ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SUMS WITH SEMI-CONVEX COEFFICIENTS

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CERTIFICATE

Certified that the dissertation entitled, **"ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SUMS WITH SEMI-CONVEX COEFFICIENTS"**, which is being submitted by **Miss Gagandeep Sharma** (Roll No. 300903003), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises candidate's own research work carried out under the supervision and guidance of Dr. S.S.Bhatia, Professor, SMCA, Thapar University, Patiala during the period from January 2011 to June 2011. The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University / Institute for the award of any degree.

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ABSTRACT

The present dissertation entitled, **"On L^1-Convergence of Certain Trigonometric Sums With Semi-Convex Coefficients"** embodies a brief account of investigations carried out by various authors and by me on L^1-convergence of trigonometric series under the supervision of Dr. S.S.Bhatia, Professor, School of Mathematics and Computer Applications, Thapar University, Patiala.

The aim of this work is, to study some classes of sequences formed by coefficients of real trigonometric series and to obtain results on L^1-convergence of Trigonometric Series with special coefficients.

The work reported in this dissertation has been divided into four chapters . The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in sequel, we have presented some known results interrelated to our results alongwith a brief plan of our results presented in the subsequent chapters. The purpose of chapter II is to study the L^1-convergence of modified sine sums introduced by Kulwinder Kaur [2003] with semi-convex coefficients. In chapter III, we have studied the L^1-convergence of the sine and cosine trigonometric sums introduced by Xhevat Z.Krasniqi [2009] with semi-convex class of coefficient sequences. We have also studied the necessary and sufficient conditions for the L^1-convergence of sine and cosine trigonometric series in this chapter.

In chapter IV, we have obtained the L^1-convergence of the sine and cosine trigonometric sums given by Ram and Kumari [1989] with semi-convex coefficients. Further in this chapter, we have also obtained the necessary and sufficient conditions for the L^1-convergence of sine and cosine trigonometric series.

Towards the end, references of various publications cited in the present dissertation have been reported.

DEDICATED TO MY PARENTS AND GOD

CONTENTS

CHAPTER I

INTRODUCTION

1.1 The present dissertation comprises the results associated with the various authors "On L^1-Convergence of certain Trigonometric Series with Special Coefficients". It is well known that if a trigonometric series converges in L^1 -metric to a function $f \in L^1(T)$, then it is the Fourier series of the function f. Riesz {[2],Vol.II,Ch.VIII§22} gave a counter example to show that in L^1-metric, the converse of the above said result does not hold good. This motivated various authors to study the L^1-convergence of trigonometric series with special coefficients.

L^1-convergence of trigonometric series with special coefficients have been studied by number of authors. The work on this topic was initiated by Young W.H.[29] and Kolmogorov A.N.[12] by taking classes of convex sequences ($\Delta^2 a_n \geq 0$) and quasi-convex sequences ($\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty$) respectively. Teljakovskiĭ S.A.[28] yet considered another class S introduced by Sidon[22] for L^1-convergence of trigonometric series. The results obtained by these authors have further been generalized and extended by Hardy G.H. and Littlewood J.E.[6], Kano T.[8], Garrett J.W. and Stanojevic C.V.([4],[5]), Ram B.([16],[18]), Bala R. and Ram B.[1], Kaur K.,Bhatia S.S. and Ram B.[10], Kaur K.([9], Xhevat Z.Krasniqi([13]) by considering various generalizations of classes of sequences mentioned above for one-dimensional trigonometric series.

During their investigations, some authors introduced modified trigonometric sums, as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in L^1-metric to the sum of trigonometric series whereas the classical series itself may not. In this concern, various authors like Rees,C.S. and Stanojevic C.V.[20], Kumari S. and Ram B.[14], Ram B. and Kumari S.[19], Hooda N., Ram B. and Bhatia S.S[7], Kaur K., Bhatia S.S and Ram B.[11] have introduced various new modified trigonometric sums and have studied their L^1- convergence under various classes of coefficient sequences. Bhatia S.S and

Ram B.[3] generalized the results of Kumari S. and Ram B.[14] and have obtained their results as corollary.

In the present dissertation , some of the results of above mentioned authors have been studied by me.

To provide sufficient background for later chapters, a summary of basic concepts,techniques and a brief chapter wise resume of the results contained in the dissertation has been given in the introductory chapter. However, some of the definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

1.2 DEFINITIONS AND NOTATIONS

Let $\{a_n\}$ be a sequence. Then we write

$$\Delta a_n = a_n - a_{n+1}$$

$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Abel's transformation([2], Vol.I, p.1): If $a_0, a_1, a_2, \ldots\ldots, v_0, v_1, \ldots\ldots v_n, \ldots.$ are any real numbers, let us assume that

$$V_n = v_0 + v_1 + \ldots.. + v_n$$

Then for any values of m and n, we find that

$$\sum_{k=m}^{n} a_k v_k = \sum_{k=m}^{n-1} \Delta a_k V_k + a_n V_n - a_m V_{m-1}$$

(under the condition that if m = 0, $V_{-1} = 0$).

Convex sequence: A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$.

Quasi-Convex sequence([2], Vol.II, p.202): A sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty.$$

Semi-convex sequence [8]: A null sequence $\{a_n\}$ is said to be semi-convex if

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty \qquad (a_0 = 0)$$

Generalized semi-convex sequence [10]: A null sequence $\{a_n\}$ is said to be generalized semi-convex, if

$$\sum_{n=1}^{\infty} n^{\alpha}|\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n| < \infty \qquad (a_0 = 0)$$

For $\alpha = 1$, this class reduces to the semi-convex sequence as in [23].

Sequence of bounded variation([2], Vol.I, p.3): A sequence $\{a_n\}$ is said to be of bounded variation if

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty.$$

We denote the class of null-sequences of bounded variation by **BV**.

Quasi-monotone sequence([21],[26]): A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone if $a_{n+1} \leq a_n(1 + \frac{\alpha}{n})$ for some $\alpha > 0$ and all $n > n_0(\alpha)$. An equivalent definition is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

O-o Relations([2]): Using the notation now accepted in mathematical literature we write

Then $\qquad u_n = \mathrm{o}(v_n)$

If $\qquad v_n \geq 0$ (n = 0,1,2,.....) and $\frac{u_n}{v_n} \to 0$ as $n \to \infty$

If $\qquad \frac{u_n}{v_n}$ is bounded, we write

$$u_n = \mathrm{O}(v_n)$$

If two positive constants A and B exist for which at sufficiently large n

$$A \leq \frac{u_n}{v_n} \leq B,$$

then we write $$u_n \sim v_n,$$

and finally $$u_n \approx v_n,$$

will signify $$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$$

Fourier series: A trigonometric series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

the coefficients a_n and b_n of which are determined by the Fourier formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \text{ dx}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \text{ dx}$$

derived from the function *f(x)*, is called the Fourier series of the function *f(x)*. We then write

$$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Dirichlet kernel([2],Vol.I,p.85):

Let $$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \ldots\ldots\ldots\ldots + \cos nx$$

then

$$2\sin\frac{x}{2}D_n(x) = \sin\frac{x}{2} + 2\sin\frac{x}{2}cosx + \ldots\ldots\ldots\ldots + 2\sin\frac{x}{2}\cos nx$$

$$= \sin\left(n+\frac{1}{2}\right)x,$$

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

This expression is known as Dirichlet's kernel. Moreover,

$$\tilde{D}_n(x) = \sin x + \sin 2x + ... + \sin nx$$

$$= \frac{\cos\dfrac{x}{2} - \cos\left(n + \dfrac{1}{2}\right)x}{2\sin\dfrac{x}{2}}$$

is called the kernel conjugate to the Dirichlet kernel.

If $x \neq 0(\mod 2\pi)$, then

$$|D_n(x)| \leq \frac{\pi}{2x}, \quad for \quad 0 < |x| \leq \pi$$

and

$$|\tilde{D}_n(x)| \leq \frac{\pi}{x}, \quad for \quad 0 < |x| \leq \pi$$

Also, we shall use the uniform estimate

$$|D_n(x)| \leq n + \frac{1}{2}, \quad for \quad any \quad x$$

Fejér kernel ([2], [30]). The Fejér kernel $K_n(x)$ is defined as

$$\begin{aligned} K_n(x) &= \frac{1}{n+1}\sum_{j=0}^{n} D_j(x) \\ &= \frac{1}{n+1}\sum_{j=0}^{n} \frac{\sin\left(j+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}. \end{aligned}$$

Using $|D_n(x)| \leq n+1$, it follows that $K_n(x) \leq n+1$.

It has the properties

$$\begin{aligned} (i) &\quad K_n(x) \geq 0, \\ (ii) &\quad \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1. \end{aligned}$$

The conjugate Fejér kernel is defined as

$$\tilde{K}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} \tilde{D}_j(x)$$

We have

$$\tilde{K}_n(x) > 0 \quad for \quad 0 < x < \pi, \quad n = 1, 2, 3, ...$$

and

$$|\tilde{K}_n(x)| < \frac{1}{2}n.$$

The Class S ([22], [28]). Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \sin nx$$

be the cosine and sine series respectively. A sequence $\{a_n\}$ belongs to class S, if $a_n \to 0 \;\; as \;\; n \to \infty$ and there exists sequence $\{A_n\}$ such that

(i) $A_n \downarrow 0$, $n \to \infty$.

(ii) $\sum_{n=0}^{\infty} A_n < \infty$,

(iii) $|\Delta a_n| \leq A_n$, for all n.

