

MULTI-OBJECTIVE INTEGER PROGRAMMING PROBLEMS

A Thesis

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by

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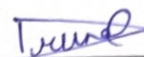
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March 26, 2024

Certificate

I hereby certify that the work, which is being presented in the thesis, entitled **MULTI-OBJECTIVE INTEGER PROGRAMMING PROBLEMS**, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work carried out during the period **July 24, 2018 to March 26, 2024** under the supervision of **Dr. Vikas Sharma**, Assistant Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

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Dr. Vikas Sharma

Supervisor

.....*dedicated to my parents*

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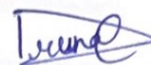
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Abstract

The present thesis aims to develop algorithms for solving various multi-objective integer programming problems that arise in multi-criteria decision-making situations. The thesis consists of seven chapters, each addressing different aspects of the multi-objective integer programming problem.

Chapter 1 is introductory, which provides an overview of multi-objective integer programming problems. An extensive literature survey related to multi-objective integer programming problems is conducted in this section. Further, the motivation for optimizing a function over the efficient set of multi-objective integer programming problem is discussed in this chapter. A brief summary of the thesis is also presented in this chapter.

In Chapter 2, a method for solving the bi-objective integer linear programming problem is discussed. The main focus of this chapter is to effectively implement the ϵ -constraint technique to produce a complete set of non-dominated points. The convergence of the algorithm has been established theoretically. Further, a comparative study of some existing algorithms has also been made.

In Chapter 3, an algorithm for finding all efficient solutions to a multi-objective integer quadratic programming problem is presented. The proposed algorithm is based on the aspect that efficient solutions of a multi-objective integer quadratic programming problem can be obtained by enumerating ranked solutions of an integer quadratic programming problem. For determining ranked solutions of an integer quadratic programming problem, we have constructed a related integer linear programming problem. Theoretically, we have shown that the developed method generates the set of all efficient solutions in a finite number of steps. Numerically, we have elaborated the working of our algorithm and compared our results with existing algorithms. Further, we have analyzed that the developed method is efficient for solving a multi-objective integer quadratic programming problem with a large number of constraints, variables and objectives.

In Chapter 4, we have discussed that a simplex like algorithm to solve an indefinite quadratic fractional programming problem proposed in [102] fails to find its optimal solution and so it may not generate the actual set of efficient solutions of the corresponding multi-objective integer indefinite quadratic fractional programs. A counter example in support of this argument is also given. Further, a novel method for solving a multi-objective integer quadratic fractional programming problem is also discussed. To tackle this problem, we have utilized a linear fractional function (or linear function), which provides a lower

bound to one of the objective functions of the multi-objective integer quadratic fractional programming problem. In the developed method, we have used successive optimization of integer linear fractional programming problem (or integer linear programming problem) for finding efficient solutions. Theoretically, we have shown that the proposed method generates the complete set of all efficient solutions in finite iterations. Numerical examples are also included in this chapter to elaborate working of the proposed algorithm. Further, we have compared the proposed method with existing methods using numerical examples. We have also incorporated computational results to analyze the performance of the developed algorithm for many variables, constraints, and objectives.

In Chapter 5, we have proposed two algorithms for optimizing a linear function over the efficient set. The optimization of a function over the efficient set is an interesting approach in decision-making situations. It helps a decision-maker to discriminate among efficient solutions and choose his preferred solution. In the first algorithm, ranked solutions of an integer linear programming problem are utilized for optimizing a linear function over the efficient set of a multi-objective integer linear programming problem. For enumerating ranked solutions of an integer linear programming problem, successive optimization of an integer linear programming problem is used. The second algorithm follow a two way procedure. From one side it finds ranked solution and from the other side efficient solutions are generated and a lower bound is calculated. This lower bound keeps on shrinking at each stage and provides an estimate on the number of ranked solutions to be calculated. The algorithm stops after calculating ranked solutions upto this lower bound. Theoretically, we have shown that the proposed algorithms optimize a linear function over the efficient set of a multi-objective integer linear programming problem in a finite number of iterations. Further, we have compared the working of proposed algorithms with existing algorithms using numerical examples taken from the literature. We have also included a comparison with existing methods based on computational experiments using large number of variables and objectives.

In Chapter 6, a novel approach to optimize a general quadratic function over the efficient set of a multi-objective integer linear programming problem is presented. The proposed algorithm obtains a globally optimal solution by systematically scanning ranked solutions of an integer quadratic programming problem until the efficiency condition is satisfied. The associated quadratic programming problem is solved by approximating its solutions from a related integer linear programming problem. The convergence of our algorithm is established theoretically. Numerical examples and computational results are also included in the chapter to elaborate on the working of the developed algorithm.

In Chapter 7, we have proposed a scheme for solving an integer quadratic fractional pro-

gramming problem over the efficient set of a multi-objective integer linear programming problem without computing all the elements of the efficient set. The proposed methodology combines a criteria space search algorithm, a ranking approach, and efficiency tests to solve the problem efficiently within a finite number of iterations. The algorithm iteratively optimizes an integer quadratic fractional programming problem by utilizing ranked solutions from a related integer linear fractional programming problem (or linear programming problem). The integration of a criteria space search algorithm, a ranking approach, and efficiency tests avoids the enumeration of all ranked solutions of the integer quadratic fractional programming problem and all efficient solutions of the multi-objective integer linear programming problem, which reduces the computational complexity. Theoretical analysis establishes the convergence of the proposed algorithm, while numerical and computational experiments validate its effectiveness.

List of Publications

1. Prerna and Vikas Sharma, *An algorithm for bi-objective integer linear programming problem*, **Filomat**, 2022, 36(16),5641-5651, (SCI, Impact factor: **0.8**).
2. Prerna Kushwah and Vikas Sharma, *A note on solving multi-objective integer indefinite quadratic fractional programs*, **Annals of Operations Research**, 2020, 289(2),459-462,doi:10.1007/s10479-019-03408-7 (SCI, Impact factor: **4.8**).
3. Prerna Kushwah and Vikas Sharma, *An algorithm to solve multi-objective integer quadratic programming problem*, **Annals of Operations Research**, 2024, 332, 433–459. (SCI, Impact factor: **4.8**).
4. Prerna and Vikas Sharma, *Optimization of a linear function over an integer efficient set*, **Journal of Industrial and Management Optimization**, 2023,19(12): 8633-8656,doi: 10.3934/jimo.2023055 (SCI, Impact factor: **1.3**).
5. Prerna and Vikas Sharma, *Optimization of a quadratic programming problem over an integer efficient set*, **Journal of Computational and Applied Mathematics**, 2024, 441, 115651, <https://doi.org/10.1016/j.cam.2023.115651> (SCI, Impact factor: **2.4**).

Communicated Papers/ Manuscripts under preparation

- Multi-objective Integer Quadratic Fractional Programming Problem.
- Optimization of a Quadratic Fractional Function over an Integer Efficient Set.

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- Presented a paper on “Multi- objective integer quadratic programming problem” in International Symposium on Optimization Theory and its Applications (ISOTA-2023), TIET, Patiala (April 13-15,2023).
- Presented a paper on “Multi- objective integer quadratic fractional programming problem” in International Symposium on Applied Optimization and Game Theoretic Models for Decision Making, **Indian Statistical Institute**, Delhi Centre (February 1-3, 2023).
- Presented a paper on “Optimization of a quadratic function over an integer efficient set” in International Conference on Computational and Mathematical Methods in Science and Engineering held at Rota, Cadiz, **Spain** (July 3-8, 2022).

- Presented a paper on “Optimization of a bi-objective linear function over an efficient set” in the 4th International Conference on Frontiers in industrial and applied mathematics organized by SLIET Longowal (December 21-22, 2021).
- Presented a paper on “Method for solving nonlinear multiobjective integer optimization problem” in International Conference on Advances in Differential Equations and Numerical Analysis (ADENA2020) held at **IIT Guwahati** (October 12 -14, 2020).
- Presented a paper on “An algorithm to solve multi-objective integer quadratic programming problem” in 23rd Punjab Science Congress held at SLIET, Longowal (February 7-9, 2020).

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List of Abbreviations

<i>DM</i>	Decision maker
<i>LP</i>	Linear Programming Problem
<i>NLP</i>	Non-Linear Programming Problem
<i>FP</i>	Fractional Programming Problem
<i>LFP</i>	Linear Fractional Programming Problem
<i>QP</i>	Quadratic Programming Problem
<i>QAP</i>	Quadratic Assignment Programming Problem
<i>QFP</i>	Quadratic Fractional Programming Problem
<i>IdQP</i>	Indefinite Quadratic Programming Problem
<i>IdQFP</i>	Indefinite Quadratic Fractional Programming Problem
<i>ILP</i>	Integer Linear Programming Problem
<i>INLFP</i>	Integer Non- Linear Fractional Programming Problem
<i>ILFP</i>	Integer Linear Fractional Programming Problem
<i>IIdQP</i>	Integer Indefinite Quadratic Programming Problem
<i>IQP</i>	Integer Quadratic Programming Problem
<i>IQFP</i>	Integer Quadratic Fractional Programming Problem
<i>SP</i>	Single objective Programming Problem
<i>BP</i>	Bi-objective Programming Problem
<i>MP</i>	Multi-objective Programming Problem
<i>MLP</i>	Multi-objective Linear Programming Problem
<i>MNLP</i>	Multi-objective Non-linear Programming Problem
<i>MFP</i>	Multi-objective Fractional Programming Problem
<i>MLFP</i>	Multi-objective Linear Fractional Programming Problem
<i>MQP</i>	Multi-objective Quadratic Programming Problem
<i>MQAP</i>	Multi-objective Quadratic Assignment Problem
<i>MQFP</i>	Multi-objective Quadratic Fractional Programming Problem
<i>SIP</i>	Single objective integer Programming Problem
<i>MIP</i>	Multi-objective Integer Programming Problem
<i>BILP</i>	Bi-objective Integer Linear Programming Problem
<i>MILP</i>	Multi-objective Integer Linear Programming Problem
<i>MINLP</i>	Multi-objective Integer Non-Linear Programming Problem
<i>MILFP</i>	Multi-objective Integer Linear Fractional Programming Problem
<i>MINLFP</i>	Multi-objective Integer Non-Linear Fractional Programming Problem
<i>MIQP</i>	Multi-objective Integer Quadratic Programming Problem

$MII dQP$	Multi-objective Integer Indefinite Quadratic Programming Problem
$BIQFP$	Bi-objective Integer Quadratic Fractional Programming Problem
$MIQFP$	Multi-objective Integer Quadratic Fractional Programming Problem
$MII dQFP$	Multi-objective Integer Indefinite Quadratic Fractional Programming Problem
ILP_E	Integer Linear Programming Problem over Efficient set
IQP_E	Integer Quadratic Programming Problem over Efficient set
$IQFP_E$	Integer Quadratic Fractional Programming Problem over Efficient set

List of Notations

$\mathbb{Z}$	set of integers.
$\mathbb{R}$	set of real numbers.
$\mathbb{R}_+$	set of positive real numbers.
$\mathbb{Z}_+$	set of positive integers.
m	number of constraints in <i>MP</i> .
n	number of variables in <i>MP</i> .
s	number of objectives in <i>MP</i> .
$X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$	variable vector (point in decision space).
$\max_{X \in S} (f^1(X), f^2(X) \dots f^s(X))$	<i>MP</i> , where $(f^1(X), f^2(X) \dots f^s(X)) \in \mathbb{R}^s$.
$E \subset S$	set of efficient solutions of <i>MP</i> .
$\max_{X \in \chi} (f^1(X), f^2(X), \dots, f^s(X))$	<i>MIP</i> , where $(f^1(X), f^2(X) \dots f^s(X)) \in \mathbb{R}^s$.
$\chi = S \cap \mathbb{Z}^n$	feasible region of <i>MIP</i> .
$A \in \mathbb{Z}^{m \times n}$	constraint matrix (in case of linear constraints).
$b \in \mathbb{Z}^m$	right hand side vector of linear constraints.
$S = \{X \in \mathbb{R}^{n \times 1} \mid AX = b, X \geq 0\}$	feasible region of <i>LP</i> , <i>QP</i> , <i>LFP</i> , <i>QFP</i> , <i>MLP</i> , <i>MQP</i> , <i>MLFP</i> and <i>MQFP</i> .
$\chi = S \cap \mathbb{Z}^n$	feasible region of <i>ILP</i> , <i>IQP</i> , <i>ILFP</i> , <i>IQFP</i> ,
(where $S = \{X \in \mathbb{R}^{n \times 1} \mid AX = b, X \geq 0\}$)	<i>MILP</i> , <i>MIQP</i> , <i>MILFP</i> and <i>MIQFP</i> .
$\max_{X \in \chi} (C^1 X, C^2 X \dots C^s X)$	<i>MILP</i> .
$C^i X$	i^{th} objective function of <i>MILP</i> .
$CX = (C^1 X, C^2 X, \dots, C^s X)$	a criterion space point of <i>MILP</i> .
$\lambda = (\lambda^1, \lambda^2, \dots, \lambda^s)$	weight vector of <i>MILP</i> .
$C = (C^1, C^2, \dots, C^s) \in \mathbb{Z}^{s \times n}$	objective function vector of <i>MILP</i> .
$\frac{C^i X + \alpha^i}{P^i X + \beta^i}$	i^{th} objective function of <i>MILFP</i> , where $C^i, P^i \in \mathbb{Z}^{1 \times n}$, $\alpha^i, \beta^i \in \mathbb{Z}$, $P^i X + \beta^i > 0$ for all $X \in \chi$.
$X^T D^i X + C^i X + \alpha^i$	i^{th} objective function of <i>MIQP</i> , where $D^i \in \mathbb{Z}^{n \times n}$, $C^i \in \mathbb{Z}^{1 \times n}$, $\alpha^i \in \mathbb{Z}$ and D^i is a symmetric matrix.
$\frac{X^T D^i X + C^i X + \alpha^i}{X^T Q^i X + P^i X + \beta^i}$	i^{th} objective function of <i>MIQFP</i> , where $D^i, Q^i \in \mathbb{Z}^{n \times n}$, $C^i, P^i \in \mathbb{Z}^{1 \times n}$, $\alpha^i, \beta^i \in \mathbb{Z}$, $X^T Q^i X + P^i X + \beta^i > 0$ for all $X \in \chi$,

$\max_{X \in \mathcal{X}} CX$	and D^i and Q^i are symmetric matrices.
$CX = C^1X$	ILP , where $C = C^1 \in \mathbb{Z}^{1 \times n}$.
$\max_{X \in S} CX$	objective function value of ILP .
$\max_{X \in S_t} CX$	LP , where $C = C^1 \in \mathbb{Z}^{1 \times n}$.
	LP^t , where $C = C^1 \in \mathbb{Z}^{1 \times n}$.
$\hat{X}$	where S^t obtained from S after adding t - Gomory cuts.
B	optimal basic feasible solution of LP^t ,
x_j	basis corresponding to $\hat{X}$.
a_j	j^{th} component of $\hat{X}$.
y_j	activity vector of x_j with respect to S^t .
I	$B^{-1}a_j$.
$\bar{I}$	$\{j \mid a_j \in B\}$.
z_j	$\{j \mid a_j \notin B\}$.
D	$\sum_{i \in I} c_i y_{i,j}$, where c_i is the i^{th} component of vector C^1 .
E_j	$\{j \in \bar{I} \mid z_j - c_j = 0\}$.
	edge incident to X for some $j \in \bar{I}$.

Chapter 1

Introduction

1.1 Introduction

In mathematical optimization one aims at minimizing or maximizing one or more than one objective functions over some feasible set. The optimization techniques have been applied in diverse fields such as transportation [10, 44, 81, 105, 129], robust portfolio optimization model [61, 128], supply chain network [21], vehicle routing problem [56, 58, 149], scheduling [79] and Covid-19 modeling [57].

In the realm of a single objective programming problem (SP), the notion of an optimal solution is straightforward. However, multiple objectives lead to many intriguing questions that are not present in SP . For example, consider a transportation problem with nine feasible solutions (as shown in Figure 1.1). In this problem, if one is only interested in minimizing the transportation cost (or transportation time), then it will be SP , whereas, if one wants to minimize both transportation time as well as transportation cost, then it becomes a multi-objective programming problem (MP). If the decision-maker (DM) selects a solution corresponding to minimum transportation time, then he may not get minimum transportation cost, and if DM selects a solution corresponding to minimum transportation cost, then he may not get minimum transportation time, as shown in Figure 1.1. Therefore, it is impossible to optimize both time and cost simultaneously.

Mathematically, MP can be expressed as :

$$(MP) \quad \begin{cases} \max (f^1(X), f^2(X) \dots f^s(X)) \\ \text{s.t} \quad X \in S \subset \mathbb{R}_+^n. \end{cases}$$

In SP , feasible region (set S) mapped to $\mathbb{R}$. However, for MP , feasible region is mapped to a space in $\mathbb{R}^s$, which is known as criterion space (or objective function space). The collection of all images of feasible solutions in criterion space forms a set, known as feasible criterion vectors of MP .

There are two types of feasible criterion vectors - Dominated Criterion Vector and Non-

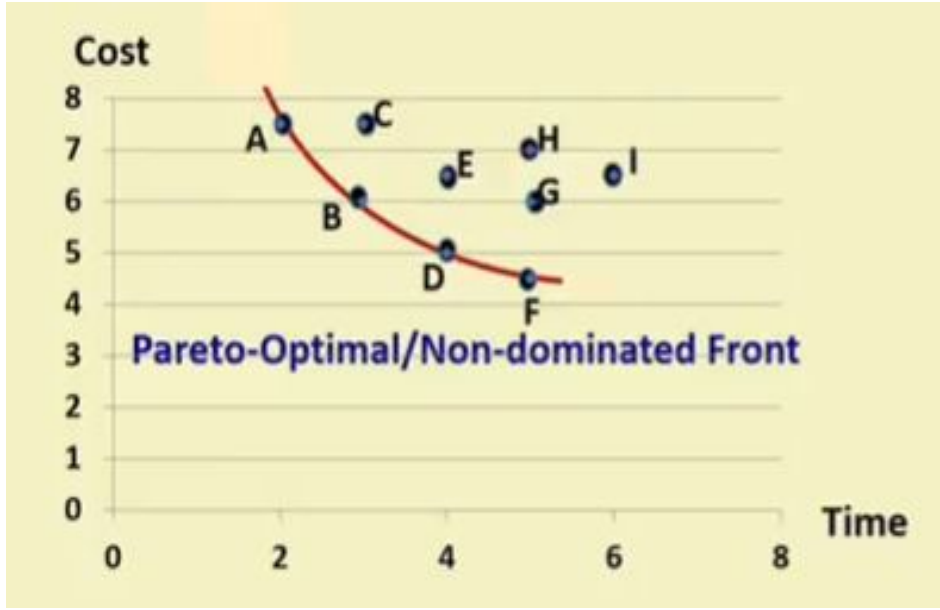


Figure 1.1: Transportation problem

Dominated Criterion vector.

Definition 1.1.1 A feasible criterion vector $(f^1(\hat{X}), f^2(\hat{X}), \dots, f^s(\hat{X})) \in \mathbb{R}^s$ is a dominated criterion vector of MP, if $(f^1(\hat{X}), f^2(\hat{X}), \dots, f^s(\hat{X})) \in \mathbb{R}^s$ is dominated by $(f^1(\hat{Y}), f^2(\hat{Y}), \dots, f^s(\hat{Y})) \in \mathbb{R}^s$ for some $\hat{Y} \in S$, i.e., $f^i(\hat{Y}) \geq f^i(\hat{X}) \forall i \in \{1, 2, \dots, s\}$ and $f^i(\hat{Y}) > f^i(\hat{X})$ for some $i \in \{1, 2, \dots, s\}$.

Definition 1.1.2 If $(f^1(\hat{X}), f^2(\hat{X}), \dots, f^s(\hat{X})) \in \mathbb{R}^s$ is not dominated by $(f^1(\hat{Y}), f^2(\hat{Y}), \dots, f^s(\hat{Y})) \in \mathbb{R}^s \forall \hat{Y} \in S$, then it is called a nondominated criterion vector.

A nondominated criterion vector represents a point, where enhancing value of one criterion is unachievable without sacrificing another criterion. The collection of all nondominated criterion vectors forms a nondominated front (or Pareto-optimal front). The nondominated front of a MP is shown in Figure 1.1.

Generally, objective functions involved in MP are conflicting, i.e., an increase in the value of one objective leads to an inferior value of another objective function. Therefore, MP focuses on generating nondominated vectors. The inverse image of a nondominated criterion vector in the feasible region is known as an efficient solution.

Definition 1.1.3 A feasible solution $\hat{Y} \in S$ is called efficient solution (Pareto optimal solution) of MP if there exists no feasible solution $\hat{X} \in S$ such that $f^i(\hat{X}) \geq f^i(\hat{Y}) \forall i \in \{1, 2, \dots, s\}$ and $f^i(\hat{X}) \neq f^i(\hat{Y})$ for at least one $i \in \{1, 2, \dots, s\}$. A feasible solution $\hat{Y} \in S$ is called a weak efficient solution of MP if there exists no feasible solution $\hat{X} \in S$

such that $f^i(\hat{X}) > f^i(\hat{Y}) \forall i \in \{1, 2, \dots, s\}$.

In criterion space, the corresponding point of an efficient solution is a nondominated vector, as any other feasible criterion vector does not dominate it. If $Y \in S$ is a (weak) efficient solution of MP , then a criterion vector $(f^1(Y), f^2(Y), \dots, f^s(Y)) \in \mathbb{R}^s$ is known as (weak) nondominated vector. Let E be the collection of all efficient solutions of MP , then the set of nondominated criterion vectors of MP is defined as

$$\{ (f^1(X), f^2(X) \dots f^s(X)) \in \mathbb{R}^s \mid X \in E \}.$$

The MP has numerous applications in various domains, including the world of the internet, bio-medicine, management studies, game theory, and engineering. But, solving MP is not a simple job. Since there are typically exponentially large efficient solutions (Pareto optimal solutions), trying to find every efficient solution is costly and time-consuming. Further, it is NP-hard to check whether a point belongs to a nondominated front, even for straightforward problems [37].

1.1.1 Multi-objective integer programming

In many decision-making situations, decision variables are restricted to be integers and decision depend upon various objectives. These types of decision-making situations can be trackled by solving multi-objective integer programming problem (MIP). Mathematically, it is defined as :

$$(MIP) \quad \max_{X \in \mathcal{X} = S \cap \mathbb{Z}^n} (f^1(X), f^2(X), \dots, f^s(X))$$

Since integer points are discrete, feasible region of MIP is non-convex. Due to the non-convexity of the feasible region, it is difficult to study MIP . However, MIP frequently arises in financial analysis, traveling salesman problems, cutting stock problems, agricultural planning, capital budgeting problems, knapsack problems, sequencing, fixed charge problems, separable programming problems, lockbox problems, and portfolio selection problems.

In MIP , if feasible region (set S) is defined by linear constraints, $s = 1$, and $f^1(X) = C^1 X$, where $C^1 \in \mathbb{R}^{1 \times n}$. Then we have an integer linear programming problem (ILP).

Integer linear programming

In mathematical terms, *ILP* can be written as :

$$(ILP) \quad \begin{cases} \max & f^1(X) = C^1 X \\ \text{s.t} & X \in \mathcal{X}, \end{cases}$$

where $\mathcal{X} = S \cap \mathbb{Z}^n$, $S = \{X \in \mathbb{R}^n \mid AX = b, X \geq 0\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $C^1 \in \mathbb{R}^{1 \times n}$.

Generally, *ILP* is solved by solving the problem, as follows:

$$(LP) \quad \begin{cases} \max & f^1(X) = C^1 X \\ \text{s.t} & X \in S. \end{cases}$$

Remark 1.1.4 *Let $\hat{Y}$ be an optimal solution for *LP*. If $\hat{Y}$ is an integer vector, then $\hat{Y}$ is an optimal solution for *ILP*.*

The most popular techniques for finding an optimal solution of *ILP* are Gomory cutting plane technique and Branch and Bound method [13].

1. Cutting plane method

A cutting plane method is the first systematic technique developed by Gomory [62] to solve *ILP*. In this method, an appropriate cut and constraint are used for solving *ILP*. The procedure starts with the optimization of *LP*. If an optimal solution of *LP* meets integer requirements, then that optimal solution of *LP* will be an optimal solution of *ILP*. However, if an optimal solution of *LP* is a non-integer, then it will not be an optimal solution of *ILP*. In this situation, the feasible region of *LP* is appended by adding a cut (called Gomory cut) to get a new linear programming problem (*LP*₁). The Gomory cut deletes a part of a feasible region without deleting any integer feasible solution. After getting *LP*₁, we solve *LP*₁ and repeat the procedure. The number of Gomory cut constraints needed to solve *ILP* is always finite. However, large Gomory cuts may be required to solve even average size *ILP*. So, this method is computationally challenging.

2. Branch and Bound Method

An optimal solution of *ILP* can also be obtained by utilizing the Branch and Bound

method [13]. In this method, firstly we solve the following problem:

$$(LP) \quad \begin{cases} \max & f^1(X) = C^1 X \\ \text{s.t} & X \in S. \end{cases}$$

Let $Y = (y^1, y^2, \dots, y^n)$ is a non-integer optimal solution of LP with non-integer variable y^p for some $p \in \{1, 2, \dots, n\}$, then the non-integer variable y^p produces two branching sub-problems I and II :

$$I \quad \begin{cases} \max & C^1 X \\ \text{s.t} & \\ & X = (x_1, x_2, \dots, x_n) \in S, \\ & x^p \leq [y^p]. \end{cases} \quad II \quad \begin{cases} \max & C^1 X \\ \text{s.t} & \\ & X = (x_1, x_2, \dots, x_n) \in S, \\ & x^p \geq [y^p] + 1. \end{cases}$$

An optimal solution of ILP is either an optimal solution of I or II . If an optimal solution of I or II is a non-integer, we repeat the procedure until all branches are fathomed.

Edge exploring technique

One can solve LP using the Gomory cutting plane approach or the Branch and Bound technique to arrive at an optimal solution of ILP . Nevertheless, there might be other optimal solutions for ILP . The edge exploring technique can be utilized to find all optimal solutions.

Suppose LP^t be the linear programming problem obtained from LP after adding t -number of Gomory cuts and its optimal solution is an integer vector. So, it can be interpreted that an optimal table of LP^t depicts a basic feasible solution of LP^t , which is also an integer vector. So, it provides an optimal solution to ILP . Call this optimal basic feasible solution of LP^t as $\hat{X}$. If in an optimal table of LP^t , $z_j - c_j = 0$ for some $j \in \bar{I}$, then LP^t may have alternate optimal solutions. The set of all integer optimal solutions of LP^t can be obtained by exploring all edges E_j incident to $\hat{X}$, for which $z_j - c_j = 0$, $j \in \bar{I}$. An edge E_{j^1} incident to X^1 for some $j^1 \in \bar{I}$ is defined in [1, 2, 45, 120] as :

$$E_{j^1} : \quad \begin{cases} x_i = x_i^1 - \theta_{j^1} y_{i,j^1} \text{ for } i \in I, \\ x_{j^1} = \theta_{j^1}, \\ x_j = 0 \text{ for } j \in \bar{I} - \{j^1\}, \end{cases} \quad (1.1.1)$$

where $0 < \theta_{j^1} \leq \min_{x_i \in B} \left\{ \frac{x_i^1}{y_{i,j^1}} \mid y_{i,j^1} > 0 \right\}$. Integer solutions on this edge can be generated by taking integral values of θ_{j^1} and $\theta_{j^1} y_{i,j^1}$. Further, the complete set of integer optimal solutions of ILP are obtained by exploring all such edges. The following example explains this process of obtaining the set of integer optimal solutions of ILP .

Example 1.1.1 Consider the problem :

$$(ILP) \quad \max_{(x_1, x_2) \in \chi} x_1 + x_2,$$

where $\chi = S \cap \mathbb{Z}^{2 \times 1}$, $S = \{(x_1, x_2) \mid 2x_1 + 2x_2 \leq 5, x_1, x_2 \geq 0\}$. To find optimal solutions of ILP , consider a relaxed problem, defined as:

$$(LP) \quad \max_{(x_1, x_2) \in S} x_1 + x_2.$$

By appending a Gomory's cut $-0.5 = x_4 - 0.5x_3$ to the set S , LP^1 is constructed from LP . An optimal solution of LP^1 is depicted in the Table 1.1.

	c_j	1	1	0	0
		x_1	x_2	x_3	x_4
$z_j - c_j$	2	0	0	0	1
x_1	2	1	1	0	1
x_3	1	0	0	1	-2

Table 1.1: Optimal Table of LP^1 (Example 1.1.1)

Since optimal solution is an integer vector, therefore $(2, 0)$ as an optimal solution of ILP . Further, it may be observed from Table 1.1 that $z_2 - c_2 = 0$ indicates the existence of alternate optimal solutions of LP^1 . The complete set of all integer optimal solutions of LP^1 can be obtained by exploring edge E_2 as given in the Formula 1.1.1.

By taking $\theta_2 = 2$ and 1, we get alternate optimal solutions of ILP as $(0, 2)$ and $(1, 1)$ respectively.

1.1.2 Multi-objective integer linear programming

The multi-objective integer linear programming problem can be written as

$$(MILP) \quad \begin{cases} \max(C^1 X, C^2 X \dots C^s X) \\ \text{s.t} \quad X \in \chi, \end{cases}$$

where $\chi = S \cap \mathbb{Z}^n$, $S = \{X \in \mathbb{R}^n \mid AX = b, X \geq 0\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $C^i \in \mathbb{R}^{1 \times n}$ for all $i \in \{1, 2, \dots, s\}$.

Airline crew scheduling problem [50] is one of the practical applications of *MILP*. In crew scheduling problems, the objective is to minimize the penalties for a schedule not being robust (i.e., likely to cause delays) and minimize the cost of the schedule.

Theorem 1.1.5 [82] *If $\hat{Y}$ is the unique optimal solution of P_i , defined as:*

$$(P_i) \quad \begin{cases} \max & f^i(X) = C^i X \\ \text{s.t.} & X \in \chi. \end{cases}$$

Then, $\hat{Y}$ is an efficient solution of *MILP*.

In the following, we discuss some of the most commonly used scalarization techniques for solving *MILP*. The scalarization technique transforms *MILP* into *ILP* that is solved repeatedly to find an efficient solution.

1. Weighted Sum Method:

In the weighted sum scalarization technique, the convex combinations of all s objective functions of *MILP* are considered as objective function of *ILP*, and feasible region of *ILP* is regarded as same as that of the feasible region of *MILP*. In the weighted sum method, the scalarized problem is formulated as

$$(P_\lambda) \quad \max_{X \in \chi} \sum_{i=1}^s \lambda^i C^i X.$$

Sylva and Crema [140, 141] in their papers used the weighted sum approach to optimize *MILP* and used the following result connecting *MILP* and P_λ .

Theorem 1.1.6 [140] *If $\lambda^i > 0$ for all i , then an optimal solution of P_λ will be an efficient solution of *MILP*.*

The converse of the above theorem might not be true. An unsupported efficient solution of *MILP*, which lies in the interior of the convex hull of S , cannot be found using this result.

Definition 1.1.7 *The efficient solution of *MILP*, which is an optimal solution of P_λ for any $\lambda > 0$ is called supported efficient solution, whereas efficient solution of *MILP* that is not an optimal solution of P_λ for any $\lambda = \{\lambda^1, \lambda^2, \dots, \lambda^s\} > 0$ is known to be non-supported efficient solution of *MILP*.*

Theorem 1.1.8 [140] Let $X^1, X^2, \dots, X^l$ be efficient solutions of $MILP$ and $D^j = \{X \in \mathbb{Z}^n | C^j X \leq C^j X^j \text{ for all } j \in \{1, 2, \dots, s\}\}$. If $\hat{Y}$ is an efficient solution of P_{D^l} :

$$(P_{D^l}) \quad \begin{cases} \max CX \\ X \in \chi - \cup_{j=1}^l D^j, \\ C = (C^1, C^2, \dots, C^s). \end{cases}$$

Then, $\hat{Y}$ is an efficient solution of $MILP$. Furthermore, if P_{D^l} is infeasible, then $\{CX^1, CX^2, \dots, CX^l\}$ is the complete set of nondominated criterion vectors for $MILP$.

Sylva and Crema [140] established a relation between an efficient solution of $MILP$ and an optimal solution of the following problem (P_{λ, D^l}) :

$$(P_{\lambda, D^l}) \quad \begin{cases} \max \sum_{i=1}^s \lambda^i C^i X \\ X \in \chi - \cup_{j=1}^l D^j. \end{cases}$$

Corollary 1.1.9 [140] Let $X^1, X^2, \dots, X^l$ be efficient solutions of $MILP$ and $D^j = \{X \in \mathbb{Z}^n | C^j X \leq C^j X^j \text{ for all } j \in \{1, 2, \dots, s\}\}$. If $\hat{X}$ is an optimal solution of P_{λ, D^l} , then $\hat{X}$ is an efficient solution of $MILP$.

In 2008, Sylva and Crema [141] established the following relation for finding efficient solutions of $MILP$.

Theorem 1.1.10 [141] Let $z = (z^1, z^2, \dots, z^s) \in \mathbb{R}^s$ be a fixed objective vector and $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^s) > 0$ is a fixed weight vector. If $\hat{Y}$ is an optimal solution of the following problem $(P_{\lambda, z})$:

$$(P_{\lambda, z}) \quad \begin{cases} \max \sum_{i=1}^s \lambda^i C^i X \\ C^i X \geq z^i \quad \forall i \in \{1, 2, \dots, s\} \\ X \in \chi, \end{cases}$$

then $\hat{Y}$ will be an efficient solution of $MILP$.

Further, Sylva and Crema [141] also discussed the following relation between the efficient set E and the set of all optimal solutions of $P_{\lambda, z}$.

Theorem 1.1.11 [141] The efficient set E is the set of all optimal solutions of $P_{\lambda, z}$ for any fixed $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^s) > 0$ and for all $z \in \mathbb{R}^s$.

2. ϵ Constraint Scalarization Method:

In this methodology, $MILP$ is transformed into ILP using the epsilon value as bound. Firstly, ϵ -constraint technique was discussed in [71]. In 1994, Neumayer and Schweigert [107] implemented this ϵ -constraint technique to the bi-objective transportation problem. Klein and Hannan [82] used the ϵ -constraint technique to sequentially generate the set of all nondominated points of $MILP$ by solving SP and by considering all other objective functions as constraints of a single objective programming problem. A variant of ϵ -constraint algorithm is discussed in [110]. In the ϵ constraint method, the scalarized problem is formulated as

$$(P_j) \quad \begin{cases} \max C^j X \\ \text{s.t} \\ C^i X \geq \epsilon^i, \quad i \neq j, \\ X \in \chi. \end{cases}$$

Since the ϵ constraint scalarization method converts $s - 1$ objectives of $MILP$ to constraints. So this method traps an efficient solution of $MILP$ in a reduced set of feasible solutions.

Theorem 1.1.12 *If the efficient set of $MILP$ is non-empty, then the ϵ constraint method generates an efficient solution. If the efficient set of $MILP$ is empty, then the ϵ constraint method obtains an unbounded objective function value.*

Theorem 1.1.13 [49] *Let $\hat{Y}$ be the unique optimal solution of P_j for some j . Then, $\hat{Y}$ is the efficient solution of $MILP$.*

Recently, this ϵ -constraint technique is discussed in [104] for solving $MILP$.

3. Benson method

In Benson [16] method, the weighted sum method is combined with the ϵ -constraint method. The scalarized problem of this method has a weighted sum objective and constraints applied on all objectives of $MILP$, and the scalarized problem is formulated as follows:

$$P(Y) \quad \begin{cases} \max \sum_{i=1}^s \Psi^i \\ C^i X - \Psi^i - C^i Y = 0, \\ \Psi^i \geq 0, \quad i \in \{1, 2, \dots, s\}, \\ X \in \chi. \end{cases}$$

Theorem 1.1.14 *A feasible solution Y of MILP is an efficient solution of MILP if and only if the optimal value of $P(Y)$ is zero.*

Theorem 1.1.15 *If $P(Y)$ has an optimal solution $(\hat{X}, \hat{\Psi})$ and optimal value of $P(Y)$ is finite, then $\hat{X}$ is an efficient solution of MILP.*

1.1.3 Multi-objective integer non-linear programming

In addition to linear case, many real-world problems can be viewed as non-linear programming problem as illustrated by [14]. Further, many decision-making situations involve multiple objectives and integer decision variables. Therefore, a multi-objective integer non-linear programming problem (*MINLP*) is an important problem. In this problem atleast one objective function $f^i(x)$ or constraint in *MIP* is non-linear.

One of the most significant and widely discussed problems in this category is *MILFP* and has unquestionable practical applications in various areas of finance [11], transportation [41] and manpower planning [84] etc. In mathematical terms, *MILFP* can be written as

$$(MILFP) \quad \begin{cases} \max f^i(X) = \frac{C^i X + \alpha^i}{P^i X + \beta^i}, & i \in \{1, 2, \dots, s\} \\ \text{s.t } X \in \chi, \end{cases}$$

where $\chi = S \cap \mathbb{Z}^n$, $S = \{X \in \mathbb{R}^n \mid AX = b, X \geq 0\}$, $C^i, P^i \in \mathbb{Z}^{1 \times n}$, $\alpha^i, \beta^i \in \mathbb{Z}$. We also assume the linear function $P^i X + \beta^i > 0$ for all $X \in \chi$.

Another important class of *MINLP* is multi-objective integer quadratic programming problem (*MIQP*). A quadratic programming problem (*QP*) is a non-linear programming problem in which all constraints are linear, and the objective function contains the quadratic term. The *QP* has wide practical applications. It finds applications in sequential quadratic programming [5, 7, 30], image and signal processing, the least square regression method, and the optimization of financial portfolios. Applications of *QP* in finance can be seen in [74]. The *QAP* found application in parallel and distributed computing, back-board wiring problems in electronics, scheduling, and location theory [115]. The linear multiplicative programs, which is related to quadratic programming used in plant layout design, VLSI chip design, robust optimization, network flow, portfolio, and financial optimization [26].

Mathematically, *MIQP* can be written as

$$(MIQP) \quad \max_{X \in \chi} (f^1(X), f^2(X), \dots, f^s(X)),$$

where $f^i(X) = X^T D^i X + C^i X + \alpha^i$, $D^i \in \mathbb{Z}^{n \times n}$, $C^i \in \mathbb{Z}^{1 \times n}$ and $\alpha^i \in \mathbb{R}$ for all $i \in \{1, 2, \dots, s\}$.

Further extension in this area is the study of multi-objective integer quadratic fractional programming problem (*MIQFP*). In this problem, objective functions are the ratio of two quadratic functions, decision variables are integers, and linear constraints define feasible region. The usage of this type of problem in the context of designing systems can be seen in [102, 131].

Mathematically, this problem can be defined as:

$$(MIQFP) \quad \begin{cases} \max (f^1(X), f^2(X), \dots, f^s(X)) \\ \text{s.t. } X \in \chi, \end{cases}$$

where $f^i(X) = \frac{X^T D^i X + C^i X + \alpha^i}{X^T Q^i X + P^i X + \beta^i}$, $\chi = S \cap \mathbb{Z}^n$, $S = \{X \in \mathbb{R}^n \mid AX = b, X \geq 0\}$, $D^i, Q^i \in \mathbb{R}^{n \times n}$, $C^i, P^i \in \mathbb{R}^{1 \times n}$ and $\alpha^i, \beta^i \in \mathbb{R}$ for all $i \in \{1, 2, \dots, s\}$. We also assume the quadratic function $X^T Q^i X + P^i X + \beta^i > 0$ for all $X \in \chi$.

MIQFP has been discussed by various researchers; among them Sharma, Dahiya, and Verma [131] used the ranking of *LFP* to solve *BIQFP*. Sharma [130] used the cutting plane technique to solve *MIQFP* and Mekhilef, Moulaï and Drici [102] used the successive optimization of *IQFP* to find all nondominated pairs of *MIQFP*.

1.1.4 Optimization of a function over the efficient set

The efficient set E of *MIP* mainly contains numerous elements. The existence of many elements in the set E confuses the decision-maker to make a decision. Also, great computational efforts are required to list efficient solutions of *MIP*. To alleviate this type of situation, optimizing a function over efficient set is valuable.

The optimization of a function over the efficient set of *MIP* is a vital approach in the field of *MIP*. In this programming problem, the function acts as a criterion for discriminating efficient solutions of *MIP*. One practical application of this is discussed in [17]. Benson [17] considered the problem of maximizing the profit of the production plan in which ten factories produce four different types of products while maintaining a high employment level at each factory. In this problem, instead of maximizing the profit, the aim is to maximize the gain over the efficient set of maximization problems whose criteria vector is the employment level at each factory. The optimization of a function over the efficient set can also be used in goal setting, ranking objective functions, and interactive algorithms. In mathematical terms, optimization of a preference function over the efficient set of *MIP*

can be written as:

$$(P_{MIP}) \quad \max_{X \in E} g^1(X)$$

where E is the efficient set of MIP and g^1 is the DM preference function. The solution space of preference functions over the efficient set would be the subset of the efficient set of MIP .

Researchers used the following efficiency test for solving P_{MIP} .

Theorem 1.1.16 *A point $Y \in \chi$ is an efficient solution of $MILP$ iff optimal value of $P(Y)$ is zero.*

$$P(Y) \quad \begin{cases} \max & \theta = \sum_{i=1}^s \Psi_i \\ s.t & \\ & CX = I\Psi + CY, \\ & X \in \chi, \Psi_i \geq 0 \forall i \in \{1, 2, \dots, s\} \end{cases}$$

where $C = (C^i)_{i=1}^s \in \mathbb{Z}^{s \times n}$, $\Psi = (\Psi^i)_{i=1}^s \in \mathbb{R}_+^s$, $I = (s \times s)$ identity matrix and $\mathbb{R}_+$ is the set of non negative real numbers.

For a point $Y \in \chi$ to be an efficient solution of $MILP$, Y must satisfy the above efficiency test discussed by Isermann [73]. Ecker and Kouada [48] also extended the Isermann [73] efficiency test to the following results.

Theorem 1.1.17 [48] *If $P(Y)$ has a finite non-zero maximum value at a point $\hat{X}$, then $\hat{X}$ is an efficient solution of $MILP$.*

Theorem 1.1.18 *If $P(Y)$ does not have a finite maximum value, then the efficient set of $MILP$ is empty.*

1.2 Literature Survey

In this section, we will discuss existing methods and algorithms related to MIP . First of all, we will describe various techniques for solving $MILP$, then we will present methods for solving $MINLP$. Further, a thorough literature review on optimizing a function over integer efficient set is presented in this section.

1.2.1 Techniques for solving MILP

In literature, researchers have developed various scalarization techniques and various interactive and non-interactive methods to tackle $MILP$.

Ranking method

To generate all efficient solutions of *MILP*, Gupta and Malhotra [68] propose a ranking approach. Their ranking method scan all integer points of the feasible region. In their method, at each stage new cuts are appended in the previous simplex table to find next best integer feasible solution. These integer solutions are obtained in ascending order of one of the objective function, then at each ranked solution objective function values is evaluated and the non dominated set is updated at each stage after removing dominated points.

In 2002, Abbas and Chaabane [1] presented a corrected form and an alternative method discussed by Gupta and Malhotra [68]. Abbas and Chaabane used simple pivoting procedures and cutting plane technique for determining efficient solutions of *MILP*. Further, they utilized an edge exploring technique for finding all alternate integer feasible solutions lying on explored edge. In their method, feasible region at each stage is reduced by appending the following cut:

$$\sum_{j \in N_k - \{j_k\}} x_j \geq 1,$$

where $j_k \in \tau_k$, $\tau_k = \{j \in N_k \mid z_{1,j} - c_{1,j} > 0 \text{ \& } z_{i,j} - c_{i,j} > 0 \text{ for at least one } i \in \{2, 3, \dots, s\}\}$, N_k is the set of non-basic variables in the simplex table corresponding to an efficient solution X^k obtained at k^{th} stage.

ϵ -constraint method

The ranking algorithm proposed by Gupta and Malhotra [68] requires to calculate all integer points of the feasible region. In order to avoid the enumeration of complete feasible region, Klein and Hannan [82] used the ϵ -constraint approach. Their method sequentially generates the set of all nondominated points of *MILP* by solving *SP* and by considering all other objective functions as constraints of *SP*. The algorithm generates a new efficient solution at each iteration by imposing constraints on the feasible region. The purpose of introducing additional constraints is to eliminate solutions dominated by known nondominated vectors. After adding a constraint, an *ILP* is solved over the truncated feasible region. The optimal solution of *ILP* over reduced feasible region provides new efficient solution.

Recently, Mesquita-Cunha, Figueira and Barbosa-Póvoa [104] have also used ϵ - constraint scalarization technique for finding efficient solutions of *MILP*.

Notably, the ϵ -constraint method has two advantageous features

1. This method generates a sequence of efficient solutions which are sorted by a selected

objective function value.

2. A representative subset of efficient solutions is generated by introducing additional constraints that permit only those feasible solutions that differ from obtained solutions in terms of criterion vector.

Sylva and Crema [140] highlighted the following limitations of Klein and Hannan [82] ϵ -constraint method.

1. In the Klein and Hannan [82] ϵ -constraint method, some desirable, efficient solutions could be generated in a later stage because efficient solutions are produced in the decreasing order of the objective function values of some fixed objective function.
2. Optimizing an objective function may not always yield an efficient solution as an alternative optimal solution could dominate the obtained optimal solution.

Weighted sum approach

Due to some limitations of Klein and Hannan [82] ϵ -constraint method, a weighted sum approach is developed by Sylva and Crema [140,141]. This method is a variation of Klein and Hannan [82] algorithm. In this method, instead of optimizing a single objective programming problem, Sylva and Crema [140] optimized a convex combination of all objectives. The first nondominated vector is generated by solving the problem, as follows:

$$(P_0) \quad \max\{\lambda^t Cx : Ax = b, x \geq 0, x \in Z^n\}$$

and after obtaining l^{th} efficient solution x^l a new problem is constructed by adding the constraint

$$(Cx)_k \geq ((Cx^l)_k + 1)y_k^l - M_k(1 - y_k^l),$$

$$\sum_{k=1}^p y_k^l \geq 1, \quad y_k^l \in \{0, 1\}, \quad \text{for } k = 1, 2, \dots, p$$

to the previously generated problem (P_{l-1}) . Here $-M_k$ is a lower bound on k^{th} objective function. At each iteration, this algorithm eliminates previously generated nondominated vectors and criterion vectors dominated by obtained nondominated vectors by introducing additional constraints into the weighted sum scalarized problem. The introduction of new constraints at each iteration enables to compute all nondominated vectors. This method of optimizing a weighted sum problem with additional constraints on the objectives is also employed in the subsequent paper [141].

Remarkably, an optimal solution of the weighted sum problem will be an efficient solu-

tion of *MILP*. So, the weighted sum approach generates an efficient solution at each iteration.

Remark 1.2.1 *If the objective function weights represents the linear estimate of the decision maker's utility function, then this method enables us to generate the most desirable solution at the first iteration, whereas the most desirable solution may be obtained by Klein and Hannan [82] ϵ -constraint method in later stage.*

A modified ϵ -constraint technique of Özlen and Azizoglu [110]

An algorithm studied by Özlen and Azizoglu [110] is a modified version of classical ϵ constraint method, which aims at generating a complete set of non dominated of *MILP*. In their algorithm, *MILP* is converted into *ILP* by considering one of the objectives as the objective function of this problem, and the rest all other objectives are considered as the constraints of *ILP*, which depends upon the objective efficiency range. For a two objective case this algorithm requires to solve a problem, defined as :

$$\begin{cases} \min f(x) + wh(x), \\ \text{Subject to} \\ h \leq l_2, \quad x \in \mathcal{X} \end{cases}$$

where w is some weight function and the bound l_2 is updated at each stage.

Two-phase method

Przybylski, Gandibleux and Ehrgott [124] used a two-phase method for finding the set of all nondominated vectors of *MILP*. In the first phase, the weighted sum scalarization approach is applied to find all supported nondominated vectors, and in the second phase, the set of all unsupported nondominated vectors is generated.

Lokman and Köksalan [95] method

Lokman and Köksalan [95] proposed two algorithms for finding nondominated points of *MILP*. Their algorithms improve the algorithms of Sylva and Crema [140, 141] by decreasing constraints and binary variables at each iteration.

A criterion space search algorithm for bi-objective integer programming

A criterion space search algorithm [23] is used for finding all nondominated criterion vectors of a bi-objective integer program. This algorithm converges quickly to the complete set of nondominated criterion vectors as compared to decision space search algorithms.

Interactive methods

In addition to non-interactive approaches [4, 8, 22, 36, 68, 73, 82, 95, 110, 112, 120, 124, 140–142], interactive methods also has been developed for solving *MILP*. For a quick review of interactive methods for solving *MILP*, one can refer to the review article of Alves and Clímaco [4].

1.2.2 Algorithms for solving MINLP

Since all multi-objective integer programming problems are not of the linear form, researchers have also drawn attention to investigate all efficient solutions of *MINLP*. Recently, Santis and Eichfelder [43] and Zerfa and Chergui [147] propose algorithms for finding nondominated vectors of the multi-objective integer convex programming problem.

The quadratic function is a non-linear function with numerous practical applications. The sequential *QP* can be utilized for finding pareto optimal solutions of *MP*, as described in [5, 7].

But, *QP* with one negative eigen value is NP-hard as explained by Pardalos and Vavasis [116]. Gupta, Bandopadhyaya, and Puri [67] used ranked solutions of a related *ILP* for enumerating ranked solutions of integer quadratic programming problem (*IQP*). The linear multiplicative programs which is related to quadratic programming is solved by Cambini, Riccardi and Scopelliti [26] using a branch-and-bound scheme.

Further, methods for solving *MIQP* are discussed in [9, 46, 109]. In [46], *IQP* is optimized using a simplex-like method for determining efficient solutions of *MIQP*. In [9], a cutting plane technique is used to find all efficient solutions of *MIQP*. In [109], a branch and cut technique and efficiency cut is used for solving *MIQP*.

Researchers have also drawn interest in fractional programming problem (*FP*). Most solution algorithms for solving linear fractional programming problem (*LFP*) use simplex-like methods, as linear fractional functions are quasi-monotone. The duality in *FP* is discussed in [32, 78]. A methodology for solving *LFP* is presented by Charnes and Cooper [33]. Charnes and Cooper converted *LFP* into *LP*. Further, authors established a relation between an optimal solution of *LFP* and *LP*. In 2005, Dahiya and Verma [40] proposed an algorithm to rank feasible solutions of the integer linear fractional programming problem (*ILFP*) with bounded variables. In 2007, Dahiya [39] presented an iterative algorithm for ranking feasible solutions of constrained *ILFP* with box constraints. Chandra and Chandramohan [31] have developed a branch and bound method for integer non-linear fractional programming problem (*INLFP*). A technique for solving a bi-criterion *ILFP* is described by Saad and Hughes [126]. Researchers have also developed various methods for solving the multi-objective fractional programming problem (*MFP*), as discussed in the review

paper of Bhati, Singh, and Arya [20]. In [89–91], duality results for *MFP* discussed. A particular class of *MFP* involving $(p, r) - \rho - (\eta, \theta)$ -invex function is studied by Jayswal, Kumar and Kumar [77]. In 2014, Jayswal [76] developed an exponential penalty function method for solving *MFP*. The optimization of *MILFP* is discussed in [35, 69]. In [69], one of the objectives of *MILFP* is considered the objective function of *ILFP*. For solving *ILFP* cutting plane technique is used. Chergui and Moulaï [35] used the optimization of *ILFP* and also used the branch and cut technique to detect all efficient solutions of *MILFP*.

Since the numerator and denominator terms in multi-objective integer fractional programming problems may not always be of linear form, researchers have also studied *MINLFP*.

In recent past, Sharma [130] presented a methodology for solving *MINLFP*, where objective functions are square of a non-negative linear fractional function. In their paper, the author used the cutting plane technique to solve the problem. Recently, a methodology for generating efficient solutions of multi-objective integer indefinite quadratic fractional programming problem (*MIIdQFP*), where quadratic functions are the product of two linear functions, which are positive in the feasible region is discussed by Mekhilef, Moulaï and Drici [102]. Their algorithm uses a simplex-like technique to solve indefinite quadratic fractional programming problem (*IdQFP*) at each iteration. Further, an efficiency cut is used to eliminate non-efficient solutions from the feasible region. Researchers [75, 131] have also developed algorithms for solving general mono-objective integer quadratic fractional programming problem (*IQFP*) and general bi-objective integer quadratic fractional programming problem (*BIQFP*). In [75], authors used ranked solutions of *ILP* or *ILFP* for determining optimal solutions of *IQFP*. This ranking approach, clubbed with the ϵ -constraint scalarization technique is employed in the subsequent paper [131] for enumerating nondominated vectors of the problem (*BIQFP*).

1.2.3 Methods for optimizing function over efficient set

Researchers have expressed concerns about the decision-making process, arguing that even though many algorithms have been developed for the comprehensive enumeration of efficient solutions, the presence of a large elements in the efficient set may make it difficult for *DM* to select his preferred solution or, in certain cases, even impossible [80]. Thus, there is growing discussion about maximizing a suitable function over the efficient set, which can distinguish between numerous efficient solutions [2, 15, 17, 24, 80, 96, 111, 111, 139, 145, 146].

Philip first addresses the problem of optimizing a function over an efficient set [117]. Many researchers have followed Philip [117], who optimized a linear function over an efficient set of *MLP* by employing an adjacent vertex search algorithm. The following categories comprise the developed methods for optimizing a continuous function over an efficient set of *MLP*: face search algorithms, adjacent and non-adjacent vertex search algorithms, dual approach, Lagrangian relaxation method, branch-and-bound based algorithm, and bisection algorithm. An overview of various methods is provided in Yamamoto [144]’s survey article.

Optimization of a linear function over integer efficient set

Now a days, researchers are showing interest in solving a function over an integer efficient set. Many researchers [2, 45, 80, 94] have developed algorithms for optimizing a linear function over an integer efficient set. The algorithms are divided into two categories: Decision space search algorithm and Criterion space search algorithm

Decision space search algorithm [2]

The Abbas and Chaabane [2] are the first to propose an algorithm for optimizing a linear function over an integer efficient set. In their paper, a decision space search algorithm is used for optimizing a linear function over an integer efficient set. Their algorithm is based on edge-exploring and edge-cutting techniques. Each iteration generates a new efficient solution by exploring edges and truncating the feasible region using various cuts. At every iteration, new cuts are imposed over the feasible region, and also the previously generated cuts are carried forward, resulting in increasing the size of the problem at each stage. Computational results regarding the algorithm are not reported. Only a small example is solved to elaborate the algorithm. So, it is a challenging task to implement it in practical situations.

Criterion space search algorithms [45, 80, 94, 145]

The decision space has more dimension than criteria space in the majority of real-world scenarios. Thus, for the purpose of optimizing a linear function over the efficient set of *MILP*, criterion space search algorithms [45, 80, 94, 145] are created. A criteria space search algorithm was initially employed by Jorge [80] to optimise a linear function over an integer efficient set. His algorithm generated a nondominated vector at each iteration by using the Sylva and Crema [140] cuts. A progressively more constrained *ILP* is built at each iteration, truncating the dominating criteria vectors from the feasible criterion space, using the Sylva and Crema [140] cuts. The feasible criterion space thus shrinks with each iteration.

Later, Djamel and Marc [45] proposed another criterion search method for solving this problem. In their method, two techniques are used. The first technique is to gradually shrink the feasible region of criterion space by using the Sylva and Crema cut [140]. Second, apply edge exploring method for searching an efficient solution over edges incident to obtained solution. The Djamel and Marc [45] algorithm is very similar to the Jorge [80] algorithm. The only difference between Jorge's [80] and Djamel and Marc's [45] algorithms is that in the Djamel and Marc [45] algorithm, edges incident to a solution are explored to obtain an efficient solution. But, in the Jorge [80] algorithm, edges are not explored.

In Jorge [80] and Djamel and Marc [45] algorithms a progressively more constrained problem is solved at each k^{th} iteration, which require large computational efforts. In order to reduce the computational burden, Lokman [94] developed two decomposition and search algorithms. Instead of solving a large constrained problem at each iteration, they decompose the criterion space into smaller spaces. A nondominated vector which optimizes objective function value is generated from each subproblem, and a nondominated vector corresponding to the maximum objective function value is selected. The criterion space search algorithms [45, 80, 94], not only provides an optimal solution of linear function, but also generates a subset of nondominated vectors.

Recently, Younes, Sarah and Djamel [145] propose a parallel algorithm for solving a linear function over the efficient set of *MILP*. Through computational results authors shown that their algorithm performs better than Jorge's method [80].

Optimization of a non-linear function over integer efficient set

Decision-makers frequently seek to optimize non-linear functions over integer efficient set in real-world scenarios. Mahdi and Chaabane [98] used classical linear programming techniques to tackle the challenge of optimizing the linear fractional function over the efficient set of *MILP*, without listing every efficient solution of *MILP*. Drici, Ouail, and Moulaï [47] presented a branch and cut technique-based approach in 2018 for optimizing an integer linear fractional function over the *MILP* efficient set.

In terms of quadratic function optimization over the efficient set of *MILP*, Moulaï and Drici [106] address this kind of problem by applying an efficiency cut, an efficiency test, and a simplex-like technique. An indefinite quadratic function (the product of two quadratic functions) was taken into consideration by Moulaï and Drici [106], who proposed optimizing it over the set of efficient solutions of a *MILP*. Their method solves the indefinite quadratic function over the relaxed polyhedron region using a method akin to that of a simplex. In addition, the branch and bound approach for integer programming is applied in conjunction

with a few effective cuts to solve this problem.

Compared to the optimization of the linear function over the efficient set of *MILP*, the linear function over the efficient set of *MILFP* has received less attention in the literature. In [146], an approach to linear function optimization over the efficient set of *MILFP* is presented.

Many times, over the efficient set of *MIP*, one or more associated decision-makers seek to maximise their individual preference functions. Cherfaoui and Moulai [34] therefore presented a technique to optimise two linear preference functions over the efficient set of *MILP* without explicitly listing all of the efficient solutions of *MILP* in order to address this kind of problem. Cherfaoui and Moulai [34] solved two linear preference functions over the efficient set of *MILP* using tests, efficient cuts, and the branch and bound procedure.

1.3 Motivation for present study

Researchers have developed various methods for finding efficient solutions of *MLP* [8, 37, 53, 72, 136]. However, research on methods for solving *MILP* has been rather limited. In literature, there are few methods [4, 36, 68, 82, 106, 110, 112, 140–142] to solve *MILP*, these methods also have their own limitations. Some methods do not generate the set of all efficient solutions of *MILP*, while others are very complicated and computationally challenging. One widely used technique to solve *MILP* is ϵ -scalarization. But, if the choice of ϵ is not properly made, it fails to give desired results. Further, Ehrgott [50] addresses various limitations and implementation issues of this technique. This methodology increases the size of the problem at each iteration, which makes it computationally difficult to handle if the problem has a large number of efficient solutions. Many researchers used the weighted sum approach to find an efficient solution of *MILP*. Some efficient solutions of *MILP*, known as non-supported efficient solutions of *MILP*, may not be an optimal solution of the problem (P_w) for any $\lambda > 0$. So, great computational efforts are required in the weighted sum approach to generate the efficient set of *MILP*. Additionally, there is a open room to develop efficient algorithms for solving *MINLP*.

Furthermore, optimizing a function over the efficient set of *MIP* is a crucial area in mathematical programming problems. An application of this type of problem in the context of network flow theory can be seen in [2, 64, 121, 132, 133, 144]. An illustration of this problem in the context of traveling salesman is discussed in [45]. In the case of continuous variables, vertex search algorithms, face search algorithms, Lagrangian relaxation method, dual approach, and many other methods are available in the literature for optimizing a

continuous function over the efficient set of MLP [144]. But, as far as optimizing a function over the efficient set of MIP is concerned, limited research is done in this domain. The significant research in this area has focused on optimizing linear function over the efficient set of $MILP$ [2, 24, 45, 80, 94, 145]. However, incorporating quadratic functions has proven increasingly difficult, despite their significant role in various engineering and optimization fields. To date, there is only one available paper by Moulaï and Drici [106] that solves a quadratic function over the efficient set, and it only considers a specific case, where the quadratic function is in the form of the product of two linear functions.

Furthermore, optimization of an integer quadratic fractional programming over the efficient set of MIP remains unexplored.

1.4 Objectives of the thesis

1. Optimization of a multi-objective integer programming problem.
2. Optimization of a linear/quadratic function over the efficient set of multi-objective integer programming problem.
3. Optimization of a fractional function over the efficient set of multi-objective integer programming problem.

1.5 Overview of the thesis

This thesis addresses the problem of multi-objective integer programming, focusing on decision-making situations where decision variables are integers. The entire thesis is organized into seven chapters, which are summarized as follows:

- Chapter 1: This chapter introduces multi-objective programming problems. It includes a comprehensive review of the existing literature on multi-objective integer programming problems. Further, it highlights the research objectives of the thesis.
- Chapter 2: This chapter focus on solving a bi-objective integer linear programming problem ($BILP$). An algorithm based on the ϵ -constraint method is discussed, which generates a complete set of nondominated points for $BILP$. The algorithm's convergence is theoretically established, and a comparative study with existing algorithms is conducted.
 - This chapter's content have been published as “An algorithm for bi-objective integer linear programming problem” in *Filomat*, 36(16):5641-5651, 2022 (SCI,

Impact factor: 0.8).

- Chapter 3: This chapter presents an algorithm for finding all efficient solutions of $MIQP$. The proposed algorithm is based on enumerating ranked solutions of IQP . The working of the method is illustrated using a numerical example. A particular case of the proposed method for solving $MILP$ is also discussed. Further, computational results show the effectiveness of the developed methodology.
 - This chapter’s content has been published as “An algorithm to solve multi-objective integer quadratic programming problem” in *Annals of Operations Research*, 2024, 332, 433–459 (SCI, Impact factor: 4.854).
 - The methodology discussed in this chapter has been presented at the International Symposium on Optimization Theory and its Applications (ISOTA 2023), TIET, Patiala (April 13-15, 2023).
 - The algorithm discussed in this chapter was also presented at the ‘International Conference on Advances in Differential Equations and Numerical Analysis ’ held at IIT Guwahati.
- Chapter 4: This chapter presents a novel methodology for finding all efficient solutions of $MIQFP$. The developed method utilizes ranked solutions of $IQFP$ to enumerate efficient solutions of $MIQFP$. The ranking of feasible solutions of a related $ILFP$ or ILP is used to find ranked solutions of $IQFP$. The workings of the proposed method are illustrated through numerical examples and theoretical results. Furthermore, comparison with existing methods and computational results are also discussed.
 - A part of this chapter has been published as “A note on solving multi-objective integer indefinite quadratic fractional programs” in *Annals of Operations Research*, 289(2):459-462, 2020 (SCI, Impact factor: 4.8).
 - The content of this chapter have also been communicated in the *Annals of Operations Research*.
 - The algorithm discussed in the chapter has been presented at the International Symposium on Applied Optimization and Game Theoretic Models for Decision Making, Indian Statistical Institute, Delhi Centre (February 1-3, 2023).
- Chapter 5: This chapter explores the optimization of a linear function over the efficient set of $MILP$. The multi-objective programming problem often contains numerous efficient solutions, which confuses DM . To assist in selecting the most desirable solution, optimizing a function over the efficient set becomes crucial. In this chapter,

we describe two new algorithms for solving ILP over the efficient set of $MILP$. The first algorithm obtains a globally optimal solution by systematically scanning ranked solutions of ILP until a efficiency condition is satisfied. In the second algorithm, in addition to ranked solutions, efficient solutions are also collected at each stage. These efficient solutions generate a lower bound, which keeps on shrinking at each stage. This lower bound also provides an estimate on the number of ranked solutions to be calculated and the algorithm stops after calculating ranked solutions upto this lower bound. The convergence of proposed algorithms are established theoretically, and steps are illustrated using numerical examples. A comparative analysis with existing algorithms is also conducted. The comparative analysis shows that the developed algorithm came out to be computationally more efficient as compared to the existing state-of-the-art algorithms.

- The content presented in this chapter has been published as “Optimization of a linear function over an integer efficient set” in *Journal of Industrial and Management Optimization*, 2023,19(12), 8633-8656, doi: 10.3934/jimo.2023055 (SCI, Impact factor: 1.3).
- Chapter 6: This chapter introduces a novel approach to optimize a general quadratic function over the efficient set of $MILP$. To solve this problem, a ranking approach and efficiency test is utilized. For generating ranked solutions, we construct a related ILP . Then, ranked solutions of ILP are used for enumerating ranked solutions of IQP . The algorithm systematically scans integer ranked solutions of IQP until the efficiency condition is met. The algorithm’s convergence is established, and its steps are illustrated using a numerical example. Further, extensive computational results demonstrate the effectiveness of our method for solving problems with large number of constraints, variables, and objective functions.
 - The content presented in this chapter has been published as “Optimization of a quadratic programming problem over an integer efficient set” in the *Journal of Computational and Applied Mathematics*, 2024, 441, 115651, (SCI, Impact factor: 2.4).
 - The algorithm discussed in this chapter is presented at the ‘International Conference on Computational and Mathematical Methods in Science and Engineering’ held at Rota, Cadiz, Spain (July 3-8, 2022).
- Chapter 7: This chapter proposes a scheme to optimize a quadratic fractional function over the efficient set of $MILP$. The proposed method provides a finite iterative process to solve the problem without relying on non-linear optimization techniques.

The developed methodology combines a criteria space search algorithm with a ranking approach and efficiency tests. The developed method follows a two-step procedure. On one side, it calculates the lower bound using optimal solutions of integer quadratic fractional programming problem, and on the other side, it calculates the upper bound using efficient solutions. The cutting plane methodology for ranking the integer feasible solutions of linear and linear fractional programming problems is used for finding optimal solutions of quadratic fractional programming problem, and the efficiency test is utilized for finding efficient solutions. The theoretical results demonstrate the convergence of the proposed algorithm, and numerical and computational experiments confirm its effectiveness in finding optimal solutions. One notable advantage of the proposed algorithm is its computational efficiency. The developed approach eliminates the requirement to list every ranked solution of *IQFP* and every efficient solution of *MILP*, reducing computational complexity. The algorithm reduces computational complexity by avoiding the need to enumerate all ranked solutions of *IQFP* and all efficient solutions of *MILP*.

By presenting various algorithms in each chapter, this thesis contributes to multi-objective integer programming. It provides valuable tools for decision-makers in multi-criteria decision-making situations.

Chapter 2

Bi-objective integer linear programming problem

2.1 Introduction

From past few decades multi-objective programming has become one of the most favored and challenging area in the field of optimization. However, the research on the multi-objective programming problems with integer variables is limited when compared with continuous variables. The introduction of integer variables makes the problem more difficult even when the objectives are linear. In past few year there has been some development in the area of *MILP*, still the research in this area is scarce. A ranking approach to generate all integer efficient solution of *MILP* has been studied by Gupta and Malhotra [68]. Klein and Hannan [82] studied an approach that sequentially generated the set of non dominated points by solving *SP* and by taking other objectives as constraints. Each time an efficient solution is generated, some additional constraints are added in the previous problem to generate next efficient solution. This is also called ϵ -constraint methodology where one objective will be used as the objective function and the remaining objectives will be treated as constraints using the epsilon value as bound. The trick in this kind of methodology is to decide the best bound so that none of the integer efficient solution is missed. In literature ϵ -constraint technique was first discussed in [71]. Later, Neumayer and Schweigert [107] implemented this approach to bi-objective transportation problem. Sylva and Crema [140] discussed a variation of Klein and Hanna [82] algorithm by using a positive combination of all objective functions instead of optimizing a single criteria problem. Özlen and Azizoglu [110] proposed a variant of ϵ -constraint algorithm, in their method *MILP* is converted into a *ILP* and other objectives are taken as constraints based on the identification of objective efficiency range. For algorithms and other developments in the field of *MILP* one can refer the survey article by Teghem and Knush [142] and by Ehrgott and Gandibleux [51, 52].

