A STUDY OF FUZZY, SUPER

AND SUPER FUZZY MATRIX THEORY

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Submitted by

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CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"A Study of Fuzzy, Super And Super Fuzzy Matrix Theory"** in partial fulfillment of the requirements for the award of degree of **Master of Science,** School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. M.A. Khan**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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PREFACE

Having paid my deep sense of gratitude, I feel myself deserve to present this thesis. The work being presented in the thesis is devoted to ‘A Study of Fuzzy, Super and Super Fuzzy Matrix Theory’. It consists of six chapters. In chapters, Tables and figures are shown not as a sub-section but as different sections and in this way, in contents, list of tables and figures are shown separately from chapters.

Starting from the first chapter on binary and fuzzy sets, which are further related to fuzzy matrix theory. Then the determinant and adjoint theories of square fuzzy matrix are included. Next, the concept of supermatrices is introduced and is further related to fuzzy supermatrices. Last chapter is ended to recall the summary of the comparative study done and to conclude the theories based on this work for further advanced study in this field and its important applications in other related specific areas.

The main body of the thesis is then followed by a list of references which by no means is a complete bibligraphy of the work, rather the work referred in this thesis has been included in this literature.

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Chapter 1

BINARY AND FUZZY SET THEORY

1.1 Introduction

The purpose of present chapter is to introduce basic concepts of fuzzy sets for the further study of fuzzy and super fuzzy matrices. In today's scientific and socio-technoligical fields, the classical approaches to decision making have been declining while innovative concepts based upon the new mathematics such as fuzzy set theory, are continually emerging.

Advances in science and technology have made our modern society very complex, and with this, decision processes have become increasingly vague and hard to analyze. The human brain possesses some special characteristics that enable it to learn and reason in a vague and fuzzy environment. It has the ability to arrive at decisions based on imprecise, qualitative data in contrast to formal mathematics and formal logic which demand precise and quantitative data. Modern binary computers possess capacity but lack human like ability. Undoubtedly, in many areas of cognition, human intelligence far excels the computer "intelligence" of today, and the development of fuzzy concepts is a step forward towards the development of tools capable of handling humanistic types of problems.

We do have sufficient mathematical tools and computer-based technology for analyzing and solving problems embedded in deterministic and uncertain (probabilistic) environment. Here uncertainty may arise from the probabilistic behaviour of certain physical phenomena in mechanistic systems. We know the important role that vagueness and inexactitude play in human decision making, but we did not know until 1965 how the vagueness arising from subjectivity which is inherent in human thought processes can be modelled and analyzed.

In 1965, Professor Loffi A. Zadeh [15] laid the foundation of fuzzy set theory. In effect, fuzzy set theory is a body of concepts and techniques that gave a form of mathematical precision to human cognitive processes that in many ways are imprecise

9and ambiguous by the standards of classical mathematics. Today, these concepts are gaining a growing acceptability among mathematicians, engineers, scientists and philosophers. Since its inception, research in the fuzzy set field has faced an increasing exponential growth. This fuzzy field has blossomed into a many faceted field of inquiry, drawing on and contributing to a wide spectrum of areas ranging from pure mathematics to human cognition, preception and judgement. Its influence in science, engineering and social sciences has been felt already and is certain to grow in the decade to come.

1.2 [7] Examples

Imprecision in characterizing the possible presence or absence of properties in a concept leads to a fuzzy set. Suppose we consider the character recognition problem of a faded typescript. We are aware of the character set available on the typewriter. Suppose a compound symbol such as "**O**" appears on the typescript. It is easily identifiable as being made up of characters "L" and "O". We would then say the compound symbol is associated with the subset $\{L,O\}$ of the character set on the typewriter.

Suppose instead a compound symbol such as "**Q**" appears. On a first glance we think we recognize a "Q", but on a closer examination we find it as superimposed "Q", "G", "O" and possible "C". We are not sure as to the presence or absence of "C" in "**Q**". Thus, if this symbol has to be represented as a subset of the symbols on the typewriter, then we either assign a "doubtful" label to "C" or decide that may be it is present. How do we represent a partial presence?

Such examples may be multiplied. How is one to measure the beauty of a snow-clad mountain? The marks assigned to the student in a course represent the degree to which his/her teacher measures his/her ability or performance in that subject. These marks may range from 0 to 100% or graded from A to F. A fuzzy concept of his/her over-all performance is obtained by his/her mark list for the term. Thus, instead of treating presence or absence in a binary manner, as in ordinary set theory, one deals with the presence in a graded manner, from 0 to 1.

Before we define the theory of fuzzy sets, let us first define the notion of the theory of binary sets starting from the definition of a set. A set is a collection of objects that are well specified and possess some common properties. These objects may represent some abstract concept, or may be a collection of some physical properties. It can be finite or infinite, enumerable or non-enumerable.

1.3 [5] Binary Set Theory

The representation of a binary set can be made in many ways, the most usual form is,

For a finite set,

$$E = \{a_1, a_2, \ldots, a_n\},$$

and for an infinite enumerable set,

$$E = \{a_1, a_2, \ldots, a_n, \ldots\}.$$

In a set E, an element a_i is called the member of the set.

In the case of an infinite non-enumerable set, real numbers are usually represented by a symbol like R (real numbers), R^+ (non-negative real numbers), R_0^+ (positive real numbers), N (natural numbers), Z (integers) etc.

Given a set E,

$E = \{a,b,c,d,e,f\}$.

A subset is a collection of some members from a set. Then the subset A of E is

$A = \{a,d,e\}$.

Symbolically, we write a subset as

$A \subset E$.

By definition, E is also a subset of itself and we can write $E \subset E$. A strict inclusion is represented by $\subset \subset$ when a set cannot be a subset of itself.

A set E and its subset A can also be represented in the following way

$E =$

a	b	c	d	e	f
1	1	1	1	1	1

A =

a	b	c	d	e	f
1	0	0	1	1	0

The numbers 0 and 1 define the membership of each element of the subset, where 1 means the element belongs to the subset and 0 means the element does not belong to the subset.

If $x \in$ A , we denote the corresponding function as

$$\forall\ x \in E : \mu_A(x) \in \{0,1\}$$

The function $\mu_A(x)$ is called the 'characteristic function' or 'membership function'. We see that a subset is the function of E on {0,1}.

Let us define the 'empty subset' ϕ as

$$\forall\ x \in E : \mu_\phi(x) = 0$$

and for the set E itself

$$\forall x \in E : \mu_E(x) = 1$$

The set of all subsets is called the 'Power Set' and is written as $P(E)$ or $\{0,1\}^E$.

For example, let $E = \{a,b,c\}$

Then power set is defined as

P(E) = {ϕ ,{a},{b},{c},{a,b},{a,c},{b,c},{a,b,c}}

1.3.1 [7] Binary Set Operations

In the theory of sets, the main operators are defined as follow:

Intersection: $\mu_{A \cap B}(X) = \mu_A(X) \wedge \mu_B(X),$

Union: $\mu_{A \cup B}(X) = \mu_A(X) \vee \mu_B(X),$

Complementation: $\mu_{\overline{A}}(X) = 1 - \mu_A(X),$

where $\wedge$ (and) means 'minimum' and $\vee$ (or) means 'maximum'. These operators are shown below in a tabular form.

$\wedge$	0	1
0	0	0
1	0	1

Intersection (Minimum)
Table 1.1 [7]

$\vee$	0	1
0	0	1
1	1	1

Union (Maximum)
Table 1.2 [7]

A	0	1
$\overline{A}$	1	0

Complementation
Table 1.3 [7]

We shall now illustrate these operators by means of an example.

Let us define a finite referential set *E* as

$E = \{a,b,c,d,e,f\},$

and the subsets A and B as

A =

a	*b*	*c*	*d*	*e*	*f*
1	0	0	1	1	0

and

B =

a	*b*	*c*	*d*	*e*	*f*
1	1	0	0	1	1

The intersection, union and complementation on these subsets are given by

$A \cap B$ =

a	*b*	*c*	*d*	*e*	*f*
1	0	0	0	1	0

$A \cup B =$

a	b	c	d	e	f
1	1	0	1	1	1

$\overline{A} =$

a	b	c	d	e	f
0	1	1	0	0	1

and

$\overline{B} =$

a	b	c	d	e	f
0	0	1	1	0	0

1.3.2 [6] Algebraic Properties of Sets

For any subset A,B,C from referential set *E*, the following properties hold:

(i) Commutativity:

$A \cup B = B \cup A$

$A \cap B = B \cap A.$

(ii) Associativity:

$(A \cap B) \cap C = A \cap (B \cap C),$

$(A \cup B) \cup C = A \cup (B \cup C).$

(iii) Idempotence:

$A \cap A = A,$

$A \cup A = A.$

(iv) Distributivity:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$

(v) Exclusion:

$A \cap \overline{A} = \phi.$

(vi) Non-contradiction:

$A \cup \overline{A} = E,$

$A \cap \phi = \phi,$

$A \cup \phi = A,$

$A \cap E = A,$

$A \cup E = E.$

(vii) Involution:

$$(\bar{\bar{A}}) = A$$

(viii) De-Morgan's theorems:

$$\overline{A \cap B} = \bar{A} \cup \bar{B},$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}.$$

Then set *E* is called a 'Boolean lattice' or a 'Distributive and Complemented lattice'.

In the next section, we will review the fuzzy set theory.

1.4 [3] Fuzzy Set Theory

Consider a referential set *E* with *x* as its element. The 'Characterstic function' or the 'membership function' of *x* is

$$\forall \; x \in E : \mu_A(x) \in [0,1]$$

where [0,1] is the segment or closed interval from 0 to 1, called fuzzy unit interval. Then subset A of *E* given by $\{(x, \mu_A(x)); x \in E, \mu_A(x) \in [0,1]\}$, is called a fuzzy set.

Thus a fuzzy set has a membership function with not only values of 0 (does not belong to) or 1 (belong to), but any number over the interval 0 and 1,

For example; 0.3,0.43,0.99,… .

To be more non-technical a fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. For Zadeh [15] introduced a theory whose objects fuzzy sets are set with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial but rather a matter of a degree. The significance of Zadeh's contribution, "Fuzzy Sets as a Basis for Theory of Possibility" [15], was that it challenged not only probability theory as a sole agent for uncertainty; but the very foundations upon which the probability theory is based, Aristotelian two-valued logic. For when A is a fuzzy set and *x* is a relevant object, the proposition *x* is a member of A, is not necessarily either true or false as required by two valued logic, but it may be true only to some degree the degree to which *x* is actually a member of A. It is most common, but not required to express degrees of membership in the fuzzy sets as well as degrees of truth of the associated propositions by numbers in the closed unit interval [0,1]. The extreme values in this interval 0 and 1, then represent respectively, the total denial and affirmation of the membership in a given fuzzy set as well as falsity or truth of the associated proposition.

The capability of fuzzy sets to express gradual transitions to membership to non-membership and vice-versa has a broad utility. This not only helps in the representation of the measurement of uncertainties but also gives a meaningful representation of vague concepts in a simple natural language.

For example; a worker wants to find the moods of his master, he cannot say cent percent in mood or 0 percent of mood or depending on the person who is going to meet the boss he can say some 20% in mood or 50% in mood or 1% in mood or 98% in mood. So even 98% in mood or 50% in mood the worker can meet with some confidence. If 1% in mood the worker may not meet his boss. 20% in mood may fear while meating him. Thus we see however that this definition eventually leads us to accept all degrees of mood of his boss as in mood no matter how gloomy the boss mood is! In order to resolve this paradox the term mood may introduce vagueness by allowing some sort of gradual transition from degrees of good mood that are considered to be in mood and those that are not. This is infant precisely the basic concept of the fuzzy set, a concept that is both simple and intuitively pleasing and that form is essence, a generalization of the classical or crisp set. The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those that certainly belong to the set) and non-members (those that certainly do not). Since full membership and full non-membership in a fuzzy set can still be indicated by the values of 1 and 0, respectively, the function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in the example. Larger values denote higher degrees of set membership. Such a function is called a membership function and the set defined by it a fuzzy set. For example:

Let E = *{a,b,c,d,e,f}* and a subset given by

A =

a	*b*	*c*	*d*	*e*	*f*
0.512	0.23	0.499	1	0	0.048

Then A represents a fuzzy set.

1.4.1 [6] Fuzzy Set operations

The intersection (minimum), union (maximum) and complementation operators defined earlier can be used also for fuzzy sets. We will show these operators by means of an example.

Let

A =

a	*b*	*c*	*d*	*e*	*f*
0.4	0.6	0.3	0	0.5	1

and

B =

a	*b*	*c*	*d*	*e*	*f*
0.7	0.9	0.5	0.8	0.4	0

The intersection of A and B is defined as

A ∩ B =

a	*b*	*c*	*d*	*e*	*f*
0.4	0.6	0.3	0	0.4	0

The union of A and B is defined as

A ∪ B =

a	*b*	*c*	*d*	*e*	*f*
0.7	0.9	0.5	0.8	0.5	1

The complements of A and B are given by

$\bar{A}$ =

a	b	c	d	e	f
0.6	0.4	0.7	1	0.5	0

and

$\bar{B}$ =

a	b	c	d	e	f
0.3	0.1	0.5	0.2	0.6	1

1.4.2 [6] Algebraic Properties of Fuzzy Sets

The algebraic properties of fuzzy subsets are the same as for ordinary subsets, except for the following relations:

$A \cap \bar{A} = \phi$ (not necessarily) and $A \cup \bar{A} = E$ (not necessarily).

For example, taking A from the above example we have

$A \cap \bar{A}$ =

a	b	c	d	e	f
0.4	0.4	0.3	0	0.5	0

$\neq \emptyset$

and

$A \cup \bar{A}$ =

a	b	c	d	e	f
0.6	0.6	0.7	1	0.5	1

$\neq E$

Chapter 2

FUZZY MATRIX THEORY

2.1 Introduction

The purpose of present chapter is to introduce the basic concept of fuzzy matrix theory. Morever, we shall consider the operations defined on these matrices for further treatment of determinant and adjoint theory of square fuzzy matrices. First of all, we shall give fuzzy matrix theory and some operations defined on fuzzy matrices. However the book of Paul Horst, "Matrix Algebra for Social Scientists", [2] would be a boon to social scientists who wish to make use of matrix theory in their analysis.

2.2 Fuzzy Matrix Theory

A fuzzy matrix is a matrix which has its elements from [0,1], called fuzzy unit interval.

Definition 2.2.1 [3]: Consider a matrix A= $\left[a_{ij}\right]_{m\times n}$ where $a_{ij} \in [0,1]$, $1 \le i \le m$ and $1 \le j \le n$. Then A is fuzzy matrix.

2.2.1 [3] Types of Fuzzy Matrices

(a) (i) Fuzzy Rectangular Matrix:

Let A= $\left[a_{ij}\right]_{m\times n}$ $(m\neq n)$ where $a_{ij} \in [0,1]$, $1 \le i \le m$ and $1 \le j \le n$

Then A is a fuzzy rectangular matrix.

For example: $\begin{bmatrix} 0 & 1 & 0.2 \\ 0.1 & 0.5 & 0.3 \end{bmatrix}$ is a 2 × 3 fuzzy rectangular matrix.

(ii) Fuzzy Square Matrix

$$\text{Let } A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{ij} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

where $a_{ij} \in [0,1], 1 \leq i, j \leq n$.

Then A is a fuzzy square matrix.

(iii) Fuzzy Row Matrix:

Let A = $\begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$, $a_j \in [0,1]$; $j=1,2,\ldots,n$.

Then A is called a $1 \times n$ fuzzy row matrix or fuzzy row vector.

For example: $\begin{bmatrix} 0.3 & 0.7 & 0.05 & 1 \end{bmatrix}$ is a 1×4 fuzzy row matrix.

(iv) Fuzzy Column Matrix:

$$\text{Let } A=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

where $a_i \in [0,1]$; $i=1,2,\ldots,m$.

Then A is called a $m \times 1$ fuzzy column matrix.