Further, letting $A_n = \sum_{k=n}^{\infty} |\triangle^2 a_k|$, we observe that every quasi-convex null sequence satisfies the class S.

Singh and Sharma [24] gave a definition of one more class namely S* in the following manner:

The class S* [24]. A sequence $\{a_n\}$ of numbers is said to belongs to class S*, if $a_n \to 0 \;\; as \;\; n \to \infty$ and there exists a sequence $\{A_n\}$ such that

(i) $\{A_n\}$ is quasi-monotone,

(ii) $\sum_{n=0}^{\infty} A_n < \infty$ and

(iii) $|\triangle a_n| \leq A_n$, for all n.

However, Leindler [15] proved that class S and class S* are equivalent.

The class $\mathcal{C}$ of Garrett and Stanojević [5]. A null sequence $\{a_n\}$ belongs to class $\mathcal{C}$ if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, independent of n, and such that

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \triangle a_k D_k(x) \right| dx < \epsilon, \quad for \ \ all \ \ n \geq 0.$$

The Class R [8]. A null sequence $\{a_n\}$ belongs to the class R, if

$$\sum_{k=1}^{\infty} k^2 \left| \triangle^2 \left(\frac{a_k}{k} \right) \right| < \infty$$

1.3 The following results about the behavior of cosine and sine series are known:

Theorem I ([2], [12], [29]). If $\{a_k\}$ is a quasi-convex null sequence, then

(1.3.1) $$f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^1[0, \pi].$$

Theorem II([2], [27]) . If $\{a_k\}$ is a quasi-convex null sequence, then

(1.3.2) $$\sum_{k=1}^{\infty} a_k \sin kx$$

is a Fourier series if and only if $\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty$.

In 1968, Kano [8] generalized Theorem I and Theorem II in the following form:

Theorem III. If $\{a_k\}$ is a null sequence such that

(1.3.3) $$\sum_{k=1}^{\infty} k^2 \left| \triangle^2 \left(\frac{a_k}{k} \right) \right| < \infty,$$

then (1.3.1) and (1.3.2) are the Fourier series, or equivalently they represent integrable functions.

We observe that Theorem I and Theorem III provide just only the sufficient conditions for the integrability of cosine series. Rees and Stanojević [20] showed that

$\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$ is a necessary and sufficient condition for $L^1[0, \pi]$ integrability but for a different type of cosine sums. They proved the following result:

Theorem IV. Let $b_k = \frac{a_k}{k} \downarrow 0$. Then

$$t(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{b_k}{2} + \left(\sum_{j=k}^{n} b_j \right) \cos kx \right]$$

exists for $x \in (0, \pi]$ and $t \in L^1[0, \pi]$ if and only if $\sum_{k=1}^{\infty} b_k < \infty$.

Ram [17] showed that the condition S is sufficient for the integrability of Rees-Stanojević sums [20]

$$t_n(x) = \frac{1}{2} \sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_j) \cos kx \tag{1.3.4}$$

He proved the following theorem:

Theorem V. Let the sequence $\{a_k\}$ satisfy the condition S. Then $t(x) = \lim_{n\to\infty} t_n(x)$ exists for $x \in (0, \pi]$, and

$$\int_0^{\pi} |t(x)| dx \le C \sum_{k=0}^{\infty} A_k.$$

The above theorem has further been studied by Ram [18], under a condition where the monotonicity of the sequence in the definition of the class S is replaced by quasi-monotonicity.

Consider the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.3.5}$$

Let the partial sums of (1.3.5) is denoted by $S_n^c(x)$ and $f(x) = \lim_{n\to\infty} S_n^c(x)$. Denote the class of sequences of Fourier coefficients $\{a_k\}$ by $\mathbf{F}$. There are subclasses of $\mathbf{F}$ for which $a_n \log n = o(1), \quad n \to \infty$ is a necessary and sufficient condition for $||S_n^c - f||_{L^1} = o(1), \quad n \to \infty$.

A subclass **G** of **F** is called a class of L^1-convergence if $||S_n^c - f||_{L^1} = o(1), \quad n \to \infty$ if and only if $a_n \log n = o(1), \quad n \to \infty$.

There are three classical examples of classes of L^1-convergence. The first one is due to Young [29], where **G** is defined to be the class of all convex sequences $\{a_k\}$. The second one is the class of all quasi-convex sequences $\{a_k\}$, introduced by Kolmogorov [12]. The third example is class S due to Teljakovskiǐ [28]. We have already pointed out that S$\subset$ **BV**. These classical classes have been extended by various authors. We present now a brief summary of the results obtained by various authors in this direction.

Concerning the L^1-convergence of the cosine series, we have the following classical result of Kolmogorov [12].

Theorem VI. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1.3.5) in the metric space L, it is necessary and sufficient that $a_k \log k = o(1), \quad k \to \infty$.

The case, in which the sequence $\{a_k\}$ is convex, of this theorem was established by Young [29].

Generalizing the above classical result, Teljakovskiǐ [28] proved the following result:

Theorem VII. If the coefficient sequence $\{a_k\}$ of the cosine series (1.3.5) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (1.3.5) is $a_k \log k = o(1), \quad k \to \infty$.

Rees and Stanojević [20] introduced modified cosine sums (1.3.4) and obtained an analogue of Theorem VII for these sums. These modified cosine sums approximate their limits better than the classical cosine series as they converge in L^1-metric to the sum of the cosine series whereas the classical cosine series itself may not. They proved the following result:

Theorem VIII. Let f be the sum of the cosine series (1.3.5). Then $t_n(x)$ converges to f in L^1-metric if and only if $\{a_k\}$ belongs to the class $\mathcal{C}$.

Ram ([16]) proved the following result on L^1-convergence of Rees-Stanojević sums (1.3.4).

Theorem IX. If (1.3.5) belongs to class S. Then

$$||f - t_n||_{L^1} = o(1), \quad n \to \infty.$$

Theorem VII of Teljakovskiĭ ([28]) follows as corollary of this theorem.

Further, Ram and Kumari ([14], [19]) introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left(\frac{a_j}{j} \right) k \cos kx \tag{1.3.6}$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left(\frac{a_j}{j} \right) k \sin kx \tag{1.3.7}$$

and studied their L^1-convergence under the condition that the coefficients belong to the classes R and S. They also deduced the results about L^1-convergence of cosine and sine series. Their results state as below:

Theorem X. Let $\{a_n\}$ belong to the class S. If $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then

$$||f - f_n||_{L^1} = o(1) \quad n \to \infty.$$

Theorem XI. Let $\{a_n\}$ belong to the class R. If $u_n(x)$ represents $f_n(x)$ and $g_n(x)$, then

$$||f - u_n||_{L^1} = o(1) \quad n \to \infty.$$

Singh and Sharma [24] proved the above theorem by replacing the monotonicity of sequence $\{A_n\}$ in the definition of class S by quasi-monotonocity of $\{A_n\}$. Their result reads as:

Theorem XII. Let $a_n \in$ S*, then $f_n(x)$ converges to $f(x)$ in L^1-metric.

Later, Hooda and Ram [7] have proved the following theorem:

Theorem XIII. Let $\{a_n\}$ belong to the class S*. Then

$$||f - f_n||_{L^1} = o(1), \quad n \to \infty.$$

Further, Kaur.K.[9] introduced new modified sine sum $K_n(x)$ defined as

$$(1.3.8) \qquad K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx$$

and studied their L^1-convergence with semi-convex coefficients and deduced the result state as below:

Theorem XIV. Let $\{a_n\}$ be a semi-convex null sequence, then $K_n(x)$ converges to *f(x)* in L^1-metric.

Theorem XV. Let a_n be a semi-convex null sequence,If $\lim_{n\to\infty} |a_n| \log n = 0$, then

$$||f - K_n||_{L}{}^1 = \mathrm{o}(1), \ \mathrm{n} \to \infty.$$

In Chapter II, we have studied the Theorems XIV, XV, for the cosine series with semi-convex coefficients by using the modified sine sums (1.3.8) of Kaur.K. [9].

Recently, Xhevat Z.Krasniqi ([13]) introduced new modified cosine and sine sums as

$$(1.3.9) \qquad H_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1} - a_{j+1})\cos jx\right]$$

$$(1.3.10) \qquad G_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta[(a_{j-1} - a_{j+1})\sin jx]$$

and studied their L^1-convergence with semi-convex coefficients and deduced the result state as below:

Theorem XVI. Let $\{a_n\}$ be a semi-convex null sequence, then $H_n(x)$ converges to *g(x)* in L^1-metric.

Theorem XVII. Let $\{a_n\}$ be a semi-convex null sequence,If $\lim_{n\to\infty} |a_{n-1}| \log n = 0$, then $||g - H_n||_{L}{}^1 = \mathrm{o}(1)$, $\mathrm{n} \to \infty$.

In Chapter III, we have studied the Theorems XVI, XVII for the sine series with semi-convex coefficients by using the modified cosine sums (1.3.9) of Xhevat Z.Krasniqi ([13]).

Theorem XVIII. Let $\{a_n\}$ be a semi-convex null sequence, then $f_n(x)$ converges to *f(x)* in L^1-metric.

Theorem XIX. Let $\{a_n\}$ be a semi-convex null sequence,If $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then $||f - f_n||_L{}^1 = \text{o}(1)$, n$\to \infty$.

In Chapter IV, we have obtained the Theorem XVIII, XIX, for the cosine series with semi-convex coefficients by using the modified cosine sums (1.3.6) of Ram and Kumari ([14],[19]).

