” *Mathematical Modelling of Lopsided Features at the Centre of Galaxies*”

Thesis submitted in partial fulfillment of the requirement for the award of the degree of

Masters of Science

in Mathematics and Computing

By

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Dedicated to my parents.

Certificate

I hereby certify that the work, which is being presented in the thesis, entitled "**Mathematical Modelling of Lopsided features at the Center of Galaxies**" in partial fulfillment of the requirements for the award of degree of **Masters of Science in Mathematics and Computing** and submitted to the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is an anthentic record of my own work carried out under the supervision **Dr. Mamta Gulati**, Assistant Professor and other research work which is duly listed in reference section.

The matter presented in this thesis has not been submitted elsewhere for the award of any other degree or diploma from any institution.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Sakshi Singla

Abstract

In this thesis, we first introduce what are galaxies and types of galaxies. Then we discuss the observations of lopsided structures seen at the centers of galaxies. A brief review of literature on the mathematical modeling of center of galaxies in last few years and the dispersion relations derived by various authors by approximating galaxy disk as fluid, softened gravity and stellar disc is then discussed. The major limitation of these models is the treatment of components of galaxies. The galaxies are a combination of gas and stars however none of the discussed model consider the dynamics of the combined disc. We propose a model for eccentric modes, appropriate to the system which have both particle and gas disc coupled together. WKB (Wentzel-Kramers- Brillouin) dispersion relation is derived in which the star and gas disc are considered as a coupled system. The perturbation occurs in both star and gas disc. We have studied the nature of perturbations. The disc are found to be stable if they satisfy the Toomre's stability criteria. We give the plot for stability region for different amount of gas content in the disc and also give the phase plots describing the nature of wave solution.

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CHAPTER 1

Introduction

The word galaxy that is derived from Greek "gala" meaning "milk", that is reference to the milky way. Galaxies are defined as a massive system of stars, dark matter, stellar remnants, and interstellar medium composed of gas and dust held together by its own gravity. Galaxies vary in shape, from disc galaxies to elliptical, and mass from 10^7 to 10^{12} solar masses. They contain large amount of mass in unseen dark matter, which contributes to major mass content of galaxies. The galaxy in which our solar system lies is known as Milky way and is a disc spiral galaxy. Night sky image of Milky way as seen from a clear sky on earth is shown in fig 1.1. The Milky way is an extremely large spiral disk galaxy with a bright central bulge. In galaxies, material contained in them is orbiting around the central core. Individual galaxies are also in motion w.r.t each other. Infact, all the galaxies in the universe are seem to be moving away from each other at very large speeds.

1.1 *Types of Galaxies*

After looking at the images of galaxies for many years, Edwin Hubble classified galaxies according to their shapes. Image of tunning fork is shown in fig 1.2. Galaxies are divided into four main categories, i.e Ellipitical , Spiral , Lenticular and Irregular.

Figure 1.1: Image of Milky way in night sky (*Image Credit : Trevor Dobson*)

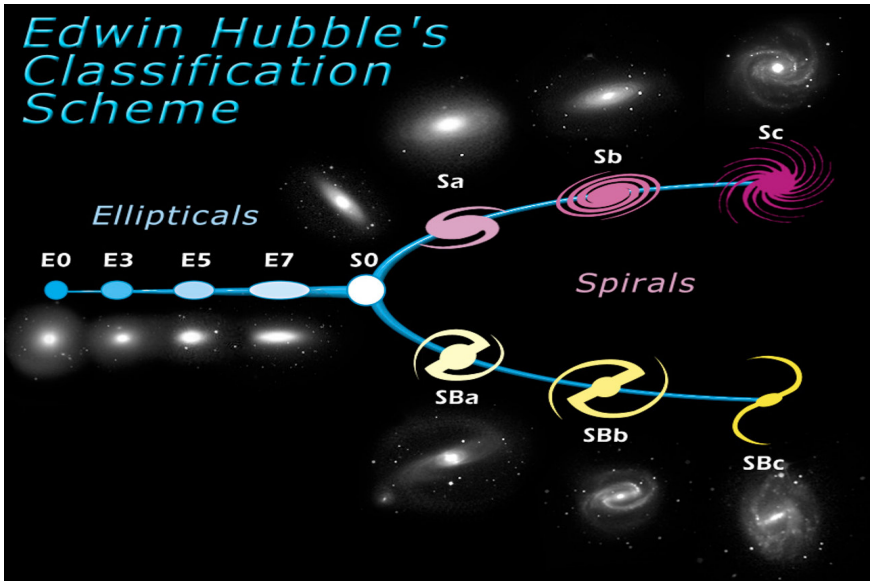

Figure 1.2: Image of Tunning Fork (*Image Credit : NASA and ESA*)

1.1.1 *Elliptical Galaxies*

The shape of Elliptical galaxies vary from completely round to extremely enlongated ovals with smooth nucleus at their center. Number of stars in elliptical galaxies vary from tens of millions to above hundred of trillion stars and these are generally older stars. The star formation activity is little as elliptical galaxies has less interstellar gas. The Elliptical galaxies are denoted by the letter E and they are divided into seven subgroups according to their shapes. They are labelled $E0$ to $E7$. The $E0$ galaxies are nearly circular in shape and $E7$ galaxies are highly enlongated. An e.g of elliptical galaxy as shown in fig 1.3.

Figure 1.3: Image of Elliptical galaxy IC 1101 (*Image Credit* : *NASA/ chandra*)

1.1.2 *Spiral Galaxies*

Spiral galaxies are classified into two groups: barred and ordinary. The ordinary group is designed by S and barred by SB. These normal galaxies contain a central bulge which is orbited by flat disk and the disk is a collection of gas, stars and dust particles. The disk is separated into arms where continuously stars are being formed. Our own galaxy, the Milky way, is also a spiral galaxy. In barred spirals there is a bar of material (gas and stars) that runs through the nucleus and the arms emerge from it. Now both of these types are further classified according to how their arms are tightly wound. $S0$ galaxies have no spiral arms and Sa galaxies have spiral arms that are tightly wound around the nucleus but the arms of Sc galaxies are much more loosely wound. An example of spiral galaxy as shown in fig 1.4.

