NEW IDENTITIES OF FRACTIONAL S-TRANSFORM WITH ITS APPLICATIONS

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CERTIFICATE

I, **Rajeev Ranjan** hereby declare that the work contained in the thesis entitled **"NEW IDENTITIES OF FRACTIONAL S-TRANSFORM WITH ITS APPLICATIONS"** being submitted by me to **Department of Electronics and Communication Engineering, Thapar Institute of Engineering and Technology, Patiala** in fulfillment of the award of the degree of **"Doctor of Philosophy"** is a record of authentic research work carried out under the supervision of **Dr. Neeru Jindal** and **Dr. A. K. Singh**. The matter presented in this thesis does not incorporate any material previously published or written by any other person except where due references are made in the text. The results obtained in this thesis have not been submitted in part or full to any other institute or university for the award of degree or diploma.

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This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge and belief. He has worked under our supervision and fulfilled the requirements for the submission of this thesis which has reached the requisite standard.

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ABSTRACT

The current work provides a comprehensive and integrated introduction to the principles, properties and applications of the S-transform (ST) and fractional S-transform (FrST). The ST, which is a significant tool in signal processing, is a conceptual version of the FT with a Gaussian window function. It has been observed from the literature study that only linearity, scaling, time-shifting and convolution theorem of ST were documented. This led to the findings of remaining properties of ST in order to establish it as a complete transform technique. Along with this, a new better definition of convolution theorem for ST has also been presented. The FrST is a generalisation of the classical ST. The FrST has demonstrated to be a valuable technique for an analysis of a non-stationary signals. The FrST also acts as a time-frequency representation method with the frequency dependent resolution. Some of the remaining properties of FrST are proposed in this work so as to establish it as a complete transform technique. The proposed properties are convolution theorem, Parseval's theorem, correlation theorem and sampling propositions. It will provide an appropriate and reasonable model for sampling and restoration of the signal for real uses. Moreover, two kinds of reconstruction error, namely truncation error and aliasing error arises due to sampling were also discussed.

Multiresolution analysis (MRA) has recently become important, and even essential, in signal analysis and image processing. As one of the famous family members of the MRA, the wavelet transform (WT) demonstrated itself in numerous successful applications in various fields, and become one of the utmost powerful tools in the fields of signal analysis and image processing. Due to the fact that only the scale info is supplied in WT, the applications with the help of WT may be restricted when the totally referenced frequency and phase information are required. The FrST is a proposed multiresolution transform that supplies the fully referenced frequency and phase information. In the areas where ST and FrFT are used, the performance can be enhanced through the use of FrST. In addition, it has a close relationship with other transforms like Fourier transform (FT), and WT. To expand the applicability of FrST as a mathematical transform tool, MRA is used. Finally, the applications of proposed convolution theorem are demonstrated on multiplicative filtering (MF) for electrocardiogram (ECG) signal and linear frequency modulated (LFM) signal under AWGN channel. The FrST can be applied for other applications of non-stationary signal analysis, radar signal processing and also in image processing.

LIST OF PUBLICATIONS

P1. Rajeev Ranjan, N. Jindal, and A.K. Singh. "A Sampling Theorem for Fractional S-transform with Error Estimation," *Digital signal processing*, vol. 93, pp.138-150, 2019. Elsevier, (SCI Indexed) (Impact Factor 2.79).

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P3. Rajeev Ranjan, A.K. Singh, and N. Jindal, "Formulation of some useful theorem for S-transform," *Optik- international journal for light and electron optics*, vol.168, pp. 913-919, 2018. Elsevier, (SCI Indexed)(Impact Factor 1.914).

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P5. Rajeev Ranjan, N. Jindal, and A.K. Singh. "Fractional S -transform and its properties- A Comprehensive Survey," *Wireless Personal Communications*, Springer, (SCI Indexed) (Impact Factor 0.929).

P6. Rajeev Ranjan, N. Jindal, and A.K. Singh, "Multiplicative filter design using S-transform," IEEE conference 'ICMETE-2018' DOI: 10.1109/ICMETE.2018.00064. (SCOUPS Indexed).

COMMUNICATED

P1. Rajeev Ranjan, A.K. Singh, and N. Jindal, "The identities of n-dimensional S-transform," Traitement du Signal**,** (SCI Indexed) (Impact Factor 0.387).

LIST OF ABBREVIATIONS

AWGN	Additive White Gaussian Noise
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DST	Discrete S-transform
DOST	Discrete Orthonormal S-transform
DFrST	Discrete Fractional S-transform
DTFrST	Discrete Time Fractional S-transform
EEG	Electroencephalogram
ECG	Electrocardiogram
FT	Fourier Transform
FrFT	Fractional Fourier Transform
FrWT	Fractional Wavelet Transform
FrSTFT	Fractional Short Time Fourier Transform
FrST	Fractional S-transform
FLOST	Fractional Lower Order ST
IDST	Inverse Discrete S-transform
IDTFrST	Inverse Discrete Fractional S-transform
IFrST	Inverse Fractional S-transform
IST	Inverse S-transform
IFLOST	Inverse Fractional Lower Order ST
LFM	Linear Frequency Modulated
LHS	Left Hand Side
MF	Multiplicative Filtering
MST	Modified S-Transform
MRA	Multiresolution Analysis
MSA	Multiscale Approximation
MSE	Mean Square Error
N-D	N-Dimension
RHS	Right Hand Side

ST	S-Transform
STFT	Short Time Fourier Transform
SDFrST	Sparse Discrete Fractional S-Transform
SNR	Signal Noise Ratio
WT	Wavelet Transform

GLOSSARY OF SYMBOLS

$m(t,f)$	Mother Wavelet
$X(f)$	Fourier Transform of $x(t)$
$X(\tau,f)$	S-Transform of $x(t)$
d	Dilation Factor
t	Time
f	Frequency
τ,ζ	Time Shift
$g(t-\tau, f)$	Gaussian Function
$x[kT]$	Discrete Version of the Signal $x(t)$
T	Transformation
δ	Delta Function
α,β	Fractional Order
v	Fractional Frequency
$\kappa_\alpha(t,v)$	Kernel of Fractional S-Transform
$X_\alpha(\tau,v)$	Fractional S-Transform of $x(t)$
$X_\alpha(v)$	Fractional Fourier Transform of $x(t)$
$*$	Complex Conjugate
$\varphi(t)$	Basis Function
$\{U_k^\alpha\}_{k\in\mathbb{Z}}$	Sequence of Subspaces
$\otimes$	Convolution
$\mathscr{S}[.]$	S-Transform
$\mathscr{S}^{-1}$	Inverse S-Transform
$\mathscr{S}^{\alpha}$	Fractional S-transform
$\tilde{\mathscr{S}}^{\alpha}$	Discrete Time Fractional S-transform
$\mathfrak{I}$	Delay and Invert
$\mathbb{R}$	Set of Real Number
$\mathbb{Z}$	Set of Integer

$\mathbb{Z}^+$	Set of Positive Integer
$L^1[0, 2\pi]$	The Space of Absolutely Integral Function on $[0, 2\pi]$
$L^\infty[0, 2\pi]$	The Space of Absolutely Integral Function on $[0,2\pi]$
$L^2[\mathbb{R}]$	The Space of all Square Integral Function on $\mathbb{R}$
$l^2\ [\mathbb{Z}]$	The Space of Entirely Square Summable Sequence on $\mathbb{Z}$
$\|\|x(t)\|\|_\infty$	Essential Supp $\|x(t)\|$
$\|\|x(t)\|\|_0$	Essential Infim $\|x(t)\|$
Supp	Supremum
Infim	Infimum
L^2 -norm	Denoted by $\|\|x\|\|^2_{L^2} = \langle x,x \rangle_{L^2}$
l^2 -norm	Denoted by $\|\|a\|\|^2_{l^2} = \langle a,a \rangle_{l^2}$
$\mathscr{H}$	Hilbert Space
$\bigcup$	Union
$\bigcap$	Intersection
$\in$	Set
$\subseteq$	Subset
$\oplus$	Orthogonal Sum
$\Phi(\tau, f)$	S-Transform of $\varphi(t)$
$\Psi(\tau,f)$	S-Transform of $\phi(t)$
γ	Set of Positive Integer with Zero
$\tilde{\Phi}(\tau, f)$	DTST of sampling sequence $\varphi[n]$
sinc(t)	Sinc Function
$\Re^n$	n- Tuples
$\forall$	For All
$\perp$	Orthogonal to
$\oplus$	Orthogonal sum
$\|\ \|$	Norm

$\mid \; \mid$	Modula's
$\rightarrow$	Tends to
Θ	Compliment
$\sim$	Discrete Time
$\langle \bullet, \bullet \rangle = 0$	Orthogonal of two function
$N^{\alpha}(\tau, v)$	Fractional S-Transform of Noise
f_0	Centre Frequency of LFM signal

LIST OF FIGURES

LIST OF TABLES

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

"***Science** is a powerful way to the systematic study of the structure and behaviour of the physical and natural world through observation, experimentation and analysis, and **Technology** is the applications of scientific knowledge for practical purposes.*"

-Anonymous

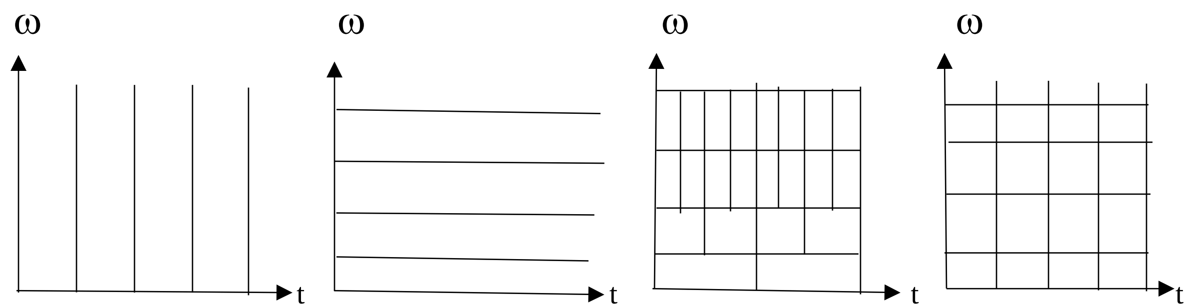
In the domain of signal processing, the term 'Transform' is frequently used. Transform is a technique to convert one form of signal to another form so that it becomes easy to analyze or process. Various types of transforms [1-3] and their applications are demonstrated by several researchers like Fourier transform (FT) [3, 4], fractional FT (FrFT) [5, 7], short-time FT (STFT) [6], Wavelet transform (WT) [7, 8], fractional WT (FrWT) [9, 10], S-transform (ST) [11, 12] and fractional ST (FrST) [13]. In the FT technique, a signal is either analyzed in the time or frequency domain. The time-domain signal exhibits facts about the signal intensities and temporal evolution. For deterministic signals, analysis is usually based on instantaneous power spectrum or energy density spectrum. For random signals, the analysis tool depends on the auto-correlation functions and the power spectrum.

1.1 HISTORICAL DEVELOPMENT OF S-TRANSFORM

The time-domain signal exhibits information about signal intensities and chronological evolution. For deterministic signal, analysis is usually based on an instantaneous power or energy density spectrum. For a random signal, the analysis tool depends on the autocorrelation function and the power spectrum. FT explores the signal at various frequencies and their relative magnitudes. However in the FT, only frequency resolution occurs but not time resolution. That means FT can analyse the signal frequency response but cannot produce their arrival time of frequency. To compensate this drawback, in few decades several transform techniques such as WT [7, 15], Wigner transform [18], STFT [6] and ST [11], and their fractional form came into existence. Where the STFT is a classical transform for the time-frequency analysis. In this, the FT of multiplication of original signal $x(t)$ with window $w(t-\tau)$ is computed. The FT changes signal domain from time to frequency by coordinating over time axis. But for non-stationary signal, (i.e., the frequency components are a function of time) at that point they can't point-out, when a specific frequency rises. The STFT overcomes this drawback of FT by presenting a window. The short time FrFT (STFrFT) is improved form of FrFT for analysis of a non-stationary signals. The thought behind STFrFT was dividing the signals by utilizing a period over narrow window and acting the FrFT spectrum for every section. As, the FrFT is calculated for every window section of the signal [86, 87], WT cannot be used appropriately because infinite storage is required. In WT, only scale information can be expected with a modulated phase information [9, 10]. Similar to FT, WT defines with the extension of a set of basis function. Unlike the FT, WT does not expand in the trigonometric polynomial forms but expands in wavelet forms. The mother wavelet is observed that every application using fast FT (FFT) is formulated by wavelet to deliver more confined temporal and frequency information. In STFT, very small-frequency components cannot be detected in the spectral because fixed-size window; however, WT overcomes this STFT problem. WT is considered to strike an equilibrium between frequency (finite bandwidth) and time (finite length) domain. The wavelet analysis is considered as a complex function and satisfy the following circumstances [30-36]. The disadvantage of the Wigner transform is cross term, which occurs due to autocorrelation function [16]. The cross-terms produce noise or distortion in the signal analysis. However, the fixed window size is disadvantage of STFT for each frequency components [17]. The STFT is usually not invertible in contrast of ST. While the drawback of the Gabor transform is a trade-off amongst time-frequency resolutions, caused due to the stable width of the windows [14]. The ST gives a time-frequency representation in contrast with frequency representation by FT [17, 18]. When the rotating machinery bearing breakdown occurs then the signal obtained by the sensor is dynamic such as- seismic signal, electrocardiograms (ECG) signal, voice signal, and genomic signal. Such signals contain time-bounded events and artefacts [19-22]. Since non-stationary signals have time-varying statistical properties, therefore time-frequency based methods are used to analyse this type of signal [23, 24]. At low frequency, multiresolution analysis is used to obtain decent frequency and deficient time resolution and vice-versa at high frequency. The concept behind the time-frequency representation is to distribute the signals into smaller parts after that parts are analyse separately. In this way, the analysed signal gives more information about different frequencies.

A time-frequency representation of FT, WT [15, 19] and ST is shown in Fig1.1.The frequency information is absolutely lost in Fig.1.1 (a), because the time axis is consistently divided. In Fig.1.1

(b) frequency axis is divided uniformly. Hence, the time axis information is entirely lost. Therefore, the frequencies resolution can be extracted, when integrated along the time axis. Fig.1.1 (c) express the time frequency pot in WT. In the expression of WT the scaling parameter ‘a’ defined is inversely proportional to the frequency. Therefore, at large frequency, small ‘a’ is used and vice versa. The additional window in ST considers time information and frequency resolution, which depends on resolution of time or size of the window. At a specific frequency range, it cannot be zoomed because the box is uniformly located as presented in Fig.1.1 (d). The time-frequency representation is control by the width of the window in the transformed domain.



(a) Basis $\delta(t - t_k)$ (b) Fourier transform (c) Wavelet transform (d) S-transform

Fig. 1.1 Time-frequency representation method

The STFT is a classical transform for time-frequency analysis. The ST mainly demonstrate some particular frequency segments in signal processing. It is a time-frequency restriction procedure with subordinate frequency resolution and compensates the drawback of STFT [18]. To control the complex Fourier, signal ST is utilized as a window. However the frequency scales height and width of the window in a relationship with wavelet [18, 19]. The ST attenuate the high-frequency signals in contrast to low-frequency signals, and its generalized form is called FrST. The FrST improves FrFT and ST adaptability of signal analysis and generalized the time-frequency demonstration to time-fractional frequency [24, 26]. Hence, FrST can deliver more space for time-frequency analysis of a signal. Next section will demonstrate various transforms with mathematical expressions.

1.2 S-TRANSFORM

The ST is conceptually a hybrid of WT and STFT in the time-frequency domain. It overcomes the drawbacks of STFT and lack of phase in WT. The ST utilizes a Gaussian function, whose width and height is constrained by frequency. The ST gives a signal clarity in contrast to different transforms since it doesn't have cross-terms issues. ST displays frequency invariant amplitude response in contrast with WT and also analyzed phase and power spectrums. The ST diminishes high-frequency signals as compared to low-frequency signals. The ST of a signal $x(t)$ is denoted by $X(\tau,f)$ and expressed as [25-28]

$$X(\tau, f) = \int_{-\infty}^{\infty} x(t)\, g(t-\tau, f)\, \exp(-j\,2\pi f t)\, dt \tag{1.1}$$

Substituting the expression of the Gaussian window, the expression of ST will become

$$X(\tau,f) = \int_{-\infty}^{\infty} x(t) \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(t-\tau)^2 f^2}{2}\right) \exp(-j2\pi f t)dt \tag{1.2}$$

where, Gaussian function $g(t-\tau,f)$ controlled by frequency ‘ f ’ and time-shift ‘ τ ’, here frequency is inversely proportional to the width of the Gaussian function. If $X(\tau,f)$ is integrated with respect to ‘ τ ’ then, it gives FT of a signal $x(t)$ is written as

$$\int_{-\infty}^{\infty} X(\tau,f)\, d\tau = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t) g(t-\tau,f) \exp(-j2\pi f t)dt\right)d\tau \tag{1.3}$$

Using the normalized condition of the Gaussian function is denoted as [26]

$$\int_{-\infty}^{\infty} g(t-\tau,f)d\tau = \int_{-\infty}^{\infty} \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(t-\tau)^2 f^2}{2}\right)d\tau = 1 \tag{1.4}$$

Therefore, (1.3) is expressed as

$$\int_{-\infty}^{\infty} X(\tau,f)d\tau = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi f t)dt = X(f) \tag{1.5}$$

where, $X(f)$ is the FT of $x(t)$. The two-dimensional (2-D) ST is defined [12, 28] as

$$X(\tau_1,\tau_2, f_1,f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2) \frac{|f_1||f_2|}{2\pi} \exp\left(-\frac{(t_1-\tau_1)^2 f_1^2}{2} - \frac{(t_2-\tau_2)^2 f_2^2}{2}\right) \exp\left(-j2\pi(f_1t_1+f_2t_2)\right)dt_1dt_2 \tag{1.6}$$

The 2-D inverse ST (IST) is expressed as

$$x(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\tau_1,\tau_2,f_1,f_2)d\tau_1 d\tau_2\right\}\exp\left(j2\pi(f_1t_1+f_2t_2)\right)df_1df_2 \tag{1.7}$$

The 2-D S-transform is a powerful tool and can be used in digital image processing.

1.2.1 DISCRETE S-TRANSFORM

Due to the advent of discrete systems, the discretization of every mathematical tool becomes necessary to increase the span of its applications. Considering the discrete form of the signal $x(t)$ as $x[kT]$ where, $k = 0,1,2,...,N-1$ and sampling time interval as T, its discrete FT(DFT) is given by [29-31]

$$X\left[\frac{n}{NT}\right]=\frac{1}{N}\sum_{k=0}^{N-1} x[kT]\exp(-\frac{j2\pi nk}{N}) \tag{1.8}$$

where, $n=0,1,2,...,N-1$. Using (1.2), the ST of a discrete case $x[kT]$ can be defined [31, 32] as

$$X\left[iT,\frac{n}{NT}\right]=\sum_{m=0}^{N-1} x\left[\frac{m+n}{NT}\right]\exp\left(-\frac{2\pi^2 m^2}{n^2}\right)\exp\left(\frac{j2\pi mi}{N}\right); \quad n \neq 0 \tag{1.9}$$

For $n = 0$, it is defined as

$$X[iT]=\frac{1}{N}\sum_{m=0}^{N-1} x\left[\frac{m}{NT}\right] \tag{1.10}$$

where, i, m and $n=0,1,2,...,N-1$. Equation (1.10) gives the continuous average of time into the zero frequency voice, thus promising the invert is same. The inverse discrete ST (IDST) is defined as

$$x[kT] = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{i=0}^{N-1} X[iT,\frac{n}{NT}]\right)\exp\left(\frac{j2\pi nk}{N}\right) \tag{1.11}$$

Subsequently, in literature, other definitions of discrete ST were also reported. Here one such definition is presented, which was called as discrete orthonormal ST (DOST) [32], which is defined in terms of 'N' unit length basis vector as

$$X[kT]_{[v,\beta,\tau]}=\frac{1}{\sqrt{\beta}}\sum_{f=v-\beta/2}^{v+\beta/2+1}\exp\left(j2\pi\frac{\tau}{\beta}f\right)\exp\left(-j2\pi\frac{k}{N}f\right)\exp(-j2\pi\tau); \quad k=0,1,2,...,N-1 \tag{1.12}$$

On simplification, it results into

$$X[kT]_{[v,\beta,\tau]}=\frac{\exp\left(-j2\alpha(\frac{2v-\beta-1}{2})\right)-\exp\left(-j2\alpha(\frac{2v+\beta-1}{2})\right)}{2\sqrt{\beta}\sin\alpha} j\exp(-j2\pi\tau) \tag{1.13}$$

where, $\alpha = \pi\left(\frac{k}{N}-\frac{\tau}{\beta}\right)$ is a center of the temporal window for k^{th} basis vector. Mathematically, these basis vectors are orthonormal as

$$\frac{1}{N}\int_0^N X[kT]_{[v',\beta',\tau']} X^*[kT]_{[v,\beta,\tau]} dk = \delta_{v'v}\delta_{\beta'\beta}\delta_{\tau'\tau} \tag{1.14}$$

where, $\delta_{v'v} = \begin{cases} 1; & v'=v \\ 0; & \text{else} \end{cases}$ is a delta function. The inverse discrete ST (IDST) is defined as

$$x[kT] = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{i=0}^{N-1} X[iT, \frac{n}{NT}]\right)\exp\left(\frac{j2\pi nk}{N}\right) \tag{1.15}$$

Now an MRA can be obtained by a set of coefficients with its corresponding frequencies information at different locations and resolutions is demonstrate in the next section.

1.3 MULTIRESOLUTION ANALYSIS

Resolution is an extensive term with diverse significances when used in various arenas of knowledge. Resolution is a quantity used to define the clarity and sharpness of a picture and image. It was needed to augment the quality of the image, and the idea of multiresolution was developed. It decomposes a single event on different scales and to analysis the event at different scales [33-36]. Multiresolution analysis (MRA) was first defined by Mallat (1987). It is a method for L^2 approximation for a multi-level representation of the signal. MRA means representing or analyzing of the signals and image at more than one resolution [34, 35]. It is a technique for the estimation of function with arbitrary precision. MRA gives an estimation on a sufficient scale and can be obtained by accumulation the details to estimation on a rough scale. It is rooted in the fact that a signal is denoted as the sum of details and approximations. The approximations does not change with the next iteration. The spaces U_k with a positive value of k are said to be high-resolution scales or fine scales and have a high estimation, for negative value of k, scales move towards becoming coarser with low determination scales are shown in Fig.1.2.

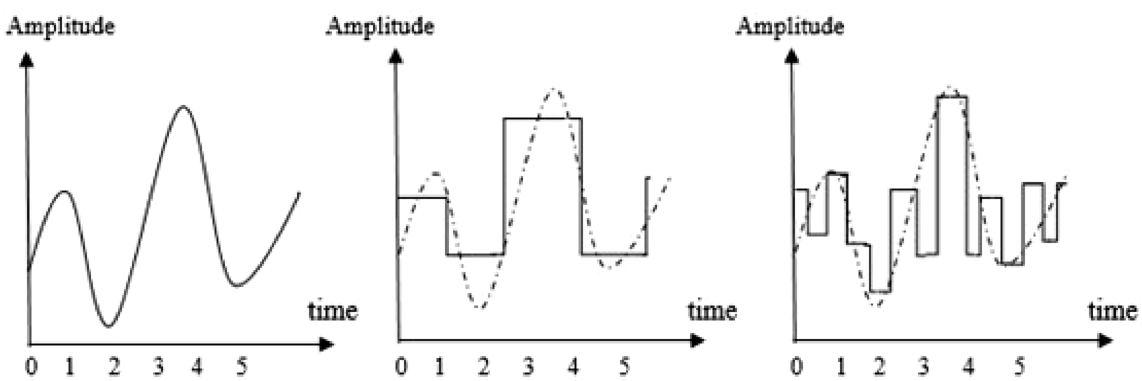


Fig. 1.2 A function $x(t)$ and its MRA calculation on the scales U_0 and U_1

In MRA, the signal processing is initiated from minimum resolution, and then the resolution can be selectively increased, when necessary. In general, MRA gives the scaling function a most substantial role in the piecewise estimation of the continuous function and depends on the scaling index. The MRA is not unique and relies on the selection of the mother wavelet functions. The selection of the scaling and mother wavelet functions is application dependent [34-36]. A multiresolution technique attempt to locate a particular frequency at a particular location, which is the primary deficiency of FT and STFT. However, it isn't conceivable to locate a particular frequency at a precise location at the same time. MRA is helpful when the signal contains low frequency segment for large interval and high frequency segments for short interval. MRA methods are more viable in picture investigation and worn out the limitation of frequency and location resolution found in FT and STFT. MRA gives an opportunity to see estimations and details of a signal and select the appropriate level of detail for further analysis.

1.4 FRACTIONAL S-TRANSFORM

Before defining the fractional ST (FrST), first, define "what is a fractional transform" and "how can make a transformation to be fractional". First, a transformation T can be described [37, 38] as

$$T\{x(t)\} = X(\tau, v)$$

where, x and X are two functions with variables t and v, τ respectively. Now another transform can be expressed as

$$T^{\alpha}\{x(t)\} = X_{\alpha}(\tau, v)$$

where, the parameter α is known as 'fractional order'. This type of transform is known as fractional transform [39-41], which fulfil the following conditions, given as

- The fraction operator is linear.
- The 1st order transform i.e. $\alpha = 1$ implies to the conventional transform and the zeroth order transform i.e. $\alpha = 0$ indicates performing no transform.
- The fractional operator is additive $T^{\alpha} T^{\beta} = T^{\alpha+\beta}$.

To further extend the application areas, a generalized version of ST is defined in a time-frequency plane to deal with non-stationary behavior of signals. This enables the definition of fractional ST (FrST). The FrST was first introduced in 2012, as a way to deal with synthetic Ricker wavelet and seismic data. The FrST is defined as a generalization of the ST with an order α. Mathematically,

α^{th} order FrST is the α power of ST. With the development of FrST and related concepts, the conventional ST becomes merely a special case of FrST. Every property and application of the conventional ST can be a distinct case of the respective property and application of FrST, if its operator is linear [42-44]. In essence, the α^{th} order FrST interpolates amongst a function $x(t)$ and it's ST. The 0^{th} order transformation is itself, whereas the first order is its ST. The 0.5^{th} order transformed in between of time and frequency domains. The FrST was introduced by integrating the concept of fractionalization with ST [1], which actually evolves as a combination of FrFT and ST. The FrST has more flexibility in signal spectrum analysis. FrST has initiated numerous applications in signal processing, image processing, bioinformatics, geo-informatics [2] and radar communication etc. FrST is an illustration of signals in the time-frequency plane as shown in Fig.1.3.

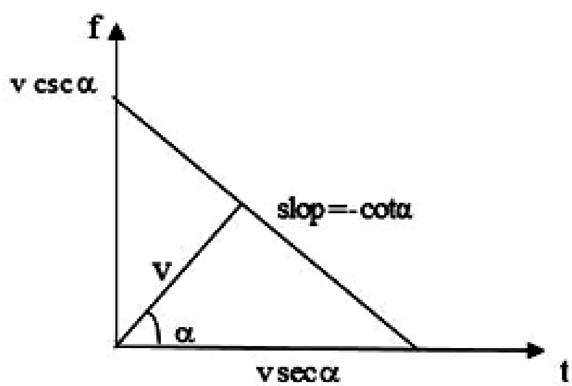


Fig.1.3 Illustration for time-frequency fractional plane

The FrST of a signal $x(t)$ with an angle α is stated [13] as

$$X_{\alpha}(\tau, v) = \int_{-\infty}^{\infty} x(t)\kappa_{\alpha}(t, v) g(t-\tau, v) dt \tag{1.16}$$

whereas,

$$\kappa_{\alpha}(t, v) = \begin{cases} B_{\alpha} \exp(j\pi(v^2+t^2)\cot\alpha - j2\pi vt(\csc\alpha)), & \alpha \neq n\pi \\ \delta(t-v), & \alpha = 2n\pi \\ \delta(t+v), & \alpha = (2n-1)\pi \end{cases} \tag{1.17}$$

where, ' v ' is fractional frequency and $B_{\alpha} = \sqrt{1-j\cot\alpha}$, $n \in \mathbb{Z}$. The Gaussian window function $g(t-\tau, v)$ is scalable with respect to fractional frequency ' v ' and time ' t '. The Gaussian window function is defined as [13]

$$g(t,v)=\frac{|v(\csc\alpha)|^{p}}{\sqrt{2\pi}q}\exp\left(\frac{-t^{2}\{v(\csc\alpha)\}^{2p}}{2q^{2}}\right) \tag{1.18}$$

The width and height of the Gaussian window are varying with respect to the fractional frequency. However, the window function with wide time domain will have low v whereas, will be narrow for high v. The shape of the window function can be acquainted by the parameter p and q with space rotation factor $\alpha=\frac{a\pi}{2}$. In the specific case of $p=q=1$ and $a=1$, the FrST convert into ST written as [13,]

$$X(\tau,v)=\int_{-\infty}^{\infty}x(t)\frac{|v|}{\sqrt{2\pi}}\exp\left(-\frac{(t-\tau)^{2}v^{2}}{2}\right)\exp(-j2\pi vt) \tag{1.19}$$

Best on the characteristic signal, the window size is altered using v, p and q. By revolving the FrST frequency, it can progress the fractional time-frequency demonstration. So FrST is improving the elasticity of time-frequency analysis and energy of signal spectra. Similarly, FrST can efficiently advance the time-frequency resolution capacity as related to ST. The inverse FrST (IFrST) is stated as [13]

$$x(t)=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}X_{\alpha}(\tau,v)d\tau\right\}\kappa_{\alpha}^{*}(t,v)dv \tag{1.20}$$

where, '$*$' represents complex conjugate. According to the marginal condition of the FrST, is further defined as [13, 42]

$$\int_{-\infty}^{\infty}X_{\alpha}(\tau,v)d\tau=X_{\alpha}(v) \tag{1.21}$$

Hence, the IFrST becomes

$$x(t)=\int_{-\infty}^{\infty}X_{\alpha}(v)\kappa_{\alpha}^{*}(t,v)dv \tag{1.22}$$

where, $X_{\alpha}(v)$ is the fractional FT of $x(t)$.

1.5 ORGANIZATION OF THE THESIS

The thesis is organized into seven chapters and its chapter-wise summary is given below:

Chapter-1: INTRODUCTION

This chapter demonstrates the historical development of the fractional S-transform (FrST). Thereafter, some basic facts of the ST, MRA and FrST have been introduced.

Chapter-2: LITERATURE REVIEW

In this chapter, a comprehensive review of related literature with their background is presented. It includes the preamble of ST, MRA, and FrST along with the motivation and objectives of the thesis. Basically, it comprises the mathematical definitions and properties of ST, MRA, and FrST. Finally, based on the literature gaps, objectives and methodology for the current work has been decided.

Chapter-3: S-TRANSFORM WITH ITS PROPERTIES

This chapter includes the proposed properties of ST with their rigorous proof. Thereafter a comparative analysis between proposed and existing methods are also presented.

Chapter-4: MULTIRESOLUTION ANALYSIS USING FRACTIONAL S-TRANSFORM

In this chapter, the necessity of MRA for reconstruction of a sampled signal is presented. Thereafter, the conditions of multiresolution analysis for FrST with its proof are also presented.

Chapter-5: FRACTIONAL S-TRANSFORM

Using the concept of MRA and as an extension of ST, various properties of FrST are derived in this chapter.

Chapter-6: APPLICATIONS OF FRACTIONAL S-TRANSFORM

This chapter presents the filtering of ECG signal and LFM signal under AWGN. Also, sampling error estimation for FrST is included here by using the proposed sampling theorem of FrST in the previous chapter.

Chapter-7: CONCLUSIONS

Finally, the summary of the proposed work and its possible future scope is documented in this last chapter of the thesis.

CHAPTER 2

LITERATURE SURVEY

"A significant literature will provide a vital feature of any thesis. An actual survey, summarizing and fusing what is known while recognizing gaps in the knowledge base, facilitating theory development, closing areas where enough research already exists, and uncovering areas where more research is needed."

J. Webster and R. Watson, 2002

Many properties of ST and FrST have been derived and established in applications of different areas like signal analysis, image processing and biomedical signal processing.

2.1 A REVIEW OF S-TRANSFORM

In FT technique, the signal is analyzed either in the time domain analysis or frequency domain analysis. The time-domain signal exhibits data about the signal intensities and temporal evolution. For deterministic signals, analysis is usually based on instantaneous power spectrum or energy density spectrum. But for random signals, the analysis tool depends on the auto-correlation functions and the power spectrum [18, 47]. FT explores the signal at various frequencies and their relative magnitudes. However, the main drawback of FT is that the time resolution does not occurs, but frequency resolution occurs. That means FT can analyse the signals frequency response but cannot predict the arrival time of frequency component. To compensate this drawback, other transforms techniques such as STFT [7], WT and ST are available in literature [45-47]. These techniques are represented as a time-frequency plane analysis tool. In WT only scale information can be expected with a modulated phase information [48, 49]. However, fixed window size is the main drawback of STFT, hence it is needed to be predefined. The STFT is usually not invertible [6]. The Stockwell transform, also called as ST, is another time-frequency analysis tool. In recent years, much work has been documented regarding the utilization of ST in time-frequency analysis. It covers the different application fields such as climate studies, geophysics [13, 50-52], seismic and bio-medical signal analysis [48, 59], image analysis [53-55, 71, 72] and speech processing and in addition broader outcomes of the local spectral analysis [47]. It uses frequency dependent variable window so that window width has a relation with frequency introduced in power signal. Hence, it gives frequency-dependent determination and consequently better signal clarity in time-frequency plane. It is well-defined as a generalization of the STFT. The S-transform can be defined in many ways. Definition and closed-form expression of ST was originated by employing a correction of phase in WT [6, 7].

Two dimensional (2-D) ST is used for the calculation of the local spectrum at each point of images. It would be more beneficial in the spectral characterization of aperiodic or random patterns [29]. Asymmetry of the Bi-Gaussian presents in subsequent time-frequency plane, with a time resolution improved in the obverse direction. The Bi-Gaussian ST is superior at determining the sharp inception of events in time series. The window which has been used in the majority of the S-transform can be defined in many ways [13]. For the study of multicomponent and non-stationary signals, traditional approaches are based on either frequency or time-domain analysis. ST gives a time-frequency representation in contrast with frequency representation by FT and it depends on Gaussian windows [5]. It has a frequency-dependent determination of time-frequency analysis and completely refers to neighborhood stage data. The ST is a suitable tool in the area of signal or image processing, Geo-informatics and bio-medical signal processing because no cross-terms appear by using ST [13, 14].