Although in literature, methods to solve bi-objective integer linear programming problem

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(*BILP*) are available but they do have their limitations. In some cases a complete set of nondominated vectors is not enumerated, in others the problem becomes very complicated and there are also instances where an explicit formula for the choice of epsilon is not given which makes it difficult to implement practically. In this chapter, an efficient ϵ -constraint method for *BP* is proposed, in which we aim at generating the set of all non-dominated vectors of *BILP* and have justified that none of the integer efficient solutions is missed. Further, the proposed methodology has two benefits. First, the constrained program solved to generate k^{th} nondominated vector does not use any previous calculations, like the constraints generated to obtain $(k-1)^{th}$ or any previous nondominated vector. Only the $(k-1)^{th}$ point is used to generate k^{th} nondominated vector, which makes the algorithm computationally efficient. The second advantage of the developed method is that one can also generate intermediate nondominated vectors. It is important to obtain intermediate nondominated vectors, as the presence of a large number of variables may generate a large number of nondominated vectors. In *MP* decision is made on the preference attitude of *DM*, for instance in a problem like financial investments, the *DM* always has in his mind the amount he wants to invest and he is aware of his risk appetite. In such a situation, it becomes irrelevant to calculate a complete set of solutions. Rather, he should be provided with solutions within his desired range that will reduce error as well as save computation time. We have made a comparative study with some of the algorithms available in the literature and have also shown by counter-example how these algorithms do not record all nondominated vectors.

This chapter is organized as follows. In the next section, we have discussed theoretical results. In Section 3, we have presented two algorithms. In Section 4, we have discussed numerical examples to elaborate the procedure, how the proposed algorithm works. Further comparison with other existing algorithms is discussed in Section 5. The computational results are presented in Section 6. In the last section, some concluding results are discussed.

2.2 Mathematical formulation

The problem under consideration is

$$(BILP) \quad \min_{X \in \mathcal{X}} (f(X), h(X)),$$

where $f(X) = C^1 X$, $h(X) = C^2 X$, $C^1, C^2 \in \mathbb{R}^{1 \times n}$.

To ensure that the feasible region contains finite points, we have made an assumption that

S is a closed and bounded set. Now consider the problem as follows:

$$(P_1) \quad \min_{X \in \chi} C^1 X.$$

Definition 2.2.1 A solution $Y^* \in \chi$ is an efficient solution if there does not exist any $Y \in \chi$ such that

$$(a) f(Y) \leq f(Y^*) \text{ and}$$

$$(b) h(Y) \leq h(Y^*)$$

and strict inequality sign either holds for (a) or (b).

Definition 2.2.2 If Y^* is an efficient solution, then $(f(Y^*), h(Y^*))$ is called a nondominated vector.

Throughout this chapter, we have used following notations:

An optimal solution of P_1 is represented by X_1 , and if there are m_1 alternate solutions, then the complete set of alternate optimal solutions is represented as $X_{1,i}$, $i = 1, 2 \dots m_1$. Similarly, an optimal solution of P_1^{k-1} for $k \geq 2$ is denoted by X_k and if P_1^{k-1} has m_k alternate optimal solutions, then the set of all optimal solutions of P_1^{k-1} is represented as $S_k = \{X_{k,i}, i = 1, 2 \dots m_k\}$. Further, we will show that $k+1^{th}$ efficient solution of $BILP$ is an optimal solution of P_1^k . So, we denote X_k as a k^{th} efficient solution of $BILP$ and $(f_k, h_k) = (f(X_k), h(X_k))$ denotes the k^{th} nondominated vector corresponding to solution X_k . P_1 always has a feasible solution, and the set of non-dominated vectors will be nonempty. The following result depicts the existence of a nondominated vector of $BILP$.

Theorem 2.2.1 [131] If the feasible region of $BILP$ is nonempty, then set of nondominated vectors is also non empty.

For some scalars $a > 0$ and $b > 0$, Consider the problem (P^*) , defined as

$$(P^*) \quad \min_{X \in \chi} (af(X), bh(X)).$$

The problem (P^*) converts objective functions with fractional coefficients to integer coefficients. Then, in the following theorem, we show that the set of non-dominated vectors of $BILP$ and P^* are the same. In the proposed methodology, we have made use of objective function cut to generate nondominated vectors and if the coefficients of the objective function are not integer then this cut may not work.

Theorem 2.2.2 There is one to one correspondence between nondominated vectors of $BILP$ and problem (P^*) .

Proof. Suppose (f, h) is nondominated vector of $BILP$, then we need to show that (af, bh) is nondominated vector of the problem P^* . Suppose on the contrary, there exist a pair (f^*, h^*) which dominates (af, bh) , then $f^* \leq af$ and $h^* \leq bh$ with strict inequality sign holding for at least one place. That means $f^*/a \leq f$ and $h^*/b \leq h$ with strict inequality at at least one place, then $\tilde{f} \leq f$ and $\tilde{h} \leq h$ where $\tilde{f} = f^*/a$ and $\tilde{h} = h^*/b$. Which contradicts the assumption that (f, h) is a nondominated vector. Similarly the converse part can be proved.

In order to find all the nondominated vectors of P_1 , we define the restrictive version of $BILP$, in which the second objective function is treated as constraint. For $k \geq 1$ we define

$$(P_1^k) \quad \begin{cases} \min f(X) \\ \text{subject to} \\ X \in \chi \\ h(X) < h_k \end{cases}.$$

Here h_k is the value of $h(X)$ at X_k . Related to the problem (P_1^k) , following notations are defined:

$S_{k+1} = \{X_{k+1,i}, i = 1, 2, \dots, m_{k+1}\}$ is the set of optimal solutions of (P_1^k) .

f_{k+1}^* is the optimal value of objective function of (P_1^k) .

Theorem 2.2.3 *An optimal solution of P_1^{k-1} gives k^{th} nondominated vector of $BILP$.*

Proof. Since optimal solution of P_1^{k-1} is denoted by S_k , yielding optimal value as f_k . Define $h_k = \min\{h(X_{k,i}), X_{k,i} \in S_k\} = h(X_{k,p})$, for some $p \in 1, 2, \dots, m_k$. then (f_k, h_k) is a nondominated vector of $BILP$. For if there exist a pair (f, h) which dominates (f_k, h_k) . Then $f \leq f_k$ and $h \leq h_k$ and either $f < f_k$ or $h < h_k$. Then either $f < f_k$ or if $f = f_k$, then $h < h_k$. Both of these situations are not possible as in the first case the optimality of f_k will be contradicted and in latter case the choice of h_k is contradicted. Therefore, (f_k, h_k) is a nondominated vector.

Remark 2.2.1 *It may be noted, if the coefficient vector C^2 in the function $h(X)$ has all the components as integer, then the constraint $h(X) < h_k$ in the problem (P_1^k) can be replaced with $h(X) \leq h_k - 1$, as X being an integer vector and none of the integer point will be missed. Further, if C^2 is not an integer vector, then solve the problem (P^*) , as evident from Theorem 2.2.2, $BILP$ and (P^*) have same set of nondominated vectors. In this case the choice of the scalar b can be suitably made so that bC^2 is an integer vector in the function $bh(X)$ and the inequality can be reduced by using the constraint $bh(X) \leq bh(X_k) - 1$.*

Remark 2.2.2 If some simplex table depicts an optimal solution of ILP with $z_j - c_j = 0$ for some nonbasic variable x_j , this indicates the presence of alternate optimal solutions. Then corresponding to each such nonbasic variable x_j , the alternate optimal integer solutions are calculated by using the formula $\hat{x}_{B_i} = x_{B_i} - \gamma_j y_{ij}; i \neq r$ and $x_{B_r} = \gamma_j$, where γ_j takes integer values such that $0 \leq \gamma_j \leq \frac{x_{B_r}}{y_{rj}} = \min_i \left\{ \frac{x_{B_i}}{y_{ij}} \mid y_{ij} > 0 \right\}$. Here $X_B = (x_{B_1}, x_{B_2}, \dots, x_{B_m})$ is a vector of basic variables in the current simplex table and $y_j = B^{-1}A_j$, with A_j as j^{th} column of matrix A . For detailed explanation one may refer Hadley [70].

2.3 Algorithm and its convergence

Based on the results proved in previous section, in the following, we have proposed an algorithm which aims at finding all nondominated vectors of problem $BILP$. In the

Algorithm 1: Algorithm to solve $BILP$

- 1 **Step 1** Initially take $\hat{E} = \emptyset$.
 - 2 Solve the problem (P_1) .
 - 3 Note all its optimal feasible solutions $X_{1,i}; i = 1, 2, \dots, m_1$, as discussed in Remark 2.6.
 - 4 Find f_1 and calculate $h_1 = \min\{h(X_{1,i}), i = 1, 2, \dots, m_1\}$, and update $\hat{E} = \hat{E} \cup (f_1, h_1)$.
 - 5 Also calculate $h_l = \min_{X \in \mathcal{X}} C^2 X$.
 - 6 **Step 2** For $k \geq 2$,
 - 7 Construct and solve the problem (P_1^{k-1}) . Find the set S_k .
 - 8 Find f_k and $h_k = \min\{h(X), X \in S_k\}$ and go to Step 3.
 - 9 **Step 3** Update $\hat{E} = \hat{E} \cup (f_k, h_k)$.
 - 10 If $h_k = h_l$ move to Step 4, else set $k = k + 1$, go to Step 2.
 - 11 **Step 4** Stop, and note $\hat{E}$ as the set of nondominated vectors.
-

following some results pertaining to the convergence of the algorithm are discussed.

Theorem 2.3.1 For $k \geq 1$, (f_k, h_k) and (f_{k+1}, h_{k+1}) are two non-dominated vectors recorded by the proposed algorithm. Then $f_k < f_{k+1}$.

Proof. The problem (P_1^k) is obtained from problem (P_1^{k-1}) by appending the constraint $h(X) < h_k$, which implies the set of feasible solutions of P_1^k is contained in the set of feasible solutions of P_1^{k-1} . Therefore, optimal value of P_1^k is lesser than the optimal value P_1^{k-1} , i.e. $f_k \leq f_{k+1}$. Further if $f_k = f_{k+1}$, then $X \in S_{k+1} \cap S_k$ such that $f(X) = f_k = f_{k+1}$ and $h(X) \geq h_k$, which is a contradiction as $h(X) < h_k$ for all $X \in S_{k+1}$. Therefore, $f_{k+1} > f_k$.

Theorem 2.3.2 The developed algorithm only records non-dominated vectors.

Proof Notably, the first non-dominated vector (f_1, h_1) is generated by using Theorem 2.2.1. As $f_2 > f_1$ (by virtue of Theorem 2.3.1) and $h_2 = \min\{h(X_{i,j}) \mid X_{i,j} \in S_2\} < h_1$ (as defined in proposed algorithm). So, (f_2, h_2) is also a nondominated vector. Continuing on the same lines, the next point obtained (f_3, h_3) is also non-dominated. Then by applying mathematical induction. It can be observed that if (f_i, h_i) , $i = 1 \dots k-1$ are previously generated nondominated vectors, then (f_k, h_k) is also a nondominated vector. Thus, the developed algorithm records only nondominated vectors.

The following theorem helps determine the algorithm's ending criteria, which states that the process ends as soon as the second objective function's minimum is reached. This is because the nondominated vectors are created in accordance with the second objective function's decreasing value.

Theorem 2.3.3 *If (f_l, h_l) is the last nondominated vector for BILP, then $h_l = \min_{X \in S} h(X)$.*

Proof Suppose (f_l, h_l) is the last nondominated vector for BILP. If possible, let h_l is not an optimal value of $h(X)$ and $\hat{h} = \min_{X \in X} h(X) = h(\hat{X})$ (say), this implies $h_l > \hat{h}$. Then either $f_l \geq \hat{f}$ or $f_l < \hat{f}$, where $\hat{f} = f(\hat{X})$. In the first case, when $f_l \geq \hat{f}$, then (f_l, h_l) is dominated by $(\hat{f}, \hat{h})$, which is false. In the latter scenario, where $f_l < \hat{f}$ as $h_l > \hat{h}$, then P_1^l solution will exist, and (f_l, h_l) will not be the last nondominated vector generated by the algorithm. Thus, our assumption was incorrect.

The ensuing theorem provides support for the claim that all non-dominated vector are included and aids in the algorithm's convergence.

Theorem 2.3.4 *The algorithm described above records all the nondominated vectors for BILP.*

Proof Let $(\bar{f}, \bar{h})$ with the corresponding solution $\bar{X}$ for BILP, be a point which is not recorded by the algorithm. Then we have $\bar{f} \geq f_1$, as f_1 being optimal value of $f(X)$. If $\bar{f} = f_1$ and the point $(\bar{f}, \bar{h})$ is not recorded, then $\bar{h} > \min\{h(X_{i,j}), X_{i,j} \in S_1\} = h_1$, where S_1 is the index set of all the optimal solutions of P_1 . This implies that $(\bar{f}, \bar{h})$ is dominated by (f_1, h_1) . If $\bar{f} > f_1$, then there exists some k for which $\bar{f} \geq f_k$. Again we have two possibilities, either $\bar{f} = f_k$ or $\bar{f} > f_k$. If $\bar{f} = f_k$ then $(\bar{f}, \bar{h})$ is dominated by (f_k, h_k) , if $\bar{f} > f_k$, in this case again either of the two subcases arises viz. $\bar{f} = f_{k+1}$ or $\bar{f} > f_{k+1}$. Continuing in this way we find that either $(\bar{f}, \bar{h})$ is a dominated point or $\bar{f} > f_l$, where $(\bar{f}, \bar{h})$ is the last nondominated vector. Also $\bar{h} \geq h_l$ as $h_l = \min_{X \in S} h(X)$, then again $(\bar{f}, \bar{h})$ is a dominated point.

Remark 2.3.1 *It may be noted that there are finite number of non-dominated points because the feasible region is closed and bounded. So, the algorithm will terminate in a finite*

step as it only records non-dominated points.

An extended (BILP)

Suppose DM wants to explore his options to take his decision, when his decision parameters with one of objectives lying within certain range. In this case, suppose DM want to find the set of nondominated vectors only in the range (γ, δ) , $\gamma, \delta \in \mathbb{R}$, of second objective function . Then, $BILP$ can be redefined as follows:

$$(\bar{P}) \quad \begin{cases} \min (f(X), h(X)) \\ \text{subject to} \\ X \in \chi \\ \gamma \leq h(X) \leq \delta \end{cases} .$$

Then the algorithm discussed Section 3, to obtain non-dominated pairs of $BILP$ can be generalized in the following way. The modified algorithm discussed below will yield only those nondominated vectors for which second objective function lies within γ and δ .

Algorithm 2: Extended algorithm to solve $BILP$

- 1 **Step 1** Initially take $\hat{E} = \emptyset$.
- 2 Solve the problem $(\bar{P}_2)$ defined as follows,

$$(\bar{P}_2) \quad \min_{X \in \chi} f(X) \quad \text{subject to} \quad \gamma \leq h(X) \leq \delta.$$

Note all its optimal feasible solutions $X_{1,i} ; i = 1, 2 \dots m_1$.

- 3 Find f_1 and calculate $h_1 = \min\{h(X_{1,i}), i = 1, 2 \dots m_1\}$.
- 4 Update $\hat{E} = \hat{E} \cup (f_1, h_1)$.
- 5 **Step 2** For $k \geq 1$, construct the problem $(\bar{P}_1^k)$ defined as follows:

$$(\bar{P}_1^k) \quad \min_{X \in \chi, h(X) \geq \gamma} f(X) \quad \text{subject to} \quad h(X) \leq h_k - 1.$$

Call its optimal solution set as S_{k+1} .

- 6 Find $f_{k+1} = f(X); X \in S_{k+1}$ and $h_{k+1} = \min\{h(X), X \in S_{k+1}\}$. and move to Step 3.
- 7 **Step 3** Update $\hat{E} = \hat{E} \cup (f_{k+1}, h_{k+1})$.
- 8 If $h_{k+1} = h_l$, where $h_l = \min_{X \in \chi, h(X) \geq \gamma} h(X)$, move to Step 4, else set $k = k + 1$, move to Step 2.

- 9 **Step 4** Stop, and note $\hat{E}$ as the set of non-dominated vectors of $\bar{P}$.
-

Notably, an optimal solution of $\bar{P}_1^k$, will yield (f_{k+1}, h_{k+1}) as the k^{th} non-dominated pair of the problem $(\bar{P})$. The convergence of the 'Extended algorithm to solve $BILP$ ' can also be done, as we did for 'Algorithm to solve $BILP$ ' in this Section.

2.4 Numerical Example

Now, we will discuss a numerical example to elaborate the procedure for finding non-dominated vectors of *BILP*. All the calculations are done by using simplex algorithm and Gomory's cutting plane technique for *ILP* as discussed in Bazaraa et al. [13] and Hadley [70].

Example 1 Consider the problem:

$$(\tilde{P}) \quad \begin{cases} \min (x_1 + \frac{x_2}{2}, \frac{-x_1}{3} + \frac{2x_2}{3}) \\ \text{Subject to} \\ 3x_1 + 2x_2 \geq 6, 4x_1 + 5x_2 \leq 20, \\ x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

It may be noted that by taking $a = 2$ and $b = 3$ we get the following problem

$$(P) \quad \begin{cases} \min (2x_1 + x_2, -x_1 + 2x_2) \\ \text{Subject to} \\ 3x_1 + 2x_2 \geq 6, 4x_1 + 5x_2 \leq 20, \\ x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

Then, by virtue of Theorem 2.2.2, it is equivalent to solve *BILP* instead of problem $(\tilde{P})$. Therefore, corresponding *BILP*, we have the following problem, as follows:

$$(P_1) \quad \begin{cases} \min f = 2x_1 + x_2 \\ \text{Subject to} \\ 3x_1 + 2x_2 \geq 6, 4x_1 + 5x_2 \leq 20, \\ x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

An optimal solution of P_1 is given in the Table 2.1. The optimal solution is given by

Table 2.1: Optimal simplex table of P_1

		c_j	2	1	0	0
c_B	B	X_B	y_1	y_2	y_3	y_4
1	x_2	3	3/2	1	-1/2	0
0	x_4	5	-7/2	0	5/2	1
$f=3$	$Z_j - c_j$		-1/2	0	-1/2	0

$X_1 = (0, 3)$, with $f(0, 3) = 3$ and $h(0, 3) = 6$ and by virtue of Theorem 2.2.1, we have

our first nondominated vector $(f_1, h_1) = (3, 6)$, with the corresponding efficient solution as $X_1 = (0, 3)$. Now for $k = 1$, construct the problem (P_1^k) , by adding second objective as an additional constraint. By using Remark 2.2.1, this additional constraint is given by $-x_1 + 2x_2 \leq 5$, and an optimal integer solution of the problem (P_1^1) is given in the Table 2.2. Since $z_3 - c_3 = 0$. It can be observed from Table 2.2, P_1^1 has alternate optimal

Table 2.2: Optimal simplex table of the problem (P_1^1)

		c_j	2	1	0	0	0	0
c_B	B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
1	x_2	0	0	1	-2	0	0	3
0	x_4	12	0	0	6	1	0	-7
2	x_1	2	1	0	1	0	0	-2
0	x_5	7	0	0	5	0	1	-8
$f=4$		$Z_j^1 - c_j$	0	0	0	0	0	-1

solutions. Since $0 \leq \gamma_3 \leq 7/5$, the integer solutions are obtained by giving $\gamma_3 = 0, 1$ in the formula $\hat{x}_{B_i} = x_{B_i} - \gamma_j y_{i,j}$, $x_3 = \gamma_j$. The optimal solutions of P_1^1 obtained are $\{X_{2,1} = (2, 0), X_{2,2} = (1, 2)\}$, with $f_2 = 4$ and $h_2 = \min\{h(2, 0), h(1, 2)\} = -2$ and second nondominated vector obtained is $(f_2, h_2) = (4, -2)$, with corresponding efficient solution as $X_{2,1} = (2, 0)$. Further, construct the problem (P_1^2) by appending the constraint $h(X) \leq -3$ and proceed as above. Continuing on the lines of algorithm we have obtained all the nondominated vectors of $BILP$, and the corresponding to $\tilde{P}$. Table 2.3 lists all non-dominated vectors of $BILP$ and $\tilde{P}$.

Table 2.3: Non-dominated vectors of $BILP$ and $\tilde{P}$

Efficient solutions	(0,3)	(2,0)	(3,0)	(4,0)	(5,0)
nondominated vectors of $BILP$	(3,6)	(4,-2)	(6,-3)	(8,-4)	(10,-5)
nondominated vectors of $(\tilde{P})$	(3/2,2)	(2,-2/3)	(3,-1)	(4,-4/3)	(5,-5/3)

Remark 2.4.1 Consider $BILP$ with one of the objective function has a fractional coefficient given as:

$$\min (2x_1 + x_2, \frac{-x_1}{3} + \frac{2x_2}{3})$$

If the coefficients of second objective function are not converted into integer by suitably chosen integer $b(=3$ in this case) as suggested in Remark 2.2.1, then the algorithm may not generate all nondominated vectors. As in this case only three nondominated vectors will be recorded by the algorithm namely $(3, 2)$, $(4, -2/3)$ and $(10, -1.6)$. But after changing the second objective function to $-x_1 + 2x_3$, the set of all nondominated vectors is generated, which includes $(6, -1)$ and $(8, -1.3)$ in addition to three already generated.

Remark 2.4.2 *It may be observed in the above example there are eleven integer points in χ , out of which only five are efficient and are obtained by the proposed algorithm and none other integer points is evaluated. Indeed some alternate optimal solutions are scanned as and when required, but to evaluate these integer solutions no extra simplex iterations are required which makes the proposed methodology more efficient.*

Example 2 Consider *BILP* with four variables, as follows:

$$(\tilde{P}) \quad \begin{cases} \min (x_1 - 2x_2 + x_3 + 2x_4, -x_1 + 2x_2 - 2x_3 + x_4) \\ \text{Subject to} \\ x_1 + 2x_2 + x_3 + 4x_4 \leq 8, -x_1 + x_2 + 2x_3 - x_4 \geq 4, \\ x_1, x_2, x_3, x_4 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

First, solve the problem, given as:

$$(P_1) \quad \begin{cases} \min f = x_1 - 2x_2 + x_3 + 2x_4 \\ \text{Subject to} \\ x_1 + 2x_2 + x_3 + 4x_4 \leq 8, -x_1 + x_2 + 2x_3 - x_4 \geq 4, \\ x_1, x_2, x_3, x_4 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

Table 2.4 depicts an optimal feasible solution to this problem. The optimal solution is given

Table 2.4: Optimal simplex table of P_1

		c_j	1	-2	1	2	0	0
c_B	B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
0	x_6	0	3/2	0	-3/2	3/2	1/2	1
-2	x_2	4	1/2	1	1/2	1/2	1/2	0
$f = 6$		$Z_j - c_j$	0	0	0	-1	-1	0

by $X_1 = (0, 4, 0, 0)$, with $f(0, 4, 0, 0) = -8$ and $h(0, 4, 0, 0) = 8$. So the first nondominated vector is $(f_1, h_1) = (-8, 8)$, with the corresponding efficient solution as $X_1 = (0, 4, 0, 0)$. Now by adding second objective as an additional constraint, we construct the problem (P_1^1) given as:

$$(P_1^1) \quad \begin{cases} \min f = x_1 - 2x_2 + x_3 + 2x_4 \\ \text{subject to} \\ x_1 + 2x_2 + x_3 + 4x_4 \leq 8, -x_1 + x_2 + 2x_3 - x_4 \geq 4, \\ -x_1 + 2x_2 - 2x_3 + x_4 \leq 7, x_1, x_2, x_3, x_4 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

An optimal integer solution of the problem (P_1^1) is $(0, 3, 1, 0)$ with $f(0, 3, 1, 0) = -5$, which is an efficient solution. The second non-dominated vector obtained at this point is $(f_2, h_2) = (-5, 4)$. Now continuing on the same lines we get the set of all efficient solutions. Notably, an optimal solution of the second objective function over $\mathcal{X}$ is $(0, 0, 8, 0)$

Table 2.5: Efficient solutions of $\tilde{P}$

S.no	1	2	3	4	5	6
X_k	$(0, 4, 0, 0)$	$(0, 3, 1, 0)$	$(0, 3, 2, 0)$	$(0, 2, 2, 0)$	$(0, 2, 3, 0)$	$(0, 2, 4, 0)$
(f_k, h_k)	$(-8, 8)$	$(-5, 4)$	$(-4, 2)$	$(-2, 0)$	$(-1, -2)$	$(0, -4)$
S.no	7	8	9	10	11	12
X_k	$(0, 1, 4, 0)$	$(0, 1, 5, 0)$	$(0, 1, 6, 0)$	$(0, 0, 6, 0)$	$(0, 0, 7, 0)$	$(0, 0, 8, 0)$
(f_k, h_k)	$(2, -6)$	$(3, -8)$	$(4, -10)$	$(6, -12)$	$(7, -14)$	$(8, -16)$

with objective function value as $h_l = -16$. Since optimal value corresponding to second objective function is attained, the algorithm stops. Table 2.5 depicts the complete set of all efficient solutions.

2.5 Comparison with some existing techniques

In the following, we have made a comparative analysis of the developed approach with existing algorithms in literature. These algorithms either don't produce a full set of nondominated vectors or, if they do, the generation process is computationally highly costly.

2.5.1 Comparison of the proposed methodology with ranking method of Gupta and Malhotra

The proposed methodology outperforms the Gupta and Malhotra [68], as it evaluates only nondominated vectors, unlike the ranking method, which scans all feasible solutions. For instance, if the ranking algorithm is applied to the Example 1 above, it will require at least eleven simplex iterations (as some alternates are available), whereas the developed algorithm terminates in five simplex calculations. Another advantage of the proposed algorithm over the ranking approach is that it requires solving problem (P_1^k) only, which is obtained by appending a constraint $h(X) < h_k$ in P_1 , whereas the ranking approach requires repeated application of Gomory cuts and ranking cuts to find all integer feasible solutions. So if a problem has eleven integer solutions, ranking algorithm requires at least eleven ranking cuts in addition to at least eleven Gomory cuts to be appended in a given problem, which is computationally very difficult to solve.

2.5.2 Comparison of proposed method with modified ϵ -constraint technique of Özlen and Azizoglu [110]

The Özlen and Azizoglu [110] algorithm is applied to a two objective problem given in Remark 2.4.1 and after making the calculations of w as given by Özlen and Azizoglu [110], the following weighted single objective integer programming problem is generated:

$$\begin{cases} \min \frac{151}{78}x_1 + \frac{44}{39}x_2 \\ \text{Subject to} \\ -\frac{1}{3}x_1 + \frac{2}{3}x_2 \leq l_2 \\ x_1, x_2 \in \chi. \end{cases}$$

Initially the l_2 is taken as 2.6 and the non-dominated pair obtained is (3,2), then l_2 is updated to $l_2 = 1$ the non-dominated pair obtained is $(4, \frac{-2}{3})$ and further l_2 is updated to $l_2 = \frac{-5}{3}$ and the non-dominated pair obtained is (10,-1.6). Here it missed two nondominated vectors (6, -1) and (8, -1.3) and these points will not be recovered by the algorithm at any later stage. However, the proposed algorithm generates the set of all nondominated points of *BILP* (discussed in Remark 2.4.1).

2.5.3 Comparison of proposed method with weighted function approach [140]

Now, look upon the example (taken from [140])

$$\begin{cases} \max (x_1 - 2x_2, -x_1 + 3x_2) \\ \text{s.t} \\ x_1 - 2x_2 \leq 0 \\ x_1, x_2 \in \{0, 1, 2\}. \end{cases}$$

In their paper, Sylva and Crema [140] obtained a set of nondominated vectors by taking $\lambda = (4, 3)$, as they claimed for any $\lambda > 0$, the algorithm will work, though it is not the case. We changed the value of λ to (1, 2) and got the problem, as follows:

$$(P_0) \quad \begin{cases} \max z = -x_1 + 4x_2 \\ \text{s.t} \\ x_1 - 2x_2 \leq 0, x_1, x_2 \in \{0, 1, 2\}. \end{cases}$$

An optimal solution of this problem is $(0, 2)$ yielding the first efficient pair as $(-4, 6)$. Then P_1 is constructed as follows

$$(P_1) \quad \begin{cases} \max & -x_1 + 4x_2 \\ \text{s.t} & \\ & x_1 - 2x_2 \leq 0 \\ & x_1 - 2x_2 \geq y_1^1 - 4, -x_1 + 3x_2 \geq 9y_2^1 - 2, \\ & y_1^1 + y_2^1 \geq 1, \\ & x_1, x_2 \in \{0, 1, 2\}, y_1^1, y_2^1 \in \{0, 1\}. \end{cases}$$

An optimal solution of P_1 is $(1, 1)$, with yielding $z = 3$ the second non-dominated pair as $(-1, 2)$. In this process an efficient pair $(2, 2)$ will never be recorded by the algorithm as it is infeasible for the current problem, but this pair was recorded with earlier choice of $\lambda = (4, 3)$.

Now we will implement this algorithm on Example 1 also. So we take given problem in maximization form given as

$$\begin{cases} \max & (-2x_1 - x_2, x_1 - 2x_2) \\ \text{s.t} & \\ & 3x_1 + 2x_2 \geq 6, 4x_1 + 5x_2 \leq 20, \\ & x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

Now construct the problem (P_0) using the positive weights $\lambda_1 = 1$ and $\lambda_2 = 4$.

$$(P_0) \quad \begin{cases} \max & z = 2x_1 - 9x_2 \\ \text{s.t} & \\ & 3x_1 + 2x_2 \geq 6, 4x_1 + 5x_2 \leq 20, \\ & x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

An optimal solution of P_0 is $x_1 = 5, x_2 = 0$, which is also an efficient solution of *BILP*. To find second efficient solution, we need to construct P_1 . For this we calculate the values of $M_1 = 10$ and $M_2 = 8$ ($-M_1$ and $-M_2$ are lower bounds on the corresponding objective

functions). Therefore P_1 becomes

$$(P_1) \quad \begin{cases} \max z = 2x_1 - 9x_2 \\ \text{s.t} \\ 3x_1 + 2x_2 \geq 6, 4x_1 + 5x_2 \leq 20, \\ 2x_1 + x_2 + y_1^1 \leq 10, -x_1 + 2x_2 + 14y_2^1 \leq 8, \\ y_1 + y_2^1 \geq 1, y_1^1, y_2^1 \in \{0, 1\}, \\ x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z}. \end{cases}$$

It may be observed that to find second efficient solution we have to add three more constraints and two more variables. Further at every subsequent stage we need to add $3k$ more constraints and $2k$ more variables in addition to the existing ones, which makes this problem very complex. In the developed algorithm, only a single constraint is introduced into the original set of constraints at each stage. Thus, our proposed algorithm is easier to implement.

2.6 Computational Results

Table 2.6: Running to solve *BILP* for different values of m and n

No.of Constraints (m)	Number of variables (n)	Efficient points (k)	Running time(s) (seconds)
6	20	2285	337.14
5	7	429	29.84
1	20	1876	178.88
9	12	665	36.77
1	14	2306	87.76
2	4	229	8.77
10	39	6742	840.74
10	6	1	0.85
5	7	844	49.2
7	4	107	6.67
2.	6	1259	69.61
3	4	1	9.73
3	6	130	6.16
7	10	963	108.19
2	10	657	30
2	2	107	4.81
5	7	560	33.61

The developed algorithm is implemented on MATLAB 2018 (Intel Core i5 processor). It can be observed that the processing time also increases by increasing variables and

constraints.

2.7 Concluding Remarks

We have presented an epsilon constraint methodology for solving *BILP*. In the proposed method, we have chosen epsilon for *BILP*. Further, we have developed a complete set of non-dominated vectors by successively solving an integer linear programming problem.

The proposed methodology can be extended to solve a bounded variable multi-objective integer linear programming problems. Further, the developed approach can also be extended to solve *ILFP*.

Chapter 3

Multi-objective integer quadratic programming problem

3.1 Introduction

Quadratic programming problem (QP) belongs to a particular type of non-linear programming problems, and finds applications in the reactive power optimization model [65], microwave imaging [12], finance, [74, 134] and various other areas [66].

Generally, decision-making problems involve various conflicting objective functions. For instance, we can consider a portfolio with n risky assets, say $a_1, a_2, \dots, a_n$ with weight $w_1, w_2, \dots, w_n$, returns $r_1, r_2, \dots, r_n$ and standard deviation $\sigma_1, \sigma_2, \dots, \sigma_n$ respectively. In the portfolio selection problem, decision-maker wants to maximize a portfolio's expected return and minimize the risk of the portfolio. The expected return (μ) is a linear function given as $\mu = m^T w$ and risk can be measured as variance of the return of portfolio, is a quadratic function given as $\sigma = w^T C w$, where $w = \text{col}(w_1, w_2, \dots, w_n)$, $m^T = (\mu_1, \mu_2, \dots, \mu_n)$, $\mu_i = E(r_i)$, $C = [c_{i,j}]$, $c_{i,j} = \text{cov}(r_i, r_j)$, $c_{i,i} = \sigma_i^2$ for all $i \in \{1, 2, \dots, n\}$ [29]. Then, the mathematical model of the portfolio selection problem can be modeled as a multi-objective programming problem that involves a quadratic function given as

$$\begin{cases} \max (m^T w, -w^T C w) \\ \text{s.t. } e^T w = 1 \end{cases}$$

which is a special class of MQP .

3.2 Literature Review

Quadratic Programming problem (QP) is a widely studied research area, and various researchers [27, 59, 60, 114] have developed methods for solving the QP . A finite algorithm

⁰The content presented in this chapter is published as “An algorithm to solve multi-objective integer quadratic programming problem”, **Annals of Operations Research**, 2024, 332, 433–459 (SCI, Impact factor: **4.8**).

for a difference of convex functions QP is presented in [27]. A portfolio selection problem that maximizes the expected portfolio return while constraining the associated risk is considered in [59]. A strategy for the exact solution of multi-parametric quadratically constrained QP is described in [114]. A convex QP with simultaneous perturbation in the objective function's linear term and the right-hand side of the constraints is considered in [60]. A sequential quadratic programming (SQP) method for constrained multi-objective optimization is developed in [6]. In SQP method, a feasible descent direction is obtained at each iteration by solving a quadratic sub-problem.

In many practical situations, decision variables are restricted to integers. So, researchers have also shown interest in the solution of IQP . An algorithm for finding the global minimum of IQP is explained in [54]. A ranking approach strategy for solving IQP is presented in [67]. An effective branch-and-bound algorithm for convex IQP is described in [25].

In addition to single objective QP , many decision-making problems involve more than one quadratic objective functions. Such type of programming problems fall in the category of Multi-objective Quadratic Programming problem (MQP). The applications of MQP and algorithms to solve MQP can be found in these papers [55, 88, 108, 119, 125, 138]. A real life example of MQP in the field of financial analysis is discussed in [125], where authors considered a portfolio selection model with multiple quadratic and linear objectives. The usage of MQP in the tri-criterion portfolio selection problem is discussed in [138]. A non-archimedean interior point method for solving lexicographic MQP is described by [55]. The pareto front of multi-objective optimization problem having convex objectives and linear constraints is obtained in [108] using multi-parametric programming by utilizing ϵ -constraint method and application of linear constraints. A fuzzy goal programming procedure for solving MQP is presented by [88, 119].

Further, the MQP with decision variables as integers is a class of $MIQP$ and researchers have also drawn attention towards investigating all the efficient solutions of $MIQP$. Unlike MQP , the class of $MIQP$ is not much studied in literature. [109] presented a method for solving a class of $MIQP$ where the Hessian matrix of objective functions are positive semidefinite. Their methodology generates efficient solutions by using efficient cuts along with a branch and bound technique. Another special case of $MIQP$ is analyzed in the papers of [9] and [46]. In these papers, authors have presented algorithms for solving multi-objective integer indefinite quadratic programming problem ($MIIdQP$). Various cuts are imposed over the feasible region in their algorithms, and indefinite quadratic programming problem ($IdQP$) is solved at each iteration using a simplex-like method. Multi-objective Quadratic Assignment Problem ($MQAP$), which is a particular type of $MIQP$, where

decision variables are either 0 or 1, is considered in [42, 83]. The application of *MQAP* in the context of military aircraft can be seen in [42]. Another practical application of *MQAP* in the context of multiobjective combinatorial landscapes can be observed in the paper of [83].

3.3 Motivation and Methodology

As far as our knowledge, only a few methods have appeared in literature that deal with a particular class of *MIQP* and no paper available in the literature that deals with general *MIQP*. Further, there are serious flaws in existing algorithms [9, 46] that solves *MIIdQP*. In both of these papers [9, 46], methodologies are based on the assumption that optimal solution of *IdQP* is always obtained at a corner point of the polyhedron region. So they used a simplex-like technique to solve *IdQP* at each iteration. A complete discussion on the class of programming problems that finds their optima at a vertex of a polyhedron region is done by [101], where the author pointed out that the vertex search method can be applied to a *NLP*, if the objective function is explicit quasi-monotone. The objective function of *IdQP* does not belong to the family of explicit quasi-monotone functions. Thus, a simplex-like technique cannot be used for finding an optimal solution of *IdQP* considered in [9, 46]. Hence, their algorithm fails to provide the correct set of all efficient solutions of *MIIdQP*. We have presented two counter examples in support of this claim in Section 3.6. Further, the extension of [46] methodology is used by [102] to solve multiobjective quadratic fractional programming problem (*MIQFP*). Since the underlying methodology for quadratic case is not working, so its generalization also fails to provide the complete efficient set of *MIQFP* as discussed by [86].

So, problem of finding efficient solutions even for the special class of *MIQP* considered in [9, 46] is still open. From all these considerations, we got a motivation to solve a general *MIQP*. The present methodology successfully generates the complete set of efficient solutions of general *MIQP* and hence also generates the complete efficient frontier of the special class of *MIQP* considered in the paper of [9, 46]. The proposed method generates the efficient set of *MIQP* by scanning ranked solutions of *IQP*. For finding ranked solutions of *IQP*, we have constructed a related *ILP*, whose objective function value provides an upper bound to the objective function value of *IQP* for all feasible solutions. The ranking approach is adopted from [67] and its generalizations are studied by [75, 131].

This chapter is divided into different sections. In Section 3.4, we have discussed the notations used throughout this chapter. In Section 3.5, we have established theoretical results and proposed an algorithm to solve *MIQP*. In Section 3.6 and 3.8, numerical and

computational results shows the effectiveness of proposed scheme. In the last section, we have discussed the concluding remarks on the developed scheme.

3.4 Notations and Preliminary Results

The problem under consideration is

$$(MIQP) \quad \max_{X \in \mathcal{X}} (f^1(X), f^2(X), \dots, f^s(X)),$$

where $f^i(X) = X^T D^i X + C^i X + \alpha^i$, $D^i \in \mathbb{Z}^{n \times n}$, $C^i \in \mathbb{Z}^{1 \times n}$ and $\alpha^i \in \mathbb{R}$ for all $i \in \{1, 2, \dots, s\}$.

Definition 3.4.1 *A point $Y \in \mathcal{X}$ is called efficient solution of $MIQP$ if there does not exist any $X \in \mathcal{X}$ such that $f^i(X) \geq f^i(Y) \forall i \in \{1, 2, \dots, s\}$ and $f^i(X) > f^i(Y)$ for at least one $i \in \{1, 2, \dots, s\}$.*

Theorem 3.4.1 *If $MIQP$ has a non-empty bounded feasible region $\mathcal{X}$, then the efficient set E of $MIQP$ is also non-empty.*

Consider a problem

$$(IQP^i) \quad \max_{X \in \mathcal{X}} f^i(X) = X^T D^i X + C^i X + \alpha^i$$

defined for any of the i^{th} objective of $MIQP$. If $MIQP$ has a non-empty bounded feasible region $\mathcal{X}$, then one of the optimal solutions of IQP^i will be an efficient solution of $MIQP$.

Next, for finding efficient solutions, we considered the problem

$$(IQP) \quad \max_{X \in \mathcal{X}} f^1(X) = X^T D^1 X + C^1 X + \alpha^1.$$

In the subsequent discussion, we need to use IQP to generate the complete set of all efficient solutions of $MIQP$, where for sake of simplicity, we have used the first objective function to define the problem IQP , but one can take any of the objective functions to define this problem.

Now, for finding an optimal solution of IQP , we have constructed the following problem, as discussed in [67] for minimization case, but for maximization case, we are redefining it.

$$(ILP) \quad \max_{X \in \mathcal{X}} g^1(X),$$

where $g^1(X) = V^1X + C^1X + \alpha^1$, $V \in \mathbb{Z}^{1 \times n}$ is constructed as described below.

Let D_j^1 be the j^{th} column of the matrix D^1 , then the j^{th} column V_j^1 of the vector V^1 is defined as

$$\begin{aligned}
V_j^1 &= \max_{X \in \chi} X^T D_j^1 \text{ for all } j \in \{1, 2, \dots, n\} \\
\implies V_j^1 &\geq X^T D_j^1 \text{ for all } X \in \chi, j \in \{1, 2, \dots, n\} \\
\implies V^1 X &\geq X^T D^1 X \text{ for all } X \in \chi \\
\implies V^1 X + C^1 X + \alpha^1 &\geq X^T D^1 X + C^1 X + \alpha^1 \forall X \in \chi \\
\text{i.e. } g^1(X) &\geq f^1(X) \forall X \in \chi
\end{aligned} \tag{3.4.1}$$

To rank the feasible solutions of ILP , we have considered the following restricted problem.

$$(ILP_l) \quad \max_{X \in \chi_l} g^1(X),$$

where $\chi_l = \{X \in \chi \mid g^1(X) \leq Z^{l-1} - 1\}$, Z^{l-1} is the objective function value at $(l-1)^{th}$ ranked solution of ILP .

Remark 3.4.1 *The l^{th} ranked solution of ILP is the optimal solution of ILP_l and objective function value at l^{th} ranked solution of ILP is equal to the optimal value of ILP_l .*

Some Notations

L^l is the set of all l^{th} ranked integer feasible solutions of ILP .

F^j is the set of all j^{th} ranked integer feasible solutions of IQP .

E is the set of all efficient solutions of $MIQP$.

$F^N = \{Y \in \chi \mid f^1(Y) = \min f^1(X) \text{ for all } X \in \chi\}$.

3.5 Theoretical development

First of all, we have derived the relation between ranked solutions of IQP and the efficient set of $MIQP$.

Theorem 3.5.1 *If $\hat{X} \in E$, then $\hat{X} \in F^j$ for some $j \in (1, 2, \dots, N)$.*

Proof As $\hat{X} \in E$ and $E \subset \chi$ therefore $\hat{X} \in \chi$. If possible let $\hat{X} \notin F^j$ for all $j \in (1, 2, \dots, N)$ therefore $\hat{X} \notin \cup_{j=1}^N F^j$. Since $\hat{X} \in \chi$ and $\hat{X} \notin \cup_{j=1}^N F^j$ therefore

$$f^1(\hat{X}) < f^1(X) \text{ for all } X \in \cup_{j=1}^N F^j \tag{3.5.1}$$

Since $F^N = \{Y \in \chi \mid f^1(Y) = \min f^1(X) \text{ for all } X \in \chi\}$ and $\hat{X} \notin F^N$ therefore

$$f^1(Y) < f^1(\hat{X}) \text{ for all } Y \in F^N$$

which contradict Equation (3.5.1). Thus if $\hat{X} \in E$, then $\hat{X} \in F^j$ for some $j \in (1, 2, \dots, N)$

By the above theorem, we concluded that the efficient solution of $MIQP$ is one of the ranked solution of IQP .

Remark 3.5.1 *If $\hat{X} \in F^j$ for some $j \in (1, 2, \dots, N)$ and there does not exist any $X \in \cup_{j=1}^N F^j$ such that $f^i(X) \geq f^i(\hat{X})$ for all $i \in \{1, 2, \dots, s\}$ and $f^i(X) > f^i(\hat{X})$ for at least one $i \in \{1, 2, \dots, s\}$, then by the definition of the efficient set $\hat{X} \in E$.*

In the light of the above discussion, it is evident that the efficient solutions of $MIQP$ are the members of the set of all ranked solutions of IQP having a non-dominated criterion vector. In the following, we'll focus our discussion on enumerating the set of ranked solutions of IQP . The following theorem shows the relation between an optimal solution of IQP and ranked solutions of ILP .

Theorem 3.5.2 [67] *If $g^1(Y) \leq \max\{f^1(X) \mid X \in \cup_{i=1}^l L^i\} = f^1(X^1)$ for some $Y \in L^l$, then X^1 is an optimal solution of IQP .*

Corollary 3.5.3 *If $g^1(Y) \leq \max\{f^1(X) \mid X \in \cup_{i=1}^l L^i\} = f^1(X^1)$ for some $Y \in L^l$, then $F^1 = \{X \in \cup_{i=1}^l L^i \mid f^1(X) = f^1(X^1)\}$.*

Proof We know F^1 is the set of all optimal solutions of IQP . Let $F = \{X \in \cup_{i=1}^l L^i \mid f^1(X) = f^1(X^1)\}$. We need to show $F = F^1$. Since $g^1(Y) \leq \max\{f^1(X) \mid X \in \cup_{i=1}^l L^i\} = f^1(X^1)$ for some $Y \in L^l$, then by Theorem 3.5.2, $X^1 \in F^1$. As $X^1 \in F^1$ and $F = \{X \in \cup_{i=1}^l L^i \mid f^1(X) = f^1(X^1)\}$. Therefore,

$$F \subseteq F^1 \tag{3.5.2}$$

Now we will show $F^1 \subseteq F$. Let $\hat{X} \in F^1$ therefore

$$f^1(\hat{X}) \geq f^1(X) \text{ for all } X \in \chi \tag{3.5.3}$$

Let, if possible $\hat{X} \notin F$ which implies $\hat{X} \notin \cup_{i=1}^l L^i$. As $\hat{X} \notin \cup_{i=1}^l L^i$ and $Y \in L^l$ therefore

$$\begin{aligned} & g^1(Y) > g^1(\hat{X}) \\ \implies & f^1(X^1) > g^1(\hat{X}) \quad (\because f^1(X^1) \geq g^1(Y)) \\ \implies & f^1(X^1) > f^1(\hat{X}) \quad (\because g^1(X) \geq f^1(X) \text{ for all } X \in \chi) \end{aligned}$$

which contradict Equation (3.5.3). Thus,

$$F^1 \subseteq F \quad (3.5.4)$$

Hence $F^1 = \{X \in \cup_{i=1}^l L^i \mid f^1(X) = f^1(X^1)\}$ (by eqns (3.5.2) and (3.5.4)).

Remark 3.5.2 *By Corollary 3.5.3, we conclude that in order to find the set of all optimal solutions of IQP we do not need to rank ILP beyond l^{th} ranked solution of ILP provided there exists some $Y \in L^l$ such that $g^1(Y) \leq \max\{f^1(X) \mid X \in \cup_{i=1}^l L^i\}$.*

Now, consider the problem :

$$(IQP_j) \quad \begin{cases} \max f^1(X) \\ \text{s.t} \\ f^1(X) < f^1(X^{j-1}) \text{ for some } X^{j-1} \in F^{j-1} \\ X \in \mathcal{X}. \end{cases}$$

We can observe that a feasible solution X^j is the j^{th} ranked solution of IQP i.e. $X^j \in F^j$ for some $j > 1$ if X^j is an optimal solution of IQP_j .

In the following theorem, we have established the relation between j^{th} ranked solution of IQP and l^{th} ranked solution of ILP.

Theorem 3.5.4 *If $g^1(Y) \leq \max\{f^1(X) \mid f^1(X) < f^1(X^{j-1}), X \in \cup_{i=1}^l L^i\} = f^1(X^j)$ for some $Y \in L^l$, $X^{j-1} \in F^{j-1}$, then X^j is the j^{th} ranked solution of IQP.*

Proof Let, if possible X^j be not a j^{th} ranked solution of IQP, therefore X^j is not an optimal solution of IQP_j . By hypothesis $X^j \in \cup_{i=1}^l L^i$ and $f^1(X^j) < f^1(X^{j-1})$ for some $X^{j-1} \in F^{j-1}$, therefore X^j is a feasible solution of the IQP_j . Since X^j is a feasible solution but not an optimal solution of the problem (IQP_j) , therefore there exists a feasible solution $\check{X}$ of the problem (IQP_j) such that

$$f^1(\check{X}) < f^1(X^{j-1}) \text{ \& } f^1(\check{X}) > f^1(X^j) \quad (3.5.5)$$

Now, there are two possibilities either $\check{X} \in \cup_{j=1}^l L^i$ or $\check{X} \in L^i$ for some $i > l$. Suppose $\check{X} \in \cup_{j=1}^l L^i$, also we have $f^1(\check{X}) < f^1(X^{j-1})$, therefore by hypothesis

$$f^1(X^j) \geq f^1(\check{X}) \quad (3.5.6)$$

Equation (3.5.6) contradicts Equation (3.5.5). So $\check{X} \in L^i$ for some $i > l$. As $Y \in L^l$ and

$\check{X} \in L^i$ for some $i > l$ therefore

$$g^1(Y) > g^1(\check{X})$$

Also, it is given that

$$f^1(X^j) \geq g^1(Y)$$

Therefore,

$$f^1(X^j) > g^1(\check{X}) \quad (3.5.7)$$

As function g provides upper bound to the values of function f for all $X \in \mathcal{X}$, therefore

$$g^1(\check{X}) \geq f^1(\check{X}) \quad (3.5.8)$$

By using eqns (3.5.7) and (3.5.8), we get

$$f^1(X^j) > f^1(\check{X}) \quad (3.5.9)$$

which contradicts Equation(3.5.5). Thus, X^j is the j^{th} ranked solution of IQP .

Corollary 3.5.5 *If $g^1(Y) \leq \max\{f^1(X) \mid f^1(X) < f^1(X^{j-1}), X \in \cup_{i=1}^l L^i\} = f^1(X^j)$ for some $Y \in L^l$, $X^{j-1} \in F^{j-1}$, then $F^j = \{X \in \cup_{i=1}^l L^i \mid f^1(X) = f^1(X^j)\}$*

Remark 3.5.3 *By Corollary 3.5.5, the set of all j^{th} ranked solutions of IQP can be obtained from ILP without finding all ranked solutions of ILP , provided there exists some $Y \in L^l$ and $X^{j-1} \in F^{j-1}$ such that $g^1(Y) \leq \max\{f^1(X) \mid f^1(X) < f^1(X^{j-1}), X \in \cup_{i=1}^l L^i\} = f^1(X^j)$. It may be noted that by all j^{th} ranked solutions, we mean all solutions (including alternate solutions) corresponding to the j^{th} ranked solution of the problem IQP , and to find these solutions we need to find ranked solutions of ILP upto a certain stage l .*

In the following discussion, we have presented an algorithm for enumerating efficient solutions of $MIQP$. As efficient points are the elements of F^j , $j = 1, 2, \dots, N$, we'll show that none of the ranked solution is missed by the algorithm. Further, we'll discuss that the proposed algorithm enumerates the complete set of all efficient solutions of $MIQP$ in a finite steps by scanning ranked solutions of IQP .

Theorem 3.5.6 *The developed algorithm generates complete set of all ranked solutions of IQP .*

Proof Let $\{F^1, F^2, \dots, F^N\}$ be the sets of all ranked solutions of IQP obtained by the proposed algorithm, and the elements of the set F^j for each $j = 1, 2, \dots, N$ be $\{X_i^j \mid i = 1, 2, \dots, j_k\}$. For the sake of simplicity, let point X^j to be the representative of the set F^j ,

Algorithm 3: Algorithm to solve $MIQP$

Data: $m, n, s \in \mathbb{Z}$, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$,
 $D^i \in \mathbb{Z}^{n \times n}$, $C^i \in \mathbb{Z}^{1 \times n}$ and $\alpha^i \in \mathbb{R} \forall i \in \{1, 2, \dots, s\}$
Result: E ▷ set of all efficient solutions of $MIQP$

```

1 Initialization  $j \leftarrow 1, l \leftarrow 1, E \leftarrow [], E^1 \leftarrow []$ 
2  $\chi \leftarrow \{X \in \mathbb{Z}^n \mid AX = b, X \geq 0\}$ 
3 for  $i \leftarrow 1 : s$  do
4    $f^i(X) \leftarrow X^T D^i X + C^i X + \alpha^i$  ▷  $IQP \quad \max_{X \in \chi} f^1(X)$ 
5 for  $i \leftarrow 1 : n$  do
6    $V^1(1, i) \leftarrow \max_{X \in \chi} X^T D^1(:, i)$ 
7  $g^1(X) \leftarrow V^1 X + C^1 X + \alpha^1$  ▷  $ILP \quad \max_{X \in \chi} g^1(X)$ 
8  $g_{min}^1 \leftarrow \min_{X \in \chi} g^1(X)$ 
9  $L^1 \leftarrow \{Y \in \chi \mid g^1(Y) = \max_{X \in \chi} g^1(X)\}$  ▷ set of optimal solutions of  $ILP$ 
10  $Z^1 \leftarrow g^1(Y)$  for  $Y \in L^1$  ▷ optimal value of  $ILP$ 
11  $f_{max}^1 \leftarrow \max\{f^1(X) \mid X \in L^1\}$ 
12 while  $g^1(Y) > f_{max}^1$  and  $g^1(Y) > g_{min}^1$  for  $Y \in L^l$  do
13    $l \leftarrow l + 1$ 
14    $\chi_l \leftarrow \{X \in \chi \mid g^1(X) \leq Z^{l-1} - 1\}$ 
15    $L^l \leftarrow \{Y \in \chi_l \mid g^1(Y) = \max_{X \in \chi_l} g^1(X)\}$  ▷ set of  $l^{th}$  ranked solutions of  $ILP$ 
16    $Z^l \leftarrow g^1(Y)$  for  $Y \in L^l$ 
17    $f_{max}^1 \leftarrow \max\{f^1(X) \mid X \in \cup_{i=1}^l L^i\}$ 
18  $F^1 \leftarrow \{Y \in \cup_{i=1}^l L^i \mid f^1(Y) = f_{max}^1\}$  ▷ set of optimal solutions of  $IQP$ 
19  $f_{min}^1 \leftarrow \min\{f^1(X) \mid X \in \cup_{i=1}^l L^i\}$ 
20 foreach  $X \in F^1$  do
21   foreach  $Y \in F^1 \cap (E^1)^c$  do
22     if  $f^i(Y) \geq f^i(X) \forall i \in \{2, 3, \dots, s\}$  and  $f^i(Y) > f^i(X)$  for some  $i \in \{1, 2, \dots, s\}$  then
23        $E^1 \leftarrow [E^1 \ X]$ 
24   if  $X \notin E^1$  then
25      $E \leftarrow [E \ X]$  ▷ efficient solutions of  $MIQP$  in  $F^1$ 
26 while  $g^1(Y) > g_{min}^1$  for some  $Y \in L^l$  or  $f^1(X^j) > f_{min}^1$  for some  $X^j \in F^j$  do
27    $j \leftarrow j + 1$ 
28   if  $\{X \mid f^1(X) < f^1(X^{j-1}), X \in \cup_{i=1}^l L^i, X^{j-1} \in F^{j-1}\} = \emptyset$  then
29      $f_{max}^1 \leftarrow -\infty$ 
30   else
31      $f_{max}^1 \leftarrow \max\{f^1(X) \mid f^1(X) < f^1(X^{j-1}), X \in \cup_{i=1}^l L^i, X^{j-1} \in F^{j-1}\}$ 
32   while  $g^1(Y) > f_{max}^1$  and  $g^1(Y) > g_{min}^1$  for  $Y \in L^l$  do
33      $l \leftarrow l + 1$ 
34      $\chi_l \leftarrow \{X \in \chi \mid g^1(X) \leq Z^{l-1} - 1\}$ 
35      $L^l \leftarrow \{Y \in \chi_l \mid g^1(Y) = \max_{X \in \chi_l} g^1(X)\}, Z^l \leftarrow g^1(Y)$  for  $Y \in L^l$ 
36      $f_{max}^1 \leftarrow \max\{f^1(X) \mid f^1(X) < f^1(X^{j-1}), X \in \cup_{i=1}^l L^i, X^{j-1} \in F^{j-1}\}$ 
37    $F^j \leftarrow \{Y \in \cup_{i=1}^l L^i \mid f^1(Y) = f_{max}^1\}$  ▷ set of  $j^{th}$  ranked solutions of  $IQP$ 
38    $f_{min}^1 \leftarrow \min\{f^1(X) \mid X \in \cup_{i=1}^l L^i\}$ 
39    $E^j \leftarrow \emptyset$ 
40   foreach  $X \in F^j$  do
41     foreach  $Y \in E \cup (F^j \cap (E^j)^c)$  do
42       if  $f^i(Y) \geq f^i(X) \forall i \in \{2, 3, \dots, s\}$  and  $f^i(Y) > f^i(X)$  for some  $i \in \{1, 2, \dots, s\}$  then
43          $E^j \leftarrow [E^j \ X]$  ▷ Set of non-efficient solutions of  $MIQP$  in  $F^j$ 
44     if  $X \notin E^j$  then
45        $E \leftarrow [E \ X]$  ▷ Set of efficient solutions of  $MIQP$  in  $\cup_{i=1}^j F^i$ 

```

for all $j \in \{1, 2, \dots, N\}$.

$$\begin{aligned}
\text{So, we have } f^1(X^1) &= \max\{f^1(X) \mid X \in \chi\} \\
f^1(X^N) &= \min\{f^1(X) \mid X \in \chi\} \\
\text{and } f^1(X^1) &> f^1(X^2) > \dots > f^1(X^N)
\end{aligned} \tag{3.5.10}$$

If possible, let $\hat{X}$ be a ranked solution of IQP , which is missed by the proposed algorithm, then the following possibilities may arise

- i) $f^1(\hat{X}) < f^1(X^N)$,
- ii) $f^1(\hat{X}) > f^1(X^1)$
- iii) $f^1(X^k) < f^1(\hat{X}) < f^1(X^{k-1})$ for some $k \in \{1, 2, \dots, N\}$

The first two possibilities cannot arise as it will straightway contradict Equation (3.5.10). Suppose $f^1(X^k) < f^1(\hat{X}) < f^1(X^{k-1})$ for some $k \in \{1, 2, \dots, N\}$. Consider the following problem

$$(IQP_k) \quad \begin{cases} \max f^1(X) \\ \text{s.t} \\ f^1(X) < f^1(X^{k-1}) \\ X \in \chi \\ \text{for some } X^{k-1} \in F^{k-1}. \end{cases}$$

As $f^1(\hat{X}) < f^1(X^{k-1})$ and $\hat{X} \in \chi$, therefore $\hat{X}$ is a feasible solution of problem (IQP_k) . Also, $X^k \in F^k$, therefore X^k is an optimal solution of problem (IQP_k) , therefore $f^1(X^k) \geq f^1(\hat{X})$, which contradicts $f^1(\hat{X}) > f^1(X^k)$.

Thus, the algorithm enumerates all ranked solutions of IQP .

In the following theorem, we will show that the proposed algorithm generates the set of all efficient solutions of $MIQP$.

Theorem 3.5.7 *The algorithm records all the efficient solutions of $MIQP$.*

Proof Let $\hat{Y} \in \chi$ be an efficient solution of $MIQP$, then by virtue of Theorem 3.5.1, $\hat{Y} \in F^j$ for some $j \in \{1, 2, \dots, N\}$. Let, if possible $\hat{Y} \in E^j$. As $\hat{Y} \in E^j$, therefore there exists some $\bar{Y} \in E \cup (F^j \cap (E^j)^c)$ such that

$$\begin{aligned} f^i(\bar{Y}) &\geq f^i(\hat{Y}) \text{ for all } i \in \{1, 2, \dots, s\} \\ f^i(\bar{Y}) &> f^i(\hat{Y}) \text{ for some } i \in \{1, 2, \dots, s\}. \end{aligned} \tag{3.5.11}$$

As $\hat{Y}$ is an efficient solutions of $MIQP$, therefore there exists no $X \in \chi$ such that

$$\begin{aligned} f^i(X) &\geq f^i(\hat{Y}) \text{ for all } i \in \{1, 2, \dots, s\} \\ f^i(X) &> f^i(\hat{Y}) \text{ for some } i \in \{1, 2, \dots, s\}. \end{aligned}$$

As $E \cup (F^j \cap (E^j)^c) \subset \chi$, therefore there exists no $X \in F^j \cup E$ such that

$$\begin{aligned} f^i(X) &\geq f^i(\hat{Y}) \text{ for all } i \in \{1, 2, \dots, s\} \\ f^i(X) &> f^i(\hat{Y}) \text{ for some } i \in \{1, 2, \dots, s\}. \end{aligned}$$

which contradicts Equation (3.5.11). So, $\hat{Y} \notin E^j$. As $\hat{Y} \in F^j$ and $\hat{Y} \notin E^j$ for some $j \in \{1, 2, \dots, N\}$. Therefore $\hat{Y}$ is recorded as an efficient solution by the algorithm. Thus, the algorithm records all the efficient solutions of $MIQP$

Remark 3.5.4 *Since the feasible regions of ILP and IQP are assumed to be closed and bounded. Therefore, there are finite integer points. Moreover, the algorithm solves at most l number of ILP, and from the solutions of these problems, a set of ranked solutions of IQP are enumerated, which are further finite in number. We deduce that the method terminates in a finite step and provides the entire efficient solutions of $MIQP$, as the set of efficient solutions is a subset of all ranked solutions.*

3.6 Numerical Section

In this section, we have solved four $MIQP$ problems using the proposed algorithm. In Example 3.6.1, we have considered a general $MIQP$ (whose objective functions cannot be expressed as a product of linear functions) of two variables and have elaborated the steps of the algorithm to solve this problem.

The Example 3.6.2 is taken from [9], it contains four variables and we have shown that our algorithm works well for this problem also. Moreover, our algorithm produces complete set of efficient solutions (total eleven efficient solutions), where as the algorithm of [9] generates only four efficient solutions.

Further, the Examples 3.6.3 & 3.6.4 serve as counter examples to the approach of [46] and [9]. The Example 3.6.3 counters the claim that their algorithm calculates all efficient solutions of the problem and Example 3.6.4 counters the claim that their algorithm generates only efficient points, whereas the proposed algorithm solve these problems correctly and generate the complete set of efficient solutions.

3.6.1 Solution of general $MIQP$ using proposed algorithm

Example 3.6.1 Consider the following problem

$$(MIQP) \quad \begin{cases} \max f^1(x_1, x_2) = x_1^2 + 9x_2^2 - 2x_1 - 16x_2 + 5 \\ \max f^2(x_1, x_2) = x_1^2 - x_2^2 + 5x_1 + 20 \\ \max f^3(x_1, x_2) = x_1^2 + 3x_2^2 + x_1x_2 + 4 \\ \text{s.t} \quad (x_1, x_2) \in \chi = S \cap \mathbb{Z}^2, \end{cases}$$

where $S = \{(x_1, x_2) \mid 2x_1 + 5x_2 \leq 16, x_1 + x_2 \leq 5, x_1, x_2 \geq 0\}$.

To find the efficient set of $MIQP$, we have used the ranking of the following integer quadratic programming problem (IQP).

$$(IQP) \quad \begin{cases} \max f^1(x_1, x_2) = x_1^2 + 9x_2^2 - 2x_1 - 16x_2 + 5 \\ \text{s.t} \quad (x_1, x_2) \in \chi. \end{cases}$$

- 1 (a) To find an optimal solution of IQP , first of all we constructed the following linear programming problem (ILP) using Equation (3.4.1)

$$(ILP) \quad \max_{X \in \chi} g^1(X),$$

where $g^1(X) = 3x_1 + 11x_2 + 5$. The optimal solution of ILP is $\{(0, 3)\}$ with objective function value 38, so $L^1 = \{(0, 3)\}$.

- (b) As $g^1(0, 3) = f^1(0, 3) = 38$. $F^1 = \{(0, 3)\}$.

- (c) The set F^1 has only one element $(0, 3)$, therefore $E^1 = \emptyset$ and $E = \{(0, 3)\}$.

- 2 (a) $g_{min}^1 = \min\{g^1(X) \mid X \in \chi\} = 5$ and $g^1(0, 3) = 38$. As $g^1(0, 3) > g_{min}^1$. We set $j = j + 1$.

- (b) Since $\{X \mid f^1(X) < f^1(0, 3), X \in L^1\} = \emptyset$, we set $f_{max}^1 = -\infty$. As $g^1(0, 3) > f_{max}^1$ and $g^1(0, 3) > g_{min}^1$, therefore we set $l = 2$ and solved the following problem (ILP_2).

$$(ILP_2) \quad \max_{X \in \chi_2} g^1(X),$$

where $\chi_2 = \{X \in \chi \mid g^1(X) \leq 37\}$. After solving problem (ILP_2), we get $L^2 = \{(3, 2)\}$, $g^1(3, 2) = 36$, $f^1(3, 2) = 12$ and $Z^2 = 36$.

- (c) $g^1(3, 2) > \max\{f^1(X) \mid f^1(X) < f^1(X^{j-1}), X \in \cup_{i=1}^2 L^i, X^{j-1} = (0, 3)\}$ and $g^1(Y) > g_{min}^1$ for all $Y \in L^2$. To find set F^2 , we set $l = 3$ and solved the following

problem ILP_3 .

$$(ILP_3) \quad \max_{X \in \chi_3} g^1(X),$$

where $\chi_3 = \{X \in \chi \mid g^1(X) \leq 35\}$. Continue in the same way, we get higher

Table 3.1: Ranked solutions of ILP (Example 3.6.1)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
L^i	(0, 3)	(3,2)	(2,2)	(1,2)	(4,1)	(0,2)	(3,1)	(2,1)	(5,0)	(1,1)	(4,0)	(0,1)	(3,0)	(2,0)	(1,0)	(0,0)
$g^1(X)$	38	36	33	30	28	27	25	22	20	19	17	16	14	11	8	5
$f^1(X)$	38	12	9	8	6	9	1	-2	20	-3	13	-2	8	5	4	5

Table 3.2: Ranked solutions of IQP and efficient set E (Example 3.6.1)

j	F^j	(f^1, f^2, f^3)	Dominated/ Non- dominated criteria vector	E
1	(0,3)	(38,11,31)	Non- dominated criteria vector	$\{(0,3)\}$
2	(5,0)	(20,70,29)	Non- dominated criteria vector	$\{(0,3),(5,0)\}$
3	(4,0)	(13,56,20)	Dominated by (20, 70, 29)	$\{(0,3),(5,0)\}$
4	(3,2)	(12,40,31)	Non- dominated criteria vector	$\{(0,3),(5,0),(3,2)\}$
5	(2,2)	(9,30,24)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
	(0,2)	(9,16,16)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
6	(1,2)	(8,22,19)	Dominated by (20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
	(3,0)	(8,44,13)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
7	(4,1)	(6,55,27)	Dominated by (20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
8	(2,0)	(5,34,8)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
	(0,0)	(5,20,4)	Dominated by (20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
9	(1,0)	(4,26,5)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
10	(3,1)	(1,43,19)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
11	(2,1)	(-2,33,13)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
	(0,1)	(-2,19,7)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$
12	(1,1)	(-3, 25,9)	Dominated by(20, 70, 29)	$\{(0,3),(5,0),(3,2)\}$

ranked solutions of ILP as shown in Table 3.1.

