SYMMETRIC AND NON-SYMMETRIC DUALITY IN MULTIOBJECTIVE PROGRAMMING PROBLEMS

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Submitted by

Madhu

Reg. No.-301303005

Under the guidance of

Dr. Navdeep Kailey

&

Dr. Vikas Sharma

School of Mathematics

Thapar University

Patiala-147004(PUNJAB)

INDIA

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Certificate

I hereby declare that the work which is being presented here in the dissertation entitled **"SYMMETRIC AND NON-SYMMETRIC DUALITY IN MULTIOBJECTIVE PROGRAMMING PROBLEMS"** in partial fulfillment of the requirement for the award of degree of **Master of Science in Mathematics and Computing** submitted in School of Mathematics, Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Navdeep Kailey**, Lecturer, SOM and **Dr. Vikas Sharma**, Assistant Professor, SOM and refer other researcher's work which are duly listed in the reference section.

The matter presented in this thesis has not been submitted to any other University/Institute for the award of my degree.

Madhu

Madhu

Dated: 14. July. 2015

Roll no. 301303005

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Dr. Vikas Sharma
Assistant Professor
SOM, Thapar University
Patiala-147004

Dr. Navdeep Kailey
Lecturer
SOM, Thapar University
Patiala-147004

Countersigned By:

Dr. A.K. Lal
Head
SOM, Thapar University
Patiala-147004

Dr. S.S. Bhatia
Dean of Academic Affairs
Thapar University
Patiala-147004

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Madhu

Madhu

Patiala

July, 2015

Abstract

The work being presented in the present thesis is devoted to the study of symmetric and non-symmetric duality results in multiobjective programming for some dual mathematical programming problems under generalized convexity assumption.

In the first chapter of the dissertation, nonlinear and multiobjective programming problem is introduced. The brief description of basic concepts, definitions that are used throughout work and detailed review of duality in single and multiobjective programming problems and summary of the thesis has also been discussed in this chapter.

In chapter 2, we have reviewed a pair of second-order Mond-Weir type symmetric duality in multiobjective programming problem considered by Suneja et al.[29] and established weak duality and strong duality under the assumption of η-bonvexity and η-pseudobonvexity.

In chapter 3, we have studied a pair of higher order Wolfe and Mond-Weir type multiobjective symmetric dual programs over arbitrary cones considered by [32]and the duality results are established under higher order (F, α, ρ, d)-convexity/pseudo-convexity assumptions.

In chapter 4, motivated by Kim et al. [33] we have discussed Mond-Weir type higher order multiobjective dual involving the nondifferentiable function and cone constraints, where every component of the objective function contains a term involving the support function of a compact convex set and established weak, strong duality theorems under the assumption of Higher order (F, ρ, α, d) type-I.

Contents

Chapter 1

Introduction

Mathematical programming is one of the most important branches of modern applied mathematics that deals with finding the best element from some set of available alternatives. Generally, a single objective mathematical programming problem can be defined as

$$(\mathbf{N}LP) \qquad \text{Minimize} \qquad f(x)$$
$$\text{subject to} \qquad x \in S = \{x \in X : g(x) \leq 0\},$$

where X is an open subset of R^n, the function $f : X \to R$ and $g : X \to R^m$ are differentiable on X. The function f is called an **objective function**, the components of g as the **constraint functions** and the corresponding inequalities as **constraints**. The set S is called the **feasible set** and any point $\bar{x} \in S$ is called a **feasible point** or simply **feasible**. If the objective function or some of the constraints are nonlinear, then the problem is called single objective nonlinear programming problem. The existence of multiple objectives leads to many interesting questions, which do not arise in single objective programming problem. Consider an example of buying a car. Ideally the first and foremost consideration would be the cost of a car but this need not be the only criteria for taking decisions as one may wish to have a car, which is powerful and fuel efficient. Then it becomes difficult to decide, when most powerful car is costly as well as the one with highest fuel assumption. In such solutions one deals with more than one objective at a time and is looking for compromising solution which is not unique. A general multiobjective nonlinear programming problem can be expressed in the form:

$$\textbf{(MP)} \quad \text{Minimize} \quad f(x) = (f_1(x), \ldots, f_k(x))$$
$$\text{Subject to} \quad x \in S = \{x \in X : g(x) \leq 0\},$$

where X is an open subset of R^n, the functions $f : X \to R^k$ and $g : X \to R^m$ are differentiable on X.

If we have more than one objective to deal with, then the motive is to find a solution which is based on the idea that there is no other alternative that is better in some aspect and not worse in every aspect of consideration. Based on this idea, efficient and weak efficient are defined as:

A point $\bar{x} \in S$ is said to be a weak efficient solution of the vector minimum problem (MP), if there exists no $x \in S$ such that

$$f(x) < f(\bar{x}).$$

A point $\bar{x} \in S$ is said to be an efficient solution of the vector minimum problem (MP), if there exists no $x \in S$ such that

$$f(x) \leq f(\bar{x}).$$

Duality plays an important role in multiobjective programming problem that connects two programs, one of which is called the primal problem and the other is called the dual problem in such a way that the existence of optimal solution to one of them guarantees an optimal solution to the other and the optimal values of the two problems are equal. A pair of dual problems is called symmetric if the dual of the dual is itself the primal.

Duality has not only used in many theoretical and computational developments but also used in economics, control theory, business problems and other diverse fields. Mangasarian [20] introduced the concept of second and higher-order duality for nonlinear problems. The study of second and higher-order duality is significant due to the computational advantage over the first-order duality as it provides tighter bounds for the value of the objective function when approximations are used. Several researchers have studied the duality relations for different dual problems under various generalized convexity assumptions. There are two dual models that are widely used namely Wolfe type model and Mond-Weir type model. Wolfe [31] formulated dual to (NLP) and proved duality theorems assuming f and g to be convex.

Mond and Weir [24] introduced dual to (NLP) and and proved duality relations by weakening the convexity assumptions of f and g to pseudoconvexity of f and quasiconvexity of $\mu^T g$.

The present chapter is divided into three sections. The first section gives important preliminaries. The second section contains a review of various developments in single and multiobjective mathematical programming which are relevant to the thesis and the last one presents a summary of the thesis.

1.1 Preliminaries

Notations and definitions

Throughout the thesis the following notations are used. The n-dimensional Euclidean space is denoted by R^n and a vector $x \in R^n$ is denoted by $x = (x_1, x_2, \ldots, x_n)$, $x_i \in R,\ i = 1, 2, \ldots, n$. Each vector x is a column vector and x^T denotes row vector where superscript T denotes the transpose of a vector. R^n_+ is the non-negative orthant of R^n and R_+ the set of nonnegative real numbers. For $x, y \in R^n$,

$$x \geqq y \Leftrightarrow x_i \geqq y_i,\ i = 1, 2, ..., n,$$

$$x \geq y \Leftrightarrow x \geqq y \text{ and } x \neq y,$$

$$x > y \Leftrightarrow x_i > y_i,\ i = 1, 2, ..., n.$$

$$x \nleq y \text{ means negation of } x \leq y.$$

The vector $\nabla f(\bar{x})$ denotes the gradient of a scalar differentiable function $f : R^n \to R$ at $\bar{x}$, and is defined as

$$\nabla f(\bar{x}) = \left[\frac{\partial}{\partial x_1} f(\bar{x}),\ \frac{\partial}{\partial x_2} f(\bar{x}), \ldots, \frac{\partial}{\partial x_n} f(\bar{x})\right]^T$$

If the function $f : R^n \to R$ is twice differentiable at $\bar{x}$, in addition to the gradient vector there exists an $n \times n$ symmetric matrix $\nabla_{xx} f$ or $\nabla^2 f$, called the Hessian matrix of f at $\bar{x}$.

Definition 1.1 [4]. A convex set C of R^n is called a **convex cone** if for each $x \in C$ and $\lambda \geqq 0$, $\lambda x \in C$.

Definition 1.2 $C^* = \{z \in R^n : x^T z \leqq 0, \text{ for all } x \in C\}$ is called the **polar cone** of C.

Definition 1.3 [23, 28]. Let B be a compact convex set in R^n. The support function of B is defined by

$$s(x|B) = \max\{x^T y : y \in B\}.$$

The support function $s(x|B)$, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that

$$s(y|B) \geqq s(x|B) + z^T(y - x) \text{ for all } y \in B.$$

Equivalently,

$$z^T x = s(x|B).$$

The subdifferential of $s(x|B)$ is given by

$$\partial s(x|B) = \{z \in B : z^T x = s(x|B)\}.$$

For any set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0 \text{ for all } z \in S\}.$$

It can be easily seen that for a compact convex set B, y is in $N_B(x)$ if and only if $s(y|B) = x^T y$, or equivalently, x is in the subdifferential of s at y.

Convex functions and its generalization

Let X be an open convex subset of R^n and the functions $f : X \to R$. Then at $u \in X$,

(i) f is said to be **Convex** if for all $x \in X$ and for all $0 \leqq \lambda \leqq 1$, we have

$$f[\lambda x + (1 - \lambda)u] \leqq \lambda f(x) + (1 - \lambda)f(u),$$

or if f is differentiable at u, then we have

$$f(x) - f(u) \geqq \nabla f(u)^T(x - u).$$

The function f is said to be strictly convex if the above conditions hold as strict inequalities for $x \neq u,\ 0 < \lambda < 1$.

(ii) f is said to be **Quasiconvex** if for all $x \in X$ and for all $0 \leqq \lambda \leqq 1$, we have

$$f(x) \leqq f(u) \Rightarrow f[\lambda x + (1 - \lambda)u] \leqq f(u),$$

or if f is differentiable at u, then we have

$$f(x) \leqq f(u) \Rightarrow \nabla f(u)^T(x - u) \leqq 0.$$

Every convex function is quasiconvex, but the converse is not true. For example $f(x) = x^3$ is quasiconvex but not convex.

(iii) f is said to be **Pseudoconvex** if f is differentiable at u and for all $x \in X$, we have

$$\nabla f(u)^T(x-u) \geqq 0 \Rightarrow f(x) \geqq f(u),$$

or equivalently, if

$$f(x) < f(u) \Rightarrow \nabla f(u)^T(x-u) < 0.$$

For example, $f(x) = x^3 + x$ is pseudoconvex.

A new class of generalized convex functions, the so-called invex functions, introduced by Hanson [15] with the aim of extending the validity of the suffciency of the Karush-Kuhn-Tucker conditions. It is obvious that the particular case of (differentiable) convex function is obtained from by choosing $\eta(x, y) = x - u$. The term invex was created by Craven [9].

(iv) f is said to be **invex** if there exists a function $\eta : X \times X \to R^n$ such that for all $x, u \in X$.

$$f(x) - f(u) \geqq \eta(x, u)^T \nabla f(u).$$

(v) f is said to be **pseudoinvex** if there exists a function $\eta : X \times X \to R^n$ such that for all $x, u \in X$

$$\eta(x, u)^T \nabla f(u) \geqq 0 \Rightarrow f(x) \geqq f(u).$$

(vi) A functional $F : X \times X \times R^n \to R$ is sublinear in the third variable, if for all $x, u \in X$

(i) $F(x, u; \xi_1 + \xi_2) \leqq F(x, u; \xi_1) + F(x, u; \xi_2),$ for all $\xi_1, \xi_2 \in R^n$,

(ii) $F(x, u; \alpha a) = \alpha F(x, u; a),$ for all $\alpha \in R_+,$ and $a \in R^n$.