Based on the theory of signal processing, ST is extending its space of square-integral functional on the real-time ($\mathbb{R}$) [52]. The discrete orthonormal ST (DOST) is an orthogonal version of the discrete ST (DST). It is used for image compression based on setting the smallest coefficients zero. DOST is also used in image restoration, filtering, and registration applications [12, 52, 53]. A new symmetric DOST that still keeps the non-redundant multi-resolution features of the DOST has the advantage of less memory and smaller computational time [53-55]. The analysis of another approach called DOST along with discrete cosine transform (DCT) for proficient representation of the electrocardiograph (ECG) signal in the time-frequency space is given in [60-62]. Subsequently, continuously efforts have been made to formulate the property set of ST. However, only a few are documented to date like; linearity property, scaling property, time-shifting property, convolution theorem [51]. This motivates us to formulate the properties like; time-reversal property, time-derivative property, complex conjugate property, an improved convolution theorem, correlation theorem and Parseval's theorems for ST in the presented work.

In [57], the system of a functions (known as DOST basis) is indeed an orthonormal basis of L^2 [0, 1] (the two-dimensional space of functions continuous on [0, 1]), which is time-frequency localized and present a fast O(N log N) algorithm that computes ST coefficients for the acceptable window. The issue of coherence and phase synchrony analysis are demonstrated using modified ST. Where the modified ST (MST) gives the idea about fluctuating cross-spectral analysis. It was observed that MST is more advantageous than standard ST [28]. Subsequently, a two-dimensional ST method is used for the analysis of the image and extract phase more correctly [29]. The another application of ST in the deformed fringe patterns for the demodulation was performed in [58], the authors have focused on examining the ST spectrum filtering system, as well as the ST edge strategy and the phase angle calculation technique. The automatic ECG signal enhancement method to eliminate noise constituents from a noisy ECG signal is obtained by performing filtering [59] and electroencephalogram (EEG) denoising using ST.

2.2 TYPES AND PROPERTIES OF S-TRANSFORM

In literature, many definitions of ST are available depending on contexts as presented below:

(a) **One-dimensional ST**

The ST of a given function, $x(t)$ belonging in $L^1[R]$ is expressed as [13]

$$X(\tau,f) = \int_{-\infty}^{\infty} x(t)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(t-\tau)^2f^2}{2}\right)\exp\left(-j2\pi f t\right)dt \tag{2.1}$$

and inverse ST (IST) is written as [27]

$$x(t) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} X(\tau,f)d\tau\right\}\exp\left(j2\pi f t\right)df \tag{2.2}$$

(b) **Two-dimensional ST**

The two-dimensional (2-D) ST is expressed as [29]

$$X(\tau_1,\tau_2,f_1,f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)\frac{|f_1||f_2|}{2\pi}\exp\left(-\frac{(t_1-\tau_1)^2f_1^2}{2}-\frac{(t_2-\tau_2)^2f_2^2}{2}\right)\exp\left(-j2\pi(f_1t_1+f_2t_2)\right)dt_1dt_2 \tag{2.3}$$

And 2-D IST is expressed as [29]

$$x(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\tau_1,\tau_2,f_1,f_2)d\tau_1d\tau_2\right\}\exp\left(j2\pi(f_1t_1+f_2t_2)\right)df_1df_2 \tag{2.4}$$

Next sub-section includes the existing properties defined for ST.

2.2.1 Existing Properties of S-transform

Based on the literature [50, 51], properties of ST identified are given in Table-2.1.

Table-2.1: Existing properties of ST

Properties	Mathematical Equations
Linearity	$\{a\,x(t) + b\,y(t)\} \overset{ST}{\Leftrightarrow} a\;X(\tau,f) + b\,Y(\tau,f)$
Scaling	$x(kt) \overset{ST}{\Leftrightarrow} \frac{1}{\lvert k\rvert}X\left(k\tau,\frac{f}{k}\right)$
Time Shifting	$X\left(t-\tau_0\right) \overset{ST}{\Leftrightarrow} \exp(-j2\pi f\,\tau_0)X\left(\tau-\tau_0,f\right)$
Convolution Theorem	$\int_{-\infty}^{\infty} p\left(t-\xi_0\right) q(\xi_0)\,d\xi_0 \overset{ST}{\Leftrightarrow} \int_{-\infty}^{\infty} P\left(\zeta-\xi_0,\,f\right) q(\xi_0)\exp(-j2\pi\xi_0 f)\,d\xi_0$

2.2.2 Relation with others Transforms

Being another mathematical tool defined in the time-frequency plane, the ST observes relation to other transform techniques defined in time-frequency plane.

(i) The relation between ST and STFT

The STFT of a signal $x(t)$ expressed as [6]

$$STFT(\tau,f) = \int_{-\infty}^{\infty} x(t)\; w(t-\tau)\; \exp(-j2\pi f\,t)\; dt \tag{2.5}$$

where, $w(t\text{-}\tau)$ represent window function.

$$X(\tau,f) = \int_{-\infty}^{\infty} x(t)\,g(t-\tau,f)\;\exp(-j2\pi f\,t)dt \tag{2.6}$$

where, the Gaussian function, $g(t\text{-}\tau,\,f)$ is controlled by frequency 'f' and time-shift 'τ'. Hence, after the comparative analysis, it is concluded that ST is a specific instance of the STFT with a Gaussian window function.

(ii) The relation between ST and WT

The WT of a signal $x(t)$ expressed as [8]

$$W(\tau,d) = \int_{-\infty}^{\infty} x(t)\; m(t-\tau,d)dt \tag{2.7}$$

where, d and τ represents, the dilation factors and the spectral localization respectively. The dilation factor decides the 'width' of the wavelet hence, it control the signal resolution. The $W(\tau,d)$ represents a scaled copy of the central mother wavelet and must have zero means [2]. Hence, after the comparative analysis, the ST is derived, when WT multiplied by the phase factor, thus the ST can be stated as

$$X(\tau,f) = W(\tau,d)\exp(-j2\pi f\tau) \quad . \tag{2.8}$$

where, the mother wavelet is expressed as

$$m(t,f) = \frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-t^2f^2}{2}\right)\exp\left(-j2\pi f t\right) \tag{2.9}$$

where, dilation factor 'd' is inversely proportional to frequency 'f'. The ST gives better, dependent frequency resolution in time-frequency analysis with least noise.

(iii) **The relation between FT and ST**

The FT of a continuous-time signal $x(t)$ is written as [13]

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi f t)dt \tag{2.10}$$

If (2.6) is integrated w.r.t 'τ' then, results into

$$\int_{-\infty}^{\infty} X(\tau, f)d\tau = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)g(t-\tau,f)\exp(-j2\pi f t)dt\right)d\tau \tag{2.11}$$

Using the normalized condition of the Gaussian function is expressed as

$$\int_{-\infty}^{\infty} g(t-\tau,f)d\tau = \int_{-\infty}^{\infty}\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(t-\tau)^2f^2}{2}\right)d\tau = 1 \tag{2.12}$$

Substituting (2.12) into (2.11) then, the result can be written as

$$\int_{-\infty}^{\infty} X(\tau, f)d\tau = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi f t)dt \tag{2.13}$$

Therefore,

$$\int_{-\infty}^{\infty} X(\tau, f)\, d\tau = X(f) \tag{2.14}$$

where, FT of a signal $x(t)$ is denoted by $X(f)$. In (i), (ii) and (iii) first give the definition of STFT, WT and FT, and discuss their basis functions. Then proceed for the relationship between these with ST.

2.3 CONCEPT OF MULTIRESOLUTION ANALYSIS

An MRA contains a sequence of nested spaces, which are traversed by translates of the scaling function φ. The scaling functions have a property that is required for the sampling system to be introduced. The concept of multiresolution is to decompose a single event on different scales and to study the event on these different scales.

The MRA for FrST collection of a sequence of subspaces $\{U_k^{\alpha}\}_{k\in\mathbb{Z}}$ is known as MRA with a scaling function $\varphi(t)$. An example is used to introduce the basic concept of MRA to scale the coefficient of FrST at different scales. It is used to achieve the high-resolution signal or data.

Example: Suppose, a square impulse function defined as

$$\varphi(t) = \begin{cases} 1\,, & 0 < t < 1 \\ 0\,, & \text{else} \end{cases} \tag{2.15}$$

This is an orthogonal function itself i.e.,

$$\langle \varphi(t), \varphi(t\text{-}k) \rangle = \delta[k] \tag{2.16}$$

Based on the orthogonality of this function, it can construct a set of scaling functions $\varphi_{0,k}(t)$ that span U_0. So that any function $x(t)$ belonging in $L^2[\mathbb{R}]$ can be approximated in this space U_0 as

$$x(t) = \sum_k a_k\ \varphi_{0,k}(t\text{-}k) \tag{2.17}$$

Substituting, $t = 2^{\gamma} t$ in $\varphi_{0,k}(t-k) = \varphi(t-k)$ and normalized by a factor $2^{\gamma/2}$, gives another set of orthonormal functions

$$\varphi_{\gamma,k}(t) = 2^{\gamma/2}\,\varphi_{0,k}(2^{\gamma} t\text{-}k) \tag{2.18}$$

As $\varphi(t) = 1\ \forall\ 0 < t < 1$, it gives, $\varphi_{\gamma,k}(t) = \varphi(2^{\gamma} t - k) = 1$ if its argument satisfies $0 < 2^{\gamma} t - k < 1$, i.e.

$$\frac{k}{2^{\gamma}} < t \quad < \frac{1+k}{2^{\gamma}} \tag{2.19}$$

Therefore, $\varphi_{\gamma,k}(t) = \varphi(2^{\gamma} t - k)$ is a square impulse of hight $\sqrt{2^{\gamma}}$ and of width $\frac{1}{2^{\gamma}}$, which is shifted by k times. Hence, these functions are also orthonormal and they span space U_{γ}.

$$\left\langle \varphi_{\gamma,k}(t), \varphi_{\gamma,p}(t\text{-}k) \right\rangle = \delta[k\text{-}p] \tag{2.20}$$

A plot of the given function and its version as a scaling function is presented in Fig. 2.1.

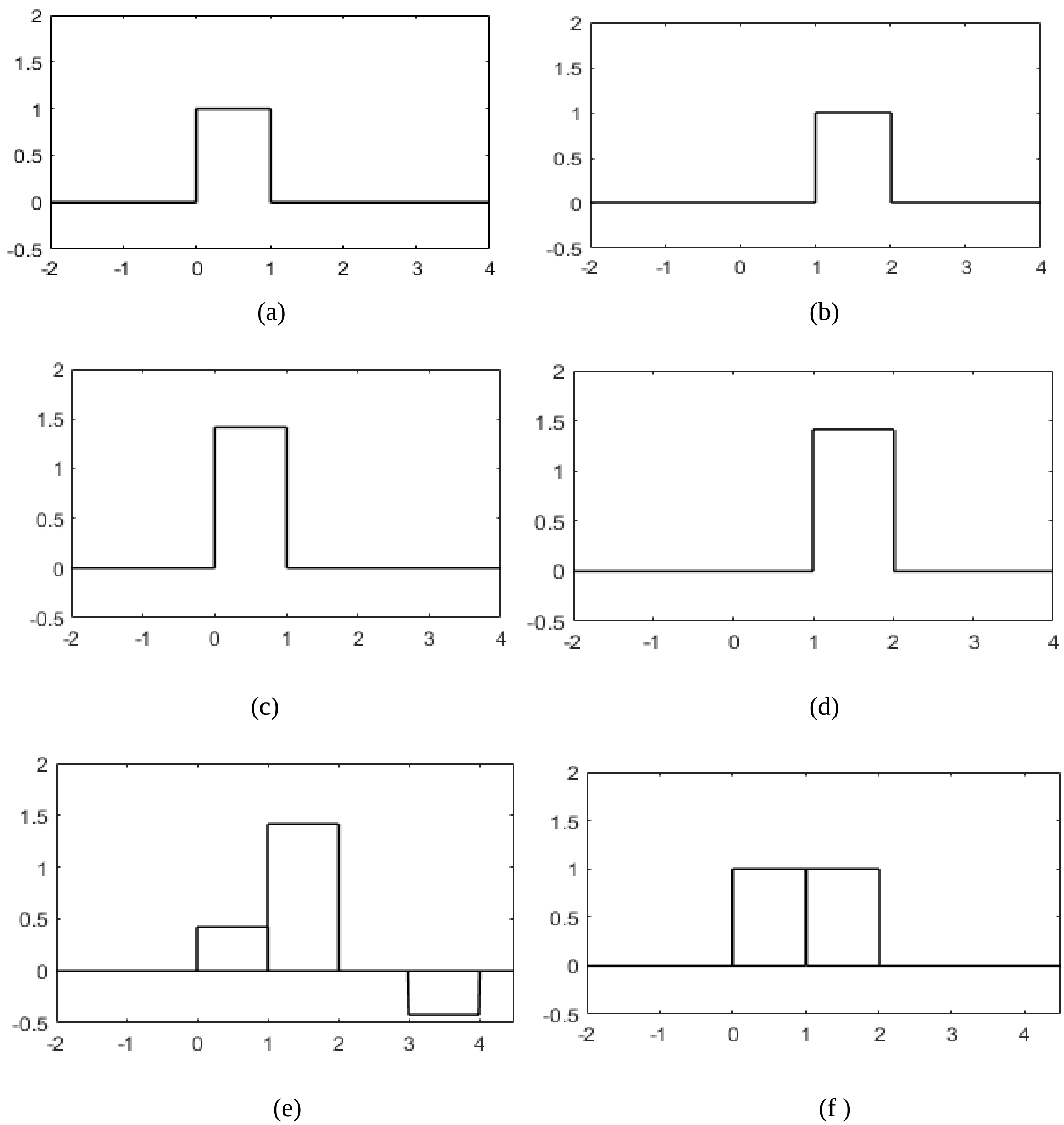


Fig.2.1 Square Impulse used as a scaling function

Fig.2.1 (a) and Fig.2.1 (b) shows the two scaling functions $\varphi(t) = \varphi_{0,0}(t)$ and $\varphi_{0,1}(t) = \varphi(t-1)$ both are in space U_0. The Fig.2.1(c) and Fig.2.1 (d) shows another two different scaling functions $\varphi_{1,0}(t) = \sqrt{2}\ \varphi(2t)$ and $\varphi_{1,1}(t) = \sqrt{2}\ \varphi(2t - 1)$ in space U_1. Fig.2.1 (e) shows a function $x(t) \in U_1$ denoted as a linear combination of the scaling functions $\varphi_{1,k}(t)$.

$$x(t) = 0.3\varphi_{1,0}(t) + \varphi_{1,1}(t) - 0.3\varphi_{1,3}(t) \tag{2.21}$$

Finally, Fig.2.1 (f) shows that a scaling functions $\varphi_{0,0}(t)$ in U_0. It can also be denoted as a linear combination of the basis functions $\varphi_{1,k}(t)$ in U_1.

$$\varphi_{0,p}(t) = c_0\ \varphi_{1,\ 2p}(t) + c_1\ \varphi_{1,\ 2p+1}(t) \tag{2.22}$$

$$= \frac{1}{\sqrt{2}}\varphi_{1,\ 2k}(t) + \frac{1}{\sqrt{2}}\ \varphi_{1,\ k+1}(t) \tag{2.23}$$

where, $c_0 = \frac{1}{\sqrt{2}} = c_1$. Finally, construct a set of scaling functions $\varphi_{0,\ k}(t)$ that span U_0, based on the orthogonality of this function. The MRA is needed for the signals which are non-stationary in nature. Low-resolution signals are suitable for compression, high-resolution signals are suitable for analysis but have poor compression capabilities [62, 63]. MRA is widely applied to solve various problems such as biomedical signal processing, noise elimination, data compression and feature extraction [62, 63]. Using MRA analysis, features of EEG signal were constructed in [60]. The multiresolution WT for evaluation and detection of the QRS complex demonstrated in [61]. Kumar et al. [63] discussed a technique for ECG signal analysis based on MRA and [62] used MRA approach to detect the R peaks of ECG signal.

2.4 FRACTIONAL S-TRANSFORM AND ITS PROPERTIES

The generalization of the ST in time-frequency plane is known as FrST. It serves as a useful analysing technique in a time-frequency localization with frequency-dependent resolution. It is used for the analysis of continuous-time function that introduces the spectrum at every point of the time axis. In [15], based on the idea of FrFT and ST of a signal, defined the FrST along with the inverse-FrST. The various properties of FrST were also documented like: linearity, scaling, inverse fractional ST and time marginal condition. Later on FrST is used extensively in many research areas. Some are presented below. After that in [45] demonstrate the FrST on space. It was observed that, the FrST is a continuous linear map of the space and defined on $\mathbb{R}$, with some properties. It is useful in the study of time-frequency performance of test function and distributions. Thereafter, generalized the results of ST on the spaces. The continuity in the results for FrST are obtained on some specific designed space. In [44], researchers extend the results of ultra-distribution for the FrST to the Bohemian spaces. Thereafter, [64] demonstrate the diverse seismic signal has various optimal fractional parameters and it is not favorable to multichannel seismic signal processing. Hence, using FrST first decomposes the common frequency, after that, it analyzes the minimum frequency. Thereafter, a combination of blind source separation and FrST is used to get the autonomous spectra of the numerous geological features. The bottom and top of a limestone reservoir were clearly identified on the common frequency segment. Hence, improving the vertical resolution of low-frequency in contrast with classical ST. Its simulation results show that the independent frequency in the time-fractional frequency plane.

In [65], a new FrST is introduced to omit the physical significance of the fractional time-frequency plane. This definition of the FrST is based on the concept of time-bandwidth product and time-frequency rotation property of the FrFT. In this method, normalized second-order central moment calculation technique was used for finding the optimal order, rather than time-bandwidth product search algorithms. The normalized second order central moment approach has higher computational efficiency. These algorithms can achieve single frequency visualization with improved time-frequency representation, thereby improving the precision of reservoir forecast. In [66] demonstrate a fractional lower order ST (FLOST) time-frequency illustration method employing fractional lower order ST and inverse FLOST (IFLOST). The FLOST time-frequency filtering technique is based on the concept of FLOST time-frequency representation techniques and IFLOST and its simulated results demonstrate that FLOST time-frequency representation algorithm is better in compared to existing ST time-frequency representation algorithm under a stable dissemination noise, that can work better under the Gaussian noise. The FLOST time-frequency representation technique can efficiently filter out a stable distribution noise and refurbish the original signal.

Another way of relating FrST and FrFT is given below. Let,

$$h(t, v, \tau) = g(t - \tau, v)\kappa_\alpha(t, v) \tag{2.24}$$

and, $$H_\alpha(\tau, v, v') = \int_{-\infty}^{\infty} h(t, v, \tau)\, \kappa_\alpha^*(t, v)\, dt \tag{2.25}$$

The FrST may be characterized as an activity on the FrFT domain [15] as

$$X_\alpha(\tau, v) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} X_\alpha(v)\, \kappa_\alpha^*(t, v')\, dv' \right\} h(t, \tau, v)\, dt \tag{2.26}$$

Interchanging the order of integration and rearranging the term (2.26) converts into

$$X_\alpha(\tau, v) = \int_{-\infty}^{\infty} X_\alpha(v)\, H_\alpha(\tau, v, v')\, dv' \tag{2.27}$$

where, $X_\alpha(\tau, v)$ and $X_\alpha(v)$ are the FrST and FrFT of a signal $x(t)$ respectively. Substituting the FrFT of signal $x(t)$ in (2.27) then FrST can also be written as [15]

$$X_\alpha(\tau, v) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(t)\, \kappa_\alpha(t, v)\, dt \right\} H_\alpha(\tau, v, v')\, dv' \tag{2.28}$$

The discrete-time FrST (DTFrST) of $x[n] \in l^2[\mathbb{Z}]$ is defined as

$$\tilde{X}_\alpha(v,\eta) = \tilde{\mathscr{S}}^\alpha\{x[n]\}(v,\eta) = \sum_{n\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} x[n]\, \kappa_\alpha(v,n)\, H_\alpha(\eta, v, v') \tag{2.29}$$

where $\tilde{\mathscr{S}}^\alpha$ represents the DTFrST and $\kappa_\alpha(.,.)$ is a kernel of FrST.

The inverse-DTFrST (IDFrST) is expressed as

$$x[n] = \int_I \int_I \tilde{X}_\alpha(v,\eta)\, \kappa_\alpha^*(v',n)\, H_\alpha^*(\eta, v, v')\, dv' dv, \qquad I \cong [0,\ 2\pi\sin\alpha] \tag{2.30}$$

The fractional convolution theorem of two continuous-time signals $x(t)$ and $y(t)$, which belonging in $L^2[\mathbb{R}]$ is defined as [70,105]

$$x(t)\ \Theta^\alpha\ y(t) = \int_{\mathbb{R}} x(\tau)\ y(t-\tau)\ \exp\left(\frac{-j(t^2-\tau^2)}{2}\right) \cot\alpha\ d\tau \tag{2.31}$$

where Θ^α denoted as an operator of fractional convolution. Thereafter, the FrST of (2.31) can be written as [70]

$$x(t)\ \Theta^\alpha\ y(t) \overset{\mathscr{S}^\alpha}{\Leftrightarrow} X_\alpha(\tau,\ v)\ Y(\tau\sin\alpha,\ v\csc\alpha) \tag{2.32}$$

where $X_\alpha(\tau, v)$ represents the FrST of $x(t)$ and $Y(\tau\sin\alpha, v\csc\alpha)$ is the ST of $y(t)$ with its argument scaled by $\sin\alpha$ and $\csc\alpha$ in time axis and frequency axis.

Similarly, a sequence $x(n)$ belonging in $l^2[\mathbb{Z}]$ and continuous signal $y(t)$ belonging in $L^2[\mathbb{R}]$, the semi-discrete fractional convolution theorem written as [105]

$$x(n)\ \overset{s}{\Theta}_{\alpha}\ y(t) = \sum_{n\in\mathbb{Z}} x(n)\ y(t-n)\ \exp\left(\frac{-j(t^2-n^2)}{2}\right) \cot\alpha \tag{2.33}$$

where $\overset{s}{\Theta}_{\alpha}$ denotes the operator of semi-discrete fractional convolution.

Subsequently, some properties of FrST were also documented, as presented in Table 2.3.

2.4.1 EXISTING PROPERTIES OF FRACTIONAL S-TRANSFORM

Based on the literature, the existing properties of FrST [13] like linearity, scaling, time reversal, time marginal condition, inverse FrST, convolution theorem, cross-correlation theorem and Parseval's theorem are presented in Table 2.2.

Table 2.2. Existing Properties of fractional ST

Properties	**Mathematical Formulation**
Linearity	If, $z(t)=a\ x(t)+by(t)$ then, $Z^{\alpha}(\tau,v)= a\ X^{\alpha}(\tau,v)+b\ Y^{\alpha}(\tau,v)$
Scaling	$X^{\alpha}(\tau,v)=\sqrt{\frac{c^2(1-j\cot\alpha)}{c^2-j\cot\alpha}}\ \exp\left(j\pi v^2\cot\alpha\left(1-\frac{\cos^2\beta}{\cos^2\alpha}\right)\right)X^{\alpha}\left(c\tau,v\frac{\sin\beta}{\cos^2\alpha}\right)$ where, $b=\frac{2\beta}{\pi}$ and $\beta=\tan^{-1}(c^2\tan\alpha)$.
Time Reversal	$X^{\alpha}_{x(-t)}(\tau,v)=X^{\alpha}(-\tau,-v)$
Time marginal condition	$\int_{-\infty}^{\infty} X^{\alpha}(\tau,v)d\tau = \int_{-\infty}^{\infty} x(t)\kappa_{\alpha}(t,v)dt = X^{\alpha}(v)$
Inverse FrST	$x(t) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X^{\alpha}(\tau,v)d\tau\right)\kappa^{*}_{\alpha}(t,v)dv$

2.5 SAMPLING AND INTERPOLATION

Sampling theorem defines the rate at which a constant time signal should be sampled such that all the data from the signal is captured with no data loss. The concept of this theorem is to rebuild a continual time signal from the sampled signal [73]. The idea in sampling is the measurement of the sampling frequencies as availability of spontaneous values of the signal at a predetermined rate allow the reconstruction of a continuous time signal from the sampled signal [67-70]. The aim is to rebuild a continuous time or discrete time signal from the sampled signal. The essential idea of the sampling is as follows; availability of several spontaneous values of some signal at a predetermined rate allows the reconstruction of the continuous time signal from the sampled signals [70-73]. However, for a perfect restoration infinite numbers of samples are required. A significant issue in sampling is the measurement of the sampling frequencies. The need for limiting the sampling frequencies is arising to diminish the information measured along, which in turn reduces the computational complexity in information handling. But choosing a low sampling frequency has a disadvantage of losing the data contained in the signal. This necessitates the seeking of the trade-off between these two limits [74-79].

Interpolation is the approximation of the lost or missing samples of a signal with help of a weighted average of a number of samples at the neighborhood points. The idea of ideal interpolation of a band-limited signal is presented and its application of interpolators include conversion of discrete-time to the continuous-time signal. The condition for the recovery of a band-limited continuous-time signal from its samples is illustrated by the Nyquist sampling theorem [78-83]. After the literature survey found that, several properties of FrST have been established earlier [15], like linearity property, shifting property, scaling property and inverse FrST. However, properties like; convolution theorem, correlation theorem, Parseval's theorem and the sampling property of FrST are still not established. Moreover, the mathematical identities and proof of these properties are more applicable, due to their vast applications in different areas. In digital signal processing, the convolution is used for filtering of a signal in fractional domain [86] and also the convolutional plays an essential role in many algorithms in edge detection and related processes [87]. In signal processing, the correlation function can provide info about reiterating events like musical beats however it cannot tell the location, in time, of the beat. The convolution can also be used to assess the pitch of a musical tone [80]. In radar signal processing, the received echo from a target is correlated with the transmitted signal to determine the distance, velocity, and acceleration of the target with respect to the receiver [84]. The correlation function is also used in power spectrum estimation [85] and in the design of matched filtering used in many communication systems [12]. Parseval's theorem is used to calculate the energy contented of pulses that are hard to determine in time-domain. It is also used to estimate the transient energy in a power distribution network [88].

2.6 MOTIVATION

After a comprehensive study of the literature, it can be observed that some research has been carried out to formulate the properties of ST and FrST. However, there is still scope for an improvement in the convolution theorem of ST because existing theorem has more computation complexity. This motivated to derive the remaining theorems of ST and FrST including a better convolution theorem.

Subsequently, an interest arises to implement some practical applications of multiplicative filtering for FrST domain. Based on the literature review and facts, an attempt is made to outline the research gaps and suggest the statement of the problem for the proposed work.

2.7 GAPS IN THE STUDY

The literature survey reported earlier mainly focused on ST, FrST and their applications. However, some gaps are observed and presented here as the motivating factor for doing further research.

- In the literature review, some properties of ST are defined, but other important properties are still undefined.
- Although some properties are documented in the case of ST for FrST, but only few properties are listed. So an extensive work is required to give closed-form expressions of the properties like convolution, correlation theorems for FrST.
- The sampling theorem shows a vital role in the discretization of systems. But its definition for both ST and FrST are not available in the literature.
- Literature survey shows a great potential of ST and FrST in their applicability. But, still, some area of signal processing can utilize these mathematical tools.

Based on these gaps, the objectives of this thesis are formed.

2.8 OBJECTIVE

After the literature study and motivation gathered by depicting gaps, the following objectives are considered for research work:

- To propose an improved method to derive the convolution theorem, correlation theorem for S-transform.
- To propose convolution, correlation and Parseval's theorem the FrST.
- To propose sampling theorem for both ST and FrST and utilize it into the determination of truncation error and aliasing error.
- To utilize the concepts of FrST in the Multiresolution analysis.

CHAPTER 3

S-TRANSFORM WITH ITS PROPERTIES

The S-transform (ST) is a time-frequency representation that presents the absolute referenced frequency and phase information. The ST is obtained as the phase enhancement of the wavelet transform with the window being Gaussian function. It is established as a scalable localizing Gaussian window and frequency dependent resolution.

3.1 Proposed properties of S-transform

It has been observed from literature study that only linearity, scaling, time-shifting and convolution theorem of ST is documented till now. This led to the finding of remaining properties of ST in order to establish it as a complete transform technique. Along with this, a new and better description of convolution theorem for ST is also presented.

3.1.1 Convolution theorem

Although a definition of convolution theorem for ST exists, as given in section 2.2, an improved definition is presented in this section. Thereafter, a related analysis of the current definition with the existing one is performed.

Definition: If $z(t)$ represents the weighted convolution of two continuous-time functions $x(t)$ and $y(t)$, is expressed as

$$z(t) = \int_{-\infty}^{\infty} x(\xi)y(t-\xi)\exp\left((\xi-\tau/2)(t-\xi-\tau/2)f^2\right)d\xi \tag{3.1}$$

Then the convolution theorem of ST can be well-defined as

$$z(t) = \int_{-\infty}^{\infty} x(\xi)y(t-\xi)\exp\left((\xi-\tau/2)(t-\xi-\tau/2)f^2\right)d\xi \overset{ST}{\Leftrightarrow} \frac{\sqrt{2\pi}}{|f|}X(\tau/2,f)\,Y(\tau/2,f) = Z(\tau,f) \tag{3.2}$$

Proof: The ST of $z(t)$ can be obtained as

$$Z(\tau,f) = \int_{-\infty}^{\infty} z(t)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(t-\tau)^2 f^2}{2}\right)\exp\left(-j2\pi f t\right)dt \tag{3.3}$$

Substituting (3.1) in (3.3), results into

$$Z(\tau, f) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x(\xi)y(t-\xi)\ \exp\left((\xi-\tau/2)(t-\xi-\tau/2)f^2\right)d\xi\right\}\frac{|f|}{\sqrt{2\pi}}\ \exp\left(\frac{-(t-\tau)^2f^2}{2}\right)\ \exp\left(-j2\pi f t\right)dt \quad (3.4)$$

Rearranging the terms in (3.4), results into

$$\int_{-\infty}^{\infty} x(\xi)\left\{\int_{-\infty}^{\infty} y(t-\xi)\frac{|f|}{\sqrt{2\pi}}\ \exp\left(\frac{-(t-\tau)^2f^2}{2}\right)\ \exp\left((\xi-\tau/2)(t-\xi-\tau/2)f^2\right)\ \exp\left(-j2\pi f t\right)dt\right\}d\xi \quad (3.5)$$

Substituting, $t-\xi=\kappa$ and $dt = d\kappa$ in (3.5), results into

$$\int_{-\infty}^{\infty} x(\xi)\left\{\int_{-\infty}^{\infty} y(\kappa)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(\kappa+\xi-\tau)^2f^2}{2}\right)\exp\left((\xi-\tau/2)(\kappa-\tau/2)f^2\right)\exp\left(-j2\pi f(\kappa+\xi)\right)d\kappa\right\}d\xi \quad (3.6)$$

Rearranging the terms and operators (3.6) converts into

$$\frac{\sqrt{2\pi}}{|f|}\left\{\int_{-\infty}^{\infty} x(\xi)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(\xi-\tau/2)^2f^2}{2}\right)\exp\left(-j2\pi f\xi\right)d\xi\right\}\left\{\int_{-\infty}^{\infty} y(\kappa)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(\kappa-\tau/2)^2f^2}{2}\right)\exp\left(-j2\pi f\kappa\right)d\kappa\right\} \quad (3.7)$$

By using the definition of ST, (3.7) converts into

$$\frac{\sqrt{2\pi}}{|f|}X(\tau/2,f)Y(\tau/2,f) \quad (3.8)$$

Thus, the proposed convolution theorem for ST will be

$$z(t) = \int_{-\infty}^{\infty} x(\xi)y(t-\xi)\ \exp\left((\xi-\tau/2)(t-\xi-\tau/2)f^2\right)d\xi \overset{ST}{\Leftrightarrow} \frac{\sqrt{2\pi}}{|f|}X(\tau/2,f)Y(\tau/2,f) = Z(\tau, f)$$

That means a time-domain weighted convolution of two functions gives the multiplication of these functions in ST domain.

3.1.1.1 Properties satisfied by convolution theorem

Convolution is a linear operator, and therefore, has a number of important properties like commutative, associative and distributive properties. The definition and interpretations of these properties in ST domain are summarized below

I. Commutative property

The ST of convolution of the functions $x(t)$ and $y(t)$ can be written as

$$\left(x\otimes y\right)(t)\overset{ST}{\Leftrightarrow}\frac{\sqrt{2\pi}}{|f|}X\left(\tau/2,f\right)Y\left(\tau/2,f\right) \quad (3.9)$$

where, $\otimes$ represents convolution of two function. Similarly, the ST of convolution of $y(t)$ and $x(t)$ can be expressed as

$$(y \otimes x)(t) \overset{ST}{\Leftrightarrow} \frac{\sqrt{2\pi}}{|f|} Y\left(\tau/2, f\right) X\left(\tau/2, f\right) \tag{3.10}$$

Since right hand side (RHS) of (3.9) and (3.10) are equal, therefore,

$$(x \otimes y)(t) = (y \otimes x)(t) \tag{3.11}$$

This property states that a system with the input $x(t)$ and response $y(t)$ behaves in exactly the same way as a scheme with the input $y(t)$ and response $x(t)$.

II. Associative property

The ST of convolution of the continuous functions $w(t)$, $x(t)$ and $y(t)$ can be stated as

$$\{(w \otimes x) \otimes y\}(t) \overset{ST}{\Leftrightarrow} \frac{\sqrt{2\pi}}{|f|} W\left(\tau/2, f\right) X\left(\tau/2, f\right) Y\left(\tau/2, f\right) \tag{3.12}$$

If rearranging the functions, then the convolution can be expressed as

$$\{w \otimes (x \otimes y)\}(t) \overset{ST}{\Leftrightarrow} \frac{\sqrt{2\pi}}{|f|} W\left(\tau/2, f\right) X\left(\tau/2, f\right) Y\left(\tau/2, f\right) \tag{3.13}$$

Since RHS of (3.12) and (3.13) are equal, therefore,

$$\{(w \otimes x) \otimes y\}(t) = \{w \otimes (x \otimes y)\}(t) \tag{3.14}$$

This property states that, if two systems with a response $x(t)$ and $y(t)$ are joined in cascade, a comparable system is one, that has a unit sample response identical to the convolution of $x(t)$ and $y(t)$.

III. Distributive properties

Using the definition of the ST, it is written as

$$\{w \otimes (x+y)\}(t) \overset{ST}{\Leftrightarrow} \frac{\sqrt{2\pi}}{|f|} W\left(\tau/2, f\right) S_{x+y}\left(\tau/2, f\right) \tag{3.15}$$

$$= \frac{\sqrt{2\pi}}{|f|} \int_{-\infty}^{\infty} w(t) \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(t-\xi)^2 f^2}{2}\right) \exp(-j2\pi f t)\, dt \int_{-\infty}^{\infty} (x+y)(t) \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(t-\xi)^2 f^2}{2}\right) \exp(-j2\pi f t)\, dt \tag{3.16}$$

$$\frac{\sqrt{2\pi}}{|f|}\int_{-\infty}^{\infty} w(t)\frac{|f|}{\sqrt{2\pi}}\exp(\frac{-(t-\xi)^2f^2}{2})\exp\left(-j2\pi f t\right)dt\left\{\int_{-\infty}^{\infty} x(t)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(t-\xi)^2f^2}{2}\right)\exp\left(-j2\pi f t\right)dt + \int_{-\infty}^{\infty} y(t)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(t-\xi)^2f^2}{2}\right)\exp\left(-j2\pi f t\right)dt\right\} \quad (3.17)$$

$$\frac{\sqrt{2\pi}}{|f|}W\left(\tau/2,f\right)\left\{X\left(\tau/2,f\right) + Y\left(\tau/2,f\right)\right\}$$

$$\frac{\sqrt{2\pi}}{|f|}W\left(\tau/2,f\right)X\left(\tau/2,f\right) + \frac{\sqrt{2\pi}}{|f|}W\left(\tau/2,f\right)Y\left(\tau/2,f\right) \quad (3.18)$$

Therefore,

$$\left\{w \otimes \left(x+y\right)\right\}(t) = (w \otimes x) + (w \otimes y) \quad (3.19)$$

This property states that, if two systems with a response $x(t)$ and $y(t)$ are joined in parallel, a comparable system is one, that has a unit sample response identical to the sum of $x(t)$ and $y(t)$.