For example: $\begin{bmatrix} 0 \\ 0.4 \\ 0.5 \end{bmatrix}$

is a 3×1 fuzzy column matrix or fuzzy column vector.

(v) Fuzzy Diagonal Matrix:

A fuzzy square matrix A= $\left[a_{ij}\right]_{n\times n}$ is said to be fuzzy diagonal matrix if a_{ij} = 0 when $i \neq j$,where $a_{ij} \in [0,1]$, $1 \leq i, j \leq n$.

For example: $\begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}$ is a fuzzy diagonal matrix of order 3.

This diagonal matrix is also denoted by $\left[0.4, 0.3, 0.9\right]$.

(vi) Fuzzy Scalar Matrix:

A fuzzy diagonal matrix is said to be fuzzy scalar matrix, if all its diagonal entries are equal.

Thus, a fuzzy square matrix A= $\left[a_{ij}\right]_{n\times n}$ is said to be a fuzzy scalar matrix if

$\begin{cases} a_{ij} = 0 \text{ when } i \neq j \\ a_{ij} = \alpha \text{ when } i = j \end{cases}$ where $\alpha \in [0,1]$, $1 \leq i, j \leq n$.

For example: $\left[0.3\right]$ and $\begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}$ are fuzzy scalar matrices of order 1 and 2 respectively.

(b) Usual identity matrix and zero matrix are fuzzy matrices as their entries are from the fuzzy crisp set {0,1}.

(c) If the entries in upper triangular matrix and lower triangular matrix are from fuzzy unit interval [0,1], then these matrices are said to be fuzzy upper triangular and fuzzy lower triangular matrices respectively.

2.2.2 Equality of Fuzzy Matrices

Two Fuzzy matrices of the same type are said to be equal iff their elements in the corresponding positions are equal.

2.3 [3] Operations on Fuzzy Matrices

2.3.1 Operations of Maximum and Minimum

We shall define the following three operations on fuzzy matrices:

(a) maximum of matrices

(b) minimum of a matrix by a scalar

(c) max min of matrices

(a) Operation I Maximum of matrices

If two fuzzy matrices are of the same type, then they are said to be conformable for addition. But the question arises that when we add two fuzzy matrices, then the resultant matrix is not a fuzzy matrix. So in case of fuzzy matrices of same type, max operation is defined.

Definition 2.3.1: Let A = $\left[a_{ij}\right]_{m\times n}$ and B = $\left[b_{ij}\right]_{m\times n}$ be two fuzzy matrices.

Then their sum, denoted by A+B, is defined as

A+B = max {A,B}

$$i.e.,\left[a_{ij}+b_{ij}\right]_{m\times n}=\left[\max(a_{ij},\ b_{ij})\right]_{m\times n};\text{for } 1\le i\le m\ ,1\le\ j\le n$$

For example: Let A = $\begin{bmatrix} 0 & 0.3 & 0.9 \\ 0.4 & 0.3 & 0.1 \\ 1 & 0.8 & 0.4 \\ 0.5 & 0.2 & 0.6 \end{bmatrix}$ and B = $\begin{bmatrix} 0.6 & 0.7 & 0 \\ 0.9 & 0.5 & 0.5 \\ 0.7 & 1 & 0 \\ 0.6 & 0.8 & 0.5 \end{bmatrix}$

Then

$$A+B = \max\{A,B\} = \begin{bmatrix} \max(0,0.6) & \max(0.3,0.7) & \max(0.9,0) \\ \max(0.4,0.9) & \max(0.3,0.5) & \max(0.1,0.5) \\ \max(1,0.7) & \max(0.8,1) & \max(0.4,0) \\ \max(0.5,0.6) & \max(0.2,0.8) & \max(0.6,0.5) \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.7 & 0.9 \\ 0.9 & 0.5 & 0.5 \\ 1 & 1 & 0.4 \\ 0.6 & 0.8 & 0.6 \end{bmatrix}$$

In a similar way, we can define the difference of two fuzzy matrices of same type as the max operation.

Thus, in case of fuzzy matrices of same type, A−B = max {A,B} = A+B.

(b) Operation II minimum of a matrix by a scalar

Definition 2.3.2: Let A= $\left[a_{ij}\right]_{m\times n}$ be any fuzzy matrix and $k \in$ F, where F = [0,1] is a fuzzy unit interval. Then scalar multiple of A by *k*, denoted by *kA* or *Ak* is given by

$$kA = Ak = \left[ka_{ij}\right]_{m\times n} = \left[\min(k, a_{ij})\right]_{m\times n} \; ; a_{ij} \in [0,1], 1 \le i \le m, 1 \le j \le n.$$

Thus *kA* or *Ak* is the matrix obtained when each entry of *A* is multiplied by *k*.

For example: $0.3 \begin{bmatrix} 0.4 & 0.5 & 1 \\ 0.2 & 0.8 & 0.6 \end{bmatrix}$

$$= \begin{bmatrix} \min(0.3,0.4) & \min(0.3,0.5) & \min(0.3,1) \\ \min(0.3,0.2) & \min(0.3,0.8) & \min(0.3,0.6) \end{bmatrix}$$

$$= \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.3 \end{bmatrix}$$

(c) Operation III max min of matrices

If we wish to find the product AB of two fuzzy matrices A and B where A and B are compatible under multiplication i.e., number of columns of A = number of rows of B ; still we may not have the product AB to be a fuzzy matrix. So in case of fuzzy matrices compatible under multiplication , max min operation is defined.

Definition 2.3.3 : Let A = $\left[a_{ij}\right]_{m\times n}$ and B = $\left[b_{jk}\right]_{n\times p}$ be two fuzzy matrices.

Then their product, denoted by AB, is defined to be the fuzzy matrix $\left[c_{ik}\right]_{m\times p}$,

where $c_{ik} = \sum_{j=i}^{n} a_{ij} b_{jk}$ = max{min (a_{ij}, b_{jk}); $1 \le i \le m$, $1 \le k \le p$} for j = 1,2,…, n.

Remark: If the fuzzy product AB is defind then BA may not be befined.

For example: Let A = $\begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.4 & 0 & 0.6 \end{bmatrix}_{2\times 3}$ and B = $\begin{bmatrix} 0 & 0.3 & 0.7 & 0.5 \\ 0.4 & 0.7 & 0.4 & 1 \\ 0.6 & 0.3 & 1 & 0.1 \end{bmatrix}_{3\times 4}$

Since no. of columns of A = no. of rows in B

Then the product AB is defined and is given by AB = $\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}_{2\times 4}$

where c_{11} = max{min(0.3,0),min(0.2,0.4),min(0.5,0.6)}

= max{0,0.2,0.5}

= 0.5

c_{12} = max{min(0.3,0.3),min(0.2,0.7),min(0.5,0.3)}

= max{0.3,0.2,0.3}

= 0.3

c_{13} = max{min(0.3,0.7),min(0.2,0.4),min(0.5,1)}
= max{0.3,0.2,0.5

= 0.5

c_{14} = max{min(0.3,0.5),min(0.2,1),min(0.5,0.1)}

$= \max\{0.3,0.2,0.1\}$

$= 0.3$

$c_{21} = \max\{\min(0.4,0),\min(0,0.4),\min(0.6,0.6)\}$

$= \max\{0,0,0.6\}$

$= 0.6$

$c_{22} = \max\{\min(0.4,0.3),\min(0,0.7),\min(0.6,0.3)\}$

$= \max\{0.3,0,0.3\}$

$= 0.3$

$c_{23} = \max\{\min(0.4,0.7),\min(0,0.4),\min(0.6,1)\}$

$= \max\{0.4,0,0.6\}$

$= 0.6$

$c_{24} = \max\{\min(0.4,0.5),\min(0,1),\min(0.6,0.1)\}$

$= \max\{0.4,0,0.1\}$

$= 0.4$

Thus AB = $\begin{bmatrix} 0.5 & 0.3 & 0.5 & 0.3 \\ 0.6 & 0.3 & 0.6 & 0.4 \end{bmatrix}$

But since, no. of columns of B $\neq$ no. of rows of A.

Thus the product BA is not defined.

Using the max-min fuction, we can find the positive integral powers of a square fuzzy matrix.

2.3.2 Transpose of fuzzy matrix

Definition 2.3.4: Let A = $\left[a_{ij}\right]_{m\times n}$ be any fuzzy matrx. Then the transpose of A, denoted by A' or A^t or A^T, is $n \times m$ fuzzy matrix obtained from A by interchanging its rows and columns.

i.e., $A' = \left[b_{ij}\right]_{n\times m}$ where $b_{ij} = a_{ji} \in [0,1]$; for $1 \le i \le n$ and $1 \le j \le m$.

Remarks: 1. The transpose of a fuzzy row matrix is a fuzzy column matrix and vice-versa.

2. The product AA' and $A'A$ of two fuzy matrices are always defined.

For example: Let A = $\begin{bmatrix} 0.2 \\ 0.6 \\ 0.7 \end{bmatrix}_{3\times 1}$ be a fuzzy column matrix.

Then $A' = \begin{bmatrix} 0.2 & 0.6 & 0.7 \end{bmatrix}_{1\times 3}$ is a fuzzy row matrix.

Observe that the products AA' and $A'A$ of two fuzzy matrices A and A' are defined.

Now $A'.A = \begin{bmatrix} 0.2 & 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.6 \\ 0.7 \end{bmatrix}$

$= \max\{0.2, 0.6, 0.7\}$

$= 0.7$

Then $A'.A$ is a singleton fuzzy matrix.

and $A.\,A' = \begin{bmatrix} 0.2 \\ 0.6 \\ 0.7 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.7 \end{bmatrix}$

$$= \begin{bmatrix} \min(0.2, 0.2) & \min(0.2, 0.6) & \min(0.2, 0.7) \\ \min(0.6, 0.2) & \min(0.6, 0.6) & \min(0.6, 0.7) \\ \min(0.7, 0.2) & \min(0.7, 0.6) & \min(0.7, 0.7) \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.6 \\ 0.2 & 0.6 & 0.7 \end{bmatrix}$$

Then $A.A'$ is a **symmetric fuzzy matrix** as the matrix obtained by interchanging its rows and columns is the matrix itself and the elements of the matrix belong to fuzzy unit interval [0,1].

Chapter 3

THE DETERMINANT THEORY OF A SQUARE FUZZY MATRIX

3.1 Introduction

The purpose of this chapter is to introduce and investigate some properties of the determinant of a square fuzzy matrix. Kim [11] defined the determinant of a square fuzzy matrix and contributed with very research work [8,9,10,11], a lot to the study of determinant theory of square fuzzy matrices.The properties of a square fuzzy matrix are somewhere analogous to the crisp case for determinant of a square matrix. We shall introduce the determinant theory of a square fuzzy matrix and their properties and investigate the elementary properties of determinant theory for fuzzy matrices in fuzzy matrix semiring M_n (F) [13].

3.2 The Determinant Theory of a Square Fuzzy Matrix

This section lays down the foundation for the determinant theory of a square fuzzy matrix where the elements takes its values from the unit interval [0,1].

Definition 3.2.1 [11]: The determinant $|A|$ of an $n \times n$ fuzzy matrix A is defined as follows:

$$|A| = \sum_{\sigma \in S_n} a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where S_n denotes the symmetric group of all permutations of the indices (1,2,…,n). We may use det(A) instead of $|A|$.We may call det(A) the permanent of A.

Preliminaries:

1. It may be noted with care that the non-square fuzzy matrices do not have determinants.

2. The elements of the determinant of a fuzzy matrix takes its values from the unit

interval [0,1].

3. Here multiplication and addition takes respectively the meanings of min. and max. operations as defined usually in fuzzy matrices.

(i) The determinant of a 1×1 fuzzy matrix $[a]$ is denoted by $|a|$ and is defined as a.

(ii) The determinant of a 2×2 fuzzy matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is defined as ad+bc or max $\{\min\{a,d\},\min\{b,c\}\}$.

(iii) The determinant of a 3×3 fuzzy matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ is denoted by } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and is defined as

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

i.e., $a_1(b_2c_3 + b_3c_2) + a_2(b_1c_3 + b_3c_1) + a_3(b_1c_2 + b_2c_1)$

or a_1 max{min(b_2, c_3) , min(b_3, c_2)} + a_2 max{min(b_1, c_3) , min(b_3, c_1)} + a_3 max{min(b_1, c_2) , min(b_2, c_1)}

= $a_1\lambda + a_2\,\mu + a_3\nu$, where λ = max{min(b_2,c_3) , min(b_3,c_2)}

μ = max{min(b_1,c_3) , min(b_3,c_1)}

ν = max{min(b_1,c_2) , min(b_2,c_1)}

= min(a_1, λ) + min(a_2, μ) + min(a_3, ν)

= max{min(a_1, λ),min(a_2, μ),min(a_3, ν)}

We can expand the determinant along any other row or column as the same value of the determinant can be obtained by expanding along any row or column. It can be easily verified that expanding along any row or column gives the value of the determinant as

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1$$

Notation 3.2.1: The determinant of a square fuzzy matrix A of order *n* is defined as follows:

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij} A_{ij}, i \in \{1, 2, \ldots, n\} \quad \text{[Expanding along } ith \text{ row]}$$

$$= \sum_{i=1}^{n} a_{ij} A_{ij}, j \in \{1, 2, \ldots, n\} \quad \text{[Expanding along } jth \text{ column]}$$

where A_{ij} is the determinant of the fuzzy matrix of order $(n-1)$ obtained from a square fuzzy matrix A of order *n* by deleting (striking out) row *i* and column *j*.

Explanation: Consider a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, A_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Then the value of the det(A) is given by

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$

We can expand the determinant along any other row or column.

The value of the determinant does not depend on along which row or column it is expanded.

Thus, we have:

$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ [Expanding by *Ist* row]

$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$ [Expanding by *IInd* row]

$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$ [Expanding by *IIIrd* row]

$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ [Expanding by *Ist* column]

$= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$ [Expanding by *IInd* column]

$= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$ [Expanding by *IIIrd* column]

Remark: It may be noted with care that the value of the determinant of a square fuzzy matrix A is one among the elements of A.

Thus $0 \le \det(A) \le 1$.

Example 3.2.1: For a square fuzzy matrix

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.8 \\ 0.6 & 0.2 & 0.9 \\ 0.0 & 0.7 & 0.4 \end{bmatrix}$$

We calculate the determinant $|A|$ as follows:

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= 0.5 \begin{vmatrix} 0.2 & 0.9 \\ 0.7 & 0.4 \end{vmatrix} + 0.3 \begin{vmatrix} 0.6 & 0.9 \\ 0.0 & 0.4 \end{vmatrix} + 0.8 \begin{vmatrix} 0.6 & 0.2 \\ 0.0 & 0.7 \end{vmatrix}$$

$= 0.5\,(0.2 + 0.7) + 0.3\,(0.4 + 0.0) + 0.8(0.6 + 0.0)$

$= 0.5(0.7) + 0.3(0.4) + 0.8(0.6)$

$= 0.5 + 0.3 + 0.6$

$= 0.6$.

It can be easily verified that the same value of the deterninant can be obtained by expanding along any other row or column.

3.3 Properties of Determinants of Square Fuzzy Matrices

The following properties of determinants are true for determinants of any order. However we shall show these for determinants of order 3. These properties are used in order to simplify the determinant before expanding it. Some properties of determinants of square fuzzy matrices are analogues to the properties of determinants of square matrices while some other properties differ a lot and a few properties when considered in fuzzy matrix semiring $M_n(F)$ give rise to some new problems and sorting out these problems establishes some new results in fuzzy matrix semiring $M_n(F)$. Before we proceed, it is worth to note that whatever operation or result is true for rows, is also true for columns.

Property 3.3.1 The value of the determinant remains unaltered by interchanging its rows and columns.