CHAPTER II

On L^1-Convergence of Modified Sine sums

2.1 Introduction

Consider the cosine trigonometric series

(2.1.1) $$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

with partial sums defined by

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

Let $$f(x) = \lim_{n\to\infty} S_n^c(x)$$

Concerning the L^1-convergence of cosine series (2.1.1) Kolmogorov ([12]) proved the following theorem.

Theorem A: If $\{a_n\}$ is a quasi-convex null sequence, then for the L^1-convergence of the cosine series (2.1.1), it is necessary and sufficient that $\lim\limits_{n\to\infty} a_n \log n = 0$.

The case in which the sequence $\{a_n\}$ is convex, of this theorem was established by Young([29]). That is why, sometimes this **Theorem A** is known as **Young-Kolmogorov Theorem**.

Definition([8]): A sequence $\{a_n\}$ is said to be semi-convex if $a_n \to 0$ as $n \to \infty$, and

(2.1.2) $$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \quad (a_0 = 0)$$

$$\text{where } \Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$$

The results reported in this chapter have been taken from : Kulwinder Kaur, SOLSTICE :Electrical Journal of Mathematics ξ Geography, 14(1)(2003),1-6.

It may be remarked here that every quasi-convex null sequence is semi-convex.

Bala R. and Ram B.([1]) have proved that **Theorem A** holds true for the cosine series with semi-convex null coefficients in the following form.

Theorem B: If $\{a_n\}$ is a semi-convex null sequence, then for the L^1-convergence of the cosine series (2.1.1),it is necessary and sufficient that $\lim_{n\to\infty} a_{n-1}\log n = o(1)$

Kaur K.([9]) introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx$$

The aim of this Chapter is to study the L^1-convergence of this modified sine sum with semi-convex coefficients and to study the above mentioned result of Bala R. and Ram B. as a corollary.

2.2 Main Result

Theorem 2.2.1: If $\{a_n\}$ is a semi-convex null sequence, then $K_n(x)$ converges to *f(x)* in L^1-norm.

Proof: We have

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n} a_k(2\cos kx \sin x)$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n} a_k(\sin(k+1)x - \sin(k-1)x)$$

$$= \frac{1}{2\sin x}\left[\sum_{k=1}^{n} a_k(\sin(k+1)x - \sum_{k=1}^{n} a_k \sin(k-1)x)\right]$$

$$= \frac{1}{2\sin x}[a_1\sin 2x + a_2\sin 3x + ... + a_{n-1}\sin nx + a_n \sin(n+1)x$$

$$-a_1\sin 0x - a_2\sin x - ... - a_n\sin(n-1)x] - a_{n+1}\sin nx + a_{n+1}\sin nx$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(a_{k-1} - a_{k+1})\sin kx\right] + a_n\frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(a_{k-1} - a_k + a_k - a_{k+1})\sin kx\right] + a_n\frac{\sin(n+1)x}{2\sin x}$$

$$+a_{n+1}\frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta a_k + \Delta a_{k-1})\sin kx\right] + a_n\frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

Applying Abel's transformation, we have

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n-1}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x) + (\Delta a_n + \Delta a_{n-1})\tilde{D}_n(x)\right]$$

$$+a_n\frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x) + (\Delta a_{n+1} + \Delta a_n)\tilde{D}_n(x)\right]$$

$$+a_n\frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

$$S_n^c(x) = \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x) + (a_n - a_{n+2})\tilde{D}_n(x)\right]$$

$$+a_n\frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} \tag{2.2.1}$$

As $\{a_n\}$ is null sequence, $\tilde{D}_n(x)$ is bounded

Therefore second, third and fourth terms tend to zero.

$$f(x) = \lim_{n\to\infty} S_n^c(x)$$

$$= \frac{1}{2\sin x}\sum_{k=0}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)$$

Now, $$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx$$

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}[\Delta a_{k-1} + \Delta a_k + \Delta a_{k+1} + \Delta a_{k+2} + ... + \Delta a_{n-1}$$

$$-\Delta a_{k+1} - \Delta a_{k+2} - .... - \Delta a_{n-1} - \Delta a_n - \Delta a_{n+1}]\sin kx$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n}[\Delta a_{k-1} + \Delta a_k - \Delta a_n - \Delta a_{n+1}]\sin kx$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n}[a_{k-1} - a_k + a_k - a_{k+1} - a_n + a_{n+1} - a_{n+1} + a_{n+2}]\sin kx$$

$$= \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(a_{k-1} - a_{k+1})\sin kx - \sum_{k=1}^{n}(a_n - a_{n+2})\sin kx\right]$$

$$= \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(a_{k-1} - a_{k+1})\sin kx - (a_n - a_{n+2})\tilde{D}_n(x)\right]$$

Applying Abel's transformation

$$= \frac{1}{2\sin x}\left[\sum_{k=1}^{n-1}(\Delta a_{k-1} - \Delta a_{k+1})\tilde{D}_k(x) + (a_{n-1} - a_{n+1})\tilde{D}_n(x) - (a_n - a_{n+2})\tilde{D}_n(x)\right]$$

$$= \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(\Delta a_{k-1} - \Delta a_{k+1})\tilde{D}_k(x) + (a_n - a_{n+2})\tilde{D}_n(x) - (a_n - a_{n+2})\tilde{D}_n(x)\right]$$

$$= \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(\Delta a_{k-1} - \Delta a_{k+1})\tilde{D}_k(x)\right]$$

$$= \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(\Delta a_{k-1} - \Delta a_k + \Delta a_k - \Delta a_{k+1})\tilde{D}_k(x)\right]$$

$$K_n(x) = \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right] \qquad (2.2.2)$$

Now,

$$f(x) - K_n(x) = \frac{1}{2\sin x}\left[\sum_{k=1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right] - \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right]$$

$$= \frac{1}{2\sin x}\left[\sum_{k=n+1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right]$$

Thus, we have

$$\int_{-\pi}^{\pi} |f(x) - K_n(x)|dx = \int_{-\pi}^{\pi} \left| \left(\sum_{k=n+1}^{\infty} (\Delta^2 a_k + \Delta^2 a_{k-1}) \frac{\tilde{D}_k(x)}{2\sin x} \right) \right| dx$$

Now,

$$\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\tilde{D}_k(x)}{2\sin x} \right| dx$$

$$= \sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\cos\frac{x}{2} - \cos(k+\frac{1}{2})x}{2\sin\frac{x}{2}\sin x} \right| dx$$

$$= \sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{-2\sin\frac{(x+kx)}{2}\sin\frac{kx}{2}}{2\sin\frac{x}{2}\sin x} \right| dx$$

$$= \sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\sin\frac{(k+1)}{2}x \sin\frac{kx}{2}}{\sin\frac{x}{2}\sin x} \right| dx$$

$$= \begin{cases} \displaystyle\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\sin\frac{(k+1)}{2}x}{\sin\frac{x}{2}} \right| \left| \frac{\sin\frac{kx}{2}}{\sin x} \right| dx & \text{(k is odd)} \\ \displaystyle\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\sin\frac{kx}{2}}{\sin\frac{x}{2}} \right| \left| \frac{\sin\frac{(k+1)x}{2}}{\sin x} \right| dx & \text{(k is even)} \end{cases}$$

$$= O\left(\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \left\{ \int_{-\pi}^{\pi} \left(\frac{\sin\frac{(k+1)}{2}x}{\sin\frac{x}{2}} \right)^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{-\pi}^{\pi} \left(\frac{\sin\frac{kx}{2}}{\sin\frac{x}{2}} \right)^2 dx \right\}^{\frac{1}{2}} \right)$$

$$= O\left(\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| k \right) = o(1) \quad \text{using}(2.1.2)$$

Therefore,

$$\int_{-\pi}^{\pi} |f(x) - K_n(x)|dx = o(1)$$

This proves the theorem.

Corollary 2.2.1: If $\{a_n\}$ is a semi-convex null sequence, then for L^1-convergence of the cosine series (2.1.1) it is necessary and sufficient that $\lim_{n\to\infty} a_n \log n = 0$

Proof: We have

$$||S_n^c(x) - f(x)|| = ||S_n^c(x) - K_n(x) + K_n(x) - f(x)||$$

$$\leq ||K_n(x) - f(x)|| + ||S_n^c(x) - K_n(x)||$$

(using (2.2.1)and (2.2.2))

$$= ||K_n(x) - f(x)|| + \left|\left|(a_n - a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}\right|\right|$$

$$\left|\left|(a_n - a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}\right|\right| = ||K_n(x) - S_n^c(x)||$$

$$\leq ||K_n(x) - f(x)|| + ||S_n^c(x) - f(x)||$$

Now,
$$|a_n - a_{n+2}| = \left|\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+2})\right|$$

$$= \left|\sum_{k=n+1}^{\infty}\frac{k}{k}(\Delta a_{k-1} - \Delta a_{k+1})\right|$$

$$\leq \frac{1}{n}\left|\sum_{k=n+1}^{\infty} k(\Delta^2 a_{k-1} + \Delta^2 a_k)\right|$$

$$= o\left(\frac{1}{n}\right)$$

and
$$\int_{-\pi}^{\pi}\frac{\tilde{D}_n(x)}{2\sin x}dx = O(n)$$

Therefore,we have

$$(a_n - a_{n+2})\int_{-\pi}^{\pi}\frac{\tilde{D}_n(x)}{2\sin x}dx = O\left((a_n - a_{n+2})n\right)$$

$$= O\left(\frac{1}{n}n\right)$$

$$= \mathrm{o}(1)$$

Moreover,

$$\int_{-\pi}^{\pi}\left|a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}\right|dx$$

$$\leq a_n\int_{-\pi}^{\pi}\left|\frac{\sin nx}{2\sin x}+\frac{\sin(n+1)x}{2\sin x}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}\left|\frac{\sin nx+\sin(n+1)x}{2\sin x}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}\left|\frac{2\sin\left(\frac{n+n+1}{2}\right)x\cos\left(\frac{-x}{2}\right)}{2\sin x}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}\left|\frac{\sin\left(n+\frac{1}{2}\right)x\cos\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}\left|\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}|D_n(x)|dx$$

$$\sim a_n\log n$$

Since $\quad ||K_n(x)-f(x)|| = o(1) \qquad$ (By Theorem(3.2.1))