3.1.1.2 Comparative analysis for convolution theorem of S-transform

The existing definition of convolution theorem of ST is given here [52] for comparative analysis, i.e.

$$\int_{-\infty}^{\infty} x\left(t-\xi\right) y(\xi)d\xi \Leftrightarrow \int_{-\infty}^{\infty} X\left(\zeta-\xi,f\right) y(\xi)\exp\left(-j2\pi\xi f\right)d\xi \quad (3.20)$$

Here, $\int_{-\infty}^{\infty} x\left(t-\xi\right) y(\xi)\, d\xi$ is the classical convolution integral of two continuous-time functions $x(t)$ and $y(t)$, and right-hand side (RHS) is the definition of ST convolution of two functions is expressed as $\int_{-\infty}^{\infty} X\left(\zeta-\xi,f\right) y(\xi)\exp\left(-j2\pi\xi f\right)d\xi$. Whereas, in the proposed convolution theorem definition, ST of two continuous-time functions is expressed as $\frac{\sqrt{2\pi}}{|f|}X(\tau/2, f)Y(\tau/2, f)$, RHS of existing convolution theorem involves integration operator, a time-domain dependent complex exponential function and multiplication of shifted ST of one function $X\left(\zeta-\xi, f\right)$ with another function equivalent to its expression in time-domain $y(t)$. Basically, this definition of RHS in existing convolution theorem is not clear, as it is a mix of convolution operation itself with FT.

This interrupts the basic requirement of a convolution theorem i.e. transform of convolution should be multiplication of transformed equivalents.

The proposed definition of convolution theorem of ST involves the multiplication of transformed equivalents. This confirms that the suggested explanation of convolution theorem for ST is realizable whereas existing definition lacks it. A block diagram representation of RHS of both existing and proposed definition is shown below in Fig.3.1.

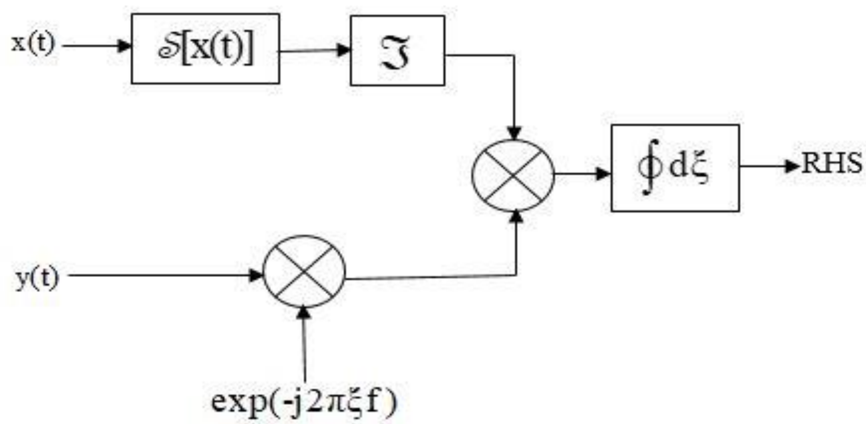


Fig. 3.1 (a) Existing convolution theorem RHS [52]

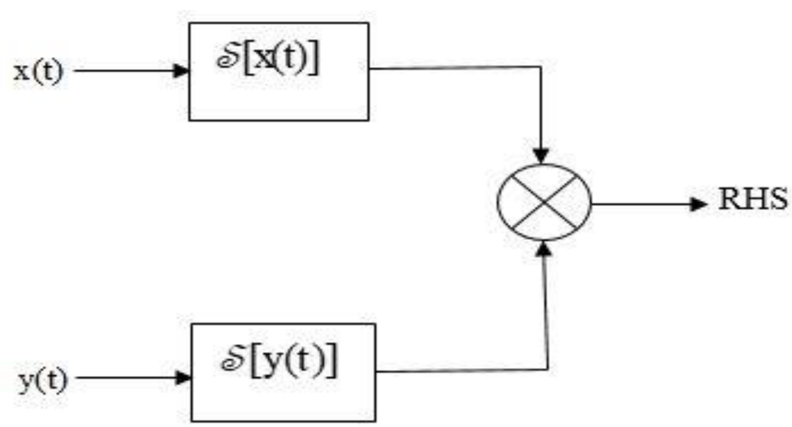


Fig. 3.1 (b) Proposed convolution theorem RHS

where, $\mathcal{S}[.]$ represents the ST of related function and $\Im$ represents the delay and invert operation. Both existing and proposed definitions are compared in terms of their computational complexity levels. In this approach, the number of complex multipliers and complex additions are calculated for the realization of RHS of both convolution theorems. Suppose the numbers of samples of function is N then the computation time for each method is obtained by considering the fact that for evaluating ST, there are N^3 complex multiplications and complex $N(N-1)$ additions. Therefore for the existing convolution theorem, a total of $N(N^3+2N)$ complex multiplications and $2N(N-1)$ complex additions are required. However, for the proposed convolution theorem, in total $2N^3+N$ complex multiplications and $2N(N-1)$ complex addition are needed. These computation complexities are given in Table 3.1.

Table- 3.1 Approximate computational complexity

Operations	Proposed convolution theorem	Existing convolution theorem [52]
Complex multiplication	$2N^3+N$	$N(N^3+2N)$
Complex addition	$2N(N-1)$	$2N(N-1)$

The information about the complex multiplication and addition required to evaluate the RHS of both convolution theorems in Table-3.1 establish that the proposed definition has a better computational complexity to achieve an equivalent convolution in the transformed domain.

3.1.2 Cross-correlation theorem

Definition: If $z(t)$ represent the weighted correlation theorem of two continuous-time functions $x(t)$ and $y(t)$, expressed as

$$z(t) = \int_{-\infty}^{\infty} x^*(\xi)\ y(t+\xi)\exp\left(-(\xi-\tau/2)\ (t+\xi-\tau/2)f^2\right)d\xi \tag{3.21}$$

Then, the cross-correlation theorem of ST can be stated as

$$z(t) = \int_{-\infty}^{\infty} x^*(\xi)\, y(t+\xi)\exp\left(-(\xi-\tau/2)(t+\xi-\tau/2)f^2\right)d\xi \overset{ST}{\Leftrightarrow} \frac{\sqrt{2\pi}}{|f|}X^*(\tau/2, f)Y(\tau/2, f) = Z(\tau, f)$$

where '$*$' represent the complex conjugate.

Proof: The ST of $z(t)$ can be obtained as

$$Z(\tau, f) = \int_{-\infty}^{\infty} z(t)\frac{|f|}{\sqrt{2\pi}}\ \exp\left(\frac{-(t-\tau)^2 f^2}{2}\right)\exp\left(-j2\pi t f\right)dt \tag{3.22}$$

Substituting (3.21) in (3.22), results into

$$Z(\tau,f)=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}x^{*}(\xi)y(t+\xi)\exp\left(-(\xi-\tfrac{\tau}{2})(t+\xi-\tfrac{\tau}{2})f^{2}\right)d\xi\right\}\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(t-\tau)^{2}f^{2}}{2}\right)\exp\left(-j2\pi ft\right)dt \quad (3.23)$$

Rearranging the terms in (3.23) gives

$$=\int_{-\infty}^{\infty}x^{*}(\xi)\left\{\int_{-\infty}^{\infty}y(t+\xi)\frac{f}{\sqrt{2\pi}}\exp\left(-(\xi-\tfrac{\tau}{2})(t+\xi-\tfrac{\tau}{2})f^{2}\right)\exp\left(\frac{-(t-\tau)^{2}f^{2}}{2}\right)\exp\left(-j2\pi ft\right)dt\right\}d\xi \quad (3.24)$$

Substituting, $t+\xi=\kappa$ and $dt=d\kappa$ in (3.24), results into

$$=\int_{-\infty}^{\infty}x^{*}(\xi)\left\{\int_{-\infty}^{\infty}y(\kappa)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(\xi-\tfrac{\tau}{2})^{2}f^{2}}{2}\right)\exp\left(\frac{-(\kappa-\tfrac{\tau}{2})^{2}f^{2}}{2}\right)\exp\left(-j2\pi f\kappa\right)\exp\left(j2\pi f\xi\right)d\kappa\right\}d\xi \quad (3.25)$$

$$=\frac{\sqrt{2\pi}}{|f|}\left\{\int_{-\infty}^{\infty}x^{*}(\xi)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(\xi-\tfrac{\tau}{2})^{2}f^{2}}{2}\right)\exp\left(j2\pi f\xi\right)d\xi\right\}\left\{\int_{-\infty}^{\infty}y(\kappa)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(\kappa-\tfrac{\tau}{2})^{2}f^{2}}{2}\right)\exp\left(-j2\pi f\kappa\right)d\kappa\right\} \quad (3.26)$$

By using the definition of ST (3.26) written as

$$z(t)\overset{ST}{\Leftrightarrow}\frac{\sqrt{2\pi}}{|f|}X^{*}(\tfrac{\tau}{2},f)Y(\tfrac{\tau}{2},f) \quad (3.27)$$

That means a time-domain weighted cross-correlation of two functions gives the multiplication of these function in ST domain.

3.1.3 Parseval's theorem

Definition: It conveys that energy possessed by a signal does not change when the signal is converted from time to frequency domain, that is

$$\int_{-\infty}^{\infty}|x(t)|^{2}dt=\int_{-\infty}^{\infty}|X(f)|^{2}df$$

where, $X(f)$ is FT of $x(t)$.

Proof: Considering the expression of IST for the function $x(t)$ obtained as

$$x(t)=\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}X(\tau,f)d\tau\}\exp\left(j2\pi tf\right)df \quad (3.28)$$

Similarly, taking the complex conjugate of both sides of (3.28) results into

$$x^*(t) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X^*(\tau, f)d\tau\right)\exp\left(-j2\pi tf\right)df \tag{3.29}$$

Now, the LHS of Parseval's theorem of the ST can be written as

$$\int_{-\infty}^{\infty} x(t)\, x^*(t)\,dt = \int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} X^*(\tau, f)d\tau\}\, \exp\left(-j2\pi f t\right)df\, dt \tag{3.30}$$

Now, using the property of convergence of ST into FT, once ST is integrated w.r.t. τ variable, that is

$$X^*(f) = \int_{-\infty}^{\infty} X^*(\tau, f)d\tau \tag{3.31}$$

Substituting (3.31) into (3.30), and rearranging the terms

$$\int_{-\infty}^{\infty} x(t)x^*(t)\,dt \ = \int_{-\infty}^{\infty} X^*(f)\int_{-\infty}^{\infty}\{x(t)\exp\left(-j2\pi f t\right)dt\}df$$

$$= \int_{-\infty}^{\infty} X(f)X^*(f)df \tag{3.32}$$

Therefore, we obtain the following

$$\int_{-\infty}^{-\infty} \left|x(t)\right|^2 dt = \int_{-\infty}^{-\infty} \left|X(f)\right|^2 df \tag{3.33}$$

Hence, this is validated that energy is preserved even after transformation.

3.1.4 Time reversal property

Definition: If the signal is reversed in time domain then its ST is obtained by taking the conjugate of ST of the original signal at a negative delay, that is

$$\text{If} \quad x(t) \overset{ST}{\Leftrightarrow} X(\tau, f)\,;\ \text{then}\ \ x(-t) \overset{ST}{\Leftrightarrow} X^*(-\tau, f)$$

Proof: The ST of $x(-t)$ can be expressed by using the definition of the ST written as

$$\mathcal{S}[x(-t)] = \int_{-\infty}^{\infty} x(-t)\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(t-\tau)^2 f^2}{2}\right)\exp\left(-j2\pi f t\right)dt \tag{3.34}$$

Substituting, $-t = t'$ and $dt = -dt'$ in (3.34), results in

$$\mathcal{S}[x(-t)] = -\int_{\infty}^{-\infty} x(t')\frac{|f|}{\sqrt{2\pi}}\exp\left(\frac{-(-t'-\tau)^2 f^2}{2}\right)\exp\left(j2\pi f\, t'\right)dt' \tag{3.35}$$

After rearranging the terms, (3.35) converts into

$$\mathcal{S}[x(-t)] = -\int_{\infty}^{-\infty} x(t') \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(-t'-\tau)^2 f^2}{2}\right) \exp\left(-j2\pi(-f)\, t'\right) dt' \tag{3.36}$$

Therefore,

$$x(-t) \overset{ST}{\Leftrightarrow} S^*(-\tau, f) \tag{3.37}$$

Therefore, this property gives that if a signal is inverted in time domain then its ST is complex conjugate with delay.

3.1.5 Time derivatives property

Definition: The effect of taking differentiation in the time domain is multiplication of imaginary radian frequency to the ST of the original signal.

$$\text{If,} \quad x(t) \overset{ST}{\Leftrightarrow} X(\tau, f) \quad \text{then,} \quad \frac{d\{x(t)\}}{dt} \overset{ST}{\Leftrightarrow} j(2\pi f) X(\tau, f)$$

Proof: Using the definition of IST of the continuous-time $x(t)$ given in (2.6) is written as

$$x(t) = \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} X(\tau, f) d\tau\} \exp\left(j2\pi f t\right) df \tag{3.38}$$

Comparing it with the definition of FT (1.5), and then differentiating (3.38) on both sides w.r.t. time written as

$$\frac{d\{x(t)\}}{dt} = \frac{d\{\int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} X(\tau, f) d\tau\}\ \exp\left(j2\pi f t\right) df\}}{dt} \tag{3.39}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{X(\tau, f) d\tau\right\}\ \frac{d}{dt}\left\{\exp\left(j2\pi f t\right)\right\} df$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{(j2\pi f) X(\tau, f) d\tau\right\}\ \exp\left(j2\pi f t\right) df\ =\ j(2\pi f)\, X(\tau, f) \tag{3.40}$$

Therefore,

$$\frac{d\{x(t)\}}{dt} \overset{ST}{\Leftrightarrow} j(2\pi f) X(\tau, f) \tag{3.41}$$

That means time-domain differentiation gives, the imaginary radian frequency multiplication in the time-frequency domain.

3.1.6 Complex conjugate property

Definition: It proposes that the ST of the complex conjugate in time domain function is also complex conjugate with negative frequency in the frequency domain.

$$\text{If} \quad x(t) \overset{ST}{\Leftrightarrow} X(\tau, f) \quad \text{then,} \quad x^*(t) \overset{ST}{\Leftrightarrow} X^*(\tau,-f)$$

where * represent complex conjugate.

Proof: Using the definition of IST of the continuous-time function $x(t)$ given in (2.6) is written as

$$x(t) = \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} X(\tau,f)d\tau\} \exp\left(j2\pi f t\right)df \tag{3.42}$$

Taking complex conjugate of (3.42) results into

$$x^*(t) = \left(\int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} X(\tau,f)d\tau\} \exp\left(j2\pi f t\right)df\right)^*$$

$$= \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} X^*(\tau,f)d\tau\} \exp\left(-j2\pi f t\right)df \tag{3.43}$$

Substituting, $f = -f^{'}$ then $df = -df^{'}$ in (3.43) and rearranging the limits of integration

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{X^*(\tau,-f^{'})d\tau\right\} \exp\left(j2\pi t f^{'}\right)df^{'} \tag{3.44}$$

Therefore,

$$x^*(t) \overset{ST}{\Leftrightarrow} X^*(\tau, -f) \tag{3.45}$$

That means time-domain complex conjugate gives complex conjugate with negative frequency in the time-frequency domain.

3.2 N-dimensional S-Transform and its Properties

The two dimensions (2-D) transforms offer numerous new and rich conceivable outcomes. Contemporary utilization of the ST is similarly prone to originate from issues in two, three, and considerably higher dimensions. One of the best applications areas of N-dimensional (N-D) transform is video processing.

In vector notations, the higher dimensional case looks like the 1-D. Suppose, $\Re^n$ as an n-tuple, say spatial variable $t=\{t_1,t_2,...,t_n\}$, $\tau=\{\tau_1,\tau_2,...,\tau_n\}$ and n-tuple of frequencies $f=\{f_1,f_2,...,f_n\}$. The dot products of a vector $\Re^n$ are expressed as

$$tf = \left(t_1f_1 + t_2f_2 + ... + t_nf_n\right) \tag{3.46}$$

where, $\Re^n$ is governed by the dot product. Let, real or complex-valued functions $h(t)$ be defined on $\Re^n$ and inscribed as $h(t)$ or $h(t_1,t_2,...,t_n)$. The ST of $h(t)$ i.e., $H(\tau, f)$ or $\mathscr{S}(\tau, f)$, can be expressed as

$$H(\tau, f) = \int_{\Re^n} h(t)\frac{|f|}{\sqrt{2\pi}}\exp\left(-\frac{(t-\tau)^2 f^2}{2}\right)\exp(-j2\pi tf)dt \tag{3.47}$$

The N-D Fourier spectrum $H(f)$ of $h(t)$ can be inscribed as

$$H(f) = \int_{\Re^n} H(\tau, f)\,dt \tag{3.48}$$

On a similar line, N-D ST can be expressed as

$$\begin{aligned} H(\tau, f) = \int_{\Re^n} H\left(\alpha_1+f_1,\alpha_2+f_2,...,\alpha_n+f_n\right)\left\{\exp\left(\frac{2\pi^2\alpha_1^2}{f_1^2}\right),\exp\left(\frac{2\pi^2\alpha_2^2}{f_2^2}\right),....,\exp\left(\frac{2\pi^2\alpha_2^2}{f_2^2}\right)\right\} \\ \exp\left\{j2\pi(\alpha_1f_1+\alpha_2f_2+...+\alpha_nf_n)\right\}d\alpha_1,d\alpha_2,...,d\alpha_n \quad ,\forall\ f_1,f_2,...,f_n \neq 0 \end{aligned} \tag{3.49}$$

The N-D ST can also be defined in terms of 'N' 1-D ST in a sequential manner as

$$H(\tau, f) = h\left\{t_1,t_2,...,t_n, f_1,f_2,...f_n\right\} \tag{3.50}$$

$$\begin{aligned} = &\int_{-\infty}^{\infty} H\left(\tau_1,f_1\right)\frac{|f_1|}{\sqrt{2\pi}}\exp\left(-\frac{(t_1-\tau_1)^2 f_1^2}{2}\right)\exp\left(-j2\pi t_1f_1\right)d\,t_1\int_{-\infty}^{\infty} H\left(\tau_2,f_2\right)\frac{|f_2|}{\sqrt{2\pi}}\exp\left(-\frac{(t_2-\tau_2)^2 f_2^2}{2}\right)\exp\left(-j2\pi t_2f_2\right)d\,t_2 \\ &... \quad \int_{-\infty}^{\infty} H\left(\tau_n,f_n\right)\frac{|f_n|}{\sqrt{2\pi}}\exp\left(-\frac{(t_n-\tau_n)^2 f_n^2}{2}\right)\exp\left(-j2\pi t_nf_n\right)d\,t_n \end{aligned} \tag{3.51}$$

$$= \int_{\Re^{n-1}}\left\{\int_{-\infty}^{\infty} H(\tau_n, f)\frac{|f_n|}{\sqrt{2\pi}}\exp\left(-\frac{(t_n-\tau_n)^2 f_n^{\,2}}{2}\right)\exp\left(-j2\pi t_nf_n\right)dt_n\right\} \tag{3.52}$$

The N-D Fourier spectrum is related to N-D ST as

$$H\left(f\right) = H\left\{f_1,f_2,...,f_n\right\} = \int_{\Re^n} H\left(\tau,f\right)dt \tag{3.53}$$

The N-D IST can be defined as

$$h(t) = \int_{\Re^n} \left\{ \int_{\Re^n} H(\tau,f)\,dt \right\} \exp(j2\pi tf)\,df \tag{3.54}$$

The dot product features of a vectors t and f to extend the definition from 1-D to higher dimensions (n-D). The integral is overall of $\Re^n$ and as an n-fold multiple integral all $\mathbf{t}_i$ or f_i.

$$\mathcal{S}(t,f) = \mathcal{S}\{s(t,f)\} = \int_{\Re^n} s(\tau,f)\frac{|f|}{\sqrt{2\pi}}\exp\left(-\frac{(t-\tau)^2 f^2}{2}\right)\exp(-j2\pi tf)\,dt \tag{3.55}$$

Substituting, $f = -f$ in both sides of (3.55), results in

$$\mathcal{S}(\tau,-f) = \mathcal{S}\{s(\tau,-f)\} = \int_{\Re^n} H(\tau,f)\frac{|f|}{\sqrt{2\pi}}\exp\left(-\frac{(t-\tau)^2(-f)^2}{2}\right)\exp(j2\pi tf)\,dt \tag{3.56}$$

$$H(\tau,f) = \mathcal{S}^{-1}(\tau,f) \tag{3.57}$$

3.2.1 S-transform be a separable function

In the separable property, a signal is said to be distinct in the event if it can be considered as a result of the 1-D function with various autonomous factors. This permits figuring the ST as a result of 1-D ST rather than multi-dimensional ST. Therefore, a function $h(t_1,t_2,...,t_n)$ of n-variable can be stated as a product of n function of one variable as

$$h(t_1,t_2,...,t_n) = h_1(t_1)h_2(t_2),...,h_n(t_n) \tag{3.58}$$

Methods of separation of variables for N-D ST can be a break in terms of 1-D ST by considering the definition of $\mathcal{S}(t,f)$ as

$$H(t,f) = \mathcal{S}\{h(t,f)\} = \int_{\Re^n} h(t,f)\frac{|f|}{\sqrt{2\pi}}\exp\left(-\frac{(\tau-t)^2 f^2}{2}\right)\exp(-j2\pi tf)\,dt \tag{3.59}$$

This separation is illustrated by considering an example of N=2, then N-D ST can be written as

$$\mathcal{S}(t_1,t_2,f_1,f_2) = \mathcal{S}\{h(t_1,t_2,f_1,f_2)\} \tag{3.60}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1,t_2,f_1,f_2)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-\{(\tau_1-t_1)^2 f_1^2+(\tau_2-t_2)^2 f_2^2\}}{2}\right)\exp(-j2\pi(t_1f_1+t_2f_2))\,dt_1dt_2 \tag{3.61}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1,f_1)h(t_2,f_2)\frac{|f_1|}{\sqrt{2\pi}}\frac{|f_2|}{\sqrt{2\pi}}\exp\left(\frac{-(\tau_1-t_1)^2 f_1^2}{2}\right)\exp\left(\frac{-(\tau_2-t_2)^2 f_2^2}{2}\right)\exp(-j2\pi t_1f_1)\exp(-j2\pi t_2f_2)\,dt_1dt_2 \tag{3.62}$$

$$= \int_{-\infty}^{\infty} h(t_1,f_1) \frac{|f_1|}{\sqrt{2\pi}} \exp\left(\frac{(\tau_1-t_1)^2 f_1^2}{2}\right) \exp\left(-j2\pi t_1 f_1\right) d t_1 \int_{-\infty}^{\infty} h(t_2,f_2) \frac{|f_2|}{\sqrt{2\pi}} \exp\left(-\frac{(\tau_2-t_2)^2 f_2^2}{2}\right) \exp\left(-j2\pi t_2 f_2\right) dt_2 \quad (3.63)$$

$$\mathcal{S}\left\{h(t_1,f_1)\right\} \mathcal{S}\left\{h(t_2,f_2)\right\} = \mathcal{S}_1\left(t_1,f_1\right) \mathcal{S}_2\left(t_2,f_2\right) \quad (3.64)$$

Which proves that N-D ST can be defined in terms of 1-D ST.

In general, for N-D ST can be defined as

$$\mathcal{S}\left\{h(t_1,t_2,...,t_n,f_1,f_2,...,f_n)\right\} = \mathcal{S}\left\{h_1(t_1,f_1)\right\} \mathcal{S}\left\{h_2(t_2,f_2)\right\} ... \mathcal{S}\left\{h_n(t_n,f_n)\right\} \quad (3.65)$$

3.2.2 Identities for N-D S-transform

Below properties of N-D ST are defined and proved:

I. Linearity Property

Definition: In the linear properties, a function $h(t)$ satisfies the superposition principle i.e. additivity and homogeneity properties of degree one.

If, $h_1(t) \overset{ST}{\Leftrightarrow} H_1(t,f)$ and $h_2(t) \overset{ST}{\Leftrightarrow} H_2(\tau,f)$

then, $\left\{\alpha_1 h_1 + \alpha_2 h_2\right\}\left(t,f\right) \overset{ST}{\Leftrightarrow} \alpha_1\left\{H_1(\tau,f)\right\} + \alpha_2\left\{H_2(\tau,f)\right\}$

Proof: Using the definition of ST can be stated as

$$\mathcal{S}\left\{\alpha_1 h_1 + \alpha_2 h_2\right\}\left(t,f\right) = \alpha_1\left\{H_1(\tau,f)\right\} + \alpha_2\left\{H_2(\tau,f)\right\} \quad (3.66)$$

$$= \int_{-\infty}^{\infty} \alpha_1 h_1(t) \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(t-\tau)^2 f^2}{2}\right) \exp\left(-j2\pi f t\right) dt + \int_{-\infty}^{\infty} \alpha_2 h_2(t) \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(t-\tau)^2 f^2}{2}\right) \exp\left(-j2\pi f t\right) d t \quad (3.67)$$

$$= \alpha_1 \int_{-\infty}^{\infty} h_1(t) \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(t-\tau)^2 f^2}{2}\right) \exp\left(-j2\pi f t\right) d t + \alpha_2 \int_{-\infty}^{\infty} h_2(t) \frac{|f|}{\sqrt{2\pi}} \exp\left(\frac{-(t-\tau)^2 f^2}{2}\right) \exp\left(-j2\pi f t\right) d t \quad (3.68)$$

$$= \alpha_1\left\{H_1(t,f)\right\} + \alpha_2\left\{H_2(t,f)\right\} \quad (3.69)$$

In general, for N-D ST can be stated as

$$\mathcal{S}\left\{\alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_n s_n\right\}\left(t,f\right) = \alpha_1\left\{H_1(t,f)\right\} + \alpha_2\left\{H_2(t,f)\right\} + ... + \alpha_n\left\{H_n(t,f)\right\} \quad (3.70)$$

II. Shifting Property

Definition: If a function is a delay in the time domain it parallels to an addition of a linear phase term in the frequency domain.

Let, N=2, N-D ST can be defined as

If $H\{\tau_1,\tau_2,f_1,f_2\} = \mathscr{S}\{h(t_1,t_2,f_1,f_2)\}$

then, $H\{\tau_1 \pm \zeta_1, \tau_2 \pm \zeta_2, f_1,f_2\} = \exp\left(\pm j2\pi(\zeta_1 f_1 + \zeta_2 f_2)\right)\mathscr{S}\{h(t_1,t_2,f_1,f_2)\}$

Proof: Using the definition of 2-D ST can be written as

$$H\{\tau_1,\tau_2,f_1,f_2\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(t_1,t_2\right)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-((t_1-\tau_1)^2 f_1^2+(t_2-\tau_2)^2 f_2^2)}{2}\right)\exp\left(-j2\pi(t_1 f_1 + t_2 f_2)\right)dt_1 dt_2 \tag{3.71}$$

Suppose, $h(t)$ a function is shifted by ζ then

$$H\{\tau_1,\tau_2,f_1,f_2\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(t_1+\zeta_1,t_2+\zeta_2\right)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-((t_1-\tau_1)^2 f_1^2+(t_2-\tau_2)^2 f_2^2)}{2}\right)\exp\left(-j2\pi(t_1 f_1 + t_2 f_2)\right)dt_1 dt_2 \tag{3.72}$$

$$= \int_{-\infty}^{\infty}\frac{|f_1|}{\sqrt{2\pi}}\exp\left(\frac{-(t_1-\tau_1)^2 f_1^2}{2}\right)\exp\left(-j2\pi t_1 f_1\right)\int_{-\infty}^{\infty} h\left(t_1+\zeta_1,t_2+\zeta_2\right)\frac{|f_2|}{\sqrt{2\pi}}\exp\left(\frac{-(t_2-\tau_2)^2 f_2^2}{2}\right)\exp\left(-j2\pi t_2 f_2\right)dt_2 dt_1 \tag{3.73}$$

Substituting, $t_2+\zeta_2=y$ and $dt_2=dy$ in (3.73), results in

$$= \int_{-\infty}^{\infty}\frac{|f_1|}{\sqrt{2\pi}}\exp\left(\frac{-(t_1-\tau_1)^2 f_1^2}{2}\right)\exp\left(-j2\pi t_1 f_1\right)\int_{-\infty}^{\infty} h\left(t_1+\zeta_1,y\right)\frac{|f_2|}{\sqrt{2\pi}}\exp\left(\frac{-(y-\zeta_2-\tau_2)^2 f_2^2}{2}\right)\exp\left(-j2\pi(y-\zeta_2)f_2\right)dy\,dt_1 \tag{3.74}$$

Substituting, $t_1+\zeta_1=x$ and $dt_2=dx$ in (3.74), results in

$$= \int_{-\infty}^{\infty}\frac{|f_1|}{\sqrt{2\pi}}\exp\left(\frac{-(x-\zeta_1-\tau_1)^2 f_1^2}{2}\right)\exp\left(-j2\pi(x-\zeta_1)f_1\right)\int_{-\infty}^{\infty} h\left(x,y\right)\frac{|f_2|}{\sqrt{2\pi}}\exp\left(\frac{-(y-\zeta_2-\tau_2)^2 f_2^2}{2}\right)\exp\left(-j2\pi(y-\zeta_2)f_2\right)dy\,dx \tag{3.75}$$

$$= \exp\left(j2\pi(\zeta f_1+\zeta_2 f_2)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(x,y\right)\frac{|f_1|}{\sqrt{2\pi}}\exp\left(\frac{-(x-\zeta_1-\tau_1)^2 f_1^2}{2}\right)\exp\left(-j2\pi x f_1\right)\frac{|f_2|}{\sqrt{2\pi}}\exp\left(\frac{-(y-\zeta_2-\tau_2)^2 f_2^2}{2}\right)\exp\left(-j2\pi y f_2\right)dy\,dx \tag{3.76}$$

$$= \exp\left(j2\pi(\zeta_1 f_1+\zeta_2 f_2)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(x,y\right)\frac{|f_1|}{\sqrt{2\pi}}\frac{|f_2|}{\sqrt{2\pi}}\exp\left(\frac{-\left\{(\tau_1+\zeta_1-x)^2 f_1^2+(\tau_2+\zeta_2-y)^2 f_2^2\right\}}{2}\right) \tag{3.77}$$

$$\exp\left(-j2\pi(f_1 x+f_2 y)\right)dydx$$

$$= \exp\left(j2\pi(\zeta_1 f_1+\zeta_2 f_2)\right)H\left\{x,y,f_1,f_2\right\} \tag{3.78}$$

Similarly, the shifting property for N-D ST can be defined as

$$H\left\{\tau_1\pm\zeta_1,\tau_2\pm\zeta_2,...,\tau_n\pm\zeta_n,\ f_1,f_2,...,f_n\right\} = \exp\left(\pm j2\pi(\zeta_1 f_1+\zeta_2 f_2+...+\zeta_n f_n)\right)\mathcal{S}\left\{h(t_1,t_2,...,t_n,f_1,f_2,...,f_n)\right\} \tag{3.79}$$

III. Scaling Property

Definition: It states that, if a signal is horizontally stretch by the factor a in the time domain, then its ST is squeeze in frequency by the same factor.

$$\text{If, } h(t)\overset{ST}{\Leftrightarrow}H(\tau,f)\ \text{ then, } h(at)\overset{ST}{\Leftrightarrow}\frac{1}{|a|}H\left(a\tau,\frac{f}{a}\right)$$

$$\text{i.e. } \mathcal{S}\left\{h(at,f)\right\} = \frac{1}{|a|}H\left\{(a\tau,\frac{f}{a})\right\}$$

For N=2, N-D ST can be written as

$$\mathcal{S}\left\{h(a_1 t_1,a_2 t_1, f_1,f_2)\right\} = \frac{1}{|a_1||a_2|}H\left\{(a_1\tau_1,a_2\tau_2,\ \frac{f_1}{a_1},\frac{f_2}{a_2})\right\}$$

Proof: Using the definition of 2-D ST can be written as

$$H\left\{\tau_1,\tau_2,f_1,f_2\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(t_1,t_2\right)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-\left\{(t_1-\tau_1)^2 f_1^2+(t_2-\tau_2)^2 f_2^2\right\}}{2}\right)\exp\left(-j2\pi(t_1 f_1+t_2 f_2)\right)dt_1 dt_2 \tag{3.80}$$

$$H\left\{a_1\tau_1,a_2\tau_2,f_1,f_2\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(a_1 t_1,a_2 t_2\right)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-\left\{(t_1-\tau_1)^2 f_1^2+(t_2-\tau_2)^2 f_2^2\right\}}{2}\right)\exp\left(-j2\pi(t_1 f_1+t_2 f_2)\right)dt_1 dt_2 \tag{3.81}$$

Substitute, $a_1 t_1 = p$, $a_2 t_2 = q$, and $dt_1=\frac{dp}{|a_1|}$, $dt_2=\frac{dq}{|a_2|}$ in (3.81), results in

$$= \frac{1}{|a_1||a_2|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(p,q\right)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-((\frac{p}{a_1}-\tau_1)^2 f_1^2+(\frac{q}{a_2}-\tau_2)^2 f_2^2)}{2}\right)\exp\left(-j2\pi\left(\frac{pf_1}{a_1}+\frac{qf_2}{a_2}\right)\right)dpdq \tag{3.82}$$

$$= \frac{1}{|a_1||a_2|}H\left\{a_1\tau_1,a_2\tau_2,\ \frac{f_1}{a_1},\frac{f_2}{a_2}\right\} \tag{3.83}$$

$$= \frac{1}{|\det M|} H\left\{(a_1\tau_1, a_2\tau_2, M^{-T}f)\right\} \tag{3.84}$$

where, $M = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ and $\det M = a_1 a_2 \neq 0$.

$$M^T = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \tag{3.85}$$

$$\text{or,} \quad M^{-T} = \begin{bmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{bmatrix} \tag{3.86}$$

where T is the transpose of the matrix.

$$\mathcal{S}\left\{h(a_1t_1, a_2t_2, f_1, f_2)\right\} = \frac{1}{|\det M|} H\left\{(a_1\tau_1, a_2\tau_2, M^{-T}f)\right\} \tag{3.87}$$

Similarly, for N-D ST, it can be defined as

$$\mathcal{S}\left\{h(a_1t_1, a_2t_2, ..., a_nt_n, f_1, f_2, ..., f_n)\right\} = \frac{1}{|a_1|, |a_2|, ..., |a_n|} H\left\{(a_1\tau_1, a_1\tau_2, ..., a_1\tau_n, \frac{f_1}{a_1}, \frac{f_2}{a_2}, ..., \frac{f_n}{a_n})\right\} \tag{3.88}$$

IV. Complex conjugate property

Definition: If the complex conjugate of time-domain function ST is also conjugate with negative frequency in other domain.

