0) = 0,$$

$$G(\bar{u}, 0) = 0, \quad \nabla_p G(\bar{u}, 0) = 0, \quad \nabla_r G(\bar{u}, 0) = 0.$$

*Then there exist  $\bar{\lambda} \in \text{int}\mathcal{K}^*$ ,  $\bar{\mu} \in \mathcal{Q}^*$ ,  $\bar{\beta} \in \mathbb{R}^k$ ,  $\bar{z} \in B_i$ ,  $\bar{v} \in D_i$ ,  $\bar{w} \in E_j$ ,  $i = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, m$  such that  $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\beta}, \bar{z}, \bar{v}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{r} = 0, \bar{s} = 0)$  is feasible solution of (NVFD) and the optimal values of primal (NVFP) and dual (NVFD) are equal. Moreover, if the conditions of weak duality are satisfied for every feasible solution of (NVFP) and every feasible solution of (NVFD) then  $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\beta}, \bar{z}, \bar{v}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{r} = 0, \bar{s} = 0)$  is an efficient solution of (NVFD).*

**Proof.** It is given that  $S(x|E_j)$ ,  $S(x|B_i)$  and  $S(x|D_i)$  are support functions of compact convex subsets  $E_j, B_i, D_i$  of  $\mathbb{R}^n$ , hence they are convex and finite. This imply that, these are locally Lipschitzian at any  $x \in \mathbb{R}^n$ . Also,  $f_i, g_i, h_j$  are continuously differentiable functions which are also locally Lipschitzian. Therefore the functions  $f_i(x) + S(x|B_i)$ ,  $g_i - S(x|D_i)$ ,  $h_j + S(x|E_j)$  are Lipschitzian. Hence we get the following due to lemma 3.4.3. If  $\bar{u}$  is a weakly efficient solution of primal (NVFP) then there exist  $\bar{\lambda} \in \text{int}\mathcal{K}^*$ ,  $\mu \in \mathcal{Q}^*$ ,  $(\bar{\lambda}, \bar{\mu}) \neq \mathbf{0}$  such

that

$$0 \in \partial^c \left[ \sum_{i=1}^k \bar{\lambda}_i \frac{f_i(\bar{u}) + S(\bar{u}|B_i)}{g_i(\bar{u}) - S(\bar{u}|D_i)} + (h_j(\bar{u}) + S(\bar{u}|E_j)) \right] \quad (3.4.15)$$

$$\sum_{j=1}^m \bar{\mu}_j (h_j(\bar{u}) + S(\bar{u}|E_j)) = 0 \quad (3.4.16)$$

As the support functions we have considered are convex, so we can write

$$\partial^c S(\bar{u}|B_i) = \partial S(\bar{u}|B_i), \partial^c S(\bar{u}|D_i) = \partial S(\bar{u}|D_i) \text{ and } \partial^c S(\bar{u}|E_j) = \partial S(\bar{u}|E_j).$$

So there exist some  $\bar{z}_i \in \partial S(\bar{u}|B_i)$ ,  $\bar{v}_i \in \partial S(\bar{u}|D_i)$ ,  $\bar{w}_j \in \partial S(\bar{u}|E_j)$  such that

$$\sum_{i=1}^k \bar{\lambda}_i \nabla \left( \frac{f_i(\bar{u}) + \bar{u}^T \bar{z}_i}{g_i(\bar{u}) - \bar{u}^T \bar{v}_i} \right) + \sum_{j=1}^m \bar{\mu}_j \nabla (h_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0 \quad (3.4.17)$$

$$\sum_{j=1}^m \bar{\mu}_j (h_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0 \quad (3.4.18)$$

$$\bar{u}^T \bar{z}_i = S(\bar{u}|B_i), \quad \bar{u}^T \bar{v}_i = S(\bar{u}|D_i), \quad \bar{u}^T \bar{w}_j = S(\bar{u}|E_j).$$

$$\bar{z}_i \in B_i, \quad \bar{v}_i \in D_i, \quad \bar{w}_j \in E_j, \quad \bar{\lambda} \in \text{int} \mathcal{K}^*, \quad \bar{\mu} \in \mathcal{Q}^*. \quad (3.4.19)$$

(3.4.17) can also be written as

$$\begin{aligned} \sum_{i=1}^k \bar{\lambda}_i \frac{\nabla (f_i(\bar{u}) + \bar{u}^T \bar{z}_i)(g_i(\bar{u}) - \bar{u}^T \bar{v}_i) - (f_i(\bar{u}) + \bar{u}^T \bar{z}_i) \nabla (g_i(\bar{u}) - \bar{u}^T \bar{v}_i)}{(g_i(\bar{u}) - \bar{u}^T \bar{v}_i)^2} \\ + \sum_{j=1}^m \bar{\mu}_j \nabla (h_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0 \end{aligned} \quad (3.4.20)$$

we denote

$$\frac{f_i(\bar{u}) + \bar{u}^T \bar{z}_i}{g_i(\bar{u}) - \bar{u}^T \bar{v}_i} = \beta_i \quad \text{and} \quad \lambda_i = \frac{\bar{\lambda}_i}{g_i(\bar{u}) - \bar{u}^T \bar{v}_i}. \quad (3.4.21)$$

Hence the necessary conditions become

$$\sum_{i=1}^k \lambda_i \nabla (f_i(\bar{u}) + \bar{u}^T \bar{z}_i - \beta_i (g_i(\bar{u}) - \bar{u}^T \bar{v}_i)) + \sum_{j=1}^m \bar{\mu}_j \nabla (h_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0, \quad (3.4.22)$$

$$f_i(\bar{u}) + \bar{u}^T \bar{z}_i - \beta_i (g_i(\bar{u}) - \bar{u}^T \bar{v}_i) = 0, \quad (3.4.23)$$

$$\sum_{j=1}^m \bar{\mu}_j (h_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0. \quad (3.4.24)$$

So we have  $(\bar{u}, \bar{\beta}, \bar{z}, \bar{v}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{r} = 0, \bar{s} = 0)$  as a feasible solution of (NVFD). Due

to (3.4.19) and (3.4.23) the objective values of both programs are equal.

Now we claim that this feasible solutions  $(\bar{u}, \bar{\beta}, \bar{z}, \bar{v}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{r} = 0, \bar{s} = 0)$  is efficient for (NVFD).

If not, then there exist some  $(u', \beta', z', v', w', p', q', r', s')$  such that,

$$(\beta' - \bar{\beta}) \in \mathcal{K} \setminus \{0\}.$$

But this contradicts weak duality theorem. Hence  $(\bar{u}, \bar{\beta}, \bar{z}, \bar{v}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{r} = 0, \bar{s} = 0)$  is an efficient solution of (NVFD).

# Chapter 4

## Higher-Order Symmetric Duality For Fractional Programming Problems Over Cones

### 4.1 Introduction

In mathematical programming problems when dual of the dual turns out to be primal again then we say the primal and dual are symmetric. A linear programming problem and its dual are always symmetric. But nonlinear programs are not symmetric in general. Several authors have studied nonlinear programming problems and their duals having symmetric nature. The symmetric quadratic programs were studied in [44, 57]. Then Dantzig [52] formulated nonlinear symmetric primal dual pair and established weak and strong duality theorems. Bazaraa and Goode [16] generalized the formulation of symmetric duality introduced in [52] to include the case where the inequality constraints are defined via convex cones and their polar cones. The symmetric programs consisting of non-differentiable functions were initially studied by Chandra et al. [32] and Chandra and Hussain [35]. Mond and Schechter [172] made an advancement into the theory of symmetric programs by adding nonsmooth functions called support function of compact convex sets.

Gulati et al. [80] introduced multiobjective symmetric programs and gave duality results for Wolfe as well as Mond-Weir type symmetric multiobjective programming problems. Kim et al. [134] obtained multiobjective symmetric programs with cone constraints using pseudo-invex and strictly pseudo-invex functions. Suneja et al. [196] studied symmetric programs over cones using  $\mathcal{K}$ -convex functions whereas Kim and Kim [136] later extended these results for Mond Weir and Wolfe type symmetric programs involving cones. The fractional symmetric programs were proposed by Chandra et al. [33] which was further extended to multiobjective fractional symmetric programs by Weir [211].

As it is a well known fact that dual gives a bound for the primal program, the second and higher-order duals give further tighter bounds due to the addition of parameters. These help in finding better approximation to the value of the primal problem. Bector and Chandra [20] introduced second order symmetric dual program for pseudobonvex and pseudoboncave functions. The multiobjective counterpart was studied by Yang and Hou [218] which was further extended to a symmetric program over cone constraints for second

order cone convex functions in [157]. Suneja et al. [200] studied symmetric second order multiobjective programs. The non-differentiable multiobjective symmetric programs are studied in [83, 86]. Gulati and Mehndiratta [81] considered a non-differentiable multiobjective symmetric pair involving arbitrary cones thus generalizing various classes of symmetric programs existing in literature.

Gulati and Gupta studied higher-order duality in symmetric programming and Chen [40] discussed about higher-order multiobjective non-differentiable symmetric program and some aspect of higher-order symmetric duality has also been discussed by Gulati and Gupta [78, 79]. Gupta et al. [87] studied Wolfe and Mond -Weir type dual symmetric models using higher-order  $(\mathcal{F}, \alpha, \rho, d)$ - convexity assumptions. Suneja and Louhan [201] studied higher-order symmetric programs with cone constraints utilizing cone invexity. Recently, Dubey and Gupta [60] gave higher-order multiobjective non-differentiable fractional symmetric programme with cone constraints.

In this Chapter, motivated by the the work of Dubey and Gupta [60] and Suneja and Louhan [201] we study fractional vector programming problems in which constraints are defined over closed convex cones. The programs involve higher-order approximations of the functions as well as support function of convex compact sets. To study the duality relations we used higher-order cone convex functions called higher-order  $\mathcal{K} - (\Phi, \rho)$ - convex functions. We simplify the fractional program by converting them into equivalent parametric programs, such that the solution sets of the corresponding programs coincide. Then we study the duality relations between this pair of parametric primal and dual programs. In this direction we state and prove weak, strong as well as converse duality theorems which imply the same relation between the corresponding fractional primal-dual symmetric pair. A brief outline of this Chapter is as follows. The Section 4.2 contains preliminaries and definitions to be used by subsequent sections. In Section 4.3 fractional symmetric programs and their equivalent parametric symmetric programs are discussed. In Section 4.4 the duality theorems are deduced.

## 4.2 Preliminaries and Definitions

Let  $\mathbb{R}^k$  be a  $k$ - dimensional Euclidean space and  $\mathbb{R}_+^k$  denote its nonnegative orthant. Let  $\mathcal{K}$  be a closed convex pointed cone in  $\mathbb{R}^k$  with non-empty interior. Consider  $F(\cdot, \cdot)$  is continuously differentiable such that  $F(x, y) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$ , then

- $\nabla_x F, \nabla_y F$  denote gradient vectors with respect to  $x, y$  respectively.
- $\nabla_{xx} F, \nabla_{yy} F$  are  $n_1 \times n_1$  and  $n_2 \times n_2$  symmetric Hessian matrices respectively.

Now we will give definition of generalized convex functions named as  $\mathcal{K} - (\Phi, \rho)$  convex

functions. First we give a brief overview of  $(\Phi, \rho)$  convexity and its generalizations. The concept of  $(\Phi, \rho)$  convexity was set forth by Caristi et al. [31] to extend the class of  $(\mathcal{F}, \rho)$  convex functions. The following definitions have made grounds for the definition introduced in this chapter.

Consider a vector valued function  $f = (f_1, f_2, \dots, f_k) : X \rightarrow \mathbb{R}^k$  differentiable on  $X$ . The components  $f_i$  are given by  $f_i : X \rightarrow \mathbb{R}$  for  $i = 1, 2, \dots, k$  and vector  $\rho = (\rho_1, \rho_2, \dots, \rho_k) \in \mathbb{R}^k$ . For a natural number  $n$  and a set  $X \subseteq \mathbb{R}^n$  consider a functional  $\Phi : X \times X \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ . Any element of  $\mathbb{R}^{n+1}$  takes the form  $(a, b)$ , where  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}$ .

**Definition 4.2.1**  $\Phi$  is convex on  $\mathbb{R}^{n+1}$  if for fixed  $x, u \in X$ , the following holds:

$$\Phi(x, u; (\lambda(\xi_1, \rho_1) + (1 - \lambda)(\xi_2, \rho_2))) \leq \lambda\Phi(x, u; (\xi_1, \rho_1)) + (1 - \lambda)\Phi(x, u; (\xi_2, \rho_2)),$$

for all  $\xi_1, \xi_2 \in \mathbb{R}^n$  and  $\rho_1, \rho_2 \in \mathbb{R}$  and  $\lambda \in [0, 1]$ . Throughout this Chapter, we assume that  $\Phi(x, u; (0, r)) \geq 0$  for  $x, u$  and  $r \in \mathbb{R}_+$ .

The  $(\Phi, \rho)$  convex functions introduced in [31] are defined as follows.

**Definition 4.2.2** For  $k = 1$ , the scalar valued function  $f : X \rightarrow \mathbb{R}$ , is  $(\Phi, \rho)$  convex on  $X$  if

$$f(x) - f(u) \geq \Phi(x, u; (\nabla f(u), \rho)), \quad \forall x \in X$$

and some  $\rho \in \mathbb{R}$ .

This class of functions were extended to cone  $(\Phi, \rho)$  convex functions by Kapoor [120] who gave the following definition.

**Definition 4.2.3** [120]  $f : X \rightarrow \mathbb{R}^k$  is  $\mathcal{K} - (\Phi, \rho)$  convex at  $u$  on  $X$  if for every  $x \in X$ , the following hold:

$$f(x) - f(u) - \Phi(x, u, (\nabla f(u), \rho)) \in \mathcal{K}.$$

In this, vector  $\Phi$  is given by

$$\Phi(x, u; (\nabla f(u), \rho)) = (\Phi(x, u; (\nabla f_1(u), \rho_1), \Phi(x, u; (\nabla f_2(u), \rho_2), \dots, \Phi(x, u; (\nabla f_k(u), \rho_k))).$$

The higher-order convex functions were defined by Tripathy and Devi [208] as in the next definition.

**Definition 4.2.4** For  $k = 1$ , the function  $f$  given by  $f : X \rightarrow \mathbb{R}$  is higher-order  $(\Phi, \rho)$ -invex at  $u \in X$  with respect to  $h(h : X \times \mathbb{R}^n \rightarrow \mathbb{R})$  if there exist real functional  $\Phi$  and scalar  $\rho \in \mathbb{R}$  such that for all  $x \in X$  we have

$$f(x) - f(u) - h(u, p) + p^T \nabla_p h(u, p) \geq \Phi(x, u; (\nabla f(u) + \nabla_p h(u, p), \rho)).$$

Now we define higher-order  $\mathcal{K} - (\Phi, \rho)$ -convex functions. For this, assume  $f, \Phi, \rho$  as defined above and  $F : X \times \mathbb{R}^n \rightarrow \mathbb{R}^k$  be a differentiable functions.

**Definition 4.2.5** A function  $f$  is higher-order  $\mathcal{K} - (\Phi, \rho)$ -convex at  $u \in X$  with respect to  $F$  and  $p$  where  $p = (p_1, p_2, \dots, p_k)$  and each  $p_i \in \mathbb{R}^n$ , if there exist real functional  $\Phi$  and  $\rho$  such that for all  $x \in X$  we have

$$\begin{pmatrix} f_1(x) - f_1(u) - F_1(u, p_1) + p_1^T \nabla_{p_1} F_1(u, p_1) - \Phi(x, u, (\nabla f_1(u) + \nabla_{p_1} F_1(u, p_1), \rho_1)) \\ f_2(x) - f_2(u) - F_2(u, p_2) + p_2^T \nabla_{p_2} F_2(u, p_2) - \Phi(x, u, (\nabla f_2(u) + \nabla_{p_2} F_2(u, p_2), \rho_2)) \\ \text{---} \\ \text{---} \\ f_k(x) - f_k(u) - F_k(u, p_k) + p_k^T \nabla_{p_k} F_k(u, p_k) - \Phi(x, u, (\nabla f_k(u) + \nabla_{p_k} F_k(u, p_k), \rho_k)) \end{pmatrix} \in \mathcal{K}$$

### Special Cases:

1. In this definition if  $k = 1$  and  $\mathcal{K} = \mathbb{R}_+$  then we have the higher-order  $(\Phi, \rho)$  convex functions defined by Tripathy and Devi [208].
2. If  $p = 0$  or no approximation functions are used then we have  $\mathcal{K} - (\Phi, \rho)$  convex functions defined by Kapoor [120]. In addition to this, if  $k = 1$  and  $\mathcal{K} = \mathbb{R}_+$  then the function reduce to  $(\Phi, \rho)$  convex functions defined by Caristi et al. [31].

We consider a non-differentiable problem in this Chapter. The non-differentiable part is due to support function. Let us recall the definition of support function and its subdifferential from previous chapters.

**Definition 4.2.6** If  $C$  is a compact and convex subset of  $\mathbb{R}^n$ , then support function of  $C$  at  $x$  is given by  $S(x|C) := \max \{x^T y | y \in C\}$ . This function being convex and finite has subdifferential

$$\partial S(x|C) := \{z \in C : x^T z = S(x|C)\}.$$

**Definition 4.2.7** If  $D \subseteq \mathbb{R}^n$  is convex, then the **normal cone** at a point  $z$  in  $D$  is defined as:

$$N_D(z) := \{y \in \mathbb{R}^n : y^T(x - z) \leq 0, \forall x \in D\}$$

If  $D$  is compact convex set then taking into consideration both the definitions one can infer that  $y \in N_D(z) \Leftrightarrow z^T y = S(y|D)$  or say  $z \in \partial S(y|D)$ .

### 4.3 Higher Order Symmetric Programs

In this section we introduce multiobjective fractional symmetric primal dual pair. Let us suppose that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are two non-empty open sets in  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2}$  respectively. Further suppose that  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are closed and convex cones in  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2}$  respectively, such that  $\mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{S}_1 \times \mathcal{S}_2$ . Consider the following non-differentiable multiobjective fractional symmetric programs (MFNSP) and (MFNSD):

Primal (MFNSP)

$$\begin{aligned}
\mathcal{K} - \text{Min} \quad & L(x, y, p) = (L_1(x, y, p), L_2(x, y, p), \dots, L_k(x, y, p)) \\
\text{where } L_i(x, y, p) = & \frac{f_i(x, y) + S(x|B_i) - y^T z_i + F_i(x, y, p_i) - p_i^T \nabla_{p_i} F_i(x, y, p_i)}{g_i(x, y) - S(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_{p_i} G_i(x, y, p_i)} \\
\text{subject to} \quad & \\
- \sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_{p_i} F_i(x, y, p_i) & \\
& - L_i(x, y, p_i) (\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))) \in \mathcal{A}_2^* \\
y^T \sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_{p_i} F_i(x, y, p_i) & \\
& - L_i(x, y, p_i) (\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))) \geq 0 \\
\lambda \in \text{int} \mathcal{K}^*, x \in \mathcal{A}_1, z_i \in D_i, r_i \in F_i, \quad i = 1, 2, \dots, k, \lambda^T e = 1.
\end{aligned}$$

Dual (MFNSD)

$$\begin{aligned}
\mathcal{K} - \text{Max} \quad & M(u, v, q) = (M_1(u, v, q), M_2(u, v, q), \dots, M_k(u, v, q)) \\
\text{where } M_i(u, v, q) = & \frac{f_i(u, v) - S(v|D_i) + u^T w_i + \bar{F}_i(u, v, q_i) - q_i^T \nabla_{q_i} \bar{F}_i(u, v, q_i)}{g_i(u, v) + S(v|F_i) - u^T t_i + \bar{G}_i(u, v, q_i) - q_i^T \nabla_{q_i} \bar{G}_i(u, v, q_i)} \\
\text{subject to} \quad & \\
\sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \bar{F}_i(u, v, q_i) & \\
& - M_i(u, v, q_i) (\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \bar{G}_i(u, v, q_i))) \in \mathcal{A}_1^* \\
u^T \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \bar{F}_i(u, v, q_i) & \\
& - M_i(u, v, q_i) (\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \bar{G}_i(u, v, q_i))) \leq 0 \\
\lambda \in \text{int} \mathcal{K}^*, v \in \mathcal{A}_2, w_i \in B_i, t_i \in E_i, \quad i = 1, 2, \dots, k, \lambda^T e = 1.
\end{aligned}$$

For  $i = 1, 2, \dots, k$  the following assumptions have been used while constructing the above programs pair:

- (1.)  $f_i, g_i : \mathcal{S}_1 \times \mathcal{S}_2 \rightarrow \mathbb{R}$  are differentiable functions,
- (2.) The differentiable functions  $F_i, G_i, \bar{F}_i, \bar{G}_i$  are such that  $F_i, G_i : \mathcal{S}_1 \times \mathcal{S}_2 \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$ , are higher-order approximation functions of  $f_i, g_i$  respectively, with respect to second variable.  $\bar{F}_i, \bar{G}_i : \mathcal{S}_1 \times \mathcal{S}_2 \times \mathbb{R}^{n_1} \rightarrow \mathbb{R}$  are higher-order approximation functions of  $f_i, g_i$  respectively, with respect to first variable.
- (3.)  $B_i, E_i$  are compact convex sets in  $\mathbb{R}^{n_1}$  and  $D_i, F_i$  are compact convex sets in  $\mathbb{R}^{n_2}$ ,
- (4.)  $p_i \in \mathbb{R}^{n_2}, q_i \in \mathbb{R}^{n_1}, \lambda \in \mathbb{R}^k$ ,
- (5.)  $\mathcal{A}_1^*, \mathcal{A}_2^*$  are positive polar cones of  $\mathcal{A}_1, \mathcal{A}_2$  respectively,
- (6.) in the feasible region the numerators and denominators are assumed to be nonnegative and positive, respectively.

### Special Cases:

1. If  $\mathcal{K} = \mathbb{R}_+^k$ ,  $F_i = H_i$ ,  $\phi_i = \bar{F}_i$ ,  $\xi_i = \bar{G}_i$  and  $C_1 = \mathcal{A}_1$ ,  $C_2 = \mathcal{A}_2$  then this above discussed model will become the one discussed in Dubey and Gupta [60].
2. If  $k = 1$ ,  $\mathcal{A}_1 = \mathbb{R}_+^n$ ,  $\mathcal{A}_2 = \mathbb{R}_+^m$ , with all the higher-order approximations are taken to be zero, or  $p_i, q_i = 0$  and the sets  $B_i = D_i = E_i = F_i = \{0\}$ . Then the symmetric programs (MFNSP) and (MFNSD) reduce to the symmetric fractional program discussed by Chandra et al. [33].

To make the theorems easier the following parametric program has been formulated. We take two parameters  $l = (l_1, l_2, \dots, l_k)$  and  $m = (m_1, m_2, \dots, m_k)$  and express the programs (MFNSP) and (MFNSD) equivalently as multiobjective non-differentiable symmetric programs (EMNSP) and (EMNSD) respectively.

Primal (EMNSP)

$$\mathcal{K} - \text{Min } l = (l_1, l_2, \dots, l_k)$$

subject to

$$\begin{aligned} f_i(x, y) + S(x|B_i) - y^T z_i + F_i(x, y, p_i) - p_i^T \nabla_{p_i} F_i(x, y, p_i) \\ - l_i(g_i(x, y) - S(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_{p_i} G_i(x, y, p_i)) = 0, \quad i = 1, 2, \dots, k \end{aligned} \quad (4.3.1)$$

$$\begin{aligned}
& - \sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_{p_i} F_i(x, y, p_i) \\
& \quad - l_i(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))) \in \mathcal{A}_2^*
\end{aligned} \tag{4.3.2}$$

$$\begin{aligned}
& y^T \sum_{i=1}^k \lambda_i (\nabla_y f_i(x, y) - z_i + \nabla_{p_i} F_i(x, y, p_i) \\
& \quad - l_i(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))) \geq 0
\end{aligned} \tag{4.3.3}$$

$$\lambda \in \text{int}\mathcal{K}^*, \quad x \in \mathcal{A}_1, \quad z_i \in D_i, \quad r_i \in F_i, \quad i = 1, 2, \dots, k, \quad \lambda^T e = 1. \tag{4.3.4}$$

Dual (EMNSD)

$$\mathcal{K} - \text{Max} \quad m = (m_1, m_2, \dots, m_k)$$

subject to

$$\begin{aligned}
& f_i(u, v) - S(v|D_i) + u^T w_i + \bar{F}_i(u, v, q_i) - q_i^T \nabla_{q_i} \bar{F}_i(u, v, q_i) \\
& \quad - m_i(g_i(u, v) + S(v|F_i) - u^T t_i + \bar{G}_i(u, v, q_i) - q_i^T \nabla_{q_i} \bar{G}_i(u, v, q_i)), \quad i = 1, 2, \dots, k
\end{aligned} \tag{4.3.5}$$

$$\begin{aligned}
& \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \bar{F}_i(u, v, q_i) \\
& \quad - m_i(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \bar{G}_i(u, v, q_i))) \in \mathcal{A}_1^*
\end{aligned} \tag{4.3.6}$$

$$\begin{aligned}
& u^T \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \bar{F}_i(u, v, q_i) \\
& \quad - m_i(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \bar{G}_i(u, v, q_i))) \leq 0
\end{aligned} \tag{4.3.7}$$

$$\lambda \in \text{int}\mathcal{K}^*, \quad v \in \mathcal{A}_2, \quad w_i \in B_i, \quad t_i \in E_i, \quad i = 1, 2, \dots, k, \quad \lambda^T e = 1. \tag{4.3.8}$$

## 4.4 Duality Results

In this Section, we validate the duality relations between the equivalent symmetric programs (EMNSP) and (EMNSD) under generalized convexity assumptions.

**Theorem 4.4.1 (Weak Duality Theorem)** *Let  $(x, y, l, z, r, \lambda, p)$  be a feasible solution of (EMNSP) and  $(u, v, m, w, t, \lambda, q)$  be a feasible of (EMNSD). Assume that:*

(i)  *$(f(\cdot, v) + (\cdot)^T w - m(g(\cdot, v) - (\cdot)^T t))$  is higher-order  $\mathcal{K} - (\Phi^1, \rho^1)$  convex, in first variable, at  $u$  for fixed  $v$ , with respect to  $\bar{F} - m\bar{G}$  and  $q$  where  $\Phi^1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1+1} \rightarrow \mathbb{R}$  and  $\rho^1 = (\rho_1^1, \dots, \rho_k^1)$ ,*

(ii)  *$(-(f(x, \cdot) + (\cdot)^T z) + l(g(x, \cdot) - (\cdot)^T r))$  is higher-order  $\mathcal{K} - (\Phi^2, \rho^2)$  convex, in second*

variable, at  $y$  for fixed  $x$ , with respect to  $-H + lG$  and  $p$  where  $\Phi^2 : \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2+1} \rightarrow \mathbb{R}$  and  $\rho^2 = (\rho_1^2, \dots, \rho_k^2)$ ,

$$(iii) \quad \Phi^1(x, u, (a, \bar{\rho})) + a^T u \geq 0 \forall a \in \mathcal{A}_1^* \text{ and } \bar{\rho} \geq 0 \ \& \ \Phi^2(v, y, (b, \tilde{\rho})) + b^T y \geq 0 \forall b \in \mathcal{A}_2^* \text{ and } \tilde{\rho} \geq 0.$$

(iv)  $g_i(x, v) + v^T r_i - x^T t_i > 0$ ,  $i = 1, 2, \dots, k$  and for some  $\omega \in \text{int}\mathbb{R}_+^k$ ,  $\kappa \in \mathcal{K} \setminus \{0\}$ ,  $\omega\kappa \in \mathcal{K} \setminus \{0\}$ .

Then  $(m - l) \notin \mathcal{K} \setminus \{0\}$ .