- (d) As $g^1(5, 0) = \max\{f^1(X) \mid f^1(X) < f^1(0, 3), X \in \cup_{i=1}^9 L^i\} = f^1(5, 0)$, therefore $F^2 = \{X \in \cup_{i=1}^9 L^i \mid f^1(X) = f^1(5, 0)\} = \{(5, 0)\}$
- (e) The set F^2 has only one element (5,0) and criteria vector (20,70,29) corresponding to (5,0) is not dominated by criteria vector (38,11,31) corresponding to (0,3), therefore $E^2 = \emptyset$ and $E = \{(0, 3), (5, 0)\}$. Continue on the same line, we have obtained the higher-ranked solutions of IQP and the efficient set E (as shown in Table 3.2). As $g^1(Y) = \min\{g^1(X) \mid X \in \chi\}$ for $Y = (0, 0) \in L^{16}$ and $f^1(X^{12}) = \min\{f^1(X) \mid X \in \cup_{i=1}^{16} L^i\}$ for $X^{12} = (1, 1) \in F^{12}$, therefore algorithm terminates, and we get $E = \{(0, 3), (5, 0), (3, 2)\}$.

3.6.2 Solution of $MIQP$ considered in [9] using proposed algorithm

Example 3.6.2 ([9]) Consider the following problem

$$(MIQP) \quad \begin{cases} \max f^1(x_1, x_2, x_3, x_4) = (x_1 + x_2 - 2x_3 - 3x_4 + 15)(-x_2 + x_3 + x_4 + 10) \\ \max f^2(x_1, x_2, x_3, x_4) = (x_1 + x_2 - x_3 - 2x_4 + 12)(-x_2 + 2x_3 + 2x_4 + 11) \\ \max f^3(x_1, x_2, x_3, x_4) = (x_1 - 3x_2 - x_3 + x_4 + 9)(2x_2 + 2x_3 + 2x_4 + 9) \\ \text{s.t.} \quad (x_1, x_2, x_3, x_4) \in \chi = S \cap \mathbb{Z}^4, \end{cases}$$

where $S = \{(x_1, x_2, x_3, x_4) \mid 3x_1 - 2x_2 + x_3 + x_4 \leq 9, x_2 + 3x_3 + x_4 \leq 7, x_1 + x_2 + x_3 + x_4 \leq 12, 2x_2 + x_3 \leq 10, 0 \leq x_1 \leq 5, 0 \leq x_2 \leq 3, 0 \leq x_3 \leq 3, 0 \leq x_4 \leq 1\}$.

To find the efficient set of $MIQP$, we have used the ranking of the following integer quadratic programming problem (IQP).

$$(IQP) \quad \begin{cases} \max_{(x_1, x_2, x_3, x_4) \in \chi} f^1(x_1, x_2, x_3, x_4) = (x_1 + x_2 - 2x_3 - 3x_4 + 15)(-x_2 + x_3 + x_4 + 10) \\ \text{s.t.} \quad (x_1, x_2, x_3, x_4) \in \chi. \end{cases}$$

- 1 (a) To find an optimal solution of IQP , first of all we constructed the following linear programming problem (ILP) using Equation (3.4.1)

$$(ILP) \quad \max_{X \in \chi} g^1(X),$$

where $g^1(X) = \frac{23x_1}{2} + 2x_3 - \frac{13x_4}{2} + 150$.

The optimal solution of ILP is $\{(5, 3, 0, 0)\}$ with objective function value 207.5, so $L^1 = \{(5, 3, 0, 0)\}$.

- (b) $g_{min}^1 = \min\{g^1(X) \mid X \in \chi\} = 143.5$. As $g^1(5, 3, 0, 0) > f^1(5, 3, 0, 0)$ and $g^1(5, 3, 0, 0) > g_{min}^1$. We set $l = 2$ and solved the following problem (ILP_2).

$$(ILP_2) \quad \max_{X \in \chi_2} g^1(X),$$

where $\chi_2 = \{X \in \chi \mid g^1(X) \leq 206\}$.

- (c) The problem ILP_2 has two optimal solutions namely $(4, 3, 1, 0)$ and $(4, 2, 1, 0)$ with objective function value 198. So $L^2 = \{(4, 3, 1, 0), (4, 2, 1, 0)\}$. $g^1(Y) > \max\{f^1(X) \mid X \in \cup_{i=1}^2 L^i\}$ and $g^1(Y) > g_{min}^1$ for all $Y \in L^2$. We set $l = 3$ and solved the following

problem ILP_3 .

$$(ILP_3) \quad \max_{X \in \chi_3} g^1(X),$$

where $\chi_3 = \{X \in \chi \mid g^1(X) \leq 197\}$.

- (c) Continue in the same way, we get the higher ranked solutions of ILP as shown in Table 3.3. $f_{max}^1 = \max\{f^1(X) \mid X \in \cup_{i=1}^9 L^i\} = 180$. As $g^1(Y) = f_{max}^1$ for $Y \in L^9$,

Table 3.3: Fourteen ranked solutions of ILP (Example 3.6.2)

1	L^l	$g^1(X)$	$f^1(X)$
1	(5, 3, 0, 0)	207.5	161
2	(4, 2, 1, 0)	198	171
	(4, 3, 1, 0)	198	160
3	(4, 2, 0, 0)	196	168
	(4, 3, 0, 0)	196	154
4	(4, 3, 1, 1)	191.5	153
5	(4, 2, 0, 1)	189.5	162
	(4, 3, 0, 1)	189.5	152
6	(3, 1, 2, 0)	188.5	165
	(3, 1, 1, 0)	186.5	170
7	(3, 2, 1, 0)	186.5	162
	(3, 3, 1, 0)	186.5	152
	(3, 0, 0, 0)	184.5	180
8	(3, 1, 0, 0)	184.5	171
	(3, 2, 0, 0)	184.5	160
	(3, 3, 0, 0)	184.5	147
	(3, 1, 1, 1)	180	154
9	(3, 2, 1, 1)	180	150
	(3, 3, 1, 1)	180	144
	(3, 1, 0, 1)	178	160
10	(3, 2, 0, 1)	178	153
	(3, 3, 0, 1)	178	144
11	(2, 0, 2, 0)	177	156
	(2, 1, 2, 0)	177	154
	(2, 0, 1, 0)	175	165
12	(2, 1, 1, 0)	175	160
	(2, 2, 1, 0)	175	153
	(2, 3, 1, 0)	175	144
	(2, 0, 0, 0)	173	170
13	(2, 1, 0, 0)	173	162
	(2, 2, 0, 0)	173	152
	(2, 3, 0, 0)	173	140
14	(2, 0, 2, 1)	170.5	130

therefore

$$F^1 = \{X \in \cup_{i=1}^9 L^i \mid f^1(X) = f_{max}^1\} = \{(3, 0, 0, 0)\},$$

(c) The set F^1 has only one element $(3,0,0,0)$, therefore $E^1 = \emptyset$ and $E = \{(3, 0, 0, 0)\}$.

2 (a) $f_{min}^1 = \min\{f^1(X) \mid X \in \cup_{i=1}^9 L^i\} = 144$, $g^1(3, 1, 1, 1) > g_{min}^1$ and $f^1(3, 0, 0, 0) > f_{min}^1$. We set $j = j + 1$.

(b) $f_{max}^1 = \max\{f^1(X) \mid f^1(X) < f^1(X^1) \mid X \in \cup_{i=1}^9 L^i, X^1 = (3, 0, 0, 0)\} = 171$. Since $g^1(3, 1, 1, 1) > f_{max}^1$ and $g^1(3, 1, 1, 1) > g_{min}^1$, therefore we set $l = 10$ and solved the following problem (ILP_{10}).

$$(ILP_{10}) \quad \max_{X \in \chi_{10}} g^1(X),$$

where $\chi_{10} = \{X \in \chi \mid g^1(X) \leq 179\}$. After solving problem (ILP_{10}), we get $L^{10} = \{(3, 1, 0, 1), (3, 2, 0, 1), (3, 3, 0, 1)\}$.

(c) Continue on the same line, we get $f_{max}^1 = \max\{f^1(X) \mid f^1(X) < f^1(3, 0, 0, 0), X \in \cup_{i=1}^{13} L^i\} = 171$, $g^1(Y) = 173$. As $g^1(Y) > f_{max}^1$ and $g^1(Y) > g_{min}^1$ for $Y \in L^{13}$, therefore we set $l = 14$ and solved the following problem (ILP_{14}).

$$(ILP_{14}) \quad \max_{X \in \chi_{14}} g^1(X),$$

where $\chi_{14} = \{X \in \chi \mid g^1(X) \leq 172\}$. After solving problem (ILP_{14}), we get $L^{14} = \{(2, 0, 2, 1)\}$.

(d) $f_{max}^1 = \max\{f^1(X) \mid f^1(X) < f^1(3, 0, 0, 0), X \in \cup_{i=1}^{14} L^i\} = 171$.

(e) Since $g^1(Y) < f_{max}^1$ for $Y \in L^{14}$ and $\{X \in \cup_{i=1}^{14} L^i \mid f^1(X) = f_{max}^1\} = \{(4, 2, 1, 0), (3, 1, 0, 0)\}$, therefore $F^2 = \{(4, 2, 1, 0), (3, 1, 0, 0)\}$. It can be seen that only fourteen ranked solutions of ILP are required for finding the complete set F^2 .

(f) The criteria vector corresponding to $(4,2,1,0)$ is $(171,187,90)$, which is not dominated by criteria vector $(180, 165,108)$ corresponding to $(3,0,0,0)$. But, criteria vector corresponding to $(3,1,0,0)$ is $(171,160,99)$, which is dominated by criteria vector $(180, 165,108)$ corresponding to $(3,0,0,0)$. Therefore $E^2 = \{(3, 1, 0, 0)\}$ and $E = \{(3, 0, 0, 0), (4, 2, 1, 0)\}$.

(g) Continue in the same way for higher values of j upto N , we get the complete efficient set $E = \{(3, 0, 0, 0), (4, 2, 1, 0), (3, 1, 1, 0), (3, 1, 2, 0), (2, 0, 1, 0), (4, 2, 0, 1), (3, 1, 0, 1), (2, 0, 2, 0), (3, 1, 1, 1), (2, 0, 1, 1), (2, 0, 2, 1)\}$

Remark 3.6.1 *It can be observed from above discussion that the proposed algorithm has generated the complete set of eleven efficient solutions of the problem discussed in the*

paper of Arora and Arora [9], where as their own algorithm has reported only four efficient solutions of this problem in their paper [9]. So, our algorithm is more efficient as compared to algorithm discussed in [9].

3.6.3 Comparison with the algorithms of [9, 46]

In the following discussion we have given two counter-examples on the algorithm of [9, 46]. Example 3.6.3 shows that their algorithms stop after calculating a single efficient solution and also miss other efficient solutions. In Example 3.6.4, we have shown that the [9, 46] algorithms declares a non-efficient solution as an efficient solution. Further, we have solved Example 3.6.3 and Example 3.6.4 using proposed algorithm and have generated the complete set of efficient solutions of these problems.

Example 3.6.3 Consider the following problem

$$(MIQP) \quad \begin{cases} \max f^1(x_1, x_2) = (x_1 + 3x_2 + 2)(-x_2 + 6) \\ \max f^2(x_1, x_2) = (x_1 + 2x_2 + 1)(-x_2 + 7) \\ \max f^3(x_1, x_2) = (x_1 + x_2 + 1)(2x_1 + x_2 + 2) \\ \text{s.t} \quad (x_1, x_2) \in \chi = S \cap \mathbb{Z}^2, \end{cases}$$

where $S = \{(x_1, x_2) \mid 3x_1 + 4x_2 \leq 21, 2x_1 + x_2 \leq 9, x_1, x_2 \geq 0\}$.

Since the functions $(x_1 + 3x_2 + 2)$, $(-x_2 + 6)$, $(x_1 + 2x_2 + 1)$, $(-x_2 + 7)$, $(x_1 + x_2 + 1)$ and $(2x_1 + x_2 + 2)$ are positive for all $(x_1, x_2) \in \chi$, so $MIQP$ satisfies the conditions of [9, 46] algorithms.

Solution using the algorithm of [46]

First of all, the following single objective problem (QP) is solved.

$$(QP) \quad \begin{cases} \max f^1(x_1, x_2) = (x_1 + 3x_2 + 2)(-x_2 + 6) \\ \text{s.t} \quad (x_1, x_2) \in S. \end{cases}$$

The optimal solution of this problem is obtained by the following result used in their paper.

Theorem 3.6.1 [46] Given a basic feasible solution (BFS) $x_B = B^{-1}b$ to the problem QP such that $\gamma_j \leq 0$ for every column $a_j \in A$ not in B . Then x_B is an optimal basic feasible solution.

Then, as per the algorithm of [46], the following conclusions correspond to the corner point (3,3) are drawn.

Table 3.4: Simplex table of Example 3.6.3 corresponding to the corner point (3,3)

x_B	x_3	x_4	RHS
x_1	-0.2	0.8	3
x_2	0.4	-0.6	3
μ_j	7.5	3.75	
$p_j^1 - z_j^{11}$	-1	1	
$q_j^1 - z_j^{21}$	0.4	-0.6	
γ_j^1	-0.4	-7.65	
$p_j^2 - z_j^{12}$	-0.6	0.4	
$q_j^2 - z_j^{22}$	0.4	-0.6	
γ_j^2	-0.2	-5.3	
$p_j^3 - z_j^{13}$	-0.2	-0.2	
$q_j^3 - z_j^{23}$	0	-1	
γ_j^3	-2.2	-8.45	

1. The point (3, 3) is declared optimal by virtue of the Theorem 2.1 [46], as $\gamma_j^1 \leq 0 \forall j$ and $f^1(3, 3) = 42$.
2. The point (3, 3) is an ideal point, as $\gamma_j^2 \leq 0$ and $\gamma_j^3 \leq 0$ for all $a_j \in A$ not in B

Further, by Proposition 3.3 [46], the point (3,3) is the only efficient point declared by the algorithm of [46].

Solution using the algorithm of [9]

Firstly, the following problem (QP) is considered.

$$(QP) \quad \begin{cases} \max f^1(x_1, x_2) = (x_1 + 3x_2 + 2)(-x_2 + 6) \\ \text{s.t} \quad (x_1, x_2) \in S, \end{cases}$$

where $S = \{(x_1, x_2) \mid 3x_1 + 4x_2 \leq 21, 2x_1 + x_2 \leq 9, x_1, x_2 \geq 0\}$. Since upper bound is infinity and lower bound is zero, therefore all non- basic variables are at its lower bound zero. An optimal solution of this problem is obtained by using the following result in their paper.

Result 3.6.1 ([9], Page 372, last para) *Any basic feasible solution to problem (QP), such that $Q_{i,j} \geq 0$ for all $j \in N^1$ and $Q_{i,j} \leq 0$ for all $j \in N^2$ is a locally optimal solution.*

Now we'll see that this optimality condition is satisfied at the corner point (3,3). For this purpose we'll calculate the value of $Q_{i,j}$, where $Q_{i,j} = L_{i,j}\phi_j$. As all non- basic variables

are at its lower bound zero, therefore

$$\begin{aligned}
L_{i,j} &= Z_{i1}(z_j^{2i} - q_j^i) + Z_{i2}(z_j^{1i} - p_j^i) - \phi_j(z_j^{1i} - p_j^i)(z_j^{2i} - q_j^i) \\
&= -\gamma_j^i \\
\phi_j &= \left\{ \frac{X_{Bi}}{y_{ij}} \mid y_{ij} > 0, y_{ij} = B^{-1}A_j \right\} \\
&= \mu_j
\end{aligned}$$

where γ_j^i and μ_j as defined in [46]. Now, corresponding to the corner point (3,3), it can be observed from Table 3.4 that $\mu_j > 0$ and $\gamma_j^1 \leq 0$ for all j , therefore $Q_{1,j} \geq 0$ for all $j \in N^1$. Also, $N^2 = \emptyset$, therefore (3,3) is declared as a global optimal solution of problem (QP) (by result [9], Page 372, last para) , which implies (3,3) is an efficient solution of problem (MIQP). It can also be observed that

$$\begin{aligned}
T^1 &= \{j \mid j \in N^1, Q_{1,j}^1 \geq 0 \text{ \& } Q_{i,j} \leq 0 \text{ for atleast one } i = 1, 2\} = \emptyset \text{ (}\because Q_{1,j} \geq 0 \text{ for all } j \in N^1) \\
T^2 &= \{j \mid j \in N^2, Q_{1,j} \leq 0 \text{ \& } Q_{1,j} \geq 0 \text{ for atleast one } i = 1, 2\} = \emptyset \text{ (}\because N^2 = \emptyset)
\end{aligned}$$

Since $T^1 \cup T^2 = \emptyset$, therefore by Proposition 1 [9], algorithm terminates. So, by [9] algorithm the set of efficient solution is $\{(3,3)\}$.

Remark 3.6.2 *It may be observed from above discussion that algorithms of [9,46], declare (3,3) as the only one efficient solution of problem MIQP. But, the feasible solution (4,1) is also an efficient solution of MIQP. As it can be seen that the criteria vector at point (3,3) is (42,40,77), and the criteria vector at point (4,1) is (45,42,66). So, both of these algorithms missed the efficient solution (4,1) of MIQP. In the following discussion we'll show that all these efficient solutions are recorded by the proposed methodology.*

Solution using the proposed algorithm

To find the efficient set of MIQP, we have used the ranking of the feasible solutions of IQP. Then proceeding on the lines of the algorithm as explained in Example 3.6.1. The complete efficient set E of MIQP is obtained as $\{(4,1), (3,3)\}$, (as shown in Table 3.5).

Example 3.6.4

$$(MIQP) \quad \begin{cases} \max f^1(x_1, x_2) = (x_1 + 3x_2 + 2)(-x_2 + 6) \\ \max f^2(x_1, x_2) = (x_1 + 2x_2 + 1)(-x_2 + 7) \\ \text{s.t} \quad (x_1, x_2) \in \chi = S \cap \mathbb{Z}^2, \end{cases}$$

where $S = \{(x_1, x_2) \mid 3x_1 + 4x_2 \leq 21, 2x_1 + x_2 \leq 9, x_1, x_2 \geq 0\}$.

Table 3.5: Ranked solutions of IQP and efficient set E (Example 3.6.3)

j	F^j	(f^1, f^2, f^3)	Dominated or non-dominated criteria vector	E
1	(4,1)	(45,42,66)	Non-dominated criteria vector	$\{(4,1)\}$
2	(3,2)	(44,40,60)	Dominated by (45,42,66)	$\{(4,1)\}$
3	(3,3)	(42,40,77)	Non-dominated criteria vector	$\{(4,1), (3,3)\}$
4	(3,1)	(40,36,45)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
	(2,2)	(40,35,40)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
5	(2,3)	(39,36,54)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
6	(4,0)	(36,35,50)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
	(1,2)	(36,30,24)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
	(1,3)	(36,32,35)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
7	(2,1)	(35,30,28)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
8	(0,3)	(33,28,20)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
9	(0,2)	(32,25,12)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
10	(3,0)	(30,28,32)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
	(1,1)	(30,24,15)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
	(1,4)	(30,30,48)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
11	(0,4)	(28,27,30)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
12	(0,1)	(25,18,6)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
13	(2,0)	(24,21,18)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
14	(1,0)	(18,14,8)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
15	(0,5)	(17,22,42)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$
16	(0,0)	(12,7, 2)	Dominated by (45,42,66)	$\{(4,1), (3,3)\}$

In this example, we have considered only the first two objectives of Example 3.6.3 to show that [9, 46] algorithms declares a non-efficient point as an efficient point.

Since the functions $(x_1 + 3x_2 + 2)$, $(-x_2 + 6)$, $(x_1 + 2x_2 + 1)$ and $(-x_2 + 7)$ are positive for all $(x_1, x_2) \in \chi$, so Example 3.6.4 satisfies the conditions of [9, 46] algorithms.

Solution using algorithm of [9, 46]

It may be observed that the two objectives of this problem are the same as that of the first two objectives of Example 3.6.3. It can also be observed from Example 3.6.3 (Table 3.4) that point $(3, 3)$ is an optimal solution for both objective functions of this problem, as it satisfies the optimality criteria given in Theorem 2.1 [46]. Also, it satisfies the optimality criteria of [9] (Page 372, last para). That means the point $(3, 3)$ is an ideal point for this problem. Therefore by Proposition 3.3 [46], the point $(3, 3)$ is the only efficient point recorded by the algorithm. Also, it can be observed that

$$T^1 = \{j | j \in N^1, Q_{1,j}^1 \geq 0 \text{ \& } Q_{i,j} \leq 0 \text{ for atleast one } i = 1, 2\} = \emptyset \text{ } (\because Q_{1,j} \geq 0 \text{ for all } j \in N^1)$$

$$T^2 = \{j | j \in N^2, Q_{1,j} \leq 0 \text{ \& } Q_{1,j}^2 \geq 0 \text{ for atleast one } i = 1, 2\} = \emptyset \text{ } (\because N^2 = \emptyset)$$

As $T^1 \cup T^2 = \emptyset$, therefore by Proposition 1 [9], algorithm terminates. Thus, by [9] algorithm the only efficient point of $MIQP$ is $(3, 3)$.

Solution using the proposed algorithm

Now, we will solve Example 3.6.4 using the proposed algorithm. The ranking of the following problem is used for finding the efficient solutions of $MIQP$.

$$(IQP) \quad \begin{cases} \max f^1(x_1, x_2) = (x_1 + 3x_2 + 2)(-x_2 + 6) \\ \text{s.t} \quad (x_1, x_2) \in \chi. \end{cases}$$

The problem (IQP) of this example is the same as that of Example 3.6.3. So, we get the same ranked solutions of IQP as obtained in Example 3.6.3. The efficient set E is obtained as $\{(4, 1)\}$ (as shown in Table 3.6).

Remark 3.6.3 *Notably, $(3, 3)$ is the only efficient solution generated by the algorithms discussed in [9, 46]. But, it can be observed that $(3, 3)$ is not an efficient solution of this example as its criteria vector $(42, 40)$ is dominated by the criteria vector $(45, 42)$ of point $(4, 1)$. That means their algorithms recorded non-efficient point $(3, 3)$ and missed the efficient point $(4, 1)$. Whereas, the proposed method records the correct efficient set $\{(4, 1)\}$ of this problem.*

Table 3.6: Ranked solutions of IQP and efficient set E (Example 3.6.4)

j	F^j	(f^1, f^2)	Dominated or non-dominated criteria vector	E
1	(4,1)	(45,42)	Non-dominated criteria vector	$\{(4,1)\}$
2	(3,2)	(44,40)	Dominated by (45,42)	$\{(4,1)\}$
3	(3,3)	(42,40)	Dominated by (45,42)	$\{(4,1)\}$
4	(3,1)	(40,36)	Dominated by (45,42)	$\{(4,1)\}$
	(2,2)	(40,35)	Dominated by (45,42)	$\{(4,1)\}$
5	(2,3)	(39,36)	Dominated by (45,42)	$\{(4,1)\}$
6	(4,0)	(36,35)	Dominated by (45,42)	$\{(4,1)\}$
	(1,2)	(36,30)	Dominated by (45,42)	$\{(4,1)\}$
	(1,3)	(36,32)	Dominated by (45,42)	$\{(4,1)\}$
7	(2,1)	(35,30)	Dominated by (45,42)	$\{(4,1)\}$
8	(0,3)	(33,28)	Dominated by (45,42)	$\{(4,1)\}$
9	(0,2)	(32,25)	Dominated by (45,42)	$\{(4,1)\}$
10	(3,0)	(30,28)	Dominated by (45,42)	$\{(4,1)\}$
	(1,1)	(30,24)	Dominated by (45,42)	$\{(4,1)\}$
	(1,4)	(30,30)	Dominated by (45,42)	$\{(4,1)\}$
11	(0,4)	(28,27)	Dominated by (45,42)	$\{(4,1)\}$
12	(0,1)	(25,18)	Dominated by (45,42)	$\{(4,1)\}$
13	(2,0)	(24,21)	Dominated by (45,42)	$\{(4,1)\}$
14	(1,0)	(18,14)	Dominated by (45,42)	$\{(4,1)\}$
15	(0,5)	(17,22)	Dominated by (45,42)	$\{(4,1)\}$
16	(0,0)	(12,7)	Dominated by (45,42)	$\{(4,1)\}$

3.7 Special case: Multi-objective Integer Linear Programming Problem

If all objective functions in $MIQP$ are linear functions (i.e. $D^i = \emptyset$ for all $i = 1, 2, \dots, s$), then $MIQP$ reduces to Multi-objective Integer Linear Programming Problem ($MILP$), which is defined as

$$(MILP) \quad \max_{X \in \mathcal{X}} f^i(X)$$

where $f^i(X) = C^i X + \alpha^i$, $C^i \in \mathbb{Z}^{1 \times n}$ & $\alpha^i \in \mathbb{R} \quad \forall i \in \{1, 2, \dots, s\}$.

For finding efficient solutions of $MILP$, we can consider any objective of $MILP$. For the sake of simplicity, we have considered the first objective $f^1(X)$. Since $D^1 = \emptyset$, therefore $V_j^1 = \max_{X \in \mathcal{X}} X^T D_j^1 = \emptyset \quad \forall j \in \{1, 2, \dots, n\}$, which implies $g^1(X) = C^1 X + \alpha^1 = f^1(X)$. So, $MILP$ can be mathematically defined as

$$(MILP) \quad \max_{X \in \mathcal{X}} g^i(X) \quad i \in \{1, 2, \dots, s\},$$

Algorithm 4: Algorithm to solve *MILP*

Data: $m, n, s \in \mathbb{Z}$, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $C^i \in \mathbb{Z}^{1 \times n}$ and $\alpha^i \in \mathbb{R} \forall i \in \{1, 2, \dots, s\}$
Result: E ▷ set of all efficient solutions of *MILP*

- 1 **Initialization** $l \leftarrow 1$, $E \leftarrow []$, $E^1 \leftarrow []$
- 2 $\chi \leftarrow \{X \in \mathbb{Z}^n \mid AX = b, X \geq 0\}$
- 3 $g^1(X) \leftarrow C^1 X + \alpha^1$ ▷ *ILP* $\max_{X \in \chi} g^1(X)$
- 4 $g_{min}^1 \leftarrow \min_{X \in \chi} g^1(X)$
- 5 $L^1 \leftarrow \{Y \in \chi \mid g^1(Y) = \max_{X \in \chi} g^1(X)\}$ ▷ set of all optimal solutions of *ILP*
- 6 $Z^1 \leftarrow g^1(Y)$ for $Y \in L^1$ ▷ optimal value of *ILP*
- 7 **foreach** $X \in L^1$ **do**
- 8 **foreach** $Y \in L^1 \cap (E^1)^c$ **do**
- 9 **if** $g^i(Y) \geq g^i(X) \forall i \in \{2, 3, \dots, s\}$ & $g^i(Y) > g^i(X)$ for some $i \in \{1, 2, \dots, s\}$ **then**
- 10 $E^1 \leftarrow [E^1 \ X]$
- 11 **if** $X \notin E^1$ **then**
- 12 $E \leftarrow [E \ X]$ ▷ efficient solutions of *MILP* in L^1
- 13 **while** $g^1(Y) > g_{min}^1$ for some $Y \in L^l$ **do**
- 14 $l \leftarrow l + 1$
- 15 $\chi_l \leftarrow \{X \in \chi \mid g^1(X) \leq Z^{l-1} - 1\}$
- 16 $L^l \leftarrow \{Y \in \chi_l \mid g^1(Y) = \max_{X \in \chi_l} g^1(X)\}$ ▷ set of l^{th} ranked solutions of *ILP*
- 17 $Z^l \leftarrow g^1(Y)$ for $Y \in L^l$
- 18 $E^l \leftarrow \emptyset$
- 19 **foreach** $X \in L^l$ **do**
- 20 **foreach** $Y \in E \cup (L^l \cap (E^l)^c)$ **do**
- 21 **if** $g^i(Y) \geq g^i(X) \forall i \in \{2, 3, \dots, s\}$ & $g^i(Y) > g^i(X)$ for some $i \in \{1, 2, \dots, s\}$ **then**
- 22 $E^l \leftarrow [E^l \ X]$ ▷ Set of non-efficient solutions of *MILP* in L^l
- 23 **if** $X \notin E^l$ **then**
- 24 $E \leftarrow [E \ X]$ ▷ Set of efficient solutions of *MILP* in $\cup_{i=1}^l L^i$

where $g^i(X) = f^i(X) = C^i X + \alpha^i$. To find the efficient solutions of *MILP*, the following problem is considered.

$$(ILP) \quad \max_{X \in \chi} g^1(X).$$

To find ranked solutions of *ILP*, we have considered

$$(ILP_l) \quad \max_{X \in \chi_l} g^1(X),$$

where $\chi_l = \{X \in \chi \mid g^1(X) \leq Z^{l-1} - 1\}$, Z^{l-1} is the objective function value at $(l-1)^{th}$ ranked solution of *ILP*. In *MILP*, $g^1(X) = f^1(X)$, therefore instead of finding sets of the kind F^j , we will only construct sets L^i to obtain efficient solutions of *MILP*. Now, using Algorithm 4, we will discuss the solution of *MILP* (taken from [141]).

Example 3.7.1 Consider the problem

$$(MILP) \quad \begin{cases} \max g^1(x_1, x_2) = x_1 - 2x_2 \\ \max g^2(x_1, x_2) = -x_1 + 3x_2 \\ s.t \quad (x_1, x_2) \in \chi = S \cap \mathbb{Z}^2, \end{cases}$$

where $S = \{(x_1, x_2) \mid x_1 - 2x_2 \leq 0, x_1, x_2 \in \{0, 1, 2\}\}$.

For determining the efficient set of $MILP$, we have used the ranking of the following problem.

$$(ILP) \quad \begin{cases} \max g^1(x_1, x_2) = x_1 - 2x_2 \\ \text{s.t} \quad (x_1, x_2) \in \chi. \end{cases}$$

1 (a) The problem ILP has two optimal solutions namely $(0,0)$ and $(2,1)$ with objective function value 0. So, $L^1 = \{(0,0), (2,1)\}$. The criteria vector corresponding to $(0,0)$ is $(0,0)$, which is dominated by criteria vector $(0, 1)$ corresponding to $(2,1)$. Therefore, $E^1 = \{(0,0)\}$ and $E = \{(2,1)\}$.

(b) $g_{min}^1 = \min\{g^1(X) \mid X \in \chi\} = -4$. As $g^1(0,0) > g_{min}^1$. We set $l = 2$ and solved the following problem (ILP_2).

$$(ILP_2) \quad \max_{X \in \chi_2} g^1(X)$$

where $\chi_2 = \{X \in \chi \mid g^1(X) \leq -1\}$.

(c) The problem ILP_2 has unique optimal solution $\{(1,1)\}$ with objective function value -1. So $L^2 = \{(1,1)\}$. The criteria vector corresponding to $(1,1)$ is $(-1,2)$, which is not dominated by criteria vector $(0, 1)$ corresponding to $(2,1)$. Therefore, $E^2 = \emptyset$ and $E = \{(2,1), (1,1)\}$. $g^1(1,1) > g_{min}^1$. We set $l = 3$ and solved the following problem ILP_3 .

$$(ILP_3) \quad \max_{X \in \chi_3} g^1(X)$$

where $\chi_3 = \{X \in \chi \mid g^1(X) \leq -2\}$.

(c) Continue in the same way, we get the higher ranked solutions of ILP and efficient set $E = \{(2,1), (1,1), (2,2), (1,2), (0,2)\}$ as shown in Table 3.7.

Table 3.7: Ranked solutions of ILP and efficient set E (Example 3.7.1)

j	L^j	(g^1, g^2)	Dominated or non-dominated criteria vector	E
1	$(2,1)$ $(0,0)$	$(0,1)$ $(0,0)$	Non-dominated criteria vector Dominated by $(0,1)$	$\{(2,1)\}$ $\{(2,1)\}$
2	$(1,1)$	$(-1,2)$	Non-dominated criteria vector	$\{(2,1), (1,1)\}$
3	$(2,2)$ $(0,1)$	$(-2,4)$ $(-2,3)$	Non-dominated criteria vector Dominated by $(-2,4)$	$\{(2,1), (1,1), (2,2)\}$ $\{(2,1), (1,1), (2,2)\}$
4	$(1,2)$	$(-3,5)$	Non-dominated criteria vector	$\{(2,1), (1,1), (2,2), (1,2)\}$
5	$(0,2)$	$(-4,6)$	Non-dominated criteria vector	$\{(2,1), (1,1), (2,2), (1,2), (0,2)\}$

Remark 3.7.1 The Example 3.7.1 is taken from the paper of Sylva and Crema [141]. The proposed algorithm generates the same efficient solutions as in [141].

3.8 Computational results

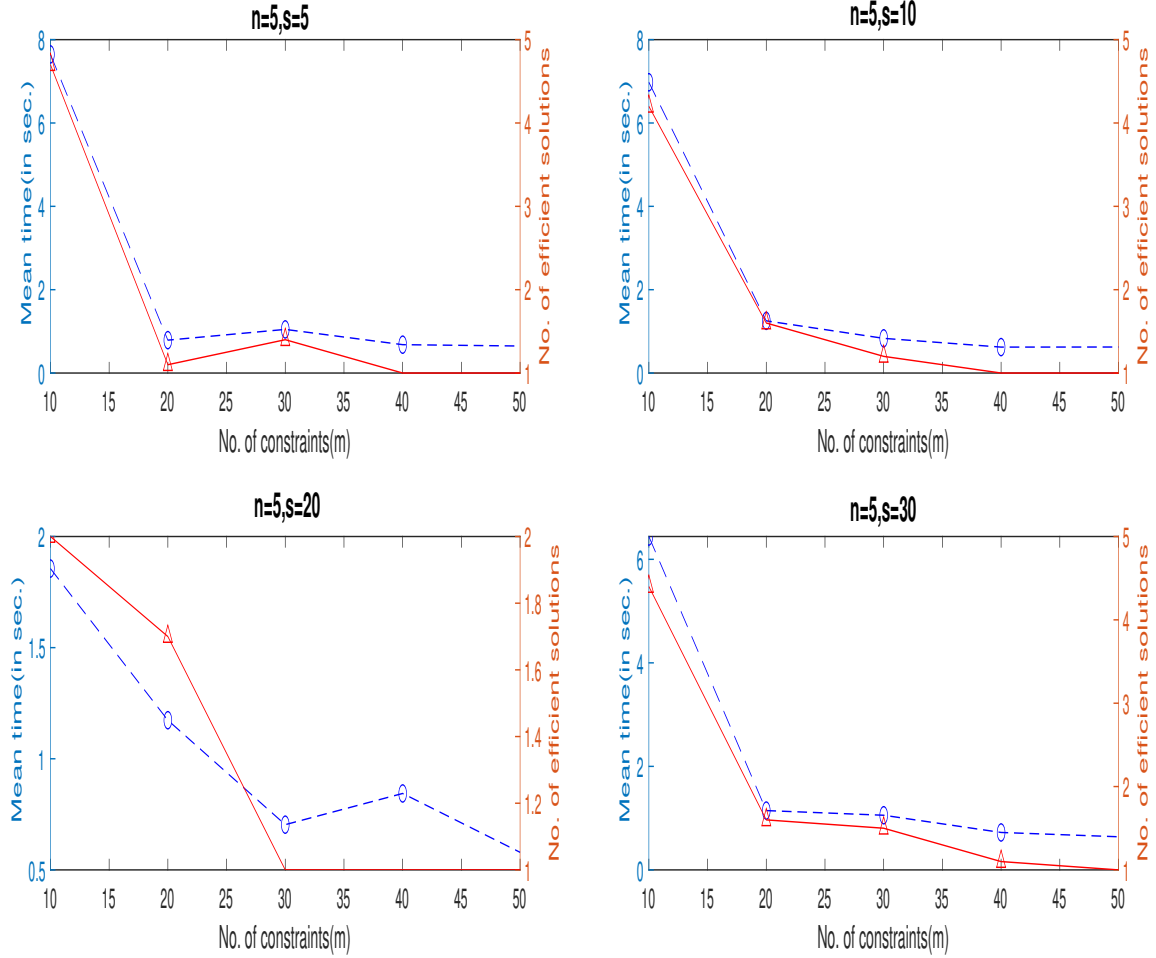


Figure 3.1: Impact of m on processing time to solve $MIQP$

The proposed algorithm was implemented in MATLAB 2019, and we obtained the computational result from an HP laptop (Intel(R), Core(TM), i3-6006U processor, CPU @ 2GHz, and 4 GB RAM). A series of twenty randomly generated $MIQP$ are solved using the proposed algorithm for each couple (m, n, s) where m , n , s represent the number of constraints, variables, and objectives of $MIQP$ respectively. The coefficient of all variables in all the constraints and all objectives of $MIQP$ have taken randomly an integer value from the set $[0, 20]$ and $[-100, 100]$ respectively, and all terms on the right side of the constraints in $MIQP$ have taken randomly an integer value from the set $[0, 100]$.

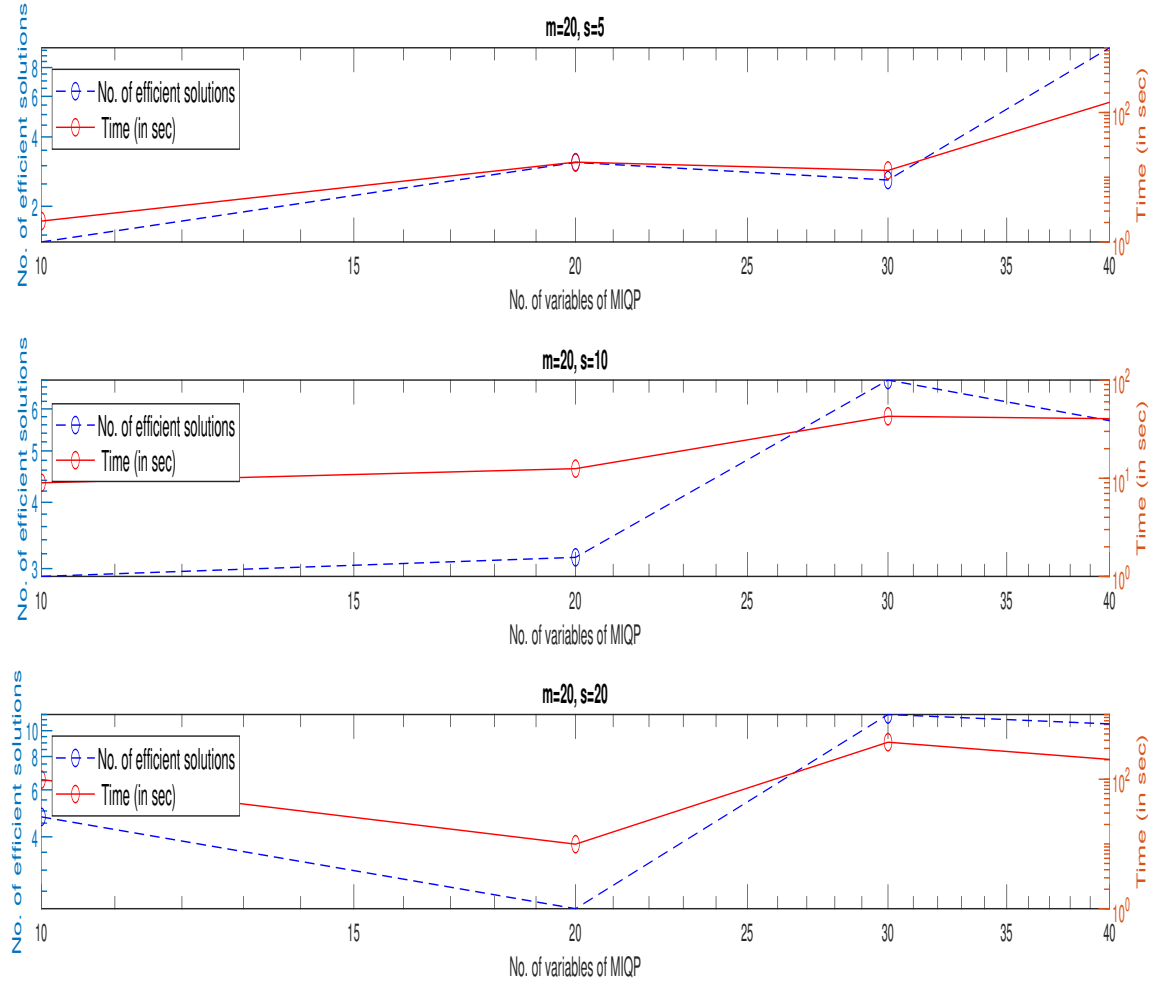


Figure 3.2: Impact of n on processing time to solve $MIQP$

Table 3.8: Number of efficient solutions of $MIQP$ and processing time to find all efficient solutions of $MIQP$ for different values of m, n and s

n	m = 20, s = 5				m = 20, s = 10				m = 20, s = 20			
	No. of efficient sol		Time (in sec)		No. of efficient sol		Time (in sec)		No. of efficient sol		Time (in sec)	
10	4	1	10	2	16	3	84	9	57	5	1819	99
20	30	3	269	17	12	3	73	13	5	2	44	10
30	16	3	146	13	32	7	254	43	167	12	6984	373
40	75	10	1398	144	34	6	328	40	87	11	2091	202

s	m = 10, n = 5			
	No. of efficient sol		Time (in sec)	
5	17	6	42	9
10	66	7	568	34
20	69	7	1521	86
30	52	7	1012	83
40	70	9	4225	257

The maximum and mean number of efficient solutions of $MIQP$ and maximum and mean processing time to solve $MIQP$ for different values of m, n and s is shown in Table 3.8.

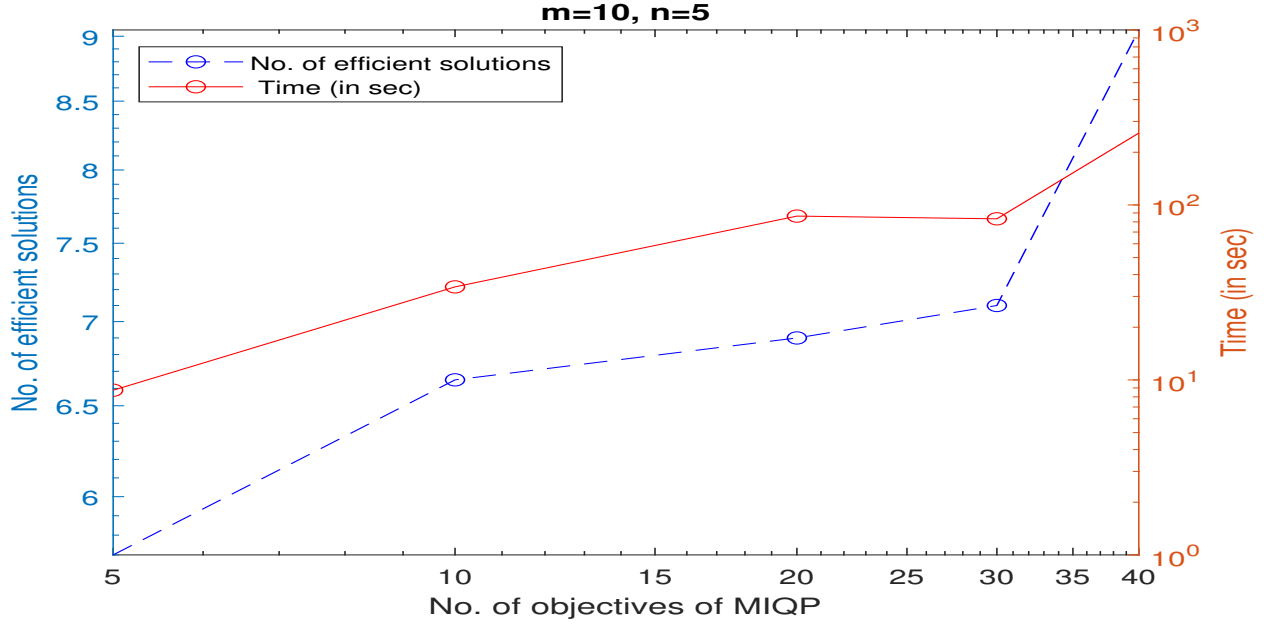


Figure 3.3: Impact of s on processing time to solve $MIQP$

Figure 3.2 and Figure 3.3 depict the effect of m , n and s on processing time to find all efficient solutions of $MIQP$. It can be observed, as efficient solutions of $MIQP$ increases, the mean processing time to solve $MIQP$ also increases.

3.9 Concluding Remarks

We have presented an algorithm to find the efficient set of $MIQP$. In the proposed algorithm, we have utilized ranked solutions of IQP for enumerating the efficient solutions of $MIQP$. Theoretically, we proved the proposed algorithm generates the complete set of all efficient solutions of $MIQP$ in finite steps. The $MIQP$ falls in the category of NLP , but we have not used any non-linear optimization technique. We have obtained the efficient set of $MIQP$ by successively solving an integer linear programming problem. Thus, fewer computational efforts require for solving $MIQP$ using the proposed algorithm.

As far as our knowledge, no algorithm available in literature that solves the general multi-objective integer quadratic programming problems, so no comparative study can be made. We have also shown that our algorithm work well on the special cases of $MIQP$ where the objectives are product of two linear functions and the existing algorithms [9, 46] cannot guarantee to find the set of efficient solutions.

Further, if we take $D^i = \emptyset$ for all $i = 1, 2, \dots, s$, the problem $MIQP$ becomes $MILP$. In this case, by taking $g(X) = f^1(X) = C^1X + \alpha^1$, the algorithm will generate the complete

efficient set of $MILP$.

The present methodology can also be extended for box constraints.

Chapter 4

Multi- objective integer quadratic fractional programming problem

4.1 Introduction

Consider the following problem

$$(P) \quad \min_{X \in S} f(X).$$

If $f(x)$ is a linear function, then it becomes LP , which is solvable by a vertex search algorithm called simplex method.

From past many decades, the simplex method has caught the attention of researchers and a number of extensions of the simplex methods to various mathematical programming problems are published. Integer programming problems, convex quadratic programming, goal programming, separable convex programming etc are some of the examples where direct application of simplex, or adjacent extreme point methods are applicable. We have also witnessed the extension of simplex method to the programming problems which are neither convex nor concave, for example, the linear fractional programming problem. Then there is an obvious question, can every non-linear programming problem of the form (P) has a vertex search algorithm? If not, then as inquired by Martos [101], in his landmark paper how far the vertex search algorithms can be extended in solving non-linear programming problems. Further he pointed out that the programming problem (P) , utilizes a vertex search algorithm if it has the following two properties.

- (a) A minimum is taken on a vertex of the convex polyhedron.
- (b) A local minimum is also a global minimum.

Then pertaining to the properties (a) and (b), Martos [101] gave the following results.

Theorem 4.1.1 *A function $f(x)$ which is continuous on S possesses Property (a) in set S and in all its convex polyhedral subsets if and only if it is quasi-concave in S .*

⁰A part of this chapter is published as “A note on solving multi-objective integer indefinite quadratic fractional programs”, **Annals of Operations Research**, 289(2):459-462, 2020 (SCI, Impact factor: **4.8**).

Theorem 4.1.2 *A function $f(x)$ which is continuous on S possesses Property (b) in set S and in all its convex polyhedral subsets if and only if $f(x)$ is explicitly quasi-convex in S .*

From these two theorems, Martos [101] gave the following corollary

Corollary 4.1.3 *Let $f(x)$ be a continuous function on the set S and $f(x)$ possesses Properties (a) and (b) in set S and in all its convex polyhedral subsets, and the problem (P) is therefore solvable by adjacent vertex methods if and only if $f(x)$ is both quasi-concave and explicitly quasi-convex in S . (In case of maximization problem $f(x)$ has to be both quasi-convex and explicitly quasi-concave.)*

Further, Martos [101] also pointed out that the result of the above corollary is valid for both maximization and minimization problem if $f(x)$ is explicitly quasi-monotone.

The purpose of this chapter is to again highlight the use of adjacent vertex search methods, as in a recently published paper [102], a vertex search algorithm to a class of programming problem (P) is discussed which fails to find an optimal solution at a vertex of the polyhedron region. Further, we will present a novel method for solving *MIQFP*.

In next section, we will discuss neither the algorithm discussed in [102] finds an optimal solution of *IdQFP* nor it will generate the correct set of efficient solutions of *MIIdQFP*.

4.2 Remarks on the algorithm in [102]

Recently an exact algorithm for enumerating efficient solutions of *MIIdQFP* has been discussed in [102] and its mathematical model is described below:

$$(MIIdQFP) \quad \max_{X \in \chi} (f^1(X), f^2(X), \dots, f^s(X)),$$

where

$$f^i(X) = \frac{(C^i X + \alpha^i)(C_0^i X + \alpha_0^i)}{(P^i X + \beta^i)(P_0^i X + \beta_0^i)} \quad \forall i \in \{1, 2, \dots, s\},$$

$\chi = S \cap \mathbb{Z}^n$, $C^i, P^i, C_0^i, P_0^i \in \mathbb{Z}^{1 \times n}$, $\alpha^i, \beta^i, \alpha_0^i, \beta_0^i \in \mathbb{Z}$ and $P^i X + \beta^i, P_0^i X + \beta_0^i > 0$ for all $X \in \chi$.

To find the set of efficient solutions, authors constructed a problem:

$$(IdQFP) \quad \max_{X \in \chi} \frac{(C^1 X + \alpha^1)(C_0^1 X + \alpha_0^1)}{(P^1 X + \beta^1)(P_0^1 X + \beta_0^1)},$$

where $C^1, P^1, C_0^1, P_0^1 \in \mathbb{Z}^{1 \times n}$, $\alpha^1, \beta^1, \alpha_0^1, \beta_0^1 \in \mathbb{Z}$ and $P^1 X + \beta^1, P_0^1 X + \beta_0^1 > 0$ for all $X \in \chi$.

In their paper, Mekhilef et al. [102] first proposed a simplex like method to find an optimal solution of *IdQFP*. Then this method is successively used at each truncation of the feasible region S , to obtain an optimal basic feasible solution of $f^1(x)$ at this truncated region. The following theorem from Mekhilef et al. [102] tells the conditions of optimality for *IdQFP*.

Theorem 4.2.1 *Let $X_B = B^{-1}b$ be a basic feasible solution for *IdQFP*, with the objective function value $f^1(X_B)$. If one the following conditions is satisfied*

- (i) $\rho_j^1 \leq 0$ for all $j \in N$
- (ii) For each $j \in N$ such that $\rho_j^1 > 0$, we have $y_{ij} \leq 0$ for all $i \in \{1, 2, \dots, m\}$.

Then X_B is an optimal basic feasible solution.

In the light of Theorems 4.1.1 and 4.1.2, the result of Theorem 4.2.1 will hold if and only if the function $f^1(x)$ satisfies both the properties (a) and (b) defined above over the feasible region S . Infact the condition (i) in above theorem talk about the local optimality of the basic feasible solution X_B . There is no discussion on global optimality of X_B is done in Mekhilef et al. [102]. The condition (ii) in the above theorem is not required for the optimality of X_B , as it indicates the problem is unbounded. In the following example we will show that none of the basic feasible solution of *IdQFP* is optimal.

Counter Example

$$(MIIdQFP) \quad \min_{X \in \chi} \frac{(x_1 + 2x_2 + 1)(-x_2 + 10)}{x_2 + 1} \frac{1}{3x_1 + 1},$$

where $\chi = S \cap, S = \{(x, y) \mid 3x + 4y \leq 32, 5x - 3y \leq 5, x, y \geq 0\}$.

It may be observed that the feasible region χ has only four corner points, namely $(0, 0)$, $(1, 0)$, $(4, 5)$ and $(0, 8)$. The objective function value at these corner points is $f(0, 0) = 10$, $f(1, 0) = 5$, $f(4, 5) = 0.9615$ and $f(0, 8) = 3.778$. Now as per Theorem 1 above, the point $(0, 0)$ should be optimal with objective function value as $f(0, 0) = 10$. But this is not a optimal solution, as it can be observed that the value of objective function at a non corner point $(0, 3)$ is 12.25. Which contradicts the optimality of a corner point $(0, 0)$.

Now consider a line segment joining the point $(0, 0)$ and $(0, 3)$ is the convex linear combination of the points $(0, 0)$ and $(0, 3)$, given as

$$\lambda(0, 0) + (1 - \lambda)(0, 3) = (0, 3 - 3\lambda), \quad 0 \leq \lambda \leq 1.$$

The value of objective function at these points is $f(0, 3 - 3\lambda) = 6\lambda + 15 + \frac{11}{3\lambda - 4} > 10$ for $0 \leq \lambda < 1$. So at every point on the line segment emanating from $(0,0)$ have value of objective function more than $f(0,0)$, which implies $(0,0)$ is not a local optimal solution. This simply means that the hypothesis of the Theorem 1 doesn't even guarantee the local optimality of a basic feasible solution.

Further, it may also be observed that according to the algorithm discussed by Mekhilef et al. [102], an optimal integer solution of $IdQFP$ gives the first efficient solutions of the corresponding $MIIdQFP$. But as observed in the example above, the simplex like method proposed by Mekhilef et al. [102] may not find the optimal integer solution of $IdQFP$, so in that case it will not record the corresponding efficient solution. Since at each iteration, the same simplex like method is applied to a more constrained problem ($IdQFP$), which means many more integer solutions will be missed and the recorded integer solutions are not actually optimal of the constrained problem ($IdQFP$). In totality, the implication of [102] algorithm will result into two possible consequences, first, the set of efficient solutions generated is not complete. Secondly, efficient solutions declared by the algorithm may not be correct, as it may be declaring non-efficient solutions as efficient.

4.3 Motivation for development of an algorithm for solving $MIQFP$

As evident from the above discussion and from the findings of Martos [101], it can simply be concluded that there cannot be a vertex search method for $IdQFP$ as both the properties (a) and (b) of Martos [101] are not satisfied. But there do exist algorithms in literature which have tackled quadratic fractional programming problem over the set χ . For instance, Sharma et al. [131] have proposed an algorithm that finds an optimal solution of a quadratic fractional programming problem by ranking feasible solutions of related LP or LFP . The $IdQFP$ discussed in [102] is a special case of $IQFP$ considered by Sharma et al. [131]. Another paper that deals with a simplex like method for a class of $IQFP$ over the set χ is discussed by Sharma [130], where the objective function is square of a non-negative linear fractional function. The simplex like method works in this case, as the objective function satisfies both the properties (a) and (b) discussed by Martos [101].

Motivation from literature, a method for solving $MIQFP$ is presented. In order to solve $MIQFP$, we have established a relation between efficient solutions of $MIQFP$ and ranked solutions of $IQFP$. In the proposed algorithm, ranking and scanning feasible solutions of ILP or $ILFP$ is used for finding ranked solutions of $IQFP$. To enumerate ranked solutions

of *ILFP*, we have adopted a ranking cut considered in [143], and for determining ranked solutions of *ILP*, we have used an objective function cut.

A special case of the proposed algorithm for solving *MILFP* is also discussed in this chapter. A similar methodology is also used in [87] for solving *MIQP*.

This chapter is divided into different sections. In the next section, notations and some preliminary results are discussed. Theoretical results and an algorithm to find the set of all efficient solutions of *MIQFP* is presented in Section 4.5. To illustrate the working of the developed algorithm, numerical results are presented in Section 4.6. A special case of the proposed algorithm for solving *MILFP* is discussed in Section 4.7. In the last section, concluding remarks are discussed.

4.4 Notations and Preliminary results

The problem under consideration is

$$(MIQFP) \quad \min_{X \in \mathcal{X}} (h^1(X), h^2(X) \dots h^s(X)),$$

where

$$h^i(X) = \frac{X^T D^i X + C^i X + \alpha^i}{X^T Q^i X + P^i X + \beta^i} \quad \text{for all } i \in \{1, 2, \dots, s\},$$

$$D^i, Q^i \in \mathbb{Z}^{n \times n}, C^i, P^i \in \mathbb{Z}^{1 \times n}, \alpha^i, \beta^i \in \mathbb{Z}.$$

We assume that quadratic function $X^T Q^i X + P^i X + \beta^i > 0$ for all $i \in \{1, 2, \dots, s\}$ and D^i and Q^i are symmetric matrices.

Definition 4.4.1 *A feasible solution Y of $MIQFP$ is called an efficient solution of $MIQFP$ iff there does not exist any other feasible solution $X \in \mathcal{X}$ such that $h^i(X) \leq h^i(Y) \forall i \in \{1, 2, \dots, s\}$ and $h^i(X) < h^i(Y)$ for atleast one $i \in \{1, 2, \dots, s\}$.*

Let E be the set of all efficient solutions of *MIQFP*.

Definition 4.4.2 *If $Y \in E$, then the vector $(h^1(Y), h^2(Y) \dots h^s(Y))$ is called a non-dominated vector.*

The following theorem discussed in [131] shows existence of a non-dominated vector of *MIQFP*.

Theorem 4.4.3 *If a feasible solution $X \in \mathcal{X}$ of $MIQFP$ is non- empty, then the set of non-dominated vectors of $MIQFP$ is also non- empty.*

By virtue of Theorem 4.4.3, the set of all efficient solutions of *MIQFP* is non- empty. To

find efficient solutions of $MIQFP$, we may take any one of the objective functions $h^i(X)$ of $MIQFP$. In the subsequent discussion, we have used the first objective function $h^1(X)$ of $MIQFP$ and consider the problem, as follows:

$$(IQFP) \quad \min_{X \in \mathcal{X}} h^1(X).$$

Let $h^1(X^1) = \min_{X \in \mathcal{X}} h(X)$, $h^1(X^q) = \max_{X \in \mathcal{X}} h^1(X)$ and k^{th} ranked solution X^k of $IQFP$ be defined as

$$h^1(X^{k-1}) < h^1(X^k) < h^1(X^{k+1}) \text{ for all } k \in \{2, 3, \dots, q-1\}.$$

Now, let H^k be the set of all k^{th} ranked solutions of $IQFP$, therefore

$$H^1 = \{X \in \mathcal{X} \mid h^1(X) = \min_{X \in \mathcal{X}} h^1(X)\}, \quad H^q = \{X \in \mathcal{X} \mid h^1(X) = \max_{X \in \mathcal{X}} h^1(X)\}. \quad (4.4.1)$$

and for all $k \in \{2, 3, \dots, q-1\}$,

$$H^k = \{X \in \mathcal{X} \mid h^1(X^{k-1}) < h^1(X) < h^1(X^{k+1}), X^{k-1} \in H^{k-1}, X^{k+1} \in H^{k+1}\}. \quad (4.4.2)$$

It can be observed that H^k ($k > 1$) is the set of all optimal solutions of the following problem:

$$(IQFP_k) \quad \begin{cases} \min & h^1(X) \\ \text{s.t} & h^1(X) > h^1(X^{k-1}) \text{ for some } X^{k-1} \in H^{k-1} \\ & X \in \mathcal{X}. \end{cases}$$

As $h^1(X)$ is a continuous function over a non-empty, closed and bounded set $\mathcal{X}$. Therefore, the set of all optimal solutions of $IQFP$ and $IQFP_k$ is also non-empty, i.e., $H^k \neq \emptyset$ for all $k \in \{1, 2, \dots, q\}$.

The following theorem discussed in [75, 131] shows existence of a linear or linear fractional function $f(X)$, which provides lower bounds to quadratic fractional function $h^1(X)$.

Theorem 4.4.4 *For every quadratic fractional function $h^1(X)$, there always exists a linear or linear fractional function $f(X)$ such that $f(X) \leq h^1(X) \forall X \in \mathcal{X}$.*

The construction of function $f(X)$ has been discussed in [75, 131] (as shown in Table 4.1).

The following notations has been used in Table 4.1 :

$$m = \min_{X \in \mathcal{X}} X^T D^1 X + C^1 X + \alpha^1, \quad M = \max_{X \in \mathcal{X}} X^T D^1 X + C^1 X + \alpha^1, \quad \mu = \min_{X \in \mathcal{X}} X^T Q^1 X + P^1 X + \beta^1,$$

$$\hat{m} = \min_{X \in \mathcal{X}} (U + C^1)X + \alpha^1, \quad \hat{M} = \max_{X \in \mathcal{X}} (U + C^1)X + \alpha^1.$$

The i^{th} column of vectors U and V for all $i \in \{1, 2, \dots, n\}$ is defined as :

$$U_i = \min_{X \in \mathcal{X}} X^T D_i^1, \quad V_j = \max_{X \in \mathcal{X}} X^T Q_j^1$$

where D_i^1 and Q_j^1 are i^{th} column of matrices D^1 and Q^1 respectively.

Table 4.1: The construction of function $f(X)$

m	M	$\hat{m}$	$\hat{M}$	$f(X)$
$m \geq 0$	—	—	—	$\frac{(U+C^1)X+\alpha^1}{(V+P^1)X+\beta}$
—	$M \leq 0$	—	—	$\frac{(U+C^1)X+\alpha^1}{(U+C^1)X+\alpha^1}$
$m < 0$	$M > 0$	—	$\hat{M} \leq 0$	$\frac{(U+C^1)X+\alpha^1}{(U+C^1)X+\alpha^1}$
—	—	$\hat{m} < 0$	$\hat{M} > 0$	$\frac{(U+C^1)X+\alpha^1-\hat{M}}{\mu}$

Remark 4.4.5 As D^1 and Q^1 are symmetric matrices and $D^1, Q^1 \in \mathbb{Z}^{n \times n}$, therefore $U, V \in \mathbb{Z}^{n \times 1}$.

It is evident by Table 4.1, $f(X)$ is either a linear function or linear fractional function. Now, consider the following problem :

$$(ILFP) \text{ or } (ILP) \quad \min_{X \in \mathcal{X}} f(X).$$

Let $f(Y^1) = \min_{X \in \mathcal{X}} f(X)$, $f(Y^r) = \max_{X \in \mathcal{X}} f(X)$ and j^{th} ranked solution Y^j of the problem $(ILFP)$ or (ILP) be defined as

$$f(Y^{j-1}) < f(Y^j) < f(Y^{j+1}) \text{ for all } j \in \{2, 3, \dots, r-1\}.$$

Let F^j be the set of all j^{th} ranked solutions of the problem $(ILFP)$ or (ILP) . To find sets $F^j \forall j \in \{1, 2, \dots, r\}$, construct the following sets:

$$\chi_1 = \mathcal{X}, \quad \chi_j = \{Y \in \mathcal{X} \mid f(Y) > f(Y^{j-1}), Y^{j-1} \in F^{j-1}\} \text{ for all } j \in \{2, 3, \dots, r\}. \quad (4.4.3)$$

It can be observed that

$$F^j = \{Y \in \chi_j \mid f(Y) = \min_{X \in \chi_j} f(X)\} \text{ for all } j \in \{1, 2, \dots, r\}. \quad (4.4.4)$$

It may be noted that members of set F^j for all $j \in \{1, 2, \dots, r\}$ are generated from function $f(X)$. It is evident by Table 4.1 that $f(X)$ is either a linear function or linear fractional function. So, two cases arises:

Case 1 $f(X)$ is a linear function :

If $f(X)$ is a linear function, then consider the following problem defined as :

$$(ILP) \quad \min_{X \in \chi} f(X).$$

It can be observed that F^j is the set of all j^{th} ranked solutions of ILP . Also, by virtue of Equation (4.4.4), F^j is the set of all optimal solutions of ILP_j defined as :

$$(ILP_j) \quad \min_{X \in \chi_j} f(X).$$

Subcase 1 (a) $\mu = 1$

If $\mu = 1$, then $f(X) \in \mathbb{Z} \forall X \in \chi$. By virtue of Equation (4.4.3), feasible region χ_j of ILP_j becomes

$$\chi_j = \{Y \in \chi \mid f(Y) \geq f(Y^{j-1}) + 1, Y^{j-1} \in F^{j-1}\} \forall j \in \{2, 3, \dots, r\}.$$

Subcase 1 (b) $\mu \neq 1$

If $\mu \neq 1$, then define $\hat{f} = \mu f$. It can be observed that $\mu f(X) \in \mathbb{Z} \forall X \in \chi$. Then, by virtue of Equation (4.4.3),

$$\begin{aligned} \chi_j &= \{Y \in \chi \mid f(Y) > f(Y^{j-1}), Y^{j-1} \in F^{j-1}\} \\ &= \{Y \in \chi \mid \mu f(Y) > \mu f(Y^{j-1}), Y^{j-1} \in F^{j-1}\} \quad (\text{because } \mu > 0) \\ &= \{Y \in \chi \mid \mu f(Y) \geq \mu f(Y^{j-1}) + 1, Y^{j-1} \in F^{j-1}\} (\text{because } \mu f(X) \in \mathbb{Z} \text{ for all } X \in \chi). \end{aligned}$$

As $f(X)$ is a continous function and feasible region χ_j of ILP_j is a closed and bounded set, therefore there exists an optimal solution of ILP_j . A branch and bound method or Gomory cutting plane technique can be utilized for finding an optimal solution of ILP_j . An edge exploring technique discussed in the paper of Prerna and Sharma [121] can be used for finding the complete set F^j .

Case 2 : $f(X)$ is a linear fractional function :

If $f(X)$ is a linear fractional function, then consider the problem defined as :

$$(ILFP) \quad \min_{X \in \mathcal{X}} f(X).$$

It is evident by Equation (4.4.4), F^j is the set of all j^{th} ranked solutions of $ILFP$. The methodology discussed in [143] can be utilized for finding ranked solutions of $ILFP$.

In order to find ranked solutions of $ILFP$, consider the problem:

$$(LFP_t) \quad \min_{X \in S_t} f(X).$$

The feasible region S_t of LFP_t is obtained by successively truncating the feasible region S of the problem, as follows:

$$(LFP) \quad \min_{X \in S} f(X).$$

Some notations

Let $\hat{X}$ be an optimal solution of LFP_t with objective function value $\hat{Z}$,

B is the basis corresponding to $\hat{X}$,

$\hat{X}_B$ is a sub vector of $\hat{X}$ associated with B ,

$C = U + C^1$, $P = V + P^1$,

C_B and P_B are vectors of the cost coefficients of basic variables associated with B in vectors C and P respectively,

x_j is the j^{th} component of $\hat{X}$,

c_j and p_j are the j^{th} component of vectors C and P respectively,

a_j is the activity vector of x_j with respect to the feasible region of LFP_t ,

$y_{i,j}$ is the i^{th} component of $B^{-1}a_j$,

$z_j^1 = \sum_{i \in \hat{I}} c_i y_{ij}$, $z_j^2 = \sum_{i \in \hat{I}} p_i y_{ij}$,

$\hat{I} = \{i \mid x_i \text{ is a basic variable corresponding to } \hat{X}\}$,

$\hat{N} = \{j \mid x_j \text{ is a non-basic variable corresponding to optimal basic feasible solution } \hat{X} \text{ with } \Delta_j < 0\}$.

The optimal value $\hat{Z}$ of LFP_t will be given by

$$\hat{Z} = \frac{Z_1}{Z_2} = \frac{C_B \hat{X}_B + \alpha^1}{P_B \hat{X}_B + \beta^1}.$$

If $\bar{X}$ be another basic feasible solution of LFP_t and $\bar{Z}$ be the objective function value at

$\bar{X}$, then

$$\bar{Z} = \frac{Z_1 - \theta_j(z_j^1 - c_j)}{Z_2 - \theta_j(z_j^2 - d_j)},$$

The change in objective function value from $\hat{Z}$ to $\bar{Z}$ is given by

$$\bar{Z} = \hat{Z} - \frac{\Delta_j \theta_j}{Z_2(Z_2 - \theta_j(z_j^2 - p_j))}, \text{ where } \Delta_j = Z_2(z_j^1 - c_j) - Z_1(z_j^2 - p_j).$$

Let $\hat{N} = \{j \mid x_j \text{ is a non-basic variable corresponding to optimal basic feasible solution } \hat{X} \text{ with } \Delta_j < 0\}$.

To find ranked solutions of $ILFP$, we have utilized the following result discussed in [143].