Proof: Let

$$\Delta = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Expanding along first row, we get

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 + b_3c_2) + a_2(b_1c_3 + b_3c_1) + a_3(b_1c_2 + b_2c_1)$$

Interchanging the rows and columns of Δ**,** then the new determinant is

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Exapanding along first column, we get:

$$\Delta' = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 + b_3c_2) + a_2(b_1c_3 + b_3c_1) + a_3(b_1c_2 + b_2c_1)$$

$$= \Delta .$$

Thus we have $\Delta' = \Delta$

Remark: Interchange of the rows and columns does not change the value of determinant, i.e., if A is a square fuzzy matrix, then det(A) = det(A') where A' denotes the transpose of the square fuzzy matrix A.

Property 3.3.2 The value of the determinant remains unaltered interchanging its any two rows (or columns).

Proof: Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along the first row, we get:

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 + b_3c_2) + a_2(b_1c_3 + b_3c_1) + a_3(b_1c_2 + b_2c_1)$$

Interchanging the first and second rows, then the new determinant is

$$\Delta' = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expanding along second row, we get:

$$\Delta' = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 + b_3c_2) + a_2(b_1c_3 + b_3c_1) + a_3(b_1c_2 + b_2c_1)$$

$$= \Delta$$

Thus we have $\Delta' = \Delta$

Property 3.3.3 If all rows (or columns) of a determinant are identical then its values is minimum element along all elements of the determinant.

Proof: Let $\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix}$

$= a(bc+\text{bc}) + b(ac+ac) + c(ab+ab)$

$= a(bc) + b(ac) + c(ab)$

$= abc + abc + abc$

$= abc$

$= \min\{a,b,c\}$

= minimum value among all elements of the determinant

Definition 3.3.1 [12]: An $m \times n$ fuzzy matrix A is said to be constant if $a_{ik} = a_{jk}$ (or $a_{ki} = a_{kj}$) for all i, j,k i.e., its rows (or columns) are equal to each-other.

Note: Thus property 3.3.3 can be stated as:

The determinant of a constant square fuzzy matrix is its minimum element.

Property 3.3.4 If all the elements of a row (or column) are equal to a (say) and all other rows (or columns) have elements ≥ a then the value of the determinant = a (which is the minimum element among all elements).

Proof: We know that each term in the determinant contains a factor of each row (or column) and hence each term contains a factor of that row (or column) in which all elements are equal to a (say) and all other elements in that term of other rows (or columns) ≥ a so that each term in the determinant of a square fuzzy matrix is equal to a and consequently $\Delta = a$.

For example: Let $\Delta = \begin{vmatrix} a & a & a \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= a(b_2c_3 + b_3c_2) + a(b_1c_3 + b_3c_1) + a(b_1c_2 + b_2c_1)$

$= a(b_2c_3 + b_3c_2 + b_1c_3 + b_3c_1 + b_1c_2 + b_2c_1)$

$= a \qquad \begin{bmatrix} \because \text{ each element } \geq a \\ \therefore \text{ min element } = a \end{bmatrix}$

Another statement of property 3.3.4[11]

Let $A_k = (a_{k1}, a_{k2}, \ldots, a_{kn})$ be the *kth* row of A. If $a_{ki} = a \, \forall i \in \{1, 2, \ldots, n\}$

and $a_{pq} \geq a \, \forall \, p, q \in \{1, 2, \ldots, n\}$

Then det(A) = *a*.

Corollary [12]: The value of the determinant containing a zero row (or column) is zero.

Proof: We know that each term in the determinant contains a factor of each row (or column) and hence contains a factor of zero row (or column) so that each term in the determinant of a square fuzzy matrix is equal to zero (min. element) and consequently $\Delta = 0$.

Example: Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix}$

$= a_1(0+0) + a_2(0+0) + a_3(0+0)$

$= a_1(0) + a_2(0) + a_3(0)$

$= 0+0+0$

$= 0$

Note: In a square fuzzy matrix, as all the elements of a zero row (or column) are equal to 0 and all other elements ≥ 0. Hence by property 3.3.4, the value of the determinant = 0.

Property 3.3.5 The determinant of a diagonal matrix is the product of its diagonal elements.

Proof: Let A = $[a_{ij}]$ be a diagonal matrix i.e., $a_{ij} = 0$ for $i \neq j$

Take a term t of $|A|$,

$$t = a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

Let $\sigma(1) \neq 1$, then $a_{1\sigma(1)} = 0$ and thus $t = 0$.

This means that each term is zero if $\sigma(1) \neq 1$.

Let now $\sigma(1) = 1$ but $\sigma(2) \neq 2$, then $a_{2\sigma(2)} = 0$ and thus $t = 0$.

This mean each term is zero if $\sigma(1) \neq 1$ or $\sigma(2) \neq 2$.

However in the similar way, we can see that each term for which $\sigma(1) \neq 1$ or $\sigma(2) \neq 2 \ldots$ or $\sigma(n) \neq n$ must be zero

Consequently $|A| = a_{11} a_{22} \ldots a_{nn} = \prod_{i=1}^{n} a_{ii}$

= Product of its diagonal elements.

Corollary 1 The determinant of a scalar matrix is the non-zero (diagonal) element.

Proof: Let Δ be the determinant of a scalar matrix with each of its non-zero diagonal element equal to k (say).

Since every scalar matrix is a diagonal matrix.

$\therefore$ By property 3.3.5, Δ = product of its diagonal elements

$= k.k. \ldots k$

$= k$.

Corollary 2 The determinant of an identity matrix is unity.

Proof: Since an identity matrix is a diagonal matrix.

$\therefore$ By property 3.3.5, $|I|$ = products of its diagonal elements

= 1.1. …1

= 1

Corollary 3 The determinant of the matrix obtained from an identity matrix by interchanging its any two rows (or columns) is unity.

Proof: By property 3.3.2, the value of determinant remains unaltered by interchanging its any two rows (or columns).

$\therefore |E_{ij}| = |I_n|$ where E_{ij} is the matrix obtained from identity matrix I_n by interchanging its rows (or columns).

i.e., $|E_{ij}| = 1$. (by corollary 2)

Remarks: 1. $\det(I_nA) = \det(A) = \det(AI_n)$

2. $\det(E_{ij}A) = \det(A) = \det(AE_{ij})$

Property 3.3.6 [12] The determinant of a triangular matrix is given by the product of its diagonal elements.

Proof: Suppose $\Delta = |a_{ij}|$ is triangular from below.

i.e., $a_{ij} = 0$ for $i < j$

Take a term t of Δ ,

$t = a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$

Let $\sigma(1) \neq 1$, so that $1 < \sigma(1)$ and so $a_{1\sigma(1)} = 0$ and thus $t = 0$.

This means that each term is zero if $\sigma(1) \neq 1$.

Let now $\sigma(1) = 1$ but $\sigma(2) \neq 2$, then $2 < \sigma(2)$ and so $a_{2\sigma(2)} = 0$

and thus $t = 0$.

This means that each term is zero if $\sigma(1) \neq 1$ or $\sigma(2) \neq 2$.

However, in a similar manner we can see that each term must be zero if $\sigma(1) \neq 1$, or $\sigma(2) \neq 2\ldots$ or $\sigma(\mathrm{n}) \neq n$.

Consequently $\Delta = a_{11}a_{22}\ldots a_{nn} = \prod_{i=1}^{n} a_{ii}$

= product of its diagonal elements.

Similar is the case with the determinant triangular from above.

Property 3.3.7 [12] If each element of a row (or column) of a determinant is multiplied by a constant $k \in$ F where F = [0,1] is the fuzzy unit interval of the real line i.e $k \in [0,1]$ then its value gets multiplied by k .

Proof: By definition of determinant,

$$\Delta = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

$$= \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)} \ldots a_{i\sigma(i)}\ldots a_{n\sigma(n)}$$

Multiply by k, the elements of the *ith* row (say) then the new determinant is

$$\Delta^{'} = \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\ldots ka_{i\sigma(i)}\ldots a_{n\sigma(n)}$$

$$= k\sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{i\sigma(i)}\ldots a_{n\sigma(n)}$$

$$= k\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

$$= k\,\Delta$$

Remark: In case of matrices kA is the matrix obtained when each entry of A is muliplied

by $k \in$ F = [0,1].

Thus if A is a square matrix of order n, then

$|kA| = k^n\ |A| = k\ |A|$ i.e., det(kA) = k^n det(A) = k det(A).

Chapter 4

THE ADJOINT THEORY OF A SQUARE FUZZY MATRIX

4.1 Introduction

In this chapter the adjoint theory of a square fuzzy matrix will be studied.The adjoint of a square fuzzy matrix is defined by Thomason [13] and Kim[11]. We state a formula for the adjoint matrix of a square fuzzy matrix and this formula shall be used anywhere in this chapter. Then we shall establish the relationship between the adjoints of two fuzzy matrices. Also we shall find the relationship between the adjoints of two fuzzy matrices corresponding the relationship between the fuzzy matrices. For a square fuzzy matrix satisfying some property, we shall verify the same property for its adjoint. In this chapter we define the symmetric, reflexive, transitive, circular and idempotent fuzzy matrices and show that some properties of a square fuzzy matrix such as symmetry, reflexivity, transitivity, circularity and idempotence are carried over to the adjoint matrix and shall understand the same with the help of illustrations. For a given square fuzzy matrix A, through the adjoint matrix *adj*(A), we shall construct a transitive fuzzy matrix A (*adj* A). Along with its illustration, we shall prove it before. We establish some results including that A(*adj*A)≥|A|I and (*adj*A)A≥ |A|I where |A| denotes the determinant of a square fuzzy matrix A and *adj*A denotes the adjoint matrix of a square fuzzy matrix A.

4.2 The Adjoint Theory of a Square Fuzzy Matrix

Let us first define the notion of adjoint of a sqaure fuzzy matrix.

Definition4.2.1 [11]: The adjoint matrix B = $[b_{ij}]$ of a square fuzzy matrix A = $[a_{ij}]$ of order n, is a square fuzzy matrix of same order n, denoted by *adj*A, is defined as $b_{ij} = \left| A_{ji} \right|$; where $\left| A_{ji} \right|$ is the determinant of the square fuzzy matrix of order $(n-1)$ obtained from a square fuzzy matrix A of order n by deleting row j and column i and B = $[b_{ij}]$ = *adj*A.

Remark: Note that $\left| A_{ji} \right|$ can be obtained from $|A|$ by replacing the element a_{ji} of A by 1 and all other row j factors a_{jk} , $k \neq i$ by 0.

Example 4.2.1: For a Square fuzzy matrix

$$A = \begin{bmatrix} 0 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.5 \\ 1 & 0.3 & 0.7 \end{bmatrix}$$

We find *adj*A as follows:

$$b_{11} = |A_{11}| = \begin{vmatrix} 0.4 & 0.5 \\ 0.3 & 0.7 \end{vmatrix} = 0.4+0.3 = 0.4$$

$$b_{12} = |A_{21}| = \begin{vmatrix} 0.3 & 0.4 \\ 0.3 & 0.7 \end{vmatrix} = 0.3+0.3 = 0.3$$

$$b_{13} = |A_{31}| = \begin{vmatrix} 0.3 & 0.4 \\ 0.4 & 0.5 \end{vmatrix} = 0.3+0.4 = 0.4$$

$$b_{21} = |A_{12}| = \begin{vmatrix} 0.2 & 0.5 \\ 1 & 0.7 \end{vmatrix} = 0.2+0.5 = 0.5$$

$$b_{22} = |A_{22}| = \begin{vmatrix} 0 & 0.4 \\ 1 & 0.7 \end{vmatrix} = 0+0.4 = 0.4$$

$$b_{23} = |A_{32}| = \begin{vmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{vmatrix} = 0+0.2 = 0.2$$

$$b_{31} = |A_{13}| = \begin{vmatrix} 0.2 & 0.4 \\ 1 & 0.3 \end{vmatrix} = 0.2+0.4 = 0.4$$

$$b_{32} = |A_{23}| = \begin{vmatrix} 0 & 0.4 \\ 1 & 0.7 \end{vmatrix} = 0+0.4 = 0.4$$

$$b_{33} = |A_{33}| = \begin{vmatrix} 0 & 0.3 \\ 0.2 & 0.4 \end{vmatrix} = 0+0.2 = 0.2$$

$$adj A = [b_{ij}] = \begin{bmatrix} 0.4 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}$$

4.3 Properties of Adjoints of Square Fuzzy Matrices

Notation 4.3.1[12]: We also can rewrite the element b_{ij} of *adj* A as

$$b_{ij} = \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} a_{t\sigma(t)}$$

where $n_j = \{1,2,\ldots,n\} \backslash \{j\}$

$n_i = \{1,2,\ldots,n\} \backslash \{i\}$

and $S_{n_j n_i}$ is the set of all permutations of set n_j over the set n_i.

Proposition 4.3.1 [12]: Comparison of the adjoints of two fuzzy matrices

For $n \times n$ fuzzy matrices A and B,

(i) A ≤ B ⇒ *adj* A ≤ *adj* B

(ii) *adj* A + *adj* B ≤ *adj* (A+B)

Proof: Let A = $\left[a_{ij}\right]$ and B = $\left[b_{ij}\right]$ where $i, j \in \{1,2,\ldots,n\}$.

(i) since A ≤ B

$\Rightarrow \; a_{ij} \leq b_{ij} \; \forall \; i, j \in \{1.2,\ldots,n\}$

$\Rightarrow \; a_{t\sigma(t)} \leq b_{t\sigma(t)}$ for every $t \neq j$, $\sigma(t) \neq i$

$$\Rightarrow \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} a_{t\sigma(t)} \leq \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} b_{t\sigma(t)}$$

⇒ *adj*A ≤ *adj*B

(ii) since A , B ≤ A + B [∵ A + B = max {A, B}]

⇒ *adj*A , *adj*B ≤ *adj*(A+B) [∵A ≤ B ⇒ *adj*A ≤ *adj*B]

⇒ *adj*A + *adj*B ≤ *adj*(A+B)

Proposition 4.3.2 [12]: The adjoint of the transpose of a matrix is the transpose of the adjoint of the matrix. i.e., for a square fuzzy matrix A of order n, $adj\ A' = (adj\ A)'$.

Proof: Let A= $\left[a_{ij}\right]$ be a $n \times n$ fuzzy matrix.

Let B = $\left[b_{ij}\right]$ = *adj* A and C= $\left[c_{ij}\right]$ = $adj\ A'$.

Then $b_{ij} = \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} a_{t\sigma(t)}$

and $c_{ij} = \sum_{\sigma \in S_{n_j n_i}} \prod_{\sigma(t) \in n_i} a_{t\sigma(t)}$, which is element b_{ji}.

Hence $adj\ A' = (adj\ A)'$, which proves the assertion.

Proposition 4.3.3 [12]: Let A be a $n \times n$ fuzzy matrix. Then

(i) A(*adj* A) $\geq |A| I_n$

(ii) (*adj* A)A $\geq |A| I_n$

where I_n is a unit matrix of order n.

Proof (i): Let A = $\left[a_{ij}\right]_{n\times n}$, then *ith* row of A is given by $(a_{i1}, a_{i2}, \ldots, a_{in})$.

Suppose B = $\left[b_{ij}\right]_{n\times n}$ = *adj*A

Then by definition of *adj*A,

the *jth* column of B = $\left[b_{ij}\right]_{n\times n}$ = *adj*A is given by

$(b_{1j}, b_{2j}, \ldots, b_{nj}) = \left(|A_{j1}|, |A_{j2}|, \ldots, |A_{jn}|\right)$

Let C = $\left[c_{ij}\right]_{n\times n}$ = A(*adj* A)

Then $(i,j)th$ element of C = $\left[c_{ij}\right]_{n\times n}$ = A(*adj* A) is given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\left|A_{jk}\right| \geq 0$$

and where $c_{ii} = \sum_{k=1}^{n} a_{ik}\left|\mathrm{A}_{ik}\right| = \left|\mathrm{A}\right|.$

Thus C = A(*adj* A) $\geq |\mathrm{A}|\, \mathrm{I}_n$, where

$$|\mathrm{A}|\, \mathrm{I}_n = \begin{bmatrix} |A| & 0 & 0 & \ldots & 0 \\ 0 & |A| & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & |A| \end{bmatrix}$$

(ii) Let A = $\left[a_{ij}\right]_{n\times n}$

Then *jth* column of A is $(a_{1j}, a_{2j}, \ldots, a_{nj})$.