Therefore $||S_n^c(x)-f(x)|| = o(1)$ if and only if $\lim_{n\to\infty} a_n\log n = 0$

This proves the corollary

CHAPTER III

On L^1-Convergence of the Sine and Cosine Trigonometric Sums with Semi-Convex Coefficients

3.1 Introduction

Let

$$\text{(3.1.1)} \qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$\text{(3.1.2)} \qquad \sum_{k=1}^{\infty} a_k \sin kx$$

be cosine and sine trigonometric series respectively with their partial sums denoted by

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

$$S_n^s(x) = \sum_{k=1}^{n} a_k \sin kx$$

Let

$$f(x) = \lim_{n\to\infty} S_n^c(x)$$

$$g(x) = \lim_{n\to\infty} S_n^s(x)$$

For convenience, in the following chapter we shall assume that a_{-1}=a_0= 0.

Semi-convex sequence [**8**]: A null sequence $\{a_n\}$ is said to be semi-convex if $a_n \to 0$, $n \to \infty$ and

$$\text{(3.1.3)} \qquad \sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + a_n| < \infty,$$

where $\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$

The results have been taken from : Xhevat.Z.Krasniqi, Int.J.Open Problems Comput.Sci.Math, Vol.2,No.2,June,(2009).

Bala R. and Ram B.[1] have proved that for series (3.1.1) with semi-convex null coefficients the following theorem holds true.

Theorem A: If $\{a_n\}$ is a semi-convex null sequence,then for the convergence of the cosine series (3.1.1) in the metric L^1, it is necessary and sufficient that $a_{n-1}\log n = o(1)$, $n \to \infty$.

Garrett J.W. and Stanojevic C.V.([4]) have introduced modified cosine sums.

$$t_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a_j \cos kx$$

Garrett and Stanojević ([5]), Ram ([16]), Singh and Sharma ([24]),and Kaur and Bhatia ([10]) studied the L^1-convergence of this cosine sum under different sets of conditions on the coefficients a_n.

Kumari and Ram ([14]) introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right) k\cos kx$$

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right) k\sin kx$$

and have studied their L^1-convergence under the condition that the coefficients $\{a_n\}$ belong to different classes of sequences. Likewise, they deduced some results about L^1-convergence of cosine and sine series as corollaries.

Later on, Kaur.K ([9]) introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx$$

and studied the L^1-convergence of this modified sine sum with semi-convex coefficients proving the following theorem.

Theorem B: Let $\{a_n\}$ be a semi-convex null sequence, then $K_n(x)$ converges to *f(x)* in L^1-norm.

Xhevat.Z.Krasniqi in 2009 introduced new modified cosine and sine sums as

$$H_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta[(a_{j-1} - a_{j+1})\cos jx]$$

$$G_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta[(a_{j-1} - a_{j+1})\sin jx]$$

The main goal of this chapter to study the L^1-convergence of these new modified sine and cosine sums with semi-convex coefficients and deduce Theorem A as a corollary.

As usually with $D_n(x)$ and $\tilde{D}_n(x)$,we shall denote the Dirichlet and its conjugate kernels respectively, defined by

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\cos kx \qquad \tilde{D}_n(x) = \sum_{k=1}^{n}\sin kx$$

3.2 Main Result

Regarding the series (3.1.2), we prove the following result:

Theorem 3.2.1: Let $\{a_n\}$ be a semi-convex null sequence, then $H_n(x)$ converges to *g(x)* in L^1-norm.

Proof: We have

$$H_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1} - a_{j+1})\cos jx\right]$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}[(a_{j-1} - a_{j+1})\cos jx - (a_j - a_{j+2})\cos(j+1)x]$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}\left[\sum_{j=k}^{n}(a_{j-1} - a_{j+1})\cos jx - \sum_{j=k}^{n}(a_j - a_{j+2})\cos(j+1)x\right]$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}[(a_{k-1}-a_{k+1})\cos kx + (a_k - a_{k+2})\cos{(k+1)x} +$$

$$(a_{k+1}-a_{k+3})\cos{(k+2)x} + (a_{k+2}-a_{k+4})\cos{(k+3)x} +$$

$$+ ... + (a_{n-1}-a_{n+1})\cos nx - (a_k - a_{k+2})\cos{(k+1)x}$$

$$-(a_{k+1}-a_{k+3})\cos{(k+2)x} - (a_{k+2}-a_{k+4})\cos{(k+3)x} - ...$$

$$-(a_{n-1}-a_{n+1})\cos nx - ... - (a_n - a_{n+2})\cos{(n+1)x}]$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}[(a_{k-1}-a_{k+1})\cos kx - (a_n - a_{n+2})\cos{(n+1)x}]$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}[(a_{k-1}-a_{k+1})\cos kx] + \frac{\cos(n+1)x}{2\sin x}(a_n - a_{n+2})\sum_{k=0}^{n}1$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}[(a_{k-1}-a_{k+1})\cos kx] + (n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}[(a_{k-1}-a_k+a_k-a_{k+1})\cos kx] + (n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}[(\Delta a_{k-1}+\Delta a_k)\cos kx] + (n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

Applying **Abel's transformation**, we get

$$= -\frac{1}{2\sin x}\left[\sum_{k=0}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)(D_k(x)-\frac{1}{2}) + (\Delta a_{n-1}+\Delta a_n)(D_n(x)-\frac{1}{2})\right]$$
$$+(n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\left[\sum_{k=0}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)D_k(x) + (\Delta a_{n-1}+\Delta a_n)D_n(x)\right] +$$

$$\frac{1}{4\sin x}\sum_{k=0}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k) + \frac{\Delta a_{n-1}+\Delta a_n}{4\sin x} + (n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x) + (\Delta a_n + \Delta a_{n+1})D_n(x)\right] +$$

$$\frac{1}{4\sin x}\sum_{k=0}^{n-1}(\Delta^2 a_{k-1}+\Delta^2 a_k)+\frac{\Delta a_{n-1}+\Delta a_n}{4\sin x}+(n+1)(a_n-a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x) + (a_n - a_{n+2})D_n(x)\right] +$$

$$\frac{1}{4\sin x}[\Delta a_{-1} + \Delta a_0 + \Delta a_1 + \Delta a_2 + ... + \Delta a_{n-2} - \Delta a_1 - \Delta a_2 -$$

$$... - \Delta a_{n-2} - \Delta a_{n-1} - \Delta a_n]+\frac{\Delta a_{n-1}+\Delta a_n}{4\sin x}+(n+1)(a_n-a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x) + (a_n - a_{n+2})D_n(x)\right]$$

$$-\frac{\Delta a_{n-1}+\Delta a_n}{4\sin x}+\frac{\Delta a_{n-1}+\Delta a_n}{4\sin x}+(n+1)(a_n-a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

$$H_n(x) = -\frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x) + (a_n - a_{n+2})D_n(x)\right]$$
$$+(n+1)(a_n-a_{n+2})\frac{\cos(n+1)x}{2\sin x} \tag{3.2.1}$$

On the other side, we have

$$S_n^s(x) = \sum_{k=1}^{n} a_k \sin kx$$

Divide and multiply by $2\sin x$

$$S_n^s(x) = \frac{1}{2\sin x}\sum_{k=1}^{n} a_k(2\sin kx \sin x)$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n} a_k\left(\cos(k-1)x - \cos(k+1)x\right)$$

$$= -\frac{1}{2\sin x}\sum_{k=1}^{n} a_k\left(\cos(k+1)x - \cos(k-1)x\right)$$

$$= -\frac{1}{2\sin x}\left[\sum_{k=1}^{n} a_k \cos(k+1)x - \sum_{k=1}^{n} a_k \cos(k-1)x\right]$$

$$= -\frac{1}{2\sin x}\,(a_1\cos 2x + a_2\cos 3x + ... + a_{n-1}\cos nx + a_n\cos(n+1)x$$

$$-a_2\cos x - a_3\cos 2x - ... - a_n\cos(n-1)x)$$

Adding and Subtracting $a_{n+1}\cos nx$

$$= -\frac{1}{2\sin x}\,(a_1\cos 2x + a_2\cos 3x + ... + a_{n-1}\cos nx + a_n\cos(n+1)x$$

$$-a_2\cos x - a_3\cos 2x - ... - a_n\cos(n-1)x) - a_{n+1}\cos nx + a_{n+1}\cos nx$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}(a_{k-1} - a_{k+1})\cos kx - a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}(a_{k-1} - a_k + a_k - a_{k+1})\cos kx - a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n}(\Delta a_{k-1} + \Delta a_k)\cos kx) - a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x}$$