Also we write it as

$$F_{x,u}(a) = F(x, u; a).$$

(vii) f is said to be F**-convex** at u if for all $x \in X$,

$$f(x) - f(u) \geqq F(x, u; \nabla f(u)).$$

(viii) f is said to be (F, ρ)**-convex** at u if there exists $d : X \times X \to R$ and $\rho \in R$ then for all $x \in X$

$$F(x, u; \nabla f(u)) + \rho d^2(x, u) \leqq f(x) - f(u).$$

(ix) f is said to be (F, α, ρ, d)**-convex** at $u \in X$ if there exists a function $\alpha : X \times X \to R_+ \setminus \{0\}$, $d : X \times X \to R$ and $\rho \in R$ then for all $x \in X$

$$f(x) - f(u) \geqq F(x, u; \alpha(x, u)\nabla f(u)) + \rho d^2(x, u).$$

Generalization of invex function

Hanson and mond [16] introduced generalization of the class of invex functions called Type I functions.

Let X be a open subset of R^n and $\eta : X \times X \to R^n$ be a vector function. Then at $u \in X$ with respect to function $\eta(x, u)$, the objective function $f : X \to R$ and constraint function $g : X \to R^m$, are said to be

(i) **Type I convex** if for all $x \in X$,

$$f(x) - f(u) \geqq \eta(x, u)^T \nabla f(u),$$

and

$$-g(u) \geqq \eta(x, u)^T \nabla g(u).$$

(ii) **Type I pseudoconvex** if for all $x \in X$,

$$\begin{aligned} &\eta(x, u)^T \nabla f(u) \geqq 0 \\ \Rightarrow\ &f(x) - f(u) \geqq 0, \end{aligned}$$

and

$$\begin{aligned} &\eta(x, u)^T \nabla g(u) \geqq 0 \\ \Rightarrow\ &-g(u) \geqq 0. \end{aligned}$$

(iii) **Type I F-convex** if for all $x \in X$,

$$f(x) - f(u) \geqq F(x, u; \nabla f(u)),$$

and

$$-g(u) \geqq F(x, u; \nabla g(u)).$$

(iv) **Type I F-pseudoconvex** if for all $x \in X$,

$$\begin{aligned} & F(x,u;\nabla f(u)) \geqq 0 \\ \Rightarrow \;\; & f(x) - f(u) \geqq 0, \end{aligned}$$

and

$$\begin{aligned} & F(x,u;\nabla g(u)) \geqq 0 \\ \Rightarrow \;\; & -g(u) \geqq 0. \end{aligned}$$

(v) **Type I (F,ρ)-convex** if there exists $d : X \times X \to R$ and $\rho \in R$, then for all $x \in X$

$$f(x) - f(u) \geqq F(x,u;\nabla f(u)) + \rho d^2(x,u),$$

and

$$-g(u) \geqq F(x,u;\nabla g(u)) + \rho d^2(x,u).$$

(v) **Type I (F,ρ)-pseudoconvex** if there exists $d : X \times X \to R$ and $\rho \in R$, then for all $x \in X$

$$\begin{aligned} & F(x,u;\nabla f(u)) \geqq -\rho d^2(x,u) \\ \Rightarrow \;\; & f(x) - f(u) \geqq 0, \end{aligned}$$

and

$$\begin{aligned} & F(x,u;\nabla g(u)) \geqq -\rho d^2(x,u) \\ \Rightarrow \;\; & -g(u) \geqq 0. \end{aligned}$$

(v) **Type I (F,ρ,α,d)-convex** at $u \in X$ if there exists a function $\alpha : X \times X \to R_+ \setminus \{0\}$, $d : X \times X \to R$ and $\rho \in R$, then for all $x \in X$

$$f(x) - f(u) \geqq F(x,u;\alpha(x,u)\nabla f(u)) + \rho d^2(x,u),$$

and

$$-g(u) \geqq F(x,u;\alpha(x,u)\nabla g(u)) + \rho d^2(x,u).$$

(v) **Type I** (F, ρ, α, d)**-pseudoconvex** at $u \in X$ if there exists a function $\alpha : X \times X \to R_+ \setminus \{0\}$, $d : X \times X \to R$ and $\rho \in R$, then for all $x \in X$

$$\begin{aligned} & F(x, u; \alpha(x, u)\nabla f(u)) \geqq -\rho d^2(x, u) \\ \Rightarrow \quad & f(x) - f(u) \geqq 0, \end{aligned}$$

and

$$\begin{aligned} & F(x, u; \alpha(x, u)\nabla g(u)) \geqq -\rho d^2(x, u) \\ \Rightarrow \quad & -g(u) \geqq 0. \end{aligned}$$

1.2 Review of related work

Symmetric Duality

In mathematical programming, a pair of primal and dual problems is called symmetric if the dual of the dual is the primal problem. Duality in linear programming is always symmetric but in non-linear programming duality is not always symmetric.

The notion of symmetric duality in which the dual of the dual is the primal, found its way in nonlinear programming in the excellent work of Dorn [12]. Dantzig et al.[10] formulated the following pair of symmetric dual programs and established weakand strong duality theorems :

$$\begin{aligned} \textbf{(PS)} \quad & \text{Minimize} \quad F(x, y) = M(x, y) - y^T \nabla_y M(x, y) \\ & \text{subject to} \quad \nabla_y M(x, y) \leqq 0, \\ & \qquad x \geqq 0, \\ & \qquad y \geqq 0. \end{aligned}$$

$$\begin{aligned} \textbf{(DS)} \quad & \text{Maximize} \quad G(u, v) = M(u, v) - u^T \nabla_x M(u, v) \\ & \text{subject to} \quad \nabla_x M(u, v) \geqq 0, \\ & \qquad u \geqq 0, \\ & \qquad v \geqq 0. \end{aligned}$$

where $M : R^n \times R^m \to R$ is a twice differentiable function.

Mond and Weir [25] considered the following pair of symmetric dual programs:

$$\textbf{(PM)} \quad \text{Minimize} \quad M(x,y)$$
$$\text{subject to} \quad \nabla_y M(x,y) \leqq 0,$$
$$y^T \nabla_y M(x,y) \geqq 0,$$
$$x \geqq 0.$$

$$\textbf{(DM)} \quad \text{Maximize} \quad M(u,v)$$
$$\text{subject to} \quad \nabla_x M(u,v) \geqq 0,$$
$$u^T \nabla_x M(u,v) \leqq 0,$$
$$v \geqq 0.$$

Symmetric duality was generalized to multiobjective case by Weir and Mond [**??**] and as well as by Gulati et al. [13].

Bazaraa and Goode [5] generalized the results in [10] to arbitrary cones. They studied the Wolfe's type symmetric dual pair over arbitrary cones :

$$\textbf{(PC)} \quad \text{Minimize} \quad F(x,y) = M(x,y) - y^T \nabla_y M(x,y)$$
$$\text{subject to} \quad \nabla_y M(x,y) \in C_2^*,$$
$$(x,y) \in C_1 \times C_2,$$

$$\textbf{(DC)} \quad \text{Maximize} \quad G(u,v) = M(u,v) - u^T \nabla_x M(u,v)$$
$$\text{subject to} \quad -\nabla_x M(u,v) \in C_1^*,$$
$$(u,v) \in C_1 \times C_2,$$

where

(i) C_1 and C_2 are closed convex cones with non-empty interiors in R^n and R^m, respectively.

(ii) For $i = 1,2,\ C_i^*$ is the polar of C_i.

(iii) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$ and $M : S_1 \times S_2 \to R$ is a twice differentiable function.

Chandra and Kumar [6] formulated the following Mond-Weir type symmetric dual programs over arbitrary cones :

$$\begin{aligned} \textbf{(PMC)} \quad & \text{Minimize} \quad && M(x,y) \\ & \text{subject to} \quad && \nabla_y M(x,y) \in C_2^*, \\ & && y^T \nabla_y M(x,y) \geqq 0, \\ & && x \in C_1. \end{aligned}$$

$$\begin{aligned} \textbf{(DMC)} \quad & \text{Maximize} \quad && M(u,v) \\ & \text{subject to} \quad && -\nabla_x M(u,v) \in C_1^*, \\ & && u^T \nabla_x M(u,v) \leqq 0, \\ & && v \in C_2. \end{aligned}$$

and proved usual duality theorems under pseudoinvexity type assumptions.

Das and Nanda [11] studied symmetric duality in multiobjective programming with cone constraints. Subsequently, Kim et al. [18] derived symmetric duality results for multiobjective programs under pseudoinvex/strictly pseudoinvex functions and arbitrary cones.

Second order Duality

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used. Mond [26] considered the following second-order symmetric dual programs:

$$\begin{aligned} \text{Minimize} \quad & M(x,y) - y^T \nabla_y M(x,y) - y^T \nabla_{yy} M(x,y)p - \frac{1}{2} p^T \nabla_{yy} M(x,y)p \\ \text{subject to} \quad & \nabla_y M(x,y) + \nabla_{yy} M(x,y)p \leqq 0, \\ & x \geqq 0. \end{aligned}$$

$$\begin{aligned} \text{Maximize} \quad & M(x,y) - x^T \nabla_x M(x,y) - x^T \nabla_{xx} M(x,y)r - \frac{1}{2} r^T \nabla_{xx} M(x,y)r \\ \text{subject to} \quad & \nabla_x M(x,y) + \nabla_{xx} M(x,y)r \geqq 0, \\ & y \geqq 0, \end{aligned}$$

Ahmad and Husain [1] formulated the following-pair of Wolfe type multiobjective second order symmetric dual programs with cone constraints:

Primal Problem (WP)

$$\text{minimize} \quad f(x,y) - [y^T \nabla_y (\lambda^T f)(x,y)]e - [y^T \nabla_{yy} (\lambda^T f)(x,y)p]e - \frac{1}{2}[p^T \nabla_{yy}(\lambda^T f)(x,y)p]e$$

subject to

$$\nabla_y (\lambda^T f)(x,y) + \nabla_{yy}(\lambda^T f)(x,y)p \in C_2^*,$$

$$x \in C_1,$$

$$\lambda > 0, \ \lambda^T e = 1.$$

Dual Problem (WD)

$$\text{maximize} \quad f(u,v) - [u^T \nabla_x (\lambda^T f)(u,v)]e - [u^T \nabla_{xx} (\lambda^T f)(u,v)q]e - \frac{1}{2}[q^T \nabla_{xx}(\lambda^T f)(u,v)q]e$$

subject to

$$-\nabla_x (\lambda^T f)(u,v) - \nabla_{xx}(\lambda^T f)(u,v)q \in C_1^*,$$

$$v \in C_2,$$

$$\lambda > 0, \ \lambda^T e = 1,$$

where $f(x,y) : S_1 \times S_2 \to R^k(S_1 \subseteq R^n, \ S_2 \subseteq R^m), \ p \in R^m, \ q \in R^n, \ \lambda \in R^k$and $e = (1,1,\ldots,1)^T \in R^k$, and usual duality results are established under second-order in- vexity assumptions.

Higher order Duality

Higher order duality in nonlinear programming has been studied by many researchers. By introducing two differentiable functions $h : R^n \times R^n \to R$ and $k : R^n \times R^n \to R^m$. Mangasarian [20] formulated the following higher order dual:

Primal problem (P5)

$$\text{Minimize} \quad f(x)$$

$$\text{subject to} \quad g(x) \leqq 0.$$

Dual problem (D5)

$$\begin{aligned} &\text{Maximize} && f(x) + h(x,p) + y^T g(x) + y^T k(x,p) \\ &\text{subject to} && \nabla_p h(x,p) + \nabla_p y^T k(x,p) = 0, \\ &&& y \geqq 0. \end{aligned}$$

where $f : R^n \to R$ and $g : R^n \to R^m$, $\nabla_p h(u,p)$ denotes the gradient of h with respect to p and $\nabla_p(y^T k(u,p))$ denotes the gradient of $y^T k$ with respect to p.

1.3 Summary of the thesis

The aim of the present thesis is to study the symmetric and nonsymmetric duality in multiobjective programming for some dual mathematical programming problems under generalized convexity assumptions.