Let, N=2, N-D ST can be defined as

$$\text{If } \mathcal{S}\left\{h(t_1, t_2)\right\} \overset{ST}{\Leftrightarrow} H\left(\tau_1, \tau_2, f_1, f_2\right),$$

$$\text{then } \mathcal{S}\left\{h^*(t_1, t_2)\right\} \overset{ST}{\Leftrightarrow} H\left(\tau_1, \tau_2, -f_1, -f_2\right)$$

Proof: Using the definition of 2-D ST can be expressed as

$$\mathcal{S}\left\{h(t_1, t_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\tau_1, \tau_2, f_1, f_2) d\tau_1 d\tau_2\right\} \exp\left(j2\pi(t_1f_1 + t_2f_2)\right) df_1 df_2 \tag{3.89}$$

Taking complex conjugate represented as '$*$' in both sides of (3.89) results into

$$\mathcal{S}^*\left\{h(t_1, t_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^*(\tau_1, \tau_2, f_1, f_2) dt_1 dt_2\right\} \exp\left(-j2\pi(t_1f_1 + t_2f_2)\right) df_1 df_2 \tag{3.90}$$

Substituting, $f_1 = -f_1$ and $f_2 = -f_2$ in (3.90), results into

$$\mathcal{S}^*\left\{h(t_1,t_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^*(\tau_1,\tau_2, -f_1,-f_2)\,dt_1 dt_2\right\}\exp\left(j2\pi(t_1f_1+t_2f_2)\right)df_1 df_2 \quad (3.91)$$

$$= \mathcal{S}^{-1}\left\{H^*(\tau_1,\tau_2, -f_1,-f_2)\right\} \quad (3.92)$$

Hence, in general, for N-dimensional ST can be defined as

$$\mathcal{S}^*\left\{h(t_1,t_2,...,t_n)\right\} = \mathcal{S}^{-1}\left\{H^*(\tau_1,\tau_2,...,\tau_n, -f_1,-f_2,...,-f_n)\right\} \quad (3.93)$$

Similar to a 1-D case, if $h(t)$ is an even, then $\mathcal{S}\{H(t, f)\}$ even and if $h(t)$ is a real and even, then $\mathcal{S}\{H(t, f)\}$ is real and even function.

V. Time reversal property

Definition: It states that, if the continuous-time signal is reversed then its ST is obtained by the complex conjugate of the original signal with negative delay, that is

Let, N=2, N-D ST can be defined as

$$\text{If, } \mathcal{S}\left\{h(t_1,t_2)\right\} \overset{ST}{\Leftrightarrow} H(\tau_1,\tau_2, f_1,f_2)$$

$$\text{then } \mathcal{S}\left\{h(-t_1,-t_2)\right\} \overset{ST}{\Leftrightarrow} H^*(-\tau_1,-\tau_2, f_1,f_2)$$

Proof: Using the definition of the ST, the function $h(-t_1,-t_2)$ is expressed as

$$\mathcal{S}\left\{h(-t_1,-t_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(-t_1,-t_2\right)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-\left\{(-t_1-\tau_1)^2f_1^2+(-t_2-\tau_2)^2f_2^2\right\}}{2}\right)\exp\left(-j2\pi(t_1f_1+t_2f_2)\right)dt_1 dt_2 \quad (3.94)$$

Substituting, $-t_1 = u_1, -t_2 = u_2$ and $dt_1 = -du_1,\ dt_2 = -du_2$ in (3.94) gives

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(u_1,u_2\right)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-\left\{(u_1-\tau_1)^2f_1^2+(u_2-\tau_2)^2f_2^2\right\}}{2}\right)\exp\left(-j2\pi(-u_1f_1-u_2f_2)\right)du_1du_2 \quad (3.95)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(u_1,u_2\right)\frac{|f_1||f_2|}{2\pi}\exp\left(\frac{-\left\{(u_1-\tau_1)^2f_1^2+(u_2-\tau_2)^2f_2^2\right\}}{2}\right)\exp\left(j2\pi(u_1f_1+u_2f_2)\right)du_1du_2 \quad (3.96)$$

Therefore,

$$\mathcal{S}\left\{h(-t_1,-t_2)\right\} \overset{ST}{\Leftrightarrow} H^*(-\tau_1,-\tau_2, f_1,f_2) \quad (3.97)$$

Similarly, for N-D the ST can be defined as

$$\mathcal{S}\{h(-t_1,-t_2,...,t_n)\} \overset{ST}{\Leftrightarrow} H^*(-\tau_1,-\tau_2,...,\tau_n,\ f_1,f_2,...,f_n) \tag{3.98}$$

VI. Time derivatives property

It is defined in [13] for N=1. Similarly, for N=2, the ST can be defined as

$$\text{If, } \mathcal{S}\{h(t_1,t_2)\} \overset{ST}{\Leftrightarrow} H(\tau_1,\tau_2,\ f_1,f_2)$$

$$\text{Then, } \frac{d^2}{dt_1 dt_2}\mathcal{S}\{h(t_1,t_2)\} \overset{ST}{\Leftrightarrow} j2\pi(f_1,f_2)\ H(\tau_1,\tau_2,\ f_1,f_2)$$

Proof: Considering the definition of n-dimensional IST [5] of $h(t)$ can be inscribed as

$$h(t,f) = \int_{\mathfrak{R}^n}\left(\int_{\mathfrak{R}^n} H(\tau,f)\,dt\right)\exp(j2\pi tf)\,df \tag{3.99}$$

For N=2, the IST of $h(t)$ can be defined as

$$h(t_1,t_2,f_1,f_2) = \int_{\mathfrak{R}^2}\left(\int_{\mathfrak{R}^2} H(\tau_1,\tau_2,f_1,f_2)\,d\tau_1 d\tau_2\right)\exp\big(j2\pi(t_1,t_2,f_1,f_2)\big)\,df_1 df_2 \tag{3.100}$$

Comparing (3.100) with the definition of Fourier transform, after differentiating it with respect to time in both sides as

$$\frac{d^2}{dt_1 dt_2}\{h(t_1,t_2,\ f_1,f_2)\} = \frac{d^2}{dt_1 dt_2}\left\{\int_{\mathfrak{R}^2}\left(\int_{\mathfrak{R}^2} H(\tau_1,\tau_2,\ f_1,f_2)\,d\tau_1 d\tau_2\right)\exp\big(j2\pi(t_1,t_2,\ f_1,f_2)\big)\,df_1 df_2\right\} \tag{3.101}$$

$$= \int_{\mathfrak{R}^2}\left(\int_{\mathfrak{R}^2} H(\tau_1,\tau_2,\ f_1,f_2)\,d\tau_1 d\tau_2\right)\frac{d^2}{dt_1 dt_2}\exp\big(j2\pi(t_1,t_2,\ f_1,f_2)\big)\,df_1 df_2 \tag{3.102}$$

$$= \int_{\mathfrak{R}^2}\left(\int_{\mathfrak{R}^2} H(\tau_1,\tau_2,\ f_1,f_2)\,d\tau_1 d\tau_2\right)\{j2\pi(f_1,f_2)\}\ \exp\big(j2\pi(t_1,t_2,\ f_1,f_2)\big)\,df_1 df_2 \tag{3.103}$$

$$= j2\pi(f_1,f_2)\ H(\tau_1,\tau_2,\ f_1,f_2) \tag{3.104}$$

Therefore,

$$\frac{d^2}{dt_1 dt_2}\{h(t_1,t_2,\ f_1,f_2)\} \overset{ST}{\Leftrightarrow} j2\pi(f_1,f_2)H(\tau_1,\tau_2,\ f_1,f_2) \tag{3.105}$$

Similarly, for N-D ST can be defined as

$$\frac{d^n}{dt_1 dt_2 ... dt_n}\left\{h\left(t_1,t_2,...,t_n,f_1,f_2,...,f_n\right)\right\} \overset{ST}{\Leftrightarrow} j2\pi(f_1,f_2,...,f_n)H\left(\tau_1,\tau_2,...,\tau_n,f_1,f_2,...,f_n\right) \quad (3.106)$$

VII. Parseval's theorem

Definition: If a signal is converted from the time to frequency domain then energy process by the signal does not change.

For N=2, the ST can be defined as

$$\int_{\Re^2} |h(t_1,t_2)|^2 dt_1 dt_2 = \int_{\Re^2} |H(f_1,f_2)|^2 df_1 df_2$$

Proof: Using the expression of inverse ST for the 2-D function of $h(t)$ [14] as

$$h\left(t_1,t_2, f_1,f_2\right) = \int_{\Re^2}\left(\int_{\Re^2} H\left(\tau_1,\tau_2, f_1,f_2\right) d\tau_1 d\tau_2\right) \exp\left(j2\pi(t_1,t_2, f_1,f_2)\right) df_1 df_2 \quad (3.106)$$

Similarly, taking the complex conjugate of (3.106) gives

$$h^*\left(t_1,t_2, f_1,f_2\right) = \int_{\Re^2}\left(\int_{\Re^2} H^*\left(\tau_1,\tau_2, f_1,f_2\right) d\tau_1 d\tau_2\right) \exp\left(-j2\pi(t_1,t_2, f_1,f_2)\right) df_1 df_2 \quad (3.107)$$

Now, the LHS of Parseval's theorem of ST is written as

$$\int_{\Re^2} h\left(t_1,t_2, f_1,f_2\right) h^*\left(t_1,t_2, f_1,f_2\right) dt_1 dt_2 = \int_{\Re^2} h\left(t_1,t_2,f_1,f_2\right) \int_{\Re^2}\left(\int_{\Re^2} H^*\left(\tau_1,\tau_2, f_1,f_2\right) d\tau_1 d\tau_2\right) \exp\left(-j2\pi(t_1,t_2, f_1,f_2)\right) df_1 df_2 dt_1 dt_2 \quad (3.108)$$

Now, by the convergence property of ST into FT, once the ST is integrated with respect to (u_1,u_2) the variable, that is

$$H^*(f_1,f_2) = \int_{\Re^2} H^*\left(\tau_1,\tau_2, f_1,f_2\right) d\tau_1 d\tau_2 \quad (3.109)$$

Substituting (3.109) into (3.108), and rearranging the terms

$$= \int_{\Re^2} H^*(f_1,f_2) \int_{\Re^2}\left\{h\left(t_1,t_2, f_1,f_2\right) \exp\left(-j2\pi(t_1,t_2, f_1,f_2)\right) dt_1 dt_2\right\} df_1 df_2$$

$$= \int_{\Re^2} H(f_1,f_2)\, H^*(f_1,f_2) df_1 df_2 \quad (3.110)$$

Therefore, after transformation, the energy is preserved. Hence, it can be written as

$$\int_{\Re^2} |h(t_1,t_2)|^2 dt_1 dt_2 = \int_{\Re^2} |H(f_1,f_2)|^2 df_1 df_2 \quad (3.111)$$

Similarly, for N-D ST can be defined as

$$\int_{\Re^n} |\,h(t_1,t_2,...,t_n)\,|^2 dt_1 dt_2 ... dt_n = \int_{\Re^n} |\,H(f_1,f_2,...,f_n)\,|^2 df_1 df_2 ... df_n \tag{3.112}$$

3.5 Summary

The main features of this chapter are:

- A convolution theorem for ST is proposed and compared with existing convolution theorem. Thereafter, the computational complexity of the proposed and existing methods are evaluated.
- After that, some more properties like cross-correlation theorem, Parseval's theorem, time-reversal property, time derivatives property, complex conjugate property and sampling theorem for ST are proposed with their analytical proofs.
- Finally, the definition of N-D ST is presented along with its properties.

CHAPTER 4

MULTIRESOLUTION ANALYSIS USING FRACTIONAL S-TRANSFORM

The time and frequency resolution complications are consequences of a physical occurrence and exist nevertheless of the transform being used. It is likely to analyze any signal with an alternative approach known as multiresolution analysis (MRA). MRA, also called multiscale approximation (MSA) analyze the signal with varying resolutions. It is intended to find a specific frequency at a specific location. MRA is a technique for the estimation of function with arbitrary accuracy. It provides an estimation in fine-scale and can be acquired by addition of details for estimation on a scratchy scale, thus also known as scaling basis functions. It is applicable on feature detection, compression and several other areas of signal analysis and image processing [61-63]. Multiresolution technique is beneficial when the signal contains low-frequency constituents for a large duration and high-frequency component for a small duration. The fundamental element of MRA is most easily recognized regarding kernel, range of vectors, and matrices. MRA of square integral function ($L^2[\mathbb{R}]$) is derived from a refiller function φ , which satisfies the condition

$$x(t) = \sum_{n \in \mathbb{Z}} a(n)\ \varphi(2t-n) \tag{4.1}$$

where, $a = \{a(n)_{n\in\mathbb{Z}} \in l(\mathbb{Z})\}$ is a sequence. MRA is an increasing sequence of subspaces $\left\{U_k^\alpha\right\}_{k\in\mathbb{Z}}$ with a positive value of k.

4.1 Time-Frequency Representation

The main drawback of an FT is that it has no time resolution but only frequency resolution. That means it will only determine the signal frequency but arrival time of that frequency will remain unknown. To reduce this drawback, in the few past decades several techniques have been established which are capable to analyze a signal in time-frequency domains [46, 50]. FrST provides a time-frequency representation in contrast to fractional FT (FrFT). MRA can also be obtained using the concept of FrST to analyze the signals in a time-frequency plane. The concept behind the time-frequency representations is to break the signal into numerous fragments and then analyze these fragments separately. It is clear that analyzing a signal in such a way will give more information about the different frequency components. The notations used in the derivation of the proposed theorem are, $\mathbb{R}$ is set of a real number, $\mathbb{Z}$ is set of integer, $\mathbb{Z}^+$ is positive integer set, $L^1[0, 2\pi]$ is the space of integral function in $[0,2\pi]$, $L^2[\mathbb{R}]$ is an space of square integral function in $\mathbb{R}$, $L^\infty[0, 2\pi]$ is the space of integral function in $[0,2\pi]$ and $l^2[\mathbb{Z}]$ is entire space square summable sequence in $\mathbb{Z}$. Continuous-Time signals are represented by parentheses, such as $x(t), t \in \mathbb{R}$ and the discrete signals as $c[n], n \in \mathbb{Z}$. The product of two continuous-time signals $x(t)$ and $y(t)$ in $L^2[\mathbb{R}]$ is written as $\langle x, y \rangle_{L^2} = \int_{\mathbb{R}} x(t)\, y^*(t) dt$. Suppose, Hilbert space is represented by $\mathscr{H}$ and its complete set of functions is $\{\psi_n(t)\}_{n \in \mathbb{Z}}$.

The main component of an MRA is a vector space because every vector space is a set of units of another vector addition and multiplication which give higher resolution until we get the highest possible resolution.

4.2 Inner product Space

A vector space with a defined inner product is termed an inner product space. If the inner product of two vectors is zero, then they are orthogonal to each other [90,106].

4.2.1 Vector Space

A continuous signal is denoted as a time function $x(t)$, or a discrete signal is represented as a vector $x = \{..., x[n], ...\}^T$ in vector space. A vector space is a set of U with two operations of vector, addition and scalar multiplication defined for its members referred to as vectors [106]. It can also be a set containing all continuous functions $x(t)$ defined over a particular interval [a b].

4.2.2 Hilbert space

The Hilbert space consists of vectors (ψ, φ), scalars (a, b) [92,106] with the following properties:

- Hilbert space is a linear vector space.
- The scalar products defined in Hilbert space are positive.
- The scalar product of two vectors ψ and φ are equals to the complex conjugate of the scalar product of the same two vectors in reverse order i.e.,

$$(\psi \ \ \varphi) = (\varphi \ \ \psi)^*$$

- The scalar product of vectors with itself is positive

$$(\psi \;\; \psi) = \|\psi\|^2 > 0$$

- The scalar product in Hilbert space is linear with respect to 2nd factor,

$$\text{If } \psi = a\psi_1 + b\psi_2 \text{, then } (\varphi \;\; \psi) = (\varphi \;\; a\psi_1 + b\psi_2) = a(\varphi \;\; \psi_1) + b(\varphi \;\; \psi_2)$$

- The scalar product in Hilbert space is nonlinear with respect to 1st factor,

$$\text{If } \varphi = a\varphi_1 + b\varphi_2 \text{, then } (\varphi \;\; \psi) = (a\varphi_1 + b\varphi_2 \;\; \psi) = a'(\varphi_1 \;\; \psi) + b'(\varphi_2 \;\; \psi)$$

where, a' and b' represent nonlinearity in space.

4.2.3 Riesz basis

A group of vectors $\{x_k\}$ in Hilbert space $\mathscr{H}$ is the Riesz basis for $\mathscr{H}$ if it is orthonormal basis under an invertible linear transform [90, 92]. Let, $\{x_k\}$ be a collection of vectors in Hilbert space $\mathscr{H}$. It has the following properties

- If $\{x_k\}$ is represents the Riesz basis for $\mathscr{H}$, then there is a unique collection $\{y_k\}$, such that $\{y_k\}$ is biorthogonal to $\{x_k\}$ then $\{y_k\}$ is also a Riesz basis.
- If $\{x_k\}$ is represents the Riesz basis for $\mathscr{H}$, then there exist the constants A and B, defined as $0 \le A \le B$ such that $\forall\, x \in \mathscr{H}$

$$A\|x\|^2 \le \sum_k |\langle x, x_k \rangle|^2 \le B\|x\|^2$$

- If $\{x_k\}$ is represents the Riesz basis for $\mathscr{H}$ and exist A and B, such that $0 \le A \le B$ then for all finite sequence $\{\alpha_k\}$,

$$A\sum_k |\alpha_k|^2 \le \left\|\sum_k \alpha_k x_k\right\|^2 \le B\sum_k |\alpha_k|^2$$

- If $\{x_k\}$ is represents the Riesz basis for $\mathscr{H}$ then for each x belonging in $\mathscr{H}$, there is a unique collection of scalars $\{\alpha_k\}$ i.e.

$$x = \sum_k \alpha_k x_k \text{ and } \sum_k |\alpha_k|^2 < \infty$$

Hilbert space can be applied to any space such as $L^2[\mathbb{R}]$ space composed of all square-integrable functions $x(t)$ defined over the interval [a b].

4.3 Multiresolution analysis with the Fractional S-transform

The concept of an MRA in vector spaces are obtain highest possible resolution for every vector space of higher resolution. Which implies that every vector space holds all other vector spaces that have lower resolution and every vector space is a scale function. A MRA for FrST is defined as an arrangement of the closed subspaces, $\{U_k^\alpha\}_{k\in\mathbb{Z}}$ where, U_k^α cover the entire square-integrable real space $L^2[\mathbb{R}]$, with the following properties:

(i) Nested vector spaces: Every vector space holds all others vector spaces that have lower resolution.

$$U_{-\infty}^\alpha \subset \ldots \subset U_{-2}^\alpha \subset U_{-1}^\alpha \subset U_0^\alpha \subset U_1^\alpha \subset U_2^\alpha \subset \ldots \subset U_\infty^\alpha$$

That fulfils certain self-similarity relation in time or space and scaled, as well as regularity and inclusiveness relations as shown in Fig 4.1.

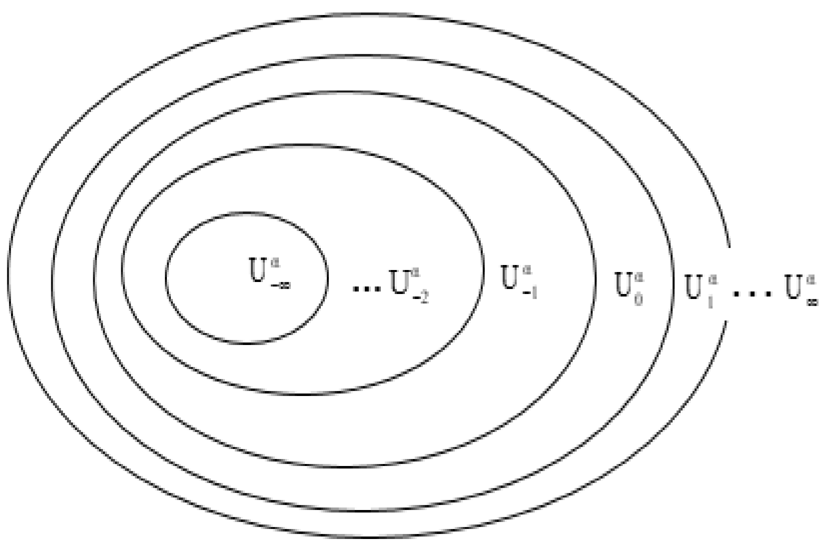


Fig. 4.1 Nested vector spaces

Given, $x \in L^2(\mathbb{R})$ and approximating it using, $x_\gamma \in U_\gamma^\alpha$.

Multiresolution approximation of $L^2(\mathbb{R})$ is $U_\gamma^\alpha = \{x(t) \;:\; x(2^{-\gamma}t) \in U_0^\alpha$

$$U_{-\infty}^{\alpha} \subset \ldots \subset U_{-2}^{\alpha} \subset U_{-1}^{\alpha} \subset U_{0}^{\alpha} \subset U_{1}^{\alpha} \subset U_{2}^{\alpha} \subset \ldots \subset U_{\infty}^{\alpha}$$

$U_{\infty}^{\alpha} \rightarrow$ dense in $L^2(\mathbb{R})$

$$U_1^{\alpha} = \{x(t) \ : \ x(2^{-1}t) \in U_1^{\alpha}$$

$$U_0^{\alpha} = \{x(t) = \sum_k \varphi(t-k) \ : \ \sum_k |a_k|^2 < \infty$$

$U_{-\infty}^{\alpha} \rightarrow \; = \{0\}$

where a_k constant and φ_k are called basis functions. Therefore, $\bigcup_{\gamma=-\infty}^{\infty} U_{\gamma}^{\alpha} \xrightarrow{\text{Dense}} L^2(\mathbb{R})$ and $\bigcap_{\gamma=-\infty}^{\infty} U_{\gamma}^{\alpha} = \{0\}$. Hence,

(ii) $\bigcup_{\gamma\in\mathbb{Z}} U_{\gamma}^{\alpha} = L^2(\mathbb{R})$,

Closing all spaces U_{γ}^{α} gives a space of finite energy $L^2(\mathbb{R})$.

(iii) $\bigcap_{\gamma\in\mathbb{Z}} U_{\gamma}^{\alpha} = \{0\}$,

That means no signal approximating function which belonging in to the space for all resolutions.

(iv) There are many sequence space U_{γ}^{α}, which complies with a condition (i)-(iii),

$$x \in U_{\gamma}^{\alpha}, \text{ therefore } x\left(2^{-\gamma}\right) \in U_0^{\alpha},$$

All spaces U_{γ}^{α} are scaled variations of spaces U_{γ}^{α} relative to the total displacements.

(v) $x \in U_0^{\alpha}$, therefore $x(\bullet - k) \in U_0^{\alpha}$ for all $k \in \mathbb{Z}$.

(vi) Function $\varphi \in L^2(\mathbb{R})$, so that U_0^{α} consists of all linear combinations of $\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$.

These properties imply that $\{\varphi_{\gamma,k}\}_{k\in\mathbb{Z}}$ are Riesz basis in U_{γ}^{α}. The nesting property gives estimations on a rough scale can be achieved from an approximation on a final scale without expressing the original function. The property (ii) and (iii) implies that approximation in $L^2[\mathbb{R}]$ with randomly small errors by projection on the subspace U_{γ}^{α}. With large positive γ of the subspace U_{γ}^{α} are said to be high-resolution scales or fine scales, and have high approximation quality. For the negative γ, the scales become rough, with low-resolution scales or low approximation quality. The property (v) and (vi) implies that the scaling function $\varphi(t)$ is belonging in $L^2[\mathbb{R}]$ such that

$\left\{ \varphi_{0,n,\alpha}(t) = \varphi(t-n) \exp(-\frac{j(t^2\text{-}n^2)}{2}\cot\alpha) \right\}_{n\in\mathbb{Z}}$ can be obtained by a sufficient shifting of the chirp modulated scaling function $\varphi(t)$ forms an orthonormal basis of a subspace U_0^α.

Assume that the set of the scaling function $\varphi(t)$ is a Riesz basis of a sequence of a subspace U_0^α, whose capacity is asserted as a scaling function of the MRA subspace U_k^α. It can be casual by expecting that a set of function $\{\varphi_{0,n,\alpha}(t)\}_{n\in\mathbb{Z}}$ to be the Riesz basis of the subspace is expressed as

$$U_0^\alpha = \left\{ \sum_{n\in\mathbb{Z}} a[n]\varphi_{0,n,\alpha}(t) \quad | \, a[n] \in l^2[\mathbb{Z}] \right\} \tag{4.2}$$

Thereafter, the set of modulated function $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ will be the Riesz basis of subspace U_0^α of $L^2[\mathbb{R}]$ and is proved as Theorem 4.1.

Theorem 4.1: A set of the continuous-time modulated function $\{\varphi_{0,n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is the Riesz basis of subspace U_0^α of $L^2[\mathbb{R}]$, if there exists constant $0 < A \le B < \infty$, such as

$$A \le \mathscr{K}_{\varphi,\alpha}^2(\tau, v) \le B, \quad \forall \, \tau, v \in I \tag{4.3}$$

where,

$$\mathscr{K}_{\varphi,\alpha}(\tau, v) = \left(\sum_{k=-\infty}^{\infty} |\Phi(\tau\sin\alpha + 2\pi k, \, v\csc\alpha + 2\pi k)|^2 \right)^{1/2} \tag{4.4}$$

and holds the condition

$$0 \le \left\| \mathscr{K}_{\varphi,\alpha}(\tau,v) \right\|_k \le \left\| \mathscr{K}_{\varphi,\alpha}(\tau,v) \right\|_{k+1} \le \infty \tag{4.5}$$

and $\varphi(\tau\sin\alpha, v\csc\alpha)$ be the FrST of the scaling function $\varphi(t)$, where the argument is scaled by $\sin\alpha$ and $\csc\alpha$ with time axis and frequency axis respectively.

Proof: For any $x(t) \in U_0^\alpha$ by (2.31) and (4.2) gives

$$x(t) = \sum_{n\in\mathbb{Z}} a[n]\varphi_{0,n,\alpha}(t) = a[n] \overset{s}{\Theta}_\alpha \varphi(t) \quad , \quad a[n] \in l^2[\mathbb{Z}] \tag{4.6}$$

Taking the FrST of $x(t)$,

$$X^\alpha(\tau,v) = \sqrt{2\pi} \, \tilde{a}_\alpha[n]\varphi(\tau\sin\alpha, \, v\csc\alpha) \tag{4.7}$$

where, $\tilde{a}_\alpha[n]$ and $\varphi(\tau\sin\alpha,\ v\csc\alpha)$ is DTFrST of $a[n]$ and the FrST (with argument scaled with $\csc\alpha$ and $\sin\alpha$ in frequency and time domain) of $\varphi(t)$ respectively. Thereafter, by using Parseval's formula of FrST

$$\|x(t)\|_{L^2}^2 = \|X_\alpha(\tau,v)\|_{L^2}^2 = 2\pi\iint_{\mathbb{R}} |\tilde{a}_\alpha[n]|^2 |\varphi(\tau\sin\alpha, v\csc\alpha)|^2\, dv d\tau \tag{4.8}$$

Since, the DTFrST has the chirp periodicity with period $2k\pi\sin\alpha$ where, $k\in\mathbb{Z}$ therefore

$$\|x(t)\|_{L^2}^2 = 2\pi\sum_{k\in\mathbb{Z}}\int_I |\tilde{a}_\alpha[v+2k\pi\sin\alpha]|^2 |\varphi(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2k\pi)|^2\, dv \tag{4.9}$$

Hence, (4.9) is rewrite as

$$\|x(t)\|_{L^2}^2 = 2\pi\int_I |\tilde{a}_\alpha[v]|^2 \mathscr{K}_{\varphi,\alpha}^2(\tau,v) dv \tag{4.10}$$

The Parseval's formula of the DTFrST is given as

$$\|a[n]\|_{l^2}^2 = \sum_{n\in\mathbb{Z}} |a[n]|^2 = \int_I |\tilde{a}_\alpha[v]|^2 dv \tag{4.11}$$

and then using (4.3), (4.6), (4.10) and (4.11), results into

$$A\|a[n]\|_{l^2}^2 \le \left\|\sum_{n\in\mathbb{Z}} a[n]\varphi_{0.n,\alpha}(t)\right\|_{L^2}^2 = B\|a[n]\|_{l^2}^2 \tag{4.12}$$

It follows from (4.12) and the properties of Riesz basis for Hilbert space (as section 4.2.3), that the modulated function $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is a Riesz basis of a subspace U_0^α of $L^2[\mathbb{R}]$. It has also produced an orthonormal basis of the subspace U_0^α spanned by $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ if $A = B = 1$, which are proved in the subsequent theorem 4.2.

Theorem-4.2 Suppose that $\{U_k^\alpha\}_{k\in\mathbb{Z}}$ is an MRA of the FrST belonging in $L^2[\mathbb{R}]$, and modulated function $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is the Riesz basis of U_0^α with bounded as $0 < A \le B < \infty$.

Suppose,
$$\Theta(\tau\sin\alpha,\ v\csc\alpha) = \frac{\varphi(\tau\sin\alpha,\ v\csc\alpha)}{\mathscr{K}_{\varphi,\alpha}(\tau,\ v)} \tag{4.13}$$

where, $\Theta(\tau\sin\alpha,\ v\csc\alpha)$ is FrST (with its argument is scaled by $\sin\alpha$ and $\csc\alpha$ in time axis and frequency axis respectively) of the function $\theta(t)$ belonging in $L^2[\mathbb{R}]$. Then the function $\theta_{0,n,\alpha}(t) = \theta(t-n)\exp\left(-\frac{j(t^2-n^2)}{2}\cot\alpha\right)$ forms an orthonormal basis for subspace U_0^α.

Proof: From the above theorem, if $A = B = 1$, then the modulated function $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is an orthonormal basis function of subspace U_0^α, therefore $\mathscr{K}_{\varphi,\alpha}(\tau,v) = 1$. Then applying (4.13) gives $\theta(t) = \varphi(t)$. Hence, $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ be an orthonormal basis function of subspace U_0^α, if $A = B = 1$. Otherwise, suppose $\theta(t)$ belonging in $L^2[\mathbb{R}]$ with $\theta(t)\exp\left(-j\frac{t^2}{2}\cot\alpha\right) \in U_0^\alpha$. Thereafter, there exists sequence $\{a[n]\}_{n\in\mathbb{Z}}$ belonging in $l^2[\mathbb{Z}]$ written as

$$\theta(t)\exp\left(-j\frac{t^2}{2}\cot\alpha\right) = \sum_{n\in\mathbb{Z}} a[n]\varphi_{0,n,\alpha}(t) \tag{4.14}$$

$$\theta(t)\exp\left(-\frac{j\,(t^2)}{2}\cot\alpha\right) = \sum_{n\in\mathbb{Z}} a[n]\,\varphi(t-n)\,\exp\left(-\frac{j(t^2-n^2)}{2}\cot\alpha\right) \tag{4.15}$$

Taking the FrST in both sides of (4.15) results in

$$\Theta(\tau\sin\alpha,\ v\csc\alpha) = \sum_{n\in\mathbb{Z}} a[n]\exp(j(\frac{n^2}{2}\cot\alpha - nv\csc\alpha))\Phi(\tau\sin\alpha,\ v\csc\alpha) \tag{4.16}$$

$$\Theta(\tau\sin\alpha,\ v\csc\alpha) = \sqrt{2\pi}\ \Xi(\tau\sin\alpha,\ v\csc\alpha)\Phi(\tau\sin\alpha,\ v\csc\alpha) \tag{4.17}$$

where $\Xi(\tau\sin\alpha,\ v\csc\alpha)$ be the DTST (with its argument scaled by $\sin\alpha$ and $\csc\alpha$ with time axis and frequency axis respectively) of $a[n]\exp\left(j(\frac{n^2}{2}\cot\alpha)\right)$ and it is periodic with $2\pi\sin\alpha$. Then using (4.17) and (4.13), results in

$$\Xi(\tau\sin\alpha,\ v\csc\alpha) = \frac{1}{\sqrt{2\pi}\ \mathscr{K}_{\varphi,\alpha}(\tau,v)} \tag{4.18}$$

Since $\Xi(\tau\sin\alpha,\ v\csc\alpha)$ is periodic with $2\pi\sin\alpha$, then applying (4.17) into (2.32), results in

$$\Theta(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2k\pi) = \sqrt{2\pi}\ \Xi(\tau\sin\alpha,\ v\csc\alpha)\Phi(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2k\pi) \tag{4.19}$$

Thereafter, squaring on both sides (4.21), results in

$$\left|\Theta(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2k\pi)\right|^2 = 2\pi\left|\Xi(\tau\sin\alpha,\ v\csc\alpha\right|^2\left|\Phi(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2k\pi)\right|^2 \tag{4.20}$$

Then summing for all k on both sides of (4.20) gives

$$\mathscr{K}^2_{\theta,\alpha}(\tau,v) = 2\pi\left|\Xi(\tau\sin\alpha,\ v\csc\alpha\right|^2 \mathscr{K}^2_{\varphi,\alpha}(\tau,v) \tag{4.21}$$

where, $\mathscr{K}_{\theta,\alpha}(\tau, v) = \left(\sum_{k\in\mathbb{Z}} \left| \Theta(\tau\sin\alpha + 2k\pi,\ v\csc\alpha + 2k\pi) \right|^2 \right)^{1/2}$. Substituting (4.18) into (4.21) yields, $\mathscr{K}^2_{\varphi,\alpha}(\tau, v) = 1$. Then, it follows from theorem 4.1, that the modulated function $\{\theta_{0,n,\alpha}(t)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for subspace U_0^α, which is justified by the theorem 4.2.

4.4. Construction of orthonormal FrST from an MRA

The establishment of an orthonormal FrST from an MRA is basically associated to the FrST. Defined a basis function for the subspace $\{M_k^\alpha\}_{k\in\mathbb{Z}}$ which is orthogonal component of U_k^α in U_{k+1}^α, defined as

$$M_k^\alpha \perp U_k^\alpha \quad ; \qquad U_{k+1}^\alpha = M_k^\alpha \oplus U_k^\alpha \tag{4.22}$$

where the symbol $\perp$ means 'orthogonal to' and $\oplus$ means 'orthogonal sum'. Then, using the definition of MRA, following condition for MRA with FrST will exist.