Applying **Abel's transformation**

$$= -\frac{1}{2\sin x}\sum_{k=0}^{n-1}\left[(\Delta^2 a_{k-1} + \Delta^2 a_k)(D_k(x) - \frac{1}{2}) + (\Delta a_{n-1} + \Delta a_n)D_n(x) - \frac{1}{2})\right]$$

$$-a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x) + (\Delta a_{n-1} + \Delta a_n)(D_n(x)\right]$$

$$+\frac{1}{4\sin x}\sum_{k=0}^{n-1}\left(\Delta^2 a_{k-1} + \Delta^2 a_k\right) + \frac{1}{4\sin x}(\Delta a_{n-1} + \Delta a_n)$$

$$-a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x}$$

$$= -\frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x) + (a_n + a_{n-2})D_n(x)\right]$$

$$-a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x}$$

$$S_n^s(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\left(\Delta^2 a_{k-1} + \Delta^2 a_k\right)D_k(x) - (a_n - a_{n-2})\frac{D_n(x)}{2\sin x}$$

$$-a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x} \tag{3.2.2}$$

Now, $$|a_n - a_{n+2}| = \left|\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+2})\right|$$

$$= \left|\sum_{k=n+1}^{\infty}\frac{k}{k}(\Delta a_{k-1} - \Delta a_{k+1})\right|$$

$$\leq \frac{1}{n}\left|\sum_{k=n+1}^{\infty}k(\Delta^2 a_{k-1} + \Delta^2 a_{k+1})\right|$$

$$= o\left(\frac{1}{n}\right)$$

and $$\int_{-\pi}^{\pi}\frac{D_n(x)}{2\sin x}dx = O(n)$$

Therefore,we have

$$(a_n - a_{n+2})\int_{-\pi}^{\pi}\frac{D_n(x)}{2\sin x}dx = O\left((a_n - a_{n+2})n\right)$$

$$= O\left(\frac{1}{n}n\right)$$

$$= \mathrm{o}(1) \tag{3.2.3}$$

Since $\{a_n\}$ is semi-convex sequence, then from (3.1.3), we have

$$|(n+1)(a_n - a_{n-2})| = (n+1)\left|\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+2})\right|$$

$$= (n+1)\left|\sum_{k=n+1}^{\infty}(\Delta a_{k-1} - \Delta a_{k+1})\right|$$

$$= (n+1)\left|\sum_{k=n+1}^{\infty}(\Delta a_{k-1} - \Delta a_k + \Delta a_k - \Delta a_{k+1})\right|$$

$$\leq \sum_{k=n+1}^{\infty} k\left|(\Delta^2 a_{k-1} + \Delta^2 a_k)\right| = \mathrm{o}(1),\, n \to \infty \tag{3.2.4}$$

Using (3.2.4), when we pass lim as $n \to \infty$ to (3.2.1) and (3.2.2) we get

$$g(x) = \lim_{n\to\infty} S_n^s(x) = \lim_{n\to\infty} H_n(x)$$

$$= -\frac{1}{2\sin x}\sum_{k=0}^{\infty}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x)$$

Applying Well known inequality $D_k(x) \leq \dfrac{1}{2} + k, \qquad k = 1,2,...$

$$\int_{-\pi}^{\pi}|g(x) - H_n(x)|dx = \int_{-\pi}^{\pi}\left|-\frac{1}{2\sin x}\left[\sum_{k=0}^{\infty}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x)\right.\right.$$

$$\left. - \sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x)\right] - (a_n - a_{n+2})\frac{D_n(x)}{2\sin x}$$

$$\left. -(n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}\right| dx$$

$$= \int_{-\pi}^{\pi}\left|-\frac{1}{2\sin x}\left[\sum_{k=n+1}^{\infty}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x)\right] - (a_n - a_{n+2})\frac{D_n(x)}{2\sin x}\right.$$

$$\left. +(n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}\right| dx$$

$$= O\left(\sum_{k=n+1}^{\infty} k\left|\Delta^2 a_{k-1} + \Delta^2 a_k\right|\right) + O\left(|(n+1)(a_n - a_{n+2})|\right) + o(1)$$

Making use of (3.2.3) and (3.2.4), we have

$$= \mathrm{o}(1),\, n \to \infty$$

$$\int_{-\pi}^{\pi} |g(x) - H_n(x)| dx = o(1)$$

which proves the theorem.

Corollary 3.2.1: Let $\{a_n\}$ be a semi-convex null sequence, then the necessary and sufficient condition for L^1-convergence of the series (3.1.2) is $a_n \log n = o(1), n \to \infty$.

Proof: Necessary condition : Let $||S_n^s(x) - g(x)|| = \text{o}(1),\ n \to \infty$

To Prove : $a_n \log n = o(1), n \to \infty$

We have,

$$||S_n^s(x) - g(x)|| = ||S_n^s(x) + H_n(x) - H_n(x) - g(x)||$$

$$\leq ||S_n^s(x) - H_n(x)|| + ||H_n(x) - g(x)||$$

$$||S_n^s(x) - g(x)|| \leq ||H_n(x) - g(x)|| + \Bigg|\Bigg| -\frac{1}{2\sin x} \sum_{k=0}^{n} (\Delta^2 a_{k-1} + \Delta^2 a_k) D_k(x) - (a_n - a_{n+2}) \frac{D_n(x)}{2\sin x}$$

$$-a_{n+1} \frac{\cos nx}{2\sin x} - a_n \frac{\cos(n+1)x}{2\sin x} + \frac{1}{2\sin x} \sum_{k=0}^{n} (\Delta^2 a_{k-1} + \Delta^2 a_k) D_k(x)$$

$$-(a_n - a_{n+2}) \frac{D_n(x)}{2\sin x} - (n+1)(a_n - a_{n+2}) \frac{\cos(n+1)x}{2\sin x} \Bigg|\Bigg| \qquad \text{(using (3.2.1)and(3.2.2))}$$

$$||H_n(x) - g(x)|| + ||S_n^s(x) - g(x)|| - \Bigg|\Bigg| 2(a_n - a_{n+2}) \frac{D_n(x)}{2\sin x} \Bigg|\Bigg| + \Bigg|\Bigg| (n+1)(a_n - a_{n+2}) \frac{\cos(n+1)x}{2\sin x} \Bigg|\Bigg|$$

$$\geq \Bigg|\Bigg| a_{n+1} \frac{\cos nx}{2\sin x} + a_n \frac{\cos(n+1)x}{2\sin x} \Bigg|\Bigg|$$

$$\geq a_n \int_{-\pi}^{\pi} \left| \frac{\cos nx}{2\sin x} + \frac{\cos(n+1)}{2\sin x} \right| dx$$

Now, $\left| \dfrac{\cos nx}{2\sin x} + \dfrac{\cos(n+1)}{2\sin x} \right| = \left| \dfrac{\cos nx + \cos(n+1)x}{2\sin x} \right|$

$$= \left| \frac{\cos(n+\frac{1}{2})x \cos\frac{x}{2}}{\sin x} \right|$$

$$= \left| \frac{\cos(n+\frac{1}{2})x \cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} \right|$$

$$= \left| -\frac{\cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}} \right|$$

$$= \left| -\frac{\cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}} - \frac{1}{2}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + \frac{1}{2}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \right|$$

$$\leq \left| \left(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{\cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right) - \frac{1}{2}\cot\frac{x}{2} \right|$$

$$= \left| \tilde{D}_n(x) - \frac{1}{2}\cot\frac{x}{2} \right|$$

$$\leq |\tilde{D}_n(x)| - \left| \frac{1}{2}\cot\frac{x}{2} \right|$$

Therefore,

$$||H_n(x)-g(x)||+||S_n^s(x)-g(x)||-\left|\left|2(a_n-a_{n+2})\frac{D_n(x)}{2\sin x}\right|\right|+\left|\left|(n+1)(a_n-a_{n+2})\frac{\cos(n+1)x}{2\sin x}\right|\right|$$

$$\geq a_n \int_{-\pi}^{\pi} \left| \tilde{D}_n(x) - \frac{1}{2}\cot\frac{x}{2} \right| dx$$

$$= a_n \left(\int_{-\pi}^{\pi} \left| \tilde{D}_n(x) \right| dx - \int_0^{\pi} \left| \cot\frac{x}{2} \right| dx \right)$$

$$= O(a_n \log n)$$

$$||H_n(x)-g(x)||+||S_n^s(x)-g(x)||-\left|\left|2(a_n-a_{n+2})\frac{D_n(x)}{2\sin x}\right|\right|+\left|\left|(n+1)(a_n-a_{n+2})\frac{\cos(n+1)x}{2\sin x}\right|\right|$$

$$\geq O(a_n \log n) \qquad (3.2.5)$$

Since $||H_n(x) - g(x)|| = \mathrm{o}(1)$, By Theorem 3.2.1

$||S_n^s(x) - g(x)|| = \mathrm{o}(1)$, By assumption of corollary

and third and fourth term at the left side of relation(3.2.5) tend to zero by (3.2.4)