In Chapter 2, we have reviewed the following pair of second-order Mond-Weir type symmetric duality in multiobjective programming problem considered by Suneja et al.[29]:

$$\begin{aligned} \textbf{(P)} \quad &\text{Minimize} && F(x,y,p) = (F_1(x,y,p), F_2(x,y,p), \ldots, F_k(x,y,p)), \\ &\text{subject to} && \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y) p_i) \leqq 0, \\ &&& y^T \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y) p_i) \geq 0, \\ &&& \lambda > 0. \end{aligned}$$

$$\begin{aligned} \textbf{(D)} \quad &\text{Maximize} && G(u,v,q) = (G_1(u,v,q), G_2(u,v,q), \ldots, G_k(u,v,q)), \\ &\text{subject to} && \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) \geqq 0, \\ &&& u^T \sum_{i=1}^{k} \lambda_i (\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) \leq 0, \\ &&& \lambda > 0. \end{aligned}$$

where

$$F_i(x,y,p) = f_i(x,y) - \frac{1}{2}p_i^T \nabla_{yy} f_i(x,y) p_i,$$
$$G_i(u,v,q) = f_i(u,v) - \frac{1}{2}q_i^T \nabla_{xx} f_i(u,v) q_i,$$
$$\lambda_i \in R, \quad p_i \in R^m, \quad q_i \in R^n, \quad i = 1,2,\ldots,k.$$

$p = (p_1, p_2, \ldots, p_k)$, $q = (q_1, q_2, \ldots, q_k)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T$.
and studied weak duality and strong duality under the assumption of η-bonvexity and η-pseudobonvexity.

In Chapter 3, we have studied the following higher order Wolfe and Mond-Weir type multiobjective higher-order symmetric dual programs considered by Gupta et al. [32]:

Wolfe type symmetric dual

Primal problem**(HWP)**

$$\text{minimize} \;\; L(x,y,\lambda,p) = \quad f(x,y) + (\lambda^T h)(x,y,p)e_k - p^T \nabla_p (\lambda^T h)(x,y,p)e_k$$
$$- y^T \nabla_y (\lambda^T f)(x,y)e_k - y^T \nabla_p (\lambda^T h)(x,y,p)e_k$$

$$\text{subject to} \quad -\{\nabla_y (\lambda^T f)(x,y) + \nabla_p (\lambda^T h)(x,y,p)\} \in C_2^*,$$
$$\lambda^T e_k = 1,$$
$$\lambda > 0, \qquad x \in C_1.$$

Dual problem**(HWD)**

$$\text{maximize} \;\; M(u,v,\lambda,r) = \quad f(u,v) + (\lambda^T g)(u,v,r)e_k - r^T \nabla_r (\lambda^T g)(u,v,r)e_k$$
$$- u^T \nabla_x (\lambda^T f)(u,v)e_k - u^T \nabla_r (\lambda^T g)(u,v,r)e_k$$

$$\text{subject to} \quad \nabla_x (\lambda^T f)(u,v) + \nabla_r (\lambda^T g)(u,v,r)\} \in C_1^*,$$
$$\lambda^T e_k = 1,$$
$$\lambda > 0, \qquad v \in C_2.$$

where

(i) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$,

(ii) $f : S_1 \times S_2 \to R^k,\ h : S_1 \times S_2 \times R^m \to R^k$ and $g : S_1 \times S_2 \times R^n \to R^k$ are differentiable functions, $e_k = (1, \ldots, 1)^T \in R^k,\ \lambda \in R^k$ and

(iii) r and p are vectors in R^n and R^m, respectively.

Mond-Weir type symmetric dual

Primal problem (HMP)

$$\begin{aligned} \text{minimize} \quad & F(x, y, p_1, p_2, \ldots, p_k) = (F_1(x, y, p_1), F_2(x, y, p_2), \ldots, F_k(x, y, p_k)) \\ \text{subject to} \quad & -\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)] \in C_2^*, \\ & y^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)] \geqq 0, \\ & \lambda > 0, \qquad x \in C_1. \end{aligned}$$

Dual problem (HMD)

$$\begin{aligned} \text{maximize} \quad & H(u, v, r_1, r_2, \ldots, r_k) = (H_1(u, v, r_1), H_2(u, v, r_2), \ldots, H_k(u, v, r_k)) \\ \text{subject to} \quad & \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + \nabla_{r_i} h_i(u, v, r_i)] \in C_1^*, \\ & u^T \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + \nabla_{r_i} h_i(u, v, r_i)] \leqq 0, \\ & \lambda > 0, \qquad v \in C_2. \end{aligned}$$

where for $i = 1, 2, \ldots, k,\ F_i(x, y, p_i) = f_i(x, y) + h_i(x, y, p_i) - p_i^T [\nabla_{p_i} h_i(x, y, p_i)]$, $H_i(u, v, r_i) = f_i(u, v) + g_i(u, v, r_i) - r_i^T [\nabla_{r_i} g_i(u, v, r_i)],\ f_i : S_1 \times S_2 \to R,\ h_i : S_1 \times S_2 \times R^m \to R$ and $h_i : S_1 \times S_2 \times R^n \to R$ are differentiable functions, $p_i \in R^m$ and $r_i \in R^n$.

Weak and strong duality theorems are studied under higher-order (F, α, ρ, d)- convexity/pseudo-convexity assumptions.

In chapter 4, we introduce the nondifferentiable higher order multiobjective problem involving cone constraints, where every component of the objective function contains a term involving the support function of a compact convex set. For this problem, Mond-Weir type dual is proposed .

Mond-Weir type duality (NMMD)

$$
\begin{aligned}
\text{Maximize} \quad & f(u) + u^T w + (\lambda^T h(u,p))e - p^T \nabla_p (\lambda^T h(u,p))e \\
\text{subject to} \quad & \lambda^T [\nabla f(u) + \nabla_p h(u,p) + w] = \nabla y^T g(u) + \nabla_p y^T k(u,p), \\
& g(u) + k(u,p) - p^T \nabla_p k(u,p) \in C_2^*, \\
& w_i \in D_i, \qquad i = 1, \dots, l, \\
& y \in C_2, \qquad \lambda > 0, \quad \lambda^T e = 1.
\end{aligned}
$$

where

(i) $f : R^n \to R^l$ and $g : R^n \to R^m$ are differentiable functions.

(ii) C_1 and C_2 are closed convex cones in R^n and R^m with nonempty interiors, respectively.

(iii) $e = (1, \dots, 1)^T$ is a vector in R^l.

(iv) $w_i (i = 1, \dots, l)$ is a vector in R^n and $D_i (i = 1, \dots, l)$ is a compact convex set in R^n, respectively.

(v) $h : R^n \times R^n \to R^l$ and $k : R^n \times R^n \to R^m$ are differentiable functions; $\nabla_p h_j(u,p)$ and $\nabla_p y^T k(u,p)$ denote the $n \times 1$ gradient of h_j and $y^T k$ with respect to p, respectively.

Kim [33] established weak, strong duality theorems for an efficient solution under higher order generalized invexity conditions. In, this dissertation, we establish weak, strong duality theorems under the assumption of Higher order (F, ρ, α, d) type-I.

Chapter 2

Second order symmetric duality in multiobjective programming

2.1 Introduction

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used. Unlike linear programming, the majority of dual formulations in nonlinear programming do not possess the symmetry property. Nonlinear programming problem and its dual are said to be symmetric if the dual of the dual is the original problem. A pair of symmetric dual programs has been formulated by Dantzig et al. [10] and established duality results for convex/concave functions by taking non-negative orthant as the cone. Later, Mishra [21] formulated a pair of multiobjective second-order symmetric dual nonlinear programming problems under second-order pseudo-invexity assumptions on the functions involved over arbitrary cones and established duality results.

In this chapter, we consider a pair of multiobjective second order symmetric dual problems of Mond-Weir type. We establish weak duality, strong duality under the assumptions of η-bonvexity and η-pseudobonvexity. The Chapter is divided into three sections. Section 2.1 is introductory, section 2.2 contains notations and definitions and in section 2.3, we consider a pair of Mond-Weir type second-order multiobjective symmetric dual programs.

2.2 Notations and definitions

Let f be a twice differentiable real valued function of x and y, where $x \in R^n$ and $y \in R^m$. Then $\nabla_x f$ and $\nabla_y f$ denote gradient vectors with respect to x and y, respectively. $\nabla_{xx} f$ and $\nabla_{yy} f$ are, respectively, the $n \times n$ and $m \times m$ symmetric Hessian matrices. $(\partial/\partial y_i)(\nabla_{yy} f)$ is the $m \times m$ matrix obtained by differentiating the elements of $(\nabla_{yy} f)$ with respect to y_i and $(\nabla_{xx} f(x,y)q)_y$ denotes the matrix whose (i,j)th element is $(\partial/\partial y_i)(\nabla_{xx} f(x,y)q)_j$, where $q \in R^n$.

Consider the multiobjective programming problem

$$\textbf{(MP)} \qquad \text{Minimize} \qquad f(x)$$
$$\text{subject to} \qquad x \in X_0,$$

where f is a k-dimensional vector function defined on R^n and $X_0 \subseteq R^n$.

Definition 2.1 A real twice differentiable function f defined on $X \times Y$, where X and Y are open sets in R^n and R^m, respectively, is said to be η_1-bonvex in the first variable at $u \in X$, if there exists a function $\eta_1 : X \times X \to R^n$ such that for $v \in Y,\ q \in R^n,\ x \in X$,

$$f(x,v) - f(u,v) \geq \eta_1^T(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v)q] - \frac{1}{2} q^T \nabla_{xx} f(u,v)q$$

and $f(x,y)$ is said to be η_2-bonvex in the second variable at $v \in Y$, if there exists a function $\eta_2 : Y \times Y \to R^m$ such that for $u \in X,\ p \in R^m,\ y \in Y$,

$$f(u,y) - f(u,v) \geq \eta_2^T(y,v)[\nabla_y f(u,v) + \nabla_{yy} f(u,v)p] - \frac{1}{2} p^T \nabla_{yy} f(u,v)p.$$

Definition 2.2 A real twice differentiable function f defined on $X \times Y$ is said to be η_1-pseudobonvex in the first variable at $u \in X$, if there exists a function $\eta_1 : X \times X \to R^n$ such that for $v \in Y,\ q \in R^n,\ x \in X$,

$$\eta_1^T(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v)q] \geq 0 \Rightarrow f(x,v) - f(u,v) + \frac{1}{2} q^T \nabla_{xx} f(u,v)q \geq 0$$

and $f(x,y)$ is said to be η_2-pseudobonvex in the second variable at $v \in Y$, if there exists a function $\eta_2 : Y \times Y \to R^m$ such that for $u \in X,\ p \in R^m,\ y \in Y$,

$$\eta_2^T(y,v)[\nabla_y f(u,v) + \nabla_{yy} f(u,v)p] \geq 0 \Rightarrow f(u,y) - f(u,v) + \frac{1}{2} p^T \nabla_{yy} f(u,v)p \geq 0.$$

2.3 Mond-Weir type symmetric duality

Consider the following pair of second order multiobjective programming problems with k-objectives:

$$
\begin{aligned}
&\textbf{(P)} \quad \text{Minimize} \quad && F(x,y,p) = (F_1(x,y,p), F_2(x,y,p), \dots, F_k(x,y,p)) \\
&\qquad \text{subject to} && \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \leqq 0, && (2.1) \\
& && y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) \geq 0, && (2.2) \\
& && \lambda > 0,
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{(D)} \quad \text{Maximize} \quad && G(u,v,q) = (G_1(u,v,q), G_2(u,v,q), \dots, G_k(u,v,q)) \\
&\qquad \text{subject to} && \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \geqq 0, && (2.3) \\
& && u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) \leq 0, && (2.4) \\
& && \lambda > 0,
\end{aligned}
$$

where

$$
\begin{aligned}
&F_i(x,y,p) = f_i(x,y) - \frac{1}{2} p_i^T \nabla_{yy} f_i(x,y) p_i, \\
&G_i(u,v,q) = f_i(u,v) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u,v) q_i, \\
&\lambda_i \in R, \quad p_i \in R^m, \quad q_i \in R^n, \quad i = 1,2,\dots,k.
\end{aligned}
$$

Also we take $p = (p_1, p_2, \dots, p_k)$, $q = (q_1, q_2, \dots, q_k)$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)^T$.