**Proof.** We validate the theorem by contradiction. Suppose that the weak duality theorem does not hold, which means that  $(m - l) \in \mathcal{K} \setminus \{0\}$ . Now for some  $\lambda \in \text{int}\mathcal{K}^*$  and due to (iv) we have

$$\sum_{i=1}^k \lambda_i (l_i - m_i) (g_i(x, v) + v^T r_i - x^T t_i) < 0. \quad (4.4.1)$$

(i) gives that

$$\left( \begin{array}{l} f_1(x, v) + x^T w_1 - m_1(g_1(x, v) - x^T t_1) - (f_1(u, v) + u^T w_1 - m_1(g_1(u, v) - u^T t_1)) \\ \quad - (\bar{F}_1(u, v, q_1) - m_1 \bar{G}_1(u, v, q_1)) + q_1^T \nabla_{q_1} (\bar{F}_1(u, v, q_1) - m_1 \bar{G}_1(u, v, q_1)) \\ - \Phi^1(x, u, (\nabla_x f_1(u, v) + w_1 - m_1(\nabla_x g_1(u, v) - t_1) + \nabla_{q_1} (\bar{F}_1(u, v, q_1) - m_1 \bar{G}_1(u, v, q_1))), \rho_1^1)) \\ \\ \\ \\ \\ \\ \\ \\ f_k(x, v) + x^T w_k - m_k(g_k(x, v) - x^T t_k) - (f_k(u, v) + u^T w_k - m_k(g_k(u, v) - u^T t_k)) \\ \quad - (\bar{F}_k(u, v, q_k) - m_k \bar{G}_k(u, v, q_k)) + q_k^T \nabla_{q_k} (\bar{F}_k(u, v, q_k) - m_k \bar{G}_k(u, v, q_k)) \\ - \Phi^1(x, u, (\nabla_x f_k(u, v) + w_k - m_k(\nabla_x g_k(u, v) - t_k) + \nabla_{q_k} (\bar{F}_k(u, v, q_k) - m_k \bar{G}_k(u, v, q_k))), \rho_k^1)) \end{array} \right) \in \mathcal{K}$$

Since  $\lambda \in \text{int}\mathcal{K}^*$  we get the following

$$\begin{aligned} & \sum_{i=1}^k \lambda_i [f_i(x, v) + x^T w_i - m_i(g_i(x, v) - x^T t_i) - (f_i(u, v) + u^T w_i - m_i(g_i(x, v) - u^T t_i)) \\ & \quad - (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i)) + q_i^T \nabla_{q_i} (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i)) \\ & \quad - \Phi^1(x, u; (\nabla_x f_i(u, v) + w_i - m_i(\nabla_x g_i(u, v) - t_i) \\ & \quad \quad + \nabla_{q_i} (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i)), \rho_i^1))] \geq 0. \end{aligned} \tag{4.4.2}$$

Divide this equation by  $\sum_{i=1}^k \lambda_i = \tau$  (it is clear that  $\sum_{i=1}^k \frac{\lambda_i}{\tau} = 1$ ) and using convexity of

$\Phi^1$  the following is deduced

$$\begin{aligned}
& \sum_{i=1}^k \frac{\lambda_i}{\tau} [f_i(x, v) + x^T w_i - m_i(g_i(x, v) - x^T t_i) - (f_i(u, v) + u^T w_i - m_i(g_i(u, v) - u^T t_i)) \\
& \quad - (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i)) + q_i^T \nabla_{q_i} (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i))] \\
& \geq \Phi^1(x, u, \frac{1}{\tau} \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + w_i - m_i (\nabla_x g_i(u, v) - t_i) \\
& \quad + \nabla_{q_i} (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i)), \rho_i^1)). \tag{4.4.3}
\end{aligned}$$

As  $a1 = \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + w_i - m_i (\nabla_x g_i(u, v) - t_i) + \nabla_{q_i} (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i))) \in \mathcal{A}_1^*$  due to (4.3.6) multiply this with  $\frac{1}{\tau} > 0$  to get  $a = \frac{a1}{\tau}$  and since  $\mathcal{A}_1^*$  is a convex cone we have  $a \in \mathcal{A}_1^*$ . Use this and  $\bar{\rho} = \sum_{i=1}^k \frac{\lambda_i}{\tau} \rho_i^1 > 0$  in hypothesis (iii) to get the following inequality

$$\begin{aligned}
& \frac{1}{\tau} \sum_{i=1}^k \lambda_i (f_i(x, v) + x^T w_i - m_i(g_i(x, v) - x^T t_i) - (f_i(u, v) + u^T w_i - m_i(g_i(u, v) - u^T t_i)) \\
& \quad - (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i)) + q_i^T \nabla_{q_i} (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i))) \\
& \geq \Phi^1(x, u, \frac{1}{\tau} \sum_{i=1}^k \lambda_i (\nabla_x f_i(u, v) + w_i - m_i (\nabla_x g_i(u, v) - t_i) \\
& \quad + \nabla_{q_i} (\bar{F}_i(u, v, q_i) - m_i \bar{G}_i(u, v, q_i)), \rho_i^1)) \\
& = \Phi^1(x, u; (a, \bar{\rho})) \\
& \geq -a^T u \\
& \geq 0, \quad (\text{due to (4.3.6)}). \tag{4.4.4}
\end{aligned}$$

On rearranging the terms in above equation and adding (4.3.5) then using feasibility conditions (4.3.4 and (4.3.8) we obtain the following

$$\sum_{i=1}^k \lambda_i (f_i(x, v) + x^T w_i - S(v|D_i) - m_i(g_i(x, v) - x^T t_i + v^T r_i)) \geq 0. \tag{4.4.5}$$

On the same lines using (ii) the following can be obtained

$$\sum_{i=1}^k \lambda_i (-f_i(x, v) + v^T z_i - S(x|B_i) + l_i(g_i(x, v) + v^T r_i - x^T t_i)) \geq 0. \tag{4.4.6}$$

Adding equations (4.4.5) and (4.4.6) we get

$$\sum_{i=1}^k \lambda_i (l_i - m_i)(g_i(x, v) + v^T r_i - x^T t_i) \geq 0.$$

This is a contradiction to equation (4.4.1). So it is concluded that the supposition was wrong and hence weak duality holds under the given set of assumptions.

**Theorem 4.4.2 (Strong Duality Theorem)** *Let  $(\bar{x}, \bar{y}, \bar{l}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$  be an efficient solution of (EMNSP) and fix  $\lambda = \bar{\lambda}$  in (EMNSD). Further, if the following assumptions hold:*

(i) *For  $i = 1, 2, \dots, k$ ,*

$$F_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} F_i(\bar{x}, \bar{y}, 0) = \nabla_x F_i(\bar{x}, \bar{y}, 0) = \nabla_y F_i(\bar{x}, \bar{y}, 0) = 0,$$

$$G_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} G_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = \nabla_y G_i(\bar{x}, \bar{y}, 0) = 0,$$

$$\bar{F}_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \bar{F}_i(\bar{x}, \bar{y}, 0) = \bar{G}_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \bar{G}_i(\bar{x}, \bar{y}, 0) = 0$$

(ii) *for any  $i = 1, 2, \dots, k$  the Hessian matrix  $\nabla_{p_i p_i}(F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i G_i(\bar{x}, \bar{y}, \bar{p}_i))$  is positive or negative definite,*

(iii) *The set of vectors  $\{\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_y F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i \nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i), \nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i \nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i), i = 1, 2, \dots, k\}$  is linearly independent.*

(iv) *if for  $\bar{p}_i \in \mathbb{R}^{n_2}$  such that  $\bar{p}_i \neq 0$  implies  $\sum_{i=1}^k \bar{p}_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_y F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i (\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))) \neq 0$*

(v)  $\bar{l}_i > 0, i = 1, 2, \dots, k$

*then the point  $(\bar{x}, \bar{y}, \bar{l}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$  is a feasible solution of the dual (EMNSD). Furthermore, if hypotheses of weak duality theorem hold for every feasible solution of dual then  $(\bar{x}, \bar{y}, \bar{l}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$  is efficient solution of (EMNSD) and objective function values of (EMNSP) and (EMNSD) are equal.*

**Proof.** Since  $(\bar{x}, \bar{y}, \bar{l}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$  is given to be an efficient solution of the primal (EMNSP), then by necessary optimality conditions [201], there exists  $\alpha \in \mathcal{K}^*, \beta \in \mathbb{R}_+^k, \gamma \in \mathcal{A}_2, \delta \in$

$\mathbb{R}_+$ ,  $\eta \in \mathbb{R}$ ,  $\bar{w}_i$  and  $\bar{t}_i \in \mathbb{R}^{n_1}$  such that the following hold

$$\begin{aligned}
& \left[ \sum_{i=1}^k \beta_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i + \nabla_x F_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{l}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i + \nabla_x G_i(\bar{x}, \bar{y}, \bar{p}_i))) \right. \\
& + (\gamma - \delta \bar{y})^T \sum_{i=1}^k \lambda_i (\nabla_{yx} f_i(\bar{x}, \bar{y}) - l_i \nabla_{yx} g_i(\bar{x}, \bar{y})) \\
& \left. + \sum_{i=1}^k \nabla_{p_i x} (F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i G_i(\bar{x}, \bar{y}, \bar{p}_i))^T ((\gamma - \delta \bar{y}) \lambda_i - \beta_i \bar{p}_i) \right]^T (x - \bar{x}) \geq 0, \forall x \in \mathcal{A}_1.
\end{aligned} \tag{4.4.7}$$

$$\begin{aligned}
& \sum_{i=1}^k \beta_i (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y F_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{l}_i (\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))) \\
& + (\gamma - \delta \bar{y})^T \sum_{i=1}^k \lambda_i \nabla_{yy} (f_i(\bar{x}, \bar{y}) - \bar{l}_i g_i(\bar{x}, \bar{y})) + \sum_{i=1}^k (\nabla_{p_i y} F_i(\bar{x}, \bar{y}, \bar{p}_i) \\
& - \bar{l}_i \nabla_{p_i y} G_i(\bar{x}, \bar{y}, \bar{p}_i)) ((\gamma - \delta \bar{y})^T \lambda_i - \beta_i \bar{p}_i) \\
& - \sum_{i=1}^k \delta \lambda_i (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y F_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{l}_i (\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0.
\end{aligned} \tag{4.4.8}$$

$$\begin{aligned}
& \alpha_i - \beta_i ((g_i(\bar{x}, \bar{y}) - S(\bar{x}|E_i) + \bar{y}^T \bar{r}_i + G_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{p}_i^T \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) \\
& - (\gamma - \delta \bar{y})^T (\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0, \quad i = 1, 2, \dots, k.
\end{aligned} \tag{4.4.9}$$

$$((\gamma - \delta \bar{y}) \lambda_i - \beta_i \bar{p}_i)^T \nabla_{p_i p_i} (F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0, \tag{4.4.10}$$

$$\beta_i \bar{y} + (\gamma - \delta \bar{y}) \lambda_i \in N_{D_i}(\bar{z}_i), \tag{4.4.11}$$

$$\beta_i \bar{l}_i \bar{y} + (\gamma - \delta \bar{y}) \bar{l}_i \lambda_i \in N_{F_i}(\bar{r}_i), \tag{4.4.12}$$

$$\begin{aligned}
& (\gamma - \delta \bar{y})^T (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) \\
& - l_i (\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) - \xi_i + \eta = 0
\end{aligned} \tag{4.4.13}$$

$$\begin{aligned}
& \gamma^T \sum_{i=1}^k \lambda_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) \\
& - l_i (\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0
\end{aligned} \tag{4.4.14}$$

$$\begin{aligned}
& \delta y^T \sum_{i=1}^k \lambda_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) \\
& - l_i (\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0
\end{aligned} \tag{4.4.15}$$

$$\xi^T \bar{\lambda} = 0, \quad \eta(\bar{\lambda} e - 1) = 0 \tag{4.4.16}$$

$$\bar{w}_i \in B_i, \quad \bar{t}_i \in E_i, \quad \bar{x}^T \bar{t}_i = S(\bar{x}|E_i), \quad \bar{x}^T \bar{w}_i = S(\bar{x}|B_i) \tag{4.4.17}$$

$$(\alpha, \beta, \gamma, \delta, \xi, \eta) \neq 0. \tag{4.4.18}$$

In (4.4.16) we have  $\xi^T \bar{\lambda} = 0$ . As  $\mathbb{R}_k^+ \subseteq \mathcal{K} \Rightarrow \mathcal{K}^* \subseteq \mathbb{R}_k^+$  we have  $\text{int} \mathcal{K}^* \subseteq \text{int} \mathbb{R}_k^+$  implies  $\bar{\lambda} > 0$ . So we have  $\xi = 0$ . From hypothesis (ii) and equation (4.4.10) we get

$$(\gamma - \delta \bar{y}) \bar{\lambda}_i = \beta_i \bar{p}_i, \quad i = 1, 2, \dots, k. \quad (4.4.19)$$

**CLAIM:**  $\beta_i \neq 0$ , for any  $i = 1, 2, \dots, k$ .

Because if  $\beta_{i_0} = 0$ , for some  $i_0 \in 1, 2, \dots, k$  we have

$$(\gamma - \delta \bar{y}) \bar{\lambda}_{i_0} = 0 \xrightarrow{\bar{\lambda}_{i_0} > 0} \gamma = \delta \bar{y} \quad (4.4.20)$$

Put this in equation (4.4.19) we get  $\beta_i \bar{p}_i = 0$  for each  $i \in 1, \dots, k$ . Using these values in (4.4.8) to get

$$\begin{aligned} & \sum_{i=1}^k \beta_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_y F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))) \\ & - \sum_{i=1}^k \delta \bar{\lambda}_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. \end{aligned} \quad (4.4.21)$$

This due to (iii) gives  $\beta_i = 0$  and  $\delta \bar{\lambda}_i = 0$ , for all  $i = 1, 2, \dots, k \Rightarrow \delta = 0, (\because \bar{\lambda} > 0)$ . Now due to (4.4.9)  $\alpha_i = 0, \forall i = 1, 2, \dots, k$ . Equation (4.4.20) gives  $\gamma = 0$  and equation (4.4.13)  $\Rightarrow \eta = 0$  respectively.  $\xi$  is already 0.  $\Rightarrow (\alpha, \beta, \gamma, \delta, \xi, \eta) = 0$ , which is a contradiction to necessary optimality conditions constructed above. So we have proved the claim that  $\beta_i \neq 0$  for any  $i = 1, 2, \dots, k$ .

Now multiply equation (4.4.13) with  $\bar{\lambda}_i$  and take its sum over the range of  $i$  to get

$$\begin{aligned} & (\gamma - \delta \bar{y})^T \sum_{i=1}^k \bar{\lambda}_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) \\ & - l_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) + \eta^T (\bar{\lambda} e) = 0 \end{aligned} \quad (4.4.22)$$

and (4.4.14)-(4.4.15)

$$= (\gamma - \delta \bar{y})^T \sum_{i=1}^k \bar{\lambda}_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) - L_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. \quad (4.4.23)$$

Now (4.4.23)-(4.4.22)

$$\begin{aligned} & = \eta^T \bar{\lambda} e = 0 \\ & \Rightarrow \eta = 0 \quad (\because \bar{\lambda} \neq 0, e \neq 0). \end{aligned} \quad (4.4.24)$$

Putting this in (4.4.13) we get

$$(\gamma - \delta \bar{y})^T (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. \quad (4.4.25)$$

So due to (4.4.19)

$$\beta_i \bar{p}_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. \quad (4.4.26)$$

As  $\beta_i \neq 0$ ,

$$\bar{p}_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. \quad (4.4.27)$$

$$\text{Or } \sum_{i=1}^k \bar{p}_i (\nabla_y f_i(\bar{x}, \bar{y}) - z_i + \nabla_{p_i} F_i(\bar{x}, \bar{y}, \bar{p}_i) - l_i(\nabla_y g_i(\bar{x}, \bar{y}) + r_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. \quad (4.4.28)$$

and due to (iv) we have each of  $\bar{p}_i = 0$ . So from (4.4.19) we get

$$\gamma = \delta \bar{y}. \quad (4.4.29)$$

By putting the obtained values in (4.4.7) and (4.4.8) we get

$$\left[ \sum_{i=1}^k \beta_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \right]^T (x - \bar{x}) \geq 0, \quad \forall x \in \mathcal{A}_1, \quad (4.4.30)$$

and

$$\sum_{i=1}^k (\beta_i - \delta \bar{\lambda}_i) (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i - \bar{l}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i))^T = 0. \quad (4.4.31)$$

Again using (iii) we have  $\beta_i - \delta \bar{\lambda}_i = 0 \Rightarrow \beta_i = \delta \bar{\lambda}_i$ . As  $\bar{\lambda}_i > 0$ ,  $\beta_i \in \mathbb{R}_+^k$  and  $\beta \neq 0$ , we have  $\delta > 0$ . Use this in (4.4.30) to get

$$\left[ \sum_{i=1}^k \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \right]^T (x - \bar{x}) \geq 0. \quad (4.4.32)$$

Put  $x = 0$  (as (4.4.32) holds for any  $x \in \mathcal{A}_1$ ) we have

$$-\left[ \sum_{i=1}^k \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \right]^T \bar{x} \geq 0. \quad (4.4.33)$$

$$\Rightarrow \left[ \sum_{i=1}^k \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \right]^T \bar{x} \leq 0. \quad (4.4.34)$$

For convex cone  $\mathcal{A}_1$ ,  $\bar{x} \in \mathcal{A}_1 \Rightarrow x + \bar{x} \in \mathcal{A}_1$ ,  $\forall x \in \mathcal{A}_1$ . So (4.4.32) becomes

$$\left[ \sum_{i=1}^k \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \right]^T x \geq 0, \forall x \in \mathcal{A}_1. \quad (4.4.35)$$

The above equation holds for every  $x \in \mathcal{A}_1$  so we can say that

$$\sum_{i=1}^k \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \in \mathcal{A}_1^*. \quad (4.4.36)$$

which is dual feasibility condition (4.3.6). Since (4.4.35) holds for every  $x \in \mathcal{A}_1$  and  $\bar{x}$  is a feasible solution of primal, so at  $x = \bar{x}$  the following is true. By putting  $x = \bar{x}$  in (4.4.35) we have

$$\left[ \sum_{i=1}^k \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \right]^T \bar{x} \geq 0. \quad (4.4.37)$$

As  $\bar{x} \in \mathcal{A}_1 \Rightarrow$  (4.4.33) and (4.4.37) give

$$\sum_{i=1}^k \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{l}_i (\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \bar{x} = 0. \quad (4.4.38)$$

Hence the third feasibility condition (4.3.7) of the dual is obtained. Now in (4.4.11), as  $\gamma = \delta \bar{y}$ ,  $\beta_i > 0 \Rightarrow \bar{y} \in N_{D_i}(\bar{z}_i)$  which means  $\bar{y}^T \bar{z}_i = S(\bar{y}|D_i)$ . Similarly due to (4.4.12), (4.4.17) and hypothesis (v) we have  $\bar{y}^T \bar{r}_i = S(\bar{y}|F_i)$ .  $p_i = 0$  as obtained above, when put in equation (4.3.1) give

$$f_i(\bar{x}, \bar{y}) + \bar{x}^T \bar{w}_i - S(\bar{y}|D_i) - \bar{l}_i (g_i(\bar{x}, \bar{y}) - \bar{x}^T \bar{t}_i + S(\bar{y}|F_i)) = 0, i = 1, 2, \dots, k. \quad (4.4.39)$$

Equation (4.4.29) gives that  $\bar{y} = \frac{\gamma}{\delta} \in \mathcal{A}_2$  From equations (4.4.36), (4.4.38) and (4.4.39) we can conclude that  $(\bar{x}, \bar{y}, \bar{l}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$  is feasible solution of dual (EMNSD) and the objective values of (EMNSP) and (EMNSD) are equal which means  $\bar{l}_i = m_i$ . Now for the second part of the theorem, if  $(\bar{x}, \bar{y}, \bar{l}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$  is not an efficient solution for dual (EMNSD), then there exist another feasible solution  $(\tilde{u}, \tilde{v}, \tilde{m}, \tilde{w}, \tilde{t}, \tilde{\lambda}, \tilde{q})$  of dual such that  $\tilde{m} - \bar{l} \in \mathcal{K} \setminus \{0\}$ . But this contradicts weak duality theorem. Hence  $(\bar{x}, \bar{y}, \bar{l}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$  is efficient for (EMNSD). Hence, the result.

**Theorem 4.4.3** (*Converse Duality Theorem*) Let  $(\bar{u}, \bar{v}, \bar{m}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q})$  be efficient solution of (EMNSD) and fix  $\lambda = \bar{\lambda}$  in (EMNSP). Then under the following set of assumptions:

(i) For  $i = 1, 2, \dots, k$ ,

$$F_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i} F_i(\bar{u}, \bar{v}, 0) = G_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i} G_i(\bar{u}, \bar{v}, 0) = 0,$$

$$\bar{F}_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \bar{F}_i(\bar{u}, \bar{v}, 0) = \nabla_x \bar{F}_i(\bar{u}, \bar{v}, 0) = \nabla_y \bar{F}_i(\bar{u}, \bar{v}, 0) = 0,$$

$$\bar{G}_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \bar{G}_i(\bar{u}, \bar{v}, 0) = \nabla_x \bar{G}_i(\bar{u}, \bar{v}, 0) = \nabla_y \bar{G}_i(\bar{u}, \bar{v}, 0) = 0$$

(ii) for any  $i = 1, 2, \dots, k$  the Hessian matrix  $\nabla_{q_i q_i}(\bar{F}_i(\bar{u}, \bar{v}, \bar{q}_i) - m_i \bar{G}_i(\bar{u}, \bar{v}, \bar{q}_i))$  is positive or negative definite,

(iii) the set of vectors  $\{\nabla_x f_i(\bar{u}, \bar{v}) + w_i + \nabla_x \bar{F}_i(\bar{u}, \bar{v}, \bar{q}_i) - m_i \nabla_x g_i(\bar{u}, \bar{v}) - t_i + \nabla_x \bar{G}_i(\bar{u}, \bar{v}, \bar{q}_i), \nabla_x f_i(\bar{u}, \bar{v}) + w_i + \nabla_{q_i} \bar{F}_i(\bar{u}, \bar{v}, \bar{q}_i) - m_i \nabla_x g_i(\bar{u}, \bar{v}) - t_i + \nabla_{q_i} \bar{G}_i(\bar{u}, \bar{v}, \bar{q}_i), i = 1, 2, \dots, k\}$  are linearly independent.

(iv) if for  $\bar{q}_i \in \mathbb{R}^{n_1}$  such that  $\bar{q}_i \neq 0$  implies  $\sum_{i=1}^k \bar{q}_i (\nabla_x f_i(\bar{u}, \bar{v}) + w_i + \nabla_x \bar{F}_i(\bar{u}, \bar{v}, \bar{q}_i) - m_i \nabla_x g_i(\bar{u}, \bar{v}) - t_i + \nabla_x \bar{G}_i(\bar{u}, \bar{v}, \bar{q}_i)) \neq 0$

(v)  $\bar{m}_i > 0, i = 1, 2, \dots, k$ .

the point  $(\bar{u}, \bar{v}, \bar{m}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q})$  is a feasible solution of (EMFNSP). Furthermore if hypothesis of weak duality theorem also hold then  $(\bar{u}, \bar{v}, \bar{m}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q})$  is an efficient solution of (EMFNSP).

**Proof.** The proof follows on the lines of Theorem 4.4.2.



# Chapter 5

## Higher-order Symmetric Duality for Fractional Variational Programs

### 5.1 Introduction

Finding an optimal solution of a definite integral under certain boundary conditions has been studied under an independent branch of mathematics called calculus of variations. A basic problem in calculus of variations is to minimize or maximize a function of functions called functional. Thus the solutions of such variational problems are in the form of curves. A curve which maximizes or minimizes a given variational problem is called the extremal. Some of the earliest problems in calculus of variations include finding a curve enclosing the largest area given with a fixed perimeter; finding the minimum time for an object of mass  $m$  to slide from one point to another on a vertical plane. The solutions of these problems is obtained using calculus of variations and the curve thus obtained for the latter problem is a cycloid and the solution of former problem is a circle. Calculus of variations was initially used to solve the physical problems in mechanics but it has a diversity of applications in mathematical physics and engineering. Some of the application areas include dynamics of rigid bodies, theory of variations and computational fluid dynamics. The problems in calculus of variations can also be interpreted as dynamic generalizations of the mathematical programming problems. Though both the fields have independent developments but the interlinking of the ideas and concepts can prove to be prolific.

A general problem in calculus of variations is to find a curve  $x(t)$  satisfying  $x(t_o) = a$  and  $x(t_1) = b$  such that it optimizes (minimize/maximize) functional  $I$  where

$$I[x(t)] = \int_{t_o}^{t_1} f(t, x(t), x'(t))dt.$$

The optimizing curve  $x(t)$  is called the extremal of this given variational problem. Hanson [91] first connected a dual program to a variational problem in order to find a lower bound to a minimizing constrained variational problem. He obtained a dual of a variational problem and used functionally convex functions (non-convex functions) to obtain duality

theorems. An example of using duality for a variational control problem is explored in [15]. The preliminary work in this field can be found in [22, 159, 169, 170].

The problems of interest in this Chapter are symmetric variational problems. Mond and Hanson [169] first associated a symmetric nonlinear program to variational problems. Kim and Lee [130] studied symmetric variational problems of Mond-Weir type and obtained duality using pseudoconvex functions. Smart and Mond [190] generalized the symmetric dual formulation for invex functions. Kim and Lee [131] studied symmetric variational programs in multiobjective case using pseudoconvex functions. Using Schaible's parametrization approach to deal with fractional programs, Kim et al. [133] considered a symmetric multiobjective fractional program pairs to give duality relations using invex functions. To further generalize the previous results, Mishra et al. [161] used support functions to the symmetric multiobjective fractional problems and obtained results using pseudo-invex functions. Kailey and Gupta [118] presented nondifferentiable symmetric duality for first order multiobjective fractional variational programs under generalized convexity assumptions. Jayswal et al. [107] studied optimality of variational programs by using a modified objective function.

The second order duality for variational programs is studied in [39, 100, 101, 113]. Recently Jayswal and Jha [108] formulated second order fractional variational symmetric programs and studied duality results. Sharma [186] defined higher-order generalized convex functionals and studied duality for a mixed-type dual of a variational control problem. For other recent work in variational programming one can refer to the work in [10, 11, 12, 13, 139, 140, 152, 163].

Motivated by the work of Jayswal and Jha [108] and Sharma [186], we discuss the duality theory for higher-order symmetric fractional variational programs. Although some literature exist for higher-order duality in variational programs but to the best of our knowledge, higher-order symmetric duality for fractional variational programs has not been studied so far. In this chapter we discuss fractional multiobjective higher-order symmetric variational problems. The higher-order approximations of the functionals are used. These approximations are built on the second order approximations used in [82]. We define two classes of higher-order generalized convex functionals named as higher-order  $(\mathcal{F}, \alpha, \rho, d)$  convex functionals and higher-order  $(\mathcal{C}, \alpha, \rho, d)$  convex functionals. We construct two nontrivial numerical examples of these functionals also. Similar to the parametrization technique used in Chapter 4, we de-fractionalize the programs to give equivalent parametric programs. Then we state and prove the duality theorems between the parametric symmetric primal and dual programs. Lastly we construct example of weak duality theorem.

The contents of this Chapter are arranged as follows. Section 5.2 includes notations and preliminaries to be used in the sections in continuation. In Section 5.2, we introduce

higher-order convex functionals along with their illustrations. In Section 5.3, higher-order multiobjective symmetric variational programs are formulated. In Section 5.4, we state and prove duality theorems.

## 5.2 Notations and Preliminaries

Let  $I = [a, b] \subset \mathbb{R}$ . For each  $t \in I$ , we consider  $x : I \rightarrow \mathbb{R}^n$  and  $y : I \rightarrow \mathbb{R}^m$  to be continuously differentiable functions with derivatives  $\dot{x}(t)$  and  $\dot{y}(t)$  respectively. For simplicity, we write  $x$  instead of a function  $x(t)$ , the difference between the numbers and functions will be made clear from their definitions. Denote by  $X(I, \mathbb{R}^n)$  the space of piecewise smooth functions  $x$  with norm  $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ . The differential operator  $D$  is given as

$$u = Dx \Leftrightarrow x(t) = x(a) + \int_a^t u(s)ds.$$

Here  $x(a)$  is a specified boundary value. Therefore,  $D \equiv \frac{d}{dt}$  except at discontinuities. In similar manner, norm of  $y$  is defined for all  $y \in Y(I, \mathbb{R}^m)$ .