Consequently, $a_n \log n = \mathrm{o}(1), n \to \infty$

Now,

Sufficient condition: Let $a_n \log n = o(1), n \to \infty$

To Prove: $||S_n^s(x) - g(x)|| = \mathrm{o}(1), n \to \infty$

We have $||S_n^s(x) - g(x)|| = ||S_n^s(x) - H_n(x) + H_n(x) - g(x)||$

$$\leq ||H_n(x) - g(x)|| + ||H_n(x) - S_n^s(x)||$$

$$= ||H_n(x) - g(x)|| + \Bigg\| -\frac{1}{2\sin x}\sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x)$$

$$-(a_n - a_{n+2})\frac{D_n(x)}{2\sin x} - a_{n+1}\frac{\cos nx}{2\sin x} - a_n\frac{\cos(n+1)x}{2\sin x}$$

$$+\frac{1}{2\sin x}\sum_{k=0}^{n}(\Delta^2 a_{k-1} + \Delta^2 a_k)D_k(x)$$

$$-(a_n - a_{n+2})\frac{D_n(x)}{2\sin x} - (n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}\Bigg\|$$

$$= ||H_n(x) - g(x)|| + \Bigg\|(n+1)(a_n - a_{n+2})\frac{\cos(n+1)x}{2\sin x}$$

$$+2(a_n - a_{n+2})\frac{D_n(x)}{2\sin x} + a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\Bigg\|$$

$$= \mathrm{o}(1) + O\left((n+1)\left|a_n - a_{n+2}\right|\right) + O(n|a_n - a_{n+2}|)$$

$$+\Bigg\|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\Bigg\|$$

$$= \text{o}(1) + \left\| a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x} \right\|$$

But,

$$\int_{-\pi}^{\pi} \left| a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x} \right| dx$$

$$\leq a_n \int_{-\pi}^{\pi} \left| \tilde{D}_n(x) - \frac{1}{2}\cot\frac{x}{2} \right| dx$$

$$= a_n \left(\int_{-\pi}^{\pi} \left| \tilde{D}_n(x) \right| dx - \int_{0}^{\pi} \left| \cot\frac{x}{2} \right| dx \right)$$

$$= O(a_n \log n) = \text{o}(1)\text{............(Given)}$$

Therefore,

We get $||S_n^s(x) - g(x)|| = o(1), n \to \infty$.which proves the corollary.

Similarly for the series (3.1.1),we have the following theorems.

Theorem 3.2.2: Let $\{a_n\}$ be a semi-convex null sequence, then $G_n(x)$ converges to *f(x)* in L^1-norm.

Proof: We have

$$G_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left[(a_{j-1} - a_{j+1})\sin jx\right]$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} \left[(a_{j-1} - a_{j+1})\sin jx - (a_j - a_{j+2})\sin(j+1)x\right]$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n} \left[\sum_{j=k}^{n}(a_{j-1} - a_{j+1})\sin jx - \sum_{j=k}^{n}(a_j - a_{j+2})\sin(j+1)x\right]$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n} \left[(a_{k-1} - a_{k+1})\sin kx + (a_k - a_{k+2})\sin(k+1)x + \right.$$

$$(a_{k+1} - a_{k+3})\sin(k+2)x + (a_{k+2} - a_{k+4})\sin(k+3)x +$$

$$+ ... + (a_{n-1} - a_{n+1}) \sin nx - (a_k - a_{k+2}) \sin (k+1)x$$

$$-(a_{k+1} - a_{k+3}) \sin (k+2)x - (a_{k+2} - a_{k+4}) \sin (k+3)x - ...$$

$$-(a_{n-1} - a_{n+1}) \sin nx - ... - (a_n - a_{n+2}) \sin (n+1)x]$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} [(a_{k-1} - a_{k+1}) \sin kx - (a_n - a_{n+2}) \sin (n+1)x]$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} [(a_{k-1} - a_{k+1}) \sin kx] - \frac{\sin(n+1)x}{2\sin x}(a_n - a_{n+2}) \sum_{k=1}^{n} 1$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} [(a_{k-1} - a_{k+1}) \sin kx] - n(a_n - a_{n+2}) \frac{\sin(n+1)x}{2\sin x}$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} [(a_{k-1} - a_k + a_k - a_{k+1}) \sin kx] - n(a_n - a_{n+2}) \frac{\sin(n+1)x}{2\sin x}$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} [(\Delta a_{k-1} + \Delta a_k) \sin kx] - n(a_n - a_{n+2}) \frac{\sin(n+1)x}{2\sin x}$$

Applying **Abel's transformation**, we get

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \tilde{D}_k(x) + (\Delta a_{n-1} + \Delta a_n) \tilde{D}_n(x) \right]$$
$$-n(a_n - a_{n+2}) \frac{\sin(n+1)x}{2\sin x}$$

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n} (\Delta^2 a_{k-1} + \Delta^2 a_k) \tilde{D}_k(x) + (\Delta a_{n+1} + \Delta a_n) \tilde{D}_n(x) \right]$$

$$-n(a_n - a_{n+2}) \frac{\sin(n+1)x}{2\sin x}$$

$$G_n(x) = \frac{1}{2\sin x} \left[\sum_{k=1}^{n} (\Delta^2 a_{k-1} + \Delta^2 a_k) \tilde{D}_k(x) \right] + (a_n - a_{n+2}) \frac{\tilde{D}_n(x)}{2\sin x}$$

$$-n(a_n - a_{n+2}) \frac{\sin(n+1)x}{2\sin x} \tag{3.2.6}$$

Now,we have

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} a_k(2\cos kx \sin x)$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} a_k(\sin(k+1)x - \sin(k-1)x)$$

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n} a_k(\sin(k+1)x - \sum_{k=1}^{n} a_k \sin(k-1)x) \right]$$

$$= \frac{1}{2\sin x} [a_1 \sin 2x + a_2 \sin 3x + ... + a_{n-1} \sin nx + a_n \sin(n+1)x$$

$$-a_1 \sin 0x - a_2 \sin x - ... - a_n \sin(n-1)x] - a_{n+1} \sin nx + a_{n+1} \sin nx$$

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n} (a_{k-1} - a_{k+1}) \sin kx \right] + a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n} (a_{k-1} - a_k + a_k - a_{k+1}) \sin kx \right] + a_n \frac{\sin(n+1)x}{2\sin x}$$

$$+a_{n+1} \frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n} (\Delta a_k + \Delta a_{k-1}) \sin kx \right] + a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x}$$

Applying Abel's transformation, we have

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n-1} (\Delta^2 a_k + \Delta^2 a_{k-1}) \tilde{D}_k(x) + (\Delta a_n + \Delta a_{n-1}) \tilde{D}_n(x) \right]$$

$$+a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n} (\Delta^2 a_k + \Delta^2 a_{k-1}) \tilde{D}_k(x) + (\Delta a_{n+1} + \Delta a_n) \tilde{D}_n(x) \right]$$

$$+a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x}$$

$$S_n^c(x) = \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x) + (a_n - a_{n+2})\tilde{D}_n(x)\right]$$

$$+a_n\frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} \tag{3.2.7}$$

As $\{a_n\}$ is null sequence, $\tilde{D}_n(x)$ is bounded

Therefore second, third and fourth terms tend to zero.

$$f(x) = \lim_{n\to\infty} S_n^c(x)$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)$$

Now,

$$f(x) - G_n(x) = \frac{1}{2\sin x}\left[\sum_{k=1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right] - \frac{1}{2\sin x}\left[\sum_{k=1}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right]$$

$$-(a_n - a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x} + n(a_n - a_{n+2})\frac{\sin(n+1)x}{2\sin x}$$

$$= \frac{1}{2\sin x}\left[\sum_{k=n+1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right] - (a_n - a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x}$$

$$+n(a_n - a_{n+2})\frac{\sin(n+1)x}{2\sin x}$$

Thus, we have

$$\int_{-\pi}^{\pi}|f(x) - G_n(x)|dx = \int_{-\pi}^{\pi}\left|\frac{1}{2\sin x}\left[\sum_{k=n+1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right] - (a_n - a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x}\right.$$

$$\left.+n(a_n - a_{n+2})\frac{\sin(n+1)x}{2\sin x}\right| dx$$

Now,

$$\sum_{k=n+1}^{\infty}|\Delta^2 a_k + \Delta^2 a_{k-1}|\int_{-\pi}^{\pi}\left|\frac{\tilde{D}_k(x)}{2\sin x}\right| dx$$

$$= \sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\cos\frac{x}{2} - \cos(k+\frac{1}{2})x}{2\sin\frac{x}{2}\sin x} \right| dx$$

$$= \sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{-2\sin\frac{(x+kx)}{2}\sin\frac{kx}{2}}{2\sin\frac{x}{2}\sin x} \right| dx$$

$$= \sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\sin\frac{(k+1)}{2}x\sin\frac{kx}{2}}{\sin\frac{x}{2}\sin x} \right| dx$$

$$= \begin{cases} \displaystyle\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\sin\frac{(k+1)}{2}x}{\sin\frac{x}{2}} \right| \left| \frac{\sin\frac{kx}{2}}{\sin x} \right| dx & \text{(k is odd)} \\ \displaystyle\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\sin\frac{kx}{2}}{\sin\frac{x}{2}} \right| \left| \frac{\sin\frac{(k+1)x}{2}}{\sin x} \right| dx & \text{(k is even)} \end{cases}$$

$$= O\left(\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \left\{ \int_{-\pi}^{\pi} \left(\frac{\sin\frac{(k+1)}{2}x}{\sin\frac{x}{2}} \right)^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{-\pi}^{\pi} \left(\frac{\sin\frac{kx}{2}}{\sin\frac{x}{2}} \right)^2 dx \right\}^{\frac{1}{2}} \right)$$

$$\text{using(3.1.3)}$$

$$= O\left(\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| k \right) = o(1) \qquad (3.2.8)$$