Figure 1.4: Spiral galaxy (*Image Credit* : *NASA*)

1.1.3 *Lenticular Galaxies*

These galaxies are transition objects between spiral and elliptical galaxies. They hold a central bulge and a rotating disk as present in spiral galaxies and similar to the elliptical galaxies, they have small amount of cool gas with nearly no recent star formation and hence contains mostly older stars. Examples : NGC 4866 as shown in Fig: 1.5.

1.1.4 *Irregular Galaxies*

As the name suggested these galaxies have no regular geometric shape or structure. They are small in size, irregular galaxies are divided into two classes i.e. 1m and 10. 1m galaxies are most common and show just little hint of structure. 10 galaxies are completely chaotic in form. Example: IC 3583 as shown in Fig: 1.6.

Figure 1.5: Lenticular galaxy NGC 4866 (*Image Credit* : *European Space Agency*)

1.2 *Observations of Centers of galaxies*

1.2.1 *Discs in Galactic Nuclei*

The galaxy centers, also called a galactic nuclei, are difficult to observe because of limitations of present telescopes. Out of few galaxies in which such observations are known, two galaxies:

1. *NGC4486B (in the virgo cluster)*
2. *M31 (closed neighbour to our galaxy)*

show an unexpected double peak distribution of stars in there nuclei as shown in Fig: (1.7), (1.8). There are important differences in double nucleus of two galaxies:

- NGC4486B is a elliptical galaxy, M31 is a spiral galaxy.
- Physical detachment of peak in NGC4486B is 5-7 times larger than that of M31.
- Both the peaks in NGC4486B are at similar distances from center, M31 the fainter peak is within 0.05" of the center.
- Central surface brightness of peaks are more closely matched in NGC4486B as compared to M31.

Figure 1.6: Irregular galaxy IC 3583 (*Image Credit : NASA and Hubble/ ESA*)

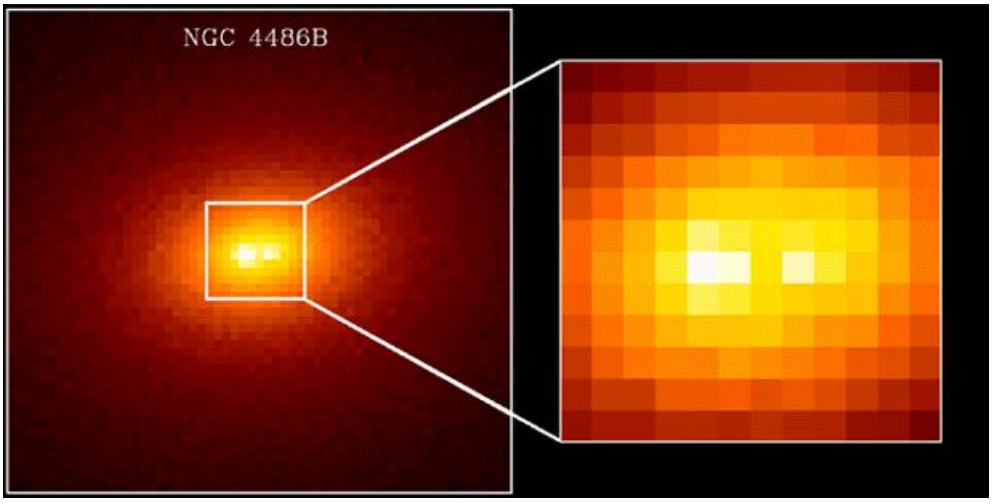

Figure 1.7: Image of NGC4486B (*Image Credit : Karl Gebhardt, Tod Lauer and NASA*)

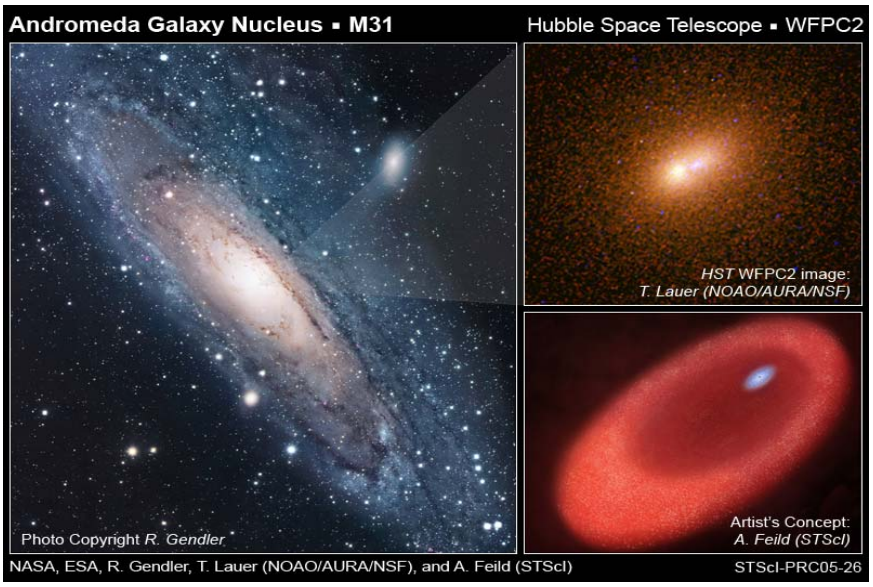

Figure 1.8: Image of Andomeda galaxy, M31 ($Image\ Credit$: $NASA$, ESA, $R.Gendler$, $T.Lauer$ $(NOAO/AURA/NSF)$, $and\ A\ Feild(STScl)$.Double peak structure is clearly revealed in this image.

1.3 *Plan of the thesis*

Uptill now we have introduced what are galaxies and types of galaxies, we have also discussed lopsided structures seen at the centers of galaxies. Below we give the plan of rest of the thesis:

- In first part, we review the literature on the mathematical modeling of center of galaxies in last few years to explain the observations of centers of galaxies. We give a brief review of the relations derived by various authors by approximating galaxy disks as fluid disc, softened gravity disks, stellar disc.
- In second part, we discuss our proposed model for the center of galaxies to explain some observations.
- We discuss the WKB dispersion relation for combined disc in which both the particle and gas disc are perturbed and further discuss its stability of these modes. We also give the phase plots of such modes, giving the structure of eigen values.
- Lastly, we provide some conclusions of the results that obtain in this thesis along some future directions.