In this paper we will deal with the functionals of the type  $f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ ,  $g^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ , with  $f^i, g^i : I \times X \times X \times Y \times Y \rightarrow \mathbb{R}$  for  $i \in K$  and  $K = \{1, 2, \dots, k\}$ . Further,  $f_x^i, f_{\dot{x}}^i, f_y^i, f_{\dot{y}}^i$  denote the gradient vectors of each functional  $f^i$  with respect to  $x, \dot{x}, y$ , and  $\dot{y}$ . We write them as

$$f_x^i = \left( \frac{\partial f^i}{\partial x_1}, \frac{\partial f^i}{\partial x_2}, \dots, \frac{\partial f^i}{\partial x_n} \right)^T, \quad f_{\dot{x}}^i = \left( \frac{\partial f^i}{\partial \dot{x}_1}, \frac{\partial f^i}{\partial \dot{x}_2}, \dots, \frac{\partial f^i}{\partial \dot{x}_n} \right)^T$$

and

$$f_y^i = \left( \frac{\partial f^i}{\partial y_1}, \frac{\partial f^i}{\partial y_2}, \dots, \frac{\partial f^i}{\partial y_m} \right)^T, \quad f_{\dot{y}}^i = \left( \frac{\partial f^i}{\partial \dot{y}_1}, \frac{\partial f^i}{\partial \dot{y}_2}, \dots, \frac{\partial f^i}{\partial \dot{y}_m} \right)^T$$

Similarly we define the gradient vectors of functional  $g^i$  with respect to the corresponding subscripts. Assume that the functionals  $f^i$  and  $g^i, i \in K$  have continuous partial derivatives of all orders. Now  $D(f_{\dot{x}}^i) = f_{\dot{x}t}^i + f_{\dot{x}x}^i \dot{x} + f_{\dot{x}\dot{x}}^i \ddot{x} + f_{\dot{x}y}^i \dot{y} + f_{\dot{x}\dot{y}}^i \ddot{y}$ . Therefore

$$\begin{aligned} \frac{\partial}{\partial x}(D(f_{\dot{x}}^i)) &= Df_{\dot{x}x}^i, & \frac{\partial}{\partial \dot{x}}(D(f_{\dot{x}}^i)) &= Df_{\dot{x}\dot{x}}^i + f_{\dot{x}x}^i, & \frac{\partial}{\partial \ddot{x}}(D(f_{\dot{x}}^i)) &= Df_{\dot{x}\ddot{x}}^i, \\ \frac{\partial}{\partial y}(D(f_{\dot{x}}^i)) &= Df_{\dot{x}y}^i, & \frac{\partial}{\partial \dot{y}}(D(f_{\dot{x}}^i)) &= Df_{\dot{x}\dot{y}}^i + f_{\dot{x}y}^i, & \frac{\partial}{\partial \ddot{y}}(D(f_{\dot{x}}^i)) &= Df_{\dot{x}\ddot{y}}^i. \end{aligned}$$

The following definitions will be used in the subsequent sections.

**Definition 5.2.1** *Let  $\mathcal{A}$  be a closed and convex cone in  $\mathbb{R}^n$  having nonempty interior, then*

its positive polar cone is defined as

$$\mathcal{A}^* = \{z \in \mathbb{R}^n : x^T z \geq 0, \forall x \in \mathcal{A}\}.$$

**Definition 5.2.2** Consider a functional  $\mathcal{F} : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ .  $\mathcal{F}$  is sublinear with respect to sixth argument, if

$$1) \mathcal{F}(t, x, \dot{x}, u, \dot{u}; a_1 + a_2) \leq \mathcal{F}(t, x, \dot{x}, u, \dot{u}; a_1) + \mathcal{F}(t, x, \dot{x}, u, \dot{u}; a_2) \text{ and}$$

$$2) \mathcal{F}(t, x, \dot{x}, u, \dot{u}; \sigma a_1) = \sigma \mathcal{F}(t, x, \dot{x}, u, \dot{u}; a_1),$$

for all  $a_1, a_2 \in \mathbb{R}^n$ ,  $\sigma \in \mathbb{R}_+$ , and  $x, \dot{x}, u, \dot{u} \in X(I, \mathbb{R}^n)$ .

Firstly we introduce a higher-order  $(\mathcal{F}, \alpha, \rho, d)$  convex functional and show its existence using a non-trivial example. Assume that  $\mathcal{F}$  is a sublinear functional with respect to sixth argument and for each  $i \in K$ ,  $f^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$  and the corresponding higher-order approximation functions (in the sense of [186])  $h^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$  are differentiable functionals. Further assume that  $\alpha, d_i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \setminus \{0\}$  and  $\rho_i \in \mathbb{R}$ .

**Definition 5.2.3** The functional  $\int_a^b f^i(t, x, \dot{x}, y, \dot{y}) dt$  is said to be higher-order  $(\mathcal{F}, \alpha, \rho_i, d_i)$ -convex at  $u, \dot{u} \in X(I, \mathbb{R}^n)$ , for fixed  $y, \dot{y} \in Y(I, \mathbb{R}^m)$  and for all  $x, \dot{x} \in X(I, \mathbb{R}^n)$ , there exist  $h^i$  and  $q_i(t) \in \mathbb{R}^n$ , satisfying

$$\begin{aligned} \int_a^b [f^i(t, x, \dot{x}, y, \dot{y}) - f^i(t, u, \dot{u}, y, \dot{y}) - h^i(t, u, \dot{u}, y, \dot{y}, q_i) + q_i^T \nabla_{q_i}(h^i(t, u, \dot{u}, y, \dot{y}, q_i) \\ - \rho_i [d_i(t, x, \dot{x}, u, \dot{u})]^2)] dt \geq \int_a^b \mathcal{F}(t, x, \dot{x}, u, \dot{u}; \alpha(t, x, \dot{x}, u, \dot{u})(f_x^i(t, u, \dot{u}, y, \dot{y}) \\ - \frac{d}{dt} f_x^i(t, u, \dot{u}, y, \dot{y}) + \nabla_{q_i} h^i(t, u, \dot{u}, y, \dot{y}, q_i))) dt. \end{aligned}$$

**Example 5.2.1** Let  $I = [0, 1]$  and  $n = m = 1$ . Let  $X(I, \mathbb{R}_+)$  denote the space of piecewise smooth functions  $x : I \rightarrow \mathbb{R}_+$  and  $Y(I, \mathbb{R})$  denote the space of piecewise smooth functions  $y : I \rightarrow \mathbb{R}$ . Consider a functional  $f : I \times X \times X \times Y \times Y \rightarrow \mathbb{R}$  given by  $f(t, x, \dot{x}, y, \dot{y}) = x^2 + 2y + \dot{y}^2$  and  $h : I \times X \times X \times Y \times Y \times \mathbb{R} \rightarrow \mathbb{R}$  given by  $h = \frac{1}{2} q^T (f_{xx} - Df_{x\dot{x}}) q = \frac{1}{2} q^T (2) q = q^T q = q^2$ . Assume that  $\rho = 0$ ,  $d(t, x, \dot{x}, u, \dot{u}) = x - u$  and  $\alpha(t, x, \dot{x}, u, \dot{u}) = 1$ . Further assume the sublinear function is given as  $\mathcal{F}(t, x, \dot{x}, u, \dot{u}; a) = a(x - u)$ . On the

curve  $u(t) = 0$

$$\begin{aligned}
A &= \int_a^b [f(t, x, \dot{x}, y, \dot{y}) - f(t, u, \dot{u}, y, \dot{y}) - h(t, u, \dot{u}, y, \dot{y}, q) + q \nabla_q(h(t, u, \dot{u}, y, \dot{y}, q)) \\
&\quad - \rho[d(t, x, \dot{x}, u, \dot{u})]^2] dt \\
&= \int_0^1 (x^2 + 2y + \dot{y}^2 - (0^2 + 2y + \dot{y}^2) - q^2 + q \cdot (2q) - 0 \cdot (x - 0)^2) dt = \int_0^1 (x^2 + q^2) dt.
\end{aligned}$$

and

$$\begin{aligned}
B &= \int_a^b \mathcal{F}(t, x, \dot{x}, u, \dot{u}; \alpha(t, x, \dot{x}, u, \dot{u})(f_x(t, u, \dot{u}, y, \dot{y}) - \frac{d}{dt} f_{\dot{x}}(t, u, \dot{u}, y, \dot{y}) \\
&\quad + \nabla_q h(t, u, \dot{u}, y, \dot{y}, q))) dt \\
&= \int_0^1 \mathcal{F}(t, x, \dot{x}, u, \dot{u}; 2u + 2q) dt = \int_0^1 \mathcal{F}(t, x, \dot{x}, u, \dot{u}; 2q) dt.
\end{aligned}$$

Now

$$\begin{aligned}
A - B &= \int_0^1 (x^2 + q^2 - \mathcal{F}(t, x, \dot{x}, u, \dot{u}; 2q)) dt \\
&= \int_0^1 (x^2 + q^2 - 2xq) dt = \int_0^1 (x - q)^2 dt \geq 0, \quad \forall x \in X(I, \mathbb{R}_+).
\end{aligned}$$

Hence  $\int_0^1 (x^2 + 2y + \dot{y}^2) dt$  is higher-order  $(\mathcal{F}, \alpha, \rho, d)$  convex at  $u(t) = 0$  with respect to  $h = q^2$  and  $q \in R$ .

Now we introduce higher-order  $(\mathcal{C}, \alpha, \rho, d)$  convex functionals. Note that, in the static case,  $\mathcal{C}$  is a convex generalization of sublinear functional  $\mathcal{F}$ . Now in dynamic case, the extension of sublinear functional is a functional  $\mathcal{C}$  which is convex in its sixth argument and is defined below.

**Definition 5.2.4** Let  $\mathcal{C} : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a functional satisfying  $\mathcal{C}(t, x, \dot{x}, u, \dot{u}; 0) = 0$  for all  $x, \dot{x}, u, \dot{u} \in X(I, \mathbb{R}^n)$ .  $\mathcal{C}$  is convex in sixth argument if for all  $x, \dot{x}, u, \dot{u} \in X(I, \mathbb{R}^n)$ ,

$$\mathcal{C}(t, x, \dot{x}, u, \dot{u}; \lambda a_1 + (1 - \lambda) a_2) \leq \lambda \mathcal{C}(t, x, \dot{x}, u, \dot{u}; a_1) + (1 - \lambda) \mathcal{C}(t, x, \dot{x}, u, \dot{u}; a_2),$$

holds for each  $a_1, a_2 \in \mathbb{R}^n$  and  $\forall \lambda \in (0, 1)$ .

Consider a functional  $h^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$  and a positive functional  $\alpha : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \setminus \{0\}$ . Further we assume another functional  $d_i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying  $d(t, x, \dot{x}, u, \dot{u}) = 0 \Leftrightarrow x(t) = u(t)$ .

**Definition 5.2.5** The functional  $\int_a^b f^i(t, x, \dot{x}, u, \dot{u})dt$  is higher-order  $(\mathcal{C}, \alpha, \rho_i, d_i)$  convex at  $u, \dot{u} \in X(I, \mathbb{R}^n)$ , for fixed  $y, \dot{y} \in Y(I, \mathbb{R}^m)$  with respect to  $h^i$  if, for all  $x, \dot{x} \in X(I, \mathbb{R}^n)$ , there exist  $\rho_i \in \mathbb{R}$ ,  $q_i(t) \in \mathbb{R}^n$  such that the following hold;

$$\begin{aligned} \int_a^b [f^i(t, x, \dot{x}, y, \dot{y}) - f^i(t, u, \dot{u}, y, \dot{y}) - h^i(t, u, \dot{u}, y, \dot{y}, q_i) + q_i \nabla_{q_i}(h^i(t, u, \dot{u}, y, \dot{y}, q_i) \\ - \rho_i [d_i(t, x, \dot{x}, u, \dot{u})]^2)] dt \geq \int_a^b \alpha(t, x, \dot{x}, u, \dot{u}) \mathcal{C}(t, x, \dot{x}, u, \dot{u}; (f_x^i(t, u, \dot{u}, y, \dot{y}) \\ - \frac{d}{dt} f_{\dot{x}}^i(t, u, \dot{u}, y, \dot{y}) + \nabla_{q_i} h^i(t, u, \dot{u}, y, \dot{y}, q_i)) dt. \end{aligned}$$

### Special Cases:

1. If  $\mathcal{C}$  is a sublinear functional, then higher-order  $(\mathcal{C}, \alpha, \rho_i, d_i)$  convex functionals become higher-order  $(\mathcal{F}, \alpha, \rho_i, d_i)$  convex functionals where  $\mathcal{F}$  is a sublinear functional as defined in Definition 5.2.2. Further,
2. if the higher-order approximation  $h^i(t, u, \dot{u}, y, \dot{y}, q_i) = 0$  or  $q_i = 0$ , then higher-order  $(\mathcal{F}, \alpha, \rho_i, d_i)$  convex functionals are called  $(\mathcal{F}, \alpha, \rho_i, d_i)$  convex as given in [118];
3. if  $\rho = 0$ ,  $\alpha(t, x, \dot{x}, u, \dot{u}) = 1$ ,  $\mathcal{F}(t, x, \dot{x}, u, \dot{u}; a) = \eta(t, x, \dot{x}, u, \dot{u})^T a$  for  $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $q_i = 0$ , then functional  $\int_a^b f^i(t, x, \dot{x}, u, \dot{u})dt$  is said to be invex at  $x, \dot{x}$  as given in [190].

**Example 5.2.2** Let  $I = [0, 1]$  and  $n = m = 1$ . Let  $X(I, \mathbb{R}_+)$  denote the space of piecewise smooth functions  $x : I \rightarrow \mathbb{R}_+$  and  $Y(I, \mathbb{R})$  denote the space of piecewise smooth functions  $y : I \rightarrow \mathbb{R}$ . Consider a functional  $f : I \times X \times X \times Y \times Y \rightarrow \mathbb{R}$  given by  $f(t, x, \dot{x}, y, \dot{y}) = x^3 - y + \dot{x}$  and  $h$  is third order approximation of functional  $f$  given as  $h(t, x, \dot{x}, y, \dot{y}, \ddot{x}, \ddot{y}) = \frac{1}{2}q^T(f_{xx} - Df_{x\dot{x}})q + \frac{1}{6}q^2(f_{xxx} - Df_{xx\dot{x}})q = \frac{1}{2}q^T(6x)q + \frac{1}{6}q^2(6)q = 3xq^2 + q^3$ . Assume that  $\rho = -1$ ,  $d(t, x, \dot{x}, u, \dot{u}) = x - u$ ,  $q(t) = t$  and  $\mathcal{C}(t, x, \dot{x}, u, \dot{u}; a) = \frac{a^2(x-u)^2}{9}$  and

$$\alpha(t, x, \dot{x}, u, \dot{u}) = \begin{cases} \frac{1}{x^2 + u^2}; & \text{if } x(t) \neq 0 \\ 1; & \text{if } x(t) = 0. \end{cases}$$

At curve  $u(t) = 0$  and if  $x(t) \neq 0$ ,

$$\begin{aligned} A = \int_a^b [f(t, x, \dot{x}, y, \dot{y}) - f(t, u, \dot{u}, y, \dot{y}) - h(t, u, \dot{u}, y, \dot{y}, q) + q \nabla_q(h(t, u, \dot{u}, y, \dot{y}, q) \\ - \rho [d(t, x, \dot{x}, u, \dot{u})]^2)] dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 (x^3 - y + \dot{x} - (u - y + \dot{u}) - 3uq^2 - q^3 + q(6uq + 3q^2) - (-1)(x - u)^2)dt \\
&= \int_0^1 (x^3 + \dot{x} + x^2 + 2t^3)dt.
\end{aligned}$$

$$\begin{aligned}
B &= \int_a^b \alpha(t, x, \dot{x}, u, \dot{u}) \mathcal{C}(t, x, \dot{x}, u, \dot{u}; (f_x^i(t, u, \dot{u}, y, \dot{y}) - \frac{d}{dt} f_x^i(t, u, \dot{u}, y, \dot{y}) \\
&\quad + \nabla_{q_i} h^i(t, u, \dot{u}, y, \dot{y}, q_i))) dt. \\
&= \int_0^1 \alpha(t, x, \dot{x}, u, \dot{u}) \mathcal{C}(t, x, \dot{x}, u, \dot{u}; 3u^2 + 3q^2) dt \\
&= \int_0^1 \frac{1}{x^2 + u^2} \mathcal{C}(t, x, \dot{x}, u, \dot{u}; 3u^2 + 3q^2) dt = \int_0^1 t^4 dt.
\end{aligned}$$

Now

$$A - B = \int_0^1 (x^3 + \dot{x} + x^2 + 2t^3 - t^4) dt \geq 0, \quad \forall x \in X(I, \mathbb{R}_+). \quad (5.2.1)$$

At  $x(t) = 0$ ,  $A = \int_0^1 2t^3 dt = 0.5$  and  $B = \int_0^1 t^4 x^2 dt = 0$ . So  $A - B = 2 \int_0^1 t^3 dt = 0.5 \geq 0$ . Hence  $\int_0^1 (x^3 - y + \dot{x}) dt$  is higher-order  $(\mathcal{C}, \alpha, \rho, d)$  convex at  $u(t) = 0$  with respect to  $h = 3xq^2 + q^3$ .

For convenience, the following notations are used.

$$\begin{aligned}
x, \dot{x}, y, \dot{y} &= \widetilde{xy}, & u, \dot{u}, v, \dot{v} &= \widetilde{uv}, \\
x, \dot{x}, v, \dot{v} &= \widetilde{xv}, & u, \dot{u}, y, \dot{y} &= \widetilde{uy}.
\end{aligned}$$

### 5.3 Higher Order Symmetric Programs

Consider two closed convex cones  $\mathcal{A}_1 \subset \mathbb{R}^n$  and  $\mathcal{A}_2 \subset \mathbb{R}^m$  such that  $\text{int}\mathcal{A}_1 \neq \emptyset$  and  $\text{int}\mathcal{A}_2 \neq \emptyset$ . The sets  $\mathcal{A}_1^*, \mathcal{A}_2^*$  are positive dual cones of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  respectively. We formulate the following pair of multiobjective fractional higher-order symmetric variational programs:

Primal (MFVP)

$$\text{Minimize} \quad \left( \frac{\int_a^b (f^1(t, \widetilde{xy}) + H^1(t, \widetilde{xy}, p_1) - p_1^T \nabla_{p_1} H^1(t, \widetilde{xy}, p_1)) dt}{\int_a^b (g^1(t, \widetilde{xy}) + G^1(t, \widetilde{xy}, p_1) - p_1^T \nabla_{p_1} G^1(t, \widetilde{xy}, p_1)) dt}, \dots, \right.$$

$$\left( \frac{\int_a^b (f^k(t, \widetilde{xy}) + H^k(t, \widetilde{xy}, p_k) - p_k^T \nabla_{p_k} H^k(t, \widetilde{xy}, p_k)) dt}{\int_a^b (g^k(t, \widetilde{xy}) + G^k(t, \widetilde{xy}, p_k) - p_k^T \nabla_{p_k} G^k(t, \widetilde{xy}, p_k)) dt} \right)$$

subject to

$$x(a) = 0, x(b) = 0, y(a) = 0, y(b) = 0,$$

$$\dot{x}(a) = 0, \dot{x}(b) = 0, \dot{y}(a) = 0, \dot{y}(b) = 0,$$

$$- \sum_{i=1}^k \lambda_i (f_y^i(t, \widetilde{xy}) - Df_y^i(t, \widetilde{xy}) + \nabla_{p_i} H^i(t, \widetilde{xy}, p_i)$$

$$- L_i(t, \widetilde{xy}, p_i)(g_y^i(t, \widetilde{xy}) - Dg_y^i(t, \widetilde{xy}) + \nabla_{p_i} G^i(t, \widetilde{xy}, p_i))) \in \mathcal{A}_2^*,$$

$$y^T \sum_{i=1}^k \lambda_i (f_y^i(t, \widetilde{xy}) - Df_y^i(t, \widetilde{xy}) + \nabla_{p_i} H^i(t, \widetilde{xy}, p_i)$$

$$- L_i(t, \widetilde{xy}, p_i)(g_y^i(t, \widetilde{xy}) - Dg_y^i(t, \widetilde{xy}) + \nabla_{p_i} G^i(t, \widetilde{xy}, p_i))) \geq 0,$$

$$\lambda \in \mathbb{R}^k, \lambda > 0, x \in \mathcal{A}_1, t \in I.$$

Dual (MFVD)

$$\text{Maximize } \left( \frac{\int_a^b (f^1(t, \widetilde{uv}) + \phi^1(t, \widetilde{uv}, q_1) - q_1^T \nabla_{q_1} \phi^1(t, \widetilde{uv}, q_1)) dt}{\int_a^b (g^1(t, \widetilde{uv}) + \xi^1(t, \widetilde{uv}, q_1) - q_1^T \nabla_{q_1} \xi^1(t, \widetilde{uv}, q_1)) dt}, \dots, \right. \\ \left. \frac{\int_a^b (f^k(t, \widetilde{uv}) + \phi^k(t, \widetilde{uv}, q_k) - q_k^T \nabla_{q_k} \phi^k(t, \widetilde{uv}, q_k)) dt}{\int_a^b (g^k(t, \widetilde{uv}) + \xi^k(t, \widetilde{uv}, q_k) - q_k^T \nabla_{q_k} \xi^k(t, \widetilde{uv}, q_k)) dt} \right)$$

subject to

$$u(a) = 0, u(b) = 0, v(a) = 0, v(b) = 0,$$

$$\dot{u}(a) = 0, \dot{u}(b) = 0, \dot{v}(a) = 0, \dot{v}(b) = 0,$$

$$\sum_{i=1}^k \lambda_i (f_x^i(t, \widetilde{uv}) - Df_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i)$$

$$- M_i(t, \widetilde{uv}, q_i)(g_x^i(t, \widetilde{uv}) - Dg_x^i(t, \widetilde{uv}) + \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i))) \in \mathcal{A}_1^*,$$

$$u^T \sum_{i=1}^k \lambda_i (f_x^i(t, \widetilde{uv}) - Df_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i)$$

$$- M_i(t, \widetilde{uv}, q_i)(g_x^i(t, \widetilde{uv}) - Dg_x^i(t, \widetilde{uv}) + \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i))) \leq 0,$$

$$\lambda \in \mathbb{R}^k, \lambda > 0, v \in \mathcal{A}_2, t \in I.$$

$$\text{where } L_i(t, \widetilde{xy}, p_i) = \frac{\int_a^b (f^i(t, \widetilde{xy}) + H^i(t, \widetilde{xy}, p_i) - p_i^T \nabla_{p_i} H^i(t, \widetilde{xy}, p_i)) dt}{\int_a^b (g^i(t, \widetilde{xy}) + G^i(t, \widetilde{xy}, p_i) - p_i^T \nabla_{p_i} G^i(t, \widetilde{xy}, p_i)) dt}.$$

$$M_i(t, \widetilde{uv}, q_i) = \frac{\int_a^b (f^i(t, \widetilde{uv}) + \phi^i(t, \widetilde{uv}, q_i) - q_i^T \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i)) dt}{\int_a^b (g^i(t, \widetilde{uv}) + \xi^i(t, \widetilde{uv}, q_i) - q_i^T \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i)) dt}.$$

In the above programs  $f^i, g^i : I \times X \times X \times Y \times Y \rightarrow \mathbb{R}$ ,  $H^i, G^i : I \times X \times X \times Y \times Y \times \mathbb{R}^m \rightarrow \mathbb{R}$  and  $\phi^i, \xi^i : I \times X \times X \times Y \times Y \times \mathbb{R}^n \rightarrow \mathbb{R}$ .  $H^i$  and  $G^i$  are higher-order approximations of the functionals  $f^i$  and  $g^i$  with respect to  $y, \dot{y}$  respectively.  $\phi^i$  and  $\xi^i$  are higher-order approximations of the functionals  $f^i$  and  $g^i$  with respect to  $x, \dot{x}$  respectively. The cones  $\mathcal{A}_1^*$  and  $\mathcal{A}_2^*$  are positive polar cones of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  respectively.  $p_i(t) \in \mathbb{R}^m$  and  $q_i(t) \in \mathbb{R}^n$  for  $i = 1, 2, \dots, k$ .

### Special Cases:

1. If  $x$  and  $y$  are independent of  $t$ , then the above programs reduce to higher-order multiobjective symmetric fractional dual programs with cone constraints discussed by Dubey and Gupta [60] in the absence of support functions.
2. If  $k = 1$ ,  $H^1(t, \widetilde{xy}, p) = \frac{1}{2}p^T A(t, \widetilde{xy}, p)p$ ,  $G^1(t, \widetilde{xy}, p) = \frac{1}{2}p^T B(t, \widetilde{xy}, p)p$ , and  $\phi^1(t, \widetilde{uv}, q) = \frac{1}{2}q^T Y(t, \widetilde{uv}, q)q$ ,  $\xi^1(t, \widetilde{uv}, q) = \frac{1}{2}q^T Z(t, \widetilde{uv}, q)q$ , where  $A$  and  $B$  are second order approximations of  $f$  and  $g$  respectively, with respect to  $y, \dot{y}$  and  $Y$  and  $Z$  are second order approximations of  $f$  and  $g$  respectively, with respect to  $x, \dot{x}$ , then the above symmetric dual pair reduces to the second order program obtained in [108]. Here functions look slightly different because we have neglected the zero terms and ignored some of the terms which cancel with each other. Ultimately the result will be unaltered.
3. If  $\mathcal{A}_1 = \mathbb{R}_+^n$ ,  $\mathcal{A}_2 = \mathbb{R}_+^m$ ,  $H^1(t, \widetilde{xy}, p) = G^1(t, \widetilde{xy}, p) = 0$  and  $\phi^1(t, \widetilde{uv}, q) = \xi^1(t, \widetilde{uv}, q) = 0$ , or  $p = 0$ ,  $q = 0$  i.e. no approximations are used, then the problems become multiobjective symmetric fractional variational problems studied in [133].

In order to simplify the programs, we introduce two parameters  $l_i$  and  $m_i$ ,  $i = 1, 2, \dots, k$ , in the sense of Dinkelbach [56]. Thus the multiobjective fractional symmetric variational programs are converted into parameterized multiobjective symmetric variational programs:

Primal (EMFVP)

Minimize  $l = (l_1, l_2, \dots, l_k)$

subject to

$$x(a) = 0, x(b) = 0, y(a) = 0, y(b) = 0,$$

$$\dot{x}(a) = 0, \dot{x}(b) = 0, \dot{y}(a) = 0, \dot{y}(b) = 0,$$

$$\begin{aligned} \int_a^b (f^i(t, \widetilde{xy}) + H^i(t, \widetilde{xy}, p_i) - p_i^T \nabla_{p_i} H^i(t, \widetilde{xy}, p_i)) dt - l_i \int_a^b (g^i(t, \widetilde{xy}) \\ + G^i(t, \widetilde{xy}, p_i) - p_i^T \nabla_{p_i} G^i(t, \widetilde{xy}, p_i)) dt = 0, \quad i = 1, 2, \dots, k, \end{aligned} \quad (5.3.1)$$

$$\begin{aligned} - \sum_{i=1}^k \lambda_i (f_y^i(t, \widetilde{xy}) - Df_y^i(t, \widetilde{xy}) + \nabla_{p_i} H^i(t, \widetilde{xy}, p_i) \\ - l_i (g_y^i(t, \widetilde{xy}) - Dg_y^i(t, \widetilde{xy}) + \nabla_{p_i} G^i(t, \widetilde{xy}, p_i))) \in \mathcal{A}_2^*, \end{aligned} \quad (5.3.2)$$

$$\begin{aligned} y^T \sum_{i=1}^k \lambda_i (f_y^i(t, \widetilde{xy}) - Df_y^i(t, \widetilde{xy}) + \nabla_{p_i} H^i(t, \widetilde{xy}, p_i) \\ - l_i (g_y^i(t, \widetilde{xy}) - Dg_y^i(t, \widetilde{xy}) + \nabla_{p_i} G^i(t, \widetilde{xy}, p_i))) \geq 0, \end{aligned} \quad (5.3.3)$$

$$\lambda \in \mathbb{R}^k, \lambda > 0, x \in \mathcal{A}_1, l \in \mathbb{R}_+^k, t \in I. \quad (5.3.4)$$

Dual (EMFVD)

Maximize  $m = (m_1, m_2, \dots, m_k)$

subject to

$$u(a) = 0, u(b) = 0, v(a) = 0, v(b) = 0,$$

$$\dot{u}(a) = 0, \dot{u}(b) = 0, \dot{v}(a) = 0, \dot{v}(b) = 0,$$

$$\begin{aligned} \int_a^b (f^i(t, \widetilde{uv}) + \phi^i(t, \widetilde{uv}, q_i) - q_i^T \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i)) dt - m_i \int_a^b (g^i(t, \widetilde{uv}) \\ + \xi^i(t, \widetilde{uv}, q_i) - q_i^T \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i)) dt = 0, \quad i = 1, 2, \dots, k, \end{aligned} \quad (5.3.5)$$

$$\begin{aligned} \sum_{i=1}^k \lambda_i (f_x^i(t, \widetilde{uv}) - Df_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i) \\ - m_i (g_x^i(t, \widetilde{uv}) - Dg_x^i(t, \widetilde{uv}) + \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i))) \in \mathcal{A}_1^*, \end{aligned} \quad (5.3.6)$$

$$\begin{aligned} u^T \sum_{i=1}^k \lambda_i (f_x^i(t, \widetilde{uv}) - Df_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i) \\ - m_i (g_x^i(t, \widetilde{uv}) - Dg_x^i(t, \widetilde{uv}) + \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i))) \leq 0, \end{aligned} \quad (5.3.7)$$

$$\lambda \in \mathbb{R}^k, \lambda > 0, v \in \mathcal{A}_2, m \in \mathbb{R}_+^k, t \in I. \quad (5.3.8)$$

## 5.4 Duality Results

In this section, we present duality theorems for symmetric variational problems (EMFVP) and (EMFVD). The dynamic generalizations of the symmetric programs for single objective problems were first introduced in [169] and further generalizations were carried out by several authors (see, [118], [130] and [136]). Second order symmetric duality relations are also explored in [109].