Theorem 4.4.6 *All integer feasible solutions of LFP_t yielding objective function value greater than the optimal value of LFP_t lie in the closed half space $\sum_{j \in \hat{N}} x_j \geq 1$.*

If $\hat{X}$ is a $(j-1)^{th}$ ranked solution of $ILFP$ and optimal basic feasible solution of LFP_t , then by virtue of Theorem 4.4.6, the set χ_j defined in Equation (4.4.3) becomes

$$\chi_j = \{Y \in \chi \mid \sum_{j \in \hat{N}} x_j \geq 1\} \text{ for all } j \in \{2, 3, \dots, r\}.$$

As $f(X)$ is a continuous function for all $X \in \chi_j$ and feasible region χ_j of $ILFP_j$ is a closed and bounded set, therefore set F^j is non-empty for all $j \in \{1, 2, \dots, r\}$.

4.5 Theoretical development

In this section, we will establish a relation between efficient solutions of $MIQFP$ and ranked solutions of $IQFP$. Further, we will discuss the results that can be used for enumerating ranked solutions of $IQFP$.

4.5.1 Relation between efficient solutions of $MIQFP$ and ranked solutions of $IQFP$

In the following theorem, we will show that the set of all feasible solutions of $MIQFP$ can be obtained by ranking feasible solutions of $IQFP$ for all $i \in \{1, 2, \dots, q\}$.

Theorem 4.5.1 *If $\hat{X}$ is a feasible solution of MIQFP, then $\hat{X} \in H^k$ for some $k \in \{1, 2, \dots, q\}$.*

Proof If $\hat{X} \in H^1$, then theorem is proved. Otherwise, if possible let $\hat{X} \notin H^1$. As $H^1 = \{X \in \chi \mid h^1(X) = \min_{X \in \chi} h^1(X)\}$, therefore

$$h^1(X^1) < h^1(\hat{X}) \text{ for all } X^1 \in H^1. \quad (4.5.1)$$

If $\hat{X} \in H^2$, i.e., $h^1(X^1) < h^1(\hat{X}) < h^1(X^3)$ for $X^1 \in H^1$, $X^3 \in H^3$, then theorem is proved. Otherwise, if possible let $\hat{X} \notin H^2$, therefore either $h^1(X^1) \geq h^1(\hat{X})$ or $h^1(\hat{X}) \geq h^1(X^3)$ for all $X^1 \in H^1$, $X^3 \in H^3$. If $h^1(X^1) \geq h^1(\hat{X})$, then it contradicts Equation (4.5.1), therefore

$$h^1(X^3) \leq h^1(\hat{X}) \text{ for all } X^3 \in H^3. \quad (4.5.2)$$

If $\hat{X} \in H^3$, i.e., $h^1(X^2) < h^1(\hat{X}) < h^1(X^4)$ for $X^2 \in H^2$, $X^4 \in H^4$, then theorem is proved. Otherwise, if possible let $\hat{X} \notin H^3$, therefore either $h^1(X^2) \geq h^1(\hat{X})$ or $h^1(\hat{X}) \geq h^1(X^4)$ for all $X^2 \in H^2$, $X^4 \in H^4$. If possible let $h^1(X^2) \geq h^1(\hat{X})$ for all $X^2 \in H^2$. As $\hat{X} \notin H^2$ and $\hat{X} \notin H^3$, therefore by using Equation (4.5.2), we get

$$h^1(X^3) < h^1(\hat{X}) < h^1(X^2) \text{ for all } X^2 \in H^2, X^3 \in H^3.$$

which implies

$$h^1(X^3) < h^1(X^2) \text{ for all } X^2 \in H^2, X^3 \in H^3. \quad (4.5.3)$$

But,

$$h^1(X^2) < h^1(X^3) \text{ for all } X^2 \in H^2, X^3 \in H^3$$

which contradicts Equation (4.5.3), therefore

$$h^1(\hat{X}) \geq h^1(X^4) \text{ for all } X^4 \in H^4.$$

Continue on the same way, if $\hat{X} \notin H^k$ for all $k \in \{1, 2, \dots, q-1\}$, then we get

$$h^1(\hat{X}) \geq h^1(X^q) \text{ for all } X^q \in H^q.$$

If $\hat{X} \in H^q$, then theorem is proved. Otherwise, if possible let $h^1(\hat{X}) > h^1(X^q)$ for $X^q \in H^q$, which contradicts $H^q = \{X \in \chi \mid h^1(X) = \max_{X \in \chi} h^1(X)\}$. Thus, $\hat{X} \in H^k$ for some $k \in \{1, 2, \dots, q\}$.

It is evident by Theorem 4.5.1,

$$\chi \subset \bigcup_{k=1}^q H^k. \quad (4.5.4)$$

Since $E \subset \chi$, therefore by Equation 4.5.4, we concluded that the set of all efficient solutions of *MIQFP* can be generated by scanning all ranked solutions of *IQFP*. Now, we will establish a necessary and sufficient condition for a point $\hat{X} \in H^k$ for some $k \in \{1, 2, \dots, q\}$ to be an efficient solution of *MIQFP*.

Theorem 4.5.2 *A point $\hat{X} \in \chi$ is an efficient solution of MIQFP iff $\hat{X} \in H^k$ for some $k \in \{1, 2, \dots, q\}$ and there does not exist any $Y \in (\bigcup_{l=1}^{k-1} E^l) \cup H^k$ such that $h^i(Y) \leq h^i(\hat{X}) \forall i \in \{2, 3, \dots, p\}$ and $h^i(Y) < h^i(\hat{X})$ for some $i \in \{1, 2, \dots, p\}$.*

Proof Let $\hat{X} \in E$. As $E \subset \chi$, therefore by virtue of Theorem 4.5.1, $\hat{X} \in H^k$ for some $k \in \{1, 2, \dots, q\}$. Since $(\bigcup_{l=1}^{k-1} E^l) \cup H^k \subset \chi$ and $\hat{X}$ is an efficient solution of *MIQFP*, therefore by definition of efficient solution there does not exist any feasible solution $Y \in (\bigcup_{l=1}^{k-1} E^l) \cup H^k$ such that

$$h^i(Y) \leq h^i(\hat{X}) \forall i \in \{1, 2, \dots, p\}, \quad h^i(Y) < h^i(\hat{X}) \text{ for at least one } i \in \{1, 2, \dots, p\}.$$

Now, let $\hat{X} \in H^k$ for some $k \in \{1, 2, \dots, q\}$ and there does not exist any $Y \in (\bigcup_{l=1}^{k-1} E^l) \cup H^k$ such that $h^i(Y) \leq h^i(\hat{X}) \forall i \in \{2, 3, \dots, p\}$ and $h^i(Y) < h^i(\hat{X})$ for some $i \in \{1, 2, \dots, p\}$.

Let if possible, $\hat{X} \notin E$, therefore there exist $Y^1 \in \chi$ such that

$$h^i(Y^1) \leq h^i(\hat{X}) \forall i \in \{1, 2, \dots, p\}, \quad h^i(Y^1) < h^i(\hat{X}) \text{ for at least one } i \in \{1, 2, \dots, p\}. \quad (4.5.5)$$

As $\hat{X} \in H^k$ and $h^i(Y^1) \leq h^i(\hat{X}) \forall i \in \{1, 2, \dots, p\}$, therefore $Y^1 \in \cup_{i=1}^k H^i$. If $Y^1 \in (\bigcup_{l=1}^{k-1} E^l) \cup H^k$, then it contradicts the hypothesis. Therefore, $Y^1 \in \cup_{i=1}^{k-1} H^i$ and $Y^1 \notin (\bigcup_{l=1}^{k-1} E^l)$, which implies Y^1 is a non-efficient solution in the set $\cup_{i=1}^{k-1} H^i$. As $Y^1 \notin E$, therefore there exist $Y^2 \in \chi$ such that

$$h^i(Y^2) \leq h^i(Y^1) \text{ for all } i \in \{1, 2, \dots, p\}, \quad h^i(Y^2) < h^i(Y^1) \text{ for at least one } i \in \{1, 2, \dots, p\}. \quad (4.5.6)$$

As $Y^1 \in \cup_{i=1}^{k-1} H^i$ and $h^1(Y^2) \leq h^1(Y^1)$, therefore $Y^2 \in \cup_{i=1}^{k-1} H^i$. By using Equation (4.5.5)

and Equation (4.5.6), we get

$$\begin{aligned} h^i(Y^2) &\leq h^i(Y^1) \leq h^i(\hat{X}) \text{ for all } i \in \{1, 2, \dots, p\} \\ h^{i_2}(Y^2) &< h^{i_2}(Y^1), \quad h^{i_1}(Y^1) < h^{i_1}(\hat{X}) \text{ for at least one } i_1, i_2 \in \{1, 2, \dots, p\}. \end{aligned} \quad (4.5.7)$$

If possible let $h^i(Y^2) = h^i(\hat{X})$ for all $i \in \{1, 2, \dots, p\}$. Then, by virtue of Equation (4.5.7), $h^i(Y^2) = h^i(Y^1) = h^i(\hat{X})$ for all $i \in \{1, 2, \dots, p\}$, which contradicts Equation (4.5.5). Therefore,

$$h^i(Y^2) < h^i(\hat{X}) \text{ for at least one } i \in \{1, 2, \dots, p\},$$

Also,

$$h^i(Y^2) \leq h^i(\hat{X}) \text{ for all } i \in \{1, 2, \dots, p\}.$$

Then, by hypothesis $Y^2 \notin (\bigcup_{l=1}^{k-1} E^l) \cup H^k$. Also, $Y^2 \in \cup_{i=1}^{k-1} H^i$, Therefore, Y^2 is a non-efficient solution in the set $\cup_{i=1}^{k-1} H^i$. Since the feasible region contains a finite number of integer points, therefore set $\cup_{i=1}^{k-1} H^i$ contains finite non-efficient solutions. Thus, the process terminates after finite (r) steps, and we get r non-efficient solutions $(Y^1, Y^2, \dots, Y^r)$ in the set $\cup_{i=1}^{k-1} H^i$ such that

$$\begin{aligned} h^i(Y^r) &\leq h^i(Y^{r-1}) \dots \leq h^i(Y^2) \leq h^i(Y^1) \leq h^i(\hat{X}) \text{ for all } i \in \{1, 2, \dots, p\} \\ h^{i_2}(Y^2) &< h^{i_2}(Y^1), h^{i_3}(Y^3) < h^{i_3}(Y^2), \dots, h^{i_r}(Y^r) < h^{i_r}(Y^{r-1}) \text{ for some } i_2, i_3, \dots, i_r \in \{1, 2, \dots, p\} \\ h^i(Y^j) &< h^i(\hat{X}) \text{ for at least one } i \in \{1, 2, \dots, p\}, \text{ for all } j \in \{1, 2, \dots, r\}. \end{aligned} \quad (4.5.8)$$

As Y^r is a non-efficient solution in the set $\cup_{i=1}^{k-1} H^i$, therefore there exist a feasible solution $\hat{Y}$ such that

$$h^i(\hat{Y}) \leq h^i(Y^r) \text{ for all } i \in \{1, 2, \dots, p\}, \quad h^i(\hat{Y}) < h^i(Y^r) \text{ for at least one } i \in \{1, 2, \dots, p\}. \quad (4.5.9)$$

Since $h^1(\hat{Y}) \leq h^1(Y^r)$ and $Y^r \in \cup_{i=1}^{k-1} H^i$, therefore $\hat{Y} \in \cup_{i=1}^{k-1} H^i$. By virtue of Equation (4.5.9), $\hat{Y} \neq Y^r$. Also, by virtue of Equation (4.5.8), $\hat{Y} \neq Y^i$ for all $i \in \{1, 2, \dots, r-1\}$. Therefore, $\hat{Y}$ is an efficient solution in the set $\cup_{i=1}^{k-1} H^i$.

Also, $h^i(\hat{Y}) \leq h^i(Y^r) \leq h^i(\hat{X})$. If $h^i(\hat{Y}) = h^i(\hat{X})$, then it implies $h^i(\hat{Y}) = h^i(Y^r) = h^i(\hat{X})$ for all $i \in \{1, 2, \dots, p\}$, which contradicts Equation (4.5.9). Therefore, $\hat{Y}$ is an efficient solution in the set $\cup_{i=1}^{k-1} H^i$ and

$$h^i(\hat{Y}) < h^i(\hat{X}) \text{ for at least one } i \in \{1, 2, \dots, p\},$$

which contradicts the hypothesis. Thus, $\hat{X} \in E$.

Now for determining ranked solution of $IQFP$, we will establish relationship between ranked solutions of $IQFP$ and set F^j , which are j^{th} ranked solution of $ILFP$ or ILP .

4.5.2 Relation between ranked solutions of $IQFP$ and set F^j

The following theorem presented in [75, 131] shows how an optimal solution of $IQFP$ can be obtained from set F^j , where $j \in \{1, 2, \dots, r\}$.

Theorem 4.5.3 *If $f(Y) \geq \min \{h^1(X) \mid X \in \bigcup_{j=1}^l F^j\} = h^1(\hat{X})$ for some $Y \in F^l$, then $\hat{X}$ is an optimal solution of $IQFP$.*

Corollary 4.5.4 *If there exist some $Y \in F^1$ such that $f(Y) = \min \{h^1(X) \mid X \in F^1\} = h^1(\hat{X})$, then $\hat{X}$ is an optimal solution of $IQFP$.*

In the following corollary, we will show that the set of all optimal solutions of $IQFP$ can be obtained without finding set F^j for all $j > l$, provided there exist some $Y \in F^l$ such that $f(Y) \geq \min \{h^1(X) \mid X \in \bigcup_{j=1}^l F^j\}$.

Corollary 4.5.5 *If there exist some $Y \in F^l$ such that $f(Y) \geq \min \{h^1(X) \mid X \in \bigcup_{j=1}^l F^j\} = h^1(\hat{X})$, then $F = \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\hat{X})\}$ is the set of all optimal solutions of $IQFP$.*

Proof Let H^1 be the set of all optimal solutions of $IQFP$. So, we need to show $F = H^1$. As $F = \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\hat{X})\}$ and by virtue of Theorem 4.5.3, $\hat{X} \in H^1$. Therefore,

$$F \subseteq H^1. \quad (4.5.10)$$

Now let $X^1 \in H^1$, then

$$h^1(X^1) = h^1(\hat{X}) \leq h^1(X) \text{ for all } X \in \chi. \quad (4.5.11)$$

If possible let $X^1 \notin F$ which implies $X^1 \notin \bigcup_{j=1}^l F^j$. As $X^1 \notin \bigcup_{j=1}^l F^j$ and $Y \in F^l$ therefore

$$\begin{aligned} f(Y) &< f(X^1) \\ \implies f(Y) &< h^1(X^1) \quad (\because f(X) \leq h^1(X) \text{ for all } X \in \chi) \\ \implies h^1(\hat{X}) &< h^1(X^1) \quad (\because f(Y) \geq h^1(\hat{X})) \end{aligned}$$

which contradict Equation (4.5.11). Thus,

$$H^1 \subseteq F. \quad (4.5.12)$$

Hence $H^1 = \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\hat{X})\}$ (by eqns (4.5.10) and (4.5.12)).

The following theorem explains how a k^{th} ranked solution of $IQFP$ can be generated from $\bigcup_{j=1}^l F^j$, where $l \in \{1, 2, \dots, r\}$.

Theorem 4.5.6 *If $f(Y) \geq \min \{h^1(X) \mid h^1(X) > h^1(X^{k-1}), X \in \bigcup_{j=1}^l F^j\} = h^1(\bar{X})$ for some $Y \in F^l$, $X^{k-1} \in H^{k-1}$, then $\bar{X}$ is a k^{th} ranked solution of $IQFP$.*

Proof Let, if possible $\bar{X}$ be not a k^{th} ranked solution of $IQFP$, therefore $\bar{X}$ is not an optimal solution of $IQFP_k$. By hypothesis $h^1(\bar{X}) > h^1(X^{k-1})$ for some $X^{k-1} \in H^{k-1}$, therefore $\bar{X}$ is a feasible solution of $IQFP_k$. Since $\bar{X}$ is a feasible solution, but not an optimal solution of $IQFP_k$, therefore there exists a feasible solution $\check{X}$ of $IQFP_k$ such that

$$h^1(\check{X}) > h^1(X^{k-1}) \text{ and } h^1(\check{X}) < h^1(\bar{X}). \quad (4.5.13)$$

Now, two cases arises either $\check{X} \in \bigcup_{j=1}^l F^j$ or $\check{X} \in F^j$ for some $j > l$.

If possible let $\check{X} \in \bigcup_{j=1}^l F^j$. Since $\check{X} \in \bigcup_{j=1}^l F^j$ and $h^1(\check{X}) > h^1(X^{k-1})$, therefore by hypothesis

$$h^1(\bar{X}) \leq h^1(\check{X}). \quad (4.5.14)$$

Equation (4.5.13) contradict Equation (4.5.14). So $\check{X} \in F^j$ for some $j > l$. As $Y \in F^l$ and $\check{X} \in F^j$ for some $j > l$. Therefore $f(Y) < f(\check{X})$. Also, $h^1(\bar{X}) \leq f(Y)$ which implies

$$h^1(\bar{X}) < f(\check{X}). \quad (4.5.15)$$

Also,

$$f(X) < h^1(X) \text{ for all } X \in \chi.$$

In particular,

$$f(\check{X}) < h^1(\check{X}). \quad (4.5.16)$$

By using eqns (4.5.15) and (4.5.16), we get

$$h^1(\bar{X}) < h^1(\check{X}) \quad (4.5.17)$$

which contradict Equation (4.5.13). So, $\bar{X}$ is an optimal solution of $IQFP_k$. As optimal solution of $IQFP_k$ is the k^{th} ranked solution of $IQFP$. Thus, $\bar{X}$ is the k^{th} ranked solution of $IQFP$.

In the following corollary, we will claim that the set H^k ($k > 1$) can be obtained without finding set F^j for all $j > l$, provided there exist some $Y \in F^l$, $X^{k-1} \in H^{k-1}$ such that $f(Y) \geq \min\{h^1(X) \mid h^1(X) > h^1(X^{k-1}), X \in \bigcup_{j=1}^l F^j\}$.

Corollary 4.5.7 *If $f(Y) \geq \min\{h^1(X) \mid h^1(X) > h^1(X^{k-1}), X \in \bigcup_{j=1}^l F^j\} = h^1(\bar{X})$ for some $Y \in F^l$, $X^{k-1} \in H^{k-1}$, then $H^k = \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\bar{X})\}$.*

Proof By hypothesis, $f(Y) \geq \min\{h^1(X) \mid h^1(X) > h^1(X^{k-1}), X \in \bigcup_{j=1}^l F^j\} = h^1(\bar{X})$. Therefore by Theorem 4.5.6, $\bar{X} \in H^k$. Since $\bar{X} \in H^k$ therefore $\{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\bar{X})\} \subseteq H^k$. Now, we want to show $H^k \subseteq \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\bar{X})\}$. Let a feasible solution $X^k \in H^k$. Now two cases arises either $X^k \in \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\bar{X})\}$ or $X^k \in \{X \in \bigcup_{j=l+1}^r F^j \mid h^1(X) = h^1(\bar{X})\}$. If possible let $X^k \in \{X \in \bigcup_{j=l+1}^r F^j \mid h^1(X) = h^1(\bar{X})\}$. Since $X^k \in \bigcup_{j=l+1}^r F^j$ and by hypothesis, $Y \in F^l$ therefore,

$$f(Y) < f(X^k). \quad (4.5.18)$$

Also, $f(X) \leq h^1(X)$ for all $X \in \mathcal{X}$ therefore

$$f(X^k) \leq h^1(X^k). \quad (4.5.19)$$

Equation 4.5.18 and Equation 4.5.19 imply that

$$f(Y) < h^1(X^k). \quad (4.5.20)$$

Since $X^k \in H^k$ and $\bar{X} \in H^k$, therefore

$$f(Y) < h^1(\bar{X}). \quad (4.5.21)$$

But,

$$f(Y) \geq \min\{h^1(X) \mid h^1(X) > h^1(X^{k-1}), X \in \bigcup_{j=1}^l F^j\} = h^1(\bar{X}). \quad (4.5.22)$$

Equation 4.5.22 contradicts Equation 4.5.21. So, $X^k \notin \{X \in \cup_{j=l+1}^r F^j \mid h^1(X) = h^1(\bar{X})\}$ which implies $X^k \in \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\bar{X})\}$. Therefore $H^k \subseteq \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\bar{X})\}$. Thus, $H^k = \{X \in \bigcup_{j=1}^l F^j \mid h^1(X) = h^1(\bar{X})\}$.

By above discussion, we concluded that the set of all k^{th} ranked solutions of IQP can be generated from set F^j .

4.5.3 Convergence of the proposed algorithm

Theorem 4.5.8 *The proposed algorithm finds the efficient set of $MIQFP$ in a finite number of steps.*

Proof The proposed algorithm finds the efficient set of $MIQFP$ by enumerating all ranked solutions of $IQFP$ and the set of all ranked solutions of $IQFP$ is obtained from ranked solutions of $ILFP$ or ILP . As feasible region χ of $ILFP$ and ILP is bounded subset of $\mathbb{Z}^n$, the number of ranked solutions of $ILFP$ and ILP is finite. Thus, the proposed algorithm terminates in a finite number of steps.

Theorem 4.5.9 *None of the efficient solution is missed by the proposed algorithm.*

Proof The proposed algorithm finds the set of all efficient solutions of $MIQFP$ by scanning all ranked solutions of $MIQFP$. It is evident by Theorem 4.5.1, the set of all efficient solutions of $MIQFP$ is contained in the set of all ranked solutions of $IQFP$. So, none of the efficient solution is missed by the proposed algorithm.

4.6 Numerical Section

This section has a twofold aim. The first aim is to elaborate on the working of the proposed algorithm, and the second is to compare with the existing methods [102, 130].

Now, we will solve four examples using the proposed algorithm. In the first example, we will consider a general $MIQFP$ containing five variables. The second example, we will consider from [102, 130], and using this example, we will compare the results of the proposed algorithm with the algorithms of [102, 130]. The first objective of the third and

Algorithm 5: Algorithm to solve *MIQFP*

Data: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^{m \times 1}$, C^i , $P^i \in \mathbb{Z}^{1 \times n}$, D^i , $Q^i \in \mathbb{Z}^{n \times n}$ and $\alpha^i, \beta^i \in \mathbb{Z}$ for $i \in \{1, 2, \dots, s\}$.

Result: $E = \bigcup_{l=1}^k E^l$ ▷ set of all efficient solutions of *MIQFP*

```

1 Initialization  $k \leftarrow 1, j \leftarrow 1, S \leftarrow \{X \in \mathbb{R}^n \mid AX = b, X \geq 0\}, \mathcal{X} \leftarrow S \cap \mathbb{Z}^n$ 
2 for  $i \leftarrow 1 : s$  do
3    $h^i(X) \leftarrow \frac{X^T D^i X + C^i X + \alpha^i}{X^T Q^i X + P^i X + \beta^i}$  ▷ IQFP  $\min_{X \in \mathcal{X}} h^1(X)$ 
4 for  $i \leftarrow 1 : n$  do
5    $U(i) \leftarrow \min_{X \in \mathcal{X}} X^T D^1(:, i), V(i) \leftarrow \max_{X \in \mathcal{X}} X^T Q^1(:, i)$ 
6   ▷  $U(i)$  and  $V(i)$  are  $i^{th}$  column of vectors  $U$  and  $V$  respectively.
7   ▷  $D^1(:, i)$  and  $Q^1(:, i)$  are  $i^{th}$  column of matrices  $D^1$  and  $Q^1$  respectively.
8  $m \leftarrow \min_{X \in \mathcal{X}} X^T D^1 X + C^1 X + \alpha^1$ 
9 if  $m \geq 0$  then
10    $f(X) \leftarrow \frac{(U+C^1)X+\alpha^1}{(V+P^1)X+\beta^1}$  ▷ ILFP  $\min_{X \in \mathcal{X}} f(X)$ 
11 else
12    $\mu \leftarrow \min_{X \in \mathcal{X}} X^T Q^1 X + P^1 X + \beta^1, M \leftarrow \max_{X \in \mathcal{X}} X^T D^1 X + C^1 X + \alpha^1,$ 
13    $\hat{m} \leftarrow \min_{X \in \mathcal{X}} (U + C^1)X + \alpha^1, \hat{M} \leftarrow \max_{X \in \mathcal{X}} (U + C^1)X + \alpha^1.$ 
14   if  $(m < 0, M > 0 \text{ and } \hat{M} \leq 0) \text{ or } (M \leq 0)$  then
15      $f(X) \leftarrow \frac{(U+C^1)X+\alpha^1}{\mu}$  ▷ ILP  $\min_{X \in \mathcal{X}} f(X)$ 
16   if  $\hat{m} < 0 \text{ and } \hat{M} > 0$  then
17      $f(X) \leftarrow \frac{(U+C^1)X+\alpha^1-\hat{M}}{\mu}$ 
18  $f_{max} \leftarrow \max_{X \in \mathcal{X}} f(X), F^1 \leftarrow \{Y \in \mathcal{X} \mid f(Y) = \min_{X \in \mathcal{X}} f(X)\}, h_{1,1} \leftarrow \min\{h^1(X) \mid X \in F^1\}$ 
19 while  $f(Y) < h_{j,1}$  and  $f(Y) < f_{max}$  for  $Y \in F^j$  do
20    $j \leftarrow j + 1, \mathcal{X}_j \leftarrow \{X \in \mathcal{X} \mid f(X) > f(Y), Y \in F^{j-1}\},$ 
21    $F^j \leftarrow \{Y \in \mathcal{X}_j \mid f(Y) = \min_{X \in \mathcal{X}_j} f(X)\}, h_{j,1} \leftarrow \min\{h(X) \mid X \in \bigcup_{i=1}^j F^i\}$ 
22  $H^1 \leftarrow \{Y \in \bigcup_{i=1}^j F^i \mid h^1(Y) = h_{j,k}\}$  ▷ set of optimal solutions of IQFP
23  $N^1 \leftarrow [], E^1 \leftarrow []$ 
24 foreach  $X \in H^1$  do
25   foreach  $Y \in H^1$  do
26     if  $h^i(Y) \leq h^i(X)$  for all  $i \in \{2, 3, \dots, s\}$  and  $h^i(Y) < h^i(X)$  for some  $i \in \{1, 2, \dots, s\}$  then
27        $N^1 \leftarrow [N^1 \ X]$  ▷ Set of non-efficient solutions of MIQFP in set  $H^1$ 
28   if  $X \notin N^1$  then
29      $E^1 \leftarrow [E^1 \ X]$  ▷ Set of efficient solutions of MIQFP in set  $H^1$ 
30 while  $f(Y) < f_{max}$  for some  $Y \in F^j$  or  $h(X^k) > h_{j,k}$  for some  $X^k \in H^k$  do
31    $k \leftarrow k + 1$ 
32   if  $\{X \mid h^1(X) > h^1(X^{k-1}), X \in \bigcup_{i=1}^j F^i, X^{k-1} \in H^{k-1}\} = \emptyset$  then
33      $h_{j,k} = \infty$ 
34   else
35      $h_{j,k} \leftarrow \min\{h(X) \mid h(X) > h(X^{k-1}), X \in \bigcup_{i=1}^j F^i \text{ for } X^{k-1} \in H^{k-1}\}$ 
36   while  $f(Y) < h_{j,k}$  and  $f(Y) < f_{max}$  for  $Y \in F^j$  do
37      $j \leftarrow j + 1, \mathcal{X}_j \leftarrow \{X \in \mathcal{X} \mid f(X) > f(Y), Y \in F^{j-1}\}$ 
38      $F^j \leftarrow \{Y \in \mathcal{X}_j \mid f(Y) = \min_{X \in \mathcal{X}_j} f(X)\}, h_{j,k} \leftarrow \min\{h(X) \mid h(X) > h(X^{k-1}), X \in \bigcup_{i=1}^j F^i \text{ for } X^{k-1} \in H^{k-1}\}$ 
39    $H^k \leftarrow \{Y \in \bigcup_{i=1}^j F^i \mid h^1(Y) = h_{j,k}\}$  ▷ set of  $k^{th}$  ranked solutions of IQFP
40    $N^k \leftarrow [], E^k \leftarrow []$ 
41   foreach  $X \in H^k$  do
42     foreach  $Y \in (\bigcup_{l=1}^{k-1} E^l) \cup H^k$  do
43       if  $h^i(Y) \leq h^i(X)$  for all  $i \in \{2, 3, \dots, s\}$  and  $h^i(Y) < h^i(X)$  for some  $i \in \{1, 2, \dots, s\}$  then
44          $N^k \leftarrow [N^k \ X]$  ▷ Set of non-efficient solutions of MIQFP in  $H^k$ 
45     if  $X \notin N^k$  then
46        $E^k \leftarrow [E^k \ X]$  ▷ Set of efficient solutions of MIQFP in set  $H^k$ 

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fourth examples are taken from [86], and these examples serve as counter-example for the Mekhilef, Moulai, and Drici [102] algorithm.

Example 4.6.1 Consider the following problem :

$$(MIQFP) \quad \min_{X \in \chi = S \cap \mathbb{Z}^5} (h^1(X), h^2(X), h^3(X)),$$

where

$$h^1(X) = \frac{-3x_3^2 - 2x_5^2 + x_2 + x_4}{2x_1x_3 + 2x_2x_3 + 2x_4x_3 + x_5^2}, \quad h^2(X) = \frac{x_2^2 - x_4^2 - 5}{x_1x_3 + x_1x_5 + 1}, \quad h^3(X) = \frac{-x_1^2 - 2x_2^2 - 3x_3^2 - 4x_4^2 - 5x_5^2}{x_3^2 + x_5},$$

$$S = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 + x_3 + x_4 + x_5 \leq 2, x_1 + x_2 + x_4 \geq 1, x_3 + x_5 \geq 1, x_1, x_2, x_3, x_4, x_5 \geq 0\}.$$

It can be observed that $2x_1x_3 + 2x_2x_3 + 2x_4x_3 + x_5^2$, $x_1x_3 + x_1x_5 + 1$, $x_3^2 + x_5 > 0$ for all $X \in \chi$. Now, consider an integer quadratic fractional problem as follows:

$$(IQFP) \quad \min_{X \in \chi} h^1(X) = \frac{-3x_3^2 - 2x_5^2 + x_2 + x_4}{2x_1x_3 + 2x_2x_3 + 2x_4x_3 + x_5^2}.$$

1. As $M = \max_{X \in \chi} -3x_3^2 - 2x_5^2 + x_2 + x_4 \leq 0$, therefore from Table 4.1 the linear function $f(X) = \frac{(U+C^1)X+\alpha^1}{\mu}$ provides lower bounds to quadratic fractional function $h^1(X)$.

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = \min_{X \in \chi} -3x_3 = -3, \quad U_4 = 0, \quad U_5 = \min_{X \in \chi} -2x_5 = -2,$$

$$\mu = \min_{X \in \chi} 2x_1x_3 + 2x_2x_3 + 2x_4x_3 + x_5^2 = 1.$$

Therefore, $f(X) = -3x_3 - 2x_5 + x_2 + x_4$.

2. The lower bound to the objective function values of the problem (IQFP) is provided by objective function values of ILP, which is defined as :

$$(ILP) \quad \min_{X \in \chi} f(X).$$

The problem (ILP) has a unique optimal solution (1,0,1,0,0) with optimal value -3, therefore $F^1 = \{(1, 0, 1, 0, 0)\}$.

3. It can be observed that $h_{1,1} = \min\{h^1(X) \mid X \in F^1\} = -1.5$ and $f_{max} = \max\{f(X) \mid X \in \chi\} = -1$. Since $f(1, 0, 1, 0, 0) < h_{1,1}$ and $f(Y) < f_{max}$ for all $Y \in F^1$. Therefore, set $j = 2$ and consider the following problem:

$$(ILP_2) \quad \min_{X \in \chi_2} f(X),$$

where $\chi_2 = \{X \in \chi \mid -3x_3 - 2x_5 + x_2 + x_4 \geq -2\}$.

The problem (ILP_2) has three optimal solutions $(0,1,1,0,0)$, $(0,0,1,1,0)$ and $(1,0,0,0,1)$, therefore $F^2 = \{(0,1,1,0,0), (0,0,1,1,0), (1,0,0,0,1)\}$.

4. It can be observed from Table 4.2 that $h_{2,1} = \min\{h^1(X) \mid X \in \bigcup_{i=1}^2 F^i\} = h^1(1,0,0,0,1) = f(Y)$ for $Y \in F^2$. So, $H^1 = \{(1,0,0,0,1)\}$.

5. As $(1,0,0,0,1)$ is the only element of set H^1 , therefore $N^1 = \{\}$ and $E^1 = \{(1,0,0,0,1)\}$.

6. Since $h(0,0,1,1,0) > h_{2,1}$, therefore set $k = 2$. It can also be observed from Table 4.2 that $h_{2,2} = \min\{h^1(X) \mid h^1(X) > h^1(1,0,0,0,1), X \in \bigcup_{i=1}^2 F^i\} = -1.5$ and $f(Y) < h_{2,2}$ for all $Y \in F^2$ and $f(Y) < f_{max}$ for all $Y \in F^2$, therefore set $j = 3$ and consider the following problem:

$$(ILP_3) \quad \min_{X \in \chi_3} f(X),$$

where $\chi_3 = \{X \in \chi \mid -3x_3 - 2x_5 + x_2 + x_4 \geq -1\}$.

The problem (ILP_3) has two optimal solutions $(0,1,0,0,1)$ and $(0,0,0,1,1)$, therefore $F^3 = \{(0,1,0,0,1), (0,0,0,1,1)\}$.

7. It can be observed from Table 4.2 that $h_{3,2} = \min\{h^1(X) \mid h^1(X) > h^1(1,0,0,0,1), X \in \bigcup_{i=1}^3 F^i\} = h^1(1,0,1,0,0)$ and $h_{3,2} \leq f(Y)$ for $Y \in F^3$. So, $H^2 = \{(1,0,1,0,0)\}$.

8. The criterion vector corresponding to $(1,0,1,0,0)$ is dominated by criterion vector corresponding to $(1,0,0,0,1)$ as shown in Table 4.3. So, $N^2 = \{(1,0,1,0,0)\}$ and $E^2 = \{\}$.

9. Since $h(0,1,1,0,0) > h_{3,2}$, therefore set $k = 3$. It can be seen from Table 4.2 that

$$h_{3,3} = \min\{h^1(X) \mid h^1(X) > h^1(1,0,1,0,0), X \in \bigcup_{i=1}^3 F^i\} = f(Y) \text{ for } Y \in F^3 \text{ and}$$

$$h_{3,3} = h^1(0,1,1,0,0) = h^1(0,0,1,1,0) = h^1(0,0,0,1,1) = h^1(0,1,0,0,1). \text{ Therefore,}$$

$$H^3 = \{(0,1,1,0,0), (0,0,1,1,0), (0,0,0,1,1), (0,1,0,0,1)\}.$$

Table 4.2: Ranked solutions of ILP (Example 4.6.1)

j	F^j	$(x_1, x_2, x_3, x_4, x_5)$	$f(x_1, x_2, x_3, x_4, x_5)$	$h^1(x_1, x_2, x_3, x_4, x_5)$	$h_{j,1}$	$h_{j,2}$	$h_{j,3}$
1	F^1	$(1, 0, 1, 0, 0)$	-3	-1.5	-1.5		
2	F^2	$(0, 1, 1, 0, 0)$	-2	-1	-2	-1.5	
		$(0, 0, 1, 1, 0)$		-1			
		$(1, 0, 0, 0, 1)$		-2			
3	F^3	$(0, 1, 0, 0, 1)$	-1	-1		-1.5	-1
		$(0, 0, 0, 1, 1)$		-1			

10. The criterion vector corresponding to $(0,0,0,1,1)$ is not dominated by criterion vector corresponding to $(1,0,0,0,1)$ and criterion vector corresponding to $(0,1,1,0,0)$, $(0, 1,$

$(0, 0, 1)$ and $(0,0,1,1,0)$ are dominated by criterion vector corresponding to $(0,0,0,1,1)$ as shown in Table 4.3. So, $N^3 = \{(0, 1, 1, 0, 0), (0, 0, 1, 1, 0), (0, 1, 0, 0, 1)\}$ and $E^3 = \{(0, 0, 0, 1, 1)\}$.

Table 4.3: Ranked solutions of $IQFP$ and efficient set of $MIQFP$ (Example 4.6.1)

H^k	$(x_1, x_2, x_3, x_4, x_5)$	(h^1, h^2, h^3)	$\bigcup_{l=1}^k E^l$
H^1	$(1, 0, 0, 0, 1)$	$(-2, -2.5, -6)$	$\{(1, 0, 0, 0, 1)\}$
H^2	$(1, 0, 1, 0, 0)$	$(-1.5, -2.5, -4)$	$\{(1, 0, 0, 0, 1)\}$
H^3	$(0, 0, 0, 1, 1)$	$(-1, -6, -9)$	$\{(1, 0, 0, 0, 1), (0, 0, 0, 1, 1)\}$
	$(0, 1, 1, 0, 0)$	$(-1, -4, -5)$	$\{(1, 0, 0, 0, 1), (0, 0, 0, 1, 1)\}$
	$(0, 0, 1, 1, 0)$	$(-1, -6, -7)$	$\{(1, 0, 0, 0, 1), (0, 0, 0, 1, 1)\}$
	$(0, 1, 0, 0, 1)$	$(-1, -4, -7)$	$\{(1, 0, 0, 0, 1), (0, 0, 0, 1, 1)\}$

11. Since $f(Y) = f_{max}$ for all $Y \in F^3$ and $h(X) = h_{3,3}$ for all $X \in F^3$, therefore algorithm terminates and set $\bigcup_{l=1}^3 E^l = \{(1, 0, 0, 0, 1), (0, 0, 0, 1, 1)\}$ is declared as an efficient set of $MIQFP$.

Example 4.6.2 Consider the following problem (taken from [102, 130])

$$(MIQFP^1) \quad \max_{X \in \chi} \left(\left(\frac{2x_1 + x_2}{x_2 + 2} \right)^2, \left(\frac{x_1}{x_2 + 1} \right)^2, \left(\frac{x_1 + 2}{2x_1 + x_2 + 2} \right)^2 \right),$$

where $\chi = S \cap \mathbb{Z}^2$, $S = \{(x_1, x_2) \mid x_1 + x_2 \leq 5, 4x_1 + x_2 \leq 8, x_1 \leq 4, x_2 \leq 2, x_1, x_2 \geq 0\}$.

First of all, maximization problem $(MIQFP^1)$ is converted into minimization problem $(MIQFP)$, which is defined as:

$$(MIQFP) \quad \min_{X \in \chi} (h^1(X), h^2(X), h^3(X)),$$

where

$$h^1(X) = \frac{-(2x_1 + x_2)^2}{(x_2 + 2)^2}, \quad h^2(X) = \frac{-(x_1)^2}{(x_2 + 1)^2}, \quad h^3(X) = \frac{-(x_1 + 2)^2}{(2x_1 + x_2 + 2)^2}.$$

It can be observed that $(x_2 + 2)^2, (x_2 + 1)^2, (2x_1 + x_2 + 2)^2 > 0$ for all $X \in \chi$. Now, consider an integer quadratic fractional problem as follows:

$$(IQFP) \quad \min_{X \in \chi} h^1(X) = \frac{-(2x_1 + x_2)^2}{(x_2 + 2)^2}.$$

1. As $M = \max_{X \in \chi} -(2x_1 + x_2)^2 \leq 0$, therefore from Table 4.1 the linear function $f(X) =$

$\frac{(U+C^1)X+\alpha^1}{\mu}$ provides lower bounds to quadratic fractional function $h^1(X)$. Since

$$U_1 = \min_{X \in \mathcal{X}} -4x_1 - 2x_2 = -8, \quad U_2 = \min_{X \in \mathcal{X}} -2x_1 - x_2 = -4, \quad \mu = \min_{X \in \mathcal{X}} (x_2 + 2)^2 = 4.$$

Therefore, $f(X) = -2x_1 - x_2$ and lower bound to the objective function values of the problem (*IQFP*) is provided by objective function values of *ILP*, which is defined as :

$$(ILP) \quad \min_{X \in \mathcal{X}} f(X) = -2x_1 - x_2.$$

2. The problem (*ILP*) has two optimal solutions (1,2) and (2,0) with objective function value -4, therefore $F^1 = \{(1, 2), (2, 0)\}$.
3. Since $f(2, 0) = h_{1,1} = h^1(2, 0) = -4$, therefore $H^1 = \{(2, 0)\}$.
4. As (2,0) is the only element of set H^1 , therefore $N^1 = \{\}$ and $E^1 = \{(2, 0)\}$.
5. As $f_{max} = \max\{f(X) \mid X \in \mathcal{X}\} = -1$ and $f(Y) < f_{max}$ for all $Y \in F^1$, therefore set $k = 2$. It can be observed that $h_{1,2} = \min\{h^1(X) \mid h^1(X) > h^1(2, 0), X \in F^1\} = -1$. As $f(1, 2) < h_{1,2}$ and $f(Y) < f_{max}$ for all $Y \in F^1$ therefore set $j = 2$. A ranking cut $f(X) \geq -3$ is introduced into the feasible region $\mathcal{X}$ and a problem (*ILP*₂) is constructed, which is defined as :

$$(ILP_2) \quad \min_{X \in \mathcal{X}_2} f(X), \quad \text{where } \mathcal{X}_2 = \{X \in \mathcal{X} \mid -2x_1 - x_2 \geq -3\}.$$

The problem (*ILP*₂) has a unique optimal solution (1,1) with objective function value -3. So, $F^2 = \{(1, 1)\}$ and $h_{2,2} = \min\{h^1(X) \mid h^1(X) > h^1(2, 0), X \in \bigcup_{i=1}^2 F^i\} = -1$.

6. Since $f(1, 1) < h_{2,2}$, therefore set $j = 3$. A ranking cut $f(X) \geq -2$ is introduced into the feasible region $\mathcal{X}$ and a problem (*ILP*₃) is constructed, which is defined as :

$$(ILP_3) \quad \min_{X \in \mathcal{X}_3} f(X), \quad \text{where } \mathcal{X}_3 = \{X \in \mathcal{X} \mid -2x_1 - x_2 \geq -2\}.$$

The problem (*ILP*₃) has two optimal solutions (1,0) and (0,2) with objective function value -2. So, $F^3 = \{(1, 0), (0, 2)\}$ and $h_{3,2} = \min\{h^1(X) \mid h^1(X) > h^1(2, 0), X \in \bigcup_{i=1}^3 F^i\} = -1$. As $f(Y) < h_{3,2}$ for all $Y \in F^3$, therefore set $j = 4$. Then, repeating the process for $j = 4$, we get $F^4 = \{(0, 1)\}$.

7. It can be observed that $h_{4,2} = \min\{h^1(X) \mid h^1(X) > h^1(2, 0), X \in \bigcup_{i=1}^4 F^i\} = -1 = f(0, 1)$. Since $h^1(1, 2) = h^1(1, 1) = h^1(1, 0) = h_{4,2} = f(0, 1)$. So, H^2

$$=\{(1, 2), (1, 0), (1, 1)\}.$$

8. The criterion vector corresponding to (1,1) and (1,2) are dominated by criterion vector corresponding to (1,0) and criterion vector corresponding to (1,0) is not dominated by criterion vector corresponding to (2,0) as shown in Table 4.5. So, $N^2 = \{(1, 1), (1, 2)\}$ and $E^2 = \{(1, 0)\}$.

Table 4.4: Ranked solutions of ILP (Example 4.6.2)

j	F^j	(x_1, x_2)	$f(x_1, x_2)$	$h^1(x_1, x_2)$	$h_{j,1}$	$h_{j,2}$	$h_{j,3}$	$h_{j,4}$	$h_{j,5}$
1	F^1	(2,0) (1,2)	-4	-4 -1	-4	-1			
2	F^2	(1,1)	-3	-1		-1			
3	F^3	(1,0) (0,2)	-2	-1 -0.25		-1			
4	F^4	(0,1)	-1	-0.111		-1	-0.25		
5	F^5	(0,0)	0	0			-0.25	-0.11	0

Table 4.5: Ranked solutions of $IQFP$ and efficient set of $MIQFP$ (Example 4.6.2)

H^k	(x_1, x_2)	(h^1, h^2, h^3)	Dominated or non-dominated vector	$\bigcup_{l=1}^k E^l$
H^1	(2, 0)	(-4,-4,-0.44)	Non-dominated vector	$\{(2,0)\}$
H^2	(1, 0)	(-1,-1,-0.56)	Non-dominated vector	$\{(2,0),(1,0)\}$
	(1, 1)	(-1,-0.25,-0.36)	Dominated by (-1,-1,-0.56)	$\{(2,0),(1,0)\}$
	(1, 2)	(-1,-0.111,-0.25)	Dominated by (-1,-1,-0.56)	$\{(2,0),(1,0)\}$
H^3	(0, 2)	(-0.25,0,-0.25)	Dominated by (-1,-1,-0.56)	$\{(2,0),(1,0)\}$
H^4	(0, 1)	(-0.111,0,-0.44)	Dominated by (-1,-1,-0.56)	$\{(2,0),(1,0)\}$
H^5	(0, 0)	(0,0,-1)	Non-dominated vector	$\{(2,0),(1,0), (0,0)\}$

9. Continue in the same way, sets H^k for all $k \in \{3, 4, 5\}$ (as shown in Table 4.5) is generated by finding set F^j for all $j \in \{3, 4, 5\}$ (as shown in Table 4.4).

10. It can be observed that the set $\bigcup_{l=1}^k E^l$ obtained at last iteration is $\{(2,0),(1,0), (0,0)\}$.

So, $\{(2,0),(1,0), (0,0)\}$ is declared as efficient set of $MIQFP$.

Remark 4.6.1 The proposed algorithm generate the efficient set E of $MIQFP$ as $\{(2, 0), (1, 0), (0, 0)\}$, which is same as obtained in [102].

Example 4.6.3 Consider the following problem ($MIQFP^1$)

$$\max_{X \in \chi} \left(\left(\frac{(x_1 + 2x_2 + 1)(-x_2 + 10)}{(x_2 + 1)(3x_1 + 1)} \right), \left(\frac{(-5x_1 + 30)(-x_2 + 10)}{(2x_1 + x_2 + 4)(x_1 + 2x_2 + 3)} \right), \left(\frac{(-5x_1 + 30)(-x_2 + 50)}{(x_2 + 3)(x_1 + 4)} \right) \right),$$

where $\chi = S \cap \mathbb{Z}^2$, $S = \{(x_1, x_2) \mid 3x_1 + 4x_2 \leq 32, 5x_1 - 3x_2 \leq 5, x_1, x_2 \geq 0\}$.

Solution using the Mekhilef, Moulai and Drici [102] algorithm:

The algorithm starts by solving the following problem :

$$(QFP^0) \quad \max_{x \in S} \frac{(x_1 + 2x_2 + 1)(-x_2 + 10)}{(x_2 + 1)(3x_1 + 1)}.$$

An optimal solution of QFP^0 is obtained by using the following theorem.

Table 4.6: Simplex table of Example 4.6.3 corresponding to corner point (0,0)

x_B	x_1	x_2	b
x_3	3	4	32
x_4	5	-3	5
μ_j	1	8	
$c_j^1 - z_j^{1(1)}$	1	2	$h_1^1 = 1$
$d_j^1 - z_j^{1(2)} j$	0	-1	$h_2^1 = 10$
$\gamma_{(1,j)}^1$	10	3	
$p_j^1 - z_j^{1(3)}$	0	1	$h_3^1 = 1$
$q_j^1 - z_j^{1(4)}$	3	0	$h_4^1 = 1$
$\gamma_{(2,j)}^1$	3	1	
ρ_j^1	-20	-7	$h^1 = 10$
$c_j^2 - z_j^{2(1)}$	-5	0	$h_1^2 = 30$
$d_j^2 - z_j^{2(2)} j$	0	-1	$h_2^2 = 10$
$\gamma_{(1,j)}^2$	-50	-30	
$p_j^2 - z_j^{2(3)}$	2	1	$h_3^2 = 4$
$q_j^2 - z_j^{2(4)}$	1	2	$h_4^2 = 3$
$\gamma_{(2,j)}^2$	12	27	
ρ_j^2	-4200	-8460	$h^2 = 25$
$c_j^3 - z_j^{3(1)}$	-5	0	$h_1^3 = 30$
$d_j^3 - z_j^{3(2)} j$	0	-1	$h_2^3 = 50$
$\gamma_{(1,j)}^3$	-250	-30	
$p_j^3 - z_j^{3(3)}$	0	1	$h_3^3 = 3$
$q_j^3 - z_j^{3(4)}$	1	0	$h_4^3 = 4$
$\gamma_{(2,j)}^3$	3	4	
ρ_j^3	-7500	-6360	$h^3 = 125$

Theorem 4.6.2 [102] *A basic feasible solution x_B of QFP^0 will be an optimal solution of QFP^0 if one of the following condition is satisfied*

1. $\rho_j^1 < 0$ for all non- basic variables.
2. For each non- basic variable such that $\rho_j^1 > 0$, all values of y_{ij} are negative.

As shown in Table 4.6, $\rho_j^1 < 0$ for all non- basic variables. Therefore, by virtue of Theorem

4.6.2, (0,0) is declared as an optimal solution of QFP^0 .

To find efficient solutions of $MIQFP^1$, the following proposition discussed in [102] is used.

Proposition 4.6.3 [102] *If X is an optimal solution of QFP^0 and $N_l = \{j \mid j \text{ is an index of non- basic variable with } \rho_j^i < 0 \text{ for some } i \in \{2, \dots, s\} \text{ or } \rho_j^i < 0 \text{ for all } i \in \{2, \dots, s\}\} = \emptyset$, then $S - \{X\}$ does not contain any efficient solution of $MIQFP^1$.*

It is evident by Table 4.6, $\rho_j^1, \rho_j^2, \rho_j^3 < 0$ for all non- basic variables. Therefore, integer vector (0,0) is an ideal point of $MIQFP^1$, which implies (0,0) is an efficient solution of $MIQFP^1$. It can also be seen from Table 4.6, $N_l = \emptyset$. Then, by virtue of proposition 4.6.3, (0,0) is the only efficient solution of $MIQFP^1$.

Observation from [86]:

The first objective of $MIQFP^1$ is taken from [86]. In their paper, Kushwah and Sharma [86] claim that (0,0) is not an optimal solution of QFP^0 as objective function value (12.25) at (0,3) is more than the objective function value (10) at (0,0). So, Mekhilef, Moulai and Drici [102] algorithm declares a non- optimal solution as optimal solution of QFP^0 .

Further, it can also be observed that a feasible solution (0,1) with criteria vector as (13.50, 10.80, 91.88) is not dominated by criteria vector (10, 25, 125) corresponding to (0,0), which contradict (0,0) is the only efficient solution of $MIQFP^1$. So, Mekhilef, Moulai and Drici [102] algorithm will not generates the set of all efficient solutions of $MIQFP$.

Now, we will find efficient solutions of $MIQFP^1$ using proposed algorithm.

Solution using the proposed algorithm:

First of all, maximization problem ($MIQFP^1$) is converted into minimization problem ($MIQFP$), which is defined as:

$$(MIQFP) \quad \min_{X \in \mathcal{X}} (h^1(X), h^2(X), h^3(X)),$$

where

$$h^1(X) = \frac{-(x_1+2x_2+1)(-x_2+10)}{(x_2+1)(3x_1+1)}, \quad h^2(X) = \frac{-(-5x_1+30)(-x_2+10)}{(2x_1+x_2+4)(x_1+2x_2+3)}, \quad h^3(X) = \frac{-(-5x_1+30)(-x_2+50)}{(x_2+3)(x_1+4)}.$$

Now, consider an integer quadratic fractional problem as follows:

$$(IQFP) \quad \min_{X \in \mathcal{X}} h^1(X) = \frac{-(x_1 + 2x_2 + 1)(-x_2 + 10)}{(x_2 + 1)(3x_1 + 1)}.$$

Then, proceeding on the lines of the proposed algorithm as explained in Example 4.6.1 and Example 4.6.2, set H^k for all $k \in \{1, 2, \dots, 25\}$ (as shown in Table 4.8) is generated by

Table 4.7: Ranked solutions of ILP (Example 4.6.3)

j	F^j	$f(x_1, x_2)$	$h^1(x_1, x_2)$
1	(0,8)	-162	-3.78
2	(1,7)	-153	-1.50
3	(4,5)	-145	-0.96
4	(2,6)	-144	-1.22
5	(0,7)	-143	-5.63
6	(3,5)	-135	-1.17
7	(1,6)	-134	-2.00
8	(2,5)	-125	-1.55
9	(0,6)	-124	-7.43
10	(3,4)	-116	-1.44
11	(1,5)	-115	-2.50
12	(2,4)	-106	-1.89
13	(0,5)	-105	-9.17
14	(1,4)	-96	-3.00
15	(2,3)	-87	-2.25
16	(0,4)	-86	-10.80
17	(1,3)	-77	-3.50
18	(2,2)	-68	-2.67
19	(0,3)	-67	-12.25
20	(1,2)	-58	-4.00
21	(0,2)	-48	-13.33
22	(1,1)	-39	-4.50
23	(0,1)	-29	-13.50
24	(1,0)	-20	-5.00
25	(0,0)	-10	-10.00

finding set F^j for all $j \in \{1, 2, \dots, 25\}$ (as shown in Table 4.7). The set $\bigcup_{l=1}^k E^l$ is updated at each iteration (as shown in Table 4.8) . It can be observed that the set $\bigcup_{l=1}^k E^l$ obtained at last iteration is $\{(0,1),(0,0)\}$. So, $\{(0,1),(0,0)\}$ is declared as efficient set of $MIQFP$.

Remark 4.6.4 *It is evident by Table 4.8 that (0,1) is an efficient solution of $MIQFP$, which is missed by Mekhilef, Moulai and Drici [102] algorithm. Thus, their algorithm will not generate the complete set of all efficient solutions of $MIQFP$, but proposed algorithm generates the complete set of all efficient solutions of $MIQFP$.*

Now, we will present an example to show that the Mekhilef, Moulai and Drici [102] algorithm declares a non- efficient solution as efficient solution of $MIQFP$.

Example 4.6.4 *Consider the following problem*

$$(MIQFP^1) \quad \max_{x \in X} \left(\left(\frac{(x_1 + 2x_2 + 1)(-x_2 + 10)}{(x_2 + 1)(3x_1 + 1)} \right), \left(\frac{(2x_1 + 3x_2 + 1)(-x_2 + 10)}{(x_2 + 1)(3x_1 + 1)} \right) \right),$$

Table 4.8: Ranked solutions of $IQFP$ (Example 4.6.3)

k	H^k	(h^1, h^2, h^3)	Dominated or non-dominated vector	$\bigcup_{l=1}^k E^l$
1	(0,1)	(-13.50,-10.80,-91.88)	Non-dominated vector	$\{(0,1)\}$
2	(0,2)	(-13.33,-5.71,-72.00)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1)\}$
3	(0,3)	(-12.25,-3.33,-58.75)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1)\}$
4	(0,4)	(-10.80,-2.05,-49.29)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1)\}$
5	(0,0)	(-10.00,-25.00,-125.00)	Non-dominated vector	$\{(0,1),(0,0)\}$
6	(0,5)	(-9.17, -1.28, -42.19)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
7	(0,6)	(-7.43, -0.80, -36.67)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
8	(0,7)	(-5.63, -0.48, -32.25)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
9	(1,0)	(-5.00, -10.42,-83.33)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
10	(1,1)	(-4.50, -5.36, -61.25)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
11	(1,2)	(-4.00, -3.13, -48.00)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
12	(0,8)	(-3.78, -0.26, -28.64)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
13	(1,3)	(-3.50, -1.94, -39.17)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
14	(1,4)	(-3.00, -1.25, -32.86)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
15	(2,2)	(-2.67, -1.78, -32.00)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
16	(1,5)	(-2.50, -0.81, -28.13)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
17	(2,3)	(-2.25, -1.16, -26.11)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
18	(1,6)	(-2.00, -0.52, -24.44)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
19	(2,4)	(-1.89, -0.77, -21.90)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
20	(2,5)	(-1.55, -0.51, -18.75)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
21	(1,7)	(-1.50, -0.32, -21.50)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
22	(3,4)	(-1.44, -0.46, -14.08)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
23	(2,6)	(-1.22, -0.34, -16.30)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
24	(3,5)	(-1.17, -0.31, -12.05)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$
25	(4,5)	(-0.96, -0.17, -7.03)	Dominated by (-13.5,-10.8,-91.88)	$\{(0,1),(0,0)\}$

where $\chi = S \cap \mathbb{Z}^2$, $S = \{(x_1, x_2) \mid 3x_1 + 4x_2 \leq 32, 5x_1 - 3x_2 \leq 5, x_1, x_2 \geq 0\}$.

Solution using the Mekhilef, Moulai and Drici [102] algorithm:

The first objective of this problem is same as that of Example 4.6.3. It is evident by Example 4.6.3, (0,0) is declared as an optimal solution of QFP^0 . It can be seen from Table 4.9, $\rho_j^1, \rho_j^2 < 0$ for all non- basic variables. Therefore, integer vector (0,0) is an ideal point of $MIQFP^1$, which implies (0,0) is an efficient solution of $MIQFP^1$. It can also be seen from Table 4.9, $N_l = \emptyset$. Then, by virtue of proposition 4.6.3, (0,0) is declared as an efficient solution of $MIQFP$ and algorithm terminates.

It can be observed that the criterion vector at (0,0) is dominated by criterion vector at (0,1), (0,2),(0,3) and (0,4). So, (0,0) is not an efficient solution of $MIQFP$. Thus, their algorithm declares a non- efficient solution as efficient solution and misses an efficient solutions of $MIQFP$. Now, we will solve Example 4.6.4 using proposed algorithm.

Solution using the proposed algorithm:

Table 4.9: Simplex table of Example 4.6.4 corresponding to corner point (0,0)

x_B	x_1	x_2	b
x_3	3	4	32
x_4	5	-3	5
μ_j	1	8	
$c_j^1 - z_j^{1(1)}$	1	2	$h_1^1 = 1$
$d_j^1 - z_j^{1(2)}j$	0	-1	$h_2^1 = 10$
$\gamma_{(1,j)}^1$	10	3	
$p_j^1 - z_j^{1(3)}$	0	1	$h_3^1 = 1$
$q_j^1 - z_j^{1(4)}$	3	0	$h_4^1 = 1$
$\gamma_{(2,j)}^1$	3	1	
ρ_j^1	-20	-7	$h^1 = 10$
$c_j^2 - z_j^{2(1)}$	2	3	$h_1^2 = 1$
$d_j^2 - z_j^{2(2)}j$	0	-1	$h_2^2 = 10$
$\gamma_{(1,j)}^2$	20	5	
$p_j^2 - z_j^{2(3)}$	0	1	$h_3^2 = 1$
$q_j^2 - z_j^{2(4)}$	3	0	$h_4^2 = 1$
$\gamma_{(2,j)}^2$	3	1	
ρ_j^2	-10	-5	$h^2 = 10$

First of all, maximization problem ($MIQFP^1$) is converted into minimization problem ($MIQFP$), which is defined as:

$$(MIQFP) \quad \min_{X \in \mathcal{X}} (h^1(X), h^2(X)),$$

where

$$h^1(X) = \frac{-(x_1 + 2x_2 + 1)(-x_2 + 10)}{(x_2 + 1)(3x_1 + 1)}, \quad h^2(X) = \frac{-(2x_1 + 3x_2 + 1)(-x_2 + 10)}{(x_2 + 1)(3x_1 + 1)}.$$

Now, consider an integer quadratic fractional problem as follows:

$$(IQFP) \quad \min_{X \in \mathcal{X}} h^1(X) = \frac{-(x_1 + 2x_2 + 1)(-x_2 + 10)}{(x_2 + 1)(3x_1 + 1)}.$$

The first objective of this example is same as considered in Example 4.6.3, therefore ranked solutions of $IQFP$ is same as obtained in Example 4.6.3. But, second objective of this problem is different from Example 4.6.3, therefore we may get different efficient set. Following on the steps of proposed algorithm (as shown in Table 4.10), the set $\bigcup_{l=1}^k E^l$ is

updated at each iteration. It can be observed that the set $\bigcup_{l=1}^k E^l$ obtained at last iteration

Table 4.10: Ranked solutions of *IQFP* (Example 4.6.4)

k	H^k	(h^1, h^2)	Dominated or non-dominated vector	$\bigcup_{l=1}^k E^l$
1	(0,1)	(-13.50,-18.00)	Non-dominated vector	$\{(0,1)\}$
2	(0,2)	(-13.33,-18.67)	Non-dominated vector	$\{(0,1),(0,2)\}$
3	(0,3)	(-12.25,-17.50)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
4	(0,4)	(-10.80,-15.60)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
5	(0,0)	(-10.00,-10.00)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
6	(0,5)	(-9.17 ,-13.33)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
7	(0,6)	(-7.43 ,-10.86)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
8	(0,7)	(-5.63 ,-8.25)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
9	(1,0)	(-5.00 ,-7.50)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
10	(1,1)	(-4.50 ,-6.75)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
11	(1,2)	(-4.00 ,-6.00)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
12	(0,8)	(-3.78 ,-5.56)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
13	(1,3)	(-3.50 ,-5.25)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
14	(1,4)	(-3.00 ,-4.50)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
15	(2,2)	(-2.67 ,-4.19)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
16	(1,5)	(-2.50 ,-3.75)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
17	(2,3)	(-2.25 ,-3.50)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
18	(1,6)	(-2.00 ,-3.00)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
19	(2,4)	(-1.89 ,-2.91)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
20	(2,5)	(-1.55 ,-2.38)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
21	(1,7)	(-1.50 ,-2.25)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
22	(3,4)	(-1.44 ,-2.28)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
23	(2,6)	(-1.22 ,-1.88)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
24	(3,5)	(-1.17 ,-1.83)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$
25	(4,5)	(-0.96 ,-1.54)	Dominated by (-13.5, -18)	$\{(0,1),(0,2)\}$

is $\{(0,1),(0,2)\}$. So, $\{(0,1),(0,2)\}$ is declared as efficient set of *MIQFP*.

Remark 4.6.5 *It is evident from Example 4.6.4, Mekhilef, Moulaï and Drici [102] algorithm may declare a non- efficient solution as efficient solution. But, the proposed algorithm generates the correct set of all efficient solutions of MIQFP.*

4.7 Special case

4.7.1 MILFP

If we take $D_i = \emptyset$ and $Q_i = \emptyset$ for all $i = 1, 2, \dots, s$, then *MIQFP* becomes *MILFP*. Now, we will present an algorithm for solving *MILFP*.

Example 4.7.1 *Consider the problem (taken from [69]) :*

$$(MILFP^1) \quad \max_{X \in \mathcal{X}} \left(\left(\frac{5x_1 + 3x_2}{5x_1 + 2x_2 + 1} \right), \left(\frac{3x_1 - 2x_2}{x_1 + 2x_2 + 2} \right), \left(\frac{x_1 + x_2}{2x_1 + x_2 + 1} \right) \right)$$

where $\mathcal{X} = S \cap \mathbb{Z}^2$, $S = \{(x_1, x_2) \mid 3x_1 + 5x_2 \leq 22, 5x_1 + 2x_2 \leq 9, x_1, x_2 \geq 0\}$.

In order to find efficient set $MILFP^1$, it is converted into the following problem :

$$(MILFP) \quad \min_{X \in \mathcal{X}} (h^1(X), h^2(X), h^3(X)),$$

where

$$h^1(X) = \frac{-(5x_1 + 3x_2)}{5x_1 + 2x_2 + 1}, \quad h^2(X) = \frac{-(3x_1 - 2x_2)}{x_1 + 2x_2 + 2}, \quad h^3(X) = \frac{-(x_1 + x_2)}{2x_1 + x_2 + 1}.$$

It can be observed that $5x_1 + 2x_2 + 1, x_1 + 2x_2 + 2, 2x_1 + x_2 + 1 > 0$ for all $X \in \mathcal{X}$.

To solve *MILFP*, consider an integer linear fractional problem as follows:

$$(ILFP) \quad \min_{X \in \mathcal{X}} h^1(X).$$

The lower bound to the objective function values of *ILFP* is provided by objective function values of *ILP*, which is defined as :

$$(ILP) \quad \min_{X \in \mathcal{X}} f(X) = -5x_1 - 3x_2. \quad .$$

Then, proceeding on the lines of the Algorithm ??, we get ranked solutions of *ILP* and

Algorithm 6: Algorithm to solve *MILFP*

Data: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^{m \times 1}$, C^i , $P^i \in \mathbb{Z}^{1 \times n}$ and $\alpha^i, \beta^i \in \mathbb{Z}$ for $i \in \{1, 2, \dots, s\}$.

Result: $E = \bigcup_{l=1}^k E^l$ ▷ set of all efficient solutions of *MILFP*