In the following theorems, we take $\eta_1 : X \times X \to R^n$, $\eta_2 : Y \times Y \to R^m$, where X and Y are open sets in R^n and R^m, respectively.

Theorem 2.1 (Weak duality). For feasible solutions (x, y, λ, p) of (P) and (u, v, λ, q) of (D), let either of the conditions hold:

(a) For $i = 1, 2, \dots, k$, f_i is η_1-bonvex in the first variable at u and $-f_i$ is η_2-bonvex in the second variable at y.

(b) $\sum_{i=1}^{k} \lambda_i f_i$ is η_1-pseudobonvex in the first variable at u and $-\sum_{i=1}^{k} \lambda_i f_i$ is η_2-pseudobonvex in the second variable at y, and

$$\eta_1(x,u)+u \geqq 0, \tag{2.5}$$

$$\eta_2(v,y)+y \geqq 0. \tag{2.6}$$

Then

$$F(x,y,p) \nleq G(u,v,q).$$

Proof. Since (u,v,λ,q) is feasible for (D), from (2.3) and (2.5), it follows that

$$[\eta_1(x,u)+u]^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v)+\nabla_{xx} f_i(u,v)q_i) \geq 0.$$

Using (2.4), we get

$$\eta_1^T(x,u) \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u,v)+\nabla_{xx} f_i(u,v)q_i) \geq 0. \tag{2.7}$$

Since (x,y,λ,p) is feasible for (P), from (2.1) and (2.6), it follows that

$$[\eta_2(v,y)+y]^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y)+\nabla_{yy} f_i(x,y)p_i) \leq 0.$$

Using (2.2), we get

$$\eta_2^T(v,y) \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x,y)+\nabla_{yy} f_i(x,y)p_i) \leq 0. \tag{2.8}$$

(a) Since f_i is η_1-bonvex in the first variable at u, we have for $i=1,2,\ldots,k$,

$$f_i(x,v)-f_i(u,v)+\frac{1}{2}q_i^T\nabla_{xx} f_i(u,v)q_i \geq \eta_1^T(x,u)[\nabla_x f_i(u,v)+\nabla_{xx} f_i(u,v)q_i].$$

As $\lambda_i > 0,\ i=1,2,\ldots,k$, on using (2.7), we get

$$\sum_{i=1}^{k} \lambda_i(f_i(x,v)-f_i(u,v)+\frac{1}{2}q_i^T\nabla_{xx} f_i(u,v)q_i) \geq 0. \tag{2.9}$$

Since $-f_i$ is η_2-bonvex in the second variable at y, we have for $i=1,2,\ldots,k$,

$$-f_i(x,v)+f_i(x,y)-\frac{1}{2}p_i^T\nabla_{yy} f_i(x,y)p_i \geq -\eta_2^T(v,y)[\nabla_y f_i(x,y)+\nabla_{yy} f_i(x,y)p_i].$$

As $\lambda_i > 0,\ i=1,2,\ldots,k$, on using (2.8), we get

$$-\sum_{i=1}^{k} \lambda_i(f_i(x,v)-f_i(x,y)+\frac{1}{2}p_i^T\nabla_{yy} f_i(x,y)p_i) \geq 0. \tag{2.10}$$

Adding (2.9) and (2.10), we get

$$\sum_{i=1}^{k} \lambda_i (f_i(x,y) - \frac{1}{2} p_i^T \nabla_{yy} f_i(x,y) p_i) \geq \sum_{i=1}^{k} \lambda_i (f_i(u,v) - \frac{1}{2} q_i^T \nabla_{xx} f_i(u,v) q_i).$$

Hence

$$F(x,y,p) \nleq G(u,v,q).$$

(b) As $\sum_{i=1}^{k} \lambda_i f_i$ is η_1-pseudobonvex in the first variable, from (2.7), we get (2.9).
As $-\sum_{i=1}^{k} \lambda_i f_i$ is η_2-pseudobonvex in the second variable, from (2.8), we get (2.10).
On adding (2.9) and (2.10), we arrive at the result as in part (a).

Theorem 2.2 (Strong duality). Let f be thrice differentiable on $R^n \times R^m$. Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ be a weak minimum of (P); fix $\lambda = \bar{\lambda}$ in (D) and suppose that

(a) $\nabla_{yy} f_i$ is non-singular for all $i = 1, 2, \ldots, k$,

(b) the set $\{\nabla_y f_1 + \nabla_{yy} f_1 \bar{p}_1, \nabla_y f_2 + \nabla_{yy} f_2 \bar{p}_2, \ldots, \nabla_y f_k + \nabla_{yy} f_k \bar{p}_k\}$, is linearly independent,

(c) the matrix $\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yy} f_i \bar{p}_i)_y$ is positive or negative definite,

where $f_i = f_i(\bar{x}, \bar{y})$, $i = 1, 2, \ldots, k$. Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)$ is feasible for (D) and $F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{q})$.

Moreover, if the hypothesis of Theorem 2.1 is satisfied for all feasible solutions of (P) and (D), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q})$ is an efficient solution for (D).

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is a weak minimum of (P), by Fritz John optimality conditions [7], there exist $\alpha \in R^k,\ \beta \in R^m,\ \gamma \in R,\ \delta \in R^k$ such that

$$\sum_{i=1}^{k} \alpha_i \left[\nabla_x f_i - \frac{1}{2}(\nabla_{yy} f_i \bar{p}_i)_x \bar{p}_i\right] + \sum_{i=1}^{k} \bar{\lambda}_i \left[\nabla_{yx} f_i + (\nabla_{yy} f_i \bar{p}_i)_x\right](\beta - \gamma \bar{y}) = 0, \tag{2.11}$$

$$\sum_{i=1}^{k} \alpha_i \left[\nabla_y f_i - \frac{1}{2}(\nabla_{yy} f_i \bar{p}_i)_y \bar{p}_i\right] + \sum_{i=1}^{k} \bar{\lambda}_i \left[\nabla_{yy} f_i + (\nabla_{yy} f_i \bar{p}_i)_y\right](\beta - \gamma \bar{y}) - \gamma \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) = 0, \tag{2.12}$$

$$(\beta - \gamma \bar{y})^T (\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) - \delta_i = 0, \quad i = 1, 2, \dots, k, \tag{2.13}$$

$$\left[(\beta - \gamma \bar{y})\bar{\lambda}_i - \alpha_i \bar{p}_i\right]^T \nabla_{yy} f_i = 0, \quad i = 1, 2, \dots, k, \tag{2.14}$$

$$\beta^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) = 0, \tag{2.15}$$

$$\gamma \bar{y} \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) = 0, \tag{2.16}$$

$$\delta^T \bar{\lambda} = 0, \tag{2.17}$$

$$(\alpha, \beta, \gamma, \delta) \geqq 0, \qquad (\alpha, \beta, \gamma, \delta) \neq 0 \tag{2.18}$$

As $\bar{\lambda} > 0$, it follows from (2.17), that $\delta = 0$. Therefore from (2.13), we get

$$(\beta - \gamma \bar{y})^T (\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) = 0, \quad i = 1, 2, \dots, k. \tag{2.19}$$

As $\nabla_{yy} f_i$ is non-singular for $i = 1, 2, \dots, k$, from (2.14), it follows that

$$(\beta - \gamma \bar{y})\bar{\lambda}_i = \alpha_i \bar{p}_i, \qquad i = 1, 2, \dots, k. \tag{2.20}$$

From (2.12), we get

$$\sum_{i=1}^{k} (\alpha_i - \gamma \bar{\lambda}_i) \nabla_y f_i + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{yy} f_i (\beta - \gamma \bar{y} - \gamma \bar{p}_i) + \sum_{i=1}^{k} (\nabla_{yy} f_i \bar{p}_i)_y \left[(\beta - \gamma \bar{y})\bar{\lambda}_i - \frac{1}{2}\alpha_i \bar{p}_i\right] = 0.$$

Using (2.20), it follows that

$$\sum_{i=1}^{k} (\alpha_i - \gamma \bar{\lambda}_i)(\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) + \frac{1}{2} \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yy} f_i \bar{p}_i)(\beta - \gamma \bar{y}) = 0. \tag{2.21}$$

Premultiplying by $(\beta - \gamma\bar{y})^T$ and using (2.19), we get

$$(\beta - \gamma\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yy} f_i \bar{p}_i)_y (\beta - \gamma\bar{y}) = 0.$$

Using the fact that $\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yy} f_i \bar{p}_i)_y$ is positive or negative definite, we get

$$\beta = \gamma\bar{y}. \tag{2.22}$$

Using (2.22) in (2.21), we get

$$\sum_{i=1}^{k} (\alpha_i - \gamma\bar{\lambda}_i)(\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) = 0.$$

By condition (b), we get

$$\alpha_i = \gamma\bar{\lambda}_i, \qquad i = 1, 2, \ldots, k. \tag{2.23}$$

If $\gamma = 0$, from (2.22) and (2.23), it follows that $\beta = 0$, $\alpha = 0$ which contradicts (2.18). Hence $\gamma > 0$. Since $\bar{\lambda}_i > 0,\ i = 1, 2, \ldots, k$, from (2.23) we have
$\alpha_i > 0,\ i = 1, 2, \ldots, k$. Using (2.22) in (2.20), we have $\alpha\bar{p}_i = 0,\ i = 1, 2, \ldots, k$, and hence $\bar{p}_i = 0,\ i = 1, 2, \ldots, k$. Using (2.22) and the fact that $\bar{p}_i = 0,\ i = 1, 2, \ldots, k$, in (2.11), it follows that

$$\sum_{i=1}^{k} \alpha_i \nabla_x f_i = 0,$$

which by (2.23) gives

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i = 0,$$

and hence we also have

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i = 0,$$

Thus it follows that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of (D) and

$$F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{q}). \tag{2.24}$$

If $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q})$ is not efficient for (D) then there exists a feasible solution $(u, v, \bar{\lambda}, q)$ of (D) such that

$$G(\bar{x}, \bar{y}, \bar{q}) \leq G(u, v, q)$$

which by (2.24) gives

$$F(\bar{x}, \bar{y}, \bar{p}) \leq G(u, v, q)$$

which is a contradiction to Theorem 2.1.

Chapter 3

Higher-order (F, α, ρ, d)-convexity and symmetric duality in multiobjective programming

3.1 Introduction

One practical advantage of higher-order duality is that it provides tighter bounds for the value of objective function of the primal problem when approximations are used because there are more parameters involved. Higher-order duality in nonlinear programming has been studied in the last few years by many researchers $[3, 2, 8, 14, 20, 27]$. A class of higher-order dual problems for nonlinear programming problems first formulated by Mangasarian [20].

In this paper, we formulate a pair of higher-order Wolfe type and Mond-Weir type multiobjective symmetric dual programs over arbitrary cones. Weak and strong theorems are proved under higher-order (F, α, ρ, d)- convexity/pseudo-convexity assumptions. The Chapter is divided into four sections. Section 3.1 is introductory, section 3.2 contains notations and definitions. In section 3.3, we consider a pair of Wolfe type Higher-order multiobjective symmetric dual programs and in section 3.4, we consider a pair of Mond-Weir type higher-order multiobjective symmetric dual programs

3.2 Notations and Preliminaries

Consider the following multiobjective programming problem:

$$\textbf{(P)} \quad \text{Minimize} \quad \phi(x)$$
$$\text{subject to} \quad -g(x) \in Q, \quad x \in C,$$

where $C \subseteq R^n,\ \phi : R^n \to R^k,\ g : R^n \to R^m,\ Q$ is closed convex cone with non-empty interior in R^m.