(a) $M_k^\alpha \perp M_l^\alpha \qquad \forall\ k \neq l$

(b) $L^2[\mathbb{R}] = \oplus_{k\in\mathbb{Z}} M_k^\alpha$

(c) $x(t) \in M_k^\alpha \Leftrightarrow \left(x(2t)\exp\left(j\dfrac{(2t)^2 - t^2}{2}\cot\alpha \right) \right) \in M_{k+1}^\alpha, \quad \forall\ k \in \mathbb{Z}$

The condition (b) can establish an orthonormal basis for $L^2[\mathbb{R}]$ by obtaining an orthonormal basis function for subspace M_k^α, (c) indicates that the task of establishment is abridged to find out the orthonormal basis for M_0^α. So, the main aim is to establish a function $\varphi(t)$ belonging in $L^2[\mathbb{R}]$ that is, $\varphi_{0,n,\alpha}(t) = \varphi(t-n)\exp\left(-\dfrac{j(t^2-n^2)}{2}\cot\alpha \right)$ forms an orthonormal basis function for M_0^α. For simplicity, suppose the chirp-modulated form of a function $p[n]$ is expressed as

$$p_\alpha[n] = p[n]\exp\left(\frac{j}{2}\left(\frac{n}{2}\right)^2 \right) \tag{4.23}$$

Since $\varphi_{0,0,\alpha}(t) \in U_0^\alpha \subseteq U_1^\alpha$ and $\{\varphi_{1,n,\alpha}(t)\}_{n\in\mathbb{Z}}$ be an orthonormal basis function for subspace U_1^α, then there exist coefficients $\{y[n]\}_{n\in\mathbb{Z}}$ such as

$$\varphi_{0,0,\alpha}(t) = \sum_{n\in\mathbb{Z}} y[n]\,\varphi_{1,n,\alpha}(t) \tag{4.24}$$

Therefore, similar to the concept of (4.16) and (4.17), which provides

$$\varphi(t)\exp\left(-j\frac{t^2}{2}\cot\alpha\right)=\sum_{n\in\mathbb{Z}} y_\alpha[n]\sqrt{2}\,\varphi(2t-n)\exp\left(-j\frac{t^2}{2}\cot\alpha\right) \tag{4.25}$$

Using the FrST on both sides of (4.25) results in

$$\Phi(\tau\sin\alpha,v\csc\alpha)=\Lambda\left(\frac{v\csc\alpha}{2}\right)\Phi\left(\frac{\tau\sin\alpha}{2},\ \frac{v\csc\alpha}{2}\right) \tag{4.26}$$

where,

$$\Lambda\left(\frac{v\csc\alpha}{2}\right)=\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} y_\alpha[n]\exp\left(-jnv\csc\alpha\right) \tag{4.27}$$

Hence, since the FrST function $\varphi_{0,0,\alpha}(t)\in M_0^\alpha\in U_1^\alpha$, there must be coefficients $y[n]$ belonging in $l^2[\mathbb{Z}]$ such that

$$\varphi_{0,0,\alpha}(t)=\sum_{n\in\mathbb{Z}} z[n]\varphi_{1,n,\alpha}(t) \tag{4.28}$$

Therefore, similar to the concept of (4.16) and (4.17), which gives

$$\psi(t)\exp\left(-j\frac{t^2}{2}\cot\alpha\right)=\sum_{n\in\mathbb{Z}} z_\alpha[n]\sqrt{2}\varphi(2t-n)\exp\left(-j\frac{t^2}{2}\cot\alpha\right) \tag{4.29}$$

Using FrST on both sides in (4.29) gives

$$\Psi(\tau\sin\alpha,\ v\csc\alpha)=\Gamma\left(\frac{v\csc\alpha}{2}\right)\Phi\left(\frac{\tau\sin\alpha}{2},\frac{v\csc\alpha}{2}\right) \tag{4.30}$$

where,

$$\Gamma\left(\frac{v\csc\alpha}{2}\right)=\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} z_\alpha[n]\exp\left(-jnv\csc\alpha\right) \tag{4.31}$$

Thereafter, the next aim is to verify whether the $\Lambda\left(v\csc\alpha\right)$ and $\Gamma\left(v\csc\alpha\right)$ are periodic with a period $2\pi\sin\alpha$ and belonging in $L^2[I]$. Since $\left\{\varphi_{0,n,\alpha}(t)\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for subspace U_0^α by Theorem 4.1, it can be expressed as

$$\sum_{k\in\mathbb{Z}}\left|\varphi(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2k\pi)\right|^2=1 \tag{4.32}$$

Likewise,

$$\sum_{k\in\mathbb{Z}}\left|\psi(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2k\pi)\right|^2=1 \tag{4.33}$$

Then, substituting (4.26) into (4.32) results in

$$1=\sum_{k\in\mathbb{Z}}\left|\varphi(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2\kappa\pi)\right|^2 \tag{4.34}$$

$$= \sum_{n\in\mathbb{Z}} \left|\Lambda\left(\frac{v\csc\alpha}{2}+k\pi\right)\right|^2 \left|\varphi\left(\frac{\tau\sin\alpha}{2}+k\pi,\ \frac{v\csc\alpha}{2}+k\pi\right)\right|^2 \tag{4.35}$$

$$= \begin{aligned}&\sum_{l\in\mathbb{Z}} \left|\Lambda\left(\frac{v\csc\alpha}{2}+2l\pi\right)\right|^2 \left|\varphi\left(\frac{\tau\sin\alpha}{2}+2l\pi,\ \frac{v\csc\alpha}{2}+2l\pi\right)\right|^2 \\ &+\sum_{l\in\mathbb{Z}} \left|\Lambda\left(\frac{v\csc\alpha}{2}+(2l+1)\pi\right)\right|^2 \left|\varphi\left(\frac{\tau\sin\alpha}{2}+(2l+1)\pi,\ \frac{v\csc\alpha}{2}+(2l+1)\pi\right)\right|^2\end{aligned} \tag{4.36}$$

$$= \begin{aligned}&\left|\Lambda\left(\frac{v\csc\alpha}{2}\right)\right|^2 \sum_{l\in\mathbb{Z}} \left|\varphi\left(\frac{\tau\sin\alpha}{2}+2l\pi,\ \frac{v\csc\alpha}{2}+2l\pi\right)\right|^2 \\ &+\left|\Lambda\left(\frac{v\csc\alpha}{2}+\pi\right)\right|^2 \sum_{l\in\mathbb{Z}} \left|\varphi\left(\frac{\tau\sin\alpha}{2}+(2l+1)\pi,\ \frac{v\csc\alpha}{2}+(2l+1)\pi\right)\right|^2\end{aligned} \tag{4.37}$$

Using (4.34) in (4.37) yields

$$\left|\Lambda\left(v\csc\alpha\right)\right|^2 + \left|\Lambda\left(v\csc\alpha+\pi\right)\right|^2 = 1 \tag{4.38}$$

Similarly substituting (4.30) into (4.33), results into

$$\left|\Gamma\left(v\csc\alpha\right)\right|^2 + \left|\Gamma\left(v\csc\alpha+\pi\right)\right|^2 = 1 \tag{4.39}$$

Moreover, since M_0^α and U_0^α are an orthogonal basis in U_1^α hence, $\left\{\psi_{0,m,\alpha}(t)\right\}_{m\in\mathbb{Z}}$ and $\left\{\varphi_{0,n,\alpha}(t)\right\}_{n\in\mathbb{Z}}$ are orthogonal function that is

$$\left\langle \varphi_{0,n,\alpha}(t),\ \psi_{0,m,\alpha}(t)\right\rangle_{L^2} = 0; \qquad \forall\ (n, m) \in \mathbb{Z} \tag{4.40}$$

Subsequently, the FrST of these function can be expressed as

$$\mathcal{S}^\alpha\left\{\varphi_{0,n,\alpha}(t)\right\}(\tau, v) = \sqrt{2\pi}\,\kappa_\alpha(v,n)\varphi(\tau\sin\alpha,\ v\csc\alpha) \tag{4.41}$$

$$\mathcal{S}^\alpha\left\{\psi_{0,m,\alpha}(t)\right\}(\tau,v) = \sqrt{2\pi}\,\kappa_\alpha(v,m)\varphi(\tau\sin\alpha,\ v\csc\alpha) \tag{4.42}$$

Then applying (4.39), (4.40), (4.41) and Parseval's theorem of FrST results into

$$0 = \left\langle \varphi_{0,n,\alpha}(t),\psi_{0,m,\alpha}(t)\right\rangle_{L^2} = \exp\left(j\frac{n^2-m^2}{2}\cot\alpha\right)\int_{\mathbb{R}} \varphi(\tau\sin\alpha,\ v\csc\alpha)\psi^*(\tau\sin\alpha,v\csc\alpha) \\ \exp\left(-j(n-m)v\csc\alpha\right)dv\ \csc\alpha \tag{4.43}$$

Since $\Lambda\left(v\csc\alpha\right)$ and $\Gamma\left(v\csc\alpha\right)$ are periodic with $2\pi\sin\alpha$, inserting (4.25), (4.28) in (4.41), results into

$$0 = \int_{\mathbb{Z}} \Lambda\left(\frac{v\csc\alpha}{2}\right)\Gamma^*\left(\frac{v\csc\alpha}{2}\right)\left|\varphi\left(\frac{\tau\sin\alpha}{2},\frac{v\csc\alpha}{2}\right)\right|^2 \exp\left(-j(n-m)v\csc\alpha\right)dv\ \csc\alpha \tag{4.44}$$

$$=\sum_{l\in\mathbb{Z}}\int_{4l\pi\sin\alpha}^{4(l+1)\pi\sin\alpha}\Lambda\left(\frac{\mathrm{v}\csc\alpha}{2}\right)\Gamma^{*}\left(\frac{\mathrm{v}\csc\alpha}{2}\right)\left|\varphi\left(\frac{\tau\sin\alpha}{2}+2l\pi,\frac{\mathrm{v}\csc\alpha}{2}+2l\pi\right)\right|^{2}\exp\left(-j(n-m)\mathrm{v}\csc\alpha\right)d\mathrm{v}\csc\alpha \quad (4.45)$$

$$=\int_{0}^{4\pi\sin\alpha}\Lambda\left(\frac{\mathrm{v}\csc\alpha}{2}\right)\Gamma^{*}\left(\frac{\mathrm{v}\csc\alpha}{2}\right)\exp\left(-j(n-m)\mathrm{v}\csc\alpha\right)d\mathrm{v}\,\csc\alpha \quad (4.46)$$

$$=\int_{I}\exp\left(-j(n-m)\mathrm{v}\csc\alpha\right)\csc\alpha\left[\Lambda\left(\frac{\mathrm{v}\csc\alpha}{2}\right)\Gamma^{*}\left(\frac{\mathrm{v}\csc\alpha}{2}\right)+\Lambda\left(\frac{\mathrm{v}\csc\alpha}{2}+\pi\right)\Gamma^{*}\left(\frac{\mathrm{v}\csc\alpha}{2}+\pi\right)\right]d\mathrm{v}\csc\alpha \quad (4.47)$$

Therefore, which obtained

$$\Lambda\left(\mathrm{v}\csc\alpha\right)\Gamma^{*}\left(\mathrm{v}\csc\alpha\right)+\Lambda\left(\mathrm{v}\csc\alpha+\pi\right)\Gamma^{*}\left(\mathrm{v}\csc\alpha+\pi\right)=0 \quad (4.48)$$

Hence, $\Lambda\left(\mathrm{v}\csc\alpha\right)$ and $\Gamma\left(\mathrm{v}\csc\alpha\right)$ are periodic with a period $2\pi\sin\alpha$ and belonging in $L^{2}[I]$. Subsequently, prove that, if $\psi(t)=\sum_{n\in\mathbb{Z}}y_{\alpha}[n]\sqrt{2}\,\varphi(2t-n)$ then the set of functions is an orthonormal basis for M_{0}^{α}, if $M\left(\mathrm{v}\csc\alpha\right)$ is a unitary matrix.

Theorem-4.3 If $\psi(t)=\sum_{n\in\mathbb{Z}}y_{\alpha}[n]\sqrt{2}\,\varphi(2t-n)$ be **t**he set function $\left\{\psi_{0,n,\alpha}(t)\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis of functions for the subspace M_{0}^{α}, if $M\left(\mathrm{v}\csc\alpha\right)$ is a unitary matrix that is

$$M\left(\mathrm{v}\csc\alpha\right)M^{\wp}\left(\mathrm{v}\csc\alpha\right)=I \quad (4.49)$$

where, I and $\wp$ are represent the identity matrix and conjugate transpose respectively.

$$M\left(\mathrm{v}\csc\alpha\right)=\begin{bmatrix}\Lambda(\mathrm{v}\csc\alpha) & \Lambda(\mathrm{v}\csc\alpha+\pi)\\ \Gamma(\mathrm{v}\csc\alpha) & \Gamma(\mathrm{v}\csc\alpha+\pi)\end{bmatrix} \quad (4.50)$$

Proof: Since $\Lambda(\mathrm{v}\csc\alpha)$ and $\Lambda(\mathrm{v}\csc\alpha+\pi)$ should not disappear together in a set of non-zero quantity due to (4.38), (4.48) implicit the existence of a $2\pi\sin\alpha$ periodic function $\gamma(\tau\sin\alpha,\ \mathrm{v}\csc\alpha)$ so that,

$$\Gamma^{*}(\mathrm{v}\csc\alpha)=\gamma(\tau\sin\alpha,\ \mathrm{v}\csc\alpha)\Lambda(\mathrm{v}\csc\alpha+\pi) \quad (4.51)$$

where, $$\gamma(\tau\sin\alpha,\ \mathrm{v}\csc\alpha)=\exp\left(j\mathrm{v}\csc\alpha\right)\xi(2\tau\sin\alpha,\ 2\mathrm{v}\csc\alpha) \quad (4.52)$$

and $$\gamma(\tau\sin\alpha,\ \mathrm{v}\csc\alpha)+\gamma(\tau\sin\alpha+\pi,\ \mathrm{v}\csc\alpha+\pi)=0 \quad (4.53)$$

where, $\xi(2\tau\sin\alpha,\ 2\mathrm{v}\csc\alpha)$ is $2\pi\sin\alpha$ periodic. Then inserting (4.52) into (4.51) gives

$$\Gamma(\mathrm{v}\csc\alpha)=\exp\left(-j\mathrm{v}\csc\alpha\right)\xi^{*}(2\tau\sin\alpha,\ 2\mathrm{v}\csc\alpha)\Lambda^{*}(\mathrm{v}\csc\alpha+\pi) \quad (4.54)$$

Choosing, $\xi(\tau\sin\alpha,\ \mathrm{v}\csc\alpha)=\exp\left(jN\mathrm{v}\csc\alpha\right),\ N\in\mathbb{Z}$ then (4.54) can be written as

$$\Gamma(\mathrm{v}\csc\alpha)=\exp\left(-j(2N+1)\mathrm{v}\csc\alpha\right)\Lambda^{*}(\mathrm{v}\csc\alpha+\pi) \quad (4.55)$$

Hence, (4.27), (4.31) and (4.55), gives

$$y_\alpha[n] = (-1)^n\, x^*_\alpha[-n+1+2N] \tag{4.56}$$

With suitably chosen $N \in \mathbb{Z}$, (4.56) gives

$$y_\alpha[n] = (-1)^n\, x^*_\alpha[-n+1] \tag{4.57}$$

Example: Suppose $\varphi(t) = \chi_{[0,1)}(t)$, where the characteristic function $[a,b)$ is represented by $\chi_{[a,b)}(t)$. In this case, $\mathscr{K}_{\varphi,\alpha}(\tau, v) = 1$, which means the modulated function $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ are orthonormal, hence by (4.25) it can be driven as

$$x_\alpha[n] = \sqrt{2}\int_{\mathbb{R}} \varphi(t)\varphi^*(2t\text{-}n)dt = \begin{cases} \frac{1}{2}, & n = 0,1 \\ 0, & \text{else} \end{cases} \tag{4.58}$$

Therefore,

$$y_\alpha[n] = (-1)^n\, x^*_\alpha[-n+1] = \begin{cases} \frac{1}{\sqrt{2}}, & n=0 \\ \frac{1}{-\sqrt{2}}, & n=1 \\ 0, & \text{else} \end{cases} \tag{4.59}$$

Consequently, using (4.29) results in $\psi(t) = \chi_{[0,1/2)}(t) - \chi_{[1/2,1)}(t)$.

4.5 Summary

In this chapter, the time-frequency representation of MRA on the idea of inner product space and Reisz basis in Hilbert space is described. MRA for FrST is well-defined as an arrangement of the closed subspaces, $\{U_k^\alpha\}_{k\in\mathbb{Z}}$ where, U_k^α cover the entire square-integrable real space $L^2[\mathbb{R}]$, with some properties.

It is proved that a set of the modulated function $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is the Riesz basis of a subspace U_0^α of $L^2[\mathbb{R}]$ an orthonormal basis of the subspace U_0^α spanned by $\{\varphi_{0.n,\alpha}(t)\}_{n\in\mathbb{Z}}$ if $A = B = 1$.

CHAPTER 5

PROPERTIES OF FRACTIONAL S-TRANSFORM

The generalization of ST is known as fractional ST (FrST). The FrST exclusively combines the frequency dependent resolution with absolutely reference phase information, therefore, the time average of FrST equals to the FrFT spectrum. The FrST is instantaneously estimated the local amplitude and the local phase spectrum, whereas a fractional wavelet method is only proficient of providing the local amplitude or power spectrum. It autonomously gives the positive frequency spectrum and negative frequency spectrum.

5.1 Proposed properties of fractional S-transform

Based on the survey, few properties of FrST have been already documented in the literature [15] which are given in chapter-2 of the Table-3. However, few properties like, convolution theorem, correlation theorem, Parseval's theorem and Sampling theorem for FrST have not been documented till now. Thereafter, the mathematical identities and proof of these properties are undertaken in the next sections.

5.1.1 Convolution theorem for fractional S-transform

Definition: If $z(t)$ represents the weighted convolution theorem of two functions $x(t)$ and $y(t)$, written as

$$z(t) = B_\alpha \int_{-\infty}^{\infty} x(\xi) y(t\text{-}\xi) \exp\left(-\left(\tau(t\text{-}\tau)+\tau\xi\text{-}2\xi(t\text{-}\xi)\text{-}\frac{\tau^2}{2} \right)\left(\frac{(v\csc\alpha)^2}{2} \right) + j\pi\left\{ v^2\text{-}2\xi(t\text{-}\xi) \right\}\cot\alpha \right) d\xi \quad (5.1)$$

where, $B_\alpha = \sqrt{1\text{-}j\cot\alpha}$. Then the convolution theorem of FrST can be well-defined as

$$z(t) = B_\alpha \int_{-\infty}^{\infty} x(\xi)\ y(t\text{-}\xi) \exp\left(-\left\{ \tau(t\text{-}\tau)+\tau\xi\text{-}2\xi(t\text{-}\xi)\text{-}\frac{\tau^2}{2} \right\}\left(\frac{(v\csc\alpha)^2}{2} \right) + j\pi\left\{ v^2\text{-}2\xi(t\text{-}\xi) \right\}\cot\alpha \right) d\xi$$

$$\overset{\mathbb{S}^\alpha}{\Leftrightarrow} \frac{\sqrt{2\pi}}{|v\csc\alpha|} X^\alpha\left(\frac{\tau}{2}, v \right) Y^\alpha\left(\frac{\tau}{2}, v \right) \quad (5.2)$$

where, FrST is denoted by $\mathbb{S}^\alpha$.

Proof: The FrST of $z(t)$ may be obtained as

$$Z^{\alpha}(\tau,v) = B_{\alpha}\int_{-\infty}^{\infty} z(t)\frac{|v\csc\alpha|}{\sqrt{2\pi}}\exp\left(\frac{-(\tau-t)^2(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\{(v^2+t^2)\cot\alpha-2vt\csc\alpha\}\right)dt;\ \alpha\neq n\pi \quad (5.3)$$

Substituting (5.1) in (5.3) results in

$$Z^{\alpha}(\tau,v) = B_{\alpha}\int_{-\infty}^{\infty}\left\{B_{\alpha}\int_{-\infty}^{\infty} x(\xi)y(t-\xi)\exp\left(-\left\{\tau(t-\tau)+\tau\xi-2\xi(t-\xi)-\frac{\tau^2}{2}\right\}\left(\frac{(v\csc\alpha)^2}{2}\right)+j\pi\left\{v^2-2\xi(t-\xi)\right\}\cot\alpha\right)d\xi\right\}$$
$$\frac{|v\csc\alpha|}{\sqrt{2\pi}}\exp\left(\frac{-(\tau-t)^2(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\left\{(v^2+t^2)\cot\alpha-2vt\csc\alpha\right\}\right)dt \quad (5.4)$$

Rearranging the terms in (5.4) results into

$$= B_{\alpha}B_{\alpha}\frac{|v\csc\alpha|}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x(\xi)y(t-\xi)\exp\left(-\left\{\tau(t-\tau)+\tau\xi-2\xi(t-\xi)-\frac{\tau^2}{2}\right\}\left(\frac{(v\csc\alpha)^2}{2}\right)+j\pi\left\{v^2-2\xi(t-\xi)\right\}\cot\alpha\right)\right\}$$
$$\exp\left(-(\tau-t)^2\frac{(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\left\{(v^2+t^2)\cot\alpha-2vt\csc\alpha\right\}\right)d\xi\, dt \quad (5.5)$$

Substituting, $t-\xi=\kappa$ and $dt=d\kappa$ in (5.5), gives

$$= B_{\alpha}B_{\alpha}\frac{|v\csc\alpha|}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x(\xi)y(\kappa)\exp\left(-\left\{\tau(\kappa+\xi-\tau)+\tau\xi-2\xi(\kappa)-\frac{\tau^2}{2}\right\}\frac{(v\csc\alpha)^2}{2}+j\pi\left\{v^2-2\xi(\kappa)\right\}\cot\alpha\right)\right\}$$
$$\exp\left(-(\tau-\kappa-\xi)^2\frac{(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\left\{(v^2+(\kappa+\xi)^2)\cot\alpha-2v(\kappa+\xi)\csc\alpha\right\}\right)d\xi\, d\kappa \quad (5.6)$$

$$= B_{\alpha}B_{\alpha}\frac{|v\csc\alpha|}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x(\xi)y(\kappa)\exp\left(-\left\{\frac{\tau^2}{2}+\kappa^2+\xi^2-\tau\kappa-\tau\xi\right\}\left(\frac{(v\csc\alpha)^2}{2}\right)\right)\right\}$$
$$\exp\left(j\pi\left\{(v^2+\kappa^2)\cot\alpha-2v\kappa\csc\alpha\right\}\right)\exp\left(j\pi\left\{(v^2+\xi^2)\cot\alpha-2v\xi\csc\alpha\right\}\right)d\xi\, d\kappa \quad (5.7)$$

$$= B_{\alpha}B_{\alpha}\frac{|v\csc\alpha|}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\xi)y(\kappa)\exp\left(-\left\{\frac{\tau^2}{4}+\kappa^2-\tau\kappa\right\}\left(\frac{(v\csc\alpha)^2}{2}\right)\right)\exp\left(-\left\{\frac{\tau^2}{4}+\xi^2-\tau\xi\right\}\frac{(v\csc\alpha)^2}{2}\right)$$
$$\exp\left(j\pi\left\{(v^2+\kappa^2)\cot\alpha-2v\kappa\csc\alpha\right\}\right)\exp\left(j\pi\left\{(v^2+\xi^2)\cot\alpha-2v\xi\csc\alpha\right\}\right)d\xi\, d\kappa \quad (5.8)$$

Rearranging the terms and operators (5.8) results into

$$= \frac{|v\csc\alpha|}{\sqrt{2\pi}}B_{\alpha}\int_{-\infty}^{\infty} x(\xi)\exp\left(-\left(\frac{\tau}{2}-\xi\right)^2\frac{(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\left\{(v^2+\xi^2)\cot\alpha-2v\xi\csc\alpha\right\}\right)d\xi$$
$$B_{\alpha}\int_{-\infty}^{\infty} y(\kappa)\exp\left(-\left(\frac{\tau}{2}-\kappa\right)^2\frac{(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\left\{(v^2+\kappa^2)\cot\alpha-2v\kappa\csc\alpha\right\}\right)d\kappa \quad (5.9)$$

By using the definition of FrST, (5.9) converts to

$$= \frac{\sqrt{2\pi}}{|v\csc\alpha|} X^{\alpha}\left(\frac{\tau}{2}, v\right) Y^{\alpha}\left(\frac{\tau}{2}, v\right) \tag{5.10}$$

Thus, the proposed convolution theorem for FrST will be

$$z(t) = B_{\alpha}\int_{-\infty}^{\infty} x(\xi)y(t-\xi)\exp\left(-\left\{\tau(t-\tau)+\tau\xi-2\xi(t-\xi)-\frac{\tau^2}{2}\right\}\frac{(v\csc\alpha)^2}{2} + j\pi\left\{v^2-2\xi(t-\xi)\right\}\cot\alpha\right)d\xi$$

$$\overset{\mathbb{S}^{\alpha}}{\Leftrightarrow} \quad \frac{\sqrt{2\pi}}{|v\csc\alpha|} X^{\alpha}\left(\frac{\tau}{2}, v\right) Y^{\alpha}\left(\frac{\tau}{2}, v\right)$$

That means a time-domain weighted convolution of two functions gives the multiplication of these function in FrST domain.

5.1.2 Cross-correlation theorem for fractional S-transform

Definition: If $z(t)$ represents the weighted correlation theorem of two functions $x(t)$ and $y(t)$, expressed as

$$z(t) = B_{\alpha}\int_{-\infty}^{\infty} x^{*}(\xi)y(t+\xi)\exp\left(-\left\{\tau t-2\tau\xi+2\xi(t-\xi)-\frac{\tau^2}{2}\right\}\frac{(v\csc\alpha)^2}{2} + j\pi\left\{v^2+2\xi(t+\xi)\right\}\cot\alpha\right)d\xi \tag{5.11}$$

Then, the cross-correlation theorem of FrST can be stated as

$$z(t) = B_{\alpha}\int_{-\infty}^{\infty} x^{*}(\xi)y(t+\xi)\exp\left(-\left\{\tau t-2\tau\xi+2\xi(t-\xi) - \frac{\tau^2}{2}\right\}\frac{(v\csc\alpha)^2}{2} + j\pi\left\{v^2+2\xi(t+\xi)\right\}\cot\alpha\right)d\xi$$

$$\overset{\mathbb{S}^{\alpha}}{\Leftrightarrow} \quad \frac{\sqrt{2\pi}}{|v\csc\alpha|} X^{\alpha}\left(\frac{\tau}{2}, v\right) Y^{\alpha}\left(\frac{\tau}{2}, v\right) \tag{5.12}$$

where $*$ represent complex conjugate and $B_{\alpha} = \sqrt{1-j\cot\alpha}$.

Proof: The FrST of $z(t)$ can be expressed as

$$Z^{\alpha}(\tau,v) = B_{\alpha}\int_{-\infty}^{\infty} z(t)\frac{|v\csc\alpha|}{\sqrt{2\pi}} \exp\left(-(\tau-t)^2\frac{(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\{(v^2+t^2)\cot\alpha-2vt\csc\alpha\}\right)dt\ ;\ \alpha \neq n\pi \tag{5.13}$$

Substituting (5.11) in (5.13), results into

$$Z^{\alpha}(\tau,v) = B_{\alpha}B_{\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{*}(\xi)y(t+\xi)\exp\left(-\left\{\tau t-2\tau\xi+2\xi(t+\xi)-\frac{\tau^2}{2}\right\}\frac{(v\csc\alpha)^2}{2} + j\pi\left\{v^2+2\xi(t+\xi)\right\}\cot\alpha\right)$$
$$\frac{|v\csc\alpha|}{\sqrt{2\pi}}\exp\left(\frac{-(\tau-t)^2(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\{(v^2+t^2)\cot\alpha-2vt\csc\alpha\}\right)d\xi dt \tag{5.14}$$

Substitute, $t+\xi=\kappa$ and $dt = d\kappa$ in (5.14), results into

$$= B_\alpha B_\alpha \frac{|v\csc\alpha|}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^*(\xi)y(\kappa)\exp\left(-\left\{\tau(\kappa-\xi)-2\tau\xi+2\xi\kappa-\frac{\tau^2}{2}\right\}\frac{(v\csc\alpha)^2}{2}+j\pi\left\{v^2+2\xi\kappa\right\}\cot\alpha\right) \exp\left(\frac{-(\tau-\kappa+\xi)^2(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\{(v^2+(\kappa-\xi)^2)\cot\alpha - 2v(\kappa-\xi)\csc\alpha\}\right)d\xi d\kappa \tag{5.15}$$

Thereafter, it is rewrite, result into

$$= B_\alpha B_\alpha \frac{|v\csc\alpha|}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^*(\xi)y(\kappa)\exp\left(-\left\{\frac{\tau^2}{2}+\kappa^2+\xi^2-\tau\kappa-\tau\xi\right\}\frac{(v\csc\alpha)^2}{2}\right)\exp\left(j\pi\left\{v^2+2\xi\kappa\right\}\cot\alpha\right) \exp\left(j\pi\{(v^2+\kappa^2+\xi^2-2\kappa\xi)\cot\alpha-2v(\kappa-\xi)\csc\alpha\}\right)d\xi d\kappa \tag{5.16}$$

$$= B_\alpha B_\alpha \frac{|v\csc\alpha|}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^*(\xi)y(\kappa)\exp\left(-\left\{\frac{\tau}{2}-\kappa\right\}^2\frac{(v\csc\alpha)^2}{2}\right)\exp\left(-\left\{\frac{\tau}{2}-\xi\right\}^2\frac{(v\csc\alpha)^2}{2}\right) \exp\left(j\pi\{(v^2+\kappa^2)\cot\alpha-2v\kappa\csc\alpha\}\right)\exp\left(j\pi\{(v^2+(-\xi)^2)\cot\alpha-2v(-\xi)\csc\alpha\}\right)d\xi d\kappa \tag{5.17}$$

Rearranging the terms and operators (5.17) converts into

$$= \frac{|v\csc\alpha|}{\sqrt{2\pi}} B_\alpha \int_{-\infty}^{\infty} p^*(\xi)\ \exp\left(-\left\{\frac{\tau}{2}-\xi\right\}^2\frac{(v\csc\alpha)^2}{2}\right)\ \exp\left(j\pi\{(v^2+(-\xi)^2)\cot\alpha-2v(-\xi)\csc\alpha\}\right)d\xi \; B_\alpha\int_{-\infty}^{\infty} y(\kappa)\exp\left(-\left\{\frac{\tau}{2}-\kappa\right\}^2\frac{(v\csc\alpha)^2}{2}\right)\ \exp\left(j\pi\{(v^2+\kappa^2)\cot\alpha-2v\kappa\csc\alpha\}\right)d\kappa \tag{5.18}$$

By using the definition of FrST, (5.18) convert into

$$= \frac{\sqrt{2\pi}}{|v\csc\alpha|}X^\alpha\left(\frac{\tau}{2},v\right)Y^\alpha\left(\frac{\tau}{2},v\right) \tag{5.19}$$

Thus, the proposed cross-correlation theorem for FrST is

$$z(t) = B_\alpha\int_{-\infty}^{\infty} x^*(\xi)y(t+\xi)\exp\left(-\left\{\tau t-2\tau\xi+2\xi(t-\xi)-\frac{\tau^2}{2}\right\}\frac{(v\csc\alpha)^2}{2}+j\pi\left\{v^2+2\xi(t+\xi)\right\}\cot\alpha\right)d\xi$$

$$\overset{\mathbb{S}^\alpha}{\Leftrightarrow} \quad \frac{\sqrt{2\pi}}{|v\csc\alpha|}X^\alpha\left(\frac{\tau}{2},v\right)Y^\alpha\left(\frac{\tau}{2},v\right)$$

It means a time-domain weighted cross-correlation of two functions gives the multiplication of these function in FrST domain.

5.1.3 Parseval's theorem for fractional S-transform

Definition: It states that energy possessed by the signal does not change once the signal is transformed from time to frequency domain as

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X_\alpha(v)|^2 \, dv \tag{5.20}$$

Proof: Considering the expression of IFrST $x(t)$ is expressed as

$$x(t) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} X(\tau, v) d\tau \right\} k_\alpha^*(t, v) \, dv \tag{5.21}$$

Similarly, taking the complex conjugate on the both sides of (5.21) gives

$$x^*(t) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} X^*(\tau, v) \, d\tau \right\} k_\alpha(t, v) \, dv \tag{5.22}$$

Now, the left-hand sides (LHS) of Parseval's theorem can articulate as

$$\int_{-\infty}^{\infty} x(t) \, x^*(t) \, dt = \int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} X^*(\tau, v) \, d\tau \right\} k_\alpha(t, v) \, dv \, dt \tag{5.23}$$

Now, using the property of convergence of ST into FT, once the ST is integrated w.r.t. ' τ ', that is

$$X_\alpha^*(v) = \int_{-\infty}^{\infty} X_\alpha^*(\tau, v) d\tau \tag{5.24}$$

Substituting, (5.24) into (5.23), and rearranging the terms

$$\int_{-\infty}^{\infty} x(t) x^*(t) \, d\,t = \int_{-\infty}^{\infty} X_\alpha^*(v) \left\{ \int_{-\infty}^{\infty} x(t) \, k_\alpha(t,v) dt \right\} dv \tag{5.25}$$

$$\int_{-\infty}^{\infty} x(t) x^*(t) \, d\,t = \int_{-\infty}^{\infty} X_\alpha^*(v) \; X_\alpha(v) \; dv \tag{5.26}$$

where, $X_\alpha(v)$ is denoted by the FrFT of $x(t)$. Thus, it can be also inscribed as

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} | X_\alpha(v) |^2 \, dv$$

Hence, it is validated that energy is preserved even after transformation.