```

1 Initialization  $k \leftarrow 1, j \leftarrow 1, S \leftarrow \{X \in \mathbb{R}^n \mid AX = b, X \geq 0\}, \mathcal{X} \leftarrow S \cap \mathbb{Z}^n$ 
2 for  $i \leftarrow 1 : s$  do
3    $h^i(X) \leftarrow \frac{C^i X + \alpha^i}{P^i X + \beta^i}$  ▷ ILFP  $\min_{X \in \mathcal{X}} h^1(X)$ 
4    $m \leftarrow \min_{X \in \mathcal{X}} C^1 X + \alpha^1$ 
5   if  $m \geq 0$  then
6      $f(X) \leftarrow \frac{C^1 X + \alpha^1}{P^1 X + \beta^1}$  ▷ ILP  $\min_{X \in \mathcal{X}} f(X)$ 
7   else
8      $\mu \leftarrow \min_{X \in \mathcal{X}} P^1 X + \beta^1, M \leftarrow \max_{X \in \mathcal{X}} C^1 X + \alpha^1, \hat{m} \leftarrow \min_{X \in \mathcal{X}} C^1 X + \alpha^1, \hat{M} \leftarrow \max_{X \in \mathcal{X}} C^1 X + \alpha^1,$ 
9     if  $M \leq 0$  then
10        $f(X) \leftarrow \frac{C^1 X + \alpha^1}{\mu}$  ▷ ILP  $\min_{X \in \mathcal{X}} f(X)$ 
11     if  $\hat{m} < 0$  and  $\hat{M} > 0$  then
12        $f(X) \leftarrow \frac{C^1 X + \alpha^1 - \hat{M}}{\mu}$ 
13    $f_{max} \leftarrow \max_{X \in \mathcal{X}} f(X), F^1 \leftarrow \{Y \in \mathcal{X} \mid f(Y) = \min_{X \in \mathcal{X}} f(X)\}, h_{1,1} \leftarrow \min\{h^1(X) \mid X \in F^1\}$ 
14   while  $f(Y) < h_{j,1}$  and  $f(Y) < f_{max}$  for  $Y \in F^j$  do
15      $j \leftarrow j + 1, \mathcal{X}_j \leftarrow \{X \in \mathcal{X} \mid f(X) > f(Y), Y \in F^{j-1}\},$ 
16      $F^j \leftarrow \{Y \in \mathcal{X}_j \mid f(Y) = \min_{X \in \mathcal{X}_j} f(X)\}, h_{j,1} \leftarrow \min\{h(X) \mid X \in \bigcup_{i=1}^j F^i\}$ 
17    $H^1 \leftarrow \{Y \in \bigcup_{i=1}^j F^i \mid h^1(Y) = h_{j,1}\}$  ▷ set of optimal solutions of ILFP
18    $N^1 \leftarrow [], E^1 \leftarrow []$ 
19   foreach  $X \in H^1$  do
20     foreach  $Y \in H^1$  do
21       if  $h^i(Y) \leq h^i(X) \forall i \in \{2, 3, \dots, s\}$  and  $h^i(Y) < h^i(X)$  for some  $i \in \{1, 2, \dots, s\}$  then
22          $N^1 \leftarrow [N^1, X]$  ▷ Set of non-efficient solutions of MILFP in set  $H^1$ 
23       if  $X \notin N^1$  then
24          $E^1 \leftarrow [E^1, X]$  ▷ Set of efficient solutions of MILFP in set  $H^1$ 
25   while  $f(Y) < f_{max}$  for some  $Y \in F^j$  or  $h(X^k) > h_{j,k}$  for some  $X^k \in H^k$  do
26      $k \leftarrow k + 1$ 
27     if  $\{X \mid h^1(X) > h^1(X^{k-1}), X \in \bigcup_{i=1}^j F^i, X^{k-1} \in H^{k-1}\} = \emptyset$  then
28        $h_{j,k} = -\infty$ 
29     else
30        $h_{j,k} \leftarrow \min\{h(X) \mid h(X) > h(X^{k-1}), X \in \bigcup_{i=1}^j F^i\}$  for  $X^{k-1} \in H^{k-1}$ 
31     while  $f(Y) < h_{j,k}$  and  $f(Y) < f_{max}$  for  $Y \in F^j$  do
32        $j \leftarrow j + 1, \mathcal{X}_j \leftarrow \{X \in \mathcal{X} \mid f(X) > f(Y), Y \in F^{j-1}\},$ 
33        $F^j \leftarrow \{Y \in \mathcal{X}_j \mid f(Y) = \min_{X \in \mathcal{X}_j} f(X)\},$ 
34        $h_{j,k} \leftarrow \min\{h(X) \mid h(X) > h(X^{k-1}), X \in \bigcup_{i=1}^j F^i\}$  for  $X^{k-1} \in H^{k-1}$ 
35      $H^k \leftarrow \{Y \in \bigcup_{i=1}^j F^i \mid h^1(Y) = h_{j,k}\}$  ▷ set of  $k^{th}$  ranked solutions of ILFP
36      $N^k \leftarrow [], E^k \leftarrow []$ 
37     foreach  $X \in H^k$  do
38       foreach  $Y \in (\bigcup_{l=1}^{k-1} E^l) \cup H^k$  do
39         if  $h^i(Y) \leq h^i(X) \forall i \in \{2, 3, \dots, s\}$  and  $h^i(Y) < h^i(X)$  for some  $i \in \{1, 2, \dots, s\}$  then
40            $N^k \leftarrow [N^k, X]$  ▷ Set of non-efficient solutions of MILFP in set  $H^k$ 
41         if  $X \notin N^k$  then
42            $E^k \leftarrow [E^k, X]$  ▷ Set of efficient solutions of MILFP in set  $H^k$ 

```

Table 4.11: Ranked solutions of ILP (Example 4.7.1)

j	F^j	$f(X)$	$h(X)$	$h_{j,1}$	$h_{j,2}$	$h_{j,3}$	$h_{j,4}$	$h_{j,5}$	$h_{j,6}$	$h_{j,7}$
1	(0 , 4)	-12	-1.33	-1.33						
2	(1 , 2)	-11	-1.10	-1.33						
3	(0 , 3)	-9	-1.29	-1.33						
4	(1 , 1)	-8	-1.00	-1.33						
5	(0 , 2)	-6	-1.20	-1.33						
6	(1 , 0)	-5	-0.83	-1.33						
7	(0 , 1)	-3	-1.00	-1.33						
8	(0 , 0)	0	0	-1.33	-1.29	-1.2	-1.1	-1	-0.83	0

Table 4.12: Ranked solutions of $ILFP$ and efficient set of $MILFP$ (Example 4.7.1)

k	H^k	$(h^1(X), h^2(X), h^3(X))$	$\bigcup_{l=1}^k E^l$
1	(0 , 4)	(-1.33 , 0.8 , -0.80)	$\{(0, 4)\}$
2	(0 , 3)	(-1.29 , 0.75 , -0.75)	$\{(0, 4), (0, 3)\}$
3	(0 , 2)	(-1.20 , 0.67 , -0.67)	$\{(0, 4), (0, 3), (0, 2)\}$
4	(1 , 2)	(-1.10 , 0.14 , -0.60)	$\{(0, 4), (0, 3), (0, 2), (1, 2)\}$
5	(1 , 1) (0 , 1)	(-1.00 , -0.2 , -0.50) (-1.00 , 0.5 , -0.50)	$\{(0, 4), (0, 3), (0, 2), (1, 2), (1, 1)\}$
6	(1 , 0)	(-0.83 , -1 , -0.33)	$\{(0, 4), (0, 3), (0, 2), (1, 2), (1, 1), (1, 0)\}$
7	(0 , 0)	(0 , 0 , 0)	$\{(0, 4), (0, 3), (0, 2), (1, 2), (1, 1), (1, 0)\}$

$ILFP$ as shown in Table 4.11 and Table 4.12. The stepwise working of the proposed algorithm for solving the problem ($MILFP$) is shown in Table 4.11 and Table 4.12. The set $E = \{(0, 4), (0, 3), (0, 2), (1, 2), (1, 1), (1, 0)\}$ is declared as efficient set $MILFP^1$, which is same as obtained by Gupta and Malhotra [69].

4.7.2 Comparative analysis of proposed algorithm with the Gupta and Malhotra [69] algorithm

The algorithm discussed in the paper of Gupta and Malhotra [69] seeks to find all efficient solutions to $MILFP$ by optimizing a linear fractional programming problem at each iteration. Moreover, the feasible region is truncated at each step by utilizing the generalization of the Dantzig cut followed by Gomory cuts, and previously generated cuts are also carried forward for the next iteration.

However, the proposed algorithm enumerates efficient solutions of the problem ($MILFP$) by finding ranked solutions of the problem ($ILFP$) or problem (ILP). If $m < 0$, then the proposed algorithm finds the efficient set by utilizing an objective function cut and optimizing an integer linear programming problem at each iteration. Further, the previously generated objective function cut does not carried forward for the next iteration.

Notably, solving *ILFP* consisting of various cuts is computationally more challenging than solving *ILP*, which only contains an additional objective function cut. So, the proposed algorithm is more efficient than the algorithm proposed by Gupta and Malhotra [69].

4.7.3 *MIQP* and *MILP*

If we take $P_i = \emptyset$ and $Q_i = \emptyset$ for all $i = 1, 2, \dots, s$, then it becomes a multi-objective integer quadratic programming problem (*MIQP*). If we consider $P_i = \emptyset, Q_i = \emptyset$ and $D_i = \emptyset$ for all $i = 1, 2, \dots, s$, then it reduces to a multi-objective integer linear programming problem (*MILP*). A particular case of the developed method for solving *MIQP* and *MILP* is discussed in [87].

4.8 Computational results

The proposed methodology is executed in MATLAB 2019 on a personal HP laptop with Intel Core i3-6006U processor running at 2GHz (4 GB of RAM). For each couple (m, n, s) , a series of twenty *MIQFP* generated at random are solved using the proposed algorithm. The coefficients of A , b , D^k , C^k , P^k , Q^k , α^k and $\beta^k \forall k \in \{1, 2, \dots, s\}$ are randomly generated integer values as follows :

- $D_{i,j}^1 \in [-100, 0] \cap \mathbb{Z}$, $C_i^1 \in [-100, 0] \cap \mathbb{Z}$ and $\alpha^1 \in [-100, -1] \cap \mathbb{Z} \forall i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, i\}$.
- $D_{i,j}^k \in [-100, 100] \cap \mathbb{Z}$, $C_i^k \in [-100, 100] \cap \mathbb{Z}$ and $\alpha^k \in [-100, 100] \cap \mathbb{Z} \forall i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, i\}$, $k \in \{2, 3, \dots, s\}$.
- $P_{i,j}^k \in [0, 100] \cap \mathbb{Z}$, $Q_i^k \in [0, 100] \cap \mathbb{Z}$, $\beta^k \in [1, 100] \cap \mathbb{Z}$, $D_{i,j}^k = D_{j,i}^k$ and $P_{i,j}^k = P_{j,i}^k \forall i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, i\}$, $k \in \{1, 2, \dots, s\}$.

The Figure 4.1 shows the effect of number of objectives and Figure 4.2 shows the effect of number of variables of *MIQFP* on processing time to find all efficient solutions of *MIQFP*.

4.9 Concluding remarks

In this chapter, we have presented a method for solving *MIQFP*. The set of all efficient solutions of *MIQFP* are obtained from ranked solutions of *IQFP*. To enumerate ranked solutions of *IQFP*, we have constructed a related problem, whose objective function values provide lower bound to the objective function values of *IQFP*. Theoretically, we have proved that ranked solutions of *IQFP* generates the set of all efficient solutions of *MIQFP*.

Table 4.13: Number of efficient solutions of *MIQFP* and processing time to find all efficient solutions of *MIQFP* for different values of m, n and s

(m, n, s)	No. of efficient solutions		Time (in sec)	
	max	mean	max	mean
(20, 5, 3)	4.0	1.5	9.3	1.9
(20, 10, 3)	6.0	1.9	19.5	3.4
(20, 15, 3)	8.0	1.9	32.0	4.8
(20, 20, 3)	5.0	1.9	29.9	7.0
(20, 25, 3)	11.0	2.9	116.7	21.4
(20, 30, 3)	14.0	2.6	773.6	48.6
(20, 5, 4)	4.0	1.5	6.2	2.0
(20, 10, 4)	7.0	1.8	16.5	3.3
(20, 15, 4)	10.0	3.2	54.1	11.8
(20, 20, 4)	11.0	3.4	82.3	15.6
(20, 25, 4)	11.0	2.7	78.8	10.9
(20, 30, 4)	9.0	2.6	62.0	12.6
(20, 5, 5)	5.0	1.6	3.5	1.4
(20, 10, 5)	3.0	1.2	8.8	2.1
(20, 15, 5)	14.0	2.9	72.6	8.7
(20, 20, 5)	7.0	2.2	26.3	6.4
(20, 25, 5)	16.0	2.6	148.1	16.6
(20, 30, 5)	16.0	3.6	219.0	28.8
(20, 5, 6)	7.0	1.7	13.7	2.7
(20, 10, 6)	10.0	2.5	22.4	4.2
(20, 15, 6)	19.0	3.1	97.5	8.6
(20, 20, 6)	32.0	4.9	605.5	47.6
(20, 25, 6)	17.0	4.7	134.2	23.6
(20, 30, 6)	33.0	5.3	857.3	71.5

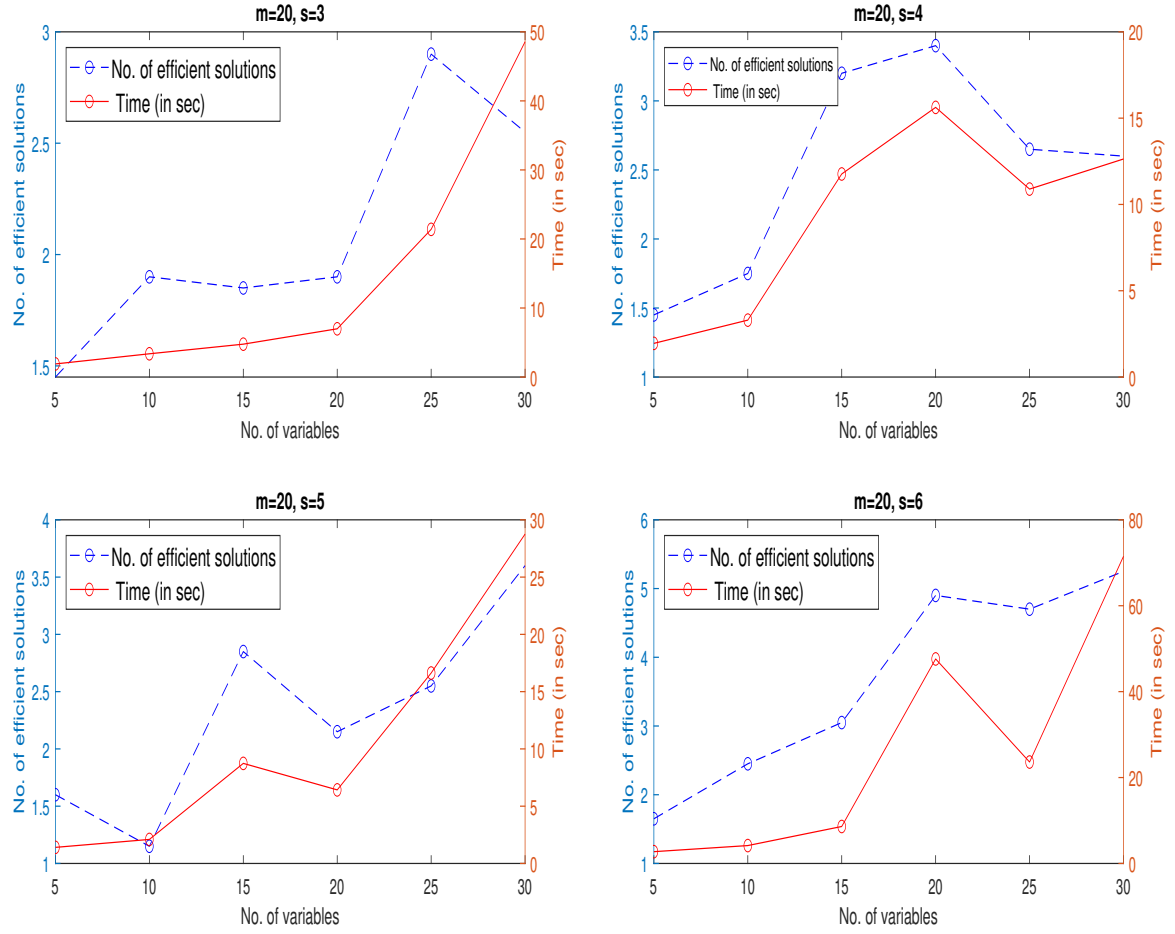


Figure 4.1: Impact of n on processing time to solve $MIQFP$

in finite number of steps. Numerically and computationally we have shown the effectiveness of the developed method.

The proposed algorithm is the only method that solves a general class of $MIQFP$. We have compared our algorithm with the recent algorithms [102, 130] for solving a particular class of $MIQFP$. Further, we have shown that the Mekhilef, Moulai and Drici [102] algorithm not always generates the set of all efficient solutions of $MIQFP$.

The developed algorithm can also be utilized for generating the complete efficient set of $MIQP$, $MILP$ and $MILFP$.

The developed method can be extended to the case with box constraints.

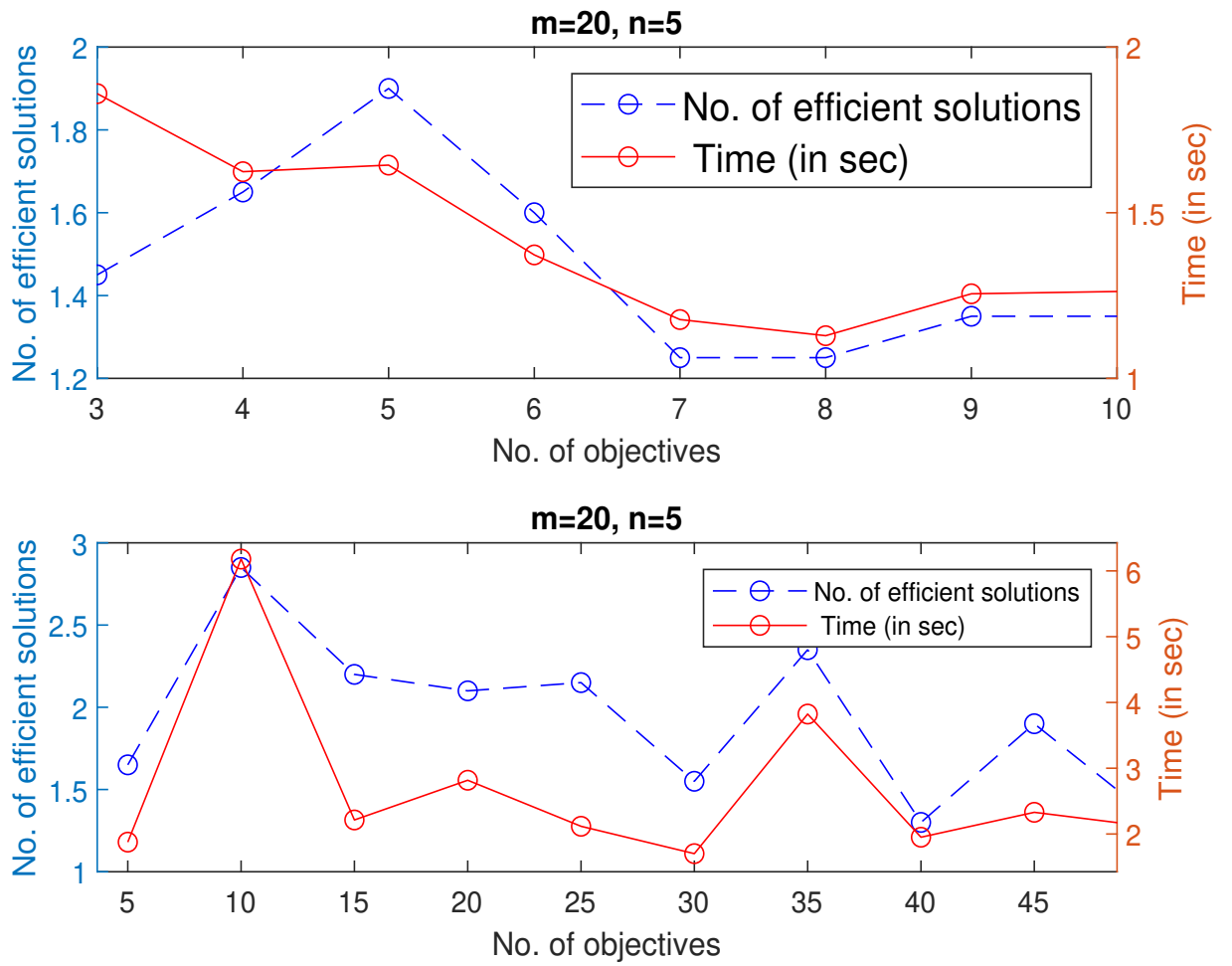


Figure 4.2: Impact of s on processing time to solve $MIQFP$

Chapter 5

Optimization of a linear function over efficient set

5.1 Introduction

Researchers have developed various methods to generate efficient solutions of *MIP* [1, 16, 50, 68, 82, 110, 112, 123, 131, 140, 142]. Mostly, the efficient set of *MIP* contains numerous elements, which confuses *DM* to take a decision [80]. Further, generating all efficient solutions is a computationally challenging task. So, enumerating all efficient solutions is sometimes not recommended for practical purposes. Optimizing a preference function over the efficient set plays a significant role in alleviating such a situation.

The optimization of a function over the efficient set has unquestionable practical applications. One such instance of this problem in the light of network flow theory is presented in [2, 64, 132, 133]. Let us consider a network $(V, s, t, A, \partial^+, \partial^-, c)$ consisting of node set V , source s , sink t , arc set $A \subseteq V \times V$ such that $|A| = m$, two incidence functions $\partial^+ : A \rightarrow V$, $\partial^- : A \rightarrow V$ and capacity c . For any arc r in arc set (A) , $\partial^+ r$ and $\partial^- r$ represent initial and terminal node of arc r , and c_r and x_r represent capacity and flow in arc r . A vector $X = (x_1, x_2, \dots, x_r, \dots, x_m)$ is called feasible flow if it satisfies capacity constraints and conservation equations.

$$0 \leq x_r \leq c_r \text{ for all } r \in A \quad (5.1.1)$$

$$\sum_{\{r|\partial^+ r=i\}} x_r = \sum_{\{r|\partial^- r=i\}} x_r \text{ for all node } i \in V - \{s, t\} \quad (5.1.2)$$

In the maximum flow problem, it is assumed that each arc flow can be freely increased and reduced as far as the capacity constraint and conservation equations are satisfied by a flow. However, under some circumstances, when an arc flow cannot be decreased further, network fails to attain a maximum flow and wind up in an undesired maximal flow. A

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feasible flow X is called maximal flow if there exists no other feasible flow $\hat{X}$ such that $\hat{X} \geq X$ and $\hat{X} \neq X$ and the flow value of a network is given as

$$g(X) = \sum_{\{r|\partial^+ r=s\}} x_r - \sum_{\{r|\partial^- r=s\}} x_r.$$

The maximal flows depend on the initial flow and way of augmentation. So, the flow value $g(X)$ plays a significant role for maximal flows.

The Shi and Yamamoto [132] are the first to raise a problem of finding the minimum flow value of maximal flows. The mathematical model of the problem is

$$\min_{X \in E} g(X),$$

here $g(X)$ is a linear function of flow, and E is the efficient set of the problem, which is defined as:

$$\max_{X \in \chi} CX,$$

where $C = I$, the identity matrix of dimension m and χ is the set of feasible flows. For detail discussion of this problem, one can refer [2, 64, 132, 133].

Another real-life application of optimization of a linear function over the efficient set of *MILP* can be seen in the paper of Benson [17], here an example is modelled in the context of a production planning problem, where the owner of a factory seeks to maximize profit (which is a linear function) while maximizing employment level in each factory (which is a *MILP*).

The optimization of a linear function over the efficient set of *MILP* is a challenging problem because of the discrete nature of decision variables and the discrete aspect of the efficient set, which is explicitly unknown.

In the case of continuous variables, vertex search algorithms, face search algorithms, Lagrangian relaxation method, dual approach, and many other methods are available in the literature. A quick review of these techniques can be found in the paper of Yamamoto [144]. As far as optimizing a linear function over the efficient set of *MILP* is concerned, limited research is done in this domain.

The Abbas and Chaabane [2] are the first to propose an algorithm for optimizing a linear function over an integer efficient set. In their paper, a decision space search algorithm is used for optimizing a linear function over an integer efficient set. Their algorithm is based on edge-exploring and edge-cutting techniques. At each iteration, a new efficient solution is generated by exploring edges and truncating the feasible region using various cuts. At

every iteration, new cuts are imposed over the feasible region, and also the previously generated cuts are carried forward, resulting in increasing the size of the problem at each stage. Computational results regarding the algorithm are not reported. Only a small example is solved to elaborate the algorithm. So, it is a challenging task to implement it in practical situations.

Later, criterion space search algorithms [45,80,94] are used for optimizing a linear function over the efficient set of *MILP*. In most practical situations, the decision space dimension is more than the criterion space. So, criterion space search algorithms perform better than decision space search algorithms. The Jorge [80] firstly used a criterion space search algorithm for solving *ILP_E*. His algorithm utilized the Sylva and Crema [140] cuts for generating a non-dominated vector at each iteration. Using the Sylva and Crema [140] cuts a progressively more constrained integer linear programming problem is constructed at each iteration, and its optimal solution is obtained. The criterion vector of *MILP* at this optimal solution is not dominated by previously generated non-dominated vector.

The Djamel and Marc [45] algorithm is very similar to the Jorge [80] algorithm. The only difference between Jorge's [80] and Djamel and Marc's [45] algorithms is that in the Djamel and Marc [45] algorithm, edges incident to a solution are explored to obtain an efficient solution. But, in the Jorge [80] algorithm, edges are not explored.

In Jorge [80] and Djamel and Marc [45] algorithms a progressively more constrained problem is solved at each k^{th} iteration, which require large computational efforts. In order to reduce the computational burden Lokman [94] developed two decomposition and search algorithms. Instead of solving a large constrained problem at each iteration, they decompose the criterion space into smaller spaces. A non-dominated vector which optimizes objective function value is generated from each subproblem, and a non-dominated vector corresponding to the maximum objective function value is selected.

The detailed working of [2, 45, 80, 94] algorithms and their comparison among each other on the examples taken from the paper [80] is included in the numerical section. This comparison will provide a quick guide of the existing state-of-the-art algorithms to the researchers working on this area.

In this chapter, two new algorithms for optimizing a linear function over the efficient set of *MILP* are also discussed. In the first algorithm, ranked solutions of *ILP* are utilized for optimizing a linear function over the efficient set of *MILP*. The ranking of integer solutions is done at every stage by using an objective function cut, which is not carried forward but regenerated at each stage. So the *ILP* solved at each stage is of same size. Further, these ranked solutions are calculated until the efficiency condition is satisfied. In

the second algorithm, in addition to ranked solutions of ILP , efficient solutions of $MILP$ are also collected at each stage. These efficient solutions generate a lower bound on the objective function value, which keeps on shrinking at each stage. This lower bound also provides an estimate on the number of ranked solutions to be calculated and the algorithm stops after calculating ranked solutions upto this lower bound. Theoretically, we have justified that the proposed algorithms can optimize a linear function over the efficient set of $MILP$ in a finite number of steps. Further, we have compared our algorithms with existing methods [2, 45, 80, 94].

In Jorge [80] and Djamal and Marc [45] algorithm an integer linear programming problem of size $(m + (2s + 1)(k - 1)) \times (n + s(k - 1))$ (where m , n and s denotes the number of constraints, variables and objectives of $MILP$ respectively) is solved at each k^{th} iteration. In Lokman [94] algorithm at most $(k - 1)^{s-1}$ integer linear programming problems of size $(m + 1 + s) \times n$ are solved at each k^{th} iteration ($k > 2$). But, in the proposed algorithms, an integer linear programming problem of size $(m + 1) \times n$ is solved at each iteration. Further, in the Abbas and Chaabane [2] algorithm, four types of cuts (Type 1 cut, Dantzig cut, Type 2 cut, and Gomory cut) are used for solving ILP_E . But in the proposed algorithms, only a single objective function cut is used, which is easier to implement and computationally efficient.

The chapter is organized as follows- In Section 5.2, all notations and preliminary results are discussed. Theoretical results and two algorithms for optimizing a linear function over the efficient set of $MILP$ are presented in Section 5.3. Numerical examples and comparison with existing methods are made in Section 5.4. Computational results are shown in Section 5.5, and concluding remarks are discussed in the last section.

5.2 Notations and Preliminary results

The following notations are used throughout the chapter.

Notations 1 $g(X) = VX$, where $V \in \mathbb{Z}^{1 \times n}$,

$G^N = \{Y \in \chi \mid g(Y) = \min_{X \in \chi} g(X)\}$,

G_E is the set of all optimal solutions of ILP_E ,

For $k \in \{1, 2, \dots, N\}$, define the following set :

$G^k = \{X_i^k \mid i = 1, 2, \dots, l_k\}$ is the set of all optimal solutions of ILP_k with objective function value Z^k . It is also the set of all k^{th} ranked solutions or k^{th} best solutions of ILP_1 .

$\chi_k = \{X \in \chi \mid g(X) \leq Z^{k-1} - 1\}$ for $k > 1$, $\chi_k = \chi$ for $k = 1$,

$G_E^k = G^k \cap E$,

$$E^k = \{Y \mid (Y, \Psi) \text{ is an optimal solution of } P(X_i^k) \text{ for some } i \in \{1, 2, \dots, l_k\}\},$$

$$g_{lower}^k = \max \{ g(Y) \mid Y \in \cup_{l=1}^k E^l \}.$$

Consider the following problem :

$$(MILP) \quad \max_{X \in \mathcal{X}} (C^1 X, C^2 X, \dots, C^s X).$$

We assume that $\mathcal{X}$ is a non-empty and bounded set.

Definition 5.2.1 Let CY and $C\hat{Y}$ be two criterion vectors in the objective function space, then CY dominates $C\hat{Y}$, iff $C^i Y \geq C^i \hat{Y}$ for all $i \in \{1, 2, \dots, s\}$ and $C^i Y \neq C^i \hat{Y}$ for atleast one $i \in \{1, 2, \dots, s\}$.

Definition 5.2.2 A feasible solution Y of MILP is called an efficient solution of MILP if there exist no other feasible solution X of MILP such that $C^i X \geq C^i Y$ for all $i \in \{1, 2, \dots, s\}$ and $C^i X \neq C^i Y$ for atleast one $i \in \{1, 2, \dots, s\}$.

Definition 5.2.3 If Y is an efficient solution of MILP, then CY is called a non-dominated vector.

Remark 5.2.1 [131] If the feasible region $\mathcal{X}$ of MILP is non-empty and bounded set, then the efficient set E of MILP is also non empty.

For a solution $\hat{X} \in \mathcal{X}$ to be an efficient solution of MILP, $\hat{X}$ must satisfy the following efficiency test discussed in [73].

Theorem 5.2.2 A feasible solution $\hat{X}$ of MILP is an efficient solution of MILP iff the optimal value of θ is zero in the following problem:

$$P(\hat{X}) \quad \begin{cases} \max & \theta(X, \Psi) = \sum_{i=1}^s \Psi_i \\ & C^i X = \Psi_i + C^i \hat{X}, \\ & X \in \mathcal{X}, \Psi_i \in \mathbb{R}_+ \text{ for all } i \in \{1, 2, \dots, s\}. \end{cases}$$

Remark 5.2.3 It may be noted that an optimal solution of $P(\hat{X})$ is a vector (Y, Ψ) , where $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_s)$. Since Ψ is a vector of surplus variable associated to the vector Y , we have collected the vector Y as an optimal solution to $P(\hat{X})$ throughout the chapter. Also, for the sake of simplicity, we will use $\theta(Y, \Psi)$ as $\theta(Y)$.

Theorem 5.2.4 [48] If $P(\hat{X})$ has a finite non-zero optimal value at Y , i.e., $\theta(Y)$ is non-zero, then Y is an efficient solution of MILP.

Remark 5.2.5 [48] If $P(\hat{X})$ doesn't have a finite optimal value, then the efficient set of MILP is empty.

Remark 5.2.6 *The problem $P(\hat{X})$ has a finite optimal value for all $\hat{X} \in \chi$, because if $P(\hat{X})$ doesn't have a finite optimal value then by Remark 5.2.5, the efficient set of MILP is empty, which contradicts Remark 5.2.1.*

5.3 Methodology for solving ILP_E

Now, we will discuss two algorithms for optimizing ILP_E .

The problem under consideration is

$$(ILP_E) \quad \max_{X \in E} g(X),$$

where $g(X) = VX$ (linear function) and E is the set of all efficient solutions of $MILP$.

To solve ILP_E , we first consider an integer linear programming problem (ILP_1) defined as:

$$(ILP_1) \quad \max_{X \in \chi = S \cap \mathbb{Z}^n} g(X).$$

An optimal solution of ILP_1 will provide an upper bound to all feasible solutions of ILP_E , in particular to an optimal solution of ILP_E . Now, we will derive an iterative scheme that will tighten this upper bound and converge to an optimal solution of ILP_E . For this purpose, we first consider a restricted version of ILP_1 defined as $(ILP_k, k > 1)$

$$(ILP_k) \quad \max_{X \in \chi_k} g(X).$$

For each $k > 1$, ILP_k is obtained from ILP_1 after appending an objective function cut $g(X) \leq Z^{k-1} - 1$ on χ . The optimal value of ILP_k is Z^k and the following relation holds between different values of Z^k for all $k \in \{1, 2, \dots, N\}$.

$$Z^1 > Z^2 > \dots > Z^N.$$

5.3.1 Description of Algorithm 7

An algorithm to find the set of all optimal solutions of ILP_E without enumerating all efficient solutions of $MILP$ is presented in this section. The proposed algorithm finds the set of all optimal solutions of ILP_E by successively solving ILP_k . Initially, set $k = 1$ and the algorithm starts by solving the following problem

$$(ILP_1) \quad \max_{X \in \chi} g(X),$$

where $\chi = S \cap \mathbb{Z}^{n \times 1}$. In order to find the set of all optimal solutions of ILP_1 consider the following relaxed linear programming problem

$$(LP_1) \quad \max_{X \in S} g(X).$$

An optimal solution of ILP_1 can be obtained by solving LP_1 using Gomory cutting plane technique or by using Branch and Bound technique [13] and the set of all optimal solutions can be obtained by exploring edges incident to obtained optimal solution as discussed in [2].

After obtaining the set of all optimal solutions of ILP_1 , our next task is to find the set of all optimal solutions of ILP_E . For this purpose, test whether any optimal solution of ILP_1 is an efficient solution of $MILP$ or not. If the optimal value of $P(X_i^1)=0$ for some $i \in \{1, 2, \dots, l_1\}$, then $X_i^1 \in G_E^1$ and X_i^1 will be declared as an optimal solution of ILP_E . But, if the optimal value of $P(X_i^1) \neq 0$ for all $i \in \{1, 2, \dots, l_1\}$, then solve the problem ILP_k for $k = k + 1$ and repeat the above process.

At each k^{th} iteration, the set G^k is constructed by solving ILP_k . Then, test whether any element of G^k is a member of E or not. For this purpose, $P(X_i^k)$ is solved for all $i \in \{1, 2, \dots, l_k\}$. If the efficiency test (Theorem 5.2.2) is satisfied for some X_i^k , where $i \in \{1, 2, \dots, l_k\}$. Then, $X_i^k \in G_E^k$ and X_i^k will be declared as an optimal solution of ILP_E . Otherwise, set $k = k + 1$ and process is repeated until the set G_E^k is non- empty.

Algorithm 7: Algorithm to solve ILP_E

Data: $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, C \in \mathbb{Z}^{s \times n}, V \in \mathbb{Z}^{1 \times n}$

Result: G_E is the set of all optimal solutions of ILP_E

```

1 Initialization  $k \leftarrow 1$ 
2  $G^1 \leftarrow \{Y \in \chi \mid g(Y) = \max_{X \in \chi} g(X)\}$  ▷ set of optimal solutions of  $ILP_1$ 
3  $Z^1 \leftarrow g(X_1^1)$  for  $X_1^1 \in G^1$  ▷ optimal value of  $ILP_1$ 
4  $G_E^1 \leftarrow \emptyset$ 
5 foreach  $X_i^1 \in G^1$  do
6   Solve  $P(X_i^1)$ 
7   if If optimal value of  $P(X_i^1) = 0$  then
8      $G_E^1 \leftarrow [G_E^1, X_i^1]$ 
9 while  $G_E^k = \emptyset$  do
10   $k \leftarrow k + 1$ 
11   $\chi_k \leftarrow \{X \in \chi \mid g(X) \leq Z^{k-1} - 1\}$ 
12   $G^k \leftarrow \{Y \in \chi_k \mid g(Y) = \max_{X \in \chi_k} g(X)\}$  ▷ set of optimal solutions of  $ILP_k$ 
13   $Z^k \leftarrow g(X_1^k)$  for  $X_1^k \in G^k$ 
14   $G_E^k \leftarrow \emptyset$ 
15  foreach  $X_i^k \in G^k$  do
16    Solve  $P(X_i^k)$ 
17    if If optimal value of  $P(X_i^k) = 0$  then
18       $G_E^k \leftarrow [G_E^k, X_i^k]$ 
19 if  $G_E^k \neq \emptyset$  then
20    $G_E \leftarrow G_E^k$  ▷ set of optimal solutions of  $ILP_E$ 

```

5.3.2 Theoretical results used in Algorithm 7

In the following discussion, we will show that the first non-empty set G_E^k is the set of all optimal solutions of ILP_E and Algorithm 7 terminates in a finite number of steps.

Theorem 5.3.1 *If the feasible region χ of MILP is non-empty, then $G_E^i \neq \emptyset$ for some $i \in \{1, 2, 3, \dots, N\}$.*

Proof As G^i is the set of all i^{th} best solution of ILP_1 for all $i \in \{1, 2, 3, \dots, N\}$, therefore $\cup_{i=1}^N G^i = \chi$, which is non-empty set. Also, efficient set E of MILP is non empty (by Remark 5.2.1), we have $\cup_{i=1}^N G^i \cap E \neq \emptyset$. Thus, $G_E^i = G^i \cap E \neq \emptyset$ for some $i \in \{1, 2, 3, \dots, N\}$.

Theorem 5.3.2 *If $\cup_{i=1}^{k-1} G_E^i = \emptyset$ and $G_E^k \neq \emptyset$ for some $k \in \{1, 2, \dots, N\}$, then G_E^k is the set of all optimal solutions of ILP_E .*

Proof Let G_E be the set of all optimal solutions of ILP_E and let $Y \in G_E$. As Y is an optimal solution of ILP_E , therefore

$$\begin{aligned} & g(Y) \geq g(X) \text{ for all } X \in E \\ \implies & g(Y) \geq g(X) \text{ for all } X \in \chi \cap E \quad (\text{because } E \subset \chi) \\ \implies & g(Y) \geq g(X) \text{ for all } X \in (\cup_{i=1}^N G^i) \cap E \quad (\text{because } \cup_{i=1}^N G^i = \chi) \\ \implies & g(Y) \geq g(X) \text{ for all } X \in \cup_{i=1}^N (G^i \cap E) \end{aligned}$$

which implies

$$g(Y) \geq g(X) \text{ for all } X \in \cup_{i=1}^N G_E^i \quad (\text{because } G_E^i = G^i \cap E) \quad (5.3.1)$$

Since $G_E^k \neq \emptyset$ for some k , therefore there exist some $\hat{X} \in G_E^k$ and by using Equation (5.3.1), we get

$$g(Y) \geq g(\hat{X}) \quad (5.3.2)$$

As $\hat{X} \in G_E^k$, therefore $\hat{X} \in G^k$ and Equation (5.3.2) implies $Y \in G^i$ for some $i \in \{1, 2, \dots, k\}$. Since $Y \in E$ also, therefore $Y \in G_E^i$ for some $i \in \{1, 2, \dots, k\}$. But, $\cup_{i=1}^{k-1} G_E^i = \emptyset$, so $Y \in G_E^k$. Thus,

$$G_E \subseteq G_E^k \quad (5.3.3)$$

Conversely, let $\bar{X} \in G_E^k$. If possible, let $\bar{X}$ be not an optimal solution of ILP_E , therefore there exists some $Y \in E$ such that

$$g(Y) > g(\bar{X}) \quad (5.3.4)$$

As $\bar{X} \in G_E^k$, therefore $\bar{X} \in G^k$ and using Equation (5.3.4), it can be seen $Y \in G^i$ for some

$i \in \{1, 2, \dots, k-1\}$. Also, $Y \in E$, therefore $Y \in G_E^i$ for some $i \in \{1, 2, \dots, k-1\}$, which contradicts $\cup_{i=1}^{k-1} G_E^i = \emptyset$. So, $\bar{X}$ is an optimal solution of ILP_E . Thus,

$$G_E^k \subseteq G_E. \quad (5.3.5)$$

By using Equations (5.3.3) & (5.3.5), we get $G_E^k = G_E$. Hence G_E^k is the set of all optimal solutions of ILP_E .

Convergence of Algorithm 7 : The Algorithm 7 terminates after obtaining a non- empty set G_E^j for some $j \in \{1, 2, 3, \dots, N\}$, whose existence is guaranteed by virtue of Theorem 5.3.1. So, our proposed algorithm terminates in a finite number of steps. Further, Theorem 5.3.2 claims that first non-empty set G_E^j is the set of all optimal solutions of ILP_E . Thus, the algorithm generates the complete set of all optimal solutions of ILP_E in a finite number of steps.

Now, we discuss another approach to solve ILP_E , which makes use of ranked solutions of ILP_1 as well as efficient solutions of $MILP$.

5.3.3 Description of Algorithm 8 for solving ILP_E

The algorithm starts by solving ILP_k (for $k = 1$) and constructs the set G^1 . Then, $P(X_i^1)$ for each $i \in \{1, 2, \dots, l_1\}$ is solved and the set E^1 is constructed. It may be noted that the set E^1 contains efficient solutions of $MILP$ (see Theorem 5.3.3). Next, calculate the objective function value at each element of the set E^1 and find g_{lower}^1 , which is the current lower bound on the objective function value of ILP_E . Now two cases arises, either $g_{lower}^1 \geq Z^1 - 1$ or $g_{lower}^1 < Z^1 - 1$.

If $g_{lower}^1 \geq Z^1 - 1$, then either $g_{lower}^1 = Z^1$ or $g_{lower}^1 = Z^2$.

1. If $g_{lower}^1 = Z^1$, then there exists an efficient solution $\hat{Y}$ of $MILP$ in the set E^1 , whose objective function value is equal to Z^1 , which implies $G_E^1 \neq \emptyset$. Therefore, by using Theorem 5.3.2, $\hat{Y}$ is an optimal solution of ILP_E .
2. If $g_{lower}^1 = Z^2$, then none of the element of the set E^1 yields objective function value Z^1 (because $Z^1 > Z^2$), which implies X_i^1 is not an optimal solution of $P(X_i^1)$ for all $i \in \{1, 2, \dots, l_1\}$. Therefore, optimal value of $P(X_i^1)$ is non- zero for all $i \in \{1, 2, \dots, l_1\}$. Then, by virtue of Theorem 5.2.2, $X_i^1 \notin E$ for all $i \in \{1, 2, \dots, l_1\}$, which implies $G_E^1 = \emptyset$. Therefore, there exists some $\hat{Y} \in E^1$ such that $g(\hat{Y}) = Z^2$, which implies $G_E^2 \neq \emptyset$. Then, by using Theorem 5.3.2 we concluded that $\hat{Y}$ is an optimal solution of ILP_E .

From above discussion, it can be seen that if $g(\hat{Y}) = g_{lower}^1 \geq Z^1 - 1$ for some $\hat{Y} \in E^1$, then

$\hat{Y}$ is declared as an optimal solution of ILP_E and algorithm terminates. If $g_{lower}^1 < Z^1 - 1$, then set $k = k + 1$ and repeat the above procedure.

Solve ILP_k , construct sets G^k and E^k , and find the value of g_{lower}^k . If $g(\hat{Y}) = g_{lower}^k \geq Z^k - 1$ for some $\hat{Y} \in \cup_{l=1}^k E^l$, then by using Theorem 5.3.6, $\hat{Y}$ is declared as an optimal solution of ILP_E and the algorithm terminates, otherwise repeat the procedure for the next value of k , until $g_{lower}^k \geq Z^k - 1$.

It may be observed that Algorithm 8 follows a two way procedure. From one side it calculate ranked solutions of ILP_1 and at the same time it update the lower bound g_{lower}^k on the objective function value of ILP_E . At every iteration, as the objective function value of ILP_1 moves down, the lower bound g_{lower}^k is tightened, i.e., $g_{lower}^{k+1} \geq g_{lower}^k$. So, at some k^{th} iteration when $g_{lower}^k \geq Z^k - 1$, the algorithm terminates.

5.3.4 Theoretical results used in Algorithm 8

In this section, we will discuss various results pertaining to the convergence of the algorithm. In the following theorem, we will claim that every optimal solution of $P(Y)$ for all $Y \in G^k$ is an efficient solution of $MILP$.

Theorem 5.3.3 *If $Y \in E^k$, then $Y \in E$.*

Proof As $Y \in E^k$, therefore Y is an optimal solution of $P(\hat{X})$ for some $\hat{X} \in G^k$. $P(\hat{X})$ always has a finite optimal value (By Remark 5.2.6) then either $\theta(Y) = 0$ or $\theta(Y) > 0$. If $\theta(Y) = 0$, then from Theorem 5.2.2 it may be observed that $\Psi_i = 0$ for all $i \in \{1, 2, \dots, s\}$, which implies $C^i Y = C^i \hat{X}$ for all $i \in \{1, 2, \dots, s\}$. As $\hat{X}$ is an efficient solution of $MILP$, therefore Y is also an efficient solution of $MILP$ i.e., $Y \in E$. Next, if $\theta(Y) > 0$, then clearly $Y \in E$, by virtue of Theorem 5.2.4.

Now, we will show that an optimal solution of ILP_k , which is also an efficient solution of $MILP$ is contained in the set E^k .

Theorem 5.3.4 *If $\bar{X} \in G_E^k$, then $\bar{X} \in E^k$.*

Proof Let $\bar{X} \in G_E^k$, therefore $\bar{X} \in E$. Let Y be an optimal solution of $P(\bar{X})$. Then, by virtue of Theorem 5.2.2, $\theta(Y) = 0$ which means all surplus variables corresponding to Y are zero, i.e., $\Psi_i = 0$ for all $i \in \{1, 2, \dots, s\}$. Now, we can see $\bar{X}$ is also an optimal solution of the problem $P(\bar{X})$, because by taking $X = \bar{X}$ in the constraint of the problem $P(\bar{X})$ yields $\Psi_i = 0$ for all $i \in \{1, 2, \dots, s\}$, which means $\theta(Y) = \theta(\bar{X})$. So, $\bar{X}$ is also an optimal solution of $P(\bar{X})$. Thus, $\bar{X} \in E^k$.

It is evident by Theorem 5.3.2 that the first non empty set G_E^k for some $k \in (1, 2, \dots, N)$

is the set of all optimal solutions of ILP_E and by Theorem 5.3.4, G_E^k is contained in the set E^k . So, the set E^k can also be use to find an optimal solution of ILP_E . Next, we will use elements of E^k to find the value of g_{lower}^k and establish a relation between g_{lower}^k , G_E^i and an optimal solution of ILP_E .

Theorem 5.3.5 *If $g_{lower}^k < Z^k - 1$, then $\cup_{i=1}^k G_E^i = \emptyset$.*

Proof If possible let $\cup_{i=1}^k G_E^i \neq \emptyset$, therefore there exists some $\hat{X} \in \cup_{i=1}^k G_E^i$. Now, let $g(\hat{Y}) = g_{lower}^k$ for some $\hat{Y} \in \cup_{i=1}^k E^i$, therefore

$$g(\hat{Y}) > g(Y) \text{ for all } Y \in \cup_{i=1}^k E^i. \quad (5.3.6)$$

Also, $g(\hat{Y}) = g_{lower}^k < Z^k - 1$, therefore

$$g(\hat{Y}) < g(X) \text{ for all } X \in G^{k+1}$$

which implies,

$$g(\hat{Y}) < g(X) \text{ for all } X \in \cup_{i=1}^{k+1} G^i. \quad (5.3.7)$$

Equation (5.3.6) and Equation (5.3.7) imply that

$$g(X) > g(Y) \text{ for all } X \in \cup_{i=1}^{k+1} G^i, Y \in \cup_{i=1}^k E^i. \quad (5.3.8)$$

As $\hat{X} \in \cup_{i=1}^k G_E^i$, therefore $\hat{X} \in \cup_{i=1}^k G^i$, which implies

$$g(\hat{X}) > g(Y) \text{ for all } Y \in \cup_{i=1}^k E^i \text{ (By Equation 5.3.8)}. \quad (5.3.9)$$

So, $\hat{X} \notin E^i$ for all $i \in \{1, 2, \dots, k\}$ (which is not possible because of Equation (5.3.9)). Also, by Theorem 5.3.4, $G_E^i \subset E^i$ for all $i \in \{1, 2, \dots, k\}$, so $\hat{X} \notin G_E^i$ for all $i \in \{1, 2, \dots, k\}$, which contradicts $\hat{X} \in \cup_{i=1}^k G_E^i$. Thus, $\cup_{i=1}^k G_E^i = \emptyset$.

It is evident by Theorem 5.3.5, as long as $g_{lower}^k < Z^k - 1$, the first non- empty set as mentioned in the Theorem 5.3.2 cannot be attained because $\cup_{i=1}^k G_E^i$ is empty. In the Theorem 5.3.6 we will show that the first non-empty set condition is no longer required as the elements of E^k give us better condition to find an optimal solution of ILP_E .

Theorem 5.3.6 *If $g(\hat{Y}) = g_{lower}^k \geq Z^k - 1$, then $\hat{Y}$ will be an optimal solution of ILP_E .*

Proof If possible, let $\hat{Y}$ be not an optimal solution of ILP_E , therefore there exists some $\hat{X} \in E$ such that

$$g(\hat{X}) > g(\hat{Y}). \quad (5.3.10)$$

Also, $g(\hat{Y}) = g_{lower}^k = \max\{g(Y) \mid Y \in \cup_{j=1}^k E^j\}$, therefore

$$g(\hat{Y}) \geq g(Y) \text{ for all } Y \in \cup_{j=1}^k E^j. \quad (5.3.11)$$

Equation (5.3.10) and Equation (5.3.11) imply that

$$g(\hat{X}) > g(Y) \text{ for all } Y \in \cup_{j=1}^k E^j. \quad (5.3.12)$$

As $g(\hat{Y}) = g_{lower}^k \geq Z^k - 1$ and $Z^k - 1 \geq Z^{k+1}$, therefore

$$g(\hat{Y}) \geq g(X) \text{ for all } X \in G^{k+1}.$$

this implies,

$$g(\hat{Y}) \geq g(X) \text{ for all } X \in \cup_{j=k+1}^N G^j \quad (5.3.13)$$

Equation (5.3.10) and Equation (5.3.13) imply that

$$g(\hat{X}) > g(X) \text{ for all } X \in \cup_{j=k+1}^N G^j. \quad (5.3.14)$$

Equation (5.3.14) implies that $\hat{X} \in \cup_{j=1}^k G^j$. Also, $\hat{X} \in E$, therefore $\hat{X} \in \cup_{j=1}^k G_E^j$. Then, by using Theorem 5.3.4, $\hat{X} \in \cup_{j=1}^k E^j$, which is not possible (using Equation (5.3.12)). Thus, $\hat{Y}$ is an optimal solution of ILP_E .

Convergence of Algorithm 8: The Algorithm 8 finds an optimal solution of ILP_E by successively solving ILP_k until $g_{lower}^k < Z^k - 1$. It can be observed by Theorem 5.3.5, if $g_{lower}^k < Z^k - 1$, then $\cup_{i=1}^k G_E^i = \emptyset$. Further, Theorem 5.3.1 implies that $G_E^i \neq \emptyset$ for some $i \in \{1, 2, 3, \dots, N\}$, so there exist some $k \in \{1, 2, 3, \dots, N\}$ such that $g_{lower}^k \geq Z^k - 1$. Thus, Algorithm 8 terminates in a finite number of steps. Also, if $g(Y) = g_{lower}^k \geq Z^k - 1$, then by Theorem 5.3.6, Y will be an optimal solution of ILP_E . So, Algorithm 8 finds an optimal solution of ILP_E in a finite number of steps.

5.3.5 Comparative analysis of Algorithm 7 and Algorithm 8

1. The working of both algorithms is based on the ranking of integer feasible solutions of ILP_1 . In the Algorithm 7, ranking of integer solutions of ILP_1 is done until a stage is reached, when $G_E^k \neq \emptyset$ for some k . In the Algorithm 8, sets E^k of efficient solutions of $MILP$ are also generated at each iteration in addition to ranked solutions of ILP_1 . Then, at k^{th} stage, g_{lower}^k is calculated from elements of E^k which provides a lower bound on the objective function values of ILP_E and it keeps on shrinking at every stage. This algorithm terminates when $g_{lower}^k \geq Z^k - 1$ for some k .

Algorithm 8: Hybrid algorithm to solve ILP_E

Data: $A \in Z^{m \times n}, b \in Z^m, C \in Z^{s \times n}, V \in Z^{1 \times n}$

Result: $G_E =$ Set of all optimal solutions of ILP_E