**Theorem 5.4.1 (Weak Duality Theorem)** *Let  $(x, y, l, \lambda, p)$  be a feasible solution of primal (EMFVP) and  $(u, v, m, \lambda, q)$  be a feasible solution of dual (EMFVD). Suppose that there exist sublinear functionals  $\mathcal{F}$  and  $\bar{\mathcal{F}}$  such that  $\mathcal{F} : I \times X \times X \times X \times X \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\bar{\mathcal{F}} : I \times Y \times Y \times Y \times Y \times \mathbb{R}^m \rightarrow \mathbb{R}$  such that for each  $i = 1, 2, \dots, k$ ,*

- (i)  $\int_a^b (f^i(t, \cdot, \cdot, v, \dot{v}) - m_i g^i(t, \cdot, \cdot, v, \dot{v})) dt$  is higher-order  $(\mathcal{F}, \alpha, \rho_i, d_i)$  convex at  $u, \dot{u}$  with respect to  $\phi^i - m_i \xi^i$  ( such that  $\phi^i - m_i \xi^i : I \times X \times X \times Y \times Y \times \mathbb{R}^n \rightarrow \mathbb{R}$  ) and  $q_i \in \mathbb{R}^n$  for fixed  $v, \dot{v}$ ,
- (ii)  $\int_a^b (-f^i(t, x, \dot{x}, \cdot, \cdot) + l_i g^i(t, x, \dot{x}, \cdot, \cdot)) dt$  is higher-order  $(\bar{\mathcal{F}}, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)$  convex at  $y, \dot{y}$  with respect to  $-H^i + l_i G^i$  ( such that  $-H^i + l_i G^i : I \times X \times X \times Y \times Y \times \mathbb{R}^m \rightarrow \mathbb{R}$  ) and  $p_i \in \mathbb{R}^m$  for fixed  $x, \dot{x}$ ,
- (iii)  $\mathcal{F}(t, \widetilde{xu}; a) + a^T u \geq 0, \forall a \in \mathcal{A}_1^*, \quad \bar{\mathcal{F}}(t, \widetilde{yv}; b) + b^T y \geq 0, \forall b \in \mathcal{A}_2^*,$
- (iv)  $\rho_i, \bar{\rho}_i \geq 0$ , and
- (v)  $g^i(t, \widetilde{xv}) > 0$ .

Then  $l \not\leq m$ .

**Proof.** Let if possible under the given assumptions the result is not true. Then  $l \leq m$  which means for  $\lambda \in \mathbb{R}^k, \lambda > 0$  and  $g^i(t, \widetilde{xv}) > 0$ , we have

$$\sum_{i=1}^k \lambda_i (l_i - m_i) \int_a^b g^i(t, \widetilde{xv}) dt < 0. \quad (5.4.1)$$

Since for  $i = 1, 2, \dots, k$ ,  $\int_a^b (f^i(t, \cdot, \cdot, v, \dot{v}) - m_i g^i(t, \cdot, \cdot, v, \dot{v})) dt$  is higher-order  $(\mathcal{F}, \alpha, \rho_i, d_i)$  convex at  $u, \dot{u}$  with respect to  $\phi^i - m_i \xi^i$  and  $q_i \in \mathbb{R}^n$  for fixed  $v, \dot{v}$ , so we have the following

$$\begin{aligned} & \int_a^b [f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv}) - (f^i(t, \widetilde{uv}) - m_i g^i(t, \widetilde{uv})) - (\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i)) \\ & \quad + q_i \nabla_{q_i} (\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i)) - \rho_i [d_i(t, \widetilde{xu})]^2] dt \\ & \geq \int_a^b \mathcal{F}(t, \widetilde{xu}; \alpha(t, \widetilde{xu}) (f_x^i(t, \widetilde{uv}) - m_i g_x^i(t, \widetilde{uv}) - \frac{d}{dt} (f_{\dot{x}}^i(t, \widetilde{uv}) - m_i g_{\dot{x}}^i(t, \widetilde{uv}))) \end{aligned}$$

$$+\nabla_{q_i}(\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i)))dt \quad (5.4.2)$$

Now as  $\lambda > 0$ , we get

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \int_a^b [f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv}) - (f^i(t, \widetilde{uv}) - m_i g^i(t, \widetilde{uv})) - (\phi^i(t, \widetilde{uv}, q_i) \\ & \quad - m_i \xi^i(t, \widetilde{uv}, q_i)) + q_i \nabla_{q_i}(\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i)) - \rho_i[d_i(t, \widetilde{xu})]^2]dt \\ & \geq \sum_{i=1}^k \lambda_i \int_a^b \mathcal{F}(t, \widetilde{xu}; \alpha(t, \widetilde{xu})(f_x^i(t, \widetilde{uv}) - m_i g_x^i(t, \widetilde{uv}) - \frac{d}{dt}(f_x^i(t, \widetilde{uv}) - m_i g_x^i(t, \widetilde{uv})) \\ & \quad + \nabla_{q_i}(\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i))))dt \end{aligned} \quad (5.4.3)$$

Since  $\mathcal{F}$  is sublinear functional with respect to sixth argument, therefore

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \int_a^b [f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv}) - (f^i(t, \widetilde{uv}) - m_i g^i(t, \widetilde{uv})) - (\phi^i(t, \widetilde{uv}, q_i) \\ & \quad - m_i \xi^i(t, \widetilde{uv}, q_i)) + q_i \nabla_{q_i}(\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i)) - \rho_i[d_i(t, \widetilde{xu})]^2]dt \\ & \geq \int_a^b \mathcal{F}(t, \widetilde{xu}; \alpha(t, \widetilde{xu}) \sum_{i=1}^k \lambda_i (f_x^i(t, \widetilde{uv}) - m_i g_x^i(t, \widetilde{uv}) - \frac{d}{dt}(f_x^i(t, \widetilde{uv}) - m_i g_x^i(t, \widetilde{uv})) \\ & \quad + \nabla_{q_i}(\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i))))dt. \end{aligned} \quad (5.4.4)$$

Setting  $\bar{a} = \sum_{i=1}^k \lambda_i (f_x^i(t, \widetilde{uv}) - m_i (g_x^i(t, \widetilde{uv})) - \frac{d}{dt}(f_x^i(t, \widetilde{uv}) - m_i (g_x^i(t, \widetilde{uv}))) + \nabla_{q_i}(\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i)))$ . Since  $\alpha(t, \widetilde{xu}) \in \mathbb{R}_+ \setminus \{0\} \Rightarrow \alpha(t, \widetilde{xu})\bar{a} = a$  (say) As  $\bar{a} \in \mathcal{A}_1^*$  by (5.3.6), gives that  $a \in \mathcal{A}_1^*$  (because  $\mathcal{A}_1^*$  is a cone). Then assumption (iii) says that  $\mathcal{F}(t, \widetilde{xu}; a) + a^T u \geq 0, \forall a \in \mathcal{A}_1^*$ . So we have

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \int_a^b [f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv}) - (f^i(t, \widetilde{uv}) - m_i g^i(t, \widetilde{uv})) - (\phi^i(t, \widetilde{uv}, q_i) \\ & \quad - m_i \xi^i(t, \widetilde{uv}, q_i)) + q_i \nabla_{q_i}(\phi^i(t, \widetilde{uv}, q_i) - m_i \xi^i(t, \widetilde{uv}, q_i)) \\ & \quad - \rho_i[d_i(t, \widetilde{xu})]^2]dt \geq 0. \end{aligned} \quad (5.4.5)$$

Then using (5.3.5) and (iv), the above inequality reduces to the following

$$\sum_{i=1}^k \lambda_i \int_a^b [f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv})] \geq 0. \quad (5.4.6)$$

Now by (ii)  $\int_a^b (-f^i(t, x, \dot{x}, \cdot, \cdot) + l_i g^i(t, x, \dot{x}, \cdot, \cdot))dt$  is higher-order  $(\bar{\mathcal{F}}, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)$  convex at  $y, \dot{y}$

with respect to  $-H^i + l_i G^i$  and  $p_i \in \mathbb{R}^m$  for fixed  $x, \dot{x}$  and  $\lambda > 0$  we obtain

$$\begin{aligned}
& \sum_{i=1}^k \lambda_i \int_a^b [-f^i(t, \widetilde{xv}) + l_i g^i(t, \widetilde{xv}) - (-f^i(t, \widetilde{xy}) + l_i g^i(t, \widetilde{xy})) \\
& \quad - (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i)) + p_i \nabla_{p_i} (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i))] dt \\
& \geq \sum_{i=1}^k \lambda_i \int_a^b \bar{\mathcal{F}}(t, \widetilde{vy}; \bar{\alpha}(t, \widetilde{yv}) (\nabla_y (-f^i(t, \widetilde{xy}) + l_i g^i(t, \widetilde{xy})) - \frac{d}{dt} (-f_y^i(t, \widetilde{xy}) + l_i g_y^i(t, \widetilde{xy})) \\
& \quad + \nabla_{p_i} (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i))) + \bar{\rho}_i [\bar{d}_i(t, \widetilde{yv})]^2) dt.
\end{aligned} \tag{5.4.7}$$

Thus using sublinearity of  $\bar{\mathcal{F}}$  we get

$$\begin{aligned}
& \sum_{i=1}^k \lambda_i \int_a^b [-f^i(t, \widetilde{xv}) + l_i g^i(t, \widetilde{xv}) - (-f^i(t, \widetilde{xy}) + l_i g^i(t, \widetilde{xy})) \\
& \quad - (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i)) + p_i \nabla_{p_i} (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i))] dt \\
& \geq \int_a^b \bar{\mathcal{F}}(t, \widetilde{vy}; \bar{\alpha}(t, \widetilde{yv}) \sum_{i=1}^k \lambda_i (\nabla_y (-f^i(t, \widetilde{xy}) + l_i g^i(t, \widetilde{xy})) - \frac{d}{dt} (-f_y^i(t, \widetilde{xy}) + l_i g_y^i(t, \widetilde{xy})) \\
& \quad + \nabla_{p_i} (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i))) + \bar{\rho}_i [\bar{d}_i(t, \widetilde{yv})]^2) dt.
\end{aligned} \tag{5.4.8}$$

Take  $\sum_{i=1}^k \lambda_i (\nabla_y (-f^i(t, \widetilde{xy}) + l_i g^i(t, \widetilde{xy})) - \frac{d}{dt} (-f_y^i(t, \widetilde{xy}) + l_i g_y^i(t, \widetilde{xy})) + \nabla_{p_i} (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i))) = \bar{b}$  which due to (5.3.2) lies in  $\mathcal{A}_2^*$ . As  $\bar{\alpha}(t, \widetilde{yv}) \in \mathbb{R}_+ \setminus \{0\}$  we have  $\bar{\alpha}(t, \widetilde{yv}) \bar{b} = b(\text{say}) \in \mathcal{A}_2^*$ . This along with (iii) and (5.3.3) implies

$$\begin{aligned}
& \sum_{i=1}^k \lambda_i \int_a^b [-f^i(t, \widetilde{xv}) + l_i g^i(t, \widetilde{xv}) - (-f^i(t, \widetilde{xy}) + l_i g^i(t, \widetilde{xy})) \\
& \quad - (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i)) + p_i \nabla_{p_i} (-H^i(t, \widetilde{xy}, p_i) + l_i G^i(t, \widetilde{xy}, p_i)) \\
& \quad - \bar{\rho}_i [\bar{d}_i(t, \widetilde{yv})]^2] dt \geq 0.
\end{aligned} \tag{5.4.9}$$

$(x, y, l, \lambda, p)$  is given to be a feasible solution of the primal, which along with (5.3.1) and assumption (iv) simplifies the above equation to the following

$$\sum_{i=1}^k \lambda_i \int_a^b [-f^i(t, \widetilde{xv}) + l_i g^i(t, \widetilde{xv})] dt \geq 0. \tag{5.4.10}$$

Adding equations (5.4.6) and (5.4.10) to obtain

$$\sum_{i=1}^k \lambda_i \int_a^b (l_i - m_i) g^i(t, \widetilde{xv}) dt \geq 0.$$

$$\Leftrightarrow \sum_{i=1}^k \lambda_i (l_i - m_i) \int_a^b g^i(t, \widetilde{xv}) dt \geq 0. \quad (5.4.11)$$

But this is a contradiction to equation (5.4.1) which means that the supposition  $l \leq m$  was wrong. Hence the result.

**Theorem 5.4.2 (Weak Duality Theorem)** *Let  $(x, y, l, \lambda, p)$  be a feasible solution of primal (EMFVP) and let  $(u, v, m, \lambda, q)$  be a feasible solution of dual (EMFVD). Suppose that there exist convex functionals  $\mathcal{C}, \bar{\mathcal{C}}$  where  $\mathcal{C} : I \times X \times X \times X \times X \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\bar{\mathcal{C}} : I \times Y \times Y \times Y \times Y \times \mathbb{R}^m \rightarrow \mathbb{R}$  such that for each  $i = 1, 2, \dots, k$  the following assumptions hold;*

- (i)  $\int_a^b f^i(t, \cdot, \cdot, v, \dot{v}) dt$  is higher-order  $(\mathcal{C}, \alpha, \rho_i, d_i)$  convex at  $u, \dot{u}$  with respect to  $\phi^i$  and  $q_i(t) \in \mathbb{R}^n$  for fixed  $v, \dot{v}$  and  $-\int_a^b g^i(t, \cdot, \cdot, v, \dot{v}) dt$  is higher-order  $(\mathcal{C}, \alpha, \rho_i, d_i)$  convex at  $u, \dot{u}$  with respect to  $-\xi^i$  and  $q_i(t) \in \mathbb{R}^n$  (such that  $\phi^i, \xi^i : I \times X \times X \times Y \times Y \times \mathbb{R}^n \rightarrow \mathbb{R}$ ) and  $q_i(t) \in \mathbb{R}^n$  for fixed  $v, \dot{v}$ ,
- (ii)  $-\int_a^b f^i(t, x, \dot{x}, \cdot, \cdot) dt$  is higher-order  $(\bar{\mathcal{C}}, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)$  convex at  $y, \dot{y}$  with respect to  $-H^i$  and  $p_i(t) \in \mathbb{R}^m$  for fixed  $x, \dot{x}$  and  $\int_a^b g^i(t, x, \dot{x}, \cdot, \cdot) dt$  is higher-order  $(\bar{\mathcal{C}}, \bar{\alpha}, \bar{\rho}_i, \bar{d}_i)$  convex at  $y, \dot{y}$  with respect to  $G^i$  and  $p_i(t) \in \mathbb{R}^m$  (such that  $H^i, G^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ ) and  $p_i \in \mathbb{R}^m$  for fixed  $x, \dot{x}$ ,
- (iii)  $\alpha(t, \widetilde{xu})\mathcal{C}(t, \widetilde{xu}, a) + a^T u \geq 0, \forall a \in \mathcal{A}_1^*$  and  $\bar{\alpha}(t, \widetilde{yv})\bar{\mathcal{C}}(t, \widetilde{yv}, b) + b^T y \geq 0, \forall b \in \mathcal{A}_2^*$ ,
- (iv)  $\rho_i, \bar{\rho}_i \geq 0$ , and
- (v)  $g^i(t, \widetilde{xv}) > 0$ .

Then  $l \not\leq m$ .

**Proof.** Since  $\int_a^b f^i(t, \cdot, \cdot, v, \dot{v}) dt$  and  $-\int_a^b g^i(t, \cdot, \cdot, v, \dot{v}) dt$  are higher-order  $(\mathcal{C}, \alpha, \rho_i, d_i)$  convex at  $u, \dot{u}$  with respect to  $\phi^i, q_i(t)$  and  $-\xi^i, q_i(t)$  respectively, for fixed  $v, \dot{v}$ , so we have the following

$$\begin{aligned} & \int_a^b [f^i(t, \widetilde{xv}) - f^i(t, \widetilde{uv}) - \phi^i(t, \widetilde{uv}, q_i) + q_i \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i) - \rho_i [d_i(t, \widetilde{xu})]^2] dt \\ & \geq \int_a^b \alpha(t, \widetilde{xu}) \mathcal{C}(t, \widetilde{xu}; (f_x^i(t, \widetilde{uv}) - \frac{d}{dt} f_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i))) dt \end{aligned} \quad (5.4.12)$$

and

$$\int_a^b [-(g^i(t, \widetilde{xv}) - g^i(t, \widetilde{uv})) + \xi^i(t, \widetilde{uv}, q_i) - q_i \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i) - \rho_i [d_i(t, \widetilde{xu})]^2] dt$$

$$\geq \int_a^b \alpha(t, \widetilde{xu}) \mathcal{C}(t, \widetilde{xu}; (-g_x^i(t, \widetilde{uv}) + \frac{d}{dt} g_x^i(t, \widetilde{uv}) - \nabla_{q_i}(\xi^i(t, \widetilde{uv}, q_i)))) dt. \quad (5.4.13)$$

It is known that in the feasible region the denominator and numerator are positive and non-negative respectively. And  $\lambda > 0$ ,  $m \geq 0$ , so multiply equation (5.4.12) by  $\frac{\lambda_i}{\tau}$  and equation (5.4.13) by  $\frac{m_i \lambda_i}{\tau}$ , where  $\tau = \sum_{i=1}^k \lambda_i + \lambda_i m_i$ . Then taking sum for  $i = 1, 2, \dots, k$  we get

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau} \int_a^b \left[ f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv}) - (f^i(t, \widetilde{uv}) - m_i g^i(t, \widetilde{uv})) \right. \\ & \quad \left. - \phi^i(t, \widetilde{uv}, q_i) + q_i \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i) + m_i (\xi^i(t, \widetilde{uv}, q_i) - q_i \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i)) \right] dt \\ & \quad - \sum_{i=1}^k \frac{\lambda_i}{\tau} \int_a^b [d_i(t, \widetilde{xu})]^2 \rho_i (1 + m_i) dt \\ & \geq \int_a^b \alpha(t, \widetilde{xu}) \sum_{i=1}^k \frac{\lambda_i}{\tau} \mathcal{C}(t, \widetilde{xu}; (f_x^i(t, \widetilde{uv}) - \frac{d}{dt} f_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i))) dt \\ & \quad + \int_a^b \alpha(t, \widetilde{xu}) \sum_{i=1}^k \frac{\lambda_i m_i}{\tau} \mathcal{C}(t, \widetilde{xu}; (-g_x^i(t, \widetilde{uv}) + \frac{d}{dt} g_x^i(t, \widetilde{uv}) \\ & \quad - \nabla_{q_i}(\xi^i(t, \widetilde{uv}, q_i)))) dt. \end{aligned} \quad (5.4.14)$$

Now if the functional  $\mathcal{C}$  is convex in sixth argument, then the following is true

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau} \mathcal{C}(t, \widetilde{xu}; (f_x^i(t, \widetilde{uv}) - \frac{d}{dt} f_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i))) dt \\ & \quad + \sum_{i=1}^k \frac{\lambda_i m_i}{\tau} \mathcal{C}(t, \widetilde{xu}; (-g_x^i(t, \widetilde{uv}) + \frac{d}{dt} g_x^i(t, \widetilde{uv}) - \nabla_{q_i}(\xi^i(t, \widetilde{uv}, q_i)))) dt \\ & \geq \mathcal{C}(t, \widetilde{xu}; \sum_{i=1}^k \frac{\lambda_i}{\tau} (f_x^i(t, \widetilde{uv}) - \frac{d}{dt} f_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i))) \\ & \quad + \sum_{i=1}^k \frac{\lambda_i m_i}{\tau} (-g_x^i(t, \widetilde{uv}) + \frac{d}{dt} g_x^i(t, \widetilde{uv}) - \nabla_{q_i}(\xi^i(t, \widetilde{uv}, q_i))). \end{aligned} \quad (5.4.15)$$

Multiplying the above inequality by a positive functional does not change the sign of the inequality. So the equation (5.4.14) takes the following form

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau} \int_a^b \left[ f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv}) - (f^i(t, \widetilde{uv}) - m_i g^i(t, \widetilde{uv})) \right. \\ & \quad \left. - \phi^i(t, \widetilde{uv}, q_i) + q_i \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i) + m_i (\xi^i(t, \widetilde{uv}, q_i) - q_i \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i)) \right] dt \end{aligned}$$

$$-\sum_{i=1}^k \frac{\lambda_i}{\tau} \rho_i(1+m_i) \int_a^b [d_i(t, \widetilde{xu})]^2 dt \geq \int_a^b \alpha(t, \widetilde{xu}) \mathfrak{C}(t, \widetilde{xu}; \frac{1}{\tau} a1) dt, \quad (5.4.16)$$

where  $a1 = \sum_{i=1}^k \lambda_i (f_x^i(t, \widetilde{uv}) - \frac{d}{dt} f_x^i(t, \widetilde{uv}) + \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i) - m_i(g_x^i(t, \widetilde{uv}) - \frac{d}{dt} g_x^i(t, \widetilde{uv}) + \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i))$ . Now due to (5.3.6)  $a1 \in \mathcal{A}_1^* \Rightarrow \frac{a1}{\tau} = a \in \mathcal{A}_1^*(\cdot: \tau > 0)$ . Use this, (iii) and (5.3.7) in (5.4.16) to get

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau} \int_a^b \left[ f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv}) - (f^i(t, \widetilde{uv}) - m_i g^i(t, \widetilde{uv})) \right. \\ & \quad \left. - \phi^i(t, \widetilde{uv}, q_i) + q_i \nabla_{q_i} \phi^i(t, \widetilde{uv}, q_i) + m_i (\xi^i(t, \widetilde{uv}, q_i) - q_i \nabla_{q_i} \xi^i(t, \widetilde{uv}, q_i)) \right] dt \\ & \quad - \sum_{i=1}^k \frac{\lambda_i}{\tau} \rho_i(1+m_i) \int_a^b [d_i(t, \widetilde{xu})]^2 dt \geq 0. \end{aligned} \quad (5.4.17)$$

This inequality, along with (5.3.5) and (iv), reduces to

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau} \int_a^b [f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv})] \geq 0 \\ & \Rightarrow \sum_{i=1}^k \lambda_i \int_a^b [f^i(t, \widetilde{xv}) - m_i g^i(t, \widetilde{xv})] \geq 0, \quad \text{as } \tau > 0. \end{aligned} \quad (5.4.18)$$

On the same lines one can obtain the following, using (ii)

$$\sum_{i=1}^k \lambda_i \int_a^b [-f^i(t, \widetilde{xv}) + l_i g^i(t, \widetilde{xv})] dt \geq 0. \quad (5.4.19)$$

Adding the inequalities (5.4.18) and (5.4.19) to get

$$\sum_{i=1}^k \lambda_i \int_a^b (l_i - m_i) g^i(t, \widetilde{xv}) dt \geq 0. \quad (5.4.20)$$

If  $l \leq m$ , it means  $l_i \leq m_i, \forall i = 1, 2, \dots, k$  and  $l_r < m_r$  for some  $r = 1, 2, \dots, k$ . This along with some  $\lambda > 0$  i.e.,  $\lambda_i > 0, \forall i = 1, 2, \dots, k$  and as  $g^i(t, \widetilde{xv}) > 0 (\Rightarrow \int_a^b g^i(t, \widetilde{xv}) dt > 0)$  give

$$\sum_{i=1}^k \lambda_i (l_i - m_i) \int_a^b g^i(t, \widetilde{xv}) dt < 0. \quad (5.4.21)$$

Equations (5.4.20) and (5.4.21) contradict each other. Hence  $l \not\leq m$ .

**Example 5.4.1** Let  $k = 1 = n = m, I = [0, 1], \mathcal{A}_1 = \mathcal{A}_2 = \mathbb{R}_+$ . So we have  $\mathcal{A}_1^* = \mathcal{A}_2^* =$

$\mathbb{R}_+$  and  $X$  and  $Y$  denote the spaces of piecewise smooth functions  $x$  and  $y$  respectively; where  $x : I \rightarrow \mathbb{R}_+$  and  $y : I \rightarrow \mathbb{R}_+$ . Now, let functions  $f, g : I \times X \times X \times Y \times Y \rightarrow \mathbb{R}$  are given by

$$f = x^3 \quad \text{and} \quad g = x + y$$

. Then second order approximations are given as  $H = 0, G = 0, \phi = 3q^2u, \xi = 0$ . Assume  $\mathcal{C}(t, x, \dot{x}, u, \dot{u}; a) = a(x - u)$  and  $\bar{\mathcal{C}}(t, y, \dot{y}, v, \dot{v}; b) = b^2(v - y)$  Consider the following pair of symmetric dual problems:

$$\begin{aligned} \text{Primal (EP)} \quad & \text{Minimize} \quad l \\ & \text{subject to} \\ & x(0) = x(1) = 0 = y(0) = y(1), \\ & \dot{x}(0) = \dot{x}(1) = 0 = \dot{y}(0) = \dot{y}(1), \\ & \int_0^1 (x^3 - l(x + y))dt = 0 \\ & -(-l(1)) \in \mathcal{A}_1^* \\ & y^T(0 - l(1)) \geq 0, \lambda > 0, x \in \mathcal{A}_1, l \in R_+, t \in I. \end{aligned}$$

And the corresponding dual model is

$$\begin{aligned} \text{Dual (ED)} \quad & \text{Maximize} \quad m \\ & \text{subject to} \\ & u(0) = u(1) = 0 = v(0) = v(1), \\ & \dot{u}(0) = \dot{u}(1) = 0 = \dot{v}(0) = \dot{v}(1) \\ & \int_0^1 (u^3 + 3uq^2 - q(6uq) - m(u + v))dt = 0, \\ & 3u^2 + 6uq - m(1) \in \mathcal{A}_1^*, \\ & u^T(3u^2 + 6uq - m(1)) \leq 0, \lambda > 0, v \in \mathcal{A}_2, m \in R_+, t \in I. \end{aligned}$$

So we have  $(x, y, l, \lambda, p) = (t^2, 0, \frac{3}{7}, 1, t)$  a feasible solution of (EP) and  $(u, v, m, \lambda, q) = (0, t^2, 0, 1, t)$  is feasible solution of (ED). Now we shall show that the assumptions of second weak duality theorem hold at these feasible solutions of primal and dual problems respectively.