5.2 Sampling theorem for fractional S-transform

If the highest frequency enclosed in a signal $x(t)$ is v_{max} and the signal is sampled at a rate $v_s > 2v_{max}$ then $x(t)$ can be exactly improved from its sample values using the interpolation function. In reconstruction, predetermined rate allows the reconstruction of the continuous-time signals from the sampled signal. The notations and definitions used in the derivation of the proposed theorem are, $\mathbb{R}$ is set of a real number, $\mathbb{Z}$ is set of integer, $\mathbb{Z}^+$ is positive integer set,

$L^1[0, 2\pi]$ is the space of integral function in $[0,2\pi]$, $L^2[R]$ is an space of square integral function in $\mathbb{R}$, $L^\infty[0, 2\pi]$ is the space of integral function in $[0,2\pi]$ and $l^2[\mathbb{Z}]$ is entire space square summable sequence in $\mathbb{Z}$. Continuous-Time signals are represented by parentheses, such as $x(t), t \in \mathbb{R}$ and the discrete signals as $c[n], n \in \mathbb{Z}$. The product of the two continuous-time signals $x(t)$ and $y(t)$ in $L^2[R]$ is expressed as $\langle x, y \rangle_{L^2} = \int_{\mathbb{R}} x(t)\, y^*(t)dt$. Similarly the sequences $a[n]$ and $b[n]$, inner product in $l^2[\mathbb{Z}]$ is expressed as $\langle a,b \rangle_{l^2} = \sum_{n\in\mathbb{Z}} a[n]\, b^*[n]$. Therefore the L^2-norm is denoted by $||x||^2_{L^2} = \langle x, x \rangle_{L^2}$ and the l^2-norms is denoted by $||a||^2_{l^2} = \langle a, a \rangle_{l^2}$. Suppose, Hilbert space is represented by $\mathscr{H}$ and its complete set of function is $\{\psi_n(t)\}_{n\in\mathbb{Z}}$. The set of function is the Riesz basis for $0<A \le B<+\infty$ such that $A||a[n]||^2_{L^2} \le ||\sum_{n\in\mathbb{Z}} a[n]\, \psi_n(t)||^2_{L^2} \le B||a[n]||^2_{l^2}$; $a[n] \in l^2[\mathbb{Z}]$ [93-95]. This equality holds, if basis function is orthonormal, that is, when $A=B=1$ exist [90-94]. For the assessable function $x(t)$ on $\mathbb{R}$, suppose, $||x(t)||_\infty$ is the essential Supp $|x(t)|$ and $||x(t)||_0$ the essential Infim $|x(t)|$ where, Supp and Infim is represented the Supremum and Infimum respectively and the characteristic function of a subset $D \subset \mathbb{R}$ is expressed as

$$\aleph_D(x) = \begin{cases} 1, & x \in D \\ 0, & \text{otherwise} \end{cases} \tag{5.27}$$

The discrete-time FrST (DTFrST) of $x[n] \in l^2[\mathbb{Z}]$ is expressed as

$$\tilde{X}_\alpha(v,\eta) = \tilde{\mathscr{S}}^\alpha\{x[n]\}(v,\eta) = \sum_{n\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x[n]\kappa_\alpha(v,n)\, H_\alpha(\eta, v, v') \tag{5.28}$$

where, $\tilde{\mathscr{S}}^\alpha$ represents the DTFrST and $\kappa_\alpha(.,.)$ is a kernel of FrST in chapter-2 section-2.4, give the expression of $\kappa_\alpha(v,n)$ and $H_\alpha(\eta, v, v')$ in discrete form. If $\alpha = \frac{\pi}{2}$, the DTFrST reduces to the DTST. The inverse DTFrST (IDFrST) is expressed as

$$x[n] = \int_I \int_I \tilde{X}_\alpha(v,\eta)\, \kappa^*_\alpha(v',n) H^*_\alpha(\eta, v, v') dv' dv, \quad I \cong [0, 2\pi \sin\alpha] \tag{5.29}$$

From the expression of $\kappa_\alpha(v,n)$ and (4.28), results in

$$\tilde{X}_\alpha(v,\eta)\exp(-j\pi v^2\cot\alpha) = \sum_{n\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a[n]\exp(j\pi n^2\cot\alpha)\exp(-j2\pi v' n\csc\alpha)H_\alpha^*(\eta, v, v') \tag{5.30}$$

where, $a[n] = B_\alpha x[n]$. Inserting the expression of $\kappa_\alpha(v,n)$ into (5.29) gives

$$a[n] = \frac{1}{2\pi\sin\alpha}\int_I\int_I \tilde{X}_\alpha(v,\eta)\exp(-j\pi v'^2\cot\alpha)\exp(-j\pi n^2\cot\alpha + j2\pi v' n\csc\alpha)H_\alpha(\eta, v, v')dvdv' \tag{5.31}$$

If $\tilde{X}_\alpha(v,\eta)$ is belonging in $L^2[I]$, then its multiplication with the chirp function $\exp(-j\pi v^2\cot\alpha)$ is also belonging in $L^2[I]$. Here (5.30) and (5.31) implies that the function is a set of $L^2[I]$ in the FrST domain.

A multiresolution analysis (MRA) with its properties associated with the FrST is discussed in chapter-4, and also included the analytical proof of a set of function $\{\varphi_{0,n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is the Riesz basis of a subspace U_0^α of $L^2[\mathbb{R}]$, if there exist the constants $0 < A \le B < +\infty$, such that

$$A \le \mathscr{K}_{\varphi,\alpha}^{-2}(\tau, v) \le B, \quad \forall \ v,\tau \in 1 \tag{5.32}$$

where,

$$\mathscr{K}_{\varphi,\alpha}(\tau, v) = \left(\sum_{k=-\infty}^{\infty} |\Phi(\tau\sin\alpha+2\pi k, v\csc\alpha+2\pi k)|^2\right)^{1/2} \tag{5.33}$$

This ensures that following condition exists

$$0 \le \left\|\mathscr{K}_{\varphi,\alpha}\tau, v)\right\|_k \le \left\|\mathscr{K}_{\varphi,\alpha}(\tau, v)\right\|_{k+1} \le \infty \tag{5.34}$$

If $\Phi(\tau\sin\alpha, v\csc\alpha)$ be the FrST of fractional scaling function $\varphi(t)$, where the argument is scaled by $\csc\alpha$ and $\sin\alpha$ for frequency and time axis respectively. Then $\mathscr{K}_{\varphi,\alpha}(\tau, v)$ is periodic with $2\pi\sin\alpha$, which is also belongs in $L^1[I]$. The $\varphi(t)$ is orthonormal if $A = B = 1$, i.e. $\mathscr{K}_{\varphi,\alpha}(\tau, v) = 1$ where, $v \in \mathbb{R}$ and each U_k^α is called a multiscale subspace of FrST.

A fractional scaling function $\varphi(t)$ is called orthonormal, if $\{\varphi_{0,n,\alpha}(t)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis of U_0^α. Suppose, function $\varphi(t)$ is a fractional scaling of the MRA $\{U_k^\alpha\}_{k\in\mathbb{Z}}$, then $\{\varphi_{1,n,\alpha}(t)\}_{n\in\mathbb{Z}}$

be the Riesz basis of U_1^α. Since, $\{\varphi_{0,n,\alpha}(t)\}_{n\in\mathbb{Z}} \in U_0^\alpha \subseteq U_1^\alpha$, then exists a sequence $g[n]$ which is set of $l^2[\mathbb{Z}]$ such as

$$\varphi_{0,0,\alpha}(t) = \sum_{n=-\infty}^{\infty} g[n]\varphi_{1,n,\alpha}(t) \tag{5.35}$$

For simplicity, define as [106],

$$\phi_{k,n,\alpha}(t) \cong 2^{\frac{k}{2}}\phi(2^k t\text{-}n)\exp\left(j\pi\{t^2\text{-}(\frac{n}{2^k})^2\}\cot\alpha\right) \tag{5.36}$$

Let, $S_0^\alpha = U_0^\alpha \; \Theta \; U_1^\alpha$ be the compliment of U_0^α in U_1^α, and $\phi(t) \in L^2[\mathbb{R}]$ such as $\phi_{0,0,\alpha}(t)$ is set of S_0^α. Then, exists a sequence $p[n]$ which in the set of $l^2[\mathbb{Z}]$ such as

$$\phi_{0,0,\alpha}(t) = \sum_{n=-\infty}^{\infty} p[n]\phi_{1,n,\alpha}(t)$$

If $\{\varphi_{0,n,\alpha}(t)\}_{n\in\mathbb{Z}}$ be the Riesz basis of S_0^α and $\phi(t)$ are a fractional function of MRA $\left\{U_k^\alpha\right\}_{k\in\mathbb{Z}}$. Hence, for any $x(t) \in U_{k+1}^\alpha = S_k^\alpha \oplus U_k^\alpha$, there must be $b[n]$ and $c[n]$ is set of $l^2[\mathbb{Z}]$ such as

$$x(t) = \sum_{n\in\mathbb{Z}} b[n]\varphi_{k,n,\alpha}(t) + \sum_{n\in\mathbb{Z}} c[n]\phi_{k,n,\alpha}(t) \tag{5.37}$$

where, $c[n]$ is the fractional ST coefficients of $x(t)$ in S_k^α. Subsequently, considering

$$\Phi(\tau\sin\alpha,\ v\csc\alpha) = \Phi\left(\frac{\tau\sin\alpha}{2},\ \frac{v\csc\alpha}{2}\right)\Lambda\left(\frac{v\csc\alpha}{2}\right) \tag{5.38}$$

$$\Psi(\tau\sin\alpha,\ v\csc\alpha) = \Phi\left(\frac{\tau\sin\alpha}{2},\ \frac{v\csc\alpha}{2}\right)\Gamma\left(\frac{v\csc\alpha}{2}\right) \tag{5.39}$$

where, $\Phi(\tau\sin\alpha,\ v\csc\alpha)$ and $\Psi(\tau\sin\alpha,\ v\csc\alpha)$ is the FrST of $\varphi(t)$ and $\phi(t)$ respectively, with its arguments, is scaled by $\sin\alpha$ and $\csc\alpha$ for time axis and frequency axis respectively, and the function $\Lambda(v\csc\alpha)$ and $\Gamma(v\csc\alpha)$ is written as

$$\Lambda(v\csc\alpha) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} g[n]\exp\left(j2\pi n v\csc\alpha\right)\exp\left(j\pi\frac{n^2}{2}\cot\alpha\right) \tag{5.40}$$

$$\Gamma(v\csc\alpha) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} p[n]\exp\left(j2\pi n v\csc\alpha\right)\exp\left(j\pi\frac{n^2}{2}\cot\alpha\right) \tag{5.41}$$

are defined in $L^{\infty}[I]$. For any $\gamma \in \mathbb{Z}^{+} \bigcup \{0\}$, iterating (5.38). It results

$$\Phi\left\{2^{\gamma}(\tau\sin\alpha, v\csc\alpha)\right\} = \Upsilon_{\gamma}(v)\Phi(\tau\sin\alpha, v\csc\alpha) \tag{5.42}$$

where, $\Upsilon_{0}(v)b = 1$ and $\Upsilon_{\gamma}(v) = \Pi_{\gamma=0}^{\gamma-1}\Lambda(2^{\gamma}v\csc\alpha)$ for $\gamma \geq 1$. It is simple to validate that

$$\Upsilon_{\gamma}(v) \hat{=} \Upsilon_{\gamma}(v+2\pi\sin\alpha) \in L^{\infty}[I] \tag{5.43}$$

and defining $E_{\gamma} = \sup \Upsilon_{\gamma}(v)$ i.e. supremum of $\Upsilon_{\gamma}(v)$ gives

$$E_{\gamma} \hat{=} \bigcap_{\gamma=0}^{\gamma-1} \sup \Lambda(2^{\gamma}v\csc\alpha); \quad \text{for } \gamma \geq 1 \tag{5.44}$$

Since,

$$\sup \Lambda(2v\csc\alpha) = \frac{1}{2}\sup \Lambda(v\csc\alpha) \tag{5.45}$$

It can also be rewritten as

$$E_{\gamma} = \bigcap_{\gamma=0}^{\gamma-1} \frac{\gamma}{2} \sup\Lambda(v\csc\alpha) \tag{5.46}$$

and defined as

$$\tilde{\Phi}_{\gamma}\left\{2^{\gamma}(\tau\sin\alpha,\ v\csc\alpha)\right\} \hat{=} \sum_{k\in\mathbb{Z}} \Phi(2^{\gamma}\tau\sin\alpha+2^{\gamma+1}k\pi,\ 2^{\gamma}v\csc\alpha+2^{\gamma+1}k\pi) \tag{5.47}$$

Adding (5.38) and Poisson's summation formula of the ST results into

$$\tilde{\Phi}_{\gamma}\left\{2^{\gamma}(\tau\sin\alpha,\ v\csc\alpha)\right\} \hat{=} \Upsilon_{\gamma}(v)\sum_{k\in\mathbb{Z}} \Phi(\tau\sin\alpha+2k\pi,\ v\csc\alpha+2k\pi) \tag{5.48}$$

$$\tilde{\Phi}_{\gamma}\left\{2^{\gamma}(\tau\sin\alpha,\ v\csc\alpha)\right\} \hat{=} \Upsilon_{\gamma}(v)\tilde{\Phi}(\tau\sin\alpha,\ v\csc\alpha) \tag{5.49}$$

where, $\tilde{\Phi}(\tau\sin\alpha,\ v\csc\alpha)$ represents the DTST of $\varphi[n]$, which is sampled form of $\varphi(t)$ with the argument is scaled by $\csc\alpha$ and $\sin\alpha$ for frequency and time axis.

The sampling of the sequence in subspace U_{0}^{α} is obtained by using a stable generator function $\varphi(t)$ and Gaussian function $g(t,v)$, defined as a set in $L^{2}[\mathbb{R}]$. Its function space is defined as

$$x(t) = \sum_{n\in\mathbb{Z}} x[n]\varphi(t-n)g(t,v)\exp(-j\pi\{t^{2}-(n)^{2}\}\cot\alpha) \tag{5.50}$$

where, $x[n]$ is a sequence defined in the set $l^2[\mathbb{Z}]$, $g(t,v) \le 1$ and $g(t,v) \in \mathbb{R}$. $x(t)$ and $g(t,v)$ belong to $L^2[\mathbb{R}]$. The $x(t)$ can be considered as pointwise converges because

$$|\sum_{n\in\mathbb{Z}} x[n]\varphi(t-n)g(t,v)\exp(-j\pi\{t^2-(n)^2\}\cot\alpha)|^2 \le \left(\sum_{n\in\mathbb{Z}}|x[n]|^2 \sum_{n\in\mathbb{Z}}|\varphi(t-n)|^2 \sum_{n\in\mathbb{Z}}|g(t,v)|^2\right) \quad (5.51)$$

Hence, without loss of generality, any continuous function $x(t) \in U_0^\alpha$ can be carried for the sampling. The sampling theorem for FrST is defined in Theorem-5.1.

Theorem 5.1: Suppose a generator function $\varphi(t)$ and $g(t,v)$ belong to $L^2[\mathbb{R}]$, be the fractional scaling signal of MRA $\{U_k^\alpha\}_{k\in\mathbb{Z}}$ related to FrST and its sampling sequence $\varphi(n)$ is an integer of $\varphi(t)$ belongs in $l^2[\mathbb{Z}]$. Then a continuous function $s(t)$, which is in $L^2[\mathbb{R}]$, can be defined with $s(t)\exp(-j\frac{t^2}{2}\cot\alpha) \in U_0^\alpha$ such that

$$x(t) = \sum_{m\in\mathbb{Z}} x[\frac{m}{2^\gamma}]\ s(2^\gamma t - m)\, g(t,v)\exp(-j\pi\{t^2 - (\frac{m}{2^\gamma})^2\}\cot\alpha) \quad (5.52)$$

where, $\gamma \in \mathbb{Z}^+ \bigcup \{0\}$ and $\forall\ x(t), g(t,v)$, are set of U_0^α. Equation (5.52) holds if

$$\frac{1}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)}\chi_{D_\gamma}(\tau,v) \in L^2[I] \quad (5.53)$$

Also, the interpolation function $s(t)$ in (5.52) contents

$$S(\tau\sin\alpha, v\csc\alpha) = \frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)}, \quad \forall\ v,\tau \in D_\gamma \quad (5.54)$$

where, $S(\tau\sin\alpha, v\csc\alpha)$ and $\Phi(\tau\sin\alpha, v\csc\alpha)$ represent the ST of $s(t)$ and $\varphi(t)$ with argument scaled by $\csc\alpha$ and $\sin\alpha$ for frequency and time axis respectively.

Proof: Assuming (5.53) to be true, $\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha) \ne 0, \forall\ v,\tau$ is set of D_γ. Using (5.29), it can be assumed that there exists the sequence $x[n]$ such that

$$\frac{1}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)}\chi_{D_\gamma}(\tau,v) = \sum_{n\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a[n]\exp(j\pi n^2\cot\alpha)\exp(-j2\pi v' n\csc\alpha)H_\alpha^*(\eta, v, v') \quad (5.55)$$

belonging in $L^2[I]$. Since $\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)$ is periodic with a period $2\pi\sin\alpha$ then (5.55) can be rewritten as

$$\int_{\mathbb{R}} \left| \frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \chi_{D_\gamma}(\tau, v) \right|^2 dv = \sum_{k\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_{I} \left| \frac{\Phi(\tau\sin\alpha + 2k\pi, v\csc\alpha + 2k\pi)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \right|^2 \chi_{D_\gamma}(\tau, v) dv \quad (5.56)$$

Substituting, (5.33) in (5.56), then the above equation can be expressed as

$$\int_{\mathbb{R}} \left| \frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \chi_{D_\gamma}(\tau, v) \right|^2 dv = \int_{I} \frac{\mathscr{K}^2_{\varphi,\alpha}(\tau, v)}{|\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)|^2} \chi_{D_\gamma}(\tau, v) dv \quad (5.57)$$

Using (5.34) and (5.57), it can be established that

$$\int_{\mathbb{R}} \left| \frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \chi_{D_\gamma}(\tau, v) \right|^2 dv \le \| \mathscr{K}^2_{\varphi,\alpha}(\tau, v) \|^2 \int_{I} \frac{1}{|\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)|^2} \chi_{D_\gamma}(\tau, v) dv \quad (5.58)$$

i.e. $\frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\sqrt{2\pi}\,\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \chi_{D_\gamma}(\tau, v) \in L^2[\mathbb{R}]$ thus, it can be derived as

$$S(\tau\sin\alpha, v\csc\alpha) = \mathscr{S}\{s(t)\}(\tau\sin\alpha, v\csc\alpha) \quad (5.59)$$

$$\hat{=}\ \chi_{D_\gamma}(\tau, v)\ \frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \quad (5.60)$$

where, $\mathscr{S}$ is ST operator. Further, it can be rearranged as

$$\Phi(\tau\sin\alpha, v\csc\alpha)\chi_{D_\gamma}(\tau, v) = \sqrt{2\pi}\ S(\tau\sin\alpha, v\csc\alpha)\ \tilde{\Phi}(\tau\sin\alpha, v\csc\alpha) \quad (5.61)$$

Substituting (5.55) into (5.60) gives

$$S(\tau\sin\alpha, v\csc\alpha) = \Phi(\tau\sin\alpha, v\csc\alpha) \sum_{n\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x[n] \exp(j\pi n^2 \cot\alpha) \exp(-j2\pi v' n \csc\alpha) H^*_\alpha(\eta, v, v') \quad (5.62)$$

Thereafter, evaluating the FrST of the modulated signal in terms of ST [15] i.e.

$$\mathscr{S}^\alpha\{s(t)\exp(-j\pi t^2\cot\alpha)\}(v, \tau) = \sqrt{2\pi} A_\alpha \exp(j\pi v^2 \cot\alpha) \mathscr{S}\{s(t)\}(\tau\sin\alpha, v\csc\alpha) \quad (5.63)$$

Substituting, (5.59) and (5.62) into (5.63) the FrST of modulated signal can be stated as

$$\mathscr{S}^\alpha\{s(t)\exp(-j\pi t^2\cot\alpha)\}(\tau, v) = \sqrt{2\pi}\Phi(\tau\sin\alpha, v\csc\alpha) \sum_{n\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a[n]\kappa_\alpha(v, n)$$

$$= \sqrt{2\pi}\ \tilde{A}_\alpha(\tau, v)\ \Phi(\tau\sin\alpha, v\csc\alpha) \quad (5.64)$$

where, $\tilde{A}_\alpha(\tau, v)$ represent the DTFrST of $a[n]$, as defined in (5.28). Subsequently, taking the IFrST in both sides of (5.64) the modulated signal can be obtained as

$$s(t)\exp(-j\pi t^2 \cot\alpha) = \sum_{n\in\mathbb{Z}} \varphi(t-n)\ a[n] \exp(-j\pi(t^2 - n^2)\cot\alpha) \quad (5.65)$$

where $s(t)\exp(-j\pi t^2 \cot\alpha)$ belong to U_0^α , such that $\varphi(t-n)\exp(-j\pi(t^2-n^2)\cot\alpha)$ is the Riesz basis of subspace U_0^α. Now adding (5.61) and (5.44), results into

$$\Upsilon_\gamma(v)\Phi(\tau\sin\alpha, v\csc\alpha)\chi_{D_\gamma}(\tau,v) = \Upsilon_\gamma(v)\sqrt{2\pi}\, S(\tau\sin\alpha, v\csc\alpha)\,\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha) \quad (5.66)$$

Then, using (5.42) and (5.48) into (5.66) and scaling by 2^γ it gives

$$\Phi(\tau\sin\alpha, v\csc\alpha) = \sqrt{2\pi}\ \tilde{\Phi}_\gamma(\tau\sin\alpha, v\csc\alpha)\ S(\frac{\tau\sin\alpha}{2^\gamma}, \frac{v\csc\alpha}{2^\gamma}) \quad (5.67)$$

Now, using the Poisson's summation formula of the ST from [96-98], (5.67) can be written as

$$\tilde{\Phi}_\gamma(\tau\sin\alpha, v\csc\alpha) = 2^{-\gamma}\sum_{n\in\mathbb{Z}}\varphi[\frac{n}{2^\gamma}]\ \exp(-j2\pi\frac{m}{2^\gamma}v\csc\alpha) \quad (5.68)$$

Thereafter, taking the IST on both sides of (5.68) results into

$$\varphi(t) = \sum_{n\in\mathbb{Z}}\varphi[\frac{n}{2^\gamma}]\ S(2^\gamma t - n) \quad (5.69)$$

However, if any function $x(t)$ is belonging to U_0^α, then there exists a sequence $d[k]$ defined in $l^2[\mathbb{Z}]$ like as

$$x(t) = \sum_{k\in\mathbb{Z}} d[k]\varphi(t-k)\exp(-j(\pi t^2 - \pi k^2)\cot\alpha) \quad (5.70)$$

Substituting (5.69) into (5.70), results in

$$x(t) = \sum_{k\in\mathbb{Z}} d[k]\sum_{n\in\mathbb{Z}}\varphi(\frac{n}{2^\gamma})\ S\{2^\gamma(t-k)-n\}\exp(-j\pi(t^2-k^2)\cot\alpha) \quad (5.71)$$

Assuming, $n = 2^\gamma k + m$ in (5.71), results into

$$x(t) = \sum_{n\in\mathbb{Z}} S(2^\gamma t + m)\sum_{k\in\mathbb{Z}} d[k]\ \varphi[\frac{n}{2^\gamma} - k]\ \exp(-j(\pi t^2 - \pi k^2)\cot\alpha) \quad (5.72)$$

From (5.70), it may be stated as

$$x[n] = \sum_{k\in\mathbb{Z}} d[k]\ \varphi[n-k]\exp(-j(\pi n^2 - \pi k^2)\cot\alpha) \quad (5.73)$$

Therefore, $x[n]$ will be a function belonging to $l^2[\mathbb{Z}]$ because it is defined in terms of $d[n]$ and $\varphi[n]$ which belongs to $l^2[\mathbb{Z}]$ sequences. It satisfies the condition of convergence, i.e.

$$x[n] \to 0 \ \text{as}, |n| \to \infty \quad (5.74)$$

Suppose, $\tilde{A}_\alpha(\tau, v)$ represents the DTFrST of $d[k]$. Since $\tilde{A}_\alpha(\tau, v)$ and $\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)$ is set of $L^1[I]$, therefore,

$$\tilde{A}_\alpha(\tau, v)\, \tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)\frac{2\pi\exp(-j\pi v^2\cot\alpha)}{\sqrt{1-j\cot\alpha}} \in L^1[I] \tag{5.75}$$

Now, evaluating the Fourier coefficients of the function in (5.75) can be obtained as

$$\frac{\csc\alpha}{2\pi}\int_I \tilde{A}_\alpha(\tau, v)\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)\frac{2\pi\exp(-j\pi v^2\cot\alpha)}{\sqrt{1-j\cot\alpha}}\exp(jnv\csc\alpha)dv \tag{5.76}$$

Substituting the expression of $\tilde{A}_\alpha(\tau, v)$ in terms of $d[k]$ in above expression results into

$$\frac{\csc\alpha}{\sqrt{1-j\cot\alpha}}\int_I \sum_{k\in\mathbb{Z}} d[k]\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)\kappa_\alpha(v,k)\exp(-j\pi v^2\cot\alpha)\exp(j2\pi vn\csc\alpha)\, dv \tag{5.77}$$

After substituting the expression of kernel factor $\kappa_\alpha(v,k)$ in (5.77), it can be solved as

$$\sum_{k\in\mathbb{Z}} d[k]\varphi[n-k]\exp(j\pi k^2\cot\alpha) = x[n]\exp(j\pi n^2\cot\alpha) \tag{5.78}$$

Substituting, $n \to \frac{n}{2^\gamma}$ in (5.73), results into

$$x[\frac{n}{2^\gamma}] = \sum_{k\in\mathbb{Z}} d[k]\varphi[\frac{n}{2^\gamma} - k]\exp(-j\pi\{(\frac{n}{2^\gamma})^2 - k^2\}\cot\alpha) \tag{5.79}$$

However, if (5.79) is substituted into (5.72), it gives results in (5.52). This actually proves the proposed sampling theorem presented in (5.52). Suppose, $s(t)$ is a set defined in $L^2[\mathbb{R}]$ with $s(t)\exp(-j\pi t^2\cot\alpha) \in U_0^\alpha$ such that (5.52) holds in $L^2[\mathbb{R}]$. Since $\varphi_{0,n,\alpha}(t) \in U_0^\alpha \quad \forall\, n \in \mathbb{Z}$, therefore, $\varphi_{0,0,\alpha}(t)$ is set in U_0^α which results in $\varphi(t)\exp(-j\pi t^2\cot\alpha) \in U_0^\alpha$. Thus, substituting $\varphi(t)\exp(-j\pi t^2\cot\alpha) \in U_0^\alpha$ for $x(t)$ in (5.52) gives

$$\varphi(t)\exp(-j\pi t^2\cot\alpha) = \sum_{n\in\mathbb{Z}} \varphi[\frac{n}{2^\gamma}]\, S(2^\gamma t + m)\exp(-j\pi t^2\cot\alpha) \tag{5.80}$$

Determining the FrST of (5.80) results into

$$\Phi(\tau\sin\alpha, v\csc\alpha) = \sqrt{2\pi}\, \tilde{\Phi}_\gamma(\tau\sin\alpha, v\csc\alpha)\, S\,(\frac{\tau\sin\alpha}{2^\gamma}, \frac{v\csc\alpha}{2^\gamma}) \tag{5.81}$$

This can be modified by using scaling operation as

$$\Phi\left\{2^{\gamma}(\tau\sin\alpha, v\csc\alpha)\right\} = \sqrt{2\pi}\ \tilde{\Phi}_{\gamma}\left\{2^{\gamma}(\tau\sin\alpha, v\csc\alpha)\right\} S(\tau\sin\alpha, v\csc\alpha) \tag{5.82}$$

Thereafter, using (5.42) and (5.49) into LHS and RHS of (5.82) results in

$$\Upsilon_{\gamma}(v)\Phi(\tau\sin\alpha, v\csc\alpha) = \Upsilon_{\gamma}(v)\sqrt{2\pi}\ \tilde{\Phi}_{\gamma}(\tau\sin\alpha, v\csc\alpha)\, S(\tau\sin\alpha, v\csc\alpha) \tag{5.83}$$

This proves the expression of interpolation function, as defined in (5.54). But this is valid for all $v, \tau \in D_{\gamma}$, thus (5.83) can be written as

$$S(\tau\sin\alpha, v\csc\alpha) = \frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)}\chi_{D_{\gamma}}(\tau, v) \tag{5.84}$$

Since, $S(\tau\sin\alpha, v\csc\alpha) \in L^2[\mathbb{R}]$ therefore, using (5.34) the boundaries of the square summable function of (5.84) can be defined as

$$\int_{\mathbb{R}} | S(\tau\sin\alpha, v\csc\alpha) |^2\, dv = \sum_{k\in\mathbb{Z}} \int_{I} \left| \frac{\Phi(\tau\sin\alpha + 2\pi k, v\csc\alpha + 2\pi k)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \right|^2$$

$$= \int_{I} \frac{\mathscr{K}^2_{\varphi,\alpha}(\tau, v)}{|\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)|^2}\ \chi_{D_{\gamma}}(\tau, v) dv\ < \infty$$

Hence,

$$0 \leq\ \left\| \mathscr{K}_{\varphi,\alpha}(\tau, v) \right\|_0^2 \int_{I} \frac{1}{|\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)|^2} \chi_{D_{\gamma}}(\tau, v)\, dv\ < \infty \tag{5.85}$$

The above expression shows that $\frac{1}{|\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)|^2}\chi_{D_{\gamma}}(\tau, v)$ should lie in between zero and infinity for all values of α in between [0, 2π], i.e. it actually represents a square integral function. Thus, it can easily construct the condition $\frac{1}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)}\chi_{D_{\gamma}}(\tau, v) \in L^2[I]$ holds, as $L^2[I]$ represents the space of all square integral function on I. This proves the condition of existence of sampling theorem for FrST, defined in (5.53). One special class of sampling for bandlimited signal in FrST domain is presented below, which is also known as an advancement of Shannon’s sampling theorem.

5.2.1 Extension of the Shannon’s sampling theorem for the FrST

Assume a function $x(t)$ is Δ_{α} bandlimited with the angle α and $\alpha \neq n\pi$ for all integer n. Then, by IFrST in (1.20), results into

$$x(t)=\sqrt{1+j\cot\alpha}\,\frac{\exp(-j\pi t^2\cot\alpha)}{\sqrt{2\pi}}\int_{-\Delta_\alpha}^{\Delta_\alpha}X_\alpha(\tau,v)(v\csc\alpha)\exp(-\frac{t^2(v\csc\alpha)^2}{2})\exp(-j\pi v^2\cot\alpha+j2\pi vt\csc\alpha)dv \tag{5.86}$$

Suppose, $$y(t)=\int_{-\Delta_\alpha}^{\Delta_\alpha}X_\alpha(\tau,v)\exp(-\frac{t^2(v\csc\alpha)^2}{2})\exp(-j\pi v^2\cot\alpha+j2\pi vt\csc\alpha)dv \tag{5.87}$$

Therefore,

$$x(t)=y(t)\sqrt{\frac{1+j\cot\alpha}{2\pi}}\exp(-j\pi t^2\cot\alpha) \tag{5.88}$$

Since $y(t)$ is $\Delta_\alpha\csc\alpha$ bandlimited in the conventional sense [17, 95, 96], hence, the Shannon sampling theorem be used on $y(t)$ [24]. In FrST domain, the generalized Shannon's sampling theorem can be expressed as

$$x(t)=\sqrt{1+j\cot\alpha}\sum_{n\in\mathbb{Z}}x(v)\operatorname{sinc}\left[(t-nT_s)\Delta_\alpha\csc\alpha\right]g(t,v)\exp(-j\pi\{t^2-(nT_s)^2\}\cot\alpha);T_s\le\frac{\pi\sin\alpha}{\Delta_\alpha} \tag{5.89}$$

When $\alpha=\frac{\pi}{2}$, the identities (5.85) is the Shannon sampling theorem for the bandlimited signal. The (5.90) holds for Δ_α-fractional band-limited signal $x(t)$ with finite energy, i.e. $x(t)$ is in $L^2[\mathbb{R}]$, those FrST $X_\alpha(\tau,v)$ has support in $[-\Delta_\alpha,\Delta_\alpha]$ with $\Delta_\alpha>0$. The Paley-Wiener (PW) space of all Δ_α-fractional band-limited signals is

$$PW_{\Delta_\alpha}\mathrel{\hat=}\{x(t)\in L^2[\mathbb{R}]\,|\,\operatorname{supp}X_\alpha(\tau,v)=[-\Delta_\alpha,\Delta_\alpha]\} \tag{5.90}$$

which is a subspace of $L^2[\mathbb{R}]$. For non-band limited signals it is not appropriate. But when choose $\Delta_\alpha=2^k\pi\sin\alpha$ with $k\in\mathbb{Z}$. In theorem-5.1, $\gamma=0$ they give the sampling theorem for FrST for fractional bandlimited signals with a specific continuity of function $\varphi(t)$ that implies sampling sequence $\varphi[n]\in l^2[\mathbb{Z}]$. By using Poisson's summation [92-94] i.e. it is obtained by shifting the function $\varphi(t)$ at a distance nT_s where $\forall\, n\in\mathbb{Z}$ and combining all this shift of FrST gives

$$\tilde{\Phi}(\tau\sin\alpha,v\csc\alpha)=\sum_{k\in\mathbb{Z}}|\,\Phi(\tau\sin\alpha+2\pi k+v\csc\alpha+2\pi k)\,| \tag{5.91}$$

Then it follows that

$$\frac{1}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha,v\csc\alpha)}=1\in L^2[I] \tag{5.92}$$

i.e.

$$S(\tau\sin\alpha, v\csc\alpha) = \frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} = \Phi(\tau\sin\alpha, v\csc\alpha); v \in \mathbb{R} \tag{5.93}$$

Suppose that, $s(t) = \text{sinc}(t)$ where, $\text{sinc}(t) \triangleq \frac{\sin \pi t}{\pi t}$ so the sampling theorem of FrST for the fractional bandlimited signal can be established. The computational complexity of DFrST is given in next section.

5.3 Computational complexity of DFrST and sparse DFrST

In the practical applications, the FrST is more important since the integration is frequency-dependent and it has degree of freedom. But the fractional-order also includes some considerable extra computational load. For the reduction of computational complexity, the sparse discrete fractional ST (SDFrST) is defined which is based on the basic concept of sparse ST (SST) and sparse discrete FrFT (SDFrFT) [99-101]. The FrST of the signal $x(t)$ is stated in (1.19). To calculate the computational complexity of FrST, it is represented in terms of FrFT, also given in (2.27) as

$$X^{\alpha}(\tau, v) = \int_{-\infty}^{\infty} X^{\alpha}(v') H_{\alpha}(\tau, v, v') dv'$$

where, $$H_{\alpha}(\tau, v, v') = \int_{-\infty}^{\infty} h(t, \tau, v)\, \kappa_{\alpha}^{*}(t, v')\, dt \tag{5.94}$$

Substituting, $h(t,\tau,v) = g(\tau\text{-}t, v)\, \kappa_{\alpha}(t,v)$ in (5.94), results in

$$H_{\alpha}(\tau, v, v') = \exp\left(\frac{-2\pi^2 (v-v')^2}{(v\csc\alpha)^2}\right) \kappa_{\alpha}(t,v)\, \kappa_{\alpha}^{*}(t,v') \tag{5.95}$$

Using (2.27) the FrST can be rewritten as [98]

$$X^{\alpha}(\tau, v) = \int_{-\infty}^{\infty} X(v')\, \exp\left(\frac{-2\pi^2 (v-v')^2}{(v\csc\alpha)^2}\right) \kappa_{\alpha}(t,v)\, \kappa_{\alpha}^{*}(t,v') \tag{5.96}$$

Thereafter, DFrST of (5.96) can be written as

$$X^{\alpha}(\tau, v) = \sum_{t=0}^{N-1} X(v') \exp\left(\frac{-2\pi^2 (v-v')^2}{(v\csc\alpha)^2}\right) \kappa_{\alpha}(t,v)\, \kappa_{\alpha}^{*}(t,v') \tag{5.97}$$

From (5.97), the complexity of DFrST can be obtained by considering the complexity of DFrST as $\left(2N + \frac{N}{2}\log_2 N\right)$[98]. Subsequently, by utilizing the fact that $X^{\alpha}(\tau,v) = X^{*\alpha}(\tau,-v)$, it is sufficient to calculate the summation from 0 to $\frac{N}{2}$. Thus the complexity of DFrST will become $\left(N^2 + \frac{N^2}{4}\log_2 N\right)$. Due to the inherent property of FrST of unaltered at a lower frequency from pixel to pixel, a sparse approximation of FrST is used to remove this redundancy. However, as given in [98], the probability that a particular pixel contains sparse approximation entry is $\frac{2\pi}{N}$. Therefore, the complexity of sparse DFrST (SDFrST) can be obtained $\frac{\pi N}{2}\left(4 + \log_2 N\right)$.