```

1 Initialization  $k \leftarrow 1$ 
2  $G^1 \leftarrow \{\hat{X} \in \mathcal{X} \mid g(\hat{X}) = \max_{X \in \mathcal{X}} g(X)\}$  ▷ set of optimal solutions of  $ILP_1$ 
3  $Z^1 \leftarrow g(X_i^1)$  for  $X_i^1 \in G^1$  ▷ optimal value of  $ILP_1$ 
4  $E^1 \leftarrow \emptyset$ 
5 foreach  $X_i^1 \in G^1$  do
6   Solve  $P(X_i^1)$ 
7   if  $Y$  is an optimal solution of  $P(X_i^1)$  then
8      $E^1 \leftarrow [E^1 \ Y]$ 
9  $g_{lower}^1 = \max\{g(Y) \mid Y \in E^1\}$  ,
10 while  $g_{lower}^k < Z^k - 1$  do
11    $k \leftarrow k + 1$ 
12    $\mathcal{X}_k \leftarrow \{X \in \mathcal{X} \mid g(X) \leq Z^{k-1} - 1\}$ 
13    $G^k \leftarrow \{\hat{X} \in \mathcal{X}_k \mid g(\hat{X}) = \max_{X \in \mathcal{X}_k} g(X)\}$  ▷ set of optimal solutions of  $ILP_k$ 
14    $Z^k \leftarrow g(X_i^k)$  for  $X_i^k \in G^k$  ▷ optimal value of  $ILP_k$ 
15    $E^k \leftarrow \emptyset$ 
16   foreach  $X_i^k \in G^k$  do
17     Solve  $P(X_i^k)$ 
18     if  $Y$  is an optimal solution of  $P(X_i^k)$  then
19        $E^k \leftarrow [E^k \ Y]$ 
20    $g_{lower}^k = \max\{g(Y) \mid Y \in \cup_{l=1}^k E^l\}$  ,
21 if  $g(\hat{Y}) = g_{lower}^k$  for some  $\hat{Y} \in \cup_{l=1}^k E^l$  then
22    $\hat{Y} \in G_E$ 

```

2. The Algorithm 8 may generate more efficient solutions than Algorithm 7 because at every stage a set E^k is generated.
3. In the Algorithm 7 only alternate solutions of ILP_k are explored, whereas in the Algorithm 8 alternate solutions of ILP_k as well as $P(Y)$ are explored for all $Y \in G^k$. So, the Algorithm 8 explore larger number of edges.

5.4 Numerical Section

In this section, we will compare the working of the proposed algorithms with existing algorithm discussed in [2,45,80,94] for solving ILP_E . Further, we will show our algorithms work better than these algorithms in terms of computational efforts. A collective comparison with [2,45,80,94] will be discussed in Table 5.4.

Example 5.4.1 Consider the problem (taken from [80]).

$$(ILP_E) \quad \max_{X \in E} g(x, y) = -x - 2y,$$

where E is the set of all efficient solutions of MILP. The problem (MILP) is defined as :

$$(MILP) \quad \max_{(x,y) \in \mathcal{X}} (x - 2y, -x + 4y),$$

where $\chi = Z^2 \cap S$, $S = \{(x, y) \in R^2 \mid -2x + y \leq 0, x \leq 3, y \leq 2 \text{ and } x, y \geq 0\}$.

1. Solution of Example 5.4.1 by using Algorithm 7

(a) Construction of set G^1 by solving ILP_1 :

$$(ILP_1) \quad \max_{(x,y) \in \chi} g(x, y) = -x - 2y.$$

This problem has a unique optimal solution $X_1^1 = (0, 0)$ and $g(0, 0) = 0$. So, $G^1 = \{(0, 0)\}$ and $Z^1 = 0$.

(b) Construction of $P(X_1^1)$ as :

$$P(X_1^1) \quad \begin{cases} \max & \theta = \Psi_1 + \Psi_2 \\ x - 2y - \Psi_1 = 0, -x + 4y - \Psi_2 = 0, \\ (x, y) \in \chi, \Psi_1, \Psi_2 \geq 0. \end{cases}$$

The optimal value of $P(X_1^1) > 0$ for $X_1^1 = (0, 0)$. So, $(0, 0) \notin E$ which implies $G_E^1 = \emptyset$. To find an optimal solution of ILP_E , set $k = 2$ and construct set G^2 .

(c) Construction of set G^2 by solving ILP_2 :

$$(ILP_2) \quad \max_{X \in \chi_2} -x - 2y,$$

where $\chi_2 = \{X \in \chi \mid -x - 2y \leq -1\}$.

This problem has a unique optimal solution $X_1^2 = (1, 0)$ and $g(1, 0) = -1$ and $Z^2 = -1$. Next, we find G_E^2 by constructing $P(X_1^2)$ for $X_1^2 = (1, 0)$.

(d) Construct $P(X_1^2)$ as :

$$P(X_1^2) \quad \begin{cases} \max & \theta = \Psi_1 + \Psi_2 \\ x - 2y - \Psi_1 = 1, -x + 4y - \Psi_2 = -1, \\ (x, y) \in \chi, \Psi_1, \Psi_2 \geq 0. \end{cases}$$

The optimal value of $P(X_1^2) > 0$ for $X_1^2 = (1, 0)$. So, $(1, 0) \notin E$ which implies $G_E^2 = \emptyset$. Now, set $k = 3$ and construct set G^3 .

(e) Construction of set G^3 by solving ILP_3 :

$$(ILP_3) \quad \max_{X \in \chi_3} -x - 2y$$

where $\chi_3 = \{X \in \chi \mid -x - 2y \leq -2\}$.

This problem has a unique optimal solution $X_1^3 = (2, 0)$ and $g(2, 0) = -2$ and $Z^2 = -2$. Next, we find G_E^3 by constructing the problem $P(X_1^3)$ for $X_1^3 = (2, 0)$.

(f) Construct $P(X_1^3)$ as :

$$P(X_1^3) \quad \begin{cases} \max & \theta = \Psi_1 + \Psi_2 \\ & x - 2y - \Psi_1 = 2, -x + 4y - \Psi_2 = -2, \\ & (x, y) \in \chi, \Psi_1, \Psi_2 \geq 0. \end{cases}$$

The optimal value of $P(X_1^3) = 0$ for $X_1^3 = (2, 0)$. So, $(2, 0) \in E$ which implies $G_E^3 = (2, 0)$. We declare $(2, 0)$ as an optimal solution of ILP_E . So, the complete set of all optimal solutions of ILP_E is $\{(2, 0)\}$.

Remark 5.4.1 *The feasible region χ of ILP_1 contains ten integer feasible solutions and efficient set E of $MILP$ contains seven efficient solutions namely $(2, 0)$, $(3, 0)$, $(2, 1)$, $(3, 1)$, $(1, 2)$, $(2, 2)$ and $(3, 2)$ (as discussed in [80]), but we have only enumerated three integer feasible solutions of ILP_1 and one efficient solution of $MILP$ for solving ILP_E .*

2. Solution of Example 5.4.1 using Algorithm 8

(a) Construction of the set G^1 by solving ILP_1 :

$$(ILP_1) \quad \max_{(x,y) \in \chi} g(x, y) = -x - 2y.$$

This problem has a unique optimal solution $X_1^1 = (0, 0)$ and $g(0, 0) = 0$. So $G^1 = \{(0, 0)\}$ and $Z^1 = 0$.

(b) Construction of the set E^1 :

Since problem $P(X_1^1)$ has two optimal solutions $(2, 1)$ and $(3, 1)$, therefore $E^1 = \{(2, 1), (3, 1)\}$ and $g_{lower}^1 = -4$. As $g_{lower}^1 < Z^1 - 1$, therefore set $k = 2$.

(c) Construction of the set E^2 :

The problem (ILP_2) has a unique optimal solution $X_1^2 = (1, 0)$ and $g(1, 0) = -1$. So $G^2 = \{(1, 0)\}$ and $Z^2 = -1$. The problem $P(X_1^2)$ has a unique optimal solution $(3, 1)$, therefore $E^2 = \{(3, 1)\}$ and $g_{lower}^2 = -4$. As $g_{lower}^2 < Z^2 - 1$, therefore set $k = 3$.

(d) Construction of the set E^3 :

The problem (ILP_3) has a unique optimal solution $X_1^3 = (2, 0)$ and $g(2, 0) = -2$.

So $G^3 = \{(2, 0)\}$ and $Z^3 = -2$. The problem $P(X_1^3)$ has a unique optimal solution $(2, 0)$, therefore $E^3 = \{(2, 0)\}$ and $\cup_{i=1}^3 E^i = \{(2, 1), (3, 1), (2, 0)\}$. The value of g_{lower}^3 is updated to -2. As $g_{lower}^3 \geq Z^3 - 1$. So, $(2, 0)$ is an optimal solution of ILP_E .

3. Solution of Example 5.4.1 using Jorge [80] algorithm

- (a) Initially, set $g^{upper} = \infty$, $g^{lower} = -\infty$ and $k = 0$. Then, we solve the problems P_1 and P_2 .

$$(P_1) \quad \min_{(x,y) \in \mathcal{X}} x - 2y, \quad (P_2) \quad \min_{(x,y) \in \mathcal{X}} -x + 4y.$$

The optimal value of problems (P_1) and (P_2) is -3. So, set $M = (-3, -3)'$

- (b) A relaxed problem (ILP) is solved.

$$(ILP) \quad \max_{(x,y) \in \mathcal{X}} g(x, y) = -x - 2y.$$

The optimal solution of ILP is $(0, 0)$.

- (c) To test the efficiency of solution $X^1 = (0, 0)$, a problem $P(X^1)$ is solved. The optimal value of the problem $P(X^1)$ is 2, which is attained at $\hat{X}^1 = (3, 1)$. Since optimal value of the problem $P(X^1)$ is greater than 0, so $X^1 = (0, 0) \notin E$. Set $g^{upper} = g(0, 0) = 0$.

- (d) Now, solve the problem T_1 define as:

$$(T_1) \quad \begin{cases} \max & g(x, y) = -x - 2y, \\ & x - 2y = 1, -x + 4y = 1, \\ & (x, y) \in \mathcal{X}. \end{cases}.$$

The size of problem T_1 is 5×2 . It can be observed that the problem T_1 is obtained from problem ILP by introducing two additional constraints $(x - 2y = 1$ and $-x + 4y = 1)$ into ILP . The optimal solution of problem T_1 is $\bar{X}^1 = (3, 1)$. Since $g(3, 1) = -5 > g^{lower}$, set $g^{lower} = -5$ and $X^{inc} = (3, 1)$.

- (e) Further, to remove solutions dominated by the efficient solution $\bar{X}^1$, some corner constraints are introduced into the feasible region, and following problem R_1 is

solved :

$$(R_1) \quad \begin{cases} \max & g(x, y) = -x - 2y \\ & x - 2y \geq 5y_1^1 - 3, -x + 4y \geq 5y_2^1 - 3, \\ & y_1^1 + y_2^1 \geq 1, y_1^1 \leq 1, y_2^1 \leq 1 \\ & y_1^1, y_2^1 \geq 0, y_1^1, y_2^1 \in \mathbb{Z}^2, \\ & (x, y) \in \chi. \end{cases}$$

The size of problem R_1 is 8×4 . It can be observed that the problem R_1 is obtained from ILP by adding two additional variables (y_1^1, y_2^1) , five additional constraints $(x - 2y \geq 5y_1^1 - 3, -x + 4y \geq 5y_2^1 - 3, y_1^1 + y_2^1 \geq 1, y_1^1 \leq 1, y_2^1 \leq 1)$ and two additional non-negativity constraints $(y_1^1 \geq 0, y_2^1 \geq 0)$ into ILP . The optimal solution of R_1 is $(2,0)$, which passes the efficiency test, so $(2,0)$ is declared as the optimal solution of ILP_E , which is same as obtained by the proposed algorithm.

4. Solution of Example 5.4.1 using Djamal and Marc [45] algorithm

- (a) Initially, set $g^{opt} = -\infty$, $k = 0$ and solve problems (P_1) and (P_2) . The optimal value of (P_1) and (P_2) is -3. Therefore, set $-M1 = -3$ and $-M2 = -3$.
- (b) A relaxed problem (ILP) is solved. The optimal solution of ILP is obtained as $(0,0)$.
- (c) To test the efficiency of solution $(0,0)$, a problem $P(Y)$ for $Y=(0,0)$ is solved. The optimal value of this problem is greater than zero, so $(0,0) \notin E$. Also, $(3,1)$ is an optimal solution of this problem. So, $(3,1)$ will be an efficient solution of $MILP$.
- (d) Next, a problem T_1 of size 5×2 is solved. An optimal solution of problem T_1 is obtained at $(3,1)$. Since $g(3,1) = -5 > g^{opt}$, set $g^{opt} = -5$ and $X^{opt} = (3,1)$.
- (e) Then, a problem R_1 of size 8×4 is constructed. The problem R_1 is obtained from ILP by adding two additional variables (y_1^1, y_2^1) , five additional constraints $(x - 2y \geq 5y_1^1 - 3, -x + 4y \geq 5y_2^1 - 3, y_1^1 + y_2^1 \geq 1, y_1^1 \leq 1, y_2^1 \leq 1)$ and two additional non-negativity constraints $(y_1^1 \geq 0, y_2^1 \geq 0)$ into ILP . The problem is solved. An optimal solution of the problem R_1 is $(2,0)$.
- (f) To test the efficiency of solution $(2,0)$, a problem $P(Y)$ for $Y=(2,0)$ is solved. The optimal value of this problem came out to be zero. Therefore, $(2,0)$ is declared as an optimal solution of ILP_E . The optimal solution $(2,0)$ obtained

using Djamal and Marc's [45] algorithm is same as obtained by the proposed algorithm.

5. Solution of Example 5.4.1 using Lokman [94] algorithm

- (a) Initially, define $g(x, y) = v^1(x - 2y) + v^2(-x + 4y)$,
 $v^1 = -3$, $v^2 = -2$, $g^* = 10^{-4}$, $p = \arg \max_{i=1,2} v^i = 2$,
 $w^1 = v^1 - \min_{i=1,2} v^i + 1 = 1$, $w^2 = v^2 - \min_{i=1,2} v^i + 1 = 2$,
 $M^1 = \min_{(x,y) \in \chi} x - 2y = -3$.

- (b) Consider the problem (*ILP*) :

$$(ILP) \quad \max_{(x,y) \in \chi} g(x, y) = -x - 2y.$$

This problem has optimal solution $X^0 = (0, 0)$ with criterion vector as $(0, 0)$ and $g^{upper} = g(0, 0) = 0$.

- (c) Next, the problem Q^0 is constructed as :

$$(Q^0) \quad \begin{cases} \max & x - 2y + 2(-x + 4y), \\ & x - 2y \geq 0, -x + 4y \geq 0, (x, y) \in \chi. \end{cases}$$

The problem (Q^0) has an optimal solution $X^0 = (2, 1)$ with criterion vector as $(0, 2)$ and $g^{lower} = g(2, 1) = -4$.

- (d) Next, a problem (P^2) is solved.

$$(P^2) \quad \max_{(x,y) \in \chi} -x + 4y.$$

An optimal solution of problem (P^2) is $(1, 2)$ with criterion vector as $(-3, 7)$ and $g(1, 2) = -5$.

- (e) Then, a problem (Q^1) is constructed :

$$(Q^1) \quad \begin{cases} \max & x - 2y + 2(-x + 4y), \\ & x - 2y \geq -3, -x + 4y \geq 7, (x, y) \in \chi. \end{cases}$$

The problem (Q^1) has an optimal solution $(1, 2)$ with criterion vector $(-3, 7)$ and $g(1, 2) = -5$.

- (f) Next, problems P_0^2 and P_1^2 are constructed to generate a solution that maximizes

C^2X and not dominated by previously generated non-dominated vector $(-3,7)$.

$$(P_0^2) \quad \begin{cases} \max & -x + 4y \\ x - 2y & \geq M^1, \\ -3(x - 2y) - 2(-x + 4y) & \geq -4 + 4g^* \\ -3(x - 2y) - 2(-x + 4y) & \leq 0 \\ (x, y) & \in \chi. \end{cases}$$

$$(P_1^2) \quad \begin{cases} \max & -x + 4y \\ x - 2y & \geq -3 + \epsilon, \\ -3(x - 2y) - 2(-x + 4y) & \geq -4 + 4g^* \\ -3(x - 2y) - 2(-x + 4y) & \leq 0 \\ (x, y) & \in \chi. \end{cases}$$

Both problems have optimal solution $(1,1)$ with criterion vector $(-1,3)$ and $g(1,1) = -3$.

(g) Then, a problem Q^2 is constructed :

$$(Q^2) \quad \begin{cases} \max & x - 2y + 2(-x + 4y), \\ x - 2y & \geq -1, -x + 4y \geq 3, (x, y) \in \chi. \end{cases}$$

The problem (Q^2) has a optimal solution $(3,2)$ with criterion vector $(-1,5)$ and $g(3,2) = -7$.

k	No. of integer linear programming problems constructed to find X'^k (problem size)	X'^k	$(C^1 X'^k, C^2 X'^k)$	$g(X'^k)$	No. of integer linear programming problems constructed to find X^k (problem size)	X^k	$(C^1 X^k, C^2 X^k)$	$g(X^k)$	g^{lower}
1	1 (3×2)	(0,0)	(0,0)	0	1 (5×2)	(2,1)	(0,2)	-4	-4
2	1 (3×2)	(1,2)	(-3,7)	-5	1 (5×2)	(1,2)	(-3,7)	-5	-4
3	2 (6×2)	(1,1)	(-1,3)	-3	1 (5×2)	(3,2)	(-1,5)	-7	-4
4	3 (6×2)	(0,0)	(0,0)	0	1 (5×2)	(2,1)	(0,2)	-4	-4
5	4 (6×2)	(1,0)	(1,-1)	-1	1 (5×2)	(3,1)	(1,1)	-5	-4
6	5 (6×2)	(2,0)	(2,-2)	-2	1 (5×2)	(2,0)	(2,-2)	-2	-2
7	6 (6×2)	-	-	-	-	-	-	-	-

Table 5.1: Demonstration of Lokman [94] algorithm on Example 5.4.1

(h) The stepwise working of the Lokman [94] algorithm is shown in Table 5.1. At sixth iteration, we get an optimal solution $(2,0)$ with criterion vector $(2,-2)$ and $g^{lower} = g(2,0) = -2$ and at the seventh iteration, the problem becomes infeasible. So, $(2,0)$ is declared as an optimal solution of $ILLP_E$, the same as obtained by the proposed algorithm.

6. Solution of Example 5.4.1 using Abbas and Chaabane [2] algorithm

(a) Initially, the problem (*ILP*) is solved. An optimal solution of this problem is obtained as (0,0), which doesn't pass the efficiency test, so $(0,0) \notin E$.

(b) Set $g^{opt} = -\infty$ and construct the problem (P^1) define as :

$$(P^1) \quad \max_{(x,y) \in \mathcal{X}} x - 2y.$$

The problem (P^1) has a unique optimal solution $X^1 = (3, 0)$ as shown in Table 5.2, so $(3, 0)$ is an efficient solution of *MILP*. Since $g(3, 0) = -3 > g^{opt}$. Set $g^{opt} = -3$ and $X^{opt} = (3, 0)$.

c_j	1	-2	0	0	0	
v_j	-1	-2	0	0	0	
B.V.	x_1	x_2	x_3	x_4	x_5	b_j
x_3	0	1	1	2	0	6
x_1	1	0	0	1	0	3
x_5	0	1	0	0	1	2
$z_j^1 - c_j$	0	2	0	1	0	3
$w_j^1 - v_j$	0	2	0	-1	0	-3

(a) Simplex table of X^1

c_j	1	-2	0	0	0	0	
v_j	-1	-2	0	0	0	0	
B.V.	x_1	x_2	x_3	x_4	x_5	x_6	b_j
x_3	0	5	1	0	0	2	4
x_1	1	2	0	0	0	1	2
x_5	0	1	0	0	1	0	2
x_4	0	-2	0	1	0	-1	1
$z_j^2 - c_j$	0	4	0	0	0	1	2
$w_j^2 - v_j$	0	0	0	0	0	-1	-2

(b) Simplex table of X^2

Table 5.2: Optimal Table corresponding to X^1 and X^2

Iteration	Edge explored	Size of integer linear programming problem constructed	Feasible solutions enumerated	Efficient solutions enumerated	g^{opt}	X^{opt}
0	-	3×2	(0,0)	-	-inf	-
1	E_4	3×5	(3,0) (0,0) (1,0),(2,0)	(3,0) (2,0)	-3 -2	(3,0) (2,0)
2	E_2 E_6	4×6	(2,0) - (0,0) ,(1,0)	(2,0) - -	-2 -2 -2	(2,0) (2,0) (2,0)
3	-	5×7	-	-	-2	(2,0)

Table 5.3: Demonstration of Abbas and Chaabane [2] algorithm on Example 5.4.1

(c) Then, a set $\tau_1 = \{j \in N^1 \mid z_j^1 - c_j \geq 0 \text{ and } w_j^1 - v_j \leq 0\} = \{4\}$ is constructed from Table 5.2. Since $\tau_1 = \{4\}$, the edge E_4 is explored. After exploring edge E_4 , we get integer feasible solutions (0,0),(1,0) and (2,0) for $\theta = 3, 2$ and 1 respectively. (0,0) and (1,0) don't satisfy the efficiency test, but solution (2,0) passes the efficiency test. Since $g(2, 0) = -2 > g^{opt}$. So, set $g^{opt} = -2$ and $X^{opt} = (2, 0)$.

(d) Set $k+2$ and a Type 2 cut $(-x_1 - 2x_2 \geq -2)$ is imposed over the feasible region, and using the dual simplex method, an optimal feasible solution $X^2 = (2, 0)$ is obtained, which passes the efficiency test. Then, a set $\tau_2 = \{j \in N^2 \mid z_j^2 - c_j \geq 0 \text{ and } w_j^2 - v_j \leq 0\} = \{2, 6\}$ is constructed from Table 5.2. Since $\beta = d_j^k - \sum_{i \in I_k} d_i^k y_{k,j} = 0$ for $j = 2$, we have only explored the edge E_6 . After exploring the

edge E_6 , we get two integer solutions namely $(0,0)$ and $(1,0)$. Both $(0,0)$ and $(1,0)$ don't satisfy the efficiency test.

- (e) We set $k = 3$ and applied the Type 1 cut ($x_2 \geq 1$) on the optimal Table corresponding to X^2 . After appending the cut, the problem becomes infeasible. So, $X^{opt} = (2,0)$ is declared as an optimal solution of problem ILP_E , which is the same as obtained by the proposed algorithm.

Comparative analysis

We have summarised finding of Jorge's [80] algorithm, Djamel and Marc's [45] algorithm, Lokman's [94] algorithm, Abbas and Chaabane's [2] algorithm, Younes, Sarah and Djamel's [145] and proposed algorithms in Table 5.4. It can be observed that our algorithms solve a lesser number of ILP for optimizing ILP_E than algorithms of [45, 80, 94, 145]. It is worth mentioning that the largest size of integer linear programming problem (ILP_k) solved by our algorithm is $(m+1) \times n$ whereas, Lokman's [94] algorithm solves at most $(k-1)^{s-1}$ integer linear problems of size $(m+1+s) \times n$ at each k^{th} iteration ($k > 2$). Further, Jorge's [80] algorithm, Younes, Sarah and Djamel's [145] and Djamel and Marc's [45] algorithm solves an integer linear programming problem (R_k) of size $(m+(2s+1)(k-1)) \times (n+s(k-1))$ at each k^{th} iteration. The reason for the larger size of problem R_k is because of the addition of $2s+1$ new constraints and s new variables at each iteration in addition to existing constraints and variables. So, at k^{th} iteration the problem will have $(m+(2s+1)(k-1))$ constraints and $(n+s(k-1))$ new variables. This makes the problem very complex, and for large-sized problems, it can become unmanageable. In the proposed algorithms, only one constraint is added at each iteration. So, the number of constraints at k^{th} iteration is $m+1$ only. Further, no additional variable is added at any iteration and previously added objective function cuts are also not carried forward. The matrix size remains $(m+1) \times n$, at each iteration. So, our proposed algorithms are computationally more efficient as compared to algorithms of [45, 80, 94, 145].

In the Abbas and Chaabane [2] algorithm, an edge exploring and edge cutting technique is used for optimizing ILP_E . A Type 1 cut ($\sum_{j \in N^k - \{j_k\}} x_j$) and Dantzig cut is utilized for truncating the whole edge and integer feasible solutions from the feasible region, and Type 2 cut is used for finding a better efficient solution. Moreover, a set $\tau_k = \{j \in N^k \mid z_{1,j}^k - c_j^1 \geq 0 \text{ and } w_j^k - d_j^k \leq 0\}$ is constructed for enumerating an integer feasible solution on incident edges. Then, efficiency test is used for testing the efficiency of integer solutions lying on the explored edge. In total, their algorithm uses three types of cuts (Type 1 cut, Dantzig cut, and Type 2 cut) for solving ILP_E . Because of these cuts, the size of the integer linear programming problem increases at every iteration (and these cuts are carried forward), and at k^{th} iteration, their algorithm will solve a problem of size at least $(m+k) \times n$. In

	Observation from example of [2]				
	Algorithm 7	Algorithm 8	Algorithms of [45, 80, 145]	Algorithm of [94]	Algorithm of [2]
No. of integer linear programming problems constructed to solve ILP_E	9	9	17	98	7
Largest size integer linear programming problems constructed to solve ILP_E	4×2	4×2	31×14	7×2	11×13
No. of efficient solutions enumerated	1	6	5	6	2
	Observation from example of [45]				
	Algorithm 7	Algorithm 8	Algorithms of [45, 80, 145]	Algorithm of [94]	Algorithm of [2]
No. of integer linear programming problems constructed to solve ILP_E	6	6	9	15	9
Largest size integer linear programming problem constructed to solve ILP_E	4×2	4×2	18×8	6×2	11×13
No. of efficient solutions enumerated	1	6	4	4	9
	Observation from example of [80]				
	Algorithm 7	Algorithm 8	Algorithms of [45, 80, 145]	Algorithm of [94]	Algorithm of [2]
No. of integer linear programming problems constructed to solve ILP_E	3	3	5	14	4
Largest size integer linear programming problem constructed to solve ILP_E	4×2	4×2	8×4	6×2	5×7
No. of efficient solutions enumerated	1	3	2	6	2

Table 5.4: Comparative analysis of proposed algorithms with [2, 45, 80, 94, 145] algorithms

the proposed algorithms, we are using only an objective function cut at each iteration, and the size of ILP_k at k^{th} iteration is $(m + 1) \times n$. So, smaller integer linear programming problems are constructed to solve ILP_E using the proposed algorithms as compared to Abbas and Chaabane's [2] algorithm.

Example 5.4.2 Consider the following problem:

$$(ILP_E) \quad \max_{X \in E} g(X) = 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5,$$

where E is the set of all efficient points of MILP.

$$(MILP) \quad \max_{X \in \chi} (x_1 + x_2 + x_3 + x_4 + x_5, x_1 - x_2 + x_3 - x_4 + x_5, 2x_2 + 4x_4),$$

where $\chi = Z^5 \cap S$, $S = \{X \in R^5 \mid (x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 + x_3 + x_4 + x_5 \leq 2, x_1 + x_3 + x_5 \leq 1, x_2 + x_4 \leq 1, x_1, x_2, x_3, x_4, x_5 \geq 0\}$.

1. Solution of Example 5.4.2 using Algorithm 7

(a) Construction of set G^1 by solving ILP_1 :

$$(ILP_1) \quad \max_{(x,y) \in \chi} g(X).$$

The problem (ILP_1) has a unique optimal solution $X_1^1 = (1, 1, 0, 0, 0)$ and $g(1, 1, 0, 0, 0) = 9$. So $G^1 = \{(1, 1, 0, 0, 0)\}$ and $Z^1 = 9$.

(b) Construction of set G_E^1 by solving $P(X_1^1)$:

The optimal value of $P(X_1^1) > 0$ for $X_1^1 = (1, 1, 0, 0, 0)$. So, $(1, 1, 0, 0, 0) \notin E$,

which implies $G_E^1 = \emptyset$. Since $G_E^1 = \emptyset$, set $k = 2$.

(c) Construction of set G^2 by solving ILP_2 :

$$(ILP_2) \quad \max_{X \in \chi_2} g(X),$$

where $\chi_2 = \{X \in \chi \mid g(X) \leq 8\}$.

This problem has two optimal solutions $X_1^2=(0,1,1,0,0)$ and $X_2^2=(1,0,0,1,0)$. So, $G^2 = \{(0, 1, 1, 0, 0), (1, 0, 0, 1, 0)\}$.

(d) Construction of set G_E^2 by solving $P(X_1^2)$ and $P(X_2^2)$:

The optimal value of $P(X_1^2) > 0$ for $X_1^2=(0,1,1,0,0)$ and optimal value of $P(X_2^2) = 0$ for $X_2^2=(1,0,0,1,0)$. So, $(1, 0, 0, 1, 0) \in E$, which implies $G_E^2 = \{(1, 0, 0, 1, 0)\}$. Thus, optimal set of problem ILP_E is $\{(1, 0, 0, 1, 0)\}$.

2. Solution of Example 5.4.2 using Algorithm 8

(a) To solve ILP_E , consider the following problem :

$$(ILP_1) \quad \max_{(x,y) \in \chi} g(X) \quad .$$

This problem has a unique optimal solution $X_1^1 = (1, 1, 0, 0, 0)$ and $g(1, 1, 0, 0, 0) = 9$. So $G^1 = \{(1, 1, 0, 0, 0)\}$ and $Z^1 = 9$.

(b) Construction of set E^1 :

The problem $P(X_1^1)$ has three optimal solutions $(1,0,0,1,0)$, $(0,0,1,1,0)$ and $(0,0,0,1,1)$, therefore $E^1 = \{(1, 0, 0, 1, 0), (0, 0, 1, 1, 0), (0, 0, 0, 1, 1)\}$ and $g_{lower}^1 = 7$. As $g_{lower}^1 < Z^1 - 1$, therefore set $k = 2$ and construct set E^2 .

(c) Construction of set E^2 :

The problem (ILP_2) has two optimal solutions $X_1^2 = (1, 0, 0, 1, 0)$ and $X_2^2 = (0, 1, 1, 0, 0)$ with optimal value 7. So $G^2 = \{(1, 0, 0, 1, 0), (0, 1, 1, 0, 0)\}$ and $Z^2 = 7$. $P(X_1^2)$ and $P(X_2^2)$ has three optimal solutions $(1,0,0,1,0)$, $(0,0,1,1,0)$ and $(0,0,0,1,1)$, therefore $E^2 = \{(1, 0, 0, 1, 0), (0, 0, 1, 1, 0), (0, 0, 0, 1, 1)\}$ and $g_{lower}^2 = 7$.

(d) As $g(1, 0, 0, 1, 0) = g_{lower}^2 > Z^2 - 1$. So, $(1, 0, 0, 1, 0)$ is an optimal solution of ILP_E .

5.5 Computational results

The Jorge's [80] algorithm, Djamel and Marc's [45] algorithm and proposed algorithms are implemented in MATLAB (R2019b) on a personal HP laptop (Intel(R), Core(TM), i3-6006U processor, CPU @ 2GHz, and 4 GB RAM). The elements of matrices A , b and C are randomly taken integer values from the set $[0,20]$, $[0,100]$ and $[-100,100]$ respectively. For each instance (m, n, s) a series of 10 randomly generated problems are solved using Jorge's [80] algorithm, Djamel and Marc's [45] algorithm and proposed algorithms.

$m = 5, n = 10$										$m = 10, n = 20$									
s	No. of ILP_k solved			Time (in sec)			max	min	mean	max	min	mean	max	min	mean	max	min	mean	max
	max	min	mean	max	min	mean													
3	10	1	3.5	8.18	0.13	1.23	14	1	3.9	2.60	0.15	0.64	14	1	3.9	2.60	0.15	0.64	14
4	17	1	2.9	2.57	0.11	0.46	2	1	1.1	0.27	0.14	0.18	2	1	1.1	0.27	0.14	0.18	2
5	4	1	1.4	0.48	0.11	0.17	4	1	1.4	0.88	0.15	0.25	4	1	1.4	0.88	0.15	0.25	4
6	68	1	7.7	9.30	0.12	1.06	2	1	1.1	0.31	0.14	0.18	2	1	1.1	0.31	0.14	0.18	2
7	2	1	1.1	0.26	0.13	0.16	18	1	2.7	3.06	0.16	0.47	18	1	2.7	3.06	0.16	0.47	18
8	9	1	1.8	1.15	0.12	0.24	1	1	1	0.21	0.15	0.17	1	1	1	0.21	0.15	0.17	1
9	1	1	1	0.19	0.12	0.14	1	1	1	0.24	0.15	0.18	1	1	1	0.24	0.15	0.18	1
10	1	1	1	0.20	0.13	0.15	2	1	1.1	0.37	0.16	0.19	2	1	1.1	0.37	0.16	0.19	2

$m = 15, n = 30$										$m = 20, n = 40$									
s	No. of ILP_k solved			Time (in sec)			max	min	mean	max	min	mean	max	min	mean	max	min	mean	max
	max	min	mean	max	min	mean													
3	6	1	1.8	0.69	0.20	0.30	9	1	1.8	1.43	0.25	0.39	9	1	1.8	1.43	0.25	0.39	9
4	5	1	1.9	1.19	0.21	0.39	5	1	1.6	0.84	0.27	0.37	5	1	1.6	0.84	0.27	0.37	5
5	1	1	1	0.25	0.22	0.24	1	1	1	0.29	0.26	0.27	1	1	1	0.29	0.26	0.27	1
6	1	1	1	0.33	0.21	0.25	1	1	1	0.35	0.26	0.30	1	1	1	0.35	0.26	0.30	1
7	1	1	1	0.37	0.23	0.28	1	1	1	0.32	0.27	0.29	1	1	1	0.32	0.27	0.29	1
8	2	1	1.1	0.56	0.30	0.35	1	1	1	0.37	0.28	0.32	1	1	1	0.37	0.28	0.32	1
9	1	1	1	0.47	0.29	0.34	1	1	1	0.33	0.29	0.31	1	1	1	0.33	0.29	0.31	1
10	1	1	1	0.33	0.27	0.31	1	1	1	0.50	0.28	0.34	1	1	1	0.50	0.28	0.34	1

$m = 25, n = 50$										$m = 30, n = 60$									
s	No. of ILP_k solved			Time (in sec)			max	min	mean	max	min	mean	max	min	mean	max	min	mean	max
	max	min	mean	max	min	mean													
3	1	1	1	0.41	0.35	0.38	4	1	1.3	0.85	0.44	0.54	4	1	1.3	0.85	0.44	0.54	4
4	4	1	1.7	1.09	0.37	0.51	2	1	1.1	0.68	0.45	0.50	2	1	1.1	0.68	0.45	0.50	2
5	1	1	1	0.49	0.38	0.42	1	1	1	0.59	0.49	0.53	1	1	1	0.59	0.49	0.53	1
6	1	1	1	0.44	0.39	0.42	1	1	1	0.77	0.45	0.53	1	1	1	0.77	0.45	0.53	1
7	1	1	1	0.46	0.38	0.41	1	1	1	0.58	0.48	0.52	1	1	1	0.58	0.48	0.52	1
8	2	1	1.1	0.82	0.41	0.51	1	1	1	0.59	0.47	0.53	1	1	1	0.59	0.47	0.53	1
9	1	1	1	0.51	0.41	0.44	1	1	1	0.62	0.50	0.55	1	1	1	0.62	0.50	0.55	1
10	1	1	1	0.69	0.47	0.57	1	1	1	0.58	0.50	0.55	1	1	1	0.58	0.50	0.55	1

m	n	s	No. of ILP_k solved			Time (in sec)		
			max	min	mean	max	min	mean
20	25	3	25	1	3.5	3.37	0.13	0.51
20	30	3	14	1	3.1	2.25	0.14	0.49
20	35	3	3	1	1.2	0.33	0.12	0.16
25	25	3	23	1	3.7	1.96	0.11	0.34
25	30	3	30	1	3.9	3.26	0.12	0.44
25	35	3	3	1	1.2	0.23	0.12	0.14
60	80	3	1	1	1	0.51	0.43	0.46
80	100	3	1	1	1	0.81	0.69	0.76
100	120	3	1	1	1	2.39	1.00	1.45
20	25	5	2	1	1.1	0.19	0.11	0.13
20	30	5	2	1	1.1	0.21	0.11	0.14
20	35	5	1	1	1	0.15	0.12	0.13
25	25	5	1	1	1	0.17	0.10	0.14
25	30	5	1	1	1	0.14	0.12	0.12
25	35	5	1	1	1	0.14	0.13	0.13
60	80	5	1	1	1	0.65	0.42	0.49
80	100	5	1	1	1	0.78	0.68	0.71
100	120	5	1	1	1	2.00	1.06	1.38
20	25	7	1	1	1	0.27	0.15	0.19
20	30	7	1	1	1	0.31	0.16	0.20
20	35	7	1	1	1	0.23	0.16	0.19
25	25	7	1	1	1	0.25	0.14	0.17
25	30	7	1	1	1	0.20	0.15	0.17
25	35	7	12	1	2.1	1.77	0.16	0.34
60	80	7	1	1	1	0.78	0.55	0.69
80	100	7	1	1	1	1.57	1.02	1.27
100	120	7	1	1	1	2.37	1.88	2.08

Table 5.5: Processing time to solve ILP_E for different values of m, n and s using Algorithm 7

The Algorithm 7 solves ILP_E by successively optimizing ILP_k until we get a first efficient

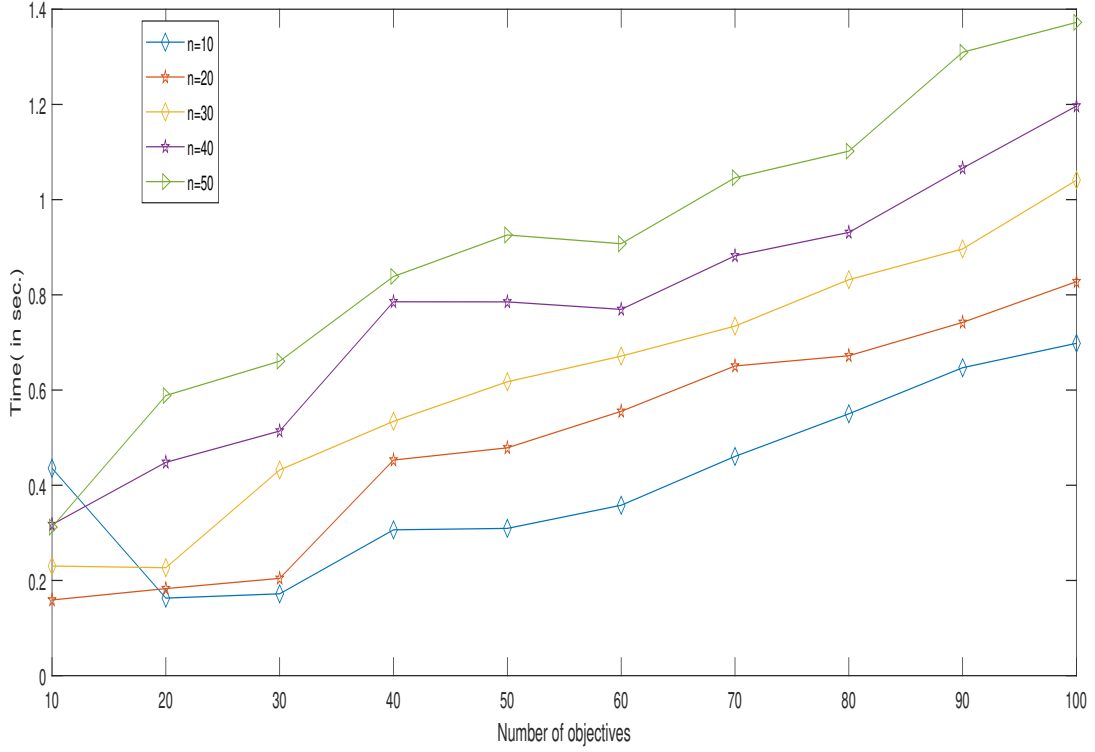


Figure 5.1: Processing time to solve ILP_E for different values of n and s using Algorithm7

$m = 5, n = 3$			$m = 5, n = 4$			$m = 5, n = 5$		
	Time (in sec) using Algorithm 7	Time (in sec) using Algorithm 8	Time (in sec) using Algorithm 7	Time (in sec) using Algorithm 8	Time (in sec) using Algorithm 7	Time (in sec) using Algorithm 8		
s	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean
3	(2.210, 0.004) 0.241	(2.716, 0.009) 0.365	(0.042, 0.004) 0.017	(0.034, 0.006) 0.015	(0.116, 0.006) 0.037	(0.547, 0.011) 0.114		
4	(0.216, 0.005) 0.032	(0.101, 0.007) 0.030	(0.124, 0.006) 0.031	(0.110, 0.005) 0.025	(0.483, 0.005) 0.110	(0.398, 0.009) 0.078		
5	(0.074, 0.004) 0.014	(0.278, 0.006) 0.037	(0.082, 0.003) 0.016	(0.225, 0.009) 0.066	(0.097, 0.006) 0.028	(0.226, 0.007) 0.067		
6	(0.020, 0.004) 0.006	(0.129, 0.007) 0.026	(0.106, 0.004) 0.019	(0.074, 0.007) 0.026	(0.692, 0.004) 0.101	(0.870, 0.007) 0.135		
7	(0.029, 0.004) 0.008	(0.038, 0.006) 0.012	(0.018, 0.004) 0.007	(0.241, 0.008) 0.033	(0.095, 0.004) 0.018	(0.024, 0.007) 0.011		
8	(0.014, 0.004) 0.007	(0.022, 0.007) 0.013	(0.007, 0.004) 0.005	(0.045, 0.006) 0.013	(0.159, 0.005) 0.055	(0.310, 0.008) 0.081		
9	(0.018, 0.004) 0.006	(0.181, 0.006) 0.026	(0.167, 0.004) 0.022	(1.967, 0.007) 0.345	(0.019, 0.005) 0.008	(0.110, 0.011) 0.022		
10	(0.006, 0.004) 0.005	(0.020, 0.007) 0.011	(0.104, 0.006) 0.018	(0.108, 0.007) 0.021	(0.162, 0.005) 0.024	(0.061, 0.007) 0.019		
$m = 10, n = 3$			$m = 10, n = 4$			$m = 10, n = 5$		
	Time (in sec) using Algorithm 7	Time (in sec) using Algorithm 8	Time (in sec) using Algorithm 7	Time (in sec) using Algorithm 8	Time (in sec) using Algorithm 7	Time (in sec) using Algorithm 8		
s	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean	(max, min) mean
3	(0.154, 0.004) 0.034	(0.184, 0.010) 0.048	(0.173, 0.005) 0.063	(0.052, 0.007) 0.019	(0.202, 0.013) 0.058	(0.099, 0.006) 0.034		
4	(0.043, 0.004) 0.011	(0.072, 0.006) 0.022	(0.059, 0.004) 0.019	(0.117, 0.007) 0.029	(0.100, 0.007) 0.024	(0.080, 0.011) 0.030		
5	(0.073, 0.004) 0.016	(0.050, 0.007) 0.018	(0.096, 0.004) 0.023	(0.056, 0.007) 0.016	(0.121, 0.006) 0.032	(0.189, 0.007) 0.037		
6	(0.034, 0.004) 0.011	(0.022, 0.006) 0.011	(0.036, 0.005) 0.009	(0.072, 0.008) 0.020	(0.061, 0.007) 0.024	(0.263, 0.007) 0.039		
7	(0.156, 0.005) 0.025	(0.028, 0.007) 0.013	(0.038, 0.004) 0.010	(0.042, 0.007) 0.014	(0.077, 0.005) 0.021	(0.053, 0.009) 0.016		
8	(0.009, 0.006) 0.007	(0.015, 0.008) 0.012	(0.034, 0.005) 0.012	(0.117, 0.012) 0.031	(0.068, 0.007) 0.024	(0.019, 0.007) 0.013		
9	(0.009, 0.006) 0.008	(0.014, 0.008) 0.012	(0.008, 0.005) 0.006	(0.013, 0.010) 0.012	(0.010, 0.005) 0.006	(0.022, 0.008) 0.014		
10	(0.039, 0.004) 0.010	(0.012, 0.007) 0.010	(0.090, 0.005) 0.015	(0.044, 0.013) 0.020	(0.142, 0.007) 0.039	(0.089, 0.008) 0.020		

Table 5.6: Processing time to solve ILP_E for different values of m, n and s using Algorithm 7 and Algorithm 8

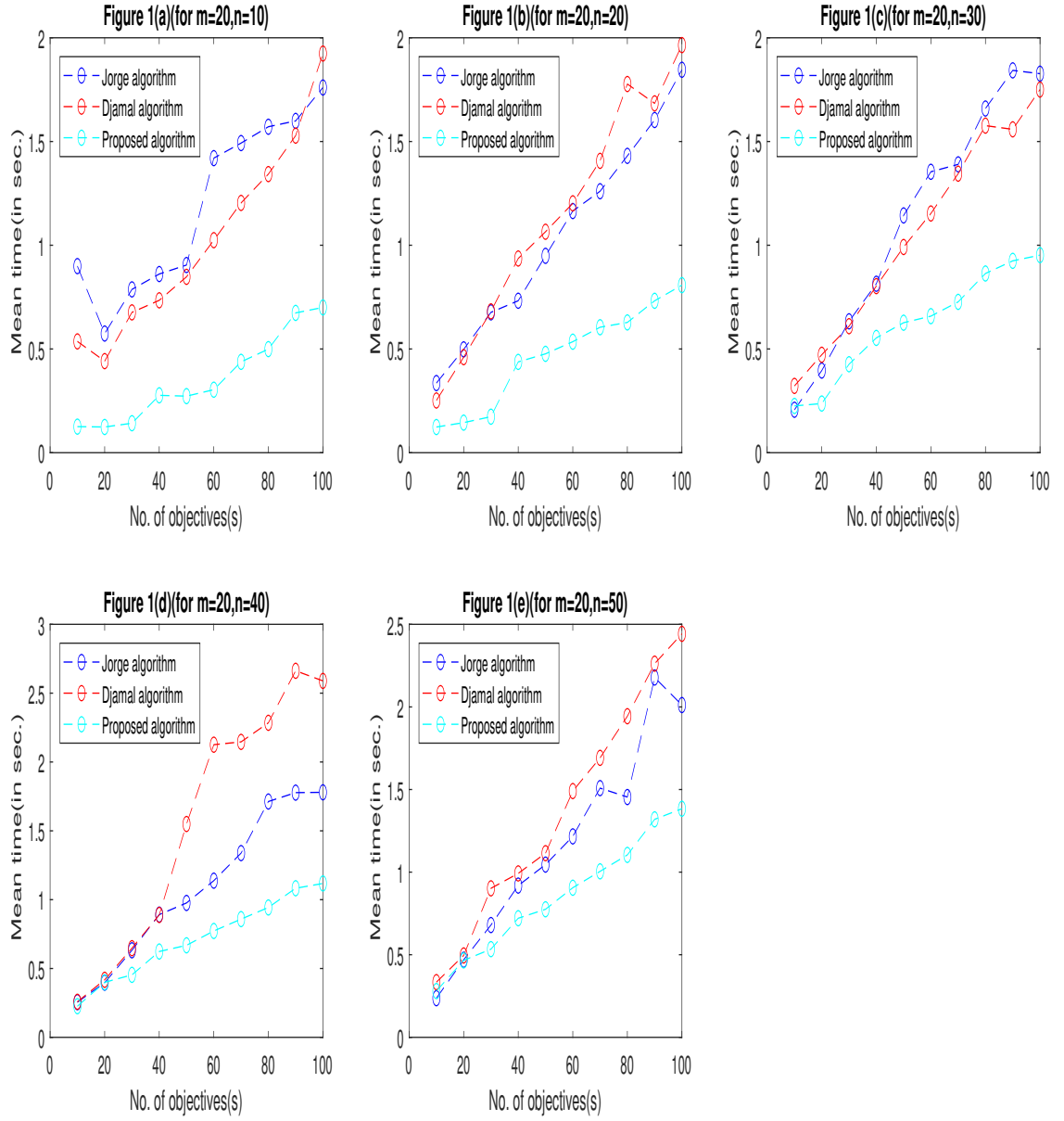


Figure 5.2: Processing time to solve ILP_E for different values of (m, n, s) using Jorge's [80] algorithm, Djamel and Marc's [45] algorithm and Algorithm 7

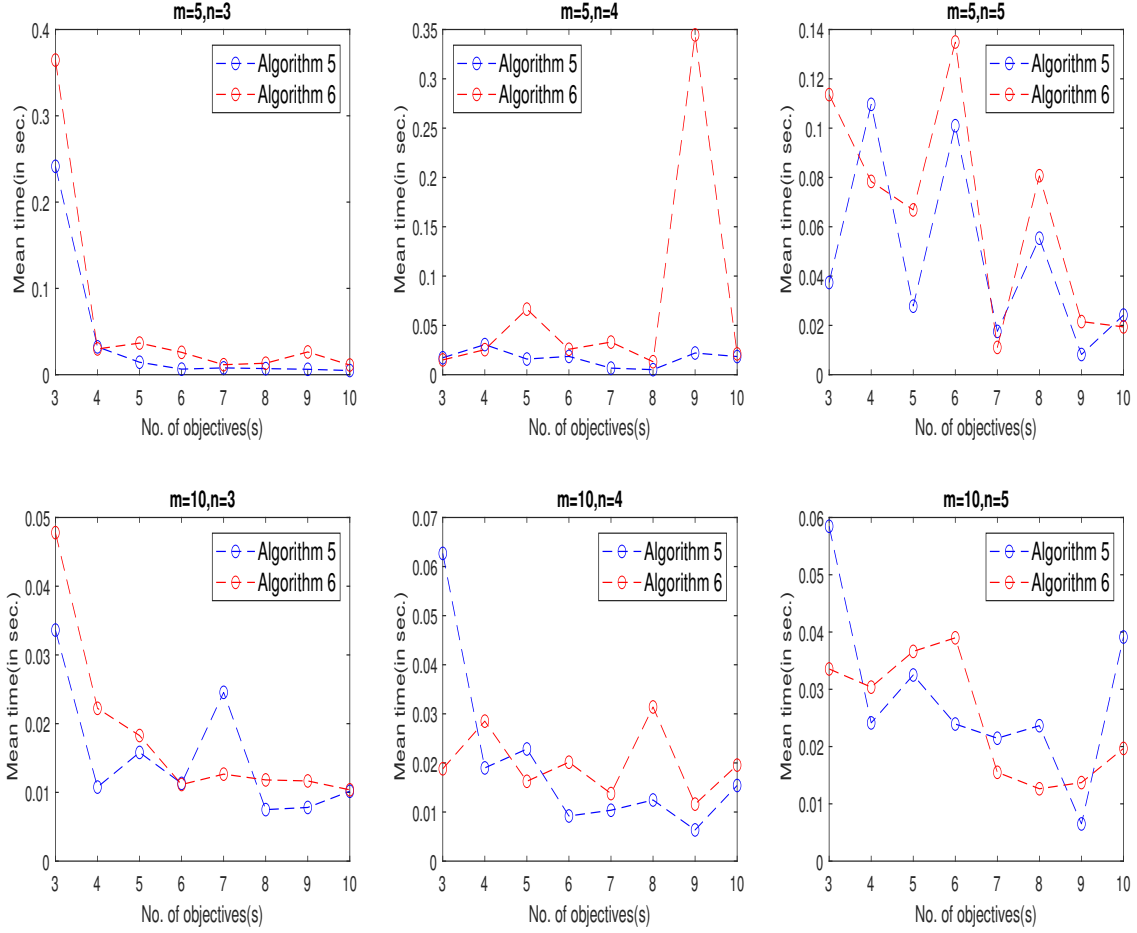


Figure 5.3: Processing time to solve ILP_E for different values of (m, n, s) using Algorithm 7 and Algorithm 8

solution. The number of ILP_k problems constructed to find an optimal set of ILP_E for different values of m, n , and s using the proposed algorithm is shown in Table 5.5. The Algorithm 7 stops after finding a first efficient solution, and number of efficient solutions of $MILP$ increases on increasing the number of objectives of $MILP$. Therefore, on increasing number of objectives (s) of $MILP$ lesser number of ILP_k problems are required to find an optimal solution of ILP_E (which can also be observed from Table 5.5). Further, it is also evident from Table 5.5 that on increasing the number of constraints (m), the number of ILP_k constructed to solve ILP_E decreases as the feasible region reduces on increasing the constraints of ILP_E .

The Figure 5.1 depicts the impact of the number of variables and number of objectives on processing time to solve ILP_E using Algorithm 7. It can be observed that on increasing the number of variables (n) and the number of objectives (s) of $MILP$, the processing

time to solve ILP_E also increases.

The Figure 5.2 shows the processing time to solve ILP_E for different values of m, n , and s using Jorge's [80] algorithm, Djamel and Marc's [45] algorithm and Algorithm 7. It can be observed from Figure 5.2 that the mean processing time to solve ILP_E for different values of m, n , and s using Algorithm 7 is lesser than the mean processing time to solve ILP_E using Jorge's [80] and Djamel and Marc's [45] algorithms. So, the proposed algorithm is computationally better than Jorge's [80] and Djamel and Marc's [45] algorithms.

The Table 5.6 and Figure 5.3 show the processing time to solve ILP_E for different values of m, n , and s using Algorithm 7 and Algorithm 8. It can be observed from Figure 5.3 and Table 5.6 that mostly mean processing time to solve ILP_E for different values of (m, n, s) using Algorithm 7 is lesser than the mean processing time to solve ILP_E using Algorithm 8.

5.6 Concluding Remarks

This chapter presents two algorithms for solving ILP_E . In these algorithms, we have used ranked solutions of ILP_1 to solve ILP_E . We have utilized the successive optimization of integer linear programming problems to enumerate ranked solutions of ILP_1 . The proposed methods avoid the enumeration of all efficient solutions of $MILP$ and all ranked solutions of ILP_1 . We have also shown that the proposed algorithms are more efficient than the existing algorithms [2, 45, 80, 94] for solving ILP_E . In Jorge's [80] and Djamel and Marc's [45] algorithms $(2s + 1)(k - 1)$ number of new corner constraints and $s(k - 1)$ number of new variables added into the problems at each k^{th} iteration, but in our algorithms, only a single constraint is added into the problem at each iteration. Further, in the Abbas and Chaabane [2] algorithm, four types of cuts (Type 1 cut, Dantzig cut, Type 2 cut, and Gomory cut) are used. However, the proposed algorithms use only an objective function cut to solve ILP_E . Computational results also show that the processing time to solve ILP_E using Algorithm 7 is lesser than the processing time to solve ILP_E using Jorge's [80] and Djamel and Marc's [45] algorithms. The Algorithm 7 solves ILP_E by successively optimizing ILP_k until we get an efficient solution. Since efficient solutions of $MILP$ increases by increasing objectives of $MILP$, fewer ILP_k problems are required for solving ILP_E . The proposed methodology can be further extended to the case when decision variables are bounded.

Chapter 6

Optimization of a quadratic function over efficient set

6.1 Introduction

Recently, there has been increasing interest in solving integer quadratic programming problem (*IQP*) due to wide applicability of *IQP* in various fields such as finance, engineering, and computer science. An application of mixed-integer quadratic programming in the context of design and analysis of an aging-aware energy management system is discussed in [85]. The usage of mixed-integer quadratic programming approach in the generation of units maintenance in combined heat and power integrated systems can be seen in [127]. An application of quadratic programming in portfolio selection can be observed in [3, 103]. Several recent papers, including the work by Bentobache, Telli and Mokhtari [18, 19], have proposed new LP-based local and global algorithm and sequential *LP* algorithm for continuous and mixed-integer nonconvex *QP*. A global solution of non-convex standard *QP* via mixed *ILP* reformulations is described in [63], and a graph-based decomposition method for mixed integer convex *QP* is presented in [92]. A ranking approach is used in [87] for solving *MIQP*.

As far as the research on methods for solving *QP* over the efficient set of *MILP* is concerned, there is not much work done in this direction and to the best of our knowledge, it is only discussed in [106]. Moulai and Drici [106] considered *IdQP* over efficient set of *MILP*. They used a simplex-like technique for solving *IdQP* over the relaxed polyhedron region. Then by using some efficient cuts along with the branch and bound method for *IIdQP*, they solved *IdQP* over the set of efficient solutions of a *MILP*.

The methodology discussed by Moulai and Drici [106] does not solve the problem. A counterexample that justifies this claim is presented in the numerical section. The reason behind the failure of their method is that they assumed that the optimal solution of *IdQP* always occurs at a corner point of the feasible region of a compact polyhedron region,

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which is not valid. A complete discussion regarding the programming problems that find their optima at a vertex of a polyhedron region is given by Martos [101] in his landmark paper. He explained that the only a class of quasi-monotone functions find their optimal solution at a vertex of a polyhedron region. The indefinite quadratic function considered by Moulai and Drici [106] fails to have both of these properties, so the simplex-like technique discussed by Moulai and Drici [106] in their paper did not work. A similar discussion about the failure of obtaining an optimal solution of *IdQP* and *IdQFP* over a convex polyhedron region has also been done in [86, 87].

The quadratic programming problem over the efficient set has unquestionable practical applications. For instance, consider a cardinality-constrained portfolio optimization problem. Suppose portfolio $V = (w_1, w_2, \dots, w_n)$ held for a fixed period of time t . At the end of this period, let $r_1, r_2, \dots, r_n$ and $\sigma_1, \sigma_2, \dots, \sigma_n$ be the return and standard deviation of assets $a_1, a_2, \dots, a_n$ respectively. In the classical Markowitz model [100], the portfolio selection problem is defined as, the problem of minimizing the portfolio's risk and at the same time maximizing the portfolio's expected return. The portfolio risk can be measured as the variance of the portfolio's return, which is a quadratic function ($\sigma = w^T C w$, where $w^T = (w_1, w_2, \dots, w_n)$, $C = [c_{i,j}]$, $c_{i,j} = \text{cov}(r_i, r_j)$ and $c_{i,i} = \sigma_i^2$ for all $i \in \{1, 2, \dots, n\}$) and portfolio's expected return is a linear function ($\mu = m^T w$, where $m^T = (\mu_1, \mu_2, \dots, \mu_n)$) as explained in [29]. The classical Markowitz model doesn't cover the issue of transaction costs, which is an important component to be considered, especially in today's era of high-frequency trading and during the continuous re-balancing of the portfolio due to the dynamic market scenarios. Nowadays, several researchers [28, 93, 99] have focused on minimizing transaction cost, in addition to minimizing the portfolio's risk and maximizing expected return. The transaction cost (T_i) of each asset a_i is proportional to the traded amount (i.e., $T_i = \bar{C} \alpha_i w_i$, where $\bar{C}$ represents the total amount invested in n assets and α_i denotes the cost rate associated with buying asset a_i for $i \in \{1, 2, \dots, n\}$). Therefore, the transaction cost (T) of the portfolio is a linear function ($T = \bar{C} \alpha^T w$, where $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_n)$). As large number of assets in the portfolio contribute to large transaction and monitoring cost, therefore many times a cardinality constraint ($\sum_{i=1}^n \phi_i \leq k$, where $\phi_i \in \{0, 1\}$) is even imposed over the assets of the portfolio. Moreover, in order to ensure that the proportion of the total investment held in each asset a_i is within specified limits (l_i and u_i), it's required to incorporate the threshold constraints ($\phi_i l_i \leq w_i \leq \phi_i u_i$) for each asset a_i , as explained in [99, 137]. Keeping in view these constraints of the portfolio, the cardinality-constrained portfolio optimization problem modeled as follows:

$$(IQP_E) \quad \min_{(\phi, w) \in E} w^T C w,$$

where E is the set of all efficient solutions to the following problem :

$$\begin{cases} \max (m^T w, -\bar{C}\alpha^T w) \\ \text{such that} \\ \sum_{i=1}^n w_i = 1, \sum_{i=1}^n \phi_i \leq k, \\ \phi_i l_i \leq w_i \leq \phi_i u_i, \phi_i \in \{0, 1\} \forall i \in \{1, 2, \dots, n\}. \end{cases}$$

Due to the practical importance of optimizing QP over the efficient set and limitation in existing method [106] for solving this type of problem, we propose a method for solving IQP_E . The proposed methodology solves the problem by enumerating ranked solutions of IQP until a ranked solution satisfies the efficiency test. Although the quadratic function is non-linear, no non-linear optimization technique is used at any stage of the developed method. The methodology to rank integer solutions of IQP is adopted from Gupta et al. [67], discussed for the minimization case. To tackle IQP , a linear function and related integer linear programming problem (ILP) is considered. Further, by appending an objective function cut to ILP , ranked solutions of ILP are enumerated. These ranked solutions are utilized for finding ranked solutions of IQP . Theoretically, we claim that one of the ranked solutions of IQP will be an efficient solution of $MILP$. So, our proposed method terminates in finite steps. Notably, using the developed methodology, we can solve IQP_E without finding all efficient solutions of $MILP$ and all feasible solutions of IQP .

In the next section, all notations and preliminary results are discussed, then all the theoretical results are established in Section 3. The methodology for solving IQP_E is elaborated in Section 4. Numerical results, computational results, and comparison with the existing algorithms method are made in the Section 5. Some concluding results are discussed in the last section.

6.2 Notations and Preliminary results

Consider the following problem

$$(MILP) \quad \max_{X \in \mathcal{X}} (C^1 X, C^2 X, \dots, C^s X),$$

where $\mathcal{X} = S \cap \mathbb{Z}^n$ and $C^i \in \mathbb{Z}^{1 \times n}$ for all $i \in \{1, 2, \dots, s\}$. For a point $\hat{Y} \in \mathcal{X}$ to be an efficient solution of $MILP$, $\hat{Y}$ must satisfies the efficiency test discussed by Isermann [73].

Theorem 6.2.1 *A point $\hat{Y} \in \mathcal{X}$ is an efficient solution of $MILP$ iff optimal value of $P(\hat{Y})$*

is zero.

$$P(\hat{Y}) \quad \begin{cases} \max & \theta = \sum_{i=1}^s \Psi_i \\ \text{s.t} & C^i X = I \Psi_i + C^i \hat{Y}, \\ & X \in \chi, \\ & \Psi_i \geq 0 \quad \forall i \in \{1, 2, \dots, s\}, \end{cases}$$

where $\Psi_i \in \mathbb{R} \quad \forall i \in \{1, 2, \dots, s\}$, $I = (s \times s)$ identity matrix.

Theorem 6.2.2 *If the feasible region χ of MILP is non-empty and bounded set, then the efficient set E of MILP is also non-empty.*

Proof Consider the problem as

$$(LP) \quad Z_1(X) = \max_{X \in \chi} C^1 X.$$

As feasible region χ is a non-empty and bounded set, therefore LP will always has an optimal solution over χ . Suppose W be the unique optimal solution of LP . Therefore,

$$C^1 W > C^1 X \text{ for all } X \in \chi. \quad (6.2.1)$$

If possible let W be not an efficient solution of $MILP$ therefore there exists $\hat{X} \in \chi$ such that

$$\begin{aligned} C^i \hat{X} &\geq C^i W \quad \forall i \in \{1, 2, \dots, s\}, \\ C^i \hat{X} &\neq C^i W \text{ for atleast one } i \in \{1, 2, \dots, s\} \end{aligned} \quad (6.2.2)$$

which contradict (6.2.1). So W is an efficient solution of $MILP$. Now, suppose that LP doesn't have a unique optimal solution, then define

$$\begin{aligned} R_1 &= \{Y \in \chi \mid C^1 Y = \max_{X \in \chi} C^1 X\} \\ R_i &= \{Y \in R_{i-1} \mid C^i Y = \max_{X \in R_{i-1}} C^i X\} \text{ for all } i \in \{2, 3, \dots, s\}. \end{aligned} \quad (6.2.3)$$

Clearly, $R_s \subset R_{s-1} \cdots \subset R_2 \subset R_1$. feasible region χ is a non-empty and bounded set, therefore $R_i \neq \emptyset \quad \forall i \in \{1, 2, \dots, s\}$. Suppose a point $\bar{X} \in R_s$, therefore $\bar{X} \in R_i$ for all $i \in \{1, 2, \dots, s\}$. If possible let $\bar{X}$ be not an efficient solution of $MILP$, which implies there exists a feasible solution $\bar{Y}$ such that

$$\begin{aligned} C^i \bar{Y} &\geq C^i \bar{X} \quad \forall i \in \{1, 2, \dots, s\}, \\ C^i \bar{Y} &\neq C^i \bar{X} \text{ for atleast one } i \in \{1, 2, \dots, s\}. \end{aligned} \quad (6.2.4)$$

Since $C^1\bar{Y} \geq C^1\bar{X}$ and $\bar{X} \in R_1$ therefore $\bar{Y} \in R_1$ and $C^1\bar{Y} = C^1\bar{X}$. As $\bar{Y} \in R_1$, $\bar{X} \in R_2$ and $C^2\bar{Y} \geq C^2\bar{X}$ therefore $\bar{Y} \in R_2$ and $C^2\bar{Y} = C^2\bar{X}$. Similarly, we can show $\bar{Y} \in R_i$ and $C^i\bar{Y} = C^i\bar{X}$ for all $i \in \{3, 4, \dots, s\}$. We conclude that $C^i\bar{Y} = C^i\bar{X}$ for all $i \in \{1, 2, \dots, s\}$, which contradict Equation (6.2.4). Therefore $\bar{X}$ is an efficient solution of *MILP*. Thus, the efficient set E of *MILP* is non-empty.

Now, we aim to solve the problem, as follows:

$$(IQP_E) \quad \max_{X \in \chi_E} h(X),$$

where $h(X) = X^T B X + U X + \alpha$, $\chi_E = \chi \cap E$, $B \in \mathbb{Z}^{n \times n}$, $U \in \mathbb{Z}^{1 \times n}$ and $\alpha \in \mathbb{R}$.

Let T_E be complete set of optimal solutions of IQP_E . For finding optimal solutions of IQP_E , we first consider the problem, defined as:

$$(IQP) \quad \max_{X \in \chi} h(X).$$

Let X^k for $k \in \{1, 2, \dots, N\}$ be a k^{th} ranked solution of IQP such that $h(X^{k-1}) > h(X^k)$, where

$$h(X^1) = \max_{X \in \chi} h(X), \quad h(X^N) = \min_{X \in \chi} h(X).$$

Let T^k be the set of all k^{th} ranked solutions of IQP , i.e.,

$$T^k = \{Y \in \chi \mid h(Y) = h(X^k)\}.$$

Therefore,

$$\chi = T^1 \cup T^2 \cup T^3 \cup \dots \cup T^N,$$

which implies

$$\begin{aligned} \chi_E &= (T^1 \cup T^2 \cup T^3 \cup \dots \cup T^N) \cap E \\ &= (T^1 \cap E) \cup (T^2 \cap E) \cup \dots \cup (T^N \cap E). \end{aligned}$$

If T_E^k denote the collection of all k^{th} ranked solutions of IQP , which are also efficient solutions of *MILP*, i.e., $T_E^k = T^k \cap E$ for $k \in \{1, 2, \dots, N\}$. Then,

$$\chi_E = T_E^1 \cup T_E^2 \cup T_E^3 \cup \dots \cup T_E^N. \quad (6.2.5)$$

For finding the set T^k , we construct the following integer linear programming (*ILP*), as

explained in [67].

$$(ILP) \quad \max_{X \in \chi} g(X),$$

where $g(X) = VX + UX + \alpha$, $V \in \mathbb{Z}^{1 \times n}$. Let B_j be the j^{th} column of the matrix B , then the j^{th} column V_j of V is defined as

$$V_j = \max_{X \in \chi} X^T B_j \quad \forall j \in \{1, 2, \dots, n\}. \quad (6.2.6)$$

Also, we can see,

$$\begin{aligned} V_j &\geq X^T B_j \quad \forall X \in \chi, j \in \{1, 2, \dots, n\} \\ \implies VX &\geq X^T BX \quad \forall X \in \chi \\ \implies VX + UX + \alpha &\geq X^T BX + UX + \alpha \quad \forall X \in \chi \\ \text{i.e. } g(X) &\geq h(X) \quad \forall X \in \chi. \end{aligned} \quad (6.2.7)$$

It may be observed from Equation 6.2.7, the objective function values of ILP provides an upper bound on objective function values of IQP for all feasible solutions. So, feasible solutions of ILP can be utilized for finding optimal and ranked solutions of IQP by systematically shrinking this bound.

Let Y^l for $l \in \{1, 2, \dots, p\}$ be a l^{th} ranked solution of ILP and $Z^l = g(Y^l)$ such that $Z^{l-1} > Z^l$, where

$$Z^1 = g(y^1) = \max_{X \in \chi} g(X), \quad Z^p = g(Y^p) = \min_{X \in \chi} g(X).$$

It can further be inferred that Y^l is an optimal solution of the problem, defined as following integer :

$$(ILP_l) \quad \max_{X \in \chi_l} g(X),$$

where $\chi_l = \{X \in \chi \mid g(X) \leq Z^{l-1} - 1\}$ and the set of all l^{th} ranked solutions of ILP can be written as:

$$L^l = \{w \in \chi \mid g(W) = g(Y^l)\}.$$

In the following section, we will discuss how these ranked solutions systematically shrink the upper bound and it will converge to an optimal solution of IQP_E .

6.3 Theoretical development

To find the set of all optimal solutions of IQP_E , we consider sets T_E^k for all $k \in \{1, 2, 3, \dots, N\}$. In the following theorem, we will discuss the existence of a non-empty set T_E^k for some $k \in \{1, 2, 3, \dots, N\}$.

Theorem 6.3.1 *If the feasible region χ of MILP is non-empty and bounded, then $T_E^k \neq \emptyset$ for some $k \in \{1, 2, 3, \dots, N\}$.*

Proof As feasible region χ of IQP is non-empty and efficient set E of $MILP$ is also non empty (by Theorem 6.2.2), therefore $\chi_E \neq \emptyset$. Since $\chi_E \neq \emptyset$ and $\chi_E = \cup_{k=1}^N T_E^k$, therefore $T_E^k \neq \emptyset$ for some $k \in \{1, 2, \dots, N\}$.

Now, we will establish the condition for a point $X^k \in T_E^k$ for some $k \in \{1, 2, \dots, N\}$ to be an optimal solution of IQP_E .

Theorem 6.3.2 *If $\cup_{i=1}^{k-1} T_E^i = \emptyset$ and $T_E^k \neq \emptyset$ for some $k \in \{1, 2, \dots, N\}$, then T_E^k is the set of all optimal solution of IQP_E .*

Proof Let T_E be the set of all optimal feasible solution of IQP_E and let $Y \in T_E$, therefore Y is a feasible solution of IQP_E . Then Equation (6.2.5) implies that $Y \in T_E^i$ for some $i \in \{1, 2, \dots, N\}$. If possible let $Y \notin T_E^k$. Also, $T_E^i = \emptyset$ for all $i \in \{1, 2, \dots, k-1\}$. Therefore $Y \in T_E^i$ for some $i \in \{k+1, k+2, \dots, N\}$. As $T_E^i = T^i \cap E$ therefore

$$Y \in T^i \text{ for some } i \in \{k+1, k+2, \dots, N\}. \quad (6.3.1)$$

Also, Y is an optimal solution of IQP_E therefore

$$\begin{aligned} h(Y) &\geq h(X) \text{ for all } X \in \chi_E \\ \implies h(Y) &\geq h(X) \text{ for all } X \in \cup_{i=1}^N T_E^i \quad (\text{Using eq (6.2.5)}). \end{aligned} \quad (6.3.2)$$

Since $T_E^k \neq \emptyset$, therefore there exists an integer feasible solution X^k of IQP such that $X^k \in T_E^k$. Equation (6.3.2) implies that

$$h(Y) \geq h(X^k). \quad (6.3.3)$$

Also, X^k is a k^{th} ranked integer feasible solution of IQP therefore

$$h(X^k) > h(X) \text{ for all } X \in \cup_{i=k+1}^N T_E^i. \quad (6.3.4)$$

Using (6.3.1) & (6.3.4) we get,

$$h(X^k) > h(Y),$$

which contradicts Equation (6.3.3) therefore $Y \in T_E^k$ and

$$T_E \subseteq T_E^k. \quad (6.3.5)$$

Now we want to show $T_E^k \subseteq T_E$. Let $X^k \in T_E^k$ therefore

$$h(X^k) \geq h(X) \text{ for all } X \in \cup_{i=k}^N T^i.$$

Since $\cup_{i=1}^{k-1} T^i = \emptyset$ and $\cup_{i=1}^N T_E^i \subseteq \cup_{i=1}^N T^i$, therefore

$$h(X^k) \geq h(X) \text{ for all } X \in \cup_{i=1}^N T_E^i$$

which implies

$$h(X^k) \geq h(X) \text{ for all } X \in \chi_E.$$

So, X^k is an optimal solution of IQP_E and

$$T_E^k \subseteq T_E. \quad (6.3.6)$$

By using (6.3.5) & (6.3.6) we get $T_E^k = T_E$. Hence T_E^k is the set of all optimal solution of IQP_E .

Corollary 6.3.3 *If $T_E^1 \neq \emptyset$, then T_E^1 is the set of all optimal solution of IQP_E .*

It is evident from Theorem 6.3.2 that an optimal solution of IQP_E is contained in the first non empty set T_E^k for some $k \in (1, 2, \dots, N)$. In the following discussion, we will discuss a systematic ranking scheme to find a ranked integer feasible solution of IQP . The following theorem establishes a relation between ranked solution of ILP and ranked solution of IQP . For the minimization case, one can refer [67], but for the sake of the reader's convenience, we have rewritten its proof for the maximization case.

Theorem 6.3.4 *If $g(Y) \leq \max \{h(X) \mid X \in \cup_{j=1}^l L^j\} = h(\hat{X})$ for some $Y \in L^l$ then $\hat{X}$ is an optimal solution of IQP .*

Proof Since $Y \in L^l$, therefore

$$g(Y) > g(X) \text{ for all } X \in L^j, j > l.$$

By hypothesis $g(Y) \leq h(\hat{X})$, therefore

$$h(\hat{X}) > g(X) \text{ for all } X \in L^j, j > l \quad (6.3.7)$$

$$\text{Also, } g(X) \geq h(X) \text{ for all } X \in \chi. \quad (6.3.8)$$

By equations (6.3.7) and (6.3.8)

$$h(\hat{X}) > h(X) \text{ for all } X \in L^j, j > l$$

$$\text{Also, } h(\hat{X}) \geq h(X) \text{ for all } X \in \cup_{j=1}^l L^j$$

$$\therefore h(\hat{X}) \geq h(X) \text{ for all } X \in \chi.$$

Thus, $\hat{X}$ is an optimal solution of IQP .

Corollary 6.3.5 *If $g(Y) \leq \max \{h(X) \mid X \in L^1\} = h(\hat{X})$ for some $Y \in L^1$, then $\hat{X}$ is an optimal solution of IQP .*

Corollary 6.3.6 *If there exist some $Y \in L^l$ such that $g(Y) \leq \max \{h(X) \mid X \in \cup_{j=1}^l L^j\} = h(\hat{X})$, then $\hat{T} = \{X \in \cup_{j=1}^l L^j \mid h(X) = h(\hat{X})\}$ is the complete set of optimal solution of IQP .*

Proof We know that the set of all optimal solution of IQP is defined as T^1 . So we need to show $\hat{T} = T^1$. Since $\hat{T} = \{X \in \cup_{j=1}^l L^j \mid h(X) = h(\hat{X})\}$ and $\hat{X}$ is an optimal solution of IQP (by Theorem 6.3.4). Therefore

$$\hat{T} \subseteq T^1. \quad (6.3.9)$$

Now let $X^1 \in T^1$, then

$$h(X^1) = h(\hat{X}) \geq h(X) \text{ for all } X \in \chi. \quad (6.3.10)$$

If possible let $X^1 \notin \hat{T}$ which implies $X^1 \notin \cup_{j=1}^l L^j$. As $X^1 \notin \cup_{j=1}^l L^j$ and $Y \in L^l$ therefore

$$\begin{aligned} & g(Y) > g(X^1) \\ \implies & h(\hat{X}) > g(X^1) \quad (\because h(\hat{X}) \geq g(Y)) \\ \implies & h(\hat{X}) > h(X^1) \quad (\because g(X) \geq h(X) \text{ for all } X \in \chi) \end{aligned}$$

which contradict equation (6.3.10). Thus,

$$T^1 \subseteq \hat{T}. \quad (6.3.11)$$

Hence $T^1 = \{X \in \cup_{j=1}^l L^j \mid h(X) = h(\hat{X})\}$ (by Equation (6.3.9) and Equation (6.3.11)).

Remark 6.3.7 By Corollary 6.3.6, we conclude that in order to find the set of all optimal solutions of IQP , we do not need to rank ILP beyond l^{th} ranked solution of ILP provided there exist some $Y \in L^l$ such that $g(Y) \leq \max \{h(X) \mid X \in \cup_{j=1}^l L^j\}$.

In order to find an optimal solution of IQP_E , we have to find all ranked solutions of IQP until a ranked solution of IQP is an efficient solution of $MILP$. So, to find ranked solutions of IQP , we consider the problem, as follows:

$$(IQP_k) \quad \begin{cases} \max h(X) \\ \text{such that } X \in \chi \\ h(X) < h(X^{k-1}) \text{ for some } X^{k-1} \in T^{k-1}. \end{cases}$$

Remark 6.3.8 It can be seen that the k^{th} ranked solution of IQP denoted by X^k is an optimal solution of IQP_k .

The following result establishes a relation between k^{th} ranked solution of IQP and j^{th} ranked solution of ILP . A similar result is proved in [67] for the minimization case. Here we have given a different proof.

Theorem 6.3.9 If $g(Y) \leq \max \{h(X) \mid h(X) < h(X^{k-1}), X \in \cup_{j=1}^l L^j\} = h(\bar{X})$ for some $Y \in L^l$, $X^{k-1} \in T^{k-1}$ then $\bar{X}$ is the k^{th} ranked solution of IQP .

Proof If possible let $\bar{X}$ be not a k^{th} ranked solution of IQP therefore $\bar{X}$ is not an optimal solution of IQP_k . By hypothesis $\bar{X} \in \cup_{j=1}^l L^j$ and $h(\bar{X}) < h(X^{k-1})$ for some $X^{k-1} \in T^{k-1}$, therefore $\bar{X}$ is a feasible solution of IQP_k . Since $\bar{X}$ is a feasible solution but not an optimal solution of IQP_k therefore there exists a feasible solution $\check{X}$ of IQP_k such that

$$h(\check{X}) < h(X^{k-1}) \text{ and } h(\check{X}) > h(\bar{X}). \quad (6.3.12)$$

Now, two cases arises either $\check{X} \in \cup_{j=1}^l L^j$ or $\check{X} \in L^j$ for some $j > l$

If possible let $\check{X} \in \cup_{j=1}^l L^j$. Since $\check{X} \in \cup_{j=1}^l L^j$ and $h(\check{X}) < h(X^{k-1})$ therefore by hypothesis

$$h(\bar{X}) \geq h(\check{X}). \quad (6.3.13)$$

Equation (6.3.13) contradict Equation (6.3.12). So $\check{X} \in L^j$ for some $j > l$. As $Y \in L^l$ and $\check{X} \in L^j$ for some $j > l$ therefore $g(Y) > g(\check{X})$, Also $h(\bar{X}) \geq g(Y)$. Therefore,

$$h(\bar{X}) > g(\check{X}). \quad (6.3.14)$$

As the function g provides upper bound to the values of function h for all $X \in \chi$, therefore

$$g(\check{X}) > h(\check{X}). \quad (6.3.15)$$

By using Equation (6.3.14) and Equation (6.3.15), we get

$$h(\bar{X}) > h(\check{X}) \quad (6.3.16)$$

which contradict Equation (6.3.12). Thus, $\bar{X}$ is the k^{th} ranked solution of IQP .

Corollary 6.3.10 *If $g(Y) \leq \max \{h(X) \mid h(X) < h(X^{k-1}), X \in \cup_{j=1}^l L^j\} = h(\bar{X})$ for some $X^{k-1} \in T^{k-1}$, $Y \in L^l$, then $T^k = \{X \in \cup_{j=1}^l L^j \mid h(X) = h(\bar{X})\}$.*

6.4 Methodology for solving IQP_E

Now, we will discuss a methodology for solving IQP_E .

To solve IQP_E , we consider a relaxed problem (IQP). In order to handle the non-linear objective function $h(X)$ of IQP , a linear function $g(X)$ and the related ILP (using Equation 6.2.6) is constructed. Then, the integer solutions of ILP is ranked by solving ILP_l . The ILP_l is constructed from ILP by introducing an objective function cut to the original feasible region. An optimal solution of ILP_l provides a ranked solution of ILP , and by exploring edges incident to the obtained ranked solution, all alternate ranked solutions are enumerated (as discussed in [120]). After finding ranked solutions of ILP , these ranked solutions are utilized for finding the following upper bound

$$h_{max}^1 = \max\{h(X) \mid X \in \cup_{j=1}^l L^j\}$$

This upper bound is updated at each iteration and converges to an optimal solution of IQP . In order to enumerate the set (T^1) consisting of all optimal solutions of IQP , we utilize a relation between set T^1 and sets L^j (Corollary 6.3.6). After finding set T^1 , an efficiency test (Theorem 6.2.1) is applied to each element of set T^1 , which means the problem $P(Y)$ is solved for all $y \in T^1$. If the optimal value of $P(Y)$ is zero for some $Y \in T^1$, then $Y \in T_E^1$ and T_E^1 is declared as the set of optimal solutions of IQP_E (by using Corollary 6.3.3) and algorithm terminates. Otherwise, we find next higher ranked solutions of IQP until we get a non-empty set T_E^k .

For finding a k^{th} ($k > 1$) ranked solution of IQP , the following upper bound h_{max}^k is

generated.

$$h_{max}^k = \max\{h(X) \mid h(X) < h(X^{k-1}), X \in \cup_{j=1}^l L^j\} \text{ for } X^{k-1} \in T^{k-1}$$

For constructing complete set $T^k (k > 1)$, the relationship between ranked solutions of ILP and IQP (Corollary 6.3.10) is utilized. It is evident by Corollary 6.3.6 and Corollary 6.3.10 that the set T^k (for any $k \in \{1, 2, \dots, N\}$) can be generated without finding all the ranked solutions of ILP . So, complete scanning of all feasible solutions of ILP is avoided in the enumeration process for finding set T^k . After finding set T^k , the problem $P(Y)$ is solved for all $y \in T^k$. If the optimal value of $P(Y)$ is zero for some $Y \in T^k$, then $Y \in T_E^k$ (by using Theorem 6.2.1) and T_E^k is declared as the set of optimal solutions of IQP_E (by using Corollary 6.3.3) and algorithm terminates. Otherwise, we repeat the step for higher ranked solutions until we obtain a non-empty set T_E^k .

Theorem 6.4.1 *The proposed method solves IQP_E in finite steps.*

Proof Since there are only N (finite) sets T_E^k , we conclude the algorithm stops within N iterations. By virtue of Theorem 6.3.1, we see that the $T_E^k \neq \emptyset$ for some $k \in \{1, 2, \dots, N\}$. Also, Theorem 6.3.2 implies that the first non-empty set T_E^k for some $k \in \{1, 2, \dots, N\}$ will be the set of optimal solutions of IQP_E .

6.5 Numerical and computational result

In this section, we will solve IQP_E using the proposed algorithm and will compare our results with the algorithm [106].

6.5.1 Optimization of IQP_E using proposed algorithm

To elaborate on the working of the proposed algorithm, we have considered the problem, as follows:

Example 6.5.1

$$(IQP_E) \quad \max_{X \in \chi_E} h(x, y) = 2x^2 - 8y^2 + 81x + 158y + 40,$$

where $\chi_E = \chi \cap E, \chi = \mathbb{Z}^2 \cap S, S = \{(x, y) \in \mathbb{R}^2 \mid 3y - x \geq 0, 16x + 3y \leq 51, x, y \geq 0\}$ and E is the set of all efficient solutions of $MILP$.

$$(MILP) \quad \max_{(x, y) \in \chi} (5x - 7y, -6x + 5y).$$

Algorithm 9: Algorithm to solve IQP_E

Data: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $C^i \in \mathbb{Z}^{1 \times n}$, $B \in \mathbb{Z}^{n \times n}$, $U \in \mathbb{Z}^{1 \times n}$ and $\alpha \in R$

Result: T_E = Set of all optimal solutions of IQP_E