- (i)  $\int_0^1 f(t, \cdot, \cdot, v, \dot{v})dt = \int_0^1 x^3 dt$  is higher-order  $(\mathcal{C}, \alpha, \rho, d)$  convex at  $u, \dot{u} = 0$  with respect to  $\phi = 3q^2u$  and  $q(t) = t$  for  $v(t) = t^2$  (here  $\alpha(t, x, \dot{x}, u, \dot{u}) = 1, d(t, x, \dot{x}, u, \dot{u}) =$

$x - u, \rho = 0$ ) as shown below

$$\begin{aligned} & \int_a^b [f(t, x, \dot{x}, y, \dot{y}) - f(t, 0, 0, y, \dot{y}) - h(t, 0, 0, y, \dot{y}, q) + q \nabla_q (h(t, 0, 0, y, \dot{y}, q) \\ & - \rho [d(t, x, \dot{x}, 0, 0)]^2) dt - \int_a^b \alpha(t, x, \dot{x}, 0, 0) \mathcal{C}(t, x, \dot{x}, 0, 0; (f_x(t, 0, 0, y, \dot{y}) \\ & - \frac{d}{dt} f_{\dot{x}}(t, 0, 0, y, \dot{y}) + \nabla_q h(t, 0, 0, y, \dot{y}, q))] dt = \int_0^1 x^3 dt \geq 0, \forall x \in X. \end{aligned}$$

Now  $-\int_0^1 g(t, \cdot, \cdot, v, \dot{v}) dt = -\int_0^1 (x+y) dt$  is higher-order  $(\mathcal{C}, \alpha, \rho, d)$  convex at  $u, \dot{u} = 0$  with respect to  $\xi$  and  $q(t) = t$  for  $v = t^2$  as shown below

$$\begin{aligned} & \int_0^1 [-(g(t, x, \dot{x}, v, \dot{v}) - g(t, x, \dot{x}, 0, 0)) + \xi(t, 0, 0, v, \dot{v}, q) - q \nabla_q \xi(t, 0, 0, v, \dot{v}, q) \\ & - \rho [d(t, x, \dot{x}, 0, 0)]^2] dt - \int_0^1 \alpha(t, x, \dot{x}, 0, 0) \mathcal{C}(t, x, \dot{x}, 0, 0; (-g_x(t, x, \dot{x}, 0, 0) \\ & + \frac{d}{dt} g_{\dot{x}}(t, 0, 0, v, \dot{v}) - \nabla_q (\xi(t, 0, 0, v, \dot{v}, q))) dt = 0, \forall x \in X. \end{aligned}$$

(ii) On the same lines we can prove that  $-\int_0^1 f(t, x, \dot{x}, \cdot, \cdot) dt$  is higher-order  $(\bar{\mathcal{C}}, \bar{\alpha}, \bar{\rho}, \bar{d})$  convex at  $y, \dot{y} = 0$  with respect to  $H$  and  $p(t) = t$  for  $x(t) = t^2$  and  $\int_0^1 g(t, x, \dot{x}, \cdot, \cdot) dt$  is higher-order  $(\bar{\mathcal{C}}, \bar{\alpha}, \bar{\rho}, \bar{d})$  convex at  $y, \dot{y} = 0$  with respect to  $G^i$  and  $p(t) = t$  for  $x(t) = t^2$  where  $H$  and  $G$  are as given above.  $\bar{\alpha}(x, \dot{x}, u, \dot{u}) = 1, \bar{\rho} = 0, \bar{d} = (x - u)$ .

(iii)  $\mathcal{C}(t, x, \dot{x}, u, \dot{u}; a) + au = a(x - u) + au = ax \geq 0, \forall a \in \mathcal{A}_1^*, x \in \mathcal{A}_1$  and  $\bar{\mathcal{C}}(t, y, \dot{y}, v, \dot{v}; b) + bx = b^2(v - y) + by = b^2v \geq 0, \forall b \in \mathcal{A}_2^*, v \in \mathcal{A}_2$ , as  $\mathcal{A}_2 = \mathbb{R}_+$ , and  $y = 0$ .

(iv) Since  $g(t, x, \dot{x}, v, \dot{v}) = x + v$  and  $x \in X, v \in Y$ , so  $\int_0^1 (x + v) dt \geq 0$  and as discussed  $\rho, \bar{\rho} = 0$ .

Now  $l - m = \frac{3}{7} \not\leq 0$ . Hence we have an example of weak duality theorem.

Now we establish the conditions under which an efficient solution of the primal (EMFVP) becomes an efficient solution of the corresponding dual (EMFVD).

**Theorem 5.4.3 (Strong Duality Theorem)** Let  $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{p})$  be an efficient solution of (EMFVP) then fix  $\lambda = \bar{\lambda}$ . If the following hold;

(i)  $H_i = G_i = 0, \nabla_y H^i = \nabla_y G^i = 0, H_{\dot{y}}^i = G_{\dot{y}}^i = 0, \nabla_x H^i = \nabla_x G^i = 0, \nabla_{p_i} H^i = \nabla_{p_i} G^i = 0, \nabla_{\dot{x}} H^i = \nabla_{\dot{x}} G^i = 0$ , at  $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{p} = 0)$  and  $\phi_i = \xi_i = 0, \nabla_{q_i} \phi^i = \nabla_{q_i} \xi^i = 0$ , for each  $i = 1, 2, \dots, k$  at  $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$ ,

- (ii) the Hessian matrices  $\nabla_{p_i p_i} H_i - l_i \nabla_{p_i p_i} G_i$  are positive or negative semi-definite,
- (iii) the set of vectors  $\{f_y^i - Df_y^i + \nabla_{p_i} H_i - l_i(g_y^i - Dg_y^i + \nabla_{p_i} G_i); i = 1, 2, \dots, k\}$  is linearly independent,
- (iv) if  $p_i \neq 0, (p_i \in \mathbb{R}^n), i = 1, \dots, k$  then  $\sum_{i=1}^k p_i^T (f_y^i - Df_y^i + \nabla_{p_i} H_i - l_i(g_y^i - Dg_y^i + \nabla_{p_i} G_i)) \neq 0$ ,

then

1.  $p_i = 0, i \in K$  and
2.  $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$  is a feasible solution of (EMFVD).

Furthermore, if the conditions of any one of the above Theorems (5.4.1 or 5.4.2) hold then  $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$  is an efficient solution of dual (EMFVD) and objective values of both the programs are equal.

**Proof.** If  $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{p})$  is an efficient solution of (EMFVP), then there exist  $\mu \in \mathbb{R}^k, \beta \in \mathbb{R}^k, \gamma : I \rightarrow \mathcal{A}_2, \psi : I \rightarrow \mathbb{R}_+, \eta \in \mathbb{R}_+^k$  such that the following equations hold at this point

$$\begin{aligned}
& \left[ \sum_{i=1}^k \beta_i (f_x^i - Df_x^i - l_i(g_x^i - Dg_x^i) + H_x^i - p_i^T \nabla_{xp} H^i - l_i(G_x^i - p_i^T \nabla_{xp} G^i)) \right. \\
& - D(H_x^i - p_i^T \nabla_{xp} H^i - l_i(G_x^i - p_i^T \nabla_{xp} G^i)) + (\gamma(t) - \psi(t)\bar{y}(t))^T \left[ \sum_{i=1}^k \lambda_i (f_{yx}^i - Df_{yx}^i \right. \\
& + \nabla_{xp} H^i - l_i(g_{yx}^i - Dg_{yx}^i + \nabla_{xp} G^i)) \left. \right] - D \left[ (\gamma(t) - \psi(t)\bar{y}(t))^T \left( \sum_{i=1}^k \lambda_i (f_{yx}^i - Df_{yx}^i \right. \right. \\
& - f_{yx}^i + \nabla_{p_i x} H^i - l_i(g_{yx}^i - Dg_{yx}^i - g_{yx}^i + \nabla_{xp} G^i)) \left. \right) \left. \right] \\
& - D^2 \left[ (\gamma(t) - \psi(t)\bar{y}(t))^T \left( \sum_{i=1}^k \lambda_i (f_{yx}^i - g_{yx}^i) \right) \right]^T (x - \bar{x}) \geq 0, \forall x \in \mathcal{A}_1. \tag{5.4.22}
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^k (\beta_i - \psi(t)\lambda_i) (f_y^i - Df_y^i + \nabla_{p_i} H^i - l_i(g_y^i - Dg_y^i + \nabla_{p_i} G^i)) + \sum_{i=1}^k \beta_i (\nabla_y H^i \\
& - \nabla_{p_i} H^i - p_i^T \nabla_{p_i y} H^i - l_i(\nabla_y G^i - \nabla_{p_i} G^i - p_i^T \nabla_{p_i y} G^i)) + (\gamma(t) - \psi(t)\bar{y})^T \sum_{i=1}^k \lambda_i (f_{yy}^i \\
& - Df_{yy}^i + \nabla_{p_i y} H^i - l_i(g_{yy}^i - Dg_{yy}^i + \nabla_{p_i y} G^i)) - D \left[ \sum_{i=1}^k \beta_i (\nabla_y H^i - p_i^T \nabla_{p_i y} H^i \right. \\
& - l_i(\nabla_y G^i - p_i^T \nabla_{p_i y} G^i)) - (\gamma(t) - \psi(t)\bar{y})^T \sum_{i=1}^k \lambda_i (f_{yy}^i - (Df_{yy}^i)_y + \nabla_{p_i y} H^i - l_i(g_{yy}^i
\end{aligned}$$

$$-(Dg_{\dot{y}}^i)_{\dot{y}} + \nabla_{p_i} G^i)) + D^2[-(\gamma(t) - \psi(t)\bar{y})^T \sum_{i=1}^k \lambda_i (Df_{\dot{y}}^i - l_i Df_{\dot{y}}^i)_{\dot{y}}] = 0. \quad (5.4.23)$$

$$(\gamma(t) - \psi(t)\bar{y}(t))^T (f_y^i - Df_{\dot{y}}^i + \nabla_{p_i} H^i - l_i (g_y^i - Dg_{\dot{y}}^i + \nabla_{p_i} G^i)) - \eta = 0, \quad i = 1, 2, \dots, k, \quad (5.4.24)$$

$$\mu_i - \beta_i (g^i + G^i - p_i^T \nabla_{p_i} G^i) - (\gamma(t) - \psi(t)\bar{y}(t))^T (g_y^i - Dg_{\dot{y}}^i + \nabla_{p_i} G^i) = 0, \quad i = 1, 2, \dots, k, \quad (5.4.25)$$

$$((\gamma(t) - \psi(t)\bar{y}(t))^T \lambda_i - \beta_i p_i) (\nabla_{p_i p_i} H^i - l_i \nabla_{p_i p_i} G^i) = 0, \quad i = 1, 2, \dots, k, \quad (5.4.26)$$

$$\gamma(t)^T \sum_{i=1}^k \lambda_i (f_y^i - Df_{\dot{y}}^i + \nabla_{p_i} H^i - l_i (g_y^i - Dg_{\dot{y}}^i + \nabla_{p_i} G^i)) = 0, \quad (5.4.27)$$

$$\psi(t)\bar{y}^T \sum_{i=1}^k \lambda_i (f_y^i - Df_{\dot{y}}^i + \nabla_{p_i} H^i - l_i (g_y^i - Dg_{\dot{y}}^i + \nabla_{p_i} G^i)) = 0, \quad (5.4.28)$$

$$\eta^T \lambda = 0, \quad (\mu, \beta, \gamma, \psi(t), \eta) \neq 0. \quad (5.4.29)$$

Since  $\lambda > 0$ , so due to (5.4.29)

$$\eta = 0. \quad (5.4.30)$$

So equation (5.4.24) becomes

$$(\gamma(t) - \psi(t)\bar{y}(t))^T (f_y^i - Df_{\dot{y}}^i + \nabla_{p_i} H^i - l_i (g_y^i - Dg_{\dot{y}}^i + \nabla_{p_i} G^i)) = 0, \quad i = 1, 2, \dots, k, \quad (5.4.31)$$

Now hypothesis (ii) and equation (5.4.26) imply

$$(\gamma(t) - \psi(t)\bar{y}(t))^T \lambda_i = \beta_i p_i, \quad i = 1, \dots, k, \quad t \in I. \quad (5.4.32)$$

Equations (5.4.31) and (5.4.32) along with  $\lambda > 0$  give that

$$\beta_i p_i^T (f_y^i - Df_{\dot{y}}^i + \nabla_{p_i} H^i - l_i (g_y^i - Dg_{\dot{y}}^i + \nabla_{p_i} G^i)) = 0, \quad i = 1, 2, \dots, k, \quad (5.4.33)$$

**CLAIM 1:**  $\beta_i \neq 0$ , for any  $i = 1, 2, \dots, k$ .

Let if possible,  $\beta_{i_0} = 0$  for some  $i_0 \in \{1, 2, \dots, k\}$ ,  $\forall t \in I$ . As  $\lambda \neq 0$  we get from equation (5.4.32) that  $\gamma(t) - \psi(t)\bar{y}(t) = 0$  for every  $t \in I$ , we can conclude the following, due to equation (5.4.32)

$$\gamma(t) - \psi(t)\bar{y}(t) = 0 \forall t \in I \Rightarrow \beta_i p_i = 0, \text{ for every } i = 1, \dots, k. \quad (5.4.34)$$

So (5.4.23) becomes

$$\begin{aligned}
& \sum_{i=1}^k (\beta_i - \psi(t)\lambda_i)(f_y^i - Df_y^i + \nabla_{p_i}H^i - l_i(g_y^i - Dg_y^i + \nabla_{p_i}G^i)) \\
& + \sum_{i=1}^k \beta_i [H_y^i - \nabla_{p_i}H^i - p_i^T \nabla_{py}H^i - l_i(G_y^i - \nabla_{p_i}G^i - p_i^T \nabla_{py}G^i)] \\
& - D \sum_{i=1}^k \beta_i [H_y^i - p_i^T \nabla_{py}H^i - l_i(G_y^i - p_i^T \nabla_{py}G^i)] = 0.
\end{aligned} \tag{5.4.35}$$

Now if  $\beta_i p_i = 0$  (i.e. either  $\beta_i = 0$  or  $p_i = 0$  for each  $i = 1, 2, \dots, k$ ) the second and third summands turn out to be zero. So one gets

$$\sum_{i=1}^k (\beta_i - \psi(t)\lambda_i)(f_y^i - Df_y^i + \nabla_{p_i}H^i - l_i(g_y^i - Dg_y^i + \nabla_{p_i}G^i)) = 0. \tag{5.4.36}$$

Hypothesis (iii) imply  $\beta_i - \psi(t)\lambda_i = 0 \Rightarrow \beta_i = \psi(t)\lambda_i, \forall i = 1, 2, \dots, k$ . Now as  $\beta_{i_0} = 0$  implies that  $\psi(t) = 0, t \in I$ . Which further implies that  $\beta_i = 0$ , for every  $i = 1, 2, \dots, k$ . This combined with equation (5.4.25) gives  $\mu_i = 0, i = 1, 2, \dots, k$ . Also equation (5.4.34) yields  $\gamma(t) = 0, t \in I$  which imply  $(\mu, \beta, \gamma, \psi, \eta) = 0$ . But this is contradiction to (5.4.29). Hence the Claim 1 is proved.

Henceforth equation (5.4.33) can be written as

$$p_i^T (f_y^i - Df_y^i + \nabla_{p_i}H^i - l_i(g_y^i - Dg_y^i + \nabla_{p_i}G^i)) = 0, \quad i = 1, 2, \dots, k, \tag{5.4.37}$$

$$\Rightarrow \sum_{i=1}^k p_i^T (f_y^i - Df_y^i + \nabla_{p_i}H^i - l_i(g_y^i - Dg_y^i + \nabla_{p_i}G^i)) = 0. \tag{5.4.38}$$

This along with (iv) gives  $p_i = 0, \forall i$ . Use this in equation (5.4.32) to get  $\gamma(t) - \psi(t)\bar{y}(t) = 0, t \in I$  putting all the obtained values in equation (5.4.23) to get the following

$$\sum_{i=1}^k (\beta_i - \psi(t)\lambda_i)(f_y^i - Df_y^i - l_i(g_y^i - Dg_y^i)) = 0. \tag{5.4.39}$$

This along with (iii) gives

$$\beta_i = \psi(t)\lambda_i, \quad \forall t \in I \tag{5.4.40}$$

**CLAIM 2:**  $\psi(t) \neq 0$ , for any  $t \in I$ .

It is obvious. Because if for some  $\bar{t} \in I, \psi(\bar{t}) = 0$ , then by (5.4.40) we have  $\beta_i = \psi(\bar{t})\lambda_i$  which implies  $\beta_i = 0$ . Which is not true as we have shown in Claim 1. So  $\psi(t) \neq 0$ .

0, for any  $t \in I$ . Now use this along with  $p_i = 0$  in equation (5.4.32) to get

$$\gamma(t) = \psi(t)\bar{y}(t) \Rightarrow \bar{y}(t) = \frac{\gamma(t)}{\psi(t)} \in \mathcal{A}_2. \quad (5.4.41)$$

Using the facts  $p_i = 0$  and  $\gamma(t) = \psi(t)\bar{y}(t)$  in equation (5.4.22) we obtain

$$(x - \bar{x})^T \sum_{i=1}^k \beta_i (f_x^i - Df_{\bar{x}}^i - l_i(g_x^i - Dg_{\bar{x}}^i)) \geq 0, \forall x \in \mathcal{A}_1. \quad (5.4.42)$$

Using  $\beta_i = \psi(t)\lambda_i$  obtained above in equation (5.4.42) to get

$$\psi(t)(x - \bar{x})^T \sum_{i=1}^k \lambda_i (f_x^i - Df_{\bar{x}}^i - l_i(g_x^i - Dg_{\bar{x}}^i)) \geq 0, \forall x \in \mathcal{A}_1. \quad (5.4.43)$$

Since  $\psi(t) > 0$  and  $\psi(t) \neq 0$  we get

$$(x - \bar{x})^T \sum_{i=1}^k \lambda_i (f_x^i - Df_{\bar{x}}^i - l_i(g_x^i - Dg_{\bar{x}}^i)) \geq 0, \forall x \in \mathcal{A}_1. \quad (5.4.44)$$

Now for some  $x \in \mathcal{A}_1$ ,  $x + \bar{x} \in \mathcal{A}_1$  ( $\mathcal{A}_1$  being convex cone). This gives

$$x^T \sum_{i=1}^k \lambda_i (f_x^i - Df_{\bar{x}}^i - l_i(g_x^i - Dg_{\bar{x}}^i)) \geq 0. \quad (5.4.45)$$

Since  $x \in \mathcal{A}_1$  we have

$$\sum_{i=1}^k \lambda_i (f_x^i - Df_{\bar{x}}^i - l_i(g_x^i - Dg_{\bar{x}}^i)) \in \mathcal{A}_1^*. \quad (5.4.46)$$

Put  $x = 0$  in (5.4.44) to get

$$-\bar{x}^T \sum_{i=1}^k \lambda_i (f_x^i - Df_{\bar{x}}^i - l_i(g_x^i - Dg_{\bar{x}}^i)) \geq 0. \quad (5.4.47)$$

And if  $x = 2\bar{x}$  equation (5.4.44) becomes

$$\bar{x}^T \sum_{i=1}^k \lambda_i (f_x^i - Df_{\bar{x}}^i - l_i(g_x^i - Dg_{\bar{x}}^i)) \geq 0. \quad (5.4.48)$$

Adding the previous two equations we get

$$\bar{x}^T \sum_{i=1}^k \lambda_i (f_x^i - Df_x^i - l_i(g_x^i - Dg_x^i)) = 0. \quad (5.4.49)$$

Thus we get that  $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$  is a feasible solution of dual. Moreover objective function values of primal and dual are equal due to (5.3.1) and (5.3.5). The efficiency is obtained from weak duality theorem.



# Chapter 6

## Higher-Order Duality for Minimax Fractional Programming Problems

### 6.1 Introduction

For minimax programming problems, the initial study of optimality and duality results was carried out by Schmitendorf [185] and Tanimoto [206] respectively. Then Yadav and Mukherjee [217] formulated dual model for fractional minimax programs with convex functions and obtained duality results. The minimax program studied by Yadav and Mukherjee [217] has the following form:

$$\begin{aligned} \text{Problem (MXFP)} \quad & \text{Minimize } \phi(x) \\ & \text{subject to } x \in X_{mx} = \{x \in X \mid g(x) \leq 0\}. \end{aligned} \tag{6.1.1}$$

The function  $\phi(\cdot)$  is as follows

$$\phi(x) = \sup_{y \in \mathcal{Y}} \frac{f(x, y)}{h(x, y)}.$$

In (MXFP),  $\mathcal{Y}$  is a compact subset of  $R^l$ ,  $f(\cdot, \cdot)$ ,  $h(\cdot, \cdot) : R^n \times R^l \rightarrow R$  are differentiable functions on  $R^n \times R^m$  and  $f(x, y) \geq 0$ ,  $h(x, y) > 0$  for each  $(x, y) \in X \times \mathcal{Y}$ .

Consider the sets

$$\mathcal{Y}(x) = \{y \in \mathcal{Y} : \frac{f(x, y)}{h(x, y)} = \sup_{z \in \mathcal{Y}} \frac{f(x, z)}{h(x, z)}\}.$$

$$\mathcal{Z} = \{(s, t, y) : s \text{ ranges over the positive integers with } 1 \leq s \leq n+1, t = (t_1, t_2, \dots, t_s),$$

$$\sum_{i=1}^s t_i = 1, t \geq 0, y = (y_1, y_2, \dots, y_s) \text{ such that each } y_i \in \mathcal{Y}(x) \subset \mathbb{R}^l,$$

so we have  $y$  as an  $ls$  – dimensional vector}.

Chandra and Kumar [36] customized the dual model given in [217] together with introducing a parametric type dual for fractional minimax programming problems. Chandra

and Kumar [36] proposed the following parametric dual model for the minimax fractional problem (MXFP):

$$\text{Problem (MXFD)} \quad \underset{(s,t,\bar{y}) \in \mathcal{Z}}{\text{Maximize}} \quad \sup_{(x,\mu,v) \in \mathcal{H}_1(s,t,\bar{y})} v$$

$\mathcal{D}_1(s,t,\bar{y})$  denote the set of triplets  $(x,\mu,v) \in \mathbb{R}^n \times \mathbb{R}^m \times R$  satisfying

$$\begin{aligned} \sum_{i=1}^s t_i \nabla(f(x, y_i)) - v h(x, y_i) + \sum_{i=1}^m \mu_i \nabla g_i(x) &= 0, \\ \sum_{i=1}^s t_i (f(x, y_i) - v h(x, y_i)) &\geq 0, \\ \sum_{i=1}^m \mu_i \nabla g_i(x) &\geq 0, \quad \mu \geq 0 \\ t_i \geq 0, i = 1, 2, \dots, s, \quad \sum_{i=1}^s t_i &= 1, \quad \{y_1, y_2, \dots, y_s\} \subset \mathcal{Y}(x). \end{aligned}$$

Similar to earlier dual model, if for an element  $(s,t,\bar{y}) \in \mathcal{Y}$  the set  $\mathcal{D}_1$  is empty, then the supremum over it will be taken as  $-\infty$ .

Various forms of parametric as well as parameter free dual programs were introduced for fractional minimax programs involving generalized convexity [149], generalized invexity [148] and  $(\mathcal{F}, \rho, \theta)$  generalized convexity [147]. Lai et al. [142] considered a non-differentiable minimax fractional program containing square root terms of positive semidefinite matrices and established N&S optimality conditions together with the construction of weak duality, strong duality and strict converse duality theorems for a parametric dual model for the said program. Another dimension into the parameter free dual models for non-differentiable fractional programs is explored in [141]. Ahmad [5] studied Mond-Weir dual for non-differentiable minimax fractional problems and obtained duality results using generalized  $(\mathcal{F}, \alpha, \rho, d)$  convex functions. Yuan et al. [221] further modified the concept of generalized convexity by introducing  $(\mathcal{C}, \alpha, \rho, d)$  convex functions and studied duality for a non-differentiable minimax fractional program.

The duality theory for second order minimax programs was studied by Ahmad et al. [8] and for various extensions of second order duality in minimax programming one can refer to ([4, 8, 21, 85, 97, 102, 103, 112, 160], and the work within). Dangar and Gupta [50] constructed necessary optimality conditions and duality theorems involving second order  $(\mathcal{C}, \alpha, \rho, d)$  convex functions for minimax fractional program containing square root term and then Gupta et al. [85] put forth second order  $(\mathcal{F}, \alpha, \rho, d)/(\mathcal{C}, \alpha, \rho, d)$  type-I functions

and studied a second order parametric dual model for minimax fractional problems.

Ahmad [3], Jayswal et al. [111] and Muhiuddin et al. [175] relaxed the convexity assumptions to construct the higher-order dual programs for non-differentiable minimax problems. Jayswal et al. [111] attained weak, strong and converse duality theorems for non-differentiable minimax program involving higher-order generalized pseudo-convex type functions. For non-differentiable minimax fractional programs, the higher-order dual and the duality theorems were studied involving higher higher-order pseudoquasi type -I functions in [3] and involving  $\eta$ -pseudoconvex functions in [175]. Stancu-Minasian et al. [195] have studied higher minimax fractional programs with infinite number of constraints, using  $(\Phi, \rho) - V$  invex functions.

Motivated by the above developments, we study minimax fractional programs which involve support functions in objective function as well as in constraints. We study optimality conditions for the said program under the assumptions of higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  convexity defined by Dubey et al. [61]. Then we construct a parametric dual model and to validate its existence duality theorems are proved.

This Chapter is divided into two sections. Section 6.2 consists of problem formulation and some preliminaries required for the subsequent sections. The optimality conditions are also given in this Section. Section 6.3 comprises of the dual model along with duality theorems.

## 6.2 Problem Formulation and Preliminaries

We consider non-differentiable minimax fractional programming problems of the following form:

Problem (MXNFP)

$$\begin{aligned} & \text{Minimize } \sup_{x \in \mathbb{R}^n} \sup_{y \in \mathcal{Y}} \frac{f(x, y) + S(x|B)}{g(x, y) - S(x|D)}, \\ & \text{subject to } h_j(x) + S(x|E_j) \leq 0, \quad j = 1, 2, \dots, m. \end{aligned}$$

We denote by  $X_{mxn}$ , the set of feasible solutions of this problem and  $X_{mxn} := \{x \in X \mid h_j(x) + S(x|E_j) \leq 0, \text{ for } j = 1, 2, \dots, m\}$ . The functions  $f, g : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}$  are given to be twice continuously differentiable functions on  $\mathbb{R}^n \times \mathbb{R}^l$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a twice continuously differentiable function on  $\mathbb{R}^n$ . Here  $B, D$  and  $E_j$  are compact convex sets in  $\mathbb{R}^n$ . The functions  $S(x|B), S(x|D)$  and  $S(x|E_j)$  denote the support functions of  $B, D$  and  $E_j$ ,  $j = 1, 2, \dots, m$  respectively. Assume that  $f(x, y) + S(x|B) \geq 0$  and  $g(x, y) - S(x|D) > 0$

for each  $(x, y) \in X \times \mathcal{Y}$ , where  $\mathcal{Y}$  is a compact subset of  $\mathbb{R}^l$ . We denote the index set  $\{1, 2, \dots, m\}$  by  $M$ . Now for each  $(x, y) \in X \times \mathcal{Y}$  consider the following sets;

- $\mathcal{J}_1(x) = \{j \in M : h_j(x) + S(x|E_j) = 0\}$
- $\mathcal{Y}_1(x) = \left\{ y \in \mathcal{Y} : \frac{f(x, y) + S(x|B)}{g(x, y) - S(x|D)} = \sup_{z \in \mathcal{Y}} \frac{f(x, z) + S(x|B)}{g(x, z) - S(x|D)} \right\}$
- $\mathcal{Z}_1(x) = \left\{ \text{the set of triplets } (s, t, \bar{y}) \in \mathbb{N} \times \mathbb{R}_+^s \times \mathbb{R}^{ls} : \text{such that } 1 \leq s \leq n+1, t = (t_1, \dots, t_s) \text{ an } s\text{-dimensional vector with } t \geq 0, \sum_{i=1}^s t_i = 1, \bar{y} = (\bar{y}_1, \dots, \bar{y}_s) \text{ is an } ls\text{-dimensional vector with } \bar{y}_i \in \mathcal{Y}(x), (i = 1, 2, \dots, s) \right\}.$

$f$  and  $g$  are given to be continuously differentiable functions and  $\mathcal{Y} \subset \mathbb{R}^l$  is a compact set, we get for each  $\bar{x} \in X$ ,  $\mathcal{Y}(\bar{x}) \neq \emptyset$  and hence for any  $\bar{y}_i \in \mathcal{Y}(x)$  we let

$$k_o = \frac{f(x, \bar{y}_i) + S(x|B)}{g(x, \bar{y}_i) - S(x|D)}.$$

In this work, a dual program for minimax fractional programming problems is proposed. Yuan et al. [221] gave definition of  $(\mathcal{C}, \alpha, \rho, d)$  by giving definition of convexity of function  $\mathcal{C}$ . These functions are defined in Chapter 1. We assume function  $\mathcal{C} : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R}$  to be convex in third argument. As in previous Chapters we assume  $\mathcal{C}_{(x, u)}(0) = 0, \forall (x, u) \in X \times X$ .

Suppose that  $H : X \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a differentiable function,  $\alpha, \gamma : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}$  and  $d : X \times X \rightarrow \mathbb{R}_+$  satisfying  $d(x, x_0) = 0 \Leftrightarrow x = x_0$  and  $p, s \in \mathbb{R}^n$ . Consider the following class of functions defined by Dubey et al. [61].