CHAPTER 2

Motivation

In this chapter we give a brief overview of models proposed for explaining the observed features at galactic nuclei. We shall also give limitations of available models, which forms the base of our model.

Tremaine [1995], it suggest that these eccentric modes are accountable for the lopsidedness of the center of M31. The question now was the generation of these eccentric modes. To which Tremaine [2001], used a softened gravity to comprehend slow modes of the keplerian discs, giving rise to eccentric modes in the disc. Further Touma [2002], it proposed that a feasible beginning of such high eccentricity orbits is instability of counter-rotating streams of stars, which could happen when a globular cluster gets disorder and is added to the starlike system. However, Touma's model assumed disc to be combination of discrete rings.
Sridhar and Saini [2010], who studied these instabilities analytically in the WKB approximation for softened gravity disc which is continuous disc and models galactic discs more accurately. There analysis had some disadvantages like they could not calculate eigen functions and only eigen values could be calculated for equal counter-rotation. Gulati et al. [2012], constructed an eigenvalue integral equation for softened gravity disk, for co-planar counter-rotating disk. These support the formation of double peak nucleus. The limitations of there model are softened gravity disk only support m=1 modes. Where m is the azimuthal wave number. These can't explain the symmetric double peak nucleus in elliptical galaxies, which may be due to m=2 or m=4 (i.e even m) modes.

Gulati and Saini [2016-2017], then proposed a model for a starlike disk with the use of Collisionless Boltzmann equation that supports all m-modes and could explain the observations in elliptical galaxies. Now for galactic system the eccentric modes till now are studied for either softened gravity disc, gas disc or collisionless stellar discs but the gas and stars co-exits in the galaxies and system should be studied as a two combined co-planar discs, one of stars and other one being gas disc.
Jalali [2013], who proposed a model that combined both gas and particle disc. Their study is relevant only under the approximation that the surface density of gas is more hold for galaxies as compare to particle surface density. Which does not propose a mathematical model for a coupled system of gas and particle (stars in our case) applicable to galactic systems.

2.1 *Review of WKB Dispersion Relation's in Literature:*

In this section we give a review of dispersion relations derived by various authors to model the asymmetric structures. Along with the review we also discuss the relevant equations and approximations we shall by using for our model.

2.1.1 Fluid Disc

Perturbation analysis is used to examine the instability in an initially axisymmetric disc. To start with, the response of galactic disk in a perturbed potential . Complete analysis would use the cylindrical coordinates. The Euler and Continuity equation are given by:

$$\frac{\partial \Sigma_d}{\partial t} + \nabla.(\Sigma_d v_0) + \nabla.(\Sigma_0 v) = 0 \tag{2.1}$$

$$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial \Phi}{\partial R} - \frac{1}{\Sigma_d}\frac{\partial p}{\partial R} \tag{2.2}$$

$$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} - \frac{1}{\Sigma_d R}\frac{\partial p}{\partial \phi} \tag{2.3}$$

Where Σ_d is disc density, v_R and v_ϕ are radial and azimuthal velocities.

Any physical quantity Y can be written as sum of unperturbed and perturbed quantity i.e Y=$Y_0 + Y_1$. Adding this in equation's $(2.1), (2.2), (2.3)$ and taking only first

order perturbed quantities, we get

$$\frac{\partial \Sigma_{d1}}{\partial t} + \Omega \frac{\partial \Sigma_{d1}}{\partial \phi} + \frac{1}{R}\frac{\partial (R v_{R1} \Sigma_0)}{\partial R} + \frac{\Sigma_0}{R}\frac{\partial v_{\phi 1}}{\partial \phi} = 0 \tag{2.4}$$

$$\frac{\partial v_{R1}}{\partial t} + \Omega \frac{\partial v_{R1}}{\partial \phi} - 2\Omega v_{\phi 1} = -\frac{\partial (\Phi_1 + h_1)}{\partial R} \tag{2.5}$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \left[\frac{d(\Omega R)}{dR} + \Omega\right] v_{R1} + \Omega \frac{\partial v_{\phi 1}}{\partial \phi} = -\frac{1}{R}\frac{\partial (\Phi_1 + h_1)}{\partial \phi} \tag{2.6}$$

Where h is enthalpy per unit mass and Ω is azimuthal frequency.

The perturbations are then assumed to be a wave like solution, Y_1=$Y_a(R)e^{i(m\phi - \omega t)}$, where m is the azimuthal wavenumber and w is the temporal frequency. Substituting these solution's in above equation's (2.4), (2.5), (2.6) and solving for v_{Ra}, $v_{\phi a}$.

$$-i(\omega - m\Omega)\Sigma_{da} + \frac{1}{R}\frac{d(R v_{Ra}\Sigma_0)}{dR} + \frac{im\Sigma_0 v_{\phi a}}{R} = 0 \tag{2.7}$$

$$v_{Ra}(R) = \frac{i}{\Delta}\left[(\omega - m\Omega)\frac{d(\phi_a + h_a)}{dR} - \frac{2m\Omega(\Phi_a + h_a)}{R}\right] \tag{2.8}$$

$$v_{\phi a(R)} = -\frac{1}{\Delta}\left[2B\frac{d(\phi_a + h_a)}{dR} + \frac{m(\omega - \Omega m)(\Phi_a + h_a)}{R}\right] \tag{2.9}$$

$$\Delta = \kappa^2 - (\omega - m\Omega)^2 \tag{2.10}$$

Where B is Oorts constant and κ is radial frequency.

The above equations determine $v_{Ra}, v_{\phi a}, \Sigma_{da}$ in terms of perturbed potential ϕ_a. Now, the potential is relate through the Poisson's equation to the surface density. Then the radial variation of perturbed quantities, Y_a(R) is written as product of slowly varying function of R and $exp[i\int^R kdR]$ which gives fast oscillatory behaviour in radial direction. k is the radial wave number here. The above equations are then solved under WKB approximation i.e $|kR| \gg m$. This gives the WKB dispersion realtion as given in section 6.2.2 of Binney and Tremaine (2008).

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\Sigma|k| + v_s^2 k^2 \tag{2.11}$$

Where v_s is sound speed.

Note that ω real gives that the perturbations are stable i.e. we get oscillatory solution and ω is imaginary gives an exponentially growing solution.