Let B = $\left[b_{ij}\right]_{n\times n}$ = *adj* A.

Then by definition of *adj* A,

The *ith* row of B = $\left[b_{ij}\right]_{n\times n}$ = *adj* A is $(b_{i1}, b_{i2}, \ldots, b_{in}) = \left(\left|A_{1i}\right|, \left|A_{2i}\right|, \ldots, \left|A_{ni}\right|\right)$

Let C = $\left[c_{ij}\right]_{n\times n}$ = (*adj*A)A

Then $(i,j)th$ element of C = $\left[c_{ij}\right]_{n\times n}$ = (*adj*A)A is $c_{ij} = \sum_{k=1}^{n}\left|\mathrm{A}_{ki}\right| a_{kj} \geq 0$ and

where $c_{ii} = \sum_{k=1}^{n}\left|\mathrm{A}_{ki}\right| a_{ki} = \left|\mathrm{A}\right|.$

Thus C = (*adj*A) A $\geq |\mathrm{A}|\, \mathrm{I}_n$ where I_n is a unit matrix of order n.

Remark: Any diagonal element of the fuzzy matrix A(*adj* A) is |A| and non-diagonal element ≥ 0.

Proposition 4.3.4 [12]: Let A be a square fuzzy matrix, then the following properties hold:

(i) If A has a zero row then (*adj*A)A = *O* (the zero matrix)

(ii) If A has a zero column then A(*adj*A) = *O* (the zero matrix)

Proof (i): Let C = $\left[c_{ij}\right]$ = (*adj*A)A, then $c_{ij} = \sum_k |A_{ki}| a_{kj}$

If the *ith* row of A is zero, then a_{kj}= 0 for every $k = i$

and for $k \neq i$, A_{ki} contains a zero row and so $|A_{ki}| = 0$ for $k \neq i$

(by corollary of property 3.3.4 of determinants)

So that $\sum_k |A_{ki}| a_{kj} = 0 \; \forall \; i, j$.

Hence (*adj*A)A = *O* (the zero matrix).

(ii) Let C = $\left[c_{ij}\right]$ = A(*adj*A),then $c_{ij} = \sum_k a_{ik} |A_{jk}|$.

If the *jth* column of A is zero, then a_{ik}= 0 for every $k = j$

and for $k \neq j$, A_{jk} contains a zero column and so $|A_{jk}| = 0$ for $k \neq j$

(by corollary of property 3.3.4 of determinants)

so that $\sum_k a_{ik} |A_{jk}| = 0 \; \forall \, i, j$

Hence A(*adj*A) = *O* (the zero matrix).

Proposition 4.3.5 [12]: Let A be an $n \times n$ constant fuzzy matrix, then

(i) $(adj\ A)'$ is constant.

(ii) C = A(*adj*A) is constant and $c_{ij} = |A|$, which is the least element in A.

Proof: Let A be an $n \times n$ constant fuzzy matrix where its all rows are equal to each-other i.e., $a_{ik} = a_{jk} \forall\ i, j$

(i) Let B = (*adj* A), then

$$b_{ij} = \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} a_{t\sigma(t)} \text{ and } b_{ik} = \sum_{\sigma \in S_{n_k n_i}} \prod_{t \in n_k} a_{t\sigma(t)}$$

Since the numbers σ(t) of columns cannot be changed in the two expansions of b_{ij} and b_{ik} as A is constant and so $b_{ij} = b_{ik}$ $\forall$ i,j,k .

In order that, $b_{ji} = b_{ki}$ $\forall$ i, j, k, we must have $(adj\ A)'$ is constant.

(ii) Since A is constant i.e., $a_{ik} = a_{jk}$ $\forall\ i, j, k$.

Then $A_{ik} = A_{jk}$ $\forall$ i,j,k and so $|A_{ik}| = |A_{jk}|$ $\forall$ i,j,k .

Let C = $\left[c_{ij}\right]$ = A(*adj*A), then $c_{ij} = \sum_{k=1}^{n} a_{ik} |A_{jk}| = \sum_{k=1}^{n} a_{ik} |A_{ik}|$

$= |A|$ $\forall$ i, j

Thus C = A(*adj*A) is constant.

Now $|A| = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$

$$= \sum_{\sigma \in S_n} a_{1\sigma(1)}\ a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

$$= a_{1\sigma(1)}\ a_{2\sigma(2)} \ldots a_{n\sigma(n)} \text{ for any } \sigma \in S_n$$

($\because$ A is constant i.e., $a_{ik} = a_{jk} \forall\ i, j, k$)

Taking σ, the identity permutation i.e., $\sigma(i) = i$ $\forall$ i,we get

$|A| = a_{11}\ a_{22} \ldots a_{nn}$; which is the least element in a constant fuzzy matrix A .

Definition 4.3.1 [14]: Let A be a square fuzzy matrix of order n , then following hold:

(i) A is said to be **reflexive** fuzzy matrix iff $A \geq I_n$ i.e., iff all diagonal elements in fuzzy matrix A are unity i.e., iff $a_{ii} = 1 \ \forall \ i$.

(ii) A is said to be **symmetric** iff $A' = A$ i.e., iff the square fuzzy matrix A remains unaltered by interchanging its rows and columns i.e., iff $a_{ij} = a_{ji} \ \forall i,j \in \{1,2,\ldots,n\}$.

(iii) A is said to be **transitive** iff $A^2 \leq A$ i.e., iff the square fuzzy matrix A multiplied by itself gives the elements less than or equal to the corresponding elements of the square fuzzy matrix A. i.e., iff $a_{ik} \, a_{kj} \leq a_{ij}$ for every $k = 1,2,\ldots,n$.

A square fuzzy matrix is **similarity (equivalence relation)** iff it is reflexive, symmetric and transitive.

Let us understand the same with the help of an example.

Example 4.3.1: Let A be a square fuzzy matrix of order 3.

(i) Consider a square fuzzy matrix

$$A = \begin{bmatrix} 1 & 0.2 & 0 \\ 0.3 & 1 & 0.4 \\ 0.9 & 0.3 & 1 \end{bmatrix}$$

Since all the diagonal elements in square fuzzy matrix are unity, then A is a reflexive fuzzy matrix.

(ii) Consider a square fuzzy matrix

$$A = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.5 & 0.1 & 1 \end{bmatrix}$$

Then

$$A' = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.5 & 0.1 & 1 \end{bmatrix} = A .$$

Thus A is a symmetric fuzzy matrix.

(iii) Consider a square fuzzy matrix

$$A = \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.7 & 0.6 \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.7 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.5 & 0.5 & 0.5 \\ 0.6 & 0.6 & 0.6 \end{bmatrix} \leq A$$

as the element in A^2 are $\leq$ the corresponding elements in A. Thus A is a transitive fuzzy matrix. Next, consider the square fuzzy matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All diagonal elements in A equal to 1 implies that A is reflxive. Further, observe that $A' = A$ so A is symmetric. Also A^2=A leads to the conclusion that A is transitive. Hence A is similarity.

Let us see how the properties of a square fuzzy matrix are carried over to its adjoint.

Theorem 4.3.1[12]: Let A be a square fuzzy matrix of order *n*, Then we have the following properties:

(i) If A is reflexive, then *adj* A is reflexive.

(ii) If A is symmetric , then *adj* A is symmetric.

(iii) If A is transitive, then *adj* A is transitive.

Proof: (i) Since A = $\left[a_{ij}\right]$ is reflexive, then $a_{ii} = 1 \; \forall \; i$

Let B = $\left[b_{ij}\right]$ = *adj* A

Then $b_{ij} = \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} a_{t\sigma(t)}$

and so $b_{ii} = \sum_{\sigma \in S_{n_i}} \prod_{t \in n_j} a_{t\sigma(t)}$

Taking only the identity permutation $\sigma(t) = t$; we get

$b_{ii} \geq a_{11}a_{22}\ldots a_{(i-1)(i-1)}a_{(i+1)(i+1)}\ldots a_{nn}$

i.e; $b_{ii} \geq 1 \ \forall \ i$ $\quad$ $(\because a_{ii} = 1 \ \forall \ i)$

and so $b_{ii} = 1 \ \forall \ i$

Hence *adj* A is reflexive.

(ii) Since A is symmetric, then $a_{ij} = a_{ji} \ \forall \ i, j$.

Let B = $[b_{ij}]$ = *adj*A

Then $b_{ij} = \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} a_{t\sigma(t)} = \sum_{\sigma \in S_{n_i n_j}} \prod_{t \in n_i} a_{t\sigma(t)}$ $[\because a_{ij} = a_{ji} \forall i, j]$

$= b_{ji}$.

Hence *adj* A is symmetric.

(iii) Since A is transitive, then $a_{ik} \, a_{kj} \leq a_{ij} \forall \, i,j$.

Let B = $[b_{ij}]$ = *adj*A.

Let D = A_{ij} , we can determine the elements of D in terms of the elements of A as follows:

$$d_{hk} = \begin{cases} a_{hk} & \text{if } h < i, k < j, \\ a_{(h+1)k} & \text{if } h \geq i, k < j, \\ a_{h(k+1)} & \text{if } h < i, k \geq j, \\ a_{(h+1)(k+1)} & \text{if } h \geq i, k \geq j. \end{cases}$$

where A_{ij} denotes the $(n-1)\times(n-1)$ fuzzy matrix obtained from A by deleting *ith* row and *jth* column.

Now we show that $A_{st}\,A_{tu} \leq A_{su}$ for every $t \in \{1,2,\ldots,n\}$.

Let $R = A_{st}$, $C = A_{tu}$, $F = A_{su}$ and $W = A_{st}A_{tu}$.

Now $w_{ij} = \sum_{k=1}^{n-1} r_{ik}\,c_{kj}$

$$= \sum_{k=1}^{n-1} a_{ik}a_{kj} \leq a_{ij} = f_{ij} \text{ if } i < s, k < t, j < u,$$

$$= \sum_{k=1}^{n-1} a_{ik}a_{k(j+1)} \leq a_{i(j+1)} = f_{ij} \text{ if } i < s, k < t, j \geq u,$$

$$= \sum_{k=1}^{n-1} a_{i(k+1)}a_{(k+1)j} \leq a_{ij} = f_{ij} \text{ if } i < s, k \geq t, j < u,$$

$$= \sum_{k=1}^{n-1} a_{i(k+1)}a_{(k+1)(j+1)} \leq a_{i(j+1)} = f_{ij} \text{ if } i < s, k \geq t, j \geq u,$$

$$= \sum_{k=1}^{n-1} a_{(i+1)k}a_{kj} \leq a_{(i+1)j} = f_{ij} \text{ if } i \geq s, k < t, j < u,$$

$$= \sum_{k=1}^{n-1} a_{(i+1)(k+1)}a_{(k+1)j} \leq a_{(i+1)j} = f_{ij} \text{ if } i \geq s, k \geq t, j < u,$$

$$= \sum_{k=1}^{n-1} a_{(i+1)(k+1)}a_{(k+1)(j+1)} \leq a_{(i+1)(j+1)} = f_{ij} \text{ if } i \geq s, k \geq t, j \geq u,$$

$$= \sum_{k=1}^{n-1} a_{(i+1)k}a_{k(j+1)} \leq a_{(i+1)(j+1)} = f_{ij} \text{ if } i \geq s, k < t, j \geq u.$$

Thus $w_{ij} \leq f_{ij}$ in every case and therefore $A_{st}\,A_{tu} \leq A_{su}$ for every $t \in \{1,2,..,n\}$

Since we know that $|AB| \geq |A||B|$; we have $|A_{st}||A_{tu}| \leq |A_{st}A_{tu}| \leq |A_{su}|$.

This means $b_{ts}b_{ut} \leq b_{us}$ i.e., $b_{ut}b_{ts} \leq b_{us}$ for every $t \in \{1,2,..,n\}$

Hence B = *adj*A is transitive.

Corollary [12]: If a square fuzzy matrix is similarity then *adj* A is also similarity.

Example 4.3.2: Consider a square fuzzy matrix A

(i) Let A = $\begin{bmatrix} 1 & 0 & 0.3 \\ 0.1 & 1 & 0 \\ 0.4 & 0.5 & 1 \end{bmatrix}$ be a reflexive fuzzy matrix, then

$adj\text{A} = \begin{bmatrix} 1 & 0.3 & 0.3 \\ 0.1 & 1 & 0.1 \\ 0.4 & 0.5 & 1 \end{bmatrix}$

Since all the diagonal elements in *adj*A are unity, then *adj* A is a reflexive fuzzy matrix.

(ii) Let A = $\begin{bmatrix} 0.2 & 0 & 0.6 \\ 0 & 1 & 0.1 \\ 0.6 & 0.1 & 0.9 \end{bmatrix}$ be a symmetric fuzzy matrix, then

$adj\text{A} = \begin{bmatrix} 0.9 & 0.1 & 0.6 \\ 0.1 & 0.6 & 0.1 \\ 0.6 & 0.1 & 0.2 \end{bmatrix}$

since $(adj\text{A})' = adj\text{A}$, then *adj*A is a symmetric fuzzy matrix.

(iii) Let A = $\begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.6 & 0.6 \end{bmatrix}$ be transitive fuzzy matrix,then

$adj\text{A} = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.7 & 0.6 \end{bmatrix}$

Now

$$(adj\mathrm{A})^2 = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.7 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.5 \\ 0.6 & 0.6 & 0.6 \end{bmatrix}$$

$\leq adj\mathrm{A}$, then $adj\mathrm{A}$ is a transitive fuzzy matrix.

Next, consider the similarity fuzzy matrix

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A is reflexive, symmetric and transitive.

Now

$$adj\mathrm{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then adjA is reflexive, symmetric and transitive.

Hence adjA is similarity fuzzy matrix.

Definition 4.3.2 [12]: A square fuzzy matrix A of order n is called **circular** fuzzy matrix iff $(\mathrm{A}^2)' \leq \mathrm{A}$ or more explicity, $a_{jk}a_{ki} \leq a_{ij}$ for every k = 1, 2, …, n.

Theorem 4.3.2 [12]: If a square fuzzy matrix A of order n is circular, then adjA is circular.

Proof: Since A = $\left[a_{ij}\right]$ is circular then $a_{jk}a_{ki} \leq a_{ij}$ $\forall$ i,j

Let B = $\left[b_{ij}\right]$ = adjA

Let D = A_{ij} , we can determine the elements of D in terms of the elements of A as follow:

$$d_{hk} = \begin{cases} a_{hk} & if\ h < i, k < j, \\ a_{(h+1)k} & if\ h \geq i, k < j, \\ a_{h(k+1)} & if\ h < i, k \geq j, \\ a_{(h+1)(k+1)} & if\ h \geq i, k \geq j. \end{cases}$$

where A_{ij} denotes the $(n-1)\times(n-1)$ fuzzy matrix of order $(n-1)$ obtained from A by deleting *ith* row and *jth* column.

Now we show that $\mathsf{A}_{st}\mathsf{A}_{tu} \leq \mathsf{A}_{us}$ for every $t \in \{1,2,\ldots,n\}$

Let $R = \mathsf{A}_{st}$,$C = \mathsf{A}_{tu}$, $F = \mathsf{A}_{us}$ and $W = \mathsf{A}_{st}\mathsf{A}_{tu}$.