Let $X^0 = \{x \in C : -g(x) \in Q\}$, be the set of all feasible solutions of (P).
Let C_1 and C_2 be closed convex cones with non-empty interiors in R^n and R^m, respectively.
Definition 3.1 The positive polar cone C_i^* of C_i $(i = 1, 2)$ is defined as

$$C_i^* = \{z \in R^n : x^T z \geqq 0,\ \text{for all } x \in C_i\}.$$

Now we consider a function $\phi = (\phi_1, \phi_2, \ldots, \phi_k) : X \to R^k$ differentiable at $u \in X$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in R^k$ and $d = (d_1, d_2, \ldots, d_k) \in R^k$.
Definition 3.2 A twice differentiable function ϕ_i over X is said to be higher-order (F, α, ρ_i, d_i)-convex at u on X with respect to $\zeta_i : X \times X \to R$, if for all $x \in X$ and $q \in R^n$, there exist a real valued function $\alpha : X \times X \to R_+ \setminus \{0\}$, a real valued function $d_i(\cdot, \cdot) : X \times X \to \mathrm{R}$ and a real number ρ_i such that

$$\phi_i(x) - \phi_i(u) - \zeta_i(u, q) + q^T \nabla_q \zeta_i(u, q) \geqq F_{x,u}\big[\alpha(x, u)(\nabla_x \phi_i(u) + \nabla_q \zeta_i(u, q))\big] + \rho_i d_i^2(x, u).$$

Definition 3.3 A twice differentiable function ϕ_i over X is said to be higher-order (F, α, ρ_i, d_i)-pseudoconvex at u on X with respect to $\zeta_i : X \times X \to R$, if for all $x \in X$ and $q \in R^n$, there exist a real valued function $\alpha : X \times X \to R_+ \setminus \{0\}$, a real valued function $d_i(\cdot, \cdot) : X \times X \to \mathrm{R}$ and a real number ρ_i such that

$$F_{x,u}\big[\alpha(x, u)(\nabla_x \phi_i(u) + \nabla_q \zeta_i(u, q))\big] + \rho_i d_i^2(x, u) \geqq 0$$
$$\to \phi_i(x) - \phi_i(u) - \zeta_i(u, q) + q^T \nabla_q \zeta_i(u, q) \geqq 0.$$

A twice differentiable vector function $\phi : X \to R^k$ is said to be higher-order (F, α, ρ, d)-convex/pseudoconvex at u, if each of its components ϕ_i is higher-order (F, α, ρ_i, d_i)-convex/pseudoconvex at u.

3.3 Wolfe type higher-order symmetric duality

In this section, we consider the following Wolfe type multiobjective higher-order symmetric dual programs:

Primal problem **(HWP)**

$$\begin{aligned} \text{Minimize } \ L(x,y,\lambda,p) \ &= \ f(x,y)+(\lambda^T h)(x,y,p)e_k - p^T\nabla_p(\lambda^T h)(x,y,p)e_k \\ &\quad -y^T\nabla_y(\lambda^T f)(x,y)e_k - y^T\nabla_p(\lambda^T h)(x,y,p)e_k \end{aligned}$$

$$\text{subject to} \qquad -\{\nabla_y(\lambda^T f)(x,y)+\nabla_p(\lambda^T h)(x,y,p)\}\in C_2^*, \tag{3.1}$$

$$\lambda^T e_k = 1, \tag{3.2}$$

$$\lambda > 0, \quad x \in C_1. \tag{3.3}$$

Dual problem **(HWD)**

$$\begin{aligned} \text{Maximize } \ M(u,v,\lambda,r) \ &= \ f(u,v)+(\lambda^T g)(u,v,r)e_k - r^T\nabla_r(\lambda^T g)(u,v,r)e_k \\ &\quad -u^T\nabla_x(\lambda^T f)(u,v)e_k - u^T\nabla_r(\lambda^T g)(u,v,r)e_k \end{aligned}$$

$$\text{subject to} \qquad \nabla_x(\lambda^T f)(u,v)+\nabla_r(\lambda^T g)(u,v,r)\in C_1^*, \tag{3.4}$$

$$\lambda^T e_k = 1, \tag{3.5}$$

$$\lambda > 0, \qquad v \in C_2. \tag{3.6}$$

where

(i) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$,

(ii) $f : S_1 \times S_2 \to R^k,\ h : S_1 \times S_2 \times R^m \to R^k$ and $g : S_1 \times S_2 \times R^n \to R^k$ are differentiable functions, $e_k = (1,\ldots,1)^T \in R^k,\ \lambda \in R^k$ and

(iii) r and p are vectors in R^n and R^m, respectively.

Theorem 3.1 (Weak duality). Let (x,y,λ,p) be feasible for the primal problem (HWP) and (u,v,λ,r) be feasible for the dual problem (HWD). Let for $i=1,2,\ldots,k$

(i) $f_i(.,v)$ be higher-order $(F,\alpha_1,\rho_i^{(1)},d_i^{(1)})$-convex at u with respect to $g_i(u,v,r)$,

(ii) $-f_i(x,.)$ be higher-order $(G,\alpha_2,\rho_i^{(2)},d_i^{(2)})$-convex at y with respect to $-h_i(x,y,p)$,

(iii) either (a) $\sum_{i=1}^{k}\lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2+\rho_i^{(2)}(d_i^{(2)}(v,y))^2]\geqq 0$ or (b) $\rho_i^{(1)}\geqq 0$ and $\rho_i^{(2)}\geqq 0$, for all i,

where the sublinear functionals $F:R^n\times R^n\times R^n\to R$ and $G:R^m\times R^m\times R^m\to R$ satisfy the following conditions:

(iv) $F_{x,u}(a)+\alpha_1^{-1}a^Tu\geqq 0$, for all $a\in C_1^*$ and

(v) $G_{v,y}(b)+\alpha_2^{-1}b^Ty\geqq 0$, for all $b\in C_2^*$.

Then

$$L(x,y,\lambda,p)\nleq M(u,v,\lambda,r). \tag{3.7}$$

Proof. Assume by contradiction that (3.7) is not true, that is

$$L(x,y,\lambda,p)\leq M(u,v,\lambda,r), \qquad \text{or}$$

$$\begin{aligned}&f(x,y)+(\lambda^Th)(x,y,p)e_k-p^T\nabla_p(\lambda^Th)(x,y,p)e_k-y^T\nabla_y(\lambda^Tf)(x,y)e_k\\&-y^T\nabla_p(\lambda^Th)(x,y,p)e_k\leq f(u,v)+(\lambda^Tg)(u,v,r)e_k-r^T\nabla_r(\lambda^Tg)(u,v,r)e_k\\&-u^T\nabla_x(\lambda^Tf)(u,v)e_k-u^T\nabla_r(\lambda^Tg)(u,v,r)e_k.\end{aligned}$$

Since $\lambda>0$and $\lambda^Te=1$,we obtain

$$\begin{aligned}&(\lambda^Tf)(x,y)+(\lambda^Th)(x,y,p)-p^T\nabla_p(\lambda^Th)(x,y,p)-y^T\nabla_y(\lambda^Tf)(x,y)\\&-y^T\nabla_p(\lambda^Th)(x,y,p)<(\lambda^Tf)(u,v)+(\lambda^Tg)(u,v,r)-r^T\nabla_r(\lambda^Tg)(u,v,r)\\&-u^T\nabla_x(\lambda^Tf)(u,v)-u^T\nabla_r(\lambda^Tg)(u,v,r).\end{aligned} \tag{3.8}$$

Since (x,y,λ,p) is feasible for the primal problem (HWP) and (u,v,λ,r) is feasible for the dual problem (HWD), $\alpha_1(x,u)>0$, by the dual constraint (3.4), the vector $a=\alpha_1(x,u)[\nabla_x(\lambda^Tf)(u,v)+\nabla_r(\lambda^Tg)(u,v,r)]\in C_1^*$ and so from the hypothesis (iv), we obtain

$$F_{x,u}(a)+\alpha_1^{-1}a^Tu\geqq 0. \tag{3.9}$$

Similarly,

$$G_{v,y}(b)+\alpha_2^{-1}b^Ty\geqq 0. \tag{3.10}$$

for the vector $b = -\alpha_2(v,y)[\nabla_y(\lambda^T f)(x,y) + \nabla_p(\lambda^T h)(x,y,p)] \in C_2^*$.
By higher-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convexity of $f_i(.,v)(1 \leq i \leq k)$ with respect to $g_i(u,v,r)$, we have

$$\begin{aligned}
&f_i(x,v) - f_i(u,v) - g_i(u,v,r) + r^T \nabla_r g_i(u,v,r) \\
&\quad \geqq F_{x,u}[\alpha_1(x,u)(\nabla_x f_i(u,v) + \nabla_r g_i(u,v,r))] + \rho_i^{(1)} \left(d_i^{(1)}(x,u) \right)^2.
\end{aligned}$$

It follows from $\lambda > 0$ and sublinearity of F that

$$\begin{aligned}
&(\lambda^T f)(x,v) - (\lambda^T f)(u,v) - (\lambda^T g)(u,v,r) + r^T \nabla_r (\lambda^T g)(u,v,r) \\
&\geqq F_{x,u}[\alpha_1(x,u)(\nabla_x(\lambda^T f)(u,v) + \nabla_r(\lambda^T g)(u,v,r))] + \sum_{i=1}^{k} \lambda_i \rho_i^{(1)} \left(d_i^{(1)}(x,u) \right)^2,
\end{aligned}$$

or

$$\begin{aligned}
&(\lambda^T f)(x,v) - (\lambda^T f)(u,v) - (\lambda^T g)(u,v,r) + r^T \nabla_r (\lambda^T g)(u,v,r) \\
&- \sum_{i=1}^{k} \lambda_i \rho_i^{(1)} \left(d_i^{(1)}(x,u) \right)^2 \geqq F_{(x,u)}(a). \qquad (3.11)
\end{aligned}$$

using (3.9) in (3.11), we have

$$\begin{aligned}
&(\lambda^T f)(x,v) - (\lambda^T f)(u,v) - (\lambda^T g)(u,v,r) + r^T \nabla_r (\lambda^T g)(u,v,r) \\
&- \sum_{i=1}^{k} \lambda_i \rho_i^{(1)} \left(d_i^{(1)}(x,u) \right)^2 \geqq -\alpha_1^{-1} u^T a. \qquad (3.12)
\end{aligned}$$

Similarly, using hypothesis (ii) and (v) along with primal constraint (3.1) and inequality (3.10), $\lambda > 0$, $\alpha_2(v,y) > 0$ and sublinearity of G, we get

$$\begin{aligned}
&(\lambda^T f)(x,y) - (\lambda^T f)(x,v) + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) \\
&- \sum_{i=1}^{k} \lambda_i \rho_i^{(2)} \left(d_i^{(2)}(v,y) \right)^2 \geqq -\alpha_2^{-1} y^T b. \qquad (3.13)
\end{aligned}$$

Adding the inequalities (3.12) and (3.13), we obtain

$$\begin{aligned}
&(\lambda^T f)(x,y) + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) + \alpha_2^{-1} y^T b - (\lambda^T f)(u,v) \\
&-(\lambda^T g)(u,v,r) + r^T \nabla_r (\lambda^T g)(u,v,r) + \alpha_1^{-1} u^T a \\
&\geqq \sum_{i=1}^{k} \lambda_i \left[\rho_i^{(1)} \left(d_i^{(1)}(x,u) \right)^2 + \rho_i^{(2)} \left(d_i^{(2)}(v,y) \right)^2 \right].
\end{aligned}$$

Further, using hypothesis (iii) in the above inequality, we get

$$(\lambda^T f)(x,y) + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) + \alpha_2^{-1} y^T b$$
$$\geqq (\lambda^T f)(u,v) + (\lambda^T g)(u,v,r) - r^T \nabla_r (\lambda^T g)(u,v,r) - \alpha_1^{-1} u^T a.$$

Finally substituting the values of a and b, we have

$$(\lambda^T f)(x,y) + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) - y^T \nabla_y (\lambda^T f)(x,y)$$
$$-y^T \nabla_p (\lambda^T h)(x,y,p) \geqq (\lambda^T f)(u,v) + (\lambda^T g)(u,v,r) - r^T \nabla_r (\lambda^T g)(u,v,r)$$
$$-u^T \nabla_x (\lambda^T f)(u,v) e - u^T \nabla_r (\lambda^T g)(u,v,r).$$

which contradicts (3.8). Hence the result.