```

1 Initialization  $k \leftarrow 1$ ,  $l \leftarrow 1$ ,  $\chi \leftarrow \{X \in \mathbb{Z}^n \mid AX = b, X \geq 0\}$ 
2  $h(X) = X^T B X + U X + \alpha$  ▷  $IQP$   $\max_{X \in \chi} h(X)$ 
3 for  $j \leftarrow 1 : n$  do
4    $V(1, j) \leftarrow \max_{X \in \chi} X^T B(n, j)$ 
5  $g(X) = V X + U X + \alpha$  ▷  $ILP$   $\max_{X \in \chi} g(X)$ 
6  $g_{min} \leftarrow \min_{X \in \chi} g(X)$ 
7  $L^1 \leftarrow \{Y \in \chi \mid g(Y) = \max_{X \in \chi} g(X)\}$  ▷ set of optimal solutions of  $ILP$ 
8  $Z^1 \leftarrow g(Y)$  for  $Y \in L^1$  ▷ optimal value of  $ILP$ 
9  $h_{max} \leftarrow \max\{h(X) \mid X \in L^1\}$ 
10 while  $g(Y) > h_{max}$  and  $g(Y) > g_{min}$  for  $Y \in L^l$  do
11    $l \leftarrow l + 1$ 
12    $\chi_l = \{X \in \chi \mid g(X) \leq Z^{l-1} - 1, Z^{l-1} = g(Y) \text{ for } Y \in L^{l-1}\}$ 
13    $L^l \leftarrow \{Y \in \chi_l \mid g(Y) = \max_{X \in \chi_l} g(X)\}$  ▷ set of  $l^{th}$  ranked solutions of  $ILP$ 
14    $Z^l \leftarrow g(Y)$  for  $Y \in L^l$ 
15    $h_{max} \leftarrow \max\{h(X) \mid X \in \cup_{j=1}^l L^j\}$ 
16  $T^1 \leftarrow \{Y \in \cup_{j=1}^l L^j \mid h(Y) = h_{max}\}$  ▷ set of optimal solutions of  $IQP$ 
17  $T_E^1 \leftarrow \emptyset$ 
18 foreach  $Y \in T^1$  do
19   Solve  $P(Y)$ 
20   if If optimal value of problem  $P(Y) = 0$  then
21      $T_E^1 \leftarrow [T_E^1 \ X]$ 
22 while  $T_E^k \leftarrow \emptyset$  do
23    $k \leftarrow k + 1$ 
24   if  $\{X \mid h(X) < h(X^{k-1}), X \in \cup_{j=1}^l L^j, X^{k-1} \in T^{k-1}\} = \emptyset$  then
25      $h_{max}^k = -\infty$ 
26   else
27      $h_{max}^k \leftarrow \max\{h(X) \mid h(X) < h(X^{k-1}), X \in \cup_{j=1}^l L^j\}$  for  $X^{k-1} \in T^{k-1}$ 
28   while  $g(Y) > h_{max}^k$  and  $g(Y) > g_{min}$  for  $Y \in L^l$  do
29      $l \leftarrow l + 1$ 
30      $\chi_l = \{X \in \chi \mid g(X) \leq Z^{l-1} - 1\}$ 
31      $L^l \leftarrow \{Y \in \chi_l \mid g(Y) = \max_{X \in \chi_l} g(X)\}$ ,  $Z^l \leftarrow g(Y)$  for  $Y \in L^l$ 
32      $h_{max}^k \leftarrow \max\{h(X) \mid h(X) < h(X^{k-1}), X \in \cup_{j=1}^l L^j\}$  for  $X^{k-1} \in T^{k-1}$ 
33    $T^k \leftarrow \{Y \in \cup_{j=1}^l L^j \mid h(Y) = h_{max}^k\}$  ▷ set of  $k^{th}$  ranked solutions of  $IQP$ 
34    $T_E^k \leftarrow \emptyset$ 
35   foreach  $Y \in T^k$  do
36     Solve  $P(Y)$ 
37     if If optimal value of problem  $P(Y) = 0$  then
38        $T_E^k \leftarrow [T_E^k \ X]$ 
39 if  $T_E^k \neq \emptyset$  then
40    $T_E \leftarrow T_E^k$  ▷ set of optimal solutions of  $IQP_E$ 

```

To solve IQP_E , we considered the problem (IQP).

$$(IQP) \quad \max_{(x,y) \in \chi} h(x,y) = 2x^2 - 8y^2 + 81x + 158y + 40.$$

Now, we will solve the problem IQP_E using proposed algorithm.

1. To find ranked solution of IQP , we have constructed the following related ILP whose objective function $g(X)$ is constructed by using the Equation 6.2.6.

$$(ILP) \quad \max_{(x,y) \in \chi} g(x,y) = 87x + 158y + 40.$$

2. Construction of set L^1 by solving ILP .

The problem (ILP) has a unique optimal solution $(0,17)$ and $g(0,17) = 2726$ and $h(0,17) = 414$. So $L^1 = \{(0,17)\}$ and $Z^1 = 2726$. As $g(0,17) > h(0,17)$. So to obtain the optimal solution of IQP , we set $l = 2$ and constructed set L^2 .

Table 6.1: Ranked solution of ILP (Example 6.5.1)

L^i	(x,y)	g(x,y)	h(x,y)
L^1	(0,17)	2726	414
L^2	(0,16)	2568	520
L^3	(0,15)	2410	610
L^4	(0,14)	2252	684
L^5	(0,13)	2094	742
L^6	(0,12)	1936	784
L^7	(1,11)	1865	893
L^8	(0,11)	1778	810
L^9	(1,10)	1707	903
L^{10}	(0,10)	1620	820
L^{11}	(1,9)	1549	897
L^{12}	(0,9)	1462	814
L^{13}	(1,8)	1391	875
L^{14}	(0,8)	1304	792
L^{15}	(1,7)	1233	837
L^{16}	(2,6)	1162	870
L^{17}	(0,7)	1146	754
L^{18}	(1,6)	1075	783
L^{19}	(2,5)	1004	800
L^{20}	(0,6)	988	700
L^{21}	(1,5)	917	713
L^{22}	(2,4)	846	714

3. Construction of set L^2 by solving ILP_2

$$(ILP_2) \quad \max_{X \in \chi_2} 87x + 158y + 40$$

where $\chi_2 = \{X \in \chi \mid 87x + 158y + 40 \leq 2725\}$

This problem has a unique optimal solution $(0,16)$ and $g(0,16) = 2568$ and $h(0,16) = 520$. So $L^2 = \{(0,16)\}$ and $Z^2 = 2568$. As $g(0,16) > \max \{h(X) \mid X \in \cup_{j=1}^2 L^j\}$. So, for finding an optimal solution of IQP , we set $l = 3$ and constructed set L^3 . Continue on the same way, we get ranked solution of ILP as shown in Table 6.1. Notably, $L^{22} = \{(2,4)\}$ and $g(2,4) = 846$. Since $g(2,4) \leq \max\{h(X) \mid X \in \cup_{j=1}^{22} L^j\} = h(\hat{X})$ for $\hat{X} = (1,10)$ therefore $\hat{X} = (1,10)$ is declared as the optimal solution of IQP . Further, $\{X \in \cup_{j=1}^{22} L^j \mid h(X) = h(\hat{X})\} = \{(1,10)\}$, so the complete set of all optimal solutions of IQP is $\{(1,10)\}$ i.e. $T^1 = \{(1,10)\}$. Next, we find the set T_E^1 by constructing problem $P(Y)$ for $Y=(1,10)$.

4. Construct the problem $P(Y)$ for $Y=(1,10)$ as

$$P(Y) \quad \begin{cases} \max & \theta = \Psi_1 + \Psi_2 \\ \text{s.t} & \\ & 5x - 7y - \Psi_1 = -65, \\ & -6x + 5y - \Psi_2 = 44, \\ & (x, y) \in \chi, \quad \Psi_1, \Psi_2 \geq 0. \end{cases}$$

As $P(Y) > 0$ for $Y=(1,10)$. So, $(1,10) \notin E$ which implies $T_E^1 = \emptyset$. Since $T_E^1 = \emptyset$. So, in order to find the optimal solution of IQP_E , we set $k = 2$ and constructed set T^2 .

5. Construction of set T^2

Since $g(2,4) \leq \max \{h(X) \mid h(X) < h(1,10), X \in \cup_{j=1}^{22} L^j\} = h(1,9)$, so $T^2 = \{X \in \cup_{j=1}^{22} L^j \mid h(X) = h(1,9)\} = \{(1,9)\}$. Next, we find the set T_E^2 by constructing the problem $P(Y)$ for $Y=(1,9)$.

6. Construct the problem $P(Y)$ for $Y=(1,9)$ as

$$P(Y) \quad \begin{cases} \max & \theta = \Psi_1 + \Psi_2 \\ \text{s.t} & \\ & 5x - 7y - \Psi_1 = -58, \\ & -6x + 5y - \Psi_2 = 39, \\ & (x, y) \in \chi, \quad \Psi_1, \Psi_2 \geq 0. \end{cases}$$

As $P(Y) > 0$ for $Y = (1,9)$. So, $(1,9) \notin E$ which implies $T_E^2 = \emptyset$.

Continuing on the same line, we found the following ranked solutions of IQP and checked the efficiency of ranked solutions of IQP . We find that the $\cup_{i=1}^6 T_E^i = \emptyset$ and the seventh

Table 6.2: Ranked solution of IQP (Example 6.5.1)

T^k	(x,y)	h(x,y)	T_E^k
T^1	(1,10)	903	$T_E^1 = \emptyset$
T^2	(1,9)	897	$T_E^2 = \emptyset$
T^3	(1,11)	893	$T_E^3 = \emptyset$
T^4	(1,8)	875	$T_E^4 = \emptyset$
T^5	(2,6)	870	$T_E^5 = \emptyset$
T^6	(1,7)	837	$T_E^6 = \emptyset$
T^7	(0,10)	820	$T_E^7 = (0, 10)$

ranked solution $(0, 10)$ passed the efficiency test i.e. optimal value of problem $P(Y) = 0$ for $Y = (0, 10) \in T^7$, so we declare the point $(0,10)$ as optimal solution of IQP_E with objective function value as 820 and the set $T_E = \{(0, 10)\}$.

In Section 6.5.2, we will apply [106] algorithm on the same example and will show that their algorithm fails in solving IQP and IQP_E .

Remark 6.5.1 *Using the proposed algorithm, we have solved the above problem (IQP_E) without finding all the efficient solutions of MILP and without computing all ranked solutions of IQP . We have only solved problems (ILP_i) iteratively for finding an optimal solution of IQP_E .*

6.5.2 A brief discussion on the algorithm [106]

In this section, we have given a counter example to show that the algorithm discussed in [106] does not solve IQP and IQP_E .

Example 6.5.2 *Consider the problem*

$$(IQP_E) \quad \max_{X \in \chi_E} h(x_1, x_2) = 2x_1^2 - 8x_2^2 + 81x_1 + 158x_2 + 40$$

where $\chi_E = \chi \cap E$, $\chi = \mathbb{Z}^2 \cap S$, $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid 3x_2 - x_1 \geq 0, 16x_1 + 3x_2 \leq 51, x_1, x_2 \geq 0\}$ and E is the set of all efficient solutions of MILP

$$(MILP) \quad \max_{(x_1, x_2) \in \chi} (5x_1 - 7x_2, -6x_1 + 5x_2).$$

The problem (IQP_E) can be expressed as:

$$(IQP_E) \quad \max_{x \in \mathcal{X}_E} h(x_1, x_2) = (2x_1 + 4x_2 + 1)(x_1 - 2x_2 + 40).$$

Since $\min_{(x_1, x_2) \in S} (2x_1 + 4x_2 + 1) = 1$ and $\min_{(x_1, x_2) \in S} (x_1 - 2x_2 + 40) = 6$. That means the functions $2x_1 + 4x_2 + 1$ and $x_1 - 2x_2 + 40$ are assuming positive values on S , so above problem (IQP_E) satisfies the conditions of [106].

Then, corresponding to the problem IQP_E , authors constructed the following relaxed problem.

$$(IQP_0) \quad \max_{(x_1, x_2) \in S} h(x_1, x_2) = (2x_1 + 4x_2 + 1)(x_1 - 2x_2 + 40).$$

To solve the problem IQP_0 , authors derived a simplex like technique and used the following result to establish the optimality of a vertex of S for the problem IQP_0 . (Theorem 2.1 of [106])

Theorem 6.5.2 *Given a basic feasible solution x_B to IQP_0 such that $\gamma_j \leq 0$ for every column in A . Then x_B is an optimal solution of IQP_0 .*

Table 6.3: Simplex table of Example 6.5.2 corresponding to solution (3,1)

x_B	s_1	s_2	RHS
x_1	0.0588	0.0588	3
x_2	-0.3137	0.0196	1
$p_j - z_j^1$	1.1373	-0.1961	
$q_j - z_j^2$	-0.6863	-0.0196	
γ_j	-0.7255	-8.0588	

- 1 Since the feasible region S of IQP_0 has only three corner points namely $(0, 0)$, $(3, 1)$ and $(0, 17)$, with objective function values as 40, 451, 414 respectively. So as per the methodology discussed in [106], the corner point $(3, 1)$ is declared optimal because it satisfies the optimality criteria given in Theorem 2.1 of [106], as shown in Table 6.3.

But $(3, 1)$ is not an optimal solution of IQP_0 . It can be seen that the objective function value at some non-corner point $x = (0, 10)$ of the feasible region is 820, which is more than $h(3, 1) = 451$. Moreover, many integer feasible solutions give better objective function value than the point $(3, 1)$ as shown in Table 6.4.

- 2 The algorithm proposed by Moulai and Drici [106] does not find an optimal solution of IQP_E . Their algorithm fails because it is based on the aspect that an optimal solution of the problem IQP_0 occurs at a vertex of the feasible region. If the optimal solution is a non-integer point, an integer solution is obtained using the branch and bound technique. Once an integer point is declared optimal by satisfying the

optimality criteria mentioned in Theorem 2.1 of [106], it is tested for efficiency. If it passes the efficiency test, this point is declared optimal for the problem IQP_E . If not, some efficiency cut (which is linear) is added to the feasible region, and the reduced region again becomes polyhedral. Now the algorithm searches for a new integer optimal solution over this polyhedral region, which will give a lesser objective function value as the feasible region is shrunk by adding various cuts. So working on the lines of their algorithm, the point (3,1) is declared optimal for IQP_0 (as discussed above). Now this point (3,1) also satisfies the efficiency test. So as per the algorithm of Moulai and Drici [106], the point (3,1) is declared optimal for IQP_E . However, the point (3,1) is not an optimal solution of IQP_E , as we have already shown that the point (0,10) obtained by the present algorithm gives a much better value ($h(0,10)=820$) of the objective function than it is obtained by the point (3,1), ($h(3,1) = 451$). Moreover, it can be seen in Table 6.4, there are a large number of points that give a better value of objective function than the point (3,1), and all of them are also efficient solutions of $MILP$.

Table 6.4: Objective function values at efficient solutions of $MILP$ (Example 6.5.2)

(x_1, x_2)	$h(x_1, x_2)$	(x_1, x_2)	$h(x_1, x_2)$	(x_1, x_2)	$h(x_1, x_2)$
(0,10)	820	(0,13)	742	(0,15)	610
(0,9)	814	(2,4)	714	(0,4)	544
(0,11)	810	(0,6)	700	(0,16)	520
(0,8)	792	(0,14)	684	(2,2)	494
(0,12)	784	(0,5)	630	(3,1)	451
(0,7)	754	(2,3)	612		

6.5.3 Computational results

Table 6.5: Number of ILP_i optimize to solve IQP_E

m	No. of ILP_i solved mean[max, min]	Time (in sec) mean[max, min]	No. of ILP_i solved mean[max, min]	Time (in sec) mean[max, min]	No. of ILP_i solved mean[max, min]	Time (in sec) mean[max, min]
$n = 10, s = 5$						
10	2 [5, 1]	3 [7, 1]	47 [440, 1]	120 [1126, 2]	39 [277, 1]	283 [2031, 7]
20	1 [5, 1]	2 [7, 1]	3 [14, 1]	7 [35, 2]	42 [326, 1]	267 [2137, 5]
30	1 [2, 1]	1 [2, 1]	1 [1, 1]	2 [3, 2]	1 [2, 1]	6 [11, 5]
50	1 [1, 1]	1 [2, 1]	1 [1, 1]	2 [3, 2]	1 [1, 1]	5 [6, 5]
$n = 10, s = 10$						
10	2 [4, 1]	3 [10, 1]	4 [9, 1]	8 [20, 2]	57 [420, 1]	439 [3248, 8]
20	1 [3, 1]	1 [4, 1]	3 [9, 1]	7 [22, 2]	7 [54, 1]	64 [459, 8]
30	1 [2, 1]	1 [2, 1]	1 [3, 1]	3 [7, 2]	2 [4, 1]	20 [53, 9]
50	1 [1, 1]	1 [1, 1]	1 [1, 1]	2 [3, 2]	1 [1, 1]	14 [27, 9]
$n = 10, s = 20$						
10	4 [18, 1]	5 [24, 1]	5 [20, 1]	10 [44, 2]	143 [991, 1]	835 [5953, 5]
20	1 [2, 1]	1 [2, 1]	2 [11, 1]	5 [24, 2]	3 [9, 1]	15 [49, 5]
30	1 [2, 1]	1 [2, 1]	2 [6, 1]	4 [13, 2]	1 [2, 1]	7 [13, 5]
50	1 [2, 1]	1 [2, 1]	1 [1, 1]	2 [2, 2]	1 [1, 1]	6 [8, 5]
$n = 10, s = 50$						
10	4 [11, 1]	6 [19, 1]	216 [1655, 1]	983 [7302, 2]	49 [405, 1]	547 [5007, 6]
20	1 [1, 1]	1 [2, 1]	2 [4, 1]	4 [10, 2]	9 [54, 1]	51 [317, 5]
30	1 [2, 1]	1 [3, 1]	1 [1, 1]	2 [3, 2]	2 [4, 1]	11 [25, 5]
50	1 [1, 1]	1 [1, 1]	1 [1, 1]	2 [3, 2]	1 [2, 1]	6 [11, 5]

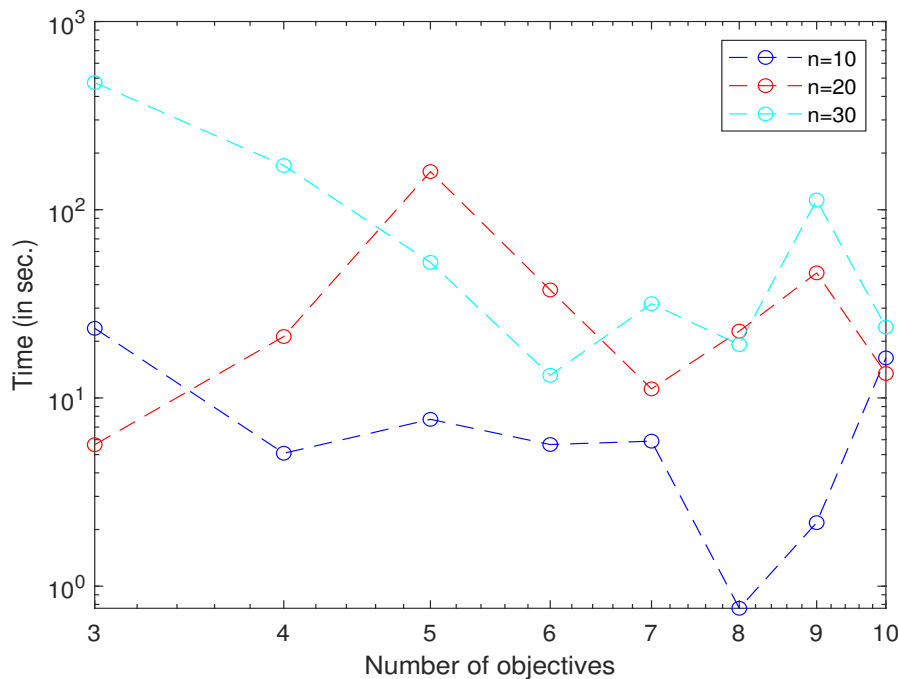


Figure 6.1: Impact of no. of objectives and no. of variables of IQP_E on CPU time

The proposed method is implemented in MATLAB (R2019b) on a personal HP laptop (Intel(R), Core(TM), i3-6006U processor, CPU @ 2GHz, and 4 GB RAM). For each instance (m, n, s) (where m , n , and s represent the numbers of constraints, variables, and objectives of $MILP$ respectively), a series of ten problems are generated. For data generation, the coefficient of variables in constraints of IQP and terms on the right-hand side of constraints of IQP are randomly given integer values from the set $[0, 20]$ and $[0, 100]$, respectively. Further, coefficients of variables in the objective function of IQP and objective functions of $MILP$ are given integer values randomly from the set $[-100, 100]$.

The number of ILP_i problems successively solved to find an optimal set of IQP_E for different values of m, n and s is shown in Table 6.5. Moreover, Table 6.5 represents the processing time to solve IQP_E for different values of m, n and s .

The Figure 6.2 depicts the impact of n and m on processing time to solve IQP_E . It can be seen from Figure 6.2, if n and s are fixed then on increasing m , the mean processing time to solve IQP_E decreases because the feasible region reduces on increasing m .

Figure 6.3 and Figure 6.1 depicts the impact of n and s on processing time to solve IQP_E . Figure 6.1, Figure 6.2 and Figure 6.3 shows that mostly the mean processing time to solve IQP_E increases as n increases.

As the mean processing time to solve IQP_E decreases as m increases and mean processing

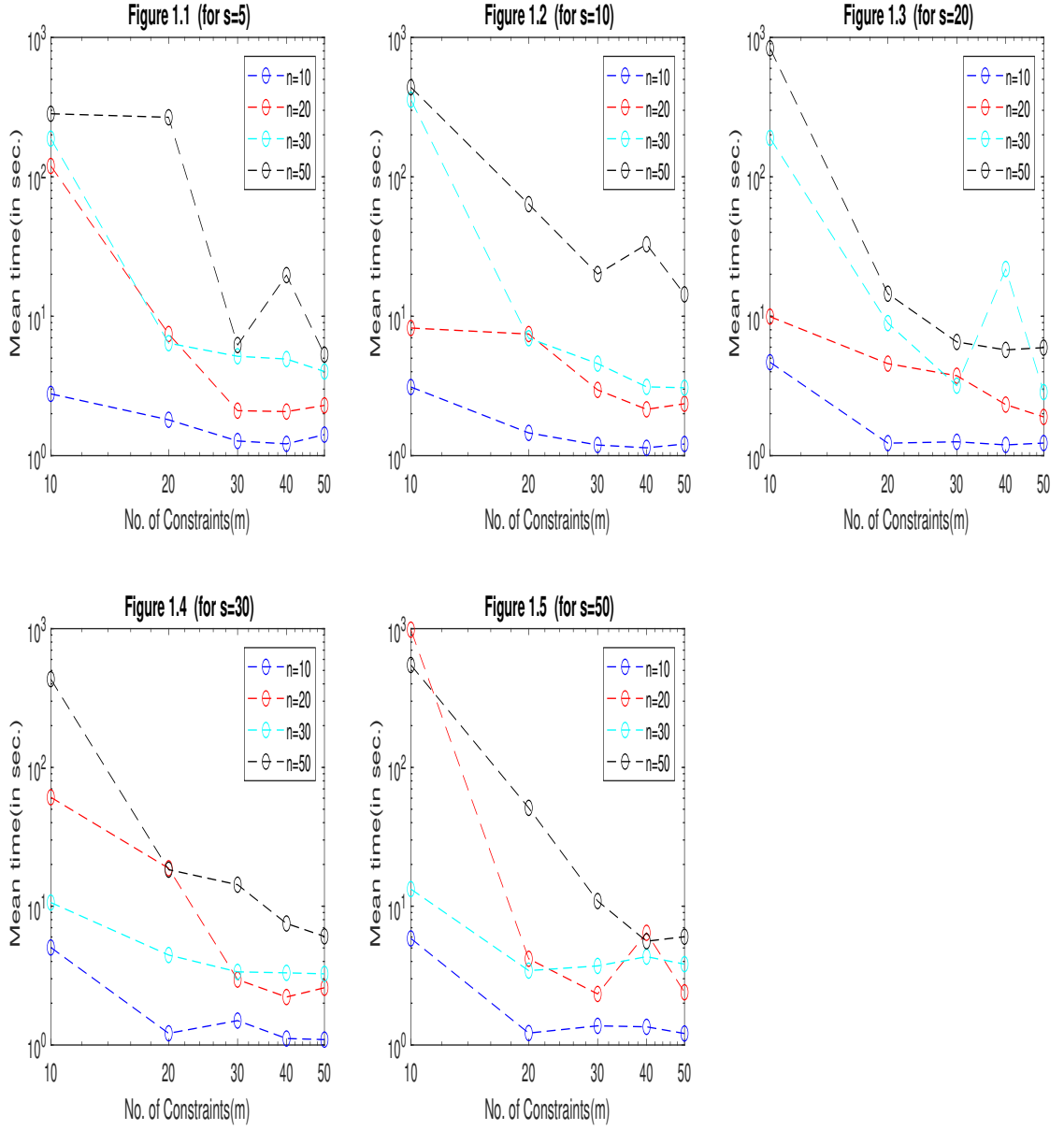


Figure 6.2: Impact of no. of variables and no. of constraints of IQP_E on CPU time

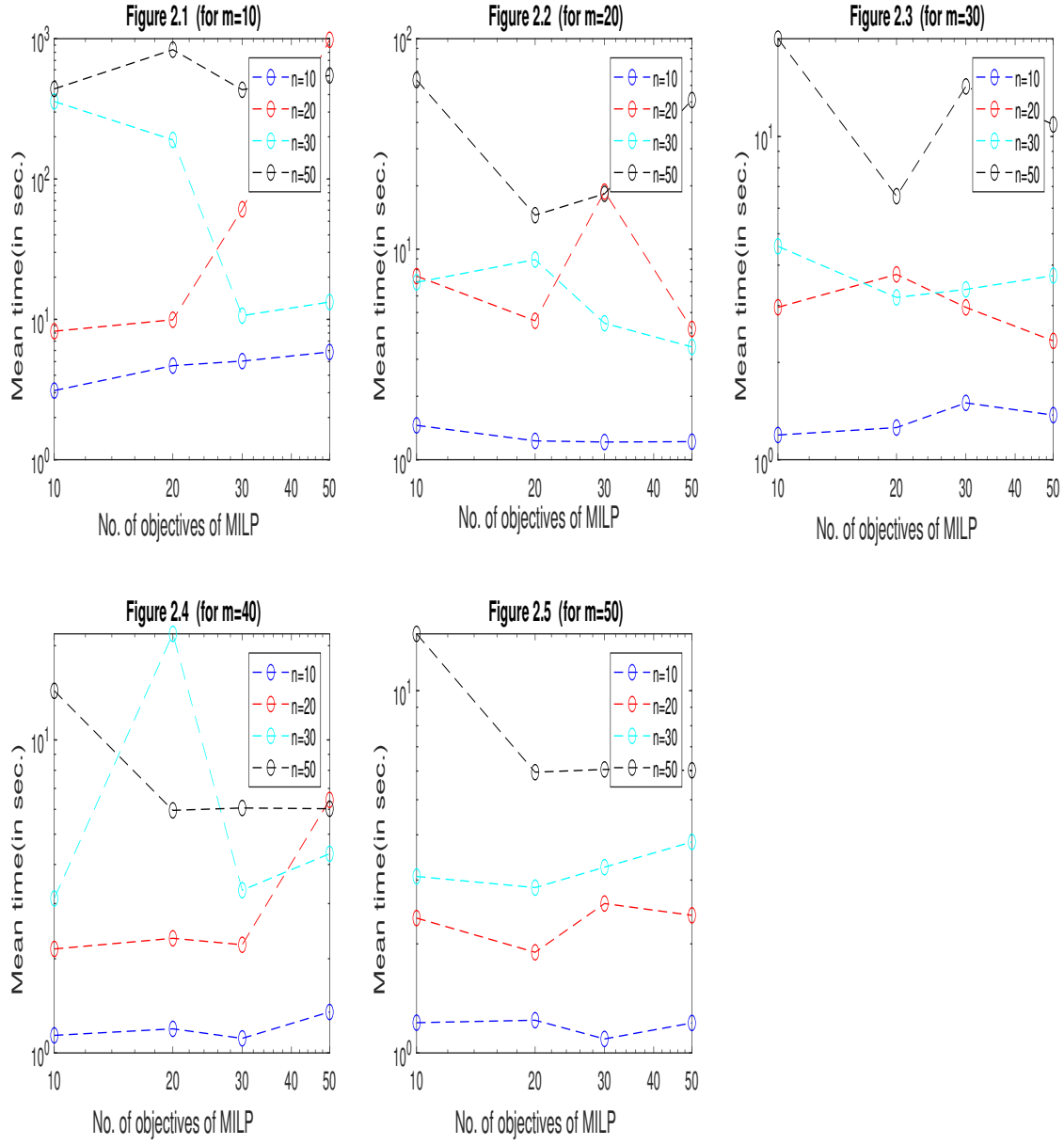


Figure 6.3: Impact of no. of variables of IQP_E and no. of objectives of $MILP$ on CPU time

time to solve IQP_E increases as n increases. So the proposed method is computationally efficient for solving IQP_E problem having a medium or large number of constraints, and a small or medium number of variables.

6.5.4 Complexity of proposed algorithm

The time complexity of the proposed algorithm depends on the number of subproblems ILP_l and $P(Y)$ constructed at each iteration. To solve these problems, we used the built-in MATLAB function ‘intlinprog,’ which has been widely used by various researchers in the literature [97,113,118,148]. For information about ‘intlinprog,’ interested readers can refer to the paper by Sokolov and Ushakov [135]. The MATLAB ‘intlinprog’ function solves mixed ILP using a branch and bound method [118]. The worst-case time complexity of this method is exponential, but in practice, the ‘intlinprog’ function typically finds an optimal solution within 10^5 major iterations, as discussed in [38].

Let p and q be the number of problems ILP_l and $P(Y)$ required for solving IQP_E . The overall worst-case time complexity of solving IQP_E using the proposed algorithm is $O(p\alpha_1^m\alpha_2^n + q\beta_1^{m+s}\beta_2^{n+s})$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 > 1$. If the value of parameter m increases, the value of p and q decreases (as shown in Table 6.5), because the feasible region of IQP reduces on increasing m .

In practice, the proposed algorithm often performs much better than the worst-case complexity, as demonstrated in Figure 6.1, Figure 6.2 and Figure 6.3.

6.6 Concluding Remarks

1. We have considered the optimization of IQP_E . The problem discussed in [106] is a particular case of the present problem. As far as we know, no other algorithm deals with this class of problem, so a comparative study cannot be made.
2. Although the objective function of IQP_E is non-linear, there is no use of any non-linear optimization technique at any stage of the proposed methodology. In contrast, an associated ILP is solved at every stage of the algorithm. The optimal solution of IQP is obtained by comparing the objective function values of ILP and IQP .
3. The ranking of the integer solutions of the quadratic programming problem is done by ranking the associated integer solutions of ILP . For this purpose, at every stage, a single objective ILP is solved with only one additional constraint (the objective function constraint) to the original set of constraints. Further, we have also shown that the k^{th} ($k > 1$) ranked solution of IQP can be obtained from associated ILP without ranking all the feasible solutions of associated ILP . So the proposed methodology is straightforward to implement.
4. Using the proposed algorithm, we are capable of optimizing IQP_E without finding

all the efficient solutions of $MILP$ and without enumerating all the feasible solutions of IQP .

5. The methodology proposed in this chapter can be extended to the problem IQP_E with box constraints.

Chapter 7

Quadratic fractional function over efficient set

7.1 Introduction

The optimization of a function over the integer efficient set was first studied by Abbas [2]. They used various cuts for solving ILP_E . Later, Jorge [80] utilized a criterion search algorithm for optimizing ILP_E . Recently, Prerna and Sharma [121] have proposed two algorithms for solving ILP_E . In their algorithms ranked solutions of a linear function are used for solving the problem. The significant research in the area of optimizing a function over integer efficient set is focused on ILP_E [2, 17, 24, 80, 121]. The problem (IQP_E) is only discussed in [106, 122]. Due to the increasing difficulties of incorporating quadratic function, this area is not much explored. Moreover, as far as the optimization of QFP over the efficient set of $MILP$ is concerned, no research has been done in this domain.

In this chapter, we have presented two methods (feasible space search algorithm and criterion space search algorithm) for solving QFP over the efficient set of $MILP$. The feasible space search algorithm is the extension of the algorithm discussed in Chapter 6. The criterion space search algorithm combines Sylva and Crema cut [140] with a ranking approach and efficiency tests. The algorithm follows a two-step procedure. On one side, it calculates the lower bound using ranked solutions of related ILP or $ILFP$, and on the other side, it calculates the upper bound using efficient solutions.

The proposed methods are straightforward to implement, as we have not utilized any non-linear optimization technique. We have utilized ranked solutions of $ILFP$ or ILP for solving $IQFP$, as discussed in the paper of Sharma, Dahiya and Verma [131].

Moreover, using the developed methods, we can also solve $ILFP_E$, IQP_E , and ILP_E .

Using theoretical results we have demonstrated the convergence of the proposed algorithms, and using numerical and computational experiments we have confirmed the algorithms' effectiveness in solving $IQFP_E$.

This chapter is divided into different sections. In the next section, we will discuss notations

and preliminary results. In Section 7.3 and Section 7.4, we will present a feasible space search algorithm and a criterion space search algorithm for solving $IQFP_E$. In the Section 7.6, we will elaborate the working of the proposed algorithms using numerical examples. In the last section, we will discuss some concluding remarks on the proposed scheme.

7.2 Notations and Preliminary Results

Consider the problem:

$$(MILP) \quad \max_{X \in \mathcal{X}} (C^1 X, C^2 X, \dots, C^s X).$$

The following theorem shows the existence of an efficient solution to $MILP$.

Theorem 7.2.1 [131] *If $MILP$ has a non- empty bounded feasible region $\mathcal{X}$, then efficient set E of $MILP$ is also non-empty.*

Let $X^1, X^2, \dots, X^l$ be efficient solutions of $MILP$ and $D^j = \{X \in \mathbb{Z}^n | C^i X \leq C^i X^j \text{ for all } i \in \{1, 2, \dots, s\}\}$. The following theorem shows a relation between an efficient solution of $MILP$ and an efficient solution of the following problem (P_{D^l}) :

$$(P_{D^l}) \quad \begin{cases} \max C^1 X, C^2 X, \dots, C^s X \\ X \in \mathcal{X} - \cup_{j=1}^l D^j. \end{cases}$$

Theorem 7.2.2 [140] *Let $X^1, X^2, \dots, X^l$ be efficient solutions of $MILP$ and $D^j = \{X \in \mathbb{Z}^n | C^i X \leq C^i X^j \text{ for all } i \in \{1, 2, \dots, s\}\}$. If $\hat{Y}$ is an efficient solution of the problem (P_{D^l}) , then $\hat{Y}$ is an efficient solution of $MILP$. Furthermore, if problem (P_{D^l}) is infeasible, then $\{(C^1 X^1, C^2 X^1, \dots, C^s X^1), (C^1 X^2, C^2 X^2, \dots, C^s X^2), \dots, (C^1 X^l, C^2 X^l, \dots, C^s X^l)\}$ is the complete set of non-dominated criterion vectors for $MILP$.*

We will describe a solution strategy for solving the problem, as follows:

$$(IQFP_E) \quad \min_{X \in E} f(X) = \frac{X^T D X + C X + \alpha}{X^T Q X + P X + \beta},$$

where $D, Q \in \mathbb{Z}^{n \times n}$, $C, P \in \mathbb{Z}^{1 \times n}$, $\alpha, \beta \in \mathbb{Z}$ and E is the efficient set of $MILP$,. We also assume $X^T Q X + P X + \beta > 0$ for all $X \in \mathcal{X}$.

The $IQFP_E$ is closely related to the following problem :

$$(IQFP) \quad \min_{X \in \mathcal{X}} f(X) = \frac{X^T D X + C X + \alpha}{X^T Q X + P X + \beta}.$$

Let X^k be a k^{th} ranked solution of $IQFP$ such that

$$f(X^{k-1}) < f(X^k) < f(X^{k+1}) \text{ for all } k \in \{2, 3, \dots, q-1\},$$

where

$$f(X^1) = \min_{X \in \mathcal{X}} f(X), \quad f(X^q) = \max_{X \in \mathcal{X}} f(X).$$

If F^k denote the complete set of all k^{th} ranked solutions of $IQFP$, then

$$F^k = \{Y \in \mathcal{X} \mid f(Y) = f(Y^k)\}.$$

Let F_E^k denotes the complete set of all k^{th} ranked solutions of $IQFP$, which are also efficient solution of $MILP$, i.e.,

$$F_E^k = F^k \cap E.$$

The following theorem discussed in [75, 131] shows the existence of a linear or linear fractional function $g(X)$, which provides lower bounds to quadratic fractional function $f(X)$.

Theorem 7.2.3 *For every quadratic fractional function $f(X)$ in $IQFP$, there always exists a linear or linear fractional function $g(X)$ such that $g(X) \leq f(X)$ for all $X \in \mathcal{X}$.*

The construction of function $g(X)$ has been discussed in [131] (as shown in Table 7.1) and following notations has been used in Table 7.1.

$$\begin{aligned} \mu &= \min_{X \in \mathcal{X}} X^T Q X + P X + \beta, \quad m = \min_{X \in \mathcal{X}} X^T D X + C X + \alpha, \quad M = \max_{X \in \mathcal{X}} X^T D X + C X + \alpha, \\ \hat{m} &= \min_{X \in \mathcal{X}} (U + C) X + \alpha, \quad \hat{M} = \max_{X \in \mathcal{X}} (U + C) X + \alpha, \quad U_i = \min_{X \in \mathcal{X}} X^T D_i, \quad V_i = \max_{X \in \mathcal{X}} X^T Q_i \quad \forall i \in \{1, 2, \dots, n\}. \end{aligned}$$

where $U, V \in \mathbb{Z}^{1 \times n}$, U_i, V_i, D_i and Q_i are i^{th} component of vectors U and V and i^{th} component of matrices D and Q respectively.

Table 7.1: The construction of function $g(X)$

m	M	$\hat{m}$	$\hat{M}$	$g(X)$
$m \geq 0$	—	—	—	$\frac{(U+C)X+\alpha}{(V+P)X+\beta}$
—	$M \leq 0$	—	—	$\frac{(U+C)X+\alpha}{(U+C)X+\alpha}$
$m < 0$	$M > 0$	—	$\hat{M} \leq 0$	$\frac{(U+C)X+\alpha}{(U+C)X+\alpha}$
—	—	$\hat{m} < 0$	$\hat{M} > 0$	$\frac{(U+C)X+\alpha-\hat{M}}{\mu}$

Now, for ranking values of $g(X)$ for all $X \in \chi$, consider the following sets:

$$G^1 = \{Y \in \chi \mid g(Y) = \min_{X \in \chi} g(X)\}, \quad G^p = \{Y \in \chi \mid g(Y) = \max_{X \in \chi} g(X)\}, \quad (7.2.1)$$

$$G^l = \{Y \in \chi \mid g(Y^{l-1}) < g(Y) < g(Y^{l+1}), Y^{l-1} \in G^{l-1}, Y^{l+1} \in G^{l+1}\} \text{ for all } l \in \{2, 3, \dots, p-1\}. \quad (7.2.2)$$

In order to find the elements of sets G^l for all $l \in \{1, 2, \dots, p\}$, construct the following sets:

$$\chi_1 = \chi, \quad \chi_l = \{Y \in \chi \mid g(Y) > g(Y^{l-1}), Y^{l-1} \in G^{l-1}\} \text{ for all } l \in \{2, 3, \dots, p\}. \quad (7.2.3)$$

It can be observed that

$$G^l = \{Y \in \chi_l \mid g(Y) = \min_{X \in \chi_l} g(X)\} \text{ for all } l \in \{1, 2, \dots, p\}. \quad (7.2.4)$$

It is evident by Table 7.1, the function $g(X)$ depends on m , M , $\hat{m}$ and $\hat{M}$. Therefore, the following cases arises:

CASE 1: $m \geq 0$:

If $m \geq 0$, then $g(X)$ will be a linear fractional function defined as $g(X) = \frac{(U+C)X+\alpha}{(V+P)X+\beta}$. In this case, for finding elements of set G^l for all $l \in \{1, 2, \dots, p\}$, we consider the problem, as follows:

$$(ILFP_l) \quad \min_{X \in \chi_l} g(X), \quad \text{where} \quad g(X) = \frac{(U+C)X+\alpha}{(V+P)X+\beta}.$$

It can be observed that G^l is the complete set of all optimal solutions of $ILFP_l$ and l^{th} ranked solutions of the problem, as follows:

$$(ILFP) \quad \min_{X \in \chi} g(X).$$

Let Y^l be a l^{th} ranked solution of $ILFP$ and an optimal basic feasible solution of the problem, as follows:

$$(LFP^t) \quad \min_{X \in S^t} g(X).$$

The feasible region S^t of LFP^t is generated by successively truncating the feasible region S of the problem, as follows:

$$(LFP) \quad \min_{X \in S} g(X).$$

Now, let Z^l be the objective function value of the problem (LFP^t) at Y^l , therefore

$$Z^l = \frac{\sum_{j=1}^n c_j x_j + \alpha}{\sum_{j=1}^n d_j x_j + \beta},$$

where x_j, c_j and d_j are the j^{th} component of vector $Y^l, (U + C)$ and $(V + P)$ respectively.

Let $\bar{Y}$ be another basic feasible solution of LFP^t with objective function value $\bar{Z}$, therefore

$$\bar{Z} = \frac{Z_1 - \theta_j(z_j^1 - c_j)}{Z_2 - \theta_j(z_j^2 - d_j)}, \quad z_j^1 = \sum_{i \in I} c_i y_{ij}, \quad z_j^2 = \sum_{i \in I} d_i y_{ij}$$

where $I = \{i \mid x_i \text{ is a basic variable corresponding to } Y^l\}$, $y_{ij} = i^{th}$ component of $B^{-l}a_j$ and $a_j = j^{th}$ component of matrix A .

The change in objective function value from Z^l to $\bar{Z}$ is given by

$$\bar{Z} = \hat{Z} - \frac{\Delta_j \theta_j}{Z_2(Z_2 - \theta_j(z_j^2 - d_j))}, \quad \text{where } \Delta_j = Z_2(z_j^1 - c_j) - Z_1(z_j^2 - d_j). \quad (7.2.5)$$

Remark 7.2.4 As Y^l is an optimal basic feasible solution of LFP^t , therefore $\Delta_j \leq 0$ for all $j \in \{1, 2, \dots, n\}$.

Let $\hat{N} = \{j \mid x_j \text{ is a non-basic variable corresponding to } Y^l \text{ with } \Delta_j = 0\}$ and $\bar{N} = \{j \mid x_j \text{ is a non-basic variable corresponding to } Y^l \text{ with } \Delta_j < 0\}$.

Remark 7.2.5 If a non-basic variable $x_j \in \hat{N}$ is introduced into the basis, we will get an alternate optimal solution of LFP^t .

For finding the set of ranked solutions of $ILFP$, we have established the following result.

Theorem 7.2.6 All feasible solutions of $ILFP$ yielding objective function value greater than Z^l lie in the closed half space $\sum_{j \in \bar{N}} x_j \geq 1$.

As Y^l is a l^{th} ranked solution of $ILFP$, therefore by virtue of Theorem 7.2.6, the set (χ_{l+1}) can be given as:

$$\chi_{l+1} = \{X \in \chi_l \mid \sum_{j \in \bar{N}_l} x_j \geq 1\} \text{ for all } l \in \{1, 2, \dots, p-1\}.$$

CASE 2: $m < 0$:

In this case, $g(X)$ will be a linear function. Since $U, V \in \mathbb{Z}^{n \times 1}$ for all $X \in \chi$ because symmetric matrices $D, Q \in \mathbb{Z}^{n \times n}$. Therefore, $\mu g(X) \in \mathbb{Z}$ for all $X \in \chi$. Let $\hat{g} = \mu g$, therefore $\hat{g}(X) \in \mathbb{Z}$ for all $X \in \chi$. Then, by virtue of Equation (7.2.3),

$$\begin{aligned}\chi_l &= \{Y \in \chi \mid g(Y) > g(Y^{l-1}), Y^{l-1} \in G^{l-1}\} \\ &= \{Y \in \chi \mid \mu g(Y) > \mu g(Y^{l-1}), Y^{l-1} \in G^{l-1}\} \quad (\text{because } \mu > 0) \\ &= \{Y \in \chi \mid \hat{g}(Y) \geq \hat{g}(Y^{l-1}) + 1, Y^{l-1} \in G^{l-1}\} \quad (\text{because } \hat{g}(X) \in \mathbb{Z} \text{ for all } X \in \chi).\end{aligned}\tag{7.2.6}$$

It is evident by Equation (7.2.6), G^l is the complete set of optimal solutions of the problem, defined as :

$$(ILP_l) \quad \min_{X \in \chi_l} g(X), \quad \text{where} \quad \chi_l = \{Y \in \chi \mid \hat{g}(Y) \geq \hat{g}(Y^{l-1}) + 1, Y^{l-1} \in G^{l-1}\}.$$

Notably, G^l is the complete set of l^{th} ranked solutions of the problem, defined as:

$$(ILP) \quad \min_{X \in \chi} g(X).$$

Since $g(X)$ is a continuous function and feasible region χ_l of ILP_l is a closed and bounded set, there exists an optimal solution of ILP_l . A branch and bound method or Gomory cutting plane technique can be utilized for finding an optimal solution to ILP_l . An edge exploring technique discussed in [121] can be utilized for enumerating all elements of the set G^l .

7.3 Theoretical development of a feasible space search algorithm

In the following theorem, we will discuss a relation between a ranked solution of $IQFP$ and a feasible solution of the problem ($IQFP_E$).

Theorem 7.3.1 *If Y is any feasible solution of $IQFP_E$, then $Y \in F_E^k$ for some $k \in \{1, 2, \dots, q\}$.*

Proof Since Y is feasible solution of $IQFP_E$, therefore $Y \in E$. As set $E \subseteq \chi$ and $\chi = \cup_{k=1}^q F^k$. So $Y \in F_E^k$ for some $k \in \{1, 2, \dots, q\}$.

Now, we will establish a condition for a feasible solution $Y \in F_E^k$ for some $k \in \{1, 2, \dots, q\}$

to be an optimal solution of $IQFP_E$.

Theorem 7.3.2 *Let $\cup_{i=1}^{k-1} F_E^i = \emptyset$ and $Y \in F_E^k$ for some $k \in \{1, 2, \dots, q\}$, then $Y \in F_E^k$ will be an optimal solution of $IQFP_E$.*

Proof Since $Y \in F_E^k$, therefore $Y \in F^k$. As F^k is the set of all k^{th} ranked solutions of $IQFP$ therefore

$$\begin{aligned} f(Y) &\leq f(X) \text{ for all } X \in \cup_{i=k}^q F^i \\ \implies f(Y) &\leq f(X) \forall X \in \cup_{i=1}^q F^i \text{ (because } \cup_{i=1}^{k-1} F^i = \emptyset) \\ \implies f(Y) &\leq f(X) \forall X \in \chi \text{ (as } \cup_{i=1}^q F^i = \chi) \\ \implies f(Y) &\leq f(X) \forall X \in E \text{ (as } E \subseteq \chi). \end{aligned}$$

So, Y is an optimal solution of $IQFP_E$.

Corollary 7.3.3 *If $\cup_{i=1}^{k-1} F_E^i = \emptyset$ and $F_E^k \neq \emptyset$ for some $k \in \{1, 2, \dots, q\}$, then F_E^k will be the set of all optimal solutions of $IQFP_E$.*

Proof Let F be the set of all optimal solutions of $IQFP_E$. It can be seen from Theorem 7.3.2, if $\cup_{i=1}^{k-1} F_E^i = \emptyset$ and $Y \in F_E^k$ for some $k \in \{1, 2, \dots, q\}$, then $Y \in F_E^k$ will be an optimal solution of $IQFP_E$. So $F_E^k \subseteq F$. Now we want to show $F \subseteq F_E^k$. If possible let $\hat{X} \in F$, but $\hat{X} \notin F_E^k$. Since $\hat{X} \in F$, therefore $\hat{X}$ is a feasible solution of $IQFP_E$. By Theorem 7.3.1, $\hat{X} \in F_E^i$ for some $i \in \{1, 2, \dots, q\}$. Since $F_E^i = \emptyset$ for all $i \in \{1, 2, \dots, k-1\}$ and $\hat{X} \notin F_E^k$. Therefore $\hat{X} \in F_E^i$ for some $i \in \{k+1, k+2, \dots, q\}$. As $F_E^i = F^i \cap E$ therefore

$$\hat{X} \in F^i \text{ for some } i \in \{k+1, k+2, \dots, q\}. \quad (7.3.1)$$

Also, $\hat{X}$ is an optimal solution of $IQFP_E$ therefore

$$\begin{aligned} f(\hat{X}) &\leq f(X) \text{ for all } X \in E \\ \implies f(\hat{X}) &\leq f(X) \text{ for all } X \in \cup_{i=1}^q F_E^i \text{ (as } F_E^i = F^i \cap E). \end{aligned} \quad (7.3.2)$$

Since $F_E^k \neq \emptyset$, therefore there exist an integer feasible solution X^k of $IQFP$ such that $X^k \in F_E^k$. Eq (7.3.2) implies that

$$f(\hat{X}) \leq f(X^k). \quad (7.3.3)$$

Also, X^k is a k^{th} ranked integer feasible solution of $IQFP$ therefore

$$f(X^k) < f(X) \text{ for all } X \in \cup_{i=k+1}^q F_E^i. \quad (7.3.4)$$

Using (7.3.1) & (7.3.4), we get

$$f(X^k) < f(\hat{X})$$

which contradict Eq (7.3.3) therefore $\hat{X} \in F_E^k$. So, $F \subseteq F_E^k$.

Corollary 7.3.4 *If $F_E^1 \neq \emptyset$, then F_E^1 will be the set of all optimal solutions of $IQFP_E$.*

By virtue of Corollary 7.3.4, if $F_E^1 \neq \emptyset$, then F_E^1 will be the set of all optimal solutions of $IQFP_E$. As F_E^1 is contained in the set F^1 . So, in order to find optimal solutions of $IQFP_E$, we have to find set F^1 i.e. the set of all optimal solutions of $IQFP$. To find set F^1 , we have utilized a relation between optimal solutions of $IQFP$ and ranked solutions of $ILFP$, discussed in [131].

Theorem 7.3.5 *[131] If $g(Y) \geq \min \{f(X) \mid X \in \cup_{j=1}^l G^j\} = f(\hat{X})$ for some $Y \in G^l$, then $\hat{X}$ is an optimal solution of $IQFP$.*

Corollary 7.3.6 *If there exist some $Y \in G^1$ such that $g(Y) = \min \{f(X) \mid X \in G^1\} = f(\hat{X})$, then $\hat{X}$ is an optimal solution of $IQFP$.*

Corollary 7.3.7 *If there exist some $Y \in G^l$ such that $g(Y) \geq \min \{f(X) \mid X \in \cup_{j=1}^l G^j\} = f(\hat{X})$, then $F = \{X \in \cup_{j=1}^l G^j \mid f(X) = f(\hat{X})\}$ is the set of all optimal solutions of $IQFP$.*

Remark 7.3.8 *By above corollary, we concluded that in order to find the set of all optimal solutions of $IQFP$, we do not need to rank $ILFP$ beyond l^{th} ranked solution of $ILFP$ provided there exist some $Y \in G^l$ such that $g(Y) \geq \min \{f(X) \mid X \in \cup_{j=1}^l G^j\}$.*

It can be seen from Corollary 7.3.3, if $\cup_{i=1}^{k-1} F_E^i = \emptyset$ and $F_E^k \neq \emptyset$ for some $k \in \{1, 2, \dots, q\}$, then F_E^k will be the set of all optimal solutions of $IQFP_E$. As $F_E^k = F^k \cap E$. So, for finding the complete set of optimal solutions of $IQFP_E$, we have to find sets F^k for all $k \in \{1, 2, \dots, q\}$ until a ranked solution $X^k \in F^k$ will be an efficient solution of $MILP$. In Corollary 7.3.7, we have discussed a ranking scheme to find set F^1 . Now, we will consider the following problem to find set F^k for all $k \in \{2, 3, \dots, q\}$.

$$(IQFP_k) \quad \begin{cases} \min f(X) \\ \text{s.t. } f(X) > f(X^{k-1}) \text{ for some } X^{k-1} \in F^{k-1} \\ X \in \chi. \end{cases}$$

In the following theorem, we have developed a relation between an optimal solution of $IQFP_k$ and ranked solution of $ILFP$.

Theorem 7.3.9 *If $g(Y) \geq \min \{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\} = f(\bar{X})$ for some $Y \in G^l$, $X^{k-1} \in F^{k-1}$, then $\bar{X}$ is an optimal solution of $IQFP_k$.*

Proof If possible let $\bar{X}$ be not an optimal solution of $IQFP_k$. By hypothesis $f(\bar{X}) > f(X^{k-1})$ for some $X^{k-1} \in F^{k-1}$, therefore $\bar{X}$ is a feasible solution of $IQFP_k$. Since $\bar{X}$ is a feasible solution, but not an optimal solution of $IQFP_k$, therefore there exists a feasible solution $\check{X}$ of $IQFP_k$ such that

$$f(\check{X}) > f(X^{k-1}) \text{ and } f(\check{X}) < f(\bar{X}). \quad (7.3.5)$$

Now, two cases arises either $\check{X} \in \cup_{j=1}^l G^j$ or $\check{X} \in F^j$ for some $j > l$.

If possible let $\check{X} \in \cup_{j=1}^l F^j$. Since $\check{X} \in \cup_{j=1}^l F^j$ and $f(\check{X}) > f(X^{k-1})$, therefore by hypothesis

$$f(\bar{X}) \leq f(\check{X}). \quad (7.3.6)$$

Equation (7.3.5) contradict Equation (7.3.6). So $\check{X} \in F^j$ for some $j > l$. As $Y \in G^l$ and $\check{X} \in F^j$ for some $j > l$. Therefore $g(Y) < g(\check{X})$. Also, $f(\bar{X}) \leq g(Y)$ which implies

$$f(\bar{X}) < g(\check{X}). \quad (7.3.7)$$

As function g provides lower bound to the values of function $f \forall X \in \chi$, therefore

$$g(\check{X}) < f(\check{X}). \quad (7.3.8)$$

By using eqns (7.3.7) and (7.3.8), we get

$$f(\bar{X}) < f(\check{X}), \quad (7.3.9)$$

which contradict Equation (7.3.5). Thus, $\bar{X}$ is an optimal solution of $IQFP_k$.

Remark 7.3.10 It may be observed that an optimal solution of $IQFP_k$ is the k^{th} ranked solution of $IQFP$. So by above theorem, we concluded that if $g(Y) \geq \min\{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\} = f(\bar{X})$ for some $Y \in G^l$, $X^{k-1} \in F^{k-1}$, then $\bar{X}$ is the k^{th} ranked solution of $IQFP$.

Now, we will establish a relation between set F^k for all $k \in \{2, 3, \dots, q\}$ and ranked solutions of $ILFP$.

Corollary 7.3.11 If $g(Y) \geq \min\{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\} = f(\bar{X})$ for some $Y \in G^l$, $X^{k-1} \in F^{k-1}$, then $F^k = \{X \in \cup_{j=1}^l G^j \mid f(X) = f(\bar{X})\}$.

Proof By hypothesis, $g(Y) \geq \min\{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\} = f(\bar{X})$. Therefore by Theorem 7.3.9, $\bar{X} \in F^k$. Since $\bar{X} \in F^k$ therefore $\{X \in \cup_{j=1}^l G^j \mid f(X) = f(\bar{X})\} \subseteq F^k$. Now, we want to show $F^k \subseteq \{X \in \cup_{j=1}^l G^j \mid f(X) = f(\bar{X})\}$. Let a feasible

solution $X^k \in F^k$. Now two cases arises either $X^k \in \{X \in \cup_{j=1}^l G^j \mid f(X) = f(\bar{X})\}$ or $X^k \in \{X \in \cup_{j=l+1}^p G^j \mid f(X) = f(\bar{X})\}$. If possible let $X^k \in \{X \in \cup_{j=l+1}^p G^j \mid f(X) = f(\bar{X})\}$. Since $X^k \in \cup_{j=l+1}^p G^j$ and by hypothesis, $Y \in G^l$ therefore,

$$g(Y) < g(X^k). \quad (7.3.10)$$

Also, $g(X) \leq f(X) \forall X \in \chi$ therefore

$$g(X^k) \leq f(X^k). \quad (7.3.11)$$

Equation 7.3.10 and Equation 7.3.11 imply that

$$g(Y) < f(X^k). \quad (7.3.12)$$

Since $X^k \in F^k$ and $\bar{X} \in F^k$, therefore

$$g(Y) < f(\bar{X}). \quad (7.3.13)$$

But,

$$g(Y) \geq \min\{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\} = f(\bar{X}). \quad (7.3.14)$$

Equation 7.3.14 contradicts Equation 7.3.13. So, $X^k \notin \{X \in \cup_{j=l+1}^p G^j \mid f(X) = f(\bar{X})\}$ which implies $X^k \in \{X \in \cup_{j=1}^l G^j \mid f(X) = f(\bar{X})\}$. Therefore $F^k \subseteq \{X \in \cup_{j=1}^l G^j \mid f(X) = f(\bar{X})\}$. Thus, $F^k = \{X \in \cup_{j=1}^l G^j \mid f(X) = f(\bar{X})\}$.

Theorem 7.3.12 *The Algorithm 10 generates the complete set of optimal solutions of $IQFP_E$.*

Proof The Algorithm 10 finds the set of all optimal solutions of $IQFP_E$ by enumerating ranked solutions of $IQFP$ until a ranked solution satisfies the efficiency test. It is evident from Corollary 7.3.3, if $\cup_{i=1}^{k-1} F_E^i = \emptyset$ and $F_E^k \neq \emptyset$ for some $k \in \{1, 2, \dots, q\}$, then F_E^k will be the optimal solutions of $IQFP_E$. Thus, Algorithm 10 generates the complete set of all optimal solutions of $IQFP_E$.

Now, we want to show that the proposed algorithm finds the set of all optimal solutions of $IQFP_E$ in a finite number of steps.

Theorem 7.3.13 *If χ is a non-empty and bounded set, then the Algorithm 10 terminates in finite steps.*

Proof As χ is a non-empty and bounded set, so $\cup_{i=1}^q F^i \neq \emptyset$. Also, Theorem 7.2.1, implies that efficient set E of $MILP$ is non-empty. Therefore $F_E^k = F^k \cap E \neq \emptyset$ for some

Algorithm 10: Algorithm to solve $IQFP_E$

Data: $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, C^i \in \mathbb{R}^{1 \times n}, D, Q \in \mathbb{Z}^{n \times n}, C, P \in \mathbb{Z}^{1 \times n}$ and $\alpha, \beta \in \mathbb{Z}$

Result: F_E = Set of all optimal solutions of $IQFP_E$

```

1  Initialization  $k \leftarrow 1, l \leftarrow 1, X \leftarrow \{X \in \mathbb{Z}^n \mid AX = b, X \geq 0\}$ 
2   $f(X) \leftarrow \frac{X^T DX + CX + \alpha}{X^T QX + PX + \beta}$   $\triangleright IQFP \quad \min_{X \in X} f(X)$ 
3   $\mu \leftarrow \min_{X \in X} X^T QX + PX + \beta, m \leftarrow \min_{X \in X} X^T DX + CX + \alpha, M \leftarrow \max_{X \in X} X^T DX + CX + \alpha,$ 
4  for  $i \leftarrow 1 : n$  do
5       $U_i \leftarrow \min_{X \in X} X^T D_i,$   $\triangleright U_i$  and  $D_i$  are  $i^{th}$  component of vector  $U$  and matrix  $D$  respectively
6       $V_i \leftarrow \max_{X \in X} X^T Q_i$   $\triangleright V_i$  and  $Q_i$  are  $i^{th}$  component of vector  $V$  and matrix  $Q$  respectively
7  if  $m \geq 0$  then
8       $g(X) \leftarrow \frac{(U+C)X+\alpha}{(V+P)X+\beta}$   $\triangleright ILFP \quad \min_{X \in X} g(X)$ 
9  else
10      $\hat{m} \leftarrow \min_{X \in X} (U+C)X + \alpha, \hat{M} \leftarrow \max_{X \in X} (U+C)X + \alpha$ 
11     if  $(m < 0, M > 0 \text{ and } \hat{M} \leq 0) \text{ or } (M \leq 0)$  then
12          $g(X) \leftarrow \frac{(U+C)X+\alpha}{\mu}$ 
13     if  $\hat{m} < 0 \text{ and } \hat{M} > 0$  then
14          $g(X) \leftarrow \frac{(U+C)X+\alpha-\hat{M}}{\mu}$ 
15   $g_{max} \leftarrow \max_{X \in X} g(X)$ 
16   $G^1 \leftarrow \{Y \in X \mid g(Y) = \min_{X \in X} g(X)\}$   $\triangleright$  set of optimal solutions of  $ILFP$ 
17   $f_{min} \leftarrow \min\{f(X) \mid X \in G^1\}$ 
18  while  $g(Y) < f_{min}$  and  $g(Y) < g_{max}$  for  $Y \in G^l$  do
19       $X_{l+1} \leftarrow \{X \in X_l \mid \sum_{j \in \bar{N}_l} x_j \geq 1\}, l \leftarrow l + 1$ 
20       $G^l \leftarrow \{Y \in X_l \mid g(Y) = \min_{X \in X_l} g(X)\}$   $\triangleright$  set of  $l^{th}$  ranked solutions of  $ILFP$ 
21       $f_{min} \leftarrow \min\{f(X) \mid X \in \cup_{j=1}^l G^j\}$ 
22   $F^1 \leftarrow \{Y \in \cup_{j=1}^l G^j \mid f(Y) = f_{min}\}$   $\triangleright$  set of optimal solutions of  $IQFP$ 
23   $F_E^1 \leftarrow \emptyset$ 
24  foreach  $Y \in F^1$  do
25      Solve  $P(Y)$ 
26      if If optimal value of problem  $P(Y) = 0$  then
27           $F_E^1 \leftarrow [F_E^1 \ X]$ 
28  while  $F_E^k \leftarrow \emptyset$  do
29       $k \leftarrow k + 1$ 
30      if  $\{X \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j, X^{k-1} \in F^{k-1}\} = \emptyset$  then
31           $f_{min}^k = -\infty$ 
32      else
33           $f_{min}^k \leftarrow \min\{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\}$  for  $X^{k-1} \in F^{k-1}$ 
34      while  $g(Y) < f_{min}^k$  and  $g(Y) > g_{max}$  for  $Y \in G^l$  do
35           $X_{l+1} \leftarrow \{X \in X_l \mid \sum_{j \in \bar{N}_l} x_j \geq 1\}, l \leftarrow l + 1$ 
36           $G^l \leftarrow \{Y \in X_l \mid g(Y) = \min_{X \in X_l} g(X)\},$ 
37           $f_{min}^k \leftarrow \min\{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\}$  for  $X^{k-1} \in F^{k-1}$ 
38       $F^k \leftarrow \{Y \in \cup_{j=1}^l G^j \mid f(Y) = f_{min}^k\}$   $\triangleright$  set of  $k^{th}$  ranked solutions of  $IQFP$ 
39       $F_E^k \leftarrow \emptyset$ 
40      foreach  $Y \in F^k$  do
41          Solve  $P(Y)$ 
42          if If optimal value of problem  $P(Y) = 0$  then
43               $F_E^k \leftarrow [F_E^k \ X]$ 
44  if  $F_E^k \neq \emptyset$  then
45       $F_E \leftarrow F_E^k$   $\triangleright$  set of optimal solutions of  $IQFP_E$ 

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$k \in \{1, 2, \dots, q\}$. So Algorithm 10 terminates in a steps.

Special cases

Algorithm 11: Algorithm to solve $ILFP_E$

Data: $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, C^i \in \mathbb{Z}^{1 \times n}, D, Q \in \mathbb{Z}^{n \times n}, C, P \in \mathbb{Z}^{1 \times n}$ and $\alpha, \beta \in \mathbb{Z}$

Result: F_E = Set of all optimal solutions of $ILFP_E$

```

1 Initialization  $k \leftarrow 1, l \leftarrow 1, \mathcal{X} \leftarrow \{X \in \mathbb{Z}^n \mid AX = b, X \geq 0\}$ 
2  $f(X) \leftarrow \frac{CX + \alpha}{PX + \beta}$   $\triangleright ILFP \quad \min_{X \in \mathcal{X}} f(X)$ 
3  $\mu \leftarrow \min_{X \in \mathcal{X}} PX + \beta, m \leftarrow \min_{X \in \mathcal{X}} CX + \alpha, M \leftarrow \max_{X \in \mathcal{X}} CX + \alpha,$ 
4 if  $m \geq 0$  then
5    $g(X) \leftarrow \frac{CX + \alpha}{PX + \beta}$ 
6 else
7    $\hat{m} \leftarrow \min_{X \in \mathcal{X}} (U + C)X + \alpha, \hat{M} \leftarrow \max_{X \in \mathcal{X}} (U + C)X + \alpha$ 
8   if  $M \leq 0$  then
9      $g(X) \leftarrow \frac{CX + \alpha}{\mu}$ 
10  if  $m < 0$  and  $M > 0$  then
11     $g(X) \leftarrow \frac{CX + \alpha - M}{\mu}$ 
12  $g_{max} \leftarrow \max_{X \in \mathcal{X}} g(X), G^1 \leftarrow \{Y \in \mathcal{X} \mid g(Y) = \min_{X \in \mathcal{X}} g(X)\},$ 
13  $f_{min} \leftarrow \min\{f(X) \mid X \in G^1\}$ 
14 while  $g(Y) < f_{min}$  and  $g(Y) < g_{max}$  for  $Y \in G^l$  do
15    $\mathcal{X}_{l+1} \leftarrow \{X \in \mathcal{X}_l \mid \sum_{j \in N_l} x_j \geq 1\}$ 
16    $l \leftarrow l + 1$ 
17    $G^l \leftarrow \{Y \in \mathcal{X}_l \mid g(Y) = \min_{X \in \mathcal{X}_l} g(X)\}$   $\triangleright$  set of  $l^{th}$  ranked solutions of  $ILFP$ 
18    $f_{min} \leftarrow \min\{f(X) \mid X \in \cup_{j=1}^l G^j\}$ 
19  $F^1 \leftarrow \{Y \in \cup_{j=1}^l G^l \mid f(Y) = f_{min}\}$   $\triangleright$  set of optimal solutions of  $ILFP$ 
20  $F_E^1 \leftarrow \emptyset$ 
21 foreach  $Y \in F^1$  do
22   Solve  $P(Y)$ 
23   if If optimal value of problem  $P(Y) = 0$  then
24      $F_E^1 \leftarrow [F_E^1 \ X]$ 
25 while  $F_E^k \leftarrow \emptyset$  do
26    $k \leftarrow k + 1$ 
27   if  $\{X \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j, X^{k-1} \in F^{k-1}\} = \emptyset$  then
28      $f_{min}^k = -\infty$ 
29   else
30      $f_{min}^k \leftarrow \min\{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\}$  for  $X^{k-1} \in F^{k-1}$ 
31     while  $g(Y) < f_{min}^k$  and  $g(Y) > g_{max}$  for  $Y \in G^l$  do
32        $\mathcal{X}_{l+1} \leftarrow \{X \in \mathcal{X}_l \mid \sum_{j \in N_l} x_j \geq 1\}$ 
33        $l \leftarrow l + 1, G^l \leftarrow \{Y \in \mathcal{X}_l \mid g(Y) = \min_{X \in \mathcal{X}_l} g(X)\},$ 
34        $f_{min}^k \leftarrow \min\{f(X) \mid f(X) > f(X^{k-1}), X \in \cup_{j=1}^l G^j\}$  for  $X^{k-1} \in F^{k-1}$ 
35      $F^k \leftarrow \{Y \in \cup_{j=1}^l G^l \mid f(Y) = f_{min}^k\}$   $\triangleright$  set of  $k^{th}$  ranked solutions of  $ILFP$ 
36      $F_E^k \leftarrow \emptyset$ 
37     foreach  $Y \in F^k$  do
38       Solve  $P(Y)$ 
39       if If optimal value of problem  $P(Y) = 0$  then
40          $F_E^k \leftarrow [F_E^k \ X]$ 
41 if  $F_E^k \neq \emptyset$  then
42    $F_E \leftarrow F_E^k$   $\triangleright$  set of optimal solutions of  $ILFP_E$ 

```

Linear fractional function over integer efficient set

If the matrices D and Q in the $IQFP_E$ are null matrices, then it reduces to a problem of optimizing a linear fractional function over integer efficient set, which is defined as :

$$(ILFP_E) \quad \min_{X \in E} f(X) = \frac{CX + \alpha}{PX + \beta},$$

where $C, P \in \mathbb{Z}^{1 \times n}$, $\alpha, \beta \in \mathbb{Z}$ and $PX + \beta > 0$ for all $X \in \chi$. The Algorithm 11 is a particular case of the Algorithm 10 for solving $ILFP_E$.