Now $w_{ij} = \sum_{k=1}^{n-1} r_{jk}c_{ki}$

$$= \sum_{k=1}^{n-1} a_{jk}a_{ki} \leq a_{ij} = f_{ij}\ if\ i < s, k < t, j < u,$$

$$= \sum_{k=1}^{n-1} a_{(j+1)k}a_{ki} \leq a_{i(j+1)} = f_{ij}\ if\ i < s, k < t, j \geq u,$$

$$= \sum_{k=1}^{n-1} a_{j(k+1)}a_{(k+1)i} \leq a_{ij} = f_{ij}\ if\ i < s, k \geq t, j < u,$$

$$= \sum_{k=1}^{n-1} a_{(j+1)(k+1)}a_{(k+1)i} \leq a_{i(j+1)} = f_{ij}\ if\ i < s, k \geq t, j \geq u,$$

$$= \sum_{k=1}^{n-1} a_{jk}a_{k(i+1)} \leq a_{(i+1)j} = f_{ij}\ if\ i \geq s, k < t, j < u,$$

$$= \sum_{k=1}^{n-1} a_{j(k+1)}a_{(k+1)(j+1)} \leq a_{(i+1)j} = f_{ij}\ if\ i \geq s, k \geq t, j < u,$$

$$=\sum_{k=1}^{n-1} a_{(j+1)(k+1)} a_{(k+1)(j+1)} \leq a_{(i+1)(j+1)} = f_{ij} \; if \; i \geq s, k \geq t, j \geq u,$$

$$=\sum_{k=1}^{n-1} a_{(j+1)k} a_{k(i+1)} \leq a_{(i+1)(j+1)} = f_{ij} \; if \; i \geq s, k < t, j \geq u.$$

Thus $w_{ij} \leq f_{ij}$ in every case and therefore $A_{st} A_{tu} \leq A_{us}$ for every $t \in \{1,2,..,n\}$

Since we know that $|AB| \geq |A||B|$; we have $|A_{st}||A_{tu}| \leq |A_{st}A_{tu}| \leq |A_{us}|$.

This means $b_{ts}b_{ut} \leq b_{su}$ i.e., $b_{ut}b_{ts} \leq b_{su}$ for every $t \in \{1,2,..,n\}$

Hence B = *adj*A is circular.

Theorem 4.3.3 [12]: To construct a transitive fuzzy matrix from a given fuzzy matrix through adjoint matrix.

For any $n \times n$ fuzzy matrix A, the fuzzy matrix A(*adj*A) is transitive.

Proof: Let C = $\left[c_{ij}\right]$ = A(*adj*A).

Then $c_{ij} = \sum_{k=1}^{n} a_{ik}|A_{jk}| = a_{if}\,|A_{if}|$ for some $f \in \{1,2,\ldots,n\}$

and $c_{ij}^2 = \sum_{s=1}^{n} c_{is}c_{sj}$

$$= \sum_{s=1}^{n}\left[\left(\sum_{l=1}^{n} a_{il}|A_{sl}|\right)\left(\sum_{t=1}^{n} a_{st}|A_{jt}|\right)\right]$$

$$= \sum_{s=1}^{n} \left(a_{ih}|A_{sh}|\right)\left(a_{su}|A_{ju}|\right)$$, for some h , $u \in \{1,2,\ldots,n\}$

$$= a_{ih}\,|A_{gh}|\;a_{gu}\,|A_{ju}|$$

$$\leq a_{ih}\left|A_{ju}\right|$$

$$\leq a_{if}\left|A_{jf}\right| = c_{ij}$$

Thus $c_{ij}^2 \leq c_{ij}$

and so $(A\,adjA)^2 \leq A\,(adjA)$.

Hence A (*adj*A) is transitive.

Example 4.3.3: For a square fuzzy matrix

$$A = \begin{bmatrix} 0.5 & 0.7 & 0.8 \\ 0.3 & 0.6 & 0.4 \\ 0.9 & 0.2 & 1 \end{bmatrix}$$

$$\text{Then } adjA = \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.4 & 0.8 & 0.4 \\ 0.6 & 0.7 & 0.5 \end{bmatrix}$$

$$\text{Now } A\,(adjA) = \begin{bmatrix} 0.5 & 0.7 & 0.8 \\ 0.3 & 0.6 & 0.4 \\ 0.9 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.4 & 0.8 & 0.4 \\ 0.6 & 0.7 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0.4 & 0.6 & 0.4 \\ 0.6 & 0.7 & 0.6 \end{bmatrix}$$

which is transitive fuzzy matrix as

$$(A(adjA))^2 = \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0.4 & 0.6 & 0.4 \\ 0.6 & 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0.4 & 0.6 & 0.4 \\ 0.6 & 0.7 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.5 \\ 0.4 & 0.6 & 0.4 \\ 0.6 & 0.6 & 0.6 \end{bmatrix}$$

$\leq A\ (adjA)$

Hence for any square fuzzy matrix A, A(adjA) is transitive.

Definition 4.3.3 [15]: An $n \times n$ fuzzy matrix A is called **idempotent** fuzzy matrix iff $A^2 = A$.

Let us understand by an example to convert reflexive fuzzy matrix into idempotent fuzzy matrix by taking its adjoint matrix:-

Example 4.3.4: For a reflxive fuzzy matrix

$$A = \begin{bmatrix} 1 & 0.1 & 0.2 \\ 0.3 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix}$$

Then

$$adjA = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix}$$

which is idempotent fuzzy matrix as

$$(adjA)^2 = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix} = adjA$$

Hence idempotent fuzzy matrix is formed by taking adjoint of reflexive fuzzy matrix.

Remark[13]: For a reflexive fuzzy matrix A of order n , $adjA = A^c$ where A^c is idempotent and $c \leq n-1$.

Example 4.3.5: For a 3×3 reflexive fuzzy matrix

$$A = \begin{bmatrix} 1 & 0.1 & 0.2 \\ 0.3 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix}$$

Then

$$adj\text{A} = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix} \text{is idempotent fuzzy matrix}$$

Now

$$\text{A}^2 = \begin{bmatrix} 1 & 0.1 & 0.2 \\ 0.3 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.1 & 0.2 \\ 0.3 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{bmatrix}$$

$$= adj\text{A}$$

Then A^2 is idempotent fuzzy matrix.

Theorem4.3.4 [12]: Let A be a $n \times n$ reflexive fuzzy matrix. If A is idempotent then $adj\text{A} = \text{A}$ is idempotent.

Proof: We know that for a reflexive fuzzy matrix A of order n, $adj\text{A} = A^c$ $(c \leq n-1)$ where A^c is idempotent.

But we have also that A is idempotent and so A^c =A.

Thus $adj\text{A} = \text{A}$.

Since A is idempotent and hence adjA is idempotent.

Theorem 4.3.5 [12]: Let A be an $n \times n$ reflexive fuzzy matrix. Then we have the following properties:

(i) $adjA^2 = (adj\text{A})^2 = adj\text{A}$

(ii) $adj\,(adj\text{A}) = adj\text{A}$

(iii) $adj\text{A} \geq \text{A}$

(iv) $\text{A}(adj\text{A}) = (adj\text{A})\text{A} = adj\text{A}$.

Proof:

(i) Since A is reflexive and thus A^2 is also reflexive.

Then $adjA^2 = \left(A^2\right)^c = \left(A^c\right)^2$ where $A^c = adj$A is idempotent.

$\therefore adj\text{A}^2 = (adj\text{A})^2$

Since adjA is idempotent and so $(adj\text{A})^2 = adj$A

Hence $adjA^2 = (adj\text{A})^2 = adj$A

(ii) Since A is reflexive and then adjA is reflexive.

Also for a reflexive fuzzy matrix A, adjA is idempotent.

Hence adj (adjA) = adjA. (by theorem 4.3.4)

(iii) Since A is reflexive, then $a_{ii} = 1\ \forall\ i$

Let B = $\left[b_{ij}\right] = adj$A

Then $b_{ij} = \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} a_{t\sigma(t)}$

Taking the permutation of set n_j over the set n_i such that σ(h) = h , $\sigma(i) = j$, $h \neq i$

i.e., the permutation

$$\begin{pmatrix} 1 & 2 & 3 & \dots & i & \dots & j-1 & j+1 & \dots & n \\ 1 & 2 & 3 & \dots & j & \dots & j-1 & j+1 & \dots & n \end{pmatrix}$$

Thus $a_{11}\ a_{22}\ a_{33} \dots a_{(j-1)(j-1)} a_{(j+1)(j+1)} \dots a_{nn}$ is a term of b_{ij} so that

$b_{ij} \geq a_{11}\ a_{22}\ a_{33} \dots a_{(j-1)(j-1)} a_{(j+1)(j+1)} \dots a_{nn} = a_{ij}$ $(\because a_{ii} = 1\ \forall\ i)$

Hence adjA $\geq$ A.

(iv) Since A = $\left[a_{ij}\right]$ is reflexive , then $a_{ii} = 1\ \forall\ i$

Let B = $\left[b_{ij}\right] = adj$A

Let C = $\left[c_{ij}\right]$ = A(adjA) and D = $\left[d_{ij}\right]$ = (adjA)A.

Then $c_{ij} = \sum_{k=1}^{n} a_{ik} \left|A_{jk}\right| \geq a_{ii} \left|\text{A}_{ji}\right| = \left|\text{A}_{ji}\right| = b_{ij} (\because a_{ii} = 1 \forall i)$

and $d_{ij} = \sum_{k=1}^{n} \left|\text{A}_{ki}\right| a_{kj} \geq \left|\text{A}_{ji}\right| a_{jj} = \left|\text{A}_{ji}\right| = b_{ij}\ (\because a_{jj} = 1 \forall j)$

Thus we have A(adjA) $\geq adj$A and (adjA)A $\geq adj$A.

But adjA = (adjA) (adjA) [by (i)]

$\geq A(adjA)$ [by (iii) and using the result $A \leq B \Longrightarrow AC \leq BC$]

So that $A(adjA) = adjA$

Also $adjA = (adjA)(adjA) \geq (adjA)\, A$

[by (i),(iii) and using the result $A \leq B \Rightarrow CA \leq BA$]

So that $(adjA)A = adjA$

Hence $A(adjA) = adjA = (adjA)A$.

Example 4.3.6: For a reflexive fuzzy matrix

$$A = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.3 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix}$$

$$\text{We have } A^2 = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.3 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.3 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix}$$

$$adjA^2 = \begin{bmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix}$$

$$adjA = \begin{bmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix} \geq A\,,\ (adjA)^2 = \begin{bmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix} = adj(adjA)$$

$$A(adjA) = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.3 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix}$$

$$(adjA)A = \begin{bmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.3 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 0.8 \\ 0.4 & 1 & 0.4 \\ 0.7 & 0.6 & 1 \end{bmatrix}$$

It is clear that this example satisfies all the statements of the above proposition. Hence $adjA^2 = (adjA)^2 = adjA = adj(adjA) = A(adjA) = (adjA)A$ and $(adjA) \geq A$.

Chapter 5

SUPER AND SUPER FUZZY MATRIX THEORY

5.1 Introduction

Since now, we are well familiar with basic and fuzzy matrix theory. In this chapter, our purpose is to compare super and super fuzzy matrix theory. Firstly, we shall study super matrices and some operations defined on super matrices. Further in this chapter, we shall study fuzzy supermatrices and some operations defined on fuzzy supermatrices. However, the concept of supermatrix for social scientists was first introduced by Paul Horst, "Matrix Algebra for Social Scientist" [15]. Let us first study the concept of supermatrices.

5.2 Supermatrix Thoery

The general rectangular or square array of numbers are known as matrices whose elements are just an ordinary number or a letter that stands for a number. In other words, the elements of a simple matrix are scalars of scalar quantities. Now we define the notion of a supermatrix.

Definition5.2.1 [4] A matrix whose elements are themselves matrices with elements that can be either scalars or other matrices is called a supermatrix.

In general the kind of supermatrices we shall deal, the matrix elements which have any scalar for their elements. Thus a supermatrix is a general rectangular or square array of matrices.

Let us understand how to construct a supermatrix from the given matrices with the help of an example. Also we shall understand the same with the help of figure and a table.

Example 5.2.1 Consider the following matrices

$$a_{11} = \begin{bmatrix} 2 & 5 & 0 \end{bmatrix}, a_{12} = \begin{bmatrix} 2 & 0 \end{bmatrix}, a_{13} = \begin{bmatrix} -1 & 2 & 5 & 7 \end{bmatrix}$$

$$a_{21} = \begin{bmatrix} 3 & 2 & 6 \\ 7 & 4 & 9 \\ 3 & 6 & 8 \end{bmatrix}, \; a_{22} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \\ 7 & 0 \end{bmatrix}, \; a_{33} = \begin{bmatrix} 2 & 3 & 6 & 1 \\ 3 & 6 & 5 & 3 \\ 6 & 3 & 2 & 9 \end{bmatrix}$$

$$a_{31} = \begin{bmatrix} 1 & -3 & 4 \\ 6 & -5 & 3 \end{bmatrix}, \; a_{32} = \begin{bmatrix} 1 & 4 \\ 6 & 6 \end{bmatrix}, \; a_{33} = \begin{bmatrix} 1 & 3 & 1 & 6 \\ 8 & 0 & 5 & 7 \end{bmatrix}$$

$$a_{41} = \begin{bmatrix} -2 & 8 & 6 \\ -1 & 3 & 3 \end{bmatrix}, \; a_{42} = \begin{bmatrix} 5 & 8 \\ 5 & 3 \end{bmatrix}, \; a_{43} = \begin{bmatrix} 4 & 1 & 7 & 9 \\ 5 & 3 & 5 & 0 \end{bmatrix}$$

Here a_{ij} denotes a matrix and not a scalar of a matrix $(1 \le i \le 4, 1 \le j \le 3)$.

Let

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

We can write out the supermatrix a in terms of the original (natural) matrix elements i.e.,

$$a = \left[\begin{array}{ccc|cc|cccc} 2 & 5 & 0 & 2 & 0 & -1 & 2 & 5 & 7 \\ \hline 3 & 2 & 6 & 1 & 3 & 2 & 3 & 6 & 1 \\ 7 & 4 & 9 & 5 & 3 & 3 & 6 & 5 & 3 \\ 3 & 6 & 8 & 7 & 0 & 6 & 3 & 2 & 9 \\ \hline 1 & -3 & 4 & 1 & 4 & 1 & 3 & 1 & 6 \\ 6 & -5 & 3 & 6 & 6 & 8 & 0 & 5 & 7 \\ \hline -2 & 8 & 6 & 5 & 8 & 4 & 1 & 7 & 9 \\ -1 & 3 & 3 & 5 & 3 & 5 & 3 & 5 & 0 \end{array} \right]$$

Here the elements are divided vertically and horizontally by thin lines. If the lines were not used, the matrix a would be read as a simple matrix.

A Diagrammatic representation of supermatrix a showing within submatrices a_{ij} along with their orders, is given by the following figure:

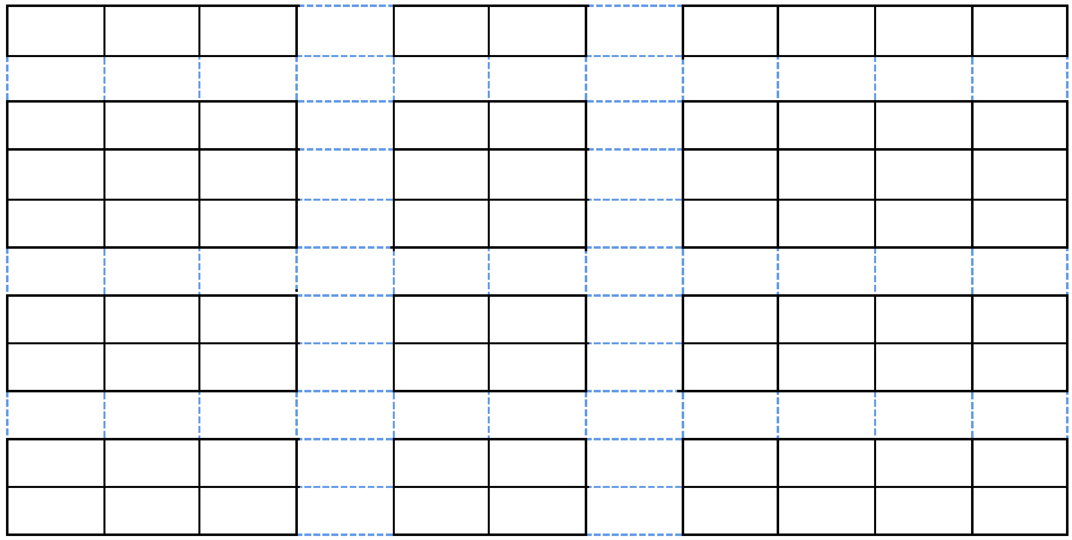

Figure 5.1 [4] A Diagrammatic representation of supermatrix a

Remark: All submatrices within a given row must have the same number of rows. Likewise all submatrices within a given column must have the same number of columns.