Therefore,

$$\sum_{k=n+1}^{\infty} |\Delta^2 a_k + \Delta^2 a_{k-1}| \int_{-\pi}^{\pi} \left| \frac{\tilde{D}_k(x)}{2\sin x} \right| dx = \text{o}(1)$$

Now,
$$|a_n - a_{n+2}| = \left| \sum_{k=n}^{\infty} (\Delta a_k - \Delta a_{k+2}) \right|$$

$$= \left| \sum_{k=n+1}^{\infty} \frac{k}{k} (\Delta a_{k-1} - \Delta a_{k+1}) \right|$$

$$\leq \frac{1}{n}\left|\sum_{k=n+1}^{\infty} k(\Delta^2 a_{k-1} + \Delta^2 a_k)\right|$$

$$= o\left(\frac{1}{n}\right)$$

and $$\int_{-\pi}^{\pi} \frac{\tilde{D}_n(x)}{2\sin x}dx = O(n)$$

Therefore,we have

$$(a_n - a_{n+2})\int_{-\pi}^{\pi} \frac{\tilde{D}_n(x)}{2\sin x}dx = O\left((a_n - a_{n+2})n\right)$$

$$= O\left(\frac{1}{n}n\right)$$

$$= \text{o}(1) \qquad (3.2.9)$$

Since $\{a_n\}$ is semi-convex sequence, then from (3.1.3), we have

$$|n(a_n - a_{n-2})| = n\left|\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+2})\right|$$

$$= n\left|\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+1} + \Delta a_{k+1} - \Delta a_{k+2})\right|$$

$$= n\left|\sum_{k=n}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k+1})\right|$$

$$\leq \sum_{k=n}^{\infty} k\left|(\Delta^2 a_k + \Delta^2 a_{k+1})\right| = \text{o}(1), n \to \infty \qquad (3.2.10)$$

Making use of (3.2.8), (3.2.9) and (3.2.10), we get

$$\int_{-\pi}^{\pi} |f(x) - G_n(x)|dx = o(1), n \to \infty$$

This proves the theorem.

Corollary 3.2.2: Let $\{a_n\}$ be a semi-convex null sequence, then the necessary and sufficient condition for L^1-convergence of the series (3.1.1) is $a_n \log n = o(1), n \to \infty$.

Proof: We have

$$||S_n^c(x) - f(x)|| = ||S_n^c(x) - G_n(x) + G_n(x) - f(x)||$$

$$\leq ||G_n(x) - f(x)|| + ||S_n^c(x) - G_n(x)||$$

(using (3.2.6)and (3.2.7))

$$= ||G_n(x) - f(x)|| + \left|\left| \sum_{k=1}^{n} (\Delta^2 a_k + \Delta^2 a_{k-1}) \frac{\tilde{D}_k(x)}{2\sin x} + (a_n - a_{n+2}) \frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x} \right.\right.$$

$$+ a_n \frac{\sin(n+1)x}{2\sin x} - \sum_{k=1}^{n} (\Delta^2 a_k + \Delta^2 a_{k-1}) \frac{\tilde{D}_k(x)}{2\sin x}$$

$$\left.\left. - (a_n - a_{n+2}) \frac{\tilde{D}_n(x)}{2\sin x} + n(a_n - a_{n+2}) \frac{\sin(n+1)x}{2\sin x} \right|\right|$$

$$\left|\left| a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x} + n(a_n - a_{n+2}) \frac{\sin(n+1)x}{2\sin x} \right|\right| = ||G_n(x) - S_n^c(x)||$$

$$\leq ||G_n(x) - f(x)|| + ||S_n^c(x) - f(x)||$$

Since $\{a_n\}$ is semi-convex sequence, then from (3.1.3), we have

$$|n(a_n - a_{n-2})| = \text{o}(1),\ n \to \infty$$

Moreover,
$$\int_{-\pi}^{\pi} \left| a_{n+1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} \right| dx$$

$$\leq a_n \int_{-\pi}^{\pi} \left| \frac{\sin nx}{2\sin x} + \frac{\sin(n+1)x}{2\sin x} \right| dx$$

$$= a_n \int_{-\pi}^{\pi} \left| \frac{\sin nx + \sin(n+1)x}{2\sin x} \right| dx$$

$$= a_n \int_{-\pi}^{\pi} \left| \frac{2\sin\left(\frac{n+n+1}{2}\right) x \cos\left(\frac{-x}{2}\right)}{2\sin x} \right| dx$$

$$= a_n \int_{-\pi}^{\pi} \left| \frac{\sin\left(n+\frac{1}{2}\right) x \cos\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} \right| dx$$

$$= a_n \int_{-\pi}^{\pi} \left| \frac{\sin\left(n+\frac{1}{2}\right) x}{2\sin\left(\frac{x}{2}\right)} \right| dx$$

$$= a_n \int_{-\pi}^{\pi} |D_n(x)| dx$$

$$\sim a_n \log n$$

Since $||G_n(x) - f(x)|| = o(1)$ (By Theorem(3.2.2))

Therefore $||S_n^c(x) - f(x)|| = o(1)$ if and only if $\lim_{n\to\infty} a_n \log n = 0$

This proves the corollary

CHAPTER IV

On L^1-Convergence of Ram and Kumari Sums with Semi-Convex Coefficients

4.1 Introduction

Consider the cosine and sine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{4.1.1}$$

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{4.1.2}$$

with partial sums denoted by

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

$$S_n^s(x) = \sum_{k=1}^{n} a_k \sin kx$$

Let
$$f(x) = \lim_{n\to\infty} S_n^c(x)$$

$$g(x) = \lim_{n\to\infty} S_n^s(x)$$

Concerning the L^1-convergence of cosine series (4.1.1), Kolmogorov ([12]) proved the following theorem.

Theorem A: If $\{a_n\}$ is a quasi-convex null sequence, then for the L^1-convergence of the cosine series (4.1.1), it is necessary and sufficient that $\lim_{n\to\infty} a_n \log n = 0$.

The case in which the sequence $\{a_n\}$ is convex, of this theorem was established by Young([29]). That is why, sometimes this **Theorem A** is known as **Young-Kolmogorov Theorem**.

Definition([8]): A sequence $\{a_n\}$ is said to be semi-convex if $a_n \to 0$ as $n \to \infty$, and

$$(4.1.3) \quad \sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \quad (a_0 = 0)$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$

Ram and Kumari ([14], [19]) introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left(\frac{a_j}{j} \right) k \cos kx$$

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left(\frac{a_j}{j} \right) k \sin kx$$

The aim of this Chapter is to obtain the L^1-convergence of modified sine sums of Ram and Kumari with semi-convex coefficients and to obtain the above mentioned result of Bala R. and Ram B. as a corollary.

4.2 Main Result

Theorem 4.2.1 : If $\{a_n\}$ is a semi-convex null sequence and if $\lim_{n\to\infty} a_n \log n = 0$, then $f_n(x)$ converges to *f(x)* in L^1-norm i.e. $||f(x) - f_n(x)||_{L^1} = o(1), n \to \infty$

Proof: We have

$$S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} a_k (2 \cos kx \sin x)$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} a_k (\sin(k+1)x - \sin(k-1)x)$$

$$= \frac{1}{2\sin x} \left[\sum_{k=1}^{n} a_k \sin(k+1)x - \sum_{k=1}^{n} a_k \sin(k-1)x \right]$$

$$= \frac{1}{2\sin x}[a_1 \sin 2x + a_2 \sin 3x + ... + a_{n-1} \sin nx + a_n \sin(n+1)x$$

$$-a_2 \sin x - ... - a_n \sin(n-1)x] - a_{n+1} \sin nx + a_{n+1} \sin nx$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(a_{k-1} - a_{k+1}) \sin kx\right] + a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(a_{k-1} - a_k + a_k - a_{k+1}) \sin kx\right] + a_n \frac{\sin(n+1)x}{2\sin x}$$

$$+a_{n+1}\frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta a_k + \Delta a_{k-1}) \sin kx\right] + a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

Applying Abel's transformation, we have

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n-1}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x) + (\Delta a_n + \Delta a_{n-1})\tilde{D}_n(x)\right]$$

$$+a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x) + (\Delta a_{n+1} + \Delta a_n)\tilde{D}_n(x)\right]$$

$$+a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

$$S_n^c(x) = \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x) + (a_n - a_{n+2})\tilde{D}_n(x)\right]$$

$$+a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x}$$

As $\{a_n\}$ is a null sequence and $\tilde{D}_n(x)$ is bounded

Therefore second,third and fourth terms tend to zero as $n \to \infty$ and so,

$$f(x) = \lim_{n\to\infty} S_n^c(x)$$

$$= \frac{1}{2\sin x}\sum_{k=0}^{\infty}\left(\Delta^2 a_k + \Delta^2 a_{k-1}\right)\tilde{D}_k(x)$$

Now,

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n}\left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + ... + \Delta\left(\frac{a_{n-1}}{n-1}\right) + \Delta\left(\frac{a_n}{n}\right)\right] k\cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n}\left[\frac{a_k}{k} - \frac{a_{k+1}}{k+1} + \frac{a_{k+1}}{k+1} - \frac{a_{k+2}}{k+2}... + \frac{a_{n-1}}{n-1} - \frac{a_n}{n} + \frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right] k\cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n}\left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right] k\cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx$$

$$= S_n^c x - \frac{a_{n+1}}{n+1}\tilde{D'}_n(x)$$

$$= \frac{1}{2\sin x}\left[\sum_{k=0}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x) + (a_n - a_{n+2})\tilde{D}_n(x)\right] + a_n\frac{\sin(n+1)x}{2\sin x}$$
$$+ a_{n+1}\frac{\sin nx}{2\sin x} - \frac{a_{n+1}}{n+1}\tilde{D'}_n(x)$$