- ***m=0 perturbations :***

  ω is real if the R.H.S of Eqn (2.11) is non-negative. Also the R.H.S is independent of m. Hence if the disc is stable for m=0, then it is stable for all m. Using m=0 in Eqn (2.11) we get,

  $$\omega^2 = \kappa^2 - 2\pi G\Sigma|k| + v_s^2 k^2 \tag{2.12}$$

  It is straightforward to prove (as in section 6.2.3 of Binney and Tremaine (2008)) that for short-wavelength the axisymmetric perturbations ,the disc is stable if

  $$Q = \frac{v_s \kappa}{\pi G\Sigma} > 1 \tag{2.13}$$

  This is the Toomre's stability criterion.

2.1.2 Softened gravity disc

The normal Newtonian potential goes as 1/R which blows up as R approaches zero. R is never zero for actual stellar systems but such situations cannot be avoided numerically. Hence to avoid the numerical singularity Miller(1961) introduced softened gravity for which potential goes as $\frac{1}{\sqrt{R^2+b^2}}$, where ε is the softening length.
The WKB dispersion of softened gravity disc follows the same steps that are used to calculate the dispersion relation for fluid disk. Under the WKB approximation the softened potential is written in the form

$$\phi_a = -\frac{2\pi G e^{-|k|\varepsilon}\Sigma_1}{|k|} \tag{2.14}$$

The WKB dispersion relation for softened gravity disc then given by (Binney and Tremaine 2008)

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\Sigma|k|F \tag{2.15}$$

Where $F = e^{-|k|\varepsilon}$, is called the reduction factor.

- ***m=0 perturbations :***

  When m=0, Eq (2.15) becomes,

  $$\omega^2 = \kappa^2 - 2\pi G\Sigma|k|F \tag{2.16}$$

  This gives that Binney and Tremaine (2008) for short-wavelength the axisymmetric perturbations ,the cold disc is stable if

  $$\varepsilon > \frac{2\pi G\Sigma}{\kappa^2 e} \tag{2.17}$$

2.1.3 Stellar disc

The dispersion relation of stellar disk is be calculated by using the same principles that we are used to obtain the WKB dispersion relation for fluid disk and softened gravity disc.

For a stellar disk with orbits of the stars given by epicyclic orbits the radial velocity perturbation is then given by (Binney and Tremaine 2008):

$$v_{Ra} = -\frac{(\omega - m\Omega)}{\Delta} k\Phi_a F \tag{2.18}$$

where $F \leq 1$ is the reduction factor.

$$F\left(\frac{\omega - m\Omega}{\kappa}, \frac{\sigma_R^2 k^2}{\kappa^2}\right) \equiv F(s, X) = \frac{1 - s^2}{sin\pi s} \int_0^\pi d\tau e^{-X(1+cos\tau)} sins\tau sin\tau \tag{2.19}$$

Under the WKB approximation, then the dispersion relation Binney and Tremaine (2008) is given by:

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\Sigma|k|F \tag{2.20}$$

- ***m=0 perturbations :***

  When m=0 Eq (2.20) becomes,

  $$\omega^2 = \kappa^2 - 2\pi G\Sigma|k|F \tag{2.21}$$

  giving us the stability condition as

  $$Q = \frac{\sigma_R \kappa}{3.36 G\Sigma} > 1 \tag{2.22}$$

This is similar to Toomre's Q parameter given in equation (2.13).

2.2 *Conclusions*

We have stated the WKB dispersion relation for gas disc, softened gravity disc and stellar disc. Also we have given their stability conditions. A closer look at equation (2.13), (2.17), (2.22) tells us that all these are quiet similar with $\kappa\varepsilon$ acting as velocity dispersion.

CHAPTER 3

Perturbative Analysis For Nearly Keplerian coupled Particle and Gas disc

3.1 *Introduction*

In this chapter, we shall drive the WKB dispersion relation of coupled gas and stellar disc. Similar work has been done by Jalali (2013), but for planetesimal disk which is not applicable to galactic disks. We shall first give the governing equations in section 3.2. We then discuss the dispersion relation where only stellar disc is perturbed in section 3.3 and its implications. Next, we drive the dispersion relation where both gas and stellar disc are perturbed we then discuss the stability of the system in section 3.4 and m=1, slow modes discuss in section 3.5, their phase plots then given in section 3.6. We offer our conclusions in section 3.7.

3.2 *Governing Equations:*

- **Gas disc**

  Gases expend above huge area's of space with only small number of particles per cm^3 and gases act like a fluids. so, we consider Euler and Continuity equations in cylindrical coordinates with disc existence in z=0 plane.

$$\frac{\partial \Sigma_{dg}}{\partial t} + \frac{1}{R}\frac{\partial (R\Sigma_{dg} v_R)}{\partial R} + \frac{1}{R}\frac{\partial (\Sigma_{dg} v_\phi)}{\partial \phi} = 0 \tag{3.1}$$

$$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{GM}{R^2} - \frac{\partial(\Phi + h)}{\partial R} \tag{3.2}$$

$$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial(\Phi + h)}{\partial \phi} \tag{3.3}$$

and Σ_{ds} = surface density, Φ = Total gravitational potential, v_ϕ and v_R = azimuthal and radial velocity,h= enthalpy per unit mass. and Σ_{dg} = surface density, h = enthalpy per unit mass, Φ = Total gravitational potential, v_ϕ and v_R = azimuthal and radial components of fluids velocity.

- **Particle disc**

  Stellar disk can be addressed in two ways:

  - *Softened gravity disc*

    Softened gravity disc is equivalent to fluid disc, So we can easily apply the fluid equations here which are given as

    $$\frac{\partial \Sigma_{ds}}{\partial t} + \frac{1}{R}\frac{\partial(R\Sigma_{ds}v_R)}{\partial R} + \frac{1}{R}\frac{\partial(\Sigma_{ds}v_\phi)}{\partial \phi} = 0 \tag{3.4}$$

    $$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{GM}{R^2} - \frac{\partial \Phi}{\partial R} \tag{3.5}$$

    $$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} \tag{3.6}$$

    and Σ_{ds} = surface density, Φ = Total gravitational potential, v_ϕ and v_R = azimuthal and radial components of fluids velocity.