**Definition 6.2.1** [61] *The function  $\phi : X \rightarrow \mathbb{R}$  is said to be higher-order  $(\mathcal{C}, \alpha, \gamma, \rho, d)$  convex at  $u$  with respect to  $H, p$  and  $s$  if for all  $x \in X$ , the following holds;*

$$\frac{1}{\alpha(x, u)}[\phi(x) - \phi(u)] \geq \mathcal{C}_{(x, u)}(\nabla \phi(u) + \nabla_p H(u, p)) + \frac{1}{\gamma(x, u)}[H(u, s) - s^T \nabla_s H(u, s)] + \frac{\rho d(x, u)}{\alpha(x, u)}.$$

## 6.2.1 Optimality Conditions

In this section, we study necessary and sufficient optimality conditions for the minimax fractional program (MXNFP).

**Theorem 6.2.1 (Necessary Optimality Conditions)** [158] *If  $\tilde{x}$  is an optimal solution of (MXNFP) satisfying  $\langle z, \tilde{x} \rangle > 0, \langle v, \tilde{x} \rangle > 0$  and if  $\nabla(h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle), j \in \mathcal{J}(\tilde{x})$  are linearly independent. Then  $\exists \mu \in \mathbb{R}_+^m, z, v, w_j \in \mathbb{R}^n$  for  $j = 1, 2, \dots, m, (s, \bar{t}, \bar{y}) \in \mathcal{Z}_1(\tilde{x}), k_o \in \mathbb{R}_+$ ,*

such that

$$\begin{aligned} & \sum_{i=1}^s \bar{t}_i (\nabla(f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle) - k_o \nabla(g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle)) \\ & + \sum_{j=1}^m \mu_j \nabla(h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) = 0, \end{aligned} \quad (6.2.1)$$

$$f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle - k_o(g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle) = 0, \quad i = 1, 2, \dots, s, \quad (6.2.2)$$

$$\sum_{j=1}^m \mu_j (h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) = 0, \quad (6.2.3)$$

$$\langle z, \tilde{x} \rangle = S(\tilde{x}|B), \quad \langle v, \tilde{x} \rangle = S(\tilde{x}|D), \quad \langle w_j, \tilde{x} \rangle = S(\tilde{x}|E_j), \quad j = 1, 2, \dots, m, \quad (6.2.4)$$

$$\bar{t}_i \geq 0, \quad i = 1, 2, \dots, s, \quad \sum_{i=1}^s \bar{t}_i = 1. \quad (6.2.5)$$

**Theorem 6.2.2 (Sufficient Optimality Conditions)** *Let  $\tilde{x}$  be a feasible solution of (MXNFP) and let there exist  $s, 1 \leq s \leq n+1, t^* \in \mathbb{R}_+^s, \bar{y}_i \in \mathcal{Y}(\tilde{x}), i = 1, 2, \dots, s, k_o \in \mathbb{R}_+$ , satisfying (6.2.1)-(6.2.5). Suppose that*

- (i)  $f(\cdot, \bar{y}_i) + \langle z, \cdot \rangle$  are higher-order  $(\mathcal{C}, \alpha_i, \gamma_i, \rho_i, d_i)$  convex at  $\tilde{x}$  with respect to  $K_i, p, r$ ,  $-(g(\cdot, \bar{y}_i) - \langle v, \cdot \rangle)$  are  $(\mathcal{C}, \alpha_i, \gamma_i, \rho_i, d_i)$  convex at  $\tilde{x}$  with respect to  $-G_i, p, r$ , for  $i = 1, 2, \dots, s$ ,
- (ii)  $h_j(\cdot) + \langle w_j, \cdot \rangle$  are  $(\mathcal{C}, \beta_j, \xi_j, \eta_j, \delta_j)$  convex at  $\tilde{x}$  with respect to  $H_j, q, r^*$ , for  $j = 1, 2, \dots, m$ ,
- (iii)  $\sum_{i=1}^s t_i \left( \frac{\rho_i d_i(x, \tilde{x})}{\alpha_i(x, \tilde{x})} + k \frac{\bar{\rho}_i \bar{d}_i(x, \tilde{x})}{\bar{\alpha}_i(x, \tilde{x})} \right) + \sum_{j=1}^m \frac{\eta_j \mu_j \delta_j(x, \tilde{x})}{\beta_j(x, \tilde{x})} \geq 0$ ,
- (iv)  $\xi_j = \bar{\xi}$  and  $\gamma_i = \bar{\gamma}$ ,  $i = 1, 2, \dots, s, j = 1, 2, \dots, m$ ,
- (v)  $\sum_{i=1}^s t_i \nabla_p(K_i(\tilde{x}, p) - kG_i(\tilde{x}, p)) + \sum_{j=1}^m \mu_j \nabla_q H_j(\tilde{x}, q) = 0$ ,  $\sum_{j=1}^m \mu_j (H_j(\tilde{x}, r^*) - r^* \nabla_{r^*} H_j(\tilde{x}, r^*)) \geq 0$  and  $\sum_{i=1}^s t_i (K_i(\tilde{x}, r) - kG_i(\tilde{x}, r) - r^T \nabla_r (K_i(\tilde{x}, r) - kG_i(\tilde{x}, r))) \geq 0$ .

Then  $\tilde{x}$  is an optimal solution of (MXNFP).

**Proof.** Suppose to the contrary  $\tilde{x}$  is not an optimal solution of (MXNFP). Then there exists  $x \in X$  such that

$$\sup_{y \in \mathcal{Y}} \frac{f(x, y) + \langle z, x \rangle}{g(x, y) - \langle v, x \rangle} < \sup_{y \in \mathcal{Y}} \frac{f(\tilde{x}, y) + \langle z, \tilde{x} \rangle}{g(\tilde{x}, y) - \langle v, \tilde{x} \rangle}.$$

For a  $\bar{y}_i \in \mathcal{Y}(\tilde{x})$

$$\sup_{y \in \mathcal{Y}} \frac{f(\tilde{x}, y) + \langle z, \tilde{x} \rangle}{g(\tilde{x}, y) - \langle v, \tilde{x} \rangle} = \frac{f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle}{g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle} = k_o.$$

Since

$$\frac{f(x, \bar{y}_i) + \langle z, x \rangle}{g(x, \bar{y}_i) - \langle v, x \rangle} \leq \sup_{y \in \mathcal{Y}} \frac{f(x, y) + \langle z, x \rangle}{g(x, y) - \langle v, x \rangle}$$

Combining the equations we obtain

$$\frac{f(x, \bar{y}_i) + \langle z, x \rangle}{g(x, \bar{y}_i) - \langle v, x \rangle} < k_o,$$

this gives

$$f(x, \bar{y}_i) + \langle z, x \rangle - k_o(g(x, \bar{y}_i) - \langle v, x \rangle) < 0.$$

As  $t_i \geq 0$ ,  $t_i \neq 0$  and  $\alpha_i(x, \tilde{x}) \in \mathbb{R}_+ \setminus \{0\}$ ,  $i = 1, 2, \dots, s$

$$\sum_{i=1}^s \frac{t_i}{\alpha_i(x, \tilde{x})} (f(x, \bar{y}_i) + \langle z, x \rangle - k_o(g(x, \bar{y}_i) - \langle v, x \rangle)) < 0. \quad (6.2.6)$$

Using hypothesis (i) and (ii) we have the following

$$\begin{aligned} \frac{1}{\alpha_i(x, \tilde{x})} [f(x, \bar{y}_i) + \langle z, x \rangle - (f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle)] &\geq \mathcal{C}_{(x, \tilde{x})}(\nabla(f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle) + \nabla_p K_i(\tilde{x}, \bar{y}_i, p)) \\ &+ \frac{1}{\gamma_i(x, \tilde{x})} [K_i(\tilde{x}, \bar{y}_i, r) - r^T \nabla_r K_i(\tilde{x}, \bar{y}_i, r)] + \frac{\rho_i d_i(x, \tilde{x})}{\alpha_i(x, \tilde{x})}. \end{aligned} \quad (6.2.7)$$

$$\begin{aligned} -\frac{1}{\bar{\alpha}_i(x, \tilde{x})} [g(x, \bar{y}_i) - \langle v, x \rangle - (g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle)] &\geq \mathcal{C}_{(x, \tilde{x})}(-\nabla(g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle) - \nabla_p G_i(\tilde{x}, \bar{y}_i, p)) \\ &- \frac{1}{\bar{\gamma}_i(x, \tilde{x})} [G_i(\tilde{x}, \bar{y}_i, r) - r^T \nabla_r G_i(\tilde{x}, \bar{y}_i, r)] + \frac{\bar{\rho}_i \bar{d}_i(x, \tilde{x})}{\bar{\alpha}_i(x, \tilde{x})}. \end{aligned} \quad (6.2.8)$$

$$\begin{aligned} \frac{1}{\beta_j(x, \tilde{x})} [h_j(x) + \langle w_j, x \rangle - (h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle)] &\geq \mathcal{C}_{(x, \tilde{x})}(\nabla(h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) + \nabla_q H_j(\tilde{x}, q)) \\ &+ \frac{1}{\xi_j(x, \tilde{x})} (H_j(\tilde{x}, r^*) - r^{*T} \nabla_{r^*} H_j(\tilde{x}, r^*)) + \frac{\eta_j \delta_j(x, \tilde{x})}{\beta_j(x, \tilde{x})}. \end{aligned} \quad (6.2.9)$$

Take  $t_i^* \times (6.2.7)$  and  $k_o t_i^* \times (6.2.8)$  and summing over the range of  $i$  where  $i = 1, 2, \dots, s$  and take the sum of  $\mu_j \times (6.2.9)$  over the range of  $j$  where  $j = 1, 2, \dots, m$ , we get

$$\begin{aligned} \sum_{i=1}^s \frac{t_i^*}{\alpha_i(x, \tilde{x})} (f(x, \bar{y}_i) + \langle z, x \rangle - (f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle) - \frac{k_o}{\alpha_i(x, \tilde{x})} (g(x, \bar{y}_i) - \langle v, x \rangle \\ - (g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle))) + \sum_{j=1}^m \frac{\mu_j}{\beta_j(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle - (h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle)) \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=1}^s t_i^* \left( \mathcal{C}_{(x, \tilde{x})}(\nabla(f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle) + \nabla_p K_i(\tilde{x}, \bar{y}_i, p)) + k_o \mathcal{C}_{(x, \tilde{x})}(-\nabla(g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle) \right. \\
&\quad \left. - \nabla_p G_i(\tilde{x}, \bar{y}_i, p)) \right) + \sum_{i=1}^s \frac{t_i^*}{\gamma_i(x, \tilde{x})} (K_i(\tilde{x}, \bar{y}_i, r) - r^T \nabla_r K_i(\tilde{x}, \bar{y}_i, r)) \\
&\quad - k_o \sum_{i=1}^s \frac{t_i^*}{\gamma_i(x, \tilde{x})} (G_i(\tilde{x}, \bar{y}_i, r) - r^T \nabla_r G_i(\tilde{x}, \bar{y}_i, r)) + \sum_{i=1}^s t_i^* (1 + k_o) \frac{\rho_i d_i(x, \tilde{x})}{\alpha_i(x, \tilde{x})} \\
&\quad + \sum_{j=1}^m \mu_j \mathcal{C}_{(x, \tilde{x})}(\nabla(h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) + \nabla_q H_j(\tilde{x}, q)) + \sum_{j=1}^m \frac{\mu_j \eta_j \delta_j(x, \tilde{x})}{\beta_j(x, \tilde{x})} \\
&\quad + \sum_{j=1}^m \frac{\mu_j}{\xi_j(x, \tilde{x})} (H_j(\tilde{x}, r^*) - r^{*T} \nabla_{r^*} H_j(\tilde{x}, r^*)). \tag{6.2.10}
\end{aligned}$$

Take  $\tau = \sum_{i=1}^s t_i^* (1 + k_o) + \sum_{j=1}^m \mu_j$  so that  $1 = \sum_{i=1}^s \frac{t_i^* (1 + k_o)}{\tau} + \sum_{j=1}^m \frac{\mu_j}{\tau}$ , use (iv) and (6.2.2) to get the following

$$\begin{aligned}
&\sum_{i=1}^s \frac{t_i^*}{\tau \alpha_i(x, \tilde{x})} (f(x, \bar{y}_i) + \langle z, x \rangle - k_o (g(x, \bar{y}_i) - \langle v, x \rangle)) + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle) \\
&\geq \sum_{i=1}^s \frac{t_i^*}{\tau} \mathcal{C}_{(x, \tilde{x})}(\nabla(f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle) + \nabla_p K_i(\tilde{x}, \bar{y}_i, p)) + k_o \mathcal{C}_{(x, \tilde{x})}(-\nabla(g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle) \\
&\quad - \nabla_p G_i(\tilde{x}, \bar{y}_i, p)) + \sum_{j=1}^m \frac{\mu_j}{\tau} \mathcal{C}_{(x, \tilde{x})}(\nabla(h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) + \nabla_q H_j(\tilde{x}, q)) \\
&\quad + \frac{1}{\tau \beta(x, \tilde{x})} \sum_{j=1}^m \mu_j (h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) + \frac{1}{\tau \xi(x, \tilde{x})} \sum_{j=1}^m \mu_j (H_j(\tilde{x}, r^*) - r^{*T} \nabla_{r^*} H_j(\tilde{x}, r^*)) \\
&\quad + \frac{1}{\tau \gamma(x, \tilde{x})} \sum_{i=1}^s t_i^* (K_i(\tilde{x}, \bar{y}_i, r) - r^T \nabla_r K_i(\tilde{x}, \bar{y}_i, r) - k_o (G_i(\tilde{x}, \bar{y}_i, r) - r^T \nabla_r G_i(\tilde{x}, \bar{y}_i, r)) \\
&\quad + \sum_{i=1}^s t_i^* (1 + k_o) \frac{\rho_i d_i(x, \tilde{x})}{\tau \alpha_i(x, \tilde{x})} + \sum_{j=1}^m \frac{\mu_j \eta_j \delta_j(x, \tilde{x})}{\tau \beta(x, \tilde{x})}. \tag{6.2.11}
\end{aligned}$$

Using (iii) and (v) to get

$$\begin{aligned}
&\sum_{i=1}^s \frac{t_i^*}{\tau \alpha_i(x, \tilde{x})} (f(x, \bar{y}_i) + \langle z, x \rangle - k_o (g(x, \bar{y}_i) - \langle v, x \rangle)) + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle) \\
&\geq \sum_{i=1}^s \frac{t_i^*}{\tau} \mathcal{C}_{(x, \tilde{x})}(\nabla(f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle) + \nabla_p K_i(\tilde{x}, \bar{y}_i, p)) + k_o \mathcal{C}_{(x, \tilde{x})}(-\nabla(g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle) \\
&\quad - \nabla_p G_i(\tilde{x}, \bar{y}_i, p)) + \sum_{j=1}^m \frac{\mu_j}{\tau} \mathcal{C}_{(x, \tilde{x})}(\nabla(h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) + \nabla_q H_j(\tilde{x}, q))
\end{aligned}$$

$$+ \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) \quad (6.2.12)$$

Now as  $\mathcal{C}$  is a functional, convex in its third argument and  $\sum_{i=1}^s \frac{t_i^*(1+k_o)}{\tau} + \sum_{j=1}^m \frac{\mu_j}{\tau} = 1$  and  $\mathcal{C}_{(x, \tilde{x})}(0) = 0, \forall (x, \tilde{x}) \in X \times X$ , equation (6.2.12) becomes

$$\begin{aligned} & \sum_{i=1}^s \frac{t_i^*}{\tau \alpha_i(x, \tilde{x})} (f(x, \bar{y}_i) + \langle z, x \rangle - k_o(g(x, \bar{y}_i) - \langle v, x \rangle)) + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle) \\ & \geq \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle). \end{aligned} \quad (6.2.13)$$

Using necessary optimality condition (6.2.3), we have

$$\sum_{i=1}^s \frac{t_i^*}{\tau \alpha_i(x, \tilde{x})} (f(x, \bar{y}_i) + \langle z, x \rangle - k_o(g(x, \bar{y}_i) - \langle v, x \rangle)) + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle) \geq 0. \quad (6.2.14)$$

Since for a feasible solution  $x$  of (MXNFP)  $(h_j(x) + S(x|E_j)) \leq 0$  which in turn implies  $h_j(x) + \langle w_j, x \rangle \leq 0$ , for  $\mu \in \mathbb{R}_+^m$ ,  $\sum_{j=1}^m \mu_j (h_j(x) + \langle w_j, x \rangle) \leq 0$ , by necessary optimality condition (6.2.5) we have the following

$$\frac{1}{\tau} \sum_{i=1}^s \frac{t_i^*}{\alpha_i(x, \tilde{x})} (f(x, \bar{y}_i) + \langle z, x \rangle - k_o(g(x, \bar{y}_i) - \langle v, x \rangle)) \geq 0. \quad (6.2.15)$$

But this is a contradiction to (6.2.6). Hence we have,  $\tilde{x}$  as an optimal solution of (MXNFP).

## 6.3 Parametric Duality

In this section we construct the higher-order dual (MXNFD) for the above considered problem (MXNFP):

Dual (MXNFD)

$$\begin{aligned} & \text{Maximize} \quad \sup_{(u, \mu, k, z, v, w) \in \mathcal{H}_1(s, t, \bar{y})} k \\ & \mathcal{H}_1(s, t, \bar{y}) \text{ is the set of all } (u, \mu, k, z, v, w) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{nm} \text{ satisfying} \\ & \sum_{i=1}^s t_i [\nabla(f(u, \bar{y}_i) + \langle z, u \rangle) - k(g(u, \bar{y}_i) - \langle v, u \rangle) + \nabla_p(K(u, \bar{y}_i, p) - kG(u, \bar{y}_i, p))] \end{aligned}$$

$$+ \sum_{j=1}^m \mu_j [\nabla(h_j(u) + \langle w_j, u \rangle) + \nabla_q H_j(u, q)] = 0 \quad (6.3.1)$$

$$\begin{aligned} \sum_{i=1}^s t_i [(f(u, \bar{y}_i) + \langle z, u \rangle) - k(g(u, \bar{y}_i) - \langle v, u \rangle) + (K(u, \bar{y}_i, p) - kG(u, \bar{y}_i, p)) \\ - r^T \nabla_r (K(u, \bar{y}_i, p) - kG(u, \bar{y}_i, p))] \geq 0 \end{aligned} \quad (6.3.2)$$

$$\sum_{j=1}^m \mu_j [h_j(u) + \langle w_j, u \rangle) + H_j(u, q) - q^T \nabla_q H_j(u, q)] \geq 0 \quad (6.3.3)$$

$$\mu_j \geq 0, \quad z \in B, \quad v \in D, \quad w_j \in E_j. \quad (6.3.4)$$

Now we state and prove following duality theorems to study the relationship between primal and the proposed dual.

**Theorem 6.3.1 (Weak Duality)** *Let  $u$  be a feasible solution of (MXNFP) and  $(u, \mu, k, z, v, w, p, q, r, r^*, s, t, \bar{y})$  be feasible solution of (MXNFD). Suppose that  $f(\cdot, \bar{y}_i) + (\cdot)^T z$  is  $(\mathcal{C}, \alpha_i, \gamma_i, \rho_i, d_i)$  convex with respect to  $K, p, r$  and  $-(g(\cdot, \bar{y}_i) - (\cdot)^T v)$  is  $(\mathcal{C}, \bar{\alpha}_i, \bar{\gamma}_i, \bar{\rho}_i, \bar{d}_i)$  convex with respect to  $-G, p, r$ . Let  $h_j(\cdot) + (\cdot)^T w_j$  be  $(\mathcal{C}, \beta_j, \xi_j, \eta_j, \delta_j)$  convex with respect to  $H_j, q, r^*$ . Further suppose that the following hold:*

$$(i) \sum_{i=1}^s t_i \left( \frac{\rho_i d_i(x, u)}{\alpha_i(x, u)} + \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)} \right) + \sum_{j=1}^m \frac{\mu_j \eta_j \delta_j(x, u)}{\beta_j(x, u)} \geq 0,$$

$$\text{or } \rho_i, \bar{\rho}_i, \eta_j \geq 0, i = 1, 2, \dots, s, j = 1, 2, \dots, m,$$

$$(ii) \alpha_i(x, u) = \bar{\alpha}_i(x, u) = \alpha(x, u), \quad \xi_j(x, u) = \xi(x, u), \quad i = 1, 2, \dots, s, \quad j = 1, 2, \dots, m.$$

Then the following holds

$$\sup_{y \in \mathcal{Y}} \frac{f(x, y) + S(x|B)}{g(x, y) - S(x|D)} \geq k. \quad (6.3.5)$$

**Proof.** We will prove this theorem by contradiction. Let if we start with that (6.3.5) is not true. It means

$$\sup_{y \in \mathcal{Y}} \frac{f(x, y) + S(x|B)}{g(x, y) - S(x|D)} < k. \quad (6.3.6)$$

Let us suppose that there exist  $\bar{y}_i \in \mathcal{Y}(x)$ , then supremum is given as

$$\sup \frac{f(x, y) + S(x|B)}{g(x, y) - S(x|D)} = \frac{f(x, \bar{y}_i) + S(x|B)}{g(x, \bar{y}_i) - S(x|D)},$$

and there exist  $z \in B$ ,  $v \in D$  such that

$$\frac{f(x, \bar{y}_i) + S(x|B)}{g(x, \bar{y}_i) - S(x|D)} = \frac{f(x, \bar{y}_i) + \langle z, x \rangle}{g(x, \bar{y}_i) - \langle v, x \rangle}.$$

Combining this with (6.3.6) we get

$$\frac{f(x, \bar{y}_i) + \langle z, x \rangle}{g(x, \bar{y}_i) - \langle v, x \rangle} < k$$

which is equivalent to

$$f(x, \bar{y}_i) + \langle z, x \rangle - k(g(x, \bar{y}_i) - \langle v, x \rangle) < 0.$$

For  $t_i \geq 0$ ,  $t_i \neq 0$  where  $i = 1, 2, \dots, s$

$$\sum_{i=1}^s t_i (f(x, \bar{y}_i) + \langle z, x \rangle - k(g(x, \bar{y}_i) - \langle v, x \rangle)) < 0. \quad (6.3.7)$$

Now using the higher-order generalized convexity assumptions the following hold:

$$\begin{aligned} \frac{1}{\alpha_i(x, u)} [f(x, \bar{y}_i) + \langle z, x \rangle - (f(u, \bar{y}_i) + \langle z, u \rangle)] &\geq \mathcal{C}_{(x, u)}(\nabla(f(u, \bar{y}_i) + \langle z, u \rangle) + \nabla_p K_i(u, \bar{y}_i, p)) \\ &+ \frac{1}{\gamma_i(x, u)} (K_i(u, \bar{y}_i, r) - r^T \nabla_r K_i(u, \bar{y}_i, r)) + \frac{\rho_i d_i(x, u)}{\alpha_i(x, u)}. \end{aligned} \quad (6.3.8)$$

$$\begin{aligned} -\frac{1}{\bar{\alpha}_i(x, u)} [g(x, \bar{y}_i) - \langle v, x \rangle - (g(u, \bar{y}_i) - \langle v, u \rangle)] &\geq \mathcal{C}_{(x, u)}(-\nabla(g(u, \bar{y}_i) - \langle v, u \rangle) - \nabla_p G_i(u, \bar{y}_i, p)) \\ &+ \frac{1}{\bar{\gamma}_i(x, u)} (G_i(u, \bar{y}_i, r) - r^T \nabla_r G_i(u, \bar{y}_i, r)) + \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)}. \end{aligned} \quad (6.3.9)$$

$$\begin{aligned} \frac{1}{\beta_j(x, u)} [h_j(x) + \langle w_j, x \rangle - (h_j(u) + \langle w_j, u \rangle)] &\geq \mathcal{C}_{(x, u)}(\nabla(h_j(u) + \langle w_j, u \rangle) + \nabla_q H_j(u, q)) \\ &+ \frac{1}{\xi_j(x, u)} (H_j(u, r^*) - r^{*T} \nabla_{r^*} H_j(u, r^*)) + \frac{\eta_j \delta_j(x, u)}{\beta_j(x, u)}. \end{aligned} \quad (6.3.10)$$

Using similar approach as in sufficient optimality conditions i.e. take  $t_i \times (6.3.8) + kt_i \times (6.3.9)$  and summing over the range of  $i$  where  $i = 1, 2, \dots, s$  and take the sum of  $\mu_j \times (6.3.10)$  over

the range of  $j$  and  $j = 1, 2, \dots, m$ , we attain

$$\begin{aligned}
& \sum_{i=1}^s \frac{t_i}{\alpha_i(x, \tilde{x})} (f(x, \bar{y}_i) + \langle z, x \rangle - (f(u, \bar{y}_i) + \langle z, u \rangle) - \frac{k}{\bar{\alpha}_i(x, \tilde{x})} (g(x, \bar{y}_i) - \langle v, x \rangle \\
& \quad - (g(u, \bar{y}_i) - \langle v, u \rangle))) + \sum_{j=1}^m \frac{\mu_j}{\beta_j(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle - (h_j(u) + \langle w_j, u \rangle)) \\
& \geq \sum_{i=1}^s t_i \left( \mathcal{C}_{(x,u)}(\nabla(f(u, \bar{y}_i) + \langle z, u \rangle) + \nabla_p K_i(u, \bar{y}_i, p)) + k \mathcal{C}_{(x,u)}(-\nabla(g(u, \bar{y}_i) - \langle v, u \rangle) \right. \\
& \quad \left. - \nabla_p G_i(u, \bar{y}_i, p)) \right) + \sum_{j=1}^m \mu_j \mathcal{C}_{(x,u)}(\nabla(h_j(u) + \langle w_j, u \rangle) + \nabla_q H_j(u, q)) \\
& \quad + \sum_{i=1}^s \frac{t_i}{\gamma_i(x, u)} (K_i(u, \bar{y}_i, r) - r^T \nabla_r K_i(u, \bar{y}_i, r)) - k \sum_{i=1}^s \frac{t_i}{\bar{\gamma}_i(x, u)} (G_i(u, \bar{y}_i, r) \\
& \quad - r^T \nabla_r G_i(u, \bar{y}_i, r)) + \sum_{i=1}^s t_i \left( \frac{\rho_i d_i(x, u)}{\alpha_i(x, u)} + k \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)} \right) + \sum_{j=1}^m \frac{\mu_j \eta_j \delta_j(x, u)}{\beta_j(x, u)} \\
& \quad + \sum_{j=1}^m \frac{\mu_j}{\xi_j(x, u)} (H_j(u, r^*) - r^{*T} \nabla_{r^*} H_j(u, r^*)). \tag{6.3.11}
\end{aligned}$$

Rearranging this equation, using (ii) and multiplying by  $\frac{1}{\tau}$  where  $\tau = \sum_{i=1}^s t_i + \sum_{i=1}^s t_i k + \sum_{j=1}^m \mu_j$  and  $1 = \sum_{i=1}^s \frac{t_i(1+k)}{\tau} + \sum_{j=1}^m \frac{\mu_j}{\tau}$ ,

$$\begin{aligned}
& \sum_{i=1}^s \frac{t_i}{\tau \alpha(x, u)} (f(x, \bar{y}_i) + \langle z, x \rangle - k(g(x, \bar{y}_i) - \langle v, x \rangle)) + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle) \\
& \geq \sum_{i=1}^s \frac{t_i}{\tau} \mathcal{C}_{(x,u)}(\nabla(f(u, \bar{y}_i) + \langle z, u \rangle) + \nabla_p K_i(u, \bar{y}_i, p)) + k \mathcal{C}_{(x,u)}(-\nabla(g(u, \bar{y}_i) - \langle v, u \rangle) \\
& \quad - \nabla_p G_i(u, \bar{y}_i, p)) + \sum_{j=1}^m \frac{\mu_j}{\tau} \mathcal{C}_{(x,u)}(\nabla(h_j(u) + \langle w_j, u \rangle) + \nabla_q H_j(u, q)) \\
& \quad + \sum_{i=1}^s \frac{t_i}{\tau \alpha(x, u)} \left( f(u, \bar{y}_i) + \langle z, u \rangle - k(g(u, \bar{y}_i) - \langle u, x \rangle) \right. \\
& \quad \left. + K_i(u, \bar{y}_i, r) - r^T \nabla_r K_i(u, \bar{y}_i, r) - k(G_i(u, \bar{y}_i, r) - r^T \nabla_r G_i(u, \bar{y}_i, r)) \right) \\
& \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(u) + \langle w_j, u \rangle) + \sum_{j=1}^m \frac{\mu_j}{\tau \xi(x, u)} (H_j(u, r^*) - r^{*T} \nabla_{r^*} H_j(u, r^*)) \\
& \quad + \sum_{i=1}^s t_i \left( \frac{\rho_i d_i(x, u)}{\tau \alpha(x, u)} + k \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\tau \alpha(x, u)} \right) + \sum_{j=1}^m \frac{\mu_j \eta_j \delta_j(x, u)}{\tau \beta(x, u)}. \tag{6.3.12}
\end{aligned}$$

Now by means of the convexity of  $\mathcal{C}_{(x,u)}(\cdot)$ , we have, using dual feasibility condition (6.3.1)

$$\begin{aligned} \mathcal{C}_{(x,u)} & \left( \sum_{i=1}^s \frac{t_i}{\tau} (\nabla(f(u, \bar{y}_i) + \langle z, u \rangle) - k(g(u, \bar{y}_i) - \langle v, u \rangle) + \nabla_p(K(u, \bar{y}_i, p) - kG(u, \bar{y}_i, p))) \right. \\ & \left. + \sum_{j=1}^m \frac{\mu_j}{\tau} (\nabla(h_j(u) + \langle w_j, u \rangle) + \nabla_q H_j(u, q)) \right) = \mathcal{C}_{(x,u)}(0) = 0. \end{aligned}$$

Using this in equation (6.3.12)

$$\begin{aligned} & \sum_{i=1}^s \frac{t_i}{\tau \alpha(x, u)} (f(x, \bar{y}_i) + \langle z, x \rangle - k(g(x, \bar{y}_i) - \langle v, x \rangle)) + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle) \\ & \geq \sum_{i=1}^s \frac{t_i}{\tau \alpha(x, u)} \left( f(u, \bar{y}_i) + \langle z, u \rangle - k(g(u, \bar{y}_i) - \langle u, x \rangle) + K_i(u, \bar{y}_i, r) - r^T \nabla_r K_i(u, \bar{y}_i, r) \right. \\ & \quad \left. - k(G_i(u, \bar{y}_i, r) - r^T \nabla_r G_i(u, \bar{y}_i, r)) \right) + \sum_{i=1}^s t_i \left( \frac{\rho_i d_i(x, u)}{\tau \alpha(x, u)} + k \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\tau \alpha(x, u)} \right) + \sum_{j=1}^m \frac{\mu_j \eta_j \delta_j(x, u)}{\tau \beta(x, u)} \\ & \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(u) + \langle w_j, u \rangle) + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, u)} (H_j(u, r^*) - r^{*T} \nabla_{r^*} H_j(u, r^*)). \end{aligned}$$

This will reduce to the following equation due to (i) and dual feasibility conditions (6.3.2) and (6.3.3)

$$\sum_{i=1}^s \frac{t_i}{\tau \alpha(x, u)} (f(x, \bar{y}_i) + \langle z, x \rangle - k(g(x, \bar{y}_i) - \langle v, x \rangle)) + \sum_{j=1}^m \frac{\mu_j}{\tau \beta(x, \tilde{x})} (h_j(x) + \langle w_j, x \rangle) \geq 0. \quad (6.3.13)$$

Since  $x$  is a feasible solution of (MXNFP) and  $(u, \mu, k, z, v, z, p, q, r, r^*, s, t, \bar{y})$  is feasible solution of (MXNFD) we obtain the following

$$\sum_{i=1}^s \frac{t_i}{\tau \alpha(x, u)} (f(x, \bar{y}_i) + \langle z, x \rangle - k(g(x, \bar{y}_i) - \langle v, x \rangle)) \geq 0. \quad (6.3.14)$$

This is a contradiction to (6.3.7) since  $\tau > 0$  and  $\alpha(x, u) \in \mathbb{R}_+ \setminus \{0\}$ . Hence we have the weak duality theorem.