If the variable λ in the problems (HWP) and (HWD) is fixed to be $\bar{\lambda}$, we shall denote these problems by $(HWP)_{\bar{\lambda}}$ and $(HWD)_{\bar{\lambda}}$.

Theorem 3.2 (Strong duality). Let $f : S_1 \times S_2 \to R^k$ be twice differentiable function and let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ be a weak efficient solution of (HWP). Suppose that

(i) the matrix $\nabla_{pp}(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})$ is non singular,

(ii) the vectors $\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})$ are linearly independent,

(iii) the vector $\{\nabla_y (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) p\} \notin$ span $\{\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\} \setminus \{0\}$,

(iv) $\nabla_y (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) \bar{p} = 0$ implies $\bar{p} = 0$ and

(v) $(\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = (\bar{\lambda}^T g)(\bar{x}, \bar{y}, 0)$, $\nabla_x (\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = \nabla_r (\bar{\lambda}^T g)(\bar{x}, \bar{y}, 0)$, $\nabla_y (\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = 0$ and $\nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, 0) = 0$.

Then $\bar{r} = 0$, $(\bar{x}, \bar{y}, \bar{r} = 0)$ is feasible for $(HWD)_{\bar{\lambda}}$, and the objective values of (HWP) and $(HWD)_{\bar{\lambda}}$ are equal. Furthermore, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (HWP) and $(HWD)_{\bar{\lambda}}$, then $(\bar{x}, \bar{y}, \bar{r} = 0)$ is an efficient solution for $(HWD)_{\bar{\lambda}}$.

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is a weak efficient solution of (HWP), by the Fritz John necessary optimality conditions [30], there exist $\bar{\alpha} \in R^k_+$, $\bar{\beta} \in C_2$, $\bar{\mu} \in R^k_+$, $\bar{\eta} \in R$, such that the following conditions are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$:

$$\{\bar{\alpha}^T \nabla_x f(\bar{x}, \bar{y}) + \nabla_x (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})(\bar{\alpha}^T e_k) + \nabla_{xy} (\bar{\lambda}^T f)(\bar{x}, \bar{y})[\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}]$$
$$+ \nabla_{px} (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})]\}(x - \bar{x}) \geq 0, \text{ for all } x \in C_1, \tag{3.14}$$

$$\nabla_y f(\bar{x}, \bar{y})[\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}] + [\nabla_y (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})](\bar{\alpha}^T e_k)$$
$$+ \nabla_{yy} (\bar{\lambda}^T f)(\bar{x}, \bar{y})[\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}]$$
$$+ \nabla_{py} (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})] = 0, \tag{3.15}$$

$$\nabla_{pp} (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})] = 0, \tag{3.16}$$

$$\nabla_y f(\bar{x}, \bar{y})[\bar{\beta} - (\bar{\alpha}^T e_k)\bar{y}] + h(\bar{x}, \bar{y}, \bar{p})(\bar{\alpha}^T e_k) - \bar{\mu} + \bar{\eta} e_k$$
$$+ \nabla_p h(\bar{x}, \bar{y}, \bar{p})[\bar{\beta} - (\bar{\alpha}^T e_k)(\bar{y} + \bar{p})] = 0, \tag{3.17}$$

$$\bar{\beta}^T [\nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})] = 0, \tag{3.18}$$

$$\bar{\mu}^T \bar{\lambda} = 0, \tag{3.19}$$

$$\bar{\eta}[\bar{\lambda}^T e_k - 1] = 0, \tag{3.20}$$

$$(\bar{\alpha}, \bar{\beta}, \bar{\mu}, \bar{\eta}) \neq 0. \tag{3.21}$$

Since $\bar{\lambda} > 0$ and $\bar{\mu} \geqq 0$, (3.19) yields $\bar{\mu} = 0$.

From (3.16) and nonsingularity of $\nabla_{pp} (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})$, we have

$$\bar{\beta} = (\bar{\alpha}^T e_k)(\bar{y} + \bar{p}). \tag{3.22}$$

If $\bar{\alpha} = 0$, then (3.17) and (3.22) yields $\bar{\eta} = 0$ and $\bar{\beta} = 0$, respectively. Consequently $(\bar{\alpha}, \bar{\beta}, \bar{\mu}, \bar{\eta}) = 0$, contradicting (3.21). Hence, $\bar{\alpha} \geq 0$ or

$$\bar{\alpha}^T e_k > 0. \tag{3.23}$$

Now, using (3.22) and (3.23) in (3.15), we get

$$\nabla_y (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) - \nabla_p (\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy} (\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p}$$
$$= -\frac{1}{\bar{\alpha}^T e_k} \nabla_y f(\bar{x}, \bar{y})[\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}]. \tag{3.24}$$

which by hypothesis (iii) and (iv) implies

$$\bar{p} = 0. \tag{3.25}$$

It follows from hypothesis (iii) and (3.24) that

$$\nabla_y f(\bar{x}, \bar{y})[\bar{\alpha} - (\bar{\alpha}^T e_k)\bar{\lambda}] = 0.$$

Since the vectors $\{\nabla_y f_1(\bar{x}, \bar{y}), \dots, \nabla_y f_k(\bar{x}, \bar{y})\}$ are linearly independent, therefore the above equation yields

$$\bar{\alpha} = (\bar{\alpha}^T e_k)\bar{\lambda}. \tag{3.26}$$

From (3.25) in (3.22), we get

$$\bar{\beta} = (\bar{\alpha}^T e_k)\bar{y}. \tag{3.27}$$

Using (3.23) and (3.25)-(3.27) in (3.14), we have

$$\{\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_x(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})\}(x - \bar{x}) \geq 0, \quad \text{for all } x \in C_1.$$

from hypothesis (v), for $\bar{r} = 0$, the above inequality yields

$$\{\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})\}(x - \bar{x}) \geq 0. \tag{3.28}$$

Let $x \in C_1$, Then $x + \bar{x} \in C_1$ and so (3.28) implies

$$\{\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})\}x \geq 0, \quad \text{for all } x \in C_1.$$

Therefore,

$$\{\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})\} \in C_1^*. \tag{3.29}$$

Also, from (3.27), we have

$$\bar{y} = \frac{\bar{\beta}}{\bar{\alpha}^T e_k} \in C_2.$$

Thus $(\bar{x}, \bar{y}, \bar{r} = 0)$ satisfies the constraint (3.4)-(3.6) in $(HWD)_{\bar{\lambda}}$, and so it is feasible for the dual problem $(HWD)_{\bar{\lambda}}$.

Now, letting $x = 0$ and $x = 2\bar{x}$ in (3.28), we get

$$\bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x}, \bar{y}, \bar{r})] = 0. \tag{3.30}$$

Further from (3.18), (3.23) and (3.27), we obtain

$$\bar{y}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_p(\bar{\lambda}^T h)(\bar{x}, \bar{y}, \bar{p})] = 0. \tag{3.31}$$

Therefore, using (3.25), (3.30), (3.31) and hypothesis (v), for $\bar{r}=0$, we get

$$f(\bar{x},\bar{y}) + (\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k - \bar{p}^T \nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k - \bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y})e_k$$
$$-\bar{y}^T \nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k = f(\bar{x},\bar{y}) + (\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k - \bar{r}^T \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k$$
$$-\bar{x}^T \nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})e_k - \bar{x}^T \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k.$$

that is, the two objective values are equal.

Now, let $(\bar{x},\bar{y},\bar{r}=0)$ is not an efficient solution of $(HWD)_{\bar{\lambda}}$, then there exist $(\bar{u},\bar{v},\bar{r}=0)$ feasible for $(HWD)_{\bar{\lambda}}$ such that,

$$f(\bar{x},\bar{y}) + (\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k - \bar{r}^T \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k - \bar{x}^T \nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})e_k -$$
$$\bar{x}^T \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})e_k \leq f(\bar{u},\bar{v}) + (\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k - \bar{r}^T \nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k$$
$$-\bar{u}^T \nabla_x(\bar{\lambda}^T f)(\bar{u},\bar{v})e_k - \bar{u}^T \nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k.$$

As $\bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})] = 0 = \bar{y}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})]$ and from hypothesis (v), for $\bar{r}=0$, we obtain

$$f(\bar{x},\bar{y}) + (\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k - \bar{p}^T \nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k - \bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y})e_k -$$
$$\bar{y}^T \nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})e_k \leq f(\bar{u},\bar{v}) + (\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k - \bar{r}^T \nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k$$
$$-\bar{u}^T \nabla_x(\bar{\lambda}^T f)(\bar{u},\bar{v})e_k - \bar{u}^T \nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})e_k.$$

which contradicts the weak duality theorem. Hence $(\bar{x},\bar{y},\bar{r}=0)$ is an efficient solution of $(HWD)_{\bar{\lambda}}$.

3.4 Mond-Weir type higher-order symmetric duality

We now formulate following pair of Mond-Weir type higher-order multiobjective symmetric dual programs over cones:

Primal problem **(HMP)**

$$\text{Minimize} \quad F(x,y,p_1,p_2,\ldots,p_k) = (F_1(x,y,p_1), F_2(x,y,p_2),\ldots,F_k(x,y,p_k))$$

$$\text{subject to} \quad -\sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x,y) + \nabla_{p_i} h_i(x,y,p_i)] \in C_2^*, \tag{3.32}$$

$$y^T \sum_{i=1}^{k} \lambda_i[\nabla_y f_i(x,y) + \nabla_{p_i} h_i(x,y,p_i)] \geqq 0, \tag{3.33}$$

$$\lambda > 0, \quad x \in C_1. \tag{3.34}$$

Dual problem **(HMD)**

$$\begin{aligned}
&\text{Maximize} && H(u,v,r_1,r_2,\ldots,r_k)=(H_1(u,v,r_1),H_2(u,v,r_2),\ldots,H_k(u,v,r_k)) \\
&\text{subject to} && \sum_{i=1}^{k}\lambda_i[\nabla_x f_i(u,v)+\nabla_{r_i} g_i(u,v,r_i)]\in C_1^*, && (3.35)\\
& && u^T\sum_{i=1}^{k}\lambda_i[\nabla_x f_i(u,v)+\nabla_{r_i} g_i(u,v,r_i)]\leqq 0, && (3.36)\\
& && \lambda>0,\quad v\in C_2. && (3.37)
\end{aligned}$$

where for $i=1,2,\ldots,k$, $F_i(x,y,p_i)=f_i(x,y)+h_i(x,y,p_i)-p_i^T[\nabla_{p_i}h_i(x,y,p_i)]$,
$H_i(u,v,r_i)=f_i(u,v)+g_i(u,v,r_i)-r_i^T[\nabla_{r_i}g_i(u,v,r_i)]$,
$f_i: S_1\times S_2\to R$, $h_i: S_1\times S_2\times R^m\to R$ and $g_i: S_1\times S_2\times R^n\to R$ are differentiable functions, $p_i\in R^m$ and $r_i\in R^n$.