  - *Collisionless particle disc*

    Fluid disc or softened gravity disc supports only m=1 modes, for other m-modes we need to treat the stellar disc as collisionless system. The stars in the disc are assumed to follow near circular orbits called Epicyclic orbits. For such system, the solution of collisionless Boltzmann equation can be written in the form of Schwarzschild distribution function given by

    $$f_0(L_R, v_R, \widetilde{v}_\phi) = \frac{\gamma\Sigma(R)}{2\pi\sigma_R^2(R)} exp\left[\frac{-(v_R^2 + \gamma^2\widetilde{V^2}_\phi)}{2\sigma_R^2(R)}\right] \tag{3.7}$$

$\Sigma(R)$ = surface density, $\sigma_R(R)$ = dispersion radial velocity, $v_c(R)$ = circular speed, $\widetilde{v}_\phi = v_\phi - v_c(R)$, $\gamma(R) = 2\Omega(R)/\kappa(R)$ [using Binney and Tremaine, 2008].

3.3 *WKB dispersion relation for perturbed gas and stellar disc:*

We shall approximate our stellar disc with softened gravity through rest of this thesis. To start with, we suppose that only the particle disc is perturbed.
The calculation of WKB dispersion relation for this was done by Amanpreet Kaur (2018) in her M.Sc thesis, The dispersion relation is given by:

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G|k|exp(-|k|b)\Sigma_d \tag{3.8}$$

This relation is same as that for a single disc discussed . The only difference is in azimuthal frequency $\Omega(R)$ which has contribution from both gas and particle disc. Hence a nearly keplerian combined disc turns out to be stable to non-axisymmetric perturbation in this case. And the lopsided distribution of particle is only long lived, if their is a continuous source of activation for such perturbations.

Next we perturb both gas and softened gravity disc and subscript 'g' terms stand for gas disc and subscript 's' is for softened gravity disc. We shall take velocity of both stars and gas as same since these are assumed to be coupled.

Gas disc : We examine the continuity and Euler equation in cylindrical polar co-ordinates with the disc being in z=0 plane.

$$\frac{\partial \Sigma_{dg}}{\partial t} + \frac{1}{R}\frac{\partial (R\Sigma_{dg} v_R)}{\partial R} + \frac{1}{R}\frac{\partial (\Sigma_{dg} v_\phi)}{\partial \phi} = 0 \tag{3.9}$$

$$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{GM}{R^2} - \frac{\partial \Phi}{\partial R} \tag{3.10}$$

$$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} \tag{3.11}$$

Particle disc : The softened gravity discs as discussed in section (3.2) and the relevant equation are

$$\frac{\partial \Sigma_{ds}}{\partial t} + \frac{1}{R}\frac{\partial (R\Sigma_{ds} v_R)}{\partial R} + \frac{1}{R}\frac{\partial (\Sigma_{ds} v_\phi)}{\partial \phi} = 0 \tag{3.12}$$

$$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{GM}{R^2} - \frac{\partial \Phi}{\partial R} \tag{3.13}$$

$$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} \tag{3.14}$$

We perturb both the particle and gas disc. The perturbation of any physical quantity Y, i.e Y=Y_0+Y_1. Adding this in equation's (3.9), (3.10), (3.11) and(3.12), (3.13), (3.14) and take only first order perturbed quantities, perturbed potential Φ_1 is contribute for the surface density of gas and star disc, we get (for both stellar and Gas disc)

$$\frac{\partial \Sigma_{1g/s}}{\partial t} + \Omega \frac{\partial \Sigma_{1g/s}}{\partial \phi} + \frac{1}{R}\frac{\partial (R v_{R1} \Sigma_0)}{\partial R} + \frac{\Sigma_0}{R}\frac{\partial v_{\phi 1}}{\partial \phi} = 0 \tag{3.15}$$

$$\frac{\partial v_{R1}}{\partial t} + \Omega \frac{\partial v_{R1}}{\partial \phi} - 2\Omega v_{\phi 1} = -\frac{\partial \Phi_1}{\partial R} \tag{3.16}$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \left[\frac{d(\Omega R)}{dR} + \Omega\right] v_{R1} + \Omega \frac{\partial v_{\phi 1}}{\partial \phi} = -\frac{1}{R}\frac{\partial \Phi_1}{\partial \phi} \tag{3.17}$$

using Y_1=$Y_a(R)e^{i(m\phi - \omega t)}$ as a solution of above equation's. Where m is the azimuthal wave number and w is the temporal frequency. Adding these solution's in above equation's (3.15), (3.16), (3.17) and solving for v_{Ra}, $v_{\phi a}$ for gas and particle disc,

$$-i(\omega - m\Omega)\Sigma_{ag/s} + \frac{1}{R}\frac{d(R v_{Ra}\Sigma_0)}{dR} + \frac{im\Sigma_0 v_{\phi a}}{R} = 0 \tag{3.18}$$

$$v_{Ra}(R) = \frac{i}{D_g}\left[(\omega - m\Omega)\frac{d\Phi_a}{dR} - \frac{2m\Omega\Phi_a}{R}\right] \tag{3.19}$$

$$v_{\phi a(R)} = -\frac{1}{D_g}\left[2B\frac{d\Phi_a}{dR} + \frac{m(\omega - \Omega m)\Phi_a}{R}\right] \tag{3.20}$$

$$v_{Ra}(R) = \frac{i}{D_s}\left[(\omega - m\Omega)\frac{d\Phi_a}{dR} - \frac{2m\Omega\Phi_a}{R}\right] \tag{3.21}$$

$$v_{\phi a(R)} = -\frac{1}{D_s}\left[\frac{\kappa^2}{2\Omega}\frac{d\Phi_a}{dR} + \frac{m(\omega - \Omega m)\Phi_a}{R}\right] \tag{3.22}$$

$$D = \kappa^2 - (\omega - m\Omega)^2 \tag{3.23}$$

The above equations discover v_{Ra}, $v_{\phi a}$, Σ_a in terms of ϕ_a.
Under the WKB approximation ,if the radial wave number k is larger (i.e. $|kR| \gg m$), then the perturbed density is of the form $\Sigma_a(R) = \Sigma_a(R)exp(i\int^R dRk(R))$ and perturbed potential is of the form $\Phi_a = F(R)e^{i\int^R kdR}$ (Binney and Tremaine 2008), which gives for gas disc:

$$\Phi_{ag} = \frac{-2\pi G\Sigma_{ag}}{|k|} \tag{3.24}$$

Particle disc:

$$\Phi_{as} = \frac{-2\pi G exp(-|k|b)}{|k|}\Sigma_{as} \tag{3.25}$$

Since the potential is scalar both these add up and the combined perturbed potential is

$$\Phi_a = \frac{-2\pi G\Sigma_{ag}}{|k|} - \frac{2\pi G exp(-|k|b)\Sigma_{as}}{|k|} \tag{3.26}$$