**Theorem 6.3.2 (Strong Duality Theorem)** *Let  $\tilde{x}$  be an optimal solution of (MXNFP) such that  $\langle z, \tilde{x} \rangle > 0$ ,  $\langle v, \tilde{x} \rangle > 0$  and the set of vectors  $\nabla(h_j + \langle w_j, \tilde{x} \rangle)$ ,  $j \in \mathcal{J}_1(\tilde{x})$  be linearly independent. If the following hold:*

$$\begin{aligned} K(\tilde{x}, y_i^*, 0) &= 0, & \nabla_p K(\tilde{x}, y_i^*, 0) &= 0, & \nabla_r K(\tilde{x}, y_i^*, 0) &= 0, \\ G(\tilde{x}, y_i^*, 0) &= 0, & \nabla_p G(\tilde{x}, y_i^*, 0) &= 0, & \nabla_r G(\tilde{x}, y_i^*, 0) &= 0, \end{aligned}$$

$$H_j(\tilde{x}, 0) = 0, \quad \nabla_q H_j(\tilde{x}, 0) = 0, \quad \nabla_{r^*} H_j(\tilde{x}, 0) = 0.$$

Then there exist  $(s, t^*, \bar{y}) \in \mathcal{Z}_1(\tilde{x})$  and  $A = (\tilde{x}, \mu, k, z, v, z, p = 0, q = 0, r = 0, r^* = 0) \in \mathcal{H}_1(s, t^*, \bar{y})$  such that  $A$  is a feasible solution of (MXNFD). Further, if the conditions of weak duality theorem hold for all feasible solutions of (MXNFD) then  $A$  is an optimal for (MXNFD).

**Proof.** As  $\tilde{x}$  is an optimal solution of the minimax program (MXNFP) and  $\nabla(h_j + \langle w_j, \tilde{x} \rangle)$ ,  $j \in \mathcal{J}_1(\tilde{x})$  are linearly independent, then by necessary optimality conditions there exist  $k_o$  such that

$$k_o = \frac{f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle}{g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle}, \quad \text{for } \bar{y}_i \in \mathcal{Y}(\tilde{x}). \quad (6.3.15)$$

and

$$\begin{aligned} \sum_{i=1}^s t_i^* \nabla(f(\tilde{x}, \bar{y}_i) + \langle z, \tilde{x} \rangle - k_o(g(\tilde{x}, \bar{y}_i) - \langle v, \tilde{x} \rangle)) \\ + \sum_{j=1}^m \mu_j \nabla(h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) = 0, \end{aligned} \quad (6.3.16)$$

$$\sum_{j=1}^m \mu_j (h_j(\tilde{x}) + \langle w_j, \tilde{x} \rangle) = 0, \quad (6.3.17)$$

$$\langle z, \tilde{x} \rangle = S(\tilde{x}|B), \quad \langle v, \tilde{x} \rangle = S(\tilde{x}|D), \quad \langle w_j, \tilde{x} \rangle = S(\tilde{x}|E_j), \quad (6.3.18)$$

$$z \in B, \quad v \in D, \quad w_j \in E_j, \quad \sum_{i=1}^s t_i^* = 1. \quad (6.3.19)$$

This gives  $(\tilde{x}, \mu, k_o, z, v, z, s, t^*, \bar{y}, p = 0, q = 0, r = 0, r^* = 0)$  is a feasible solution of (MXNFD). (6.3.15) and (6.3.18) give that the objective function values of (MXNFP) and (MXNFD) are equal. Further we claim that  $(\tilde{x}, \mu, k_o, z, v, z, s, t^*, \bar{y}, p = 0, q = 0, r = 0, r^* = 0)$  is an optimal solution of (MXNFD). For that let us say it is not an optimal solution of (MXNFD), then there exists another feasible solution say  $(x', \mu', k', z', v', z', s', t', y', p', q', r', r')$  such that

$$k' > k_o. \quad (6.3.20)$$

But this is a contradiction to weak duality theorem.

**Theorem 6.3.3 (Strict Converse Duality)** *Let  $\bar{x}$  be a feasible solution of (MXNFP) and  $(\bar{u}, \bar{\mu}, \bar{k}, \bar{z}, \bar{v}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{r}^*, \bar{s}, \bar{t}, \bar{y})$  be feasible solution of (MXNFD). Assume that the hypothesis of Theorem 6.3.2 is fulfilled at  $\bar{x}$ . If any one of the  $f(\cdot, \bar{y}_i) + (\cdot)^T z$  is higher-order  $(\mathcal{C}, \alpha_i, \gamma_i, \rho_i, d_i)$  strictly convex with respect to  $K, p, r$ , or  $\sum_{i=1}^s \bar{t}_i \left( \frac{\rho_i d_i(\bar{x}, \bar{u})}{\alpha_i(\bar{x}, \bar{u})} + \frac{\bar{\rho}_i \bar{d}_i(\bar{x}, \bar{u})}{\bar{\alpha}_i(\bar{x}, \bar{u})} \right) +$*

$\sum_{j=1}^m \frac{\mu_j \eta_j \delta_j(\bar{x}, \bar{u})}{\beta_j(\bar{x}, \bar{u})} > 0$ , then  $\bar{x} = \bar{u}$ , that is  $\bar{u}$  is an (MXNFP) optimal solution.

**Proof.** Suppose to the contrary  $\bar{x} \neq \bar{u}$ . Since  $(\bar{u}, \bar{\mu}, \bar{k}, \bar{z}, \bar{v}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{r}^*, \bar{s}, \bar{t}, \bar{y})$  is a feasible solution of dual problem (MXNFD) and  $\bar{x}$  is feasible for (MXNFP), using the similar argument as in the proof of Weak Duality Theorem we have the following

$$\frac{1}{\tau} \sum_{i=1}^s \frac{\bar{t}_i}{\alpha_i(x, \bar{x})} (f(\bar{x}, \bar{y}_i) + \langle \bar{z}, \bar{x} \rangle - \bar{k}(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x} \rangle)) > 0. \quad (6.3.21)$$

Knowing that  $\bar{t} \geq 0$  and  $\sum_{i=1}^s \bar{t}_i = 1$ , we can conclude that there exist some  $i_o \in \{1, 2, \dots, s\}$ , such that

$$f(\bar{x}, \bar{y}_{i_o}) + \langle \bar{z}, \bar{x} \rangle > \bar{k}(g(\bar{x}, \bar{y}_{i_o}) - \langle \bar{v}, \bar{x} \rangle). \quad (6.3.22)$$

It follows that

$$\sup_{y \in \mathcal{Y}} \frac{f(\bar{x}, \bar{y}_i) + \langle \bar{z}, \bar{x} \rangle}{(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x} \rangle)} > \bar{k}. \quad (6.3.23)$$

On the other hand, the strong duality theorem gives that

$$\sup_{y \in \mathcal{Y}} \frac{f(\bar{x}, \bar{y}_i) + \langle \bar{z}, \bar{x} \rangle}{(g(\bar{x}, \bar{y}_i) - \langle \bar{v}, \bar{x} \rangle)} = \bar{k}. \quad (6.3.24)$$

So we have arrived at a contradiction. Which means  $\bar{x} = \bar{u}$  and hence  $\bar{u}$  is a (MXNFP) optimal solution.

### Special Cases:

1. If  $B = \{B_1 y : y^T B_1 y \leq 0\}$ ,  $D = \{D_1 x : x^T D_1 x \leq 0\}$  and  $E_j = \{0\}$ ,  $j \in M$  such that  $B_1$  and  $D_1$  are positive semi-definite matrices of order  $n \times n$ ,  $K(u, y_i, p) = \frac{1}{2} p^T \nabla_{xx} f(u, y_i) p$ ,  $G(u, y_i, p) = \frac{1}{2} p^T \nabla_{xx} g(u, y_i) p$  and  $H_j(u, q) = \frac{1}{2} q^T \nabla_{xx} h_j(u) q$  then the problems (MXNFP) and (MXNFD) reduce to (PP) and its second order parametric dual problem (DP) considered in [85] for second order  $(\mathcal{C}, \alpha, \rho, d)$  type-I functions.
2. For same  $B, D$  and  $E_j$  as in first part,  $p = 0, q = 0$  the problems (MXNFP) and (MXNFD) reduce to primal (P) and its first order dual (D1) considered in [142] under convexity and pseudoconvexity assumptions.
3. If  $B = \{0\}, D = \{0\}$  and  $E_j = 0$ ,  $j \in M$ , and  $p = 0, q = 0$  and the functions  $f, h, g$  are locally Lipschitzian then our problems (MXNFP) and (MXNFD) become (P) and corresponding first order dual (DI) studied in [221].

# Chapter 7

## Optimality and Duality for Semi-infinite Programming Problems Under Data Uncertainty

### 7.1 Introduction

The feasible set contains all those points which satisfy constraints of the problem and the constraints further depend on the availability of the resources and environment. In the real life optimization problems, the input information may come from an uncertain or incomplete source. For example, multiobjective optimization problems arising in financial market may involve varying costs, returns and future demands that might be unknown or hidden at the time of the decision making. So these parameters have to be replaced with some initial forecast or measurements. Therefore the efficient techniques, which involve all the possible risks to be encountered while studying the optimization problem, are required. Robust optimization approach which is a deterministic approach, has come out to be a potent method for studying mathematical programming under uncertainty, associating a robust counterpart to an uncertain optimization problem. This method intends to find the solution considering all the possible values of the parameters within their prescribed set of uncertainty including the worst case of all the existing scenarios. A detailed study of robust optimization can be found in [24]. Several other researchers have also explored diverse aspects of such problems.

Goberna et al. [72] studied radius of feasibility, optimality and duality for a robust counterpart of a linear multiobjective problem affected by data uncertainty. Goberna et al. [71] obtained characterization of efficient, properly efficient as well as strongly efficient solutions of a linear semi-infinite multiobjective and an application to robust programs. Goberna et al. [73] characterized weakly efficient solutions of a multiobjective convex uncertain problem by estimating bounds on the radius of feasibility. Huang and Chen [98] studied optimality of a convex multiobjective problem with data uncertainty in both objectives and constraints via weighted sum approach. Jeyakumar et al. [114] also studied optimality of an uncertain convex program using robust optimization approach by exploring various properties of optimal solutions of the robust problem. Lee and Lee [144] studied approximated( $\epsilon$ -) solutions of a robust program obtained from a data uncertainty problem.

They studied robust N&S optimality conditions and duality results for  $\epsilon$ -solutions of the robust program.

Duality theory has played a key role in convex programming and enough literature is available for programs in which data uncertainty has not been incorporated. But since last decade, the study of duality theory for programming problems with uncertain parameters has gained interest of several authors. Beck and Ben-Tal [17] first studied duality results for a robust problem linked to a convex data uncertainty problem. Then Kim [135] studied Mond-Weir type dual problem for a robust multiobjective program associated with an uncertainty problem with several objectives. To get further insight into development of duality theory in robust programs, one can refer to the work of Jeyakumar and Li [116], Jeyakumar et al. [115], Choung [42] and Chen et al. [38] respectively.

Semi-infinite programming problems involve optimizing a finite number of objectives with finite variables in the presence of infinite number constraints. Recently Lee and Lee [143] studied optimality and duality for a robust semi-infinite programming problem. The function involved were assumed to be convex which is a generalization of the robust linear programs studied in [71] and [72]. In this Chapter, motivated by the work of Lee and Lee [143] we study the optimality and duality for robust semi-infinite programs under non-convex settings. We use the pseudoconvexity and quasiconvexity assumptions on the objective and constraint functions respectively to study the mentioned results.

A brief outline of this Chapter is as follows. In Section 7.2 we give problem formulation and preliminaries. In Section 7.3 we give sufficient optimality conditions for the existence of weakly efficient solutions, efficient solutions and properly efficient solutions. Section 7.4 contains the dual model along with weak, strong and strict converse duality theorems. In Section 7.5 we study a fractional multiobjective semi-infinite program and its optimality and duality results using generalized convex functions.

## 7.2 Problem Formulation and Preliminaries

In this section, we consider a mathematical programming problem with data uncertainty in constraints along with some notations and definitions to be used throughout the Chapter. In mathematical programming, semi-infinite programming problem is finding optimal of an objective function (finitely many components) subject to infinite number of constraints. A semi-infinite multiobjective programming problem may be formulated as follows:

$$\begin{array}{ll} \text{Problem (SIP)} & \text{Minimize } f(x) = (f_1(x), f_2(x), \dots, f_k(x)) \\ & \text{subject to } g_t(x) \leq 0, \forall t \in I, \end{array}$$

here for  $i = 1, 2, \dots, k$ ,  $t \in I$ ,  $f_i, g_t : \mathbb{R}^n \rightarrow \mathbb{R}$  are continuously differentiable functions. The set  $I$  is an arbitrary index set (possibly infinite) and the feasible set of (SIP) is given by  $F_S = \{x \in \mathbb{R}^n : g_t(x) \leq 0, \forall t \in I\}$ .

In this Chapter, we study semi-infinite programming problems with data uncertainty in constraints. It is assumed that the uncertainty factors lie in some compact sets and uncertainty in each of the constraints is independent of the uncertainty in rest of constraints. Consider the following semi-infinite multiobjective program under data uncertainty (USIP):

$$\begin{array}{ll} \text{Problem (USIP)} & \text{Minimize } f(x) = (f_1(x), f_2(x), \dots, f_k(x)) \\ & \text{subject to } g_t(x, v_t) \leq 0, \forall t \in I, \end{array}$$

here  $f_i$  for  $i = 1, 2, \dots, k$  are same as in (SIP) and  $g_t : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$ ,  $t \in I$ , are continuously differentiable functions. In (USIP),  $v_t \in \mathbb{R}^q$  is an uncertainty parameter which lies in some convex and compact  $\mathcal{V}_t \subset \mathbb{R}^q$ ,  $t \in I$ .

This set  $\mathcal{V}_I$  depends on  $I$ . The set valued mapping  $\mathcal{V} : I \rightrightarrows \mathbb{R}^q$  is defined as  $\mathcal{V}(t) := \mathcal{V}_t$  for all  $t \in I$ . We say  $v$  is an element of  $\mathcal{V}$ , when  $v$  is a selection from the elements of  $\mathcal{V}$ . This occurs in such a way that  $v : I \rightrightarrows \mathbb{R}^q$  and  $v_t \in \mathcal{V}_t$  for all  $t \in I$ . So the points in the graph of  $\mathcal{V}_t$  are of the type  $(t, v_t)$ .

Following the robust optimization approach to deal with data uncertainty, we frame the robust counterpart of (USIP) as:

$$\begin{array}{ll} \text{Problem (RSIP)} & \text{Minimize } f(x) = (f_1(x), f_2(x), \dots, f_k(x)) \\ & \text{subject to } g_t(x, v_t) \leq 0, \forall v_t \in \mathcal{V}_t, t \in I. \end{array}$$

The functions  $f_i, g_t$  are same as defined in (USIP). The feasible solutions are those elements of  $\mathbb{R}^n$  which are feasible for any possible situation of uncertainty from the given uncertain sets. The set of robust feasible solutions, namely robust feasible set, is defined as

$$F := \{x \in \mathbb{R}^n : g_t(x, v_t) \leq 0, \forall t \in I \text{ and } v_t \in \mathcal{V}_t\}.$$

An element  $x \in F$  is called a robust feasible solution of (USIP). Now we give some basic notations and definitions which will be used in the subsequent sections.

**Definition 7.2.1**  $u \in F$  is called weakly robust efficient solution of (USIP), when it is weakly efficient solution of (RSIP) means; if there does not exist any  $x \in F$  such that  $f(x) < f(u)$ .

**Definition 7.2.2**  $u \in F$  is called robust efficient solution of (USIP) or an efficient solution

of (RSIP), if there does not exist any  $x \in F$  such that  $f(x) \leq f(u)$ .

Similarly properly robust efficient solution of (USIP) can be defined.

We denote by  $\mathbb{R}^n$  and  $\mathbb{R}_+^n$  an  $n$ -dimensional Euclidean space and its nonnegative orthant, respectively. Now let us denote by  $\mathbb{R}^I$  the following space

$$\mathbb{R}^I := \{\lambda = (\lambda_t)_{t \in I} : \lambda_t = 0 \ \forall t \in I, \text{ except for finitely many } t \in I\},$$

or it can also be defined as a linear space of mappings of the type  $\mu \in \mathbb{R}^I$  such that  $\{t \in I; \mu_t \neq 0\}$  is a finite set.

The non-negative cone of  $\mathbb{R}^I$  is defined as follows:

$$\mathbb{R}_+^I = \{\lambda = (\lambda_t)_{t \in I} \in \mathbb{R}^I : \lambda_t \geq 0, t \in I\}.$$

The following existing definitions of generalized convex functions will be used in achieving the main results of this Chapter.

**Definition 7.2.3** Assume a vector function  $f : X \rightarrow \mathbb{R}^k$  where  $X \subseteq \mathbb{R}^n$ .

(i)  $f$  is said to be **pseudoconvex** at  $u \in X$  if  $\forall x \in X$ , we have

$$f(x) < f(u) \Rightarrow \nabla f(u)(x - u) < 0.$$

(ii)  $f$  is said to be **quasiconvex** at  $u \in X$  if  $\forall x \in X$ , we have

$$f(x) \leq f(u) \Rightarrow \nabla f(u)(x - u) \leq 0.$$

(iii)  $f$  is said to be **strongly pseudoconvex** at  $u \in X$  if  $\forall x \in X$ , we have

$$f(x) \leq f(u) \Rightarrow \nabla f(u)(x - u) \leq 0.$$

(iv)  $f$  is said to be **weak strictly pseudoconvex** at  $u \in X$  if  $\forall x \in X$ , we have

$$f(x) \leq f(u) \Rightarrow \nabla f(u)(x - u) < 0.$$

**Definition 7.2.4** [114, 143] Let  $x \in F$  and  $I_1(x) = \{t \in I : \exists v_t \in \mathcal{V}_t \text{ s.t. } g_t(x, v_t) = 0\}$  and let  $\mathcal{V}_t(x) = \{v_t \in \mathcal{V}_t : g_t(x, v_t) = 0\}$ . We say that the Extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ) is satisfied at  $x \in F$  if and only if there exist,

$$d \in \mathbb{R}^n \text{ s.t. } \nabla_x g_t(x, v_t)^T d < 0, \ \forall t \in I_1(x), v_t \in \mathcal{V}_t(x).$$

### 7.3 Sufficient Robust Optimality Conditions

This part corresponds to the sufficient optimality conditions for (RSIP). The necessary optimality conditions for a robust semi-infinite program has already been obtained in [143]. We present the sufficient optimality conditions under which a feasible solution becomes (a weakly/properly) efficient solution.

**Theorem 7.3.1** *Let  $\bar{x}$  be a robust feasible solution of (USIP). Suppose that there exist  $\mu \geq 0$ ,  $(\lambda_t)_{t \in I} \in \mathbb{R}_+^I$  and  $v_t \in \mathcal{V}_t$ ,  $t \in I$  such that*

$$\sum_{i=1}^k \mu_i \nabla f_i(\bar{x}) + \sum_{t \in I} \lambda_t \nabla_x g(\bar{x}, v_t) = 0, \quad (7.3.1)$$

$$\sum_{t \in I} \lambda_t g_t(\bar{x}, v_t) = 0. \quad (7.3.2)$$

*Further suppose that, for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $\bar{x}$  and the function  $f$  is pseudoconvex at  $\bar{x}$ . Then  $\bar{x}$  is a weakly robust efficient solution of (USIP).*

**Proof.** Assume to the contrary that,  $\bar{x}$  is not a weakly robust efficient solution of (USIP), i.e.,  $\bar{x}$  is not a weakly efficient solution of (RSIP). This means that there is  $x \in F$  such that

$$f(x) < f(\bar{x}). \quad (7.3.3)$$

Since  $f$  is pseudoconvex at  $\bar{x}$ , (7.3.3) implies

$$\nabla f(\bar{x})^T (x - \bar{x}) < 0. \quad (7.3.4)$$

It follows by  $\mu \geq 0$  that

$$\mu^T \nabla f(\bar{x})^T (x - \bar{x}) < 0 \Leftrightarrow \sum_{i=1}^k \mu_i \nabla f_i(\bar{x})^T (x - \bar{x}) < 0. \quad (7.3.5)$$

Now as  $x \in F$  and for  $(\lambda_t)_{t \in I} \in \mathbb{R}_+^I$ , we get that  $\lambda_t g_t(x, v_t) \leq 0$ ,  $t \in T$ . This along with (7.3.2) gives

$$\sum_{t \in I} \lambda_t g_t(x, v_t) - \sum_{t \in I} \lambda_t g_t(\bar{x}, v_t) \leq 0, \quad t \in T. \quad (7.3.6)$$

The quasiconvexity of  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  at  $\bar{x}$  implies

$$\sum_{t \in I} \lambda_t \nabla_x g_t(\bar{x}, v_t)(x - \bar{x}) \leq 0, \quad t \in T, \quad (7.3.7)$$

which along with (7.3.1) entails

$$\sum_{i=1}^k \mu_i \nabla f_i(\bar{x})(x - \bar{x}) = - \sum_{t \in I} \lambda_t \nabla_x g(\bar{x}, v_t)(x - \bar{x}) \geq 0. \quad (7.3.8)$$

But (7.3.8) is a contradiction to (7.3.5). Hence  $\bar{x}$  is a weakly robust efficient solution of (USIP).

We now discuss an example to validate the above result.

**Example 7.3.1** Consider the following robust bi-objective program:

$$\begin{aligned} (BP) \quad & \text{Minimize} \quad f(x) = (x^3 + x, -2x) \\ & \text{subject to} \quad 2xv_t - 3 \leq 0, \\ & \quad \quad \quad v_t \in [-t, t+1], t \in I = [1, 2], \quad x \in \mathbb{R}, \end{aligned}$$

where the set of feasible solutions is given by

$$F_{BP} = \{x \in \mathbb{R}, 2xv_t - 3 \leq 0, v_t \in [-t, t+1], t \in I = [1, 2]\}.$$

We obtain the feasible region to be  $[-\frac{3}{4}, \frac{1}{2}]$  for the set  $F_{BP}$ . We consider a point  $\bar{x} = \frac{1}{2}$  such that

$$\begin{aligned} \sum_{t \in I} \lambda_t g_t(x, v_t) - \sum_{t \in I} \lambda_t g_t(\frac{1}{2}, v_t) &= \sum_{t \in I} \lambda_t (2xv_t - 3 - (2(\frac{1}{2})(v_t) - 3)) \\ &= \sum_{t \in I} \lambda_t v_t (2x - 1) \leq 0. \end{aligned} \quad (7.3.9)$$

Now

$$\begin{aligned} \nabla_x \left( \sum_{t \in I} \lambda_t g_t(x, v_t) \right) (x - \bar{x}) &= \nabla_x \left( \sum_{t \in I} \lambda_t (2xv_t - 3) \right) (x - \frac{1}{2}) \\ &= \sum_{t \in I} \lambda_t (2v_t) (x - \frac{1}{2}) \\ &= \sum_{t \in I} \lambda_t v_t (2x - 1) \\ &\leq 0, \quad (\text{by (7.3.9)}). \end{aligned} \quad (7.3.10)$$

Hence  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $\bar{x} = \frac{1}{2}$ . The function  $f(x) = (x^3 + x, -2x)$  is pseudoconvex at  $\bar{x} = \frac{1}{2}$ . For  $\bar{x} = \frac{1}{2}$ ,  $\mu = (\frac{1}{2}, \frac{1}{2})$ ,  $\lambda_t = \frac{1}{48}$  for  $t = 2$  and  $\lambda_t = 0 \forall t \in [1, 2)$  and  $v_t = 3$ . So the equations (7.3.1) and (7.3.2) are satisfied. Hence, we say that  $\bar{x} = \frac{1}{2}$  is a weakly efficient solution of (BP).

**Theorem 7.3.2** Let  $\bar{x}$  be a robust feasible solution of (USIP). Assume that, there exist  $\mu \geq 0$  and  $(\lambda_t)_{t \in I} \in \mathbb{R}_+^I$  and  $v_t \in \mathcal{V}_t$ ,  $t \in I$  such that (7.3.1) and (7.3.2) hold at  $\bar{x}$ . Further, suppose that for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $\bar{x}$ . If any of the following assumptions hold:

- (i)  $f$  is weak strictly pseudoconvex at  $\bar{x}$ , or
- (ii)  $f$  strongly pseudoconvex at  $\bar{x}$  and  $\mu > 0$ .

Then  $\bar{x}$  is a robust efficient solution of (USIP).