Now let us show the orders of submatrices a_{ij} $(1 \le i \le 4,\ 1 \le j \le 3)$ within supermatrix a, in the tabular form as below:

Column (j) → Row (i) ↓	1	2	3
1	1×3	1×2	1×4
2	3×3	3×2	3×4
3	2×3	2×2	2×4
4	2×3	2×2	2×4

Table 5.1 An index for the orders of submatrices a_{ij} $(1 \le i \le 4\ ,\ 1 \le j \le 3)$

The Order of a Supermatrix [4] is defined in the same way as that of a simple matrix. The height of a supermatrix is the number of rows of submatrices in it and the width of a supermatrix is the number of columns of submatrices in it. Thus we have for this supermatrix 4 rows and 3 columns and so order of supermatrix is 4 × 3. However the order of the corresponding simple matrix (considering the matrix without horizontal and vertical thin lines) is 8 × 9.

Observations:

1. The order of a supermatrix tells us nothing about the orders of the submatrices which by general rectangular or square array construct a supermatrix.
2. The order of a supermatrix tells us nothing about the order of the simple matrix which on dividing the elements by horizontal and vertical thin lines construct a supermatrix.

This process of dividing the elements of a simple matrix by horizontal and vertical thin lines to construct a supermatrix is called **partition [4].**

Note: Different supermatrices constructed from a simple matrix by different partitions are all equal. This we shall discuss later on while dealing with type II supervectors.

From a given simple matrix, different supermatrices either of same order or of different orders can be constructed, each time partitioning between different rows and columns in any way that happens to suit our purpose.

Example 5.2.2 Consider a 5 × 5 simple matrix

$$A = \begin{bmatrix} 2 & 7 & 2 & 4 & 7 \\ 5 & 8 & 2 & 7 & 0 \\ 3 & 2 & 1 & 4 & 7 \\ 4 & 6 & 2 & 0 & 3 \\ 6 & 5 & 1 & 2 & 8 \end{bmatrix}$$

This matrix can be partitioned by drawing thin lines in a number of ways to obtain different supermatrices.

Let us partition between the columns one and two and three and four. Also let us partition between the rows one and two and three and four.

Then the supermatrix is obtained is

$$a_s = \left[\begin{array}{cc|cc|c} 2 & 7 & 2 & 4 & 7 \\ \hline 5 & 8 & 2 & 7 & 0 \\ 3 & 2 & 1 & 4 & 7 \\ \hline 4 & 6 & 2 & 0 & 3 \\ 6 & 5 & 1 & 2 & 8 \end{array}\right]$$

which is symmetrically partitioned.

By a **symmetric partition [4]** of a matrix, we mean that the rows and columns are partitioned in exactly the same way. According to this rule of symmetric partition, only square simple matrix can be symmetrically partitioned. Now we proceed on to recall the notion of **symmetric partition of a symmetric simple matrix [4].**

Example 5.2.3 Consider a fourth order symmetric matrix and partition it between the second and third rows and also between the second and third columns. Then the second order supermatrix obtained is

$$a = \left[\begin{array}{cc|cc} 4 & 6 & 7 & 2 \\ 6 & 9 & 6 & 5 \\ \hline 7 & 6 & 0 & 2 \\ 2 & 5 & 1 & 3 \end{array}\right]$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a'_{12} & a_{22} \end{bmatrix}$$

The general expression for a symmetrically partitioned symmetric matrix is given by

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a'_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a'_{13} & a'_{23} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a'_{1n} & a'_{2n} & a'_{3n} & \dots & a_{nn} \end{bmatrix}$$

The simple rule about the matrix elements of a symmetrically partitioned symmetric simple matrix are

1. The diagonal submatrices of the supermatrix are all symmetric matrices i.e., $a_{ii}^{t} = a_{ii}$.
2. The non-diagonal submatrices are symmetric about symmetric about the diagonal. In other words the matrix elements below the diagonal are the transposes of the corresponding elements above the diagonal and vice-versa.

   i.e. $a_{ij}^{t} = a_{ji}$ and $a_{ji}^{t} = a_{ij}$ for $i \neq j$.

The general experssion for a **symmetrically partitioned simple diagonal matrix [4]** is given by

$$D = \begin{bmatrix} D_{11} & 0 & \dots & 0 \\ 0' & D_{12} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0' & 0' & \dots & D_{nn} \end{bmatrix}$$

0 denote the matrices with zero as all entries and $0'$ only represents the order is reversed or transformed. We can write simply,

$$D = \begin{bmatrix} D_{1} & 0 & \dots & 0 \\ 0 & D_{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & D_{n} \end{bmatrix}$$

which is reffered to as the super diagonal matrix.

Example 5.2.4 Consider a simple partitioned symmetrically diagonal matrix

$$D = \left[\begin{array}{ccc|cc} 4 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ \hline 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right] = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

The Identity Matrix $I = \begin{bmatrix} I_s & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_r \end{bmatrix}$

where s, t and r denote the orders of the identity matrices I_s , I_t and I_r respectively, is a **super identity matrix [4]**.

Example 5.2.5 Consider a super identity matrix

$$I = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} I_4 & 0 \\ 0 & I_2 \end{bmatrix}$$

Now we define the notion of **partial triangular matrix as a supermatrix [4].**

Example 5.2.6 Let $u = \left[\begin{array}{ccc|cc} 2 & 7 & 8 & 3 & 4 \\ 0 & 6 & 4 & 3 & 5 \\ 0 & 0 & 8 & 1 & 8 \end{array}\right] = \left[T \mid a'\right]$

where T is a upper triangular submatrix.

Then u is a **partial upper triangular matrix partitioned as a supermatrix [4]**.

Example 5.2.7 Let

$$\boldsymbol{l} = \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 8 & 5 & 4 & 0 \\ 3 & 1 & 4 & 6 \\ \hline 4 & 9 & 6 & 2 \\ 1 & 9 & 7 & 6 \end{array}\right] = \left[\begin{array}{c} T \\ \hline a' \end{array}\right]$$

where T is a lower triangular submatrix.

Then l is a **partial lower triangular matrix partitioned as a supermatrix [4]**.

Now we define the notion of supervectors of type I and type II.

A simple vector is a vector each of whose elements is a scalar. In this way a supervector is a vector each of whose elements is either a scalar or a vector.

Definition 5.2.2 [4] Type I Column supervector

Let $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

where each v_i $(1 \le i \le n)$ is a column subvector of the column vector v. Then v is a type I column supervector.

Example 5.2.8 Let $v = \left[\begin{array}{c} 6 \\ 7 \\ 4 \\ \hline 3 \\ 6 \end{array}\right] = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

where v_1 , v_2 are column subvectors.

Then v is a type I column supervector.

Definition 5.2.3 [4] Type I row supervector

Let $v' = \begin{bmatrix} v_1' & v_2' & \ldots & v_n' \end{bmatrix}$ where each v_i' $(1 \le i \le n)$ is a row subvector of the row vector v.

Then v is a type I row super vector.

Example 5.2.9 Let $v' = \left[\begin{array}{ccc|cccc} 2 & 5 & 1 & 5 & 4 & 7 & 5 \end{array}\right] = \begin{bmatrix} v_1' & v_2' \end{bmatrix}$

where v_1' and v_2' are row subvector of the row vector v'.

Then v' is a type I row supervector.

Next we recall the definition of type II supervectors.

Definition 5.2.4 [4] Type II column and row supervector

The general m × n supermatrix is given by

$$a = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

If

$$a_1^1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$$

$$a_2^1 = \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}$$

………………………..

$$a_m^1 = \begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Then

$$a = \begin{bmatrix} a_1^1 \\ a_2^1 \\ \vdots \\ a_m^1 \end{bmatrix}_n$$

is defined to be the **type II column supervector**. Similarly if

$$a^1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, a^2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, a^n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then $a = \begin{bmatrix} a^1 & a^2 & \dots & a^n \end{bmatrix}_m$ is defined to be the **type II row supervector.**

Clearly

$$a = \begin{bmatrix} a_1^1 \\ a_2^1 \\ \vdots \\ a_m^1 \end{bmatrix}_n = \begin{bmatrix} a^1 & a^2 & \dots & a^n \end{bmatrix}_m$$

shows the **equality of supermatrices [4].**

Remark: Different supervectors (supermatrices) constructed by different partitions from a simple matrix are all equal. In other words, two supermatrices are equal iff their corresponding simple forms are equal.

Now we define the notion of **transpose of a supermatrix [4]**.

The transpose of the general $m \times n$ supermatrix a, denoted by a' is given by

$$a' = \begin{bmatrix} a'_{11} & a'_{21} & \dots & a'_{m1} \\ a'_{12} & a'_{22} & \dots & a'_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a'_{1n} & a'_{2n} & \dots & a'_{mn} \end{bmatrix}$$

Then a' is a $n \times m$ supermatrix obtained by taking the transpose of each element i.e., the submatrices of a.

Next we will find the **transpose of supermatrix constructed by symmetrically partitioned symmetric simple matrix.**

The transpose of a symmetrically partitioned $n \times n$ symmetric simple matrix denotedby a' is given by

$$a' = \begin{bmatrix} a'_{11} & (a'_{12})' & (a'_{13})' & \dots & (a'_{1n})' \\ a'_{12} & a'_{22} & (a'_{23})' & \dots & a'_{2n} \\ a'_{13} & a'_{23} & a'_{33} & \dots & (a'_{3n})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a'_{1n} & a'_{2n} & a'_{3n} & \dots & a'_{nn} \end{bmatrix}$$

(1) The diagonal submatrices are all symmetric matrices and so are unaltered by transposition i.e., $a'_{ii} = a_{ii}$.

(2) Recall also the transpose of a transpose is the original matrix i.e., $(a'_{ij})' = a_{ij}$

Thus the transpose of supermatrix constructed by symmetrically partitioned symmetric simple matrix a, denoted by a' is given by

$$a' = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a'_{12} & a_{22} & a_{23} & \ldots & a_{2n} \\ a'_{13} & a'_{23} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a'_{1n} & a'_{2n} & a'_{3n} & \ldots & a_{nn} \end{bmatrix} = a$$

Hence the transpose of supermatrix constructed by symmetrically partitioned symmetric simple matrix is the supermatrix itself.

Similarly the **transpose of symmetrically partitioned diagonal matrix and identity matrix** are the original supermatrices itself.

Also it can be easily verified that

- The transpose of a partial triangular supermatrix is the partial lower triangular supermatrix and vice versa.
- The transpose of type I column supervecrtor is a type I row supervector and vice versa.
- The transpose of type II column supervector is a type II row supervector and vice versa.

Now we proceed on to define the notion of minor and major product of supervectors. Firstlly we recall the definition of minor and major product of type I supervectors.

Definition 5.2.5[4] Minor product of type I supervectors:

Suppose

$v_a = \begin{bmatrix} v_{a_1} \\ v_{a_2} \\ \vdots \\ v_{a_n} \end{bmatrix}$ and $v_b = \begin{bmatrix} v_{b_1} \\ v_{b_2} \\ \vdots \\ v_{b_n} \end{bmatrix}$ be two column supervectors of type I.

The minor product is defined as

$$v_a' v_b = \begin{bmatrix} v_{a_1}' & v_{a_2}' & \dots & v_{a_n}' \end{bmatrix} \begin{bmatrix} v_{b_1} \\ v_{b_2} \\ \vdots \\ v_{b_n} \end{bmatrix}$$

$$= v_{a_1}' v_{b_1} + v_{a_2}' v_{b_2} + \dots + v_{a_n}' v_{b_n}$$

Example 5.2.10 Consider two type I supervectors

$$v_a = \begin{bmatrix} v_{a_1} \\ v_{a_2} \\ v_{a_3} \end{bmatrix} \text{ and } v_b = \begin{bmatrix} v_{b_1} \\ v_{b_2} \\ v_{b_3} \end{bmatrix}$$

where

$$v_{a_1} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, v_{a_2} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, v_{a_3} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$v_{b_1} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, v_{b_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, v_{b_3} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$v_a' v_b = \begin{bmatrix} v_{a_1}' & v_{a_2}' & v_{a_3}' \end{bmatrix} \begin{bmatrix} v_{b_1} \\ v_{b_2} \\ v_{b_3} \end{bmatrix}$$

$$= v_{a_1}' v_{b_1} + v_{a_2}' v_{b_2} + v_{a_3}' v_{b_3}$$

$$= \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= (2+3) + (4+12) + (-4+1)$$

$$= 5+16-3$$

$$= 18$$

It can be easily verified that $v_a' v_b = v_b' v_a$.

Example 5.2.11 We just recall if

$$v = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

is a column vector and v' , the transpose of v is a row vector then we have

$$v'v = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$= 1^2 + 3^2 + 4^2$$

$$= 1+9+16$$

$$= 26$$

Thus if

$v_x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector , then

$$v_x' \, v_x = \begin{bmatrix} x_1 & x_2 & \dots x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1^2 + x_2^2 + \dots + x_n^2$$

Also $\begin{bmatrix} x_1 & x_2 & \dots x_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + \dots + x_n \end{bmatrix}$

and $\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + \dots + x_n \end{bmatrix}$

i.e., $v_x'.1 = 1'.v_x = \sum x_i$.

Definition 5.2.6[4] Major product of type I supervectors

Suppose $v_a = \begin{bmatrix} v_{a_1} \\ v_{a_2} \\ \vdots \\ v_{a_m} \end{bmatrix}$ and $v_b = \begin{bmatrix} v_{b_1} \\ v_{b_2} \\ \vdots \\ v_{b_n} \end{bmatrix}$

be any two supervectors of type I

The major product is defined as

$$v_a v_b' = \begin{bmatrix} v_{a_1} \\ v_{a_2} \\ \vdots \\ v_{a_m} \end{bmatrix} . \begin{bmatrix} v_{b_1}' & v_{b_2}' & \dots & v_{b_n}' \end{bmatrix}$$

$$= \begin{bmatrix} v_{a_1}v_{b_1}' & v_{a_1}v_{b_2}' & \cdots & v_{a_1}v_{b_n}' \\ v_{a_2}v_{b_1}' & v_{a_2}v_{b_2}' & \cdots & v_{a_2}v_{b_n}' \\ \vdots & \vdots & \vdots & \vdots \\ v_{a_m}v_{b_1}' & v_{a_m}v_{b_2}' & \cdots & v_{a_m}v_{b_n}' \end{bmatrix}$$

Example 5.2.12 Consider two type I supervectors

$$v_a = \begin{bmatrix} v_{a_1} \\ v_{a_2} \\ v_{a_3} \end{bmatrix} \text{ and } v_b = \begin{bmatrix} v_{b_1} \\ v_{b_2} \\ v_{b_3} \\ v_{b_4} \end{bmatrix}$$

where $v_{a_1} = [3]$, $v_{a_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $v_{a_3} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

and $v_{b_1} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, $v_{b_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_{b_3} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}$, $v_{b_4} = [2]$

Now $v_a v_b' = \begin{bmatrix} v_{a_1} \\ v_{a_2} \\ v_{a_3} \end{bmatrix} \begin{bmatrix} v_{b_1}' & v_{b_2}' & v_{b_3}' & v_{b_4}' \end{bmatrix}$

$$= \begin{bmatrix} v_{a_1}v_{b_1}' & v_{a_1}v_{b_2}' & v_{a_1}v_{b_3}' & v_{a_1}v_{b_4}' \\ v_{a_2}v_{b_1}' & v_{a_2}v_{b_2}' & v_{a_2}v_{b_3}' & v_{a_2}v_{b_4}' \\ v_{a_3}v_{b_1}' & v_{a_3}v_{b_2}' & v_{a_3}v_{b_3}' & v_{a_3}v_{b_4}' \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c|c} [3]\,[5\;\;1\;\;0] & [3]\,[2\;\;1] & [3]\,[3\;\;1\;\;0\;\;6] & [3]\,[2] \\ \hline \begin{bmatrix}1\\3\end{bmatrix}[5\;\;1\;\;0] & \begin{bmatrix}1\\3\end{bmatrix}[2\;\;1] & \begin{bmatrix}1\\3\end{bmatrix}[3\;\;1\;\;0\;\;6] & \begin{bmatrix}1\\3\end{bmatrix}[2] \\ \hline \begin{bmatrix}1\\2\\0\end{bmatrix}[5\;\;1\;\;0] & \begin{bmatrix}1\\2\\0\end{bmatrix}[2\;\;1] & \begin{bmatrix}1\\2\\0\end{bmatrix}[3\;\;1\;\;0\;\;6] & \begin{bmatrix}1\\2\\0\end{bmatrix}[2] \end{array}\right]$$

$$= \left[\begin{array}{ccc|cc|cccc|c} 15 & 5 & 0 & 6 & 3 & 9 & 3 & 0 & 18 & 6 \\ \hline 5 & 1 & 0 & 2 & 1 & 3 & 1 & 0 & 6 & 2 \\ 15 & 3 & 0 & 6 & 3 & 9 & 3 & 0 & 18 & 6 \\ \hline 5 & 1 & 0 & 2 & 1 & 3 & 1 & 0 & 6 & 2 \\ 10 & 2 & 0 & 4 & 2 & 6 & 2 & 0 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It can be easily verified that $(v_a v_b^{'})^{'} = v_b v_a^{'}$.