The term with $\frac{d}{dR}$ in R.H.S of equations (3.18), (3.19), (3.20), (3.21), (3.22) is $O(|kR|/m)$ larger than the others and thus, neglecting the latter. we get,

Gas disc:

$$\Sigma_{ag} = -\frac{k^2\Sigma_{0g}\Phi_a}{D_g} \tag{3.27}$$

$$v_{Ra} = -\frac{(\omega - m\Omega)}{D_g}k\Phi_a \tag{3.28}$$

$$v_{\phi a} = -\frac{2iB}{D_g}k\Phi_a \tag{3.29}$$

Particle disc:

$$\Sigma_{as} = -\frac{k^2\Sigma_{0s}\Phi_a}{D_s} \tag{3.30}$$

$$v_{Ra} = -\frac{(\omega - m\Omega)}{D_s}k\Phi_a \tag{3.31}$$

$$v_{\phi a} = -\frac{i\kappa^2}{2\Omega D_s}k\Phi_a \tag{3.32}$$

Substitute (3.27), (3.30) in equation (3.26) and gives WKB Dispersion Relation,

$$D_s D_g = 2\pi G|k|exp(-|k|b)\Big[D_g\Sigma_{0s} + D_s\Sigma_{0g}exp(|k|b)\Big] \tag{3.33}$$

3.4 *Stability Analysis*

- ***m=0 perturbation:***

  We have $D = D_g = D_s = \kappa^2 - \omega^2$, and equation (3.33) becomes,

  $$D = 2\pi G|k|\Sigma_{0s}[exp(-|k|b) + n] \tag{3.34}$$

  suppose that $\Sigma_{0g}/\Sigma_{0s} \cong n$, where n is constant and put the value of D in equation (3.34), we get

  $$\omega^2 = \kappa^2 - 2\pi G|k|\Sigma_{0s}[exp(-|k|b) + n] \tag{3.35}$$

If ω^2 is negative, then the disc are unstable where as ω^2 positive implies that the discs are stable. So, the line of neutral stability is

$$\kappa^2 = 2\pi G|k|\Sigma_{0s}[exp(-|k|b) + n] \tag{3.36}$$

The disk is stable when solutions for $|k| > 0$ exists, implying

$$|k| = \frac{\kappa^2 eb - 2\Pi G\Sigma_{0s}}{2\pi G\Sigma_{0s} neb} > 0 \tag{3.37}$$

Solving this we get that the axisymmetric stability requires,

$$Q = \frac{\kappa^2 eb}{2\Pi G\Sigma_{0s}} > 1 \tag{3.38}$$

where, Q is the Toomre's Q parameter for softened gravity disk (Binney & Tremaine, 2008). The line of neutral stability defined by eqn (3.36) is drawn in Fig: (3.1) in terms of dimensionless ratios λ/λ_c and Q. Where $\lambda = \frac{2\pi}{k}$ and $\lambda_c = \frac{4\pi^2 G\Sigma}{\kappa^2}$.

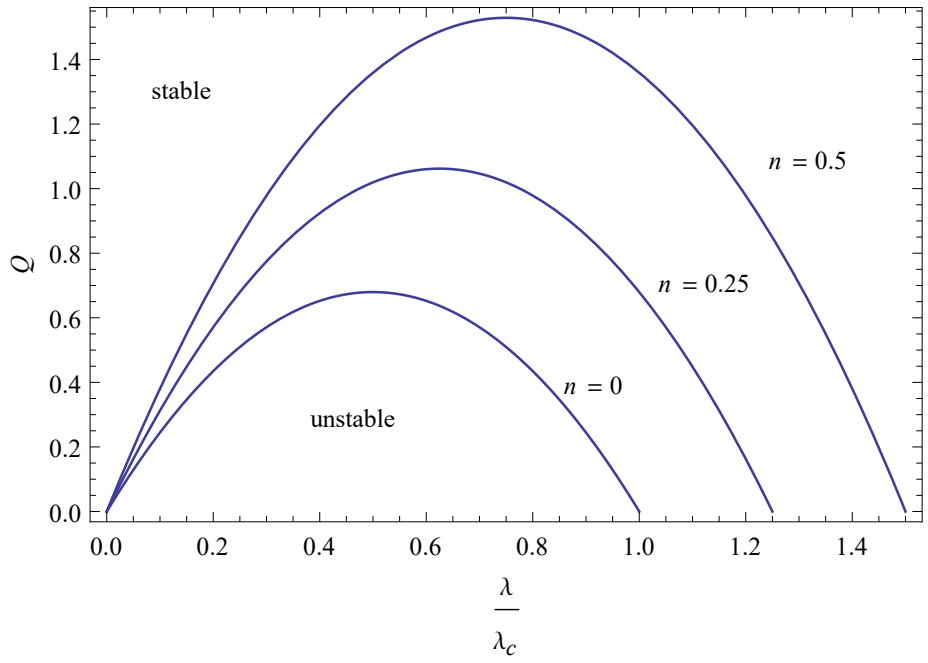

Figure 3.1: Combined plot for constant values of n. The plot drawn between ratios λ/λ_c and Q by using the dimensionless equation (3.36).