**Proof.** Suppose for the contradiction that  $\bar{x}$  is not a robust efficient solution of (USIP). This means that there is  $x \in F$  such that

$$f(x) \leq f(\bar{x}). \quad (7.3.11)$$

- (i) Since  $f$  is weak strictly pseudoconvex at  $\bar{x}$ , we get,  $\nabla f(\bar{x})^T(x - \bar{x}) < 0$ . This along with  $\mu \geq 0$  implies

$$\sum_{i=1}^k \mu_i \nabla f_i(\bar{x})(x - \bar{x}) < 0. \quad (7.3.12)$$

- (ii) In this part, since  $f$  is strongly pseudoconvex at  $\bar{x}$ , (7.3.11) implies  $\nabla f(\bar{x})^T(x - \bar{x}) \leq 0$ . This along with  $\mu > 0$  gives

$$\sum_{i=1}^k \mu_i \nabla f_i(\bar{x})(x - \bar{x}) < 0. \quad (7.3.13)$$

Now using the same approach as in Theorem 7.3.1, we arrive at

$$\sum_{i=1}^k \mu_i \nabla f_i(\bar{x})(x - \bar{x}) = - \sum_{t \in I} \lambda_t \nabla_x g(\bar{x}, v_t)(x - \bar{x}) \geq 0, \quad (7.3.14)$$

which is a contradiction to the equations (7.3.12) and (7.3.13). This means that  $\bar{x}$  is a robust efficient solution of (USIP).

The following lemma is given by Geoffrion [68] and is used for obtaining properly efficient solutions of (RSIP) or properly robust efficient solutions of (USIP).

**Lemma 7.3.3** [68] *Given a set  $F \subseteq \mathbb{R}^n$  of feasible points and a function  $h : \mathbb{R}^n \rightarrow \mathbb{R}^k$ . If for some fixed  $\mu(> 0)$  in  $\mathbb{R}^k$ ,  $\bar{x}$  is an optimal solution of the following scalar minimization problem*

$$(P_\lambda) \quad \text{Min} \quad \mu^T h(x) \quad \text{subject to } x \in F.$$

*Then  $\bar{x}$  is proper efficient solution of the following multiobjective problem (P):*

$$(P) \quad \text{Min} \quad h(x) = (h_1(x), h_2(x), \dots, h_k(x)) \quad \text{subject to } x \in F.$$

**Theorem 7.3.4** *Let  $\bar{x}$  be a feasible solution of (RSIP). Suppose that there exist  $\mu > 0$ ,  $(\lambda_t)_{t \in I} \in \mathbb{R}_+^I$  and  $v_t \in \mathcal{V}_t$ ,  $t \in I$  such that (7.3.1) and (7.3.2) hold at  $\bar{x}$ . Further suppose that for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $\bar{x}$  and the function  $\mu^T f$  is pseudoconvex at  $\bar{x}$ . Then  $\bar{x}$  is a properly efficient solution of (RSIP).*

**Proof.** Since  $\bar{x} \in F$  and  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $\bar{x}$ , then on the lines of proof of Theorem 7.3.1, we get

$$\sum_{i=1}^k \mu_i \nabla f_i(\bar{x})(x - \bar{x}) = - \sum_{t \in I} \lambda_t \nabla_x g(\bar{x}, v_t)(x - \bar{x}) \geq 0. \quad (7.3.15)$$

Equation (7.3.15) can be rewritten as

$$(x - \bar{x})^T \nabla \mu^T f(\bar{x}) \geq 0. \quad (7.3.16)$$

This along with pseudoconvexity of  $\mu^T f$  provides us with the following inequality

$$\mu^T f(x) \geq \mu^T f(\bar{x}), \quad \forall x \in F. \quad (7.3.17)$$

This implies that  $\bar{x}$  is an optimal solution of  $(\text{RSIP}_\lambda)$  which in turn implies that  $\bar{x}$  is a properly robust efficient solution of (RSIP). Hence the theorem.

## 7.4 Robust Dual Model

We present the following Mond-Weir type dual for robust semi-infinite program (RSIP):

$$\begin{aligned} \text{Dual (RSIMWD)} \quad & \text{Maximize} \quad f(u) = (f_1(u), f_2(u), \dots, f_k(u)) \\ & \text{subject to} \quad \sum_{i=1}^k \alpha_i \nabla f_i(u) + \sum_{t \in I} \lambda_t \nabla_x g(u, v_t) = 0, \end{aligned}$$

$$\sum_{t \in I} \lambda_t g_t(u, v_t) \geq 0, \quad t \in I,$$

$$\lambda_t \in \mathbb{R}_+^I, \quad \alpha \in \mathbb{R}_+^k \setminus \{0\}, \quad v_t \in \mathcal{V}_t.$$

Now we discuss the weak, strong and strict converse duality theorems under the assumptions of pseudoconvexity and quasiconvexity.

**Theorem 7.4.1 (Weak Duality)** *Let  $x$  be a feasible solution of (RSIP) and  $(u, \alpha, \lambda_t, v_t)$  be feasible solution of (RSIMWD). Assume that, for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $u$ . Furthermore,*

(i) *if  $f(\cdot)$  is pseudoconvex at  $u$ , then the following can not hold*

$$f(x) < f(u); \quad (7.4.1)$$

(ii) *if  $f(\cdot)$  is strongly pseudoconvex at  $u$  and  $\alpha > 0$ , then the following can not hold*

$$f(x) \leq f(u); \quad (7.4.2)$$

(iii) *if  $f(\cdot)$  is weak strictly pseudoconvex at  $u$ , then the following can not hold*

$$f(x) \leq f(u). \quad (7.4.3)$$

**Proof.** Since  $x \in F$ , it follows by  $\lambda_t \in \mathbb{R}_+^I$  that  $\sum_{t \in I} \lambda_t g(x, v_t) \leq 0$ .  $(u, \alpha, \lambda_t, v_t)$  being feasible for the dual problem, give  $\sum_{t \in I} \lambda_t g_t(u, v_t) \geq 0$ . Combining, we get

$$\sum_{t \in I} \lambda_t g(x, v_t) - \sum_{t \in I} \lambda_t g(u, v_t) \leq 0. \quad (7.4.4)$$

Since  $\sum_{t \in I} \lambda_t g(\cdot, v_t)$  is quasiconvex at  $u$  for each  $v_t \in \mathcal{V}_t$  and  $t \in I$ , we get

$$\nabla_x \left( \sum_{t \in I} \lambda_t g(u, v_t) \right) (x - u) \leq 0.$$

$$\Leftrightarrow \sum_{t \in I} \lambda_t \nabla_x g(u, v_t) (x - u) \leq 0. \quad (7.4.5)$$

Now by first dual feasibility condition, we have

$$\sum_{i=1}^k \alpha_i \nabla f_i(u) = - \sum_{t \in I} \lambda_t \nabla_x g(u, v_t) \quad (7.4.6)$$

$$\begin{aligned} \Leftrightarrow \sum_{i=1}^k \alpha_i \nabla f_i(u)(x-u) &= - \sum_{t \in I} \lambda_t \nabla_x g(u, v_t)(x-u) \\ &\geq 0, \quad (\text{by (7.4.5)}). \end{aligned} \quad (7.4.7)$$

Now we justify (i)-(iii).

- (i) Suppose to the contrary, we assume that  $f(x) - f(u) < 0$ . The pseudoconvexity of  $f$  at  $u$  implies

$$\begin{aligned} \nabla f(u)(x-u) &< 0. \\ \Rightarrow \sum_{i=1}^k \alpha_i \nabla f_i(u)(x-u) &< 0, \quad (\text{for } \alpha \in \mathbb{R}_+^k \setminus \{0\}). \end{aligned} \quad (7.4.8)$$

But (7.4.8) and (7.4.7) contradict each other. So, we conclude that  $f(x) - f(u) < 0$  is not true. Hence we have proved the first part of theorem.

- (ii) In this part also, if we say that the result does not hold, i.e.,  $f(x) \leq f(u)$ . As  $f$  is strongly pseudoconvex at  $u$ , we have

$$\nabla f(u)(x-u) \leq 0.$$

This along with  $\alpha > 0$  implies

$$\sum_{i=1}^k \alpha_i \nabla f_i(u)(x-u) < 0. \quad (7.4.9)$$

Rest of the proof is same as done in part (i).

- (iii) The proof of this part is same as that of part (i), hence omitted.

**Example 7.4.1** Consider the same problem as discussed in Example 7.3.1. We take the following problem (BP) as the primal problem:

$$\begin{aligned} (BP) \quad & \text{Minimize} \quad f(x) = (x^3 + x, -2x) \\ & \text{subject to} \quad 2xv_t - 3 \leq 0, \quad v_t \in [-t, t+1], \\ & \quad \quad \quad t \in I = [1, 2], \quad x \in \mathbb{R}. \end{aligned}$$

The corresponding robust Mond-Weir type dual for the above primal (BP) is written as follows:

$$(BD) \quad \text{Maximize} \quad f(x) = (u^3 + u, -2u)$$

$$\begin{aligned}
\text{subject to} \quad & 3\alpha_1 u^2 + \alpha_1 - 2\alpha_2 + 2 \sum_{t \in I} \lambda_t v_t = 0, \\
& \sum_{t \in I} \lambda_t (2xv_t - 3) \geq 0, \\
& v_t \in [-t, t + 1], \quad t \in [1, 2], \\
& x \in \mathbb{R}, \quad \alpha_1, \alpha_2 \geq 0, \quad \alpha \neq 0.
\end{aligned}$$

$\bar{x} = \frac{1}{2}$  is a feasible solution of (BP) and  $(\frac{1}{2}, (\frac{1}{2}, \frac{1}{2}), \frac{1}{48}, 3)$  satisfies the feasibility conditions of the dual model (BD). The assumptions of weak duality theorem part (i) hold at these points (as discussed in Example 7.3.1). Hence the weak duality relation holds between (BP) and (BD).

**Theorem 7.4.2 (Strong Duality)** *Let  $\bar{u}$  be a weakly efficient solution of (RSIP) and for each  $x \in \mathbb{R}^n$  and  $t \in I$ ,  $g_t(x, \cdot)$  be concave on  $\mathcal{V}_t$ . Also assume that (EMFCQ) hold at  $\bar{u}$ . Then there exist  $(\bar{\alpha}, \bar{\lambda}_t, \bar{v}_t) \in \mathbb{R}_+^k \setminus \{0\} \times \mathbb{R}_+^I \times \mathcal{V}_t$  such that  $(\bar{u}, \bar{\alpha}, \bar{\lambda}, \bar{v}_t)$  is a feasible solution of the Mond-Weir dual (RSIMWD). Further, if for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $\bar{u}$  and  $f$  is pseudoconvex at  $\bar{u}$ , then  $(\bar{u}, \bar{\alpha}, \bar{\lambda}, \bar{v}_t)$  is also a weakly efficient solution of (RSIMWD).*

**Proof.** If  $\bar{u}$  is a weakly efficient solution of (RSIP) and the constraint qualification (EMFCQ) is satisfied at  $\bar{u}$ , then by necessary optimality conditions [143], there exist  $\bar{\alpha} \geq 0$ ,  $\bar{\lambda}_t \in \mathbb{R}_+^I$ ,  $\bar{v}_t \in \mathcal{V}_t$ ,  $t \in I$  and  $\sum_{i=1}^k \bar{\alpha}_i = 1$  such that

$$\sum_{i=1}^k \bar{\alpha}_i \nabla f_i(\bar{u}) + \sum_{t \in I} \bar{\lambda}_t \nabla_x g_t(\bar{u}, \bar{v}_t) = 0. \quad (7.4.10)$$

$$\sum_{t \in I} (\bar{\lambda}_t) g_t(\bar{u}, \bar{v}_t) = 0, \quad t \in I. \quad (7.4.11)$$

Equations (7.4.10) and (7.4.11) give that  $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$  is a feasible solution of the dual (RSIMWD). Now we will show that this point is a weakly efficient solution of (RSIMWD). The proof is as follows:

If we say that it is not a weakly efficient solution of the dual. Then there exist some feasible solution of dual (RSIMWD), say  $(\tilde{u}, \tilde{\alpha}, \tilde{\lambda}_t, \tilde{v}_t)$ , such that  $f(\tilde{u}) > f(\bar{u})$ . In wake of  $\bar{u}$  being a feasible solution of primal, we get a contradiction to weak duality theorem 7.4.1(i). Hence  $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$  is a weakly efficient solution of dual (RSIMWD). This completes the theorem.

**Theorem 7.4.3 (Strict Converse Duality)** *Let  $\bar{x}$  be a feasible solution of (RSIP) and  $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$  be a feasible solution of (RSIMWD). Let for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,*

$\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  be quasiconvex at  $\bar{u}$  and  $\bar{\alpha}^T f$  be strictly pseudoconvex at  $\bar{u}$ . If

$$\sum_{i=1}^k \bar{\alpha}_i f_i(\bar{x}) = \sum_{i=1}^k \bar{\alpha}_i f_i(\bar{u}), \quad (7.4.12)$$

then  $\bar{x} = \bar{u}$ .

**Proof.** If  $\bar{x} \neq \bar{u}$ . Since  $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$  is a feasible solution of dual (RSIMWD) and  $\bar{x}$  is feasible solution of (RSIP), for  $\bar{\lambda}_t \in \mathbb{R}_+^I$  we obtain

$$\sum_{t \in I} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) - \sum_{t \in I} \bar{\lambda}_t g_t(\bar{u}, \bar{v}_t) \leq 0, \quad t \in I. \quad (7.4.13)$$

The above inequality along with the quasiconvexity of  $\sum_{t \in I} \bar{\lambda}_t g_t(\cdot, \bar{v}_t)$  give

$$\sum_{t \in I} \bar{\lambda}_t \nabla_x g_t(\bar{u}, \bar{v}_t)(\bar{x} - \bar{u}) \leq 0, \quad t \in I. \quad (7.4.14)$$

Equation (7.4.14) and first dual feasibility condition yield

$$\sum_{i=1}^k \bar{\alpha}_i \nabla f_i(\bar{u})(\bar{x} - \bar{u}) \geq 0. \quad (7.4.15)$$

Since  $\bar{\alpha}^T f$  is strictly pseudoconvex at  $\bar{u}$ , (7.4.15) implies

$$\sum_{i=1}^k \bar{\alpha}_i f_i(\bar{x}) > \sum_{i=1}^k \bar{\alpha}_i f_i(\bar{u}). \quad (7.4.16)$$

Combining (7.4.12) with (7.4.16) we arrive at a contradiction. Thus the proof is completed.

## 7.5 Application to Fractional Programming

In this section, we consider a fractional counterpart of the semi-infinite multiobjective program under data uncertainty. In this problem each objective function is a ratio of two functions. Consider the following fractional semi-infinite programming problem (UFSIP):

$$\begin{array}{ll} \text{Problem (UFSIP)} & \text{Minimize } \bar{f}(x) = \left( \frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \dots, \frac{f_k(x)}{h_k(x)} \right) \\ & \text{subject to } g_t(x, v_t) \leq 0, \quad t \in I. \end{array}$$

where  $f_i, h_i : \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable functions.  $v_t$  and  $g_t$  are same as in (USIP). Assume that  $f_i \geq 0$  and  $h_i > 0$  in  $\mathbb{R}^n$ . The corresponding robust problem is written as:

$$\begin{array}{ll} \text{Problem (RFSIP)} & \text{Minimize } \bar{f}(x) = \left( \frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \dots, \frac{f_k(x)}{h_k(x)} \right) \\ & \text{subject to } g_t(x, v_t) \leq 0, \forall v_t \in \mathcal{V}_t, t \in I. \end{array}$$

The set of robust feasible solutions is given by

$$F := \{x \in \mathbb{R}^n, g_t(x, v_t) \leq 0, \forall t \in I \text{ and } v_t \in \mathcal{V}_t\}.$$

Here the functions  $f_i, h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , for  $i = 1, 2, \dots, k$ , are continuously differentiable functions and it is assumed that in the feasible region  $f_i$  is nonnegative and  $h_i$  is positive for each  $i = 1, 2, \dots, k$ .

**Theorem 7.5.1 (Sufficient Optimality Condition)** *Let  $\bar{x}$  be a robust feasible solution of (UFSIP). Suppose that there exist  $\mu \geq 0$  and  $(\lambda_t)_{t \in I} \in \mathbb{R}_+^I$  and  $v_t \in \mathcal{V}_t$ ,  $t \in I$  such that*

$$\sum_{i=1}^k \mu_i \nabla \left( \frac{f_i(\bar{x})}{h_i(\bar{x})} \right) + \sum_{t \in I} \lambda_t \nabla_x g(\bar{x}, v_t) = 0, \quad (7.5.1)$$

$$\sum_{t \in I} \lambda_t g_t(\bar{x}, v_t) = 0, \quad t \in I. \quad (7.5.2)$$

*If for each  $i = 1, 2, \dots, k$ ,  $f_i$  is convex at  $\bar{x}$ ,  $h_i$  is concave at  $\bar{x}$ . Further if, for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $\bar{x}$ . Then  $\bar{x}$  is a weakly robust efficient solution of (UFSIP).*

**Proof.** Suppose if possible  $\bar{x}$  is not a weakly robust efficient solution of (RFSIP), then there is an  $x \in F$  such that

$$\frac{f_i(x)}{h_i(x)} < \frac{f_i(\bar{x})}{h_i(\bar{x})}, \text{ for each } i = 1, 2, \dots, k. \quad (7.5.3)$$

If  $f_i$  is convex at  $\bar{x}$ ,  $h_i$  is concave at  $\bar{x}$ ,  $f_i \geq 0, h_i > 0$ , then  $\frac{f_i}{h_i}$  is pseudoconvex at  $\bar{x}$ . Then pseudoconvexity of  $\frac{f_i}{h_i}$  for each  $i = 1, 2, \dots, k$  at  $\bar{x}$  and (7.5.3) give

$$\nabla \left( \frac{f_i(\bar{x})}{h_i(\bar{x})} \right) (x - \bar{x}) < 0.$$

This along with  $\alpha \geq 0$  gives

$$\sum_{i=1}^k \alpha_i \nabla \left( \frac{f_i(\bar{x})}{h_i(\bar{x})} \right) (x - \bar{x}) < 0. \quad (7.5.4)$$

The feasibility of  $x$  for (RFSIP) along with (7.5.2) yields

$$\sum_{t \in I} \lambda_t g_t(x, v_t) - \sum_{t \in I} \lambda_t g_t(\bar{x}, v_t) \leq 0, \quad t \in I. \quad (7.5.5)$$

By the assumption that  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex for each  $v_t \in \mathcal{V}_t$ ,  $t \in I$  at  $\bar{x}$ , we get

$$\sum_{t \in I} \lambda_t \nabla_x g_t(\bar{x}, v_t) (x - \bar{x}) \leq 0, \quad t \in I. \quad (7.5.6)$$

The equations (7.5.1) and (7.5.6) give

$$\sum_{i=1}^k \alpha_i \nabla \left( \frac{f_i(\bar{x})}{h_i(\bar{x})} \right) (x - \bar{x}) = - \sum_{t \in I} \lambda_t \nabla_x g_t(\bar{x}, v_t) (x - \bar{x}) \geq 0. \quad (7.5.7)$$

But (7.5.7) is contradiction to (7.5.4). So  $\bar{x}$  is a weakly robust efficient solution of (UFSIP) and a weakly efficient solution of (RFSIP).

**Theorem 7.5.2 (Sufficient Optimality Condition)** *Let  $\bar{x}$  be a robust feasible solution of (UFSIP). Suppose that there exist  $\mu \geq 0$  and  $(\lambda_t)_{t \in I} \in \mathbb{R}_+^I$  and  $v_t$ ,  $t \in I$  such that the conditions (7.5.1) and (7.5.2) hold at  $\bar{x}$ . If the vector valued function  $\bar{f}$  is weak strictly pseudoconvex at  $\bar{x}$ . Further if, for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasi-convex at  $\bar{x}$ , then  $\bar{x}$  is a robust efficient solution of (UFSIP).*

**Proof.** On the contrary, assume that  $\bar{x}$  is not a robust efficient solution of (UFSIP). Then there is an  $x \in F$  such that

$$\frac{f(x)}{h(x)} \leq \frac{f(\bar{x})}{h(\bar{x})}.$$

Since the vector function  $\bar{f} = (\frac{f_1}{h_1}, \frac{f_2}{h_2}, \dots, \frac{f_k}{h_k})$  is weak strictly pseudoconvex, we get

$$\nabla \bar{f}(\bar{x})(x - \bar{x}) < 0.$$

This along with  $\alpha \geq 0$  gives

$$\sum_{i=1}^k \alpha_i \nabla \left( \frac{f_i(\bar{x})}{h_i(\bar{x})} \right) (x - \bar{x}) < 0. \quad (7.5.8)$$

The rest of the proof follows on the lines of the proof of Theorem 7.5.1.

### 7.5.1 Dual Model

The Mond Weir dual program for the robust fractional semi-infinite programming problem (RFSIP) is formulated as follows:

$$\begin{aligned}
\text{Dual (RFSID)} \quad & \text{Maximize} \quad \bar{f}(u) = \left( \frac{f_1(u)}{h_1(u)}, \frac{f_2(u)}{h_2(u)}, \dots, \frac{f_k(u)}{h_k(u)} \right) \\
& \text{subject to} \\
& \sum_{i=1}^k \alpha_i \nabla \left( \frac{f_i(u)}{h_i(u)} \right) + \sum_{t \in I} \lambda_t \nabla_x g_t(u, v_t) = 0, \quad (7.5.9) \\
& \sum_{t \in I} \lambda_t g_t(u, v_t) \geq 0, \quad t \in I, \quad (7.5.10) \\
& v_t \in \mathcal{V}_t, \lambda_t \in \mathbb{R}_+^I, \alpha \in \mathbb{R}_+^k \setminus \{0\}.
\end{aligned}$$

Now we establish the duality relations between (RFSIP) and (RFSID).

**Theorem 7.5.3 (Weak Duality)** *Let  $x$  be a feasible solution of (RFSIP) and let  $(u, \alpha, \lambda_t, v_t)$  be a feasible solution of (RFSID). If for each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ ,  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  is quasiconvex at  $u$  and the vector valued function  $(\frac{f_1}{h_1}, \frac{f_2}{h_2}, \dots, \frac{f_k}{h_k})$  is weak strictly pseudoconvex at  $u$ , then the following can not hold:*

$$\left( \frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \dots, \frac{f_k(x)}{h_k(x)} \right) \leq \left( \frac{f_1(u)}{h_1(u)}, \frac{f_2(u)}{h_2(u)}, \dots, \frac{f_k(u)}{h_k(u)} \right). \quad (7.5.11)$$

**Proof.** We will prove this by contradiction. Suppose if possible the result is not true, i.e.,

$$\left( \frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \dots, \frac{f_k(x)}{h_k(x)} \right) \leq \left( \frac{f_1(u)}{h_1(u)}, \frac{f_2(u)}{h_2(u)}, \dots, \frac{f_k(u)}{h_k(u)} \right). \quad (7.5.12)$$

Using (7.5.12) and weak strictly pseudoconvexity of  $\bar{f} = (\frac{f_1}{h_1}, \frac{f_2}{h_2}, \dots, \frac{f_k}{h_k})$  we get

$$\nabla \bar{f}(u)(x - u) < 0. \quad (7.5.13)$$

Since  $\alpha \in \mathbb{R}_+^k \setminus \{0\}$ , we obtain

$$\sum_{i=1}^k \alpha_i \nabla \left( \frac{f_i(u)}{h_i(u)} \right) (x - u) < 0. \quad (7.5.14)$$

As  $x$  is a feasible solution of (RFSIP),  $(\lambda_t)_{t \in I} \in \mathbb{R}_+^I$  yields  $\sum_{t \in I} \lambda_t g_t(x, v_t) \leq 0$  whereas a feasible solution  $(u, \alpha, \lambda_t, v_t)$  of the dual satisfies (7.5.10), so we have

$$\sum_{t \in I} \lambda_t g_t(x, v_t) - \sum_{t \in I} \lambda_t g_t(u, v_t) \leq 0. \quad (7.5.15)$$

This along with quasiconvexity of  $\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  at  $u$  gives

$$\sum_{t \in I} \lambda_t \nabla_x g_t(u, v_t)(x - u) \leq 0. \quad (7.5.16)$$

Adding (7.5.14) and (7.5.16) we get

$$\left( \sum_{i=1}^k \alpha_i \nabla \left( \frac{f_i(u)}{h_i(u)} \right) + \sum_{t \in I} \lambda_t \nabla_x g_t(u, v_t) \right) (x - u) < 0. \quad (7.5.17)$$

But this contradicts dual feasibility condition (7.5.9). Hence we have the theorem.

**Theorem 7.5.4 (Strong Duality)** *Suppose that  $u$  is a weakly efficient solution of (RFSIP) and for each  $x \in \mathbb{R}^n$  and for each  $t \in I$ ,  $g_t(x, \cdot)$  is concave on  $\mathcal{V}_t$ . Also assume that (EMFCQ) hold at  $u$ . Then there exist  $(\bar{\alpha}, \bar{\lambda}_t, \bar{v}_t) \in \mathbb{R}_+^k \setminus \{0\} \times \mathbb{R}_+^I \times \mathcal{V}_t$  such that  $(u, \bar{\alpha}, \bar{\lambda}, \bar{v}_t)$  is a feasible solution of (RFSID). If the assumptions of weak duality theorem 7.5.3 hold for each of the feasible solutions of (RFSIP) and (RFSID) respectively, then  $(u, \bar{\alpha}, \bar{\lambda}, \bar{v}_t)$  is also an efficient solution of (RFSID).*

**Proof.** If  $u$  is a weakly efficient solution of (RFSIP), then by the necessary optimality conditions [143], there exist  $(\bar{\alpha}, \bar{\lambda}_t, \bar{v}_t) \in \mathbb{R}_+^k \setminus \{0\} \times \mathbb{R}_+^I \times \mathcal{V}_t$ ,  $t \in I$  such that  $(u, \bar{\alpha}, \bar{\lambda}, \bar{v}_t)$  becomes feasible solution of the dual (RFSID).

Now we claim that this point is also an efficient solution of the dual. Because, if  $(u, \bar{\alpha}, \bar{\lambda}, \bar{v}_t)$  is not an efficient solution of the dual, then for some feasible solution  $(\tilde{x}, \tilde{\alpha}, \tilde{\lambda}, \tilde{v}_t)$  of the dual following holds

$$\frac{f(u)}{h(u)} \leq \frac{f(\tilde{x})}{h(\tilde{x})}.$$

But this is a contradiction to weak duality theorem 7.5.3. Hence  $(u, \bar{\alpha}, \bar{\lambda}, \bar{v}_t)$  is an efficient solution of (RFSID). This completes the strong duality theorem.

**Theorem 7.5.5 (Strict Converse Duality)** *If  $\bar{x}$  is a feasible solution of (RFSIP) and  $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$  is a feasible solution of (RFSID). For each  $v_t \in \mathcal{V}_t$  and for each  $t \in I$ , let*

$\sum_{t \in I} \lambda_t g_t(\cdot, v_t)$  be quasiconvex at  $\bar{u}$  and  $\bar{\alpha}^T(\frac{f}{h})$  be strictly pseudoconvex at  $\bar{u}$ . If

$$\bar{\alpha}^T\left(\frac{f(\bar{x})}{h(\bar{x})}\right) = \bar{\alpha}^T\left(\frac{f(\bar{u})}{h(\bar{u})}\right), \quad (7.5.18)$$

then  $\bar{x} = \bar{u}$ .

**Proof.** The proof follows on the lines of Theorem 7.4.3.

## 7.6 Conclusion and Further Research

In this Chapter, a robust counterpart of an uncertain semi-infinite multiobjective programming problem is studied. By using pseudoconvexity and quasiconvexity assumptions on the functions, we devised conditions which identify the efficient solutions of the given program. We have studied sufficient optimality conditions for three types of efficient solutions of (RSIP) namely, weakly robust efficient solution, robust efficient solution and properly robust efficient solution. Then we constructed the Mond-Weir type dual program for robust semi-infinite programming problem and studied weak, strong and strict converse duality theorems to establish the relation between the primal and its dual problem. A numerical example of the weak duality theorem was also provided. We concluded the Chapter with an extension of the above discussed program to fractional programming problems. We studied sufficient optimality conditions under suitable function assumptions. Then dual model was constructed for fractional program and duality theorems were also stated and proved. The results of this Chapter were obtained by taking suitable generalized convexity assumptions on the functions involved. This Chapter opens different avenues for the future work of the researchers. For example, this work can be extended to the study of a problem with uncertainty into the objective function also. Also a new program can be introduced which has nondifferentiable objective function and constraints or adding some nondifferentiable functions to the respective functions. In addition to that these results can be generalized for vector valued programs which involve ordering with respect to a given set.



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