Quadratic function over integer efficient set

If the matrices P and Q in $IQFP_E$ are null matrices, then it reduces to a problem of optimizing a quadratic function over integer efficient set, which is defined as :

$$(IQP_E) \quad \min_{X \in E} f(X) = X^T DX + CX + \alpha,$$

where E is the efficient set of $MILP$, $D \in \mathbb{Z}^{n \times n}$, $C \in \mathbb{Z}^{1 \times n}$ and $\alpha \in \mathbb{Z}$.

In this problem, the value of $\mu = \beta$ and $V_i = 0 \forall i \in \{1, 2, \dots, n\}$, where V_i is i^{th} component of vector $V \in \mathbb{Z}^{1 \times n}$. Therefore, function $g(X)$ is defined as follows:

Table 7.2: The construction of function $g(X)$

m	M	$\hat{m}$	$\hat{M}$	$g(X)$
$m \geq 0$	—	—	—	$\frac{(U+C)X+\alpha}{\beta}$
—	$M \leq 0$	—	—	$\frac{(U+C)X+\alpha}{\beta}$
$m < 0$	$M > 0$	—	$\hat{M} \leq 0$	$\frac{(U+C)X+\alpha}{\beta}$
—	—	$\hat{m} < 0$	$\hat{M} > 0$	$\frac{(U+C)X+\alpha-\hat{M}}{\beta}$

It can be seen from Table 7.2, $g(X)$ is a linear function. A particular case of the developed method for solving IQP_E is discussed in [122].

Linear function over integer efficient set

If the matrices D , P , and Q in $IQFP_E$ are null matrices, then it reduces to the problem of optimizing ILP over the efficient set. A particular case of the proposed method for solving a linear function over the efficient set of $MILP$ can be seen in [121].

7.4 Theoretical development of a criterion space search algorithm

It may be noted that the Algorithm 10 finds the set of all optimal solutions of $IQFP_E$ by enumerating ranked solutions of $IQFP$ and terminates when obtained ranked solution satisfies the efficient test. Now, we will develop a criterion space search algorithm to find the set of all optimal solutions of $IQFP_E$. In the criterion space search algorithm, we will utilize a ranking approach, efficiency test, and Sylva and Crema [140] cut for solving $IQFP_E$.

7.4.1 Description of criterion space search algorithm

1. Initial Step:

Set $j = 1$, $f_{lower} = -\infty$, $f_{upper} = \infty$ and $\chi^1 = \chi$.

2. General Step:

Consider the following problem:

$$(IQFP^j) \quad \min_{X \in \chi^j} f(X).$$

Let F_j be the set of optimal solutions of $IQFP^j$.

(a) To find the set of optimal solutions of $IQFP^j$, consider the following problem:

$$(ILP^j) \text{ or } (ILFP^j) \quad \min_{X \in \chi^j} g^j(X).$$

The construction of function $g^j(X)$ is shown in Table 7.3 and following notations has been used in Table 7.3.

$$\begin{aligned} \mu^j &= \min_{X \in \chi^j} X^T Q X + P X + \beta, \\ m^j &= \min_{X \in \chi^j} X^T D X + C X + \alpha, \quad M^j = \max_{X \in \chi^j} X^T D X + C X + \alpha, \\ \hat{m}^j &= \min_{X \in \chi^j} (U^j + C) X + \alpha, \quad \hat{M}^j = \max_{X \in \chi^j} (U^j + C) X + \alpha, \\ U_i^j &= \min_{X \in \chi^j} X^T D_i, \quad V_i^j = \max_{X \in \chi^j} X^T Q_i \quad \forall i \in \{1, 2, \dots, n\}. \end{aligned}$$

where $U^j, V^j \in \mathbb{Z}^{1 \times n}$, U_i^j, V_i^j, D_i and Q_i are i^{th} component of vectors U^j and V^j and i^{th} component of matrices D and Q respectively.

Table 7.3: The construction of function $g^j(X)$

m^j	M^j	$\hat{m}^j$	$\hat{M}^j$	$g^j(X)$
$m^j \geq 0$	—	—	—	$\frac{(U^j+C)X+\alpha}{(V^j+P)X+\beta}$
—	$M^j \leq 0$	—	—	$\frac{(U^j+C)X+\alpha}{\mu^j}$
$m^j < 0$	$M^j > 0$	—	$\hat{M}^j \leq 0$	$\frac{(U^j+C)X+\alpha}{\mu^j}$
—	—	$\hat{m}^j < 0$	$\hat{M}^j > 0$	$\frac{(U^j+C)X+\alpha-\hat{M}^j}{\mu^j}$

Now, for ranking values of $g^j(X)$ for all $X \in \chi^j$, consider the following sets:

$$G_j^1 = \{Y \in \chi^j \mid g^j(Y) = \min_{X \in \chi^j} g^j(X)\}, \quad G_j^p = \{Y \in \chi^j \mid g^j(Y) = \max_{X \in \chi^j} g^j(X)\} \quad (7.4.1)$$

For all $l \in \{2, 3, \dots, p-1\}$,

$$G_j^l = \{Y \in \chi \mid g^j(Y^{l-1}) < g^j(Y) < g^j(Y^{l+1}), Y^{l-1} \in G_j^{l-1}, Y^{l+1} \in G_j^{l+1}\}. \quad (7.4.2)$$

In order to find the elements of sets G_j^l for all $l \in \{1, 2, \dots, p\}$, construct the following sets:

$$\chi_1^j = \chi^j, \quad \chi_l^j = \{Y \in \chi^j \mid g^j(Y) > g^j(Y^{l-1}), Y^{l-1} \in G_j^{l-1}\} \text{ for all } l \in \{2, 3, \dots, p\}. \quad (7.4.3)$$

It can be observed that

$$G_j^l = \{Y \in \chi_l^j \mid g^j(Y) = \min_{X \in \chi_l^j} g^j(X)\} \text{ for all } l \in \{1, 2, \dots, p\}. \quad (7.4.4)$$

It is evident by Table 7.3 that the function $g^j(X)$ is either a linear function or a linear fractional function. For finding ranked solutions of problem (ILP^j) or $(ILFP^j)$, an objective function cut or a ranking cut (as discussed in Section 7.2) can be utilized.

(b) The relation discussed in Corollary 7.3.7 can be utilized for finding set F_j .

(c) After finding elements of set F_j and G_j^i for all $i \in \{1, 2, \dots, l\}$, construct the set:

$$E^j = \{Y \mid Y \text{ optimal solution of } P(X) \text{ for some } X \in \cup_{i=1}^l G_j^i\}.$$

(d) Then, compute the value of

$$f_{lower}^j = \min \{f(X) \mid X \in G_j^l\}, \quad f_{upper}^j = \min \{f(Y) \mid Y \in \cup_{i=1}^j E^i\}.$$

(e) Update the value of $f_{lower} = f_{lower}^j$ and $f_{upper} = f_{upper}^j$.

(f) If $f_{upper} > f_{lower}$, set $j = j + 1$ and construct the set χ^j using Equation 7.4.5.

$$\chi^j = \{X \in \chi^{j-1} \mid C^i X \geq (C^i Y_j + 1)\rho + (1 - \rho)C_{min}^i, Y_j \in E_j \text{ for all } i \in \{1, 2 \dots s\}\}, \quad (7.4.5)$$

where $C_{min}^i = \min_{X \in \chi} C^i$ for all $i \in \{1, 2 \dots s\}$, $\rho \in \{0, 1\}$.

3. Repeat Step 2 until $f_{upper} > f_{lower}$.

4. **Terminal Step:**

If $f_{upper} \leq f_{lower}$, Then, $F_E = \{X \in \cup_{i=1}^j E^i \mid f(X) = f_{upper}^j\}$ is declared as the set of optimal solutions of $IQFP_E$

Theorem 7.4.1 [121] *If $Y \in E^j$, then $Y \in E$.*

It is evident by above theorem, every element of set E^j is efficient solution of $MILP$, i.e.,

$$E^j \subset E.$$

7.5 Comparative analysis of feasible space search algorithm and criterion space search algorithm

1. In the Algorithm 10, ranking of integer feasible solutions of $IQFP$ are used for solving $IQFP_E$, whereas Algorithm 12 utilizes ranked solutions of $IQFP$ as well as efficient solutions of $MILP$ for solving $IQFP_E$.
2. The Algorithm 10 terminates after generating an efficient solution. But, Algorithm 12 may generate more efficient solutions.
3. Both algorithms solve $IQFP_E$ in a finite number of steps. The working of both algorithms is based on the ranking of integer feasible solutions of $IQFP$. In the Algorithm 10, ranking of integer solutions of $IQFP$ is done until a stage is reached when $F_E^k \neq \emptyset$ for some k . In the Algorithm 12, sets E^j of efficient solutions of $MILP$ are also generated at each iteration and at each j^{th} stage, f_{upper}^j is calculated from elements of E^j which provides an upper bound to $IQFP_E$ and it keeps on shrinking at every stage. This algorithm terminates when $f_{lower}^j \geq f_{upper}^j$ for some j .
4. The Algorithm 12 solves $IQFP_E$ by iteratively solving $IQFP^j$, whereas Algorithm 10 solves $IQFP_E$ by iteratively solving $IQFP_j$ until an optimal solution of $IQFP_j$

Algorithm 12: Hybrid algorithm to solve $IQFP_E$

Data: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $C^i \in \mathbb{R}^{1 \times n}$, $D, Q \in \mathbb{Z}^{n \times n}$, $C, P \in \mathbb{Z}^{1 \times n}$ and $\alpha, \beta \in \mathbb{Z}$

Result: F_E = Set of all optimal solutions of $IQFP_E$

```

1 Initialization  $\mathcal{X} \leftarrow \{X \in \mathbb{Z}^n \mid AX = b, X \geq 0\}$ 
2  $f(X) \leftarrow \frac{X^T DX + CX + \alpha}{X^T QX + PX + \beta}$   $\triangleright IQFP \quad \min_{X \in \mathcal{X}} f(X)$ 
3 for  $i \leftarrow 1 : s$  do
4    $C_{min}^i \leftarrow \min_{X \in \mathcal{X}} C^i$   $\triangleright C_{min}^i$  is the  $i^{th}$  component of vector  $C_{min}$ 
5  $j \leftarrow 1, \mathcal{X}^j \leftarrow \mathcal{X}, f_{lower} \leftarrow -\infty, f_{upper} \leftarrow \infty$ 
6 while  $f_{upper} > f_{lower}$  do
7    $\mu^j \leftarrow \min_{X \in \mathcal{X}^j} X^T QX + PX + \beta, m^j \leftarrow \min_{X \in \mathcal{X}^j} X^T DX + CX + \alpha, M^j \leftarrow \max_{X \in \mathcal{X}^j} X^T DX + CX + \alpha,$ 
8   for  $i \leftarrow 1 : n$  do
9      $U_i^j \leftarrow \min_{X \in \mathcal{X}^j} X^T D_i,$   $\triangleright U_i^j$  and  $D_i$  are  $i^{th}$  component of vector  $U^j$  and matrix  $D$  respectively
10     $V_i^j \leftarrow \max_{X \in \mathcal{X}^j} X^T Q_i$   $\triangleright V_i^j$  and  $Q_i$  are  $i^{th}$  component of vector  $V^j$  and matrix  $Q$  respectively
11    if  $m^j \geq 0$  then
12       $g^j(X) \leftarrow \frac{(U^j + C)X + \alpha}{(V^j + P)X + \beta}$   $\triangleright ILFP^j \quad \min_{X \in \mathcal{X}} g^j(X)$ 
13    else
14       $\hat{m}^j \leftarrow \min_{X \in \mathcal{X}^j} (U^j + C)X + \alpha, \hat{M}^j \leftarrow \max_{X \in \mathcal{X}^j} (U^j + C)X + \alpha$ 
15      if  $(m^j < 0, M^j > 0 \text{ and } \hat{M}^j \leq 0)$  or  $(M^j \leq 0)$  then
16         $g^j(X) \leftarrow \frac{(U^j + C)X + \alpha}{\mu^j}$ 
17        if  $\hat{m}^j < 0$  and  $\hat{M}^j > 0$  then
18           $g^j(X) \leftarrow \frac{(U^j + C)X + \alpha - \hat{M}^j}{\mu^j}$ 
19       $g_{max}^j \leftarrow \min_{X \in \mathcal{X}^j} g^j(X), l \leftarrow 1$ 
20       $\mathcal{X}_l^j = \mathcal{X}^j$ 
21       $G_j^l \leftarrow \{Y \in \mathcal{X}_l^j \mid g^j(Y) = \min_{X \in \mathcal{X}_l^j} g^j(X)\}, f_{lower}^j \leftarrow \min\{f(X) \mid X \in G_j^l\}$ 
22      while  $g^j(Y) < f_{lower}^j$  and  $g^j(Y) < g_{max}^j$  for  $Y \in G_j^l$  do
23        if  $m^j \geq 0$  then
24           $\mathcal{X}_{l+1}^j \leftarrow \{X \in \mathcal{X}_l^j \mid \sum_{i \in \bar{N}_l} x_i \geq 1\}$   $\triangleright$  Ranking cut discussed in [131]
25        else
26           $\mathcal{X}_{l+1}^j = \{X \in \mathcal{X}_l^j \mid g^j(X) \geq g^j(Y) - 1 \text{ for } Y \in G_j^l\}$   $\triangleright$  Objective function cut
27         $l \leftarrow l + 1$ 
28         $G_j^l \leftarrow \{Y \in \mathcal{X}_l^j \mid g^j(Y) = \min_{X \in \mathcal{X}_l^j} g^j(X)\}$ 
29         $f_{lower}^j \leftarrow \min\{f(X) \mid X \in \cup_{i=1}^l G_j^i\}$ 
30       $E^j = \{Y \mid Y \text{ is an optimal solution of } P(X) \text{ for some } X \in \cup_{i=1}^l G_j^i\}$ 
31       $f_{upper}^j = \min\{f(Y) \mid Y \in \cup_{i=1}^j E^i\}.$ 
32       $F_j \leftarrow \{Y \in \cup_{i=1}^j G_j^i \mid f(Y) = f_{lower}^j\}$   $\triangleright$  set of optimal solutions of  $IQFP^j$ 
33       $f_{upper} \leftarrow f_{upper}^j, f_{lower} \leftarrow f_{lower}^j$ 
34      if  $f_{upper} \leq f_{lower}$  then
35         $F_E \leftarrow \{X \in \cup_{i=1}^j E^i \mid f(X) = f_{upper}^j\}$   $\triangleright$  set of optimal solutions of  $IQFP_E$ 
36      else
37         $j \leftarrow j + 1$ 
38         $\mathcal{X}^j \leftarrow \{X \in \mathcal{X}^{j-1} \mid C^i X \geq (C^i Y_j + 1)\rho + (1 - \rho)C_{min}^i, \rho^i \in \{0, 1\}, Y_j \in F_j \text{ for all } i \in \{1, 2, \dots, s\}, \sum_{i=1}^s \rho^i = 1\}$   $\triangleright$  Sylva and Crema cut [140]
39        Consider  $(IQFP^j)$   $\min_{X \in \mathcal{X}^j} f(X)$ 
40

```

satisfies efficiency test. So, Algorithm 10 terminates after finding an efficient solution.

5. The Algorithm 12 may generate more efficient solutions than Algorithm 10 because at every stage a set E^j of efficient solutions is generated.
6. In the Algorithm 10 only alternate solutions of ILP or $ILFP$ are explored, whereas in the Algorithm 12 alternate solutions of ILP or $ILFP$ as well as $P(Y)$ are explored for all $Y \in G_j^l$. So, the Algorithm 12 explore larger number of edges.

7. The Algorithm 12 also provides information regarding the number of iterations required for solving the problem. At each iteration, we get an idea about the upper bound on the number of iterations required to terminate the algorithm. At each j^{th} iteration, this bound is given by $l^j = f_{upper}^j - f_{lower}^j$, which keeps on tightening at every stage. But, using Algorithm 10, we will not get any idea about the iterations required for solving $IQFP_E$ without reaching the last iteration.

7.6 Numerical Section

To elaborate the working of the proposed algorithms, we will solve numerical examples using developed methods. Further, we will compare the working of the feasible space search algorithm and criterion space search algorithm.

Example 7.6.1

$$(IQFP_E) \quad \min_{X \in E} f(X) = \frac{2x_1^2 + 4x_1x_2 - x_1 + x_2}{2x_1x_2 + x_2^2 + x_1},$$

where E is the set of all efficient solutions of the problem, as follows:

$$(MILP) \quad \min_{X \in \chi} (x_1 - 7x_2, -8x_1 - 5x_2),$$

where $\chi = \{(x_1, x_2) \mid 3x_1 + 2x_2 \geq 6, 4x_1 + 5x_2 \leq 20, x_1, x_2 \geq 0, \text{integers}\}$.

The problem $(IQFP_E)$ is closely related to the following integer quadratic fractional programming $IQFP$. So, ranked solutions of $IQFP$ can be used for solving $IQFP_E$.

$$(IQFP) \quad \min_{X \in \chi} f(X) = \frac{2x_1^2 + 4x_1x_2 - x_1 + x_2}{2x_1x_2 + x_2^2 + x_1}.$$

As $m = \min_{X \in \chi} 2x_1^2 + 4x_1x_2 - x_1 + x_2 \geq 0$, therefore using Table 7.1, we get

$$g(X) = \frac{3x_1 + x_2}{5x_1 + 5x_2}.$$

Now, we will solve the problem $(IQFP_E)$ using Algorithm 10.

1. Search for set F^1

- (a) To find set F^1 , we solved the problem, as follows:

$$(ILFP_1) \quad \min_{X \in \chi_1} g(X) = \frac{3x_1 + x_2}{5x_1 + 5x_2}, \quad \text{where } \chi_1 = \chi.$$

It can be seen from above table, $\Delta_j \leq 0$ for all $j \in \{1, 2, 3, 4\}$. So, $(0, 4)$ will be

Table 7.4: Optimal table of $ILFP_1$

d_B	c_B	x_B	x_1	x_2	x_3	x_4	b_j
0	0	x_3	-1.4	0	1	0.4	2
5	1	x_2	0.8	1	0	0.2	4
		$z_j^1 - c_j$	-2.2	0	0	0.2	$Z_1 = 4$
		$z_j^2 - d_j$	-1	0	0	1	$Z_2 = 20$
		Δ_j	-40	0	0	0	

an optimal solution of $ILFP_1$ with objective function value 0.2. Since $\hat{N}_l = \{4\}$, so to get an alternate optimal solution x_4 is introduced into the basis. After introducing x_4 into the basis, we get the solution $(0, 3)$. So, $G^1 = \{(0, 4), (0, 3)\}$. Since $f(0, 4) = 0.25$, $f(0, 3) = 0.333$ and $g(Y) = 0.2$ for all $Y \in G^1$. So, $g(Y) < \min\{f(X) \mid X \in G^1\}$ for all $Y \in G^1$. Since $g(Y) < \min\{f(X) \mid X \in G^1\}$ for all $Y \in G^1$. Therefore, for finding set F^1 , set $l=l+1$.

(b) Solve the following problem:

$$(ILFP_2) \quad \min_{X \in \chi_2} g(X) = \frac{3x_1 + x_2}{5x_1 + 5x_2} \quad \text{where } \chi_2 = \{X \in \chi_1 \mid x_1 \geq 1\}.$$

After applying simplex method on problem $(ILFP_2)$, we get a non-integer

Table 7.5: Optimal simplex table of $ILFP_2$

d_B	c_B	x_B	x_1	x_2	x_3	x_4	x_5	b_j
0	0	x_3	0	0	1	0.4	-1.4	3.4
5	1	x_2	0	1	0	0.2	0.8	3.2
5	3	x_1	1	0	0	0	-1	1
		$z_j^1 - c_j$	0	0	0	0.2	-2.2	$Z_1 = 6.2$
		$z_j^2 - d_j$	0	0	0	1	-1	$Z_2 = 21$
		Δ_j	0	0	0	-2	-40	

optimal solution $(1, 3.2)$ (as shown in Table 7.5). For finding an integer optimal solution of $ILFP_2$, we have used the branch and bound technique. The branch and bound technique produces two nodes $N_1 : x_2 \leq 3$ and $N_2 : x_2 \geq 4$. If a constraint $x_2 \geq 4$ introduced to simplex table (Table 7.5), then problem $(ILFP_2)$ will become infeasible and if a constraint $x_2 \leq 3$ is introduced to simplex table (Table 7.5), we get a unique integer optimal solution $(1, 3)$ of problem $(ILFP_2)$ (as shown in Table 7.6). So, $G^2 = \{(1, 3)\}$. As $g(1, 3) = 0.3$, $f(\bar{X}) = \min\{f(X) \mid X \in \cup_{j=1}^2 G^j\} = 0.25$. So $g(Y) > \min\{f(X) \mid X \in$

$\cup_{j=1}^2 G^i\}$ for $Y \in G^2$ which implies $F^1 = \{X \in \cup_{j=1}^2 G^j | f(X) = 0.25\} = \{(0, 4)\}$.

Table 7.6: Optimal solution at node N_1

d_B	c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	b_j
0	0	x_3	0	0	1	0	-3	2	3
5	1	x_2	0	1	0	0	0	1	3
5	3	x_1	1	0	0	0	-1	0	1
0	0	x_4	0	0	0	1	4	-5	1
		$z_j^1 - c_j$	0	0	0	0	-3	1	$Z_1 = 6$
		$z_j^2 - d_j$	0	0	0	0	-5	5	$Z_2 = 20$
		Δ_j	0	0	0	0	-30	-10	

2. To obtain set F_E^1 , we solved the following problem $P(Y)$ for $Y = (0, 4)$.

$$P(Y) \quad \begin{cases} \max & \theta = \sum_{i=1}^2 \Psi^i \\ \text{s.t} & x_1 - 7x_2 - \Psi^1 = -28, \\ & -8x_1 - 5x_2 - \Psi^2 = -20, \\ & \Psi^1, \Psi^2 \geq 0, \quad X \in \chi. \end{cases}$$

The optimal value of problem $P(Y) \neq 0$ for $Y = (0, 4)$ which implies $F_E^1 = \emptyset$. Since $F_E^1 = \emptyset$. Therefore, for finding optimal solution of $IQFP_E$, find higher ranked solution of $IQFP$.

3. Search for set F^2

As $g(Y) < \min \{f(X) \mid f(X) > f(0, 4), X \in \cup_{j=1}^l G^j\}$ for all $Y \in G^l$ so to obtain set F^2 , we solved the following problem:

$$(ILFP_3) \quad \min_{X \in \chi_3} g(X) = \frac{3x_1 + x_2}{5x_1 + 5x_2} \quad \text{where} \quad \chi_3 = \{X \in \chi \mid x_5 + x_6 \geq 1\}.$$

The problem $(ILFP_3)$ has unique optimal solution $(1, 2)$ which implies $G^3 = \{(1, 2)\}$. As $g(1, 2) = 0.333, f(\bar{X}) = \min \{f(X) \mid f(X) > f(0, 4), X \in \cup_{j=1}^3 G^j\} = f(0, 3) = 0.333$, So $g(Y) = \min \{f(X) \mid f(X) > f(0, 4), X \in \cup_{j=1}^3 G^j\} = f(0, 3)$ for $Y = (1, 2)$ which implies $F^2 = \{(0, 3)\}$.

4. To obtain set F_E^2 , we solved the following problem $P(Y)$ for $Y = (0, 3)$.

$$P(Y) \quad \begin{cases} \max & \theta = \sum_{i=1}^2 \Psi^i \\ \text{s.t.} & x_1 - 7x_2 - \Psi^1 = -21, \\ & -8x_1 - 5x_2 - \Psi^2 = -15, \\ & X \in \chi, \quad \Psi^1, \Psi^2 \geq 0. \end{cases}$$

As $P(Y) = 0$ for $Y = (0, 3)$. So, $(0, 3)$ is an optimal solution of $IQFP_E$.

Remark 7.6.1 *Using the Algorithm 10, we have solved the Example 7.6.1. Notably, we haven't computed all efficient solutions of MILP and all ranked solutions of IQFP.*

Now, we will consider an example of IQP_E , and we will solve the problem using the proposed algorithms

Example 7.6.2

$$(IQP_E) \quad \min_{(x_1, x_2) \in \chi_E} f(x_1, x_2) = -2x_1^2 + 8x_2^2 - 81x_1 - 158x_2 - 40,$$

where $\chi_E = \chi \cap E$, $\chi = \mathbb{Z}^2 \cap S$, $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 + x_2 \leq 6, x_1 - x_2 \leq 2, x_2 \leq 4, (x_1, x_2) \geq 0\}$ and E is the set of all efficient solutions of MILP.

$$(MILP) \quad \max_{(x_1, x_2) \in \chi} (5x_1 - 7x_2, -2x_1 - x_2).$$

The objective of IQP_E and first objective of $MILP$ is taken from the paper of Prerna and Sharma [121].

Solution of Example 7.6.2 using Algorithm 10

The problem (IQP_E) is closely related to the following problem:

$$(IQP) \quad \min_{(x_1, x_2) \in \chi} f(x_1, x_2).$$

Now, for finding ranked solutions of IQP , consider the problem:

$$(ILP) \quad \min_{(x_1, x_2) \in \chi} g(x_1, x_2).$$

As $M = \max_{(x_1, x_2) \in \chi} f(x_1, x_2) \leq 0$, therefore using Table 7.2, we get

$$g(x_1, x_2) = -85x_1 - 158x_2 - 40.$$

Then, sets F^k for all $k \in \{1, 2, \dots, 10\}$ (as shown in Table 7.7) is generated by finding the ranked solutions of ILP . It can be observed from Table 7.7 that $\cup_{i=1}^9 F_E^i = \emptyset$ and

Table 7.7: Ranked solutions of IQP (Example 7.6.2)

F^k	(x_1, x_2)	$f(x_1, x_2)$	F_E^k
F^1	(1,4)	-627	$F_E^1 = \emptyset$
F^2	(0,4)	-544	$F_E^2 = \emptyset$
F^3	(1,3)	-525	$F_E^3 = \emptyset$
F^4	(2,2)	-494	$F_E^4 = \emptyset$
F^5	(0,3)	-442	$F_E^5 = \emptyset$
F^6	(1,2)	-407	$F_E^6 = \emptyset$
F^7	(2,1)	-360	$F_E^7 = \emptyset$
F^8	(0,2)	-324	$F_E^8 = \emptyset$
F^9	(1,1)	-273	$F_E^9 = \emptyset$
F^{10}	(2,0)	-210	$F_E^{10} = \{(2,0)\}$

$F_E^{10} = \{(2,0)\}$, so (2,0) is declared as an optimal solution of the problem IQP_E .

Remark 7.6.2 *The feasible region of Example 7.6.2 contains thirteen feasible solutions namely (1,4),(0,4),(1,3),(2,2),(0,3),(1,2),(2,1),(0,2),(1,1),(2,0),(0,1),(1,0) and (0,0) and efficient set E of IQP_E contains three elements (2,0),(1,0) and (0,0). Using the Algorithm 10, we have solved the above problem (IQP_E) without finding all the efficient solutions of $MILP$ and without enumerating all feasible solutions of IQP .*

Solution of Example 7.6.2 using Algorithm 12

Now, we will apply Algorithm 12 on the same example.

1. Initially, we set $j = 1$, $f_{lower} = -\infty$, $f_{upper} = \infty$ and $\chi^1 = \chi$.
2. Consider the problem, as follows:

$$(IQP^j) \quad \min_{(x_1, x_2) \in \chi^j} f(x_1, x_2).$$

- (a) Construct the function $g^j(x_1, x_2)$ by using Table 7.3 and consider the following problem:

$$(ILP^j) \quad \min_{(x_1, x_2) \in \chi^j} g^j(x_1, x_2).$$

where

$$g^j(x_1, x_2) = -85x_1 - 158x_2 - 40.$$

- (b) To construct the set F_j , the set G_j^l for $l = 1, 2, 3$ is generated (as shown in Table 7.8).

Table 7.8: Ranked solutions of ILP^1 (Example 7.6.2)

G_l^j	(x_1, x_2)	$g^j(x_1, x_2)$	$f(x_1, x_2)$
G_1^j	(1,4)	-757	-627
G_2^j	(0,4)	-672	-544
G_3^j	(1,3)	-599	-525

- (c) As $g^j(x_1, x_2) < \min\{f(x_1, x_2) \mid (x_1, x_2) \in \cup_{l=1}^2 G_l^j\}$ and $g^j(x_1, x_2) \geq \min\{f(x_1, x_2) \mid (x_1, x_2) \in \cup_{l=1}^3 G_l^j\} = f(1, 4)$. Therefore, $F_1 = \{(1, 4)\}$.
- (d) $E^j = \{(x_1, x_2) \mid (x_1, x_2) \text{ is an optimal solution of } P(x_1, x_2) \text{ for some } (x_1, x_2) \in \cup_{i=1}^3 G_j^i\} = \{(2, 0)\}$.
- (e) $f_{lower}^j = \min\{f(x_1, x_2) \mid (x_1, x_2) \in G_j^l\} = -627$ and $f_{upper}^j = \min\{f(x_1, x_2) \mid (x_1, x_2) \in \cup_{i=1}^j E^i\} = -210$.
- (f) Update $f_{lower} = f_{lower}^j = -627$ and $f_{upper} = f_{upper}^j = -210$.
- (g) As $f_{upper} > f_{lower}$. Therefore, set $j = j + 1$ and construct the set χ^j using Equation 7.4.5.
- (h) Then, Step 2 is repeated for higher values of j .

Table 7.9: Ranked solutions of ILP^j for $j = 2, 3$

$j = 2$				$j = 3$			
G_l^j	(x_1, x_2)	$g^j(x_1, x_2)$	$f(x_1, x_2)$	G_l^j	(x_1, x_2)	$g^j(x_1, x_2)$	$f(x_1, x_2)$
G_1^j	(0,3)	-514	-442	G_1^j	(0,1)	-198	-190
G_2^j	(0,2)	-356	-324	G_2^j	(0,0)	-40	-40

Table 7.10: The values of f_{lower}^j and f_{upper}^j for $j = 1, 2, 3$

j	F_j	f_{lower}^j	f_{upper}^j	f_{lower}	f_{upper}
1	$\{(1, 4)\}$	-627	-210	-627	-210
2	$\{(0, 3)\}$	-442	-210	-442	-210
3	$\{(0, 1)\}$	-190	-210	-190	-210

The construction of sets G_j^l and F_j for $l = 1, 2$ and $j = 2, 3$ is shown in Table 7.9. Further, values of f_{lower}^j , f_{upper}^j for $j = 1, 2, 3$ and f_{lower} , f_{upper} is updated at each step as shown in Table 7.10.

3. For $j = 3$, $f_{upper} < f_{lower}$, Therefore, $F_E = \{X \in \cup_{i=1}^3 E^i \mid f(X) = f_{upper}^3\} = \{(2, 0)\}$ and $(2, 0)$ is declared as optimal solution of IQP_E .

Remark 7.6.3 Using Algorithm 12 and Algorithm 10, we have obtained the optimal solution $(2,0)$ of IQP_E . Further, same optimal solution $(2,0)$ can also generated using the algorithm of Prerna and Sharma [121].

Table 7.11: Comparative analysis of sets F_j and F^k

F_j	(x_1, x_2)	F^k	(x_1, x_2)
F_1	$\{(1, 4)\}$	F^1	$(1, 4)$
F_2	$\{(0, 3)\}$	F^2	$(0, 4)$
F_3	$\{(0, 1)\}$	F^3	$(1, 3)$
		F^4	$(2, 2)$
		F^5	$(0, 3)$
		F^6	$(1, 2)$
		F^7	$(2, 1)$
		F^8	$(0, 2)$
		F^9	$(1, 1)$
		F^{10}	$(2, 0)$
		F^{11}	$(0, 1)$

It can be observed from Table 7.11 that F_2 is the set of fifth-ranked solutions of IQP , and F_3 is the set of all eleventh-ranked solutions of IQP .

7.7 Computational results

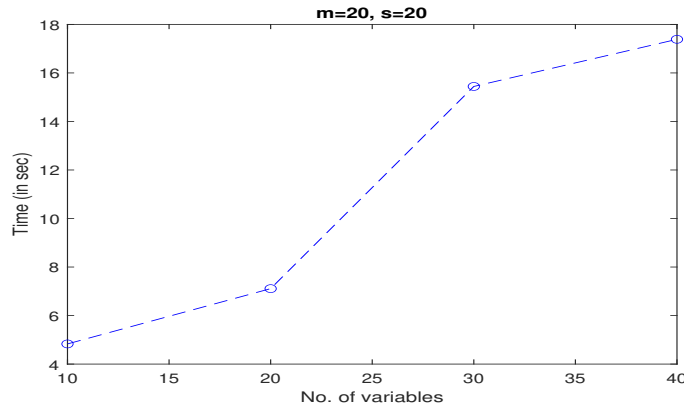


Figure 7.1: Impact of no. of variables of $IQFPE$ on CPU time

The Algorithm 10 is implemented in MATLAB 2019 on a personal HP laptop (with 4 GB of RAM, Intel Core i3-6006U processor running at 2GHz). For each couple (m, n, s) , a series of twenty problems ($IQFPE$) generated at random are solved using the proposed algorithm. The coefficients of A , b , C , D , P , Q , α and β are randomly generated integer values from set $[0, 20]$, $[0, 100]$, $[-100, 0]$, $[-100, 0]$, $[0, 100]$, $[0, 100]$, $[-100, -1]$ and $[1, 100]$ respectively. Further, C^i for all $i \in \{1, 2, \dots, s\}$ are randomly generated integer values

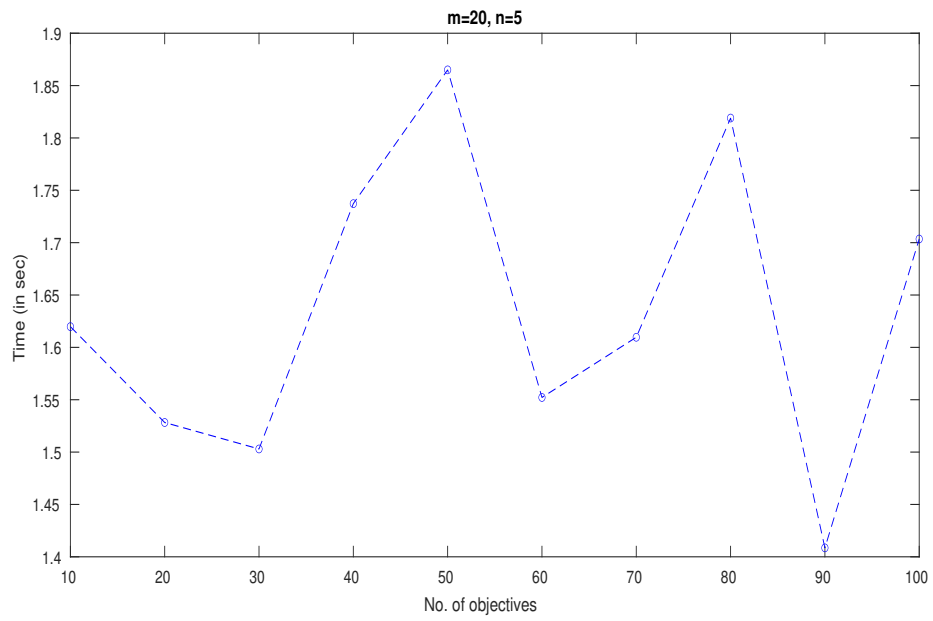


Figure 7.2: Impact of no. of objectives of $IQFP_E$ on CPU time

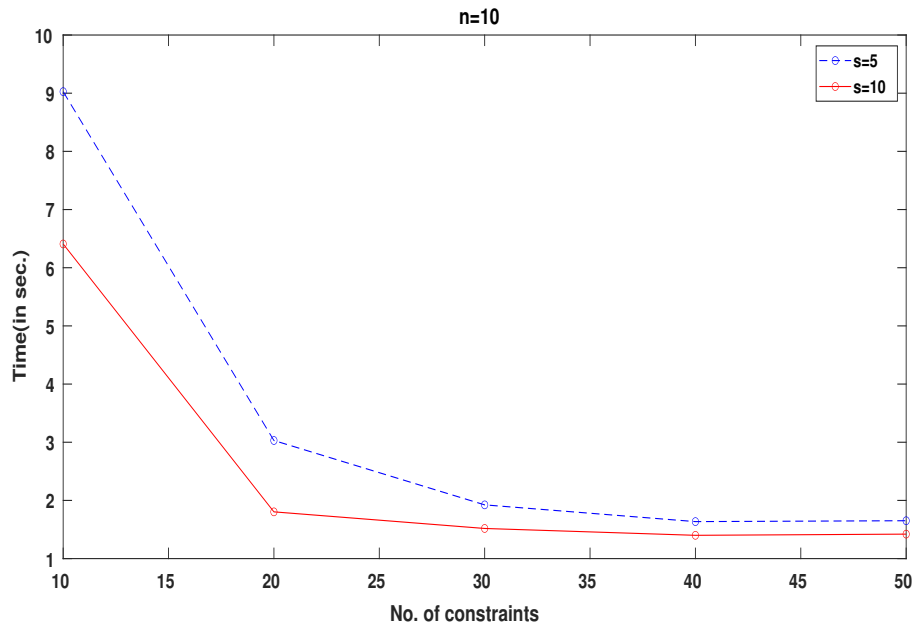


Figure 7.3: Impact of no. of constraints of $MILP$ on CPU time

from set $[-100,100]$. The Figure 7.1, Figure 7.2 and Figure 7.3 show the effect of variables, objectives and constraints of $MILP$ on processing time to solve $IQFP_E$ respectively.

7.8 Concluding remarks

1. We have developed two algorithms (feasible space search algorithm and criterion space search algorithm) for solving $IQFP_E$.
2. The feasible space search algorithm is the extension of the algorithm discussed in Chapter 6 for solving IQP_E .
3. The criterion space search follows a two-step procedure. On one side, it calculates the lower bound using ranked solutions of ILP^j or $ILFP^j$, and on the other side, it calculates the upper bound using efficient solutions.
4. The developed algorithms avoid the need to find all ranked solutions of $IQFP$ and all efficient solutions of $MILP$.
5. Although the quadratic fractional function is non-linear, and the efficient set of $MILP$ is non-convex still without using any non-linear optimization technique, we have solved $IQFP_E$.
6. The developed method can also be extended to the case with the box constraints.

Conclusion and Future Scope

Conclusion

This thesis contributes significantly to the field of MP by offering techniques to address a range of issues that arise in scenarios involving multiple criteria and integer decision variables. The seven chapters of the thesis address various aspects of MIP , including $MIQP$, $MIQFP$, ILP_E , IQP_E and $IQFP_E$. The algorithms presented in each chapter are theoretically analyzed and validated through numerical examples and computational results. Further, we have compared our proposed algorithms with existing algorithms. The developed algorithms came out to be more efficient as compared to the existing methods in terms of computational efforts.

Overall, this thesis expands the current knowledge and understanding of multi-objective integer programming and equips decision-makers with valuable tools for addressing decision-making scenarios involving discrete variables. The algorithms presented in this thesis can assist decision-makers in selecting desired efficient solutions.

Future Scope

Despite, this thesis has made significant advancements in the field of multi-objective integer programming, there are several avenues for future research and development. The future research in this field could concentrate on the following aspects:

1. **Refining algorithm efficiency:** There is room for further improving the efficiency of the developed algorithms. One potential direction is to incorporate criterion space search methods into the proposed algorithms. This could lead to faster computation time.
2. **Exploring new problem formulations:** Future research could focus on exploring new problem formulations. For example, extending the methodology discussed in this thesis to solve bi-level quadratic and quadratic fractional programming problems could broaden the applicability of the algorithms.
3. **Handling bounded variables:** The developed methodology can also be extended to handle MIP with bounded variables. This would provide decision-makers with more flexibility in real-world scenarios where variables have specific bound.

4. **Applying the algorithms to real-world scenarios:** It would be valuable to incorporate real-world case studies and practical applications to validate the effectiveness of the developed algorithms. This would provide a deeper understanding of their performance and relevance in practical decision-making contexts.
5. **Integration with existing methods:** The proposed methods can be combined with existing methods for improving efficiency and effectiveness of the developed algorithms. Exploring how these algorithms can complement and enhance existing approaches would be beneficial.

In summary, future research in the field of multi-objective integer programming should focus on refining algorithm efficiency, exploring new problem formulations, extending the methods to handle bounded variables, applying the algorithms to real-world scenarios, and integrating them with existing methods. These efforts would enhance the practicality and applicability of the algorithms and further advance the field.

Bibliography

- [1] M. Abbas and D. Chaabane. An algorithm for solving multiple objective integer linear programming problem. *RAIRO Operations Research*, 36(4):351–364, 2002.
- [2] M. Abbas and D. Chaabane. Optimizing a linear function over an integer efficient set. *European Journal of Operational Research*, 174(2):1140–1161, 2006.
- [3] M. A. Akbay, C. B. Kalayci, and O. Polat. A parallel variable neighborhood search algorithm with quadratic programming for cardinality constrained portfolio optimization. *Knowledge-Based Systems*, 198:105944, 2020.
- [4] M. J. Alves and J. Clímaco. A review of interactive methods for multiobjective integer and mixed-integer programming. *European Journal of Operational Research*, 180(1):99–115, 2007.
- [5] M. A. T. Ansary and G. Panda. A sequential quadratically constrained quadratic programming technique for a multi-objective optimization problem. *Engineering Optimization*, 51(1):22–41, 2019.
- [6] M. A. T. Ansary and G. Panda. A sequential quadratic programming method for constrained multi-objective optimization problems. *Journal of Applied Mathematics and Computing*, 64(1-2):379–397, 2020.
- [7] M. A. T. Ansary and G. Panda. A globally convergent sqcqp method for multiobjective optimization problems. *SIAM Journal on Optimization*, 31(1):91–113, 2021.
- [8] C. H. Antunes, M. J. Alves, and J. Clímaco. *Multiobjective linear and integer programming*. Springer, 2016.
- [9] R. Arora and S. Arora. A cutting plane approach for multi-objective integer indefinite quadratic programming problem. *Opsearch*, 52:367–381, 2015.
- [10] R. Arora and S. Arora. Solution of bilevel linear fractional transportation problem with pythagorean fuzzy numbers. In *Combinatorial Optimization Under Uncertainty*, pages 45–58. CRC Press, 2023.
- [11] D. Ashton and D. R. Atkins. Multicriteria programming for financial planning. *Journal of the Operational Research Society*, 30(3):259–270, 1979.

- [12] A. C. Batista, L. S. Batista, and R. Adriano. A quadratic programming approach for microwave imaging. *IEEE Transactions on Antennas and Propagation*, 69(8):4923–4934, 2021.
- [13] M. S. Bazaraa, J. J. Jarvis, and H. D. Sherali. *Linear programming and network flows*. John Wiley & Sons, 2008.
- [14] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear programming: theory and algorithms*. John wiley & sons, 2013.
- [15] H. Belkhiri, M. E.-A. Chergui, and F. Z. Ouail. Optimizing a linear function over an efficient set. *Operational Research*, pages 1–19, 2022.
- [16] H. P. Benson. Existence of efficient solutions for vector maximization problems. *Journal of optimization Theory and Applications*, 26(4):569–580, 1978.
- [17] H. P. Benson. Optimization over the efficient set. *Journal of Mathematical Analysis and Applications*, 98(2):562–580, 1984.
- [18] M. Bentobache, M. Telli, and A. Mokhtari. A sequential linear programming algorithm for continuous and mixed-integer nonconvex quadratic programming. In *Optimization of Complex Systems: Theory, Models, Algorithms and Applications*, pages 26–36. Springer, 2020.
- [19] M. Bentobache, M. Telli, and A. Mokhtari. New LP-based local and global algorithms for continuous and mixed-integer nonconvex quadratic programming. *Journal of Global Optimization*, 82(4):659–689, 2022.
- [20] D. Bhati, P. Singh, and R. Arya. A taxonomy and review of the multi-objective fractional programming (mofp) problems. *International Journal of Applied and Computational Mathematics*, 3:2695–2717, 2017.
- [21] T. K. Bhatia, A. Kumar, S. S. Appadoo, Y. Gajpal, and M. K. Sharma. Mehar approach for finding shortest path in supply chain network. *Sustainability*, 13(7):4016, 2021.
- [22] G. R. Bitran. Linear multiple objective programs with zero–one variables. *Mathematical Programming*, 13:121–139, 1977.
- [23] N. Boland, H. Charkhgard, and M. Savelsbergh. A criterion space search algorithm for biobjective integer programming: The balanced box method. *INFORMS Journal on Computing*, 27(4):735–754, 2015.

- [24] N. Boland, H. Charkhgard, and M. Savelsbergh. A new method for optimizing a linear function over the efficient set of a multiobjective integer program. *European Journal of Operational Research*, 260(3):904–919, 2017.
- [25] C. Buchheim, A. Caprara, and A. Lodi. An effective branch-and-bound algorithm for convex quadratic integer programming. *Mathematical programming*, 135:369–395, 2012.
- [26] R. Cambini, R. Riccardi, and D. Scopelliti. Solving linear multiplicative programs via branch-and-bound: a computational experience. *Computational Management Science*, 20(1):38, 2023.
- [27] R. Cambini and C. Sodini. A finite algorithm for a particular dc quadratic programming problem. *Annals of Operations Research*, 117:33–49, 2002.
- [28] X. Cao and S. Li. Neural networks for portfolio analysis with cardinality constraints. *IEEE Transactions on Neural Networks and Learning Systems*, 2023.
- [29] M. Capinski and T. Zastawniak. *Mathematics for finance*. Springer, 2003.
- [30] S. K. Chakraborty and G. Panda. Two-phase-sqp method with higher-order convergence property. *Journal of the Operations Research Society of China*, 4(3):385–396, 2016.
- [31] S. Chandra and M. Chandramohan. A branch and bound method for integer non-linear fractional programs. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 60(12):735–737, 1980.
- [32] S. Chandra and T. Gulati. A duality theorem for a nondifferentiable fractional programming problem. *Management Science*, 23(1):32–37, 1976.
- [33] A. Charnes and W. W. Cooper. Programming with linear fractional functionals. *Naval Research logistics quarterly*, 9(3-4):181–186, 1962.
- [34] Y. Cherfaoui and M. Moulaï. Biobjective optimization over the efficient set of multiobjective integer programming problem. *Journal of Industrial & Management Optimization*, 17(1):117, 2021.
- [35] M. E.-A. Chergui and M. Moulaï. An exact method for a discrete multiobjective linear fractional optimization. *Journal of Applied Mathematics & Decision Sciences*, 2008(1), 2008.
- [36] M. E.-A. Chergui, M. Moulaï, and F. Z. Ouail. Solving the multiple objective integer linear programming problem. In *MCO*, pages 69–76. Springer, 2008.

- [37] A. Chinchuluun and P. M. Pardalos. A survey of recent developments in multiobjective optimization. *Annals of Operations Research*, 154(1):29–50, 2007.
- [38] S. Cicalo and V. Tralli. Joint admission control and resource allocation for d2d communications with qos constraints. In *2015 IEEE Globecom Workshops (GC Wkshps)*, pages 1–6. IEEE, 2015.
- [39] K. Dahiya. Constrained integer fractional programming problem with box constraints. *SeMA Journal*, 74:441–455, 2017.
- [40] K. Dahiya and V. Verma. Ranking in integer programming with bounded variables. *Opsearch*, 42:252–279, 2005.
- [41] N. Datta and D. Bhatia. Algorithm to determine an initial efficient basic solution for a linear fractional multiple objective transportation problem. *Cahiers du Centre d’études de recherche opérationnelle*, 26(1-2):127–136, 1984.
- [42] R. O. Day, M. P. Kleeman, and G. B. Lamont. Solving the multi-objective quadratic assignment problem using a fast messy genetic algorithm. In *The 2003 Congress on Evolutionary Computation, 2003. CEC’03.*, volume 4, pages 2277–2283. IEEE, 2003.
- [43] M. De Santis and G. Eichfelder. A decision space algorithm for multiobjective convex quadratic integer optimization. *Computers & Operations Research*, 134:105396, 2021.
- [44] S. Dhruv, R. Arora, and S. Arora. An algorithm for fully intuitionistic fuzzy multiobjective transportation problem with a goal programming perspective. In *International Conference on Intelligent and Fuzzy Systems*, pages 540–549. Springer, 2023.
- [45] C. Djamel and P. Marc. A method for optimizing over the integer efficient set. *Journal of industrial and management optimization*, 6(4):811, 2010.
- [46] W. Drici and M. Moulai. An exact method for solving multi-objective integer indefinite quadratic programs. *Optimization Methods and Software*, pages 1–15, 2018.
- [47] W. Drici, F. Z. Ouail, and M. Moulai. Optimizing a linear fractional function over the integer efficient set. *Annals of Operations Research*, 267:135–151, 2018.
- [48] J. G. Ecker and I. Kouada. Finding efficient points for linear multiple objective programs. *Mathematical Programming*, 8(1):375–377, 1975.
- [49] M. Ehrgott. *Multicriteria optimization*, volume 491. Springer Science & Business Media, 2005.
- [50] M. Ehrgott. A discussion of scalarization techniques for multiple objective integer programming. *Annals of Operations Research*, 147(1):343–360, 2006.

- [51] M. Ehrgott and X. Gandibleux. A survey and annotated bibliography of multiobjective combinatorial optimization. *OR-spektrum*, 22:425–460, 2000.
- [52] M. Ehrgott and X. Gandibleux. Approximative solution methods for multiobjective combinatorial optimization. *Top*, 12(1):1–63, 2004.
- [53] M. Ehrgott, H. W. Hamacher, K. Klamroth, S. Nickel, A. Schöbel, and M. M. Wiecek. Equivalence of balance points and pareto solutions in multiple-objective programming. *Journal of Optimization Theory and Applications*, 92:209–212, 1997.
- [54] S. Erenguc and H. Benson. An algorithm for indefinite integer quadratic programming. *Computers & Mathematics with Applications*, 21(6-7):99–106, 1991.
- [55] L. Fiaschi and M. Cococcioni. A non-archimedean interior point method and its application to the lexicographic multi-objective quadratic programming. *Mathematics*, 10(23):4536, 2022.
- [56] Y. Gajpal, M. Abdulkader, S. Zhang, and S. Appadoo. Optimizing garbage collection vehicle routing problem with alternative fuel-powered vehicles. *Optimization*, 66(11):1851–1862, 2017.
- [57] Y. Gajpal, S. Appadoo, V. Shi, and G. Hu. Optimal multi-stage group partition for efficient coronavirus screening. *Annals of Operations Research*, pages 1–17, 2022.
- [58] Y. Gajpal, V. Roy, and B. Sahay. Vehicle routing for a mid-day meal delivery distribution system. *Heliyon*, 5(1), 2019.
- [59] S. Geissel, H. Graf, J. Herbinger, and F. T. Seifried. Portfolio optimization with optimal expected utility risk measures. *Annals of Operations Research*, 39(1):1–19, 2022.
- [60] A. Ghaffari-Hadigheh, O. Romanko, and T. Terlaky. Bi-parametric convex quadratic optimization. *Optimization Methods & Software*, 25(2):229–245, 2010.
- [61] A. Goel and A. Mehra. Robust omega ratio optimization using regular vines. *Optimization Letters*, 15:2067–2108, 2021.
- [62] R. E. Gomory. An algorithm for integer solutions to linear programs. *Recent advances in mathematical programming*, 64:269–302, 1963.
- [63] J. Gondzio and E. A. Yildirim. Global solutions of nonconvex standard quadratic programs via mixed integer linear programming reformulations. *Journal of Global Optimization*, 81(2):293–321, 2021.

- [64] J. Y. Gotoh, N. Van Thoai, and Y. Yamamoto. Global optimization method for solving the minimum maximal flow problem. *Optimization Methods and Software*, 18(4):395–415, 2003.
- [65] N. Grudin. Reactive power optimization using successive quadratic programming method. *IEEE Transactions on Power Systems*, 13(4):1219–1225, 1998.
- [66] O. K. Gupta. Applications of quadratic programming. *Journal of Information and Optimization Sciences*, 16(1):177–194, 1995.
- [67] R. Gupta, L. Bandopadhyaya, and M. C. Puri. Ranking in quadratic integer programming problems. *European Journal of Operational Research*, 95(1):231–236, 1996.
- [68] R. Gupta and R. Malhotra. Multi-criteria integer linear programming problem. *Cahiers du Centre d'études de recherche opérationnelle*, 34(1):51–68, 1992.
- [69] R. Gupta and R. Malhotra. Multi-criteria integer linear fractional programming problem. *Optimization*, 35(4):373–389, 1995.
- [70] G. Hadley. Linear programming, addison-wesley. *Reading, Mass*, 1962.
- [71] Y. Haimes. On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE transactions on systems, man, and cybernetics*, (3):296–297, 1971.
- [72] Y. Y. Haimes. *Multiobjective optimization in water resources systems: The surrogate worth trade-off method*. Elsevier, 2011.
- [73] H. Isermann. Proper efficiency and the linear vector maximum problem. *Operations Research*, 22(1):189–191, 1974.
- [74] M. Jackson and M. Staunton. Quadratic programming applications in finance using excel. *Journal of the Operational Research Society*, 50(12):1256–1266, 1999.
- [75] E. Jain, K. Dahiya, and V. Verma. Integer quadratic fractional programming problems with bounded variables. *Annals of Operations Research*, 269(1):269–295, 2018.
- [76] A. Jayswal and S. Choudhury. Convergence of exponential penalty function method for multiobjective fractional programming problems. *Ain Shams Engineering Journal*, 5(4):1371–1376, 2014.
- [77] A. Jayswal, R. Kumar, and D. Kumar. Multiobjective fractional programming problems involving (p, r) - $\rho(\eta, \theta)$ -invex function. *Journal of Applied Mathematics and Computing*, 39(1):35–51, 2012.

- [78] A. Jayswal, K. Kummari, and I. Ahmad. Sufficiency and duality in minimax fractional programming with generalized (ϕ, ρ) -invexity. *Mathematical Reports*, 17(2):183–200, 2015.
- [79] R. Jha, Y. Gajpal, M. Chattopadhyay, and X. Yang. Multiple operation theatre scheduling for mitigating the disturbance caused by emergency patients. *Systems and Soft Computing*, 5:200058, 2023.
- [80] J. M. Jorge. An algorithm for optimizing a linear function over an integer efficient set. *European Journal of Operational Research*, 195(1):98–103, 2009.
- [81] P. Kaur, A. Sharma, V. Verma, and K. Dahiya. An alternate approach to solve two-level hierarchical time minimization transportation problem. *4OR*, pages 1–39, 2022.
- [82] D. Klein and E. Hannan. An algorithm for the multiple objective integer linear programming problem. *European Journal of Operational Research*, 9(4):378–385, 1982.
- [83] J. D. Knowles and D. Corne. Towards landscape analyses to inform the design of hybrid local search for the multiobjective quadratic assignment problem. *HIS*, 87:271–279, 2002.
- [84] J. Kornbluth. Ratio goals in manpower planning models. *INFOR: Information Systems and Operational Research*, 21(2):151–154, 1983.
- [85] V. Kumtepli, Y. Zhao, M. Naumann, A. Tripathi, Y. Wang, A. Jossen, and H. Hesse. Design and analysis of an aging-aware energy management system for islanded grids using mixed-integer quadratic programming. *International Journal of Energy Research*, 43(9):4127–4147, 2019.
- [86] P. Kushwah and V. Sharma. A note on solving multi-objective integer indefinite quadratic fractional programs. *Annals of Operations Research*, 289(2):459–462, 2020.
- [87] P. Kushwah and V. Sharma. An algorithm to solve multi-objective integer quadratic programming problem. *Annals of Operations Research*, 332:433–459, 2024.
- [88] K. Lachhwani. Fuzzy goal programming approach to multi objective quadratic programming problem. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 82:317–322, 2012.
- [89] C. Lalitha, P. K. Garg, and S. Khurana. Duality for multi objective fractional programming via linearization and scalarization approaches. *Opsearch*, 42:37–54, 2005.

- [90] C. Lalitha, S. Suneja, and S. Khurana. Symmetric duality involving invexity in multiobjective fractional programming. *Asia-Pacific Journal of Operational Research*, 20(1), 2003.
- [91] Z.-A. Liang, H.-X. Huang, and P. M. Pardalos. Efficiency conditions and duality for a class of multiobjective fractional programming problems. *Journal of Global Optimization*, 27:447–471, 2003.
- [92] P. Liu, S. Fattahi, A. Gómez, and S. Küçükyavuz. A graph-based decomposition method for convex quadratic optimization with indicators. *Mathematical Programming*, pages 1–33, 2022.
- [93] M. S. Lobo, M. Fazel, and S. Boyd. Portfolio optimization with linear and fixed transaction costs. *Annals of Operations Research*, 152:341–365, 2007.
- [94] B. Lokman. Optimizing a linear function over the nondominated set of multiobjective integer programs. *International Transactions in Operational Research*, 28(4):2248–2267, 2021.
- [95] B. Lokman and M. Köksalan. Finding all nondominated points of multi-objective integer programs. *Journal of Global Optimization*, 57:347–365, 2013.
- [96] K. Lu, S. Mizuno, and J. Shi. A new mixed integer programming approach for optimization over the efficient set of a multiobjective linear programming problem. *Optimization Letters*, 14(8):2323–2333, 2020.
- [97] A. Maggi, M. Wenzel, and K. Sundmacher. Mixed-integer linear programming (milp) approach for the synthesis of efficient power-to-syngas processes. *Frontiers in Energy Research*, 8:161, 2020.
- [98] S. Mahdi and D. Chaabane. A linear fractional optimization over an integer efficient set. *RAIRO Operations Research*, 49(2):265–278, 2015.
- [99] R. Mansini, W. ,odzimierz Ogryczak, M. G. Speranza, and E. T. A. of European Operational Research Societies. *Linear and mixed integer programming for portfolio optimization*, volume 21. Springer, 2015.
- [100] H. M. Markowitz. Foundations of portfolio theory. *The journal of finance*, 46(2):469–477, 1991.
- [101] B. Martos. The direct power of adjacent vertex programming methods. *Management Science*, 12(3):241–252, 1965.

- [102] A. Mekhilef, M. Moulai, and W. Drici. Solving multi-objective integer indefinite quadratic fractional programs. *Annals of Operations Research*, 296:821–840, 2021.
- [103] L. Mencarelli and C. d’Ambrosio. Complex portfolio selection via convex mixed-integer quadratic programming: a survey. *International Transactions in Operational Research*, 26(2):389–414, 2019.
- [104] M. Mesquita-Cunha, J. R. Figueira, and A. P. Barbosa-Póvoa. New ϵ - constraint methods for multi-objective integer linear programming: A pareto front representation approach. *European Journal of Operational Research*, 306(1):286–307, 2023.
- [105] A. Mishra, A. Kumar, and S. Appadoo. A note on “an algorithmic approach to solve unbalanced triangular fuzzy transportation problems”. 2023.
- [106] M. Moulai and W. Drici. An indefinite quadratic optimization over an integer efficient set. *Optimization*, 67(8):1143–1156, 2018.
- [107] P. Neumayer and D. Schweigert. Three algorithms for bicriteria integer linear programs. *Operations-Research-Spektrum*, 16:267–276, 1994.
- [108] R. Oberdieck and E. N. Pistikopoulos. Multi-objective optimization with convex quadratic cost functions: A multi-parametric programming approach. *Computers & Chemical Engineering*, 85:36–39, 2016.
- [109] F. Z. Ouail and M. E.-A. Chergui. A branch-and-cut technique to solve multiobjective integer quadratic programming problems. *Annals of Operations Research*, 267:431–446, 2018.
- [110] M. Ozlen and M. Azizoglu. Multi-objective integer programming: A general approach for generating all non-dominated solutions. *European Journal of Operational Research*, 199(1):25–35, 2009.
- [111] M. Ozlen, M. Azizoglu, and B. A. Burton. Optimising a nonlinear utility function in multi-objective integer programming. *Journal of Global Optimization*, 56(1):93–102, 2013.
- [112] M. Ozlen, B. A. Burton, and C. A. MacRae. Multi-objective integer programming: An improved recursive algorithm. *Journal of optimization Theory and Applications*, 160(2):470–482, 2014.
- [113] A. Papen, R. Vandenhoeck, J. Bolting, and F. Defaÿ. Collision-free rendezvous maneuvers for formations of unmanned aerial vehicles. *IFAC-PapersOnLine*, 50(1):282–289, 2017.

- [114] I. Pappas, N. A. Diangelakis, and E. N. Pistikopoulos. The exact solution of multi-parametric quadratically constrained quadratic programming problems. *Journal of Global Optimization*, 79(1):59–85, 2021.
- [115] P. M. Pardalos, F. Rendl, and H. Wolkowicz. The quadratic assignment problem: A survey and recent developments. 1994.
- [116] P. M. Pardalos and S. A. Vavasis. Quadratic programming with one negative eigenvalue is np-hard. *Journal of Global optimization*, 1(1):15–22, 1991.
- [117] J. Philip. Algorithms for the vector maximization problem. *Mathematical programming*, 2:207–229, 1972.
- [118] R. Picelli, R. Sivapuram, and Y. M. Xie. A 101-line matlab code for topology optimization using binary variables and integer programming. *Structural and Multidisciplinary Optimization*, 63:935–954, 2021.
- [119] S. Pramanik and P. P. Dey. Multi-objective quadratic programming problem: a priority based fuzzy goal programming. *international journal of computer Applications*, 26(10):30–35, 2011.
- [120] Prerna and V. Sharma. An algorithm for bi-objective integer linear programming problem. *Filomat*, 36(16):5641–5651, 2022.
- [121] Prerna and V. Sharma. Optimization of a linear function over an integer efficient set. *Journal of Industrial and Management Optimization*, 19(12):8633–8656, 2023.
- [122] Prerna and V. Sharma. Optimization of a quadratic programming problem over an integer efficient set. *Journal of Computational and Applied Mathematics*, 441:115651, 2024.
- [123] A. Przybylski, X. Gandibleux, and M. Ehrgott. A recursive algorithm for finding all nondominated extreme points in the outcome set of a multiobjective integer programme. *INFORMS Journal on Computing*, 22(3):371–386, 2010.
- [124] A. Przybylski, X. Gandibleux, and M. Ehrgott. A two phase method for multi-objective integer programming and its application to the assignment problem with three objectives. *Discrete Optimization*, 7(3):149–165, 2010.
- [125] Y. Qi. On the criterion vectors of lines of portfolio selection with multiple quadratic and multiple linear objectives. *Central European Journal of Operations Research*, 25:145–158, 2017.

- [126] O. M. Saad and J. B. Hughes. Bicriterion integer linear fractional programs with parameters in the objective functions. *Journal of Information and optimization Sciences*, 19(1):97–108, 1998.
- [127] O. Sadeghian, A. Moradzadeh, B. Mohammadi-Ivatloo, M. Abapour, and F. P. Garcia Marquez. Generation units maintenance in combined heat and power integrated systems using the mixed integer quadratic programming approach. *Energies*, 13(11):2840, 2020.
- [128] R. Sehgal and A. Mehra. Robust portfolio optimization with second order stochastic dominance constraints. *Computers & Industrial Engineering*, 144:106396, 2020.
- [129] S. Sharma and S. Arora. Bi-objective capacitated transportation problem with bounds over distributions and requirement capacities. *International Journal of Applied and Computational Mathematics*, 7(3):115, 2021.
- [130] V. Sharma. Multiobjective integer nonlinear fractional programming problem: A cutting plane approach. *Opsearch*, 49:133–153, 2012.
- [131] V. Sharma, K. Dahiya, and V. Verma. A ranking algorithm for bi-objective quadratic fractional integer programming problems. *Optimization*, 66(11):1913–1929, 2017.
- [132] J. Shi and Y. Yamamoto. A global optimization method for minimum maximal flow problem. *Acta Mathematica Vietnamica*, 22(1):271–287, 1997.
- [133] M. Shigeno, I. Takahashi, and Y. Yamamoto. Minimum maximal flow problem: an optimization over the efficient set. *Journal of Global Optimization*, 25(4):425–443, 2003.
- [134] J. K. Shim. A survey of quadratic programming applications to business and economics. *International Journal of Systems Science*, 14(1):105–115, 1983.
- [135] B. V. Sokolov and V. A. Ushakov. Formation reachability area as a data vector using a dynamic model for controlling information processes in the automated control system for moving objects. *Models and Methods for Researching Information Systems in Transport 2020 (MMRIST 2020)*, (1):67–75, 2020.
- [136] R. E. Steuer. Multiple criteria optimization. *Theory, Computation, and Application*, 1986.
- [137] R. E. Steuer, Y. Qi, and M. Wimmer. Computing cardinality constrained portfolio selection efficient frontiers via closest correlation matrices. *European Journal of Operational Research*, 313(2):628–636, 2024.

- [138] R. E. Steuer, M. Wimmer, and M. Hirschberger. Overviewing the transition of markowitz bi-criterion portfolio selection to tri-criterion portfolio selection. *Journal of Business Economics*, 83:61–85, 2013.
- [139] E. Sun. On optimization over the efficient set of a multiple objective linear programming problem. *Journal of optimization Theory and Applications*, 172(1):236–246, 2017.
- [140] J. Sylva and A. Crema. A method for finding the set of non-dominated vectors for multiple objective integer linear programs. *European Journal of Operational Research*, 158(1):46–55, 2004.
- [141] J. Sylva and A. Crema. Enumerating the set of non-dominated vectors in multiple objective integer linear programming. *RAIRO-Operations Research*, 42(3):371–387, 2008.
- [142] J. Teghem and P. Kunsch. A survey of techniques for finding efficient solutions to multi-objective integer linear programming. *Asia-Pacific Journal of Operational Research*, 3(2):95–108, 1986.
- [143] V. Verma, H. Bakhshi, and M. Puri. Ranking in integer linear fractional programming problems. *Zeitschrift für Operations Research*, 34:325–334, 1990.
- [144] Y. Yamamoto. Optimization over the efficient set: overview. *Journal of Global Optimization*, 22(1-4):285–317, 2002.
- [145] D. Younes, H. Sarah, and C. Djamel. Parallel optimization over the integer efficient set. *International Transactions in Operational Research*, 31(3):1949–1974, 2024.
- [146] O. Zerdani and M. Moulai. Optimization over an integer efficient set of a multiple objective linear fractional problem. *Applied Mathematical Sciences*, 5(50):2451 – 2466, 2011.
- [147] L. Zerfa and M. E.-A. Chergui. Finding non dominated points for multiobjective integer convex programs with linear constraints. *Journal of Global Optimization*, 84(1):95–117, 2022.
- [148] D. Zhang, S. Jiang, J. Liu, L. Wang, Y. Chen, Y. Xiao, S. Jiao, Y. Xie, Y. Zhang, and M. Li. Stochastic optimization operation of the integrated energy system based on a novel scenario generation method. *Processes*, 10(2):330, 2022.
- [149] S. Zhang, Y. Gajpal, S. Appadoo, and M. Abdulkader. Electric vehicle routing problem with recharging stations for minimizing energy consumption. *International journal of production economics*, 203:404–413, 2018.