Theorem 3.3 (Weak Duality). Let $(x,y,\lambda,p_1,p_2,\ldots,p_k)$ be feasible for the primal problem (HMP) and $(u,v,\lambda,r_1,r_2,\ldots,r_k)$ be feasible for the dual problem (HMD). Let for $i=1,2,\ldots,k$

(i) $f_i(.,v)$ be higher-order $(F,\alpha_1,\rho_i^{(1)},d_i^{(1)})$-convex at u with respect to $g_i(u,v,r_i)$,

(ii) $-f_i(x,.)$ be higher-order $(G,\alpha_2,\rho_i^{(2)},d_i^{(2)})$-convex at y with respect to $-h_i(x,y,p_i)$,

(iii) either (a) $\sum_{i=1}^{k}\lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2+\rho_i^{(2)}(d_i^{(2)}(v,y))^2]\geqq 0$ or (b) $\rho_i^{(1)}\geqq 0$ and $\rho_i^{(2)}\geqq 0$, for all i,

where the sublinear functionals $F: R^n\times R^n\times R^n\to R$ and $G: R^m\times R^m\times R^m\to R$ satisfy the following conditions:

(iv) $F_{x,u}(a)+\alpha_1^{-1}a^Tu\geqq 0$, for all $a\in C_1^*$ and

(v) $G_{v,y}(b)+\alpha_2^{-1}b^Ty\geqq 0$, for all $b\in C_2^*$.

Then

$$F(x,y,p_1,p_2,\ldots,p_k)\nleq H(u,v,r_1,r_2,\ldots,r_k).$$

Proof. Suppose, to the contrary, that

$$F(x,y,p_1,p_2,\ldots,p_k)\leq H(u,v,r_1,r_2,\ldots,r_k).$$

Since $\lambda > 0$, we obtain

$$\sum_{i=1}^{k} \lambda_i[f_i(x,y) + h_i(x,y,p_i) - p_i^T(\nabla_{p_i} h_i(x,y,p_i))] <$$
$$\sum_{i=1}^{k} \lambda_i[f_i(u,v) + g_i(u,v,r_i) - r_i^T(\nabla_{r_i} g_i(u,v,r_i))] \tag{3.38}$$

Since $(x, y, \lambda, p_1, p_2, \ldots, p_k)$ is feasible for the primal problem (HMP) and $(u, v, \lambda, r_1, r_2, \ldots, r_k)$ is feasible for the dual problem (HMD), $\alpha_1(x,u) > 0$, by the dual constraint (3.35), the vector $a = \alpha_1(x,u)\{\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_{r_i} g_i(u,v,r_i)]\} \in C_1^*$ and so from the hypothesis (iv), we obtain

$$F_{x,u}(a) \geqq -\alpha_1^{-1} u^T a.$$

Substituting the value of a in the above inequality, we obtain

$$\begin{aligned} & F_{x,u}\Bigg(\alpha_1(x,u)\Bigg\{\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_{r_i} g_i(u,v,r_i)]\Bigg\}\Bigg) \\ & \geqq -u^T \sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_{r_i} g_i(u,v,r_i)] \\ & \geqq 0 \text{ [by dual constraint(3.36)].} \end{aligned} \tag{3.39}$$

By higher-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convexity of $f_i(., v)(1 \leq i \leq k)$ with respect to $g_i(u,v,r_i)$, we have

$$\begin{aligned} & f_i(x,v) - f_i(u,v) - g_i(u,v,r_i) + r_i^T \nabla_{r_i} g_i(u,v,r_i) \\ & \quad \geqq F_{x,u}[\alpha_1(x,u)(\nabla_x f_i(u,v) + \nabla_{r_i} g_i(u,v,r_i))] + \rho_i^{(1)}\Big(d_i^{(1)}(x,u)\Big)^2. \end{aligned}$$

Using the sublinearity of the functional F about the third variable, and multiplying each inequality by λ_i and summing over i, we obtain

$$\begin{aligned} & \sum_{i=1}^{k} \lambda_i\Bigg[f_i(x,v) - f_i(u,v) - g_i(u,v,r_i) + r_i^T(\nabla_{r_i} g_i(u,v,r_i)) - \rho_i^{(1)}\Big(d_i^{(1)}(x,u)\Big)^2\Bigg] \\ & \geqq F_{x,u}\Bigg(\alpha_1(x,u)\Bigg\{\sum_{i=1}^{k} \lambda_i[\nabla_x f_i(u,v) + \nabla_{r_i} g_i(u,v,r_i)]\Bigg\}\Bigg). \\ & \geqq 0 \text{ [by (3.39)].} \end{aligned} \tag{3.40}$$

Similarly using hypothesis (ii) and (v) along with primal constraint (3.32) and (3.33), $\alpha_2(v,y) > 0$ and sublinearity of G, we get

$$\sum_{i=1}^{k} \lambda_i \left[f_i(x,y) - f_i(x,v) + h_i(x,y,p_i) - {p_i}^T(\nabla_{p_i} h_i(x,y,p_i)) - \rho_i^{(2)} \left(d_i^{(2)}(v,y) \right)^2 \right] \geqq 0. \quad (3.41)$$

Adding the inequalities (3.40) and (3.41), we obtain

$$\sum_{i=1}^{k} \lambda_i [f_i(x,y) + h_i(x,y,p_i) - {p_i}^T(\nabla_{p_i} h_i(x,y,p_i)) - f_i(u,v) - g_i(u,v,r_i)$$
$$+ r_i^T(\nabla_{r_i} g_i(u,v,r_i))] \geqq \sum_{i=1}^{k} \lambda_i \left[\rho_i^{(1)} \left(d_i^{(1)}(x,u) \right)^2 + \rho_i^{(2)} \left(d_i^{(2)}(x,u) \right)^2 \right]. \quad (3.42)$$

Using hypothesis (iii) in (3.42),we get

$$\sum_{i=1}^{k} \lambda_i \left[f_i(x,y) + h_i(x,y,p_i) - {p_i}^T(\nabla_{p_i} h_i(x,y,p_i)) \right]$$
$$\geqq \sum_{i=1}^{k} \lambda_i \left[f_i(u,v) + g_i(u,v,r_i) - r_i^T(\nabla_{r_i} g_i(u,v,r_i)) \right]. \quad (3.43)$$

which contradicts (3.38). Hence the result.

Theorem 3.4 (Strong duality). Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p_1}, \bar{p_2}, \ldots, \bar{p_k})$ be an efficient solution of (HMP), $f_i : S_1 \times S_2 \to R$ is a thrice differentiable function at $(\bar{x}, \bar{y})$, $h_i : S_1 \times S_2 \times R^m \to R$ is a twice differentiable function at $(\bar{x}, \bar{y}, \bar{p_i})$, $g_i : S_1 \times S_2 \times R^n \to R$ is differentiable at $(\bar{x}, \bar{y}, \bar{r_i})$, $i = 1, 2, \ldots, k$. If the following conditions hold:

(i) $h_i(\bar{x}, \bar{y}, 0) = 0,\ g_i(\bar{x}, \bar{y}, 0) = 0,\ \nabla_x h_i(\bar{x}, \bar{y}, 0) = \nabla_{r_i} g_i(\bar{x}, \bar{y}, 0),\ \nabla_{p_i} h_i(\bar{x}, \bar{y}, 0) = 0,\ \nabla_y h_i(\bar{x}, \bar{y}, 0) = 0,\ i = 1, 2, \ldots, k$.

(ii) for all $i = 1, 2, \ldots, k$, the Hessian matrix $\nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p_i})$ is positive or negative definite,

(iii) the set of vectors $\{\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p_i})\},\ i = 1, 2, \ldots, k$, is linearly independent,

(iv) the set of vectors $\{\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p_i}), \nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p_i})\},\ i = 1, 2, \ldots, k$, is linearly independent,

(v) for some $\alpha \in R^k (\alpha > 0)$ and $p_i \in R^m$, $p_i \neq 0$ $(i = 1, 2, \ldots, k)$ implies that $\sum_{i=1}^{k} \alpha_i {p_i}^T [\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p_i})] \neq 0$.

Then $\bar{r}_i = 0$ $(i = 1, 2, \ldots, k)$, $(\bar{x}, \bar{y}, \bar{r}_1 = \bar{r}_2 = \cdots = \bar{r}_k = 0)$ is feasible for $(HMD)_{\bar{\lambda}}$, and the two objectives values are equal. Furthermore, if the hypothesis of Theorems 3.3 is satisfied, then $(\bar{x}, \bar{y}, \bar{r}_1 = \bar{r}_2 = \cdots = \bar{r}_k = 0)$ is an efficient solution of $(HMD)_{\bar{\lambda}}$.

Proof. It follows on the lines of Theorem 3.2 in [3] on taking $K = R_+^k$ and omitting the nondifferentiable terms.

Chapter 4

Nondifferentiable higher order duality in multiobjective programming involving cones

4.1 Introduction

In this chapter, we consider the nonlinear programming problem

(P) Minimize $f(x)$

subject to $g(x) \geqq 0,$

where f and g are twice differentiable functions from R^n into R and R^m, respectively.

Higher order duality in nonlinear programming has been studied by many researchers. Mangasarian [20] introduced higher-order duality in nonlinear programming by introducing twice differentiable functions $h : R^n \times R^n \to R$ and $k : R^n \times R^n \to R^m$. Mishra and Rueda [22] introduced higher order type I functions and established various higher order duality results involving these functions.

In this chapter, we introduce the nondifferentiable higher order multiobjective problem involving cone constraints, where every component of the objective function contains a term involving the support function of a compact convex set. For this problem, Mond-Weir type dual is proposed. We establish weak, strong duality theorems for an efficient solution under higher order generalized invexity conditions. The Chapter is divided into three sections.

Section 4.1 is introductory, section 4.2 contains notations and definitions. In section 4.3, we consider a pair of Mond-Weir type nondifferentiable higher-order multiobjective dual programs.

4.2 Notations and definitions

Consider the following non-differentiable multiobjective programming problem:

$$\textbf{(NMP)} \quad \text{Minimize} \quad f(x) + s(x|D) = \{f_1(x) + s(x|D_1), f_2(x) + s(x|D_2), \ldots, f_l(x) + s(x|D_l)\}$$

$$\text{subject to} \quad -g(x) \in C_2^*, \quad x \in C_1.$$

where $f : R^n \to R^l,\ g : R^n \to R^m,\ C_1$ and C_2 are closed convex cones with nonempty interiors in R^n and R^m, respectively.

Definition 4.1 Let $F : X \times X \times R^n \to R$ (where $X \subseteq R^n$) is sublinear functional, $\alpha : X \times X \to R_+ \setminus \{0\},\ p \in R^n,\ \rho \in R$ and $d : X \times X \to R$ be a metric.