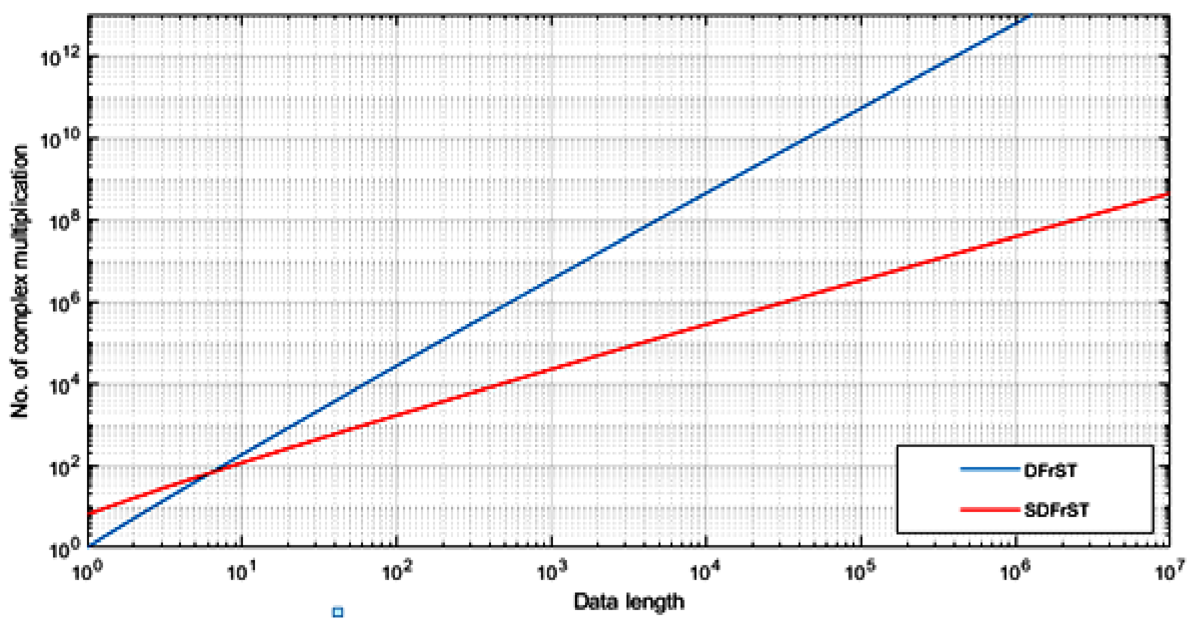


Fig. 5.1 Computation complexity between DFrST and SDFrST

As it is clear from Fig. 5.1, the computational complexity required for evaluating sparse-DFrST is significantly less than the conventional DFrST once the data length is larger than a few samples, as always is the case. This established the superiority of SDFrST over DFrST.

5.4 Summary

The main features of this chapter are:

- First the FrST along with its existing properties are presented, thereafter, a new definition of convolution theorem for FrST is proposed.
- Subsequently, some other important identities for FrST like; cross-correlation theorem, Parseval's theorem and sampling theorem are proposed along with their analytical proofs.
- Finally, the computational complexity of DFrST and SDFrST are evaluated and compared.

CHAPTER 6
APPLICATIONS OF FRACTIONAL S-TRANSFORM

The S-transform (ST) bridges the gap between STFT and WT, it has larger application areas due to its superiority over these two transforms. The generalisation of ST is known as FrST, which is more flexible than ST. It is useful for the analysis of time-frequency response of the test function and distributions. Foster [106] extends the results of ultra-distribution for the FrST given to the Bohemian spaces. Since the diverse seismic signal has distinctive ideal fractional parameters which are not helpful for multichannel seismic data analysis. Therefore utilizing FrST first decay the basic frequency after that, they analysed the minimum frequency [13]. The FrST depends on the concept of time-bandwidth product and the time-frequency rotation property of FrFT, and furthermore, present the standardized second-order pivotal moment calculation technique for finding the optimal order; in contrast of time-bandwidth product [102]. The reason for establishing these theorems is due to their vast use in various applications. In digital signal processing, convolution is used for filtering of a signal in the fractional domain and also the convolutional plays an essential role in many algorithms in edge detection and related applications. In the field of probability theory, the sum of two autonomous arbitrary variables probability distribution is the convolution of their individual distributions. In radar signal processing, the received echo from the target is correlated with the transmitted signal to determine the distance, velocity, and acceleration of the target with respect to the receiver. The correlation function is also used in power spectrum estimation and in the design of matched filtering used in many communication systems. Whereas, the Parseval's theorem is used to calculate the energy that are hard to determine in time-domain. It is also used to compute the transient energy in control distribution network. The FrST are applicable in the different area of signal analysis, image processing, and Bio-medical signal processing [102, 103], Radar signal and communication [89, 97,110] and filtering of the signals or images [107-109].

6.1 Filtering using FrST

Proposed convolution theorem is useful for designing of a multiplicative filter in FrST domain. Let the test signal $x_{in}(t)$ be comprises of the noise signal $n(t)$ and preferred signal $f(t)$. $\mathcal{S}^{\alpha}(\tau, v)$,

$N^{\alpha}(\tau, v)$ are the FrST of $x_{in}(t)$ and $n(t)$ respectively. Since FrST is a linear transform having no cross-terms antiquity, the spectrum of FrST in a region $[v_1, v_2]$ of signal $f(t)$ and filter impulse response $h(t)$ are analyzed. Here, $H^{\alpha}(\tau, v)$ is considered to be unity in the region $[v_1, v_2]$ and zero otherwise, i.e.

$$H^{\alpha}(\tau, v) = \begin{cases} 1, & v \in [v_1, v_2] \\ 0, & v \notin [v_1, v_2] \end{cases} \tag{6.1}$$

Based on (6.1) and its inverse FrST (IFrST), the model of FrST domain MF is shown in Fig.6.1.

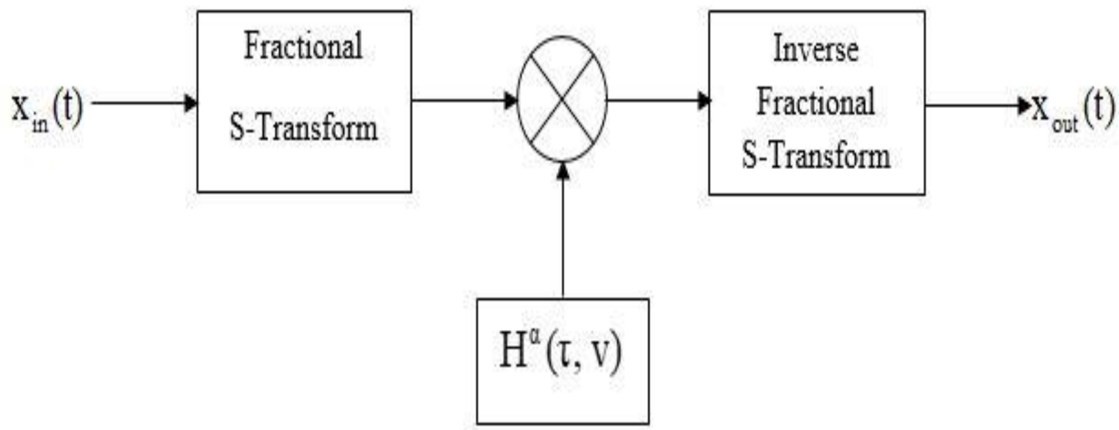


Fig. 6.1. The FrST domain multiplicative filter

The output signal of the multiplicative filter is expressed as

$$x_{out}(t) = \mathcal{S}^{-1}(\mathcal{S}\{x_{in}(t)\} H^{\alpha}(\tau, v)) \tag{6.2}$$

6.1.1 Filtering of ECG signals

The above presented multiplicative filtering process is used to filter the electrocardiogram (ECG) signal and linear frequency modulated (LFM) signal under the presence of noise. The ECG signal data is taken from the MIT-BIH arrhythmia database. The FrST of the ECG signal at an angle $\alpha = 32^0$ is shown in Fig.6.2. The filter response is chosen based on the FrST of the original ECG signal. Thereafter, noise is augmented in the ECG signal and its FrST is evaluated. After the FrST, the transformed noisy signal is multiplied with a mask whose boundaries are chosen based on FrST of the original signal.

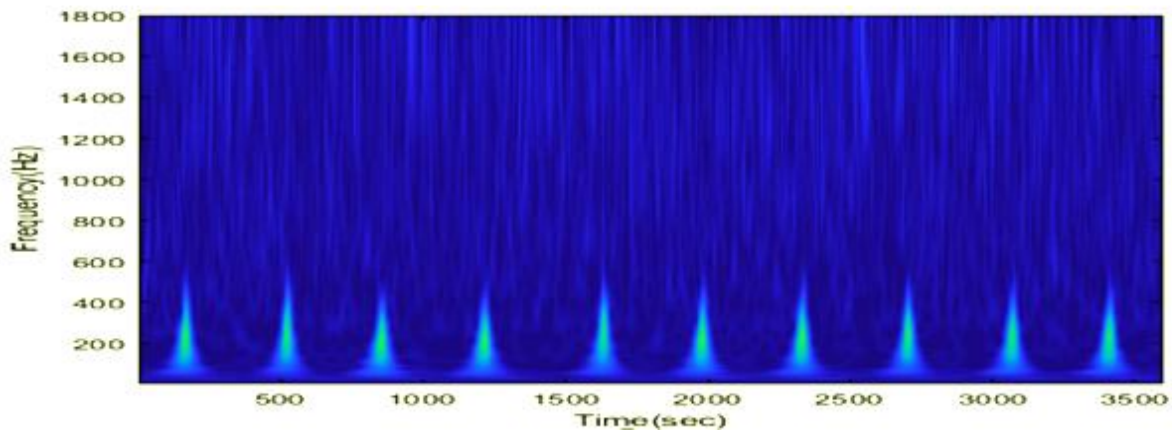


Fig.6.2 Illustration of the ECG signal in the time-frequency domain

Once masking is done, the output ECG signal is acquired by taking IFrST of the signal. Fig.6.3 displays the original ECG signal, noisy ECG signal and filtered ECG signal at an angle $\alpha = 32^0$.

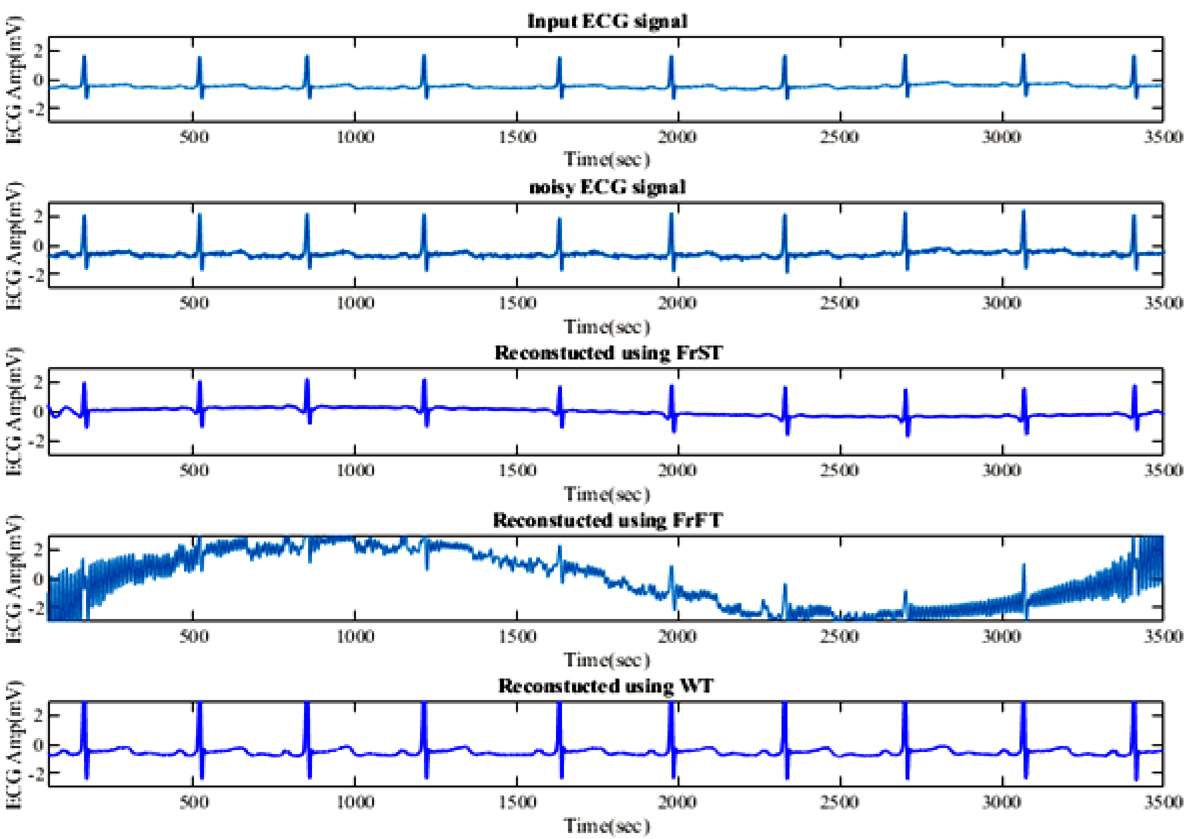


Fig.6.3 ECG signal responses

Thereafter, the mean square error (MSE) between filtered ECG and the original ECG signal is schemed by varying the angle parameter (α) associated with FrST, as shown in Fig.6.4.

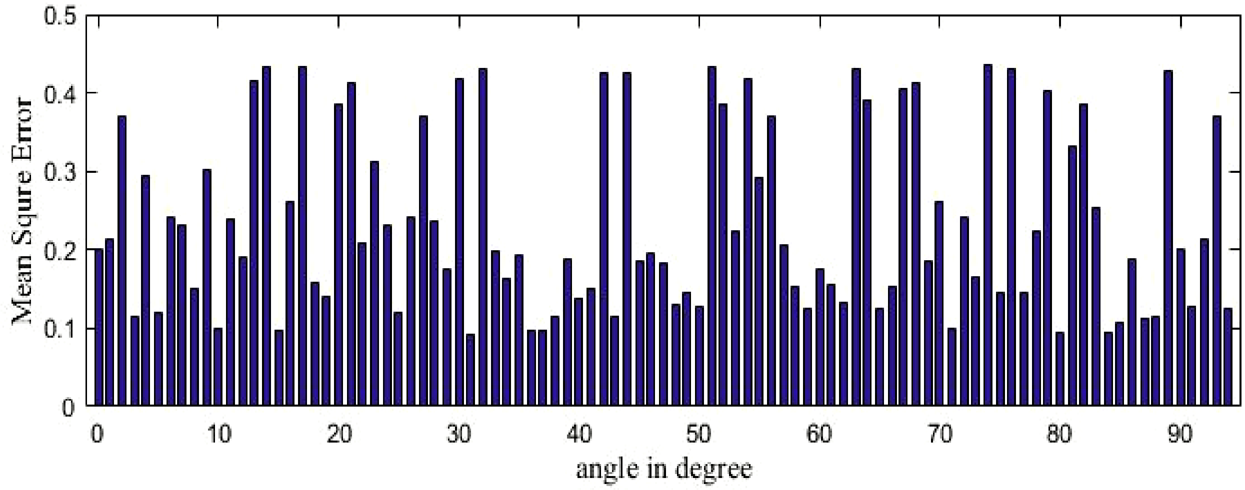


Fig.6.4 Error varies with respect to the fractional angle

It clearly shows that MSE is minimum for $\alpha = 32^0$, which is lesser than MSE for $\alpha = 90^0$, i.e. FrST at $\alpha = 32^0$ outperforms ST (FrST at $\alpha = 90^0$). Thereafter, the presentation of FrST based filtering technique is matched with FrFT and WT based filtering methods by plotting the error between original and filtered signal as shown in Fig.6.5.

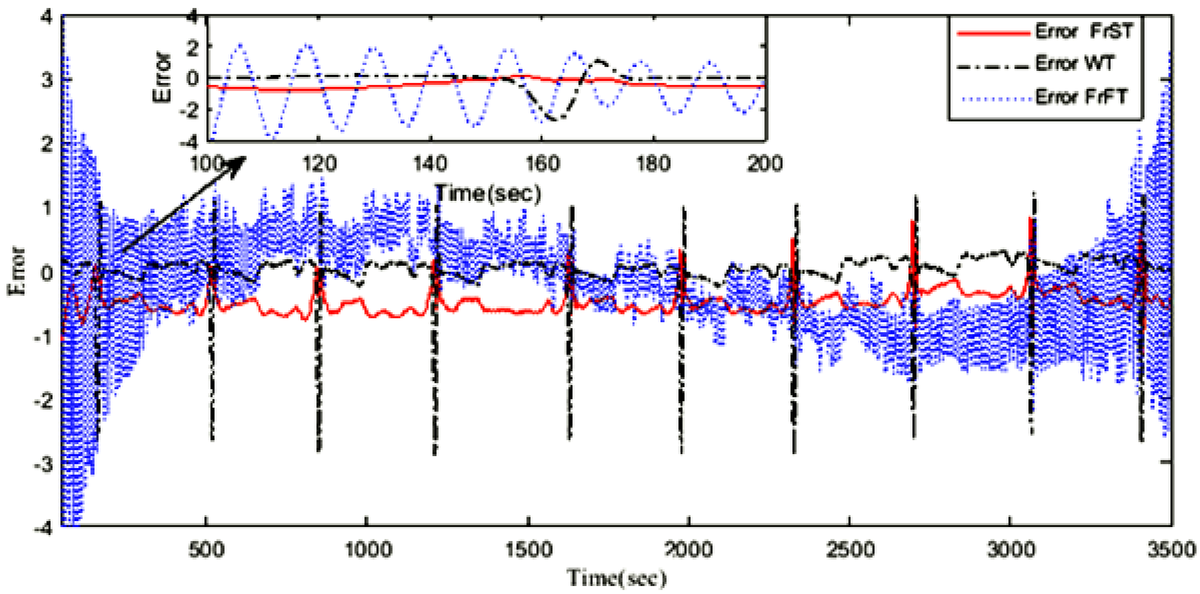


Fig.6.5 Reconstructed error of ECG signal

To highlight the difference between performance of FrST, FrFT and WT base filtering methods, a portion of MSE waveforms (i.e., the region of interest in ECG signal) is zoomed as shown in Fig.6.5. Above results validate that FrST based filtering method outperforms the FrFT and WT based methods.

6.1.2 Filtering of LFM signal

Proposed convolution theorem is also utilized in the filtering of the LFM signal under AWGN channel. Here, the LFM signal is considered as [110]

$$x_{in} = \cos(2\pi f_0 t + 2\pi k t^2)) \tag{6.3}$$

where, $f_0 = 50$ MHz and $k = 0.001$ are centre frequency and chirp rate respectively. For the demonstration purpose, the MSE between LFM signal and filtered signal is compared for the FrST domain filtering and the WT filtering are shown in Fig.6.6.

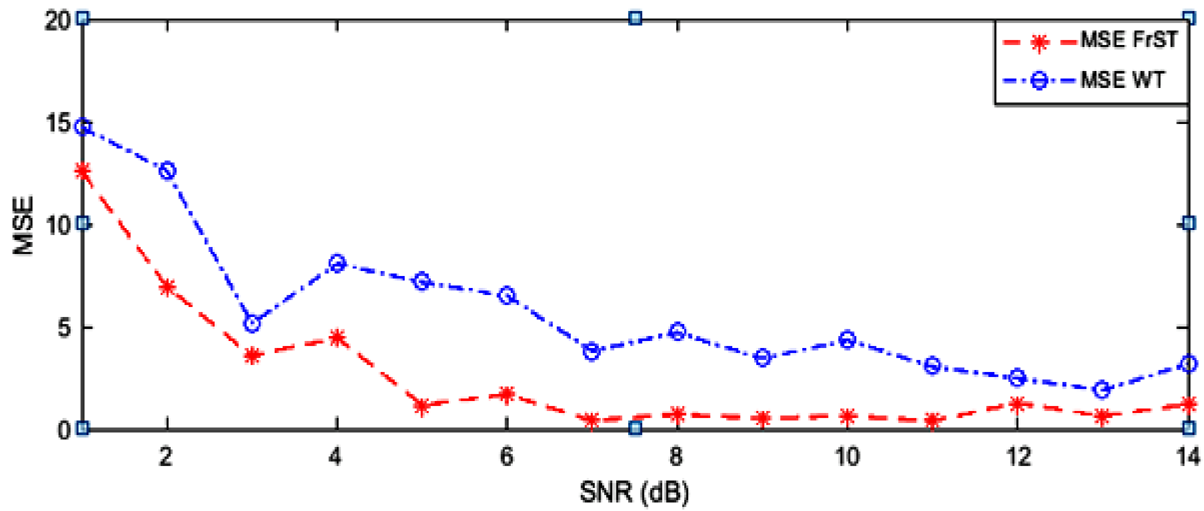


Fig.6.6 Mean Square Error for FrST and WT filtering

The MSE for FrST domain filtering is smaller than WT based filtering for varying SNR values. Based upon the proposed convolution theorem in the filtering of ECG signal and LFM signal under AWGN channel are demonstrated and it is concluded that FrST outperforms FrFT and WT based methods.

6.2 Reconstruction of signal from sampled version

The sampling formula of the FrST for a bandlimited signal expresses that the Nyquist rate for the sampling of a chirp modulated signal is quite smaller than the fundamental Fourier theory because the chirp signal is bandlimited in the FrST domain. The continuous function $x(t)$ may be resulting in

$$x(t) = \sum_{n\in\mathbb{Z}} x[n]\operatorname{sinc}(t-n)\, g(\tau, v)\exp(-j\pi(t^2 - n^2)\cot\alpha) \tag{6.4}$$

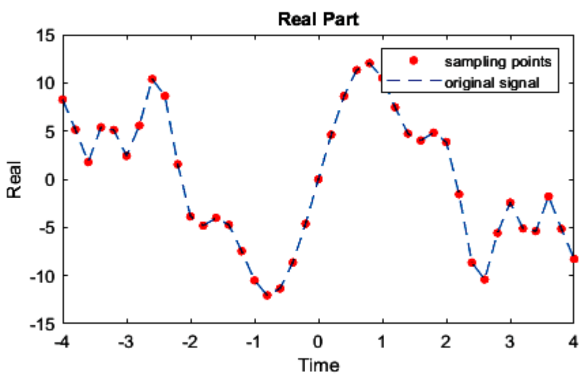
Based on Theorem 5.1, (6.4) can also be observed as the special case of (5.52) with the function $\varphi(t)$ selected as sinc(t) and $\gamma = 0$. The main aim is to select a generator function $\varphi(t)$ of U_0^{α} that has the faster decay than the standard sinc function.

6.2.1 Reconstruction of the multi-tone signal

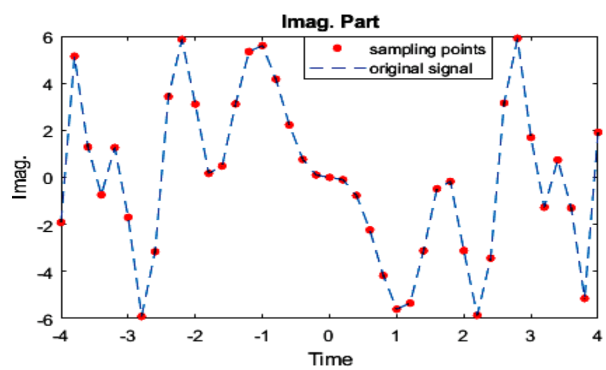
For the purpose of demonstration, a continuous signal $x(t)$ is taken as [103,104]

$$x(t) = \left(2\sin(0.4\pi t) + 5\sin(0.5\pi t) + \sin(0.6\pi t)\right)\exp(\text{-}j0.5kt^2) \tag{6.5}$$

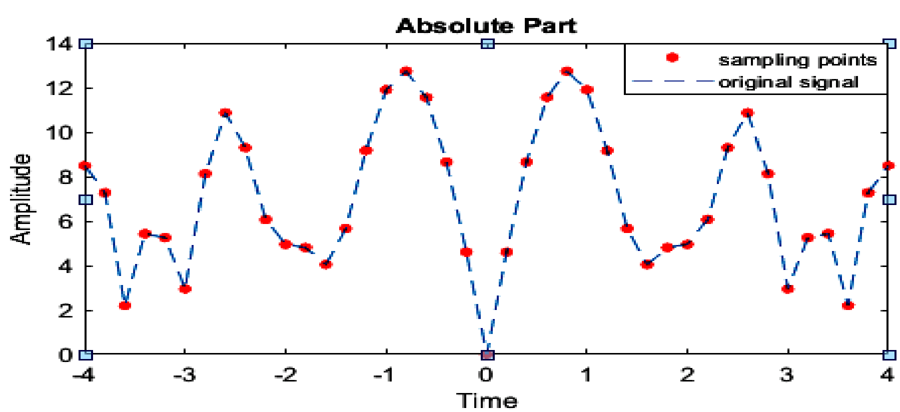
where, k =2. This is band-limited in FrST domain with optimum $\alpha = \cot^{-1}(k)$. The maximum frequency value of the signal is $0.6\pi\sin\alpha$. The sampling rate Δ_{α} should satisfy $\Delta_{\alpha} \leq \frac{\pi\sin\alpha}{1.2\pi\sin\alpha}$. In this example, it is taken as $\Delta_{\alpha} = \frac{1}{2}$. The sampled signal $x(t)$ represented by a dotted line and corresponding sampling points of the signal are presented in Fig.6.7.



(a)



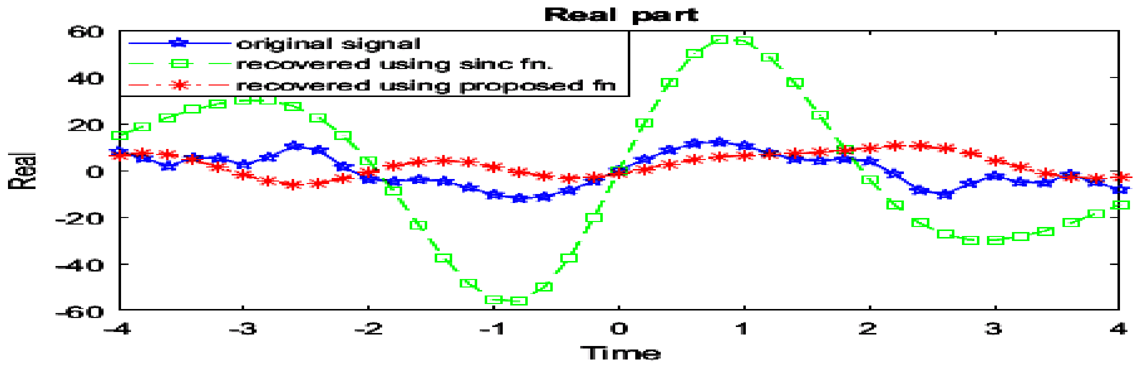
(b)



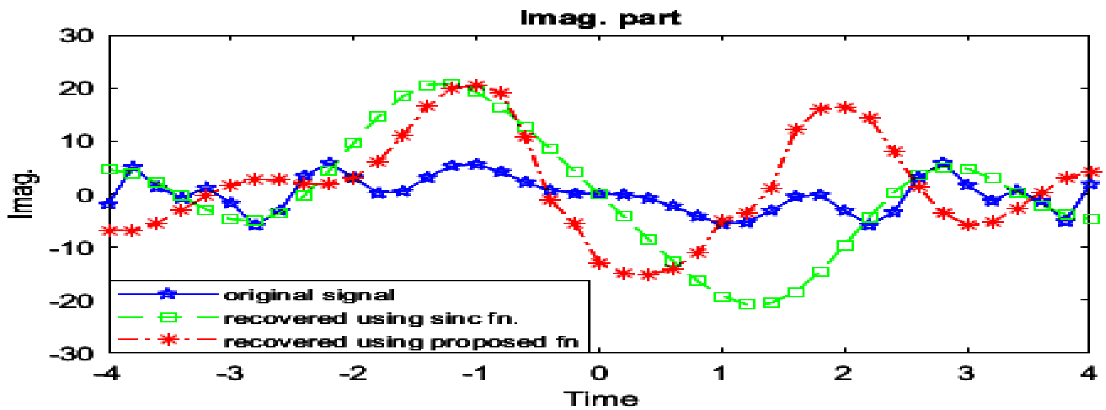
(c)

Fig. 6.7. The sampled and sampling points: (a) Real part (b) Imaginary part (c) Absolute part

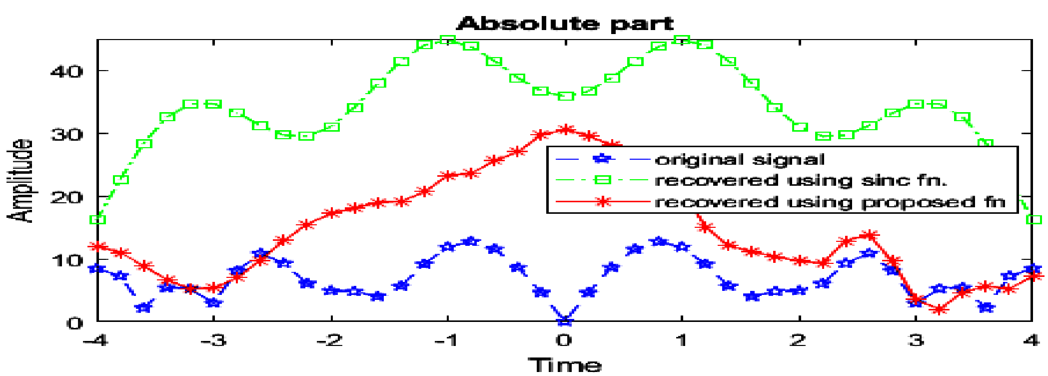
Now, the reconstruction is performed by using the proposed method (5.52) and its presentation is compared with the reconstruction using sinc function. Fig.6.8, shows the simulated results of sampled and recovered signals.



(a)



(b)



(c)

Fig. 6.8. The sampled and recovered signals: (a) Real part (b) Imaginary part (c) Absolute part

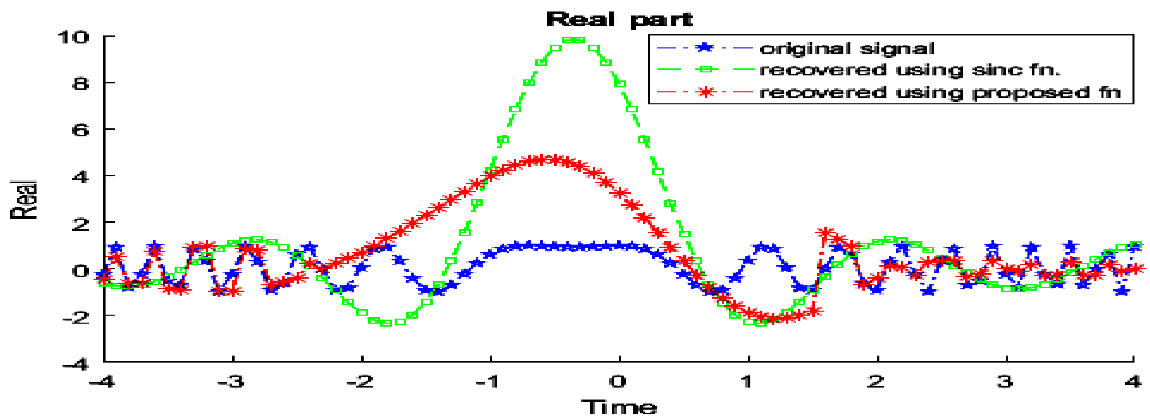
The above plot clearly establishes the benefit of using the proposed method to reconstruct the signal from its sampled version because it resulting in less error.

6.2.2 Reconstruction of LFM signal

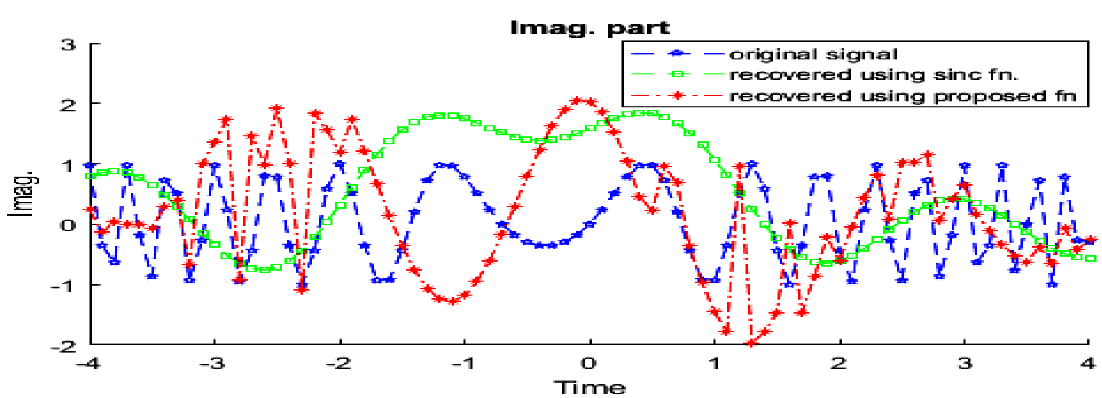
The linear frequency modulated (LFM) signal is an important non-stationary signal that has been broadly used in sonar, radar and other communication systems. In general, the LFM signal is a time-limiting and eventually band-limiting signal. In this case, the LFM signal is considered as a chirp periodic with a bandlimited signal. The reconstruction of this signal from a finite set of samples is explored with the help of FrST sampling and results obtained are shown in Fig.4. For the purpose of demonstration, the chirp signal is considered as [109]

$$y(t) = \exp(j\pi(2f_0 t + k t^2)) \tag{6.6}$$

where, k and f_0 denote the chirp rate and centre frequency of the radar respectively. Here, the parameter value of the radar signal is $f_0 = 1/3$ MHz, $k = 20/21$ Hz/Sec. Again a similar type of exercise is performed to obtain the reconstructed signal from the sampled signal by using the proposed method and sinc interpolation method. The result obtained is displayed in Fig.6.9.



(a)



(b)

Fig. 6.9. The sampled and recovered signals: (a) Real part (b) Imaginary part

For a finite number of samples, the simulated outcomes clearly show that the anticipated sampling and reconstructed method clearly outperform the sinc interpolation method.

6.3 Sampling error estimation for FrST

The aim of this section is to compute the sampling error for FrST to approximate the linear band-limited signal. The error estimation is a technique to ensure the approximation of the assumed signal with given accuracy and reliability to achieve the actual band-limited signal. The techniques used for error estimation of the sampling theorem are known as truncation error and aliasing error.

6.3.1 Truncation error

As already depicted, the perfect reconstruction of the signal requires an immeasurable number of samples of the signal. But due to realization aspect, reconstruction is performed from a finite number of samples only. However, truncating infinite samples into finite samples results into loss of information, which is known as truncation error and can be conveyed as

$$e(t) = \sum_{|n| \geq N} x[\frac{n}{2^\gamma}]\, s(2^\gamma t - n)\, g(\tau, v) \exp\left(-j\pi\{t^2 - (\frac{n}{2^\gamma})\}\cot\alpha \right) \tag{6.7}$$

Where, $x(t)$ which is defined as a set in U_0^α should be estimated prior to determination of (6.1). This definition of truncation error is used to define the error bounds of it, as given in Theorem-6.1.