Next we recall the definition of minor and major product of type II supervectors.

Definition 5.2.7 [4] Minor and Major Product of Type II Supervectors

Consider two general supermatrices

$$a = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ and } b = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1s} \\ b_{21} & b_{22} & \dots & b_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ b_{r1} & b_{r2} & \dots & b_{rs} \end{bmatrix}$$

$$\text{Then } a = \begin{bmatrix} a_1^1 \\ a_2^1 \\ \vdots \\ a_m^1 \end{bmatrix}_n = \begin{bmatrix} a^1 & a^2 & \dots & a^n \end{bmatrix}_m \text{ and } b = \begin{bmatrix} b_1^1 \\ b_2^1 \\ \vdots \\ b_r^1 \end{bmatrix}_s = \begin{bmatrix} b^1 & b^2 & \dots & b^s \end{bmatrix}_r$$

The **minor product of type II supervectors** is given by

$$ab = \begin{bmatrix} a^1 & a^2 & \ldots & a^n \end{bmatrix}_m \begin{bmatrix} b_1^1 \\ b_2^1 \\ \vdots \\ b_r^1 \end{bmatrix}_s$$

$$= \begin{bmatrix} a^1 b_1^1 + a^2 b_2^1 + \ldots + a^n b_r^1 \end{bmatrix}_{ms}$$

and the **major product of type II supervectors** is given by

$$ab = \begin{bmatrix} a_1^1 \\ a_2^1 \\ \vdots \\ a_m^1 \end{bmatrix}_n . \begin{bmatrix} b^1 & b^2 & \ldots & b^s \end{bmatrix}_r$$

$$= \begin{bmatrix} a_1^1 b^1 & a_1^1 b^2 & \ldots & a_1^1 b^s \\ a_2^1 b^1 & a_2^1 b^2 & \ldots & a_2^1 b^s \\ \vdots & \vdots & \vdots & \vdots \\ a_m^1 b^1 & a_m^1 b^2 & \ldots & a_m^1 b^s \end{bmatrix}_{ms}$$

Example 5.2.13 The minor product of type II supervectors

$$a = \begin{matrix} a^1 & a^2 \\ \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 1 & 4 \end{bmatrix} \end{matrix} \text{ and } b = \begin{bmatrix} 1 & 7 \\ 2 & 0 \end{bmatrix} \begin{matrix} b_1^1 \\ b_2^1 \end{matrix}$$

is given by

$$ab = \begin{bmatrix} a^1 & a^2 \end{bmatrix} \begin{bmatrix} b_1^1 \\ b_2^1 \end{bmatrix}$$

$$= \begin{bmatrix} a^1 b_1^1 + a^2 b_2^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ 2 & 14 \\ 1 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 12 & 0 \\ 8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 7 \\ 14 & 14 \\ 9 & 0 \end{bmatrix}$$

And the major product of type II supervector

$$a = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 1 & 4 \end{bmatrix} \begin{matrix} a_1^1 \\ a_2^1 \\ a_3^1 \end{matrix} \text{ and } b = \begin{matrix} b^1 & b^2 \\ \begin{bmatrix} 1 & 7 \\ 2 & 0 \end{bmatrix} \end{matrix}$$

is given by

$$ab = \begin{bmatrix} a_1^1 \\ a_2^1 \\ a_3^1 \end{bmatrix} \begin{bmatrix} b^1 & b^2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^1 b^1 & a_1^1 b^2 \\ a_2^1 b^1 & a_2^1 b^2 \\ a_3^1 b^1 & a_3^1 b^2 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 7 \\ 14 & 14 \\ 9 & 7 \end{bmatrix}$$

It is easily verified that the product of two matrices a and b as a minor product of type II supervectors coincides with the major product.

Next, Let us discuss the concept of fuzzy supermatrices.

5.3 Super Fuzzy Matrix Theory

We are well femiliar with fuzzy matrices whose elements belong to [0,1]. In this section of our dealing, we for the first time introduce the notion of fuzzy supermatrices and operations on them. Throughout this section we consider matrices with entries only from the fuzzy interval [0,1]. Thus all matrices in this section unless we make a specific partition or mention of them will be fuzzy matrices. Now we define the notion of a fuzzy supermatrix

Definition 5.3.1[4] Let us consider a fuzzy matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

where A_{ij} ($1 \le i \le m$ and $1 \le j \le n$) are fuzzy submatrices of A; with entries from [0,1] such that number of rows in fuzzy submatrices A_{i1} , A_{i2}, …,A_{in} for each i=1,2,…,m are equal and similarly number of columns in fuzzy submatrices A_{1j} , A_{2j}, …,A_{mj} for each j=1,2,…,n are equal.

Then A is a general super fuzzy matrix or a general fuzzy supermatrix.

Now we define the notion of **transpose of a super fuzzy matrix [4]**.

The transpose of the fuzzy matrix A, denoted by A' is given by

$$A' = \begin{bmatrix} A'_{11} & A'_{21} & \cdots & A'_{m1} \\ A'_{12} & A'_{22} & \cdots & A'_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ A'_{1n} & A'_{2n} & \cdots & A'_{mn} \end{bmatrix}$$

where A'_{ij} are transpose of A_{ij} ($1 \le i \le m$ and $1 \le j \le n$).

Example 5.3.1 Consider a super fuzzy matrix

$$A = \left[\begin{array}{cc|cccc|ccc} 0 & 1 & 0.4 & 0.7 & 0.6 & 0.2 & 0 & 0.9 & 1 \\ 0.4 & 0.1 & 0.6 & 0.5 & 0.4 & 0.6 & 0.7 & 0.6 & 0.3 \\ \hline 0.6 & 0.5 & 0.7 & 0.4 & 0.4 & 0.3 & 1 & 0.3 & 0.4 \\ 0.5 & 0.2 & 0.7 & 0.9 & 0.4 & 0.1 & 0.3 & 0 & 0.3 \\ 0.7 & 0.1 & 0.4 & 0.4 & 0.1 & 0.6 & 0.7 & 0.4 & 0.6 \\ \hline 0.8 & 0.9 & 0.8 & 0.1 & 0.3 & 0.4 & 0.6 & 0.3 & 1 \\ 0 & 1 & 0.3 & 0.4 & 0.6 & 0.1 & 0.4 & 0.2 & 0.3 \\ \hline 1 & 0 & 0.5 & 0.6 & 0.6 & 0.2 & 1 & 0.4 & 0.4 \end{array}\right]$$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix}$$

where A_{ij} are fuzzy submatrices of A with entries from [0,1] such that A_{i1} , A_{i2},A_{i3} have same number of rows for i =1,2,3,4 and similarily A_{1j} , A_{2j},A_{3j},A_{4j} have same number of columns for j = 1, 2, 3.

The transpose of the fuzzy supermatrix A denoted by A' is given by

$$A' = \begin{bmatrix} A'_{11} & A'_{21} & A'_{31} & A'_{41} \\ A'_{12} & A'_{22} & A'_{33} & A'_{42} \\ A'_{13} & A'_{23} & A'_{33} & A'_{43} \end{bmatrix}$$

$$= \left[\begin{array}{cc|ccc|cc|c} 0 & 0.4 & 0.6 & 0.5 & 0.7 & 0.8 & 0 & 1 \\ 1 & 0.1 & 0.5 & 0.2 & 0.1 & 0.9 & 1 & 0 \\ \hline 0.4 & 0.6 & 0.7 & 0.7 & 0.4 & 0.8 & 0.3 & 0.5 \\ 0.7 & 0.5 & 0.4 & 0.9 & 0.4 & 0.1 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.4 & 0.4 & 0.1 & 0.3 & 0.6 & 0.6 \\ 0.2 & 0.6 & 0.3 & 0.1 & 0.6 & 0.4 & 0.1 & 0.2 \\ \hline 0 & 0.7 & 1 & 0.3 & 0.7 & 0.6 & 0.4 & 1 \\ 0.9 & 0.6 & 0.3 & 0 & 0.4 & 0.3 & 0.2 & 0.4 \\ 1 & 0.3 & 0.4 & 0.3 & 0.6 & 1 & 0.3 & 0.4 \end{array}\right]$$

Next we defne the notion of fuzzy super row matrix, fuzzy super column matrix and their transpose.

Definition 5.3.2 [4] Fuzzy Super Row Vector (Fuzzy Super Row Matrix)

Let $A = [A_1 \mid A_2 \mid \ldots A_n]$ or simply $[A_1 \quad A_2 \ldots. A_n]$ ($n>1$) where each A_i, $i=1,2,\ldots,n$. is a fuzzy row vector (fuzzy row matrix). Then A is called as the fuzzy super row vector or fuzzy super row matrix.

Definition 5.3.3 [4] Fuzzy Super Column Vector (Fuzzy Super Column Matrix)

Let $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$ $(m > 1)$

where each A_i ($i=1, 2, \ldots, m$) is a fuzzy column matrix (fuzzy column vector).
Then A is called the super fuzzy column matrix or super fuzzy column vector.

Now we define the notion of **transpose of super fuzzy row matrix and super fuzzy column matrix [4]**.

The transpose of super fuzzy row matrix A , denoted by A' is a super fuzzy column matrix is given by

$$A' = \begin{bmatrix} A'_1 \\ A'_2 \\ \vdots \\ A'_n \end{bmatrix}$$

where each A_i' is fuzzy column matrix.

Similarly the transpose of super fuzzy column matrix A, denoted by A' is a super fuzzy row matrix given by $A' = \left[A_1' \quad A_2' \quad \ldots A_m'\right]$ where each A_i' is a fuzzy row matrix.

Example 5.3.2 Let A = [A_1 A_2 A_3 A_4] be a super fuzzy row vector where A_1 = [0 0.2 0.3], A_2 = [0.4 0.2 0 0.3], A_3 = [1 0] , A_4 = [0.2 0.3 0.2]

i.e., A = [0 0.2 0.3 | 0.4 0.2 0 0.3 1 | 0 0.2 | 0.3 0.2]

The transpose of super fuzzy row vector A is given by

$$A' = \begin{bmatrix} A_1' \\ A_2' \\ A_3' \\ A_4' \end{bmatrix}$$

where $A_1' = \begin{bmatrix} 0 \\ 0.2 \\ 0.3 \end{bmatrix}$, $A_2' = \begin{bmatrix} 0.4 \\ 0.2 \\ 0 \\ 0.3 \end{bmatrix}$, $A_3' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A_4' = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.2 \end{bmatrix}$

$$\text{Thus } A' = \begin{bmatrix} 0 \\ 0.2 \\ 0.3 \\ \hline 0.4 \\ 0.2 \\ 0 \\ 0.3 \\ \hline 1 \\ 0 \\ \hline 0.2 \\ 0.3 \\ 0.2 \end{bmatrix}$$

Example 5.3.3 Let A = $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$ be a super fuzzy column vector, where $A_1 = \begin{bmatrix} 1 \\ 0.3 \\ 0.6 \end{bmatrix}$,

$A_2 = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.7 \\ 0 \end{bmatrix}$

i.e., $A = \begin{bmatrix} 1 \\ 0.3 \\ 0.6 \\ \hline 0.3 \\ 0.5 \\ \hline 0.3 \\ 0.4 \\ 0.7 \\ 0 \end{bmatrix}$

The transpose of super fuzzy column vector A is given by

$$A' = \begin{bmatrix} A_1' & A_2' & A_3' \end{bmatrix}$$

where $A_1' = \begin{bmatrix} 1 & 0.3 & 0.6 \end{bmatrix}$, $A_2' = \begin{bmatrix} 0.3 & 0.5 \end{bmatrix}$, $A_3' = \begin{bmatrix} 0.3 & 0.4 & 0.7 & 0 \end{bmatrix}$

Thus $A' = \begin{bmatrix} 1 & 0.3 & 0.6 \mid 0.3 & 0.5 \mid 0.3 & 0.4 & 0.7 & 0 \end{bmatrix}$

is a super fuzzy row matrix.

Next we define the two types of products of these fuzzy super column and row matrices i.e. $A \,.\, A'$ and where $A'. A$ is a fuzzy super row matrix.

Firstly, Let us define $A \,.\, A'$ where A is a fuzzy super row matrix.

Definition 5.3.4 Let $A= [A_1 \quad A_2 \ldots. A_n]$ be a super fuzzy super row matrix i.e., each A_i is a $1 \times t_i$ fuzzy row submatrix of A , $i=1,2,\ldots,n$ and A' be the transpose of A.

$$\max\{A. A'\} = \max \left\{ \begin{bmatrix} A_1 & A_2 \ldots. & A_n \end{bmatrix} . \begin{bmatrix} A_1' \\ A_2' \\ \vdots \\ A_n' \end{bmatrix} \right\}$$

$$= \max \{A_1.A_1', A_2.A_2', \ldots, A_n.A_n'\},$$

which is known as the minor product of two fuzzy supermatrices.

i.e., $\max\{A. A'\} = \max \{\max \min(a_{i_1}, a'_{i_1}), \max \min(a_{i_2}, a'_{i_2}), \ldots, \max \min(a_{i_n}, a'_{i_n})\}$

$1 < i_1, i_1' < t_1, 1 < i_2, i_2' < t_2, \ldots, 1 < i_n, i_n' < t_n$ where

$$A_1 = \begin{bmatrix} a_1, a_2, \ldots, a_{t_1} \end{bmatrix} \text{ and } A_1' = \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_{t_1}' \end{bmatrix},$$

$$A_2 = \begin{bmatrix} a_1, a_2, \ldots, a_{t_2} \end{bmatrix} \text{ and } A_2' = \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_{t_2}' \end{bmatrix},$$

…..…………………………………..

$$A_n = \begin{bmatrix} a_1, a_2, \ldots, a_{t_n} \end{bmatrix} \text{ and } A_n' = \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_{t_n}' \end{bmatrix}.$$

Thus $\max\{A. A'\} = a \in [0,1]$.

This fuzzy supermatrix operation is the usual operation with product of a_i , a_j' replaced by minimum of (a_i , a_j') and the sum of the elements a_{i_1} +…+ a_{i_n} replaced by the maximum of (a_{i_1}, …, a_{i_n}).

Thus given any fuzzy super row matrix A, its transpose A' is the fuzzy super column matrix such that the product A. A', defined as the product of two fuzzy super row matrix (vector) and fuzzy super column matrix, is always an element from the fuzzy interval [0,1].