(i) A differentiable function f and the constraint function g are said to be of higher order (F, α, ρ, d) type I-Convex at $u \in X$, with respect to $h : X \times X \to R^l$ and $k : X \times X \to R^m$, if for all $x \in X$, the following inequalities hold:

$$f(x) + x^T w - f(u) - u^T w \geqq F[x, u; \alpha(x, u)(\nabla f(u) + \nabla_p h(u, p) + w)]$$
$$+h(u, p) - p^T \nabla_p h(u, p) + \rho d^2(x, u),$$

and

$$-g(u) \geqq F[x, u; \alpha(x, u)(\nabla g(u) + \nabla_p k(u, p))] + k(u, p) - p^T \nabla_p k(u, p)$$
$$+\rho d^2(x, u).$$

(ii) A differentiable function f and the constraint function g are said to be of higher order (F, α, ρ, d) type I-pseudoconvex at $u \in X$, with respect to $h : X \times X \to R^l$ and $k : X \times X \to R^m$, if for all $x \in X$, the following inequalities hold:

$$F[x, u; \alpha(x, u)(\nabla f(u) + \nabla_p h(u, p) + w)] \geqq -\rho d^2(x, u)$$
$$\Rightarrow \quad f(x) + x^T w - f(u) - u^T w - h(u, p) + p^T \nabla_p h(u, p) \geqq 0,$$

and

$$F[x,u;\alpha(x,u)(\nabla g(u)+\nabla_p k(u,p))] \geqq -\rho d^2(x,u)$$
$$\Rightarrow \quad -g(u)-k(u,p)+p^T\nabla_p k(u,p) \geqq 0.$$

4.3 Mond-Weir type duality

In this section, we propose the following dual problem **(NMMD)** to **(NMP)**:
(NMMD)

$$\begin{aligned}
\text{Maximize} \quad & f(u)+u^Tw+(\lambda^T h(u,p))e - p^T\nabla_p(\lambda^T h(u,p))e \\
\text{subject to} \quad & \lambda^T[\nabla f(u)+\nabla_p h(u,p)+w] = \nabla y^T g(u)+\nabla_p y^T k(u,p), && (4.1)\\
& g(u)+k(u,p)-p^T\nabla_p k(u,p) \in C_2^*, && (4.2)\\
& w_i \in D_i, \quad i=1,\dots,l, && (4.3)\\
& y\in C_2, \quad \lambda>0, \quad \lambda^T e = 1. && (4.4)
\end{aligned}$$

where

(i) $f: R^n \to R^l$ and $g: R^n \to R^m$ are differentiable functions.

(ii) C_1 and C_2 are closed convex cones in R^n and R^m with nonempty interiors, respectively.

(iii) C_1^* and C_2^* are polar cones of C_1 and C_2, respectively.

(iv) $e=(1,\dots,1)^T$ is a vector in R^l.

(v) $w_i (i=1,\dots,l)$ is a vector in R^n and $D_i (i=1,\dots,l)$ is a compact convex set in R^n, respectively.

(vi) $h: R^n\times R^n \to R^l$ and $k: R^n\times R^n\to R^m$ are differentiable functions; $\nabla_p h_j(u,p)$, $j=1,2,\dots,l$ and $\nabla_p y^T k(u,p)$ denote the $n\times 1$ gradient of h_j and y^Tk with respect to p, respectively.

Now we establish the duality theorems of **(NMP)** and **(NMMD)**.

Theorem 4.1 (Weak duality). Let x and (u,y,λ,w,p) be feasible solutions of (NMP) and

(NMMD), respectively. Assume that

$$\lambda^T[f(\cdot)+(\cdot)^T w] \text{ is higher order } (F,\rho^1,\alpha^1,d) \text{ type-I convex and } -y^T g(\cdot)$$
$$\text{is higher order } (F,\rho^2,\alpha^2,d) \text{ type-I convex with } \rho^1 \geqq 0 \text{ and } \rho^2 \geqq 0,$$

Then

$$f(x)+s(x|D) \nleq f(u)+u^T w+(\lambda^T h(u,p))e-p^T\nabla_p(\lambda^T h(u,p))e.$$

Proof. Assume to the contrary that

$$f(x)+s(x|D) \leq f(u)+u^T w+(\lambda^T h(u,p))e-p^T\nabla_p(\lambda^T h(u,p))e.$$

since $\lambda > 0$

$$\lambda^T[f(x)+s(x|D)] < \lambda^T[f(u)+u^T w]+\lambda^T h(u,p)-p^T\nabla_p\lambda^T h(u,p). \tag{4.5}$$

By the assumption, we have

$$\begin{aligned}&\lambda^T[f(x)+x^T w]-\lambda^T[f(u)+u^T w]-\lambda^T h(u,p)+p^T\nabla_p\lambda^T h(u,p)\\&\quad\geqq F[x,u;\alpha^1(x,u)(\nabla f(u)+\nabla_p\lambda^T h(u,p)+\lambda^T w)]+\rho^1 d^2(x,u).\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &\lambda^T[f(x)+x^T w]-\lambda^T[f(u)+u^T w]-\lambda^T h(u,p)+p^T\nabla_p\lambda^T h(u,p)\\&\geqq \alpha^1(x,u)F[x,u;(\nabla f(u)+\nabla_p\lambda^T h(u,p)+\lambda^T w)]+\rho^1 d^2(x,u).\end{aligned}$$

Multiplying both sides by $\alpha^2(x,u)$, we get

$$\begin{aligned}&\alpha^2(x,u)\{\lambda^T[f(x)+x^T w]-\lambda^T[f(u)+u^T w]-\lambda^T h(u,\ p)+p^T\nabla_p\lambda^T h(u,p)\}\\&\geqq \alpha^1(x,u)\alpha^2(x,u)F[x,u;(\nabla f(u)+\nabla_p\lambda^T h(u,p)+\lambda^T w)]+\alpha^2(x,u)\rho^1 d^2(x,u).\end{aligned} \tag{4.6}$$

and also

$$\begin{aligned}&y^T g(u)+y^T k(u,p)-p^T\nabla_p y^T k(u,p)\\&\quad\geqq F[x,u;\alpha^2(x,u)(-(\nabla y^T g(u)+\nabla_p y^T k(u,p)))]+\rho^2 d^2(x,u).\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &y^T g(u)+y^T k(u,p)-p^T\nabla_p y^T k(u,p)\\&\quad\geqq \alpha^2(x,u)F[x,u;(-(\nabla y^T g(u)+\nabla_p y^T k(u,p)))]+\rho^2 d^2(x,u).\end{aligned}$$

Multiplying both sides by $\alpha^1(x,u)$, we get

$$\alpha^1(x,u)\{y^T g(u) + y^T k(u,p) - p^T \nabla_p y^T k(u,p)\}$$
$$\geqq \alpha^1(x,u)\alpha^2(x,u)F[x,u;(-(\nabla y^T g(u) + \nabla_p y^T k(u,p)))] + \alpha^1(x,u)\rho^2 d^2(x,u). \tag{4.7}$$

Summing (4.6) and (4.7) and using sublinearity of $F(x,u;\cdot)$, we have

$$\alpha^2(x,u)\{\lambda^T[f(x)+x^T w] - \lambda^T[f(u)+u^T w] - \lambda^T h(u,p) + p^T\nabla_p\lambda^T h(u,p)\}$$
$$-\alpha^1(x,u)[-y^T g(u) - y^T k(u,p) + p^T \nabla_p y^T k(u,p)]$$
$$\geqq \alpha^1(x,u)\alpha^2(x,u)F[x,u;(\nabla f(u) + \nabla_p\lambda^T h(u,p) + \lambda^T w) - (\nabla y^T g(u) + \nabla_p y^T k(u,p))] +$$
$$\alpha^1(x,u)\rho^2 d^2(x,u) + \alpha^2(x,u)\rho^1 d^2(x,u).$$

By using (4.1), we get

$$\Rightarrow \alpha^2(x,u)\{\lambda^T[f(x)+x^T w] - \lambda^T[f(u)+u^T w] - \lambda^T h(u,p) + p^T\nabla_p\lambda^T h(u,p)\}$$
$$\geqq \alpha^1(x,u)[-y^T g(u) - y^T k(u,p) + p^T \nabla_p y^T k(u,p)] +$$
$$d^2(x,u)[\alpha^1(x,u)\rho^2 + \alpha^2(x,u)\rho^1].$$

Since $-y^T[g(u)+k(u,p)-p^T\nabla_p k(u,p)] \geqq 0$, we have

$$\Rightarrow \alpha^2(x,u)\{\lambda^T[f(x)+x^T w] - \lambda^T[f(u)+u^T w] - \lambda^T h(u,p) + p^T\nabla_p\lambda^T h(u,p)\}$$
$$\geqq d^2(x,\ u)[\alpha^1(x,u)\rho^2 + \alpha^2(x,u)\rho^1].$$
$$\Rightarrow \alpha^2(x,u)\{\lambda^T[f(x)+x^T w] - \lambda^T[f(u)+u^T w] - \lambda^T h(u,p) + p^T\nabla_p\lambda^T h(u,p)\} \geqq 0$$
$$\Rightarrow \lambda^T[f(x)+x^T w] - \lambda^T[f(u)+u^T w] - \lambda^T h(u,p) + p^T\nabla_p\lambda^T h(u,p) \geqq 0.$$

Using the fact that $s(x|D) \geqq x^T w$, we get

$$\lambda^T[f(x)+s(x|D)] - \lambda^T[f(u)+u^T w] - \lambda^T h(u,\ p) + p^T\nabla_p\lambda^T h(u,p)\} \geqq 0.$$
$$\lambda^T[f(x)+s(x|D)] \geqq \lambda^T[f(u)+u^T w] + \lambda^T h(u,p) - p^T\nabla_p\lambda^T h(u,p)\}.$$

which contradicts to (4.5).

Lemma 4.1 if $\bar{x}$ is an efficient solution of (NMP) at which constraint qualification [19] be satisfied. Then there exists $\bar{w}_i \in D_i (i=1,\ldots,l)$, $\bar{\lambda} > 0$ and $\bar{y} \in C_2$ with $(\bar{\lambda},\bar{y}) \neq 0$ such that

$$[\bar{\lambda}^T(\nabla f(\bar{x}) + \bar{w}) - \nabla \bar{y}^T g(\bar{x})^T](x-\bar{x}) \geqq 0, \quad \text{for all } x \in C_1,$$
$$\bar{y}^T g(\bar{x}) = 0,$$
$$\bar{w}_i \in D_i, \quad s(\bar{x}|D_i) = \bar{x}^T \bar{w}_i, \quad i=1,\ldots,l.$$

Theorem 4.2 (Strong Duality). If $\bar{x}$ is an efficient solution of (NMP) at which constraint qualification [19] be satisfied. Let

$$h(\bar{x},0)=0, \qquad k(\bar{x},0)=0, \qquad \nabla_p h(\bar{x},0)=\nabla f(\bar{x}), \qquad \nabla_p k(\bar{x},0)=\nabla g(\bar{x}). \tag{4.8}$$

Then there exists $\bar{\lambda}>0$, $\bar{y}\in C_2$ and $\bar{w}_i \in D_i (i=1,\dots,l)$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)$ is feasible for (NMMD) and the objective values of (NMP) and (NMMD) are equal. If the assumption of Theorem 4.1 is satisfied, then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)$ is an efficient solution of (NMMD).

Proof. Since $\bar{x}$ is an efficient solution of (NMP), by lemma 4.1, then there exist $\bar{w}_i\in D_i, i=1,\dots,l$, $\bar{\lambda}>0$ and $\bar{y}\in C_2$ with $(\bar{\lambda},\bar{y})\neq 0$ such that

$$[\bar{\lambda}^T(\nabla f(\bar{x})+\bar{w})-\bar{y}^T\nabla g(\bar{x})]^T(x-\bar{x}) \geqq 0, \qquad \text{for all } x\in C_1, \tag{4.9}$$

$$\bar{y}^T g(\bar{x})=0, \tag{4.10}$$

$$s(\bar{x}|D_i)=\bar{x}^T\bar{w}_i, \qquad i=1,\dots,l. \tag{4.11}$$

Since $x\in C_1$, $\bar{x}\in C_1$ and C_1 is a closed convex cone, we have $x+\bar{x}\in C_1$ and thus the inequality (4.9) implies

$$(\bar{\lambda}^T(\nabla f(\bar{x})+\bar{w})-\bar{y}^T\nabla g(\bar{x})]^T x \geqq 0, \qquad \text{for all } x\in C_1,$$

$$i.e., \ \ \bar{\lambda}^T(\nabla f(\bar{x})+\bar{w})-\nabla\bar{y}^T g(\bar{x})=0.$$

And (4.10) implies $\bar{y}^T g(\bar{x})\leqq 0$, then $g(\bar{x})\in C_2^*$. Clearly, using (4.8) and (4.11), $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)$ is feasible for (NMMD) and corresponding values of (NMP) and (NMMD) are equal. If the assumption of Theorem 4.1 is satisfied , then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)$ is an efficient solution of (NMMD).

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