Theorem 6.1: Suppose, $\varphi(t)$ is a fractional continuous scaling function in a set of $L^2[\mathbb{R}]$ and the fractional scaling functions of MRA $\left\{U_k^\alpha\right\}_{k\in\mathbb{Z}}$ associated with the FrST, then the sampling sequence $\{\varphi[n]\}_{n\in\mathbb{Z}} \in l^2[\mathbb{Z}]$ and $\frac{1}{\sqrt{2\pi\, \tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)}}\chi_{D_\gamma}(\tau,v)$ belonging to $L^\infty(\mathbb{R})$. Then the bounds of truncation error is given as follows

$$\| e(t) \|_{L^2} \leq 2^{-\frac{\gamma}{2}} \left(\sqrt{\sum_{|n|\geq N} |x[\frac{n}{2^\gamma}]|^2} \right) \left\| \frac{\mathscr{K}_{\varphi,\alpha}(\tau, v)}{\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \chi_{D_\gamma}(\tau, v) \right\|_\infty \tag{6.8}$$

Proof: Taking the FrST of (6.8) on both sides, gives

$$\mathscr{E}_\alpha(\tau,v) = \mathscr{S}^\alpha\{e(t)\}(\tau,v) = \sqrt{2\pi}\, 2^{-\gamma} \sum_{|n|\geq N} x[\frac{n}{2^\gamma}]\, \kappa_\alpha(v, \frac{n}{2^\gamma}) S(\frac{\tau\sin\alpha}{2^\gamma}, \frac{v\csc\alpha}{2^\gamma}) \tag{6.9}$$

Using the Parseval's theorem of FrST, it follows that

$$||e(t)||^2_{L^2} = \frac{\csc\alpha}{2^{2\gamma}} \int_{\mathbb{R}} \left| \sum_{|n|\geq N} x[\frac{n}{2^\gamma}] \exp\left(j\pi(n/2^\gamma)^2 \cot\alpha\right) \exp\left(j2\pi v \frac{n}{2^\gamma} \csc\alpha|\right) \right|^2 \left| S\left(\frac{\tau\sin\alpha}{2^\gamma}, \frac{v\csc\alpha}{2^\gamma}\right)\right|^2 dv \quad (6.10)$$

Suppose, $\tilde{x}[n] \triangleq x[n]\exp(j\pi n^2\cot\alpha)$ and substituting $v' = v/2^\gamma$ in (6.10). Since, $\exp(-j2\pi v \frac{n}{2^\gamma}\csc\alpha)$ is periodic with $2\pi\sin\alpha$ then it gives,

$$||e(t)||^2_{L^2} = \frac{\csc\alpha}{2^\gamma} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} | \sum_{|n|\geq N} \tilde{x}[\frac{n}{2^\gamma}] \exp(-j2\pi nv'\csc\alpha)|^2 \, | S(\tau'\sin\alpha, v'\csc\alpha)|^2 dv' \quad (6.11)$$

$$||e(t)||^2_{L^2} = \frac{\csc\alpha}{2^\gamma} \sum_{k\in\mathbb{Z}} \int_{I} | \sum_{|n|\geq N} \tilde{x}[\frac{n}{2^\gamma}] \exp(-j2\pi nv'\csc\alpha)|^2 \, | S(\tau'\sin\alpha+2\pi k, v'\csc\alpha+2\pi k)|^2 dv' \quad (6.12)$$

$$= \frac{\csc\alpha}{2^\gamma} \int_{I} | \sum_{|n|\geq N} \tilde{x}[\frac{n}{2^\gamma}] \exp(-j2\pi nv'\csc\alpha)|^2 \sum_{k\in\mathbb{Z}} |S(\tau'\sin\alpha+2\pi k, v'\csc\alpha+2\pi k)|^2 dv' \quad (6.13)$$

Then, from (5.84), (5.34) and using the Parseval's theorem of the DTFrST, (6.13) can be expressed as

$$||e(t)||^2_{L^2} = \frac{1}{2^\gamma} \int_{I} | \frac{1}{\sqrt{2\pi}} \sum_{|n|\geq N} \tilde{x}[\frac{n}{2^\gamma}] \exp(-j2\pi nv'\csc\alpha)|^2 \csc\alpha \frac{\mathcal{K}^2_{\varphi,\alpha}(v',\eta)}{|\tilde{\Phi}(\tau'\sin\alpha, v'\csc\alpha)|^2} \chi_{D_\gamma}(\tau',v') dv'$$

$$\leq \int_{I} | \frac{1}{\sqrt{2\pi}} \sum_{|n|\geq N} \tilde{x}[\frac{n}{2^\gamma}] \exp(-j2\pi nv'\csc\alpha)|^2 dv' \; \frac{\csc\alpha}{2^\gamma} \left\| \frac{\mathcal{K}_{\varphi,\alpha}(v',\eta)}{\tilde{\Phi}(\tau'\sin\alpha, v'\csc\alpha)} \chi_{D_\gamma}(\tau',v') \right\|^2_\infty$$

$$= \frac{1}{2^\gamma} \sum_{|n|\geq N} | \tilde{x}[\frac{n}{2^\gamma}]|^2 \left\| \frac{\mathcal{K}_{\varphi,\alpha}(v',\eta)}{\tilde{\Phi}(\tau'\sin\alpha, v'\csc\alpha)} \chi_{D_\gamma}(\tau',v') \right\|^2_\infty \quad (6.14)$$

The above equation (6.14) validates that the truncation error is bounded by (6.8). Similarly, the truncation error for any $x(t) \in U^\alpha_k$ where, k belongs $\mathbb{Z}^+$ with zero, can be stated as

$$e(t) = \sum_{|n|\geq N} x[\frac{n}{2^\gamma}] \, s(2^\gamma t - n) \exp\left(-j\pi\{t^2 - (\frac{n}{2^\gamma})\}\cot\alpha\right) \quad (6.15)$$

is a fractional,

$$\|e(t)\|_{L^2} \leq 2^{-\frac{\gamma+k}{2}} \left(\sqrt{\sum_{|n|\geq N} | x[\frac{n}{2^{\gamma+k}}]|^2} \right) \left\| \frac{\mathcal{K}_{\varphi,\alpha}(\tau, v)}{\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \chi_{D_\gamma}(\tau, v) \right\|_\infty \quad (6.16)$$

The equation (6.16) signifies as a bound of the truncation error for any $x(t) \in U^\alpha_k$.

6.3.2 Aliasing error

The aliasing error obtained from the input signal that is not Ω-band-limited but rather than Ω'-band-limited for $\Omega' > \Omega$. The sampling theorem for aliasing error derived in Theorem 5.1 arises, where the function $x(t)$ belongs to U_k^α for rather than U_0^α for positive k. Such type of error can be stated as

$$e_A(t) = x(t) - \sum_{m\in\mathbb{Z}} x[\frac{m}{2^\gamma}]\, s(2^\gamma t - m) g(t,v) \exp(-j\pi\{t^2 - (\frac{m}{2^\gamma})^2\}\cot\alpha) \tag{6.17}$$

By calculating the aliasing error, one can choose a fractional scaling with a very little aliasing error to recreate input signals.

Theorem 6.2. Suppose, $\varphi(t)$ is in the set $L^2[\mathbb{R}]$ be a fractional scaling function of an MRA U_k^α, such that the sampling sequence $\varphi[n]$ is in the set $l^2[\mathbb{Z}]$ and $\frac{1}{\sqrt{2\pi}\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)}\chi_{D_\gamma}(\tau,v) \in L^2[I]$ for some $\gamma \in \mathbb{Z}^+ \bigcup \{0\}$. Hence, the aliasing error of the sampling theorem defined in (6.11) is bounded as

$$\| e_A(t)\|_{L^2} \le \sqrt{2\pi}\, 2^{\frac{\gamma+\delta_\gamma}{2}} \sqrt{\sum_{m\in\mathbb{Z}} |d[n]|^2}\, \|\{\frac{\tilde{\Phi}(\tau\sin\alpha+\pi, v\csc\alpha+\pi)}{\tilde{\Phi}(2\tau\sin\alpha, 2v\csc\alpha)} \det W(v\csc\alpha)\}^{\delta_\gamma}\{\Gamma(v\csc\alpha)\}^{1-\delta_\gamma}$$
$$\mathscr{K}_{\varphi,\alpha}(\frac{\tau}{2^{\gamma+\delta_\gamma-1}}, \frac{v}{2^{\gamma+\delta_\gamma-1}})\{\Pi_{\gamma=1}^{\gamma-1}\Lambda(\frac{v\csc\alpha}{2^\gamma})\}^{1-\delta_\gamma-\delta_{\gamma-1}}\|_\infty \tag{6.18}$$

where δ_γ is the Dirac delta function and $d[n]$; $n\in\mathbb{Z}$ are the FrST coefficients of function $x(t)$ in W_0^α and $W(v\csc\alpha)$ is defined as [19]

$$W(v\csc\alpha) = \begin{bmatrix} \Lambda(v\csc\alpha) & \Lambda(v\csc\alpha+\pi) \\ \Gamma(v\csc\alpha) & \Gamma(v\csc\alpha+\pi) \end{bmatrix} \tag{6.19}$$

Proof: Suppose that $W_0^\alpha = U_1^\alpha \Theta U_0^\alpha$ be the direct complement of U_0^α in U_1^α, from (5.52), now it is needed to verify that (6.18) is satisfied for any $x(t)\in W_0^\alpha$. Let, $\varphi(t)$ is in the set W_0^α being the FrST coefficients of the MRA $\{U_k\}_{k\in\mathbb{Z}}$. Because $\varphi_{0,n}(t) = \{\varphi(t-n)\ \exp(-j\pi(t^2-n^2)\cot\alpha)\}_{n\in\mathbb{Z}}$ form the Riesz basis of W_0^α, and $d[n]\in l^2[\mathbb{Z}]$ such that,

$$x(t)g(t,v) = \sum_{n\in\mathbb{Z}} d[n]\varphi(t-n)\exp(-j\pi(t^2-n^2)\cot\alpha) \tag{6.20}$$

Suppose, $X_\alpha(\tau, v)$ represents the FrST of $x(t)$ and $\tilde{D}_\alpha(\tau, v)$ represents the DTFrST of the product of $d[n]$ and $g(t, v)$. Now, by taking FrST on both sides of (6.17) and also using (5.84), gives

$$X_\alpha(\tau, v)G_\alpha(\tau, v) = \sqrt{2\pi}\ \tilde{D}_\alpha(v, \tau)\Psi(\tau, v) = \sqrt{2\pi}\tilde{D}_\alpha(\tau, v)\Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2})\Gamma(\frac{v\csc\alpha}{2}) \quad (6.21)$$

Now, taking the FrST on both sides of (6.17) and using (5.84) gives

$$E_{A,\alpha}(v, \tau) = X_\alpha(v, \tau)G_\alpha(v, \tau) - \sqrt{2\pi}2^{-\gamma}\sum_{n\in\mathbb{Z}} x[\frac{n}{2^\gamma}]g(\frac{n}{2^\gamma}, v)\kappa_\alpha\left(v, \frac{n}{2^\gamma}\right)S(\frac{v\csc\alpha}{2^\gamma}, \frac{\tau\sin\alpha}{2^\gamma})$$

$$= X_\alpha(\tau, v)G_\alpha(\tau, v) - \sqrt{2\pi}2^{-\gamma}\sum_{n\in\mathbb{Z}} x[\frac{n}{2^\gamma}]g(\frac{n}{2^\gamma}, v)\kappa_\alpha\left(v, \frac{n}{2^\gamma}\right)\frac{\Phi(\frac{\tau\sin\alpha}{2^\gamma}, \frac{v\csc\alpha}{2^\gamma})}{\sqrt{2\pi}\tilde{\Phi}(\frac{\tau\sin\alpha}{2^\gamma}, \frac{v\csc\alpha}{2^\gamma})}\aleph_{D_\gamma}(\frac{\tau}{2^\gamma}, \frac{v}{2^\gamma}) \quad (6.22)$$

where $E_A(\tau, v)$ represent the FrST of $e_A(t)$. By Poisson summation [96, 97], results into

$$2^{-\gamma}\sum_{n\in\mathbb{Z}} x[\frac{n}{2^\gamma}]g(\frac{n}{2^\gamma}, v)\kappa_\alpha\left(v, \frac{n}{2^\gamma}\right) = \exp(j\pi v^2\cot\alpha)\sum_{n\in\mathbb{Z}} X_\alpha(\tau + 2^{\gamma+1}n\pi\sin\alpha, v + 2^{\gamma+1}n\pi\sin\alpha) \exp(-j\pi(v + 2^{\gamma+1}n\pi\sin\alpha)^2\cot\alpha) \quad (6.23)$$

Inserting (6.21) into (6.23), results into

$$2^{-\gamma}\sum_{n\in\mathbb{Z}} x[\frac{n}{2^\gamma}]g(\frac{n}{2^\gamma}, v)\kappa_\alpha\left(v, \frac{n}{2^\gamma}\right) = \sqrt{2\pi}\exp(j\pi v^2\cot\alpha)\sum_{n\in\mathbb{Z}}\tilde{D}_\alpha(\tau + 2^{\gamma+1}n\pi\sin\alpha, v + 2^{\gamma+1}n\pi\sin\alpha) \exp(-j\pi(v + 2^{\gamma+1}n\pi\sin\alpha)^2\cot\alpha)\Psi(\tau + n\pi 2^{\gamma+1}\sin\alpha, v + n\pi 2^{\gamma+1}\sin\alpha) \quad (6.24)$$

Which can be simplified as

$$2^{-\gamma}\sum_{n\in\mathbb{Z}} x[\frac{n}{2^\gamma}]g(\frac{n}{2^\gamma}, v)\kappa_\alpha\left(v, \frac{n}{2^\gamma}\right) = \sqrt{2\pi}\,\tilde{D}_\alpha(\tau, v)\sum_{n\in\mathbb{Z}}\Psi(\tau\sin\alpha + n\pi 2^{\gamma+1}, v\csc\alpha + n\pi 2^{\gamma+1}) \quad (6.25)$$

Then inserting (6.21) and (6.25) into (6.22) yields

$$E_{A,\alpha}(\tau, v) = \sqrt{2\pi}\tilde{D}_\alpha(\tau, v)\Gamma(\frac{v\csc\alpha}{2})\Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2}) - \sqrt{2\pi}\tilde{D}_\alpha(\tau, v) \sum_{n\in\mathbb{Z}}\Psi(\tau\sin\alpha + n\pi 2^{\gamma+1}, v\csc\alpha + n\pi 2^{\gamma+1})\frac{\Phi(\frac{\tau\sin\alpha}{2^\gamma}, \frac{v\csc\alpha}{2^\gamma})}{\tilde{\Phi}(\frac{\tau\sin\alpha}{2^\gamma}, \frac{v\csc\alpha}{2^\gamma})}\aleph_{D_\gamma}(\frac{\tau}{2^\gamma}, \frac{v}{2^\gamma}) \quad (6.26)$$

Case 1: when $\gamma = 0$ Adding (6.26) and the Parseval's theorem of the FrFT results into

$$\| e_A(t) \|_{L^2}^2 = 2\pi \| \tilde{D}_\alpha(\tau, v)\Gamma(\frac{v\csc\alpha}{2})\Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2}) - \tilde{D}_\alpha(\tau, v) \sum_{n\in\mathbb{Z}} \Psi(\tau\sin\alpha + 2n\pi, v\csc\alpha + 2n\pi) \frac{\Phi(\tau\sin\alpha, v\csc\alpha)}{\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \|_{L^2}^2 \tag{6.27}$$

Then using (5.38) and (6.27), it can be written as

$$\| e_A(t) \|_{L^2}^2 = 2\pi \| \tilde{D}_\alpha(\tau, v)\{\Gamma(\frac{v\csc\alpha}{2})\Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2}) - \frac{\sum_{n\in\mathbb{Z}} \Psi(\tau\sin\alpha + 2n\pi, v\csc\alpha + 2n\pi)}{\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2})\Lambda(\frac{v\csc\alpha}{2})\} \|_{L^2}^2 \tag{6.28}$$

Suppose, $I_m = [0, 2^{m+1}\pi\sin\alpha]$ then

$$\| e_A(t) \|_{L^2}^2 = 2\pi \sum_{k\in\mathbb{Z}} \int_{I_1} | \Phi(\frac{\tau\sin\alpha}{2} + 2\pi k, \frac{v\csc\alpha}{2} + 2\pi k) |^2 \, | \tilde{D}_\alpha(\tau, v) |^2 | \Gamma(\frac{v\csc\alpha}{2}) - \frac{\sum_{n\in\mathbb{Z}} \Psi(\tau\sin\alpha + 2n\pi, v\csc\alpha + 2n\pi)}{\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \Lambda(\frac{v\csc\alpha}{2}) |^2 \, dv \tag{6.29}$$

$$\| e_A(t) \|_{L^2}^2 = 2\pi \int_{I_1} \mathscr{K}_{\varphi,\alpha}^2(\frac{\tau}{2}, \frac{v}{2}) | \tilde{D}_\alpha(\tau, v) |^2 | \Gamma(\frac{v\csc\alpha}{2}) - \frac{\sum_{n\in\mathbb{Z}} \Psi(\tau\sin\alpha + 2\pi n, v\csc\alpha + 2\pi n)}{\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \Lambda(\frac{v\csc\alpha}{2}) |^2 \, dv \tag{6.30}$$

Now, it follows from (5.38) and (5.39), results in

$$\sum_{n\in\mathbb{Z}} \Psi(\tau\sin\alpha + 2\pi n, v\csc\alpha + 2\pi n) = \sum_{n\in\mathbb{Z}} \Gamma(\frac{v\csc\alpha}{2} + n\pi)\Phi(\frac{\tau\sin\alpha}{2} + n\pi, \frac{v\csc\alpha}{2} + n\pi) \tag{6.31}$$

$$= \sum_{k\in\mathbb{Z}} \Gamma(\frac{v\csc\alpha}{2} + n\pi)\, \Phi(\frac{\tau\sin\alpha}{2} + n\pi, \frac{v\csc\alpha}{2} + n\pi) + \sum_{k\in\mathbb{Z}} \Gamma(\frac{v\csc\alpha}{2} + \pi(2k+1))\Phi(\frac{\tau\sin\alpha}{2} + \pi(2k+1), \frac{v\csc\alpha}{2} + \pi(2k+1))$$

$$= \Gamma(\frac{v\csc\alpha}{2})\tilde{\Phi}(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2}) + \tilde{\Phi}(\frac{\tau\sin\alpha}{2} + \pi, \frac{v\csc\alpha}{2} + \pi)\Gamma(\frac{v\csc\alpha}{2} + \pi) \tag{6.32}$$

and $\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha) = \sum_{n\in\mathbb{Z}} \Phi(\tau\sin\alpha + 2\pi n, v\csc\alpha + 2\pi n)$

$$= \tilde{\Phi}(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2})\Lambda(\frac{v\csc\alpha}{2}) + \tilde{\Phi}(\frac{\tau\sin\alpha}{2}+\pi, \frac{v\csc\alpha}{2}+\pi)\Lambda(\frac{v\csc\alpha}{2}+\pi) \quad (6.33)$$

Then, substituting (6.32) and (6.33) into (6.31) yields

$$\| e_A(t) \|_{L^2}^2 = 2\pi \int_{I_1} \mathscr{K}_{\varphi,\alpha}^2(\frac{\tau}{2}, \frac{v}{2}) \,|\, \tilde{D}_\alpha(\tau, v)|^2 \;\Big| \frac{\tilde{\Phi}(\frac{\tau\sin\alpha}{2}+\pi, \frac{v\csc\alpha}{2}+\pi)}{\tilde{\Phi}(v\csc\alpha, \tau\sin\alpha)} \{\Gamma(\frac{v\csc\alpha}{2})\Lambda(\frac{v\csc\alpha}{2}+\pi) - \Gamma(\frac{v\csc\alpha}{2}+\pi)\Lambda(\frac{v\csc\alpha}{2})\}\Big|^2 dv$$

$$\le 2\pi \| \mathscr{K}_{\varphi,\alpha}^2(\frac{\tau}{2}, \frac{v}{2}) \,|\, \frac{\tilde{\Phi}(\frac{\tau\sin\alpha}{2}+\pi, \frac{v\csc\alpha}{2}+\pi)}{\tilde{\Phi}(\tau\sin\alpha, v\csc\alpha)} \det W(\frac{v\csc\alpha}{2}) \|_\infty^2 \int_{I_1} |\tilde{D}_\alpha(\tau, v)|^2 dv$$

$$= 4\pi \| \mathscr{K}_{\varphi}^2(\frac{\tau}{2}, \frac{v}{2}) \,|\, \frac{\tilde{\Phi}(\tau\sin\alpha+\pi, v\csc\alpha+\pi)}{\tilde{\Phi}(2\tau\sin\alpha, 2v\csc\alpha)} \det W(v\csc\alpha) \|_\infty^2 \sum_{n\in Z} |d[n]|^2 \quad (6.34)$$

Case II. When $\gamma = 1$ adding (6.26) and the Parseval's theorem of the FrST gives

$$\| e_A(t) \|_{L^2}^2 = 2\pi \| \tilde{D}_\alpha(\tau, v)\Gamma(\frac{v\csc\alpha}{2})\Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2}) - \tilde{D}_\alpha(\tau, v) \sum_{n\in\mathbb{Z}} \Psi(v\csc\alpha + 4\pi n) \frac{\Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2})}{\tilde{\Phi}(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2})} \aleph_{2\,\mathrm{supp}\Lambda(v\csc\alpha)}(\tau, v) \|_{L^2}^2 \quad (6.35)$$

Now using (5.39), it gives

$$\| e_A(t) \|_{L^2}^2 = 2\pi \| \tilde{D}_\alpha(\tau, v)\Gamma(\frac{v\csc\alpha}{2})\Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2}) - \tilde{D}_\alpha(\tau, v)\Gamma(\frac{v\csc\alpha}{2}) \sum_{n\in\mathbb{Z}} \Phi(\frac{\tau\sin\alpha}{2}+2\pi n, \frac{v\csc\alpha}{2}+2\pi n)\{\frac{\Phi(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2})}{\tilde{\Phi}(\frac{\tau\sin\alpha}{2}, \frac{v\csc\alpha}{2})} \aleph_{2\,\mathrm{supp}\,\Lambda(v\csc\alpha)}(\tau, v)\} \|_{L^2}^2 \quad (6.36)$$

Using (6.7) and (6.33) results into

$$\| e_A(t) \|_{L^2}^2 = 2\pi \| \tilde{D}_\alpha(\tau,v)\Gamma(\frac{v\csc\alpha}{2})\Phi(\frac{\tau\sin\alpha}{2},\frac{v\csc\alpha}{2}) \quad - \tilde{D}_\alpha(\tau,v)\Gamma(\frac{v\csc\alpha}{2})$$

$$\Phi(\frac{\tau\sin\alpha}{2},\frac{v\csc\alpha}{2})\{\frac{\Phi(\frac{\tau\sin\alpha}{2},\frac{v\csc\alpha}{2})}{\tilde{\Phi}(\frac{\tau\sin\alpha}{2},\frac{v\csc\alpha}{2})}\aleph_{2\,\mathrm{supp}\,\Lambda(v\csc\alpha)}(\tau,v)\}\|_{L^2}^2 \tag{6.37}$$

Thereafter, it can be expressed as

$$\| e_A(t) \|_{L^2}^2 = 2\pi \| \tilde{D}_\alpha(\tau,v)\Gamma(\frac{v\csc\alpha}{2})\Phi(\frac{\tau\sin\alpha}{2},\frac{v\csc\alpha}{2})\{1-\aleph_{2\,\mathrm{supp}\,\Lambda(v\csc\alpha)}(\tau,v)\}\|_{L^2}^2$$

$$= 2\pi\sum_{k\in\mathbb{Z}}\int_{I_1}|\tilde{D}(\tau,v)|^2|\Gamma(\frac{v\csc\alpha}{2})|^2|\Phi(\frac{\tau\sin\alpha}{2}+2\pi k,\frac{v\csc\alpha}{2}+2\pi k)|^2|1-\aleph_{2\,\mathrm{supp}\,\Lambda(v\csc\alpha)}(\tau,v)|\,dv$$

$$= 2\pi\int_{I_1}|\tilde{D}_\alpha(\tau,v)|^2|\Gamma(\frac{v\csc\alpha}{2})|^2\mathscr{K}_{\varphi,\alpha}^2(\frac{\tau}{2},\frac{v}{2})|1-\aleph_{\mathrm{supp}\,\Lambda(v\csc\alpha/2)}(\tau,v)|\,dv$$

$$\le 2\pi\int_{I_1}|\tilde{D}_\alpha(\tau,v)|^2\,dv\,\|\Gamma(\frac{v\csc\alpha}{2})\mathscr{K}_{\varphi,\alpha}(\frac{\tau}{2},\frac{v}{2})(1-\aleph_{\mathrm{supp}\,\Lambda(v\csc\alpha/2)}(\tau,v))\|_\infty^2$$

$$= 4\pi\sum_{n\in\mathbb{Z}}|d[n]|^2\,\|\Gamma(v\csc\alpha)\mathscr{K}_{\varphi,\alpha}(\tau,v)\aleph_{\mathbb{R}\Theta\,\mathrm{supp}\,\Lambda(v\csc\alpha)}(\tau,v)\|_\infty^2$$

Therefore,

$$\| e_A(t) \|_{L^2}^2 \quad \le \; 4\pi\sum_{n\in\mathbb{Z}}|d[n]|^2\,\|\Gamma(v\csc\alpha)\mathscr{K}_{\varphi,\alpha}(\tau,v)\|_\infty^2 \tag{6.38}$$

Case III. For $\gamma > 2$, Using (6.26) and the Parseval's theorem of FrST gives

$$\| e_A(t) \|_{L^2}^2 = 2\pi \| \tilde{D}_\alpha(\tau,v)\Gamma(\frac{v\csc\alpha}{2})\prod_{J=2}^{J}\Lambda(\frac{v\csc\alpha}{2^J})\Phi(\frac{\tau\sin\alpha}{2^J},\frac{v\csc\alpha}{2^J}) - \tilde{D}_\alpha(\tau,v)\Gamma(\frac{v\csc\alpha}{2})$$

$$\prod_{\gamma=2}^{\gamma}\Lambda(\frac{v\csc\alpha}{2^\gamma})\tilde{\Phi}(\frac{\tau\sin\alpha}{2^\gamma},\frac{v\csc\alpha}{2^\gamma})\frac{\Phi(\frac{\tau\sin\alpha}{2^\gamma},\frac{v\csc\alpha}{2^\gamma})}{\tilde{\Phi}(\frac{\tau\sin\alpha}{2^\gamma},\frac{v\csc\alpha}{2^\gamma})}\aleph_{\cap_{\gamma=1}^{\gamma}2^\gamma\,\mathrm{supp}\,\Lambda(v\csc\alpha)}(\tau,v)\|_{L^2}^2$$

$$= \; 2\pi\|\tilde{D}_\alpha(\tau,v)\Gamma(\frac{v\csc\alpha}{2})\prod_{\gamma=2}^{\gamma}\Lambda(\frac{v\csc\alpha}{2^\gamma})\Phi(\frac{\tau\sin\alpha}{2^\gamma},\frac{v\csc\alpha}{2^\gamma})(1-\aleph_{\cap_{\gamma=1}^{\gamma}2^\gamma\,\mathrm{supp}\,\Lambda(v\csc\alpha)}(\tau,v)\|_{L^2}^2 \tag{6.39}$$

Thus it can be written as

$$\| e_A(t) \|_{L^2}^2 = 2\pi \int_{I_\gamma} | \tilde{D}_\alpha(\tau, v) \Gamma(\frac{v \csc\alpha}{2}) \prod_{\gamma=2}^{\gamma} \Lambda(\frac{v \csc\alpha}{2^\gamma}) |^2$$

$$\sum_{k\in\mathbb{Z}} | \Phi(\frac{\tau \sin\alpha}{2^\varphi} + 2k\pi, \frac{v \csc\alpha}{2^\gamma} + 2k\pi) |^2 \, | 1 - \aleph_{\cap_{\gamma=1}^{\gamma} 2^\gamma \operatorname{supp} \Lambda(v \csc\alpha/2)}(\tau, v) | \, dv$$

$$\leq 2\pi \int_{I_\gamma} | \tilde{D}_\alpha(\tau, v) |^2 \, dv \, \| \Gamma(\frac{v \csc\alpha}{2}) \mathscr{K}_{\varphi,\alpha}(\frac{\tau \sin\alpha}{2^\gamma}, \frac{v \csc\alpha}{2^\gamma}) \prod_{\gamma=2}^{\gamma} \Lambda(\frac{v \csc\alpha}{2^\gamma}) | 1 - \aleph_{\cap_{\gamma=1}^{\gamma} 2^\gamma \operatorname{supp} \Lambda(v \csc\alpha/2)}(\tau, v) \|_\infty^2$$

$$2\pi \; 2^\gamma \sum_{n\in\mathbb{Z}} | d[n] |^2 \| \Gamma(v \csc\alpha) \mathscr{K}_{\varphi,\alpha}(\frac{\tau}{2^{\gamma-1}}, \frac{v}{2^{\gamma-1}}) \prod_{\gamma=1}^{\gamma-1} \Lambda(\frac{v \csc\alpha}{2^\gamma}) \; \aleph_{\cap_{\gamma=0}^{\gamma-1} 2^\gamma \operatorname{supp} \Lambda(v \csc\alpha)}(\tau, v) \|_\infty^2$$

$$\leq 2\pi 2^\gamma \sum_{n\in\mathbb{Z}} | d[n] |^2 \| \Gamma(v \csc\alpha) \mathscr{K}_{\varphi,\alpha}(\frac{\tau}{2^{\gamma-1}}, \frac{v}{2^{\gamma-1}}) \prod_{\gamma=1}^{\gamma-1} \Lambda(\frac{v \csc\alpha}{2^\gamma}) \|_\infty^2 \quad (6.40)$$

All the above three cases are consider as a bound of aliasing error defined in (6.18). For sampling in U_k^α, the corresponding aliasing error can be expressed as

$$e_A(t) = x(t) - \sum_{m\in\mathbb{Z}} x[\frac{m}{2^{\gamma+k}}] s(2^{\gamma+k} t - m) g(t, v) \exp(-j\pi\{t^2 - (\frac{m}{2^{\gamma+k}})^2\} \cot\alpha) \quad (6.41)$$

For all elements in U_{k+1}^α. Similarly to analysing the other case of the sample selection in U_0^α, the aliasing error bounded by the condition

$$\| e_A(t) \|_{L^2} \leq \sqrt{2\pi} 2^{\frac{\gamma+\delta_\gamma-k}{2}} \sqrt{\sum_{m\in\mathbb{Z}} | d[n] |^2} \, \| \{ \frac{\tilde{\Phi}(\tau \sin\alpha + \pi, v \csc\alpha + \pi)}{\tilde{\Phi}(2\tau \sin\alpha, 2v \csc\alpha)} \det W(v \csc\alpha) \}^{\delta_\gamma} \{ \Gamma(v \csc\alpha) \}^{1-\delta_\gamma}$$

$$\mathscr{K}_{\varphi,\alpha}(\frac{\tau}{2^{\gamma+\delta_\gamma-1}}, \frac{v}{2^{\gamma+\delta_\gamma-1}}) \{ \prod_{\gamma=1}^{\gamma-1} \Lambda(\frac{v \csc\alpha}{2^\gamma}) \}^{1-\delta_\gamma-\delta_{\gamma-1}} \|_\infty \quad (6.42)$$

where, $\{d[n]\}_{n\in\mathbb{Z}}$ are the FrST coefficient of the function $x(t)$ in W_k^α. That suggests that the aliasing error can be very little as one wishes, if k is large for a fixed γ.

6.4 Summary

The main features of this chapter are:

- First the filtering of ECG signal and LFM signal under AWGN noise is performed by using the multiplicative filtering in FrST domain.
- Thereafter, the proposed technique of filtering for FrST is compared to the existing filtering techniques based on WT and FrFT.

- Subsequently, the proposed sampling theorem for FrST is used to reconstruct the signal from its samples. Thereafter, a comparison is made between proposed method of reconstruction and sinc-interpolation method by evaluating the difference between the original and reconstructed signal as a parameter.
- Finally, the proposed sampling theorem of FrST is used to determine the error bounds of truncation error and aliasing error.

CHAPTER 7

CONCLUSIONS AND FUTURE SCOPE

From the analysis and simulated results being performed and reported in the previous chapters, conclusions have been drawn here. The concluding notes are presented in the following section along with the future scope of the work.

7.1 CONCLUSION

The fractional S-transform (FrST) has been verified as a better mathematical means for study of non-stationary signals in the time-frequency plane. Several methods are available in the literature to evaluate the FrST identities and their applications. The ST is also mentioned because it is defined as a special case of FrST at an angle $\alpha = \frac{\pi}{2}$. As the generalization of ST, FrST is a useful mathematical tool for digital signal analysis. Since the tractability of FrST is improved than compared to conventional ST, many difficulties that may not be overcome by conventional ST are solved using FrST. In chapter 3, some identities of ST are proposed such as; convolution theorem, time-reversal, time-derivatives, complex conjugate, correlation theorem and Parseval's theorem. The proposed convolution theorem is compared with existing convolution theorem on the basis of its capability to convert in classical convolution theorem at $\alpha = \frac{\pi}{2}$ and computational complexity. This comparative analysis established the superiority of the proposed definition of convolution theorem of ST over existing theorem. Also, to further increase the span of application region the N-dimensional ST with its properties are presented. In chapter 4, the concept of orthogonal MRA for FrST has been discussed. In chapter 5, the need for defining FrST identities is presented followed by the definitions and proofs of convolution theorem, correlation theorem, and Parseval's theorem. Along with this, the concept of multiresolution analysis with its properties for FrST has been discussed, which enables the formation of an orthogonal kernel of FrST. The detailed analytical framework of MRA and the orthogonal kernel of FrST are analyzed. Subsequently, the computational complexity of DFrST and SDFrST are also documented. It is shown that for data length larger than six, SDFrST is a better algorithm to compute the FrST. Finally, the truncation error and the aliasing error with their bounds are determined for the proposed sampling theorem of FrST. The proposed methods are utilized in the restoration of the continuous signal from its sampled version and compared with cubic spline, and sinc function. A comparative analysis to justify the superiority of the proposed approach over the existing is included in chapter 6. Finally, ST and FrST is an efficient, more flexible and more powerful tool for applications in non-stationary signal processing.

7.2 FUTURE SCOPE OF WORK

In the future, more efficient ways for implementing the FrST in different fields of signal processing can be investigated. Moreover, an effort should also made be to find fresh uses of FrST since some of the applications include digital signal, image, and biomedical signal processing.

Similar to FrST, the DFrST can be developed for efficient implementation of digital image processing applications, such as image compression, encryption, enhancement, rotation and filtering of the image.

The FrST can also be used for other applications of non-stationary signal analysis like bio-medical and radar signals analysis.

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