Now,

$$f(x) - f_n(x) = \left[\frac{1}{2\sin x}\sum_{k=1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right] - \left[\frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right.$$
$$\left. + (a_n - a_{n-2})\frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x} - \frac{a_{n+1}}{(n+1)}\tilde{D'}_n(x)\right]$$

$$= \sum_{k=n+1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\frac{\tilde{D}_k(x)}{2\sin x} - (a_n - a_{n-2})\frac{\tilde{D}_n(x)}{2\sin x} - a_{n+1}\frac{\sin nx}{2\sin x}$$
$$- a_n\frac{\sin(n+1)x}{2\sin x} + \frac{a_{n+1}}{(n+1)}\tilde{D'}_n(x)$$

Therefore,

$$\int_{-\pi}^{\pi}|f(x) - f_n(x)| = \int_{-\pi}^{\pi}\left|\sum_{k=n+1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\frac{\tilde{D}_k(x)}{2\sin x} - (a_n - a_{n-2})\frac{\tilde{D}_n(x)}{2\sin x} - a_{n+1}\frac{\sin nx}{2\sin x}\right.$$
$$\left. - a_n\frac{\sin(n+1)x}{2\sin x} + \frac{a_{n+1}}{(n+1)}\tilde{D'}_n(x)\right| dx$$

$$\leq \int_{-\pi}^{\pi}\left|\sum_{k=n+1}^{\infty}(\Delta^2 a_k+\Delta^2 a_{k-1})\frac{\tilde{D}_k(x)}{2\sin x}\right|dx+\int_{-\pi}^{\pi}\left|(a_n-a_{n-2})\frac{\tilde{D}_n(x)}{2\sin x}\right|dx$$

$$+\int_{-\pi}^{\pi}\left|a_n\frac{\sin(n+1)x}{2\sin x}+a_{n+1}\frac{\sin nx}{2\sin x}\right|dx+\int_{-\pi}^{\pi}\left|\frac{a_{n+1}}{(n+1)}\tilde{D}'_n(x)\right|dx$$

Now,

$$\sum_{k=n+1}^{\infty}|\Delta^2 a_k+\Delta^2 a_{k-1}|\int_{-\pi}^{\pi}\left|\frac{\tilde{D}_k(x)}{2\sin x}\right|dx$$

$$=\sum_{k=n+1}^{\infty}|\Delta^2 a_k+\Delta^2 a_{k-1}|\int_{-\pi}^{\pi}\left|\frac{\cos\frac{x}{2}-\cos(k+\frac{1}{2})x}{2\sin\frac{x}{2}\sin x}\right|dx$$

$$=\sum_{k=n+1}^{\infty}|\Delta^2 a_k+\Delta^2 a_{k-1}|\int_{-\pi}^{\pi}\left|\frac{-2\sin\frac{(x+kx)}{2}\sin\frac{kx}{2}}{2\sin\frac{x}{2}\sin x}\right|dx$$

$$=\sum_{k=n+1}^{\infty}|\Delta^2 a_k+\Delta^2 a_{k-1}|\int_{-\pi}^{\pi}\left|\frac{\sin\frac{(k+1)}{2}x\sin\frac{kx}{2}}{\sin\frac{x}{2}\sin x}\right|dx$$

$$=\begin{cases}\displaystyle\sum_{k=n+1}^{\infty}|\Delta^2 a_k+\Delta^2 a_{k-1}|\int_{-\pi}^{\pi}\left|\frac{\sin\frac{(k+1)}{2}x}{\sin\frac{x}{2}}\right|\left|\frac{\sin\frac{kx}{2}}{\sin x}\right|dx & \text{(k is odd)}\\ \displaystyle\sum_{k=n+1}^{\infty}|\Delta^2 a_k+\Delta^2 a_{k-1}|\int_{-\pi}^{\pi}\left|\frac{\sin\frac{kx}{2}}{\sin\frac{x}{2}}\right|\left|\frac{\sin\frac{(k+1)x}{2}}{\sin x}\right|dx & \text{(k is even)}\end{cases}$$

$$=O\left(\sum_{k=n+1}^{\infty}|\Delta^2 a_k+\Delta^2 a_{k-1}|\left\{\int_{-\pi}^{\pi}\left(\frac{\sin\frac{(k+1)}{2}x}{\sin\frac{x}{2}}\right)^2dx\right\}^{\frac{1}{2}}\left\{\int_{-\pi}^{\pi}\left(\frac{\sin\frac{kx}{2}}{\sin\frac{x}{2}}\right)^2dx\right\}^{\frac{1}{2}}\right)$$

$$=O\left(\sum_{k=n+1}^{\infty}|\Delta^2 a_k+\Delta^2 a_{k-1}|k\right)=o(1)\quad \text{using(4.1.3)}$$

Now

$$|a_n - a_{n+2}| = \left|\sum_{k=n}^{\infty}(\Delta a_k - \Delta a_{k+2})\right|$$

$$= \left|\sum_{k=n+1}^{\infty}\frac{k}{k}(\Delta a_{k-1} - \Delta a_{k+1})\right|$$

$$\leq \frac{1}{n}\left|\sum_{k=n+1}^{\infty}k(\Delta^2 a_{k-1} + \Delta^2 a_k)\right|$$

$$= o\left(\frac{1}{n}\right)$$

Since $\displaystyle\int_{-\pi}^{\pi}\frac{\tilde{D}_n(x)}{2\sin x}dx = O(n)$

$$(a_n - a_{n+2})\int_{-\pi}^{\pi}\frac{\tilde{D}_n(x)}{2\sin x}dx = O\left((a_n - a_{n+2})n\right) = O\left(\frac{1}{n}n\right) = \mathrm{o}(1)$$

Moreover $\displaystyle\int_{-\pi}^{\pi}\left|a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}\right|dx$

$$\leq a_n\int_{-\pi}^{\pi}\left|\frac{\sin nx}{2\sin x} + \frac{\sin(n+1)x}{2\sin x}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}\left|\frac{\sin nx + \sin(n+1)x}{2\sin x}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}\left|\frac{2\sin\left(\frac{n+n+1}{2}\right)x\cos\left(\frac{-x}{2}\right)}{2\sin x}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}\left|\frac{\sin\left(n+\frac{1}{2}\right)x\cos\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}\left|\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}\right|dx$$

$$= a_n \int_{-\pi}^{\pi} |D_n(x)|dx$$

$$\sim a_n \log n$$

Therefore

$$\int_{-\pi}^{\pi} |f(x) - f_n(x)|dx \quad \leq O\left[\sum_{k=n+1}^{\infty} k(\Delta^2 a_k + \Delta^2 a_{k-1})\right] + o(1) + a_n \log n + \frac{a_{n+1}}{(n+1)} \log n$$

$$\therefore \int_{-\pi}^{\pi} |f(x) - f_n(x)| = o(1), \text{ as } k \to \infty.$$

Corollary 4.2.1: If $\{a_n\}$ is a semi-convex null sequence,then the necessary and sufficient condition for L^1-convergence of the cosine series (4.1.1) is $\lim_{n\to\infty} a_{n+1} \log n = 0$

Proof : We have,

$$||S_n^c(x) - f(x)|| = ||S_n^c(x) - f_n(x) - f_n(x) - f(x)||$$

$$\leq ||S_n^c(x) - f_n(x)|| + ||f_n(x) - f(x)||$$

$$= ||f_n(x) - f(x)|| + \Bigg|\Bigg|\sum_{k=1}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\frac{\tilde{D}_k(x)}{2\sin x} + (a_n - a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x}$$

$$+ a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x} - \frac{a_{n+1}}{n+1}\tilde{D'}_n(x) - \sum_{k=1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\frac{\tilde{D}_k(x)}{2\sin x}\Bigg|\Bigg|$$

$$= ||f_n(x) - f(x)|| + \Bigg|\Bigg|\sum_{k=n+1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\frac{\tilde{D}_k(x)}{2\sin x} + (a_n - a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x}$$

$$+ a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x} - \frac{a_{n+1}}{n+1}\tilde{D'}_n(x)\Bigg|\Bigg|$$

and $$\int_{-\pi}^{\pi} \left|(a_n - a_{n-2})\frac{\tilde{D}_n(x)}{2\sin x}\right| dx = o(1), n \to \infty$$

Moreover, $$\int_{-\pi}^{\pi}\left|a_{n+1}\frac{\sin nx}{2\sin x}+a_n\frac{\sin(n+1)x}{2\sin x}\right|dx$$

$$= a_n\int_{-\pi}^{\pi}|D_n(x)|dx$$

$$\sim a_n\log n$$

and $$\int_{-\pi}^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right|dx$$

$$= \frac{a_{n+1}}{n+1}\int_{-\pi}^{\pi}\left|\tilde{D}'_n(x)\right|dx$$

$$= \frac{a_{n+1}}{n+1}\log n$$

$$= C\quad a_{n+1}\log n \text{ where C} = \frac{1}{n+1}$$

$$\sim a_{n+1}\log n$$

Since $$||f_n(x) - f(x)|| = o(1)$$ (By Theorem (4.2.1)

Therefore,

$$||S_n^c(x) - f(x)|| = o(1), \text{ and } \lim_{n\to\infty} a_{n+1}\log n = 0, \text{ as } n\to\infty$$

which proves the corollary.

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