Example 5.3.4 Let $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ be a super fuzzy row matrix.

where $A_1 = \begin{bmatrix} 0.1 & 0.4 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.3 \end{bmatrix}$

$A' = \begin{bmatrix} A_1' \\ A_2' \end{bmatrix}$ is a fuzzy super column matrix, where

$A'_1 = \begin{bmatrix} 0.1 \\ 0.4 \\ 0 \end{bmatrix}$ and $A'_2 = [0.3]$

Now A. A' = max min{A , A'}

(This is a notational convenience of the product)

$$= \max\min\left\{[A_1 \quad A_2], \begin{bmatrix} A'_1 \\ A'_2 \end{bmatrix}\right\}$$

$$= \max\{A_1 . A'_1., A_2. A'_2\}$$

$$= \max\left\{[0.1 \;\; 0.4 \quad 0]\begin{bmatrix} 0.1 \\ 0.4 \\ 0 \end{bmatrix}, [0.3]\,[0.3]\right\}$$

$$= \max\{\min(0.1,0.4),\min(0.4,0.4),\min(0,0)\},\max\{\min(0.3,0.3)\}$$

$$= \max(0.1,0.4,0),\max(0.3)$$

$$= 0.4,0.3$$

Then max{ A.A'} = max {0.4,0.1}

= 0.4

The way in which max {A.A'}= max {max min{A , A'}} is defined as peculiar which may be defined as the super pseudo product of the fuzzy supermatrix A and the transpose of the fuzzy supermatrix A'.

Now how is $A'A$ defined where A' is a $n \times 1$ fuzzy super column vector and A is a $n \times 1$ fuzzy super row vector.

As we have defined A.A' as the super pseduo product, now we define A'.A as a super fuzzy $n \times n$ matrix.

Definition 5.3.5 [4] Let A = (a_1 ,a_2 ,…,a_n) = $(A_1 | A_2 | ... | A_n)$ be a super fuzzy row matrix and A' be the transpose of A

$$\min\{A', A\} = \min\left\{\begin{bmatrix} a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{bmatrix}[a_1 \quad a_2 \quad ...a_n]\right\}$$

$$\min\{A',A\} = \begin{bmatrix} \min(a'_1, a_1) & \min(a'_1, a_2) & \ldots & \min(a'_1, a_n) \\ \min(a'_2, a_1) & \min(a'_2, a_2) & \ldots & \min(a'_2, a_n) \\ \vdots & \vdots & \vdots & \vdots \\ \min(a'_n, a_1) & \min(a'_n, a_1) & \ldots & \min(a'_n, a_n) \end{bmatrix}.$$

Now min{A′,A} is a $n \times n$ square fuzzy matrix. It is partitioned as per the divison of rows and columns of A and A′ respectively. It is important to note min{A′,A} is a symmetric matrix about the diagonal.

$$\min\{A',A\} = \begin{bmatrix} \min(a_1, a_1) & \min(a_1, a_2) & \ldots & \min(a_1, a_n) \\ \min(a_2, a_1) & \min(a_2, a_2) & \ldots & \min(a_2, a_n) \\ \vdots & \vdots & \vdots & \vdots \\ \min(a_n, a_1) & \min(a_n, a_1) & \ldots & \min(a_n, a_n) \end{bmatrix}$$

where $a'_i = a_i$ for the elements remain as it is while transposing the elements. Clearly since min(a_2,a_1) = min (a_1,a_2) , we get min{A′,A} matrix to be a symmetric matrix.

Further if A = (A_1 A_2 … A_n) with number of elements in A_i (i=1,2,…,n) is t_i ($1 < t_i < n$), min{A′,A} is a super fuzzy matrix with $i \times i$ fuzzy submatrices i = 1,2,…,n. and min {A′,A}is a $n \times n$ fuzzy matrix.

We illustrate this by the following example.

Example 5.3.5 Let A = $\left[0.2 \;\; 0.3 \;\; 0 \,\middle|\, 0.3 \;\; 0.6 \,\middle|\, 0.4 \;\; 0.7 \;\; 0.7 \;\; 0.5\right]$ = [A_1 | A_2 | A_3]

be a fuzzy super row matrix with

$A_1 = [0.2 \;\; 0.3 \;\; 0]$, $A_2 = [0.3 \;\; 0.6]$ and $A_3 = [0.4 \;\; 0.7 \;\; 0.7 \;\; 0.5]$

and A', = $\begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix}$; where $A'_1 = \begin{bmatrix} 0.2 \\ 0.3 \\ 0 \end{bmatrix}$, $A'_2 = \begin{bmatrix} 0.3 \\ 0.6 \end{bmatrix}$ and $A'_3 = \begin{bmatrix} 0.4 \\ 0.7 \\ 0.7 \\ 0.5 \end{bmatrix}$

we find $\min\{A',A\} = \min \left\{ \left[\begin{array}{c} 0.2 \\ 0.3 \\ 0 \\ \hline 0.3 \\ 0.6 \\ \hline 0.4 \\ 0.7 \\ 0.7 \\ 0.5 \end{array}\right], \left[\begin{array}{ccc|cc|cccc} 0.2 & 0.3 & 0 & 0.3 & 0.6 & 0.4 & 0.7 & 0.7 & 0.5 \end{array}\right] \right\}$

$$\min\{A',A\} = \begin{bmatrix} \min(0.2,0.2) & \min(0.2,0.3) & \dots & \min(0.2,0.7) & \min(0.2,0.5) \\ \min(0.3,0.2) & \min(0.3,0.3) & \dots & \min(0.3,0.7) & \min(0.3,0.5) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \min(0.7,0.2) & \min(0.7,0.3) & \vdots & \min(0.7,0.7) & \min(0.7,0.5) \\ \min(0.5,0.2) & \min(0.5,0.3) & \vdots & \min(0.5,0.7) & \min(0.5,0.5) \end{bmatrix}$$

Here min $\{A',A\}$ is a 9 × 9 fuzzy supermatrix partitioned between 3rd and 4th row , 5th and 6th row and similarly between 3rd and 4th column, 5th and 6th column.

$$\min\{A',A\} = \left[\begin{array}{ccc|cc|cccc} 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0.2 & 0.3 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0 & 0.3 & 0.6 & 0.4 & 0.6 & 0.6 & 0.5 \\ \hline 0.2 & 0.3 & 0 & 0.3 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.2 & 0.3 & 0 & 0.3 & 0.6 & 0.4 & 0.7 & 0.7 & 0.5 \\ 0.2 & 0.3 & 0 & 0.3 & 0.6 & 0.4 & 0.7 & 0.7 & 0.5 \\ 0.2 & 0.3 & 0 & 0.3 & 0.5 & 0.4 & 0.5 & 0.5 & 0.5 \end{array}\right]$$

$$= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B'_{12} & B_{22} & B_{23} \\ B'_{13} & B'_{23} & B_{33} \end{bmatrix}$$

where

$$B_{11} = \min\{A_1', A_1\}$$
$$B_{12} = \min\{A_1', A_2\}$$
$$B_{13} = \min\{A_1', A_3\}$$
$$B_{12}' = \min\{A_2', A_1\} = \min\{\min(A_1', A_2)'\}$$
$$B_{22} = \min\{A_2', A_2\}$$
$$B_{23} = \min\{A_2', A_3\}$$
$$B_{13}' = \min\{A_3', A_1\} = \min\{\min(A_1', A_3)'\}$$
$$B_{23}' = \min\{A_3', A_2\} = \min\{\min(A_2', A_3)'\}$$
$$B_{33} = \min\{A_3', A_3\}$$

Thus $A'.A$ is a 9 × 9 super fuzzy matrix. Further $A'.A$ under the super pseduo product is a symmetric fuzzy square supermatrix i.e., $A'.A$ is a symmetric fuzzy supermatrix. Next we define the notion of square fuzzy symmetric supermatrix but before, firstly let us define the notion of square fuzzy supermatrix.

Definition 5.3.6 [4]: Square Fuzzy Supermatrix

Let A is $n \times n$ square fuzzy matrix. Then A is called a square fuzzy supermatrix or super fuzzy square matrix, if A can be partitioned arbitrarily between the columns i_1 and i_{1+1} , i_2 and i_2 and i_{2+1}, …,i_r and i_{r+1} and similarly between the rows i_1 and i_{1+1} , i_2 and i_2 and i_{2+1}, …,i_r and i_{r+1} (r+1< n) We illustrate this by a following example:

Example 5.3.6: Let A be a fuzzy square matrix

$$A = \left[\begin{array}{cc|c|ccc} 0.2 & 0.4 & 1 & 0 & 0.3 & 0.9 \\ 0.3 & 0.5 & 0.1 & 0.2 & 0.3 & 0.8 \\ \hline 0.5 & 0.6 & 0.3 & 0.4 & 0.1 & 0.3 \\ \hline 0.3 & 0.4 & 0.1 & 0.4 & 0.5 & 0.4 \\ 0.4 & 0 & 0.3 & 0 & 0.6 & 0.3 \\ 0.3 & 0.5 & 0.9 & 0.6 & 0 & 1 \end{array}\right]$$

The 6 ×6 square fuzzy matrix has been partitioned between the columns 2nd and 3rd, 3rd and 4th and also partitioned between the rows 2nd and 3rd, 3rd and 4th . Thus A is a square fuzzy supermatrix. Now having defined a square fuzzy supermatrix, we next proceed on to define the notion of symmetric square fuzzy supermatrix.

Definition 5.3.7 [4]: Square Fuzzy Symmetric Supermatrix

Let A be a fuzzy super square matrix or fuzzy square supermatrix. Then A is called as a symmetyric fuzzy super square matrix or a fuzzy symmetric super square matrix or a symmetric square super fuzzy matrix , if

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

is such that A_{ii} are square fuzzy matrices and each of these fuzzy square matrices are symmetric square matrices and diagonal submatrices A_{ii} are symmetric square fuzzy matrices i.e., $A'_{ii} = A_{ii}$ and non diagonal fuzzy submatrices are symmetric about the diagonal i.e., $A'_{ij} = A_{ji}$ for $1 \le i, j \le n$.

We illustrate this by the following example:

$$A = \left[\begin{array}{ccc|cc|cccc} 0.3 & 0.4 & 0.5 & 0.3 & 0 & 0.3 & 0 & 0.3 & 0.8 \\ 0.3 & 0.6 & 0.2 & 0.5 & 0.2 & 0.4 & 0.1 & 0.5 & 0.7 \\ 0.4 & 0.5 & 0.1 & 0.2 & 0.2 & 0.6 & 0.5 & 0.2 & 0.6 \\ \hline 0.4 & 0.7 & 0.1 & 0.3 & 0.2 & 0.4 & 0.3 & 0.4 & 0.3 \\ 1 & 0.2 & 1 & 0.4 & 0.4 & 0.1 & 0.2 & 0.7 & 0.3 \\ \hline 0.2 & 0.3 & 0.1 & 0.6 & 0.7 & 0 & 1 & 0.9 & 0.4 \\ 0.5 & 0.6 & 0.5 & 0.1 & 0.2 & 0.2 & 0.1 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.8 & 0.6 & 0.2 & 0.3 & 0.2 & 0.4 \\ 1 & 0 & 0.3 & 0.4 & 0.3 & 0.5 & 0.4 & 0 & 0.5 \end{array}\right]$$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Clearly A is a square fuzzy supermatrix. For it is 9×9 fuzzy matrix partitioned between the columns 3rd and 4th , 5th and 6th and also partitioned the rows 3rd and 4th , 5th and 6th . Further the diagonal fuzzy matrices $A_{11}, A_{22}, \ldots, A_{33}$ are all symmetric fuzzy matrices and non-diagonal fuzzy matrices are symmetric about the diagonal i.e.,

$A'_{12} = A_{21}$, $A'_{13} = A_{31}$, $A'_{23} = A_{32}$.

Thus A is symmetric fuzzy supermatrix.

Also we have shown that if A is a fuzzy super row vector and A', the transpose of A is a fuzzy super column vector then $A'A$ under the super pseduo product is a symmetric fuzzy square supermatrix i.e., $A'A$ is symmetric fuzzy supermatrix.

Thus we can say if one wants to construct fuzzy symmetric supermatrices then one can take a fuzzy super row vector A and find the pseudo product of A' and A. Then $A'A$ will always be a symmetric fuzzy supermatrix.

Chapter 6

SUMMARY AND CONCLUSION

6.1 Introduction

George Cantor (1845-1918), a renowned German mathematician, was the first to start the formulation of set theory. The first axiomatic study was attempted by G. Frege in 1879. But in 1902, this theory received a setback. Bertrand Russel, an English mathematician and Philosopher, showed that Frege's system was inconsistent by devising a contradiction, now called Russel's paradox. In 1908, Zermelo proposed a theory of sets which was later improved by Fraenkel and Skolem. Most parts of present day mathematics are based on Zermelo's set theory.

Considering the important role that vagueness and inexactitude play in human decision making, Professor Loffi A. Zadeh [15] through a seminal paper in 1965, laid the foundation of fuzzy set theory which gave a form of mathematical precision to human cognitive processes that in many ways are imprecise and ambiguous by the standards of classical mathematics.

The fuzzy set theory is further related to fuzzy matrix in the same way as the set theory is related to matrix theory. Kim [11] defined the determinant of a square fuzzy matrix. The adjoint of a square matrix theory. The adjoint of a square fuzzy matrix is defined by Thomason [13] and Kim [11].

The concept of supermatrix for social scientists was first introduced by Paul Horst, "Matrix Algebra for social scientists" [2]. This concept was then observed in the light of fuzzy matrices to introduce fuzzy supermatrices.

6.2 Summary

This section recalls the summary of the work done in the previous five chapters as follow:

Chapter 1 (Binary And Fuzzy Set Theory) sets the platform for matrix theory by introducing set theory. The aspects of binary set theory involving set operations and algebraic properties of sets are explained which in essence, from the prerequisites for the similar understanding of fuzzy sets. Some examples have also been shared to understand fuzzy set theory.

Chapter 2 (Fuzzy Matrix Theory) consists of fuzzy matrix theory involving the types and equality of matrices. Moreover, operations on these matrices including transpose of these matrices are explained.

Chapter 3 (The Determinant Theory of a Square Fuzzy Matrix) contains the determinant theory of square fuzzy matrix along with their properties including another statement, illustration, corollary parts and remarks, wherever observed. Moreover, some theorems for fuzzy matrices is considered.

Chapter 4 (The Adjoint Theory of a Square Fuzzy Matrix) deals with the adjoint theory of square fuzzy matrix with their properties including corollary parts and remarks, wherever observed. Moreover, some special properties of square fuzzy matrices such as symmetry, reflexivity, transitivity, circularity and idempotence are dealt with and carried over to the adjoint matrix, which can be easily understand with the help of given examples.

Chapter 5 (Super and Super Fuzzy Matrix Theory) concerns with supermatrix and fuzzy supermatrix theory. Moreover operations defined on these matrices including transpose and then two types of products are also explained. Most of the chapter is tried to develop and explain with the help of examples for the sake of simplification.

6.3 Conclusion

The work being presented in the thesis is devoted to ‘The Study of Fuzzy, Super and Super Fuzzy Matrix Theory’. The study being comparative shows vividly the difference between fuzzy, super and super fuzzy matrices. Moreover operations defined on fuzzy, super and super fuzzy matrices are involved which shows the comparative working of these matrix theories. This comparative study also includes the determinant and adjoint theory of square fuzzy matrix. Only those operations on super fuzzy matrices are provided which are essential for developing super fuzzy multi expert models.

The Comparative study done in this illustration is complete in itself but not complete in all regards. Beyond the thesis work, we can describe simple fuzzy matrix model, fuzzy cognitive maps, bidirectional associative memories model, fuuzy associative memories model and comparatively new super fuzzy relational maps model, new super fuzzy bidirectional and associative memories models can be introduced which will be highly useful to social scientists who wish to work in future with multi expert models to analyze their problems. All in all, this will certainly be a boon not only to social scientists but also to mathematicians, engineers, doctors, researchers and students. Its influence is certain to grow more in the decade to come.

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