3.5 *m=1, Slow modes*

We have $D = D_g = D_s = \kappa^2 - (\Omega - \omega)^2$,

$$D = 2\pi G|k|\Sigma_{0s}[exp(-|k|b) + n] \tag{3.39}$$

putting the value of D in eqn (3.39) we get,

$$\kappa^2 - (\Omega - \omega)^2 = 2\pi G|k|\Sigma_{0s}[\exp(-|k|b) + n] \tag{3.40}$$

From here on we will use a kuzmin discs profile for our unperturbed surface density of stars. It being central concentrated works as a reasonable approximation. Moreover, earlier investigations of slow modes (Tremaine,2001; Sridhar and Saini, 2010) have explored modes in Kuzmin discs. The surface density for Kuzmin disc is given by

$$\Sigma_{0s} = \frac{aM_d}{2\pi(R^2 + a^2)^{3/2}} \tag{3.41}$$

Now, define $\widetilde{w} = (\Omega - \kappa)$ then the equation (3.40) becomes,

$$(\omega - \widetilde{w}) = \frac{\pi G|k|\Sigma_{0s}}{\Omega}[\exp(-|k|b) + n] \tag{3.42}$$

where $\widetilde{w}$ is a precession rate. Which for Kuzmin disc profile is given by (Gulati et. al 2012).

$$\widetilde{w} = -\frac{3GM_d a^2}{2\Omega(R)(R^2 + a^2)^{5/2}} \tag{3.43}$$

Putting the values of $\widetilde{w}, \Sigma_{0s}, \omega$ by using (Binney and Tremine (2008)). After solving and put the different values of n and convert into dimensionless form of above equation as

$$w = \left(-\frac{3}{2}\frac{x^{3/2}}{(x^2+1)^{5/2}} + \frac{|k|x^{3/2}}{2(x^2+1)^{3/2}}[\exp(-|k|b) + n]\right) \tag{3.44}$$

3.6 *Phase Plots*

To have an idea about the nature of modes we make phase plots of dispersion relation derived in equation (3.44). The different plots made are as follows:

- For **n=0** , Contour plots are for the case when there is no gas content. This is same as studied by Gulati et.al 2012 and Tremaine 2001 and that is shown in fig(3.2).
- For **n=0.25**, which means gas corresponds to 25% of mass, are shown in fig(3.3).
- For **n=0.5**, the case of equal gas and star mass, are shown in fig(3.4).

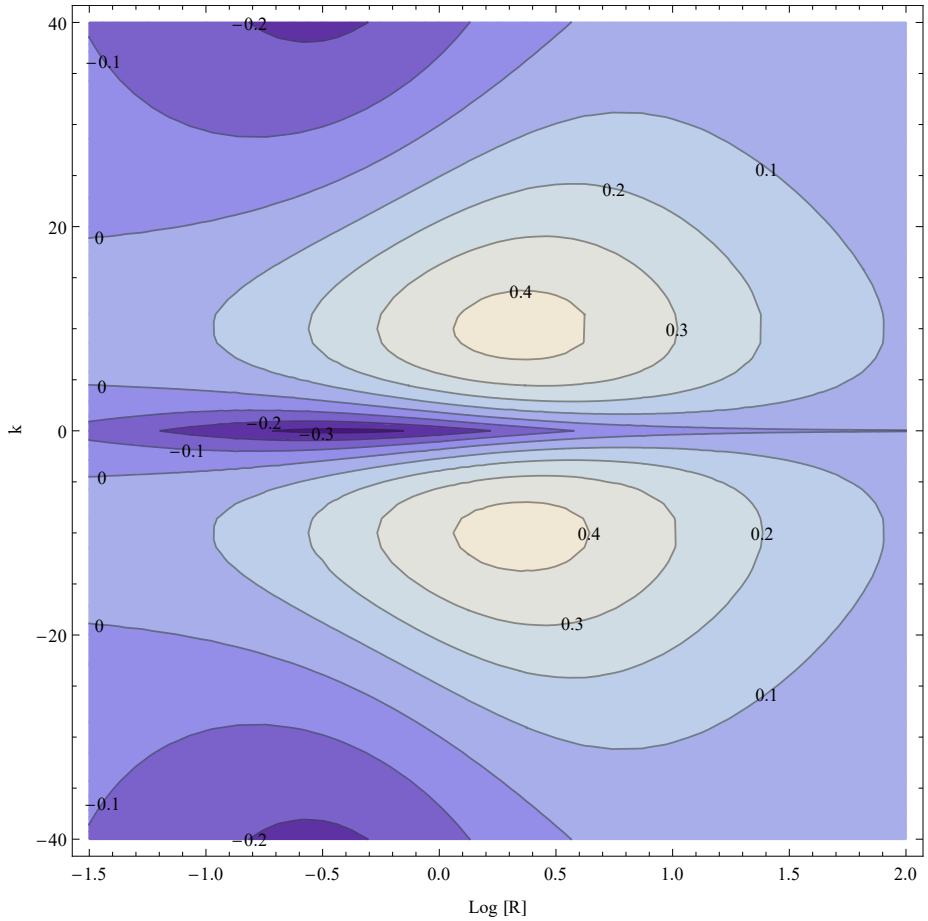

Figure 3.2: Contour plot for constant values of ω. Lighter shades are positive ω and darker for negative ω and this plot is drawn at n=0 by using the dimensionless equation (3.44).

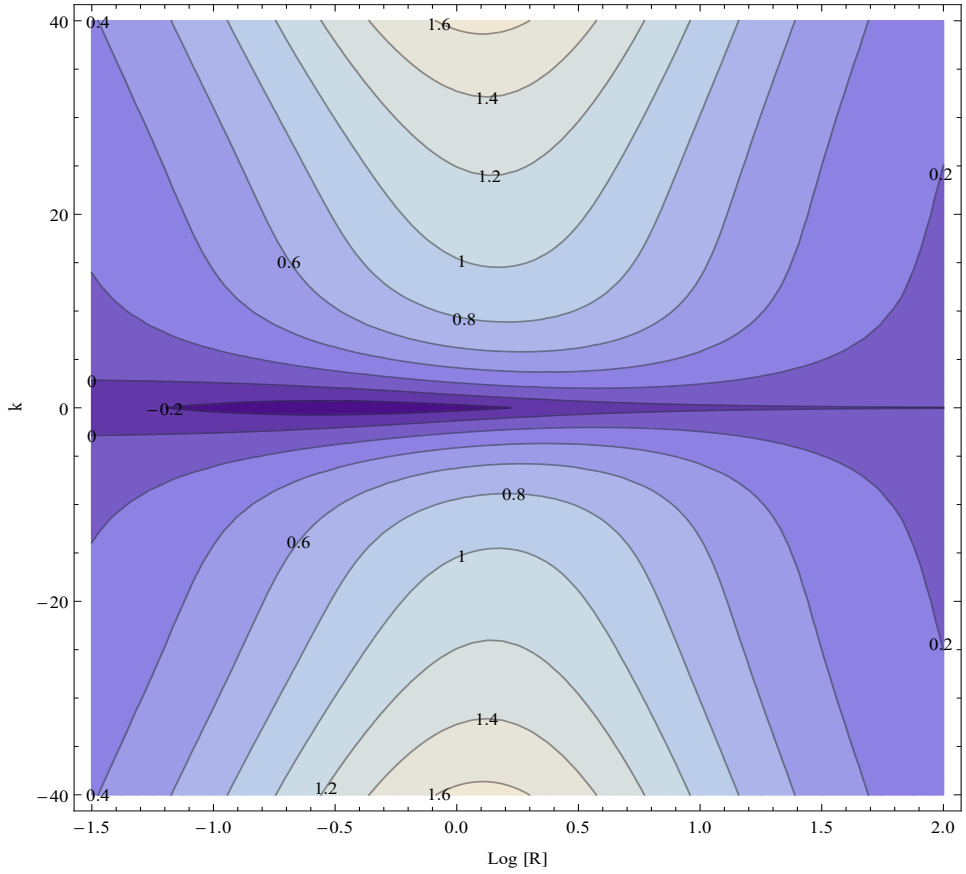

Figure 3.3: Contour plot for constant values of ω. Lighter shades are positive ω and darker for negative ω and this plot is drawn at n=0.25 by using the dimensionless equation (3.44).

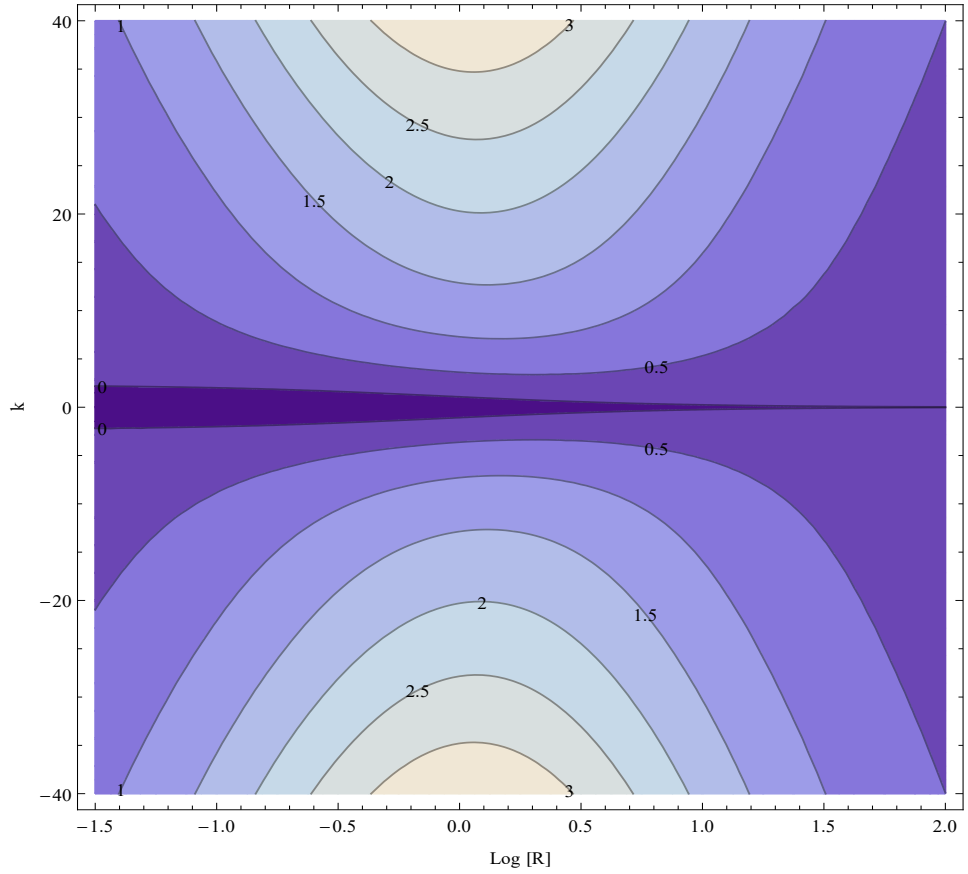

Figure 3.4: Contour plot for constant values of ω. Lighter shades are positive ω and darker for negative ω and this plot is drawn at n=0.5 by using the dimensionless equation (3.44).

3.6.1 Types of contour

- At k=0, closed contours situated on every side.
- The closed contours are appear on both sides of k=0. When the disc rotates in the direction of growing azimuthal angle, then $k > 0$ is for trailing waves and $k < 0$ is for leading waves.
- The open contours (dark shades) for both negative and positive k in the direction of the highest value of $|k|$.

3.7 *Results and Discussion*

We give our model to explain the eccentric modes, appropriate to the system which have both particle and gas disc coupled together. Now, we derive a WKB dispersion relation in which the star and gas disc are considered as a coupled system. The perturbation occurs in both star and gas disc. We study, the nature of eigenvalues, the disc are found to be stable if they satisfy the Toomre's stability criteria. We give the plot for stability region for different amount of gas content in the disc. We conclude that layer is the gas content leads to instabilities in the disc. We also give the phase plots of the dispersion relation discussing different types of wave that exists.

CHAPTER 4

Conclusion and future directions

In this thesis, we discuss the mathematical modeling of galactic systems. Behind this, the basic motivation was the counter-intuitive observations that we see at the center of galaxies. In last years, many works have been done to explain these observations but each theory has there own drawback which we have talk about in detailed at various steps in this thesis. In the present work, we focus to expand the theory further to model the actual systems of galaxies which consist of coupled both gas and stars.

- Chapter 1, consists of a review of galaxy and its types and details about the observation of center of galaxy.
- Chapter 2, is a review of the literature on the mathematical modeling of center of galaxies in last few years.
- Chapter 3, give our model to explain the eccentric modes, appropriate to the system which have both particle and gas disc coupled together. Now, we derive a WKB dispersion relation in which the star and gas disc are considered as a coupled system. The perturbation occurs in both star and gas disc. We study, the nature of eigenvalues, the disc are found to be stable if they satisfy the Toomre's stability criteria. We give the plot for stability region for different amount of gas content in the disc. We conclude that layer is the gas content leads to instabilities in the disc. We also give the phase plots of the dispersion relation discussing different types of wave that exists.

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